Minimum Covariance Bounds for the Fusion under Unknown Correlations

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***Abstract*—One of the key challenges in distributed linear estimation is the systematic fusion of estimates. While the fusion gains that minimize the mean squared error of the fused estimate for known correlations have been established, no analogous statement could be obtained so far for unknown correlations. In this contribution, we derive the gains that minimize the bound on the true covariance of the fused estimate and prove that Covariance Intersection (CI) is the optimal bounding algorithm for two estimates under completely unknown correlations. When combining three or more variables, the CI equations are not necessarily optimal, as shown by a counterexample.**

***Index Terms*—Data Fusion, Distributed Estimation, Kalman filtering, Covariance Intersection**

I. Introduction

In decentralized target tracking, spatially distributed nodes maintain local estimates of the same or overlapping states. When nodes communicate with each other, information is exchanged and estimates are systematically fused. Quite often, especially in linear estimation, the quality of point estimates is assessed by means of covariances and hence, the objective is to derive fusion algorithms that minimize a cost function of these covariances. Depending on communication structure and processing type of the nodes, estimates are combined pairwise or several estimates are collected and fused batchwise.

For (exactly) known correlations, the linear gains minimizing the mean squared error have been derived for two [1] and arbitrary many [24] estimates. However, in the considered tracking scenario with distributed nodes, correlations emerge between estimates due to past data exchanges and common process noise [1]. The evolution of these cross-covariance matrices depends on filter and fusion transformations of remote nodes [16], which are typically only known locally. To the authors' knowledge, even for linear systems with white noise that are observed by two sensors, the distributed calculation of cross-covariance matrices requires to store process noise covariances separately, resulting in an ever-increasing number of terms until the estimates are fused.

Hence, different strategies have been pursued to cope with unavailable cross-covariance matrices. A simple technique is to ignore the correlations, as it is proposed for the simple convex combination [4]. Consensus [19] and diffusion [2] approaches optimize weights according to sensor network parameters. Alternatively, the lack of knowledge about the correlations can be explicitly modeled. As the covariance of the fused estimate varies with the (unknown) underlying cross-covariance matrices between the estimates, all permissible cross-covariance matrices must be considered, which, in turn, leads to a set of possible covariances for the fused estimate.

The fusion under unknown correlations was first considered with Covariance Intersection (CI) [12]. Since then, a variety of methods has been proposed [3], [5], [8], [13], [18], [21], [25], and the techniques have been applied, e.g., to distributed estimation [10]. The key feature of these approaches is to provide a covariance bound, i.e., a covariance which overestimates the true covariance of the fused estimate and thus, allows to pursue consistent estimation without processing cross-covariance matrices [25].

Techniques that aim at reducing the computational effort of CI have been discussed in [6], [17], [26]. As covariance bounds by definition are significantly larger than covariances provided by the fusion under known correlations [1], [24], more general approaches have been derived that permit to shrink bounds by including additional information in the fusion process. One way is to assume that the local errors consist of two independent parts and correlations between one of the parts is exactly known [8], [18] or zero [13]. Alternatively, possible correlations are bounded by means of a scalar factor [7], [21]. If the lack of knowledge about cross-covariance matrices can be modeled by means of additive norm-bounded terms, the linear combination that provides the minimal worst-case bound on the mean squared error is obtained as the solution of a semidefinite programming problem [20].

Recently, an alternative to CI, termed Ellipsoidal Intersection, has been presented [23], which provides smaller covariances than the bounds obtained with CI. However, although simulations justify the use of Ellipsoidal Intersection, a consequence of the results from this paper is that the covariances obtained with Ellipsoidal Intersection underestimate the true error for some cross-covariance matrices and thus, the obtained estimates are inconsistent.

In this contribution, we derive the fusion gains for two estimates that minimize the covariance bound of the fused estimate under unknown correlations subject to a whole class of cost functions – including trace and determinant. As it turns out, the optimal gains are given by CI and therefore, we prove that CI is the optimal bounding technique for two estimates under completely unknown correlations.

