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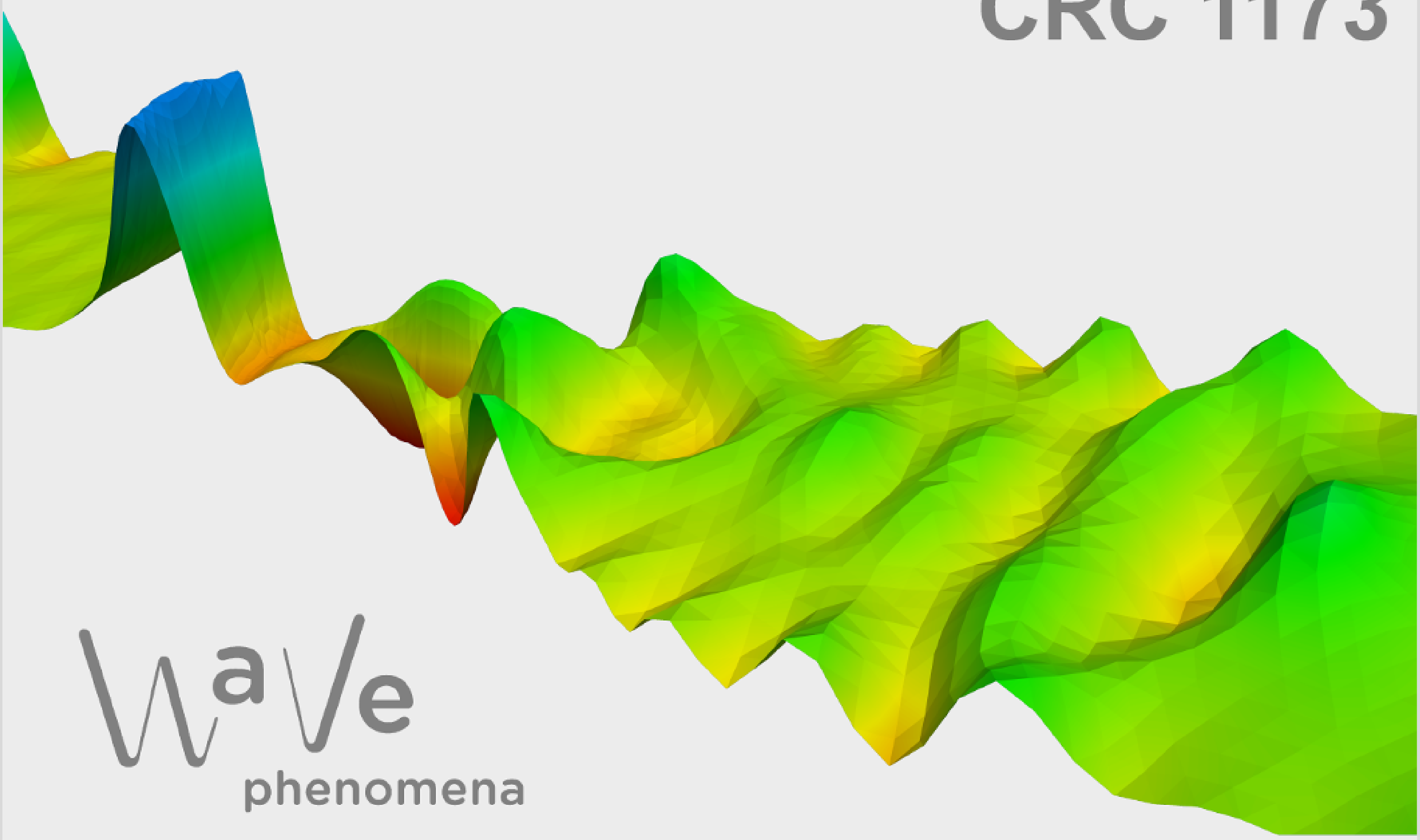

Wave phenomena

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INVERSE PROBLEMS FOR ABSTRACT EVOLUTION EQUATIONS WITH APPLICATIONS IN ELECTRODYNAMICS AND ELASTICITY.

ANDREAS KIRSCH AND ANDREAS RIEDER

Abstract. It is common knowledge – mainly based on experience – that parameter identification problems in partial differential equations are ill-posed. Yet, a mathematical sound argumentation is missing, except for some special cases. We present a general theory for inverse problems related to abstract evolution equations which explains not only their local ill-posedness but also provides the Fréchet derivative of the corresponding parameter-to-solution map which is needed, e.g., in Newton-like solvers. Our abstract results are applied to inverse problems related to the following first order hyperbolic systems: Maxwell's equation (electromagnetic scattering in conducting media) and elastic wave equation (seismic imaging).

1. Introduction

In this paper we consider parameter identification problems related to first order hyperbolic systems such as the electromagnetic or the elastic wave systems. Especially, we show that these inverse problems are locally ill-posed anywhere no matter how many measurements are available. Further, we characterize the Fréchet derivative of the parameter-to-solution map which is an essential ingredient of iterative regularization schemes.

Our approach is based on abstract evolution equations in Hilbert spaces,

$$Bu'(t) + Au(t) = f(t)\,, \quad u(0) = u_0\,,$$

with a maximal monotone operator A and a positive definite operator B. Existence, uniqueness, and regularity of the solution follow from the famous Hille-Yosida theorem. As we think that most of our intended readers are not familiar with operator semigroup theory we collect basic facts with some proofs in the next section.

The operator B is the "parameter" to be identified from (partial) knowledge of u. Thus, $F\colon B \mapsto u$ is the parameter-to-solution map for which we validate Fréchet differentiability (Section 3) and local ill-posedness (Section 4). Finally, we apply our abstract theory to inverse electromagnetic scattering in time domain to identify spatial dependent electric permittivities and magnetic permeabilities (Section 5). A second application concerns seismic imaging where the governing equation is the elastic wave equation in hyperbolic system formulation (Section 6). Here, the mass density and the two Lamé parameters are sought.

Fréchet differentiability of parameter-to-solution maps of abstract first order hyperbolic systems has been studied before by Blazek et al. [1] using the technique of weak solutions. Indeed, our research was triggered by reading their article and with the present paper we complement and extend their work. Please consult [1] also for an overview on prior and related work in this direction.

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In their recent work [15] Lechleiter and Schlasche identify Lamé parameters of the second order elastic wave equation from boundary measurements. They set up an inexact Newton iteration to this end validating the Fréchet differentiability of the parameter-to-solution map in the spirit of [12]. Boehm and Ulbrich [2] attack the same problem with a semi-smooth Newton iteration also providing an expression for the Fréchet derivative. By eliminating the stress tensor from the first order elastic wave equation, the representation of the Fréchet derivatives in [2] and [15] can be obtained within our setting, however, under weaker assumptions, see Section 7.

2. Evolution Equations

In the first part we recall the basic facts from the abstract theory of evolution equations. Although this is very well known to the experts (see, e.g., [5, 16]) we recall the rather elementary approach as in [4] for the convenience of the reader.

Definition 2.1. *Let X be a Hilbert space and $A : X \supset \mathcal{D}(A) \to X$ a linear operator with domain of definition $\mathcal{D}(A) \subset X$. The operator A is called* ***monotone*** *if*

$$(Ax, x)_X \geq 0 \quad \text{for all } x \in \mathcal{D}(A)\,.$$

A is called ***maximal monotone*** *if it is monotone and $I + A$ is surjective as an operator from $\mathcal{D}(A)$ onto X. Here, I denotes the identity operator in X.*

We note that the maximal monotonicity of the operator A implies already denseness of the domain of definition $\mathcal{D}(A)$ in X and closedness of the operator (see [4]).

These assumptions on A are already sufficient for the well-posedness of the abstract evolution equation.

Theorem 2.2. *(Hille–Yosida) Let $A : X \supset \mathcal{D}(A) \to X$ be a linear maximal monotone operator and $u_0 \in \mathcal{D}(A)$. Then there exists a unique $u \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(A)\big)$ with*

$$u'(t) \;=\; -Au(t)\,,\ t \geq 0\,, \quad u(0) = u_0\,. \tag{1}$$

Here, $\mathcal{D}(A)$ is equipped with the graph norm; that is, $\|v\|_{\mathcal{D}(A)} = \big[\|v\|_X^2 + \|Av\|_X^2\big]^{1/2}$ for $v \in \mathcal{D}(A)$. Furthermore, the stability result holds in the form

$$\|u(t)\|_X \;\leq\; \|u_0\|_X \quad \text{for all } t \geq 0\,. \tag{2}$$

For a proof we refer to [4], Theorem 7.4. This theorem guarantees that the operator $S(t)$ which maps $u_0 \in \mathcal{D}(A)$ to $u(t)$ is bounded in X and thus has a bounded extension into all of X with $\|S(t)\|_{\mathcal{L}(X)} \leq 1$. The following lemma is also part of the Theorem by Hille-Yosida. (For this part see, e.g., [18], where A and λ are changed into $-A$ and into $1/\lambda$, respectively.)

Lemma 2.3. *Let $S(t) : X \to X$ be defined as above. Then A coincides with the operator B, defined by*

$$Bv \;=\; \lim_{h\to 0+} \frac{1}{h}\,[S(h)v - v]\,, \quad v \in \mathcal{D}(B) := \Big\{ v \in X : \lim_{h\to 0+} [S(h)v - v]/h \text{ exists in } X \Big\}$$

(which is called the infinitesimal generator of the semigroup $S(t)$). In particular, the domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ coincide.

The inhomogeneous evolution equation is solved by the variation-of-constant formula:

Theorem 2.4. *Let $A : X \supset \mathcal{D}(A) \to X$ be a linear maximal monotone operator.*
(a) Let $u_0 \in X$, and $f \in L^1\big((0,\infty), X\big)$. The function $u \in C\big([0,\infty), X\big)$, defined by

$$u(t) \;=\; S(t)u_0 \;+\; \int_0^t S(t-s)\, f(s)\,\mathrm{d}s\,, \quad t \ge 0\,, \tag{3}$$

is called the mild solution *of*

$$u'(t) \;=\; -Au(t) \;+\; f(t)\,, \quad t \ge 0\,, \quad u(0) = u_0\,. \tag{4}$$

It satisfies the estimate

$$\|u(t)\|_X \;\le\; \|u_0\|_X \;+\; \|f\|_{L^1((0,t),X)} \quad \text{for all } t \ge 0\,. \tag{5a}$$

(b) Let u be given by (3) and $u_0 \in \mathcal{D}(A)$ and $f \in W^{1,1}\big((0,\infty), X\big)$. Then $u \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(A)\big)$. Furthermore, u is the unique classical solution of (4), and the following stability estimates hold:

$$\|u'(t)\|_X \;\le\; \|Au_0 - f(0)\|_X \;+\; \|f'\|_{L^1((0,t),X)}\,, \tag{5b}$$

$$\|Au(t)\|_X \;\le\; \|u'(t)\|_X \;+\; \|f(t)\|_X \tag{5c}$$

for $t \ge 0$. We note that $f \in W^{1,1}\big((0,\infty), X\big)$ is continuous and $\|f(t)\|_X \le 2\|f\|_{W^{1,1}(0,\infty),X)}$ for all $t \ge 0$. We can combine these estimates in the form

$$\|u\|_{C^1([0,\infty),X)} \;+\; \|u\|_{C([0,\infty),\mathcal{D}(A))} \;\le\; c\big[\|u_0\|_{\mathcal{D}(A)} + \|f\|_{W^{1,1}((0,\infty),X)}\big] \tag{6}$$

where c depends only on A.

Proof: (a) u is well defined and continuous by the continuity of the semigroup $S(t) : X \to X$ and the assumption on f. The estimate (5a) follows directly from the fact that $\|S(t)\|_{\mathcal{L}(X)} \le 1$ for all t.
(b) First we show that the integral $v(t) := \int_0^t S(t-s)\, f(s)\,\mathrm{d}s$ is in $\mathcal{D}(A)$. For this we write $v(t)$ as $v(t) = \int_0^t S(s)\, f(t-s)\,\mathrm{d}s$ and observe that v is differentiable a.e. by the assumption on f and $v'(t) = S(t)f(0) + \int_0^t S(s)\, f'(t-s)\,\mathrm{d}s = S(t)f(0) + \int_0^t S(t-s)\, f'(s)\,\mathrm{d}s$. Since the right hand side is continuous we conclude that v is differentiable for every $t \ge 0$. Therefore, for $h \ne 0$ the term

$$\begin{aligned}
\frac{1}{h}\big(v(t+h) - v(t)\big) \;&=\; \frac{1}{h}\int_t^{t+h} S(t+h-s)\, f(s)\,\mathrm{d}s \\
&\quad + \frac{1}{h}\int_0^t \big[S(t+h-s) - S(t-s)\big]\, f(s)\,\mathrm{d}s \\
&=\; \frac{1}{h}\int_t^{t+h} S(t+h-s)\, f(s)\,\mathrm{d}s \;+\; \frac{1}{h}\big[S(h) - I\big]\, v(t)
\end{aligned}$$

converges to $v'(t)$ as h tends to zero. Since the first term on the right hand side converges to $S(0)f(t) = f(t)$ also the second term converges which, by the previous lemma, yields $v(t) \in \mathcal{D}(A)$ for all $t \ge 0$ and $v'(t) = f(t) - Av(t)$. Therefore, if $u(t)$ denotes the right hand side of (3), then $u'(t) = -AS(t)u_0 + f(t) - Av(t) = f(t) - Au(t)$. This shows that the right hand side of (3) solves (4).
To show the estimate (5b) we note that

$$u'(t) \;=\; S(t)\,\big[-Au_0 + f(0)\big] \;+\; \int_0^t S(t-s)\, f'(s)\,\mathrm{d}s\,. \tag{7}$$

This proves the estimate (5b). Estimate (5c) follows obviously. □

Remark 2.5. *If $f \in C^1\big([0,\infty),X\big)$ then, in general, $f \notin W^{1,1}\big((0,\infty),X\big)$, and this theorem is not directly applicable. However, the right hand sides of (5a) – (5c) depend only on f on the interval $(0,t)$. Therefore, if we multiply f by a smooth function ϕ of compact support with $\phi(t)=1$ on $[0,T]$ then one can replace f by $f\phi$. On the bounded interval $[0,T]$ the estimates (5a) – (5c) hold without modification. Using $\|f\|_{L^1((0,t),X)} \le t\|f\|_{C([0,t],X)}$ (and analogously for the derivative) yields an extra factor T in the estimates for $t\in[0,T]$ when using the maximum norm.*

Corollary 2.6. *Let $u_0 \in X$ and $f \in L^1\big((0,\infty),X\big)$. The mild solution $u \in C\big([0,\infty),X\big)$ of (3) is the weak solution; that is,*

$$\frac{d}{dt}\big(u(t),\psi\big)_X \;=\; -\big(u(t),A^*\psi\big)_X \;+\; \big(f(t),\psi\big)_X \quad \text{for a.a. } t\ge 0 \text{ and } \psi\in\mathcal{D}(A^*) \tag{8}$$

where $A^*: X \supset \mathcal{D}(A^*) \to X$ denotes the adjoint of A.

