A Collocation Method for Integral Equations with Super-Algebraic Convergence Rate

T. Arens

T. Rösch

INSTITUT FÜR WISSENSCHAFTLICHES RECHNEN UND MATHEMATISCHE MODELLBILDUNG

Anschriften der Verfasser:

PD Dr. Tilo Arens
Institut für Angewandte und Numerische Mathematik
Karlsruher Institut für Technologie (KIT)
D-76128 Karlsruhe

Thomas Rösch MSc
Institut für Angewandte und Numerische Mathematik
Karlsruher Institut für Technologie (KIT)
D-76128 Karlsruhe

A Collocation Method for Integral Equations with Super-Algebraic Convergence Rate

Tilo Arens, Thomas Rösch

Karlsruhe Institute of Technology, Karlsruhe, Germany

Email: tilo.arens@kit.edu, thomas.roesch@kit.edu

Abstract

We consider biperiodic integral equations of the second kind with weakly singular kernels such as they arise in boundary integral equation methods. The equations are solved numerically using a collocation scheme based on trigonometric polynomials. The weak singularity is removed by a local change to polar coordinates. The resulting operators have smooth kernels and are discretized using the tensor product composite trapezodial rule. We prove stability and convergence of the scheme under suitable parameter choices, achieving algebraic convergence of any order under appropriate regularity assumptions. The method can be applied to typical boundary value problems such as potential and scattering problems both for bounded obstacles and for periodic surfaces. We present numerical results demonstrating that the expected convergence rates can be observed in practice.

Keywords: numerical methods for integral equations, collocation method, super-algebraic convergence rate, Laplace equation, Helmholtz equation

AMS-Classification: 65R20

1 Introduction

The boundary integral equation method for either an interior or an exterior boundary value problem for an elliptic partial differential equation such as Laplace's equation or the Helmholtz equation consists in reformulating the problem equivalently as a boundary integral equation

$$\lambda\varphi(x) - \int_{\partial D} K(x,y)\,\varphi(y)\,\mathrm{d}s(y) = \psi(x)\,, \qquad x \in \partial D\,. \tag{1}$$

This approach has attracted continuous interest over the past decades, in particular for exterior problems. Projection methods, both of collocation and Galerkin type are the most popular numerical solution techniques. Matrix compression schemes such as the Fast Multipole Method or H-Matrix calculus have played an important part, keeping the overall complexity of such approaches comparable to Finite Element Methods applied to the original boundary value problem. It is state of the art to achieve low algebraic convergence orders with linearly or

close to linearly growing operation count and memory requirement. On the other hand, for two-dimensional problems, based on original work by Kussmaul [12] and Martensen, Nyström methods can be applied that achieve exponential convergence rates at a $\mathrm{O}(N^2)$ operation count. At least for smooth, globally parametrizable boundaries, such methods are particularly easy to implement. They involve only the composite trapezoidal rule and a second quadrature rule for the singularity with the same quadrature points and weights given by a simple formula.

It is not so easy to achieve a comparable result in 3D: The singularity in the kernels not only depends on distance from the singular point but also on the direction from which this point is approached. This makes it hard to find a quadrature rule with the points of a composite trapezoidal rule that achieves high order convergence. For the case of an integral equation of the second kind, i.e. (1) with $\lambda = 1$, which we will also consider throughout the paper, a successfull approach was suggested in the doctoral thesis of Wienert [14]. It is limited to smooth surfaces globally parametrizable over a sphere, and in a nutshell, the method consists of applying a Galerkin method using spherical harmonics. By a rotation of the parametrization sphere, the singularity can be removed by transforming the integral. A full convergence analysis was given later in [8, 9].

A different approach was suggested by Bruno and Kunyanski in the papers [5,6]. Let us assume for simplicity that ∂D can be globally parametrized by a map $\eta : Q = (-\pi, \pi)^2 \to \partial D$. This gives rise to an integral equation

$$\varphi(t) - \int_Q k(t, \tau)\, \varphi(\tau)\, \mathrm{d}\tau = \psi(t)\,, \qquad t \in Q\,. \tag{2}$$

Using a cut-off function χ with $\chi(t) = 0$ for $|t| \geq \varrho$, the singularity is isolated,

$$\varphi(t) - \int_Q k(t, \tau)\, \chi(\tau - t)\, \varphi(\tau)\, \mathrm{d}\tau + \int_Q k(t, \tau)\, (1 - \chi(\tau - t))\, \varphi(\tau)\, \mathrm{d}\tau = \psi(t) \tag{3}$$

for $t \in Q$. A transformation of the first integral in polar coordinates centered on t removes the singularity, making all integrals computable to high order by the composite trapezoidal rule. The integral equation is solved numerically by a Nyström method. Although the initial idea is rather simple, there are a number of technicalities to be addressed in the implementation. The original papers give some numerical results, however no stability or convergence analysis was included relating achievable convergence rates to the overall complexity of the algorithm. Over the past decade, there have been some contributions to closing this gap: In his thesis [10], Heinemeyer related the approach to the method of locally corrected weights as proposed by [7]. He was able to prove point-wise convergence of the discrete operators with super-algebraic convergence rates but did not give a convergence rate of the overall scheme. The most complete analysis is given in [4], but the authors limit themselves to only considering scattering problems rather than more general boundary integral equations.

An alternative approach to [5, 6] was taken by one of the authors of this paper in [1], in interpreting the scheme as a collocation method based on trigonometric polynomials rather than a Nyström method. Instead of applying a heuristic approach to obtain approximate density values in the polar coordinate grid points, the scheme evaluates the trigonometric polynomials in these points exactly. Stability and a super-algebraic convergence rate at a computational complexity quadratic in the number of unknowns were shown for a semi-discrete scheme. However, the implementation of a fully discrete scheme leads to a much less favorable complexity estimate.

In the present paper, we improve the scheme of [1], with an emphasis on reducing the overall complexity. This involves coupling the choice of ϱ, the parameter characterizing the size of the support of χ in (3), to the total number of unknowns. The main work lies in explicitly deriving the dependence of constants in stability and convergence estimates on ϱ.

Before we start with this analysis, we give some basic facts on Sobolev spaces of biperiodic functions and on interpolation by trigonometric polynomials in such spaces in Section 2. The main analytical results are all contained in Section 3. We analyse the mapping properties of the integral operators and their discrete approximations. The main results are Theorems 3.11 and 3.12 establishing stability and convergence of the overall scheme with ϱ coupled to the grid size h. Assuming that the transformed kernel functions are infinitely often continuously differentiable, any algebraic convergence rate is achieved.

In Section 4, we consider three different applications for the solver in the context of boundary integral equations: an interior boundary value problem for Laplace's equation, a scattering problem for a bounded object and a scattering problem for a periodic surface. Numerical results are presented in Section 5.

Let us emphasize that the present work only represents an intermediate step on the way to a complete solution strategy: Firstly, we consider only a single integral equation limiting ourselves to relatively simple settings and geometries. Even though the extension to systems is relatively straight-forward, it did not seem feasible to include this material into the present paper. Moreover, we focus only on questions of stability and convergence rates for the full linear system with the critical issue being the analysis of approximating the weakly singular integral operator. Schemes as considered here have to compete against simpler strategies which just remove the singularity. As shown in [3], such a strategy can achieve 4th order convergence by neglecting the singularity in a clever way.

For a full solution strategy, the question of approximating the matrix representing the integral operator with a smooth kernel by possibly a block-wise low-rank matrix needs to be addressed as well. The question of how to achieve this while maintaining the overall high convergence rate will be the subject of furture research.

2 Periodic Sobolev Spaces and Interpolation

The numerical method introduced in this paper is applicable to weakly singular integral operators with kernel functions that are biperiodic with respect to both arguments. We will make the notion precise: Set $Q = (-\pi, \pi)^2$. We call a function $u : \mathbb{R}^2 \to \mathbb{C}$ Q-periodic if

$$u(t) = u(t_1 + 2\nu_1\pi, t_2 + 2\nu_2\pi), \quad t = (t_1, t_2)^\top \in \mathbb{R}^2, \qquad \nu = (\nu_1, \nu_2)^\top \in \mathbb{Z}^2.$$

Particular examples of Q-periodic functions are the trigonometric monomials,

$$T^{(\nu)}(t) = \frac{1}{2\pi} \exp\left(\mathrm{i}\,\nu \cdot t\right), \qquad t \in \mathbb{R}^2, \quad \nu \in \mathbb{Z}^2.$$

The trigonometric monomials span spaces of trigonometric polynomials $\mathcal{T}_N = \mathrm{span}\{T^{(\nu)} : \nu \in \mathbb{Z}_N^2\}$ where $N = (N_1, N_2) \in \mathbb{N}^2$ and $\mathbb{Z}_N^2 = \{\mu \in \mathbb{Z}^2 : -N_1 < \mu_1 \leq N_1, \ -N_2 < \mu_2 \leq N_2\}$. They also form a complete orthonormal system in $L^2(Q)$, i.e. every $u \in L^2(Q)$ can be expanded

into a Fourier series

$$u = \sum_{\nu \in \mathbb{Z}^2} u_\nu \, T^{(\nu)} \,.$$

The Fourier coefficients u_ν are given by

$$u_\nu = \left(u, T^{(\nu)}\right)_{L^2(Q)} = \int_Q u(t) \, T^{(-\nu)}(t) \, \mathrm{d}t \,, \qquad \nu \in \mathbb{Z}^2 \,.$$

Definition 2.1 *Let $s \geq 0$, Q as above. The Sobolev space H_Q^s is given by*

$$H_Q^s = \{u \in L^2(Q) : \sum_{\nu \in \mathbb{Z}^2} (1 + |\nu|^2)^s \, |u_\nu|^2 < \infty\}$$

with the inner product $(u,v)_{H_Q^s} = \sum\limits_{\nu \in \mathbb{Z}^2} (1 + |\nu|^2)^s \, u_\nu \, \overline{v_\nu}$ and norm $\|u\|_{H_Q^s} = (u,u)_{H_Q^s}^{1/2}$.

The 1D analogue of H_Q^s has been studied in depth in [11] and [13], the latter reference also containing some results on multivariate functions. We remark that H_Q^s is a Hilbert space. For $s = 0$, there holds $H_Q^s = L^2(Q)$. For $s > 1$, by Sobolev's imbedding theorem, H_Q^s is compactly imbedded in $(C_{\mathrm{per}}, \| \cdot \|_\infty)$, the space of Q-periodic continuous functions with the maximum norm. It makes sense, then, to introduce an operator for interpolation by trigonometric polynomials. We introduce a grid of interpolation points on Q, setting

$$t_\mu^N = (t_{\mu,1}^N, t_{\mu,2}^N)^\top = \left(\frac{\mu_1 \pi}{N_1}, \frac{\mu_2 \pi}{N_2}\right)^\top, \quad \mu \in \mathbb{Z}_N^2 \,.$$

Lemma 2.2 (Lemma 5.1 in [1]) *Suppose that $s > 1$ and $0 \leq \sigma \leq s$. Given $u \in H_Q^s$, for every $N \in \mathbb{N}^2$, there is a unique interpolation polynomial $P_N u \in \mathcal{T}_N$ such that*

$$u(t_\mu^N) = P_N u(t_\mu^N)\,, \qquad \mu \in \mathbb{Z}_N^2 \,.$$

The linear operator $P_N : H_Q^s \to H_Q^\sigma$ is bounded with

$$\|P_N u - u\|_{H_Q^\sigma} \leq C \, \frac{(\max\{N_1, N_2\})^\sigma}{(\min\{N_1, N_2\})^s} \, \|u\|_{H_Q^s},$$

where $C > 0$ is a constant depending on σ and s.

An alternative way to express the interpolation operator is using the Lagrange basis representation,

$$P_N u = \sum_{\mu \in \mathbb{Z}_N^2} u(t_\mu^N) \, L_\mu^N \,, \tag{4}$$

with the Lagrange basis functions given by

$$L_\mu^N(t) = \frac{\pi}{2 \, N_1 \, N_2} \sum_{\nu \in \mathbb{Z}_N^2} T^{(\nu)}(t - t_\mu^N) \,, \qquad t \in \mathbb{R}^2 \,.$$

For $t \in Q \setminus \{t_\mu^N\}$, there also holds the expression

$$L_\mu^N(t) = \frac{1}{4\,N_1\,N_2} \prod_{j=1}^{2} \sin\left(N_j\,(t_j - t_{\mu,j}^N)\right) \left[\mathrm{i} + \cot \frac{t_j - t_{\mu,j}^N}{2}\right].$$

This follows from the corresponding one-dimensional result in [11, Section 11.3] with some obvious modifications due to a slightly different choice of the space $\mathcal{T}_N$.

Lemma 2.3 *The set $\{L_\mu^N : \mu \in \mathbb{Z}_N^2\}$ is an orthogonal basis of $(\mathcal{T}_N, \|\cdot\|_{L^2(Q)})$ with*

$$\left(L_\mu^N, L_\alpha^N\right)_{L^2(S)} = \frac{\pi^2}{N_1\,N_2}\,\delta_{\alpha,\mu}, \qquad \mu, \alpha \in \mathbb{Z}_N^2\,.$$

Proof: From $P_N(\mathcal{T}_N) = \mathcal{T}_N$ and (4) it follows, that $\mathcal{T}_N = \mathrm{span}\{L_\mu^N : \mu \in \mathbb{Z}_N^2\}$. Moreover,

$$
\begin{aligned}
(L_\mu^N, L_\alpha^N)_{L^2(Q)} &= \frac{\pi^2}{4\,N_1^2\,N_2^2} \sum_{\nu,\iota \in \mathbb{Z}_N^2} \left(T^{(\nu)}(\cdot - t_\mu^N), T^{(\iota)}(\cdot - t_\alpha^N)\right)_{L^2(Q)} \\
&= \frac{(\pi^2)^2}{N_1^2\,N_2^2} \sum_{\nu,\iota \in \mathbb{Z}_N^2} \left(T^{(\nu)}, T^{(\iota)}\right)_{L^2(Q)} T^{(\iota)}(t_\alpha^N)\,T^{(\nu)}(-t_\mu^N) \\
&= \frac{\pi^3}{2\,N_1^2\,N_2^2} \sum_{\nu \in \mathbb{Z}_N^2} T^{(\nu)}(t_\alpha^N - t_\mu^N) = \frac{\pi^2}{N_1\,N_2}\,L_\mu^N(t_\alpha^N) = \frac{\pi^2\,\delta_{\alpha,\mu}}{N_1\,N_2}\,.
\end{aligned}
$$

$\blacksquare$

In some instances, products of functions from H_Q^s with smooth functions occur. For $m \in \mathbb{N}_0$ and $\psi \in C_{\mathrm{per}}^m$, we set

$$\|\psi\|_{\infty;m} := \sup_{t \in Q} |\psi(t)| + \max_{|\beta|=m} \sup_{t \in Q} |\partial^\beta \psi(t)|.$$

Lemma 2.4 *Let $s \geq 0$ and $\sigma \in \mathbb{N}_{\geq s}$. Suppose $\varphi \in H_Q^s$ and let $\psi \in C_{\mathrm{per}}^\sigma$. Then $\psi\varphi \in H_Q^s$ and*

$$\|\psi\varphi\|_{H_Q^s} \leq C\|\psi\|_{\infty;\sigma}\|\varphi\|_{H_Q^s},$$

where the constant $C > 0$ is independent of φ and ψ.

Proof: The assertion follows from the equivalence of the norm $\|\cdot\|_{H_Q^s}$ with the Sobolev-Slobodeckiĭ norm (see [11, Section 11.3] for a detailed exposition in 1D). $\blacksquare$

In particular, we are interested in an estimate of this kind when the smooth factor is a trigonometric monomial.

