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# The Griffith relation – a historical review

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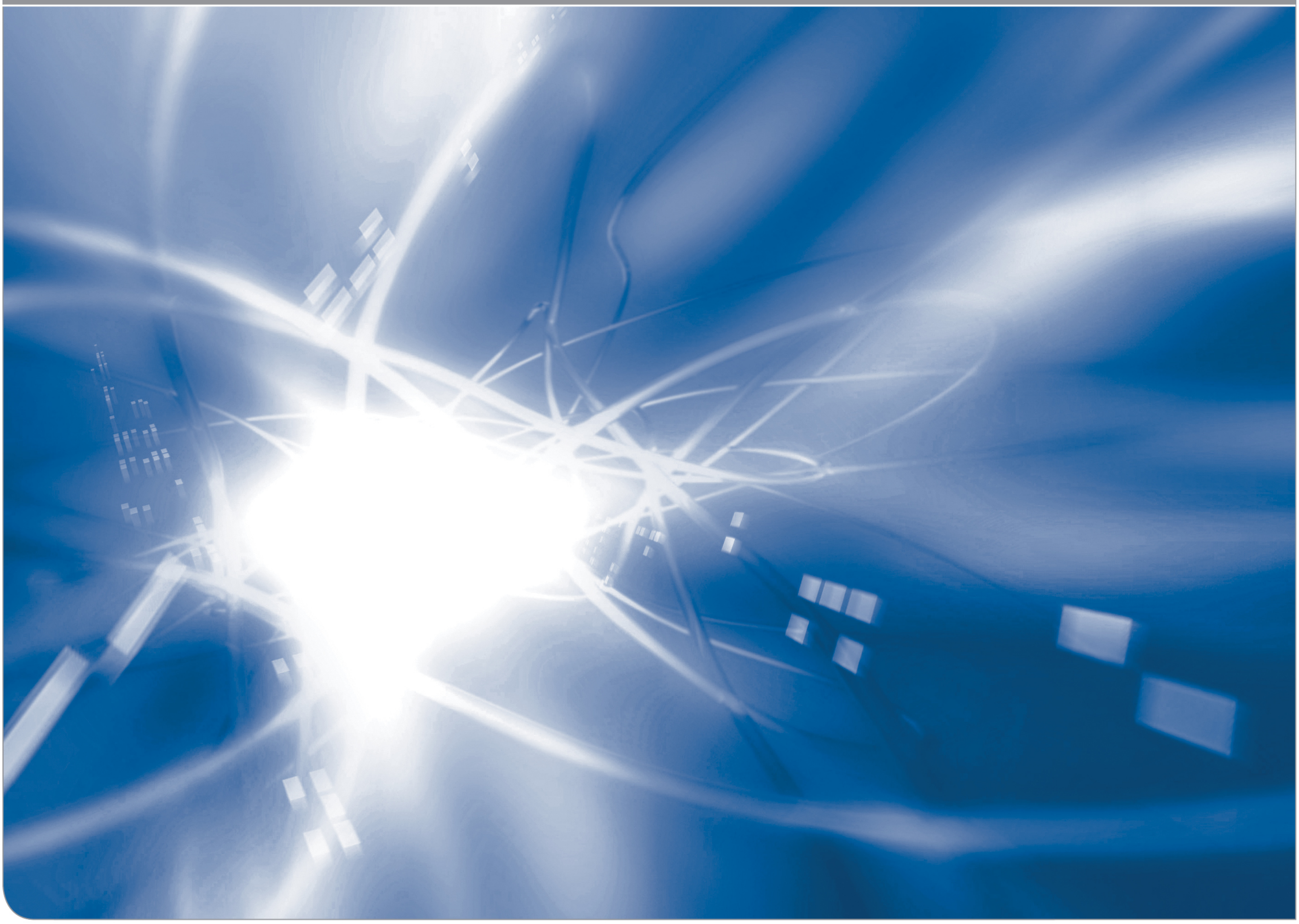

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## Abstract

The equation between the fracture strength and the crack size published by Griffith in his famous paper in 1921 is not correct. Griffith corrected the equation in a conference volume in 1924 without a sufficient explanation. Afterwards several authors published papers with correct and not correct derivation of the Griffith relation. These papers are critically reviewed and the flaw in the first paper of Griffith is analysed.

## Contents

## Nomenclature

| | |
|---|---|
| $a$ | long semi-axis of an ellipse |
| $b$ | short semi-axis of an ellipse |
| $c$ | crack length |
| $E$ | Young's modulus |
| $F$ | area, stress function used by Spencer |
| $G$ | energy release rate |
| $h$ | abbreviation defined by eq.(7), plate thickness |
| $K_{Ic}$ | fracture toughness |
| $M_F$ | Folias factor |
| $p$ | pressure |
| $P$ | periphery at which externally load is applied |
| $r$ | radius (polar coordinates) |
| $R_i$ | inner tube radius |
| $t$ | tube wall thickness |
| $u$ | displacement |
| $U$ | total surface energy, system energy |
| $w$ | crack opening displacement |
| $W$ | strain energy |
| $Z$ | Westergaard stress function |
| $x$ | Cartesian coordinate |
| $y$ | Cartesian coordinate |

| | |
|---|---|
| $\alpha$ | curvilinear coordinate |
| $\beta$ | curvilinear coordinate |
| $\gamma$ | specific surface energy |
| $\varepsilon$ | strain, biaxiality ratio of applied stresses, maximum crack opening displacement |
| $\varphi$ | angle |
| $\kappa$ | $3-4\nu$ (plane strain), $(3-\nu)/(1+\nu)$ (plane stress) |
| $\lambda$ | abbreviation defined by eq.(73) |
| $\theta$ | polar angle |
| $\mu$ | shear modulus |
| $\nu$ | Poisson's ratio |
| $\rho$ | radius at ellipse end |
| $\sigma$ | stresses (normal and tangential) |
| $\sigma_\infty$ | remote stress |
| $\sigma_c$ | strength |
| $\tau$ | shear stress |

## 1. Introduction

In 1921[1], Griffith [1] published his seminal paper about fracture of solids in which he developed a relation between the fracture stress and the size of a flaw in the material. The fundamental idea was that the energy necessary to increase the crack area is obtained from a change of the strain energy during crack extension. In a second paper [2] published in1924 Griffith revised his relation for the fracture stress $\sigma_c$ of an infinite plate with an internal crack of length $2c$ under plane stress state to

$$\sigma_c = \sqrt{\frac{2E\gamma}{\pi c}} \tag{1}$$

with the Young's modulus E and the surface energy $\gamma$. This relation was given without any satisfactory explanation. Other authors [3-11] also solved the problem partly by applying later developed methods (stress function method of Mushelishvili, fracture mechanics) and came to correct and incorrect solutions. In this article these papers will be reviewed and a correct derivation of the Griffith relation will be presented based exclusively on the relations given in [1].