Proof: Let $\psi \in \mathcal{D}(A^*)$ and $\varphi \in C_0^\infty[0,\infty)$. For $u_0\in\mathcal{D}(A)$ and $f \in W^{1,1}\big([0,\infty),X\big)$ we multiply (4) by $\varphi(t)\psi$, integrate from 0 to ∞, use partial integration and the definition of the adjoint. This yields

$$-\int_0^\infty \big(u(t),\psi\big)_X\,\varphi'(t)\,\mathrm{d}t \;=\; -\int_0^\infty \big(u(t),A^*\psi)\big)_X\,\varphi(t)\,\mathrm{d}t \;+\; \int_0^\infty \big(f(t),\psi\big)_X\,\varphi(t)\,\mathrm{d}t$$

By the denseness of $\mathcal{D}(A)$ in X and $W^{1,1}\big([0,\infty),X\big)$ in $L^1\big((0,\infty),X\big)$ and the stability estimate (5a) we conclude that this formula holds also if u is only the mild solution. Now (8) follows from a standard argument (see, e.g., [9], Theorem 2.18[1].). □

We will need the following regularity result (see [4], Theorem 7.5 for the case $f=0$).

Theorem 2.7. *Let $A: X\supset\mathcal{D}(A)\to X$ be a linear maximal monotone operator and, for some $k\in\mathbb{N}_{\ge 1}$, let $f\in W^{k,1}\big((0,\infty),X\big)$ and*

$$u_{0,\ell} \;:=\; (-A)^\ell u_0 \;+\; \sum_{j=0}^{\ell-1}(-A)^j f^{(\ell-1-j)}(0) \;\in\; \mathcal{D}(A) \quad \text{for } \ell=0,\dots,k-1\,. \tag{9}$$

(In the case $\ell=0$ the sum is set to zero.) Let $u\in C^1\big([0,\infty),X\big)\cap C\big([0,\infty),\mathcal{D}(A)\big)$ be the unique solution of (4).
Then $u \in C^k\big([0,\infty),X\big)\cap C^{k-1}\big([0,\infty),\mathcal{D}(A)\big)$. Furthermore, the stability estimates hold in the forms

$$\|u^{(\ell)}(t)\|_X \;\le\; \|u_{0,\ell}\|_X \;+\; \|f^{(\ell)}\|_{L^1((0,t),X)}\,,\quad t\ge 0\,,\ \ell=0,\dots,k, \tag{10}$$

where $u_{0,k} := -Au_{0,k-1} + f^{(k-1)}(0)\in X$. Note that $f^{(k-1)}$ is continuous.

Proof: First we note that the $u_{0,\ell}$'s satisfy the recursion formula: $u_{0,0}=u_0$ and $u_{0,\ell} = -Au_{0,\ell-1} + f^{(\ell-1)}(0)$, $\ell=1,\dots,k$. Next we show

$$u^{(\ell)}(t) \;=\; S(t)u_{0,\ell} \;+\; \int_0^t S(t-s)\,f^{(\ell)}(s)\,\mathrm{d}s\,,\quad t\ge 0\,,\quad \ell=0,\dots,k\,, \tag{11}$$

by induction with respect to k. The case $k=0$ reduces to (3). Now we assume that these formulas are true for $\ell=0,\dots,k$ (under the assumption (9)) and we assume that

[1]This argument is sometimes called the fundamental lemma of calculus of variations, see [10], Lemma 1.2.1

$f \in W^{k+1,1}\big((0,\infty),X\big)$ and (9) holds for $k+1$. Then $u_{0,k} \in \mathcal{D}(A)$ and (11) holds for $\ell = k$; that is,

$$u^{(k)}(t) \;=\; S(t)u_{0,k} \;+\; \int_0^t S(s)\, f^{(k)}(t-s)\,\mathrm{d}s\,, \quad t \geq 0\,.$$

The additional differentiability of f yields that $u^{(k)}$ is differentiable and thus, as in (7),

$$\begin{aligned} u^{(k+1)}(t) \;&=\; -AS(t)u_{0,k} \;+\; S(t)f^{(k)}(0) \;+\; \int_0^t S(s)\, f^{(k+1)}(t-s)\,\mathrm{d}s \\ &=\; S(t)\left[-Au_{0,k} + f^{(k)}(0)\right] \;+\; \int_0^t S(t-s)\, f^{(k+1)}(s)\,\mathrm{d}s \end{aligned}$$

which is formula (11) for $\ell = k+1$.
From this representation (11) the stability estimates follow immediately. □

3. Differentiability With Respect to Parameters

In this section we consider a class of evolution equations, depending on a parameter, and will show continuity and differentiability properties of the parameter-to-solution map. The parameters are modeled by the set of self adjoint and uniformly bounded and coercive operators in X. We define the set

$$\begin{aligned} \mathcal{B} \;&=\; \mathcal{B}(\gamma_-,\gamma_+) \\ &=\; \big\{B \in \mathcal{L}(X) : B \text{ self adjoint}\,, \gamma_-\|x\|_X^2 \leq (Bx,x)_X \leq \gamma_+\|x\|_X^2 \text{ for all } x \in X\big\} \end{aligned}$$

where $0 < \gamma_- < \gamma_+$ and $\mathcal{L}(X)$ denotes the space of linear and bounded operators from X into X. Let again $A : X \supset \mathcal{D}(A) \to X$ be a maximal monotone operator. The following result which we have found in [19] assures that also $A+B$ is surjective for every $B \in \mathcal{B}$ as operators from $\mathcal{D}(A)$ onto X. We include a direct proof for the convenience of the reader.

Lemma 3.1. *Let $A : X \supset \mathcal{D}(A) \to X$ be a linear maximal monotone operator and $B \in \mathcal{B}$. Then $A+B$ is surjective.*

Proof: In the first part we prove that the adjoint A^* is monotone as well. We note that this is not true in general; that is, without the assumption that A is maximal, as the example $A = -\Delta$ with $X = L^2(D)$ and $\mathcal{D}(A) = H_0^2(D) = \{u \in H^2(D) : u = \partial_\nu u = 0 \text{ on } \partial D\}$ shows. First we note (see [4], Proposition 7.1) that the maximal monotonicity implies that $(A+rI)^{-1}$ exists and is bounded from X into itself with range $\mathcal{D}(A)$ for all $r \in (0,1]$. We define $\tilde{A} = \left[(A+rI)^{-1}\right]^* : X \to X$ as its adjoint; that is, $(\tilde{A}z,y)_X = \big(z,(A+rI)^{-1}y\big)_X$ for all $z,y \in X$; that is, $\big(\tilde{A}z,(A+rI)x\big)_X = (z,x)_X$ for all $z \in X$ and $x \in \mathcal{D}(A)$. From this we conclude that $\tilde{A}z \in \mathcal{D}(A^*)$ and $(A^*+rI)\tilde{A}z = z$ for all $z \in X$. Furthermore, from $\big((A^*+rI)y,z\big)_X = \big(y,(A+rI)z\big)_X$ and the surjectivity of $A+rI$ we observe that A^*+rI is injective for all $r \in (0,1]$. We observe that $(A^*+rI)\big[\tilde{A}(A^*+rI)y - y\big] = (A^*+rI)y - (A^*+rI)y = 0$ and thus $\tilde{A}(A^*+rI)y = y$ for all $y \in \mathcal{D}(A^*)$; that is, we have shown that $\tilde{A} = \left[(A+rI)^{-1}\right]^* = (A^*+rI)^{-1}$.
Now we use that $\big((A+rI)x,x\big)_X \geq 0$ for all $x \in \mathcal{D}(A)$ and $r \geq 0$ and thus $\big(y,(A+rI)^{-1}y\big)_X \geq 0$ for all $y \in X$; that is, $\big((A^*+rI)^{-1}y,y\big)_X \geq 0$ for all $y \in X$; that is, $\big(z,(A^*+rI)z\big)_X \geq 0$ for all $z \in \mathcal{D}(A^*)$ and all $r \in (0,1]$. For $r \to 0$ we arrive at $(z,A^*z)_X \geq 0$ for all $z \in \mathcal{D}(A^*)$; that is, the monotonicity of A^*.

Let now $B \in \mathcal{B}$. Then $A^* + B$ is one-to-one by the monotonicity of A^*. Therefore, the range of $A + B$ is dense. Indeed, from $(z, Ax + Bx)_X = 0$ for all $x \in \mathcal{D}(A)$ we conclude that $(z, Ax)_X = -(z, Bx)_X = -(Bz, x)$ for all $x \in \mathcal{D}(A)$. Therefore, $z \in \mathcal{D}(A^*)$ and $A^*z = -Bz$ and thus $z = 0$ which shows the denseness of the range. Furthermore, the range of $A+B$ is also closed. This follows from the estimate $\gamma_-\|x\|_X^2 \le \big((A+B)x, x\big)_X \le \|(A + B)x\|_X\|x\|_X$ for all $x \in \mathcal{D}(A)$; that is, $\gamma_-\|x\|_X \le \big\|(A + B)x\big\|_X$ for all $x \in \mathcal{D}(A)$. Indeed, let $(A+B)x_j \to z$ for some sequence $x_j \in \mathcal{D}(A)$. The estimate implies that $\{x_j\}$ is a Cauchy sequence and thus convergent $x_j \to x$ for some $x \in X$. Therefore, $Bx_j \to Bx$ and thus $Ax_j \to z - Bx$. The closedness of A yields $x \in \mathcal{D}(A)$ and $Ax = z - Bx$. This shows that z is in the range of $A + B$ and finishes the proof. □

We note the following equivalent interpretation of this result. If we define, for $B \in \mathcal{B}$, the weighted inner product $(\cdot,\cdot)_B$ in X by

$$(x, y)_B \;=\; (Bx, y)_X\,, \quad x, y \in X\,,$$

then the operator $B^{-1}A$ is maximal monotone with respect to this weighted inner product. The corresponding norm $\|\cdot\|_B$ is equivalent to the ordinary norm because obviously

$$\gamma_-\|x\|_X^2 \;\le\; \|x\|_B^2 \;\le\; \gamma_+\|x\|_X^2\,, \quad x \in X\,.$$

Therefore, for any $B \in \mathcal{B}$, $u_0 \in \mathcal{D}(A) = \mathcal{D}(B^{-1}A)$, and $f \in W^{1,1}\big((0,\infty), X\big)$ there exists a unique solution $u \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(A)\big)$ with

$$Bu'(t) \;=\; -Au(t) \;+\; f(t)\,, \quad t \ge 0\,, \quad u(0) = u_0\,. \tag{12}$$

In the estimates of Theorems 2.4 and 2.7 one has to replace f by $B^{-1}f$ and $\|\cdot\|_X$ by $\|\cdot\|_B$ – or compensate the use of $\|\cdot\|_X$ by introducing the constants $\sqrt{\gamma_-}$ or $\sqrt{\gamma_+}$.

First we show that under certain regularity assumptions the mapping $F : B \mapsto u$ is (locally) Lipschitz continuous on $\mathcal{B}$.