Lemma 2.5 *Let $\sigma \in \mathbb{N}$. Then $\|T^{(\mu)}\|_{\infty;\sigma} \leq \frac{1}{2\pi}\left(1 + |\mu|^\sigma\right)$ for all $\mu \in \mathbb{Z}^2$.*

Proof: Let $\beta \in \mathbb{N}_0^2$ with $|\beta| = \sigma$. Then, for $\mu \in \mathbb{Z}^2$ and $t \in \overline{Q}$,

$$\partial^\beta T^{(\mu)}(t) = \mathrm{i}^{|\beta|}\,\mu_1^{\beta_1}\,\mu_2^{\beta_2}\,T^{(\mu)}(t),$$

and hence
$$\left|\partial^\beta T^{(\mu)}(t)\right| \leq \frac{|\mu_1|^{\beta_1}\,|\mu_2|^{\beta_2}}{2\pi} \leq \frac{|\mu|_\infty^{|\beta|}}{2\pi}\,.$$

Since $|\mu|_\infty \leq |\mu|$, we finally obtain
$$\|T^{(\mu)}\|_{\infty;\sigma} \leq \frac{1}{2\pi}\left(1 + |\mu|^\sigma\right).$$

∎

In the later analysis, functions which are Q-periodic with respect to several independent variables will occur. Such functions can be expanded into a Fourier series with respect to one of these variables. The behaviour of the Fourier coefficients in such expansions will be of importance.

Lemma 2.6 *Let $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ be Q-periodic with respect to both arguments. Then*
$$F(t,\tau) = \sum_{\lambda \in \mathbb{Z}^2} F^{(\lambda)}(t)\,T^{(\lambda)}(\tau), \qquad t,\tau \in \mathbb{R}^2,$$

holds pointwise, where $(F^{(\lambda)})_{\lambda \in \mathbb{Z}^2} \subseteq C_{\mathrm{per}}^\infty$. Moreover, for any $m \in \mathbb{N}_0$ and any multi-index $\beta \in \mathbb{N}_0^2$ there exists a constant $C > 0$ such that
$$\sup_{\lambda \in \mathbb{Z}^2}\sup_{t \in \mathbb{R}^2}\left(1 + |\lambda|^2\right)^m \left|\partial^\beta F^{(\lambda)}(t)\right| \leq C\,\|F\|_{\infty;|\beta|+2m}\,.$$

Proof: The series representation of F in the lemma is obtained by expanding F into a Fourier series with respect to the second argument; here pointwise convergence holds due to the smoothness of $F(t,\cdot)$ for all $t \in \mathbb{R}^2$. In particular, we have
$$F^{(\lambda)}(t) = \int_Q F(t,\tau)\,T^{(-\lambda)}(\tau)\,\mathrm{d}\tau, \qquad \lambda \in \mathbb{Z}^2, \quad t \in \mathbb{R}^2,$$

and well-known facts about parameter-dependent integrals yield $F^{(\lambda)} \in C_{\mathrm{per}}^\infty$. Furthermore,
$$\left|\partial^\beta F^{(\lambda)}(t)\right| \leq 2\pi\,\|F\|_{\infty;|\beta|}, \qquad t \in \mathbb{R}^2, \quad \beta \in \mathbb{N}_0^2.$$

Now, let $m \in \mathbb{N}_0$. In the remainder of the proof, we make use of the estimate,
$$\left(1 + |\lambda|^2\right)^m \leq 2^m\left(1 + |\lambda|^{2m}\right), \qquad \text{for all } \lambda \in \mathbb{Z}^2,$$

and the identity
$$|\lambda|^{2m} = \sum_{k=0}^m \binom{m}{k}\lambda_1^{2k}\lambda_2^{2(m-k)} = \sum_{\alpha \in \mathbb{N}_0^2:|\alpha|=m} \tfrac{m!}{\alpha_1!\alpha_2!}\lambda_1^{2\alpha_1}\lambda_2^{2\alpha_2}.$$

Let $\lambda \in \mathbb{Z}^2$, $t \in \mathbb{R}^2$, $\beta \in \mathbb{N}_0^2$ and $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = m$. Then
$$\tfrac{m!}{\alpha_1!\alpha_2!}\lambda_1^{2\alpha_1}\lambda_2^{2\alpha_2}\left|\partial^\beta F^{(\lambda)}(t)\right|$$
$$= \tfrac{m!}{\alpha_1!\alpha_2!}\left|\int_Q \partial_t^\beta F(t,\tau)\underbrace{\left(-\mathrm{i}\right)^{2|\alpha|}\lambda_1^{2\alpha_1}\lambda_2^{2\alpha_2}\,T^{(-\lambda)}(\tau)}_{=\partial^{(2\alpha_1,2\alpha_2)}T^{(-\lambda)}(\tau)}\,\mathrm{d}\tau\right|$$

6

$$= \frac{m!}{\alpha_1!\alpha_2!} \left| \int_Q \partial_t^\beta \partial_\tau^{(2\alpha_1, 2\alpha_2)} F(t,\tau) \, T^{(-\lambda)}(\tau) \, \mathrm{d}\tau \right| \leq 2\pi \, \frac{m!}{\alpha_1!\alpha_2!} \|F\|_{\infty; |\beta|+2m} \, ,$$

where we used integration by parts in the third line. Hence,

$$|\lambda|^{2m} \big| \partial^\beta F^{(\lambda)}(t) \big| = \sum_{|\alpha|=m} \frac{m!}{\alpha_1!\alpha_2!} \lambda_1^{2\alpha_1} \lambda_2^{2\alpha_2} \big| \partial^\beta F^{(\lambda)}(t) \big|$$

$$\leq 2\pi \, \|F\|_{\infty; |\beta|+2m} \sum_{|\alpha|=m} \frac{m!}{\alpha_1!\alpha_2!} = 2^{m+1} \, \pi \, \|F\|_{\infty; |\beta|+2m} \, .$$

From these we obtain

$$\big(1 + |\lambda|^2\big)^m \big| \partial^\beta F^{(\lambda)}(t) \big| \leq 2^m \big(1 + |\lambda|^{2m}\big) \big| \partial^\beta F^{(\lambda)}(t) \big|$$

$$\leq 2^{m+1} \pi \left(\|F\|_{\infty; |\beta|} + 2^m \, \|F\|_{\infty; |\beta|+2m} \right) .$$

Since $\lambda \in \mathbb{Z}^2$ and $t \in \mathbb{R}^2$ were chosen arbitrarily, the proof is completed by observing the boundedness of the imbedding of $C_{\mathrm{per}}^{|\beta|+2m}$ into $C_{\mathrm{per}}^{|\beta|}$.

∎

3 The Approach for a Single Biperiodic Integral Equation

We now return to considering the integral equation (2). We will impose the following assumptions on the kernel function and the right hand side:

Assumption 3.1 *The kernel function has the representation $k = k_1 + k_2$, where $k_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and Q-periodic with respect to both variables while k_1 is Q-periodic with respect to both arguments and $k_1 \in C^\infty(\overline{Q} \times \overline{Q} \setminus \{(t,t) : t \in \overline{Q}\})$. Moreover, for every multi-index $\alpha \in \mathbb{N}_0^2$, the estimate*

$$|\partial^\alpha k_1(t,\tau)| \leq \frac{C}{\min_{\nu \in \mathbb{Z}^2} |t - \tau - 2\pi \, \nu|^{1+|\alpha|}} \, , \qquad t, \tau \in \mathbb{R}^2 \, ,$$

is satisfied.

For some $0 < \varrho_0 < \pi$, setting $\ell(t, r, v) = |r| \, k_1(t, t + rv)$, $t \in Q$, $r \in [-\varrho_0, \varrho_0]$, $v \in \mathbb{S}^1$, we assume that $\ell \in C^\infty(Q \times [-\varrho_0, \varrho_0] \times \mathbb{S}^1)$.

We also assume $\psi \in H_Q^s$.

Hence, k_1 is assumed to be weakly singular, but with a special type of singularity that can be removed by a transformation to polar coordinates around the singularity. In particular, many boundary integral operators exhibit this type of singularity, as will be discussed in Section 4.

To make use of the assumption in the numerical method, we require appropriate cut off functions. For $0 < \delta < \varepsilon < \pi$ define

$$\chi_{\delta,\varepsilon}(\tau) = \begin{cases} 1 \, , & |\tau| \leq \delta \, , \\ \tilde{\chi}\left(\frac{\varepsilon - |\tau|}{\varepsilon - \delta}\right) , & \delta < |\tau| < \varepsilon \, , \\ 0 \, , & \tau \in Q, \ |\tau| \geq \varepsilon \, , \end{cases}$$

with

$$\tilde{\chi}(s) = \frac{e^{-1/s}}{e^{-1/s} + e^{-1/(1-s)}}, \quad s \in (0, 1).$$

On all of $\mathbb{R}^2$, $\chi_{\delta,\varepsilon}$ is assumed to be Q-periodic. Furthermore, an argument by induction shows that for any $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = m$,

$$\partial^\alpha \chi_{\delta,\varepsilon}(\tau) = \sum_{\ell=1}^m \frac{p_\ell^\alpha(\tau)}{(\delta - \varepsilon)^\ell |\tau|^{2m-\ell}} \tilde{\chi}^{(\ell)}\left(\frac{\varepsilon - |\tau|}{\varepsilon - \delta}\right),$$

where p_ℓ^α are either homogeneous polynomials of degree m or the zero function. From this representation, we obtain the estimate

$$|\partial^\alpha \chi_{\delta,\varepsilon}(t)| \le C_\alpha \sum_{\ell=1}^m \frac{\varepsilon^m}{\delta^{2m-\ell} (\varepsilon - \delta)^\ell}, \quad t \in \mathbb{R}^2. \tag{5}$$

Usually, we will fix numbers $0 < \delta_1 < \delta_2$ and $0 < \varrho < \pi/\delta_2$ and consider $\chi_{\delta_1\varrho,\delta_2\varrho}$. In this case, (5) simplifies to

$$|\partial^\alpha \chi_{\delta_1\varrho,\delta_2\varrho}(t)| \le C_{\alpha,\delta_1,\delta_2} \varrho^{-m}, \tag{6}$$

with a constant $C_{\alpha,\delta_1,\delta_2}$ independent of ϱ.

With the help of these cut-off functions, the integral operator from (2) can be split into a weakly singular operator localized around the singularity and a globally acting operator with a smooth kernel. Fixing numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $0 < \varrho < \varrho_0$, we write

$$\int_Q k(t, \tau) \varphi(\tau) \, d\tau$$

$$= \int_Q k_1(t, \tau) \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t) \varphi(\tau) \, d\tau$$

$$+ \int_Q [k_1(t, \tau)(1 - \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t)) + k_2(t, \tau)] \varphi(\tau) \, d\tau$$

$$= \int_Q k_1(t, \tau) \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t) \chi_{\varepsilon_2\varrho,\varrho}(\tau - t) \varphi(\tau) \, d\tau$$

$$+ \int_Q [k_1(t, \tau)(1 - \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t)) + k_2(t, \tau)] \varphi(\tau) \, d\tau.$$

Note that at this point, $\chi_{\varepsilon_2\varrho,\varrho}$ serves no purpose as it is identical to one on the support of $\chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}$. Its significance will become clear later when we discuss approximations of the weakly singular integral operator.

Setting

$$k_{\text{smooth}}(t, \tau) = k_1(t, \tau)(1 - \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t)) + k_2(t, \tau), \tag{7}$$

and introducing the operators

$$J_1 \varphi(t) = \int_Q k_1(t, \tau) \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t) \chi_{\varepsilon_2\varrho,\varrho}(\tau - t) \varphi(\tau) \, d\tau,$$

$$\hspace{6cm} t \in Q,$$

$$J_2 \varphi(t) = \int_Q k_{\text{smooth}}(t, \tau) \varphi(\tau) \, d\tau,$$

we have written the integral equation in the form

$$\varphi - J_1\varphi - J_2\varphi = \psi \qquad \text{on } Q. \tag{8}$$

We proceed by rewriting $J_1\varphi(t)$ via a transformation in polar coordinates around t. We set

$$\Pi(p) = r \frac{\varrho}{\pi} \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix}, \qquad p = (r,\vartheta)^\top \in Q,$$

and

$$k_{\text{polar}}(t,p) = \frac{|r|\,\varrho^2}{2\,\pi^2}\, k_1(t, t + \Pi(p))\, \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\Pi(p)), \qquad t, p = (r,\vartheta)^\top \in Q.$$

Substituting $\tau = t + \Pi(p)$ in the expression for the operator J_1 gives

$$J_1\varphi(t) = \int_Q k_{\text{polar}}(t,p)\, \chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\, \varphi(t + \Pi(p))\,\mathrm{d}p, \qquad t \in Q. \tag{9}$$

Because of Assumption 3.1 and $k_{\text{polar}}(t,p) = 0$ for $|\Pi(p)| \geq \varepsilon_2\varrho$, we have that $k_{\text{polar}} \in C^\infty(\overline{Q} \times \overline{Q})$ and that this function can be extended Q-periodically to $\mathbb{R}^2$ with the same smoothness with respect to both arguments. Note that $\chi_{\varepsilon_2\varrho,\varrho}(\Pi(\cdot))\,\varphi(t + \Pi(\cdot))$ can also be Q-periodically extended to $\mathbb{R}^2$ without loss of regularity.

We want to solve the integral equation (8) numerically using a collocation method on the space $\mathcal{T}_N$. Thus, the semidiscrete problem is to find $\varphi_N \in \mathcal{T}_N$ such that

$$\varphi_N - P_N J_1\varphi_N - P_N J_2\varphi_N = P_N\psi \qquad \text{on } Q. \tag{10}$$

A fully discrete method is obtained in several steps. Firstly, both integrals are replaced by composite trapezoidal rules which are highly efficient for periodic functions. For $M, N \in \mathbb{N}^2$, we set for $\varphi \in H_Q^s$

$$\begin{aligned}
J_{1,M}\varphi(t) &= \int_Q P_M \left[k_{\text{polar}}(t,\cdot)\, \chi_{\varepsilon_2\varrho,\varrho}(\Pi(\cdot))\, \varphi(t + \Pi(\cdot)) \right](p)\,\mathrm{d}p \\
&= \frac{\pi^2}{M_1 M_2} \sum_{\nu \in \mathbb{Z}_M^2} k_{\text{polar}}(t, t_\nu^M)\, \chi_{\varepsilon_2\varrho,\varrho}(\Pi(t_\nu^M))\, \varphi(t + \Pi(t_\nu^M)) \tag{11}
\end{aligned}$$

$$\begin{aligned}
J_{2,N}\varphi(t) &= \int_Q P_N \left[k_{\text{smooth}}(t,\cdot)\, \varphi \right](\tau)\,\mathrm{d}\tau \\
&= \frac{\pi^2}{N_1 N_2} \sum_{\nu \in \mathbb{Z}_N^2} k_{\text{smooth}}(t, t_\nu^N)\, \varphi(t_\nu^N). \tag{12}
\end{aligned}$$

While both operators are discrete in principle, only $J_{2,N}$ can be used directly. The expression for $J_{1,M}$ involves the evaluation of $\varphi(t + \Pi(t_\nu^M))$. An exact evaluation requires the knowledge of $L_\mu^N(\Pi(t_\nu^M))$ for all $\mu \in \mathbb{Z}_N^2$, $\nu \in \mathbb{Z}_M^2$ which amounts to $\mathrm{O}(N_1 N_2 M_1 M_2)$ operations. In [4–6] the quadrature rule in radial direction is slightly perturbed and the values of $\varphi(t + \Pi(\cdot))$ in the quadrature points are obtained to high accuracy by fixed degree polynomial interpolation. However, this approach limits the asymptotic convergence rate.