## 2. The approach of Griffith

Griffith [1] calculated the elastic energy W of an infinite plate with an internal crack of length $2c$ under equi-biaxial external tension. He applied the stress distribution calculated by Inglis [12] for a plate with an elliptical hole in an infinite plate (Fig. 1). To solve this problem, Inglis used curvilinear co-ordinates $\alpha, \beta$, such that the relations with Cartesian co-ordinates are

$$\begin{aligned} x &= c\cosh\alpha\cos\beta \\ y &= c\sinh\alpha\sin\beta \end{aligned} \tag{2}$$

$\alpha$ = constant corresponding to the ellipse

$$\frac{x^2}{c^2\cosh^2\alpha} + \frac{y^2}{c^2\sinh^2\alpha} = 1 \tag{3}$$

with the semi-axes $a = c\cosh\alpha$, $b = c\sinh\alpha$. The contour of the elliptical hole is characterised by $\alpha = \alpha_0$. For a plate with a tensile stress $\sigma_\infty$ in all directions, Inglis obtained the following stress components

$$\sigma_\alpha = \frac{\sigma_\infty \sinh(2\alpha)\left[\cosh(2\alpha) - \cosh(2\alpha_0)\right]}{\left[\cosh(2\alpha) - \cos(2\beta)\right]^2} \tag{4a}$$

$$\sigma_\beta = \frac{\sigma_\infty \sinh(2\alpha)\left[\cosh(2\alpha) + \cosh(2\alpha_0) - 2\cos(2\beta)\right]}{\left[\cosh(2\alpha) - \cos(2\beta)\right]^2} \tag{4b}$$

[1] There is some confusion about the year of the publication of the paper of Griffith. He presented his results to the Royal Society in 1920 and the paper was published in 1921

$$\tau_{\alpha\beta} = \frac{\sigma_\infty \sin(2\beta)\left[\cosh(2\alpha) - \cosh(2\alpha_0)\right]}{\left[\cosh 2\alpha - \cos(2\beta)\right]^2} \tag{4c}$$

Note that in the paper of Inglis an $\alpha$ is missing in the equation for $\tau_{\alpha\beta}$.
If the elliptical hole degenerates to a crack, b = 0, , $\alpha_0 = 0$ and $a = c$.
The equations then change to

$$\sigma_\alpha = \frac{\sigma_\infty \sinh(2\alpha)\left[\cosh(2\alpha) - 1\right]}{\left[\cosh(2\alpha) - \cos(2\beta)\right]^2} \tag{5a}$$

$$\sigma_\beta = \frac{\sigma_\infty \sinh(2\alpha)\left[\cosh(2\alpha) + 1 - 2\cos(2\beta)\right]}{\left[\cosh(2\alpha) - \cos(2\beta)\right]^2} \tag{5b}$$

$$\tau_{\alpha\beta} = \frac{\sigma_\infty \sin(2\beta)\left[\cosh(2\alpha) - 1\right]}{\left[\cosh 2\alpha - \cos(2\beta)\right]^2} \tag{5c}$$

The corresponding displacements also derived by Inglis [12] were used by Griffith in the form of

$$\frac{u_\alpha}{h} = \frac{c^2 \sigma_\infty}{8\mu}\left[(\kappa - 1)\cosh(2\alpha) - (\kappa + 1)\cos(2\beta) + 2\cosh(2\alpha_0)\right] \tag{6a}$$

$$\frac{u_\beta}{h} = 0 \tag{6b}$$

with

$$h = \left[\frac{2}{c^2(\cosh(2\alpha) - \cos(2\beta))}\right] \tag{7}$$

and $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress ($\nu$ : Poisson ratio, $\mu$ : shear modulus).

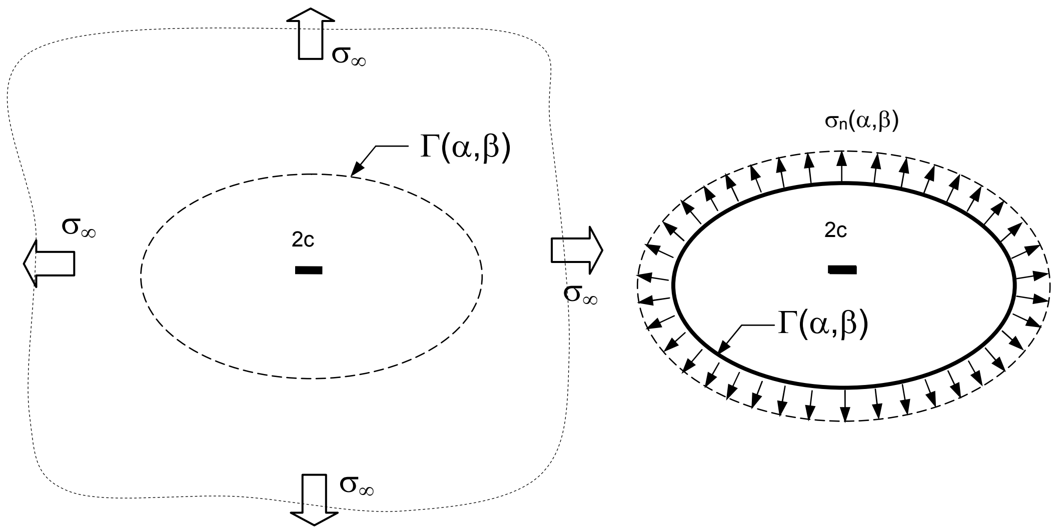


**Fig.1** a) Biaxially loaded infinite plate, b) finite elliptical plate cut out of the infinite plate and loaded by tractions with the normal stresses occurring in the infinite plate at the cutting contour.

The strain energy per unit thickness of the plate is

$$W = \frac{1}{2}\int_0^{2\pi} \frac{u_\alpha}{h}\sigma_\alpha d\beta + \frac{1}{2}\int_0^{2\pi} \frac{u_\beta}{h}\tau_{\alpha\beta} d\beta \tag{8}$$

According to Griffith the strain energy for large approaches the value 0f

$$W_{total} = \frac{\pi c^2 \sigma_\infty^2}{8\mu}\left[\frac{1}{2}(\kappa - 1)e^{2\alpha} + (3-\kappa)\cosh(2\alpha_0)\right] \tag{9}$$

In this context, it should be mentioned that eq.(9) is not the energy of a finite elliptic plate having a size parameter α, the Inglis stress solution is not correct since for a finite plate. Equation (9) rather represents the amount of energy stored in the finite region defined by $0 \leq \alpha < \infty$ of an infinite plate. This energy is the same as for a finite elliptic plate with a size of α, which is loaded by normal tractions $\sigma_n = \sigma_\alpha \neq \sigma_\infty$, along the circumference, i.e. by the stresses $\sigma_\alpha$ occurring in the infinite plate at the prospective circumference at which the finite plate is cut out from a fictitious infinite plate loaded by remote tractions $\sigma_x = \sigma_y = \sigma_\infty$. This fact will be addressed in more detail in Section 4.