Lemma 3.2. *Let $A : X \supset \mathcal{D}(A) \to X$ be maximal monotone, $u_0 \in \mathcal{D}(A)$, $\hat{B} \in \mathcal{B}$ and $B \in \mathcal{L}(X)$ such that $\hat{B} + B \in \mathcal{B}$. Furthermore, let $f \in W^{2,1}\big((0,\infty), X\big)$ and $\hat{v}_0 := \hat{B}^{-1}\big(Au_0 - f(0)\big) \in \mathcal{D}(A)$. Let $\hat{u}, \tilde{u} \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(A)\big)$ be the solutions of*

$$\hat{B}\hat{u}'(t) \;+\; A\hat{u}(t) \;=\; f(t) \quad \textit{and} \quad (\hat{B} + B)\tilde{u}'(t) \;+\; A\tilde{u}(t) \;=\; f(t)\,, \quad t \ge 0\,,$$

and $\hat{u}(0) = \tilde{u}(0) = u_0$, respectively. Then there exists c, depending only on A, γ_+, γ_-, $\hat{v}_0$, and f, such that for all $T > 0$:

$$\|\hat{u}(t) - \tilde{u}(t)\|_X \;+\; \|\hat{u}'(t) - \tilde{u}'(t)\|_X \;\le\; c(1 + T)\,\|B\|_{\mathcal{L}(X)} \quad \textit{for } 0 \le t \le T\,.$$

Proof: The difference $\hat{u} - \tilde{u}$ satisfies

$$(\hat{B} + B)\big(\hat{u}'(t) - \tilde{u}'(t)\big)(t) \;+\; A(t)\big(\hat{u}(t) - \tilde{u}(t)\big) \;=\; B\hat{u}'(t)\,,\; t \ge 0\,,$$

and $\hat u(0)-\tilde u(0)=0$. We note that $\hat u \in C^2\big([0,\infty),X\big)\cap C^1\big([0,\infty),\mathcal{D}(A)\big)$ by the assumptions on f and u_0 and Theorem 2.7 for $k=2$ and thus

$$\begin{aligned}
\|\hat u'(t)\|_X &\le \frac{1}{\sqrt{\gamma_-}}\,\|\hat u'(t)\|_{\hat B} \le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\|\hat v_0\|_X + \|\hat B^{-1}f'\|_{L^1((0,t),X)}\Big]\\
&\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\|\hat v_0\|_X + \frac{1}{\gamma_-}\,\|f'\|_{L^1((0,t),X)}\Big] \le c_1 \quad \text{and analogously}\\
\|\hat u''(t)\|_X &\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\|\hat B^{-1}\big(f'(0)+A\hat v_0\big)\|_X + \|\hat B^{-1}f''\|_{L^1((0,t),X)}\Big]\\
&\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\frac{1}{\gamma_-}\,\|f'(0)+A\hat v_0\|_X + \frac{1}{\gamma_-}\,\|f''\|_{L^1((0,t),X)}\Big] \le c_2
\end{aligned}$$

for $t\ge 0$ where c_1,c_2 depend only on A, γ_+, γ_-, $\hat v_0$, and f. Therefore, Theorem 2.4 is applicable to $\hat u-\tilde u$ (see Remark 2.5) and yields the stability estimates (note that $\hat u(0)-\tilde u(0)=0$)

$$\begin{aligned}
\|\hat u(t)-\tilde u(t)\|_X &\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\|(\hat B+B)^{-1}B\hat u'\|_{L^1((0,t),X)}\Big] \le c_1\,\frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\,\frac{T}{\gamma_-}\,\|B\|_{\mathcal{L}(X)}\,,\\
\|\hat u'(t)-\tilde u'(t)\|_X &\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\Big[\|(\hat B+B)^{-1}B\hat u'(0)\|_X + \|(\hat B+B)^{-1}B\hat u''\|_{L^1((0,t),X)}\Big]\\
&\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\,\frac{1}{\gamma_-}\,\|B\|_{\mathcal{L}(X)}\,(c_1+Tc_2)
\end{aligned}$$

for $0\le t\le T$. □

Remark 3.3. *In general, the Lipschitz constant $c(1+T)$ depends obviously on T but also on $\hat B$ through $\hat v_0$; that is, the Lipschitz continuity holds only locally with respect to time and B. However, under the stronger assumption $Au_0=f(0)$ the constant c is universal for all $\hat B,\hat B+B\in\mathcal{B}$ because in this case $\hat v_0=0$. For fixed $T>0$ the assumption on f is obviously too strong. It is sufficient to assume that $f\in W^{2,1}\big((0,T),X\big)$. Indeed, one can extend f to $f\in W^{2,1}\big((0,\infty),X\big)$, and the solutions corresponding to these extensions coincide on $[0,T]$ by (3), compare also with Remark 2.5.*

Theorem 3.4. *Let $T>0$, $f\in W^{1,1}\big((0,T),X\big)$, and $u_0\in\mathcal{D}(A)$. Then the mapping $B\mapsto u$ is continuous from $\mathcal{B}$ into $C^1\big([0,T],X\big)$.*

Proof: Let $B,B_n\in\mathcal{B}$ with $B_n\to B$ in $\mathcal{L}(X)$. Define the sequence of linear operators $P_n:\mathcal{D}(A)\times W^{1,1}\big((0,T),X\big)\to C^1\big([0,T],X\big)$ by $P_n(u_0,f)=u_n-u$ where $u_n,u\in C^1\big([0,T],X\big)\cap C\big([0,T],\mathcal{D}(A)\big)$ solve (12) for B_n and B, respectively, and u_0 and f. By the previous lemma (and Remark 3.3) we have that $P_n(u_0,f)\to 0$ in $C^1\big([0,T],X\big)$ as $n\to\infty$ for $(u_0,f)\in D:=\big\{(u_0,f)\in\mathcal{D}(A)\times W^{2,1}\big((0,T),X\big): B^{-1}\big(Au_0-f(0)\big)\in\mathcal{D}(A)\big\}$. The space D is dense in $\mathcal{D}(A)\times W^{1,1}\big((0,T),X\big)$. Indeed, let $(u_0,f)\in\mathcal{D}(A)\times W^{1,1}\big((0,T),X\big)$. Choose sequences $u_0^j\in\mathcal{D}\big((B^{-1}A)^2\big)$, $z^j\in\mathcal{D}(A)$, and $\tilde f^j\in W^{2,1}\big((0,T),X\big)$ with $u_0^j\to u_0$ in $\mathcal{D}(A)$ (possible because $\mathcal{D}\big((B^{-1}A)^2\big)$ is dense in $\mathcal{D}(B^{-1}A)=\mathcal{D}(A)$), $z^j\to B^{-1}f(0)$ in X and $\tilde f^j\to f$ in $W^{1,1}\big((0,T),X\big)$. Furthermore, choose $\phi\in C^\infty[0,\infty)$ with $\phi(0)=1$ and $\phi(t)=0$ for $t\ge t_0$ for some $t_0\in(0,T)$. Define $f^j\in W^{2,1}\big((0,T),X\big)$ by $f^j(t)=\tilde f^j(t)+\phi(t)\big[Bz^j-\tilde f^j(0)\big]$, $t\ge 0$. Then $B^{-1}f^j(0)=z^j\in\mathcal{D}(A)$ and $f^j\to f$ in $W^{1,1}\big((0,T),X\big)$

as $j \to \infty$. Furthermore, $B^{-1}Au_0^j \in \mathcal{D}(B^{-1}A) = \mathcal{D}(A)$ and thus $(u_0^j, f^j) \in D$ which shows denseness of D in $\mathcal{D}(A) \times W^{1,1}\big((0,T),X\big)$. Furthermore, for $(u_0, f) \in \mathcal{D}(A) \times W^{1,1}\big((0,T),X\big)$ we have by Theorem 2.4 that

$$\big\|P_n(u_0,f)\big\|_{C^1([0,T],X)} \;\le\; \big\|u_n\big\|_{C^1([0,T],X)} + \big\|u\big\|_{C^1([0,T],X)} \;\le\; c\big[\|u_0\|_{\mathcal{D}(A)} + \|f\|_{W^{1,1}((0,T),X)}\big]$$

where c depends only on A, γ_-, and γ_+. Therefore, $\|P_n\|$ is uniformly bounded, and a density argument implies that $P_n(u_0,f) \to 0$ in $C^1\big([0,T],X\big)$ for all $(u_0,f) \in \mathcal{D}(A) \times W^{1,1}\big((0,T),X\big)$. □

Next we show differentiability of this mapping $F : B \mapsto u$ from $\mathcal{B}$ into $C\big([0,T),X\big)$.

Theorem 3.5. *Let $T > 0$, $f \in W^{1,1}\big((0,T),X\big)$, and $u_0 \in \mathcal{D}(A)$. Then $F : \mathcal{B} \to C\big([0,T),X\big)$ is Fréchet differentiable at $\hat{B} \in \mathrm{int}(\mathcal{B})$ and $F'(\hat{B})B = \overline{u}$ where $\overline{u} \in C\big([0,T],X\big)$ is the mild solution of*

$$\hat{B}\overline{u}'(t) \;+\; A\overline{u}(t) \;=\; -B\hat{u}'(t)\,,\ t \in [0,T]\,, \quad \overline{u}(0) = 0\,. \tag{13}$$

Here, $\hat{u} \in C^1\big([0,T],X\big) \cap C\big([0,T],\mathcal{D}(A)\big)$ is the (classical) solution of $\hat{B}\hat{u}'(t) + A\hat{u}(t) = f(t)$, $t \in [0,T]$, $\hat{u}(0) = u_0$.

Proof: First we note that the source term in (13) is in $C\big([0,T],X\big)$ which implies the existence of a mild solution $\overline{u} \in C\big([0,T],X\big)$ of (13) (we refer again to Remark 3.3). Let $B \in \mathcal{L}(X)$ such that $\hat{B} + B \in \mathcal{B}$ and let $\tilde{u} \in C^1\big([0,T],X\big) \cap C\big([0,T],\mathcal{D}(A)\big)$ be the solution of $(\hat{B}+B)\tilde{u}'(t) + A\tilde{u}(t) = f(t)$, $t \in [0,T]$, and $\tilde{u}(0) = u_0$. Writing this as $\tilde{u}'(t) + \hat{B}^{-1}A\tilde{u}(t) = \hat{B}^{-1}\big[f(t) - B\tilde{u}'(t)\big]$ we have

$$\begin{aligned}
\tilde{u}(t) &= \hat{S}(t)u_0 \;+\; \int_0^t \hat{S}(t-s)\,\hat{B}^{-1}\big[f(s) - B\tilde{u}'(s)\big]\,ds\,,\\
\hat{u}(t) &= \hat{S}(t)u_0 \;+\; \int_0^t \hat{S}(t-s)\hat{B}^{-1}f(s)\,ds\,,\\
\overline{u}(t) &= -\int_0^t \hat{S}(t-s)\,\hat{B}^{-1}B\hat{u}'(s)\,ds\,,
\end{aligned}$$

for $t \in [0,T]$ where $\hat{S}(s)$ denotes the semigroup corresponding to $\hat{B}^{-1}A$. Therefore,

$$\tilde{u}(t) - \hat{u}(t) - \overline{u}(t) \;=\; \int_0^t \hat{S}(t-s)\,\hat{B}^{-1}B\big(\hat{u}'(s) - \tilde{u}'(s)\big)\,ds\,, \quad t \in [0,T]\,,$$

and thus

$$\begin{aligned}
\|\tilde{u}(t) - \hat{u}(t) - \overline{u}(t)\|_X &\le \frac{1}{\sqrt{\gamma_-}}\int_0^t \underbrace{\|\hat{S}(t-s)\|_{\hat{B}}}_{\le 1}\,\|\hat{B}^{-1}B\big(\hat{u}'(s) - \tilde{u}'(s)\big)\|_{\hat{B}}\,ds\\
&\le \frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\,t\,\|B\|_{\mathcal{L}(X)}\max_{0\le s\le t}\|\hat{u}'(s) - \tilde{u}'(s)\|_X\\
&\le T\,\frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\,\|B\|_{\mathcal{L}(X)}\,\|\hat{u}' - \tilde{u}'\|_{C([0,T],X)} \quad \text{for } t \in [0,T]\,.
\end{aligned}$$

Therefore,

$$\frac{1}{\|B\|_{\mathcal{L}(X)}}\,\|\tilde{u} - \hat{u} - \overline{u}\|_{C([0,T],X)} \;\le\; T\,\frac{\sqrt{\gamma_+}}{\sqrt{\gamma_-}}\,\|\hat{u}' - \tilde{u}'\|_{C([0,T],X)}$$

which ends the proof because $\|\hat{u}' - \tilde{u}'\|_{C([0,T],X)} \to 0$ as $\|B\|_{\mathcal{L}(X)} \to 0$ by Theorem 3.4. $\square$

Remarks 3.6. *(a) Again, the mild solution is also a weak solution in the sense of (8); that is,*

$$\frac{d}{dt}\big(\hat{B}\overline{u}(t),\psi\big)_X + \big(\overline{u}(t),A^*\psi\big)_X = -\frac{d}{dt}\big(B\hat{u}(t),\psi\big)_X \quad \text{for a.a. } t\in[0,T] \text{ and } \psi\in\mathcal{D}(A^*)\,.$$

(b) Under the stronger assumptions $f\in W^{2,1}\big((0,T),X\big)$ and $\hat{B}^{-1}[Au_0 - f(0)]\in\mathcal{D}(A)$ the mild solution $\overline{u}$ is a classical solution $\overline{u}\in C^1\big([0,T],X\big)\cap C\big([0,T],\mathcal{D}(A)\big)$ because the source term in (13) is in $C^1\big([0,T],X\big)$ by Theorem 2.7 for $k=2$.

(c) If we consider the mapping $F: B\mapsto u$ from $\mathcal{B}$ into the canonical space $C^1\big([0,T],X\big)\cap C\big([0,T],\mathcal{D}(A)\big)$ rather that into $C\big([0,T],X\big)$ we would need even stronger regularity assumptions. Indeed, if we use the notations of the previous proof we note that $v=\tilde{u}-\hat{u}-\overline{u}$ satisfies also

$$(\hat{B}+B)v'(t) + Av(t) = -B\overline{u}'(t)\,,\ t\in[0,T]\,.$$

Therefore, in order to estimate $\|v\|_{C^1([0,T],X)}$ and $\|v\|_{C([0,T],\mathcal{D}(A))}$ the source term $B\overline{u}'$ has to be in $W^{1,1}\big((0,T),X\big)$ which requires $\hat{u}\in C^3\big([0,T],X\big)$. By Theorem 2.7 for $k=3$ this requires the additional assumptions $f\in W^{3,1}\big((0,T),X\big)$, $\hat{v}_0 := \hat{B}^{-1}[Au_0-f(0)]\in\mathcal{D}(A)$, and $\hat{B}^{-1}[A\hat{v}_0 - f'(0)]\in\mathcal{D}(A)$.

(d) We have shown differentiability of F as a mapping from $\mathcal{B}$ into $C\big([0,T],X\big)$. This implies that the mapping is also differentiable as a mapping into the more appropriate (w.r.t. the applications) space $L^2\big((0,T),X\big)$.