The approach of [1] is a collocation method and uses the exact values of $\varphi(t + \Pi(\cdot))$ in the quadrature points. Here, we modify the scheme by reducing the cost in the approximation of J_1. We require the orthogonal projection O_M from $L^2(Q)$ onto $\mathcal{T}_M$,

$$O_M v = \sum_{\mu \in \mathbb{Z}_M^2} \left(v, T^{(\mu)} \right)_{L^2(Q)} T^{(\mu)} = \frac{M_1 M_2}{\pi^2} \sum_{\mu \in \mathbb{Z}_M^2} (v, L_\mu^M)_{L^2(Q)} L_\mu^M \tag{13}$$

for $v \in L^2(Q)$, where the second representation is due to Lemma 2.3. Let $1 < \varepsilon_3$ denote a number such that $\varepsilon_3 \varrho \leq \varrho_0$. A scaled projection for functions on $Q_\varrho = (-\varepsilon_3 \varrho, \varepsilon_3 \varrho)^2$ is given by

$$\tilde{O}_M v = O_M \left[v \left(\frac{\varrho}{\pi} \cdot \right) \right] \left(\frac{\pi}{\varrho} \cdot \right), \qquad v \in L^2(Q_\varrho).$$

We define for $M, \tilde{M} \in \mathbb{N}^2$,

$$J_{1,M,\tilde{M}} \varphi(t) = \int_Q P_M \Big[k_{\text{polar}}(t, \cdot)$$

$$O_M \left[\left\{ \chi_{\varepsilon_2 \varrho, \varrho} \tilde{O}_{\tilde{M}} \left[\chi_{\varrho, \varepsilon_3 \varrho} \varphi(t + \cdot) \right] \right\} \circ \Pi \right] \Big] (p) \, \mathrm{d}p. \tag{14}$$

As we will see in the following analysis, the reasons for introducing the additional approximations are the possibility for proving a convergence and stability theorem in the case of O_M and the reduction of computational complexity in the case of $\tilde{O}_{\tilde{M}}$.

For the remaining part of this section, we focus on the convergence analysis of the approach introduced above. We will start with properties of the operators J_2 and $J_{2,N}$ which are simpler to analyse.

Theorem 3.2 *Let $s \geq 0$. Then $J_2 : H_Q^s \to H_Q^{s+1}$ is a well-defined, bounded linear operator with $\|J_2\| \leq C \varrho^{-\max\{5, s+3\}}$ for all $\varrho \leq \varrho_0$ with C dependent on k, ε_1 and ε_2.*

Proof: We write k_{smooth} using its Fourier series representation from Lemma 2.6,

$$k_{\text{smooth}}(t, \tau) = \sum_{\lambda \in \mathbb{Z}^2} k_{\text{smooth}}^{(\lambda)}(t) \, T^{(\lambda)}(\tau), \qquad t, \tau \in \mathbb{R}^2.$$

Let $\varphi \in H_Q^s$ and denote by $\sigma = \lfloor s \rfloor$ the largest integer smaller or equal to s. By Lemma 2.4, there holds

$$\left\| k_{\text{smooth}}^{(\lambda)} \right\|_{H_Q^{s+1}} = 2\pi \left\| k_{\text{smooth}}^{(\lambda)} T^{(0)} \right\|_{H_Q^{s+1}}$$

$$\leq C \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{\infty; \sigma+2} \left\| T^{(0)} \right\|_{H_Q^{s+1}} \leq C \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{\infty; \sigma+2}$$

for all $\lambda \in \mathbb{Z}^2$. Therefore,

$$\|J_2 \, \varphi\|_{H_Q^{s+1}} \leq \sum_{\lambda \in \mathbb{Z}^2} \left| \int_Q T^{(\lambda)}(\tau) \, \varphi(\tau) \, \mathrm{d}\tau \right| \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{H_Q^{s+1}}$$

$$= \sum_{\lambda \in \mathbb{Z}^2} |\varphi_{-\lambda}| \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{H_Q^{s+1}}$$

$$\leq C \left\| \varphi \right\|_{H_Q^s} \left(\sum_{\lambda \in \mathbb{Z}^2} \left(1 + |\lambda|^2 \right)^{-\sigma} \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{\infty;\sigma+2}^2 \right)^{1/2}$$

$$\leq C \left\| \varphi \right\|_{H_Q^s} \sup_{\lambda \in \mathbb{Z}^2} \left[\left(1 + |\lambda|^2 \right)^{\max\{(2-\sigma)/2,0\}} \left\| k_{\text{smooth}}^{(\lambda)} \right\|_{\infty;\sigma+2} \right]$$

$$\times \left(\sum_{\lambda \in \mathbb{Z}^2} \left(1 + |\lambda|^2 \right)^{-2} \right)^{1/2}$$

$$\leq C \left\| \varphi \right\|_{H_Q^s} \left\| k_{\text{smooth}} \right\|_{\infty;\sigma+2+\max\{2-\sigma,0\}}$$

$$= C \left\| \varphi \right\|_{H_Q^s} \left\| k_{\text{smooth}} \right\|_{\infty;\max\{4,\sigma+2\}}.$$

where the last estimate is due to Lemma 2.6. Here, and throughout the paper, we denote by C a generic constant that may be different in each occurence.

Define the set $\Omega_\varrho = \{(t,\tau) \in \mathbb{R}^2 \times \mathbb{R}^2 : |t - \tau + 2\pi\nu| \geq \varepsilon_2 \varrho \text{ for all } \nu \in \mathbb{Z}^2\}$. We proceed to bound for $m \in \mathbb{N}_0$ using Assumption 3.1 and (6)

$$\left\| k_{\text{smooth}} \right\|_{\infty;m} \leq \left\| k_2 \right\|_{\infty;m} + \left\| k_1 \right\|_\infty$$
$$+ C \sum_{|\alpha|+|\beta|=m} \left\| \partial_t^\alpha k_1(\cdot,\cdot) \right\|_{\infty;\Omega_\varrho} \left\| \partial^\beta (1 - \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}) \right\|_{\infty;\mathbb{R}^2} \leq C \varrho^{-m-1}$$

for $\varrho \leq \varrho_0$, which completes the proof. ∎

Theorem 3.3 *Let $s > 1$ and $t \in [0,s]$. Then $J_{2,N} : H_Q^s \to H_Q^{t+1}$ is a well-defined, bounded linear operator. Moreover,*

$$\left\| (J_2 - J_{2,N}) \varphi \right\|_{H_Q^{t+1}} \leq C \, \varrho^{-2s-6} \, \frac{(\max\{N_1, N_2\})^t}{(\min\{N_1, N_2\})^s} \left\| \varphi \right\|_{H_Q^s}$$

for all $\varphi \in H_Q^s$, $\varrho \leq \varrho_0$ and all $N \in \mathbb{N}^2$, where C depends on k, ε_1 and ε_2.

Proof: Let $\sigma \in \mathbb{N}_{\geq s}$. From Lemma 2.2, we conclude

$$\left| \int_Q \left(T^{(\lambda)} \varphi - P_N[T^{(\lambda)}\varphi] \right)(\tau) \, d\tau \right| \leq 2\pi \left\| T^{(\lambda)}\varphi - P_N[T^{(\lambda)}\varphi] \right\|_{H_Q^t}$$

$$\leq C \, \frac{(\max\{N_1, N_2\})^t}{(\min\{N_1, N_2\})^s} \left\| T^{(\lambda)}\varphi \right\|_{H_Q^s}.$$

From Lemmas 2.4 and 2.5, we obtain

$$\left\| T^{(\lambda)}\varphi \right\|_{H_Q^s} \leq C \left\| T^{(\lambda)} \right\|_{\infty;\sigma} \left\| \varphi \right\|_{H_Q^s} \leq C \left(1 + |\lambda|^\sigma \right) \left\| \varphi \right\|_{H_Q^s} \leq C \left(1 + |\lambda|^2 \right)^{\frac{\sigma}{2}} \left\| \varphi \right\|_{H_Q^s}$$

so that

$$\left| \int_Q \left(T^{(\lambda)}\varphi - P_N[T^{(\lambda)}\varphi] \right)(\tau) \, d\tau \right| \leq C \, \frac{(\max\{N_1, N_2\})^t}{(\min\{N_1, N_2\})^s} \left(1 + |\lambda|^2 \right)^{\sigma/2} \left\| \varphi \right\|_{H_Q^s}.$$

11

Thus

$$\|(J_2 - J_{2,N})\,\varphi\|_{H_Q^{t+1}} \le \sum_{\lambda \in \mathbb{Z}^2} \left| \int_Q \big(T^{(\lambda)}\varphi - P_N[T^{(\lambda)}\varphi]\big)(\tau)\,\mathrm{d}\tau \right| \big\|k_{\mathrm{smooth}}^{(\lambda)}\big\|_{H_Q^{t+1}}$$

$$\le C\,\frac{(\max\{N_1, N_2\})^t}{(\min\{N_1, N_2\})^s}\,\|\varphi\|_{H_Q^s} \sum_{\lambda \in \mathbb{Z}^2} \big(1 + |\lambda|^2\big)^{\sigma/2}\,\big\|k_{\mathrm{smooth}}^{(\lambda)}\big\|_{\infty;\sigma+1}$$

and again Lemma 2.6 completes the proof as the remaining argument is very similar to that at the end of Theorem 3.2. $\blacksquare$

The derivation of a similar result for the approximation $J_{1,M,\tilde{M}}$ of J_1 as introduced in (14) is more complicated. The coordinate transform in polar coordinates around the singularity has removed the singularity. However, the integral operator now takes on a non-standard form which makes the analysis of its mapping properties much more involved.

To simplify the considerations, let us rewrite J_1 in terms of expressions that are easier to analyse. Writing k_{polar} as a Fourier series with respect to p,

$$k_{\mathrm{polar}}(t, p) = \sum_{\lambda \in \mathbb{Z}^2} k_{\mathrm{polar}}^{(\lambda)}(t)\,T^{(\lambda)}(p), \qquad t, p \in Q\,, \tag{15}$$

we formally have

$$J_1\varphi(t) = \sum_{\lambda \in \mathbb{Z}^2} k_{\mathrm{polar}}^{(\lambda)}(t) \int_Q T^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,\varphi(t + \Pi(p))\,\mathrm{d}p\,.$$

The later analysis will show that interchanging integration and summation is indeed justified. Recalling $Q_\varrho = (-\varepsilon_3\varrho, \varepsilon_3\varrho)^2$, and defining the scaled trigonometric monomials

$$T_{Q_\varrho}^{(\nu)}(\tau) = \frac{1}{2\varepsilon_3\varrho}\,\exp\left(\mathrm{i}\,\frac{\pi}{\varepsilon_3\varrho}\,\tau \cdot \nu\right), \qquad \tau \in Q_\varrho\,,$$

consider functions u of $t \in Q$, $\tau \in Q_\varrho$. These can be expanded into Fourier series with respect to both variables,

$$u(t, \tau) = \sum_{\mu,\nu \in \mathbb{Z}^2} u_{\mu,\nu}\,T^{(\mu)}(t)\,T_{Q_\varrho}^{(\nu)}(\tau)\,.$$

For $s \ge 0$, we introduce the vector space

$$\mathcal{H}_{Q,Q_\varrho}^s = \Big\{ u \in L^2(Q \times Q_\varrho) :$$

$$\sum_{\mu,\nu \in \mathbb{Z}^2} (1 + |\mu|^2)^s (1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^\sigma |u_{\mu,\nu}|^2 < \infty \text{ for all } \sigma \ge 0 \Big\}.$$

Remark 3.4 *For all $0 \le t \le s$, $\mathcal{H}_{Q,Q_\varrho}^s$ is a subspace of $\mathcal{H}_{Q,Q_\varrho}^t$.*

For convenience, set for $u \in \mathcal{H}_{Q,Q_\varrho}^s$ and $\sigma \ge 0$

$$p_{s,\sigma}(u) := \sum_{\mu,\nu \in \mathbb{Z}^2} (1 + |\mu|^2)^s (1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^\sigma |u_{\mu,\nu}|^2 \qquad \text{and}$$

12

$$q_{s,\sigma}(u) := \sum_{\mu,\nu\in\mathbb{Z}^2} (1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^s (1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^\sigma |u_{\mu,\nu}|^2.$$

Between $p_{s,\sigma}$ and $q_{s,\sigma}$, there holds a certain equivalence relation. For $u \in \mathcal{H}^s_{Q,Q_\varrho}$ and $\sigma \geq 0$, we estimate

$$p_{s,\sigma}(u) = \sum_{\mu,\nu\in\mathbb{Z}^2} \left(1 + |\mu|^2\right)^s \left(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^\sigma |u_{\mu,\nu}|^2$$

$$\leq 2^s \sum_{\mu,\nu\in\mathbb{Z}^2} \left(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^s \left(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^\sigma |u_{\mu,\nu}|^2$$

$$\leq 2^s \sum_{\mu,\nu\in\mathbb{Z}^2} \left(1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^s \left(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^{\sigma+s} |u_{\mu,\nu}|^2.$$

Thus

$$p_{s,\sigma}(u) \leq 2^s\, q_{s,\sigma+s}(u)\,, \tag{16}$$

and by similar arguments also $q_{s,\sigma}(u) \leq 2^s\, p_{s,\sigma+s}(u)$.

Two technical lemmas yield most results required to establish the mapping properties of J_1.

Lemma 3.5 *Denote by $\widehat{\chi_{\varrho,\varepsilon_3\varrho}}$ the Fourier transform of the extension of $\chi_{\varrho,\varepsilon_3\varrho}|_Q$ to $\mathbb{R}^2$ by 0. Then for any $\sigma \in \mathbb{N}_0$ and $\varepsilon_3\varrho \leq \varrho_0$,*

$$\sup_{x\in\mathbb{R}^2} \left[(1+|x|^2)^\sigma \left| \widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) \right| \right] \leq C\, \varrho^{-2\sigma+2}\,,$$

where the constant C depends only on σ and ε_3.

Proof: We note

$$\chi_{\varrho,\varepsilon_3\varrho}(\varrho t) = \chi_{1,\varepsilon_3}(t)\,, \qquad t \in Q\,.$$

Let $B_\delta = \{ t \in \mathbb{R}^2 : |t| < \delta \}$. Then

$$\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) = \int_{B_{\varepsilon_3\varrho}} \chi_{\varrho,\varepsilon_3\varrho}(t)\, \mathrm{e}^{-\mathrm{i}t\cdot x}\, \mathrm{d}t = \varrho^2 \int_{B_{\varepsilon_3}} \chi_{1,\varepsilon_3}(t)\, \mathrm{e}^{-\mathrm{i}\varrho\, t\cdot x}\, \mathrm{d}t\,.$$

Let $R > 0$ and consider $x = |x|\, \hat{x}$ with $|x| \geq R$. We rewrite the integral using the divergence theorem as

$$\int_{B_{\varepsilon_3}} \chi_{1,\varepsilon_3}(t)\, \mathrm{e}^{-\mathrm{i}\varrho\, t\cdot x}\, \mathrm{d}t$$

$$= \int_{B_{\varepsilon_3}} \left\{ \frac{\hat{x}\cdot\nabla\chi_{1,\varepsilon_3}(t)\, \mathrm{e}^{-\mathrm{i}\varrho|x|\, t\cdot\hat{x}}}{\mathrm{i}\,|x|\,\varrho} - \nabla_t\cdot\left[\frac{\hat{x}\,\chi_{1,\varepsilon_3}(t)\, \mathrm{e}^{-\mathrm{i}\varrho|x|\, t\cdot\hat{x}}}{\mathrm{i}\,|x|\,\varrho} \right] \right\} \mathrm{d}t$$

$$= \frac{1}{\mathrm{i}\varrho|x|} \int_{B_{\varepsilon_3}} \hat{x}\cdot\nabla\chi_{1,\varepsilon_3}(t)\, \mathrm{e}^{-\mathrm{i}\varrho|x|\, t\cdot\hat{x}}\, \mathrm{d}t\,.$$

We repeat this argument $2\sigma - 1$ times to obtain

$$\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) = \frac{\varrho^2}{(\mathrm{i}\varrho|x|)^{2\sigma}} \int_{B_{\varepsilon_3}} h(t)\, \mathrm{e}^{-\mathrm{i}\varrho|x|\, t\cdot\hat{x}}\, \mathrm{d}t$$

with some function h depending on ε_3 and continuously on $\hat{x}$. The assertion follows by applying the triangular inequality for integrals and taking the maximum with respect to $\hat{x}$.

