L^q-Helmholtz decomposition and L^q-spectral theory for the Maxwell operator on periodic domains

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Abstract

We prove the existence of the Helmholtz decomposition

$$L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) = L^q_\sigma(\Omega_{\mathrm{p}}) \oplus G^q(\Omega_{\mathrm{p}}),$$

for periodic domains $\Omega_{\mathrm{p}} \subseteq \mathbb{R}^d$ with respect to a lattice $L \subseteq \mathbb{R}^d$, i.e. $\Omega_{\mathrm{p}} = \Omega_{\mathrm{p}} + z$ for all $z \in L$, and for all $q \in I = (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$, where $\varepsilon = \varepsilon(\Omega_{\mathrm{p}})$ is given by

- $\varepsilon = \infty$ if $\Omega_{\mathrm{p}} \in C^1$, i.e. $I = (1, \infty)$.
- $\varepsilon > 1$ if Ω_{p} is Lipschitz and $d = 2$.
- $\varepsilon > 0$ if Ω_{p} is Lipschitz and $d \geq 3$.

The proof of the Helmholtz decomposition builds upon [Bar13], where periodic operators are extended continuously from L^2 to operators defined on L^q by using Bloch theory and Fourier multiplier theorems. Here, we prove that the Helmholtz projection P_2 on L^2 has an extension to a bounded linear operator on L^q.
The same approach yields the existence of the Leray decomposition

$$L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) = L^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}}) \oplus G^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}}),$$

which is related to the weak Dirichlet problem, while the Helmholtz decomposition is related to the weak Neumann problem.

Following the approach in [KU15a], we define the Maxwell operator on periodic domains by using the form method, and this allows us to extend the operator on L^q-spaces, where $q \in [3/2, 3]$ if $\partial\Omega_{\mathrm{p}}$ is Lipschitz and $q \in [6/5, 6]$ if $\partial\Omega_{\mathrm{p}} \in C^{1,1}$, by using generalized Gaussian estimates. Furthermore, we show spectral independence of q for the Maxwell operator and prove a spectral multiplier theorem for a shifted version of the operator.

Furthermore, we get that the Stokes operator with Dirichlet boundary conditions generates an analytic semigroup for periodic domains of C^3-class, and are able to show several results for the incompressible Navier-Stokes equations which make use of [GHHS12] and [GK15].

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Contents

CHAPTER 1

Introduction

In 1858 von Helmholtz [Hel58] introduced the vector field decomposition

$$u = \nabla(\text{div } F * u) - \text{curl }(\text{curl } F * u), \qquad u \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^d),$$
$$F(x) := \begin{cases} \frac{1}{2\pi} \log |x|, & d = 2, \\ \frac{1}{d(2-d)|B(0,1)|} |x|^{2-d}, & d = 3, \end{cases}$$

where F is the fundamental solution of the Laplace operator. Since div curl $= 0$, the decomposition consists of a gradient part and a solenoidal vector field. This ansatz can be used to prove the existence of the Helmholtz decomposition on $L^q(\mathbb{R}^d, \mathbb{C}^d)$ for all $q \in (1, \infty)$, i.e. a unique decomposition into a gradient part and a solenoidal vector field. From there on mathematicians analysed this decomposition and generalized it to arbitrary domains $\Omega \subseteq \mathbb{R}^d$. This decomposition, if it exists, is known as the Helmholtz decomposition and one common today's formulation for a domain $\Omega \subseteq \mathbb{R}^d$ looks as follows.
We say that the Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^d)$, if for all $f \in L^q(\Omega, \mathbb{C}^d)$ there exist unique functions $\nabla p \in G^q(\Omega)$, $g \in L^q_\sigma(\Omega)$ so that

$$f = g + \nabla p, \qquad \text{and} \qquad ||g||_{L^q(\Omega, \mathbb{C}^d)} + ||\nabla p||_{L^q(\Omega, \mathbb{C}^d)} \le C ||f||_{L^q(\Omega, \mathbb{C}^d)},$$

where $C = C(q, \Omega)$. Here, the spaces $L^q_\sigma(\Omega)$ and $G^q(\Omega)$ are defined as follows:

$$L^q_\sigma(\Omega) = \overline{C^\infty_{c,\sigma}(\Omega)}^{||\cdot||_q} = \overline{\{u \in C_c^\infty(\Omega, \mathbb{C}^d) \mid \text{div } u = 0 \text{ in } \Omega\}}^{L^q(\Omega, \mathbb{C}^d)},$$
$$G^q(\Omega) = \{\nabla p \in L^q(\Omega, \mathbb{C}^d) \mid p \in W^{1,q}_{loc}(\Omega)\} = \{\nabla p \in L^q(\Omega) \mid p \in L^1_{loc}(\Omega)\}.$$

If the decomposition exists, there is a bounded projection operator $P_q \colon L^q(\Omega, \mathbb{C}^d) \to L^q(\Omega, \mathbb{C}^d)$ satisfying kernel$(P_q) = G^q(\Omega)$ and image$(P_q) = L^q_\sigma(\Omega)$. P_q is called the Helmholtz projection. The Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^d)$ if and only if the weak formulation of the classical Neumann problem

$$\Delta p = \text{div } u \qquad \text{in } \Omega,$$
$$\frac{\partial p}{\partial \nu} = u \cdot \nu \qquad \text{on } \partial\Omega.$$

has for all $u \in L^q(\Omega, \mathbb{C}^d)$ a unique solution $p \in G^q(\Omega)$, compare Theorem 2.43 and Remark 2.44.
The Helmholtz decomposition can be used in physical models considering source-free vector fields, which is in fact a very common assumption. In this work we examine the Maxwell equations (note that the magnetic field is source-free) and the incompressible Navier-Stokes equations for fluids. We discuss below (p.4 ff.) how the Helmholtz projection enters in these systems. We shall come back to applications, but we first outline the existence theory of the Helmholtz decomposition.

On half-spaces there is, as in the case of $\Omega = \mathbb{R}^d$, a fundamental solution for the Laplace operator with Neumann boundary condition, which yields the existence of Helmholtz decomposition for all $q \in (1, \infty)$. From a mathematical point of view the Hilbert space case $q = 2$ is by far the easiest, and in fact one can easily prove the existence of the Helmholtz decomposition on any domain $\Omega \subseteq \mathbb{R}^d$ in that case by using the Lax-Milgram lemma, compare Theorem 2.45.
In the study of (nonlinear) differential equations it often does not suffice to consider the case $q = 2$. In the theory of the three dimensional Navier-Stokes equations the spaces L^q for $q \geq 3$ play an important role. For example, an important tool in the study is the Fujita-Kato iteration scheme [FK64], which yields a unique mild solution in the case $q > d = 3$. Here, the iterations enlarges step by step the interval on which the mild solution is unique. The exponent $q > d$ is needed to have the embedding of $W^{1,q}$ into the space of continuous functions. Also the case $q = d$ is of special interest, cf. for example [Gig86], because one can use the scaling invariance of the Navier-Stokes equations on L^d under the scaling

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

Furthermore, in nonlinear Maxwell equations and heat equations nonlinear terms behaving like $u|u|^{p-1}$, $p > 1$ have to be studied. So here, one also is obviously reliant on using L^q-theory.

At least since 1986 [Bou86] it became clear that the Helmholtz decomposition fails for some unbounded (even C^∞-) domains and also fails for some bounded Lipschitz domains for some range of q. So, the following question attracted and is still attracting many mathematicians:

Under which assumptions on Ω exists the Helmholtz decomposition, and on which L^q-spaces?

In the 1990's Simader and Sohr [SS92] and Fabes, Mitrea and Mendez [FMM98] gave a satisfactory answer to this question for domains having compact boundary, i.e. for bounded and exterior domains. The Helmholtz decomposition exists for all $q \in (1, \infty)$ if Ω has a compact C^1-boundary and for all $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3 + \varepsilon)$, where $\varepsilon = \varepsilon(\Omega) > 0$, if Ω has a compact Lipschitz boundary. Furthermore, the result for Lipschitz domains is sharp, i.e. for all $q < 3/2$ and $q > 3$ there exists a domain Ω for which the Helmholtz decomposition fails on $L^q(\Omega, \mathbb{C}^d)$. Still, there is no general theory treating domains having non-compact boundary. For some unbounded domains, the Helmholtz decomposition is known to exist for some $q \neq 2$. We present them along with references in Theorem 2.46. However, Farwig, Kozono and Sohr presented in 2007 the following approach to general unbounded domains [FKS07], which are uniformly C^1. They proved the existence of the Helmholtz decomposition for all

$q \in (1,\infty)$ on the spaces $\widetilde{L}^q(\Omega,\mathbb{C}^d)$, which are defined as $L^q(\Omega,\mathbb{C}^d)\cap L^2(\Omega,\mathbb{C}^d)$ for $q \in [2,\infty)$ and by $L^q(\Omega,\mathbb{C}^d) + L^2(\Omega,\mathbb{C}^d)$ for $q \in (1,2)$.

In this thesis, we present a new class of (unbounded) domains on which the (classical) Helmholtz decompositions exists, namely domains periodic with respect to a lattice $L \subseteq \mathbb{R}^d$, i.e. domains $\Omega \subseteq \mathbb{R}^d$ satisfying $\Omega = \Omega + z$ for all $z \in L$. By transforming the basis of L to the standard basis, it suffices to consider $\mathbb{Z}^d$-periodic domains, cf. Remark 2.13 and Remark 3.36. Therefore we only examine ($\mathbb{Z}^d$-)periodic domains, which we denote by Ω_p. The main result reads as follows:

Theorem: Helmholtz decomposition of L^q-vector fields on periodic domains
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain. The Helmholtz decomposition on $L^q(\Omega_\mathrm{p},\mathbb{C}^d)$ exists for all $q \in I$, where

- *$I = (1,\infty)$, if $\partial\Omega_\mathrm{p} \in C^1$.*
- *$I = (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$, for some $\varepsilon = \varepsilon(\Omega_\mathrm{p}) > 0$, if $\partial\Omega_\mathrm{p}$ is Lipschitz. If $d = 2$, then $\varepsilon > 1$.*

The whole chapter 3 deals with the proof of this theorem. Note that the interval on which the Helmholtz decomposition exists on periodic domains is looking as in the bounded case. In fact, we will reduce the case of periodic domains to the case of bounded domains by using the Bloch transform.

Furthermore, we also take a look at a similar decomposition, the Leray decomposition, and prove the existence of the Leray decomposition on periodic domains for the same range of q as in the Helmholtz case. While the Helmholtz decomposition is related to the weak Neumann problem, the Leray decomposition is related to the weak Dirichlet problem. We give a precise definition of the decomposition at the end of the introduction.

We shortly summarize the structure of the remaining part of the introduction now. We start by giving the motivation for studying periodic domains, i.e. we expound the physical background on periodic domains. Having studied the physical background, we turn our attention to applications, namely the Maxwell equations and the incompressible Navier-Stokes equations with Dirichlet boundary conditions. After that, we come back to the Helmholtz decomposition on periodic domains and summarize the main steps of the proof. Finally, we discuss the Leray case, too. We present the spaces involved in the Leray decomposition and formulate the analogue of the theorem above in this case. At the end, we shortly summarize the content of the single chapters and their sections.

Photonic crystals

Periodic domains attracted much attention in recent years, mainly because they play a key role in photonic crystals, which became very popular, compare e.g. [HP01].
For the introduction of photonic crystals we follow [JJWM08, Introduction]. The main idea behind photonic crystals is to build materials with special optical properties. The goal is to construct them in such a way that certain frequencies are reflected and some ranges of frequencies pass the material in a particular direction, or even some frequencies might be confined in a chosen domain.
The original motivation for this are observations on the propagation of electrons in a periodic

potential, which is realised in semiconductor crystals, where exactly these phenomena occur. So, depending on the energy, some electrons pass without scattering at all, while some electrons are not able to propagate in certain directions. And furthermore, there arise even complete band gaps, which means that no electrons at all are able to pass through. The explanation for these phenomena comes from quantum mechanics, since electrons behave like waves in that case. In photonic crystals, one considers the propagation of electromagnetic waves instead of the propagation of electrons. Mathematically, one detects passing and reflecting frequencies by looking at the spectrum, respectively the resolvent set of the Maxwell operator. There is a series of papers by Figotin and Kuchment [FK94, FK95, FK96, FK98] concerning the band-gap structure of the Maxwell operator in periodic media. In [KK99] numerical results are analysed. So, in [FK95] the spectrum of the Maxwell operator for two dimensional photonic crystals is examined. They proved the band-gap structure of the spectrum, which is a typical application of Bloch theory, cf. Remark 4.1. In [HPW09] the existence of a band-gap is proven by using numerical approximations of the eigenvalues. These band-gaps are of special interest, because in that frequency range the photons are not able to spread and this can be used to lead light through solids. In order to achieve band gaps, one takes periodic macroscopic media with different dielectric constants instead of atoms and molecules. Instead of a periodic potential, we face a periodic permittivity here.

Photonic crystals appear in nature and were observed in wings of butterflies, opals and feathers. They also have applications in technology. One of the most important is the possible use in telecommunication. Its striking advantages are its superior speed and better stability despite smaller dimensions. Besides, light waves do less interfere than electron waves. Physicists are optimistic to use them to build optical transistors [NSM$^+$05, VRW$^+$12].

A step further they might even help to build a quantum computer. By the rules of quantum mechanics, this computer would be able to do lots of calculations at the same time, which would lead to a tremendous speed. Another futuristic application is quantum cryptography, compare for example [CD05], which also uses photons. This encryption uses quantum mechanics, too and is by Heisenberg's uncertainty principle absolutely safe. In recent time, there were already first successful realizations of quantum computers and quantum cryptography.

For further explanations and more about the physical background we refer to [Mit09, Sou12].

The Maxwell operator for photonic crystals

We are getting to the mathematical modelling of photonic crystals. The mathematical description of photonic crystals goes back to Yablonovitch [Yab87] and John [Joh87]. In applications, the wavelengths are about some hundreds of nanometres (e.g. blue light $\sim 400-500$ nm, green light $\sim 520-570$ nm, yellow light $\sim 580-600$ nm, red light $\sim 650-750$ nm). This number is in comparison to the atomic level quite large, so we can take the macroscopic Maxwell equations instead of the microscopic ones. We examine both types of equation systems in more detail in the physical appendix.

After integrating the properties of photonic crystals like linearity, non-homogeneity and perfect conductor boundary conditions, which as well are studied in the physical appendix and at the beginnig of Chapter 4, we get by the time harmonic ansatz $H(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}H(x)$ (i.e. we assume a monochromatic wave form) the following system of equations for the magnetic field strength H:

$$\begin{aligned}
\operatorname{curl}(\varepsilon^{-1}\operatorname{curl} H) &= \omega^2\mu_0^{-1}H, && \text{in} \quad \Omega,\\
\operatorname{div} H &= 0, && \text{in} \quad \Omega,\\
\nu\cdot H &= 0, && \text{on } \partial\Omega,\\
\nu\times\varepsilon^{-1}\operatorname{curl} H &= 0, && \text{on } \partial\Omega.
\end{aligned}$$

Because of its essentiality for the theory, the first equation is sometimes called the master equation. Throughout this work we focus on periodic domains Ω_{p}. We analyse the related Maxwell operator $M = \operatorname{curl}\varepsilon^{-1}\operatorname{curl}$ by using the form method and incorporate the other three equations in the definition of the domain. More precisely, in $L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d)$ we examine the form

$$a(u,v) = \int_{\Omega_{\mathrm{p}}} \varepsilon(\cdot)^{-1}\operatorname{curl} u\,\overline{\operatorname{curl} v}\,dx + \int_{\Omega_{\mathrm{p}}} \operatorname{div} u\,\overline{\operatorname{div} v}\,dx,$$

for $u,v\in V(\Omega_{\mathrm{p}})$, where

$$V(\Omega_{\mathrm{p}}) = \{u\in L^2(\Omega_{\mathrm{p}},\mathbb{C}^d) \mid \operatorname{div} u\in L^2(\Omega_{\mathrm{p}},\mathbb{C}),\ \operatorname{curl} u\in L^2(\Omega_{\mathrm{p}},\mathbb{C}^d),\ \nu\cdot u\,|_{\partial\Omega_{\mathrm{p}}}=0\}.$$

We deduce the embedding of $V(\Omega_{\mathrm{p}})$ into $H^1(\Omega_{\mathrm{p}})$ if Ω_{p} has $C^{1,1}$-boundary and into $H^{1/2}(\Omega_{\mathrm{p}})$ if it has Lipschitz boundary from the bounded case, where this is a well known fact. As we shall see this yields $A_2u = \operatorname{curl}\varepsilon^{-1}\operatorname{curl} u - \nabla\operatorname{div} u$ for $u\in D(A_2)$, and $u\in D(A_2)$ satisfies automatically the boundary conditions $\nu\times\varepsilon^{-1}\operatorname{curl} u = 0$ on $\partial\Omega_{\mathrm{p}}$. In fact, we get the following characterization for the domain of A_2, provided functions in $H^1(\Omega_{\mathrm{p}})\cap V(\Omega_{\mathrm{p}})$ with bounded support are dense. This is for instance the case if $\partial\Omega_{\mathrm{p}}\in C^{1,1}$. Then,

$$D(A_2) = \{u\in V(\Omega_{\mathrm{p}}) \mid \operatorname{curl}\varepsilon^{-1}\operatorname{curl} u\in L^2(\Omega_{\mathrm{p}},\mathbb{C}^3),\ \operatorname{div} u\in H^1(\Omega_{\mathrm{p}}),\ \nu\times\varepsilon^{-1}\operatorname{curl} u|_{\partial\Omega_{\mathrm{p}}} = 0\}.$$

The operator associated to the form is not yet the Maxwell operator, since we have an extra divergence term here. To get rid of the divergence term and take into account the second equation we use the Helmholtz projection. By using the Helmholtz projection we restrict the domain of A_2 to solenoidal vector fields and obtain the Maxwell operator

$$M_2 := A_2, \qquad D(M_2) = P_2D(A_2) = D(A_2)\cap L^2_\sigma(\Omega_{\mathrm{p}}).$$

Here $P_2D(A_2) = D(A_2)\cap L^2_\sigma(\Omega_{\mathrm{p}})$, since A_2 and P_2 commute, which is due to the boundary conditions, see [KU15a, Lemma 3.6]. This is quite handy, because we can assign many results we shall prove for A_2 to the Maxwell operator M_2.

The definition of the Maxwell operator on L^q requires the extension of both operators to L^q, namely the operator A_2 and the Helmholtz projection P_2. As already mentioned, the operator $-A_2$ generates a bounded analytic semigroup, and by using generalized Gaussian estimates we show that the semigroup generated by $-A_2-\lambda$, $\lambda>0$, extends to a bounded analytic semigroup of angle $\pi/2$ on L^q with generator $-A_{q,\lambda}$. Since we have proven the existence of the Helmholtz projection on $L^q(\Omega_{\mathrm{p}},\mathbb{C}^3)$ we can define in the same manner the Maxwell operator on L^q. Besides, we are able to transfer a spectral multiplier theorem for

operators fulfilling generalized Gaussian estimates [KU15b, Theorem 5.4] and the spectral independence of q from $A_{q,\lambda}$ to the Maxwell operator $M_{q,\lambda}$. Here, a decisive ingredient is that the operator $A_{q,\lambda}$ and P_q also commute.
Summing up, we are able to extend the theory for bounded domains from [KU15a] to periodic domains.

Navier-Stokes equations

Besides the Maxwell equations, the Helmholtz decomposition has a key role in the theory of the Navier-Stokes equations, which describe the motion of viscous fluids and gases. The Navier-Stokes equations is one of the most important and most studied system of partial differential equations. Nevertheless, many aspects of the existence-, uniqueness- and regularity theory are still not satisfactorily answered. The case of the three dimensional incompressible system on $\mathbb{R}^3$ or the torus Π^3 is one of the famous open Millennium Prize Problems, see [Wie99] for a discussion.
At the very end of this thesis, we have a physical appendix, where we give short information about the background of the Maxwell and Navier-Stokes equations. The incompressible Navier-Stokes equations with no-slip boundary condition are given by (where $\partial_t u := \frac{\partial u}{\partial t}$)

$$\begin{aligned} \varrho\big(\partial_t u - \mu\Delta u + (u\cdot\nabla)u\big) + \nabla p &= f, && \text{in } \Omega\times(0,T),\\ \operatorname{div} u &= 0, && \text{in } \Omega\times(0,T),\\ u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\times(0,T),\\ u(0,\cdot) &= u_0. \end{aligned}$$

Here Δ denotes the vectorial Laplacian, ϱ the density, u the flux, μ the viscosity, ∇p the pressure and f the external force. In the application part we only consider the case of a constant density and we consider the Stokes system for high-viscous media and scale in such a way that we get

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f, && \text{in } \Omega\times(0,T),\\ \operatorname{div} u &= 0, && \text{in } \Omega\times(0,T),\\ u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\times(0,T),\\ u(0,\cdot) &= u_0. \end{aligned}$$

For this simplified system, the L^2-theory is well-known, since the negative of the Stokes operator $A_2 = -P_2\Delta$ is, as in the Maxwell case, self-adjoint and semibounded and hence generates an analytic semigroup. Using a form ansatz as in the Maxwell case, the Stokes operator can be defined on general three dimensional domains Ω, compare [Mon06]. For the definition of the Stokes operator on L^q-spaces one is of course reliant on the existence of the Helmholtz decomposition on these spaces. On the other hand, there are several results of the form 'let us assume the Helmholtz decomposition exists on L^q, then...', so since we prove the existence of the Helmholtz decomposition on L^q (e.g. for all $q \in (1,\infty)$ if $\partial\Omega_{\mathrm{p}} \in C^1$), we get some results without much further effort. We apply the results from [GHHS12] and [GK15], which describe applications in the L^q-case for general domains $\Omega \subseteq \mathbb{R}^d$ with uniform C^3-boundary, in the case that the Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^d)$ for some $q \in (1,\infty)$. We begin by defining for smooth periodic domains the Stokes operator

A_q on $L^q_\sigma(\Omega_\mathrm{p})$ by

$$\begin{aligned} D(A_q) &:= W^{2,q}(\Omega_\mathrm{p}) \cap W^{1,q}_0(\Omega_\mathrm{p}) \cap L^q_\sigma(\Omega_\mathrm{p}), \\ A_q u &:= -P_q \Delta u. \end{aligned}$$

Next, we see that $-A_q$ is the generator of an analytic semigroup on periodic domains having uniform C^3-boundary, admitting even maximal L^p-L^q-regularity. In particular, the Cauchy problem

$$\begin{aligned} \partial_t u + A_q u &= f, \\ u(0) &= u_0, \end{aligned}$$

is uniquely solvable. Following this, we consider the Stokes resolvent problem

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f, && \text{in } \Omega, \\ \operatorname{div} u &= 0, && \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega. \end{aligned}$$

By defining $u := (\lambda + A_q)^{-1} P_q f$ and $\nabla p = (Id - P)(f + \Delta u)$ we get a solution pair of the Stokes resolvent problem. In addition, we obtain a norm estimate, too.
Furthermore, we get by applying the results from [GK15] that the Stokes operator $\lambda + A_q$ admits for all $q \in (1, \infty)$ a bounded $\mathcal{H}^\infty$-calculus for some $\lambda > 0$.
We prove that the incompressible Stokes system has a unique solution pair $(u, \nabla p)$ in the L^q-setting. Once more, we get accompanying norm estimates. Finally, we apply a theorem [GHHS12, Theorem 3.2] and obtain a unique mild solution for the full non-linear incompressible Navier-Stokes equations in the case of fluids with constant density and no external force, which are given by

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= 0, && \text{in } \Omega_\mathrm{p} \times (0, T), \\ \operatorname{div} u &= 0, && \text{in } \Omega_\mathrm{p} \times (0, T), \\ u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega_\mathrm{p} \times (0, T), \\ u(0, \cdot) &= u_0, && \text{in } \Omega_\mathrm{p}. \end{aligned}$$

More precisely, we have a unique mild solution in some time interval $(0, T_0)$, where T_0 depends on u_0, provided $u_0 \in L^q_\sigma(\Omega_\mathrm{p})$, $q \in (d, \infty)$ and Ω_p has uniform C^3-boundary.

Sketch of the proof of the L^q-Helmholtz decomposition on periodic domains

Next, we outline the main steps of the existence proof of the Helmholtz decomposition on periodic domains for $q \neq 2$. The rough idea is to use adapted versions of Bloch multiplier theorems from [Bar13], which are applicable thanks to the periodicity of the domain and the periodicity of the Helmholtz projection. We prove that this yields an extension of the Helmholtz projection operator from $\mathcal{L}(L^2(\Omega_\mathrm{p}, \mathbb{C}^d))$ to a bounded operator in $\mathcal{L}(L^q(\Omega_\mathrm{p}, \mathbb{C}^d))$. Afterwards, we prove that this operator actually defines the Helmholtz projection on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$. Since a periodic domain has non-compact boundary, there arise problems, which do not occur for bounded or exterior domains. For example one cannot exclude in general in the class of unbounded (even C^∞) domains that the following inclusions of spaces of solenoidal vector fields and gradients on $\Omega \subseteq \mathbb{R}^d$ are strict.

$$\begin{aligned} &L^q_\sigma(\Omega) \subsetneq \widehat{L}^q_\sigma(\Omega) := \{u \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} u = 0 \text{ in } \Omega,\ \nu \cdot u = 0 \text{ on } \partial\Omega\}, \\ &\widehat{G}^q(\Omega) := \overline{\nabla C_c^\infty(\overline{\Omega})}^{\|\cdot\|_{L^q(\Omega,\mathbb{C}^d)}} \subsetneq G^q(\Omega). \end{aligned}$$

However, we prove that these spaces coincide for all periodic Lipschitz domains Ω_{p} and $q \in (1, \infty)$. It turns out that besides the boundary regularity, the equality of these spaces is a crucial property for our proof of the existence of the Helmholtz decomposition on $L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$. On the one hand, we use this equality to prove a representation for the fibre operators (compare the forthcoming lines for the definition of them), on the other hand this property is essential to prove that the operator obtained by continuous extension from L^2 to L^q coincides with the Helmholtz projection on L^q.
We give shortly the main idea of the proof for the equality of these spaces. For this, we use the fact, that it suffices to extend functions $p \in L^q_{\mathrm{loc}}(\Omega)$ with $\nabla p \in L^q(\Omega, \mathbb{C}^d)$ to functions $\widetilde{p}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ with $\nabla \widetilde{p} \in L^q(\mathbb{R}^d, \mathbb{C}^d)$ for all $q \in (1, \infty)$. This suffices, since $\widehat{G}^q(\mathbb{R}^d) = G^q(\mathbb{R}^d)$. The equality $L^q_\sigma(\Omega_{\mathrm{p}}) = \widehat{L}^q_\sigma(\Omega_{\mathrm{p}})$ follows then by duality. The extension can be realized by using extension theorems for $W^{1,q}$-functions, which go back to Calderón, Stein and Jones, together with the Poincaré-inequality and the periodic structure of the domain.

Now, we give more details about the sub-steps. First of all, we observe that the Helmholtz projection P_2 and the associated projection $Q_2 := Id - P_2$ are periodic operators, which means that they interchange with translations having integer entries. We define the manifold with boundary $\Omega_\# := \Omega_{\mathrm{p}}/\mathbb{Z}^d$ representing one periodicity cell. Then, the idea is to use an isometric isomorphism, the so-called Bloch transform, given by

$$\begin{aligned} &\Phi\colon L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d) \to L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)), \\ &((\Phi f)(\theta))(x) = \sum_{k \in \mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i}\theta\cdot(x-k)} f(x-k), \qquad \theta \in B^d,\ x \in \Omega_\#,\ f \in C_c^\infty(\Omega_{\mathrm{p}}, \mathbb{C}^d), \end{aligned}$$

to transform the operator Q_2 into a family of fibre operators $Q(\theta) \in \mathcal{L}(L^2(\Omega_\#, \mathbb{C}^d))$, where $\theta \in B^d = [-1/2, 1/2)^d$, which are uniquely determined by the equation

$$Q(\theta)(\Phi g(\theta)) = (\Phi(Qg))(\theta), \qquad g \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d).$$

By periodicity arguments, it suffices to consider them on one unit cube, and we choose B^d. The fibre operators $Q(\theta)$ exist since Q_2 is periodic [Bar13] and we obtain representations that are explicit enough to apply appropriate variants of the Bloch multiplier theorems from Barth's thesis [Bar13, Theorem 4.22], which yield the extension property of Q_2 on L^q-spaces and read as follows. In fact, we use the following general theorem for the special case that $Q_f = Q(\theta)$ from above.

Theorem: Bloch multiplier theorem
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain, $q \in (1,\infty)$, $\Omega_\# = \Omega_\mathrm{p}/\mathbb{Z}^d$ and E_1, E_2 be UMD-spaces. Let $Q_\mathrm{f}\colon B^d \to \mathcal{L}(L^q(\Omega_\#, E_1), L^q(\Omega_\#, E_2))$. If

(i) $\theta \mapsto Q_\mathrm{f}(\theta) \in C^d(B^d, \mathcal{L}(L^q(\Omega_\#, E_1), L^q(\Omega_\#, E_2)))$, or if,

(ii) $\theta \mapsto Q_\mathrm{f}(\theta) \in C^d(B^d \setminus \{0\}, \mathcal{L}(L^q(\Omega_\#, E_1), L^q(\Omega_\#, E_2)))$ and

$$\tau := \{|\theta|^{|\alpha|}\partial^\alpha Q(\theta) \colon \theta \in B^d \setminus \{0\},\ \alpha \leq (1,\dots,1)\} \subseteq \mathcal{L}(L^q(\Omega_\#, E_1), L^q(\Omega_\#, E_2))$$

is R-bounded,

then $Q := \Phi\, Q_\mathrm{f}\, \Phi^{-1}$ defines a translationsinvariant operator in $\mathcal{L}(L^q(\Omega_\mathrm{p}, E_1), L^q(\Omega_\mathrm{p}, E_2))$.

To get a representation for the fibre operators, we have to take a look at the following equivalent characterization for the existence of the Helmholtz decomposition. The Helmholtz decomposition exists on $L^q(\Omega_\mathrm{p},\mathbb{C}^d)$ if and only if the operator $N_q = (\nabla_{q'})^*\nabla_q$ is an isomorphism, where

$$\dot{W}^{1,q}(\Omega_\mathrm{p}) \xrightarrow{\nabla_q} L^q(\Omega_\mathrm{p}, \mathbb{C}^d) \cong (L^{q'}(\Omega_\mathrm{p}, \mathbb{C}^d))^* \xrightarrow{(\nabla_{q'})^*} (\dot{W}^{1,q'}(\Omega_\mathrm{p}))^*,$$

and $\dot{W}^{1,q}(\Omega) := \{u \in L^q_{loc}(\Omega) \mid \nabla u \in L^q(\Omega, \mathbb{C}^d)\}/\mathbb{C}$ is endowed with $||u|| := ||\nabla u||_q$. Furthermore, if the Helmholtz decomposition exists, the projection Q_q on the gradient part is given by $Q_q := \nabla_q N_q^{-1}(\nabla_q)^*$. Motivated by the fact that $\nabla + 2\pi\mathrm{i}\theta$ are the fibre operators associated with ∇, we prove that for $\theta \in B^d \setminus \{0\}$ the fibre operators for Q_2 are given by $Q(\theta) = (\nabla + 2\pi\mathrm{i}\theta)((\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta))^{-1}(\nabla + 2\pi\mathrm{i}\theta)^*$, where

$$W^{1,2}(\Omega_\#) \xrightarrow{\nabla + 2\pi\mathrm{i}\theta} L^2(\Omega_\#, \mathbb{C}^d) \xrightarrow{(\nabla + 2\pi\mathrm{i}\theta)^*} (W^{1,2}(\Omega_\#))^*.$$

The proof of this formula relies on the equality $L^q_\sigma(\Omega_\mathrm{p}) = \widehat{L}^q_\sigma(\Omega_\mathrm{p})$ and $G^q(\Omega_\mathrm{p}) = \widehat{G}^q(\Omega_\mathrm{p})$, since this allows us to restrict ourselves to the dense subspaces consisting of C_c^∞-functions. An essential observation in the study of the fibre operators is that the operators $\nabla + 2\pi\mathrm{i}\theta \colon W^{1,2}(\Omega_\#) \to L^2(\Omega_\#, \mathbb{C}^d)$ are injective for $\theta \in B^d \setminus \{0\}$. So $\theta = 0$ has a special role. In fact, this singularity in $\theta = 0$ prevents the usage of the condition (i) in the Bloch multiplier theorem stated above. On the other hand, the condition (ii) of the Bloch multiplier theorem requires an R-boundedness condition, and the verification is thus more involved. To overcome this problem, we split $Q(\theta)$ into a sum

$$Q(\theta) = Q_0(\theta) + T(\theta),$$

where $Q_0(\theta)$ is defined as $Q(\theta)$, but with $W^{1,2}(\Omega_\#)$ replaced by $W^{1,2}(\Omega_\#)_0 = \{u \in W^{1,2}(\Omega_\#) \mid \int_{\Omega_\#} u\,dx = 0\}$, and T is the rest term. The mapping $\theta \mapsto Q_0(\theta)$ is real analytic, even at $\theta = 0$. So, part (i) of the Bloch multiplier theorem from above applies for Q_0. A calculation shows that there is a concrete formula for the rest term T, which is an orthogonal projection onto a one-dimensional subspace. This can be used to prove that this part fulfils the assumption of the part (ii) in the Bloch multiplier theorem, which means that we can check the R-boundedness assumption here. Note that the space $W^{1,2}(\Omega_\#)$ is by one dimension larger than $\dot{W}^{1,2}(\Omega_\#)$, which is a heuristic explanation why the rest term

T has an one-dimensional range. Finally, the arguments prove that Q_2 has a continuous extension to an operator $Q_q \in \mathcal{L}(L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d))$.

It remains to prove that the operator Q_q on L^q obtained by continuous extension from L^2 coincides with the projection onto $G^q(\Omega)$ and $Id - Q_q$ defines the Helmholtz projection. This is easy to prove since $\widehat{G}^q(\Omega_{\mathrm{p}}) = G^q(\Omega_{\mathrm{p}})$ and $\widehat{L}^q_\sigma(\Omega_{\mathrm{p}}) = L^q_\sigma(\Omega_{\mathrm{p}})$. Otherwise, i.e. without the equality of these spaces, this would not be clear.

The Leray decomposition

Although we only focus on applications using the Helmholtz decomposition, we also consider another Helmholtz-type decomposition, the Leray decomposition. We recall that the Helmholtz decomposition is connected to the weak Neumann problem, whereas the Leray decomposition is related to the weak Neumann problem. We prove that the Leray decomposition exists on $L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$, where

$$\begin{aligned}
L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) &= L^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}}) \oplus G^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}}),\\
L^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}}) &= \overline{C^\infty_{c,\sigma}(\mathbb{R}^d)|_{\Omega_{\mathrm{p}}}}^{\|\cdot\|_q} = \{u \in L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) \mid \operatorname{div} u = 0\},\\
G^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}}) &= \overline{\nabla C^\infty_c(\Omega_{\mathrm{p}})}^{\|\cdot\|_q} = \nabla W^{1,q}_0(\Omega_{\mathrm{p}})\\
&= \{\nabla p \in L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) \mid p \in W^{1,q}_{\mathrm{loc}}(\Omega_{\mathrm{p}}),\ p|_{\partial\Omega_{\mathrm{p}}} = 0\},
\end{aligned}$$

and the range of q is depending on the regularity of $\partial\Omega_{\mathrm{p}}$ in the same manner as in the Helmholtz case. In the statement above we already used the equality of spaces $L^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}}) = \widehat{L}^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}})$ and $G^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}}) = \widehat{G}^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}})$, which is not true in general for unbounded domains, see the discussion in Section 3.5. Here, these equalities for periodic domains are due to a Poincaré inequality on Ω_{p}, which is applicable thanks to the Dirichlet boundary conditions. This proof requires much less technical tools in comparison to the same proof in the Helmholtz case.
The existence proof of the Leray decomposition on $L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$ follows the method of the existence proof in the Helmholtz case. In fact, this proof is also much easier, since in contrast to the weak Neumann problem no singularity for the fibre operators at $\theta = 0$ occurs. So we are not obliged to use version (ii) of the Bloch multiplier theorem, i.e. we do not have to prove an R-boundedness condition, which had been the most technical step in the Helmholtz case.

Scheme of the thesis

In the following we give a brief summary of this thesis.

Chapter 2 consists of a collection of well-known facts used in the chapters to follow. We start with the notation used in this work. Afterwards, we analyse semigroups. Both the Maxwell and the Stokes operator will be proven to generate analytic semigroups. Following this, we consider sesquilinear forms, which will be used to define the Maxwell operator. In Section 2.4 we consider differentiability and analyticity of Banach space valued maps. In Section 2.5 we regard R-boundedness. In Section 2.6 we take a brief look at the $\mathcal{H}^\infty$-calculus. At the end of the chapter 2, we summarize the main known theorems and open problems concerning the Helmholtz decomposition.

The Helmholtz decomposition on periodic domains is the content of **Chapter 3**. In Section 3.1 we prove $L^q_\sigma(\Omega_\mathrm{p}) = \widehat{L}^q_\sigma(\Omega_\mathrm{p})$ and $G^q(\Omega_\mathrm{p}) = \widehat{G}^q(\Omega_\mathrm{p})$ for all periodic Lipschitz domains and all $q \in (1, \infty)$. Subsequently, we summarize basics of the Bloch theory and give appropriate variants of the multiplier theorem from [Bar13]. In Section 3.3 we decompose the projection Q onto the gradient space $G^2(\Omega_\mathrm{p})$ under the Bloch transform into the fibre operators $Q(\theta)$ and prove a concrete representation for them. Finally, we are ready to prove the existence of the Helmholtz decomposition on L^q for the stated range of q on periodic domains in Section 3.4. To this end, we have to consider the fibre operators on L^q and check the assumptions of the Bloch multiplier theorem. At the end of the chapter, we consider another Helmholtz-type decomposition, the Leray decomposition, which is related to the weak Dirichlet problem, while the Helmholtz decomposition is related to the weak Neumann problem. We prove that an analogous result holds true for the Leray decomposition of L^q vector fields.

We examine applications in **Chapter 4** and start with considering photonic crystals and the Maxwell operator on $L^2(\Omega_\mathrm{p})$. In Section 4.3 we establish the Maxwell operator on L^q. Between these two sections, we prove Gaussian estimates, which we need to extend the Maxwell operator on L^q. In Section 4.4 we give properties of the L^q Stokes operator with no-slip boundary condition on periodic domains, where $q \neq 2$.

Chapter 5 is the physical appendix. There we present the Maxwell and the Navier-Stokes equations from a physical point of view. Our focus lies on the explanation of physical terms and the mathematical consequences for the equations.

CHAPTER 2

Preliminaries

2.1 Basic Notations

We introduce the following notations. We denote by C a positive constant which might change from line to line. Let $A \subseteq \mathbb{R}^d$. By $\overline{A}$ we denote the closure of A, and by $|A|$ we denote the volume, if A is measurable. Let $\mathbb{N}= \{1, 2 \dots, \}$ be the natural numbers, $\mathbb{N}_0= \mathbb{N} \cup \{0\}$, and let $\mathbb{Z}$ denote the integers. We denote the open and closed balls as follows:

$$
\begin{aligned}
B_d(x,r) &= \{y \in \mathbb{R}^d \mid |x-y|_2 < r\}, \\
\overline{B}_d(x,r) &= \{y \in \mathbb{R}^d \mid |x-y|_2 \leq r\}.
\end{aligned}
$$

Let $M \subseteq B$ be a subset of a Banach space B. By $\overline{M}^{\|\cdot\|_B}$ we denote the norm closure of M in B. Let B_1, B_2 be Banach spaces. We denote

$$
\mathcal{L}(B_1, B_2) := \{T \colon B_1 \to B_2 \mid T \text{ is linear and bounded}\}, \qquad \mathcal{L}(B_1) := \mathcal{L}(B_1, B_1).
$$

All frequently used symbols are listed at the end of the thesis. We tried to make things as self-contained as possible. For more details, further explanations and the proofs we refer to [Soh01, Chapter I and II], [Neč12] and [Tem77, Chapter I], where most of the following statements were taken from. A domain $\Omega \subseteq \mathbb{R}^d$ is an open, connected subset of $\mathbb{R}^d$. We start with the introduction of boundary regularity classes, compare [Soh01, Chapter I, section 3.2].

Definition 2.1 *(boundary regularity classes)*
We say that a domain $\Omega \subseteq \mathbb{R}^d$ has C^α-regularity, $\alpha \geq 0$, if for all $x \in \partial\Omega$, there is a coordinate transform $\psi \colon \mathbb{R}^d \to \mathbb{R}^d$, which is given by a concatenation of a rotation and a translation, such that $x = \psi(0, y_d)$ and that there are constants $r, \beta > 0$ and a C^α-regular function $\Phi_x \colon B_{d-1}(0,r) \to \mathbb{R}$ satisfying

$$
\begin{aligned}
\partial\Omega \cap U_x &= \{\psi(y_1, \dots, y_d) \mid y_d = \Phi_x(y_1, \dots, y_{d-1})\}, \\
\Omega \cap U_x &= \{\psi(y_1, \dots, y_d) \mid y_d > \Phi_x(y_1, \dots, y_{d-1})\},
\end{aligned}
$$

where $U_x := \psi(B_{d-1}(0,r) \times (-\beta + y_d, \beta + y_d))$ and $\partial\Omega$ denotes the boundary of Ω. Ω has

local Lipschitz boundary, if Φ_x can be chosen Lipschitz continuous for all $x \in \partial\Omega$. If Φ_x can be chosen $k-$times differentiable with Lipschitz continuous partial derivative of order k for all x, then Ω is said to be a $C^{k,1}$-domain.
Ω has a uniform C^k, respectively Lipschitz boundary, if the constants r, β can be chosen independent of x and if the C^k-norm, respectively Lipschitz-norm, of Φ_x can be estimated by a constant independent of x.

Sometimes one finds a weaker definition of Lipschitz domains, which uses bilipschitz maps. For the definition and examples that this class contains more domains, see [QHMS04, Definition 2.1, Example 2.2]. Therefore the last definition is sometimes called weakly Lipschitz and the definition above is called strongly Lipschitz. However, in our investigation at the beginning of Chapter 3, the following definition [AF03, chapter 4, (4.9)] for uniform Lipschitz domains will be used. It requires besides the uniformity of the Lipschitz constant M and of the locally minimally covered size constant δ, the existence of a so called overlap number R which states that nowhere more than R sets of the covering are intersecting.

Definition 2.2 *(strongly local Lipschitz boundary)*
A domain $\Omega \subseteq \mathbb{R}^d$ is called strongly local Lipschitz, if there are constants $\delta, M > 0$ such that there exists a locally finite open covering $\{U_j\}$ of $\partial\Omega$, and, for each j there is a function $f_j = f_j(x_1, \ldots, x_{d-1})$ with the following properties:

a) There is a $R \in \mathbb{N}$ such that every collections of $R+1$ of the sets U_j has empty intersection.

b) For all $x, y \in \Omega_\delta = \{z \in \Omega \mid dist(z, \partial\Omega) < \delta\}$ with $|x - y| < \delta$ there is a j with

$$x, y \in V_j := \{z \in U_j \mid dist(z, \partial U_j) > \delta\}.$$

c) Every function f_j is Lipschitz continuous with constant M, which means

$$|f_j(x_1, \ldots, x_{d-1}) - f_j(y_1, \ldots, y_{d-1})| \leq M|(x_1 - y_1, \ldots, x_{d-1} - y_{d-1})|.$$

d) There is a Cartesian coordinates system $x_{j,1}, \ldots, x_{j,d}$ in U_j such that

$$\Omega \cap U_j = \{x_{j,d} \mid x_{j,d} < f_j(x_{j,1}, \ldots, x_{j,d-1})\}.$$

This definition is also called minimally smooth boundary condition [Ste70, chapter 6, section 3.3]. Note that Definition 2.1 and Definition 2.2 coincide for bounded domains [AF03]. There, one finds a detailed description of the theory. We remark that our attention in this thesis lies on periodic domains, and there the two definitions do also coincide because of the periodic structure of these domains.

Lemma 2.3 *(existence of the exterior unit vector)*
If Ω has local Lipschitz boundary, the exterior unit vector ν exists for almost all $x \in \partial\Omega$.

Proof: See [Soh01, Chapter I, Section 3.4] □

For the sake of completeness and to facilitate the overview, we give some basics about Lebesgue spaces, Sobolev spaces and dual spaces.