(e) We note that in applications (see Sections 5 and 6 below) the operator B is just a multiplication operator with some L^∞−function. Therefore, the assumptions $\hat{v}_0\in\mathcal{D}(A)$ and $\hat{B}^{-1}\big(A\hat{v}_0 - f'(0)\big)\in\mathcal{D}(A)$ include smoothness assumptions on $\hat{B}$.

4. Local Ill-posedness

We recall from [7] that a (nonlinear) equation $F(x)=y$ is *locally ill-posed* at $x^+\in\mathcal{D}(F)$ satisfying $F(x^+)=y$ if in any neighborhood of x^+ there exists a sequence $\{x_k\}_{k\in\mathbb{N}}\subset\mathcal{D}(F)$ such that

$$\lim_{k\to\infty}\|F(x_k)-F(x^+)\|_Y = 0\,,\ \text{however } \|x_k - x^+\|_X \not\to 0 \text{ for } k\to\infty.$$

For fixed $f\in W^{1,1}\big((0,T),X\big)$ we consider the mapping $F:\mathcal{B}\supset\mathcal{D}(F)\to L^2\big((0,T),X\big)$ from the previous section; that is, $F(B)=u$ and $u\in C^1\big([0,T],X\big)\cap C\big([0,T],\mathcal{D}(A)\big)$ satisfies (12); that is,

$$Bu'(t) = -Au(t) + f(t)\,,\quad t\in[0,T]\,,\quad u(0)=u_0\,. \tag{14}$$

We note that ill-posedness of an equation depends on the space of parameters B. In particular, the ill-posedness may disappear if the set is shrinked too much. Therefore, it is important to prove ill-posedness of the equation $F(B)=u$ on a suitable subsets $\mathcal{D}(F)$ of $\mathcal{B}$.

Theorem 4.1. *Let $u_0\in\mathcal{D}(A)$ and $f\in W^{1,1}\big((0,T),X\big)$. Then the equation $F(B)=u$ is locally ill-posed at any $B^+\in\mathcal{D}(F)$ satisfying $F(B^+)=u$ if for any $r\in(0,1]$ there exists $\hat{r}\in(0,r)$ and a sequence of bounded, symmetric and monotone operators $E_k: X\to X$ with $B^+ + E_k\in\mathcal{D}(F)$ and $\hat{r}\le\|E_k\|\le r$ for all $k\in\mathbb{N}$ and $\lim_{k\to\infty}E_k v = 0$ for all $v\in X$.*

Proof: Let $B^+ \in \mathcal{D}(F)$ and $0 < r \leq 1$ be arbitrary and $E_k : X \to X$ a sequence with the above property. Then $\gamma_- \|v\|_X^2 \leq \|v\|^2_{B+E_k} \leq (\gamma_+ + r)\|v\|_X^2$ where we have used the notation $\|v\|^2_{B+E_k} = \big((B+E_k)v, v\big)_X$.
Let $u = J(B^+)$ and $u_k = J(B^+ + E_k)$; that is, $u_k, u \in C^1\big([0,\infty), X\big) \cap C\big([0,\infty), \mathcal{D}(A)\big)$ solve

$$\begin{aligned} (B^+ + E_k)u_k'(t) &= -Au_k(t) + f(t)\,, \\ B^+ u'(t) &= -Au(t) + f(t)\,, \end{aligned}$$

and $u(0) = u_k(0) = u_0$. From the stability estimates and the above mentioned fact that $\|\cdot\|_X$ and $\|\cdot\|_{B+E_k}$ are equivalent norms we get the existence of $c > 0$ with $\|u_k\|_{C([0,T],X)} \leq c$ for all $k \in \mathbb{N}$. Therefore, $v_k = u - u_k$ solves $v_k(0) = 0$ and

$$(B^+ + E_k)v_k'(t) = -Av_k(t) + E_k u'(t)\,.$$

Multiplication with $v_k(t)$ and the monotonicity of A yields

$$\frac{1}{2}\frac{d}{dt}\|v_k(t)\|^2_{B^+ + E_k} = \frac{1}{2}\frac{d}{dt}\big((B^+ + E_k)v_k(t), v_k(t)\big)_X \leq \big(E_k u'(t), v_k(t)\big)_X\,,$$

and thus

$$\begin{aligned} \frac{1}{2}\|v_k(t)\|^2_{B^+ + E_k} &= \frac{1}{2}\int_0^t \frac{d}{ds}\|v_k(s)\|^2_{B^+ + E_k} \mathrm{d}s \leq \int_0^t \big(E_k u'(s), v_k(s)\big)_X \mathrm{d}s \\ &\leq \|v_k\|_{C([0,T],X)} \int_0^t \|E_k u'(s)\|_X \,\mathrm{d}s \\ &\leq \big[c + \|u\|_{C([0,T],X)}\big] \int_0^t \|E_k u'(s)\|_X \,\mathrm{d}s\,. \end{aligned}$$

The integrand converges pointwise to zero for every $s \in [0,t]$ and is uniformly bounded by $\|u'\|_{C([0,T],X)}$. Therefore, the integral converges to zero; that is, we have pointwise convergence $u_k(t) \to u(t)$ for every t. This implies also convergence in $L^2\big((0,T), X\big)$ because u_k and u are uniformly bounded. Therefore we have shown that $F(B^+ + E_k) \to F(B^+)$ in $L^2\big((0,T), X\big)$ and $\hat{r} \leq \|(B^+ + E_k) - B^+\|_{\mathcal{L}(X)} \leq r$ for all k. □

Remark: In our previous paper [13] (Proposition 2.1) we presented a criterion for local ill-posedness which requires compactness and weak-*-weak continuity of the underlying operator F. The above theorem does not need these strong assumptions if F is the parameter-to-solution map of the first order system (14).

5. Application to the Maxwell System

We want to apply the abstract results of the previous sections to the following Maxwell system:

$$\mu(x)\,\frac{\partial \mathbf{H}}{\partial t}(t,x) + \operatorname{curl}\mathbf{E}(t,x) = \mathbf{J}_m(t,x)\,, \tag{15a}$$

$$\varepsilon(x)\,\frac{\partial \mathbf{E}}{\partial t}(t,x) - \operatorname{curl}\mathbf{H}(t,x) = -\mathbf{J}_e(t,x) - \sigma(x)\,\mathbf{E}(t,x)\,, \tag{15b}$$

for $(t,x) \in (0,\infty) \times D$ with boundary conditions

$$\boldsymbol{\nu}(x) \times \mathbf{E}(t,x) = 0 \quad \text{for } (t,x) \in (0,\infty) \times \partial D\,, \tag{15c}$$

and initial conditions

$$\mathbf{E}(0,x) = \mathbf{e_0}(x) \quad \text{and} \quad \mathbf{H}(0,x) = \mathbf{h_0}(x) \quad \text{for } x \in D\,. \tag{15d}$$

Here, $D \subset \mathbb{R}^3$ is some Lipschitz domain which is either bounded or the complement of a bounded domain. We note that (in the case $\sigma = 0$) the conservation equations $\frac{\partial}{\partial t} \operatorname{div}\big(\mu(x)\mathbf{H}(t,x)\big) - \operatorname{div} \mathbf{J}_m(t,x) = 0$ and $\frac{\partial}{\partial t} \operatorname{div}\big(\varepsilon(x)\mathbf{E}(t,x)\big) + \operatorname{div} \mathbf{J}_e(t,x) = 0$ follow directly from (15a) and (15b), respectively. If $\operatorname{div} \mathbf{J}_e = 0$ then $\operatorname{div}\big(\varepsilon(x)\mathbf{E}(t,x)\big) = 0$ follows provided one assumes $\operatorname{div}(\varepsilon \mathbf{e_0}) = 0$ for the initial field. Analogously, the same arguments hold for the magnetic field. The additional boundary condition $\boldsymbol{\nu} \cdot \mathbf{H} = 0$ on ∂D (in the physically relevant case $\mathbf{J}_m = 0$) follows from

$$\frac{\partial}{\partial t}(\mu\,\boldsymbol{\nu} \cdot \mathbf{H}) \;=\; -\boldsymbol{\nu} \cdot \operatorname{curl} \mathbf{E} \;=\; -\operatorname{Div}(\boldsymbol{\nu} \times \mathbf{E}) \quad \text{on } \partial D$$

and the boundary condition (15c). Here, Div denotes the surface divergence (see, e.g., [11]).

We make the following assumptions on the data:

Assumption 5.1.

- *$\varepsilon, \mu \in L^\infty(D)$ such that $c_\varepsilon \le \varepsilon(x) \le c_\varepsilon^{-1}$ and $c_\mu \le \mu(x) \le c_\mu^{-1}$ on D for some $c_\varepsilon, c_\mu > 0$ (then also $c_\varepsilon \le \varepsilon(x)^{-1} \le c_\varepsilon^{-1}$ and $c_\mu \le \mu(x)^{-1} \le c_\mu^{-1}$ on D),*
- *$\sigma \in L^\infty(D)$ and $\sigma(x) \ge 0$ on D,*
- *$\mathbf{J}_e, \mathbf{J}_m \in W^{1,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$,*
- *$\mathbf{e_0} \in H_0(\operatorname{curl}, D)$ and $\mathbf{h_0} \in H(\operatorname{curl}, D)$.*

To treat this system by the abstract theory we set $X = L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)$, $\mathcal{D}(A) = H_0(\operatorname{curl}, D) \times H(\operatorname{curl}, D)$, and

$$A \;=\; \begin{pmatrix} \sigma I & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}, \qquad B\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon \mathbf{E} \\ \mu \mathbf{H} \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{16}$$

As already done in Assumption 5.1 we identify functions $\mathbf{v} : \mathbb{R}_{\ge 0} \times D \to \mathbb{R}^3$ of two variables with Hilbert-space valued functions $\mathbf{v} : \mathbb{R}_{\ge 0} \to L^2(D,\mathbb{R}^3)$ of one variable and set $u = (\mathbf{E},\mathbf{H})^\top$ and $u_0 = (\mathbf{e_0},\mathbf{h_0})^\top$ and $f = (-\mathbf{J}_e, \mathbf{J}_m)^\top$. Then the system (15a)–(15d) can be written as $Bu'(t) = -Au(t) + f(t)$, $t > 0$, and $u(0) = u_0$.

Lemma 5.2. *The operator A is maximal monotone in the sense of Definition 2.1.*

Proof: For $(\mathbf{E},\mathbf{H})^\top \in \mathcal{D}(A)$ we have

$$\left(A\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right)_X = \int_D \big[(\sigma \mathbf{E} - \operatorname{curl} \mathbf{H}) \cdot \mathbf{E} + \operatorname{curl} \mathbf{E} \cdot \mathbf{H}\big]\, \mathrm{d}x = \int_D \sigma\, |\mathbf{E}|^2 dx \;\ge\; 0$$

by Green's theorem. Note that no boundary term appears because $\mathbf{E} \in H_0(\operatorname{curl}, D)$.
It remains to show surjectivity of $A + I$. For any $\mathbf{J}_e, \mathbf{J}_m \in L^2(D,\mathbb{R}^3)$ we have to find $\mathbf{E} \in H_0(\operatorname{curl}, D)$ and $\mathbf{H} \in H(\operatorname{curl}, D)$ with

$$\sigma\, \mathbf{E} - \operatorname{curl} \mathbf{H} + \mathbf{E} = \mathbf{J}_e \quad \text{and} \quad \operatorname{curl} \mathbf{E} + \mathbf{H} = \mathbf{J}_m\,. \tag{17}$$

For any $\boldsymbol{\psi} \in H_0(\operatorname{curl}, D)$ we multiply the first equation by $\boldsymbol{\psi}$ and the second by $\operatorname{curl} \boldsymbol{\psi}$, add the equations and integrate over D. Noting that $\int_D [\boldsymbol{\psi} \cdot \operatorname{curl} \mathbf{H} - \mathbf{H} \cdot \operatorname{curl} \boldsymbol{\psi}]\, \mathrm{d}x = 0$ we arrive at

$$\int_D [\operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \boldsymbol{\psi} + (\sigma + 1)\, \mathbf{E} \cdot \boldsymbol{\psi}]\, \mathrm{d}x \;=\; \int_D \big[\mathbf{J}_m \cdot \operatorname{curl} \boldsymbol{\psi} + \mathbf{J}_e \cdot \boldsymbol{\psi}\big]\, \mathrm{d}x\,.$$

The theorem of Lax-Milgram in $H_0(\mathrm{curl}, D)$ implies existence of a solution $\mathbf{E} \in H_0(\mathrm{curl}, D)$. Finally we define $\mathbf{H} = \mathbf{J}_m - \mathrm{curl}\,\mathbf{E}$. Then the second equation of (17) is satisfied and the variational equation takes the form

$$\int_D \big[(\sigma+1)\,\mathbf{E}\cdot\boldsymbol{\psi} - \mathbf{H}\cdot\mathrm{curl}\,\boldsymbol{\psi}\big]\mathrm{d}x \;=\; \int_D \mathbf{J}_e\cdot\boldsymbol{\psi}\,\mathrm{d}x$$

which is the weak form of the first equation of (17). □

Application of Theorems 2.4 and 2.7 for $k = 2$ and 3 yields:

Theorem 5.3.
(a) Under Assumption 5.1 there exists a unique solution $\mathbf{E} \in C\big([0,\infty), H_0(\mathrm{curl}, D)\big) \cap C^1\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$ *and* $\mathbf{H} \in C\big([0,\infty), H(\mathrm{curl}, D)\big) \cap C^1\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$ *of (15a)–(15d).*
(b) Let $\mathbf{J}_e, \mathbf{J}_m \in W^{2,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$ *and* $\hat{\mathbf{e}}_\mathbf{0} := \frac{1}{\varepsilon}\big[\mathrm{curl}\,\mathbf{h_0} - \sigma\mathbf{e_0} - \mathbf{J}_e(0)\big] \in H_0(\mathrm{curl}, D)$ *and* $\hat{\mathbf{h}}_\mathbf{0} := \frac{1}{\mu}\big[\mathbf{J}_m(0) - \mathrm{curl}\,\mathbf{e_0}\big] \in H(\mathrm{curl}, D)$.
Then $\mathbf{E} \in C^1\big([0,\infty), H_0(\mathrm{curl}, D)\big) \cap C^2\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$ *and* $\mathbf{H} \in C^1\big([0,\infty), H(\mathrm{curl}, D)\big) \cap C^2\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$.
(c) Let in addition to the assumptions of part (b) $\mathbf{J}_e, \mathbf{J}_m \in W^{3,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$ *and* $\frac{1}{\varepsilon}\big[\mathrm{curl}\,\hat{\mathbf{h}}_\mathbf{0} - \sigma\hat{\mathbf{e}}_\mathbf{0} - \mathbf{J}_e'(0)\big] \in H_0(\mathrm{curl}, D)$ *and* $\frac{1}{\mu}\big[\mathbf{J}_m'(0) - \mathrm{curl}\,\hat{\mathbf{e}}_\mathbf{0}\big] \in H(\mathrm{curl}, D)$.
Then $\mathbf{E} \in C^2\big([0,\infty), H_0(\mathrm{curl}, D)\big) \cap C^3\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$ *and* $\mathbf{H} \in C^2\big([0,\infty), H(\mathrm{curl}, D)\big) \cap C^3\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$.