For $|x| \leq R$, the assertion follows from $|\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x)| \leq C\, (\varepsilon_3\varrho)^2$ and $\varepsilon_3\varrho \leq \varrho_0$. ∎

Lemma 3.6 *Let $s \geq 0$, $\varepsilon_3 \varrho \leq \varrho_0$.*

(a) *For $\varphi \in H_Q^s$ define*
$$\mathcal{M}\varphi(t, \tau) := \chi_{\varrho, \varepsilon_3 \varrho}(\tau)\, \varphi(t + \tau)\,, \qquad (t, \tau) \in Q \times Q_\varrho.$$

Then $\mathcal{M}\varphi \in \mathcal{H}_{Q,Q_\varrho}^s$, and for all $\sigma \geq 0$,
$$p_{s,\sigma}(\mathcal{M}\varphi) \leq C\, \varrho^{-2\sigma-2}\, \|\varphi\|_{H_Q^s}^2\,,$$

where the constant C depends only on σ and ε_3.

(b) *For $\lambda \in \mathbb{Z}^2$ and $u \in \mathcal{H}_{Q,Q_\varrho}^s$, set*
$$\mathcal{J}^{(\lambda)}u(t) := \int_Q T^{(\lambda)}(p)\, \chi_{\varepsilon_2 \varrho, \varrho}(\Pi(p))\, u(t, \Pi(p))\, \mathrm{d}p, \qquad t \in Q.$$

Then $(\mathcal{J}^{(\lambda)})_{\lambda \in \mathbb{Z}^2}$ is a family of linear operators mapping $\mathcal{H}_{Q,Q_\varrho}^s \to H_Q^{s+1}$. Moreover
$$\|\mathcal{J}^{(\lambda)}u\|_{H_Q^{s+1}} \leq \frac{c}{\varrho}\,(1 + |\lambda|^2)\, \sqrt{p_{s,3}(u)}, \qquad \text{for all } u \in \mathcal{H}_{Q,Q_\varrho}^s \text{ and } \lambda \in \mathbb{Z}^2,$$

where $c > 0$ is a constant only depending on ε_2, ε_3 and s.

(c) *$(\mathcal{J}^{(\lambda)} \circ \mathcal{M})_{\lambda \in \mathbb{Z}^2}$ is a family of linear and bounded operators mapping $H_Q^s \to H_Q^{s+1}$. In particular,*
$$\|\mathcal{J}^{(\lambda)}\mathcal{M}\,\varphi\|_{H_Q^{s+1}} \leq C\, \varrho^{-5}\,(1 + |\lambda|^2)\|\varphi\|_{H_Q^s}, \qquad \text{for all } \varphi \in H_Q^s$$

and all $\lambda \in \mathbb{Z}^2$, the constant $C > 0$ only depending on ε_2, ε_3 and s.

Proof: (a) Let $s \geq 0$ and $\varphi \in H_Q^s$. In a first step, we calculate the Fourier-coefficients $u_{\mu,\nu}$ of $u = \mathcal{M}\varphi$. Therefore, let $\mu, \nu \in \mathbb{Z}^2$. Then
$$
\begin{aligned}
u_{\mu,\nu} &= \int_Q \int_{Q_\varrho} u(t, \tau)\, T^{(-\mu)}(t)\, T_{Q_\varrho}^{(-\nu)}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t \\
&= \frac{1}{2\varepsilon_3 \varrho} \int_{Q_\varrho} \chi_{\varrho, \varepsilon_3 \varrho}(\tau)\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{\varepsilon_3 \varrho}\nu \cdot \tau} \left(\frac{1}{2\pi} \int_Q \varphi(t + \tau)\, \mathrm{e}^{-\mathrm{i}\mu \cdot (t + \tau - \tau)}\, \mathrm{d}t \right) \mathrm{d}\tau \\
&= \frac{1}{2\varepsilon_3 \varrho} \int_{\mathbb{R}^2} \chi_{\varrho, \varepsilon_3 \varrho}(\tau)\, \mathrm{e}^{-\mathrm{i}(\frac{\pi}{\varepsilon_3 \varrho}\nu - \mu) \cdot \tau} \left(\frac{1}{2\pi} \int_{\tau + Q} \varphi(t')\, \mathrm{e}^{-\mathrm{i}\mu \cdot t'}\, \mathrm{d}t' \right) \mathrm{d}\tau \\
&= \frac{1}{2\varepsilon_3 \varrho}\, \widehat{\chi_{\varrho, \varepsilon_3 \varrho}}\left(\tfrac{\pi}{\varepsilon_3 \varrho}\nu - \mu\right) \varphi_\mu,
\end{aligned}
$$

where the last step holds due to the Q-periodicity of φ. Now, in a second step, for $\sigma \geq 0$, there holds
$$p_{s,\sigma}(u) = \frac{1}{(2\varepsilon_3 \varrho)^2} \sum_{\mu,\nu \in \mathbb{Z}^2} \left(1 + |\mu|^2\right)^s \left(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho}\nu|^2\right)^\sigma |\varphi_\mu|^2 \left|\widehat{\chi_{\varrho, \varepsilon_3 \varrho}}\left(\tfrac{\pi}{\varepsilon_3 \varrho}\nu - \mu\right)\right|^2$$

14

$$= \frac{1}{(2\varepsilon_3\varrho)^2} \sum_{\mu\in\mathbb{Z}^2} \left(1+|\mu|^2\right)^s |\varphi_\mu|^2$$

$$\times \left(\sum_{\nu\in\mathbb{Z}^2} \frac{\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^{\sigma+2}}{\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^2} \left|\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(\tfrac{\pi}{\varepsilon_3\varrho}\nu-\mu)\right|^2 \right).$$

From

$$\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^2}\mathrm{d}x \geq h^2 \sum_{\substack{\nu\in\mathbb{Z}^2 \\ \nu_1,\nu_2\neq 0}} \frac{1}{(1+|h\nu|^2)^2}\,, \qquad h>0\,,$$

and similar estimates for the remaining terms in the sum, we see that the value of the series $\sum_{\nu\in\mathbb{Z}^2}\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^{-2}$ is uniformly bounded in μ and ϱ for $\varepsilon_3\varrho\leq\varrho_0$. Thus from Lemma 3.5, the assertion follows.

(b) Using the Fourier series expansion of u, there holds

$$\mathcal{J}^{(\lambda)}u = \sum_{\mu,\nu\in\mathbb{Z}^2} u_{\mu,\nu} \int_Q T^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,T_{Q_\varrho}^{(\nu)}(\Pi(p))\,\mathrm{d}p\,T^{(\mu)}.$$

Suppose $\nu\neq 0$. We write $(\nu_1,\nu_2)^\top = q_\nu(\cos\vartheta_\nu,\sin\vartheta_\nu)^\top$ for some $q_\nu > 0$ and some $\vartheta_\nu\in(-\pi,\pi]$, and obtain

$$T_{Q_\varrho}^{(\nu)}(\Pi(p)) = \frac{1}{2\varepsilon_3\varrho}\exp\left(\mathrm{i}\,q_\nu\,(r/\varepsilon_3)\,\cos(\vartheta-\vartheta_\nu)\right), \qquad p=(r,\vartheta)\in Q.$$

Hence, the substitution $\vartheta' = \vartheta - \vartheta_\nu$ and the 2π-periodicity with respect to ϑ yield

$$\int_Q T^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,T_{Q_\varrho}^{(\nu)}(\Pi(p))\,\mathrm{d}p$$

$$= \frac{1}{2\varepsilon_3\varrho}\mathrm{e}^{\mathrm{i}\,\lambda_2\,\vartheta_\nu}\int_Q T^{(\lambda)}(r,\vartheta')\,[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi]\,(r,\vartheta'+\vartheta_\nu)\mathrm{e}^{\mathrm{i}\,q_\nu\,(r/\varepsilon_3)\,\cos\vartheta'}\,\mathrm{d}(r,\vartheta').$$

The behaviour of the integral in this expression with respect to λ and ν can be estimated by the method of stationary phase. A detailed proof is given in [1, Lemma 6.2]. We obtain

$$\left|\int_Q T^{(\lambda)}(p)\left[\chi_{\varepsilon_2\varrho,\varrho}T_{Q_\varrho}^{(\nu)}\right]\circ\Pi(p)\,\mathrm{d}p\right| \leq C\,\frac{\left\|T^{(\lambda)}\left[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi\right](\cdot,\cdot+\vartheta_\nu)\right\|_{\infty;2}}{q_\nu}\,.$$

Similarly as in the proof of Lemma 3.5 we observe that

$$[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi]\,(r,\vartheta+\vartheta_\nu) = \chi_{\varepsilon_2,1}\left(\frac{r}{\pi}\begin{pmatrix}\cos(\vartheta+\vartheta_\nu)\\\sin(\vartheta+\vartheta_\nu)\end{pmatrix}\right)$$

is independent of ϱ. Hence

$$\left|T^{(\lambda)}(p)\left[\chi_{\varepsilon_2\varrho,\varrho}T_{Q_\varrho}^{(\nu)}\right]\circ\Pi(p)\,\mathrm{d}p\right| \leq C\,\frac{\left\|T^{(\lambda)}\right\|_{\infty;2}}{q_\nu} \leq \sqrt{2}\,C\,\frac{\left\|T^{(\lambda)}\right\|_{\infty;2}}{(1+|\nu|^2)^{1/2}}\,. \tag{17}$$

Note that the final estimate is also true for $\nu=0$. Now, using Lemma 2.5, gives

$$\left|\int_Q T^{(\lambda)}(p)\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))T_{Q_\varrho}^{(\nu)}(\Pi(p))\,\mathrm{d}p\right| \leq C\,\frac{1+|\lambda|^2}{(1+|\nu|^2)^{1/2}}\,.$$

15

We proceed with

$$\left\|\mathcal{J}^{(\lambda)}u\right\|^2_{H^{s+1}_Q}$$

$$\leq C^2 \sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^{s+1}\left(\sum_{\nu\in\mathbb{Z}^2}\tfrac{1+|\lambda|^2}{(1+|\nu|^2)^{1/2}}|u_{\mu,\nu}|\right)^2$$

$$= C^2(1+|\lambda|^2)^2\sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^s\left(\sum_{\nu\in\mathbb{Z}^2}\tfrac{(1+|\mu|^2)^{1/2}}{(1+|\nu|^2)^{1/2}}|u_{\mu,\nu}|\right)^2$$

$$\leq C^2(1+|\lambda|^2)^2\sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^s$$

$$\times\left(\sum_{\nu\in\mathbb{Z}^2}\tfrac{(1+|\frac{\pi}{\varepsilon_3\varrho}\nu|^2)^{1/2}}{(1+|\nu|^2)^{1/2}}(1+|\mu-\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^{1/2}|u_{\mu,\nu}|\right)^2$$

$$\leq \frac{C^2}{\varrho^2}(1+|\lambda|^2)^2\sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^s\left(\sum_{\nu\in\mathbb{Z}^2}\tfrac{(1+|\mu-\frac{\pi}{\varepsilon_3\varrho}\nu|^2)^{3/2}}{1+|\mu-\frac{\pi}{\varepsilon_3\varrho}\nu|^2}|u_{\mu,\nu}|\right)^2. \tag{18}$$

As in the proof of part (a), the series $\sum_{\nu\in\mathbb{Z}^2}\left(1+|\mu-\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\right)^{-2}$ is bounded independently of μ and $\varrho<1$, so that we can apply the Hölder inequality for ℓ^2-series to obtain

$$\left\|\mathcal{J}^{(\lambda)}u\right\|^2_{H^{s+1}_Q} \leq \frac{2\,C^2}{(\varepsilon_3\varrho)^2}(1+|\lambda|^2)^2\sum_{\mu,\nu\in\mathbb{Z}^2}(1+|\mu|^2)^s(1+|\mu-\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2)^3|u_{\mu,\nu}|^2$$

$$= \frac{C^2}{\varrho^2}(1+|\lambda|^2)^2\,p_{s,3}(u)\,.$$

(c) The assertion follows directly by combining (a) and (b). ∎

With these preliminary considerations, we are now able to investigate the mapping properties of J_1.

Theorem 3.7 *Let $s\geq 0$. Then*

$$J_1 : H^s_Q \to H^{s+1}_Q$$

defined in (9) is a bounded linear operator with

$$\|J_1\varphi\|_{H^{s+1}_Q} \leq C\,\varrho^{-5}\,\|\varphi\|_{H^s_Q}$$

for $\varepsilon_3\varrho\leq\varrho_0$ with C depending only on s, ε_1, ε_2 and ε_3.

Proof: By definition, J_1 is a linear integral operator. It remains to show its boundedness from H^s_Q to H^{s+1}_Q. We rewrite J_1 slightly by inserting another cut-off function. We then expand k_{polar} into its Fourier series (15) and use the operators from Lemma 3.6 to obtain

$$J_1\varphi(t) = \int_Q k_{\text{polar}}(t,p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,\chi_{\varrho,\varepsilon_3\varrho}(\Pi(p))\,\varphi(t+\Pi(p))\,\mathrm{d}p$$

$$= \sum_{\lambda \in \mathbb{Z}^2} k_{\text{polar}}^{(\lambda)}(t) \, \mathcal{J}^{(\lambda)} \, \mathcal{M} \varphi(t) \, .$$

This is justified by the estimates for any $\sigma \in \mathbb{N}_{\geq s+1}$ using Lemma 3.6

$$\|J_1 \varphi\|_{H_Q^{s+1}} \leq C \sum_{\lambda \in \mathbb{Z}^2} \|k_{\text{polar}}^{(\lambda)}\|_{\infty;\sigma} \, \big\| \mathcal{J}^{(\lambda)} \mathcal{M} \, \varphi \big\|_{H_Q^{s+1}}$$

$$\leq C \, \varrho^{-5} \, \|\varphi\|_{H_Q^s} \sum_{\lambda \in \mathbb{Z}^2} \big(1 + |\lambda|^2 \big) \|k_{\text{polar}}^{(\lambda)}\|_{\infty;\sigma}$$

$$\leq C \, \varrho^{-5} \, \|\varphi\|_{H_Q^s} \sum_{\lambda \in \mathbb{Z}^2} \frac{1}{(1 + |\lambda|^2)^2} \big(1 + |\lambda|^2 \big)^3 \|k_{\text{polar}}^{(\lambda)}\|_{\infty;\sigma} \, .$$

The series converges as the two last factors are bounded by Lemma 2.6 with

$$\sup_{\lambda \in \mathbb{Z}^2} \big(1 + |\lambda|^2 \big)^3 \|k_{\text{polar}}^{(\lambda)}\|_{\infty;\sigma} \leq C \, \|k_{\text{polar}}\|_{\infty,\sigma+6} \, .$$

With $t, \, p = (r, \vartheta)^\top \in Q$ and setting $\hat{p} = (\cos \vartheta, \sin \vartheta)^\top$, we write $k_{\text{polar}}(t, p)$ as

$$k_{\text{polar}}(t, p) = \frac{\varrho}{2\pi} \, \ell \left(t, \frac{\varrho \, r}{\pi}, \hat{p} \right) \chi_{\varepsilon_1, \varepsilon_2} \left(\frac{r}{\pi} \hat{p} \right)$$

with the function ℓ from Assumption 3.1. It follows that $\|k_{\text{polar}}\|_{\infty,\sigma+6} \leq C$ uniformly for $\varepsilon_3 \varrho \leq \varrho_0$ with C depending only on σ, ε_1 and ε_2. This completes the proof. $\blacksquare$

We next wish to derive an analogue of Theorem 3.3 for J_1, i.e. an estimate for the difference $J_1 - J_{1,M,\tilde{M}}$. We do this in two steps, writing

$$J_1 - J_{1,M,\tilde{M}} = \left[J_1 - \tilde{J}_{1,\tilde{M}} \right] + \left[\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}} \right],$$

where

$$\tilde{J}_{1,\tilde{M}} \varphi(t) = \int_Q k_{\text{polar}}(t, p) \left[\left\{ \chi_{\varepsilon_2 \varrho, \varrho} \, \tilde{O}_{\tilde{M}} \left[\chi_{\varrho, \varepsilon_3 \varrho} \, \varphi(t + \cdot) \right] \right\} \circ \Pi \right] (p) \, \mathrm{d}p \tag{19}$$

for $t \in Q$, and bounding the two differences separately. Note, that using the projection $\mathcal{O}_{\tilde{M}}$,

$$\mathcal{O}_{\tilde{M}} u(\cdot, \cdot\cdot) = \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}_{\tilde{M}}^2} u_{\mu,\nu} \, T^{(\mu)}(\cdot) \, T_{Q_\varrho}^{(\nu)}(\cdot\cdot), \qquad u \in \mathcal{H}_{Q,Q_\varrho}^s, \tag{20}$$

we can write $\tilde{J}_{1,\tilde{M}}$ as

$$\tilde{J}_{1,\tilde{M}} \varphi = \sum_{\lambda \in \mathbb{Z}^2} k_{\text{polar}}^{(\lambda)} \, \mathcal{J}^{(\lambda)} \mathcal{O}_{\tilde{M}} \mathcal{M} \, \varphi$$

Some technical tools are collected in the next lemma.