Definition and Remark 2.4 *(Lebesgue and Sobolev spaces)*
Let $(\Omega, \mathbb{A}, \mu)$ be a measure space, $q \in [1, \infty)$ and $K = \mathbb{C}^d$ or $K = \mathbb{R}^d$. After identifying functions, which coincide almost everywhere, we get the L^q-spaces

$$L^q(\Omega, K) = \{[f\colon \Omega \to K] \mid f \text{ is measurable and } \int\limits_{\Omega} |f(x)|^q dx < \infty\},$$

endowed with the norm $||f|| := (\int_\Omega |f(x)|^q dx)^{1/q}$. In most cases we consider $\Omega \subseteq \mathbb{R}^d$ measurable, as $\mathbb{A}$ the Borel σ-algebra, and μ is the Lebesgue measure. In that case we define

$$L^q_{loc}(\Omega) := \{[f\colon \Omega \to K] \mid f \text{ is measurable, } f|_{\Omega'} \in L^q(\Omega') \text{ for all compact } \Omega' \text{ with } \Omega' \subseteq \Omega\}.$$

The dual space of a complex Banach space E is defined by $E' := \mathcal{L}(E, \mathbb{C})$. Furthermore the anti-dual space E^* of E consists of all continuous antilinear forms, i.e. $\varphi(\alpha f) = \overline{\alpha}\varphi(f)$ for all $\alpha \in \mathbb{C}$, $f \in E$. For all $q \in [1, \infty)$ we have $L^q(\Omega, K)' \cong L^{q'}(\Omega, K)$, where the canonical isomorphism is given by

$$L^{q'}(\Omega, K) \to L^q(\Omega, K)', \qquad g \mapsto [f \mapsto \int_\Omega fg\, dx].$$

For $\Omega \subseteq \mathbb{R}^d$ open, $f \in L^1_{\text{loc}}(\Omega)$ a function $g \in L^1_{\text{loc}}(\Omega)$ is called the weak j-th partial derivative of f if

$$\int\limits_{\Omega} f\partial_j\varphi dx = \int\limits_{\Omega} g\varphi dx, \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We define weak derivatives of higher order iteratively. For $\Omega \subseteq \mathbb{R}^d$ open, $1 \leq q < \infty$, $k \in \mathbb{N}$, we consider the Sobolev spaces

$$W^{k,q}(\Omega) = \{f \in L^q(\Omega) \mid \text{ all weak derivatives of } f \text{ of order } \leq k \text{ belong to } L^q(\Omega)\},$$

which are complete for the norm $||f|| = (\sum_{|\alpha| \leq k} |||\partial^\alpha f||^q_{L^q(\Omega)})^{1/q}$. We further define

$$W^{k,q}_0(\Omega) := \overline{C_c^\infty(\Omega)}^{||\cdot||_{W^{k,q}(\Omega)}} \subseteq W^{k,q}(\Omega),$$
$$W^{k,q}_{loc}(\Omega) := \{f \in L^q_{loc}(\Omega) \mid D^l f \in L^q_{loc}(\Omega) \text{ for all } 0 \leq |l| \leq m\}.$$

It is also possible to define Sobolev spaces of negative index, namely

$$W^{-k,q}(\Omega) = (W^{k,q'}_0(\Omega))' = \big(\overline{C_c^\infty(\Omega)}^{||\cdot||_{W^{k,q'}(\Omega)}}\big)'.$$

Next, we define the surface integral. We follow the approach in [Soh01, Chapter I, section 3.4]. Let Ω be a bounded Lipschitz domain. Since $\partial\Omega$ is compact, we find by definition a finite open covering $U_1, \ldots, U_m$ of $\partial\Omega$ and functions ψ_j, Φ_j as in Definition 2.1. Furthermore, there is a decomposition of unity on $\partial\Omega$, given by functions $\varphi_j \in C_c^\infty(U_j)$ satisfying

$$0 \leq \varphi_j \leq 1, \qquad \sum_{j=1}^m \varphi_j(x) = 1 \text{ for all } x \in \partial\Omega.$$

Now, we are ready to define the surface integral $\int_{\partial\Omega} u\, dS$ and function spaces on $\partial\Omega$.

Definition 2.5 *(boundary function spaces)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. A function $u\colon \partial\Omega \to \mathbb{C}$ is called measurable/integrable if

$$y \mapsto u(\psi_j(y, \Phi_j(y)))\varphi_j(\psi_j(y, \Phi_j(y)))(1 + |\nabla\Phi_j(y)|^2)^{1/2}$$

is measurable/integrable for all $j = 1, \dots, m$. We define $dS := (1 + |\nabla\Phi_j(y)|^2)^{1/2}dy$, which yields to the definition of the surface integral

$$\int_{\partial\Omega} u\, dS := \sum_{j=1}^{m} \int_{\partial\Omega\cap U_j} u\varphi_j\, dS.$$

This definition is well-defined and we define the spaces $L^q(\partial\Omega)$ consisting of equivalence classes of measurable functions, whose q-th power is integrable. We define for $\beta \in (0,1)$, $q \in (1,\infty)$ the Besov spaces

$$\begin{aligned} B^q_\beta(\partial\Omega) &:= \{u \in L^q(\partial\Omega) \mid ||u||_{B^{\beta,q}(\partial\Omega)} < \infty\}, \\ ||u||_{B^q_\beta(\partial\Omega)} &= \Big(||u||^q_{L^q(\partial\Omega)} + \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{d-1+\beta q}} dS_x dS_y\Big)^{1/q}. \end{aligned}$$

These spaces and their duals $B^q_{-\beta}(\partial\Omega) = (B^{q'}_\beta(\partial\Omega))'$ are important tools in [FMM98], where the solvability of the weak Neumann problem is proven for bounded Lipschitz domains on an interval of the form $(\frac{3+\varepsilon}{2+\varepsilon}, 3 + \varepsilon)$.

Lemma 2.6 *(trace operator)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $q \in (1,\infty)$.

(a) On bounded Lipschitz domains exists the trace operator

$$\Gamma_{\text{trace}}\colon W^{1,q}(\Omega) \to B^q_{1-1/q}(\partial\Omega),$$

which defines generalized boundary values. In particular, $\Gamma_{\text{trace}}u = u|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$. Furthermore, Γ_{trace} is bounded and surjective.

(b) There is a bounded, linear extension operator $\Gamma_{\text{ext}}\colon B^q_{1-1/q}(\partial\Omega) \to W^{1,q}(\Omega)$ satisfying $\Gamma_{\text{trace}}\Gamma_{\text{ext}} = Id$ on $B^q_{1-1/q}(\partial\Omega)$.

(c) $\text{kernel}\,(\Gamma_{\text{trace}}) = W^{1,q}_0(\Omega)$.

Proof: See [Neč12, Chapter II, Theorem 5.7 and Chapter II, Theorem 4.10]. □

Next, we introduce the Poincaré inequality, which will be used frequently in Section 3.1. The proof can be found in most of the standard analysis books. Nevertheless, we present the proof here.

Lemma 2.7 *(Poincaré inequality)*
Let Ω be a bounded Lipschitz domain and $q \in [1, \infty)$. We consider the following classes:

- $K_1 = \{u \in W^{1,q}(\Omega) \mid \int_\Omega u \, dx = 0\}$.
- $K_2 = \{u \in W^{1,q}(\Omega) \mid u|_M = 0\}$, *where $M \subseteq \Omega$ has positive measure.*
- $K_3 = \{u \in W^{1,q}(\Omega) \mid \Gamma_{\text{trace}} u|_N = 0\}$, *where $N \subseteq \partial\Omega$ is a non-empty relatively open subset.*

On all these classes, the Poincaré inequality applies, i.e. there are constants $C_i > 0$ satisfying

$$||u||_{L^q(\Omega)} \leq C_i ||\nabla u||_{L^q(\Omega, \mathbb{C}^d)}, \qquad u \in K_i.$$

In particular,

$$||u - m(u)||_{L^q(\Omega)} \leq C_1 ||\nabla u||_{L^q(\Omega, \mathbb{C}^d)}, \qquad u \in W^{1,q}(\Omega),$$

where $m(u)$ denotes the mean of u.

Proof: We suppose such a constant does not exist. Then, there is a sequence $(w_n)_{n \in \mathbb{N}}$ with the demanded properties such that

$$||w_n||_q = 1, \qquad ||\nabla w_n||_q \leq 1/n.$$

By the Rellich-Kondrachov theorem [Neč12, Chapter 2, Theorem 6.3] the embedding $W^{1,q} \hookrightarrow L^q$ is compact, and since the sequence is bounded in $W^{1,q}(\Omega)$, there is a subsequence $\{w_{j_k}\}_{k \in \mathbb{N}}$, $w \in W^{1,q}(\Omega)$ such that $w_{j_k} \to w$ in $L^q(\Omega)$ and $\nabla w_{j_k} \to \nabla w$ weakly in $L^q(\Omega)$. This yields

$$||w||_q = \lim_{k \to \infty} ||w_{j_k}||_q = 1, \qquad ||\nabla w||_q \ \leq \lim_{k \to \infty} \inf ||\nabla w_{j_k}||_q = 0.$$

Hence, $\nabla w = 0$ and w is constant, since Ω is connected. By assumption, only $w = 0$ is possible, but that contradicts $||w||_q = 1$. $\square$

Next, we give a general formulation of the integration by part/Green's formula. For this purpose, we first define spaces of functions having integrable divergence/curl. On these spaces, we are able to define normal/tangential components for functions at the boundary by extending the integration by part/Greens' formula.

Definition and Remark 2.8 *(spaces of div and curl)*
Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $q \in (1, \infty)$. We define the following spaces:

$$\begin{aligned} \mathrm{Div}_q(\Omega) &= \{v \in L^q(\Omega, \mathbb{C}^d) \mid \mathrm{div}\, u \in L^q(\Omega)\}, \\ \mathrm{Curl}_q(\Omega) &= \{v \in L^q(\Omega, \mathbb{C}^d) \mid \mathrm{curl}\, u \in L^q(\Omega, \mathbb{C}^d)\}, \end{aligned}$$

Here, div and curl are defined in distributional sense. In the case $q = 2$ these spaces are usually denoted by $H(\mathrm{div}\,, \Omega)$ and $H(\mathrm{curl}\,, \Omega)$. The norms on the Banach spaces Div_q and Curl_q are defined by

$$||v||_{\mathrm{Div}_q(\Omega)} = \left(||v||_q^q + ||\mathrm{div}\ v||_q^q\right)^{1/q},$$
$$||v||_{\mathrm{Curl}_q(\Omega)} = \left(||v||_q^q + ||\mathrm{curl}\ v||_q^q\right)^{1/q}.$$

Furthermore, it is worth to mention that $C^\infty(\overline{\Omega})$ is dense in these spaces if $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain, cf. [FM77] and [Tem77, Chapter I, Theorem 1.1].

Now, we are in the position to consider generalized traces, which are defined as follows.

Lemma 2.9 *(normal component)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $q \in (1, \infty)$. There is a bounded, linear operator

$$\Gamma_N \colon \mathrm{Div}_q(\Omega) \to B^q_{-1/q}(\partial\Omega) := \left(B^{q'}_{1/q}(\partial\Omega)\right)',$$

which is given as the continuous extension of the mapping $\Gamma_N \colon u \mapsto u \cdot \nu$ from the dense subspace $C_c^\infty(\overline{\Omega})$ to Div_q. The operator is called the generalized trace operator.

Proof: See [SS92, Theorem 5.3]. □

The density results from Definition and Remark 2.8 yield the following general formulation of the integration by parts formula.

Lemma 2.10 *(integration by parts)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $d \geq 2$, $q \in (1, \infty)$, $u \in W^{1,q}(\Omega)$, $v \in \mathrm{Div}_{q'}(\Omega)$. Then we have

$$\langle u, \mathrm{div}\ v\rangle_\Omega = \langle u, v \cdot \nu\rangle_{\partial\Omega} - \langle \nabla u, v\rangle_\Omega.$$

Note that $\langle u, v \cdot \nu\rangle_{\partial\Omega}$ is well-defined since

$$\Gamma_N v \in B^{q'}_{-1/q'}(\partial\Omega), \qquad u|_{\partial\Omega} \in B^q_{1-1/q}(\partial\Omega).$$

Proof: See [Soh01, Chapter II, Lemma 1.2.3]. □

In the three dimensional case an analogous approach yields the existence of the tangential component and an integral formula for the operator curl, which we summarize in the next lemma, compare e.g. [GR12, Chapter I, Theorem 2.11]. To distinguish between the two appearing integral formulas, we call the equation below Green's formula and the formula from Lemma 2.10 integration by parts.

Lemma 2.11 *(tangential component and Green's formula)*
Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Then, the mapping $\Gamma_T \colon u \mapsto u \times \nu|_{\partial\Omega}$ defined on $C_c^\infty(\overline{\Omega}, \mathbb{C}^3)$ can be extended to a bounded linear operator, still denoted by Γ_T, where

$$\Gamma_T \colon \mathrm{Curl}_q(\Omega) \to B^q_{-1/q}(\partial\Omega).$$

Furthermore, we have the following Green's formula

$$\langle \mathrm{curl}\ u, v\rangle - \langle u, \mathrm{curl}\ v\rangle = \langle \Gamma_T u, v\rangle_{\partial\Omega}, \qquad \textit{for all } u \in \mathrm{Div}_q(\Omega),\ v \in W^{1,q'}(\Omega).$$

We arrive at the definition of a periodic domain, which is of greatest importance in this work.

Definition 2.12 *(periodic domains)*
A lattice $L \subseteq \mathbb{R}^d$ is a discrete subgroup of $(\mathbb{R}^d, +)$ with rang $L = d$. In other words,

$$L = \{\sum_{j=1}^{d} z_j b_j \mid z_j \in \mathbb{Z}\},$$

where $\{b_1, \ldots, b_d\}$ form a basis of $\mathbb{R}^d$. A domain $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ is called periodic with respect to a given lattice L, if $\Omega_\mathrm{p} = \Omega_\mathrm{p} + p$ for all $p \in L$. If $L = \mathbb{Z}^d$, we call Ω_p a $\mathbb{Z}^d$-periodic domain or just a periodic domain. A periodic domain is characterized by one periodicity cell

$$\Omega_0 := \Omega \cap [0,1]^d.$$

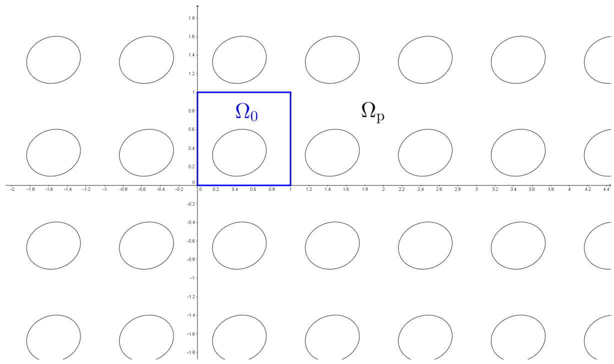


Figure 2.1: *A two dimensional ($\mathbb{Z}^d$-)periodic domain.*

Remark 2.13 (reduction to $\mathbb{Z}^d$-periodic domains)
Let Ω_p be a periodic domain with respect to a given lattice L and $B = \{b_1, \ldots, b_d\}$ be a basis of L. We denote by M the (invertible) matrix mapping B to the standard basis $\{e_1, \ldots, \mathrm{e}_d\}$. Clearly, the transformation is C^∞, and invertible with inverse map M^{-1}. In particular, the transformation is bilipschitz and the results in Section 3.1 remains untouched under this transformation, cf. Remark 3.1. Also, the Bloch theory can be adapted to an arbitrary periodicity cell, cf. Remark 3.36. Therefore we restrict ourselves to the case of $\mathbb{Z}^d$-periodic domains. From now on, a periodic domain is defined as a $\mathbb{Z}^d$-periodic domain and is denoted by Ω_p.

Definition and Remark 2.14 *(definition of $\Omega_\#$)*
If $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ is a periodic domain, we define the manifold $\Omega_\# := \Omega_\mathrm{p}/\mathbb{Z}^d$. So, $\Omega_\#$ is given by Ω_0, where the opposite edges are identified. Note that $\Omega_\#$ is a flat manifold with boundary.

Note that $L^q(\Omega_0, \mathbb{C}^d) = L^q(\Omega_\#, \mathbb{C}^d)$, but $W^{1,q}(\Omega_\#)$ is much smaller than $W^{1,q}(\Omega_0)$, since functions from $W^{1,q}(\Omega_0)$ are only in $W^{1,q}(\Omega_\#)$ if they fulfil $\mathbb{Z}^d$-periodic boundary conditions.

2.2 Semigroups

Let X be a Banach space. We follow [EN00, Chapter I, §5 and Chapter II, §1]. Alternatively, the basic theory of semigroups can be found in [Lun95], too.

Definition 2.15 *(semigroups)*
A C_0-semigroup or strongly continuous semigroup is a map $T\colon [0,\infty) \to \mathcal{L}(X)$ having the following properties:

a) For all $x \in X$ the map $[0,\infty) \to X,\ t \mapsto (T(t))(x)$ is continuous.

b) We have $T(t+s) = T(t)T(s)$ for all $t, s \in [0,\infty)$.

c) We have $T(0) = Id$.

Since all semigroups appearing in this work are strongly continuous we spare to mention the C_0-property of the semigroup. So, by just writing semigroup we actually denote a strongly continuous semigroup.

Definition and Remark 2.16 *(generator of a semigroup)*
Let T be a semigroup. The generator A of the semigroup is defined by

$$Ax = \lim_{h \to 0^+} \frac{1}{h}(T(h)x - x),$$
$$D(A) = \{x \in X \mid \lim_{h \to 0^+} \frac{1}{h}(T(h)x - x) \text{ exists}\}.$$

It follows that every generator of a semigroup is densely defined and closed.

Let us give the most basic results on this subject [Sch12b, chapter I], [EN00, Chapter II]. The theory of semigroups was developed in the late 1940' and early 1950' among others by Feller, Hille, Miyadera, Phillips and Yosida.

Proposition 2.17 *(basics on semigroups and their generators)*
Let $T(\cdot)$ be a semigroup. Then, there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||T(t)|| \leq M e^{\omega t}, \qquad t \geq 0.$$

The infimum over all possible choices ω is called the growth bound of T and denoted by $\omega_0(T)$ or $\omega_0(A)$. A linear operator generates a semigroup $T(\cdot)$ with growth constants M, ω if and only if A is closed, densely defined, $(\omega, \infty) \subseteq \varrho(A)$ and

$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda - \omega)^n}, \qquad \text{for all } \lambda > \omega,\ n \in \mathbb{N}.$$

If $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0(T)$, then $\lambda \in \varrho(A)$ and

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt.$$

Besides, semigroups have the following connection to the well-posedness of the Cauchy problem.

Proposition 2.18 *(Cauchy problem)*
A closed operator on a Banach space X generates a semigroup if and only if the Cauchy problem

$$u'(t) = Au(t), \quad t > 0,$$
$$u(0) = u_0,$$

is well-posed, i.e. $D(A)$ is dense in X, for all $u_0 \in D(A)$ the problem is uniquely solvable, the solution $u(t, u_n)$ converges to $u(t, u_0)$ uniformly for t in compact subsets of $\mathbb{R}$, whenever $u_n, u_0 \in D(A)$ and $u_n \to u$ in X. Furthermore, in that case, the solution is given by $u = T(\cdot)u_0$.

Example 2.19 (holomorphic functional calculus)
Let $A \in \mathcal{L}(X)$, hence $\sigma(A)$ is compact. We choose a smooth path Γ around the spectrum with winding number equal to one like in the following picture.

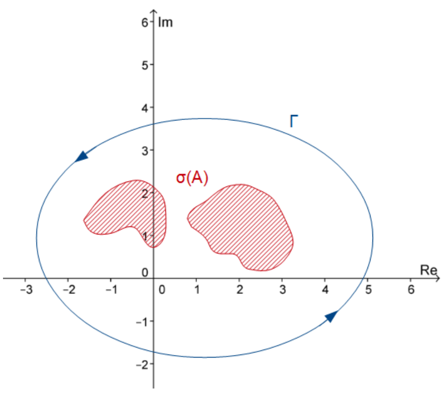


Figure 2.2: *The path Γ encloses the spectrum of A*

By using the holomorphic functional calculus, we define the family of operators $T(t) \in \mathcal{L}(X)$ by

$$T(t) := \mathrm{e}^{tA} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{\lambda t} R(\lambda, A) d\lambda.$$

By the well known results for the holomorphic functional calculus, it is not hard to prove that T is a uniformly continuous semigroup with generator A, i.e. $t \mapsto T(t) \in \mathcal{L}(X)$ is continuous with respect to the uniform operator topology cf. [EN00, Chapter I, Prop 3.5].

There is a family of closed operators, for which the approach with the contour integral works. We consider an operator A having $\sigma(A)$ lying in some sector and examine the following path.
We choose arbitrary $\varphi' \in (\varphi, \pi)$, $r > 0$ and define the path $\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$, where

$$\Gamma_1 := \{-t\mathrm{e}^{\mathrm{i}\varphi'} \mid t \in (-\infty, -r]\},$$
$$\Gamma_2 := \{r\mathrm{e}^{-\mathrm{i}t} \mid t \in [-\varphi', \varphi']\},$$

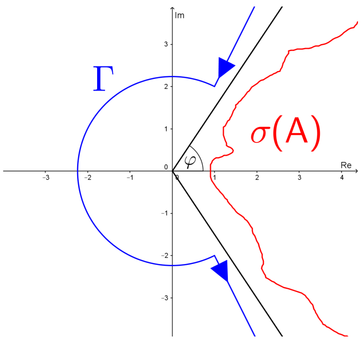


Figure 2.3: *Γ is orientated counterclockwise. The spectrum is contained in the sector Σ_φ lying in the right half-plane, and since zero might be in the spectrum one has to steer clear of zero.*

$$\Gamma_3 := \{te^{-i\varphi'} \mid t \in [r, \infty)\}.$$

Assuming a resolvent estimate, the path integral from Example 2.19 becomes convergent, compare Proposition 2.21 below. This property allows to extend semigroups on certain sectors compare [Sch12b, Definition 2.11] or [EN00, Chapter II, 4.1 Definition, 4.5 Definition]. We remark that these authors use a mirrored version for sectorial operators by considering $-A$ instead of A, in particular they consider the supremum instead of the infimum as angle and the value of the angle $w(A)$ changes to $\pi - w(A)$.

Definition 2.20 *(sectorial operators and analytic semigroups)*
A closed operator A is called sectorial if there is a $\varphi \in (0, \pi)$ such that $\sigma(A) \subseteq \overline{\Sigma_\varphi}$, where

$$\Sigma_\varphi := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg(\lambda)| < \varphi\},$$

and

$$||R(\lambda, A)|| \leq \frac{C}{|\lambda|},$$

for all $\varphi' \in (\varphi, \pi]$, $0 \neq \lambda$ with $|arg(\lambda)| > \varphi'$, where $C = C(\varphi')$. The infimum of all such φ is called the angle of A and is denoted by $w(A)$.
An analytic semigroup of angle $\theta \in (0, \pi]$ is a family of operators $\{T(z) \mid z \in \{0\} \cup \Sigma_\theta\}$ satisfying

- *T is a semigroup and the semigroup law extends to Σ_θ.*
- *$T\colon \Sigma_\theta \to \mathcal{L}(X)$ is analytic.*
- *$T(z)x \to x$ in X as $z \to 0$ in Σ_Θ for all $0 < \Theta < \theta$.*

If, in addition $\sup_{z \in \Sigma_\Theta} ||T(z)|| < \infty$ for all $0 < \Theta < \theta$, T is called a bounded analytic semigroup of angle θ.

Next, we study equivalent descriptions of analyticity and give helpful properties using analyticity.

Proposition 2.21 *(characterizations of analytic semigroups)*
Let A be a closed operator on a Banach space X. The following are equivalent:

- *A generates a bounded analytic semigroup of angle $\theta \in (0, \pi/2]$.*

- *A is densely defined and $-A$ is sectorial of angle $w(-A) < \pi/2$.*
- *A is densely defined, $\{\lambda \in \mathbb{C} \mid Re(\lambda) > 0\} =: \mathbb{C}_+ \subseteq \varrho(A)$ and*

$$||R(\lambda, A)|| \leq \frac{C}{Re\ \lambda}, \qquad ||R(\lambda, A)|| \leq \frac{C}{|Im\ \lambda|},$$

  for all $\lambda \in \mathbb{C}_+$ and a constant $C > 0$.
- *There is a $\Theta \in (0, \pi/2)$ such that the operators $\mathrm{e}^{\pm \mathrm{i}\Theta}A$ generate bounded semigroups.*
- *A generates a semigroup T such that $T(t)X \subseteq D(A)$ for all $t > 0$ and*

$$||AT(t)|| \leq \frac{M_1}{t},$$

  where $M_1 > 0$ is a constant.
  Furthermore, $T \in C^1((0,\infty), \mathcal{L}(X))$ and $\frac{d}{dt}T(t) = AT(t)$ for all $t > 0$.

Proof: See [Sch12b, Theorem 2.12] or [EN00, Chapter II, 4.6 Theorem]. □

Proposition 2.22 *(representation of the semigroup by using the Laplace transform)*
Let A be the generator of a bounded analytic semigroup of angle $\theta \in (0, \pi/2]$ and Γ be the mirrored (at the y-axis) path of the path in Figure 2.3. The semigroup T is given by

$$T(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{\lambda t} R(\lambda, A) d\lambda, \qquad t > 0.$$

Here, the integral has to be understood as a Bochner integral, which is defined by approximating with simple functions. Hence, it is the generalisation of the Lebesgue integral, compare [AE08, chapter X] for the details.
We have $T(t)X \subseteq D(A^n)$ and even $T \in C^\infty((0,\infty), \mathcal{L}(X))$, where $\frac{d^n}{dt^n}T(t) = A^nT(t)$, $||A^nT(t)|| \leq \frac{M_n}{t^n}$ for all $n \in \mathbb{N}$ and constants M_n depending on C and n.

Proof: See [Sch12b, Theorem 2.12] or [EN00, Chapter II, §4]. □

At the end of the section we just want to mention that the smoothing property mentioned in Proposition 2.22 is useful in applications. It yields better regularity of the solution of the homogeneous and inhomogeneous Cauchy problem. Besides, analytic semigroups are known to behave better in perturbation theory. For more details concerning semigroups we refer to standard literature concerning this topic like [EN00].

2.3 Sesquilinear forms

At the beginning of this section we have to warn the reader. The notation of the following terms is not consistent in the literature. So, other authors may, for example call coercive, what we call shifted ellipticity condition. For more details concerning forms, see [Sch12a, chapter 11].

Definition 2.23 *(terms used for sesquilinear forms)*
We consider a sesquilinear form $a(u, v)$, defined on $V \overset{i}{\hookrightarrow} H$, where V, H are complex Hilbert spaces and the embedding has to be continuous and dense.

(a) a is called continuous if there is a constant $C > 0$ satisfying

$$|a(u, v)| \leq C||u||_V ||v||_V, \qquad \text{for all } u, v \in V.$$

(b) a is called coercive if there is a constant $C > 0$ satisfying

$$|a(u, u)| \geq C||u||_V^2, \qquad \text{for all } u \in V.$$

(c) a is called elliptic if there is a constant $C > 0$ satisfying

$$\mathit{Re}\ a(u, u) \geq C||u||_V^2, \qquad \text{for all } u \in V.$$

(d) a fulfils the shifted ellipticity condition if there are constants $C_1, C_2 \geq 0$ such that

$$\mathit{Re}\ a(u, u) + C_1||u||_H^2 \geq C_2||u||_V^2, \qquad \text{for all } u \in V.$$

(e) a is called symmetric, if

$$a(u, v) = \overline{a(v, u)}, \qquad \text{for all } u, v \in V.$$

To avoid confusion the notation V-continuous, V-coercive etc. is common.

Obviously, every elliptic sesquilinear form is coercive. If $V \overset{i}{\hookrightarrow} H$ is continuous and dense, then the map

$$j\colon H \hookrightarrow V^*, \qquad h \mapsto [v \mapsto \langle h, v\rangle_H]$$

is continuous, injective, linear and the image is dense in V^*. So we have $V \overset{i}{\hookrightarrow} H \overset{j}{\hookrightarrow} V^*$, which is the so-called Gelfand triple. By using the isomorphism φ between H and H^* this can be written as

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*.$$

By using the Riesz representation theorem it is quite easy to define an operator associated with the form.

Definition and Lemma 2.24 *(operator associated to a sesquilinear form)*
Let $V \overset{i}{\hookrightarrow} H \overset{j}{\hookrightarrow} V^*$ be a Gelfand triple, where V, H are complex Hilbert spaces. Besides, let $a(\cdot, \cdot)$ be a V-continuous sesquilinear form. We define the operator A associated with the form by

$$u \in D(A),\ Au = f \Leftrightarrow u \in V,\ a(u, v) = \langle f, v\rangle_H,\ \text{for all } v \in V.$$

Then, we have

$$\begin{aligned} D(A) &= \{u \in V \mid \text{ there is a } h \in H \text{ such that } a(u,v) = \langle h, v\rangle, \text{ for all } v \in V\} \\ &= \{u \in V \mid \text{ there is a } C_u \geq 0 \text{ so that } |a(u,v)| \leq C_u ||v||_H, \text{ for all } v \in V\}. \end{aligned}$$

The operator is densely defined on H if $a(\cdot,\cdot)$ is V-coercive or if $a(\cdot,\cdot)$ fulfils the shifted ellipticity condition. In the latter case $-A$ generates an analytic semigroup. If $a(\cdot,\cdot)$ is even V-elliptic, the analytic semigroup generated by $-A$ is contractive. If in addition $a(\cdot,\cdot)$ is symmetric, then A is self-adjoint.

Sketch of proof: By considering the adjoint sesquilinear form and using Lax-Milgram it follows that A is densely defined [Ban10, Chapter 11, Theorem 3]. Next, we prove that $-A$ generates a contractive semigroup if a is V-elliptic. For this purpose, we use the Lumer-Philips theorem [EN00, Theorem 3.15]. So, we have to check that $-A$ is dissipative and $\lambda Id + A$ is surjective for some $\lambda > 0$. $-A$ is dissipative since

$$\text{Re } \langle -Ax, x\rangle = -\text{Re } a(x,x) \leq -C||x||_V^2 \leq 0, \qquad \text{for all } x \in D(A).$$

By using a value $\lambda \in (0, C)$, the Lax-Milgram lemma for coercive sesquilinear forms yields the second part. One gets the analyticity of the semigroup by showing resolvents estimates of the form $||R(\lambda, A)|| \leq \frac{C}{|\text{Im}(\lambda)|}$ for some $C > 0$. This technical calculation involves the consideration of an operator $\widehat{A}\colon V \to V^*$ as operator on V^*, compare [Ban10, chapter 13, Theorem 7], [IK02, chapter 3, 3.6 Theorem]. If a fulfils the shifted ellipticity condition with constants C_1, C_2 we consider the form

$$\widetilde{a}(u,v) := a(u,v) + C_1\langle u, v\rangle_H.$$

This is obviously elliptic and the operator associated with the form is given by $\widetilde{A} := A + C_1 Id$ and $D(\widetilde{A}) = D(A)$. Hence, $-(A + C_1)$ generates an analytic contraction semigroup, in particular $-A$ generates an analytic semigroup. Let a be in addition symmetric. One can prove that A^* is the operator associated with the dual form a^* of a, and hence $A = A^*$. □

Lemma 2.25 *(shifted coercivity implies discrete spectrum if $V \hookrightarrow H$ is compact)*
Let $a(\cdot,\cdot)$ be a continuous symmetric densely defined sesquilinear form which fulfils the shifted coercivity condition, i.e.

$$|a(u,u)| + ||u||_H^2 \geq C||u||_V^2, \qquad \text{for all } u \in V.$$

Furthermore, let the embedding of $V \hookrightarrow H$ be compact. Then, the spectrum of the associated operator A is discrete, consisting only of eigenvalues. The statement is true for shifted ellipticity instead of shifted coercivity, too.

Proof: By shifting, we can restrict to the case of an coercive sesquilinear form, because the spectrum shifts, too. So, [DL00, chapter II, Theorem 7], which uses Fredholm-theory, yields the desired properties. □

2.4 Differentiable and analytical maps

In this section we consider differentiability and analyticity of maps from a domain $U \subseteq \mathbb{R}^d$ into an arbitrary Banach space E. Note that there are different approaches to differentiability. We will use the Fréchet differentiability, since it occurs in Weis' generalization of Mikhlin's multiplier theorem and in related theorems such as the Bloch multiplier theorem, which we examine in Section 3.2.

Definition 2.26 *(Fréchet differentiability)*
Let $U \subseteq \mathbb{R}^d$ be open and E be an arbitrary Banach space. A map $f\colon U \to E$ is called Fréchet differentiable at $x_0 \in U$ if there is a bounded, linear map $\Psi\colon \mathbb{R}^d \to E$ satisfying

$$\lim_{h\to 0} \frac{||f(x_0+h) - f(x_0) - \Psi h||}{|h|} = 0.$$

If the derivative exists, we denote it by $f'(x_0) := \Psi$. If the derivative exists for all $u \in U$ the map f is called differentiable. If furthermore,

$$f'\colon U \to \mathcal{L}(\mathbb{R}^d, E), \qquad x_0 \mapsto f'(x_0)$$

is continuous, then f is called continuously differentiable. In that case we use the notation $f \in C^1(U,E)$. Since $\mathcal{L}(\mathbb{R}^d, E)$ is a Banach space again, we can define $C^2(U,E)$ as the set of C^1-functions, whose derivative is C^1 again. Inductively, we define $C^k(U,E)$ for arbitrary $k \in \mathbb{N}$. f is called smooth, if $f \in C^\infty(U,E)$.

We start with the definition of analyticity of a family of operators.

Definition 2.27 *(analyticity of operators)*
Let B_1, B_2 be complex Banach spaces and $Q\colon \mathbb{C}^d \supseteq D \to \mathcal{L}(B_1,B_2)$, $\theta \mapsto Q(\theta)$, where D is a complex domain. The family of operators $Q(\theta)$ is called analytic, if the map

$$\theta \mapsto g(Q(\theta)f)$$

is complex analytic for all $f \in B_1$ and $g \in B_2'$. This is equivalent [Bar13, Corollary 5.6] to the following: The map Q is analytic at all $\theta_0 \in D$, which means that for fixed θ_0 there is a family of operators Q_α, $\alpha \in \mathbb{N}_0^d$ and a neighbourhood U_{θ_0} satisfying

$$Q(\theta) = \sum_{\alpha \in \mathbb{N}_0^d} (\theta - \theta_0)^\alpha Q_\alpha, \qquad \theta \in U_{\theta_0},$$

and the series converges absolutely.
A function $T\colon \mathbb{R}^d \supseteq U \to \mathcal{L}(B_1,B_2)$ is called real analytic at x, if the power series representation above is true locally on a neighbourhood $O_x \subseteq \mathbb{R}^d$ of x. These are exactly those functions having locally an extension to analytic functions.

We give a very simple example for an analytic function, which becomes important when we analyse the fibre operators of the Helmholtz projection later. After that, we state some basic facts about analyticity, which will be needed in the next chapter, too.

Example 2.28 (analyticity of $\nabla + 2\pi \mathrm{i}\theta$)
Let Ω_{p} be a $\mathbb{Z}^d$ periodic domain and $\Omega_{\#} = \Omega_{\mathrm{p}}/\mathbb{Z}^d$, cf. Definition and Remark 2.14. We consider $S\colon \mathbb{C}^d \to \mathcal{L}(W^{1,q}(\Omega_{\#}), L^q(\Omega_{\#}, \mathbb{C}^d))$ defined by

$$S(\theta) := \nabla + 2\pi \mathrm{i}\theta.$$

Let $e_j \in \mathbb{R}^d$ denote the j-th standard basis vector. We define $S_j \in \mathcal{L}(W^{1,q}(\Omega_{\#}), L^q(\Omega_{\#}, \mathbb{C}^d))$ by $S_j(f) := 2\pi \mathrm{i} f e_j$, $j = 1, \ldots, d$. Then,

$$S(\theta) = \nabla + \sum_{j=1}^{d} \theta_j S_j,$$

so S is analytic. We consider $T(\theta) := (\nabla + 2\pi \mathrm{i}\theta)^*\colon \mathbb{C}^d \to \mathcal{L}(L^q(\Omega_{\#}, \mathbb{C}^d), (W^{1,q'}(\Omega_{\#}))^*)$ and define $T_j \in \mathcal{L}(L^q(\Omega_{\#}, \mathbb{C}^d), (W^{1,q'}(\Omega_{\#}))^*)$ by $T_j(f)(v) := \langle f, 2\pi \mathrm{i} v e_j \rangle$. We get

$$T(\theta) = \nabla^* + \sum_{j=1}^{d} \overline{\theta_j} T_j.$$

Hence, $T|_{\mathbb{R}^d}$ is real analytic, but T is not complex analytic. That is because complex conjugation is not analytic. In fact, the function $\theta \mapsto T(\overline{\theta})$ is the analytic extension of $T|_{\mathbb{R}^d}$.

Corollary 2.29 *(analyticity implies differentiability)*
It is not hard to see that real analytic functions $f\colon \mathbb{R}^d \supseteq U \to E$ are continuously differentiable. In fact, the derivative of an absolutely convergent power series yields again an absolutely convergent power series. Hence, real analytic functions are C^∞-functions.

Lemma 2.30 *(analyticity of an inverse map)*
Let $Q\colon \mathbb{R}^d \supseteq U \to \mathcal{L}(E_1, E_2)$ be real analytic and $Q(\theta)$ bijective for all $\theta \in U$. In that case $R\colon \mathbb{R}^d \subseteq U \to \mathcal{L}(E_2, E_1)$ defined by $\theta \mapsto Q(\theta)^{-1}$ is real analytic as well.

Corollary 2.31 *(composition of analytic maps)*
Let $Q\colon \mathbb{R}^d \supseteq U \to \mathcal{L}(E_1, E_2)$ and $T\colon \mathbb{R}^d \supseteq U \to \mathcal{L}(E_2, E_3)$ be real analytic. Then, $S\colon \mathbb{R}^d \supseteq U \to \mathcal{L}(E_1, E_3)$ defined by $\theta \mapsto T(Q(\theta))$ is real analytic, too.

2.5 R-boundedness

We present the classical approach to R-boundedness, but take only a short glance at this topic. For further reading we refer to [KW04], where all statements in this section were taken from. R-boundedness was used first to formulate a vector-valued version of Mikhlin's theorem. Furthermore, it yields a criterion for proving maximal L^q-regularity.
The most important statement in this work using R-boundedness is Theorem 3.20, which is essentially taken from [Bar13]. This theorem will lead to the extension of the Helmholtz decomposition on L^q-spaces. In [Bar13], R-boundedness is used to prove L^q-boundedness

for certain classes of periodic operators.

Definition 2.32 *(R-boundedness)*
Let X, Y be Banach spaces. A set of operators $\tau \subseteq \mathcal{L}(X,Y)$ is called R-bounded, if there is a constant $C > 0$ such that

$$||\sum_{n=1}^{m} r_n T_n x_n||_{L^2([0,1],Y)} \leq C||\sum_{n=1}^{m} r_n x_n||_{L^2([0,1],X)},$$

for all $m \in \mathbb{N}$, $T_1, \dots, T_m \in \tau, x_1, \dots, x_m \in X$. The functions $r_n(t) = sign\ (sin(2^n \pi t))$ are called Rademacher functions. If τ is R-bounded, we denote the infimum of all constants C fulfilling the inequality by $R(\tau)$.

Remark 2.33 (Rademacher functions)
Note that the Rademacher functions are an orthonormal sequence in $L^2([0,1])$ with mean value 0.

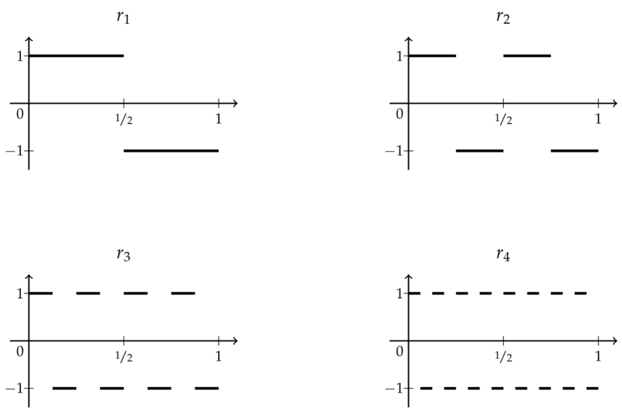


Figure 2.4: *Illustration of the first four Rademacher functions, visualizing the orthogonality (from [Bar13, p.35]).*

So, they can be seen as a sequence of identically distributed, stochastically independent functions with values in $\{-1, 1\}$. In fact, one can replace them in the definition by any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of identically distributed, stochastically independent functions with values in $\{-1, 1\}$ and mean value zero [KW04, Remark 2.6b)]. It is not hard to see that scalar multiples, sums and concatenations of R-bounded sets are R-bounded [vG06, Prop 2.1].

The forthcoming Remark 2.35 yields a useful criterion to prove R-boundedness. We will apply it to prove R-boundedness of the fibre operators associated to the Helmholtz decomposition in the next chapter. It is an easy consequence of the following two basic estimates, which we state before the remark.

Proposition 2.34 *(a) (Khinchine's inequality)*
For $q \in [1, \infty)$ there is a constant $C_q > 0$ satisfying

$$\frac{1}{C_q}\Big(\sum_n |a_n|^2\Big)^{1/2} \leq \Big|\Big|\sum_n r_n a_n\Big|\Big|_{L^q([0,1])} \leq C_q\Big(\sum_n |a_n|^2\Big)^{1/2}.$$

(b) *(Kahane's inequality)*
Let X be a Banach space, $q \in [1,\infty)$. There is a constant $C_q > 0$ satisfying

$$\frac{1}{C_q}||\sum_n r_n x_n||_{L^2([0,1],X)} \leq ||\sum_n r_n x_n||_{L^q([0,1],X)} \leq C_q||\sum_n r_n x_n||_{L^2([0,1],X)}.$$

Proof: See [KW04, 2.2. Khinchine's inequality and Theorem 2.4]. □

Therefore, it is possible to replace the exponent 2 in Definition 2.32 by any $q \in [1,\infty)$.
Let $X = Y = L^q(\Omega)$ and $q \in [1,\infty)$. We calculate for $x_1, \dots, x_m \in X$

$$||\sum_{n=1}^m r_n x_n||_{L^2([0,1],X)} \sim ||\big(\sum_{n=1}^m |x_n|^2\big)^{1/2}||_X.$$

So we get the following equivalent characterization for R-boundedness of $\tau \subseteq \mathcal{L}(X)$, which we use in the proof of Theorem 3.35.

Remark 2.35 (characterization of R-boundedness)
Let $X = Y = L^q(\Omega,\mu)$, (Ω,μ) a σ-additive measure space and $q \in [1,\infty)$. $\tau \subseteq \mathcal{L}(X)$ is R-bounded if and only if

$$||\big(\sum_{n=1}^m |T_n x_n|^2\big)^{1/2}||_X \leq C||\big(\sum_{n=1}^m |x_n|^2\big)^{1/2}||_X, \qquad T_1, \dots, T_m \in \tau.$$

Proof: This is an easy consequence of Kahane's inequality, Khinchine's equality and Fubini's theorem, see [KW04, Remark 2.9]. □

Now, we introduce the UMD-property and explain its significance within this topic.

Definition 2.36 *(UMD-spaces)*
A Banach space E is called UMD-space, if the Hilbert transform

$$\mathbb{H}f(t) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{|y|>\varepsilon} \frac{f(x-y)}{y} dy$$

extends from the Schwartz space $S(\mathbb{R},E)$ to a bounded linear operator on the whole space $L^q(\mathbb{R},E)$ for one or equivalently for all $q \in (1,\infty)$. Once again, the integral has to be understood as a Bochner integral.

Remark 2.37 (UMD and martingales)
The notation UMD refers to 'unconditional martingale differences' and in fact UMD-spaces were first described by the following equivalent definition. E is a UMD-space if for all

probability spaces $(\Omega, \mathcal{A}, P)$ there is a $C > 0$ such that

$$||\sum_{k=1}^{n} \varepsilon_k (u_k - u_{k-1})||_{L^2(\Omega,E)} \le C||\sum_{k=1}^{n}(u_k - u_{k-1})||_{L^2(\Omega,E)}, \qquad \text{for all } n \in \mathbb{N},\ \varepsilon_k \in \{\pm 1\},$$

where (u_k) are arbitrary E-valued martingales. As in the case of the definition of R-boundedness it is possible to replace 2 by any $q \in (1,\infty)$ (of course, $q = 1$ is not possible since the Hilbert transform is not bounded on L^1).

Before we state the main results we introduce maximal regularity.

Definition 2.38 *(maximal regularity)*
Let X be a Banach space, $(T(t))_{t\ge 0}$ be an analytic semigroup with generator A. Consider for a given $f \in L^q(\mathbb{R}_+, X)$ the inhomogeneous Cauchy problem

$$y'(t) = Ay(t) + f(t), \qquad t \ge 0, \qquad y(0) = 0.$$

The operator A is said to have maximal L^q-regularity if the unique solution y satisfies

$$||y'||_{L^q(\mathbb{R}_+,X)} + ||Ay||_{L^q(\mathbb{R}_+,X)} \le C||f||_{L^q(\mathbb{R}_+,X)}.$$

If X is an L^r-space, this is called maximal L^q-L^r-regularity.

A first result on this topic was the result by [dS64], where maximal regularity for arbitrary analytic generators in the Hilbert space case was shown. Bourgain [Bou86] showed a vector-valued generalization of Mikhlin's multiplier theorem employable for multipliers of the form $M(t) = m(t)Id$ and $q \in (1,\infty)$, where m is a scalar-valued function. This result was extended from $\mathbb{R}$ to $\mathbb{R}^d$ by [Zim89]. For a more detailed overview, cf. [KW04, page 2]. Maximal regularity has many applications, we only state two important results from [Wei01]. The first one is a general Banach space valued version of Mikhlin's multiplier theorem and the second one gives a characterization of maximal regularity by using R-boundedness.

Theorem 2.39 *(a) (Mikhlin-type multiplier theorem)*
Let X, Y be UMD-spaces and $M\colon \mathbb{R}\setminus\{0\} \to \mathcal{L}(X,Y)$ be differentiable such that the sets

$$\{M(t)\colon t \in \mathbb{R}\setminus\{0\}\}, \quad \{tM'(t)\colon t \in \mathbb{R}\setminus\{0\}\},$$

are R-bounded. Then it follows that $F^{-1}\circ M \circ F$ extends to a bounded linear operator $K\colon L^q(\mathbb{R}, X) \to L^q(\mathbb{R}, Y)$ for all $q \in (1,\infty)$.

(b) (characterizations of maximal regularity)
Let X be a UMD-space and $T(t)$ a bounded analytic semigroup with generator A. The following are equivalent:

- *A has maximal L^q-regularity.*
- *There is a constant $C > 0$ satisfying*

$$R(\{a^{2n}R(\mathrm{i}a2^n, A) \mid n \in \mathbb{Z}\}) \le C, \qquad 1 \le |a| \le 2.$$

- *There is a $\theta > 0$ such that the following set is R-bounded.*

$$\{\lambda R(\lambda, A) \mid \lambda \in \Sigma(\pi/2 + \theta)\}.$$

- *There is a $\theta > 0$ such that the following set is R-bounded.*

$$\{T(z) \mid z \in \Sigma(\theta)\}.$$

- *There are $\theta, C > 0$ such that for all $a \in [1,2]$, $|\varphi| \leq \theta$*

$$R(\{T_{a2^n e^{i\varphi}} \mid n \in \mathbb{Z}\}) \leq C.$$

Proof: See [Wei01, Theorem 3.4 and Theorem 4.2] or [KW04]. We sketch shortly below where the R-boundedness conditions have their origin, cf. [KW04, Discussion 1.5]. □

Let A be a generator of a bounded analytic semigroup. The unique mild solution of the Cauchy problem

$$y' - Ay = f, \qquad y(0) = x_0$$

is given by the variation of constants formula

$$y(t) = T(t)x_0 + \int_0^t T(t-s)(f(s))ds.$$

Taking $x_0 = 0$ and differentiating yields

$$y'(t) = \int_0^t AT(t-s)f(s)ds + f(t).$$

The operator K, first defined for $f \in C_c(\mathbb{R}_+, D(A))$, by

$$Kf(t) = \int_0^t AT(t-s)f(s)ds,$$

is a convolution operator with the singular kernel $AT(t)$, whose norm behaves like $1/t$. Therefore, it is natural to take the Fourier transform $\mathcal{F}(AT(t))$ of $AT(t)$. Since A generates a bounded analytic semigroup, the formula for the resolvents from Proposition 2.17 formally leads to

$$m(u) := \mathcal{F}(AT(t))(u) = AR(iu, A) = iuR(iu, A) - Id.$$

The functions m and

$$um'(u) = -iuAR(iu, A)^2 = [uR(iu, A)]^2 + iuR(iu, A)$$

are, due to the analyticity of the semigroup, both bounded on $\mathbb{R}\setminus\{0\}$. The theorems we use later in this work are due to [Bar13] and use R-boundedness to extend periodic operators from L^2 to L^q-spaces. We analyse them in detail in Section 3.2.