Proof: For parts (b) and (c) we have to translate the assumptions of Theorem 3.5 into the special case of the Maxwell system. Here, $f \in W^{\ell,1}\big((0,\infty), X\big)$ corresponds to $\mathbf{J}_e, \mathbf{J}_m \in W^{\ell,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$. The assumption $\hat{B}^{-1}(Au_0 - f(0)) \in \mathcal{D}(A)$ translates into $\frac{1}{\varepsilon}\big[\sigma\mathbf{e_0} - \mathrm{curl}\,\mathbf{h_0} + \mathbf{J}_e(0)\big] \in H_0(\mathrm{curl}, D)$ and $\frac{1}{\mu}\big[\mathrm{curl}\,\mathbf{e_0} - \mathbf{J}_m(0)\big] \in H(\mathrm{curl}, D)$. The assumption $\hat{B}^{-1}(A\hat{v}_0 - f'(0)) \in \mathcal{D}(A)$ translates into $\frac{1}{\varepsilon}\big[\mathrm{curl}\,\hat{\mathbf{h}}_\mathbf{0} - \sigma\hat{\mathbf{e}}_\mathbf{0} - \mathbf{J}_e'(0)\big] \in H_0(\mathrm{curl}, D)$ and $\frac{1}{\mu}\big[\mathbf{J}_m'(0) - \mathrm{curl}\,\hat{\mathbf{e}}_\mathbf{0}\big] \in H(\mathrm{curl}, D)$. These are exactly the assumptions made for this theorem. □

We note again that it is sufficient to make the assumptions on $\mathbf{J}_e$ and $\mathbf{J}_m$ on the finite interval $(0,T)$ only if one is interested in the finite time case.

For fixed $T > 0$ and $\sigma \geq 0$ we will now consider the mapping properties of the parameter-to-solution operator $\tilde{F} : \mathcal{P} \to C\big([0,T], L^2(D,\mathbb{R}^3)\big) \times C\big([0,T], L^2(D,\mathbb{R}^3)\big)$ defined by $\tilde{F}(\varepsilon,\mu) = (\mathbf{E},\mathbf{H})^\top$ where $\mathcal{P} = \{(\varepsilon,\mu)^\top \in L^\infty(D) \times L^\infty(D) : c_\varepsilon^{-1} \leq \varepsilon(x) \leq c_\varepsilon\,,\ c_\mu^{-1} \leq \mu(x) \leq c_\mu \text{ on } D\}$ for some $c_\varepsilon, c_\mu > 1$ denotes the set of parameters and $(\mathbf{E},\mathbf{H})^\top$ is the solution of (15a)–(15d). We note that this operator is slightly different from the operator F of the previous section which maps *any* symmetric and coercive operator B from $L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)$ into itself to the solution. Since we consider special multiplication operators B we introduce the linear and bounded operator $V : \mathcal{P} \to \mathcal{L}\big(L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)\big)$, defined by $V(\varepsilon,\mu)(\mathbf{u},\mathbf{v}) = (\varepsilon\mathbf{u}, \mu\mathbf{v})^\top$ for $(\mathbf{u},\mathbf{v})^\top \in L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)$ and $(\varepsilon,\mu)^\top \in \mathcal{P}$. Then $\tilde{F} = F \circ V$ and thus $\tilde{F}'(\varepsilon,\mu) = F'(V(\varepsilon,\mu)) \circ V$ because V is obviously linear as an operator from $L^\infty(D) \times L^\infty(D)$ into $\mathcal{L}\big(L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^3)\big)$. Application of Theorem 3.5 yields directly the following result.

Theorem 5.4. *Let the Assumptions 5.1 hold and denote by* $(\hat{\mathbf{E}},\hat{\mathbf{H}})$ *the solution of the system (15a)–(15d) corresponding to* $(\hat\varepsilon,\hat\mu)\in\operatorname{int}\mathcal{P}$, $\hat\sigma$, $\mathbf{e_0}$, $\mathbf{h_0}$, $\mathbf{J}_e$, *and* $\mathbf{J}_m$. *Then the mapping* $\tilde F:(\varepsilon,\mu)^\top\mapsto(\mathbf{E},\mathbf{H})^\top$ *from* $\mathcal{P}$ *into* $C\big([0,T],L^2(D,\mathbb{R}^3)\big)\times C\big([0,T],L^2(D,\mathbb{R}^3)\big)$ *is Fréchet–differentiable at* $(\hat\varepsilon,\hat\mu)$ *and* $\tilde F'(\hat\varepsilon,\hat\mu)(\varepsilon,\mu)=(\overline{\mathbf{E}},\overline{\mathbf{H}})$ *where* $(\overline{\mathbf{E}},\overline{\mathbf{H}})$ *is the mild solution of the system*

$$\hat\mu\,\overline{\mathbf{H}}'(t)\;+\;\operatorname{curl}\overline{\mathbf{E}}(t)\;=\;-\mu\,\hat{\mathbf{H}}'(t)\,,\quad t\in[0,T]\,,\tag{18a}$$

$$\hat\varepsilon\,\overline{\mathbf{E}}'(t)\;-\;\operatorname{curl}\overline{\mathbf{H}}(t)\;+\;\hat\sigma\,\overline{\mathbf{E}}(t)\;=\;-\varepsilon\,\hat{\mathbf{E}}'(t)\,,\quad t\in[0,T]\,,\tag{18b}$$

$$\overline{\mathbf{E}}(0)\;=\;\overline{\mathbf{H}}(0)\;=\;0\,.\tag{18c}$$

Remarks 5.5. *(a) The mild solution of (18a) – (18c) is also the weak solution; that is,*

$$\frac{d}{dt}\big(\hat\mu\,\overline{\mathbf{H}}(t),\psi\big)_{L^2(D)}\;+\;\big(\overline{\mathbf{E}}(t),\operatorname{curl}\psi\big)_{L^2(D)}\;=\;-\frac{d}{dt}\big(\mu\,\hat{\mathbf{H}}(t),\psi\big)_{L^2(D)}$$

for all $\psi\in H(\operatorname{curl},D)\big)$ *and almost all* $t\in[0,T]$ *and*

$$\frac{d}{dt}\big(\hat\varepsilon\,\overline{\mathbf{E}}(t),\phi\big)_{L^2(D)}\;-\;\big(\overline{\mathbf{H}}(t),\operatorname{curl}\phi\big)_{L^2(D)}\;+\;\big(\hat\sigma\,\overline{\mathbf{E}}(t),\phi\big)_{L^2(D)}\;=\;-\frac{d}{dt}\big(\epsilon\,\hat{\mathbf{E}}(t),\phi\big)_{L^2(D)}$$

for all $\phi\in H_0(\operatorname{curl},D)\big)$ *and almost all* $t\in[0,T]$.

(b) Under the additional regularity assumptions of part (b) of Theorem 5.3 the mild solution $(\overline{\mathbf{E}},\overline{\mathbf{H}})$ *is a classical solution.*

(c) Under the additional regularity assumptions of part (c) of Theorem 5.3 the mapping $\tilde F$ *is also Fréchet differentiable as a mapping from* $\mathcal{P}$ *into* $\big[C^1\big([0,T],L^2(D,\mathbb{R}^3)\big)\cap C\big([0,T],H_0(\operatorname{curl},D)\big)\big]\times\big[C^1\big([0,T],L^2(D,\mathbb{R}^3)\big)\cap C\big([0,T],H(\operatorname{curl},D)\big)\big]$.

(d) The differentiability with respect to σ *can not be treated analogously by the abstract theory. Instead, one has to consider abstract evolution equations of the form* $u'(t)=-Au(t)+Bu(t)+f(t)$, $t\geq 0$, *and* $u(0)=u_0$ *with* $B\in\mathcal{L}(X)$. *Therefore,* u *satisfies the fixed point equation* $u(t)=S(t)u_0+\int_0^t S(t-s)Bu(s)\mathrm{d}s+\int_0^t S(t-s)f(s)\mathrm{d}s$, $t\geq 0$. *Properties of the mapping* $B\mapsto u$ *as, e.g. differentiability, can then be treated quite analogously.*

Finally, we show that the inverse problem, to determine the coefficients ε and μ from $\mathbf{E}$ and $\mathbf{H}$ is locally ill-posed by applying Theorem 4.1.

Theorem 5.6. *Let the Assumptions 5.1 hold and let* $\tilde F:\mathcal{P}\to L^2\big((0,T)\times D,\mathbb{R}^3\big)\times L^2\big((0,T)\times D,\mathbb{R}^3\big)$ *by the parameter-to-solution map of the previous theorem. Then the equation* $\tilde F(\varepsilon,\mu)=(\mathbf{E},\mathbf{H})$ *is locally ill-posed at any* $(\varepsilon^+,\mu^+)^\top\in\operatorname{int}\mathcal{P}$.

Proof: Fix a point $\hat x\in D$ and define balls $K_n=\{y\in\mathbb{R}^3:|y-\hat x|\leq\delta/n\}$ where $\delta>0$ is small enough such that $K_n\subset D$ for all $n\in\mathbb{N}$. Let χ_n be the characteristic function of K_n; that is, $\chi_n(x)=1$ if $|x-\hat x|\leq\delta/n$ and 0 else. Let $r>0$ be so small such that $(\varepsilon^++r\chi_n,\mu^++r\chi_m)^\top\in\mathcal{P}$ for all $n,m\in\mathbb{N}$. Then we write $\tilde F(\varepsilon^++r\chi_n,\mu^++r\chi_m)=F\big(V(\varepsilon^+,\mu^+)+E_{n,m}\big)$ with the operator V defined above as $V(\varepsilon,\mu)(\mathbf{u},\mathbf{v})=(\varepsilon\mathbf{u},\mu\mathbf{v})$ for $(\mathbf{u},\mathbf{v})\in L^2(D,\mathbb{R}^3)\times L^2(D,\mathbb{R}^3)$ and $\varepsilon,\mu\in L^\infty(D)$ and $E_{n,m}\in\mathcal{L}\big(L^2(D)\times L^2(D)\big)$ defined by $E_{n,m}(\mathbf{u},\mathbf{v})=rV(\chi_n,\chi_m)(\mathbf{u},\mathbf{v})=r(\chi_n\mathbf{u},\chi_m\mathbf{v})$. Then $\|E_{n,m}\|_{\mathcal{L}(L^2(D)\times L^2(D))}=r$ (that is, $\hat r=r$ in Theorem 4.1) and $\|E_{n,m}(\mathbf{u},\mathbf{v})\|_{L^2(D)\times L^2(D)}\to 0$ as $n,m\to\infty$. Indeed, $\|\chi_n\mathbf{u}\|_{L^2(D)}\leq\|\mathbf{u}\|_{L^2(D)}$ is obvious and $\|\chi_n^2\|_{L^2(D)}=\|\chi_n\|_{L^2(D)}$ which yields $\|E_{n,m}\|_{\mathcal{L}(L^2(D)\times L^2(D))}=r$. Furthermore, $\|\chi_n\mathbf{u}\|_{L^2(D)}^2=\int_{K_n}|\mathbf{u}|^2\mathrm{d}x\to 0$ as n

tends to infinity. Therefore, the operators $E_{n,m}$ satisfy the assumptions of Theorem 4.1. This ends the proof. $\square$

6. Application to the Elastic Wave Equation

We apply the abstract results to the elastic wave equation in the reference domain $D \subset \mathbb{R}^3$ which we assume to be Lipschitz and either be bounded or the exterior of a bounded domain.