Although statements concerning the tightness of CI in the joint space have been made before [25], optimality of the fusion result could not be proven so far. The reason is that positive definite matrices feature inner dependencies between entries such that the set of possible joint covariance matrices exhibits a complicated structure [11]. Hence, checking all possible fusion outcomes for arbitrary gains and providing optimal bounds analytically is not feasible, and the alternative, i.e., finding a bound on possible covariances in the joint space, is not guaranteed to provide the optimal result in the fused

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space. Chen et. al. [3] focused on a family of scalar inflated covariances and showed that the optimal gains in this family are given by CI. However, the proof is based on a specific trace structure of the covariance of the fused estimate, which is not satisfied for arbitrary linear combinations. Hence, optimality holds only within the considered family of scalar inflated covariances.

In our proof, the complicated set of possible joint covariance matrices is shown to define a tractable necessary condition for bounds of the fused estimate. By means of a result from set theory, the necessary condition can be formulated in terms of ellipsoids. Only because the bounds obtained with CI satisfy this necessary condition, i.e., they define ellipsoids that tightly circumscribe the intersection of ellipsoids, we are able to prove optimality. Unfortunately, for more than two estimates, CI provides larger bounds than those defined by the necessary condition as demonstrated in a counterexample and thus, the proof does not generalize to more than two estimates.

II. Problem Formulation

Consider two unbiased estimates $\hat{\underline{\boldsymbol{x}}}_i \in \mathbb{R}^n$ of a common state $\underline{\boldsymbol{x}} \in \mathbb{R}^n$ with covariances $\mathbf{P}_i$, $i \in \{1,2\}$ and cross-covariance matrix $\mathbf{P}_{12}$. Throughout the paper, let $\mathbf{P}_1$ and $\mathbf{P}_2$ be positive definite and let $J(\cdot)$ denote an arbitrary strictly monotonically increasing cost function, i.e., it satisfies $\mathbf{P}_1 > \mathbf{P}_2 \Rightarrow J(\mathbf{P}_1) > J(\mathbf{P}_2)$, such as trace or determinant.

The unbiased linear combination[1] of two estimates in the Kalman filter framework boils down to finding gains $\mathbf{K}_i$ such that the fused estimate $\hat{\underline{\boldsymbol{x}}}_c = \mathbf{K}_1\hat{\underline{\boldsymbol{x}}}_1 + \mathbf{K}_2\hat{\underline{\boldsymbol{x}}}_2$ is optimized according to a cost function of the covariance $J(\mathbf{P}_c)$, i.e.,

$$\arg\min_{\mathbf{K}_1,\mathbf{K}_2} J(\mathbf{P}_c) \ , \tag{1}$$

where the covariance of the fused estimate is given as $\mathbf{P}_c = \mathrm{E}\{(\hat{\underline{\boldsymbol{x}}}_c - \underline{\boldsymbol{x}})(\hat{\underline{\boldsymbol{x}}}_c - \underline{\boldsymbol{x}})^\top\} = \mathbf{K}\mathbf{P}(\mathbf{K})^\top$ with $\mathbf{K} = \begin{pmatrix}\mathbf{K}_1 & \mathbf{K}_2\end{pmatrix}$ and joint covariance matrix $\mathbf{P} = \begin{pmatrix}\mathbf{P}_1 & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_2\end{pmatrix}$. If the cross-covariance matrix $\mathbf{P}_{12}$ is known, the solution to this problem is given by the Bar-Shalom/Campo formulas [1].

In this paper, we seek to find fusion gains when $\mathbf{P}_{12}$ is unknown to the fuser. Indeed, the covariance of the fused estimate $\mathbf{P}_c$ depends on the underlying true but unknown $\mathbf{P}_{12}$ and therefore, cannot be calculated. However, the possible cross-covariance matrices are bounded [11], which, in turn, restricts the possible outcomes of $\mathbf{P}_c$ to a bounded set. It has already been shown that consistent estimation is feasible, if a covariance bound $\mathbf{B}_c$ with $\mathbf{B}_c \geq \mathbf{P}_c$ is provided as a substitute for the unknown true covariance $\mathbf{P}_c$ [25], where $\geq$ is to be understood in the positive semi-definite sense. Therefore, the equivalent to the optimization (1) for the fusion under unknown correlations is given by

$$\arg\min_{\mathbf{K}_1,\mathbf{K}_2,\mathbf{B}_c} J(\mathbf{B}_c) \ \text{ with } \mathbf{B}_c \geq \mathbf{P}_c \text{ for all possible } \mathbf{P}_{12} \ . \tag{2}$$