2.6 The $\mathcal{H}^\infty$-calculus

In Section 2.2 we met the Dunford type integral

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \mathrm{e}^{\lambda t} R(\lambda, A) d\lambda, \qquad t > 0,$$

which gave a formula for the semigroup generated by A. In this section, we consider for sectorial operators A of angle $\varphi \in (0, \pi/2)$ general integrals of the form

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} f(\lambda) R(\lambda, A) d\lambda,$$

where Γ is defined as in Figure 2.3, $f \in \mathcal{H}^\infty(\Sigma_\sigma) := \{g \colon \Sigma_\sigma \to \mathbb{C} \mid g \text{ is bounded analytic}\}$, and $\varphi < \varphi' < \sigma < \pi$. In general, this integral is not well-defined. Since $||R(\lambda, A)|| \cong |\lambda|^{-1}$ on $\partial\Sigma_{\varphi'}$ the integral exists for all bounded analytic functions f satisfying

$$|f(\lambda)| \leq C \Big|\frac{\lambda}{(1+\lambda)^2}\Big|^\varepsilon, \qquad \text{for some } C, \varepsilon > 0.$$

We denote the space of such functions by $\mathcal{H}_0^\infty(\Sigma_\sigma)$, i.e.

$$\mathcal{H}_0^\infty(\Sigma_\sigma) := \{f \colon \Sigma_\sigma \to \mathbb{C} \mid f \text{ bounded analytic, } |f(\lambda)| \leq C \Big|\frac{\lambda}{(1+\lambda)^2}\Big|^\varepsilon \text{ for some } C, \varepsilon > 0\}.$$

Roughly saying, we say that A admits a bounded $\mathcal{H}^\infty$-calculus, if this can be extended to a bounded algebra homomorphism with values in $B(X)$ for all bounded analytic functions defined on Σ_σ, where $\sigma > \varphi'$. More exactly, we have the following definition of the $\mathcal{H}^\infty$-calculus. The holomorphic functional calculus was introduced by McIntosh in 1986 [McI86]. For more details on the construction we refer to [KW04, section 2.9].

Definition 2.40 *($\mathcal{H}^\infty$-calculus)*
Let $A \colon D(A) \subseteq X \to X$ be a closed sectorial operator of angle $\varphi < \pi/2$ and $\sigma > \nu > \varphi$. A has bounded $\mathcal{H}^\infty$-calculus, if the map $\widetilde{T}_A$ mapping $f \mapsto f(A) \in \mathcal{L}(X)$ can be extended from $\mathcal{H}_0^\infty(\Sigma_\sigma)$, i.e. from all bounded analytic functions f satisfying

$$|f(\lambda)| \leq C \Big|\frac{\lambda}{(1+\lambda)^2}\Big|^\varepsilon, \qquad \textit{for some } C, \varepsilon > 0,$$

to a bounded map $T_A \colon \mathcal{H}^\infty(\Sigma_\sigma) \to \mathcal{L}(X)$ satisfying the following properties:

- *T_A is linear and multiplicative, i.e. $T_A(f \cdot g) = T_A(f)T_A(g) = T_A(g)T_A(f)$.*

- $T_A((\mu - \cdot)^{-1}) = R(\mu, A)$ *for all* $\mu \notin \overline{\Sigma_\sigma}$.

Here, the norm on $\mathcal{H}^\infty$ *is given by the* L^∞*-norm. This is equivalent to [KW04, chapter II, Remark 9.11]*

$$||\Phi_A(f)||_{\mathcal{L}(X)} \leq C||f||_{\mathcal{H}^\infty(\Sigma_\sigma)}, \qquad \text{for all } f \in \mathcal{H}_0^\infty(\Sigma_\sigma).$$

The infimum over all such ω *is called the* $\mathcal{H}^\infty$*-angle of* A *and is denoted by* $w_{\mathcal{H}^\infty}(A)$.

There is an extensive theory concerning characterizations and examples for operators admitting an $\mathcal{H}^\infty$-calculus, comprising many classes of differential operators, compare [KW04, chapter 2]. Besides, one finds there a very detailed discussion of the $\mathcal{H}^\infty$-calculus [KW04, chapter 2, sections 9-15]. We just mention one fundamental theorem and one important application here, namely the 'sum theorem', which implies maximal regularity for R-sectorial operators on UMD-spaces [KW01]. We defined the terms UMD-space, R-boundedness and maximal regularity in Definition 2.36, Definition 2.32 and Definition 2.38.

Theorem 2.41 *(sum theorem and maximal regularity)*
Let X *be a Banach space and* B *a closed,* R*-sectorial operator on* X *of angle* $\varphi < \pi/2$*, i.e. the operators* $\{\lambda R(\lambda, B)\}$ *on the sector are not only bounded, but* R*-bounded.*

(a) Let A *be a closed operators on* X *such that the resolvents of* A *and* B *commute. We assume that* A *has a bounded* $\mathcal{H}^\infty$*-calculus. Then,* $A + B$ *is closed on* $D(A) \cap D(B)$ *and*

$$||Ax|| + ||Bx|| \leq C||(A + B)x||, \qquad x \in D(A) \cap D(B).$$

(b) If X *is a UMD-space, then* B *has maximal* L^q*-regularity for all* $q \in (1, \infty)$.

2.7 The Helmholtz decomposition

We already mentioned in the introduction that the Helmholtz decomposition is named after von Helmholtz, who introduced the decomposition of a vector field on $\mathbb{R}^3$ into a gradient part and a curl part. We remark that a vector field on a simply connected domains Ω is solenoidal if and only if it is the curl of another vector field. If Ω is a bounded domain, which is not simply connected, then there exist L^2 vector fields, which cannot be describes as a sum of a gradient field and a curl of another vector field. In fact, the orthogonal complement of the gradient space consists of solenoidal vector fields and this space can be splitted into a curl part and some (on bounded domains finite dimensional) rest part, cf. the concluding remark in Chapter 3. Nevertheless, for $\Omega = \mathbb{R}^3$ or $\Omega \subseteq \mathbb{R}^3$ a simply connected smooth domain it is quite helpful to know that every sufficiently decreasing vector field u is uniquely determined by its divergence and its curl. For $\Omega = \mathbb{R}^3$, we recall the decomposition (for $u \in C_c^\infty$)

$$u(r) = -\nabla\Big(\frac{1}{4\pi}\int \frac{\operatorname{div} u(x)}{|r - x|}dx\Big) + \operatorname{curl}\Big(\frac{1}{4\pi}\int \frac{\operatorname{curl} u(x)}{|r - x|}dx\Big),$$

and consider the following example from electrodynamics [Gri99, App. B]. The physical motivation and definitions behind the theory are given in the appendix. We have div $E = \frac{\varrho}{\varepsilon_0}$ and curl $E = 0$, so the electromagnetic field is given by

$$E(r) = -\nabla\Big(\frac{1}{4\pi\varepsilon_0}\int \frac{\varrho(x)}{|r-x|}dx\Big).$$

Analogously, we get a formula for the magnetic field (here $\partial_t E = 0$)

$$B(r) = \text{curl}\,\Big(\frac{\mu_0}{4\pi}\int \frac{\text{curl}\ J(x)}{|r-x|}dx\Big),$$

since div $B = 0$ and curl $B = \mu_0 J$. We discuss additional applications of the Helmholtz decomposition later, namely the Stokes operator and the Maxwell operator. Now we are coming to the mathematical details of the Helmholtz decomposition.

Definition 2.42 *(Helmholtz decomposition and Helmholtz projection)*
Let $\Omega \subseteq \mathbb{R}^d$ be a domain. We say that the Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^d)$ if there are for all $f \in L^q(\Omega, \mathbb{C}^d)$ unique functions $\nabla p \in G^q(\Omega)$, $g \in L^q_\sigma(\Omega)$ with

$$f = g + \nabla p, \qquad \textit{and} \qquad ||g||_{L^q(\Omega,\mathbb{C}^d)} + ||\nabla p||_{L^q(\Omega,\mathbb{C}^d)} \le C||f||_{L^q(\Omega,\mathbb{C}^d)},$$

where $C = C(q,\Omega)$. Here, the spaces $L^q_\sigma(\Omega)$ and $G^q(\Omega)$ are defined as follows:

$$L^q_\sigma(\Omega) = \overline{C^\infty_{c,\sigma}(\Omega)}^{||\cdot||_q} = \overline{\{u \in C^\infty_c(\Omega,\mathbb{C}^d) \mid \text{div}\ u = 0\ \textit{in}\ \Omega\}}^{L^q(\Omega,\mathbb{C}^d)},$$
$$G^q(\Omega) = \{\nabla p \in L^q(\Omega) \mid p \in L^1_{loc}(\Omega)\} = \{\nabla p \in L^q(\Omega,\mathbb{C}^d) \mid p \in W^{1,q}_{loc}(\Omega)\}.$$

If the decomposition exists, there is a bounded projection operator $P_q \colon L^q(\Omega,\mathbb{C}^d) \to L^q(\Omega,\mathbb{C}^d)$ satisfying kernel$(P_q) = G^q(\Omega)$ *and* image$(P_q) = L^q_\sigma(\Omega)$. *$P_q$ is called the Helmholtz projection. Hence, $Q_q = Id - P_q$ defines the projection onto the gradient part of the decomposition.*

There are more equivalent characterizations, which we present in the following theorem.

Theorem 2.43 *(characterizations for the Helmholtz decomposition)*
Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain and $q \in (1,\infty)$. Then are equivalent:

a) The Helmholtz decomposition exists on $L^q(\Omega,\mathbb{C}^d)$.

b) For all $u \in L^q(\Omega,\mathbb{C}^d)$ there is exactly one

$$p \in \dot{W}^{1,q}(\Omega) := \{u \in L^q_{loc}(\Omega) \mid \nabla u \in L^q(\Omega,\mathbb{C}^d)\}/\mathbb{C}$$

solving the problem $\int_\Omega (\nabla p - u)\cdot\nabla\varphi = 0$ for all $\varphi \in \dot{W}_{1,q'}(\Omega)$ and $||\nabla p||_q \le C||u||_q$. This problem is also called the weak formulation of the Neumann problem. The definition of the classical Neumann problem is stated in the forthcoming remark.

c) The operator $N_q = (\nabla_{q'})^*\nabla_q$ is bijective, hence an isomorphism. Here:

$$\dot{W}^{1,q}(\Omega) \xrightarrow{\nabla_q} L^q(\Omega,\mathbb{C}^d) \cong (L^{q'}(\Omega,\mathbb{C}^d))' \xrightarrow{(\nabla_{q'})^*} (\dot{W}^{1,q'}(\Omega))^*.$$

Here, $||u||_{\dot{W}^{1,q}(\Omega)} := ||\nabla u||_q$. In that case, the projection onto the space $G^q(\Omega)$ is given by $\nabla_q N_q^{-1} \nabla_{q'}^*$.

d) The estimate

$$||\nabla p||_{G^s(\Omega)} \leq C \sup_{0 \neq \nabla v \in G^{s'}(\Omega)} \frac{|\langle \nabla v, \nabla p \rangle|}{||\nabla v||_{s'}}, \qquad \text{for all } \nabla p \in G^s(\Omega)$$

holds true for $s = q, q'$ and $C = C(d, q, \Omega)$.

e) The estimate

$$||g||_{L^s_\sigma(\Omega, \mathbb{C}^d)} \leq C \sup_{0 \neq h \in L^{s'}_\sigma(\Omega)} \frac{|\langle g, h \rangle|}{||h||_{s'}}, \qquad \text{for all } g \in L^s_\sigma(\Omega, \mathbb{C}^d)$$

holds true for $s = q, q'$ and $C = C(d, q, \Omega)$.

Proof: See [SSV14, Theorem 2.2 and Theorem 2.3] and combine it with Corollary 2.50 to get the equivalence of a),d) and e).
The equivalence of a) and b) directly follows from $L^q_\sigma(\Omega) = (G^{q'}(\Omega))^\perp$, cf. Proposition 2.49. To get the Helmholtz decomposition from c) (cf. [HK12, Theorem 2.6]) we define $p := (\nabla_{q'}^* \nabla_q)^{-1} \nabla_{q'}^* f$ and get $\nabla_{q'}^*(\nabla p - f) = 0$, which means $\nabla p - f \in L^q_\sigma(\Omega)$. It remains to show a)$\Rightarrow$c). If $\nabla_{q'}^* \nabla_q p = 0$, it follows $\nabla p \in L^q_\sigma(\Omega)$, hence $\nabla p = 0$, so $\nabla_{q'}^* \nabla_q$ is injective. Let $\psi \in (\dot{W}^{1,q'}(\Omega))^*$ be arbitrary. Recalling that $\nabla_{q'}^*$ is surjective we find $f \in L^q(\Omega, \mathbb{C}^d)$ satisfying $\nabla_{q'}^* f = \varphi$. We can decompose $f = g + \nabla p$. Since $\nabla_{q'} g = 0$, the proof is finished. $\square$

Remark 2.44 (a) (classical Neumann problem)
The problem in part b) of Theorem 2.43 is called the weak Neumann problem. If Ω and u are smooth enough, the solution p is a solution of the classical Neumann problem, which is given as follows:

$$\begin{aligned} \Delta p &= \operatorname{div} u && \text{in } \Omega, \\ \frac{\partial p}{\partial \nu} &= u \cdot \nu && \text{on } \partial\Omega. \end{aligned}$$

Note that all solutions of the classical Neumann problem are also weak solutions by the integration by parts formula.

(b) (remark on the space $\dot{W}^{1,q}(\Omega)$)
Note that the way the space $\dot{W}^{1,q}(\Omega)$ is defined, is such that $\nabla \dot{W}^{1,q}(\Omega) = G^q(\Omega)$. By factorizing out the constant functions, ∇ is made injective. It is always possible to replace in the definition of $\dot{W}^{1,q}(\Omega)$ the space L^q_{loc} by L^1_{loc} or by distributions.
Moreover, we have [Neč12, Chapter II, Theorem 7.6] for local Lipschitz domains

$$\dot{W}^{1,q}(\Omega) = \{u \in L^q_{loc}(\overline{\Omega}) \mid \nabla u \in L^q(\Omega, \mathbb{C}^d)\}/\mathbb{C}.$$

In particular, we have for bounded Lipschitz domains domains

$$\dot{W}^{1,q}(\Omega) = W^{1,q}(\Omega)/\mathbb{C}.$$

Up to now, we did not mention in which cases the Helmholtz decomposition exists. Because of Proposition 2.49 below, there is one very easy special case, namely the Hilbert space case $L^2(\Omega, \mathbb{C}^d)$. It is also not hard to prove directly the existence of the Helmholtz decomposition for $q = 2$ on any domain $\Omega \subseteq \mathbb{R}^d$.

Theorem 2.45 *(Helmholtz decomposition on $L^2(\Omega, \mathbb{C}^d)$)*
Let $\Omega \subseteq \mathbb{R}^d$ be any domain. The Helmholtz decomposition exists on $L^2(\Omega, \mathbb{C}^d)$, and the subspaces $L^2_\sigma(\Omega)$ and $G^2(\Omega)$ are orthogonal complements. We write $P = P_2$ and $Q = Q_2$ for the associated projections.

Sketch of proof: We give the very short proof here. For a given $u \in L^2(\Omega, \mathbb{C}^d)$ we consider the weak Neumann problem

$$\langle \nabla p, \nabla \psi \rangle = \langle u, \nabla \psi \rangle, \qquad \text{for all } \nabla \psi \in G^2(\Omega).$$

By the Lax-Milgram lemma, there is a unique solution $\nabla p \in G^2(\Omega)$ and the solutions satisfies $||\nabla p||_2 \leq C||u||_2$, so the Helmholtz decomposition exists, compare Theorem 2.43. □

The non-Hilbert space case $L^q(\Omega, \mathbb{C}^d)$, $q \neq 2$, is far more difficult. We state the most important historical existence results in the following theorem.

Theorem 2.46 *(existence results for the Helmholtz decomposition and counterexamples)*

(a) $\Omega = \mathbb{R}^d$*: This is in historical terms the first case considered. We considered this case at the beginning of the introduction using the fundamental solution F of Δ.*

(b) $\Omega = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d \mid x_d > 0\}$ *is a half space: The Helmholtz decomposition exists for all $q \in (1, \infty)$. Here again, we have an explicit formula. The Neumann Green's function for the half space is given by*

$$G(x, y) = F(x - y) - F(x - y^*), \qquad y^* = (y_1, \ldots, y_{d-1}, -y_d),$$

where F denotes the fundamental solution from (a). So, the gradient part is given by

$$p = \int_{\mathbb{R}^d_+} G(x, y) \operatorname{div} u(y) dy, \qquad x \in \mathbb{R}^d_+.$$

(c) Bended half space Ω: That means Ω lies above a graph of a C^1-function with compact support. The easiest proof for this statement is given by combining [SS92] and Theorem 2.43.

(d) Ω *bounded: The first results for bounded domains are due to [FM77] and [CM85]. They applied more general results known for elliptic problems from [LM62, Theorem 4.1],[Sch63b, Sch63a] and [Mir78, § 57].*
The strongest results are showed in [SS92] and [FMM98]. The first one states the existence of the Helmholtz decomposition for all $q \in (1, \infty)$ *if* Ω *is a bounded* C^1*-domain.*
The second concerns Lipschitz domains $\Omega \subseteq \mathbb{R}^d$*, where* $d \geq 3$*. There exists an* $\varepsilon > 0$ *such that the Helmholtz decomposition exists for all* $q \in [\frac{3+\varepsilon}{2+\varepsilon}, 3 + \varepsilon]$*. It is proven that this result is sharp in the following sense: For every* $\varepsilon > 0$ *there is a Lipschitz domain* Ω_ε*, where the Helmholtz decomposition does not hold if* $q > 3 + \varepsilon$ *or if* $q < 3/2 - \varepsilon$*, compare (h). For the proof, concrete representations by using the single layer potential, were used. A similar approach for exterior and bounded domains can be found already in [vW90], which yields interesting maximal regularity results, too. The case* $d = 2$ *is discussed together with the counterexamples in (h) below.*
Furthermore the Helmholtz decomposition does hold for all bounded convex domains [GS10].

(e) Ω *exterior, i.e.* Ω *unbounded and* $\partial\Omega$ *compact: There are results in [Sol77] and [Miy82]. Again, the Helmholtz decomposition exists for the full range* $q \in (1, \infty)$*, provided* Ω *has* C^1*-boundary [SS92].*
The same techniques as in the bounded Lipschitz case, can be used to show the existence of the Helmholtz decomposition on exterior domains for $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$*, compare [LM06, Theorem 6.1]. As in the bounded case the range of* q *is known to be optimal [LM06, Theorem 7.3].*

(f) Ω *is an infinite cylinder or an infinite layer: See [ST98, Miy94]. In [Far03] weighted versions are considered.*

(g) Ω *is an aperture domain: See [FS96]. For the definition of an aperture domains, see Remark 3.1.*

(h) *There are also domains known, where the Helmholtz decomposition does not exist. There are unbounded domains with even* C^∞*-boundary known, where the Helmholtz decomposition does not exist for some range of* q*. The first example was discovered 1986 by Bogovskii and Maslennikova [MB86, Example 3], consisting of a complement of a smoothed angle, where* $\theta > \pi$ *and* $d = 2$*.*

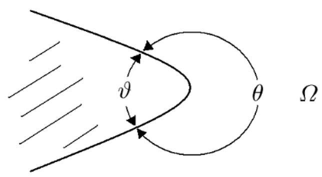


Figure 2.5: *The existence range of* q*, for which the Helmholtz decomposition exists, depends on the angle* θ*, and also on the smoothness of the angle (from [Gal11, p. 153]).*

The interval on which the Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^2)$ *depends on the angle* θ*. One can show (in the two dimensional case) that the weak Neumann problem has no solution if* $1 < q < \frac{2}{1+\frac{\pi}{\theta}}$*. In return, it loses uniqueness if* $q > \frac{2}{1-\frac{\pi}{\theta}}$*. We remark*

that it matters for the problem if the 'corner is smooth'. Astoundingly, the existence interval enlarges if the corner is sharp. In that case the Helmholtz decomposition exists for all $q \in (1,\infty)\setminus\{2/(1 \pm \frac{\pi}{\theta})\}$. *As we shall see below in the discussion after Corollary 2.50 the disconnectedness of the existence range is a phenomena, which does not occur on uniform* C^1*-domains since there the Helmholtz projections are consistent.*
The result for bounded Lipschitz domains is sharp in the following sense: For all $q < 3/2$ *and* $q > 3$ *there are bounded Lipschitz domains* $\Omega \subseteq \mathbb{R}^3$ *so that the Helmholtz decomposition does not exist on* $L^q(\Omega,\mathbb{C}^d)$.
The 'optimal counterexample' looks as follows, cf. [JK89] and [FMM98, Theorem 12.2 and Corollary 12.3]. It suffices to find for $q > 3$ *a Lipschitz domain* Ω *and a function* $v \in W^{1,2}(\Omega)$ *satisfying* $\Delta v \in C_c^\infty(\overline{\Omega})$, $v|_{\partial\Omega} = 0$, *but* $v \notin W^{1,q}(\Omega)$. *This domain can be constructed by choosing* Ω *as the intersection of a ball and the complement of a circular cone of angle* α *having its apex in the center of the ball. By using some radial symmetry arguments, once can prove the existence of such a function* v, *provided* α *is small enough.*

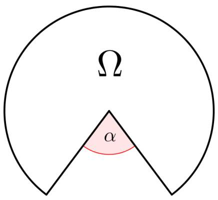


Figure 2.6: *two dimensional visualization: The existence interval of the Helmholtz decomposition shrinks to* $(3/2,3)$ *if the angle* α *tends to zero.*

Counterexamples for Lipschitz domains were known even before.

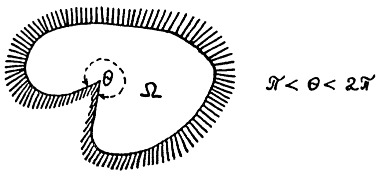


Figure 2.7: *Two dimensional counterexample, also yielding a 'sharp negative result' (from [MB86, Example 11]).*

Here, the Helmholtz decomposition on the two dimensional domain Ω *exists if* $p \in (\frac{2}{1+\pi/\theta}, \frac{2}{1-\pi/\theta})$ *and does not exist for* $p < \frac{2}{1+\pi/\theta}$ *or* $p > \frac{2}{1-\pi/\theta}$. *Note that this example yields that the best possible general bound for two dimensional Lipschitz domains cannot be better than* $(\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$. *As in the three dimensional case the result is sharp, i.e. for all bounded Lipschitz domains* Ω *there is an* $\varepsilon > 0$ *such that the Helmholtz decomposition exists for all* $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$, *cf. [Gen12, Theorem 1.2].*

Proof: The proofs can be found in the mentioned references. We just want to remark that for different domains different approaches lead to the proof. Therefore, it is quite helpful to have the equivalent characterizations at hand. □

In the following we want to take a closer look at other possible characterizations of the involved spaces in the Helmholtz decomposition. It turns out, that one has to be quite careful. Once more we want to mention that it is possible to consider more general domains as strongly local Lipschitz domains.

Definition and Remark 2.47 *(the spaces $\widehat{L}^q_\sigma(\Omega)$ and $\widehat{G}^q(\Omega)$)*
Let $\Omega \subseteq \mathbb{R}^d$ be a strongly local Lipschitz domain. We define

$$\begin{aligned}
\widehat{L}^q_\sigma(\Omega) &= \{f \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} f = 0 \textit{ in } \Omega,\ \nu \cdot f = 0 \textit{ on } \partial\Omega\},\\
\widehat{G}^q(\Omega) &= \overline{\nabla C_c^\infty(\overline{\Omega})}^{||\cdot||_q}.
\end{aligned}$$

Obviously, $\widehat{G}^q(\Omega) \subseteq G^q(\Omega)$ and $L^q_\sigma(\Omega) \subseteq \widehat{L}^q_\sigma(\Omega)$. Here, $\nu \cdot f$ has to be understood in the generalized trace sense, which is possible since div $f = 0 \in L^q(\Omega)$*, cf. Lemma 2.9.*

It is quite an important question, if these spaces are equal or not. A long time it was taken for granted that these spaces are equal. Heywood was the first one who detected that these spaces might be different. We shall not take a closer look on the historical development of this theory. For further reading we suggest [MB81] and the literature mentioned therein. Concerning the question of this coincide, there are still a lot of unsolved problems. We will discuss them at the beginning of Chapter 3.

The orthogonality of the spaces L^2_σ and G^2, compare Theorem 2.45, has a natural extension on L^q. Before, we consider de Rham's argument, which is essential for the whole theory and is a main part of the subsequent proof.

Lemma 2.48 *(de Rham's argument)*
Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Suppose that $u \in L^1_{loc}(\Omega)$ verifies

$$\int_\Omega u \cdot w = 0, \qquad \textit{for all } w \in C^\infty_{c,\sigma}(\Omega),$$

then there is a function $p \in W^{1,1}_{loc}(\Omega)$ such that $u = \nabla p$.

Proof: See [Gal11, Lemma III.1.1] □

We remark that there is also a variant of this lemma on the distributional level, cf. [Soh01, II.2.2.1 Lemma]. We recall

$$\begin{aligned}
L^q_\sigma(\Omega) &= \overline{\{f \in C_c^\infty(\Omega, \mathbb{C}^d) \mid \operatorname{div} f = 0\}}^{||\cdot||_q}\\
G^q(\Omega) &= \{\nabla p \in L^q(\Omega, \mathbb{C}^d) \mid p \in L^q_{loc}(\Omega)\}\\
\widehat{L}^q_\sigma(\Omega) &= \{f \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} f = 0 \text{ in } \Omega,\ \nu \cdot f = 0 \text{ on } \partial\Omega\},\\
\widehat{G}^q(\Omega) &= \overline{\nabla C_c^\infty(\overline{\Omega})}^{||\cdot||_q}.
\end{aligned}$$

Proposition 2.49 *(annihilator relations)*
Let $\Omega \subseteq \mathbb{R}^d, d \geq 2$, be any domain and $q \in (1, \infty)$. The following annihilator relations hold:

$$L^q_\sigma(\Omega) = G^{q'}(\Omega)^\perp, \qquad G^q(\Omega) = L^{q'}_\sigma(\Omega)^\perp$$

If Ω has local Lipschitz boundary, we have

$$\widehat{L}^q_\sigma(\Omega) = \widehat{G}^{q'}(\Omega)^\perp, \qquad \widehat{G}^q(\Omega) = \widehat{L}^{q'}_\sigma(\Omega)^\perp.$$

Proof: For $L^q_\sigma(\Omega) = (G^{q'}(\Omega))^\perp$ cf. [Gal11, Lemma III.2.1], [HK12, Proposition 2.5]. The proof that $G^q(\Omega)$ is closed works as follows, compare [Gal11, Lemma II.6.2]. Since this result is fundamental for the theory, we give the sketch of the proof here.
Let $(\nabla p_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $L^q(\Omega,\mathbb{C}^d)$ converging to g with $p_i \in L^q_{loc}(\Omega)$. There is [Gal11, Lemma II.1.1] a covering D of Ω with at most countable many open balls $\{B_k\}_{k\in I}$ such that for all families $F = \{B_l\}_{l\in I'}$, $I' \subsetneq I$ there is a $B \in D\setminus F$ so that $(\cup_{l\in I'} B_l)\cap B \neq \emptyset$. Let $B_0 \in D$ be arbitrary. By using Poincaré's inequality it follows that $p_i - \mathrm{mean}(p_i|_{B_0})$ converges to some $u_0 \in L^q(B_0)$. Furthermore, by definition of the weak derivative, $\nabla u_0 = g|_{B_0}$ almost everywhere. By using the property of the covering we find B_1 satisfying $B_0 \cap B_1 \neq \emptyset$. As above, we construct $u_1 \in L^q(B_1)$. Since $B_0 \cap B_1 \neq \emptyset$ we get $u_0 = u_1 + c$ on $B_0 \cap B_1 \neq \emptyset$. Without restriction, we can assume $u_0 = u_1$ on $B_0 \cap B_1 \neq \emptyset$. So there is a function $u_{0,1} \in L^q(B_0 \cup B_1 =: B_{0,1})$ satisfying $\nabla u_{0,1} = g|_{B_{0,1}}$. By using another property of the covering [Gal11, Lemma II.1.1(iii)], this argument can be extended inductively until Ω is totally covered. All in all, this yields a function $u \in L^q_{loc}(\Omega)$ satisfying $\nabla u = g$.
So, we have $G^q(\Omega) = G^q(\Omega)^{\perp\perp} = L^{q'}_\sigma(\Omega)^\perp$. For a detailed direct proof of this annihilator equality not using a duality argument we refer to [FM77, Lemma 7]. The main issues in the proofs are the integration by part formula and de Rham's argument. The other annihilator relations can be proven in a similar way by using the integration by parts formula in Ω. Therefore we need the local Lipschitz boundary condition. □

An immediate consequence of Proposition 2.49 is the duality property of the weak Neumann problem.

Corollary 2.50 *(duality of the weak Neumann problem)*
Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. The Helmholtz decomposition exists on $L^q(\Omega,\mathbb{C}^d)$ if and only if it exists on $L^{q'}(\Omega,\mathbb{C}^d)$.

Proof: This follows directly from Proposition 2.49. See [GHHS12, Lemma 5.1] for another proof using the duality between the weak Neumann problems on L^q and $L^{q'}$. □

We want to remark that, although there is in some sense a duality of the problem, there is no interpolation of the problem. So, the set

$$M = \{q \in (1,\infty) \mid \text{Helmholtz decomposition exists on } L^q(\Omega,\mathbb{C}^d)\}$$

is not always an interval. In fact, we considered the example of an unbounded domain with sharp angle in Theorem 2.46(h), where the Helmholtz decomposition exists on $(1,\infty)$ up to two isolated points. Besides, it seems not to be clear, if M is always an open set.

On the other hand, if Ω is an arbitrary domain of uniform C^1-class, then M is an interval. To get this description for M one proves consistency of the Helmholtz projections. To get this result, one uses the uniqueness of the Helmholtz decomposition on $L^2 \cap L^q$ for $q \geq 2$ from Remark 2.52 below, compare [GK15, Proposition 2.1].

Moreover, even if the Helmholtz projection extends from L^2 to a bounded operator on L^q, it is not certain that the extended operator defines the Helmholtz projection on L^q.
Since our approach is based on this extension property, we need a feature of the domain guaranteeing that the extended operator defines indeed the Helmholtz projection on L^q. More precisely, we have the following criterion.

Lemma 2.51 *(Helmholtz decomposition on L^q by extending P_2)*
Let $q \in (1, \infty)$, $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain with strongly local Lipschitz boundary such that $L^r_\sigma(\Omega) = \widehat{L}^r_\sigma(\Omega)$ and $G^r(\Omega) = \widehat{G}^r(\Omega)$ for $r = 2, q$. Furthermore, we assume that the Helmholtz decomposition $P := P_2 \in \mathcal{L}(L^2(\Omega, \mathbb{C}^d))$ extends to an operator $P_q \in \mathcal{L}(L^q(\Omega, \mathbb{C}^d))$. Then, the Helmholtz decomposition exists on $L^q(\Omega, \mathbb{C}^d)$ and P_q coincides with the Helmholtz projection on $L^q(\Omega, \mathbb{C}^d)$.

Proof: Clearly, we have $P_q^2 = P_q$. We show that

$$\widehat{G}^q(\Omega) \subseteq \operatorname{kernel}(P_q) \subseteq G^q(\Omega),$$
$$L^q_\sigma(\Omega) \subseteq \operatorname{image}(P_q) \subseteq \widehat{L}^q_\sigma(\Omega),$$

which yields, together with the projection property, the stated. We observe that $f \in \operatorname{kernel}(P_q)$ if and only if there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(\Omega, \mathbb{C}^d) \cap L^q(\Omega, \mathbb{C}^d)$ satisfying $f_n \to f$ in $L^q(\Omega, \mathbb{C}^d)$ and $P_2 f_n \to 0$ in $L^q(\Omega, \mathbb{C}^d)$. In that case, $g_n := f_n - P_2 f_n$ satisfies $g_n \to f$ in $L^q(\Omega, \mathbb{C}^d)$ and $P_2 g_n = 0$. So, by Remark 2.44(b),

$$\begin{aligned}
\operatorname{kernel}(P_q) &= \overline{\{f \in L^2(\Omega) \cap L^q(\Omega) \mid P_2 f = 0\}}^{\|\cdot\|_q} \\
&= \overline{\{\nabla p \in L^2(\Omega) \cap L^q(\Omega) \mid p \in L^1_{loc}(\Omega)\}}^{\|\cdot\|_q}.
\end{aligned}$$

We calculate

$$\begin{aligned}
\widehat{G}^q(\Omega) &= \overline{\nabla C_c^\infty(\overline{\Omega})}^{\|\cdot\|_q} \subseteq \overline{\{f \in L^2(\Omega) \cap L^q(\Omega) \mid P_2 f = 0\}}^{\|\cdot\|_q} = \operatorname{kernel}(P_q), \\
\operatorname{kernel}(P_q) &= \overline{\{\nabla p \in L^2(\Omega) \cap L^q(\Omega) \mid p \in L^1_{loc}(\Omega)\}}^{\|\cdot\|_q} \\
&\subseteq \overline{\{\nabla p \in L^q(\Omega) \mid p \in L^1_{loc}(\Omega)\}}^{\|\cdot\|_q} \\
&= \overline{G^q(\Omega)}^{\|\cdot\|_q} = G^q(\Omega), \\
L^q_\sigma(\Omega) &= \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_q} \subseteq \overline{\{P_2 f \mid f \in L^2(\Omega) \cap L^q(\Omega)\}}^{\|\cdot\|_q} = \operatorname{image}(P_q), \\
\operatorname{Im}(P_q) &= \overline{\{g \in L^2(\Omega) \cap L^q(\Omega) \mid \operatorname{div} g = 0,\ \nu \cdot g = 0\}}^{\|\cdot\|_q} \\
&\subseteq \overline{\{g \in L^q(\Omega) \mid \operatorname{div} g = 0,\ \nu \cdot g = 0\}}^{\|\cdot\|_q} \\
&= \overline{\widehat{L}^q_\sigma(\Omega)}^{\|\cdot\|_q} = \widehat{L}^q_\sigma(\Omega).
\end{aligned}$$

□

Note that we were not able to find a proof for the above lemma, which does not require the equality of the gradient and solenoidal vector field spaces. The statement of Lemma 2.51

might be wrong without that assumption.
At the end of this section we want to mention another approach [FKS07] to the Helmholtz decomposition, which has the big advantage to work on arbitrary unbounded domains of uniform C^1-class.

Remark 2.52 (variant of the Helmholtz decomposition)
Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain of uniform C^1-class. We define

$$\widetilde{L}^q(\Omega, \mathbb{C}^d) = \begin{cases} L^q(\Omega, \mathbb{C}^d) \cap L^2(\Omega, \mathbb{C}^d), & 2 \le q < \infty, \\ L^q(\Omega, \mathbb{C}^d) + L^2(\Omega, \mathbb{C}^d), & 1 < q < 2, \end{cases}$$

$$\widetilde{L}^q_\sigma(\Omega) = \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty, \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2, \end{cases}$$

$$\widetilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) + G^2(\Omega), & 2 \le q < \infty, \\ G^q(\Omega) \cap G^2(\Omega), & 1 < q < 2. \end{cases}$$

The norm on these spaces is given by

$$||f||_{\widetilde{L}^q_\sigma(\Omega)} = \begin{cases} \max\{||f||_{L^q(\Omega,\mathbb{C}^d)}, ||f||_{L^2(\Omega,\mathbb{C}^d)}\}, & 2 \le q < \infty, \\ \inf\{||f_1||_{L^q(\Omega,\mathbb{C}^d)} + ||f_2||_{L^2(\Omega,\mathbb{C}^d)} \mid f = f_1 + f_2\}, & 1 < q < 2. \end{cases}$$

The main result [FKS07, Theorem 1.2] reads as follows: Let $q \in (1, \infty)$. Each $f \in \widetilde{L}^q(\Omega, \mathbb{C}^d)$ has a unique decomposition

$$f = g + \nabla p, \qquad g \in \widetilde{L}^q_\sigma(\Omega),\ \nabla p \in \widetilde{G}^q(\Omega)$$

satisfying the estimate

$$||g||_{\widetilde{L}^q} + ||\nabla p||_{\widetilde{L}^q} \le C||f||_{\widetilde{L}^q}$$

for some C independent of f. In particular, there is a continuous Helmholtz projection defined on $\widetilde{L}^q(\Omega, \mathbb{C}^d)$ onto the space $\widetilde{L}^q_\sigma(\Omega)$.

Proof: See [FKS07]. □

It is an interesting open question if a similar result holds true for uniform Lipschitz domains and a range of q which depends on the uniform constants.

CHAPTER 3

The Helmholtz decomposition on periodic domains

In the first part of this chapter we prove the space equalities $L^q_\sigma(\Omega_\mathrm{p}) = \widehat{L}^q_\sigma(\Omega_\mathrm{p})$ and $G^q(\Omega_\mathrm{p}) = \widehat{G}^q(\Omega_\mathrm{p})$ for all periodic Lipschitz domains and all $q \in (1, \infty)$.
Following this, we prove the validity of the (adapted) Bloch multiplier theorems [Bar13] in the case of a periodic domains instead of $\mathbb{R}^d$. Thereupon, we calculate the fibre operators for the Helmholtz projection on $L^2(\Omega_\mathrm{p}, \mathbb{C}^d)$ and prove existence and boundedness of these operators in the L^q-setting. By applying the multiplier theorems we prove that P_2 extends to a bounded operator on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$, which implies the existence of the Helmholtz decomposition on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ by Lemma 2.51.
In the last section of this chapter we consider the weak Dirichlet problem and the inherent projection, the Leray projection. We prove similar statements as in the weak Neumann/Helmholtz decomposition case.

3.1 Equality of two solenoidal vector field spaces on periodic domains

At the beginning of this chapter we examine the not yet considered question in which cases the two spaces of solenoidal vector fields and the two spaces of gradients introduced in Definition and Remark 2.47 coincide. We show that this is always the case for periodic Lipschitz domains.
The proof of this statement uses a quite easy criterion taken from [MB83]. It says that is suffices to extend functions $p \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})$ with $\nabla p \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ to functions in $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ with gradient in $L^q(\mathbb{R}^d, \mathbb{C}^d)$. This is because the equality of the two spaces is known on $\mathbb{R}^d$ and the C_c^∞ approximations of the extended functions can be used to approximate the function on the subset Ω, too. Naturally, the periodicity of the domain has a large share on the extension property. Furthermore, a combination of Stein's extension theorem and the Poincaré inequality are an important tool for the proof. Before we go into details about the case of periodic domains we consider general results concerning the question in which cases the two spaces coincide.

Remark 3.1 Let Ω be a bounded Lipschitz domain, $q \in [1, \infty)$. Then, the two spaces of gradients and solenoidal functions coincide [MB83, Corollary of Theorem 1]. In particular,

$$L^q_\sigma(\Omega) = \{f \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} f = 0 \text{ in } \Omega,\ \nu \cdot f = 0 \text{ on } \partial\Omega\}.$$

One gets this equality even for arbitrary domains with compact Lipschitz boundary [MB81, Theorem 2], which includes the case of exterior domains. We remark that this case is also trivially included by the approach for periodic domains in this work, which we consider after Corollary 3.4. Besides, the codimension is not only depending on Ω, but also highly depending on q, cf. [MB83, last remark]. So, for all unbounded strongly local Lipschitz domains we have $L^q_\sigma(\Omega) = \widehat{L}^q_\sigma(\Omega)$ for all $q \in [1, d/(d-1)]$, and consequently $G^q(\Omega) = \widehat{G}^q(\Omega)$ for all $q \in [d, \infty)$ by duality, cf. [MB83, Theorem 5]. These results are sharp in the general context. More exact, for all $q \in (\frac{d}{d-1}, \infty)$ the codimension of $L^q_\sigma(\Omega) \subseteq \widehat{L}^q_\sigma(\Omega)$ might be any natural number, or even infinite, where the boundary can even be chosen smooth [MB83, last remark].
The first counterexample was discovered by Heywood in 1976 [Hey76]. Later, Ladyzhenskaya, Solonnikov, Maslennikova, Bogovskii and Kapitanskii studied this problem intensively. We want to give a concrete example. We consider the aperture domain

$$\Omega = \big(\mathbb{R}^d \setminus \{x \in \mathbb{R}^d \mid x_d \in [-1/2, 1/2]\}\big) \cup (-1,1)^d,$$

consisting of two half spaces which are connected by an aperture. Here, the codimension is one if $d = 3$ and $q = 2$, see [Gal11, Theorem III.4.4] for a detailed proof. One explanation is that there are two exits at infinity. In general, if there are m exits at infinity, then the codimension is $m - 1$, cf. [LS76, MB83]. We remark that the codimension for a cylinder is zero, in that case the exits are too 'small'. Up to now, it is not known which exits are huge enough too raise the codimension. We refer to [Gal11, Section III.4.3] concerning this discussion. By all yet known results, it is quite plausible that the size of the codimension depends only on the shape of Ω and on q, and is independent of the boundary regularity within the Lipschitz class of the associated domain, cf. [MB81, page 242]. On any account, this property persists under a bilipschitz transform $\Phi\colon \Omega_1 \to \Omega_2$, i.e. the spaces coincide on Ω_1 if and only if they coincide on Ω_2, cf. the arguments in [MB83, Theorem 7]. Here, bilipschitz means that Φ is bijective, Lipschitz and the inverse is Lipschitz, too.

Now we turn our attention to periodic domains. First, we give the main tools for the approach. Since the proof is actually very technical, we illustrate the approach by concrete examples first, including some pictures. After that, we give full technical details. We start with the following fundamental lemma [MB83, Theorem 6].

Lemma 3.2 *Let $q \in (1, \infty)$ and $\Omega \subseteq \mathbb{R}^d$ be a strongly local Lipschitz domain admitting the extensions of functions $p \in L^q_{\mathrm{loc}}(\Omega)$ with $\nabla p \in L^q(\Omega, \mathbb{C}^d)$ to functions in $p \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$ with $\nabla p \in L^q(\mathbb{R}^d, \mathbb{C}^d)$. Then $G^q(\Omega) = \widehat{G}^q(\Omega)$ and $L^{q'}_\sigma(\Omega) = \widehat{L}^{q'}_\sigma(\Omega)$, i.e.*

$$G^q(\Omega) = \overline{\nabla C^\infty_c(\overline{\Omega})}^{\|\cdot\|_q},$$
$$L^{q'}_\sigma(\Omega) = \{u \in L^{q'}(\Omega, \mathbb{C}^d) \mid \operatorname{div} u = 0 \textit{ in } \Omega,\ \nu \cdot u = 0 \textit{ on } \partial\Omega\}.$$

Proof: The relation $\widehat{G}^q(\Omega) \subseteq G^q(\Omega)$ is trivially always true. So, let $p \in L^q_{\text{loc}}(\Omega)$ with $\nabla p \in L^q(\Omega, \mathbb{C}^d)$. We denote by $\widetilde{p}$ the extension of p on $\mathbb{R}^d$ given by assumption. Note that we have $G^q(\mathbb{R}^d) = \overline{\nabla C_c^\infty(\mathbb{R}^d)}^{||\cdot||_q}$. Hence, $\nabla\widetilde{p}$ can be approximated by C_c^∞ functions, which trivially approximate ∇p on Ω, too. The statement $L^{q'}_\sigma(\Omega) = \widehat{L}^{q'}_\sigma(\Omega)$ follows by duality, compare Proposition 2.49. □

The second important tool for the forthcoming proof of the coincide for periodic domains is the Calderón-Stein theorem.

Theorem 3.3 *(Calderón-Stein theorem)*
Let $\Omega \subseteq \mathbb{R}^d$ be any domain having strongly local Lipschitz boundary and $q \in [1, \infty]$. Then there is a linear, continuous extension operator $E\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^d)$, i.e. $Eu|_\Omega = u$. By continuity,

$$||Eu||_{W^{1,q}(\mathbb{R}^d)} \le C_{\Omega,q}||u||_{W^{1,q}(\Omega)}, \qquad u \in W^{1,q}(\Omega).$$

Proof: The proof in the case $q \in (1, \infty)$ goes back to Calderón [Cal61]. A proof including the endpoints $q = 1, \infty$ was discovered by Stein [Ste70, chapter VI], using a completely different approach. □

Since we are interested in extensions of functions p where only $||\nabla p||_q$ is finite, we do not use Theorem 3.3 directly, but the following corollary concerning only gradient estimates.