Let $\boldsymbol{\sigma}\colon [0,\infty) \times D \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be the stress tensor and $\mathbf{v}\colon [0,\infty)\times D \to \mathbb{R}^3$ be the velocity field. Then,

$$\partial_t \boldsymbol{\sigma}(t,x) = C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}(\mathbf{v}(t,x)) \quad \text{in } [0,\infty)\times D\,, \tag{19a}$$

$$\varrho(x)\partial_t \mathbf{v}(t,x) = \operatorname{div}\boldsymbol{\sigma}(t,x) + \mathbf{f}(t,x) \quad \text{in } [0,\infty)\times D\,, \tag{19b}$$

where $\varrho\colon D \to \mathbb{R}$ is the mass density, $\mathbf{f}\colon [0,\infty)\times D \to \mathbb{R}^3$ is a volume force and

$$C(m,\ell)\boldsymbol{\varepsilon} = 2m\,\boldsymbol{\varepsilon} + \ell\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I}\,, \quad \boldsymbol{\varepsilon} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\,,\ m,\ell \in \mathbb{R}\,,$$

is Hooke's law with Lamé parameters $m = \mu(x)$ and $\ell = \lambda(x)$. Finally, $\varepsilon(\mathbf{v}) := \frac{1}{2}\big[(\nabla_x \mathbf{v})^\top + \nabla_x \mathbf{v}\big]$ is the (linearized) strain. Initial and boundary conditions will be specified below. We consider C as a mapping from $\mathcal{D}(C) = \big\{(m,\ell)^\top \in \mathbb{R}^2 : \mathrm{c}^{-1} \le 2m + 3\ell \le \mathrm{c},\ \mathrm{c}^{-1} \le m \le \mathrm{c}\big\}$ into $\mathrm{Aut}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Here, $\mathrm{c} > 1$ is some constant, and $\mathrm{Aut}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ is the space of automorphisms in $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$; that is, isomorphisms from $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ onto itself. Indeed, the inverse $\big[C(m,\ell)\boldsymbol{\varepsilon}\big]^{-1}$ of $C(m,\ell)\boldsymbol{\varepsilon}$ is given by

$$\begin{aligned}\tilde{C}(m,\ell)\boldsymbol{\varepsilon} &:= \big[C(m,\ell)\boldsymbol{\varepsilon}\big]^{-1} = \frac{1}{2m}\boldsymbol{\varepsilon} - \frac{\ell}{2m\big(3\ell+2m\big)}\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbf{I} \\ &= C\left(\frac{1}{4m}, -\frac{\ell}{2m(3\ell+2m)}\right)\boldsymbol{\varepsilon}\end{aligned} \tag{20}$$

for $\boldsymbol{\varepsilon} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, provided $(m,\ell)^\top \in \mathcal{D}(C)$. Then (19a) is equivalent to

$$\tilde{C}\big(\mu(x),\lambda(x)\big)\partial_t\boldsymbol{\sigma}(t,x) = \boldsymbol{\varepsilon}(\mathbf{v}(t,x)) \quad \text{in } [0,\infty)\times D\,.$$

We make the assumption that $(\mu,\lambda,\varrho)^\top \in \mathcal{P}$ where

$$\mathcal{P} := \big\{(\mu,\lambda,\varrho)^\top \in L^\infty(D)^3 : \mathrm{c}^{-1} \le \varrho, \mu \le \mathrm{c},\ \mathrm{c}^{-1} \le 2\mu + 3\lambda \le \mathrm{c} \text{ a.e. in } D\big\}\,. \tag{21}$$

Introducing the standard inner product

$$\boldsymbol{\sigma} : \boldsymbol{\psi} := \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\sigma}_{i,j}\boldsymbol{\psi}_{i,j} \quad \text{for matrices } \boldsymbol{\sigma},\boldsymbol{\psi} \in \mathbb{R}^{3\times 3}$$

we have

$$\begin{aligned} C(\mu,\lambda)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} &= 2\mu\sum_{i\ne j}\varepsilon_{ij}^2 + \mathrm{c}\sum_{i=1}^{3}\varepsilon_{ii}^2 + (2\mu - \mathrm{c})\sum_{i=1}^{3}\varepsilon_{ii}^2 + \lambda\left(\sum_{i=1}^{3}\varepsilon_{ii}\right)^2 \\ &\ge 2\mu\sum_{i\ne j}\varepsilon_{ij}^2 + \mathrm{c}\sum_{i=1}^{3}\varepsilon_{ii}^2 + \frac{1}{3}\underbrace{(2\mu - \mathrm{c} + 3\lambda)}_{\ge\, 0}\left(\sum_{i=1}^{3}\varepsilon_{ii}\right)^2 \\ &\ge \mathrm{c}\sum_{i,j=1}^{3}\varepsilon_{ij}^2 = \mathrm{c}\,|\boldsymbol{\varepsilon}|_F^2 \end{aligned} \tag{22}$$

because $\left(\sum_{i=1}^3 \varepsilon_{ii}\right)^2 \le 3\sum_{i=1}^3 \varepsilon_{ii}^2$. Here $|\cdot|_F$ denotes the Frobenius norm for matrices; that is, $|\boldsymbol{\varepsilon}|_F = \sqrt{\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}} = \sqrt{\sum_{j,j=1}^3 \varepsilon_{ij}^2}$. Furthermore,

$$C(\mu,\lambda)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \;=\; 2\mu \sum_{i,j=1}^{3} \varepsilon_{ij}^2 \;+\; \lambda \left(\sum_{i=1}^{3} \varepsilon_{ii}\right)^2 \;\le\; (2\mu + 3\lambda)\,|\boldsymbol{\varepsilon}|_F^2 \;\le\; \mathrm{c}|\boldsymbol{\varepsilon}|_F^2 .$$

Therefore,

$$\mathrm{c}^{-1}|\boldsymbol{\varepsilon}|_F^2 \le C(\mu,\lambda)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \le \mathrm{c}|\boldsymbol{\varepsilon}|_F^2 \text{ and thus } \mathrm{c}^{-1}|\boldsymbol{\sigma}|_F^2 \le C(\mu,\lambda)^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma} \le \mathrm{c}|\boldsymbol{\sigma}|_F^2 \tag{23}$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $(\mu,\lambda)^\top \in \mathcal{D}(C)$. Also, this implies

$$\mathrm{c}^{-1}|\boldsymbol{\varepsilon}|_F \;\le\; |C(\mu,\lambda)\boldsymbol{\varepsilon}|_F \;\le\; \mathrm{c}|\boldsymbol{\varepsilon}|_F \quad \text{and} \quad \mathrm{c}^{-1}|\boldsymbol{\sigma}|_F \;\le\; |\tilde{C}(\mu,\lambda)\boldsymbol{\sigma}|_F \;\le\; \mathrm{c}|\boldsymbol{\sigma}|_F$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $(\mu,\lambda)^\top \in \mathcal{D}(C)$.

Next we want to formulate (19a) and (19b) as an abstract evolution equation. Let $X = L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^3)$ with inner product

$$\big((\boldsymbol{\sigma}, \mathbf{v})^\top, (\boldsymbol{\psi}, \mathbf{w})^\top\big)_X \;:=\; \int_D \big(\boldsymbol{\sigma} : \boldsymbol{\psi} + \mathbf{v}\cdot\mathbf{w}\big)\,\mathrm{d}x\,, \quad (\boldsymbol{\sigma}, \mathbf{v})^\top, (\boldsymbol{\psi}, \mathbf{w})^\top \in X\,.$$

For fixed $(\mu, \lambda, \varrho)^\top \in \mathcal{P}$ we define $B \in \mathcal{L}(X)$ by

$$B\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} \;:=\; \begin{pmatrix}\tilde{C}(\mu,\lambda) & \mathbf{0}\\ \mathbf{0} & \varrho\, I\end{pmatrix}\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} \tag{24}$$

(pointwise for almost all $x \in D$) which is self adjoint and uniformly positive definite by (22). With

$$A \;:=\; -\begin{pmatrix}0 & \boldsymbol{\varepsilon}\\ \mathrm{div} & 0\end{pmatrix} \tag{25}$$

the system (19a) and (19b) with initial conditions reads as

$$B\,\partial_t\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} \;=\; -A\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix} \;+\; \begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{\sigma}(0,\cdot)\\ \mathbf{v}(0,\cdot)\end{pmatrix} \;=\; \begin{pmatrix}\boldsymbol{\sigma}_0\\ \mathbf{v}_0\end{pmatrix}. \tag{26}$$

To define the domain of definition $\mathcal{D}(A)$ of A we split $\partial D = \partial D_D \,\dot\cup\, \partial D_N$ into disjoint parts where ∂D_D has positive 2-dimensional volume. Let $\mathbf{n}$ be the outer normal vector on ∂D_N. Then

$$\mathcal{D}(A) \;=\; \Big\{(\boldsymbol{\sigma}, \mathbf{v})^\top \in H\big(\mathrm{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big) \times H^1_D(D, \mathbb{R}^3) : \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \text{ on } \partial D_N\Big\} \tag{27}$$

where $H^1_D(D,\mathbb{R}^3) = \{\mathbf{v} \in H^1(D,\mathbb{R}^3) : \mathbf{v} = 0 \text{ on } \partial D_D\}$ and $H\big(\mathrm{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big) = \big\{\boldsymbol{\sigma} \in L^2\big(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}\big) : \mathrm{div}\,\boldsymbol{\sigma}_{*,j} \in L^2(D),\ j = 1,2,3\big\}$. We note that the traces $\boldsymbol{\sigma}_{*,j}\cdot\mathbf{n}$ exist in $H^{-1/2}(\mathrm{Div}, \partial D)$ because $\boldsymbol{\sigma}_{*,j} \in H\big(\mathrm{div}, D\big)$ (see, e.g., [17]).

Lemma 6.1. *The operator A is maximal monotone in the sense of Definition 2.1.*

Proof: The operator A is skew-symmetric, see, e.g., [20], and, as such, is monotone: $\big(A(\boldsymbol{\sigma}, \mathbf{v})^\top, (\boldsymbol{\sigma}, \mathbf{v})^\top\big)_X = 0$. Indeed, using the identities

$$\mathrm{div}(\boldsymbol{\sigma}\mathbf{v}) = \mathrm{div}\,\boldsymbol{\sigma}\cdot\mathbf{v} + \boldsymbol{\sigma} : \nabla\mathbf{v} \;\text{ and }\; \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma} = \nabla\mathbf{v} : \boldsymbol{\sigma}\,,$$

as well as the divergence theorem we find for $(\boldsymbol{\sigma},\mathbf{v})^\top$, $(\boldsymbol{\psi},\mathbf{w})^\top \in \mathcal{D}(A)$ that

$$\begin{aligned}
\big(A(\boldsymbol{\sigma},\mathbf{v}),(\boldsymbol{\psi},\mathbf{w})^\top\big)_X &= -\int_D \big(\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\psi}+\operatorname{div}(\boldsymbol{\sigma})\cdot\mathbf{w}\big)\mathrm{d}x \\
&= -\int_D \big(\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\psi}+\operatorname{div}(\boldsymbol{\sigma}\mathbf{w})-\boldsymbol{\sigma}:\nabla\mathbf{w}\big)\mathrm{d}x \\
&= -\int_D \big(\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\psi}-\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\sigma}\big)\mathrm{d}x + \underbrace{\int_{\partial D}(\boldsymbol{\sigma}\mathbf{w})\cdot\mathbf{n}\,\mathrm{d}s}_{=\,0} \\
&= \int_D \big(\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\sigma}-\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\psi}\big)\mathrm{d}x + \underbrace{\int_{\partial D}(\boldsymbol{\psi}\mathbf{v})\cdot\mathbf{n}\,\mathrm{d}s}_{=\,0} \\
&= \int_D \big(\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\sigma}+\operatorname{div}(\boldsymbol{\psi}\mathbf{v})-\boldsymbol{\psi}:\nabla\mathbf{v}\big)\mathrm{d}x \\
&= \int_D \big(\boldsymbol{\varepsilon}(\mathbf{w}):\boldsymbol{\sigma}+\operatorname{div}(\boldsymbol{\psi})\cdot\mathbf{v}\big)\mathrm{d}x \;=\; \big((\boldsymbol{\sigma},\mathbf{v})^\top,-A(\boldsymbol{\psi},\mathbf{w})^\top\big)_X\,.
\end{aligned}$$

Next we show that $I+A$ is surjective. To this end let $(\boldsymbol{\psi},\mathbf{g})^\top \in X$. We have to solve the equations

$$\boldsymbol{\sigma}-\boldsymbol{\varepsilon}(\mathbf{v})=\boldsymbol{\psi} \quad\text{and}\quad \mathbf{v}-\operatorname{div}\boldsymbol{\sigma}=\mathbf{g} \tag{28}$$

for $(\boldsymbol{\sigma},\mathbf{v})^\top \in \mathcal{D}(A)$. We multiply the second equation by some test function $\mathbf{w} \in H^1_D(D,\mathbb{R}^3)$, integrate over D and use the divergence theorem. This yields

$$\begin{aligned}
\int_D \mathbf{g}\cdot\mathbf{w}\,\mathrm{d}x &= \int_D \big(\mathbf{v}\cdot\mathbf{w}-\operatorname{div}\boldsymbol{\sigma}\cdot\mathbf{w}\big)\,\mathrm{d}x \\
&= \int_D \big(\mathbf{v}\cdot\mathbf{w}+\boldsymbol{\sigma}:\nabla\mathbf{w}\big)\,\mathrm{d}x \;+\; \underbrace{\int_{\partial D}(\boldsymbol{\sigma}\mathbf{w})\cdot\mathbf{n}\,\mathrm{d}s}_{=\,0}\,.
\end{aligned}$$

Now we use the first equation and arrive at

$$\int_D \mathbf{g}\cdot\mathbf{w}\,\mathrm{d}x \;=\; \int_D \big[\mathbf{v}\cdot\mathbf{w}+\big(\boldsymbol{\varepsilon}(\mathbf{v})+\boldsymbol{\psi}\big):\nabla\mathbf{w}\big]\,\mathrm{d}x\,;$$

that is,

$$\int_D \big[\mathbf{v}\cdot\mathbf{w}+\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varepsilon}(\mathbf{w})\big]\,\mathrm{d}x \;=\; \int_D \big[\mathbf{g}\cdot\mathbf{w}-\boldsymbol{\psi}:\nabla\mathbf{w}\big]\,\mathrm{d}x \quad \text{for all } \mathbf{w}\in H^1_D(D,\mathbb{R}^3)\,.$$

This variational equation is known as the pure displacement ansatz in elasticity which has a unique solution $\mathbf{v} \in H^1_D(D,\mathbb{R}^3)$ see, e.g., [3]. Finally, set $\boldsymbol{\sigma} := \boldsymbol{\psi}+\boldsymbol{\varepsilon}(\mathbf{v})$. Thus $\boldsymbol{\sigma}=\boldsymbol{\sigma}^\top$ and, as above,

$$\int_D \mathbf{g}\cdot\mathbf{w}\,\mathrm{d}x \;=\; \int_D \big(\mathbf{v}\cdot\mathbf{w}+\boldsymbol{\sigma}:\nabla\mathbf{w}\big)\,\mathrm{d}x \;+\; \int_{\partial D_N}(\boldsymbol{\sigma}\mathbf{w})\cdot\mathbf{n}\,\mathrm{d}s$$

for all $\mathbf{w} \in H^1_D(D,\mathbb{R}^3)$. This is the variational form of $\boldsymbol{\sigma}\mathbf{n}=\mathbf{0}$ on ∂D_N and $\operatorname{div}\boldsymbol{\sigma} = \mathbf{v}-\mathbf{g} \in L^2(D,\mathbb{R}^3)$ which yields that $\boldsymbol{\sigma}\in H\big(\operatorname{div},D,\mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$. Altogether we have constructed $(\boldsymbol{\sigma},\mathbf{v})^\top \in \mathcal{D}(A)$ satisfying (28). $\square$

Therefore, the operators B and A defined in (24) and (25), respectively, fulfill the requirements of our abstract theory of the previous sections, and the following theorem holds.