[1] Let $\hat{\underline{\boldsymbol{x}}}$ denote a biased estimator with $\mathrm{E}\{\hat{\underline{\boldsymbol{x}}}\} = \mathrm{E}\{\underline{\boldsymbol{x}}\} + \underline{b}$. Then, $\mathrm{E}\{(\hat{\underline{\boldsymbol{x}}} - \underline{\boldsymbol{x}})(\hat{\underline{\boldsymbol{x}}} - \underline{\boldsymbol{x}})^\top\} = \mathbf{P} + \underline{b}\underline{b}^\top$, where $\mathbf{P}$ denotes the covariance of the unbiased estimator $\hat{\underline{\boldsymbol{x}}} - \underline{b}$ and $\underline{b}\underline{b}^\top$ is a positive (semi-)definite matrix. Therefore, biased linear combinations with $\mathbf{K}_1 + \mathbf{K}_2 \neq \mathbf{I}$ yield estimates with a higher MSE than their unbiased counterparts and are therefore not considered in this contribution.

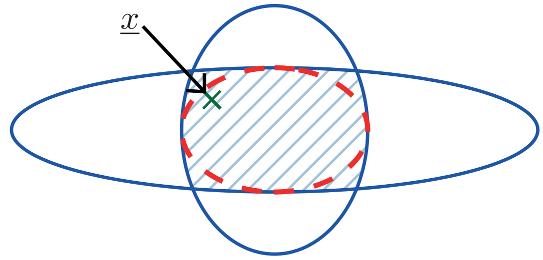

Figure 1. Two centered ellipses and their intersection in shaded light blue. The dashed red ellipse depicts the covariance of the fused estimate from Lemma 1 for $\mathbf{P}_{12} = \mathbf{0}$. The green cross depicts an arbitrary point $\underline{x}$ from the intersection.

In this contribution we solve (2).

III. Minimal Covariance Bounds

In the following, statements and properties from linear fusion theory, set theory, and bounding theory are utilized, which will be stated before the main theorem. The connection between set theory and estimation theory is established by means of centered (multidimensional) ellipsoids

$$\mathcal{E} = \{\underline{x}\,|\,\underline{x}^\top\mathbf{P}^{-1}\underline{x} \leq 1,\ \underline{x} \in \mathbb{R}^n\} \ ,$$

which are utilized as the geometric counterpart of positive definite covariances $\mathbf{P}$. These ellipsoids are in particular useful to illustrate positive definite relations as $\mathbf{P}_1 \leq \mathbf{P}_2 \Leftrightarrow \mathcal{E}_1 \subseteq \mathcal{E}_2$ [5]. The lemmata and statements are already informally motivated and give a rough structure of the final proof. For clarity, we prove them in the Appendix.

The optimal fusion gains and covariances under known correlations, i.e., the solution to (1), are well known in literature as the Bar-Shalom/Campo formulas [1].

Lemma 1 *Let*

$$\mathbf{K}_1^* = (\mathbf{P}_2 - \mathbf{P}_{21})(\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1}$$

and $\mathbf{K}_2^* = \mathbf{I} - \mathbf{K}_1^*$*, then* $\hat{\underline{\boldsymbol{x}}}_c^* = \mathbf{K}_1^*\hat{\underline{\boldsymbol{x}}}_1 + \mathbf{K}_2^*\hat{\underline{\boldsymbol{x}}}_2$ *with covariance* $\mathbf{P}_c^* = \mathbf{P}_1 - \mathbf{K}_2^*(\mathbf{P}_1 - \mathbf{P}_{12})^\top$ *is the solution to* (1) *with* $\mathbf{P}_c^* \leq \mathbf{P}_c$ *for covariances* $\mathbf{P}_c$ *of any other linear combination.*

As it becomes apparent in the formulas of Lemma 1, the gains as well as the covariance of the fused estimate depend on the cross-covariance matrix $\mathbf{P}_{12}$, amounting to different combination rules subject to different cross-covariance matrices. An example that illustrates Lemma 1 is given in Fig. 1.

By means of $\mathbf{P}_1$ and $\mathbf{P}_2$, the set of possible cross-covariance matrices and thus, the set of covariances that result from the optimal fusion under known correlations, can be bounded.