Corollary 3.4 *(gradient estimates using the Calderón-Stein theorem)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $M \subseteq \mathbb{R}^d$ be bounded and measurable. Then, there is an extension operator $\widetilde{E}\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^d)$ as in Theorem 3.3 and a constant $C = C(\Omega, M, q)$ such that

$$||\nabla(\widetilde{E}u)||_{L^q(M,\mathbb{C}^d)} \le C||\nabla u||_{L^q(\Omega,\mathbb{C}^d)}, \qquad u \in W^{1,q}(\Omega).$$

Proof: Starting from the extension operator E given by Theorem 3.3, we construct the operator $\widetilde{E}$. We have to ensure that $\widetilde{E}c|_M = c$, where c denotes the constant function with value $c \in \mathbb{C}$. We take a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \equiv 1$ on $M \cup \Omega$ and define

$$\widetilde{E}u = E(u - u_\Omega) + \varphi u_\Omega,$$

where u_Ω denotes the mean of u on Ω. Obviously, $\widetilde{E}\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^d)$ is linear and continuous, where the continuity constant depends on the continuity constant of E and on the choice of φ. Hence,

$$||\nabla(\widetilde{E}u)||_{L^q(M)} \le C_{M,\Omega,q}\big(||u||_{L^q(\Omega)} + ||\nabla u||_{L^q(\Omega)}\big), \qquad u \in W^{1,q}(\Omega).$$

Now let $v := u - u_\Omega$. By linearity and the choice of φ we get $\widetilde{E}v = \widetilde{E}u - u_\Omega$ on M. Using the Poincaré inequality on Ω this leads for any $u \in W^{1,q}(\Omega)$ to

$$||\nabla(\widetilde{E}u)||_{L^q(M)} = ||\nabla(\widetilde{E}v)||_{L^q(M)} \le C_{\Omega,M,q}\big(||u - u_\Omega||_{L^q(\Omega)} + ||\nabla u||_{L^q(\Omega)}\big) \le C_{\Omega,M,q}||\nabla u||_{L^q(\Omega)}.$$

□

Now, we explain descriptively how Corollary 3.4 can be used to extend gradients in the setting of periodic domains. For this purpose we recall from Remark 2.44(b) that restrictions of functions from $\{p \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p}) \mid \nabla p \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)\}$ on bounded subdomains Ω are actually $W^{1,q}(\Omega)$ functions, i.e.

$$W^{1,q}(\Omega) = \{u \in L^q_{\mathrm{loc}}(\Omega) \mid \nabla u \in L^q(\Omega, \mathbb{C}^d)\}.$$

Now, we have the main tools at hand to explain how we extend functions from $G^q(\Omega_\mathrm{p})$ onto $G^q(\mathbb{R}^d)$. We start with the easy case that $\partial(0,1)^d \subseteq \Omega_\mathrm{p}$ and define $\Omega_1 = (0,1)^d \cap \Omega_\mathrm{p}$ and $M_1 = (0,1)^d \setminus \Omega_1$.

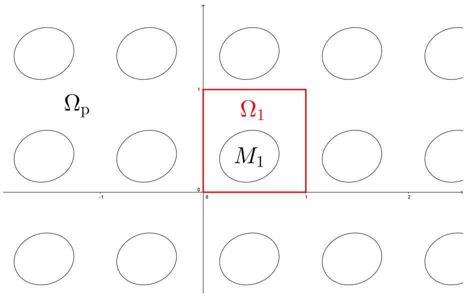


Figure 3.1: *Using Corollary 3.4 we can extend p from Ω_1 to M_1. Note that the values of p outside of Ω_1 are irrelevant for this extension. Besides we can estimate the gradient norm on M_1 by that of Ω_1.*

So, let $p \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})$ with $\nabla p \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$. By Remark 2.44(b) we have $p|_{\Omega_1} \in W^{1,q}(\Omega_1)$, hence Corollary 3.4 yields an extension $\widetilde{E}(p|_{\Omega_1}) \in W^{1,q}(\mathbb{R}^d)$ onto M_1 such that

$$||\nabla(\widetilde{E}(p|_{\Omega_1}))||_{L^q(M_1,\mathbb{C}^d)} \leq C||\nabla p||_{L^q(\Omega_1,\mathbb{C}^d)}.$$

We define

$$p_1(x) := \begin{cases} p(x), & \text{if } x \in \Omega_\mathrm{p}, \\ \widetilde{E}(p|_{\Omega_1})(x), & \text{if } x \in M_1. \end{cases}$$

By Corollary 3.4, this yields a function $p_1 \in W^{1,q}(\Omega_1 \cup M_1)$ accompanied by the estimate

$$||\nabla p_1||_{M_1} \leq C||\nabla p||_{\Omega_1}.$$

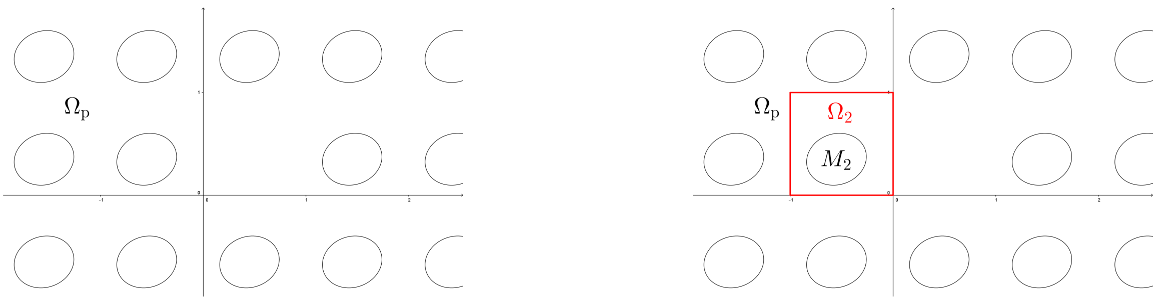


Figure 3.2: *Left: The set $\Omega_\mathrm{p} \cup M_1$. Right: In a next step, we pick a neighboured cell and continue as before.*

We consider a neighboured periodicity cell Ω_2 and by the same procedure we get an extension p_2 of p_1 on $\Omega_\mathrm{p} \cup M_1 \cup M_2$ satisfying

$$||\nabla p_2||_{M_2} \leq C||\nabla p||_{\Omega_2}.$$

Note that, by the periodicity of the domain, the constant C is the same as in the first estimate. Proceeding this approach allows us to get an extension $\widetilde{p}$ of p defined on the whole space $\mathbb{R}^d$. Besides, the gradient norm on M_z, $z \in \mathbb{Z}^d$ can be estimated by the gradient norm of the surrounding cell by Corollary 3.4 and hence $||\nabla\widetilde{p}||_{L^q(\mathbb{R}^d,\mathbb{C}^d)} \leq C||\nabla p||_{L^q(\Omega,\mathbb{C}^d)} < \infty$. Altogether, we get $\widetilde{p} \in \{u \in L^q_{\mathrm{loc}}(\mathbb{R}^d) \mid \nabla u \in L^q(\mathbb{R}^d, \mathbb{C}^d)\}$, i.e. $\nabla\widetilde{p} \in G^q(\mathbb{R}^d)$, which is what we needed.

It is possible to use slightly different variants of this procedure. Any bounded Lipschitz domain, which fully surrounds a periodically repeated compact component works, too. In particular, any periodic Lipschitz domain Ω_p whose complement consists of compact isolated components can be treated by this approach. Note that this is always the case for $d = 2$. Besides, the approach can be transferred easily to periodic domains with respect to any given lattice L. By Remark 3.1 the equality of spaces property remains true under any bilipschitz transformation, so it suffices anyhow to consider only $\mathbb{Z}^d$-periodic domains, cf. Remark 2.13.
If $d \geq 3$ the complement of Ω_p might have unbounded connected components. For example, the domain may consist of periodically arranged tubes of the same length.

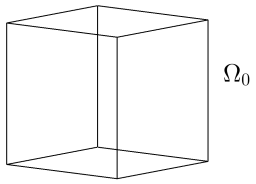


Figure 3.3: *Possible shape of a (three dimensional) periodicity cell, where the tubes are arranged as edges of a cube with non-zero diameter.*

The case considered above was easy to handle because the sets M_j and M_k had a positive distance for $j \neq k$. In the case that these sets do overlap, it is not possible to extend the function p independently on both, since it is not ensured that the extensions coincide on the intersection. In the next lines we introduce an approach which helps us to overcome this problem.

To illustrate our approach in that case we consider the following periodic open set $\Omega_\mathrm{p} \subseteq \mathbb{R}^2$. Note that Ω_p is not a domain, but this is not important for the illustration of the approach.

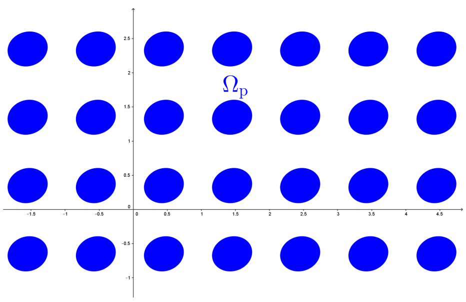


Figure 3.4: Ω_p *is a periodic open set with connected complement.*

So, let $p \in \{u \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})|\nabla u \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)\}$. We will extend p onto the whole space, accompanied by a gradient estimate. By Corollary 3.4 we can extend p in a first step on any bounded set M_1. For this extension we have to choose a bounded Lipschitz sub domain $\Omega \subseteq \Omega_{\mathrm{p}}$ fulfilling $\overline{\Omega} \cap \overline{M_1} = \overline{\Omega_{\mathrm{p}}} \cap \overline{M_1}$. For example, we chose $\Omega = (0,1)^d \cap \Omega_{\mathrm{p}}$ in the case above. In a second step we can extend p on a bounded set M_2. The problem is that we need countable many steps to cover Ω_p^C, and hence it is not ensured that the gradient norm of the extended function is finite (whereas in the case above it was possible, since there the gradient norms of M_j only depended on the gradient norm of Ω_j and in particular did not depend on each other). We are searching for an approach such that Ω_p^C is covered in finitely many steps. For this purpose we remark that we can extend p simultaneously on translated sets $M_i + z_i$, provided they do not intersect. A possible choice can be seen in the following picture.

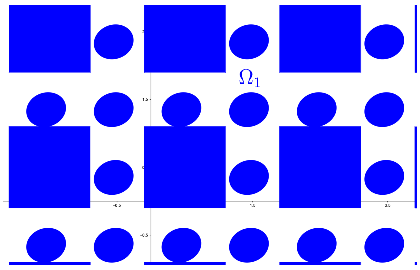


Figure 3.5: *Using Corollary 3.4 we have extended p onto the blue marked set with finite gradient norm. Note that we covered 'more than a fourth of the complement.'*

Here, we take a set M_1 satisfying $[0,1]^2 \subseteq M_1 \subseteq [-1/3, 4/3]^2$ and choose a bounded Lipschitz sub domain $\Omega \subseteq \Omega_{\mathrm{p}}$ satisfying $\left(\Omega_{\mathrm{p}} \cap (-1/3, 4/3)^2\right) \subseteq \Omega$. Furthermore, we consider translations of the form $M_1 + z$ and $\Omega + z$, where $z \in (2\mathbb{Z})^2$. This yields an extension p_1 of p on the set $\Omega_1 := \Omega_{\mathrm{p}} \cup \bigcup_{z \in (2\mathbb{Z})^2} (M_1 + z)$, and

$$||\nabla p_1||_{L^q(\cup_z M_1 + z)} \leq C ||\nabla p||_{L^q(\Omega_{\mathrm{p}})} < \infty.$$

In the next step we extend p on a set M_2 containing the neighboured cell $[0,1] \times [1,2]$ and translations of the form $M_2 + z$, $z \in (2\mathbb{Z})^2$. Once more, we choose the Lipschitz domain $\Omega \subseteq \Omega_1$ in Corollary 3.4 so that it touches the whole boundary of M_2. So, this approach yields an extension of p with finite gradient L^q norm defined on $\Omega_2 := \Omega_1 \cup \bigcup_{z \in (2\mathbb{Z})^2} (M_2 + z)$.

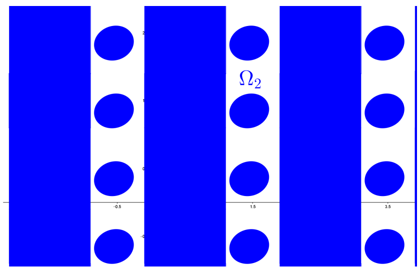


Figure 3.6: *We have extended p on a set, which is 'bigger than the half of Ω_{p}^C.'*

Next, we choose M_3 such that it contains $[1,2] \times [0,1]$, $\Omega \subseteq \Omega_2$ surrounding M_3 and the related $(2\mathbb{Z})^2$ translations. This yields the extension of p defined on $\Omega_3 := \Omega_2 \cup \bigcup_{z \in (2\mathbb{Z})^2} (M_3 + z)$.

Finally, we arrive again in the setting that the complement consists of separated compact sets, cf. Figure 3.7. Hence, we can extend p to $\widetilde{p}$ on the whole space. Note that $||\nabla(\widetilde{p})||_q$ is

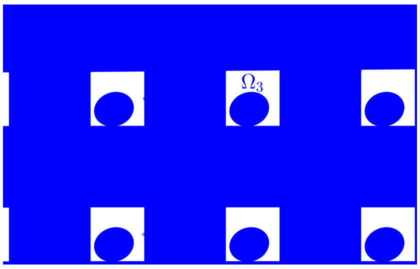


Figure 3.7: *We covered more than* $3/4$ *of* Ω_{p}^C. *Besides, the still missing parts are isolated components.*

finite, thanks to the estimate in Corollary 3.4, and hence $\nabla\widetilde{p} \in G^q(\mathbb{R}^d)$.

This approach works in higher dimensions, too. More exact we start with covering $[0,1]^d$ with M_1 and consider all translations of the form $M_1 + z$, $z \in (2\mathbb{Z})^d$. Afterwards we cover all neighboured cells of $[0,1]^d$ gradually. So, in total we need 2^d steps.

Now, we are coming to the technical details of the approach. We note that we have to choose the sets M_i in such a way that $\Omega_{\mathrm{p}} \cup M_1 \cup \ldots \cup M_i$ still has enough boundary regularity to allow the use of Corollary 3.4. In general, it is not possible to use any cuboid M_i, for example there might appear cusps or touching boundaries as in the following sketch.

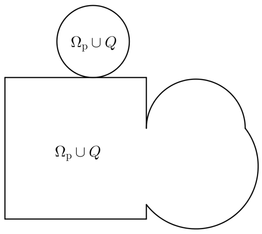


Figure 3.8: *Here, the union is not only not of Lipschitz class, it is even known that cusps obstruct the existence of an extension operator on Sobolev spaces [Jon81].*

To overcome this difficulty we enlarge the cube Q by adding some small cuboid upon the graph of the Lipschitz boundary of Ω_{p} at all points where ∂Q and $\partial\Omega_{\mathrm{p}}$ intersect, as sketched below.

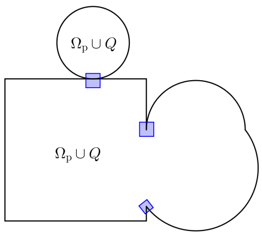


Figure 3.9: *By adding some small extra cuboids at every intersecting point of* Ω_{p} *and* Q *we again get a Jones domain as we shall see below. Besides, the compactness of the intersection at the boundary will yield that it suffices to add finitely many cuboids.*

Since the union of cuboids is not a Lipschitz domain in general, we work instead with domains of a weaker boundary regularity class, the so-called locally uniform domains. For this purpose we show that we have a statement analogous to Corollary 3.4 for locally uniform domains. Afterwards, we construct the sets M_i concretely as suggested in such a way that $\Omega_{\mathrm{p}} \cup M_1 \cup \ldots \cup M_i$ is locally uniform. We start with the introduction of locally uniform domains. Note that there is no explicit boundary regularity assumption stated in the definition. In fact, within this class are e.g. some fractals, cf. Figure 3.10 below.

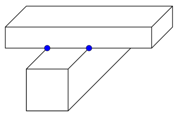
Definition 3.5 *(uniform and Jones domains)*
A domain $\Omega \subseteq \mathbb{R}^d$ is called (ε,δ)-locally uniform, where $\varepsilon \in (0,\infty)$, $\delta \in (0,\infty]$ if for any $x_1, x_2 \in \Omega$ satisfying $|x_1 - x_2| \leq \delta$ there is a rectifiable arc $\gamma \in \Omega$ of length $l(\gamma) \leq |x_1 - x_2|/\varepsilon$ such that for all $z \in \gamma$

$$dist(z, \partial\Omega) \geq \frac{\varepsilon|z - x_1||z - x_2|}{|x_1 - x_2|}.$$

In the case $\delta = \infty$, Ω is also called Jones domain or just uniform domain.

Obviously, every uniform domain is locally uniform. Note that in the bounded case the reverse statement is true, too [EE04, page 187]. That means every bounded locally uniform domain is uniform. Therefore, in the case of a bounded domain, we only use the notation Jones domain.

It is known that local uniformly Lipschitz domains are (ε,δ)-locally uniform for some values of ε, δ. Besides, the class is larger than the class of Lipschitz domains, for example the following domains are locally uniform.



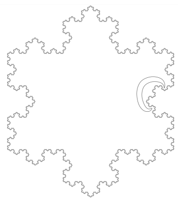
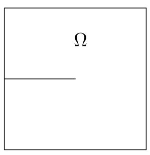


Figure 3.10: *Left: The boundary of the three dimensional polyhedron cannot be represented locally by a Lipschitz graph at the indicated points.*
Right: The Koch snowflake, which is a fractal (from [Rog04, page 10]).

Typical examples for domains which are not locally uniform are domains lying on both sides of the boundary or domains having cusps, compare the following pictures.



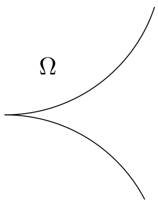


Figure 3.11: *Left: Domain lying on both sides of the boundary.*
Right: Domain having a cusp.

So, while extending the functions from Ω_p to $\Omega_\mathrm{p} \cup M_1 \cup \ldots \cup M_i$, we still have to avoid these cases. Before we do this, we show that we can use the theory of the beginning of this section for locally uniform domains instead of Lipschitz domains, which mainly means that the extension property and the Poincaré inequality hold.

Theorem 3.6 *(extension operator on Jones domains)*
Let $\Omega \subseteq \mathbb{R}^d$ be an (ε,δ)-locally uniform domain, $q \in [1,\infty]$. Then, there is a bounded, linear extension operator $E\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^d)$ and the norm bound depends on $q, d, \varepsilon, \delta$.

Proof: See [Jon81]. □

For an analogous use of Corollary 3.4 we need the Poincaré estimate to hold on bounded locally uniform domains.

Lemma 3.7 *(Poincaré inequality on Jones domains)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Jones domain, $q \in [1, \infty)$. Then there is a constant $C > 0$ such that

$$||u - u|_\Omega||_{L^q(\Omega)} \leq C||\nabla u||_{L^q(\Omega,\mathbb{C}^d)}, \qquad u \in W^{1,q}(\Omega).$$

Proof: A Jones domain is a John domain [SS90, page 2] in the sense defined in [SS90] and such domains fulfil the Poincaré estimate [SS90, Theorem 10]. □

Corollary 3.8 *(gradient estimate on Jones domains)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Jones domain and $M \subseteq \mathbb{R}^d$ be bounded and measurable. Then, there is an extension operator $\widetilde{E}\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^d)$ as in Theorem 3.3 and a constant $C = C(\Omega, M, q)$ such that

$$||\nabla(\widetilde{E}u)||_{L^q(M,\mathbb{C}^d)} \leq C||\nabla u||_{L^q(\Omega,\mathbb{C}^d)}, \qquad u \in W^{1,q}(\Omega).$$

Proof: Since we have the extension property for $W^{1,q}$-functions and the Poincaré inequality, compare Theorem 3.6 and Lemma 3.7, we may copy the proof from Corollary 3.4. □

We have the following easy criterion to prove that a bounded domain is a Jones domain.

Lemma 3.9 *(sufficient criterion for being a Jones domain)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain such that there are $\delta, C > 0$ with the following properties: For all $x, y \in \Omega$ with $|x - y| \leq \delta$ there is an rectifiable arc $\gamma \subseteq \Omega$ connecting x and y of length $l(\gamma) \leq C|x - y|$ such that for all $z \in \gamma$ at least one of the following two conditions holds.

- $|y - z| \leq Cd(z, \partial\Omega)$.
- $|x - z| \leq Cd(z, \partial\Omega)$.

Then, Ω is a Jones domain.

Proof: Let $x, y \in \Omega$ and $\gamma \subseteq \Omega$ a rectifiable arc given by the assumption and $z \in \gamma$. In the first case we have since $|z - x| \leq l(\gamma)$ the inequalities

$$\frac{|z - x||z - y|}{|x - y|} \leq C^2 \frac{|z - x|}{l(\gamma)} d(z, \partial\Omega) \leq C^2 d(z, \partial\Omega).$$

The estimate in the second case can be proven analogously. □

As already mentioned the union of two intersecting cuboids is a Jones domain. We give a short proof below.

Lemma 3.10 *Let $Q_1, Q_2 \subseteq \mathbb{R}^d$ be two intersecting cuboids. Then, $Q := Q_1 \cup Q_2$ is a Jones domain.*

Proof: Let $x, y \in Q$. The only interesting case is when $x \in Q_1$ and $y \in Q_2$. We can apply the previous Lemma. In fact, we can choose the arc lying in one plane as sketched below.

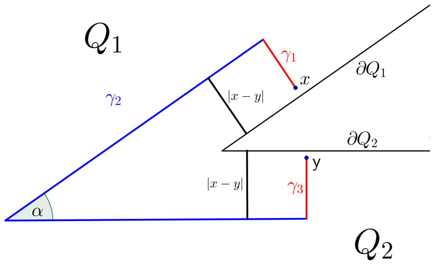


Figure 3.12: *Path connecting two points within two intersecting cuboids, which fulfils the inequalities from Lemma 3.9*

Note that we have to choose $\delta > 0$ so small that the whole arc γ is contained in Q and far enough away from other parts of the boundary. We get $l(\gamma) \leq C|x-y|$, where C only depends on the dimension and the intersection angle α of the two cuboids. Clearly, $|z-x| \leq d(z, \partial\Omega)$ for $z \in \gamma_1$ and $|z-y| \leq d(z,\partial\Omega)$ for $z \in \gamma_3$. For $z \in \gamma_2$ we have $|z-x| \leq l(\gamma) \leq C|x-y| \leq Cd(z,\partial\Omega)$. $\square$

We recall Figure 3.9. The last lemma stated that the union of two cuboids is a Jones domain, and the next lemma ensures that the union of some (small) cuboid added upon a small boundary part of a Lipschitz domains, is a Jones domain, cf. Figure 3.13.

Lemma 3.11 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $0 \in \partial\Omega$ and $d \geq 2$. We assume that there are $r, \beta > 0$ and $f\colon (-r,r)^{d-1} \to (-\beta,\beta)$ Lipschitz with $f(0) = 0$ such that*

$$\Omega \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{x \in (-r,r)^{d-1} \times (-\beta,\beta) \mid x_d < f(x_1,\dots,x_{d-1})\}.$$

Then, $\widetilde{\Omega} := \Omega \cup Q := \Omega \cup ((-r/2,r/2)^{d-1} \times (-\beta,\beta))$ is a Jones domain.

Proof: We will prove the fact constructively, by defining an arc with the properties demanded in Lemma 3.9. So, let $x, y \in \widetilde{\Omega}$. It suffices to prove the property locally, so we can assume $|x-y| < \delta < r/2$. Besides, there is nothing to prove if $x, y \in \Omega$ or $x, y \in (-r/2,r/2)^{d-1} \times (-\beta,\beta)$. So, let $x \in \Omega$ and $y \in \big((-r/2,r/2)^{d-1} \times (-\beta,\beta)\big) \setminus \Omega$.

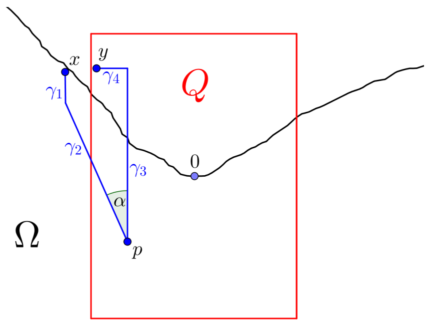


Figure 3.13: *The incline of γ_2 is determined by the Lipschitz constant L. This is important to keep distance to the boundary and hence allows us the use of Lemma 3.9. Besides, we choose $l(\gamma_4) = l(\gamma_1) = |x-y|$.*

The path is constructed as follows (cf. Figure 3.13): We divide the path γ in four parts: $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. We choose $\gamma_1(t) = x - t(0,0,\dots,0,1)$, where $t \in [0, |x-y|]$. The part γ_4 is defined by $\gamma_4(t) := y - t(y_1,\dots,y_{d-1},0)$, where $t \in [0, |x-y|/|(y_1,\dots,y_{d-1},0)|]$. We define $p \in \mathbb{R}^d$ as the point of intersection of the straight line $g := \gamma_4(|x-y|) + s(0,\dots,0,1)$,

$s \in \mathbb{R}$ and the cone surface

$$K := \{u \in \mathbb{R}^d \mid u_d - (\gamma_1(|x-y|))_d = -L|(u_1 - (\gamma_1(|x-y|))_1, \dots, u_{d-1} - (\gamma_1(|x-y|))_{d-1}|\},$$

where L denotes the Lipschitz constant of f. The parts γ_2 and γ_3 are defined as the line segments connecting p and $\gamma_1(|x-y|)$, $\gamma_4(|x-y|)$, respectively. We choose $\delta > 0$ so small that we have $p \in Q$ for all choices of $x \in \Omega$, $y \in Q \setminus \Omega$ with $|x-y| \leq \delta$ and that we keep away far enough from the remaining boundary (i.e. the boundary not occurring in Figure 3.13). We have $l(\gamma_1) = l(\gamma_4) = |x-y|$. To estimate the total length of γ we consider the following sketch.

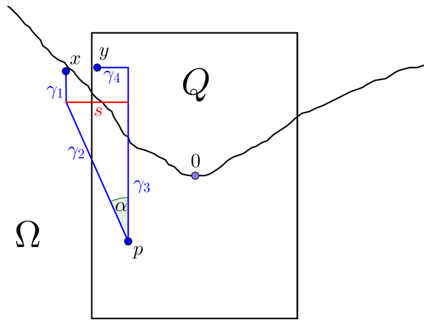


Figure 3.14: *By the triangle equality $|s| \leq 2|x-y|$ and hence the lengths of γ_2 and γ_3 are smaller than $C|x-y|$.*

We have $|s| \leq 2|x-y|$ and hence $l(\gamma_2), (\gamma_3) \leq C|x-y|$, where C only depends on the angle α and the angle α is independent of x, y, since it only depends on the Lipschitz constant L of the function f. This shows $l(\gamma) \leq C|x-y|$ for a constant $C > 0$, which is independent of the choice of x and y. Besides, $d(z, \partial\widetilde{\Omega}) \geq C_3|x-y|$ for all $z \in \gamma_2 \cup \gamma_3$ by construction and C_3 depends only on the Lipschitz constant of f. Furthermore, we have

$$|x-z| \leq C_4 d(z, \partial\widetilde{\Omega})$$

for $z \in \gamma_1$, where C_4 depends on the Lipschitz constant L and

$$|z-y| \leq d(z, \partial\widetilde{\Omega})$$

for $z \in \gamma_4$. All in all, we have verified the assumptions of Lemma 3.9. □

We remark that $\widetilde{\Omega}$, as defined in Lemma 3.11, is in fact even a Lipschitz domain. It is possible to represent the boundary at every point $x \in \partial\widetilde{\Omega}$ locally by the graph of a Lipschitz function. In fact, if $x \in \partial\Omega \cap \partial Q$ we rotate the coordinate system slightly and get, since the function f is Lipschitz, a representation of the boundary as a graph of a Lipschitz function. Now, we are able to prove the equality of gradient spaces on periodic domains. By duality we also get the coincide of the solenoidal vector field spaces.

Theorem 3.12 *(equality of spaces for periodic domains)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic Lipschitz domain. Then, we have $L^q_\sigma(\Omega_\mathrm{p}) = \widehat{L}^q_\sigma(\Omega_\mathrm{p})$ and $G^q(\Omega_\mathrm{p}) = \widehat{G}^q(\Omega_\mathrm{p})$ for all $q \in (1,\infty)$, i.e.

$$L^q_\sigma(\Omega) = \{f \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} f = 0 \text{ in } \Omega,\ \nu \cdot f = 0 \text{ on } \partial\Omega\},$$
$$G^q(\Omega) = \overline{\nabla C_c^\infty(\overline{\Omega})}^{\|\cdot\|_q}.$$

Proof: We apply Lemma 3.2, so we have to prove that every function $p \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})$ with $\nabla p \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ has an extension to a function $\widetilde{p} \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$ with $\nabla\widetilde{p} \in L^q(\mathbb{R}^d, \mathbb{C}^d)$. So, let $p \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})$ with $\nabla p \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$.
Let $\Omega_b \subseteq \Omega_\mathrm{p}$ be a bounded Lipschitz domain satisfying $\Omega_\mathrm{p} \cap [-10,10]^d \subseteq \Omega_b$. Note that Ω_b is just an auxiliary tool, which we need because Corollary 3.8 requires a bounded domain. Besides, we have to ensure that Ω_b has the same shape as Ω_p everywhere, where the boundary of Ω_{per} touches $\overline{M}$. Note that M denotes the set on which we extend p, see Corollary 3.8. We refer also to the explanation of the approach subsequent to Corollary 3.4.
We define $\widetilde{M} := (-4/3, 4/3)^d$ and consider the compact set $K := \partial\widetilde{M} \cap \partial\Omega_\mathrm{p} = \partial\widetilde{M} \cap \partial\Omega_b$. By using a small rotation, it is possible to find for all $x \in K$ one common coordinate system, where $\partial\widehat{M}$ and $\partial\Omega_\mathrm{p}$ are both locally described by Lipschitz functions, compare [GQ12, Proof of Lemma 4.7]. So, for all $x \in K$ there is a transformation of coordinates T consisting of rotations and translations such that $T(x) = 0$, $r, \beta \in (0, 1/15)$ and Lipschitz functions $f, g\colon (-r,r)^{d-1} \to (-\beta,\beta)$ such that

$$T(\Omega_\mathrm{p}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d < f(z_1 \dots, z_{d-1})\},$$
$$T(\partial\Omega_\mathrm{p}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d = f(z_1 \dots, z_{d-1})\},$$

and either

$$T(\widetilde{M}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d < g(z_1 \dots, z_{d-1})\},$$
$$T(\partial\widetilde{M}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d = g(z_1 \dots, z_{d-1})\},$$

or

$$T(\widetilde{M}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d > g(z_1 \dots, z_{d-1})\},$$
$$T(\partial\widetilde{M}) \cap \big((-r,r)^{d-1} \times (-\beta,\beta)\big) = \{y \in (-r,r)^{d-1} \times (-\beta,\beta) \mid z_d = g(z_1 \dots, z_{d-1})\}.$$

We fix for all $x \in K$ the cuboid K_x given by $K_x := T^{-1}\big((-r/2, r/2)^{d-1} \times (-\beta,\beta)\big)$. Since K is compact we find a finite covering of K by $K_{x_1}, \dots, K_{x_n}$, $x_1 \dots, x_n \in K$. We have $K_{x_1}, \dots K_{x_n} \subseteq [-7/5, 7/5]^d$, since $4/3 + 1/15 = 7/5$. We define $M_1 := \widetilde{M} \cup K_{x_1} \cup \ldots \cup K_{x_n}$. By Lemma 3.10 and Lemma 3.11 it follows that

$$\Omega_{1,0} := \Omega_b \cup M_1$$

is a Jones domain, since we can choose $\delta > 0$ so small that we have, for $x, y \in \Omega_{1,0}$ with $|x - y| \le \delta$, only to consider the following cases: $x, y \in \Omega_b$, x, y lying in one or two cuboids, or $x \in \Omega_b$ and y lying in a intersecting cuboid, constructed as above. In all these cases the existence of a connecting arc with the demanded properties is already proven, compare

Lemma 3.10 and Lemma 3.11. By Lemma 3.8 we can extend $p|_{\Omega_b}$ to $p_{1,0}$ defined on $\Omega_{1,0}$ and

$$||\nabla p_{1,0}||_{L^q(M_1,\mathbb{C}^d)} \leq C||\nabla p|_{\Omega_b}||_{L^q(\Omega_b,\mathbb{C}^d)}.$$

This approach yields an extension of p to $\Omega_\mathrm{p} \cup M_1$. By using translations of the form $\Omega_b + z$, $z \in (2\mathbb{Z})^d$, we can in an analogous manner, extend p on all sets $M_1 + z$ simultaneously. Altogether this yields a function $p_1 \in L^q_{\mathrm{loc}}(\Omega_1)$, where $\Omega_1 := \Omega_\mathrm{p} \cup \bigcup_{z\in(2\mathbb{Z})^d}(M_1 + z)$. Note that the $||p_1||_{L^q(M_1+z)}$ can be estimated by a constant independent of $z \in \mathbb{Z}^d$ times the gradient norm of p on the bounded surrounding $\Omega_b + z$.
Let $V^{1,q}(\Omega) := \{u \in L^q_{\mathrm{loc}}(\Omega) \mid \nabla u \in L^q(\Omega, \mathbb{C}^d)\}$.
Hence, there is a $C_1 > 0$ and an extension operator $E_1 \colon V^{1,q}(\Omega_\mathrm{p}) \to V^{1,q}(\Omega_1)$ such that

$$||\nabla(E_1 p)||_{L^q(\Omega_1,\mathbb{C}^d)} \leq C_1||\nabla p||_{L^q(\Omega_\mathrm{p},\mathbb{C}^d)}, \qquad p \in V^{1,q}(\Omega_\mathrm{p}).$$

By the same procedure we find a set M_2 with $(-1/3, 7/3) \times (-4/3, 4/3)^{d-1} \subseteq M_2 \subseteq [-2/5, 12/5] \times [-7/5, 7/5]^{d-1}$ such that $\Omega_2 := \Omega_1 \cup \bigcup_{k\in(2\mathbb{Z})^d}(M_2 + k)$ is locally a Jones domain and there is a $C_2 > 0$ and an extension operator $E_2 \colon V^{1,q}(\Omega_1) \to V^{1,q}(\Omega_2)$ satisfying

$$||\nabla(E_2 p)||_{L^q(\Omega_2,\mathbb{C}^d)} \leq C_2||\nabla p||_{L^q(\Omega_1,\mathbb{C}^d)}, \qquad p \in V^{1,q}(\Omega_1).$$

There are $M_3, M_4, \dots, M_{2^d}$, accompanied by extension operators $E_3, \dots, E_{2^d}$, covering, together with M_1, M_2 and all $(2\mathbb{Z}^d)$-translations of these sets, in total the whole $\mathbb{R}^d$.
Hence, we have proved the existence of an extension operator $E := E_{2^d} \circ \ldots \circ E_1 \colon V^{1,q}(\Omega_\mathrm{p}) \to \dot{W}^{1,q}(\mathbb{R}^d)$ such that

$$||\nabla(Ep)||_{L^q(\mathbb{R}^d)} \leq C_1 \cdot C_2 \ldots \cdot C_{2^d}||\nabla p||_{L^q(\Omega_\mathrm{p})}, \qquad p \in V^{1,q}(\Omega_\mathrm{p}),$$

and so the assertion follows from Lemma 3.2. □

By combining Remark 3.1 and Remark 2.13 the statements remain true for periodic domains with respect to any lattice L.

3.2 The Bloch transform and multiplier theorems

In the next lines we introduce the Bloch transform and show that we can adapt the results from [Bar13] to our case. This transform was already used by Bloch [Blo29] to study crystal lattices. Even before Floquet, Lyapunov and Hill used similar techniques to consider periodic structures. Therefore, other nomenclatures for the Bloch transform are common, too.

Besides, we assume throughout this section Ω_p to be a periodic domain and $q \in [1, \infty)$. We define the unit cube $B^d := [-1/2, 1/2)^d$. Before we come to the exact definition, we remark that the Bloch transform is a very common tool when considering all variants of periodic operators. We refer to [Kuc15] for a comprehensive overview. Roughly speaking, the idea of the Bloch theory is to transform functions on the whole periodic domain into a sequence of functions defined on one periodicity cell and apply Fourier theory there. The just mentioned transform has the following representation.

Lemma 3.13 *The operator* $\Gamma\colon L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) \to l^q(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d))$, *given by*

$$(\Gamma f)(z) = f(\cdot - z)|_{\Omega_\#},$$

is an isometric isomorphism. The inverse mapping is given by

$$(\Gamma^{-1}\varphi)(x) = \sum_{z\in\mathbb{Z}^d} E(\varphi(z))(x+z),$$

where $E\colon L^q(\Omega_\#, \mathbb{C}^d) = L^q(\Omega_0, \mathbb{C}^d) \to L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$ *denotes the zero extension for functions on* Ω_0 *to functions on* Ω_{p}.

We define

$$\begin{aligned} s(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d)) &:= \{\varphi \in l^\infty(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d)) \mid \sup_{z\in\mathbb{Z}^d} ||z^\alpha \varphi(z)||_q < \infty \text{ for all } \alpha \in \mathbb{N}_0^d\}, \\ L^q_s(\Omega_{\mathrm{p}}, \mathbb{C}^d) &:= \Gamma^{-1} s(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d)). \end{aligned}$$

Note that, for all $q \in [1, \infty)$, these space are dense in $l^q(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d))$, $L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$, respectively, since functions with finite respectively compact support are dense.

Definition 3.14 *We define the operator*

$$\begin{aligned} \Xi\colon L^q(B^d, L^q(\Omega_\#, \mathbb{C}^d)) &\to L^q(B^d, L^q(\Omega_\#, \mathbb{C}^d)), \\ f &\mapsto [\theta \mapsto [x \mapsto \mathrm{e}^{-2\pi\mathrm{i}\theta\cdot x} f(\theta, x)]]. \end{aligned}$$

Note that Ξ *is for all* $q \in [1, \infty]$ *an isometric isomorphism with inverse*

$$\begin{aligned} \Xi^{-1}\colon L^q(B^d, L^q(\Omega_\#, \mathbb{C}^d)) &\to L^q(B^d, L^q(\Omega_\#, \mathbb{C}^d)), \\ f &\mapsto [\theta \mapsto [x \mapsto \mathrm{e}^{2\pi\mathrm{i}\theta\cdot x} f(\theta, x)]]. \end{aligned}$$

Definition and Remark 3.15 *(Bloch transform)*
We define

$$\begin{aligned} \mathcal{F}^{-1}\colon s(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d)) &\to L^\infty(B^d, L^q(\Omega_\#, \mathbb{C}^d)), \\ [\mathcal{F}^{-1}g](\theta) &:= \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{2\pi\mathrm{i}z\cdot\theta} g(z). \end{aligned}$$

We denote by $D(B^d, L^q(\Omega_\#, \mathbb{C}^d))$ *the image of* $\mathcal{F}^{-1}$. *There is an explicit characterization [Bar13, Lemma 2.9] for that dense subspace of* $L^q(B^d, L^q(\Omega_\#, \mathbb{C}^d))$ *and one can show that*

$$\begin{aligned} \mathcal{F}\colon D(B^d, L^q(\Omega_\#, \mathbb{C}^d)) &\to s(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d)), \\ (\mathcal{F}u)(z) &:= \int_{B^d} \mathrm{e}^{-2\pi\mathrm{i}z\cdot\theta} u(\theta) d\theta \end{aligned}$$

is well-defined and is the inverse mapping to $\mathcal{F}^{-1}$ [Bar13, Lemma 2.17].

Hence, the Zak transform

$$Z\colon L^q_s(\Omega_{\mathrm{p}}, \mathbb{C}^d) \to D(B^d, L^q(\Omega_\#, \mathbb{C}^d)),$$
$$Zf := \mathcal{F}^{-1}\Gamma f,$$

is bijective, but not necessarily continuous. We define the Bloch transform

$$\Phi\colon L^q_s(\Omega_{\mathrm{p}}, \mathbb{C}^d) \to D(B^d, L^q(\Omega_\#, \mathbb{C}^d)),$$
$$\Phi f := \Xi\mathcal{F}^{-1}\Gamma f.$$

Φ is bijective, too, cf. Defintion 3.14. For $q = 2$, thanks to Plancherel's theorem, these operators extend to isometric isomorphisms on the whole space [Bar13, Lemma 2.18]. Note that Φ has the following description:

$$\Phi\colon L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d) \to L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)),$$
$$((\Phi f)(\theta))(x) = \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\theta\cdot(x-k)} f(x-k), \qquad \theta \in B^d,\ x \in \Omega_\#,\ f \in C_c^\infty(\Omega_{\mathrm{p}}, \mathbb{C}^d).$$

Note that Φ is quasi-periodic in θ and periodic in x, i.e.

$$((\Phi f)(\theta + z))(x) = \mathrm{e}^{2\pi\mathrm{i}\theta z}((\Phi f)(\theta))(x), \qquad ((\Phi f)(\theta))(x+z) = ((\Phi f)(\theta))(x),$$

and this is the reason why we restrict ourselves to values $\theta \in B^d = [-1/2, 1/2)^d$ and $x \in \Omega_\# = \Omega_{\mathrm{p}}/\mathbb{Z}^d$.

Note that it is also possible to define the Bloch transform associated to other lattices L. Using the adapted version of the Bloch transform, the whole forthcoming theory can be developed in an analogous manner for periodic domains with respect to any lattice. Since there occur enough other technical issues, we ignore this fact firstly und just refer to Remark 3.36

The Bloch transform Φ and the Zak transform Z are isometrically isomorphic, since Ξ is an isometric isomorphism. The advantage of the Zak transform is that the the fibre operators (cf. Definition 3.25) associated to the Zak transform are independent of θ, whereas the domains of the fibre operators depend on θ. In contrast, the fibre operators associated to the Bloch transform depend on θ, but the domains do not, cf. [Bar13, page 149]. In the forthcoming, we only use the Bloch transform.

The most common application of the Bloch transform is the decomposition of the spectrum of a periodic differential operator into bands. We refer to the discussion in Remark 3.36 and focus on Bloch multiplier theorems here. Before we state the adapted main theorems from [Bar13], we give a short summary of the approach there. A given operator T defined on L^q-spaces is periodic if and only if the operator $\widetilde{T} := \Gamma T \Gamma^{-1}$ defined on the sequence space l^q is translation invariant. Furthermore, as in the Fourier theory, translation invariant operators are exactly those operators which are given as a convolution operator, cf. [Bar13, Theorem 3.7]. This correlation can be used to show that, at least in some cases like Hilbert

spaces or subspaces of L^q, translation invariant operators can be described by L^∞ Fourier multipliers, cf. [Bar13, Theorem 3.12 and Theorem 3.18]. One possible criterion to transfer these results from subspaces to the whole space is the case of UMD-spaces and C^d-regularity of the multiplier on B^d, alternatively C^d-regularity of the multiplier on $B^d \setminus \{0\}$ and some R-boundedness condition, cf. [Bar13, Theorem 4.22] or the related version Theorem 3.22 in this work.
So, as we want to apply this machinery, we have at first to check that our underlying operator, the Helmholtz projection, is periodic. Furthermore, we have to check the UMD-property of the involved space. It is known that reflexivity of the Banach space is a necessary condition for the UMD-property. On the other hand, the most common spaces in analysis are UMD-spaces, if they are reflexive [KW04, page 11]. For that reason, the following lemma is no surprise.

Lemma 3.16 *(UMD-property of $L^q(\Omega_\#, \mathbb{C}^d)$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain, $q \in (1,\infty)$. Then, $L^q(\Omega_\#, \mathbb{C}^d)$ is a UMD-space.

Proof: See [Bar13, Prop 2.61 (iii)]. □

Remark 3.17 (periodicity of the Helmholtz projection)
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be periodic and P_2 the Helmholtz projection. The operator P_2 is $\mathbb{Z}^d$-periodic, which means

$$P_2 \tau_z f = \tau_z P_2 f, \qquad \text{for all } f \in L^2(\Omega_\mathrm{p}, \mathbb{C}^d),\ z \in \mathbb{Z}^d.$$

This can be seen directly. If $f = g + \nabla p$ is the Helmholtz decomposition of f, then

$$f(\cdot - z) = g(\cdot - z) + \nabla p(\cdot - z),$$

and $g(\cdot - z)$, $\nabla p(\cdot - z)$ belong to $L^2_\sigma(\Omega_\mathrm{p})$, $G^2(\Omega_\mathrm{p})$, respectively, thanks to the periodicity of the domain.

Definition 3.18 *(space of Fourier multipliers)*
Let E_1, E_2 be Banach spaces. The space $M_q(\mathbb{Z}^d, E_1, E_2)$ consists of all $m\colon B^d \to \mathcal{L}(E_1, E_2)$ such that, $T_m := \mathcal{F} \circ M_m \circ \mathcal{F}^{-1}$, first defined for $\varphi \in s(\mathbb{Z}^d, E_1)$, extends to a bounded, linear operator

$$T_m\colon l^q(\mathbb{Z}^d, E_1) \to l^q(\mathbb{Z}^d, E_2).$$

In that case, T_m is translation invariant, cf. [Bar13, Remark 4.20].
Here, $M_m\colon L^q(B^d, E_1) \to L^q(B^d, E_2)$ is defined by

$$(M_m f)(\theta) := m(\theta) f(\theta).$$

Theorem 3.19 *(Fourier multiplier theorem)*
Let $m \in C^d(B^d, \mathcal{L}(E_1, E_2))$ and E_1, E_2 UMD-spaces. Then, $m \in M_q(\mathbb{Z}^d, E_1, E_2)$ for all $q \in (1,\infty)$.

Proof: See [Bar13, Theorem 4.17 (i)]. □

Theorem 3.20 *(Mikhlin type Fourier multiplier theorem)*
Let E_1, E_2 be UMD-spaces and $m \in C^d(B^d \setminus \{0\}, \mathcal{L}(E_1, E_2))$ be such that the set

$$\tau = \{|\theta|^{|\alpha|}\partial^\alpha m(\theta) \colon \theta \in B^d \setminus \{0\},\ \alpha \le (1, \dots, 1)\}$$

is an R-bounded subset of $\mathcal{L}(E_1, E_2)$. Then, $m \in M_q(\mathbb{Z}^d, E_1, E_2)$ for all $q \in (1, \infty)$ and

$$||T_m||_{\mathcal{L}(l^q(\mathbb{Z}^d, E_1, E_2))} \le CR_q(\tau),$$

where $C = C(q, d, E_1, E_2)$.

Proof: See [Bar13, Theorem 4.19]. □

As noted in [Bar13, bottom of page 44] most of the analysis takes place in the sequence spaces. Therefore, it is possible to transfer the results from $\mathbb{R}^d/\mathbb{Z}^d$ to arbitrary $\Omega_\# = \Omega_\mathrm{p}/\mathbb{Z}^d$, where Ω_p is a periodic domain. Nevertheless, we give the details here. The next theorem is a variant of [Bar13, Theorem 4.21] for periodic domains and states that Fourier multipliers are Zak multipliers.

Theorem 3.21 *(Zak multiplier theorem)*
Let $q \in (1, \infty)$ and $m \in M_q(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d), L^q(\Omega_\#, \mathbb{C}^d))$ be bounded and measurable. Then, m is a Zak multiplication function, i.e.

$$Z_m f := Z^{-1} M_m Z f$$

first defined for $f \in L^q_s(\Omega_\mathrm{p}, \mathbb{C}^d)$, extends to a bounded, linear and translation invariant operator in $\mathcal{L}(L^q(\Omega_\mathrm{p}, \mathbb{C}^d))$.