Theorem 6.2. *(a) Let* $(\mu,\lambda,\varrho)^\top \in \mathcal{P}$, $\mathbf{f} \in W^{1,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$, *and* $(\boldsymbol{\sigma}_0, \mathbf{v}_0)^\top \in \mathcal{D}(A)$ *where* $\mathcal{P}$ *and* $\mathcal{D}(A)$ *have been defined in (21) and (27), respectively. Then there exists a unique solution* $(\boldsymbol{\sigma},\mathbf{v})^\top \in C\big([0,\infty),\mathcal{D}(A)\big) \cap C^1\big([0,\infty),X\big)$ *of (19a), (19b) with* $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ *and* $\mathbf{v}(0) = \mathbf{v}_0$. *Here, again,* $X = L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D,\mathbb{R}^3)$.

(b) Let in addition $\mathbf{f} \in W^{2,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$, $\hat{\mathbf{v}}_0 := \varrho^{-1}\big[\operatorname{div}\boldsymbol{\sigma}_0 + \mathbf{f}(0)\big] \in H^1_D(D,\mathbb{R}^3)$, $\widehat{\boldsymbol{\sigma}}_0 := C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0) \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, *and* $C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0)\mathbf{n} = \mathbf{0}$ *on* ∂D_N.
Then $(\boldsymbol{\sigma},\mathbf{v})^\top \in C^1\big([0,\infty),\mathcal{D}(A)\big) \cap C^2\big([0,\infty),X\big)$.

(c) Let in addition to the assumptions of part (b) $\mathbf{f} \in W^{3,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$, $\varrho^{-1}\big[\operatorname{div}\widehat{\boldsymbol{\sigma}}_0 + \mathbf{f}'(0)\big] \in H^1_D(D,\mathbb{R}^3)$, $C(\mu,\lambda)\boldsymbol{\varepsilon}(\hat{\mathbf{v}}_0) \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *and* $C(\mu,\lambda)\boldsymbol{\varepsilon}(\hat{\mathbf{v}}_0)\mathbf{n} = \mathbf{0}$ *on* ∂D_N.
Then $(\boldsymbol{\sigma},\mathbf{v})^\top \in C^2\big([0,\infty),\mathcal{D}(A)\big) \cap C^3\big([0,\infty),X\big)$.

Proof: We have again to check the conditions of Theorem 2.7 for $k=2$ and $k=3$. We have: $f \in W^{\ell,1}((0,\infty),X)$ translates into $\mathbf{f} \in W^{\ell,1}\big((0,\infty), L^2(D,\mathbb{R}^3)\big)$, $B^{-1}(Au_0 - f(0)) \in \mathcal{D}(A)$ reads as $C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0) \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, $\varrho^{-1}\big[\operatorname{div}\boldsymbol{\sigma}_0 + \mathbf{f}(0)\big] \in H^1_D(D,\mathbb{R}^3)$, and $C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0)\mathbf{n} = \mathbf{0}$ on ∂D_N. Furthermore, $\hat{B}^{-1}(A\hat{u}_0 - f'(0)) \in \mathcal{D}(A)$ translates into $C(\mu,\lambda)\boldsymbol{\varepsilon}(\hat{\mathbf{v}}_0) \in H(\operatorname{div}, D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, $\varrho^{-1}\big[\operatorname{div}\widehat{\boldsymbol{\sigma}}_0 + \mathbf{f}'(0)\big] \in H^1_D(D,\mathbb{R}^3)$, and $C(\mu,\lambda)\boldsymbol{\varepsilon}(\hat{\mathbf{v}}_0)\mathbf{n} = \mathbf{0}$ on ∂D_N. $\square$

We note again that it is sufficient to make the assumptions on $\mathbf{f}$ on the finite interval $(0,T)$ only if one is interested in the finite time case.

In particular, the mapping $\tilde{F} : (\mu,\lambda,\varrho)^\top \mapsto (\boldsymbol{\sigma},\mathbf{v})^\top$ is well defined from the set $\mathcal{P}$ of parameters into $C\big([0,T],X\big)$ for any fixed $T > 0$.

We will now express this operator $\tilde{F}$ in terms of the operator $F\colon B \mapsto u$ of the abstract theory. To this end we introduce, analogously to the previous section, $V\colon L^\infty(D)^3 \supset \mathcal{P} \to \mathcal{L}(X)$ as the mapping $(\mu,\lambda,\varrho)^\top \mapsto \begin{pmatrix} \tilde{C}(\mu,\lambda) & 0 \\ 0 & \varrho I \end{pmatrix}$ where we interpret the application of $\tilde{C}(\mu,\lambda)$ pointwise a.e. Then $\tilde{F} = F \circ V$ on $\mathcal{P}$. To compute the derivative of $\tilde{F}$ we have to use the chain rule.

Lemma 6.3. *Let* $\tilde{C} : \mathcal{D}(C) \to \operatorname{Aut}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *be the mapping defined in (20). Its Fréchet derivative at* $(\mu,\lambda)^\top \in \operatorname{int}\mathcal{D}(C)$ *is given by*

$$\tilde{C}'(\mu,\lambda)(m,\ell) \;=\; -\tilde{C}(\mu,\lambda) \circ C(m,\ell) \circ \tilde{C}(\mu,\lambda)\,, \quad (m,\ell)^\top \in \mathbb{R}^2\,.$$

Proof: First, we note that C is a linear operator and $\tilde{C}(\mu,\lambda) \circ C(\mu,\lambda) = C(\mu,\lambda) \circ \tilde{C}(\mu,\lambda) = I$. Then we have for sufficiently small m, ℓ

$$\begin{aligned}
&\tilde{C}(\mu+m,\lambda+\ell) - \tilde{C}(\mu,\lambda) + \tilde{C}(\mu,\lambda) \circ C(m,\ell) \circ \tilde{C}(\mu,\lambda) \\
&\quad= \tilde{C}(\mu+m,\lambda+\ell) \circ \big[C(\mu,\lambda) - C(\mu+m,\lambda+\ell)\big] \circ \tilde{C}(\mu,\lambda) \\
&\qquad + \tilde{C}(\mu,\lambda) \circ C(m,\ell) \circ \tilde{C}(\mu,\lambda) \\
&\quad= \big[\tilde{C}(\mu,\lambda) - \tilde{C}(\mu+m,\lambda+\ell)\big] \circ C(m,\ell) \circ \tilde{C}(\mu,\lambda)
\end{aligned}$$

and thus

$$\|\tilde{C}(\mu+m,\lambda+\ell) - \tilde{C}(\mu,\lambda) + \tilde{C}(\mu,\lambda) \circ C(m,\ell) \circ \tilde{C}(\mu,\lambda)\|_{\mathrm{Aut}}$$

$$\leq \; c\|\tilde{C}(\mu,\lambda)\|_{\mathrm{Aut}}\|\tilde{C}(\mu,\lambda)-\tilde{C}(\mu+m,\lambda+\ell)\|_{\mathrm{Aut}}\big(|m|+|\ell|\big) \;=\; o\big(|m|+|\ell|\big)$$

as $|m|+|\ell|\to 0$. This proves the assertion. □

Theorem 6.4. *Let $T>0$, $(\mu,\lambda,\varrho)^\top\in\mathcal{P}$, $\mathbf{f}\in W^{1,1}\big([0,T],L^2(D,\mathbb{R}^3)\big)$, and $(\boldsymbol{\sigma}_0,\mathbf{v}_0)^\top\in\mathcal{D}(A)$. Then the mapping $\tilde{F}\colon L^\infty(D)^3\supset\mathcal{P}\to C\big([0,T],X\big)$, $(\mu,\lambda,\varrho)^\top\mapsto(\boldsymbol{\sigma},\mathbf{v})^\top$, where $(\boldsymbol{\sigma},\mathbf{v})$ solves (19a), (19b) w.r.t. the parameters (μ,λ,ϱ) and initial values $(\boldsymbol{\sigma}_0,\mathbf{v}_0)$, is Fréchet differentiable at $(\hat\mu,\hat\lambda,\hat\varrho)^\top$. In fact, we have that $\tilde{F}'(\hat\mu,\hat\lambda,\hat\varrho)(\mu,\lambda,\varrho)=(\overline{\boldsymbol{\sigma}},\overline{\mathbf{v}})^\top$ where $(\overline{\boldsymbol{\sigma}},\overline{\mathbf{v}})^\top\in C\big([0,T],X\big)$ is the mild solution of*

$$\partial_t\overline{\boldsymbol{\sigma}}(t,x) \;=\; C\big(\hat\mu(x),\hat\lambda(x)\big)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}(t,x)) \;+\; C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}(\widehat{\mathbf{v}}(t,x))\,, \tag{29a}$$

$$\hat\varrho(x)\partial_t\overline{\mathbf{v}}(t,x) \;=\; \operatorname{div}\overline{\boldsymbol{\sigma}}(t,x) \;-\; \rho(x)\partial_t\widehat{\mathbf{v}}(t,x) \tag{29b}$$

in $[0,T]\times D$ and $\overline{\boldsymbol{\sigma}}(0)=\mathbf{0}$, $\overline{\mathbf{v}}(0)=\mathbf{0}$. Here, $\widehat{\boldsymbol{\sigma}}$ and $\widehat{\mathbf{v}}$ correspond to the parameters $(\hat\mu,\hat\lambda,\hat\varrho)^\top$.

Proof: We have that $\tilde{F}(\mu,\lambda,\varrho)=F\big(V(\mu,\lambda,\varrho)\big)$ where $F\colon\mathcal{L}(X)\supset\mathcal{B}\to C\big([0,T],X\big)$, $B\mapsto(\boldsymbol{\sigma},\mathbf{v})^\top$ and $(\boldsymbol{\sigma},\mathbf{v})$ solves (26) w.r.t. B. Thus, $\tilde{F}'(\mu,\lambda,\varrho)=F'\big(V(\mu,\lambda,\varrho)\big)V'(\mu,\lambda,\varrho)$. We determine the derivative of F with Theorem 3.5.

$$\tilde{F}'(\hat\mu,\hat\lambda,\hat\varrho)(\mu,\lambda,\varrho) \;=\; F'\big(V(\hat\mu,\hat\lambda,\hat\varrho)\big)\big(\tilde{C}'(\hat\mu,\hat\lambda)(\mu,\lambda)\,,\,\varrho\big) \;=\; (\overline{\boldsymbol{\sigma}},\overline{\mathbf{v}})^\top$$

where $\overline{\boldsymbol{\sigma}}(0)=\mathbf{0}$, $\overline{\mathbf{v}}(0)=\mathbf{0}$, and

$$\begin{pmatrix}\tilde{C}(\hat\mu,\hat\lambda) & 0\\ 0 & \hat\varrho I\end{pmatrix}\partial_t\begin{pmatrix}\overline{\boldsymbol{\sigma}}\\ \overline{\mathbf{v}}\end{pmatrix} \;=\; \begin{pmatrix}0 & \boldsymbol{\varepsilon}\\ \operatorname{div} & 0\end{pmatrix}\begin{pmatrix}\overline{\boldsymbol{\sigma}}\\ \overline{\mathbf{v}}\end{pmatrix} \;-\; \begin{pmatrix}\tilde{C}'(\hat\mu,\hat\lambda)(\mu,\lambda)\partial_t\widehat{\boldsymbol{\sigma}}\\ \varrho\widehat{\mathbf{v}}\end{pmatrix}$$

that is, using Lemma 6.3,

$$\begin{aligned}\partial_t\overline{\boldsymbol{\sigma}}(t,x) \;&=\; C\big(\hat\mu(x),\hat\lambda(x)\big)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}(t,x)) \;+\; C\big(\mu(x),\lambda(x)\big)\tilde{C}\big(\hat\mu(x),\hat\lambda(x)\big)\partial_t\widehat{\boldsymbol{\sigma}}(t,x)\\ &=\; C\big(\hat\mu(x),\hat\lambda(x)\big)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}(t,x)) \;+\; C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}(\hat{\mathbf{v}}(t,x))\\ \hat\varrho(x)\partial_t\overline{\mathbf{v}}(t,x) \;&=\; \operatorname{div}\overline{\boldsymbol{\sigma}}(t,x) \;-\; \varrho(x)\partial_t\widehat{\mathbf{v}}(t,x)\end{aligned}$$

which proves the theorem. □

Remarks 6.5. *(a) As before, the mild solution is also the weak solution for the elastic equation; that is (compare with the proof of Lemma 6.1)*

$$\frac{d}{dt}\big(\tilde{C}(\hat\mu,\hat\lambda)\overline{\boldsymbol{\sigma}}(t),\boldsymbol{\psi}\big)_{L^2} \;=\; -\big(\overline{\mathbf{v}}(t),\operatorname{div}\boldsymbol{\psi}\big)_{L^2} \;+\; \big(\tilde{C}(\hat\mu,\hat\lambda)C(\mu,\lambda)\boldsymbol{\varepsilon}(\widehat{\mathbf{v}}(t)),\boldsymbol{\psi}\big)_{L^2}$$

for almost all $t\in[0,T]$ and all $\boldsymbol{\psi}\in H(\operatorname{div},D,\mathbb{R}^{3\times3}_{sym})$ with $\boldsymbol{\psi}\mathbf{n}=\mathbf{0}$ on ∂D_N and

$$\frac{d}{dt}\big(\hat\rho\,\overline{\mathbf{v}}(t),\boldsymbol{\phi}\big)_{L^2} \;=\; -\big(\overline{\boldsymbol{\sigma}}(t),\boldsymbol{\varepsilon}(\boldsymbol{\phi}(t))\big)_{L^2} \;-\; \frac{d}{dt}\big(\rho\,\hat{\mathbf{v}}(t),\boldsymbol{\phi}\big)_{L^2}$$

for almost all $t\in[0,T]$ and all $\boldsymbol{\phi}\in H^1_D(D,\mathbb{R}^3)$.