Lemma 3.8 *Let $s \geq 0$, $\varepsilon_3 \varrho \leq \varrho_0$, $M = (M_1, M_2)^\top \in \mathbb{N}^2$ and recall the definitions of $\mathcal{J}^{(\lambda)}$ and $\mathcal{M}$ from Lemma 3.6.*

(a) *For all $u \in \mathcal{H}_{Q,Q_\varrho}^s$ and $\lambda \in \mathbb{Z}^2$,*

$$\|\mathcal{J}^{(\lambda)} u\|_{H_Q^{s+1}} \leq \frac{C}{\varrho} \big(1 + |\lambda|^2 \big) \sqrt{q_{s,s+3}(u)} \, ,$$

where C depends only on s, ε_2 and ε_3.

(b) For $0 \leq t \leq s$, $\sigma \geq 0$ and all $u \in \mathcal{H}^s_{Q,Q_\varrho}$,

$$q_{t,\sigma}\big((\mathcal{I} - \mathcal{O}_M)u\big) \leq \Big(\sqrt{2}\,\min\{M_1, M_2\}\Big)^{2(t-s)} q_{s,\sigma}(u)\,.$$

Here $\mathcal{I}$ denotes the identity operator.

(c) Let $0 \leq t \leq s$, $\lambda \in \mathbb{Z}^2$. Then $\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_M)\mathcal{M} : H^s_Q \to H^{t+1}_Q$ is bounded with

$$\|\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_M)\mathcal{M}\,\varphi\|_{H^{t+1}_Q} \leq \frac{C}{\varrho^{2s+5}}\,(1 + |\lambda|^2)\,(\min\{M_1, M_2\})^{t-s}\,\|\varphi\|_{H^s_Q}$$

for all $\varphi \in H^s_Q$, where the constant $C > 0$ only depends on s, t, ε_2 and ε_3.

Proof: (a) This follows from Lemma 3.6 (b) together with (16).

(b) Let $0 \leq t \leq s$ and $\sigma \geq 0$. Then

$$
\begin{aligned}
& q_{t,\sigma}\big((\mathcal{I} - \mathcal{O}_M)u\big) \\
&= \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_M} \big(1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^t \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\
&= \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_M} \big(1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^{t-s}\big(1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\
&\leq \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_M} \big(1 + |\nu|^2\big)^{t-s}\big(1 + |\tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\
&\leq \Big(\sqrt{2}\,\min\{M_1, M_2\}\Big)^{2(t-s)} q_{s,\sigma}(u)
\end{aligned}
$$

holds for all $u \in \mathcal{H}^s_{Q,Q_\varrho}$.

(c) Let $\varphi \in H^s_Q$ and set $u = \mathcal{M}\varphi$. Then, by Lemma 3.6 (a) and Remark 3.4, $u \in \mathcal{H}^t_{Q,Q_\varrho}$, and hence also $(\mathcal{I} - \mathcal{O}_M)u \in \mathcal{H}^t_{Q,Q_\varrho}$. From part (a) and (b) together with (16) and Lemma 3.6 (a), we obtain the estimate

$$
\begin{aligned}
\|\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_M)\,u\|_{H^{t+1}_Q} &\leq \frac{C}{\varrho}\,(1 + |\lambda|^2)\,\sqrt{q_{t,t+3}\big((I - \mathcal{O}_M)u\big)} \\
&\leq \frac{C}{\varrho}\,(1 + |\lambda|^2)\,(\min\{M_1, M_2\})^{t-s}\,\sqrt{q_{s,s+3}(u)} \\
&\leq \frac{C}{\varrho}\,(1 + |\lambda|^2)\,(\min\{M_1, M_2\})^{t-s}\,\sqrt{p_{s,2s+3}(u)} \\
&\leq \frac{C}{\varrho^{2s+5}}\,(1 + |\lambda|^2)\,(\min\{M_1, M_2\})^{t-s}\,\|\varphi\|_{H^s_Q},
\end{aligned}
$$

which is the desired result. $\blacksquare$

Theorem 3.9 Let $\tilde{M} \in \mathbb{N}^2$, $s \geq 0$ and $t \in [0, s]$. Then $\tilde{J}_{1,\tilde{M}} : H^s_Q \to H^{t+1}_Q$ defined in (19) is a well-defined, linear and bounded operator with

$$\|(J_1 - \tilde{J}_{1,\tilde{M}})\varphi\|_{H^{t+1}_Q} \leq \frac{C}{\varrho^{2s+5}}\,(\min\{\tilde{M}_1, \tilde{M}_2\})^{t-s}\|\varphi\|_{H^s_Q}$$

for all $\varphi \in H^s_Q$ and all $\varepsilon_3\varrho \leq \varrho_0$, where the constant $C > 0$ only depends on s, t, ε_2 and ε_3.

18

Proof: Let $\varphi \in H^s_Q$ and $\sigma \in \mathbb{N}_{\geq s+1}$. Proceeding analogously as in the proof of Theorem 3.7, from Lemma 3.8 (c) we obtain

$$\|(J_1 - \tilde{J}_{1,\tilde{M}})\varphi\|_{H^{t+1}_Q} \leq C \sum_{\lambda \in \mathbb{Z}^2} \|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma} \left\|\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_{\tilde{M}})\mathcal{M}\,\varphi\right\|_{H^{t+1}_Q}$$

$$\leq \frac{C}{\varrho^{2s+5}}\,(\min\{\tilde{M}_1, \tilde{M}_2\})^{t-s}\|\varphi\|_{H^s_Q} \sum_{\lambda \in \mathbb{Z}^2}\left(1 + |\lambda|^2\right)\|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma}\,.$$

The remainder of the proof is identical to the last arguments in the proof of Theorem 3.7. ∎

Theorem 3.10 *Let $M, \tilde{M} \in \mathbb{N}^2$, $s \geq 0$ and $t \in [0, s]$. Then $J_{1,M,\tilde{M}} : H^s_Q \to H^{t+1}_Q$ defined in (14) is a well-defined, linear and bounded operator. Moreover, there is some $\tau > 0$ such that*

$$\|(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}})\varphi\|_{H^{t+1}_Q} \leq C\,\varrho^{-4}\,\frac{(\max\{M_1, M_2\})^\tau}{(\min\{M_1, M_2\})^{s-t+\tau}}\,\|\varphi\|_{H^s_Q}$$

for all $\varphi \in H^s_Q$ and all $\varepsilon_3\varrho \leq \varrho_0$, where the constant C only depends on s, t, τ, ε_2 and ε_3.

Proof: We follow the proof of Theorem 6.5 in [1]. Let $\varphi \in H^s_Q$. We set

$$v(p) = \left\{\chi_{\varepsilon_2\varrho,\varrho}\,\tilde{O}_{\tilde{M}}[\chi_{\varrho,\varepsilon_3\varrho}\,\varphi(t + \cdot)]\right\} \circ \Pi(p), \qquad p \in Q,$$

and write the operators as

$$\tilde{J}_{1,\tilde{M}}\varphi = \sum_{\lambda \in \mathbb{Z}^2} k^{(\lambda)}_{\text{polar}} \int_Q T^{(\lambda)}(p)\,v(p)\,\mathrm{d}p\,,$$

$$J_{1,M,\tilde{M}}\varphi = \sum_{\lambda \in \mathbb{Z}^2} k^{(\lambda)}_{\text{polar}} \int_Q P_M\left[T^{(\lambda)}O_Mv\right](p)\,\mathrm{d}p\,.$$

A central observation regarding this representation of $J_{1,M,\tilde{M}}$ is

$$\int_Q P_M\left[T^{(\lambda)}O_Mv\right](p)\,\mathrm{d}p = \sum_{\iota \in \mathbb{Z}^2_M} T^{(\lambda)}(t^M_\iota)\,O_Mv(t^M_\iota)\int_Q L^M_\iota(p)\,\mathrm{d}p$$

$$= \sum_{\iota \in \mathbb{Z}^2_M} T^{(\lambda)}(t^M_\iota)\frac{\pi^2}{M_1 M_2}O_Mv(t^M_\iota)$$

$$= \sum_{\iota \in \mathbb{Z}^2_M} T^{(\lambda)}(t^M_\iota)\frac{\pi^2}{M_1 M_2}\Big(\sum_{\nu \in \mathbb{Z}^2_M}\frac{M_1 M_2}{\pi^2}\int_Q L^M_\nu(p)\,v(p)\,\mathrm{d}p\,L^M_\nu(t^M_\iota)\Big)$$

$$= \sum_{\iota \in \mathbb{Z}^2_M} T^{(\lambda)}(t^M_\iota)\int_Q L^M_\iota(p)\,v(p)\,\mathrm{d}p$$

$$= \int_Q v(p)\,P_M T^{(\lambda)}(p)\,\mathrm{d}p\,,$$

so that we obtain

$$(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}})\varphi = \sum_{\lambda \in \mathbb{Z}^2} k^{(\lambda)}_{\text{polar}} \int_Q v(p)\left[T^{(\lambda)}(p) - P_M T^{(\lambda)}(p)\right]\mathrm{d}p\,. \tag{21}$$

Moreover, let $\tau > 3$ and $\omega \geq s - t + \tau$. By Sobolev's Imbedding Theorem, the space H_Q^τ is continuously imbedded in the space of twice continuously differentiable Q-periodic functions. Hence, by Lemma 2.2

$$
\begin{aligned}
\left\| T^{(\lambda)} - P_M T^{(\lambda)} \right\|_{\infty;2} &\leq C \left\| T^{(\lambda)} - P_M T^{(\lambda)} \right\|_{H_Q^\tau} \\
&\leq C \frac{(\max\{M_1, M_2\})^\tau}{(\min\{M_1, M_2\})^\omega} \left\| T^{(\lambda)} \right\|_{H_Q^\omega} \leq C \frac{(\max\{M_1, M_2\})^\tau}{(\min\{M_1, M_2\})^{s-t+\tau}} \left(1 + |\lambda|^2\right)^{\omega/2}.
\end{aligned}
$$

Setting $u = \mathcal{M}\varphi$ with $\mathcal{M}$ from Lemma 3.6 and recalling $\mathcal{O}_{\tilde{M}}$ from (20), we obtain

$$
\int_Q \left(T^{(\lambda)}(p) - P_M T^{(\lambda)}(p) \right) \chi_{\varepsilon 2\varrho, \varrho}(\Pi(p)) \, \mathcal{O}_{\tilde{M}} u(\cdot, \Pi(p)) \, \mathrm{d}p
$$
$$
= \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}_{\tilde{M}}^2} u_{\mu,\nu} \int_Q \left(T^{(\lambda)} - P_M T^{(\lambda)} \right)(p) \left[\chi_{\varepsilon 2\varrho, \varrho} T_{Q_\varrho}^{(\nu)} \right] \circ \Pi(p) \, \mathrm{d}p \, T^{(\mu)}.
$$

Hence, by a slight modification of the estimate in (17), we can proceed as in (18) to obtain

$$
\left\| \int_Q \left(T^{(\lambda)}(p) - P_M T^{(\lambda)}(p) \right) \chi_{\varepsilon 2\varrho, \varrho}(\Pi(p)) \, \mathcal{O}_{\tilde{M}} u(\cdot, \Pi(p)) \, \mathrm{d}p \right\|_{H_Q^{t+1}}
$$
$$
\leq C \frac{(\max\{M_1, M_2\})^\tau (1+|\lambda|^2)^{\omega/2}}{(\min\{M_1, M_2\})^{s-t+\tau}} \left(\sum_{\mu \in \mathbb{Z}^2} \left(1 + |\mu|^2\right)^t \left(\sum_{\nu \in \mathbb{Z}_{\tilde{M}}^2} \frac{(1+|\mu|^2)^{1/2}}{(1+|\nu|^2)^{1/2}} |u_{\mu,\nu}| \right)^2 \right)^{1/2}
$$
$$
\leq C \frac{(\max\{M_1, M_2\})^\tau (1+|\lambda|^2)^{\omega/2}}{(\min\{M_1, M_2\})^{s-t+\tau}} \left(\sum_{\substack{\mu \in \mathbb{Z}^2 \\ \nu \in \mathbb{Z}_{\tilde{M}}^2}} \left(1 + |\mu|^2\right)^t \left(1 + |\mu - \tfrac{\pi}{\varepsilon 3\varrho}\nu|^2\right)^3 |u_{\mu,\nu}|^2 \right)^{1/2}
$$
$$
\leq C \frac{(\max\{M_1, M_2\})^\tau}{(\min\{M_1, M_2\})^{s-t+\tau}} \left(1 + |\lambda|^2\right)^{\omega/2} \sqrt{p_{t,3}(u)}.
$$

Now, by setting $\sigma = \lfloor t \rfloor$ from (21), we arrive at

$$
\left\| \left(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}} \right) \varphi \right\|_{H_Q^{t+1}}
$$
$$
\leq C \sum_{\lambda \in \mathbb{Z}^2} \| k_{\text{polar}}^{(\lambda)} \|_{\infty;\sigma+2} \left\| \int_Q \left(T^{(\lambda)}(p) - P_M T^{(\lambda)}(p) \right) \chi_{\varepsilon 2\varrho, \varrho}(\Pi(p)) \right.
$$
$$
\left. \times \, \mathcal{O}_{\tilde{M}} u(\cdot, \Pi(p)) \, \mathrm{d}p \right\|_{H_Q^{t+1}}
$$
$$
\leq C \frac{(\max\{M_1, M_2\})^\tau}{(\min\{M_1, M_2\})^{s-t+\tau}} \sqrt{p_{t,3}(u)} \sum_{\lambda \in \mathbb{Z}^2} \| k_{\text{polar}}^{(\lambda)} \|_{\infty;\sigma+2} \left(1 + |\lambda|^2\right)^{\omega/2}.
$$

Using Lemma 3.6 (a), Lemma 2.6 and arguing as in the proof of Theorem 3.7, we establish the bound

$$
\left\| \left(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}} \right) \varphi \right\|_{H_Q^{t+1}} \leq C \, \varrho^{-4} \, \| \varphi \|_{H_Q^t}.
$$

The assertion follows from the continuous imbedding of H_Q^s in H_Q^t. ∎

We now consider the approximation of the solution of the integral equation (8) by the fully discrete version of (10) which is to find $\varphi_N \in \mathcal{T}_N$ such that

$$\varphi_N - P_N \left(J_{1,M,\tilde{M}} + J_{2,N} \right) \varphi_N = P_N \psi \,. \tag{22}$$

The results on operator approximations allow us to prove stability and convergence for (22). To simplify expressions in these results, let us assume $N_1 = N_2$ and introduce the meshsize $h = \pi/N_1$. We next set $\tilde{M}_1 = \tilde{M}_2 = \lfloor \varrho/h \rfloor$ and $M = \tilde{M}$. Further set

$$A = J_1 + J_2 \,, \qquad A_h = P_N \left(J_{1,M,\tilde{M}} + J_{2,N} \right) \,.$$

We will assume that $I - A$ is boundedly invertible on any H_Q^s, $s \geq 0$.