Lemma 2 *Let* $\mathcal{E}_1, \mathcal{E}_2$ *denote the ellipsoids for* $\mathbf{P}_1$ *and* $\mathbf{P}_2$*, respectively. It holds*

$$\underline{x} \in \mathcal{E}_1 \cap \mathcal{E}_2 \Leftrightarrow \text{there is a valid } \mathbf{P}_{12} \text{ with } \underline{x} \in \mathcal{E}_c^* \ ,$$

where $\mathcal{E}_c^*$ *denotes the ellipsoid for* $\mathbf{P}_c^*$ *from Lemma 1.*

Hence, for all points $\underline{x}$, there is a (possible) cross-covariance matrix $\mathbf{P}_{12}$ such that the covariance from Lemma 1 defines an ellipsoid that contains it. Vice versa, all ellipsoids from Lemma 1 are contained in the intersection of the input ellipsoids. This relation is depicted in Fig. 1.

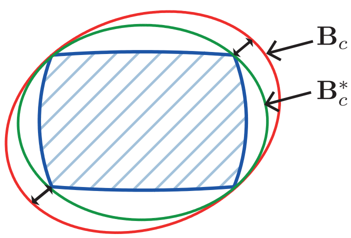

Figure 2. The intersection of two centered ellipses in shaded light blue. If a bound $\mathbf{B}_c$ is not tight, a smaller bound $\mathbf{B}_c^*$ from the set defined in Theorem 3 with $\mathbf{B}_c^* \leq \mathbf{B}_c$ can be found.

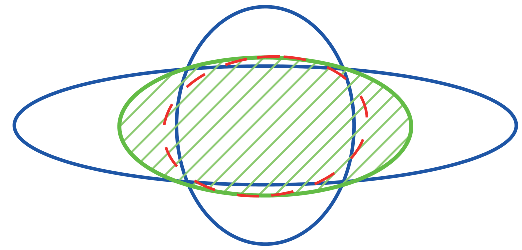

Figure 3. The ellipse of a bound of the fused estimate in green. The ellipses for specific cross-covariance matrices, e.g., the one in dashed red, are enclosed by the ellipse of the bound.

Note that $\mathcal{E}_1 \cap \mathcal{E}_2$ defines a set of covariances that are obtained with different gains, which, in turn, are individually optimized with respect to the corresponding known cross-covariance matrices. A fusion algorithm that solves (2) must choose a specific pair of gains – irrespective of the true cross-covariance matrix. Therefore, bounding the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ is only a necessary but not a sufficient condition to guarantee that the true covariance of the fused estimate is bounded. Still, finding the best representative from the set of ellipsoids that circumscribes the intersection of two centered ellipsoids is a known problem from set theory. An illustration of the problem is given in Fig. 2. The solution has been derived by Kahan [14], [15].

Theorem 3 *Let $\mathcal{E}_1, \mathcal{E}_2$, $\mathcal{E}_c^*$ denote the centered ellipsoids of covariances $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{B}_c^*$, respectively. When $\mathcal{E}_c^*$ tightly circumscribes the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$, i.e., $\mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \mathcal{E}_c \subseteq \mathcal{E}_c^* \Rightarrow \mathcal{E}_c = \mathcal{E}_c^*$ for an arbitrary ellipsoid $\mathcal{E}_c$, then*

$$(\mathbf{B}_c^*)^{-1} = \omega \mathbf{P}_1^{-1} + (1-\omega)\mathbf{P}_2^{-1} \ , \omega \in [0,1] \ . \tag{3}$$

Let $\mathbf{K}$ and $\mathbf{P}$ denote the joint space matrix vectors and covariances from the problem formulation. Eventually, the challenge is to derive fusion gains such that the true covariance of the fused estimate $\mathbf{P}_c = \mathbf{K}\mathbf{P}(\mathbf{K})^\top$ is bounded by $\mathbf{B}_c^*$ for all cross-covariance matrices $\mathbf{P}_{12}$. To this end, we note that joint space bounds imply bounds on the fused estimate.

Lemma 4 *Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{R}^{n \times n}$ denote fusion gains and let*

$$\begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix} \geq \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_2 \end{pmatrix}$$

denote a bound on the true joint covariance matrix. Then, $\mathbf{B}_c = \mathbf{K}_1\mathbf{B}_1(\mathbf{K}_1)^\top + \mathbf{K}_2\mathbf{B}_2(\mathbf{K}_2)^\top$ is a bound on the covariance of the fused estimate $\hat{\underline{\boldsymbol{x}}}_c = \mathbf{K}_1\hat{\underline{\boldsymbol{x}}}_1 + \mathbf{K}_2\hat{\underline{\boldsymbol{x}}}_2$.