Griffith then claims that the increase of the strain energy due to the cavity $\alpha_0$ would be given by

$$W = \frac{\pi c^2 \sigma_\infty^2}{8\mu}(3-\kappa)\cosh(2\alpha_0), \tag{10}$$

For the crack with $\alpha_0 = 0$

$$W = \frac{\pi c^2 \sigma_\infty^2 (3-\kappa)}{8\mu} \tag{11}$$

The change of this energy by an infinitesimal crack extension - later by Irwin called strain energy release rate [13] or crack extension force [14] *G* - is

$$G = \frac{dW}{dc} = \frac{\pi c \sigma_\infty^2 (3-\kappa)}{4\mu} \tag{12}$$

This energy release rate increases with increasing external stress applied. Crack propagation starts when the energy release rate is equal to the energy necessary to create a new surface. This energy was called surface energy $\gamma$ by Griffith (in the paper of Griffith: *T*). The surface energy of a crack of length 2c is

$$U = 4c\gamma \tag{13}$$

Thus, the criterion for crack extension is

$$\frac{dW}{dc} = \frac{dU}{dc} \tag{14}$$

leading to the relations for the critical stress $\sigma_\infty = \sigma_c$

$$\sigma_c = \sqrt{\frac{8E\gamma(1+\nu)}{\pi\nu c}} \tag{15a}$$

for plain strain and

$$\sigma_c = \sqrt{\frac{2E\gamma}{\pi\nu c}} \tag{15b}$$

for plain stress.

To verify his calculations, Griffith performed tests on glass tubes and glass bulbs of diameter $D$ with cracks of length $2c$ under internal pressure. The fracture stress is shown in Tables 1 and 2. The log-log plot in Fig.2 shows that the results follow a slope of -1/2 as predicted. The surface energy obtained for plane stress is 0.442 N/m and 0.436 N/m, respectively. Griffith measured the surface energy between 745°C and 1100°C by a procedure called "Quincke's drop method" and obtained a value of 0.543 N/m by linear extrapolation at 15°C. From the fairly good agreement with the values obtained from his fracture tests, he concluded the correctness of his theory. Expressed in terms of later developed fracture mechanics, the corresponding fracture toughness results

$$K_{Ic} = \sqrt{2E\gamma} \tag{16}$$

are 0.234 $MPam^{1/2}$ and 0.233 $MPam^{1/2}$ with $E$=62 GPa as measured by Griffith.

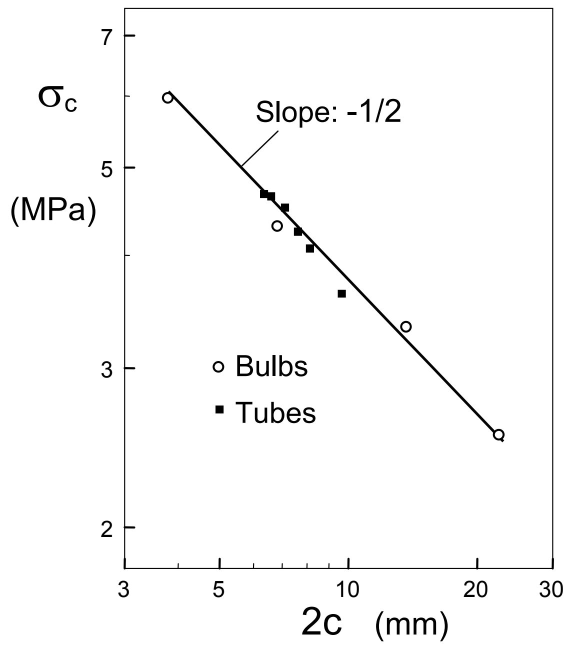


**Fig. 2** Strength $\sigma_c$ of glass tubes and bulbs with through-the-thickness cracks of length 2c.

Griffith added a note at the end of his paper (***statement I***):

> "It has been found that the method of calculating the strain energy of a cracked plate, which is used in Section 3 of this paper, requires correction. The correction affects the numerical values of all quantities calculated from

equations [his numbers are changed to the corresponding numbers in this paper] (9), (10), (11), (15a), (15b), but not their order of magnitude. The main argument of the paper is therefore not impaired, since it deals only with the order of magnitude of the results involved, but some reconsideration of the experimental verification of the theory is necessary".

He did not explain the correction.

In connection with the preceding evaluation, let us make a first remark on the strength problem. For the determination of the strength $\sigma_c$ from the burst tests, Griffith computed the stresses in the tubes via a simple 2-D relation implying an axisymmetrical geometry. The so computed stresses are present in the test specimens in the absence of a crack only. After introducing a starter crack of length 2c, the axisymmetry is lost.. Now, a highly complicated 3-D problem must be solved. This, of course, could hardly be done with the analytical methods available in 1920-24.

From this point of view it would be very astonishing if a good agreement between computations (even by application of the later addressed correct equation from 1924) and the experiments would result. We will come back to this problem in Section 12.

## 3. Griffith's revision of his first paper

In 1924, Griffith published a revision of the main equation of his first paper in a conference volume [2]. He wrote (***statement II***)

> "in the solution there given the calculation of the strain energy was erroneous, in that the expression used for the stresses gave values at infinity differing from the postulated uniform stress at infinity by an amount which, though infinitesimal, yet made a finite contribution to the energy when integrated round the infinite boundary. This difficulty has been overcome by slightly modifying the expressions for the stresses, so as to make this contribution to the energy vanish".

Griffith did not explain what was wrong in the results of Inglis and he did not mention what he changed in the stress expressions. Actually, the stresses given in eqs. (4) and (5) approach $\sigma_\infty$ for $\alpha \to \infty$. Griffith only mentions that he replaced $3-\kappa$ in the strain energy – eq.(11) – by $\kappa+1$, leading to the following fracture stress for plane stress

$$\sigma_c = \sqrt{\frac{2E\gamma}{\pi c}}\,, \tag{17}$$

which is the well-known and correct equation. For plane strain, Griffith erroneously gave the equation

$$\sigma_c = \sqrt{\frac{2E(1-\nu^2)\gamma}{\pi c}} \tag{18}$$

The correct equation is

$$\sigma_c = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)c}}\ . \tag{19}$$

The authors of this paper cannot understand how Griffith obtained the correct solution. The same conclusion was drawn by others:

Swedlow [7]: "no further detail was given as to how (9) (*the Inglis stress equations*) should be modified to obtain this second, and more familiar result".

Spencer [8]: "Griffith gave correct results, but no details of his calculations"

Shih, Liebowitz [9]: "…the details of his calculations were not published. As a consequence, many of the investigators in fracture mechanics have attempted to fill in the missing details and have even cast doubt on such a revision."

Yatomi [11]: "Griffith noted that he made an error in [1] and presented, without proof, the correct formula for the fracture load"

In Tables 1 and 2, the surface energies calculated for plane stress according to eq.(17) are given. The average values are 1.76 N/m and 1.74 N/m corresponding to $K_{Ic}$ of 0.468 MPam$^{1/2}$ and 0.465 MPam$^{1/2}$. Griffith had a problem with the disagreement of the new surface energy of 1.75 N/m with the "correct" value of 0.543 N/m. He argued in the following way: To reduce possible internal stresses, Griffith had annealed the tubes and bulbs at 450°C. For not annealed specimens he obtained surface energies between 0.218 N/m and 0.368 N/m. These values were too low because of internal stresses. According to Griffith the 1.75 N/m of the annealed specimens were too high because of an enlargement of the crack tip radius during annealing. Therefore, Griffith performed new tests, where the annealing procedure was changed. Heating was shut off immediately after reaching 450°C. From the fracture tests, he then obtained a surface energy of 0.50 N/m in agreement with the "correct" value. It will be shown in section 12 that another explanation is possible.