Proof: We recall that $L^q(\Omega_\#, \mathbb{C}^d)$ is a UMD-space and that $\Gamma L^q_s(\Omega_\mathrm{p}, \mathbb{C}^d) = s(\mathbb{Z}^d, L^q(\Omega_\#, \mathbb{C}^d))$ by definition. Furthermore,

$$Z_m f = \Gamma^{-1} T_m \Gamma f.$$

Since both, Γ and Γ^{-1} are isometric isomorphisms, Z_m inherits the properties of T_m. The periodicity follows from the translation invariance of T_m and $\Gamma \tau_z f = \tau_{-z} \Gamma f$, plus $\Gamma^{-1}\tau_{-z} f = \tau_z \Gamma^{-1} f$. □

This theorem does not work in full generality for the Bloch transform. But in combination with Theorem 3.20 or respectively Theorem 3.19 it works.

Theorem 3.22 *(Bloch multiplier theorem)*
Let m be as in Theorem 3.20 or as in Theorem 3.19 and $q \in (1, \infty)$. Then,

$$B_m f := \Phi^{-1} M_m \Phi f, \qquad \text{first defined for } f \in L^q_s(\Omega_\mathrm{p}, \mathbb{C}^d),$$

extends to a bounded, linear and periodic operator $B_m \in \mathcal{L}(L^q(\Omega_\mathrm{p}, \mathbb{C}^d))$.

Proof: The case m given as in Theorem 3.19, works as in the proof of [Bar13, Theorem 4.22(i)] by applying Theorem 3.19 and Theorem 3.21 to the function $\Xi^{-1} m \Xi$.

The second case, works as in [Bar13, proof of Theorem 4.22(ii)], too. The main idea is to apply Theorem 3.20 to $\theta \mapsto \Xi^{-1}(\theta)m(\theta)\Xi(\theta)$ and afterwards Theorem 3.21, which proves the assertion, since $B_m = Z^{-1}T_{\Xi^{-1}m\Xi}Z$. So we have to show the premises of Theorem 3.20. The functions Ξ and Ξ^{-1} belong to $C^\infty(B^d, \mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d)))$ and the derivatives are given by

$$\partial^\alpha \Xi(\theta) = M_{x \mapsto (-2\pi \mathrm{i} x)^\alpha \mathrm{e}^{-2\pi \mathrm{i} x\cdot\theta}},$$
$$\partial^\alpha \Xi^{-1}(\theta) = M_{x \mapsto (2\pi \mathrm{i} x)^\alpha \mathrm{e}^{2\pi \mathrm{i} x\cdot\theta}},$$

which shows that $\partial^\alpha\Xi$ and $\partial^\alpha\Xi^{-1}$ are scalar multiplications on $L^q(\Omega_\#, \mathbb{C}^d)$ bounded by $(2\pi)^{|\alpha|}$. By [Bar13, Lemma 2.58(a)], the sets

$$\kappa_1 := \{|\theta|^{|\beta|}\partial^\beta \Xi(\theta) \mid \theta \in B^d,\ \beta \le (1, \dots, 1)\},$$
$$\kappa_2 := \{|\theta|^{|\beta|}\partial^\beta \Xi^{-1}(\theta) \mid \theta \in B^d,\ \beta \le (1, \dots, 1)\},$$

are R-bounded subsets of $\mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d))$ with constant $R_q(\kappa_i) \le 2(2\pi)^d$.
We split $|\theta|^{|\alpha|}[\partial^\alpha \Xi^{-1} m \Xi](\theta)$ into a finite sum with terms of the form

$$|\theta|^{|\gamma_1|}[\partial^{\gamma_1}\Xi^{-1}](\theta) \circ |\theta|^{|\gamma_1|}[\partial^{\gamma_2} m](\theta) \circ |\theta|^{|\gamma_1|}[\partial^{\alpha-\gamma_1-\gamma_2}\Xi](\theta),$$

where $\gamma_1 \le \alpha$ and $\gamma_2 \le \alpha - \gamma_1$. Since sums and concatenations of R-bounded sets are again R-bounded, we are done. $\square$

3.3 Fibre operators for the Helmholtz projection on $L^2(\Omega_\#, \mathbb{C}^d)$

At first, we explain the definition of fibre operators in an abstract setting.

Definition 3.23 *Let A, B, C be any sets and $Q\colon maps(A, B) \to maps(A, C)$. A family of operators $\{Q_{\mathrm{f}}(a)\colon B \to C\}_{a\in A}$ is called fibre operators of Q, if*

$$Q_{\mathrm{f}}(a)(\varphi(a)) = (Q\varphi)(a), \qquad \textit{for all } \varphi\colon A \to B,\ a \in A.$$

In our concrete situation we have L^q-spaces instead of general maps. The following easy standard example is taken from [Bar13, Theorem 1.2].

Example 3.24 (fibre operators associated to the gradient operator)
We consider the periodic operator $Au := \nabla u$ defined on $D(A) = H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$. We want to calculate the fibre operators of the operator $\Phi \circ A \circ \Phi^{-1}$, where the Bloch transform Φ is given as in Definition and Remark 3.15. We calculate

$$\begin{aligned}(\Phi A f)(\theta, x) &= \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i}\theta\cdot(x-z)}(\nabla f)(x-z) \\ &= \nabla \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i}\theta\cdot(x-z)} f(x-z)\end{aligned}$$

$$
\begin{aligned}
&= (\nabla + 2\pi \mathrm{i}\theta) \sum_{z \in \mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i}\theta\cdot(x-z)} f(x-z) \\
&= (\nabla + 2\pi \mathrm{i}\theta)(\Phi f)(\theta, x), \qquad\qquad f \in C_c^\infty(\mathbb{R}^d).
\end{aligned}
$$

Moreover, we get for $f \in D(A)$ and for almost all $\theta \in B^d = [-1/2, 1/2)^d$ that $\Phi f(\theta, \cdot) \in D(A(\theta)) := H^1_{\mathrm{per}}([0,1)^d)$. It follows that $A(\theta) = (\nabla + 2\pi \mathrm{i}\theta)$ are the fibre operators of the operator A, compare the forthcoming Definition 3.25. Note that the periodic boundary conditions of $D(A(\theta))$ are a consequence of the correlation between translation invariance and periodicity.
In the case of a periodic domain Ω_{p} this correlation will result in the fact that we consider the manifold $\Omega_\# = \Omega_{\mathrm{per}}/\mathbb{Z}^d$ instead of Ω_0.

As we know, the Helmholtz decomposition exists on $L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d)$ and the projection $Q = Q_2$ onto the gradient part defines a periodic operator in $\mathcal{L}(L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d))$. We take a closer look at the following diagram.

$$
\begin{tikzcd}
L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d) \arrow[r, shift left, "\Gamma"] \arrow[d, "Q"'] & l^2(\mathbb{Z}^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "\Gamma^{-1}"] \arrow[r, shift left, "F^{-1}"] & L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "F"] \arrow[r, shift left, "\Xi"] & L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "\Xi^{-1}"] \arrow[d, "\widetilde{Q}"] \\
L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d) \arrow[r, shift left, "\Gamma"] & l^2(\mathbb{Z}^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "\Gamma^{-1}"] \arrow[r, shift left, "F^{-1}"] & L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "F"] \arrow[r, shift left, "\Xi"] & L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)) \arrow[l, shift left, "\Xi^{-1}"]
\end{tikzcd}
$$

We recognize the Bloch transform $\Phi = \Xi \circ \mathcal{F}^{-1} \circ \Gamma$, which is an isometric isomorphism and define $\widetilde{Q} := \Phi \circ Q \circ \Phi^{-1}$. Therefore, it is quite natural to make the following definition.

Definition 3.25 *(fibre operators in the Bloch setting)*
The fibre operators of $\widetilde{Q}$ are called the fibre operators of Q, too. So, $Q(\theta)\colon L^2(\Omega_\#, \mathbb{C}^d) \to L^2(\Omega_\#, \mathbb{C}^d)$, $\theta \in B^d$ are the fibre operators associated to Q if and only if

$$
Q(\theta)(f(\theta)) = ((\Phi \circ Q \circ \Phi^{-1}) f)(\theta), \qquad f \in L^2(B^d, L^2(\Omega_\#, \mathbb{C}^d)),
$$

or equivalently, if and only if

$$
Q(\theta)(\Phi g(\theta)) = (\Phi(Qg))(\theta), \qquad\qquad g \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d).
$$

Once more, we recall that the existence of the fibre operators of the Helmholtz projection is due to the periodicity of the operator. This is because operators on the sequence spaces are given as multiplication operators if and only if they are translation invariant and we know that Γ transforms periodic operators into translation invariant operators and backwards. The theorem for bounded operators can be found in [Bar13, Theorem 3.12]. We remark that there is also a version for unbounded operators [Bar13, Theorem 3.26].
Next, we deduce a concrete formula for the fibre operators. For this purpose, we recall at first that the projection Q onto the gradient part is given by $Q = \nabla((\nabla)^*(\nabla))^{-1}\nabla^*$, cf. Theorem 2.43. Motivated by Example 3.24 it is natural to expect the following form for the fibre operators: $Q(\theta) = (\nabla + 2\pi \mathrm{i}\theta)((\nabla + 2\pi \mathrm{i}\theta)^*(\nabla + 2\pi \mathrm{i}\theta))^{-1}(\nabla + 2\pi \mathrm{i}\theta)^*$. Here, the problem of choosing the right domain for the operators $\nabla + 2\pi \mathrm{i}\theta$ arises. Clearly, we expect

periodic boundary conditions for the domain. We realize this by using the flat manifold with boundary $\Omega_\# = \Omega_\mathrm{p}/\mathbb{Z}^d$. In fact, we will see that the spaces $W^{1,2}(\Omega_\#)$ are the right choice, at least for $\theta \neq 0$. We observe that $\theta = 0$ is somehow a special point, since only there $\nabla + 2\pi\mathrm{i}\theta\colon W^{1,2}(\Omega_\#) \to L^2(\Omega_\#, \mathbb{C}^d)$ is not injective (actually the operator is not injective for all $\theta \in \mathbb{Z}^d$, but we are only interested in $\theta \in B^d = [-1/2, 1/2)^d$), cf. the proof of Lemma 3.27. For determining the fibre operators, it is helpful to consider $\nabla + 2\pi\mathrm{i}\theta$ on

$$W^{1,2}(\Omega_\#)_0 := \{f \in W^{1,2}(\Omega_\#) \mid \int_{\Omega_\#} f\,dx = 0\} \cong W^{1,2}(\Omega_\#)/\mathbb{C},$$

too. Note that we have $\dot{W}^{1,2}(\Omega_\#) = W^{1,2}(\Omega_\#)/\mathbb{C} \cong W^{1,2}(\Omega_\#)_0$ by Remark 2.44(b). We recall that these are the spaces occurring in the Helmholtz decomposition. Besides, the fibre operators for $\theta = 0$ are related to these spaces as we shall see below.
In the forthcoming diagram we chose to use $W^{1,2}(\Omega_\#)_0$ instead of $\dot{W}^{1,2}(\Omega_\#)$ (only on the first space $\nabla + 2\pi\mathrm{i}\theta$ is well-defined) and arrive at the following diagram.

$$\begin{tikzcd}
W^{1,2}(\Omega_\#)_0^* \arrow[dd, shift right, "R_0(\theta)"'] & & W^{1,2}(\Omega_\#)^* \arrow[ll, "\varrho"'] \arrow[dd, shift left, "R(\theta)"] \\
& L^2(\Omega_\#, \mathbb{C}^d) \arrow[ul, "(\nabla+2\pi\mathrm{i}\theta)_0^*"'] \arrow[ur, "(\nabla+2\pi\mathrm{i}\theta)^*"] & \\
W^{1,2}(\Omega_\#)_0 \arrow[uu, shift right, "N_0(\theta)"'] \arrow[ur, "(\nabla+2\pi\mathrm{i}\theta)_0"] \arrow[rr, shift left, "i"] & & W^{1,2}(\Omega_\#) \arrow[uu, shift left, "N(\theta)"] \arrow[ul, "\nabla+2\pi\mathrm{i}\theta"'] \arrow[ll, shift left, "\widetilde{m}"]
\end{tikzcd}$$

Figure 3.15: *Diagram showing fibre operators of Q. We will prove that these are given by $Q(\theta) = (\nabla + 2\pi\theta)R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^*$ for $\theta \in B^d \setminus \{0\}$ and by $Q_0(0) = \nabla_0 R_0(0)\nabla_0^*$ for $\theta = 0$.*

Here, we denote by i the inclusion operator and by ϱ the restriction operator. We denote by $m\colon W^{1,2}(\Omega_\#) \to \mathbb{C}$ the mean of a function and define the operator $\widetilde{m}$ by $\widetilde{m}(u) := u - m(u)$. Before we take a closer look at Figure 3.15, we first prove the existence of the inverse operators for $N_0(\theta) = (\nabla + 2\pi\mathrm{i}\theta)_0^*(\nabla + 2\pi\mathrm{i}\theta)_0$ and $N(\theta) = (\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta)$, which we denote by $R_0(\theta)$ and $R(\theta)$. For this purpose we need the following compactness theorem.

Lemma 3.26 *(Rellich-Kondrachov theorem)*
Let Ω_p be a periodic local Lipschitz domain and $q \in [1, \infty)$. Then, the embedding $W^{1,q}(\Omega_\#) \hookrightarrow L^q(\Omega_\#)$ is compact.

Proof: For $q \leq d$ the compactness of the embedding follows from

$$W^{1,q}(\Omega_\#) \subseteq W^{1,q}(\Omega_0) \hookrightarrow L^q(\Omega_0) = L^q(\Omega_\#),$$

and the Rellich-Kondrachov embedding theorem [Neč12, Chapter 2, Theorem 6.3], and for $q > d$ it is even more simple. In that case the space is compactly embedded into the space of continuous functions. Obviously, convergence in $||\cdot||_\infty$-norm on a bounded domain implies L^q convergence. $\square$

Lemma 3.27 *(invertibility of $N_0(\theta)$ and $N(\theta)$)*
The operators $N_0(\theta)$ and $N(\theta)$ are invertible for all $\theta \in B^d \setminus \{0\}$. Besides, $N_0(0)$ is invertible.
Furthermore, we have

$$L^2(\Omega_\#, \mathbb{C}^d) = (Kern(\nabla + 2\pi \mathrm{i}\theta)^*) \oplus (\nabla + 2\pi \mathrm{i}\theta) W^{1,2}(\Omega_\#),$$

and

$$\begin{aligned} Q(\theta) &= (\nabla + 2\pi \mathrm{i}\theta)((\nabla + 2\pi \mathrm{i}\theta)^*(\nabla + 2\pi \mathrm{i}\theta))^{-1}(\nabla + 2\pi \mathrm{i}\theta)^* \\ Q_0(\theta) &= (\nabla + 2\pi \mathrm{i}\theta)_0((\nabla + 2\pi \mathrm{i}\theta)_0^*(\nabla + 2\pi \mathrm{i}\theta)_0)^{-1}(\nabla + 2\pi \mathrm{i}\theta)_0^* \end{aligned}$$

are orthogonal projections in $L^2(\Omega_\#, \mathbb{C}^d)$.

Proof: We consider the sesquilinear form associated to $N(\theta)$, which is given by

$$a(u, v) = \langle (\nabla + 2\pi \mathrm{i}\theta)u, (\nabla + 2\pi \mathrm{i}\theta)v \rangle, \qquad u, v \in W^{1,2}(\Omega_\#) \subseteq L^2(\Omega_\#).$$

Hence,

$$a(u, u) = ||\nabla u||_2^2 + 4\pi^2|\theta|^2||u||_2^2 + 2\pi \int_\Omega \mathrm{i}\theta(u\overline{\nabla u} - \overline{u}\nabla u)dx.$$

We use the following easy consequence of the Cauchy-Schwarz and the binomial inequality

$$|\int_\Omega \mathrm{i}\theta(u\overline{\nabla u} - \overline{u}\nabla u)dx| \le 2\int_\Omega |\theta||u||\nabla u|dx \le 2|\theta|\varepsilon^2||\nabla u||_2^2 + 2\varepsilon^{-2}|\theta|||u||_2^2,$$

which holds true for all $\varepsilon > 0$. Hence,

$$a(u, u) \ge ||\nabla u||_2^2 + 4\pi^2|\theta|^2||u||_2^2 - 4\pi|\theta|\varepsilon^2||\nabla u||_2^2 - 4\pi\varepsilon^{-2}|\theta|||u||_2^2.$$

We choose $\varepsilon > 0$ such that $1 - 4\pi|\theta|\varepsilon^2 > 0$ holds true. Altogether, after a suitable choice of t, C we get

$$a(u, u) + t||u||_2^2 \ge C||u||_{W^{1,2}}^2 = C(||u||_2^2 + ||\nabla u||_2^2).$$

Because the domain of the form is compactly embedded into $L^2(\Omega_\#)$, cf. Lemma 3.26, $N(\theta) + t$ has discrete spectrum consisting of eigenvalues, cf. Lemma 2.25.
The solution $f(x) = \mathrm{e}^{-2\pi \mathrm{i}\theta \cdot x}$ of the equation $\nabla f = -2\pi \mathrm{i} f\theta$ is not $\mathbb{Z}^d$-periodic, since $\theta \notin \mathbb{Z}^d$. So, we know that 0 cannot be an eigenvalue of $N(\theta)$, and it follows that $N(\theta)$ is bijective. The bijectivity of $N_0(\theta)$ can be proven analogously. As already mentioned, $\nabla_0(0)$ is injective, whereas $\nabla(0)$ is not injective. That is why $N_0(0)$ is invertible and $N(0)$ is not. The projection property, which implies the decomposition of $L^2(\Omega_\#, \mathbb{C}^d)$ is obvious. Note that

$$Q(\theta)^* = (\nabla + 2\pi \mathrm{i}\theta)(((\nabla + 2\pi \mathrm{i}\theta)^*(\nabla + 2\pi \mathrm{i}\theta))^{-1})^*(\nabla + 2\pi \mathrm{i}\theta)^* = Q(\theta).$$

□

Using $i^* = \varrho$ and $(\nabla + 2\pi \mathrm{i}\theta)_0 = (\nabla + 2\pi \mathrm{i}\theta) \circ i$ we get

$$\begin{aligned}(\nabla + 2\pi \mathrm{i}\theta)_0^* &= \mathrm{i}^* \circ (\nabla + 2\pi \mathrm{i}\theta)^* = \varrho \circ (\nabla + 2\pi \mathrm{i}\theta)^*,\\ N_0(\theta) &= (\nabla + 2\pi \mathrm{i}\theta)_0^* \circ (\nabla + 2\pi \mathrm{i}\theta)_0 = \varrho \circ N(\theta) \circ i.\end{aligned}$$

Further we have for $\varphi \in W^{1,2}(\Omega_\#)^*$ with $R(\theta)\varphi \in W^{1,2}(\Omega_\#)_0$ the following relationship:

$$\begin{aligned}N_0(\theta)R(\theta)(\varphi) &= \varrho(\varphi),\\ R(\theta)(\varphi) &= (R_0(\theta) \circ \varrho)(\varphi).\end{aligned}$$

Note that $\widetilde{m}$ is an orthogonal projection in $W^{1,2}(\Omega_\#)$ with respect to the scalar product

$$\langle u, v\rangle_{W^{1,2}} := \langle \nabla u, \nabla v\rangle_{L^2} + \langle u, v\rangle_{L^2}$$

and $\widetilde{m}W^{1,2}(\Omega_\#) = W^{1,2}(\Omega_\#)_0$. The kernel of $\widetilde{m}$ is one-dimensional, more precisely it consists of all constant functions. Now, we are able to formulate and prove the following theorem, which describes the fibre operators of Q.

Theorem 3.28 *(representation for the fibre operators)*
Let Ω_p be a periodic local Lipschitz domain and $Q\colon L^2(\Omega_\mathrm{p}, \mathbb{C}^d) \to L^2(\Omega_\mathrm{p}, \mathbb{C}^d)$ be the projection onto the gradient part. Then, Q is periodic and the operators $Q(\theta) \in \mathcal{L}(L^2(\Omega_\#, \mathbb{C}^d))$ given by

$$Q(\theta) = (\nabla + 2\pi \mathrm{i}\theta)((\nabla + 2\pi \mathrm{i}\theta)^*(\nabla + 2\pi \mathrm{i}\theta))^{-1}(\nabla + 2\pi \mathrm{i}\theta)^*, \qquad \theta \in B^d \setminus \{0\},$$

are the fibre operators associated with Q. The fibre operator for $\theta = 0$ is the operator

$$Q_0(0) = \nabla_0(\nabla_0^*\nabla_0)^{-1}\nabla_0^*.$$

Here, $\nabla + 2\pi \mathrm{i}\theta\colon W^{1,2}(\Omega_\#) \to L^2(\Omega_\#, \mathbb{C}^d)$ and $(\nabla + 2\pi \mathrm{i}\theta)^*\colon L^2(\Omega_\#, \mathbb{C}^d) \to \big(W^{1,2}(\Omega_\#, \mathbb{C}^d)\big)^*$.

Proof: We start with $\theta \in B^d \setminus \{0\}$. We already know by Theorem 3.12 that $C^\infty_{c,\sigma}(\Omega_\mathrm{p}, \mathbb{C}^d)$ and $\nabla C^\infty_c(\overline{\Omega_\mathrm{p}})$ are dense in $L^2_\sigma(\Omega_\mathrm{p})$ and $G^2(\Omega_\mathrm{p})$, respectively. Hence the sum of the two orthogonal spaces is dense in $L^2(\Omega_\mathrm{p}, \mathbb{C}^d)$, too. So, it suffices to consider

$$f = f_0 + \nabla p,$$

where $f_0 \in C^\infty_{c,\sigma}(\Omega_\mathrm{p}, \mathbb{C}^d)$ and $p \in C^\infty_c(\overline{\Omega_\mathrm{p}})$. We use the Bloch transform Φ and get

$$\Phi f(\theta, \cdot) = (\Phi f_0)(\theta, \cdot) + \Phi(\nabla p)(\theta, \cdot).$$

At first, we show

$$\Phi(\nabla p)(\theta, \cdot) = (\nabla + 2\pi \mathrm{i}\theta)(\Phi p)(\theta, \cdot).$$

Since $p \in C_c^\infty(\overline{\Omega_{\mathrm{p}}})$ the following sum is finite and we calculate, cf. Example 3.24,

$$\begin{aligned}\Phi(\nabla p)(\theta,\cdot) &= \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot-z)}(\nabla p)(\cdot-z)\\ &= \sum_{z\in\mathbb{Z}^d} (\nabla+2\pi\mathrm{i}\theta)\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot-z)}p(\cdot-z)\\ &= (\nabla+2\pi\mathrm{i}\theta)\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot-z)}p(\cdot-z)\\ &= (\nabla+2\pi\mathrm{i}\theta)(\Phi p)(\theta,\cdot).\end{aligned}$$

If we can show that $(\Phi f_0)(\theta,\cdot)$ is lying in kernel$(\nabla+2\pi\mathrm{i}\theta)^*$, then

$$Q(\theta)\Phi(\nabla p)(\theta,\cdot) = Q(\theta)(\nabla+2\pi\mathrm{i}\theta)(\Phi p)(\theta,\cdot) = (\nabla+2\pi\mathrm{i}\theta)(\Phi p)(\theta,\cdot),$$

implies that

$$\begin{aligned}(\Phi Qf)(\theta,\cdot) &= Q(\theta)(\Phi f)(\theta,\cdot),\\ (\Phi Pf)(\theta,\cdot) &= P(\theta)(\Phi f)(\theta,\cdot),\end{aligned}$$

which is, because of Definition 3.25 exactly what we stated. We recall that $P = Id - Q$ denotes the Helmholtz projection on $L^2(\Omega_{\mathrm{p}},\mathbb{C}^d)$.

So we show that $(\Phi f_0)(\theta,\cdot) \in$ kernel$(\nabla+2\pi\mathrm{i}\theta)^*$. There is a function $\varphi \in C_c^\infty(\overline{\Omega_{\mathrm{p}}})$ satisfying $\sum_{k\in\mathbb{Z}^d}\varphi(\cdot-k) = 1$ on Ω_{p}. Let $g \in C_{per}^\infty(\overline{\Omega_{\mathrm{p}}})$. By periodicity, $g = \sum_k(\varphi g)(\cdot-k)$ on Ω_{p}. Hence, the set of functions $\{g|_{\Omega_\#} \mid g = \sum_k g_0(\cdot-k),\ g_0 \in C_c^\infty(\overline{\Omega_{\mathrm{p}}})\}$ is dense in $W^{1,2}(\Omega_\#)$. Let g be such a function. We calculate using integration by part

$$\begin{aligned}&\langle(\nabla+2\pi\mathrm{i}\theta)^*(\Phi f_0)(\theta,\cdot), g\rangle_{\Omega_\#}\\ &\quad= \langle(\Phi f_0)(\theta,\cdot), (\nabla+2\pi\mathrm{i}\theta)g\rangle_{\Omega_\#}\\ &\quad= \langle\sum_{k\in\mathbb{Z}^d}\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot-k)}f_0(\cdot-k), \sum_{k\in\mathbb{Z}^d}((\nabla+2\pi\mathrm{i}\theta)g_0)(\cdot-k)\rangle_{\Omega_\#}\\ &\quad= \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0, (\nabla+2\pi\mathrm{i}\theta)g_0\rangle_{\Omega_{\mathrm{p}}}\\ &\quad= \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0, \nabla g_0\rangle_{\Omega_{\mathrm{p}}} + \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0, 2\pi\mathrm{i}g_0\theta\rangle_{\Omega_{\mathrm{p}}}\\ &\quad= -\langle\mathrm{div}\,(\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0), g_0\rangle_{\Omega_{\mathrm{p}}} + \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0, 2\pi\mathrm{i}g_0\theta\rangle_{\Omega_{\mathrm{p}}}\\ &\quad= \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}2\pi\mathrm{i}\theta f_0, g_0\rangle_{\Omega_{\mathrm{p}}} + \langle\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0, 2\pi\mathrm{i}g_0\theta\rangle_{\Omega_{\mathrm{p}}}\\ &\quad= 0.\end{aligned}$$

We note that there are no boundary terms occurring, since f_0 has compact support in Ω_{p}. In the calculation above we used the relation

$$\mathrm{div}\,(\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}f_0) = \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\mathrm{div}\,f_0 - 2\pi\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\mathrm{i}\theta\cdot f_0 = -2\pi\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\mathrm{i}\theta\cdot f_0.$$

In the case $\theta = 0$ we do the same approach as above, and observe that

$$Q_0(0)(\Phi f)(0) = \nabla_0(\nabla_0^*\nabla_0)^{-1}\nabla_0^*\nabla_0\widetilde{m}(\Phi p)(0) = \nabla_0\widetilde{m}(\Phi p)(0) = \Phi(\nabla p)(0) = \Phi(Qf)(0).$$

$\square$

Now we are ready to give a decomposition of the fibre operators into a sum of two operators. The advantage of this decomposition is that the first summand consists of a family of operators, which is real analytic in θ for all $\theta \in B^d$, thus including $\theta = 0$. The second summand are projections onto a one-dimensional subspace, and therefore much better to handle. In fact, we are able to show the R-boundedness assumption of Theorem 3.22 for these operators.

Lemma 3.29 *(decomposition of the fibre operators)*
Let Ω_{p} be a periodic Lipschitz domain, Q be the projection on the gradient part defined on $L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d)$ and $Q(\theta)$ the fibre operators from Theorem 3.28 defined on $L^2(\Omega_\#, \mathbb{C}^d)$. Then, we have for $\theta \in B^d \setminus \{0\}$ and $f \in L^2(\Omega_\#, \mathbb{C}^d)$

$$\begin{aligned}
Q(\theta)f &= Q_0(\theta)f + m\big(R(\theta)(\nabla + 2\pi \mathrm{i}\theta)^* f\big)(2\pi \mathrm{i}\theta - Q_0(\theta)2\pi \mathrm{i}\theta) \\
&= Q_0(\theta)f + \frac{\langle f, P_0(\theta)2\pi \mathrm{i}\theta\rangle}{||P_0(\theta)(2\pi \mathrm{i}\theta)||^2} P_0(\theta)(2\pi \mathrm{i}\theta) \\
&= Q_0(\theta)f + T(\theta)f,
\end{aligned}$$

where

$$T(\theta)f := m\big(R(\theta)(\nabla + 2\pi \mathrm{i}\theta)^* f\big)(2\pi \mathrm{i}\theta - Q_0(\theta)2\pi \mathrm{i}\theta) = \frac{\langle f, P_0(\theta)2\pi \mathrm{i}\theta\rangle}{||P_0(\theta)(2\pi \mathrm{i}\theta)||^2} P_0(\theta)(2\pi \mathrm{i}\theta)$$

is an orthogonal projection. Note that $2\pi \mathrm{i}\theta$ is considered as a constant vector field here.

Proof: We take $\theta \neq 0$, $f \in L^2(\Omega_\#, \mathbb{C}^d)$ and define $\varphi := (\nabla + 2\pi \mathrm{i}\theta)^* f \in W^{1,2}(\Omega_\#)^*$, $u := R(\theta)(\nabla + 2\pi \mathrm{i}\theta)^* f = R(\theta)\varphi$. In particular we have $(\nabla + 2\pi \mathrm{i}\theta)u = Q(\theta)f$. To get to the right side of Figure 3.15 we consider $\widetilde{m}(u)$. We have

$$\begin{aligned}
(\nabla + 2\pi \mathrm{i}\theta)_0(\widetilde{m}(u)) &= (\nabla + 2\pi \mathrm{i}\theta)u - (\nabla + 2\pi \mathrm{i}\theta)m(u) \\
&= (\nabla + 2\pi \mathrm{i}\theta)u - m(u)2\pi \mathrm{i}\theta.
\end{aligned}$$

Hence

$$N_0(\theta)(\widetilde{m}(u)) = \varrho N(\theta)u - m(u)(\nabla + 2\pi \mathrm{i}\theta)_0^*(2\pi \mathrm{i}\theta),$$

which implies

$$\widetilde{m}(u) = R_0(\theta)\varrho N(\theta)u - m(u)R_0(\theta)(\nabla + 2\pi \mathrm{i}\theta)_0^*(2\pi \mathrm{i}\theta).$$

Now, we use the representation $u = R(\theta)\varphi$ and obtain

$$\widetilde{m}(R(\theta)\varphi) = R_0(\theta)\varrho\varphi - m(R(\theta)\varphi)R_0(\theta)(\nabla + 2\pi \mathrm{i}\theta)_0^* 2\pi \mathrm{i}\theta,$$

or equivalently

$$R(\theta)\varphi = R_0(\theta)\varrho\varphi + m(R(\theta)\varphi)(\mathbb{1} - R_0(\theta)(\nabla + 2\pi \mathrm{i}\theta)_0^* 2\pi \mathrm{i}\theta).$$

We use the relation $\varphi = (\nabla + 2\pi \mathrm{i}\theta)^* f$, and arrive after applying $\nabla + 2\pi\mathrm{i}\theta$ from the left side and using $\varrho \circ (\nabla + 2\pi\mathrm{i}\theta)^* = (\nabla + 2\pi\mathrm{i}\theta)_0^*$ at

$$Q(\theta)f = Q_0(\theta)f + m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* f)(2\pi\mathrm{i}\theta - Q_0(\theta)2\pi\mathrm{i}\theta)$$

for all $f \in L^2(\Omega_\#, \mathbb{C}^d)$ and $\theta \in B^d \setminus \{0\}$. The projection property of $T(\theta)$ follows from

$$T(\theta)^2 = (Q(\theta) - Q_0(\theta))^2 = Q(\theta)^2 - Q(\theta)Q_0(\theta) - Q_0(\theta)Q(\theta) + Q_0(\theta)^2 = Q(\theta) - Q_0(\theta) = T(\theta).$$

Here we used $Q(\theta)Q_0(\theta) = Q_0(\theta)$ and $Q_0(\theta)Q(\theta) = Q_0(\theta)$. The first one immediately follows by definition, since $(\nabla + 2\pi\mathrm{i}\theta)f = (\nabla + 2\pi\mathrm{i}\theta)_0 f$ for all $f \in W^{1,2}(\Omega_\#)_0$. The second follows from

$$Q_0(\theta)Q(\theta) = (Q_0(\theta)Q(\theta))^{**} = (Q(\theta)^* Q_0(\theta)^*)^* = (Q(\theta)Q_0(\theta))^* = Q_0(\theta)^* = Q_0(\theta).$$

Furthermore the projection is even orthogonal since

$$T(\theta)^* = Q(\theta)^* - Q_0(\theta)^* = Q(\theta) - Q_0(\theta) = T(\theta).$$

There is another way of describing this situation. Since $T(\theta)$ is the orthogonal projection onto the one dimensional subspace spanned by $P_0(\theta)2\pi\mathrm{i}\theta$ it must be of the following form:

$$T(\theta)f = \frac{\langle f, P_0(\theta)2\pi\mathrm{i}\theta\rangle}{||P_0(\theta)(2\pi\mathrm{i}\theta)||^2} P_0(\theta)(2\pi\mathrm{i}\theta).$$

$\square$

Remark 3.30 It is possible to show the projection property of $T(\theta)$ directly. By using $m(R_0(\theta)...) = 0$, which holds since $R_0(\theta)$ is mapping into $W^{1,2}(\Omega_\#)_0$, we calculate

$$\begin{aligned}
&m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^*(Id - Q_0(\theta))2\pi\mathrm{i}\theta) \\
&\quad = m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^*(Id - Q_0(\theta))(\nabla + 2\pi\mathrm{i}\theta)\mathbb{1}) \\
&\quad = m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta)\mathbb{1} - R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* Q_0(\theta)(\nabla + 2\pi\mathrm{i}\theta)\mathbb{1}) \\
&\quad = m(\mathbb{1} - R_0(\theta)(\nabla + 2\pi\mathrm{i}\theta)_0^*(\nabla + 2\pi\mathrm{i}\theta)\mathbb{1}) \\
&\quad = 1.
\end{aligned}$$

This yields

$$\begin{aligned}
T(\theta)(T(\theta)f) &= m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* f)P_0(\theta)2\pi\mathrm{i}\theta)P_0(\theta)(2\pi\mathrm{i}\theta) \\
&= m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* f) m(R(\theta)(\nabla + 2\pi\mathrm{i}\theta)^* P_0(\theta)2\pi\mathrm{i}\theta)P_0(\theta)(2\pi\mathrm{i}\theta) \\
&= T(\theta)f.
\end{aligned}$$

3.4 The fibre operators on $L^q(\Omega_\#, \mathbb{C}^d)$

In the previous section we proved a representation for the fibre operators $Q(\theta)$ of the projection Q onto the gradient space $G^2(\Omega_\mathrm{p})$. The fibre operators turned out to be orthogonal projections.

In this section we consider the situation on $L^q(\Omega_\#, \mathbb{C}^d)$. By applying perturbation theory, we prove existence and boundedness for the fibre operators on $L^q(\Omega_\#, \mathbb{C}^d)$ for some range of q depending on the boundary regularity as in Theorem 2.46.

Roughly speaking, the operator $\nabla + 2\pi\mathrm{i}\theta$ is a compact perturbation of the operator ∇, so Fredholm theory is applicable. Furthermore, we will show that we can apply Theorem 3.20, which will lead to the central results of this work, the Helmholtz decomposition on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ for periodic domains Ω_p. We recall Figure 3.15, the diagram for the fibre operators on L^2, and consider therefore for $\theta \in \mathbb{R}^d$ the following operator:

$$\begin{aligned} N_q(\theta) = (\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta)\colon W^{1,q}(\Omega_\#) &\to (W^{1,q'}(\Omega_\#))^*, \\ (N_q(\theta)(u))(v) &= \langle(\nabla + 2\pi\mathrm{i}\theta)u, (\nabla + 2\pi\mathrm{i}\theta)v\rangle. \end{aligned}$$

If q is fixed, we just will write $N(\theta)$. Analogously, we define

$$N_{0,q}(\theta)\colon W^{1,q}(\Omega_\#)_0 \to (W^{1,q'}(\Omega_\#)_0)^*,$$

where $W^{1,q}(\Omega_\#)_0 := \{u \in W^{1,q}(\Omega_\#) \mid \int_{\Omega_\#} u \, dx = 0\}$.
At first, before we use perturbation arguments, we recall the existence results for the Helmholtz decomposition on bounded domains, which yield the bijectivity of the operator $N_{0,q}(0)$.

Theorem 3.31 *(Helmholtz decomposition on $L^q(\Omega_\#)$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be periodic. Then, $N_{0,q}(0)\colon W^{1,q}(\Omega_\#)_0 \to (W^{1,q'}(\Omega_\#)_0)^*$ is an isomorphism for all $q \in (1,\infty)$ if $\Omega_\#$ has C^1-boundary. If $\Omega_\#$ has Lipschitz boundary, there is an $\varepsilon = \varepsilon(\Omega_\mathrm{p}) > 0$ such that the statement is true for $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$. If $d = 2$, then $\varepsilon > 1$.

Proof: Note that $\dot{W}^{1,q}(\Omega_\#) \cong W^{1,q}(\Omega_\#)_0$ by Remark 2.44b), and hence $N_{0,q}(0)$ is an isomorphism if and only if the Helmholtz decomposition exists on $L^q(\Omega_\#)$. The existence of the Helmholtz decomposition has been proven in [SS92] for C^1-domains and in [FMM98] for Lipschitz domains, compare Theorem 2.46. Both make use of the localization procedure, so it causes no problem that we consider the flat manifold with boundary $\Omega_\#$ instead of a domain in $\mathbb{R}^d$. For $d = 2$, cf. Theorem 2.46. □

Proposition 3.32 *(well-definedness of the fibre operators on L^q)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ a periodic domain with C^1-boundary and $q \in (1,\infty)$ or having Lipschitz boundary and $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$, where $\varepsilon > 0$ is given as in Theorem 3.31. The operators $Q(\theta)$ are even well-defined elements of $\mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d))$ for all $\theta \in B^d \setminus \{0\}$, i.e.

$$(\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta)\colon W^{1,q}(\Omega_\#) \to (W^{1,q'}(\Omega_\#))^*$$

are isomorphisms. The operators $Q_0(\theta)\colon W^{1,q}(\Omega_\#)_0 \to (W^{1,q'}(\Omega_\#))^*_0$ are well-defined for all $\theta \in B^d$.

Before we can do the proof of the proposition, we have to establish two Lemmas.

Lemma 3.33 *(compactness result)*
Let Ω_p and q be as in Proposition 3.32. Then, $N(\theta) - N(0)\colon W^{1,q}(\Omega_\#) \to (W^{1,q'}(\Omega_\#))^*$ is compact for all $\theta \in \mathbb{R}^d$. The same is true for $N_0(\theta) - N_0(0)\colon W^{1,q}(\Omega_\#)_0 \to (W^{1,q'}(\Omega_\#)_0)^*$.

Proof: Let $\theta \in \mathbb{R}^d$. By Lemma 3.26, the map $2\pi\mathrm{i}\theta\colon W^{1,q}(\Omega_\#) \to L^q(\Omega_\#, \mathbb{C}^d)$ is compact. By Schauder's theorem $(2\pi\mathrm{i}\theta)^*\colon L^q(\Omega_\#, \mathbb{C}^d) \to (W^{1,q'}(\Omega_\#))^*$ is compact, too. We note that $N(\theta) - N(0)$ is given by

$$N(\theta) - N(0) = (2\pi\mathrm{i}\theta)^*\nabla + \nabla^*(2\pi\mathrm{i}\theta) + (2\pi\mathrm{i}\theta)^*(2\pi\mathrm{i}\theta),$$

and hence is compact. We can replace N by N_0 and $W^{1,q}()$ by $W^{1,q}()_0$ to prove the last assertion. □

Lemma 3.34 *Let Ω_p and q be as in Proposition 3.32. Then, $N(0)$ is a Fredholm operator of index zero. More precisely,*

$$\begin{aligned}\mathrm{kernel}(N(0)) &= \{u \in W^{1,q}(\Omega_\#) \mid u \textit{ is constant}\},\\ \mathrm{image}(N(0)) &= \{\varphi \in (W^{1,q'}(\Omega_\#))^* \mid \varphi(u) = 0 \textit{ for all constant functions } u\}.\end{aligned}$$

Proof: Clearly, every constant function is in $\mathrm{kernel}(N(0))$ by definition of $N(0)$. Now, let $u \in W^{1,q}(\Omega_\#)$ be not constant, hence $u - m(u) \in W^{1,q}(\Omega_\#)_0$ is not zero. By Theorem 3.31 there is an $0 \neq v \in W^{1,q'}(\Omega_\#)_0$ satisfying $(N(0)(u - m(u)))(v) \neq 0$. From $N(0)(u) = N(0)(u - m(u))$ we get that u is not in $\mathrm{kernel}(N(0))$, so the first assertion is proved.
Again by definition of $N(0)$ it is clear, that $\varphi \in \mathrm{image}(N(0))$ satisfies $\varphi(u) = 0$ for all constant functions u. Now, let φ be in $(W^{1,q'}(\Omega_\#))^*$ such that $\varphi(u) = 0$ for all constant functions u. We can restrict φ to $W^{1,q'}(\Omega_\#)_0$. Again, by Theorem 3.31 we know that there is a function $u \in W_0^{1,q}(\Omega_\#)$ satisfying $\varphi(v) = \langle \nabla u, \nabla v\rangle$ for all $v \in W_0^{1,q'}(\Omega_\#)$. Since we know that $\nabla w = \varphi(w) = 0$ holds for all constant functions $w \in W^{1,q'}(\Omega_\#)$ this equality extends to all $v \in W^{1,q'}(\Omega_\#)$, since $W^{1,q}(\Omega_\#) = W^{1,q}(\Omega_\#)_0 \oplus \mathbb{C}$. So, we have shown that the dimension of the kernel and the cokernel is one, hence $N(0)$ is a Fredholm operator of index zero. □

Proof of Proposition 3.32: By Lemma 3.33 and Lemma 3.34 we get that $N(\theta)$ is a Fredholm operator of index zero. So, it suffices to show that $N(\theta)$ is injective or surjective. If $N(\theta)$ is injective, so is $N_0(\theta)$. Hence, it is enough to consider $N(\theta)$.
We begin with the case $q \geq 2$. Let $u \in \mathrm{kernel}(N_q(\theta))$. Then, $u \in W^{1,q}(\Omega_\#) \subseteq W^{1,2}(\Omega_\#) \subseteq W^{1,q'}(\Omega_\#)$ by Hölder's inequality. We have

$$||(\nabla + 2\pi\mathrm{i}\theta)u||_2^2 = \langle(\nabla + 2\pi\mathrm{i}\theta)u, (\nabla + 2\pi\mathrm{i}\theta)u\rangle = 0,$$

so $(\nabla + 2\pi\mathrm{i}\theta)u = 0$ and thus $u = 0$, compare the proof of Lemma 3.27. Hence $N(\theta)$ is injective. Now, we consider the case $q < 2$. We use a duality argument as follows.
We know that $N_{q'}(\theta)$, defined on $W^{1,q'}(\Omega_\#)$, is an isomorphism. In the next lines, we prove that $N_q(\theta)^*$ and $N_{q'}(\theta)$ are the same up to an isometric isomorphism, so $N_q(\theta)^*$ is an isomorphism and hence $N_q(\theta)$, too. We have $N_q(\theta)^*\colon (W^{1,q}(\Omega_\#))^{**} \to (W^{1,q'}(\Omega_\#))^*$. The

space $W^{1,q'}(\Omega_\#)$ is reflexive, so

$$J_{q'}: W^{1,q'}(\Omega_\#) \to (W^{1,q'}(\Omega_\#))^{**}, \quad w \mapsto [\varphi \mapsto \overline{\varphi(w)}]$$

is an isometric isomorphism. We show that $N_q(\theta)^* \circ J_{q'} = N_{q'}(\theta)$ holds. For this purpose, let $u \in W^{1,q'}(\Omega_\#)$, $v \in W^{1,q}(\Omega_\#)$. We have by definition

$$\begin{aligned}(N_q(\theta)^* \circ J_{q'}(u))(v) &= (N_q(\theta)^*(\varphi \mapsto \overline{\varphi(u)}))(v) = (\varphi \mapsto \overline{\varphi(u)})(N_q(\theta)(v)) \\ &= \overline{(N_q(\theta)(v))(u)} = \overline{\langle(\nabla + 2\pi\mathrm{i}\theta)v, (\nabla + 2\pi\mathrm{i}\theta)u\rangle} \\ &= N_{q'}(\theta)(u)(v).\end{aligned}$$

$\square$

Theorem 3.35 *(regularity of the fibre operators)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain with Lipschitz boundary, $q \in [2, 3+\varepsilon)$, where $\varepsilon > 0$ is given as in Theorem 3.31, or a C^1-domain and $q \geq 2$. We recall for $\theta \in B^d \setminus \{0\}$ the decomposition

$$\begin{aligned}Q(\theta)f &= Q_0(\theta)f + \frac{\langle f, P_0(\theta)2\pi\mathrm{i}\theta\rangle}{||P_0(\theta)(2\pi\mathrm{i}\theta)||^2} P_0(\theta)(2\pi\mathrm{i}\theta) \\ &= Q_0(\theta)f + T(\theta)f.\end{aligned}$$

(a) The operator Q_0 is real analytic in θ, in particular $Q_0 \in C^d(B^d, \mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d)))$. So, Theorem 3.22 applies to the operator Q_0.

(b) We have $T \in C^d(B^d \setminus \{0\}, \mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d)))$. Furthermore,

$$\tau := \{|\theta|^{|\alpha|}\partial^\alpha T(\theta) \mid \theta \in B^d,\ \alpha \leq (1, \dots, 1)\}$$

is an R-bounded subset of $\mathcal{L}(L^q(\Omega_\#, \mathbb{C}^d))$. So again, Theorem 3.22 applies to the operator T.

Proof: (a) The real analyticity follows by combining Example 2.28, Lemma 2.30 and Corollary 2.31. Furthermore we know that real analytic functions are C^∞, see Corollary 2.29.