Note that the other direction does not hold, i.e., a bound on the covariance of the fused estimate does not imply a bound in the joint space in general. In order to derive the optimal solution to (2), it is not even sufficient to find a tight bound in the joint space. However, when a joint space bound with appropriate fusion gains yields the ellipsoids from Theorem 3, which define a necessary size of the bound, the result is optimal.

Lemma 5 *Let $\omega \in (0,1)$, then*

$$\begin{pmatrix} \frac{1}{\omega}\mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \frac{1}{1-\omega}\mathbf{P}_2 \end{pmatrix} \geq \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_2 \end{pmatrix} \ .$$

Combining the joint space bounds from Lemma 5 with appropriate fusion gains results in our main theorem.

Theorem 6 (Optimal Covariance Bounding) *Let $\mathbf{B}_c$ denote a bound obtained with arbitrary fusion gains and let*

$$\mathbf{K}_1^* = \omega\mathbf{B}_c^*\mathbf{P}_1^{-1} \text{ and } \mathbf{K}_2^* = (1-\omega)\mathbf{B}_c^*\mathbf{P}_2^{-1} \tag{4}$$

denote specific fusion gains with $\mathbf{B}_c^*$ from (3). Then, $\mathbf{B}_c^*$ defines a bound on the fused estimate and $\mathbf{B}_c \leq \mathbf{B}_c^*$ implies $\mathbf{B}_c = \mathbf{B}_c^*$. The solution to (2) is given by (3) and (4) with

$$\omega^* = \arg\min_{\omega} J\left(\mathbf{B}_c^*\right) \ . \tag{5}$$

PROOF. First we note that for each cross-covariance matrix $\mathbf{P}_{12}$, the covariance of the optimally fused estimates $\mathbf{P}_c^*$ is given by Lemma 1. As the optimality holds in the positive definite sense, the combination of estimates by means of any other gains yields a covariance $\mathbf{P}_c$ that is larger in the positive definite sense. In other words, the ellipsoid $\mathcal{E}_c^*$ of $\mathbf{P}_c^*$ is contained in the ellipsoid $\mathcal{E}_c$ of $\mathbf{P}_c$, i.e., $\mathcal{E}_c^* \subseteq \mathcal{E}_c$.

Hence, a necessary (but not sufficient!) condition is that a covariance bound $\mathbf{B}_c$ must be larger than $\mathbf{P}_c^*$ for all possible cross-covariance matrices in order to guarantee that $\mathbf{B}_c \geq \mathbf{P}_c \geq \mathbf{P}_c^*$, where $\mathbf{P}_c$ is the covariance of the fused estimate subject to the gains used in (2). According to Lemma 2, the set of optimal covariances for all possible cross-covariance matrices is depicted by the ellipsoidal intersection $\mathcal{E}_1 \cap \mathcal{E}_2$, where $\mathcal{E}_i$ is the ellipsoid of covariance $\mathbf{P}_i$, $i \in \{1,2\}$. From $\mathbf{P}_c^* \leq \mathbf{B}_c$, it follows that the ellipsoid which depicts the optimal bound $\mathbf{B}_c^*$ must contain the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$.

According to Theorem 3, for all bounds $\mathbf{B}_c$ not described by $(\omega\mathbf{P}_1^{-1} + (1-\omega)\mathbf{P}_2^{-1})^{-1}$, $\omega \in [0,1]$, a smaller covariance with $\mathbf{B}_c^* < \mathbf{B}_c$ can be derived. Hence, no bounds obtained with arbitrary fusion gains can be smaller than $\mathbf{B}_c^*$ in the positive definite sense. In particular, if gains can be found so that the true covariances are bounded by $\mathbf{B}_c^*$, it is a consequence of the strict monotonicity of the cost function that $J\left(\mathbf{B}_c^*\right) < J\left(\mathbf{B}_c\right)$ and thus, the solution to (2) is found within these gains and bounds.