It is, however, known that the fracture energy of glass is even higher than 1.75 N/m. Values of $K_{Ic}$ between 0.85 MPam$^{1/2}$ and 0.96 MPam$^{1/2}$ corresponding to $\gamma$ between 4.4 N/m and 5.6 N/m for a comparable glass have been measured [15].

Consequently two questions require an answer:

- What is wrong in the calculations of the first paper of Griffith and what are the modifications of Griffith in his second paper?
- If the relation in the second paper is correct – and it will be shown that it is correct for a crack in an infinite plate - why then is there a disagreement with the real fracture energy?

## 4. Correct derivation of the Griffith relation

Like Griffith, we consider an infinite plate with an internal crack of length $2c$ under equi-biaxial loading. Now we cut along the elliptical contour $\Gamma$ and obtain a finite elliptical plate as illustrated in Fig. 1b. At the circumference of the plate normal tractions are applied which are identical to the stresses $\sigma_\alpha$ present at the contour $\Gamma$ of the infinite plate. Consequently, also the displacements $u_\alpha$ are the same as in the case of the infinite plate at Γ.

The energy $W_{\text{total}}$ stored in the cracked elliptical plate under this special load is given by eq.(8) leading to

$$W_{total} = \frac{\sigma_\infty^2 c^2 \pi e^{-2\alpha}\left[-(e^{2\alpha}-1)^4 + \kappa(e^{4\alpha} 1)^2\right]}{16\mu(e^{2\alpha}+1)^2} \tag{20}$$

i.e. $W_{total}$ now is also valid for the finite plate with $\alpha < \infty$.

The series expansion of eq.(20) with respect to exp(-2$\alpha$) reads

$$W_{total} = \frac{\sigma_\infty^2 c^2 \pi}{8\mu}\left[\frac{(\kappa-1)e^{2\alpha}}{2} + 3 - \kappa - \frac{(17-\kappa)e^{-2\alpha}}{2} + O(e^{-4\alpha})\right] \tag{21}$$

In his paper Griffith presented an asymptotic solution, eq.(9), which for $\alpha_0$ (the crack) is identical to the first two terms in the bracket of eq.(21).

This equation approaches eq.(20) for large $\alpha$. In the uncracked elliptic plate with the same outer contour, loaded by constant tractions $\sigma_n=\sigma_\infty$, the stored energy results as

$$W_0 = (\kappa-1)\pi\sigma_\infty^2 \frac{c^2}{8\mu}\sinh(2\alpha) \tag{22}$$

The energy difference is

$$\Delta W = W_{total} - W_0 = \frac{\pi\sigma_\infty^2 c^2}{8\mu}\left[3-\kappa-(1-\kappa)e^{-2\alpha}\right] \tag{23}$$

For $\alpha \to \infty$ this is the relation of Griffith in his first paper, eq.(11). This energy, however, is not the excess energy $W_{\text{crack}}$ caused by the presence of the crack, as was implicitly assumed by Griffith. The two energies are obtained under two slightly different loadings at the circumference (namely $\sigma = \sigma_\alpha < \sigma_\infty$ for the cracked and $\sigma = \sigma_\infty$ for the uncracked plate). In order to determine the contribution of the crack, the energy in the uncracked plate must be computed also under the load $\sigma_\alpha < \sigma_\infty$. This will be done below.

In the following computations we focus on the special case of large but finite elliptic plates (size large compared with the crack length 2$c$). This implies: $\alpha$ >>1.

A finite elliptical plate with the boundary $\Gamma(\alpha,\beta)$ may be loaded by normal tractions $\sigma_n(\alpha,\beta)$ as occurring in the infinite plate at the same contour Γ.

This distribution of normal tractions will differ slightly from the remote stress $\sigma_\infty$ and can be expressed by

$$\sigma_n(\alpha,\beta) = \sigma_\infty + \Delta\sigma_n \tag{24}$$

with the difference stress term $\Delta\sigma_n$ as

$$\Delta\sigma_n = \sigma_\infty\left(\frac{\sinh 2\alpha(\cosh 2\alpha - 1)}{(\cosh 2\alpha - \cos 2\beta)^2} - 1\right) = \sigma_\infty(-2 + 4\cos 2\beta)\exp[-2\alpha] + O(\exp[-4\alpha]) \quad (25)$$

This stress term is plotted in Fig. 3 in normalised form.

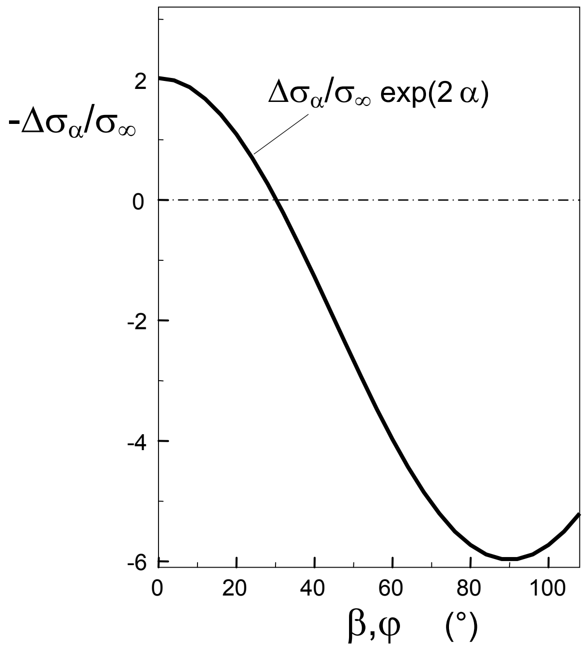


**Fig. 3** Normal tractions $\Delta\sigma_\alpha = \Delta\sigma_n$ necessary to obtain the same displacements $u_\alpha$ for a large finite elliptical plate cut out of the infinite plate under equi-biaxial load.

Since the tractions on the boundary of the uncracked finite plate differ slightly from the tractions on the cracked plate (same loading conditions) also the displacements must differ slightly in both cases. It must hold for the displacements $u_0$ of the uncracked plate

$$u_0(\alpha,\beta) = u_\alpha(\alpha,\beta) + \Delta u(\alpha,\beta) \quad (26)$$

with the displacements $u_\alpha$ of the cracked plate according to eq. (6a) and $\Delta u_\alpha << u_\alpha$ for $\alpha >> 1$. For this difference term, we also have to expect the asymptotic behaviour of $\Delta u/u_\alpha \propto O(\exp[-2\alpha])$. This can be concluded in the following way.