(b) The real analyticity and hence the C^d-regularity follows as in (a). For the R-boundedness condition, we start with the case $\alpha = 0$. For proving it, we use Remark 2.35, so we have to show

$$||\Big(\sum_j |T(\theta_j)f_j|^2\Big)^{1/2}||_{L^q(\Omega_\#)} \leq C||\Big(\sum_j |f_j|^2\Big)^{1/2}||_{L^q(\Omega_\#)}, \qquad \theta_j \in B^d \setminus \{0\},\ f_j \in L^q(\Omega_\#, \mathbb{C}^d).$$

We define $g(\theta) = P_0(\theta)2\pi\mathrm{i}\theta$. Note that $\theta \mapsto g(\theta)$ is real analytic for all $\theta \in B^d$, hence in particular C^∞. Our starting point is the following representation of T.

$$T(\theta)f = \frac{\langle f, g(\theta)\rangle g(\theta)}{||g(\theta)||_2^2}.$$

By definition of T and Hölder's inequality we have

$$||T(\theta_j)f_j||_q = ||\langle f_j, g(\theta_j)\rangle g(\theta_j)||_q ||g(\theta_j)||_2^{-2} \leq ||f_j||_{q'} ||g(\theta_j)||_q^2 ||g(\theta_j)||_2^{-2}.$$

Since $q \geq 2$ we have by the Minkowski inequality

$$||\big(\sum_j |T(\theta_j)f_j|^2\big)^{1/2}||_q^2 = ||\sum_j |T(\theta_j)f_j|^2||_{q/2} \leq \sum_j || \, |T(\theta_j)f_j|^2 \, ||_{q/2} = \sum_j ||T(\theta_j)f_j||_q^2,$$

and hence

$$\begin{aligned} ||\big(\sum_j |T(\theta_j)f_j|^2\big)^{1/2}||_q &\leq \big(\sum_j ||T(\theta_j)f_j||_q^2\big)^{1/2} \\ &\leq \sup_j ||g(\theta_j)||_q^2 ||g(\theta_j)||_2^{-2} \big(\sum_j ||f_j||_{q'}^2\big)^{1/2} \\ &\leq \sup_j ||g(\theta_j)||_q^2 ||g(\theta_j)||_2^{-2} ||\big(\sum_j |f_j|^2\big)^{1/2}||_{q'} \\ &\leq C \sup_j ||g(\theta_j)||_q^2 ||g(\theta_j)||_2^{-2} ||\big(\sum_j |f_j|^2\big)^{1/2}||_q. \end{aligned}$$

Here we used the reverse Minkowski inequality (note that $q'/2 \leq 1$) and Hölder's inequality in the last steps. Finally, we have to estimate $\sup_j ||g(\theta_j)||_q^2 ||g(\theta_j)||_2^{-2}$ by a constant. We assume that to be impossible, i.e.

$$\sup_{\theta \in B^d \setminus \{0\}} \frac{||P_0(\theta)2\pi \mathrm{i}\theta||_q}{||P_0(\theta)2\pi \mathrm{i}\theta||_2} = \infty.$$

Since B^d is bounded, we find a converging sequence $\theta_k \to \theta_0$ satisfying $||P_0(\theta)2\pi\mathrm{i}\theta_k||_q \geq k||P_0(\theta_k)2\pi\mathrm{i}\theta_k||_2$ or equivalently $||P_0(\theta)\frac{\theta_k}{|\theta_k|}||_q \geq k||P_0(\theta_k)\frac{\theta_k}{|\theta_k|}||_2$. By analyticity of $P_0(\cdot)$ on B^d it follows that only $\theta_0 = 0$ is possible. Once more by compactness of $\overline{B^d}$, we can without restriction assume that $\frac{\theta_k}{|\theta_k|} \to \eta_0$. So we get $P_0(0)\eta_0 = 0$, which is equivalent to $\eta_0 = \nabla p$ for some function $p \in W^{1,q}(\Omega_\#)$. Since η_0 is a constant function, it follows that $p(x) = c_1x_1 + c_2x_2 + \ldots c_dx_d - c$ for some constants $c, c_1, \ldots c_d \in \mathbb{C}$. Since $\eta_0 \neq 0$ not all constants $c_1, \ldots, c_d$ are zero. But such a function is not periodic and hence not an element of $W^{1,q}(\Omega_\#)$. Consequently, we have a contradiction and therefore

$$\sup_{\theta \in B^d \setminus \{0\}} \frac{||P_0(\theta)2\pi \mathrm{i}\theta||_q}{||P_0(\theta)2\pi \mathrm{i}\theta||_2} < \infty.$$

So, the case $\alpha = 0$ is proven. We will prove the assertion by induction. For better transparency, we present the details of the proof in the case $|\alpha| = 1$, too.
We have to show

$$||\big(\sum_k | \, |\theta_k|^{|e_j|}\partial_j T(\theta_k)f_k \, |^2\big)^{1/2}||_q \leq C||\big(\sum_k |f_k|^2\big)^{1/2}||_q.$$

Since $q \geq 2$ the estimate from above yields

$$||\Big(\sum_k |\,|\theta_k|^{|e_j|}\partial_j T(\theta_k) f_k\,|^2\Big)^{1/2}||_q \leq \Big(\sum_k ||\,|\theta_k|^{|e_j|}\partial_j T(\theta_k) f_k\,||_q^2\Big)^{1/2},$$

so, again by the same estimates as above, it is sufficient to prove the estimate

$$||\,|\theta|^{|e_j|}\partial_j T(\theta) f\,||_q \leq C||f||_{q'}, \qquad \theta \in B^d \setminus \{0\},\ f \in L^q(\Omega_\#)$$

for a constant C independent of θ and f. We calculate

$$\begin{aligned}\partial_j T(\theta) f = &\frac{\langle f, \partial_j g(\theta)\rangle g(\theta) + \langle f, g(\theta)\rangle \partial_j g(\theta)}{||g(\theta)||_2^2} \\ &- \frac{\langle \partial_j g(\theta), g(\theta)\rangle + \langle g(\theta), \partial_j g(\theta)\rangle}{||g(\theta)||_2^4}\langle f, g(\theta)\rangle g(\theta),\end{aligned}$$

and arrive at

$$||\,|\theta|^{|e_j|}\partial_j T(\theta) f\,||_q \leq C\Big(\frac{||g(\theta)||_q}{||g(\theta)||_2}\frac{||\,|\theta|^{|e_j|}\partial_j g(\theta)\,||_q}{||g(\theta)||_2} + \frac{||g(\theta)||_q^3}{||g(\theta)||_2^3}\frac{||\,|\theta|^{|e_j|}\partial_j g(\theta)\,||_q}{||g(\theta)||_2}\Big)||f||_{q'}.$$

Since we already know that $\frac{||g(\theta)||_q}{||g(\theta)||_2}$ is bounded, it remains to prove

$$\sup_{\theta \in B^d \setminus \{0\}} \frac{||\,|\theta|^{|e_j|}\partial_j g(\theta)\,||_q}{||g(\theta)||_2} < \infty.$$

Again, we do a proof by contradiction and assume without restriction of generality that we have a sequence $\theta_k \to 0$ with $\theta_k/|\theta_k| \to \eta_0$ and $\frac{||\,|\theta_k|^{|e_j|}\partial_j g(\theta_k)||_q}{||g(\theta_k)||_2} \geq k$. Note that $|\theta_k|^{|e_j|} = |\theta_k|$. We divide nominator and denominator by $|\theta_k|$ and get that the fraction converges to $\frac{||\partial_j g(0)||_q}{||P_0(0)\eta_0||_2}$, which is finite.
The same fundamental idea is used in the induction for higher orders of partial derivatives. We are coming to the induction step now. For this purpose, let $0 \neq \alpha \in \mathbb{N}_0^d$ be a multi-index. Then we have using $h(\theta) = \langle g(\theta), g(\theta)\rangle^{-1}$ the representation

$$\partial^\alpha T(\theta) f = \sum_{\substack{\beta,\gamma \in \mathbb{N}_0^d \\ \beta+\gamma \leq \alpha}} C_{\beta,\gamma}\langle f, \partial^\beta g(\theta)\rangle \partial^\gamma g(\theta) \partial^{\alpha-\beta-\gamma} h(\theta)$$

for some constants $C_{\beta,\gamma}$. We point out the following characterization for the partial derivatives of h. For all multi-indexes α with $|\alpha| \geq 1$ it holds true

$$\partial^\alpha h(\theta) = \sum_{d=1}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_d=\alpha} C_{\beta_1,\ldots,\beta_d}\frac{\partial^{\beta_1}\langle g(\theta), g(\theta)\rangle \ldots \partial^{\beta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{d+1}}.$$

We prove it by induction over $|\alpha|$. We have

$$\partial_j h(\theta) = -\frac{\partial_j \langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^2}.$$

Now, we are coming to the induction step:

$$\begin{aligned}
&\partial_j \frac{\partial^{\beta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{\beta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)^{d+1}} \\
&= \frac{\partial^{e_j+\beta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{\beta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{d+1}} + \dots + \frac{\partial^{\beta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{e_j+\beta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{d+1}} \\
&- (d+1)\frac{\partial^{\beta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{\beta_d}\langle g(\theta), g(\theta)\rangle \langle g(\theta), g(\theta)\rangle^d \partial_j \langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{2d+2}}.
\end{aligned}$$

Hence, the induction for the representation of the derivatives of h is complete. So, $\partial^\alpha T(\theta) f$ has the representation

$$\sum_{\substack{\beta,\gamma \in \mathbb{N}_0^d \\ \beta+\gamma \leq \alpha}} C_{\beta,\gamma} \langle f, \partial^\beta g(\theta)\rangle \partial^\gamma g(\theta) \sum_{d=1}^{|\alpha|-|\beta|-|\gamma|} \sum_{\substack{\eta \in \mathbb{N}_0^d \\ \eta_1+\dots+\eta_d = \alpha-\beta-\gamma}} C_\eta h^{d+1}(\theta) \partial^{\eta_1} h^{-1}(\theta) \dots \partial^{\eta_d} h^{-1}(\theta).$$

To prove the stated R-boundedness condition we show

$$||\, |\theta|^{|\alpha|} S||_q \leq C||f||_{q'}$$

for every summand S of $\partial^\alpha T(\theta) f$ and a constant independent of θ and f. So, we look at just one summand of the form

$$\langle f, \partial^\beta g(\theta)\rangle \partial^\gamma g(\theta) \frac{\partial^{\eta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{\eta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{d+1}},$$

where $\eta_1 + \dots + \eta_d + \beta + \gamma = \alpha$. We estimate

$$\begin{aligned}
&||\, |\theta|^{|\alpha|} \langle f, \partial^\beta g(\theta)\rangle \partial^\gamma g(\theta) \frac{\partial^{\eta_1}\langle g(\theta), g(\theta)\rangle \dots \partial^{\eta_d}\langle g(\theta), g(\theta)\rangle}{\langle g(\theta), g(\theta)\rangle^{d+1}} \,||_q \\
&\leq \frac{||f||_{q'} ||\, |\theta|^{|\beta|}\partial^\beta g(\theta)\, ||_q \, ||\, |\theta|^{|\gamma|}\partial^\gamma g(\theta)\, ||_q \, \big|\, |\theta|^{|\eta_1|}\partial^{\eta_1}\langle g(\theta), g(\theta)\rangle \,\big| \dots \big|\, |\theta|^{|\eta_d|}\partial^{\eta_d}\langle g(\theta), g(\theta)\rangle \,\big|}{||g(\theta)||_2 ||g(\theta)||_2 ||g(\theta)||_2^{2d}}.
\end{aligned}$$

We know that we have

$$\sup_{\theta \in B^d \setminus \{0\}} \frac{||\, |\theta|^{|\beta|}\partial^\beta g(\theta)\, ||_q}{||g(\theta)||_2} < \infty, \qquad \sup_{\theta \in B^d \setminus \{0\}} \frac{||\, |\theta|^{|\gamma|}\partial^\gamma g(\theta)\, ||_q}{||g(\theta)||_2} < \infty,$$

by using the same contradiction proof as for $\alpha = 0$ and $|\alpha| = 1$. So, it remains to prove

$$\sup_\theta \frac{\big||\theta|^{|\eta|}\partial^\eta \langle g(\theta), g(\theta)\rangle\big|}{||g(\theta)||_2^2} < \infty.$$

The case $|\eta| \leq 1$ was already examined. If $|\eta| \geq 2$ we note that $\partial^\eta \langle g(\theta), g(\theta)\rangle$ is a sum of terms $\langle \partial^{z_1} g(\theta), \partial^{z_2} g(\theta)\rangle$, where $z_1 + z_2 = \eta$. We estimate

$$\frac{\big||\theta|^{|\eta|}\langle \partial^{z_1} g(\theta), \partial^{z_2} g(\theta)\rangle\big|}{||g(\theta)||_2^2} \leq |\theta|^{|\eta|-2} \frac{||\,|\theta|\partial^{z_1} g(\theta)\,||_2}{||g(\theta)||_2} \frac{||\,|\theta|\partial^{z_2} g(\theta)\,||_2}{||g(\theta)||_2},$$

which is bounded. □

Finally, we are ready to state the main theorem, the existence of the Helmholtz decomposition on periodic domains. Before we state the theorem, we shortly remark that this approach works also for periodic domains with respect to a lattice $L \neq \mathbb{Z}^d$.

Remark 3.36 Let L be any lattice with basis $\{b_1, \dots, b_d\}$ and $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ periodic with respect to L. Here, the Bloch transform is defined by

$$\begin{aligned} &\Phi\colon L^2(\Omega_\mathrm{p},\mathbb{C}^d) \to L^2(B_L^d, L^2(\Omega_{L,\#}, \mathbb{C}^d)),\\ &((\Phi f)(\theta))(x) = |B_L^d|^{-1} \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\theta\cdot(x-Mk)} f(x-Mk), \quad \theta \in B_L^d,\ x \in \Omega_{L,\#},\ f \in C_c^\infty(\Omega_\mathrm{p}, \mathbb{C}^d), \end{aligned}$$

where $\Omega_{L,\#} = \Omega_\mathrm{p}/L$, M is the matrix consisting of the basis vectors, and $|B_L^d|$ denotes the volume of one fundamentally mesh, the so-called Brillouin zone, of the reciprocal lattice B_L^d. The reciprocal lattice B_L^d of L is defined as the lattice having a basis $\{a_1, \dots, a_d\}$ so that $a_i \cdot b_j = \delta_{ij}$. The whole theory from Section 3.2 up to here could be adapted in an analogous manner to arbitrary (not necessarily $\mathbb{Z}^d$)-periodic domains. For example, $\nabla + 2\pi\mathrm{i}\theta\colon W^{1,q}(\Omega_{L,\#}) \to L^q(\Omega_{L,\#}, \mathbb{C}^d)$ is injective if and only if $\theta \notin B_L^d$.

Theorem 3.37 *(Helmholtz decomposition of L^q-vector fields on periodic domains)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain (with respect to some lattice L). The Helmholtz decomposition on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ exists for all $q \in I$, where

- $I = (1, \infty)$, *if* $\partial\Omega_\mathrm{p} \in C^1$.
- $I = (\frac{3+\varepsilon}{2+\varepsilon}, 3 + \varepsilon)$, *if $\partial\Omega_\mathrm{p}$ is Lipschitz and $\varepsilon = \varepsilon(\Omega_\mathrm{p}) > 0$. If $d = 2$, then $\varepsilon > 1$.*

Proof: By using Theorem 3.12 and Lemma 2.51 it suffices to check that P_2 extends to an operator $P_q \in \mathcal{L}(L^q(\Omega_\mathrm{p}, \mathbb{C}^d))$. Moreover, the duality argument from Corollary 2.50 yields that it is sufficient to prove the case $q \geq 2$.
By Lemma 3.29 and by combining the parts a) and b) from Theorem 3.35, the Bloch multiplier Theorem 3.22 yields that $Q \in \mathcal{L}(L^q(\Omega_\mathrm{p}, \mathbb{C}^d))$, so the proof is already finished, since $Q = Id - P_2 \in \mathcal{L}(L^2(\Omega_\mathrm{p}, \mathbb{C}^d))$ denotes the projection on the gradient part. □

3.5 The weak Dirichlet problem on periodic domains

At the end of this chapter we consider decompositions similar to the Helmholtz decomposition. In many applications one considers a system of equations containing only the equation div $u = 0$ but with different boundary conditions. The main result of this section is that there is a projection

$$L_q\colon L^q(\Omega_{\mathrm{p}},\mathbb{C}^d) \to \{u \in L^q(\Omega_{\mathrm{p}},\mathbb{C}^d) \mid \operatorname{div} u = 0\},$$

provided the periodic domain Ω_{p} has C^1-boundary and $q \in (1,\infty)$ or Lipschitz boundary and $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ if $d \geq 3$ and $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$ if $d = 2$, where $\varepsilon = \varepsilon(\Omega_{\mathrm{p}}) > 0$.
We call this projection Leray projection. Also the name Helmholtz-Leray projection or Helmholtz projection or Weyl projection are used for it. Naturally, since there is no restriction for the solenoidal vector fields at the boundary, it is straightforward to assume the complement to consist of gradients ∇p having zero trace.

In fact, Simader and Sohr [SS96] proved for bounded C^1-domains $\Omega \subseteq \mathbb{R}^d$, $q \in (1,\infty)$ the decomposition

$$\begin{aligned} L^q(\Omega,\mathbb{C}^d) &= \{u \in L^q(\Omega,\mathbb{C}^d) \mid \operatorname{div} u = 0\} \oplus \nabla W_0^{1,q}(\Omega) \\ &= \overline{C_{c,\sigma}^\infty(\mathbb{R}^d)|_\Omega}^{||\cdot||_q} \oplus \{\nabla p \in L^q(\Omega,\mathbb{C}^d) \mid p \in W_{\mathrm{loc}}^{1,q}(\overline{\Omega}),\ p|_{\partial\Omega} = 0\} \end{aligned}$$

and the spaces of solenoidal functions and gradients coincide, respectively. The Leray decomposition is related to the weak Dirichlet problem. As in the Helmholtz case, one has to be careful with the definition of the underlying spaces. We discuss this problem in Remark 3.39. So next, we give the definition of the emerging spaces.

Definition 3.38 *(spaces for the weak Dirichlet problem)*
Let $\Omega \subseteq \mathbb{R}^d$ be a strongly local Lipschitz domain and $q \in (1,\infty)$. We define

$$\begin{aligned} L_{\sigma,\mathrm{Dir}}^q(\Omega) &= \overline{C_{c,\sigma}^\infty(\mathbb{R}^d)|_\Omega}^{||\cdot||_q}, \\ G_{\mathrm{Dir}}^q(\Omega) &= \{\nabla p \in L^q(\Omega,\mathbb{C}^d) \mid p \in W_{\mathrm{loc}}^{1,q}(\overline{\Omega}),\ p|_{\partial\Omega} = 0\}, \\ \widehat{L}_{\sigma,\mathrm{Dir}}^q(\Omega) &= \{u \in L^q(\Omega,\mathbb{C}^d) \mid \operatorname{div} u = 0\}, \\ \widehat{G}_{\mathrm{Dir}}^q(\Omega) &= \overline{\nabla C_c^\infty(\Omega)}^{||\cdot||_q}. \end{aligned}$$

We say that the Leray decomposition exists on $L^q(\Omega,\mathbb{C}^d)$ if

$$L^q(\Omega,\mathbb{C}^d) = L_{\sigma,\mathrm{Dir}}^q(\Omega) \oplus G_{\mathrm{Dir}}^q(\Omega),$$

and the projections onto the subspaces are continuous.

As in the Helmholtz case, we give an overview of the theory in the bounded and exterior domain case before focussing on periodic domains. Then, we prove the equality of these spaces for periodic Lipschitz domains. As in the case of the Helmholtz decomposition, this is the 'door opener' for the proof of the L^q Leray decomposition on periodic domain. We summarize the main facts in the following remark. A main difference to the Helmholtz case is that here, the vector spaces do never coincide on exterior domains (for some $q \in (1,\infty)$).

Remark 3.39 (main known results for the weak Dirichlet problem)
As in Proposition 2.49 we see for strongly local Lipschitz domains Ω and $q \in (1,\infty)$ the relations

$$L^q_{\sigma,\mathrm{Dir}}(\Omega) = \big(G^{q'}_{\mathrm{Dir}}(\Omega)\big)^{\perp}, \qquad G^q_{\mathrm{Dir}}(\Omega) = \big(L^{q'}_{\sigma,\mathrm{Dir}}(\Omega)\big)^{\perp},$$
$$\widehat{L}^q_{\sigma,\mathrm{Dir}}(\Omega) = \big(\widehat{G}^{q'}_{\mathrm{Dir}}(\Omega)\big)^{\perp}, \qquad \widehat{G}^q_{\mathrm{Dir}}(\Omega) = \big(\widehat{L}^{q'}_{\sigma,\mathrm{Dir}}(\Omega)\big)^{\perp}.$$

In particular, the Leray projection exists on $L^2(\Omega,\mathbb{C}^d)$ if Ω is a strongly local Lipschitz domain and the spaces $L^2_{\sigma,\mathrm{Dir}}(\Omega), G^2_{\mathrm{Dir}}(\Omega)$ are orthogonal.
Clearly, there are equivalent characterizations for the existence of the Leray decomposition, compare Theorem 2.43. The existence of the Leray decomposition is equivalent to the following statements:

- For all $f \in L^q(\Omega,\mathbb{C}^d)$ there is a unique $\nabla p \in G^q_{\mathrm{Dir}}(\Omega)$ such that $\langle f - \nabla p, \nabla g\rangle = 0$ for all $\nabla g \in G^{q'}_{\mathrm{Dir}}(\Omega)$ and $||\nabla p|| \le C||f||$ for some constant $C > 0$ independent of f. This is the weak formulation of the classical Dirichlet problem.
- The operator $(\nabla_{q'})^*\nabla_q$ is an isomorphism, where

$$\dot{W}^{1,q}_{\mathrm{Dir}}(\Omega) \xrightarrow{\nabla_q} L^q(\Omega,\mathbb{C}^d) \cong (L^{q'}(\Omega,\mathbb{C}^d))^* \xrightarrow{(\nabla_{q'})^*} (\dot{W}^{1,q'}_{\mathrm{Dir}}(\Omega))^*,$$

  and $\dot{W}^{1,q}_{\mathrm{Dir}}(\Omega) = \{p \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}) \mid \nabla p \in L^q(\Omega,\mathbb{C}^d), p|_{\partial\Omega} = 0\}$ is equipped with the norm $||p|| := ||\nabla p||_q$.
- There is a constant $C > 0$ such that

$$||\nabla p||_{L^r(\Omega,\mathbb{C}^d)} \le C \sup_{0\neq g\in G^{r'}_{\mathrm{Dir}}} \frac{\langle \nabla p, \nabla g\rangle}{||\nabla g||_{r'}} \qquad \text{for all } \nabla p \in G^r_{\mathrm{Dir}}(\Omega)$$

  where $r = q, q'$.

Let Ω be a bounded Lipschitz domain and $q \in (1,\infty)$. Then, $L^q_{\sigma,\mathrm{Dir}}(\Omega) = \widehat{L}^q_{\sigma,\mathrm{Dir}}(\Omega)$ and $G^q_{\mathrm{Dir}}(\Omega) = \widehat{G}^q_{\mathrm{Dir}}(\Omega) = \nabla W^{1,q}_0(\Omega)$ [Soh01, Chapter II, 2.2.3. Lemma]. Besides, the Leray decomposition exists, and as in the case of the Helmholtz decomposition, this can be proven by using localization arguments [SS96, page 17].
If Ω is a bounded Lipschitz domain, then the Leray decomposition exists for $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ if $d = 3$ and for $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$ id $d = 2$, where $\varepsilon = \varepsilon(\Omega) > 0$ and these results are sharp in the same sense as for the weak Neumann problem. The examination of the weak Dirichlet problem for bounded Lipschitz domains goes back to Jerison and Kenig, compare [JK95, Theorem 0.5] for the positive results and [JK95, Theorem A.1] for the negative results.
In the case of exterior domains Ω the spaces $G^q_{\mathrm{Dir}}(\Omega)$ and $\widehat{G}^q_{\mathrm{Dir}}(\Omega)$ are equal for $q \ge d$ and they are not equal if $q \in [1,d)$. In fact, in the latter case $\nabla p \in \widehat{G}^q_{\mathrm{Dir}}(\Omega)$ implies $p \in L^{dq/(d-q)}(\Omega)$ [SS96, Chapter I, Theorem 2.8]. Hence, every function $p \in C^\infty(\Omega)$ with $p \equiv 1$ on $B(0,r)^C$ for some $r >> 0$ and $p|_{\partial\Omega} = 0$ lies in $G^q_{\mathrm{Dir}}(\Omega)$ but not in $\widehat{G}^q_{\mathrm{Dir}}(\Omega)$. This is the only counterexample in the following sense: For all exterior domains we have the

decomposition [SS96, Chapter I, Theorem 2.15]

$$G^q_{\text{Dir}}(\Omega) = \widehat{G}^q_{\text{Dir}}(\Omega) \oplus [\nabla p],$$

where $[\nabla p]$ denotes the one dimensional subspace spanned by a function as mentioned above.
Simader and Sohr demonstrated that the variational inequality is not satisfied on $\widehat{G}^q_{\text{Dir}}(\Omega)$ for $q \in (d, \infty)$ on exterior domains Ω. More precisely, there is a function $\nabla h \in \widehat{G}^q_{\text{Dir}}(\Omega)$ such that

$$\sup_{0 \neq \nabla p \in \widehat{G}^{q'}_{\text{Dir}}(\Omega)} \frac{\langle \nabla h, \nabla p \rangle}{||\nabla p||_{q'}} = 0,$$

and hence $L^q(\Omega, \mathbb{C}^d) \neq \widehat{L}^q_{\sigma,\text{Dir}}(\Omega) \oplus \widehat{G}^q_{\text{Dir}}(\Omega)$. However, the variational inequality holds true on $G^q_{\text{Dir}}(\Omega)$ for all exterior C^1-domains and all $q \in (1, \infty)$ [SS96, Chapter II, Theorem 1.1]. Consequently, the Leray decomposition

$$L^q(\Omega, \mathbb{C}^d) = L^q_{\sigma,\text{Dir}}(\Omega) \oplus G^q_{\text{Dir}}(\Omega)$$

holds. Once more we want to warn the reader that the notation is not consistent in the literature. So, other authors may write that the Leray respectively Weyl decomposition fails on exterior domains and they refer to the spaces with hat in our notation.

One of the main steps in the case of periodic domains was the proof of the density of C_c^∞-functions in the function spaces of gradients, respectively spaces of solenoidal vector fields. In the Dirichlet setting, this is also true and the proof of this statement is much easier since we can show a Poincaré inequality on the unbounded domain Ω_{p}, thanks to the Dirichlet boundary conditions, see Corollary 3.41 below.

Lemma 3.40 *(Poincaré implies $L^{q'}_{\sigma,\text{Dir}}(\Omega) = \widehat{L}^{q'}_{\sigma,\text{Dir}}(\Omega)$ and $G^q_{\sigma,\text{Dir}}(\Omega) = \widehat{G}^q_{\sigma,\text{Dir}}(\Omega)$)*
Let $\Omega \subseteq \mathbb{R}^d$ be a strongly local Lipschitz domain and $q \in (1, \infty)$. We recall

$$\dot{W}^{1,q}_{\text{Dir}}(\Omega) = \{p \in W^{1,q}_{\text{loc}}(\overline{\Omega}) \mid \nabla p \in L^q(\Omega, \mathbb{C}^d), p|_{\partial\Omega} = 0\}.$$

If the Poincaré estimate is fulfilled on Ω, i.e.

$$||u||_q \leq C||\nabla u||_q \qquad \textit{for all } u \in \dot{W}^{1,q}_{\text{Dir}}(\Omega),$$

then $L^{q'}_{\sigma,\text{Dir}}(\Omega) = \widehat{L}^{q'}_{\sigma,\text{Dir}}(\Omega)$ and $G^q_{\sigma,\text{Dir}}(\Omega) = \widehat{G}^q_{\sigma,\text{Dir}}(\Omega) = \nabla W^{1,q}_0(\Omega)$.

Proof: Clearly, by the Poincaré inequality we have

$$\widehat{G}^q_{\text{Dir}}(\Omega) \subseteq G^q_{\text{Dir}}(\Omega) = \nabla W^{1,q}_0(\Omega).$$

Since $\overline{C_c^\infty(\Omega_{\text{p}})}^{||\cdot||_{W^{1,q}}} = \{u \in W^{1,q}(\Omega_{\text{p}}) \mid u|_{\partial\Omega_{\text{p}}} = 0\}$, compare Lemma 2.6, the relation $\nabla W^{1,q}_0(\Omega) \subseteq \widehat{G}^q_{\text{Dir}}(\Omega)$ is trivial, too. The assertion about the $L^q_{\sigma,\text{Dir}}$-spaces follows by duality. $\square$

Corollary 3.41 *($L^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) = \widehat{L}^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p})$ and $G^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) = \widehat{G}^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p})$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic Lipschitz domain, $q \in (1,\infty)$. Then, the Poincaré inequality is fulfilled, in particular we have the space equalities

$$L^{q'}_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) = \widehat{L}^{q'}_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}), \qquad G^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) = \widehat{G}^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) = \nabla W^{1,q}_0(\Omega_\mathrm{p}).$$

Proof: Let $u \in L^q_{\mathrm{loc}}(\Omega_\mathrm{p})$ with $u|_{\Omega_\mathrm{p}} = 0$ and $\nabla u \in L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$. Let $\Omega_b \subseteq \Omega_\mathrm{p}$ be a bounded Lipschitz domain with $\Omega_\mathrm{p} \cap [0,1]^d \subseteq \Omega_b$. Since u is zero at some non-trivial part of the boundary, we can apply the Poincaré inequality from Lemma 2.7, i.e. there is a constant $C > 0$ such that

$$||u||_{L^q(\Omega_b)} \leq C||\nabla u||_{L^q(\Omega_b,\mathbb{C}^d)}.$$

By periodicity, this inequality applies on all sets $\Omega_b + k$, $k \in \mathbb{Z}^d$, i.e.

$$||u||_{L^q(\Omega_b+k)} \leq C||\nabla u||_{L^q(\Omega_b+k,\mathbb{C}^d)}, \qquad k \in \mathbb{Z}^d,$$

and C is independent of $k \in \mathbb{Z}^d$. By summing up, it follows $u \in L^q(\Omega_\mathrm{p})$ with accompanying Poincaré estimate and thus $u \in W^{1,q}(\Omega_\mathrm{p})$. □

In particular, we have for periodic Lipschitz domains $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ the orthogonal decomposition

$$L^2(\Omega_\mathrm{p}, \mathbb{C}^d) = L^2_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) \oplus G^2_{\mathrm{Dir}}(\Omega_\mathrm{p}) = L^2_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) \oplus \nabla W^{1,2}_0(\Omega_\mathrm{p}).$$

From now on we focus on the case of periodic domains $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ of Lipschitz class. We denote by Q_{Dir} the orthogonal projection $Q_{2,\mathrm{Dir}}$ mapping $L^2(\Omega_\mathrm{p}, \mathbb{C}^d)$ onto $G^2_{\mathrm{Dir}}(\Omega_\mathrm{p})$. The approach for the existence proof of the Leray decomposition on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$ is the same as in the Helmholtz decomposition case. So, we just recall the main steps and mention the differences occurring in comparison to the Helmholtz case.
As in the case of the Helmholtz decomposition the main idea is to extend the operator $Q_{2,\mathrm{Dir}} = Id - P_{2,\mathrm{Dir}}$ to a bounded operator on $L^q(\Omega_\mathrm{p}, \mathbb{C}^d)$. The following lemma states that this yields the Leray projection on L^q. Once more the crucial factor for this statement is the equality of the solenoidal vector field spaces and the gradient spaces.

Lemma 3.42 *Let $\Omega \subseteq \mathbb{R}^d$ be a strongly local Lipschitz domain, $q \in (1,\infty)$ such that $L^r_{\sigma,\mathrm{Dir}}(\Omega) = \widehat{L}^r_{\sigma,\mathrm{Dir}}(\Omega)$ and $G^r_{\mathrm{Dir}}(\Omega) = \widehat{G}^r_{\mathrm{Dir}}(\Omega)$ for $r = 2, q$. Besides we assume that the Leray projection $P_{2,\mathrm{Dir}}$ has a continuous extension to a bounded operator $P_{q,\mathrm{Dir}}$ in $\mathcal{L}(L^q(\Omega, \mathbb{C}^d))$. Then, the Leray decomposition exists on $L^q(\Omega, \mathbb{C}^d)$ and the Leray projection is given by $P_{q,\mathrm{Dir}}$.*

Proof: The proof can be copied line by line from Lemma 2.51. □

We continue by computing the fibre operators of the translation invariant operator Q_{Dir}. Not surprisingly, we will prove that they are given by

$$Q_{\mathrm{Dir}}(\theta) = (\nabla + 2\pi\mathrm{i}\theta)((\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta))^{-1}(\nabla + 2\pi\mathrm{i}\theta)^*,$$

where $\nabla + 2\pi\mathrm{i}\theta \colon W_0^{1,2}(\Omega_\#) \to L^2(\Omega_\#, \mathbb{C}^d)$ and $(\nabla + 2\pi\mathrm{i}\theta)^* \colon L^2(\Omega_\#, \mathbb{C}^d) \to \big(W_0^{1,2}(\Omega_\#)\big)^*$. At first, we examine the operator $N_{\mathrm{Dir}}(\theta) = (\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta)$.

Lemma 3.43 *Let $\Omega_{\mathrm{p}} \subseteq \mathbb{R}^d$ be a periodic Lipschitz domain and $\Omega_\# = \Omega_{\mathrm{p}}/\mathbb{Z}^d$. Then, the operators $N_{\mathrm{Dir}}(\theta)$, given as in the following diagram*

$$\begin{tikzcd}
& (W_0^{1,2'}(\Omega_\#))^* \\
L^2(\Omega_\#, \mathbb{C}^d) \arrow[ur, "(\nabla+2\pi\mathrm{i}\theta)^*"] & \\
& W_0^{1,2}(\Omega_\#)\ , \arrow[ul, "\nabla+2\pi\mathrm{i}\theta"'] \arrow[uu, "N_{\mathrm{Dir}}(\theta)"']
\end{tikzcd}$$

are invertible for all $\theta \in B^d$. More exact, we have

$$L^2(\Omega_\#, \mathbb{C}^d) = \mathrm{kernel}\,(\nabla + 2\pi\mathrm{i}\theta)^* \oplus (\nabla + 2\pi\mathrm{i}\theta) W_0^{1,2}(\Omega_\#),$$

and the operators

$$Q_{\mathrm{Dir}}(\theta) = (\nabla + 2\pi\mathrm{i}\theta)((\nabla + 2\pi\mathrm{i}\theta)^*(\nabla + 2\pi\mathrm{i}\theta))^{-1}(\nabla + 2\pi\mathrm{i}\theta)^*, \qquad \theta \in B^d,$$

are orthogonal projections.

Proof: Since the embedding $W_0^{1,2}(\Omega_\#) \hookrightarrow L^2(\Omega_\#)$ is compact, we can copy the proof from Lemma 3.27 for $\theta \neq 0$. We admit the case $\theta = 0$ here, and this case corresponds directly to the Leray projection on $L^2(\Omega_\#, \mathbb{C}^d)$ □

In contrast to the Helmholtz decomposition case, we do not have technical difficulties for $\theta = 0$. This is because $\nabla \colon W_0^{1,2}(\Omega_\#) \to L^2(\Omega_\#, \mathbb{C}^d)$ is already injective thanks to the Dirichlet boundary conditions. Note that in the case of the weak Neumann problem we had to factor out the constants, since the weak Neumann problem is only uniquely solvable up to an additive constant. Now, we can prove the representation of the fibre operators.

Lemma 3.44 *(representation of the fibre operators)*
Let Ω_{p} be a periodic Lipschitz domain. Then, the fibre operators of $Q_{\mathrm{Dir}} = Id - P_{\mathrm{Dir}} \in \mathcal{L}(L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d))$, $\theta \in B^d$ are given by

$$\begin{aligned} Q_{\mathrm{Dir}}(\theta) &= (\nabla + 2\pi\mathrm{i}\theta) N_{\mathrm{Dir}}(\theta)^{-1} (\nabla + 2\pi\mathrm{i}\theta)^*, \ \textit{i.e.} \\ (\Phi Q_{\mathrm{Dir}} f)(\theta, \cdot) &= Q_{\mathrm{Dir}}(\theta)(\Phi f)(\theta, \cdot). \end{aligned}$$

Proof: Here again, we nearly copy the proof from Theorem 3.28. We consider the Leray decomposition $f = f_0 + \nabla p$ in $L^2(\Omega_{\mathrm{p}}, \mathbb{C}^d)$. By density of the C_c^∞-functions we only need to consider $f_0 \in C_{c,\sigma}^\infty(\overline{\Omega_{\mathrm{p}}})$, $p \in C_c^\infty(\Omega_{\mathrm{p}})$. Clearly, $\Phi(\nabla p)(\theta, \cdot) = (\nabla + 2\pi\mathrm{i}\theta)(\Phi p)(\theta, \cdot)$. Next we show that $(\Phi f_0)(\theta, \cdot) \in \mathrm{kernel}\,(\nabla + 2\pi\mathrm{i}\theta)^*$. For this purpose we still proceed as in the proof of Theorem 3.28. Since, on a first glance, one cannot exclude the occurrence of non-zero

boundary integrals, we give full details. We shall see that in fact the boundary integrals vanish.
Let $\varphi \in C_c^\infty(\overline{\Omega_\mathrm{p}})$ be a function satisfying $\sum_{k\in\mathbb{Z}^d}\varphi(\cdot - k) = 1$ on Ω_p and $g \in C_{per}^\infty(\Omega_\mathrm{p})$ with $g|_{\partial\Omega_\mathrm{p}} = 0$ and $\operatorname{dist}(\operatorname{supp}(g), \partial\Omega_\mathrm{p}) > 0$. By periodicity $g = \sum_k (\varphi g)(\cdot - k)$ on Ω_p. Hence, the set of functions $\{g|_{\Omega_\#} \mid g = \sum_k g_0(\cdot - k),\ g_0 \in C_c^\infty(\Omega_\mathrm{p})\}$ is dense in $W_0^{1,2}(\Omega_\#)$. Let g be such a function. We calculate using integration by part

$$\begin{aligned}
&\langle(\nabla + 2\pi\mathrm{i}\theta)^*(\Phi f_0)(\theta,\cdot), g\rangle_{\Omega_\#}\\
&\quad = \langle(\Phi f_0)(\theta,\cdot), (\nabla + 2\pi\mathrm{i}\theta)g\rangle_{\Omega_\#}\\
&\quad = \langle \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot-k)} f_0(\cdot - k), \sum_{k\in\mathbb{Z}^d} ((\nabla + 2\pi\mathrm{i}\theta)g_0)(\cdot - k)\rangle_{\Omega_0}\rangle_{\Omega_\#}\\
&\quad = \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0, (\nabla + 2\pi\mathrm{i}\theta)g_0\rangle_{\Omega_\mathrm{p}}\\
&\quad = \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0, \nabla g_0\rangle_{\Omega_\mathrm{p}} + \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0, 2\pi\mathrm{i} g_0\theta\rangle_{\Omega_\mathrm{p}}\\
&\quad = -\langle \operatorname{div}(\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0), g_0\rangle_{\Omega_\mathrm{p}} + \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0, 2\pi\mathrm{i} g_0\theta\rangle_{\Omega_\mathrm{p}}\\
&\quad = \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} 2\pi\mathrm{i}\theta f_0, g_0\rangle_{\Omega_\mathrm{p}} + \langle \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0, 2\pi\mathrm{i} g_0\theta\rangle_{\Omega_\mathrm{p}}\\
&\quad = 0.
\end{aligned}$$

There are no boundary terms occurring, since g_0 has compact support in Ω_p. We recall that the boundary terms also vanished in the Helmholtz projection case, but there f had compact support. In the calculation above we used the relation

$$\operatorname{div}(\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)} f_0) = \mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\operatorname{div} f_0 - 2\pi\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\mathrm{i}\theta \cdot f_0 = -2\pi\mathrm{e}^{-2\pi\mathrm{i}\theta(\cdot)}\mathrm{i}\theta \cdot f_0.$$

□

Now we are ready to consider the fibre operators on L^q. In comparison to the Helmholtz case there are less technical difficulties to overcome, so we treat this topic quite briefly. We start with the bijectivity of the fibre operators for $\theta = 0$. The following theorem conforms Theorem 3.31 in the Helmholtz setting.

Theorem 3.45 *(Leray projection on $L^q(\Omega_\#, \mathbb{C}^d)$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic Lipschitz domain. Then,

$$N_{\mathrm{Dir},q}(0) = (\nabla_{q'})^*\nabla_q \colon W_0^{1,q}(\Omega_\#) \to (W_0^{1,q'}(\Omega_\#))^*$$

is an isomorphism for all $q \in (1,\infty)$ if Ω_p is a C^1-domain. In the Lipschitz case the statement holds for $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ if $d = 3$ and for $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$ if $d = 2$, where $\varepsilon = \varepsilon(\Omega_\mathrm{p}) > 0$.

Proof: The operator $N_{\mathrm{Dir},q}(0)$ is an isomorphism if and only if the Leray decomposition exists on $L^q(\Omega_\#, \mathbb{C}^d)$. The Leray projection exists for all bounded C^1-domains Ω by [SS96], and since the proof uses a localization procedure, the projections also exist for the flat manifold with C^1-boundary $\Omega_\#$.
The theory in the Lipschitz case goes back to [JK95], compare also Remark 3.39. The

problem is also covered by the work of Fabes, Mitrea and Mendez [FMM98, Theorem 11.2]. In particular, this problem localizes, too. □

Next, we prove that the fibre operators, as defined in the L^2-case, are well-defined on L^q-spaces, compare Lemma 3.33 for the analogue in the Helmholtz case.

Corollary 3.46 *(well-definedness of the fibre operators on L^q)*
The operators $Q_{\mathrm{Dir},q}(\theta)$ are well-defined operators in $\mathcal{L}(L^q(\Omega_\#,\mathbb{C}^d))$ for all $\theta \in B^d$, which means that the operators

$$(\nabla + 2\pi \mathrm{i}\theta)^*(\nabla + 2\pi \mathrm{i}\theta)\colon W_0^{1,q}(\Omega_\#) \to (W_0^{1,q'}(\Omega_\#))^*$$

are isomorphisms.

Proof: Using the compactness of $2\pi \mathrm{i}\theta\colon W_0^{1,q}(\Omega_\#) \hookrightarrow L^q(\Omega_\#,\mathbb{C}^d)$ and the compactness of $(2\pi \mathrm{i}\theta)^*\colon\ L^q(\Omega_\#,\mathbb{C}^d) \hookrightarrow (W_0^{1,q'}(\Omega_\#))^*$, it follows that $N_{\mathrm{Dir},q}(\theta) - N_{\mathrm{Dir},q}(0)$ is compact. Since $N_{\mathrm{Dir},q}(0)$ is an isomorphism, $N_{\mathrm{Dir},q}(\theta)$ is a Fredholm operator of index zero. So, it suffices to prove that $N_{\mathrm{Dir},q}(\theta)$ is injective for $q > 2$ and surjective for $q < 2$. This can be done by copying the proof of Proposition 3.32. □

So, the operators

$$Q_{\mathrm{Dir},q}(\theta) = (\nabla_q + 2\pi \mathrm{i}\theta)((\nabla_{q'} + 2\pi \mathrm{i}\theta)^*(\nabla_q + 2\pi \mathrm{i}\theta))^{-1}(\nabla_{q'} + 2\pi \mathrm{i}\theta)^*$$

are well-defined operators in $\mathcal{L}(L^q(\Omega_\#,\mathbb{C}^d))$. Now, we are in a position to show the regularity conditions for the mapping $\theta \mapsto Q_{\mathrm{Dir},q}(\theta)$, which guarantee that the operator $Q_{\mathrm{Dir},q}$ extends from L^2 to a bounded operator on L^q.

Theorem 3.47 *(analyticity of the fibre operators)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic Lipschitz domain and $Q_{\mathrm{Dir},q}(\theta) \in \mathcal{L}(L^q(\Omega_\#,\mathbb{C}^d)$ as defined above. Then, the mapping $\theta \mapsto Q_{\mathrm{Dir},q}(\theta)$ is real analytic in θ, in particular $Q_{\mathrm{Dir},q} \in C^d(B^d,\mathcal{L}(L^q(\Omega_\#,\mathbb{C}^d))$, compare Corollary 2.29.

Proof: The real analyticity follows directly by combining Example 2.28, Lemma 2.30 and Corollary 2.31, compare also the proof of the (a)-part of Theorem 3.35. □

We remark once more that this proof is in fact much easier than the analogue in the Helmholtz case, because it suffices to apply the (a)-part of the multiplier Theorem 3.35. This is because the fibre operators of the Leray projection do not have a singularity at zero. In particular, there is no R-boundedness condition to prove here. As in the Helmholtz case, the theory works also on periodic domains with respect to any lattice L, cf. Remark 3.36.

Theorem 3.48 *(Leray decomposition on periodic domains)*
Let Ω_p be a periodic (w.r.t. some lattice L) C^1-domain and $q \in (1,\infty)$. Then, the Leray decomposition

$$L^q(\Omega_\mathrm{p},\mathbb{C}^d) = L^q_{\sigma,\mathrm{Dir}}(\Omega_\mathrm{p}) \oplus G^q_{\mathrm{Dir}}(\Omega_\mathrm{p})$$

exists for all $q \in (1, \infty)$*. If* $\Omega_{\mathrm{p}} \subseteq \mathbb{R}^d$ *has Lipschitz boundary, there is an* $\varepsilon > 0$ *such that the Leray decomposition exists for all* $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ *if* $d \geq 3$ *and* $q \in (\frac{4+\varepsilon}{3+\varepsilon}, 4+\varepsilon)$ *if* $d = 2$*. Furthermore, we have*

$$\begin{aligned} L^q_{\sigma,\mathrm{Dir}}(\Omega_{\mathrm{p}}) &= \overline{C^\infty_{c,\sigma}(\mathbb{R}^d)|_{\Omega_{\mathrm{p}}}}^{\|\cdot\|_q} = \{u \in L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) \mid \operatorname{div} u = 0\}, \\ G^q_{\mathrm{Dir}}(\Omega_{\mathrm{p}}) &= \overline{\nabla C^\infty_c(\Omega_{\mathrm{p}})}^{\|\cdot\|_q} = \nabla W^{1,q}_0(\Omega_{\mathrm{p}}) \\ &= \{\nabla p \in L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d) \mid p \in L^q_{\mathrm{loc}}(\Omega_{\mathrm{p}}),\ p|_{\partial\Omega_{\mathrm{p}}} = 0\}. \end{aligned}$$

Proof: Theorem 3.47 allows us to use the multiplier Theorem 3.22, which proves that Q_{Dir} extends to a bounded operator in $\mathcal{L}(L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d))$. Since the stated characterizations for $L^q_{\sigma,\mathrm{Dir}}(\Omega)$ and $G^q_{\mathrm{Dir}}(\Omega)$ hold true for periodic domains (cf. Corollary 3.41), Lemma 3.42 yields the existence of the decomposition. □

We remark that there are more interesting problems in decomposition theory for vector fields. So, one could examine periodic domains with periodically repeated mixed boundary conditions, see for example [MM07]. There, the connected boundary splits into two parts, on one part one considers Neumann boundary conditions, on the other part Dirichlet boundary conditions. Note that this also leads to a Poincaré inequality, which gives hope to be able to prove a Helmholtz type decomposition, where the gradient space and solenoidal vector field space have to be equipped locally with the correct boundary conditions.