Let $\omega \in (0,1)$ be fixed. The inflated covariance from Lemma 5 combined with the gains (4) yield

$$\omega\mathbf{B}_c^*\mathbf{P}_1^{-1}(\mathbf{B}_c^*)^\top + (1-\omega)\mathbf{B}_c^*\mathbf{P}_2^{-1}(\mathbf{B}_c^*)^\top = \mathbf{B}_c^* \ .$$

According to Lemma 4, the true covariance of the fused estimate with gains (4) is smaller than $\mathbf{B}_c^*$. Therefore, $\mathbf{B}_c^*$ specifies a consistent bound. For $\omega \in \{0,1\}$, one of the gains is zero and the fusion result corresponds to a prior estimate, which is bounded by the corresponding covariance trivially.

As discussed above, the solution to (2) is found within the gains (4) and bounds (3), and is therefore given by the optimization (5). □

Indeed, fusion gains and bound of Theorem 6 correspond to the CI formulas. An implication of this result is that algorithms that provide smaller covariances than CI and operate under unknown correlations cannot satisfy $\mathbf{B}_c \geq \mathrm{E}\{\underline{\hat{e}}_c(\underline{\hat{e}}_c)^\top\}$.

Note that the fusion gains are not only calculated without knowledge of the cross-covariance matrices but are also the same for all possible cross-covariance matrices. Hence, it seems as if the bound on the fused estimate should be much larger than the set $\mathcal{E}_1 \cap \mathcal{E}_2$ that is obtained based on individual optimizations considering known cross-covariance matrices. Although the covariance of the fused estimate is indeed worse than the theoretic optimum under known correlations, the true covariance is still bounded by the smallest ellipsoid enclosing the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ as it is depicted in Fig. 3.

Moreover, the result raises the question whether the natural generalization of CI to more than two estimates satisfies similar optimality properties. For N estimates, it has been proposed, e.g., in [25], to inflate covariances $\mathbf{P}_i$, $i = 1, \ldots, N$ with scalar factors $\frac{1}{\omega_i}$ so that $\sum_{i=1}^N \omega_i = 1$ is retained. Utilizing appropriate gains, the covariance $\mathbf{B}_c^{-1} = \sum_{i=1}^N \omega_i \mathbf{P}_i^{-1}$ is obtained as a bound on the fused estimate. However, consider the covariances

$$\mathbf{P}_1^{-1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 4.1 \end{pmatrix}, \mathbf{P}_2^{-1} = \begin{pmatrix} 3.1 & \sqrt{3} \\ \sqrt{3} & 1.1 \end{pmatrix}, \mathbf{P}_3^{-1} = \begin{pmatrix} 3.1 & -\sqrt{3} \\ -\sqrt{3} & 1.1 \end{pmatrix},$$

with (almost) ribbon shaped ellipses as discussed in Example 1 in [15]. Then, the trace minimization of $\mathbf{B}_c$ leads to a circle with radius ≈ 0.69. Indeed, as depicted in Fig. 4, the intersection of the three ellipses, i.e., the hexagon in the center, is circumscribed by a circle with radius ≈ 0.58, which is strictly smaller. As the ellipses obtained by the optimal fusion [24] lie within this hexagon, an optimality proof for the generalization must be conceptually different from the one proposed in this paper. In fact, the counterexample even suggests that there may exist linear combinations of more than two estimates under unknown correlations that yield a smaller bound than the CI generalization.

IV. Conclusion

In this contribution, we proved that covariance intersection (CI) provides the optimal bound in the fusion of two estimates under unknown correlations subject to strictly monotonically increasing cost functions.

A generalization of the procedure to more than two estimates is still an open research question. In particular, a statement about the tightness of ellipsoids for the intersection of more than two centered ellipsoids has not been provided yet [15].

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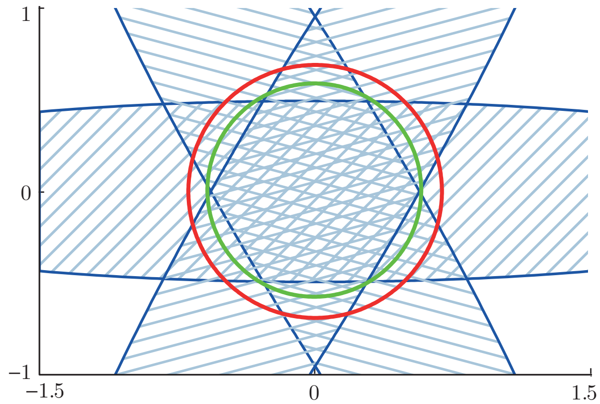

Figure 4. The illustration of the slightly adapted Example 1 from [15]. The covariances $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ are depicted in blue, the bound obtained by the natural CI generalization for more than two estimates in red, and the (optimal) tight bound in green.