From the stresses of the order $O(\exp[-2\alpha])$ it follows via Hook's law that also the strains must be of the same order. Integration of strains results in the displacements which are, consequently, of the same order

$$\frac{\Delta\sigma}{\sigma_\alpha} \underbrace{\equiv}_{Hooke} \frac{\Delta\varepsilon}{\varepsilon_\alpha} \underbrace{\equiv}_{Integration} \frac{\Delta u}{u_\alpha} = O(\exp[-2\alpha]) \quad (27)$$

The energy in the uncracked plate is given by

$$W_{uncracked} = W_0 + \Delta W = (\kappa - 1)\pi\sigma_\infty^2 \frac{c^2}{8\mu}\sinh(2\alpha) + \Delta W \quad (28)$$

with

$$\Delta W=\int_0^{2\pi}u_0\Delta\sigma_n\,/\,hd\beta\;=\int_0^{2\pi}(u_\alpha+\Delta u)\Delta\sigma_n\,/\,hd\beta\;=\underbrace{\int_0^{2\pi}u_\alpha\Delta\sigma_n\,/\,hd\beta}_{W_1}\;+\underbrace{\int_0^{2\pi}\Delta u\Delta\sigma_n\,/\,hd\beta}_{W_1\times O(\exp[-2\alpha])}$$

$$\Rightarrow\Delta W=\int_0^{2\pi}u_\alpha\Delta\sigma_n\,/\,hd\beta \tag{29a}$$

In addition to the normal stresses $\sigma_\alpha=\sigma_n$, small shear tractions exist at the contour Γ. A similar consideration of shear tractions and displacements results in

$$\Delta W_{shear}=\int_0^{2\pi}\underbrace{u_\beta}_{\propto\exp[-2\alpha]}\underbrace{\Delta\tau_{\alpha\beta}\sqrt{h/2}d\beta}_{\propto\exp[-2\alpha]}\propto O(\exp[-4\alpha])\rightarrow 0 \tag{29b}$$

Introducing eqs.(6a) and (25) in (29) yields

$$\Delta W=\int_0^{2\pi}u_\alpha\Delta\sigma_n\,/\,hd\beta\;=\tfrac{1}{4}\frac{c^2\sigma_\infty^2}{\mu}\left(\frac{\exp(4\alpha)-6\exp(2\alpha)+1}{(1+\exp(2\alpha))^2}-\kappa\right)\rightarrow\tfrac{\pi}{4}\frac{c^2\sigma_\infty^2}{\mu}(1-\kappa) \tag{30}$$

The energy contribution due to the presence of a crack is

$$W_{crack}=W_{total}-W_{uncracked}=W_{total}-W_0-\Delta W \tag{31}$$

resulting in

$$\begin{aligned}W_{crack}&=\frac{c^2\sigma_\infty^2\pi}{8\mu}[3-\kappa-\underbrace{(1-\kappa)\cosh(2\alpha)+(1-\kappa)\sinh(2\alpha)}_{\rightarrow 0}-2(1-\kappa)]=\\&=\frac{c^2\sigma_\infty^2\pi}{8\mu}(3-\kappa-2+2\kappa)=\frac{c^2\sigma_\infty^2\pi}{8\mu}(\kappa+1)\end{aligned} \tag{32}$$

This result was reported in the second Griffith paper.

There is a second possibility of how Griffith may have improved his first analysis. Again for this attempt only the stress/displacement solutions reported in [1] are needed exclusively.

The displacements of the crack surfaces ($\alpha$ =0) follow from eq.(6a) as

$$\frac{u}{h}=\frac{c^2\sigma_\infty}{8\mu}(\kappa+1)(1-\cos2\beta)$$

$$u=\frac{c^2\sigma_\infty}{8\mu}(\kappa+1)(1-\cos2\beta)\sqrt{\frac{2}{c^2(1-\cos2\beta)}}=\frac{\sqrt{2}c\sigma_\infty}{8\mu}(\kappa+1)\sqrt{1-\cos2\beta} \tag{33}$$

In order to determine the energy contribution caused by the crack, let us close the open crack (Fig.4a) by application of a constant tensile stress $\sigma_{crack}$ acting on the crack faces (Fig. 4b). If this stress is increased up to $\sigma_\infty$, the stress in the uncracked plate normal to the prospective crack plane, the crack must be closed (Fig. 4c).
In the case $\sigma_{crack}=\sigma_\infty$ the energy results as

$$W_{closure} = \tfrac{1}{2}\int_0^{2\pi} u/h\sigma_\infty d\beta = \tfrac{1}{2}\frac{c^2\sigma_\infty^2}{8\mu}(\kappa+1)\underbrace{\int_0^{2\pi}(1-\cos 2\beta)d\beta}_{2\pi} = \frac{c^2\sigma_\infty^2\pi}{8\mu}(\kappa+1) \tag{34}$$

This energy has to be introduced into the plate. The same amount of energy is released of course, when a crack is generated, i.e. $|W_{\text{crack}}|=|W_{\text{closure}}|$. Finally, it should be noted that if Griffith used this approach, no additional condition like the superposition principle etc. was necessary.

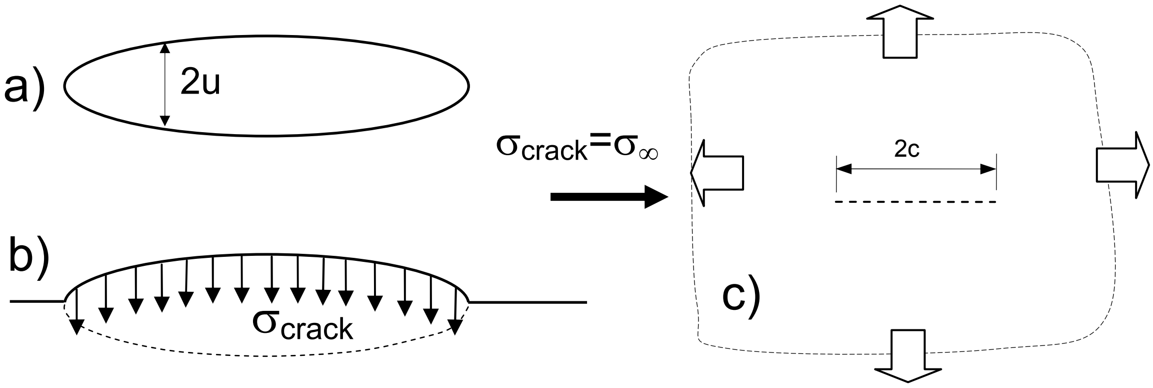


**Fig. 4** Closure of the crack by application of crack-face tensile stresses: a) displacements in the infinite plate under remote biaxial load $\sigma_\infty$, b) application of crack-face loading $\sigma_{\text{crack}}$, c) stress state of the uncracked infinite plate reached for $\sigma_{\text{crack}}=\sigma_\infty$.

Griffith mentions in the second statement (statement II in section 3): "… the expression used for the stresses gave values at infinity differing from the postulated uniform stress". In this respect it seems to be highly probable that Griffith considered the infinitesimal correction stress $\Delta\sigma_n$, i.e. the deviation of the stress in an infinite plate at a finite, but large distance from the crack, as we did in our eqs.(24, 25).
In the following sections, results of other authors who discussed the papers of Griffith are critically reviewed.