Theorem 3.11 *Let $t > 1$ and assume that $\varrho = h^\alpha$ for some $\alpha \in (0, 1/(2t + 6))$. Then there exists $h_0 > 0$ such that $I - A_h : H_Q^t \to H_Q^t$ has a bounded inverse for any $0 < h \leq h_0$ with norm bounded independently of h.*

Proof: We write

$$A - A_h = \left(J_1 - J_{1,M,\tilde{M}} \right) + \left(J_2 - J_{2,N} \right) + \left(I - P_N \right) \left(J_{1,M,\tilde{M}} + J_{2,N} \right) \,.$$

From Theorems 3.3, 3.10 and 3.9, we have the estimates

$$\| (J_1 - J_{1,M,\tilde{M}}) \varphi \|_{H_Q^t} \leq C \, h \left(\frac{1}{\varrho^{2t+5}} + \frac{1}{\varrho^4} \right) \| \varphi \|_{H_Q^t} \,,$$

$$\| (J_1 - J_{1,M,\tilde{M}}) \varphi \|_{H_Q^{t+1}} \leq C \left(\frac{1}{\varrho^{2t+5}} + \frac{1}{\varrho^4} \right) \| \varphi \|_{H_Q^t} \,,$$

$$\| (J_2 - J_{2,N}) \varphi \|_{H_Q^t} \leq \frac{C \, h}{\varrho^{2t+6}} \| \varphi \|_{H_Q^t} \,,$$

$$\| (J_2 - J_{2,N}) \varphi \|_{H_Q^{t+1}} \leq \frac{C}{\varrho^{2t+6}} \| \varphi \|_{H_Q^t} \,.$$

By Lemma 2.2, $I - P_N : H_Q^{t+1} \to H_Q^t$ with operator norm bounded by Ch. Thus

$$\| (A - A_h) \, \varphi \|_{H_Q^t} \leq \frac{C \, h}{\varrho^{2t+6}} \| \varphi \|_{H_Q^t} \longrightarrow 0 \quad (h \to 0) \,.$$

The assertion now follows from standard results for operator approximation. ∎

Theorem 3.12 *Let $\alpha \in (0, 1/2)$ and $\varrho = h^\alpha$. Assume $t \geq 0$ and $s > \max\{1, t, \frac{12\alpha + 2\alpha t + t}{1 - 2\alpha}\}$. Assume further that (22) is a stable approximation of (8) in H_Q^s, i.e. $\| \varphi_h \|_{H_Q^s} \leq C \| \varphi \|_{H_Q^s}$ for sufficiently small h. Then there exists $h_0 > 0$ such that*

$$\| \varphi - \varphi_h \|_{H_Q^t} \leq C \, h^{(s-t)(1-2\alpha)/2} \| \varphi \|_{H_Q^s}$$

for all $0 < h \leq h_0$.

Proof: From
$$\varphi_h = P_N \psi + A_h \varphi_h = P_N(\varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h)$$

we obtain

$$
\begin{aligned}
&(I - A)(\varphi - \varphi_h) \\
&= \varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h - \varphi_h - (J_{1,M,\tilde{M}} + J_{2,N} - A)\varphi_h \\
&= (I - P_N)(\varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h) - (J_{1,M,\tilde{M}} + J_{2,N} - A)\varphi_h \, .
\end{aligned}
$$

From Theorems 3.3, 3.9 and 3.10, we have

$$\left\| J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h \right\|_{H_Q^s} \leq \left(\|A\| + \frac{C\,h}{\varrho^{2s+6}} \right) \|\varphi_h\|_{H_Q^s} \, .$$

Similarly, we have

$$\left\| (J_{1,M,\tilde{M}} + J_{2,N} - A)\,\varphi_h \right\|_{H_Q^1} \leq \frac{C\,h^s}{\varrho^{2s+6}} \|\varphi_h\|_{H_Q^s} \, .$$

Thus from the boundedness of $(I + A)^{-1}$ in $L^2(Q)$, Lemma 2.2 and the stability estimate, we conclude

$$\|\varphi - \varphi_h\|_{L^2(Q)} \leq C \left\| (I + A)(\varphi - \varphi_h) \right\|_{L^2(Q)} \leq \frac{C\,h^s}{\varrho^{2s+6}} \|\varphi\|_{H_Q^s}$$

for all $h \leq h_0$ such that also $\varrho \leq \varrho_0$.

For the general result, we observe that for $T \in \mathcal{T}_N$, the estimate $\|T\|_{H_Q^t} \leq C\,h^{-t} \|T\|_{L^2(Q)}$ follows directly from the definition of the norm in H_Q^t. Using the orthogonal projection O_N, we have

$$
\begin{aligned}
\|\varphi - \varphi_h\|_{H_Q^t} &\leq \|\varphi - O_N\varphi\|_{H_Q^t} + \|O_N\varphi - \varphi_h\|_{H_Q^t} \\
&\leq \|\varphi - O_N\varphi\|_{H_Q^t} + C\,h^{-t} \|O_N\varphi - \varphi_h\|_{L^2(Q)} \\
&\qquad\qquad \leq \|\varphi - O_N\varphi\|_{H_Q^t} + C\,h^{-t} \|\varphi - \varphi_h\|_{L^2(Q)} \, ,
\end{aligned}
$$

where the last estimate follows from the Pythagorean theorem. For $\varphi - O_N\varphi$ the same bounds as for $\varphi - P_N\varphi$ have been shown as part of the proof of Lemma 2.2. Thus

$$\|\varphi - \varphi_h\|_{H_Q^t} \leq C\,h^{s-t} \left(1 + \frac{1}{\varrho^{2s+6}} \right) \|\varphi\|_{H_Q^s} \, .$$

From $s \geq \frac{12\alpha + 2\alpha t + t}{1 - 2\alpha}$ follows $(s - t)(1 - 2\alpha)/2 \geq 2\alpha(t + 3)$. Thus

$$\frac{h^{s-t}}{\varrho^{2s+6}} = h^{s-t-\alpha(2s+6)} = h^{(s-t)(1-2\alpha)}\,h^{-2\alpha(t+3)} \leq h^{(s-t)(1-2\alpha)/2} \, .$$

This concludes the proof. ■

4 Applications

In this section we will consider some applications for the method analysed in Section 3. There are two basic classes of examples: boundary value problems for bounded domains that are globally parametrizable over Q and boundary value problems in periodic media. More general boundary value problems can be treated by special surface representations and lead to systems of Q-periodic integral equations. We will discuss such cases in a forthcoming paper.

Example 4.1 Consider $D \subseteq \mathbb{R}^3$ such that ∂D is the image of a C^∞ smooth regular Q-periodic parametrization $\eta : \mathbb{R}^2 \to \mathbb{R}^3$, i.e. D has the shape of a torus. The Dirichlet boundary value problem

$$
\begin{aligned}
\Delta u &= 0 \qquad \text{in } D\,, \\
u &= f \qquad \text{on } \partial D\,,
\end{aligned}
\tag{23}
$$

can be solved by a double layer potential ansatz,

$$
u(x) = \text{DL}\,\tilde\varphi(x) = \int_{\partial D} \frac{n(y) \cdot (x - y)}{4\pi\,|x - y|^3}\,\tilde\varphi(y)\,\mathrm{d}s(y)\,, \qquad x \in D\,,
$$

where $n(y)$ denotes the outward drawn unit normal to ∂D in y. Through the jump relations for the double layer potential we see that u is a solution to (23) if $\tilde\varphi$ is a solution to the integral equation

$$
\tilde\varphi(x) - \int_{\partial D} \frac{n(y) \cdot (x - y)}{2\pi\,|x - y|^3}\,\tilde\varphi(y)\,\mathrm{d}s(y) = -2\,f(x)\,, \qquad x \in \partial D\,.
$$

Inserting η and setting $\varphi(t) = \tilde\varphi(\eta(t))$, we obtain the Q-periodic integral equation

$$
\varphi(t) - \int_Q \frac{(\partial_1\eta(\tau) \times \partial_2\eta(\tau)) \cdot (\eta(t) - \eta(\tau))}{2\pi\,|\eta(t) - \eta(\tau)|^3}\,\varphi(\tau)\,\mathrm{d}\tau = -2\,f(\eta(t))\,, \qquad t \in Q\,. \tag{24}
$$

Lemma 4.2 *The kernel of the integral operator in* (24) *satisfies Assumption 3.1 with* $k_2 = 0$.

Proof: We denote the kernel of (24) by $k(\cdot, \cdot\cdot)$. The bound on derivatives of k is obvious from iterated applications of the quotient rule.

Representing $\tau = t + r\,v$, $t \in Q$, $r \in \mathbb{R}$, $v \in \mathbb{S}^1$, we can apply various variants of the residual representation in Taylor's theorem to obtain

$$
\begin{aligned}
\eta(t) - \eta(\tau) &= r \int_0^1 \left.\frac{\partial}{\partial s}\eta(\tau - s\,v)\right|_{s = \sigma r} \mathrm{d}\sigma \\
&= -r\,\eta'(\tau)\,v + r^2 \int_0^1 (1 - \sigma)\,\left.\frac{\partial^2}{\partial s^2}\eta(\tau - sv)\right|_{s = \sigma r} \mathrm{d}\sigma\,.
\end{aligned}
\tag{25}
$$

Note that

$$
(\partial_1\eta(\tau) \times \partial_2\eta(\tau)) \cdot [\eta'(\tau)\,v] = 0\,,
$$

as this is a scalar product of the normal and a tangential vector to ∂D. It follows that both enumerator and denominator in $|r|\,k(t, t + rv)$ are C^∞.

It remains to show that $|\eta(t+rv)-\eta(t)|/r$ is uniformly bounded away from zero. Fixing any $r_0 > 0$, this is clear for $r_0 \le r \le \varrho_0 = \pi/2$. From the first representation in (25), we see

$$\frac{|\eta(t+rv)-\eta(t)|^2}{r^2} = \int_0^1 \int_0^1 v^\top \eta'(t+\sigma_1 r\, v)^\top \, \eta'(t+\sigma_2 r\, v)\, v\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1\,.$$

The matrix $\eta'(t)^\top \eta'(t)$ is the first fundamental form of ∂D, which is uniformly positive definite as η is regular. Hence we can choose r_0 such that the integrand above is bounded away from 0 uniformly in $t \in Q$, $v \in \mathbb{S}^1$ and $r \le r_0$. Thus ℓ from Assumption 3.1 is C^∞ smooth. ∎

Example 4.3 Consider the scattering of a time-harmonic acoustic wave by a smooth bounded obstacle. Such a problem can be modelled as an exterior boundary value problem for the Helmholtz equation. Again, we assume that $D \subseteq \mathbb{R}^3$ is such that ∂D is the image of a C^∞ smooth regular Q-periodic parametrization $\eta : \mathbb{R}^2 \to \mathbb{R}^3$. For a wavenumber $k > 0$, we assume that the total field u satisfies the Helmholtz equation outside D and a Dirichlet boundary condition on ∂D, i.e.

$$\begin{aligned} \Delta u + k^2 u &= 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \\ u &= 0 \qquad \text{on } \partial D\,. \end{aligned} \tag{26}$$

Given the incident field u^i, which is assumed to be a solution to the Helmholtz equation in all of $\mathbb{R}^3$, we additionally assume that the scattered field $u^s = u - u^i$ satisfies the Sommerfeld radiation condition at infinity.

A solution to this problem is given by the combined potential layer ansatz,

$$u^s(x) = \mathrm{DL}\,\tilde{\varphi}(x) - \mathrm{i}\lambda\,\mathrm{SL}\,\tilde{\varphi}(x) = \int_{\partial D} \left[\frac{\partial \Phi(x,y)}{\partial n(y)} - \mathrm{i}\lambda\,\Phi(x,y)\right]\tilde{\varphi}(y)\,\mathrm{d}s(y)$$

for $x \in \mathbb{R}^3 \setminus \overline{D}$ provided that $\tilde{\varphi}$ is a solution to the boundary integral equation

$$\tilde{\varphi}(x) + 2\int_{\partial D}\left[\frac{\partial \Phi(x,y)}{\partial n(y)} - \mathrm{i}\lambda\,\Phi(x,y)\right]\tilde{\varphi}(y)\,\mathrm{d}s(y) = -2\,u^i(x)\,, \qquad x \in \partial D.$$

Here, λ is some complex number satisfying $\mathrm{Re}(\lambda) \ne 0$ and Φ denotes the fundamental solution to the Helmholtz equation

$$\Phi(x,y) = \frac{\mathrm{e}^{\mathrm{i}k\,|x-y|}}{4\pi\,|x-y|}\,, \qquad x \ne y\,.$$

After inserting the parametrization and substituting $\varphi(t) = \tilde{\varphi}(\eta(t))$, we obtain the Q-periodic integral equation

$$\varphi(t) + \int_Q \frac{\mathrm{e}^{\mathrm{i}k\,|\eta(t)-\eta(\tau)|}}{2\pi}\left[\frac{(\partial_1\eta(\tau)\times\partial_2\eta(\tau))\cdot(\eta(t)-\eta(\tau))}{|\eta(t)-\eta(\tau)|^3}\left(1 - \mathrm{i}k\,|\eta(t)-\eta(\tau)|\right)\right.$$
$$\left.- \mathrm{i}\lambda\frac{|\partial_1\eta(\tau)\times\partial_2\eta(\tau)|}{|\eta(t)-\eta(\tau)|}\right]\varphi(\tau)\,\mathrm{d}\tau = -2u^i(\eta(t))\,, \qquad t \in Q\,. \tag{27}$$

Lemma 4.4 *The kernel function in* (27) *satisfies Assumption 3.1.*

Proof: We set

$$K_1(x, y) = \frac{1}{2\pi} \left[\frac{D(y)\, n(y) \cdot (x - y)}{|x - y|^3} \left(\cos(k\,|x - y|) + k\,|x - y|\,\sin(k|x - y|) \right) \right.$$

$$\left. - \,\mathrm{i}\lambda\, \frac{D(y)}{|x - y|}\, \cos(k\,|x - y|) \right],$$

$$K_2(x, y) = \frac{1}{2\pi} \left[\frac{D(y)\, n(y) \cdot (x - y)}{|x - y|^3} \left(\mathrm{i}\,\sin(k\,|x - y|) - \mathrm{i}k\,|x - y|\,\cos(k|x - y|) \right) \right.$$

$$\left. + \,\lambda\, \frac{D(y)}{|x - y|}\, \sin(k\,|x - y|) \right],$$

where $D(y) = |\partial_1 \eta(\tau) \times \partial_2 \eta(\tau)|\big|_{y = \eta(\tau)}$ is the determinant of the first fundamental form. We denote the kernel of (27) by

$$k(t, \tau) = k_1(t, \tau) + k_2(t, \tau) = K_1(\eta(t), \eta(\tau)) + K_2(\eta(t), \eta(\tau))\,.$$

Note that

$$K_2(x, y) = D(y)\, n(y) \cdot (x - y)\, p_1(k^2\,|x - y|^2) + \lambda\, D(y)\, p_2(k^2\,|x - y|^2)$$

with analytic functions p_1, p_2. The smoothness of $D(\eta(\tau))$ follows again from the positive definiteness of the first fundamental form. Thus k_2 is smooth.