Appendix

Proof of Lemma 1. As a bias in the fusion gains leads to a positive definite residual and thus, suboptimal fusion gains, we confine our attention to unbiased combinations with $\mathbf{K}_2 = \mathbf{I} - \mathbf{K}_1$. Then, $\mathbf{K}\mathbf{P}(\mathbf{K})^\top = \mathbf{P}_2 + \mathbf{K}_1\mathbf{A}(\mathbf{K}_1)^\top - \mathbf{K}_1(\mathbf{B})^\top - \mathbf{B}(\mathbf{K}_1)^\top = (\mathbf{K}_1 - \mathbf{B}\mathbf{A}^{-1})\mathbf{A}(\mathbf{K}_1 - \mathbf{B}\mathbf{A}^{-1})^\top + \mathbf{P}_2 - \mathbf{B}\mathbf{A}^{-1}(\mathbf{B})^\top$ where $\mathbf{A} = \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{12} - \mathbf{P}_{21}$ and $\mathbf{B} = \mathbf{P}_2 - \mathbf{P}_{12}$. As $(\mathbf{K}_1 - \mathbf{B}\mathbf{A}^{-1})\mathbf{A}(\mathbf{K}_1 - \mathbf{B}\mathbf{A}^{-1})^\top \geq \mathbf{0}$, the covariance is minimized in the positive definite sense by $\mathbf{K}_1 = \mathbf{B}\mathbf{A}^{-1}$. □

Proof of Lemma 2. $\Leftarrow$: As according to Lemma 1, the fused covariance for known $\mathbf{P}_{12}$ is $\mathbf{P}_c^* = \mathbf{P}_1 - \mathbf{A}_1$ with $\mathbf{A}_1 = (\mathbf{P}_1 - \mathbf{P}_{12})(\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{12} - \mathbf{P}_{21})^{-1}(\mathbf{P}_1 - \mathbf{P}_{12})^\top \geq \mathbf{0}$, $\mathbf{P}_c^* \leq \mathbf{P}_1$. An analogous derivation proves $\mathbf{P}_c^* \leq \mathbf{P}_2$.
$\Rightarrow$: statement **(2)** in [3]. □

Proof of Lemma 4. A result from linear algebra states that $\mathbf{B} \geq \mathbf{P} \Rightarrow \mathbf{K}\mathbf{B}(\mathbf{K})^\top \geq \mathbf{K}\mathbf{P}(\mathbf{K})^\top$ for $\mathbf{K} \in \mathbb{R}^{n \times m}$ with $m \leq n$ [9]. Let $\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & \mathbf{K}_2 \end{pmatrix}$, then the joint matrix inequality implies

$$\mathbf{K} \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix} (\mathbf{K})^\top \geq \mathbf{K} \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_2 \end{pmatrix} (\mathbf{K})^\top .$$

As the left hand side amounts to $\mathcal{E}_c$ and the right hand side of the inequality denotes the true covariance of the fused estimate $\underline{\hat{\boldsymbol{x}}}_c$, the claims follows. □

Proof of Lemma 5. The inequality is equivalent to

$$\begin{pmatrix} \frac{1}{\omega}\mathbf{P}_1 - \mathbf{P}_1 & -\mathbf{P}_{12} \\ -\mathbf{P}_{21} & \frac{1}{1-\omega}\mathbf{P}_2 - \mathbf{P}_2 \end{pmatrix} \geq \mathbf{0} .$$

According to Theorem 7.7.3 from [9] in combination with the exercise following Theorem 7.7.6, this inequality is satisfied for positive definite $\mathbf{P}_1$ and $\mathbf{P}_2$ if and only if

$$\frac{\omega}{1-\omega}\mathbf{P}_2 \geq \mathbf{P}_{21}\left(\frac{1-\omega}{\omega}\mathbf{P}_1\right)^{-1}\mathbf{P}_{12} \Leftrightarrow \mathbf{P}_2 \geq \mathbf{P}_{21}\mathbf{P}_1^{-1}\mathbf{P}_{12}$$

which, in turn, proves the lemma for positive semi-definite joint covariance matrices $\mathbf{P}$. Note that this result was originally proven for ellipsoids in set theory [22]. □

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