## 5. The calculations of Wolf

In 1922Wolf [3] published a paper in German, where he claimed that the calculations of Griffith contained an error. The paper of Wolf also contained several errors (or misprints). Like Griffith, he applied the stress relations of Inglis [12] and considered a plane stress state. For the calculation of the strain energy, he gave the relation (his equ.(2))

$$W = \frac{1}{2E}\int\left[(\sigma_\alpha+\sigma_\beta)^2+(1+\nu)(\tau_{\alpha\beta}^2-\sigma_\alpha\sigma_\beta)\right]dF = \frac{1}{2E}\int(\sigma_\alpha^2+\sigma_\beta^2+(1-\nu)\sigma_\alpha\sigma_\beta+(1+\nu)\tau_{\alpha\beta}^{2)})dF \tag{35}$$

The correct relation is

$$W = \frac{1}{2E}\int(\sigma_\alpha^2+\sigma_\beta^2-2\nu\sigma_\alpha\sigma_\beta+2(1+\nu)\tau_{\alpha\beta}^2)dF \tag{36}$$

Wolf then argued that the energy should be proportional to the square of the half crack length c, leading to

$$W=\frac{2\sigma_0^2c^2}{E}\int_0^{\alpha}d\alpha\int_0^{\pi/2}d\beta\frac{\cosh(2\alpha)-\cos(2\beta)}{2}\left\{\begin{array}{l}\dfrac{4\sinh^2(2\alpha)}{(\cosh(2\alpha-\cos(2\beta))^2}+\dfrac{(1+\nu)(\cosh(2\alpha)-1)}{(\cosh(2\alpha-\cos(2\beta))^2}\\ \left[\begin{array}{l}\sin^2(2\beta)(\cosh(2\alpha)-1)-\sinh^2(2\alpha)(\cosh(2\alpha)\\ -2\cos(2\beta)+1)\end{array}\right]\end{array}\right\} \quad (37)$$

For the ß-integration, Wolf obtained

$$W=\frac{\sigma_0^2\pi c^2}{E}\int_0^{\alpha}\left[e^{2\alpha}-e^{-2\alpha}+\frac{1+\nu}{4}\left(-e^{2\alpha}+9e^{-2\alpha}+8\frac{-4e^{2\alpha}+9-4e^{-2\alpha}-2e^{-4\alpha}+8e^{-8\alpha}}{e^{6\alpha}-3e^{2\alpha}+3e^{-2\alpha}-e^{-6\alpha}}\right)\right]d\beta \quad (38)$$

This relation cannot be correct (maybe a misprint), because the integrand approaches infinity for $\alpha\to 0$ (the denominator in the bracket term disappears)
For the $\alpha$ -integration Wolf obtained

$$W=\frac{\pi\sigma_0^2c^2}{E}\left\{\frac{1}{2}\left(e^{2\alpha}+e^{-2\alpha}-2\right)+\frac{1+\nu}{4}\left[-\frac{1}{2}e^{2\alpha}-\frac{1}{2}e^{-2\alpha}+2-\frac{4}{e^{-2\alpha}+1}+\frac{4}{(e^{-2\alpha}+1)^2}\right]\right\} \quad (39)$$

This relation follows from eq.(37). Then, Wolf argued that the terms approaching infinity for $\alpha\to\infty$ should be omitted and the terms approaching zero should also be neglected. He then obtained the result (after changing the sign)

$$W=\frac{\pi\sigma_0^2c^2\nu}{E} \quad (40)$$

But actually, the result

$$W=\frac{\pi\sigma_0^2c^2(1-\nu)}{2E} \quad (41)$$

is obtained from eq. (39). Finally, Wolf mentioned in a footnote that Griffith had obtained half the value of eq.(40), which is not correct. If Wolf would have applied the correct strain energy density, he would have obtained the same erroneous equation as Griffith in his first paper.
In 1922 Smekal [16] mentioned the paper of Griffith with the remark (translated into English) "The details of the calculations by Griffith proved to be not quite correct. Prof. Wolf, however, had the kindness to perform the rectifications and to give them to the author". From a dimensional analysis, Smekal concluded that the strain energy is proportional to $\sigma_\infty^2c^2$ and, therefore,

$$\sigma_c\sqrt{c}=C \quad (42)$$

From the tests of Griffith Smekal obtained $C$ = 26.8 kg/cm$^{3/2}$ = 8.31 N/m$^{3/2}$. From the theory of Wolf, he obtained $C$ = 25.2 kg/cm$^{3/2}$ = 7.81 N/mm$^{3/2}$. Smekal did not provide any equation of Wolf. The result of Smekal at least is not in agreement with the published result of Wolf.

## 6. The calculations of Horowitz and Pines

Horowitz and Pines [4] published a paper in 1928 in which they criticised the approach of Griffith. They did not give a reference to the Griffith paper. Therefore we do not know whether they were aware of the second paper of Griffith. The authors presented a new approach, which is not easy to follow. They applied some arbitrary choices of parameters in their calculations. Finally, they came to the "correct" relation.

## 7. The calculations of Orowan

Orowan [5] calculated the fracture stress in a different way, not applying the energy criterion. He considered the maximum stress for an elliptical cavity of length $2c$ and the radius $\rho$ at the corner of the ellipse. According to Inglis the maximum stress is

$$\sigma_{\max} = 2\sigma_{\infty}\sqrt{\frac{c}{\rho}} \tag{43}$$

For the radius of a crack Orowan assumed the atomic distance $a$, i.e. $c \approx a$. The fracture stress is reached as soon as the maximum stress reaches the theoretical strength $\sigma_{th}$. According to Orowan it is given by

$$\sigma_{th} = \sqrt{\frac{4\gamma E}{a}} \tag{44}$$

leading to

$$\sigma_c = \sqrt{\frac{\gamma E}{c}} \tag{45}$$

which differs by a factor $\sqrt{2/\pi} = 0.8$ from the revised equation of Griffith.

## 8. The calculations of Sneddon

Sneddon [6] obviously was the first who gave a correct and conclusive derivation of the Griffith relation. He did not consider the case of an externally loaded plate. He considered a plate with an internal crack under an internal pressure $p_0$. He applied the stress function of Westergaard [17] modified for internal pressure:

$$Z = p_0\left(\frac{z}{\sqrt{z^2 - c^2}} - 1\right) \tag{46a}$$

with the complex coordinate $z = x + \mathrm{i}y$. From

$$\sigma_x = \operatorname{Re} Z - y \operatorname{Im} Z' \tag{47a}$$

$$\sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z' \tag{47b}$$

$$\sigma_{xy} = -y \operatorname{Re} Z' \tag{47c}$$

Sneddon obtained the stress distribution and the crack opening displacement *w*:

$$\left(\frac{x}{c}\right)^2 + \left(\frac{w}{\varepsilon}\right)^2 = 1\,, \tag{48}$$

where the maximum crack opening displacement at $x = 0$, ε, is given by

$$\varepsilon = \frac{2(1-\nu^2)p_0 c}{E} \tag{49}$$

for plane strain. Equation (48) is identical to eq.(33) (*u* in eq.(33) corresponds to *w* in eq.(48)). While the internal pressure increases from 0 to $p_0$, the linear relation between *p* and $\varepsilon$ reads

$$p = \frac{p_0}{\varepsilon(p_0)}\varepsilon(p) \tag{50}$$

If the minor axis is increased by $d\varepsilon$ , the work done for an element d*x* is

$$dW = 2pdxdw = \frac{E\varepsilon}{(1-\nu^2)c}dx\sqrt{1-x^2/c^2}\,d\varepsilon \tag{51}$$

and the energy of the crack then is

$$W = \frac{2E}{(1-\nu^2)c}\int_0^c \sqrt{1-x^2/c^2}\,dx \int_0^\varepsilon \varepsilon d\varepsilon = \frac{\pi E\varepsilon^2}{4(1-\nu^2)} = \frac{\pi(1-\nu^2)p_0^2c^2}{E} \tag{52}$$

In a second step, Sneddon superimposed a second (constant) stress function term $\Delta Z$

$$\Delta Z = +p_0 \;\Rightarrow\; Z_1 = Z + p_0 \tag{46b}$$

to the internally pressured crack, resulting in the total stress function $Z_1$. As can be seen by introducing the constant term $p_0$ in (47a-47c), the additional stress function term only generates a location-independent equi-biaxial stress $\sigma_x = \sigma_y = p_0$. Consequently, the normal tractions at the former free plate boundary now become $\sigma_n = p_0$ and those at the crack surfaces balance the former tractions $-p_0$ from the crack-face pressure. In total, traction-free crack faces are obtained. Since the superimposed stress function now exactly satisfies the boundary conditions of the cracked plate under pure biaxial tension, the solution holds for the Griffith problem.
Thus, Sneddon provided the first correct derivation of the Griffith relation.