Similarly, we have

$$K_1(x, y) = \frac{D(y)\, n(y) \cdot (x - y)}{|x - y|^3}\, p_3(k^2\,|x - y|^2) + \lambda\, \frac{D(y)}{|x - y|}\, p_4(k^2\,|x - y|^2)$$

with analytic functions p_3, p_4. The smoothness of ℓ now follows with similar arguments as in the proof of Lemma 4.2. ∎

Example 4.5 As a third example we consider scattering of a plane acoustic wave by a biperiodic smooth surface. The problem formulation is similar to that of Example 4.3, with D given by

$$D = \{x = (t, x_3)^\top \in \mathbb{R}^3 : t \in \mathbb{R}^2\,,\ x_3 < f(t)\}\,,$$

where $f \in C^\infty(\mathbb{R}^2)$ is Q-periodic, and the incident field is given by

$$u^i(x) = \mathrm{e}^{\mathrm{i}\,k\,d \cdot x}\,, \qquad x \in \mathbb{R}^3\,, \quad d \in \mathbb{S}^2\,, d_3 < 0\,.$$

An important feature of this field and correspondingly also the scattered field is that it is kd-quasi-periodic, i.e. $u^i(t + 2\pi\,\nu, x_3) = \exp(\mathrm{i}\,2\pi k\,(d_1, d_2)^\top \cdot \nu)\, u^i(t, x_3)$.

The Sommerfeld radiation condition has also got to be replaced by the condition that u^s is a linear superposition of upward propagating plane waves and evanescent waves, see [1] for details. The scattered field can again be found as a combined double- and single-layer potential,

$$u^s(x) = \mathrm{DL}\,\tilde{\varphi}(x) - \mathrm{i}\lambda\,\mathrm{SL}\,\tilde{\varphi}(x) = \int_{\partial D} \left[\frac{\partial G(x, y)}{\partial n(y)} - \mathrm{i}\lambda\, G(x, y) \right] \tilde{\varphi}(y)\,\mathrm{d}s(y)$$

for $x \in \mathbb{R}^3 \setminus \overline{D}$. Here G is the kd-quasi-periodic fundamental solution to the Helmholtz equation and $\tilde{\varphi}$ is a kd quasi-periodic density. Representations and algorithms for the efficient evaluation of this function are discussed in [1, 2]. The unit normal $n(y)$ to ∂D at y is assumed to point upward, i.e. $n_3 > 0$.

Setting $\varphi(t) = \exp(-\mathrm{i} k (d_1, d_2)^\top \cdot t) \, \tilde{\varphi}(t, f(t))$, we obtain a Q-periodic density φ. u^s is a solution to the scattering problem, if φ solves the integral equation

$$
\varphi(t) + \int_Q e^{\mathrm{i} k (d_1, d_2)^\top \cdot (\tau - t)} \left[\frac{\partial G(x, y)}{\partial n(y)} - \mathrm{i} \lambda \, G(x, y) \right]
$$
$$
\times \sqrt{1 + |\nabla f(\tau)|^2} \, \varphi(\tau) \, \mathrm{d}\tau = -2 \, e^{-\mathrm{i} k (d_1, d_2)^\top \cdot t} \, u^i(t, f(t)), \qquad t \in Q. \quad (28)
$$

For simplicity, we have used the abbreviations $x = (t, f(t))$, $y = (\tau, f(\tau))$ here.

From [1, Theorem 3.8] we have the representation

$$
G(x, y) = \frac{\cos(k |x - y|)}{4\pi |x - y|} + P(k^2 |x - y|^2), \qquad |x - y| \le \frac{\pi}{2},
$$

with an analytic function P. Hence, by a similar analysis as in Lemma 4.4, we see that the kernel in (28) satisfies Assumption 3.1.

5 Implementation and Numerical Examples

The implementation of the operator $J_{2,N}$ is already given in (12): It is simply the application of the tensor-product composite trapezoidal rule to the integral representing J_2. The implementation of $J_{1,M,\tilde{M}}$ as given by (14) is more complicated as it involves an approximation of the two orthogonal projections. To simplify notation, we define

$$
w_{\tilde{M}}^\mu = \tilde{O}_{\tilde{M}} \left[\chi_{\varrho, \varepsilon 3 \varrho} \, \varphi(t_\mu^N + \cdot) \right] = \sum_{\iota \in \mathbb{Z}_{\tilde{M}}^2} w_{\tilde{M}}^{\mu, \iota} \, T_{Q_\varrho}^{(\iota)},
$$

$$
v_{M,\tilde{M}}^\mu = O_M \left[\left\{ \chi_{\varepsilon 2 \varrho, \varrho} \, w_{\tilde{M}}^\mu \right\} \circ \Pi \right],
$$

so that

$$
J_{1,M,\tilde{M}} \varphi(t_\mu^N) = \int_Q P_M \left[k_{\text{polar}}(t_\mu^N, \cdot) \, v_{M,\tilde{M}}^\mu \right] (p) \, \mathrm{d}p
$$
$$
= \frac{\pi^2}{M_1 M_2} \sum_{\nu \in \mathbb{Z}_M^2} k_{\text{polar}}(t_\mu^N, t_\nu^M) \, v_{M,\tilde{M}}^\mu(t_\nu^M), \qquad \mu \in \mathbb{Z}_N^2. \quad (29)
$$

From (13), using the Lagrange basis function, we obtain for $\mu \in \mathbb{Z}_N^2$, $\nu \in \mathbb{Z}_M^2$,

$$
\frac{\pi^2}{M_1 M_2} v_{M,\tilde{M}}^\mu(t_\nu^M) \quad (30)
$$
$$
= \sum_{\lambda \in \mathbb{Z}_M^2} \sum_{\iota \in \mathbb{Z}_{\tilde{M}}^2} w_{\tilde{M}}^{\mu, \iota} \int_Q \chi_{\varepsilon 2 \varrho, \varrho}(\Pi(p)) \, T_{Q_\varrho}^{(\iota)}(\Pi(p)) \, \overline{L_\lambda^M(p)} \, \mathrm{d}p \, L_\lambda^M(t_\nu^M)
$$

Preparation (carry out once)

    – compute $b_{\mu,\nu} = k_{\text{polar}}(t_\mu^N, t_\nu^M), \quad \mu \in \mathbb{Z}_N^2, \quad \nu \in \mathbb{Z}_M^2$

    – compute $c_{\nu,\iota} \approx \int_Q \left[\left\{ \chi_{\varepsilon 2\varrho, \varrho} T_{Q_\varrho}^{(\iota)} \right\} \circ \Pi \, \overline{L_\nu^M} \right](p) \, \mathrm{d}p, \quad \nu \in \mathbb{Z}_M^2, \quad \iota \in \mathbb{Z}_{\tilde{M}}^2$

Matrix-Vector-Multiplication $(\varphi(t_\mu^N))_\mu \mapsto (\psi_\mu)_\mu$

    – for each $\mu \in \mathbb{Z}_N^2$

        - compute $d_{\mu,\lambda} = \chi_{\varrho, \varepsilon 3\varrho}(t_\lambda^N) \, \varphi(t_\mu^N + t_\lambda^N), \quad \lambda \in \mathbb{Z}_{\tilde{M}}^2$

        - perform $(w_{\tilde{M}}^{\mu,\iota})_\iota \longleftarrow \text{FFT}_{\tilde{M}}(d_{\mu,\lambda})_\lambda$

        - compute $v_{M,\tilde{M},\nu}^\mu = \sum\limits_{\iota \in \mathbb{Z}_{\tilde{M}}^2} c_{\nu,\iota} \, w_{\tilde{M}}^{\mu,\iota}, \quad \nu \in \mathbb{Z}_M^2$

        - compute $\psi_\mu = \sum\limits_{\nu \in \mathbb{Z}_M^2} b_{\mu,\nu} \, v_{M,\tilde{M},\nu}^\mu$

Figure 1: Algorithm for Matrix-Vector-Multiplication $\boldsymbol{\psi} \leftarrow \mathbf{A}_1 \boldsymbol{\varphi}$

$$= \sum_{\iota \in \mathbb{Z}_{\tilde{M}}^2} w_{\tilde{M}}^{\mu,\iota} \int_Q \chi_{\varepsilon 2\varrho, \varrho}(\Pi(p)) \, T_{Q_\varrho}^{(\iota)}(\Pi(p)) \, \overline{L_\nu^M(p)} \, \mathrm{d}p \,. \tag{31}$$

For an implementation of this formula, the integrals need to be computed accurately. It is helpful that $\chi_{\varepsilon 2\varrho, \varrho}$ is radially symmetric: applying the Jacobi-Anger expansion for $T_{Q_\varrho}^{(\iota)}(\Pi(\cdot))$, gives a sum of products of two 1D integrals. The integration over p_2 can be carried out analytically, while for the integration over p_1 we apply the composite trapezodial rule. The computation of an approximation of the Fourier coefficients $w_{\tilde{M}}^{\mu,\iota}$ is done by an interpolation instead of the orthogonal projection. The convergence rate of both operations are identical for functions in H_Q^s.

For a description of the implementation, we assume again that $N_1 = N_2 = \pi/h$ and that $\varrho = h^\alpha$ for some fixed $\alpha \in (0,1)$. Denote by $\mathbf{N} = 4N_1 N_2$ the total number of unknowns. With $\tilde{M}_j = M_j = \lfloor \varrho/h \rfloor$, we have $\#\mathbb{Z}_M^2 = \#\mathbb{Z}_{\tilde{M}}^2 = \mathrm{O}(\mathbf{N}^{1-\alpha})$.

In the following arguments, we will use indices in $\mathbb{Z}_N^2$ for the coefficients of vectors in $\mathbb{C}^{\mathbf{N}}$, the map from $\mathbb{Z}_N^2 \to \{1, \ldots, \mathbf{N}\}$ being implicit, and likewise for matrices in $\mathbb{C}^{\mathbf{N} \times \mathbf{N}}$. We write the fully discrete version of (22) as the linear system

$$(\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2)\boldsymbol{\varphi} = \boldsymbol{\psi} \,,$$

where $\boldsymbol{\varphi}_\mu = \varphi(t_\mu^N)$, $\boldsymbol{\psi}_\mu = \psi(t_\mu^N)$, $\mu \in \mathbb{Z}_N^2$, $\mathbf{I}$ denotes the identity matrix in $\mathbb{C}^{\mathbf{N} \times \mathbf{N}}$ and $\mathbf{A}_j = (a_{j,\mu,\nu}) \in \mathbb{C}^{\mathbf{N} \times \mathbf{N}}$ denotes the discretization of J_j, $j = 1, 2$.

The linear system will be solved by an iterative method, so that algorithms for the computation of $\mathbf{A}_j \boldsymbol{\varphi}$ are required. The computation of $a_{2,\mu,\nu}$ is obvious from (7) and (12), so that this matrix in principle can be pre-calculated and used for matrix vector multiplication. The operation count and storage requirements for this are $\mathrm{O}(\mathbf{N}^2)$. However, various approaches are known to reduce this considerably, such as Fast Multipole Methods, H-Matrix calculus, etc. The focus of the present paper does not lie in this aspect.
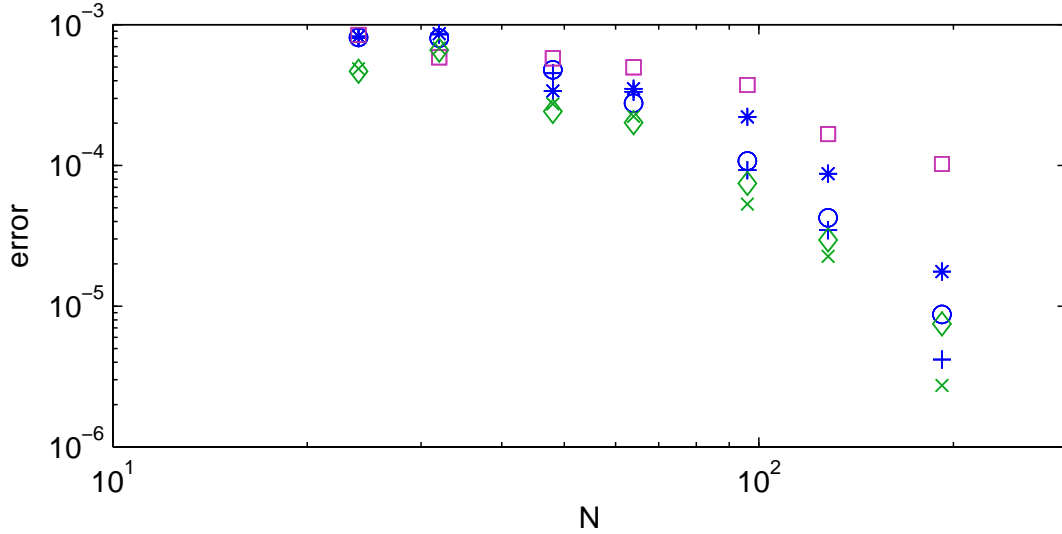
Figure 2: Errors for Dirichlet Problem for Laplace's equation. Green symbols are used for $\varepsilon_1 = 0.1667$ with $\alpha = 0.15$ ($\times$), $\alpha = 0.25$ ($\diamond$), blue symbols for $\varepsilon_1 = 0.6667$, with $\alpha = 0.15$ ($+$), $\alpha = 0.25$ ($\circ$), $\alpha = 0.4$ ($*$), magenta symbols are used for $\varepsilon_1 = 0.6667$ and $\alpha = 0.6$ ($\square$).

The matrix vector multiplication $\mathbf{A}_1\boldsymbol{\varphi}$ can be carried out using (29)–(31). The algorithm is described in detail in Figure 1. From this description, we see that the preparation step requires $O(\mathbf{N}^{2-\alpha} + \mathbf{N}^{2-2\alpha})$ memory locations for storage and $O(\mathbf{N}^{2-\alpha} + S\mathbf{N}^{2-2\alpha})$ operations, where S denotes the number of operations for acurate evaluation of the integrals in (31). For the matrix vector multiplication itself, only the operational count is of interest. It amounts to

$$O(\mathbf{N}\,(\mathbf{N}^{1-\alpha} + \mathbf{N}^{1-\alpha}\,\log(\mathbf{N}^{1-\alpha}))) = O(\mathbf{N}^{2-\alpha}\,(1 + (1-\alpha)\log(\mathbf{N}))$$

operations.

In order for the preparation step not to dominate the overall complexity, it is necessary that S is at most proportional to $\mathbf{N}^\alpha$. This requires some careful balancing. Note however also that (31) only involves the parameters ϱ and ε_j, but is independent of the actual problem. Hence, these calculations can be carried out once for a given parameter set and the resulting values stored in a data base.