Sometimes, it is of interest to decompose the spaces $L^q_\sigma(\Omega)$ and $L^q_{\sigma,\mathrm{Dir}}(\Omega)$ into two subspaces, splitting out a curl part. More exactly, one can prove

$$\begin{aligned} L^q_\sigma(\Omega) = {}& \{u \in L^q_\sigma(\Omega) \mid \operatorname{curl} u = 0\} \\ & \oplus \operatorname{curl} \{u \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} u = 0, \operatorname{curl} u \in L^q(\Omega, \mathbb{C}^d),\ \nu \times u = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

and

$$\begin{aligned} L^q_{\sigma,\mathrm{Dir}}(\Omega) = {}& \{u \in L^q(\Omega, \mathbb{C}^d) \mid \operatorname{div} u = \operatorname{curl} u = 0,\ \nu \times u = 0 \text{ on } \partial\Omega\} \\ & \oplus \operatorname{curl} \{u \in L^q_\sigma(\Omega) \mid \operatorname{curl} u \in L^q(\Omega)\} \end{aligned}$$

for bounded C^∞-domains $\Omega \subseteq \mathbb{R}^3$ and all $q \in (1, \infty)$ [KY09, Corollary 1.1].

These decompositions are related to the so-called Hodge decompositions

$$L^q(\Lambda^l M) = dW^{1,q}_{\mathrm{bc1}}(\Lambda^{l-1} M) \oplus d^* W^{1,q}_{\mathrm{bc2}}(\Lambda^{l+1} M) \oplus \mathcal{H}^q_{\mathrm{bc3}}(\Lambda^l M),$$

for differential forms, which exist on bounded manifolds M with sufficient smooth boundary. Here, bc1, bc2 and bc3 denote some (suitable) boundary conditions. The space $\mathcal{H}$ consists of differential form u with zero Hodge-Laplacian, i.e. $d^*d + dd^*u = 0$. An important statement within the theory is that the dimension of the space $\mathcal{H}^q_{\mathrm{bc3}}(\Lambda^l M)$ is finite, and the dimension is related to the topology of the domain. Note that in the three dimensional case, one can interpret 1-forms and 2-forms as vector fields, zero forms as scalar functions, d_0 as ∇, d_1^* as curl and the Hodge-Laplacian $d_1^* d_1 + d_0 d_0^*$ equals the vectorial Laplacian. Since the Hodge decomposition is related to the Helmholtz decomposition, it is an important tool in the study of the Navier-Stokes equation on Riemann manifolds, cf. [MT01]. We did not examine the question if such types of decompositions exist for periodic domains Ω_{p}. Clearly, one cannot expect the dimension of $\mathcal{H}^q_{\mathrm{bc3}}(\Lambda^l M)$ to be finite, since Ω_{p} might have infinitely many holes. The Hodge-decomposition was developed by Hodge, Kodaira and de Rham, cf. [Mit04] for more information.

CHAPTER 4

Applications

This chapter is structured as follows. We start with the mathematical description of photonic crystals and establish the Maxwell equations in this setting. Once more, we remark that the physical background can be found in the physical appendix, while the physical motivation of periodic domains can be found in the introduction.
Here, we use a form method ansatz to define a self-adjoint, positive operator A_2 given by $A_2\, u := \text{curl } \varepsilon^{-1}\text{curl } u - \nabla\text{div } u$, and define the Maxwell operator $M_2 = A_2$ on $L^2_\sigma(\Omega_{\mathrm{p}})$. Furthermore, $-A_2$ generates a bounded analytic contraction semigroup of angle $\pi/2$. We prove that M_2 inherits these properties, which is essentially true since A_2 and P_2 commute. This fact in turn holds since the boundary conditions of $D(A_2)$ match to the Helmholtz projection.

We use generalized Gaussian estimates to extend the semigroup generated by a shifted version of A_2 to a semigroup on L^q, where $q \in [3/2, 3]$ if $\partial\Omega_{\mathrm{p}}$ is Lipschitz, and $q \in [6/5, 6]$ if $\partial\Omega_{\mathrm{p}} \in C^{1,1}$. So, in Section 4.2, we formulate the generalized Gaussian estimates and show their validity in our application case. Besides, they lead to a spectral multiplier theorem for the shifted operator.

In section 4.3, we define the Maxwell operator on L^q by using the Helmholtz projection, and prove that the validity of the spectral multiplier theorem remains true. Besides, we show spectral independence of q for the Maxwell operator.

The last section in this chapter deals with the incompressible Navier-Stokes equations. Thanks to the existence of the Helmholtz decomposition, we obtain that the Stokes operator generates an analytic semigroup on L^q for all $q \in (1, \infty)$ if $\partial\Omega_{\mathrm{p}} \in C^3$. Besides, we are able to show several results for the incompressible Navier-Stokes equations. These applications make use of results from [GHHS12].

4.1 Photonic crystals and the Maxwell operator on $L^2(\Omega_{\mathrm{p}})$

We already discussed in the introduction possible applications and the properties of periodic domains in the theory of photonic crystals. In this section, we focus on the mathematical side. In the following, we only consider the physical relevant case $d = 3$. Photonic crystals are realized by arranging dielectrics in spatially periodic structures, most often by using a lattice structure. One distinguishes between one-, two- and three-dimensional photonic crystals, compare the following illustration.

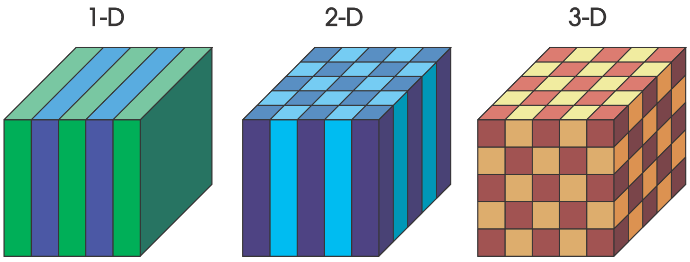


Figure 4.1: *Illustration of the different types of photonic crystals (from [JJWM08, page 4])*

So, if ε only depends on one variable, i.e. $\varepsilon = \varepsilon(x_1)$, the medium is called one-dimensional photonic crystals. If $\varepsilon = \varepsilon(x_1, x_2)$ then it has a two-dimensional structure. We discuss these simplified structures in more detail in Remark 4.1. We continue with the main object in this chapter, the Maxwell operator for a dielectric medium. Since the wavelengths are about some hundred nanometres [JJWM08], the starting point are the macroscopic Maxwell equations

$$\begin{aligned} \operatorname{div} B &= 0, \\ \operatorname{div} D &= \varrho_f, \\ \operatorname{curl} H &= J_f + \partial_t D, \\ \operatorname{curl} E &= -\partial_t B, \end{aligned}$$

where $D = \varepsilon_0 E + P$ and $H = \frac{1}{\mu} B - M$, which we consider in detail in the physical appendix. We are interested in the propagation of light in a dielectric periodic medium without free charges and free currents, so we have $\varrho_f = J_f = 0$. Furthermore, in photonic crystals one uses non-magnetic materials, so we have $M = 0$ and get

$$\begin{aligned} \operatorname{div} B &= 0, \\ \operatorname{div} D &= 0, \\ \operatorname{curl} H &= \partial_t D, \\ \operatorname{curl} E &= -\partial_t B, \end{aligned}$$

plus $D = \varepsilon_0 E + P$ and $H = \frac{1}{\mu} B$.
Next, we transform the equations in such a way that only the vector fields E and H occur. We assume to have a linear, non-dispersive material. By linearity of the material we get $D = \varepsilon_0(1 + \chi)E = \varepsilon E$. We neither assume the material to be homogeneous nor to be isotropic. We assume the material to be little magnetic. Since there is no magnetism (or to be precise, the magnetic effect is negligible, i.e. $\mu_r \approx 1$) we have $B = \mu_0 H$, and hence $H = \mu_0^{-1} B$. Since the material is linear and non-dispersive, we have $D = \varepsilon E$ and $\partial_t(\varepsilon E) = \varepsilon \partial_t E$. Maxwell's equations thus transform to

$$\operatorname{div} H = 0,$$

$$\begin{aligned}\operatorname{div}(\varepsilon E) &= 0,\\ \operatorname{curl} H &= \varepsilon\partial_t E,\\ \operatorname{curl} E &= -\mu_0\partial_t H.\end{aligned}$$

Next, we use the time-harmonic separation ansatz

$$H(x,t) = \mathrm{e}^{-i\omega t}H(x), \qquad E(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}E(x),$$

i.e. we assume E and H to be monochromatic waves. In a photonic crystal this means that the light wave has a fixed frequency. This equals the approach of using the Fourier transform F in time [KH15, section 1.3.3]. By the rule $F(u') = -\mathrm{i}\omega F u$ we get the time harmonic form of Maxwell's equations

$$\begin{aligned}\operatorname{div} H &= 0,\\ \operatorname{div}(\varepsilon E) &= 0,\\ \operatorname{curl} H &= -\mathrm{i}\omega\varepsilon E,\\ \operatorname{curl} E &= \mathrm{i}\omega\mu_0 H.\end{aligned}$$

This representation is also called the phasor form of Maxwell's equations. Note that $E = \frac{\mathrm{i}}{\omega}\varepsilon^{-1}\operatorname{curl} H$, in particular we have $\operatorname{div}(\varepsilon E) = 0$ unconditionally. Motivated by the equations above, one often finds the following definition for the Maxwell operator (also for non-constant μ instead of μ_0):

$$M = \begin{pmatrix} 0 & \mathrm{i}\varepsilon^{-1}\operatorname{curl} \\ -\mathrm{i}\mu^{-1}\operatorname{curl} & 0 \end{pmatrix}.$$

Hence $M(E,H) = (\mathrm{i}\varepsilon^{-1}\operatorname{curl} H, -\mathrm{i}\mu^{-1}\operatorname{curl} E)$ and one can easily prove the self-adjointness of M on some weighted L^2-spaces, cf. [BS87].

We consider another approach. More precisely, we combine the last two equations and get the eigenvalue problem

$$\operatorname{curl}(\varepsilon^{-1}\operatorname{curl} H) = \frac{\omega^2}{\mu_0}H.$$

The operator $\operatorname{curl}(\varepsilon^{-1}\operatorname{curl}\,)$ is what we call the Maxwell-operator. Note that we also obtain the operator by squaring the matrix above. Just as well we could use $H = -\frac{\mathrm{i}}{\omega}\mu_0^{-1}\operatorname{curl} E$ and get

$$\varepsilon^{-1}\operatorname{curl}\operatorname{curl} E = \omega^2\mu_0 E.$$

Obviously, the operator $\mu^{-1}\varepsilon^{-1}\operatorname{curl}\operatorname{curl}$ is the other non-zero entry of the matrix M^2. The two equations are unitarily equivalent, so we restrict ourselves to the first equation involving the magnetic field. We refer to [JJWM08, page 16ff] for the proof of the unitary equivalence and the discussion of advantages and disadvantages of the two systems. Before we continue with our approach to the Maxwell operator, we state the band-gap structure of the Maxwell operator. This is a consequence of the Bloch theory, which works for general strongly elliptic differential operators.

Remark 4.1 (band-gap structure and Bloch waves)
In this work, we used the Bloch multiplier theorems to prove the existence of the Helmholtz decomposition. The most common application of Bloch theory in the study of periodic strongly elliptic differential operators (see [DLP+11, Section 3.2] for the exact formulation of the needed assumptions) with continuous coefficients is another, which we present now. By using the Bloch transform it is possible to decompose the spectrum of a periodic differential operator A into bands. More exactly, one transforms the eigenvalue problem on the whole space to an eigenvalue problem on one periodicity cell (for fixed $\theta \in B^d$, where B^d denotes the reciprocal lattice). By using compactness of embeddings which are due to the boundedness of one periodicity cell, the spectrum of this reduced operator consists of a sequence of eigenvalues $\lambda_1(\theta) < \ldots < \lambda_n(\theta) < \ldots$, where $\lambda_n(\theta) \to +\infty$ for $n \to \infty$. The corresponding eigenfunctions are called Bloch waves and satisfy a completeness property in L^2 [DLP+11, Section 3.5]. The eigenvalues are continuous in θ, i.e. $\lambda_j(\theta_n) \to \lambda_j(\theta)$ if $\theta_n \to \theta$. In particular, the sets $I_j := \cup_{\theta\in\overline{B^d}}\lambda_j(\theta)$ are compact intervals. The main result of the theory is that the spectrum of A is given as the union of all these bands, i.e.

$$\sigma(A) = \bigcup_{j\in\mathbb{N}} I_j.$$

A band gap is a gap in the spectrum, which appears if I_j and I_{j+1} do not overlap for some $j \in \mathbb{N}$. To put it another way, a band gap appears if $\sup_{\theta\in B^d}\lambda_j(\theta) < \inf_{\theta\in B^d}\lambda_{j+1}(\theta)$. In praxis, one tries to build materials (photonic crystals) having some band gaps, cf. also the discussion in the introduction. This problem can be simplified by considering one- and two-dimensional photonic crystals. In fact, one can show that it suffices in these cases to consider scalar-valued problems to determine the spectrum of the Maxwell operator. In fact [FK95, Section 7], it suffices to consider the cases of so-called transverse magnetic and transverse electric waves, i.e. $H = H(x_1, x_2)$, respectively $E = E(x_1, x_2)$. The examples mentioned in the introduction are also in the two-dimensional setting. For more details we refer to the mentioned references and [DLP+11, Chapter 2, 3].

We continue with the Maxwell operator. In addition to the stated assumptions, we assume to have perfect conductor boundary conditions, which means that we have a perfect conductor on one side of the boundary, so $\sigma = \infty$ there. This corresponds to total reflection. Since the flux cannot be infinity, Ohm's law shows that $E = 0$ inside a perfect conductor, which implies $D = H = B = 0$, too. This implies that the boundary conditions are given by

$$n \times E|_{\partial\Omega_\mathrm{p}} = 0, \qquad n \cdot D|_{\partial\Omega_\mathrm{p}} = \varrho|_{\partial\Omega_\mathrm{p}}, \qquad n \cdot B|_{\partial\Omega_\mathrm{p}} = 0, \qquad n \times H|_{\partial\Omega_\mathrm{p}} = J|_{\partial\Omega_\mathrm{p}},$$

cf. [ZL13, section 1.2.2] or [DLP+11, section 1.1.6] for comprehensive explanations. Here, the charges distribute on the boundary of the domain in such a way that the second and the last equation hold.

We now give the precise definition of the Maxwell operator on $L^2(\Omega_\mathrm{p})$. We assume $\varepsilon(\cdot) \in L^\infty(\Omega_\mathrm{p}, \mathbb{C}^{3\times3})$ to be almost everywhere in Ω_p positive definite, Hermitian matrices. Besides, we assume $\varepsilon(\cdot)^{-1} \in L^\infty(\Omega_\mathrm{p}, \mathbb{C}^{3\times3})$. Note that the positive definiteness of ε corresponds to transparency of the underlying material, compare the physical appendix. As a consequence of the assumptions on ε, the uniform ellipticity condition holds, i.e.

$$\varepsilon(x)\zeta\overline{\zeta} \geq \varepsilon_0|\zeta|^2$$

for all $\zeta \in \mathbb{C}^3$ and almost all $x \in \Omega_{\mathrm{p}}$, where ε_0 is independent of x, ζ.
It might be tempting to take the form

$$a(u,v) = \int_{\Omega_{\mathrm{p}}} \varepsilon(\cdot)^{-1} \mathrm{curl}\ u\ \overline{\mathrm{curl}\ v}\ dx.$$

But, we get better properties by adding a divergence term. Note that the first Maxwell equation implies that the added term will vanish. Mathematically, this will be realized by using the Helmholtz projection. Concretely, we take a look at the following densely defined symmetric sesquilinear form

$$a(u,v) = \int_{\Omega_{\mathrm{p}}} \varepsilon(\cdot)^{-1} \mathrm{curl}\ u\ \overline{\mathrm{curl}\ v}\ dx + \int_{\Omega_{\mathrm{p}}} \mathrm{div}\ u\ \overline{\mathrm{div}\ v}\ dx,$$

where $u, v \in V(\Omega_{\mathrm{p}})$ and

$$V(\Omega_{\mathrm{p}}) = \{u \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^3) \mid \mathrm{div}\ u \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}),\ \mathrm{curl}\ u \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^3),\ \nu \cdot u\ |_{\partial\Omega_{\mathrm{p}}} = 0\}.$$

This form fulfils the shifted ellipticity condition

$$\mathrm{Re}\ a(u,u) + C_1||u||^2_{L^2(\Omega_{\mathrm{p}},\mathbb{C}^3)} \geq C_2||u||^2_{V(\Omega_{\mathrm{p}})}.$$

Before we define the operator associated to this form, we discuss properties of the form domain $V(\Omega_{\mathrm{p}})$.

Remark 4.2 (discussion about $V(\Omega)$)
Let $\Omega \subseteq \mathbb{R}^3$ be a strongly local Lipschitz domain. We define the space $X(\Omega) = \mathrm{Div}_2(\Omega) \cap \mathrm{Curl}_2(\Omega)$, cf. Definition and Remark 2.8. Then, functions $u \in X(\Omega)$ belong to $H^1_{\mathrm{loc}}(\Omega)$ [GR12, Chapter I, Corollary 2.10]. In fact, we even have $X(\mathbb{R}^3) = H^1(\mathbb{R}^3)$, but problems occur at the boundary of Ω. In general, the space $X(\Omega)$ is not embedded in $H^1(\Omega)$, even if Ω has smooth boundary.
Under the additional assumption that the normal component vanishes, we arrive at the space $V(\Omega) = \{u \in X(\Omega) \mid u \cdot \nu = 0\}$ and this space can be embedded into $H^1(\Omega)$, if Ω is a bounded $C^{1,1}$ domain and into $H^{1/2}(\Omega)$, if Ω is a bounded Lipschitz domain, see Lemma 4.6 below. We prove in Lemma 4.7 that these embeddings are still valid in the case of periodic domains. We use these embeddings in the proof of the generalized Gaussian estimates in Section 4.2. A main step in the proof of the embedding property on bounded domains is to show at first that $H^1 \cap V(\Omega)$ is dense in $V(\Omega)$. As we shall see immediately this fact remains true on unbounded domains and we even can use H^1-functions with bounded support. This density result will be used below in the proof of the domain characterization for A_2.

Lemma 4.3 *(density result for $V(\Omega)$)*
Let $\Omega \subseteq \mathbb{R}^3$ be any domain of class $C^{1,1}$. Then, $H^1_b(\Omega) \cap V(\Omega)$ is dense in $V(\Omega)$, where

$$H^1_b(\Omega) := \{u \in H^1(\Omega) |\ supp(u) \text{ is bounded}\}.$$

If $\Omega \subseteq \mathbb{R}^3$ is strongly local Lipschitz, then $H^{1/2}_b(\Omega) \cap V(\Omega)$ is dense in $V(\Omega)$.

Proof: See [ABDG98, Lemma 2.10] for the bounded case. If Ω is unbounded, we consider $u_k := u \cdot \varphi_k \to u \in V(\Omega)$, where $(\varphi_k)_{k\in\mathbb{N}} \subseteq C^\infty_c(\mathbb{R}^3)$ are chosen such that $\varphi_k = 1$ on $[-k, k]^3$. Note that $u_k \in V(\Omega)$ for all $k \in \mathbb{N}$ since scalar multiplication preserves the direction and hence the boundary conditions are respected. □

We get the following properties for the operator associated to this form:

Proposition and Definition 4.4 *(operator A_2)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic Lipschitz domain. The operator A_2 associated to the form $a(\cdot,\cdot)$ is defined by

$$u \in D(A_2),\ A_2 u = f \Leftrightarrow u \in V(\Omega_\mathrm{p}),\ a(u, v) = \langle f, v\rangle_{L^2(\Omega_\mathrm{p},\mathbb{C}^3)},\ v \in V(\Omega_\mathrm{p}),$$

compare Definition and Lemma 2.24. Then, $\langle A_2 u, u\rangle \geq 0$, A_2 is self-adjoint and $-A_2$ generates a bounded analytic contraction semigroup on $L^2(\Omega_\mathrm{p}, \mathbb{C}^3)$ of angle $\pi/2$.

Proof: By Definition and Lemma 2.24 we get that A_2 is self-adjoint and generates an analytic contraction semigroup. We know even more. By definition $\langle A_2 u, u\rangle = a(u, u) \geq 0$ for all $u \in D(A_2)$. $-A_2$ is upper semibounded, i.e. $\langle -Au, u\rangle \leq w||u||^2$ for some $w \in \mathbb{R}$ (here we have $w = 0$), and self-adjoint, and hence generates a bounded analytic semigroup of angle $\pi/2$, see [EN00, chapter II, Corollary 4.7ff]. □

Before we continue with the definition of the Maxwell operator, we take a closer look at the domain of A_2 and prove a concrete representation.

Remark 4.5 Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic Lipschitz domain so that $H^1_b(\Omega_\mathrm{p}) \cap V(\Omega_\mathrm{p})$ is dense in $V(\Omega_\mathrm{p})$. This is for example the case if $\partial\Omega_\mathrm{p} \in C^{1,1}$. Then,

$$A_2 u = \mathrm{curl}\, \varepsilon^{-1} \mathrm{curl}\, u - \nabla \mathrm{div}\, u,$$
$$D(A_2) = \{u \in V(\Omega_\mathrm{p}) \mid \mathrm{curl}\, \varepsilon^{-1}\mathrm{curl}\, u \in L^2(\Omega_\mathrm{p}, \mathbb{C}^3),\ \mathrm{div}\, u \in H^1(\Omega_\mathrm{p}),\ \nu \times \varepsilon^{-1}\mathrm{curl}\, u|_{\partial\Omega_\mathrm{p}} = 0\}.$$

We start with proving the stated assertions for functions in $D(A_2)$. If $u \in D(A_2)$ we have $\langle Au, v\rangle = \langle \varepsilon^{-1}\mathrm{curl}\, u, \mathrm{curl}\, v\rangle + \langle \mathrm{div}\, u, \mathrm{div}\, v\rangle$ for all $v \in C^\infty_c(\Omega_\mathrm{p})$ by definition, and hence $A_2 u = \mathrm{curl}\, \varepsilon^{-1}\mathrm{curl}\, u - \nabla\mathrm{div}\, u$ in distributional sense. Besides, we have by Definition and Lemma 2.24

$$D(A_2) = \{w \in V(\Omega_\mathrm{p}) \mid \text{There is } C_w \geq 0 \text{ so that } |a(w, v)| \leq C_w||v||_{L^2} \text{ for all } v \in V(\Omega_\mathrm{p})\}.$$

Thus, we have by Riesz' representation theorem curl ε^{-1}curl $u \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^3)$ and div $u \in H^1(\Omega_{\mathrm{p}})$. Using Lemma 2.10 and Lemma 2.11 we get for $u \in D(A_2), v \in C_c^\infty(\overline{\Omega_{\mathrm{p}}}) \cap V(\Omega)$

$$\begin{aligned}\langle A_2 u, v\rangle = a(u,v) &= \langle \varepsilon^{-1}\text{curl } u, \text{curl } v\rangle + \langle \text{div } u, \text{div } v\rangle \\ &= \langle \text{curl } \varepsilon^{-1}\text{curl } u, v\rangle - \int\limits_{\partial\Omega_{\mathrm{p}}} (\nu \times \varepsilon^{-1}\text{curl } u) v d\sigma - \langle \nabla \text{div } u, v\rangle,\end{aligned}$$

so $\nu \times \varepsilon^{-1}$curl $u = 0$ for $v \in C_c^\infty(\Omega_{\mathrm{p}})$ with $v \cdot \nu = 0$. Note that $(\nu \times \varepsilon^{-1}\text{curl } u) \cdot \nu = 0$, so it suffices to consider $v \in C_c^\infty(\overline{\Omega_{\mathrm{p}}})$ with $v \cdot \nu = 0$ to prove $\nu \times \varepsilon^{-1}$curl $u = 0$. Hence,

$$D(A_2) \subseteq \{u \in V(\Omega_{\mathrm{p}}) \mid \text{curl } \varepsilon^{-1}\text{curl } u \in L^2(\Omega_{\mathrm{p}}, \mathbb{C}^3),\ \text{div } u \in H^1(\Omega_{\mathrm{p}}),\ \nu \times \varepsilon^{-1}\text{curl } u|_{\partial\Omega_{\mathrm{p}}} = 0\}.$$

On the other hand, „ $\supseteq$ “ follows since $H_b^1(\Omega_{\mathrm{p}})$ is dense in $V(\Omega_{\mathrm{p}})$, cf. Lemma 4.3. More exact, let $u \in V(\Omega_{\mathrm{p}})$ satisfying curl ε^{-1}curl $u \in L^2(\Omega_{\mathrm{p}})$, div $u \in H^1(\Omega_{\mathrm{p}})$ and $\nu \times \varepsilon^{-1}$curl $u|_{\partial\Omega_{\mathrm{p}}} = 0$. We have to show

$$\langle \text{curl } \varepsilon^{-1}\text{curl } u - \nabla \text{div } u, v\rangle = a(u,v), \qquad \text{for all } v \in V(\Omega_{\mathrm{p}}).$$

Since $H_b^1(\Omega_{\mathrm{p}}) \cap V(\Omega_{\mathrm{p}})$ is dense in $V(\Omega_{\mathrm{p}})$ it suffices to consider $v \in H_b^1(\Omega_{\mathrm{p}}) \cap V(\Omega_{\mathrm{p}})$. By using the integration by parts/Green's formulas, cf. Lemma 2.10 and Lemma 2.11, we get

$$\langle \text{curl } \varepsilon^{-1}\text{curl } u - \nabla \text{div } u, v\rangle = \langle \text{curl } u, \text{curl } v\rangle + \langle \text{div } u, \text{div } v\rangle = a(u,v),$$

where we note that the boundary integrals vanish since $\nu \times \varepsilon^{-1}$curl $u|_{\partial\Omega_{\mathrm{p}}} = 0$ and $\nu \cdot v = 0$.

We still have to incorporate the condition div $u = 0$. Consequently, we define the Maxwell operator by projecting the operator A_2 onto $L^2_\sigma(\Omega_{\mathrm{p}})$, using the Helmholtz decomposition on $L^2(\Omega_{\mathrm{p}})$. To be precise, we define the Maxwell operator on $L^2(\Omega_{\mathrm{p}})$ by $M_2 := A_2$ with domain $D(M_2) = P_2 D(A_2) = D(A_2) \cap L^2_\sigma(\Omega_{\mathrm{p}})$. The last equation is due to the following observations.
Thanks to the boundary conditions of $V(\Omega_{\mathrm{p}})$, the Helmholtz projection $P = P_2$ leaves the domain of the form $V(\Omega_{\mathrm{p}})$ invariant. Furthermore, it follows that the operator A_2 and the Helmholtz operator P_2 commute, see [KU15a, Lemma 3.6]. This is quite handy, because we can assign many known results for A_2 to the Maxwell operator M_2.

We want to examine the Maxwell operator not only on L^2, but also on L^q. We proved in Chapter 3 the existence of the Helmholtz projection on periodic Lipschitz domains for $q \in (\frac{3+\varepsilon}{2+\varepsilon}, 3+\varepsilon)$ and on periodic C^1-domains for all $q \in (1, \infty)$.
Besides, for the definition of the Maxwell operator on $L^q(\Omega_{\mathrm{p}})$ we need to extend the operator A_2 onto L^q, too. This will be realized by extending the semigroup generated by $-A_2 - \lambda$, $\lambda > 0$ to L^q via generalized Gaussian estimates.
So, before we resume the theory for the Maxwell operator, we introduce generalized Gaussian estimates, and prove the extension of the semigroup under the assumptions made above.

4.2 Gaussian estimates and a spectral multiplier theorem

The foundation of the extension result for the semigroup $\mathrm{e}^{-A_2 t}$ is the forthcoming Theorem 4.9 taken from [KU15b], which uses generalized Gaussian estimates. We refer to [KU15b] for a discussion of the underlying theory. Here, we just recall that the generalized Gaussian estimates (GGE) have their origin in pointwise Gaussian estimates [SV94, Dav95b], which are given by

$$|p_t(x,y)| \leq C|B(x,t^{1/m})|^{-1}\mathrm{e}^{-b\frac{|x-y|^{m/(m-1)}}{t^{1/(m-1)}}},$$

where $p_t(x,y)$ denotes the kernel of the semigroup e^{-tL}. The spectral multiplier results for non-negative self-adjoint operators L fulfilling pointwise Gaussian estimates are due to Duong, Ouhabaz and Sikora [DOS02].

Before we start looking at the spectral multiplier theorem, we present the following embedding lemma for the form domain, which is essential for the proof of GGE. In the proof we resort to the bounded case by using a decomposition of unity, so we quote the results for the bounded case before.

Proposition 4.6 *(embedding result for $V(\Omega)$)*
If $\Omega \subseteq \mathbb{R}^3$ is a bounded Lipschitz domain, we have $V(\Omega) \hookrightarrow H^{1/2}(\Omega, \mathbb{C}^3)$ and if Ω has $C^{1,1}$-boundary, we have $V(\Omega) \hookrightarrow H^1(\Omega, \mathbb{C}^3)$.

Proof: See [Cos90] for the case that $\partial\Omega$ is a connected Lipschitz domain and [MMT01, p. 87] for the full Lipschitz case. A proof in the case $\partial\Omega \in C^{1,1}$ can be found in [ABDG98, Theorem 2.9]. □

Lemma 4.7 *(embedding result for $V(\Omega_\mathrm{p})$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ a periodic domain with $C^{1,1}$-boundary. Then,

$$V(\Omega_\mathrm{p}) \hookrightarrow H^1(\Omega_\mathrm{p}) \hookrightarrow L^6(\Omega_\mathrm{p}).$$

If Ω_p has a local Lipschitz boundary, we have

$$V(\Omega_\mathrm{p}) \hookrightarrow H^{1/2}(\Omega_\mathrm{p}) \hookrightarrow L^3(\Omega_\mathrm{p}).$$

Proof: The idea is to reduce this problem to one periodicity cell, since there we can use the embedding property for bounded domains, compare Proposition 4.6. Let $0 \leq \varphi \in C_c^\infty(\mathbb{R}^3)$ be a function such that $\sum_{k\in\mathbb{Z}^d} \varphi_k = 1$, where $\varphi_k := \varphi(\cdot - k)$ and $\Omega \subseteq \Omega_\mathrm{p}$ be a bounded Lipschitz domain containing $\mathrm{supp}(\varphi)$. We set $\Omega_k := \Omega + k,\ k \in \mathbb{Z}^d$.
Let $u = (u_1, \ldots, u_d)^T \in V(\Omega_\mathrm{p})$ and N be the number of Ω_k intersecting Ω_0. We estimate

$$\begin{aligned} ||u||^2_{H^1(\Omega_\mathrm{p},\mathbb{C}^3)} &= ||\sum_{k\in\mathbb{Z}^d} u\varphi(\cdot - k)||^2_{H^1(\Omega_\mathrm{p},\mathbb{C}^3)} = \sum_{i=1}^d ||\sum_{k\in\mathbb{Z}^d} u_i\varphi_k||^2_{H^1(\Omega_\mathrm{p})} \\ &\leq N\sum_{i=1}^d \sum_{k\in\mathbb{Z}^d} ||u_i\varphi_k||^2_{H^1(\Omega_k)} = N\sum_{k\in\mathbb{Z}^d} ||u\varphi(\cdot - k)||^2_{H^1(\Omega_k,\mathbb{C}^3)} \end{aligned}$$

$$\leq NC \sum_{k\in\mathbb{Z}^d} ||u\varphi(\cdot - k)||^2_{V(\Omega_k)} \leq NC||u||^2_{V(\Omega_\mathrm{p})}.$$

In the calculation above we used the embedding in the bounded case in the second estimate. The first inequality is a consequence of the simple inequality

$$(\sum_{i=1}^N a_i)^2 \leq N \sum_{i=1}^N a_i^2,$$

and due to the fact that at every point at most N of the Ω_k's are overlapping. The last inequality is a conclusion of

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} ||f\varphi_k||_2^2 &\leq \int_\Omega \sum_{k\in\mathbb{Z}} |f(x)|^2 \varphi_k(x)^2 dx \\ &\leq \int_\Omega |f(x)|^2 (\sum_{k\in\mathbb{Z}} \varphi_k(x))^2 dx = ||f||_2^2.\end{aligned}$$

For estimating the div and curl parts of the $V(\Omega)$ norm, we use the relations

$$\text{curl } (\varphi_k u) = \varphi_k \text{curl } u + (\nabla\varphi_k) \times u, \qquad \text{div } (\varphi_k u) = \varphi_k \text{div } u + \nabla\varphi_k \cdot u,$$

together with the estimate $|\nabla\varphi_k| \leq C$, which imply

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} ||(\nabla\varphi_k) \times u||_2^2 &\leq C^2 \sum_{k\in\mathbb{Z}^d} ||u||^2_{L^2(\Omega_k)} \leq C^2 N ||u||_2^2, \\ \sum_{k\in\mathbb{Z}^d} ||(\nabla\varphi_k) \cdot u||_2^2 &\leq C^2 \sum_{k\in\mathbb{Z}^d} ||u||^2_{L^2(\Omega_k)} \leq C^2 N ||u||_2^2.\end{aligned}$$

In the same way, by reducing to the bounded case, the embedding $V(\Omega_\mathrm{p}) \hookrightarrow H^{1/2}(\Omega_\mathrm{p})$ can be proven. The embeddings $H^{1/2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ follow by Sobolev's inequality [AF03, Theorem 4.31]. By using the Calderón-Stein extension Theorem 3.3 these embeddings hold for arbitrary strongly local Lipschitz domains. □

We follow [KU15a, §2]. For this approach we need a space of homogeneous type in the sense of Coifman and Weiss. (X, d, μ) is a space of homogeneous type, if (X, d) is a non-empty metric space endowed with a σ-finite measure μ satisfying $\mu(X) > 0$. In addition, the doubling condition must be satisfied, which means

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \qquad x \in X, \ r > 0,$$

where C is independent of r, x. Examples are $\mathbb{R}^d$ and arbitrary bounded open subsets with Lipschitz boundary endowed with Euclidean metric and Lebesgue measure.

Furthermore, periodic domains with Lipschitz boundary fulfil the doubling condition, too. For small values of r, say $r \leq 4$, the doubling condition can be reduced to the case of a bounded Lipschitz domain by using periodicity.

Let $x \in \Omega_{\mathrm{p}}$, $r \geq 4$ and $\Omega_k = \Omega_0 + k$, $k \in \mathbb{Z}^d$. Recall that $\Omega_0 = \Omega_{\mathrm{p}} \cap [0,1]^d$. Let

$$c_{k,x,r} := \begin{cases} 1, & \text{if} \quad B(x,r) \cap \Omega_k \neq \emptyset, \\ 0, & \text{if} \quad B(x,r) \cap \Omega_k = \emptyset, \end{cases}$$

$$d_{k,x,r} := \begin{cases} 1, & \text{if} \quad \Omega_k \subseteq B(x,r), \\ 0, & \text{if} \quad \Omega_k \nsubseteq B(x,r), \end{cases}$$

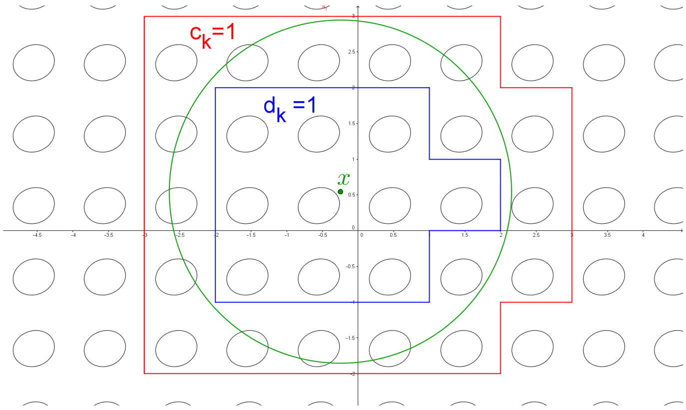


Figure 4.2: $d_k = 1$ *if the periodicity cell is fully contained in the ball, while* $c_k = 1$ *if the ball intersects the periodicity cell.*

We have

$$|B(x,2r) \cap \Omega_{\mathrm{p}}| \leq \sum_{k \in \mathbb{Z}^d} c_{k,x,r} |\Omega_0| \leq (2r+1)^d |\Omega_0|,$$

and

$$|B(x,r) \cap \Omega_{\mathrm{p}}| \geq \sum_{k \in \mathbb{Z}^d} d_{k,x,r} |\Omega_0| \geq (r/2)^d |\Omega_0|,$$

and hence Ω_{p} is of homogeneous type.

Now, we give the spectral multiplier theorem for operators fulfilling GGE in the special case of a domain in $\mathbb{R}^d$.

Definition 4.8 *(generalized Gaussian estimates)*
Let $X \subseteq \mathbb{R}^d$ be a non-empty domain, d the Euclidean metric and μ the Lebesgue measure. Furthermore, let A be a non-negative, self-adjoint operator on $L^2(X)$, $p \in [1,2]$ and $q \in [2,\infty]$. A satisfies generalized Gaussian (p,q)-estimates if there are $b, C > 0$ such that

$$||\mathbb{1}_{B(x,t^{1/2})} \mathrm{e}^{-tA} \mathbb{1}_{B(y,t^{1/2})}||_{L^p \to L^q} \leq C t^{-d/2(\frac{1}{p}-\frac{1}{q})} \mathrm{e}^{-b\frac{|x-y|^2}{t}}, \qquad t > 0,$$

where

$$||\mathbb{1}_{E_1} \mathrm{e}^{-tA} \mathbb{1}_{E_2}||_{p \to q} := \sup_{||f||_p \leq 1} ||\mathbb{1}_{E_1} \mathrm{e}^{-tA} \mathbb{1}_{E_2} f||_q,$$

for Borel sets E_1, E_2. In that case, we say that A fulfils GGE (p,q).

There is the following spectral multiplier theorem for positive self-adjoint operators fulfilling GGE. In this theorem there appears a non-negative cut-off function $\omega \in C_c^\infty((0,\infty))$ such that $\operatorname{supp} \omega \subseteq (1/4,1)$, $\sum_{n\in\mathbb{Z}} \omega(2^{-n}\lambda) = 1$ for all $\lambda > 0$.

Theorem 4.9 *(spectral multiplier theorem for operators satisfying GGE)*
Let $X \subseteq \mathbb{R}^d$ be a non-empty domain admitting the doubling condition, d the Euclidean metric and μ the Lebesgue measure. Furthermore, let A be a positive, self-adjoint operator on $L^2(X)$ fulfilling GGE (p_0, p_0'), $p_0 \in [1,2)$. Then, for all $p \in (p_0, p_0')$, and all bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ satisfying $\sup_{n\in\mathbb{Z}} ||\omega F(2^n \cdot)||_{C^s} < \infty$ for some $s > d|\frac{1}{p} - \frac{1}{2}|$, we get $F(A) \in \mathcal{L}(L^p(X))$ with accompanying norm estimate

$$||F(A)||_{L^p\to L^p} \leq C_p\big(\sup_{n\in\mathbb{Z}} ||\omega F(2^n\cdot)||_{C^s} + |F(0)|\big).$$

Furthermore, the analytic semigroup generated by $-A$ has an extension to an analytic semigroup on $L^p(X)$ for $p \in [p_0, p_0']$.

Proof: See [KU15b, Theorem 5.4] for the first part, where $p \in (p_0, p_0')$, and [Blu07, Theorem 1.1] for the second part. For $p = p_0$ and $p = p_0'$, cf. Theorem 4.13. □

Now we are ready to prove generalized Gaussian estimates for a shifted version of the operator A_2.

Theorem 4.10 *(GGE for $A_2 + \lambda$)*
For all $\lambda > 0$ the operator $A_{2,\lambda} := A_2 + \lambda Id$ fulfils GGE $(3/2,3)$-estimates if Ω_p has local Lipschitz boundary. The interval extends to $(6/5,6)$ if Ω_p has $C^{1,1}$-boundary.

Proof: By using duality and the semigroup law it suffices to prove GGE $(2,3)$, respectively GGE $(2,6)$. The proof of GGE $(2,2)$ alias Davies-Gaffney estimates for

$$(\mathrm{e}^{-tA_2})_{t>0}, \quad (t^{1/2}\mathrm{curl}\, \mathrm{e}^{-tA_2})_{t>0}, \quad (t^{1/2}\mathrm{div}\, \mathrm{e}^{-tA_2})_{t>0}$$

is exactly the same as in the proof of [KU15a, Steps 1-3 in the proof of Theorem 3.2] for bounded domains and uses Davies' perturbation method. Nevertheless, we give the main ideas of the approach, which consists in studying of 'twisted' forms

$$a_{\varrho,\varphi}(u,v) := a(\mathrm{e}^{\varrho\varphi}u, \mathrm{e}^{-\varrho\varphi}v),$$

where $u, v \in V(\Omega_\mathrm{p})$, $\varrho \in \mathbb{R}$ and

$$\varphi \in \mathcal{E} = \{\varphi \in C_c^\infty(\overline{\Omega_\mathrm{p}}, \mathbb{R}) \mid ||\partial_j\varphi||_\infty \leq 1 \text{ for } j = 1,2,3.\}.$$

This is well-defined since $\mathrm{e}^{\varrho\varphi}V(\Omega_\mathrm{p}) \subseteq V(\Omega_\mathrm{p})$ and one can prove for all $\gamma \in (0,1)$ the existence of a constant $\omega_0 \geq 0$ such that

$$|a_{\varrho\varphi}(u,u) - a(u,u)| \leq \gamma a(u,u) + \omega_0\varrho^2||u||_2^2, \qquad u \in V(\Omega_\mathrm{p}),\ \varrho \in \mathbb{R},\ \varphi \in \mathcal{E}.$$

In a next step it follows that the operator $A_{\varrho\varphi\omega}$, where $\omega > \omega_0$, associated to the form $a_{\varrho\varphi\omega} := a_{\varrho\varphi} + \omega\varrho^2$ is sectorial, and that the bounded analytic semigroup generated by $-A_{\varrho\varphi\omega}$ is contractive. This can be used to prove

$$||\mathrm{e}^{-\varrho\varphi}\mathrm{e}^{-zA_2}\mathrm{e}^{\varrho\varphi}||_{\mathcal{L}(L^2(\Omega_\mathrm{p},\mathbb{C}^3))} \leq C\mathrm{e}^{\omega\varrho^2 \mathrm{Re}\, z},$$

which implies the Davies-Gaffney estimates for A_2. The Davies-Gaffney estimates for $(t^{1/2}\mathrm{curl}\ \mathrm{e}^{-tA_2})_{t>0}$, $(t^{1/2}\mathrm{div}\ \mathrm{e}^{-tA_2})_{t>0}$ also follow by considering twisted forms. We just remark that the term $t^{1/2}$ appears since the estimate

$$||A_{\varrho\varphi\omega}\mathrm{e}^{-tA_{\varrho\varphi\omega}}||_{\mathcal{L}(L^2(\Omega_\mathrm{p},\mathbb{C}^3))} \leq \frac{1}{t\sin\theta_0},$$

is used, where θ_0 denotes the angle of $A_{\varrho\varphi\omega}$.