## 9. The calculations of Spencer

Spencer [8] realised that the calculations of Griffith in his first paper were incorrect. He mentions that "for a valid comparison (*of strain energy of cracked and an uncracked plate)*, it is necessary to impose boundary conditions for the uniformly stressed plate". This is in agreement with Section 4. Doing this, Spencer obtained the correct strain energy due to the crack. In his Section 2 he calculated the stress distribution for a crack in an infinite body under biaxial loading from a stress function

$$F = \sigma_\infty c^2 (\sinh 2\alpha - 2\alpha)/4 + \sigma_0 (\varepsilon - 1) y^2 / 2 \tag{53}$$

where $\sigma_\infty$ is the remote stress in x-direction and $\varepsilon\sigma_\infty$ is the remote stress in y-direction. With this stress function, the stress relations of Inglis can be verified. In his further calculations Spencer made transformations of the stress function to a polar coordinate system and obtained

$$F = \frac{1}{4}\sigma_\infty r^2 \left[ 1 + \varepsilon + (1-\varepsilon)\cos 2\theta - \frac{c^2}{r^2}\left\{ \ln \frac{4c^2}{r^2} + \cos 2\theta \right\} + O(c^2/r^2) \right] \tag{54}$$

This stress function did not result in the stresses $\sigma_r, \sigma_\theta, \tau_{r\theta}$ as given by Spencer. The authors repeated the somewhat complicated coordinate transformations and found a mistake in the logarithm term of Spencer's equation (13), which correctly must read

$$F = \frac{1}{4}\sigma_\infty r^2 \left[ 1 + \varepsilon + (1-\varepsilon)\cos 2\theta - \frac{c^2}{r^2}\left\{ \ln \frac{4r^2}{c^2} + \cos 2\theta \right\} + O(c^2/r^2) \right] \tag{54a}$$

Since by using this corrected relation, the stresses given by Spencer can be derived, eq.(Spencer 13) has to be interpreted as a typing error. In the further computations, Spencer determines additional stresses at the circumference to satisfy constant tractions or constant displacement boundary conditions at the circumference of the cracked finite circular plate. This rather lengthy analysis yields the correct Griffith equation. As outlined in Section 4,such extended calculations are not necessary, since the only required properties are the stresses and displacements for the cracked infinite plate as given by Inglis.

## 10. The calculations of Swedlow

In 1965 [7] published a paper "on Griffith's theory of fracture". In the first part of his paper Swedlow considered like Griffith an infinite plate with an internal crack of length $2c$ under equi-biaxial loading. The basic concept is the relation

$$\partial U_{system} / \partial c = 0 \tag{55}$$

with the system energy defined as

$$U_{\mathrm{system}} = U_{\mathrm{loading}} + U_{\mathrm{strain}} + U_{\mathrm{surface}} \tag{56}$$

For constant applied stresses

$$U_{loading} = U_0 - h \int_P (\sigma_\alpha u_\alpha + \tau_{\alpha\beta} u_\beta) dP \tag{57}$$

where $P$ is the periphery, at which the external stresses are applied, $h$ is the plate thickness, and $U_0$ an arbitrary constant energy contribution. The strain energy is

$$U_{strain} = \frac{1}{2}\int_{vol}(\sigma_\alpha \varepsilon_\alpha + \sigma_{\alpha\beta}\gamma_{\alpha\beta} + \sigma_\beta \varepsilon_\beta)dvol \tag{58}$$

The surface energy is given by

$$U_{surface} = \gamma(2A + Ph + 4ch), \tag{59}$$

where $A$ is the surface of the plate.

Swedlow applied the relations for an elliptical hole, described by $\alpha = \alpha_0$, and obtained for the system energy

$$U_{system} = U_0 + \gamma(2A + Ph + 4ch - \lim_{\alpha_0 \to 0} \frac{c^2 h}{16\mu}\int_{\alpha_0}^{\infty}\int_0^{2\pi}\left[\frac{\kappa - 1}{2}(\sigma_\alpha + \sigma_\beta)^2 + (\sigma_\alpha - \sigma_\beta)^2 + (2\sigma_{\alpha\beta})^2\right]Hd\beta d\alpha \tag{60}$$

According to Swedlow the integration of this relation for $\alpha_0 \to 0$ leads to

$$U_{system} = U_0 + \gamma(2A + Ph + 4hc) - \frac{Ah}{4\mu}(\kappa - 1)\sigma_0^2 - \frac{\pi c^2 h}{8\mu}(3 - \kappa)\sigma_0^2 \tag{61}$$

It is not obvious where the third term including the plate surface $A$ is coming from. Applying eq. (37) leads to

$$\gamma = \frac{\pi(3 - \kappa)c\sigma_0^2}{16\mu} \tag{62}$$

which is the same result as Griffith found in his first paper.

Swedlow then mentions that Griffith claimed this result to be erroneous in his second paper. He stated "No further detail was given as to how (9) [the relations of Inglis] should be modified to obtain this second and more familiar result". Swedlow obviously believed that his result and hence the result of Griffith in his first paper would be correct.

In the second part of his paper, Swedlow considered an infinite plate with an internal crack loaded by internal pressure $p_0$. For this loading case, he modified the stress distribution of Inglis by adding $-p_0$ to the stress components $\sigma_\alpha$ and $\sigma_\beta$.

For $\alpha_0 \to 0$, the integration then leads to

$$U_{system} = U_0 + \gamma(2A + Ph + 4hc) - \frac{\pi(\kappa + 1)c^2 h}{8\mu} \tag{63}$$

resulting in

$$\gamma = \frac{\pi(\kappa + 1)c\sigma_0^2}{16\mu} \tag{64}$$

in agreement with the result of Sneddon for the same loading case and with the result of Griffith in his second paper for the plate under external loading. Swedlow obviously thinks that the two loading cases – internal pressure and external loading - lead to different solutions, which is not correct. In the rest of the paper, he considers the superposition of external loading and internal pressure by applying the different equations for these loading cases. Due to the incorrect solution for external loading, also these results are not correct.