As a first example, we consider the Dirichlet boundary value problem for Laplace's equation from Example 4.1. As the domain D, we choose a torus given by the parametrization

$$\eta(t) = (R_1 + R_2\,\cos(t_2))\begin{pmatrix}\cos(t_1)\\\sin(t_1)\\0\end{pmatrix} + R_2\begin{pmatrix}0\\0\\\sin(t_2)\end{pmatrix}, \qquad t = (t_1, t_2)^\top \in \mathbb{R}^2, \qquad (32)$$

with $R_1 = 1.0$ and $R_2 = 0.25$. We have carried out the case, where the exact solution is $u(x) = \left(4\pi\,\left|x - (2,1,1)^\top\right|\right)^{-1}$, $x \in D$. The integral equation (24) was solved numerically for various sets of parameters and the double layer potential with the resulting density was evaluated at a number of points $x^{(\ell)}$, $\ell = 1, ..., P$, in the domain.

Tests were run with $\varrho = 0.15\,(20/N)^\alpha$ for various $\alpha \in (0, 1)$ and for two choices for ε_1. In all cases $\varepsilon_2 = 0.8333$, $\varepsilon_3 = 2.0$ as well as $\tilde{M} = \text{ceil}(\varrho/h)$ and M proportional to $\tilde{M}$ with a

28

| | total # of | $\varepsilon_1 = 0.1667$ | | | |
N	unknowns	$\alpha = 0.15$		$\alpha = 0.25$	
24	2304	3		3	
			-1.1848		-1.1996
32	4096	4		4	
			2.2589		2.4786
48	9216	5		5	
			0.7081		0.6326
64	16384	6		6	
			3.5330		2.4481
96	36864	9		8	
			2.9765		3.2272
128	65536	11		10	
			5.2162		3.3908
192	147456	16		13	

| | total # of | $\varepsilon_1 = 0.6667$ | | | | | | | |
N	unknowns	$\alpha = 0.15$		$\alpha = 0.25$		$\alpha = 0.4$		$\alpha = 0.6$	
24	2304	3		3		3		3	
			-0.2149		0.0386		-0.1475		1.2710
32	4096	4		4		4		3	
			1.5702		1.2810		2.3161		0.0382
48	9216	5		5		5		3	
			1.0533		1.8965		-0.1309		0.5098
64	16384	6		6		5		4	
			3.1685		2.3366		1.1316		0.7219
96	36864	9		8		6		4	
			3.4198		3.2304		3.2355		2.7811
128	65536	11		10		7		5	
			5.2133		3.8985		3.9480		1.2084
192	147456	16		13		11		6	

Table 1: Estimated orders of convergence for the Dirichlet problem for Laplace's equation.

fixed constant. Figure 2 shows the maximum error Err_N for evaluations of the double layer potential in the points $x^{(\ell)}$. Table 1 lists estimated orders of convergence

$$\mathrm{EOC} = \frac{\log(\mathrm{Err}_{N_1} / \mathrm{Err}_{N_2})}{\log(N_2/N_1)}$$

for subsequent values of N. The total number of unknowns for each problem is also listed.

For the choices of $\alpha < 0.5$ we see that the increasing convergence rates to be expected from Theorem 3.12 are indeed achieved. On the other hand, the result for $\alpha = 0.6$ does not exhibit an increasing convergence rate which may be an indication that Theorem 3.12 is sharp in this respect.

As the second example, we carried out computations for a scattering problem from Example 4.3 for the wave numbers $k_1 = 2\sqrt{2}$ and $k_2 = 4\sqrt{2}$. As the obstacle, the torus defined by (32) was used and ρ was chosen by the same recipe as before. After solution of the boundary integral equation, the field was evaluated on a 51×51 grid G in the horizontal plane $x_3 = -1$ beneath the obstacle. To assess the achievable convergence rates, we used the "incident field" $u^i = \Phi(\cdot, z)$ with $z = (0.9, 0.0, 0.15)^\top \in D$. In this case $u^s = -u^i$ in $\mathbb{R}^3 \setminus D$, so that we know the scattered field exactly. We denote the field computed by our method for a given value of N by u_N and evaluated

$$\mathrm{Err}_N = \max_{x \in G} |u_N^s(x) + u^i(x)|.$$

The values obtained have been plotted in Figure 3. In Table 2 we report on estimated convergence rates computed as in the previous example for the case $k = 2\sqrt{2}$. Corresponding rates for $k = 4\sqrt{2}$ are presented in Table 3.
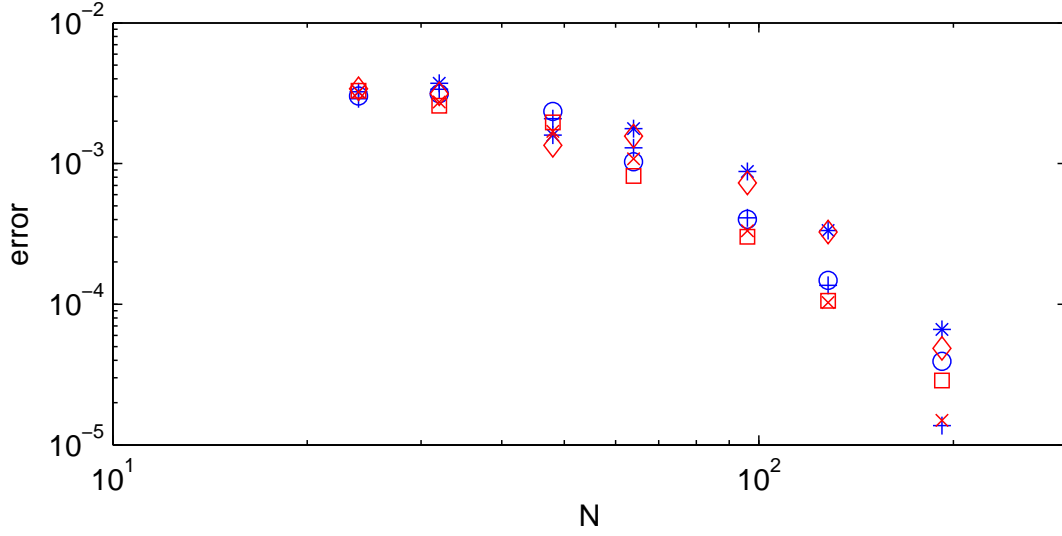
Figure 3: Errors for the scattering problem. Blue symbols used for $k = 2\sqrt{2}$ with $\alpha = 0.15$ (+), $\alpha = 0.25$ (∘), $\alpha = 0.4$ (∗), red symbols are used for $k = 4\sqrt{2}$ with $\alpha = 0.15$ (×), $\alpha = 0.25$ (□), $\alpha = 0.4$ (◇).

N	total # of unknowns	$\alpha = 0.15$		$\alpha = 0.25$		$\alpha = 0.4$	
		M	EOC	M	EOC	M	EOC
24	2304	3		3		3	
			-0.5278		-0.1241		-0.4843
32	4096	4		4		4	
			1.1799		0.7187		2.0920
48	9216	5		5		5	
			1.6587		2.8525		-0.3633
64	16384	6		6		5	
			2.8249		2.3226		1.7294
96	36864	9		8		6	
			3.8533		3.4798		3.3664
128	65536	11		10		7	
			5.6597		3.2667		3.9864
192	147456	16		13		11	

Table 2: Estimated orders of convergence for the scattering problem with $k = 2\sqrt{2}$.

N	total # of unknowns	$\alpha = 0.15$		$\alpha = 0.25$		$\alpha = 0.4$	
		M	EOC	M	EOC	M	EOC
24	2304	3		3		3	
			0.5676		0.8457		0.2843
32	4096	4		4		4	
			1.1900		0.6782		2.0810
48	9216	5		5		5	
			1.5574		3.0471		-0.5332
64	16384	6		6		5	
			2.9099		2.4527		1.8940
96	36864	9		8		6	
			4.0797		3.6469		2.7877
128	65536	11		10		7	
			4.7555		3.2173		4.6971
192	147456	16		13		11	

Table 3: Estimated orders of convergence for the scattering problem with $k = 4\sqrt{2}$.
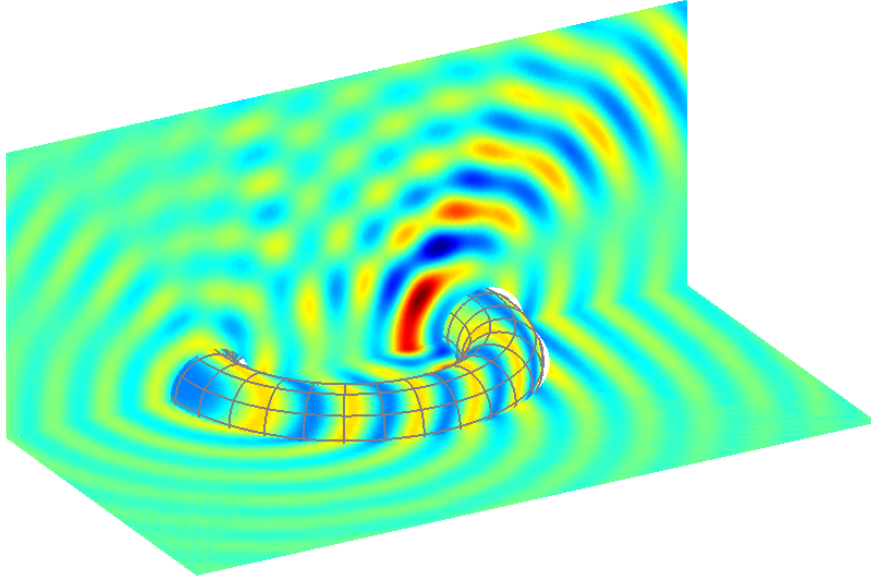
Figure 4: Scattering of a plane wave by a torus at $k = 12\sqrt{2}$.

A true scattering problem is displayed in Figure 4. A plane wave with incident direction $d = (\sqrt{3}/2, 0, -1/2)^\top$ is scattered by the torus given by (32). The wavenumber is $k = 12\sqrt{2}$, the other parameters are chosen as $N = 128$, $M = 11$ and $\varrho = 0.35671$.

The torus is visibly enhanced by a superimposed grid that is in no way related to the computational grid used. On the surface of the torus, the real part of $-u^i$ is displayed, on the horizontal and vertical plane, the potential evaluated for the density computed with our solver is presented.

Finally, we also carried out computations for scattering problems involving periodic surfaces as described in Example 4.5. Specifically, we used the surface given as the graph of the function

$$f(t) = \frac{1}{6} \cos(t_1) \exp(\sin(t_1)), \qquad t \in \mathbb{R}.$$

Computations were carried out at $k = 2\sqrt{2}$ with the plane wave $u^i(x) = \exp(\mathrm{i}k\, x \cdot d)$ with $d = (\sqrt{3}/2, 0, -1/2)^\top$ used as the incident field. Figure 5 shows the real part of the scattered field in this situation. The parameters used for this plot are $N = 48$, $M = 5$ and $\varrho = 0.41325$. Two periods of the surface are shown in both directions. The field on the biperiodic surface is simply the Dirichlet boundary values, on the vertical planes the potential with the computed density was evaluated.

In conclusion, these examples show comprehensively that the super-algebraic convergence rates predicted by Theorem 3.12 are achievable in practice in a variety of applications. Moreover, there is some numerical evidence, that the bound $\alpha < 1/2$ necessary in the proof of this theorem is indeed sharp.
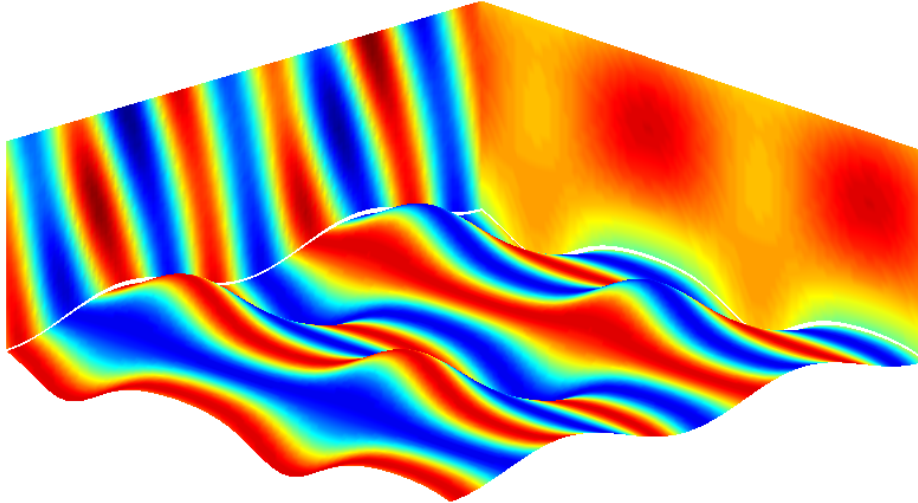
Figure 5: Scattering of a plane wave by a periodic surface at $k = 2\sqrt{2}$. The scale in the vertical direction is doulbe that in the horizontal directions.

Acknowledgements

References

[1] Arens, T.: Scattering by biperiodic layered media: The integral equation approach (2010). Habilitation Thesis, Karlsruhe Institute of Technology

[2] Arens, T., Lechleiter, A., Sandfort, K., Schmitt, S.: Analysing ewald's method for the evaluation of green's functions for periodic media. IMA J. Appl. Math. **78**, 405–431 (2013). DOI 10.1093/imamat/hxr057

[3] Beale, J.T.: A grid-based boundary integral method for elliptic problems in three dimensions. SIAM J. Numer. Anal. **42**, 599–620 (2004)

[4] Bruno, O., Dominguez, V., Sayas, F.: Convergence analysis of a high-order Nystrom integral-equation method for surface scattering problems. Numer. Math. **124**, 603–645 (2013)

[5] Bruno, O.P., Kunyanski, L.A.: A fast high order algorithm for the solution of surface scattering problems: Basic implementation, tests and applications. J. Comp. Phys. **169**, 80–110 (2001)

[6] Bruno, O.P., Kunyanski, L.A.: Surface scattering in three dimensions: an accelerated high-order solver. Proc. R. Soc. Lond. A **457**, 2921–2934 (2001)

[7] Canino, L., Ottusch, J., Stalzer, M., Visher, J., S.M. Wandzura, S.: Numerical solution of the Helmholtz equation in 2D and 3D using a high-order nystroem discretization. J. Comp. Phys. **146**, 627–663 (1998)

[8] Ganesh, M., Graham, I.: A high-order algorithm for obstacle scattering in three dimensions (198), 211–242 (2004)

[9] Graham, I., Sloan, I.: Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in $\mathbb{R}^3$. Numer. Math. **92**(2), 289–323 (2002). DOI 10.1007/s002110100343

[10] Heinemeyer, E.: Integral equation methods for rough surface scattering problems in three dimensions. Ph.D. thesis, Georg-August-Universität Göttingen (2008)

[11] Kress, R.: Linear integral equations, 2. ed. edn. Springer (1999)

[12] Kussmaul, R.: Ein numerisches Verfahren zur Lösung des Neumannschen Außenraumproblems für die Helmholtzsche Schwingungsgleichung. Computing **4**, 246–273 (1969)

[13] Saranen, J., Vainikko, G.: Periodic integral and pseudodifferential equations with numerical approximation. Springer monographs in mathematics. Springer, Berlin (2002)

[14] Wienert, L.: Die numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im $\mathbb{R}^3$. Ph.D. thesis, Univ. Göttingen (1990)

IWRMM-Preprints seit 2012

Eine aktuelle Liste aller IWRMM-Preprints finden Sie auf:

www.math.kit.edu/iwrmm/seite/preprints