(b) We note that the regularity assumptions in Theorem 6.4 are much weaker than in, e.g., [15] *or* [2]*, see Section 7. This is due to the fact that we show differentiability of $\tilde{F}$ only as a mapping from $\mathcal{P}$ into $C\big([0,T],X\big)$ and use the concept of mild solutions. Under the additional regularity assumptions of part (b) of Theorem 6.2 the mild solution is also a classical solution. Under the assumptions of part (c) of this theorem the mapping $\tilde{F}$ is differentiable from $\mathcal{P}$ into $C^1\big([0,T],X\big)\cap C\big([0,T],\mathcal{D}(A)\big)$.*

Finally, we prove the local ill-posedness of $\tilde{F}(\mu,\lambda,\varrho)=(\boldsymbol{\sigma},\mathbf{v})^\top$.

Theorem 6.6. *The equation $\tilde{F}(\mu,\lambda,\varrho) = (\boldsymbol{\sigma},\mathbf{v})^\top$ is locally ill-posed at any interior point $(\mu,\lambda,\varrho)^\top$ of $\mathcal{P}$.*

Proof: Let $r \in (0,1]$ and $\chi_n \in L^\infty(D)$ as in the proof of Theorem 5.6. Choose $r_1, r_2, r_3 \in [0,r]$ with $r_1 + r_2 + r_3 > 0$ and $\mathrm{c}^{-4}[2r_1 + 3r_2]^2 + r_3^2 \le r^2$ and such that $(\mu_n,\lambda_n,\varrho_n)^\top \in \mathcal{P}$ for all n where $\mu_n = \mu + r_1\chi_n$, $\lambda_n = \lambda + r_2\chi_n$, and $\varrho_n = \varrho + r_3\chi_n$. We show that the operators $E_n = V(\mu_n,\lambda_n,\varrho_n) - V(\mu,\lambda,\varrho) \in \mathcal{L}(X)$ satisfy the assumptions of Theorem 4.1 for some $\hat{r} \in (0,r)$. Here again, $V\colon L^\infty(D)^3 \supset \mathcal{P} \to \mathcal{L}(X)$ is defined as the mapping $(\mu,\lambda,\varrho)^\top \mapsto \begin{pmatrix} \tilde{C}(\mu,\lambda) & \mathbf{0} \\ \mathbf{0} & \varrho\mathbf{I} \end{pmatrix}$. This mapping V is matrix-valued, and we consider first the component $\tilde{C}$. For $\boldsymbol{\sigma} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and fixed $x \in D$ (where we write μ instead of $\mu(x)$, etc.) we compute as in Lemma 6.3

$$\begin{aligned}
&\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma} - \tilde{C}(\mu,\lambda)\boldsymbol{\sigma} \\
&\quad = \tilde{C}(\mu_n,\lambda_n)\big[C(\mu,\lambda) - C(\mu_n,\lambda_n)\big]\tilde{C}(\mu,\lambda)\boldsymbol{\sigma} \\
&\quad = 2(\mu-\mu_n)\tilde{C}(\mu_n,\lambda_n)\tilde{C}(\mu,\lambda)\boldsymbol{\sigma} \;+\; \frac{\lambda-\lambda_n}{(3\lambda+2\mu)(3\lambda_n+2\mu_n)}\,\mathrm{trace}(\boldsymbol{\sigma})\,\mathbf{I}
\end{aligned}$$

because $\mathrm{trace}\big(\tilde{C}(\mu,\lambda)\boldsymbol{\sigma}\big) = \frac{1}{3\lambda+2\mu}\mathrm{trace}(\boldsymbol{\sigma})$ and $\tilde{C}(\mu_n,\lambda_n)\mathbf{I} = \frac{1}{3\lambda_n+2\mu_n}\mathbf{I}$.
Using $|\mathrm{trace}(\boldsymbol{\sigma})\,\mathbf{I}|_F \le 3|\boldsymbol{\sigma}|_F$ we conclude that

$$\big|\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma} - \tilde{C}(\mu,\lambda)\boldsymbol{\sigma}\big|_F \;\le\; \big[2|\mu-\mu_n| + 3|\lambda-\lambda_n|\big]\,\mathrm{c}^2|\boldsymbol{\sigma}|_F \;=\; \mathrm{c}^2[2r_1+3r_2]\,\chi_n\,|\boldsymbol{\sigma}|_F\,.$$

Let now $\boldsymbol{\sigma} \in L^2\big(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$. Then we conclude from the last estimate that

$$\big\|\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma} - \tilde{C}(\mu,\lambda)\boldsymbol{\sigma}\big\|^2_{L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})} \;\le\; \mathrm{c}^4[2r_1+3r_2]^2\int_D \chi_n(x)\,|\boldsymbol{\sigma}(x)|_F^2\mathrm{d}x$$

which converges to zero as n tends to infinity as in the proof of Theorem 5.6. Furthermore, in the same way one shows that $\|\varrho_n\mathbf{v} - \varrho\mathbf{v}\|^2_{L^2(D,\mathbb{R}^3)} = r_3^2\int_D \chi_n(x)\,|\mathbf{v}(x)|^2\mathrm{d}x \to 0$ as n tends to infinity and thus $\left\|\big[V(\mu_n,\lambda_n,\varrho_n) - V(\mu,\lambda,\varrho)\big]\begin{pmatrix}\boldsymbol{\sigma}\\ \mathbf{v}\end{pmatrix}\right\|_X \to 0$ as n tends to infinity.
Furthermore, we note that

$$\big\|V(\mu_n,\lambda_n,\varrho_n) - V(\mu,\lambda,\varrho)\big\|^2_{\mathcal{L}(X)} \;\le\; \mathrm{c}^4[2r_1+3r_2]^2 + r_3^2 \;\le\; r^2\,.$$

On the other hand we set $\boldsymbol{\sigma}_n = \chi_n\mathbf{I}$ and have, omitting again the argument x,

$$\begin{aligned}
&\big|\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma}_n - \tilde{C}(\mu,\lambda)\boldsymbol{\sigma}_n\big|_F|\boldsymbol{\sigma}_n|_F \;\ge\; \big|\big[\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma}_n - \tilde{C}(\mu,\lambda)\boldsymbol{\sigma}_n\big] : \boldsymbol{\sigma}_n\big| \\
&\quad = 2r_1\chi_n\,\big(\tilde{C}(\mu,\lambda)\boldsymbol{\sigma}_n\big) : \big(\tilde{C}(\mu_n,\lambda_n)\boldsymbol{\sigma}_n\big) \;+\; \frac{r_2\chi_n}{(3\lambda+2\mu)(3\lambda_n+2\mu_n)}\big(\mathrm{trace}(\boldsymbol{\sigma}_n)\big)^2 \\
&\quad = \frac{6r_1+9r_2}{(3\lambda+2\mu)(3\lambda_n+2\mu_n)}\,\chi_n(x) \;=\; \frac{2r_1+3r_2}{(3\lambda+2\mu)(3\lambda_n+2\mu_n)}\,|\boldsymbol{\sigma}_n|_F^2
\end{aligned}$$

by using again $\tilde{C}(\mu,\lambda)\mathbf{I} = \frac{1}{3\lambda+2\mu}\mathbf{I}$ and $|\boldsymbol{\sigma}_n(x)|_F^2 = 3\chi_n(x)$. Therefore,

$$\big\|\tilde{C}(\mu_n,\lambda_n) - \tilde{C}(\mu,\lambda)\big\|_{\mathcal{L}(L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}}))} \;\ge\; [2r_1+3r_2]\,\mathrm{c}^{-2}$$

for all n. Using this in the definition of the mapping V yields

$$\big\|V(\mu_n,\lambda_n,\varrho_n) - V(\mu,\lambda,\varrho)\big\|^2_{\mathcal{L}(X)} \;\ge\; \mathrm{c}^{-4}\,[2r_1+3r_2]^2 + r_3^2 \;=:\; \hat{r}^2$$

for all n. Therefore, the operators $E_n = V(\mu_n,\lambda_n,\varrho_n) - V(\mu,\lambda,\varrho)$ satisfy the assumptions of Theorem 4.1. □

7. Final Remarks

In [2, 15] the following second order initial-boundary value problem has been considered as the model in seismology.

$$\rho(x)\,\partial_{tt}\mathbf{v}(t,x) \;=\; \operatorname{div}\big[C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}\big(\mathbf{v}(t,x)\big)\big] \;+\; \mathbf{g}(t,x) \text{ for } (t,x)\in[0,T]\times D\,, \tag{30}$$

$\mathbf{v}(0,\cdot)=\mathbf{v}_0$ in D, $\partial_t\mathbf{v}(0,\cdot)=\mathbf{v}_1$ in D, $\mathbf{v}=\mathbf{0}$ on $[0,T]\times\partial D_D$, $C(\mu,\lambda)\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{n}=\mathbf{0}$ on $[0,T]\times\partial D_N$. It is easy to see that if $\mathbf{v}$ satisfies (30) with the initial and boundary conditions then $(\mathbf{v},\boldsymbol{\sigma})$ solves

$$\partial_t\boldsymbol{\sigma}(t,x) \;=\; C\big(\mu(x),\lambda(x)\big)\boldsymbol{\varepsilon}\big(\mathbf{v}(t,x)\big)\,, \tag{31a}$$

$$\rho(x)\,\partial_t\mathbf{v}(t,x) \;=\; \operatorname{div}\boldsymbol{\sigma}(t,x) \;+\; \int_0^t \mathbf{g}(s,x)\,\mathrm{d}s \;+\; \rho(x)\mathbf{v}_1(x) \tag{31b}$$

for $(t,x)\in[0,T]\times D$ with $\mathbf{v}(0,\cdot)=\mathbf{v}_0$ in D, $\boldsymbol{\sigma}(0,\cdot)=\mathbf{0}$ in D, $\mathbf{v}=\mathbf{0}$ on $[0,T]\times\partial D_D$, $\boldsymbol{\sigma}\mathbf{n}=\mathbf{0}$ on $[0,T]\times\partial D_N$. Here, $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma}(t,x) \;=\; C\big(\mu(x),\lambda(x)\big)\int_0^t \boldsymbol{\varepsilon}\big(\mathbf{v}(s,x)\big)\,\mathrm{d}s\,, \quad (t,x)\in[0,T]\times D\,. \tag{31c}$$

On the other hand, if $(\mathbf{v},\boldsymbol{\sigma})$ solves (31a), (31b) with the initial and boundary conditions then $\mathbf{v}$ solves (30).

We translate the requirements for the Fréchet derivatives of (31a), (31b) for the case considered in this paper; that is, for the parameter-to-solution operator $\tilde F$ from $\mathcal{P}$ into $L^2\big((0,T);L^2(D)\big)$. Comparing (31b) to (19b) we observe that $\mathbf{f}(t,x)=\int_0^t\mathbf{g}(s,x)\,\mathrm{d}s+\rho(x)\mathbf{v}_1(x)$. Therefore, in order to satisfy the assumptions of Theorem 6.4 we have to assume that $\mathbf{g}\in L^1\big((0,T);L^2(D,\mathbb{R}^3)\big)$, $\mathbf{v}_0\in H^1_D(D,\mathbb{R}^3)$, $\mathbf{v}_1\in L^2(D,\mathbb{R}^3)$, and $C(\hat\mu,\hat\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0)\in H\big(\operatorname{div},D,\mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$ with $C(\hat\mu,\hat\lambda)\boldsymbol{\varepsilon}(\mathbf{v}_0)\mathbf{n}=\mathbf{0}$. These conditions are substantially weaker than the assumptions made in, e.g., [2, 15]. We recall, however, that we consider the parameter-to-solution map $(\mu,\lambda,\rho)\mapsto\mathbf{v}$ from $\mathcal{P}$ into $C\big([0,T],L^2(D,\mathbb{R}^3)\big)$ rather than into the smaller space $C^1\big([0,T],L^2(D,\mathbb{R}^3)\big)\cap C\big([0,T],H^1_D(D,\mathbb{R}^3)\big)$ with the stronger topology.

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