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A fixed point theorem for monotone operators in ordered Banach spaces

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Let $(E, \|\cdot\|)$ be a real Banach space ordered by a cone K. A cone K is a closed convex subset of E such that $\lambda K \subseteq K$ $(\lambda \geq 0)$ and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$, and for $a, b \in E$ let $[a, b]$ denote the order interval of all $x \in E$ with $a \leq x \leq b$. A function $g : D \to E$, $D \subseteq E$ is called (monotone) increasing if $x, y \in D$, $x \leq y \Rightarrow g(x) \leq g(y)$.

We assume K to be a normal cone, i.e.,

$$\exists\, \gamma \geq 1 : \ 0 \leq x \leq y \ \Rightarrow \ \|x\| \leq \gamma \|y\|,$$

and we assume that $\Psi : K \to [0, \infty)$ is a given functional having the following property:

(P) If (x_n) is a decreasing sequence in K, then

$$\Psi(x_n) \to 0 \ (n \to \infty) \ \Rightarrow \ x_n \to 0 \ (n \to \infty).$$

Functionals with this property always exist, take $\Psi = \|\cdot\|$ for example, but there are more interesting examples:

Consider the Banach space $E = CB([0, \infty), \mathbb{R})$ of all bounded continuous functions on $[0, \infty)$ endowed with the supremum-norm $\|\cdot\|_\infty$ and ordered by the normal cone $K = \{x \in E : x(t) \geq 0 \ (t \geq 0)\}$. Then

$$(1) \qquad \Psi(x) = \max\{\max\{e^{-t} x(t) : t \geq 0\}, \limsup_{t \to \infty} x(t)\} \quad (x \in K)$$

is a functional with property (P).

Another example is $E = C([0,1],\mathbb{R})$ endowed with the maximum-norm $\|\cdot\|_\infty$ and ordered by the normal cone $K = \{x \in E : x(t) \geq 0\ (0 \leq t \leq 1)\}$. Let $h : [0,1] \to (0,\infty)$ be any bounded function and let $q : [0,\infty) \to [0,\infty)$ be a continuous function with $q(0) = 0$ and $q(s) > 0\ (s > 0)$. Then

$$\Psi(x) = \sup\{h(t)q(x(t)) : 0 \leq t \leq 1\} \quad (x \in K)$$

has property (P); this can be shown by using Dini's Theorem.

In the following theorem a combination of monotonicity and contraction requirements for a function g leads to the existence of a unique fixed point. Starting with a Theorem of Ran and Reurings [7] several fixed point theorems in ordered metric spaces under contraction conditions related to the ordering are known, see [1, 2, 3, 4, 5, 6] and the references given there. Here we use a rather mild contraction condition with respect to the given functional Ψ (having property (P)).

Theorem 1 *Let $a, b \in E$, $a \leq b$ and let $g : [a,b] \to [a,b]$ have one of the following properties:*

1. *g is increasing and $\exists L \in [0,1)\ \forall x, y \in [a,b] : x \leq y \Rightarrow \Psi(g(y)-g(x)) \leq L\Psi(y-x)$, or*
2. *g is decreasing and $\exists L \in [0,1)\ \forall x, y \in [a,b] : x \leq y \Rightarrow \Psi(g(x)-g(y)) \leq L\Psi(y-x)$.*

Then g has a unique fixed point $z \in [a,b]$ and $g^{(n)}(x) \to z\ (n \to \infty)$ for each $x \in [a,b]$.

Proof: First, let g be increasing. Then the sequences

$$(x_n) := (g^{(n)}(a))_{n=0}^\infty \text{ and } (y_n) := (g^{(n)}(b))_{n=0}^\infty$$

are increasing and decreasing, respectively, and

$$x_n \leq y_n \quad (n \in \mathbb{N}_0 = \{0,1,2,\dots\}).$$

Therefore $(y_n - x_n)$ is a decreasing sequence in K, and *1* gives

$$\Psi(y_n - x_n) \leq L^n\Psi(b-a) \to 0\ (n \to \infty).$$

Hence $y_n - x_n \to 0\ (n \to \infty)$. Next, for $n, m \in \mathbb{N}_0$

$$0 \leq y_n - y_{n+m} \leq y_n - x_{n+m} \leq y_n - x_n \ \Rightarrow\ \|y_n - y_{n+m}\| \leq \gamma\|y_n - x_n\|.$$

Thus (y_n) is a Cauchy sequence, and so (y_n) and (x_n) are convergent and have the same limit z, say. Since for all $x \in [a,b]$ and $n \in \mathbb{N}_0$

$$0 \leq g^{(n)}(x) - x_n \leq y_n - x_n \ \Rightarrow \|g^{(n)}(x) - x_n\| \leq \gamma \|y_n - x_n\|,$$

we have $g^{(n)}(x) \to z$ $(n \to \infty)$ as well. To see that z is a fixed point consider

$$x_n \leq z \leq y_n \ \Rightarrow \ x_{n+1} = g(x_n) \leq g(z) \leq g(y_n) = y_{n+1} \quad (n \in \mathbb{N}_0),$$

and as $n \to \infty$ we obtain $g(z) = z$.