## 11. The calculations of Shih and Liebowitz

As did Spencer, Shih and Liebowitz [10] realised the flaw in the first paper of Griffith. They first solved the problem of a plate with a circular hole under biaxial loading. They applied the relations for the stress components $\sigma_r$ and $\tau_{r\theta}$ and for the displacements $u_r$ and $v_\theta$ in polar coordinates and calculated the strain energy from

$$W = \frac{1}{2}\int_0^{2\pi}\left[\sigma_r u_r + \tau_{r\theta} v_r\right]_{r=c} c\,d\theta \tag{65}$$

From the result, they subtracted the energy for the plate with a hole and obtained for large $c$:

$$\Delta W = \frac{\pi(1+\nu)\sigma_0^2 a^2}{8E}\left[(3-\kappa)(1+\varepsilon)^2 + 2(\kappa-1)(1-\varepsilon)^2\right] \tag{66}$$

For equi-biaxial loading ($\varepsilon = 1$)

$$\Delta W = \frac{\pi(1+\nu)\sigma_0^2 a^2 (3-\kappa)}{8E} \tag{67}$$

This is the relation corresponding to the crack relation in the first Griffith paper and hence it is not correct.

To obtain the correct solution, the authors solved the problem for a concentric annulus loaded by surface tractions corresponding to the stresses in an infinite plate under biaxial loading

$$\sigma_r = \frac{\sigma_0}{2}\left[(1+\varepsilon) - (1-\varepsilon)\cos 2\theta\right] \tag{68a}$$

$$\tau_{r\theta} = \frac{\sigma_0}{2}(1-\varepsilon)\sin 2\theta \tag{68b}$$

For this loading, they calculated the stresses and the displacements in the annulus and from both, the strain energy. For the outer radius of the annulus → $\infty$ they obtained a relation from which they subtracted the energy for the plate without a hole:

$$\Delta W = \frac{\pi(1+\nu)\sigma_0^2 a^2}{8E}(\kappa+1)\left[(1+\varepsilon)^2 + 2(1-\varepsilon)^2\right] \tag{69}$$

and for equi-biaxial loading

$$\Delta W = \frac{\pi(1+\nu)\sigma_0^2 a^2}{2E}(\kappa+1) \tag{70}$$

which is in agreement with the corresponding result for a crack in Griffith's second paper.

In the second part, the authors considered an elliptical hole and applied the same procedure as Spencer. For the crack, the correct Griffith equation was obtained.

## 12. Disagreement between the surface energy from Griffith experiments and his measured surface energy

In the first paper of Griffith he obtained from eq.(15b) a surface energy of 0.436 N/m which was in fairly good agreement with his measured surface energy of 0.543 N/m. The fracture toughness $K_{Ic}$ can be calculated from the surface energy by eq.(16). The obtained value of $K_{Ic}$ = 0.233 MPam$^{1/2}$ is far below the values measured by usual fracture mechanics methods for glass between 0.7 and 1 MPam$^{1/2}$. When applying the "correct" equation, the surface energy is 1.74 N/m and $K_{Ic}$ = 0.465 MPm$^{1/2}$, which still is too low. The reason for the discrepancy is the difference between a plate under uniaxial or biaxial loading and a bulb or a tube under internal pressure. The internal pressure leads to bulging of the crack border which is treated by the Folias factor $M_F$. The correct equation for tubes under internal pressure is

$$K = p\frac{R_i}{t}\sqrt{\pi c}M_F = \sigma_0\sqrt{\pi c}M_F \tag{71}$$

with [18]

$$M_F = [1 + 0.38\lambda^2 - 0.00124\lambda^4]^{1/2} \tag{72}$$

and

$$\lambda = \frac{c}{\sqrt{R_i t}}[12(1-\nu^2)]^{1/4} \tag{73}$$

$t$ = wall thickness, $R_i$ = inner radius. The calculated fracture toughness is shown in Table 3. The average value is 1.06 MPam$^{1/2}$, which is in fairly good agreement with other measurements of glass. The corresponding surface energy is 9.0 N/m, which is larger than the value Griffith obtained experimentally by a factor of 17. It is well-known that the real fracture surface energy is much larger than the theoretical surface energy.

## 13. Conclusion

The following conclusions can be drawn from the previous sections:

1. The calculations in the first paper of Griffith [1] are not correct because of different boundary conditions for the cracked and the uncracked plate.
2. The equation of the strain energy caused by the crack in the second paper of Griffith [2] is correct. Griffith did, however, not give an explanation as to how he derived the equation.

3. Sneddon [6] made the first correct calculation of the relation between fracture stress and the size of the crack for a crack under internal pressure. He then applied the principle of superposition and thus obtained the correct solution for a plate under external tension loading.
4. Spencer [8] and Shih and Liebowitz [10] also derived the correct equation in a somewhat complicated way.
5. A much simpler derivation is given in Section 4.

## 14. References


[1] A.A. Griffith, The phenomena of rupture and flow in solids, Phil. Trans. Roy. Soc. 221A, 1920, 163-198.

[2] A.A. Griffith, The theory of rupture, Proc. Int. Congr. Appl. Mech. 1924, 56-63.

[3] K. Wolf, Zur Bruchtheorie von A. Griffith, Z. Angew. Math. und Mech. 3, 1923, 107-112.

[4] G. Horowitz, B. Pines, Beitrag zur Griffithschen Bruchtheorie, Zeitschrift für Physik 47, 1928, 904-911.

[5] E. Orowan, Die mechanische Festigkeitseigenschaften und die Realstruktur der Kristalle, Z. Kristallographie 89, 1934, 327-343.

[6] I.N. Sneddon, The distribution of stress in the neighbourhood of a crack in an elastic solid, Proc. Roy. Soc. A, 1946, 229-260.

[7] J.L. Swedlow, On Griffith’s theory of fracture, Int. J. Fract. 1, 1965, 210-216.

[8] A.J.M. Spencer, On the energy of the Griffith crack, Int. J. Engng. Sci. 3, 1965, 441-449.

[9] G.C. Shih, H. Liebowitz, On the energy criterion for brittle fracture, Int. J. Sol. Struct. 3, 1967, 1-22.

[10] G.C. Shih, H. Liebowitz, in Fracture, Vol. II, Academic Press, 1968, 67-190.

[11] C. Yatomi, On Griffith’s theory, Int. J. Fract. 16, 1980. R239-R242.

[12] C.E. Inglis, Stresses in a plate due to the presence of cracks and sharp notches, Trans. Roy. Inst. Naval Architects 55, 1913, 219-241.

[13] G.R. Irwin, J.A. Kies, H.L. Smith, Fracture strength relative to onset and arrest of crack propagation, Proc. ASTM 58, 1958, 640-660.

[14] G.R. Irwin, Structural aspects of brittle fracture, Applied Materials Research 3, 1964, 65-81.

[15] J. Mencik, Strength and Fracture of Glass and Ceramics, Glass Science and Technology 12, Elsevier 1992

[16] A. Smekal, Technische Festigkeit und molekulare Festigkeit, Die Naturwissenschaften 37, 1922, 102-104.

[17] H.M. Westergaard, Bearing pressure and cracks, Trans. ASMRE, J. Appl. Mech. 61, 1939, A-49 – A-58.

[18] J.F. Kiefner, W.A. Maxey, R.J. Eibner, R.J. Duffy, Failure stress loads of flows in pressurized cylinders, ASTM STP 536 (1973), 461-481.
[19] A.J. Holland, W.E.S. Turner, The breaking strength of glass. The effect of flaws and scratches, J. Soc. Glass Techn. 20, 1936, 279-302.

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