We use Lemma 4.7, which yields

$$||u||_{L^r(\Omega_\mathrm{p},\mathbb{C}^3)} \leq C\big(||u||_{L^2(\Omega_\mathrm{p},\mathbb{C}^3)} + ||\mathrm{div}\ u||_{L^2(\Omega_\mathrm{p},\mathbb{C})} + ||\mathrm{curl}\ u||_{L^2(\Omega_\mathrm{p},\mathbb{C}^3)}\big),$$

where $r = 3$ if Ω_p is a Lipschitz domain and $r = 6$ if Ω_p is a $C^{1,1}$-domain. Now, we proceed as in [KU15a, Step 5 in the proof of Theorem 3.2]. The argument has its origin in [MM09, Section 5]. So, let $t > 0$, $x, y \in \Omega_\mathrm{p}$ and $f \in C_c^\infty(\Omega_\mathrm{p}, \mathbb{C}^3)$ with $\mathrm{supp}(f) \subseteq B(y, t^{1/2})$. We put $\Omega_x := B(x, 2t^{1/2}) \cap \Omega_\mathrm{p}$ and choose a cut-off function $\eta \in C_c^\infty(\Omega_0, \mathbb{R})$ satisfying

$$0 \leq \eta \leq 1, \qquad \eta = 1 \ \text{ on } B(x, t^{1/2}), \qquad ||\nabla\eta||_\infty \leq 2t^{-1/2}.$$

This implies, together with the product rules for div and curl, which we already used in the proof of Lemma 4.7, the estimates

$$\begin{aligned}||\mathrm{div}\ (\eta\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C})} &\leq C\big(||\mathrm{div}\ (\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C})} + t^{-1/2}||\mathrm{e}^{-tA_2}f||_{L^2(\Omega_x,\mathbb{C}^3)}\big),\\ ||\mathrm{curl}\ (\eta\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C}^3)} &\leq C\big(||\mathrm{curl}\ (\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C}^3)} + t^{-1/2}||\mathrm{e}^{-tA_2}f||_{L^2(\Omega_x,\mathbb{C}^3)}\big).\end{aligned}$$

Note that we have $\mathrm{Range}(\mathrm{e}^{-tA_2}) \subseteq D(A_2)$ by Proposition 2.21, which is due to the fact that $-A_2$ generates a bounded analytic semigroup. And since we used the form ansatz we get by definition $D(A_2) \subseteq V(\Omega_\mathrm{p})$, compare Proposition and Definition 4.4. By using Lemma 4.7 and the GGE (2,2) we get

$$\begin{aligned}&||\mathrm{e}^{-tA_2}f||_{L^r(B(x,t^{1/2}),\mathbb{C}^3)} \leq ||\eta\mathrm{e}^{-tA_2}f||_{L^r(\Omega_\mathrm{p},\mathbb{C}^3)}\\ &\quad\leq C\big(||\eta\mathrm{e}^{-tA_2}f||_{L^2(\Omega_x,\mathbb{C}^3)} + ||\mathrm{div}\ (\eta\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C})} + ||\mathrm{curl}\ (\eta\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C}^3)}\big)\\ &\quad\leq C\big((1+2t^{-1/2})||\mathrm{e}^{-tA_2}f||_{L^2(\Omega_x,\mathbb{C}^3)} + ||\mathrm{div}\ (\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C})} + ||\mathrm{curl}\ (\mathrm{e}^{-tA_2}f)||_{L^2(\Omega_x,\mathbb{C}^3)}\big)\\ &\quad\leq C(1+t^{-1/2})\mathrm{e}^{-b\frac{|x-y|^2}{t}}||f||_{L^2(B(y,t^{1/2}),\mathbb{C}^3)},\end{aligned}$$

where $r = 6$ if $\partial\Omega_\mathrm{p} \in C^{1,1}$ and $r = 3$ if $\partial\Omega_\mathrm{p}$ is Lipschitz. Now, let $\lambda > 0$ and $A_{2,\lambda} := A_2 + \lambda Id$. First, we consider the case $\partial\Omega_\mathrm{p} \in C^{1,1}$ and estimate

$$\begin{aligned}&||\mathrm{e}^{-tA_{2,\lambda}}f||_{L^6(B(x,t^{1/2}),\mathbb{C}^3)} = e^{-\lambda t}||\mathrm{e}^{-tA_2}f||_{L^6(B(x,t^{1/2}),\mathbb{C}^3)}\\ &\quad\leq C(1+t^{-1/2})\mathrm{e}^{-\lambda t}\mathrm{e}^{-b\frac{|x-y|^2}{t}}||f||_{L^2(B(y,t^{1/2}),\mathbb{C}^3)} \leq Ct^{-1/2}\mathrm{e}^{-b\frac{|x-y|^2}{t}}||f||_{L^2(B(y,t^{1/2}),\mathbb{C}^3)}.\end{aligned}$$

Hence $A_{2,\lambda}$ fulfils GGE (2,6). Now let $\partial\Omega_{\mathrm{p}}$ be Lipschitz. There is a constant $C > 0$ independent of $t \geq 1$ satisfying

$$(1+t^{-1/2})\mathrm{e}^{-\lambda t} \leq Ct^{-1/4}, \qquad t \in [1,\infty),$$

so we get

$$||\mathrm{e}^{-tA_{2,\lambda}} f||_{L^3(B(x,t^{1/2}),\mathbb{C}^3)} \leq Ct^{-1/4}\mathrm{e}^{-b\frac{|x-y|^2}{t}}||f||_{L^2(B(y,t^{1/2}),\mathbb{C}^3)}, \qquad t \in [1,\infty).$$

The case $t \leq 1$ requires another approach. This is the easier case, because this case can be treated as in the case of a bounded domain, thanks to the periodicity of the domain. So, let $t \in (0,1]$ and $x_0 \in \Omega$. By periodicity of the domain we can assume that $\Omega_x = B(x, 2t^{1/2}) \subseteq \Omega_{\mathrm{p}}$ lies in a bounded Lipschitz domain Ω independent of x. We know that

$$||w||_{L^r(\Omega,\mathbb{C}^3)} \leq C\big(||w||_{L^2(\Omega,\mathbb{C}^3)} + ||\mathrm{div}\ w||_{L^2(\Omega,\mathbb{C})} + ||\mathrm{curl}\ w||_{L^2(\Omega,\mathbb{C}^3)}\big),$$

where $r = 3$ if $\partial\Omega_{\mathrm{p}}$ is Lipschitz and $r = 6$ if $\partial\Omega_{\mathrm{p}}$ is of class $C^{1,1}$. Besides, the embedding constant only depends on $\partial\Omega$ and the diameter of Ω. In particular, rescaling [MM09, page 3145] leads to (where we note that $\mathrm{diam}(\Omega_x) = 4t^{1/2}$)

$$||w||_{L^r(\Omega_x,\mathbb{C}^3)} \leq Ct^{-1/4}\big(||w||_{L^2(\Omega_x,\mathbb{C}^3)} + t^{1/2}||\mathrm{div}\ w||_{L^2(\Omega_x,\mathbb{C})} + t^{1/2}||\mathrm{curl}\ w||_{L^2(\Omega_x,\mathbb{C}^3)}\big),$$

where C only depends on the Lipschitz character of Ω_x, which is controlled by that of Ω_{p}. Hence, we get

$$\begin{aligned}
&||\mathrm{e}^{-tA_2} f||_{L^3(B(x,t^{1/2}),\mathbb{C}^3)} \leq ||\eta \mathrm{e}^{-tA_2} f||_{L^3(\Omega_x,\mathbb{C}^3)} \\
&\leq Ct^{-1/4}\big(||\eta \mathrm{e}^{-tA_2} f||_{L^2(\Omega_x,\mathbb{C}^3)} + t^{1/2}||\mathrm{div}\ (\eta \mathrm{e}^{-tA_2} f)||_{L^2(\Omega_x,\mathbb{C})} + t^{1/2}||\mathrm{curl}\ (\eta \mathrm{e}^{-tA_2} f)||_{L^2(\Omega_x,\mathbb{C}^3)}\big) \\
&\leq Ct^{-1/4}\big(3||\mathrm{e}^{-tA_2} f||_{L^2(\Omega_x,\mathbb{C}^3)} + t^{1/2}||\mathrm{div}\ (\mathrm{e}^{-tA_2} f)||_{L^2(\Omega_x,\mathbb{C})} + t^{1/2}||\mathrm{curl}\ (\mathrm{e}^{-tA_2} f)||_{L^2(\Omega_x,\mathbb{C}^3)}\big) \\
&\leq Ct^{-1/4}\mathrm{e}^{-b\frac{|x-y|^2}{t}}||f||_{L^2(B(y,t^{1/2}),\mathbb{C}^3)},
\end{aligned}$$

which is the inequality we need for GGE (2,3). Note that, in the case $t \leq 1$, we can even forego to add λId to A_2. □

We would be in the position to skip the shift in Theorem 4.10, if we could prove the embedding $V(\Omega_n) \hookrightarrow H^{1/2}(\Omega_n)$, or $V(\Omega) \hookrightarrow H^1(\Omega_n)$, respectively with an embedding constant independent of $n \in \mathbb{N}$.

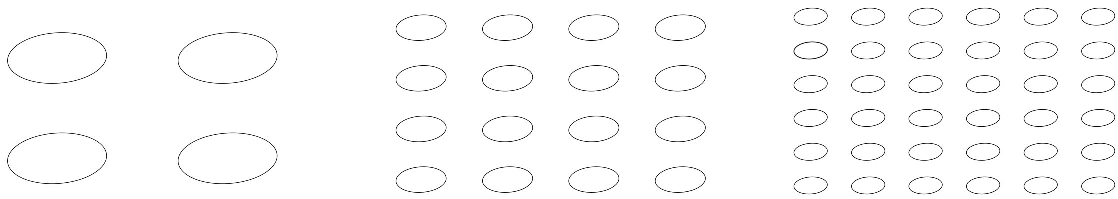


Figure 4.3: *We use the domain introduced following Remark 2.13 and scale the cube $[-n,n]^d$ to size one for $n = 1,2,3$.*

This is because we are reliant upon the independence of some scaling constant, compare the proof of the theorem above. By combining Theorem 4.9 and Theorem 4.10 we immediately get the following spectral multiplier theorem.

Theorem 4.11 *(spectral multiplier theorem for $A_{2,\lambda}$)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic domain with local Lipschitz boundary $\lambda > 0$, $q \in (3/2, 3)$, $s > 3|1/q - 1/2|$ and ω be a cut-off function as in Theorem 4.9. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ satisfying $\sup_{n\in\mathbb{Z}} ||\omega F(2^n \cdot)||_{C^s} < \infty$, the operator $F(A_{2,\lambda})$ is bounded on $L^q(\Omega_\mathrm{p}, \mathbb{C}^3)$ and there is a constant $C_q > 0$ such that

$$||F(A_{2,\lambda})||_{L^q \to L^q} \leq C_q\big(\sup_{n\in\mathbb{Z}} ||\omega F(2^n \cdot)||_{C^s} + |F(0)|\big).$$

If $\partial\Omega_\mathrm{p} \in C^{1,1}$, then the statement holds for all $q \in (6/5, 6)$.

4.3 The Maxwell operator on $L^q(\Omega_\mathrm{p})$

We follow [KU15a] and use the same notation as there. The approach there worked for bounded domains and we will transfer it to periodic domains. We recall that we have to shift the operator A_2 to get Gaussian estimates since we cannot use the localization procedure for large values of t, as it is used in [KU15a, Proof of Theorem 3.2, step 4].
By using Theorem 4.10 and Theorem 4.9 we can extend the bounded analytic semigroup generated by the operator $-A_{2,\lambda} = -A_2 - \lambda Id$, $\lambda > 0$ to a bounded analytic semigroup on L^q, where $q \in [3/2, 3]$ if $\partial\Omega_\mathrm{p}$ is Lipschitz and $q \in [6/5, 6]$ if $\partial\Omega_\mathrm{p} \in C^{1,1}$. We denote by $-A_{q,\lambda}$ the generator of the extended semigroup. Of course, by the scaling properties of semigroups, we have $A_{q,\lambda_1} + (\lambda_2 - \lambda_1)Id = A_{q,\lambda_2}$ for all $\lambda_1, \lambda_2 > 0$. In particular, it is possible to define $A_{q,0}$ by $A_{q,0} := A_{q,\lambda} - \lambda Id$ and $D(A_{q,\lambda})$ is independent of the choice of λ. We recall that $\nu \cdot (P_q u) = 0$ for all $u \in L^q(\Omega_\mathrm{p})$. This matches the boundary conditions of $V(\Omega_\mathrm{p})$, and so one can show that A_2 and the Helmholtz projection P_2 are commuting, and this property extends to A_q and P_q, if these operators exist. We introduce the following notation for the common interval on which the Helmholtz decomposition and the analytic semigroup exist.

Definition 4.12 *(interval I_{Ω_p})*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic domain with Lipschitz boundary. We denote by $I_{\Omega_\mathrm{p}} \subseteq (1,\infty)$ the largest subinterval containing 2 and consisting of all q, on which both, the Helmholtz decomposition on $L^q(\Omega_\mathrm{p})$ and the extension of the analytic semigroup generated by $-A_2 - \lambda$, $\lambda > 0$, to a bounded analytic semigroup on $L^q(\Omega_\mathrm{p})$ exist.

The following theorem [Blu07, Theorem 1.1] implies that GGE do not only guarantee that the bounded analytic semigroup $\mathrm{e}^{-(A_2+\lambda)t}$ extends on L^q, but, moreover, the angle $\pi/2$ remains, too.

Theorem 4.13 *(bounded analyticity of e^{-tA} on L^q)*
Let (X, d, μ) be as in Definition 4.8 and A be a non-negative selfadjoint operator on $L^2(X)$ satisfying GGE (p, p'), where $p \in [1, 2)$. Then, the semigroup generated by $-A$ extends to a

bounded analytic semigroup of angle $\pi/2$ on $L^q(X)$ for all $q \in [p, p']$ and

$$||\mathrm{e}^{-zA}||_{q\to q} \leq C\Big(\frac{|z|}{\mathrm{Re}(z)}\Big)^{d(\frac{1}{p}-\frac{1}{2})}.$$

From now on, let $\lambda > 0$ be arbitrary, but fixed. $A_{q,\lambda}$ inherits more properties from $A_{2,\lambda}$, if $q \in I_{\Omega_\mathrm{p}}$. Now, we collect the most important properties of $A_{q,\lambda}$ in the following proposition.

Proposition 4.14 *(properties of A_q)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic domain with Lipschitz boundary and $q \in I_{\Omega_\mathrm{p}}$. The semigroup generated by $-A_{q,\lambda}$ is bounded analytic of angle $\pi/2$. Besides, we have $\sigma(A_{q,\lambda}) = \sigma(A_{2,\lambda})$, so the spectrum is independent of $q \in I_{\Omega_\mathrm{p}}$. The operators $A_{q,\lambda}$ and P_q are commutating. More exact, $P_q(D(A_{q,\lambda})) \subseteq D(A_{q,\lambda})$ and

$$P_q(A_{q,\lambda}u) = A_{q,\lambda}(P_q u), \qquad u \in D(A_{q,\lambda}).$$

Proof: The first assertion follows from Theorem 4.13. Equality of spectra holds since we continued the semigroup using generalized Gaussian estimates, cf. [BK05, Proposition 1.4, Corollary 1.5]. The last assertion can be proven as [KU15a, Lemma 3.6] using density arguments and consistency of the resolvents. Let us give some details here. Thanks to the boundary conditions of $V(\Omega_\mathrm{p})$ we get $P_2 V(\Omega_\mathrm{p}) \subseteq V(\Omega_\mathrm{p})$. By using the self-adjointness of P_2 in L^2 and $a(u, P_2 v) = a(P_2 u, v)$, the assertion follows for $q = 2$. To prove the assertion for $q \in I_{\Omega_\mathrm{p}}$ we consider the following consistency argument. Since the resolvents $R(-A_{q,\lambda}, \mu)$ are for all $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$ given by the formula in 2.17 we get $P_q(\mu + A_{q,\lambda})^{-1} = (\mu + A_{q,\lambda})^{-1} P_q$ on $L^2(\Omega_\mathrm{p}, \mathbb{C}^3) \cap L^q(\Omega_\mathrm{p}, \mathbb{C}^3)$. Note that all these resolvents exist since $-A_{2,\lambda}$ and $-A_{q,\lambda}$ are both generator of bounded analytic semigroups of angle $\pi/2$. The assertion follows by using boundedness of the resolvents and the density of $L^2 \cap L^q$ in L^q. Note that the Helmholtz projections are also consistent by construction, compare Lemma 2.51 and the discussion before. □

Definition 4.15 *(Maxwell operator)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^3$ be a periodic Lipschitz domain, $q \in I_{\Omega_\mathrm{p}}$. We define the (shifted) Maxwell operator $M_{q,\lambda}$ on $L^q_\sigma(\Omega_\mathrm{p})$ by

$$\begin{aligned} D(M_{q,\lambda}) &:= P_q D(A_q) = D(A_q) \cap L^q_\sigma(\Omega_\mathrm{p}), \\ M_{q,\lambda} u &:= A_{q,\lambda} u \qquad \textit{for } u \in D(M_q). \end{aligned}$$

The operator $M_q := M_{q,0} = M_{q,\lambda} - \lambda Id$ is called the Maxwell operator.

We can extend the multiplier theorem from Theorem 4.11 to the operators $M_{q,\lambda}$ for all $\lambda > 0$ since $A_{q,\lambda}$ and P_q commute.

Theorem 4.16 *(spectral multiplier theorem for the Maxwell operator)*
Let $\lambda > 0$ and q, $F\colon [0,\infty) \to \mathbb{C}$, Ω_{p} as in Theorem 4.9. Then, the operator $F(M_{2,\lambda})$ defines a bounded operator on $L^q_\sigma(\Omega_{\mathrm{p}})$ and there is a constant $C_q > 0$ such that

$$||F(M_{2,\lambda})||_{L^q_\sigma \to L^q_\sigma} \leq C_q(\sup_{n\in\mathbb{Z}} ||\omega F_\lambda(2^n \cdot)||_{C^s} + |F(0)|).$$

Proof: This follows from Theorem 4.11 and Proposition 4.14. □

We already have seen that the spectrum of $A_{q,\lambda}$ is independent of q, but we get this result for the Maxwell operator, too. Note that the spectrum of $M_{q,\lambda}$ might be smaller than the spectrum of $A_{q,\lambda}$.

Proposition 4.17 *(spectral independence of q for the Maxwell operator)*
Let $\Omega_{\mathrm{p}} \subseteq \mathbb{R}^3$ be a periodic domain with Lipschitz boundary. Then, the spectrum of $M_{q,\lambda}$ (and hence also of $M_q = M_{q,0}$) is independent of $q \in I_{\Omega_{\mathrm{p}}}$.

Proof: We have $z \in \mathbb{R} \cap \varrho(M_{q,\lambda})$ if and only if there is an $\varepsilon > 0$ and a function $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi = 1$ on $(z-\varepsilon, z+\varepsilon)$ so that $f(M_{q,\lambda}) = 0$, compare [Dav95a, Lemma 4]. Since we have consistency of $f(M_{q,\lambda}) = 0$ for all $q \in I_{\Omega_{\mathrm{p}}}$, and since $L^p \cap L^q$ is dense in L^q for all $p, q \in I_{\Omega_{\mathrm{p}}}$, we get the assertion. □

4.4 Navier-Stokes equations and the Stokes operator

We consider the physical aspect of the Navier-Stokes equations in the appendix. From a mathematical point of view, the Navier-Stokes equations fascinate and attract mathematicians since a long time. Leray was able to prove existence in the weak sense already at the beginning of the 1930's [Ler33, Ler34]. It is nearly impossible to give all of the most important results here. We refer to [Tem77, GR12] for a comprehensive treatment of the Navier-Stokes equations implying numerical theory, too. Another recommendation including the historical development of the theory is [Kye12]. Although already more than 15 years old [Wie99] is quite interesting to read, because many aspects and not yet solved problems are summarized. Still, there are many open problems in the theory. The most famous and important one is part of the Millennium problems. The matter of the problem is the existence proof of strong, regular global solutions in the three dimensional setting.

Since periodic domains are not that important for the incompressible Navier-Stokes equations as in the case of Maxwell equations, we do not make a comprehensive analysis of the theory here.
We clarify the importance of the Helmholtz projection within the theory and apply the results from [GHHS12] to the case of periodic domains.
If $\partial\Omega_{\mathrm{p}} \in C^3$ this leads to analyticity of the semigroup generated by the Stokes operator on L^q and we even obtain maximal L^r-L^q regularity for the solution pair $(u, \nabla p)$. In addition, we get a unique mild solution of the incompressible Navier-Stokes equations on L^q, provided $\partial\Omega_{\mathrm{p}} \in C^3$ and $q > d$.

Furthermore, [GK15, Theorem 1.1] implies for all $q \in (1, \infty)$, and for all periodic domains Ω_p with uniform C^3-boundary that the Stokes operator $\lambda_0 + A_q$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^q_\sigma(\Omega_\mathrm{p})$ for some $\lambda_0 > 0$. Besides, the function calculi are consistent.

We start with the definition of the Stokes operator.

Definition 4.18 *Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain with uniform $C^{1,1}$-boundary and $q \in (1, \infty)$. We define the Stokes operator $A_q := -P_q \Delta$ by*

$$\begin{aligned} D(A_q) &:= W^{2,q}(\Omega_\mathrm{p}, \mathbb{C}^d) \cap W^{1,q}_0(\Omega_\mathrm{p}, \mathbb{C}^d) \cap L^q_\sigma(\Omega_\mathrm{p}, \mathbb{C}^d), \\ A_q u &:= -P_q \Delta u. \end{aligned}$$

We observe that the Stokes operator is a special case of the Maxwell operator, namely by putting $\varepsilon \equiv 1$, but with Dirichlet boundary conditions instead of Neumann boundary conditions. So, by using the form method ansatz, it is possible to define the Stokes operator with less smooth boundary.
As in the Maxwell case, the Hilbert space case L^2 is much easier to handle than the L^q-case. So, the negative of the Stokes operator is known to be self-adjoint and generator of an analytic semigroup, even on general domains $\Omega \subseteq \mathbb{R}^3$, cf. [Soh01, Mon06].
In the bounded case, it had been known for a long time, that $-A_q$ generates a bounded analytic semigroup on L^q_σ for all $q \in (1, \infty)$, if $\partial\Omega$ is smooth, compare e.g. [Gig81].
In contrast, the case of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ turned out to be much more difficult. Shen [She12] proved the bounded analyticity, the so-called Taylor's conjecture, for q satisfying $|\frac{1}{q} - \frac{1}{2}| < \frac{1}{2d} + \varepsilon$, where $d \geq 3$. Note that for $d = 3$ the interval corresponds to $(\frac{3+\varepsilon}{2+\varepsilon}, 3 + \varepsilon)$ which correlates to the existence interval of the Helmholtz decomposition.
The theory for unbounded domains is more complicated, particularly because the Helmholtz decomposition might fail.
In the L^q-case for periodic domains, we use the results from [GHHS12] for general unbounded domains, on which the Helmholtz decomposition exists on L^q.

Proposition 4.19 *(Stokes operator and the inhomogeneous problem)*
Let $\Omega_\mathrm{p} \subseteq \mathbb{R}^d$ be a periodic domain with C^3-boundary, $J = (0, T)$ for some $T > 0$ and $r, q \in (1, \infty)$. Then, the negative of the Stokes operator $-A_q$ generates an analytic semigroup on $L^q_\sigma(\Omega_\mathrm{p}, \mathbb{C}^d)$. Besides, the solution u of the inhomogeneous problem

$$\begin{aligned} \partial_t u + A_q u(t) &= f(t), \qquad t > 0, \;\; f \in L^r(J, L^q_\sigma(\Omega_\mathrm{p})), \\ u(0) &= u_0, \end{aligned}$$

satisfies

$$||\partial_t u||_{L^r(J, L^q(\Omega_\mathrm{p}))} + ||A_q u||_{L^r(J, L^q(\Omega_\mathrm{p}))} \leq C\big(||f||_{L^r(J, L^q(\Omega_\mathrm{p}))} + ||u_0||_{X_0}\big)$$

where C is independent of $f \in L^r(J, L^q(\Omega_\mathrm{p}))$ and $u_0 \in X_0$. Here, X_0 is the real interpolation space

$$X_0 = (L^q_\sigma(\Omega_\mathrm{p}), D(A_q))_{1-1/r, r}$$

given by the K-Method [BL76, chapter 3], [Lun95, Section 1.2].

Proof: We can apply [GHHS12, Corollary 2.2] since the Helmholtz decomposition exists on $L^q(\Omega_{\mathrm{p}}, \mathbb{C}^d)$, see Theorem 3.37.
Note that the interpolation space is a standard tool in the study of space-time regularity for evolution equations, compare [Lun95, Section 2.2.1 and Section 4.3.2]. □

The resolvents of the Stokes operator are an interesting object. We consider the so-called Stokes resolvent problem

$$\begin{aligned} \lambda u - \Delta u + \nabla p = f, &\qquad \text{in } \Omega_{\mathrm{p}} \times (0,T), \\ \operatorname{div} u = 0, &\qquad \text{in } \Omega_{\mathrm{p}} \times (0,T), \\ u = 0, &\qquad \text{on } \partial\Omega_{\mathrm{p}} \times (0,T). \end{aligned}$$

First, we want to clarify the connection between the Helmholtz projection, the Stokes resolvent system and the resolvents of the Stokes operator. Therefore, we give the following formal calculation, which reveals the approach. We recall that a vector field is a gradient if and only if it is of the form $(Id-P_q)(\dots)$. Furthermore we know that $P_q u = u$, $(Id-P_q)u = 0$, so we rewrite the system above and get

$$(Id - P_q)(\lambda u - \Delta u + \nabla p) + P_q(\lambda u - \Delta u + \nabla p) = (Id - P_q)f + P_q f,$$

which can be simplified to

$$\lambda u + A_q u + \nabla p = P_q f + (Id - P_q)(f + \Delta u).$$

So, the idea is to find a solution u of the Stokes resolvent problem

$$\lambda u + A_q u = P_q f,$$

and define subsequently

$$\nabla p = (Id - P_q)(f + \Delta u).$$

This is possible since $Id - P_q$ maps onto $G^q(\Omega_{\mathrm{p}})$. We get the following statement [GHHS12, Corollary 2.3] concerning the Stokes resolvent system.

Proposition 4.20 *(Stokes resolvent system)*
Let Ω_{p} be as in Proposition 4.19, $q \in (1,\infty)$ and $0 \neq \lambda \in \Sigma_\theta$, where $\theta \in (0,\pi)$. Then, there is a $\theta_0 \in \mathbb{R}$ such that for all $\lambda \in \theta_0 + \Sigma_\theta$ and $f \in L^q_\sigma(\Omega_{\mathrm{p}})$ there is a unique pair $(u, \nabla p)$ in $(W^{2,q}(\Omega_{\mathrm{p}}) \cap W^{1,q}_0(\Omega_{\mathrm{p}}) \cap L^q_\sigma(\Omega_{\mathrm{p}})) \times G^q(\Omega_{\mathrm{p}})$ solving the Stokes resolvent system. Furthermore, the inequality

$$|\lambda| ||u||_{L^q(\Omega_{\mathrm{p}})} + ||\Delta u||_{L^q(\Omega_{\mathrm{p}})} + ||\nabla p||_{L^q(\Omega_{\mathrm{p}})} \leq C ||f||_{L^q(\Omega_{\mathrm{p}})}, \qquad f \in L^q(\Omega_{\mathrm{p}}),$$

holds for some $C > 0$ independent of $\lambda \in \theta_0 + \Sigma_\theta$.

In addition, the Stokes operator admits a bounded $\mathcal{H}^\infty$-calculus, more exactly we get the following.

Proposition 4.21 *($\mathcal{H}^\infty$-calculus for Stokes resolvent)*
Let Ω_p be as in Proposition 4.19 and $q \in (1,\infty)$. Then, there exists $\lambda_0 > 0$ such that the Stokes operator $\lambda_0 + A_q$ admits a bounded $\mathcal{H}^\infty$-calculus in $L^q_\sigma(\Omega_\mathrm{p})$. Besides, the functional calculi are consistent for all $q \in (1,\infty)$.

Proof: We use Theorem 3.37 and [GK15, Theorem 1.1]. □

Next, we give a unique solvability theorem for the time dependent Stokes equations with no-slip boundary condition, $u_0 = 0$ and $\mu \equiv 1$. Furthermore, we even get maximal regularity for the solution. We recall that this system is given by

$$\begin{aligned} \partial_t u - \Delta u + \nabla p = f, &\qquad \text{in } \Omega_\mathrm{p} \times (0,T),\\ \operatorname{div} u = 0, &\qquad \text{in } \Omega_\mathrm{p} \times (0,T),\\ u|_{\partial\Omega} = 0, &\qquad \text{on } \partial\Omega_\mathrm{p} \times (0,T),\\ u(0,\cdot) = u_0. & \end{aligned}$$

Theorem 4.22 *(theorem for the time dependent Stokes system)*
Let $\Omega_\mathrm{p}, J, T, r, q, f$ be as in Proposition 4.19. Then, the time dependent Stokes system from above has a unique solution pair

$$(u, \nabla p) \in W^{1,r}(J, L^q(\Omega_\mathrm{p})) \cap L^r(J, W^{2,q}(\Omega_\mathrm{p}) \cap W^{1,q}_0(\Omega_\mathrm{p}) \cap L^q_\sigma(\Omega_\mathrm{p})) \times L^r(J, G^q(\Omega_\mathrm{p})),$$

and there is a constant $C > 0$ such that (where $X = L^r(J, L^q(\Omega_\mathrm{p}))$)

$$||\partial_t u||_X + ||u||_X + ||\Delta u||_X + ||\nabla p||_X \le C||f||_X.$$

Proof: We use Theorem 3.37 and [GHHS12, Theorem 2.1]. □

Remark 4.23 There is a similar theory [FKS05, FKS08, FKS09] for arbitrary domains, on which the L^q Helmholtz decomposition might fail, by using the alternative Helmholtz decomposition in $\widetilde{L}^q$ from Remark 2.52. In [FKS09] there was only needed $\partial\Omega$ to be uniformly of class $C^{1,1}$. So, there is hope to weaken the boundary regularity in the Theorem above, for example to demand only $\partial\Omega$ uniformly C^2 or even $C^{1,1}$.

After having discussed the linear case, we now take a look at the full Navier-Stokes system. We recall that the incompressible Navier-Stokes equations for fluids with constant density are given by

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla p = f, &\qquad \text{in } \Omega_\mathrm{p} \times (0,T),\\ \operatorname{div} u = 0, &\qquad \text{in } \Omega_\mathrm{p} \times (0,T),\\ u|_{\partial\Omega} = 0, &\qquad \text{on } \Omega_\mathrm{p} \times (0,T),\\ u(0,\cdot) = u_0, & \end{aligned}$$

where we used for simplicity once more $\mu \equiv 1$. As above, by using the Helmholtz decomposition, it suffices to solve

$$\partial_t u - A_q u = P_q f - P_q((u \cdot \nabla)u),$$

and subsequently set

$$\nabla p = (Id - P_q)(\Delta u - (u \cdot \nabla)u)).$$

We consider the case of no external force $f = 0$, which leads to

$$\begin{aligned}
\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p &= 0, && \text{in } \Omega_{\mathrm{p}} \times (0,T),\\
\mathrm{div}\; u &= 0, && \text{in } \Omega_{\mathrm{p}} \times (0,T),\\
u|_{\partial\Omega} &= 0, && \text{on } \Omega_{\mathrm{p}} \times (0,T),\\
u(0,\cdot) &= u_0.
\end{aligned}$$

So we have to solve the system

$$\begin{aligned}
\partial_t u = -A_q u - P_q((u \cdot \nabla)u), &\qquad \text{in } \Omega_{\mathrm{p}} \times (0,T),\\
u(0,\cdot) = u_0.&
\end{aligned}$$

Here, [GHHS12] yields the existence of a continuous local mild solution. Roughly speaking, a mild solution is a function, which is given by the variation of constants formula. This solution concept is weaker than the classical solution concept, compare [EN00, section VI.7].

A local mild solution of the system above is a function $u \in C([0,T), L^q_\sigma(\Omega_{\mathrm{p}}))$ satisfying

$$u(t) = \mathrm{e}^{-tA_q} u_0 - \int_0^t \mathrm{e}^{-(t-s)A_q} P_q((u(s) \cdot \nabla)u(s))ds, \qquad 0 \le t < T,$$

for some $T > 0$. Observe that [GHHS12] use

$$\mathrm{div}\;(u(s) \otimes u(s)) = (u(s) \cdot \nabla)u(s),$$

which is true since div $u(s) = 0$. Here, we refer to [Way05, page 6] for the details. By applying Theorem 3.37 and [GHHS12, Theorem 3.2] we get immediately the following theorem.

Theorem 4.24 *(mild solution for the Navier-Stokes system)*
Let $\Omega_{\mathrm{p}} \subseteq \mathbb{R}^d$ be a periodic domain with C^3-boundary. Then, for $q \in (d,\infty)$ and $u_0 \in L^q_\sigma(\Omega_{\mathrm{p}})$, there exists $T_0 > 0$ and a unique mild solution $u \in C([0,T_0), L^q_\sigma(\Omega_{\mathrm{p}}))$ of the system

$$\begin{aligned}
\partial_t u = -A_q u - P_q((u \cdot \nabla)u), &\qquad \textit{in } \Omega_{\mathrm{p}} \times (0,T),\\
u(0,\cdot) = u_0,&
\end{aligned}$$

which is given by

$$u(t) = \mathrm{e}^{-tA_q}u_0 - \int\limits_0^t \mathrm{e}^{-(t-s)A_q}P_q((u(s)\cdot\nabla)u(s))ds, \qquad 0 \le t < T_0.$$

Overall we have seen that the existence of the Helmholtz decomposition on L^q-spaces directly has consequences on linear and nonlinear Navier-Stokes equations.

CHAPTER 5

Physical Appendix

5.1 Maxwell's equations

We start with the Maxwell's equations. Throughout this section we use SI units. We follow [JJWM08]. The Maxwell equations are the fundamental equations in classical electrodynamics and optics. They describe the correlation of electric and magnetic fields in dependence of each other and of currents plus charges. In literature, one finds two different formulations of the Maxwell equations. We consider both of them without going too much into details from physics.

On atomic level one uses the microscopic Maxwell Equations, which are given by

$$\begin{aligned} \operatorname{div} E &= \varrho/\varepsilon_0, && \text{(Gauß' law)},\\ \operatorname{div} B &= 0, && \text{(Gauß' law for magnetism)},\\ \operatorname{curl} E &= -\partial_t B, && \text{(Faraday's law of induction)},\\ \operatorname{curl} B &= \mu_0 J + \mu_0\varepsilon_0\partial_t E, && \text{(Ampére's circuital law)}. \end{aligned}$$

Here, E, B denote the electric and the magnetic field, ϱ is the given electric charge density, J is the electric current density. The constants ε_0 and μ_0 are the vacuum permittivity and the vacuum permeability. We explain the origin of the equations very briefly now. The first equation, the Gauss law, states that charges are the sources of the electric field. The second equation states that there are no sources of magnetic fields, since there are no magnetic monopoles. In consequence, the magnetic field lines are always closed. The third equation, Faraday's induction law, states that a changing magnetic field generates an electric eddy current field. The last equation, Ampère's law, states that electric changes raise to a magnetic eddy field. This law includes the Maxwell's displacement current.

In applications, it is often more convenient to use the macroscopic Maxwell equations, which we consider now. If one is interested in the behaviour of a material, there occur bound electrons, which have to be considered, too. Note that they are not free, so it is useful to distinguish between free charges and bound charges. One further effect is that also bound charges can change their position slightly. The strength of this effect depends on the material and is denoted as polarization. Analogously, there might be magnetic effects in the material. Consequently, we have to add the polarisation P to the electric field, and the magnetism M to the magnetic field. This yields the electric displacement field D and the magnetizing field H which are given by

$$D := \varepsilon_0 E + P, \qquad H := \frac{1}{\mu_0} B - M.$$

For more details we refer to [Gri99, pages 328-330]. The macroscopic Maxwell equations are given by

$$\begin{aligned}
\operatorname{div} B &= 0,\\
\operatorname{div} D &= \varrho_{\mathrm{f}},\\
\operatorname{curl} H &= J_{\mathrm{f}} + \partial_t D,\\
\operatorname{curl} E &= -\partial_t B.
\end{aligned}$$

We have

$$\varrho = \varrho_{\mathrm{f}} + \varrho_{\mathrm{b}}, \qquad J = J_f + J_b,$$

where

$$\varrho_{\mathrm{b}} = -\operatorname{div} P, \qquad J_{\mathrm{b}} = \operatorname{curl} M + \partial_t P,$$

and ϱ_{f}, J_{f} denote the free parts and ϱ_{b} and J_{b} the bound parts. J_b consists of movements of electric and magnetic dipole moments. At a first glance, this system of equations seems more difficult to manage. However, in many application cases the system simplifies. If the material is not magnetic, then $M = 0$. Then, we have $B = \mu_0 \mu_{\mathrm{r}} H$, where μ_{r} denotes the permeability relative to the vacuum. A value $0 \leq \mu_{\mathrm{r}} << 1$ corresponds to a material, which counteracts an external magnetic field. These materials are called diamagnets. Without external magnetic field such materials are not magnetic. They are used in superconductors. A large value of μ_{r} corresponds to ferromagnets, which are strongly attracted by an external magnetic field. In fact, these materials often are magnets, for example iron is a ferromagnet. They are used among others as motors and transformers. Besides there are many materials having $\mu_{\mathrm{r}} \approx 1$ (in fact it is usually slightly bigger than), for example the materials air, concrete, wood, aluminium, Teflon, sapphire and water. In these materials there is only a small (negligibly strengthening) effect to external magnetic fields. The magnetic susceptibility $\chi_{\mathrm{m}} := \mu_{\mathrm{r}} - 1$ is also commonly used to measure the magnetizability of a material. Hence

$$\mu = \mu_0 \mu_r = \mu_0 (1 + \chi_{\mathrm{m}}).$$

Besides, many materials are linear, which means that the polarisation is linear to the electric field, which results in the linear connection

$$P = \varepsilon_0 \chi_{\mathrm{e}} E, \qquad D = \varepsilon_0 (1 + \chi_{\mathrm{e}}) E.$$

Linearity is a typical and common assumption. Nonlinear media are used in nonlinear optics and are realized by using very high intensities. Here, the electric susceptibility χ_{e} is not scalar, but matrix-valued, and the term $\varepsilon_{\mathrm{r}} := (1 + \chi_{\mathrm{e}})$ denotes the permittivity relative to the vacuum. If, in addition, χ_{e} is scalar-valued, which means that E and P are parallel, then the material is called isotropic. The permittivity ε is defined by

$$\varepsilon := \varepsilon_0(1 + \chi_\mathrm{e}).$$

If χ_e is location-independent, the material is called homogeneous. The real part of ε yields information about the dispersion. For example, a high value corresponds to transparent material, and a very negative value corresponds to metal. For metals [JJWM08, page 8] one even takes the limit $\varepsilon \to -\infty$. Whereas the imaginary part of ε yields information about the absorption, compare [Jac06, section 7.5, page 359]. So, positive values for ε correspond to the transparency of the underlying material. In contrast, a negative value would correspond to a material where light is being lost. In praxis, every material is not steady in all places, so homogeneity is an idealization, but one which works for many materials in good approximation. Furthermore, if the material changes, this will be reflected in discontinuity of ε. A material is called non-dispersive, if the value of P at the time t depends only on $E(t)$, and not on the values of $E(t_0)$ for $t_0 < t$. This allows to pull ε outside of the time derivative $\partial_t(\varepsilon E)$.

Since we used a huge amount of physical terms here and in the last chapter, we give here explanations of all these terms. More details can be found in classical books considering electrodynamics like [Jac99].

- **E: Electric field strength or Electric field:** The vector field stating the electric force a test particle would be exposed. The force is caused by electric charges or by varying magnetic fields.
- **B: Magnetic flux density or magnetic field:** The vector field stating the force on a moving charged particle such that the Lorentz force law is satisfied. It can be characterized alternatively by the torque it produces on a magnetic dipole. The field is generated by electric currents and magnetic materials.
- **D: Electric displacement field:** The electric displacement field describes the density of the electric field lines. In doing so, it regards the generated fields of free as well as bound charges.
- **H: Magnetic field strength or Magnetizing field:** The vector field which describes the strength and direction of the magnetic field generated by magneto-motive force. It states the magnetic effect of external currents without considering the magnetism of the underlying material, while the B-field describes the sum of both.
- **ϱ: electric charge density:** ϱ describes the distribution of charges.
- **J: Electric current density:** This vector field just describes the density of the electric current.
- **M: Magnetism:** A magnet is a material, which attracts or rejects other magnets.
- **Dielectric:** A dielectric medium is a material, which is an insulator, but can be polarised by an electric field. Examples of such materials are glass, most of the gases and most of the plastics.
- **P: Polarisation:** This vector field describes the density of electric dipole moments

in a dielectric. Polarisation appears when a dielectric is put in an electric field. The dielectric can not transport charges, but the charges within the material change the position slightly, and this effect is called polarisation.

- ε**: permittivity:** The permittivity describes the permeability of the material for electric fields. It measures the ability of a material to support the formation of an electric field.
- ε_0**: vacuum permittivity:** The permittivity of the vacuum.
- ε_r**: relative permittivity:** In most cases the permittivity is expressed relative to the vacuum permittivity, so $\varepsilon_\mathrm{r} = \varepsilon/\varepsilon_0$.
- χ_e**: electric susceptibility:** The electric susceptibility ε_r is defined by $\chi_\mathrm{e} = \varepsilon/\varepsilon_0 - 1$ and describes as ε_r the permittivity of the material. Hence, it describes how strong the polarisation effect is.
- μ**: permeability:** The permeability describes the permeability of the material for magnetic fields. It is the magnetic analogue to the (electric) permittivity.
- μ_0**: vacuum permeability:** The permeability of the vacuum.
- μ_r**: relative permeability:** The permeability relative to the vacuum.
- χ_m**: magnetic susceptibility:** χ_m is defined by $\chi_\mathrm{m} = \mu/\mu_0 - 1$ and describes as μ_r the permeability of the material.
- σ**: conductivity:** The material's ability to conduct electric current.

5.2 The Navier-Stokes equations

The origin of the Navier-Stokes equations is based on the equation of motion and the continuity equation applied to viscous fluids. Note that all involved functions may depend on time, which will be denoted by the first variable, and space, the second variable. If the time-dependence of the used derivation operation is not explicitly stated, we assume derivation operations like div, ∇ to be applied on the space variables. Although it is possible to consider the equations in d dimensions, we restrict ourselves to the by far most relevant case $d = 3$.

In [Red08, section 8.1.2] there is a list of stunningly 22 equations of viscous fluids with 22 variables. The Navier-Stokes equations are a combination of four of them, the continuity equation, the equations of motion, the constitutive equation and the rate of deformation-velocity equations, which are looking as follows:

$$\begin{aligned}
\partial_t \varrho &= -\mathrm{div}\ (\varrho u), && \text{(Continuity equation)},\\
\nabla\sigma + \varrho f &= \varrho(\partial_t u + (u\cdot\nabla)u), && \text{(Equations of motion)},\\
\sigma &= 2\mu D + \lambda(\mathrm{tr}\ D)Id - p\,Id, && \text{(Constitutive equation)},\\
D &= 1/2(\nabla u + (\nabla u)^T), && \text{(Rate of deformation-velocity equations)}.
\end{aligned}$$

Here, u describes the speed of the fluid, ϱ the density, p the pressure and f the body force density. Here, the operator λ and μ denote the Lamé constants of the fluid. Without going into details, we just mention that they are used to describe Hook's law, which describes the stress in dependence of the strain tensor [Red08, section 6.3.3]. We note that we have tr $D =$ div u, since $D_{ii} = \partial_i x_i$ for $i = 1, 2, 3$. Besides we have $2D_{ij} = \partial_i u_j + \partial_j u_i$ for $i, j \in \{1, 2, 3\}$, so one easily computes $\nabla 2D = \Delta u + \nabla \text{div } u$, where Δ, in literature also denoted by ∇^2, is the vectorial Laplacian, which is given by

$$u = (\Delta u_1, \ldots, \Delta u_d)^T.$$

In the case $d = 3$ we have the representation

$$\Delta u = \nabla(\text{div } u) - \text{curl } (\text{curl } u).$$

So, by combing the last three equations [Red08, section 8.1.3] we get the momentum equation

$$\varrho\big(\partial_t u + (u \cdot \nabla)u\big) = -\nabla p + \mu \Delta u + (\lambda + \mu)\nabla(\text{div } u) + \varrho f,$$

The continuity equation is given by

$$\partial_t \varrho + \text{div } j = 0,$$

where $j = \varrho u$ is the current density of the fluid and hence

$$\partial_t \varrho + \nabla \varrho \cdot u = -\varrho \text{div } u.$$

By using notation from differential geometry we have

$$\partial_t(\varrho(t, u(t))) = -\varrho \text{div } u.$$

We only consider the most common variant of the Navier-Stokes equations, the incompressible Navier-Stokes equations. In the proper sense a flow is incompressible if the density of the fluid is independent of the pressure at the same temperature. In many applications one ignores the influence of the temperature, since it is small in comparison to the effect of the pressure. This yields to the idealisation that a media is incompressible if the density is constant along every trajectory, which is mathematically reflected in

$$\partial_t(\varrho(t, u(t))) = 0.$$

The continuity equation simplifies in that case to

$$\text{div } u = 0.$$

Furthermore, this also simplifies the momentum equation to

$$\varrho\big(\partial_t u + (u \cdot \nabla)u\big) = -\nabla p + \mu \Delta u + \varrho f.$$

If we, in addition, ignore thermal influences on the density, assume incompressibility and assume that the density ϱ is even constant, we can divide the equation by ϱ and get

$$\partial_t u + (u \cdot \nabla)u = -\nabla \widetilde{p} + \widetilde{\mu}\Delta u + f,$$

where $\widetilde{p} = p/\varrho$ and $\widetilde{\mu} = \mu/\varrho$. Note that incompressibility does not imply constant density. For example the density of water might depend on the salinity, which is not everywhere constant. We want to give curt physical explanations of the appearing terms. Clearly, $\partial_t u$ describe the changing rate of the speed at the time t at the position x. So, we examine the driving forces causing the changes of the speed. As already mentioned the term $-\nabla p$ describes the effect of the pressure. The term $\mu\Delta u$ is related to internal friction, so the value of μ measures the size of this effect, and is the called viscosity. The term $(u \cdot \nabla)u$ describes the convective acceleration and is consequence of the arising inertia forces. The missing term f describes all exterior forces, which can be for example gravitation or electric fields.

In applications, the viscosity μ of a material is usually assumed to be constant, although it might depend on the temperature of the material. In the case of Non-Newtonian fluids, the viscosity depends in addition on the shear rate. Most of the industrial products are Non-Newtonian fluids, while water, milk or mineral oil are Newtonian fluids. Since we ignore once more the thermal effect, we assume μ to be constant for Newtonian fluids.

To get a complete description of a system, we add boundary conditions and an initial value u_0 to the system. A typical boundary condition consists for instance of Dirichlet boundary conditions, also called no slip boundary condition and conforms adhesion of the fluid on the boundary. This boundary conditions turn out to appear in many physical applications. In that case one has the following formulation for the incompressible Navier-Stokes equations with constant density, for Newton fluids with negligibly thermal effects and no slip boundary condition.

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &= f, && \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0, && \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} &= 0, && \text{on } \Omega \times (0,T), \\ u(0,\cdot) &= u_0. \end{aligned}$$

Depending on the chosen fluid, the shape of the domain Ω and the mathematical modelling other boundary conditions also appear, for example one might choose Neumann boundary conditions, pressure boundary conditions or permeable wall boundary conditions. This are by far not all thinkable boundary conditions. Besides, mixtures are possible. There are cases, where this system, which still is mathematically difficult to solve, can be further simplified. From a mathematical point of view, the non linear inertia term $(u \cdot \nabla)u$ often causes difficulties. In the case of a Stokes flow, which means that the cinematic viscosity is appreciable bigger than the inertia term and also thermal influences are negligible, one gets the time dependent Stokes system (with no slip boundary condition)

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f, && \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0, && \text{in } \Omega \times (0,T), \end{aligned}$$

$$\begin{aligned} u|_{\partial\Omega} &= 0, \qquad \text{on } \Omega \times (0,T), \\ u(0,\cdot) &= u_0. \end{aligned}$$

In this system, the non-linear term has vanished and we see in Section 4.4 how one can use the Helmholtz projection to simplify this system further, slinging out the pressure of the equation. Examples of materials having high viscosity are syrup, honey and solid materials. In contrast, most of the gases, water, alkanes or alcohols have a small viscosity. For systems which have very small viscosity, i.e. fluids with minor internal friction, one can ignore the term $\mu\Delta u$ and this yields to the Euler equation

$$\partial_t u + (u \cdot \nabla)u + \nabla p = f.$$

Since we only consider incompressible Navier-Stokes equations in this work, we do not take a closer look at the (more complicated) compressible case here.

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