Now, if g is decreasing, we can apply the increasing case to $G := g \circ g : [a,b] \to [a,b]$, since by *2*

$$\forall x, y \in [a,b] : \ x \leq y \ \Rightarrow \ \Psi(G(y) - G(x)) \leq L^2 \Psi(y - x).$$

Thus G has a unique fixed point z, which is the unique fixed point of g since $G(g(z)) = g(z)$. Moreover, for each $x \in [a,b]$ we have

$$g^{(2n)}(x) = G^{(n)}(x) \to z \quad (n \to \infty),$$

thus $g^{(2n+1)}(x) = g^{(2n)}(g(x)) \to z$ $(n \to \infty)$, so $g^{(n)}(x) \to z$ $(n \to \infty)$ for each $x \in [a,b]$. ∎

Example: Consider the functional equation

$$h(t)\arctan(x(t^2/(1+t))) + u(t) = x(t) \quad (t \geq 0), \tag{2}$$

with any $u, h \in CB([0,\infty), \mathbb{R})$, $h(t) \geq 0$ $(t \in \mathbb{R})$ such that

$$q_1 := \sup\{e^{-t/(1+t)} h(t) : t \geq 0\} < 1 \ \text{ and } \ q_2 := \limsup_{t \to \infty} h(t) < 1.$$

Consider the order interval

$$I := [u - \frac{\pi}{2}h, u + \frac{\pi}{2}h] \subseteq CB([0,\infty), \mathbb{R}).$$

Then $g : I \to I$, defined by

$$g(x)(t) = h(t)\arctan(x(t^2/(1+t))) + u(t)$$

is increasing. We consider Ψ from (1) and set

$$\Psi_1(x) = \max\{e^{-t}x(t) : t \geq 0\}, \quad \Psi_2(x) = \limsup_{t \to \infty} x(t).$$

Note that $t \mapsto \arctan(t)$ is Lipschitz continuous with Lipschitz constant 1, and

$$\frac{t^2}{1+t} - t = -\frac{t}{1+t} \ (t \geq 0), \quad \frac{t^2}{1+t} \to \infty \ (t \to \infty).$$

Let $x, y \in I$, $x \leq y$. We have

$$\begin{aligned} e^{-t}(g(y) - g(x))(t) &\leq e^{-t} h(t)(y-x)(t^2/(1+t)) \\ &\leq e^{-t} e^{t^2/(1+t)} h(t) \Psi_1(y-x) \\ &= e^{-t/(1+t)} h(t) \Psi_1(y-x) \leq q_1 \Psi_1(y-x) \quad (t \geq 0) \end{aligned}$$

and therefore

$$\Psi_1(g(y) - g(x)) \leq q_1 \Psi_1(y-x).$$

Moreover

$$(g(y) - g(x))(t) \leq h(t)(y-x)(t^2/(1+t)) \quad (t \geq 0).$$

Thus

$$\begin{aligned} \Psi_2(g(y) - g(x)) &\leq \limsup_{t\to\infty} h(t) \limsup_{t\to\infty} (y-x)(t^2/(1+t)) \\ &= q_2 \limsup_{t\to\infty} (y-x)(t^2/(1+t)) \\ &= q_2 \limsup_{t\to\infty} (y-x)(t) = q_2 \Psi_2(y-x). \end{aligned}$$

Summing up,

$$\Psi(g(y) - g(x)) \leq \max\{q_1, q_2\} \Psi(y-x) \quad (x, y \in I, \ x \leq y).$$

Now Theorem 1 applies and equation (2) has a unique solution in I; hence also exactly one solution in $CB([0,\infty), \mathbb{R})$.

Theorem 1 is applicable to functions f with the property that $f + \lambda id_E$ is increasing for some $\lambda \geq 0$. We now assume that $\Psi : E \to \mathbb{R}$ is a sublinear functional, i.e.,

$$\Psi(x+y) \leq \Psi(x) + \Psi(y), \quad \Psi(\alpha x) = \alpha \Psi(x) \quad (x, y \in E, \alpha \geq 0),$$

and that $\Psi_{|K}$ has property (P).

Theorem 2 *Suppose $\Psi : E \to \mathbb{R}$ to be sublinear, $\Psi_{|K} : K \to [0,\infty)$ having property* (P). *Let $a, b \in E$, $a \leq b$ and let $f : [a,b] \to E$ have the following properties:*

1. $\exists\lambda \geq 0 : f + \lambda id_E$ *is increasing,*

2. $\exists L \in [0,1)\ \forall x, y \in [a,b] : x \leq y \Rightarrow \Psi(f(y) - f(x)) \leq L\Psi(y-x)$*,*

3. $f(a) \geq a$ *and* $f(b) \leq b$*.*

Then f *has a unique fixed point* $z \in [a,b]$*.*

Proof: We choose $\mu \in [0,1)$ such that $\mu/(1-\mu) \geq \lambda$. Then $g : [a,b] \to E$ defined as

$$g(x) = (1-\mu)f(x) + \mu x$$

is increasing, $g(a) \geq (1-\mu)a + \mu a = a$, $g(b) \leq b$ and hence $g([a,b]) \subseteq [a,b]$. Moreover

$$\begin{aligned}\Psi(g(y) - g(x)) &\leq (1-\mu)\Psi(f(y) - f(x)) + \mu\Psi(y-x)\\ &\leq ((1-\mu)L + \mu)\Psi(y-x) \quad (x, y \in [a,b],\ x \leq y),\end{aligned}$$

with $(1-\mu)L + \mu < 1$. According to Theorem 1 there is a unique fixed point $z \in [a,b]$ of g which is the unique fixed point of f. ∎

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