The Factorization Method for Conducting Transmission Conditions

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Preface

In this work we consider an inverse problem arising from electromagnetic scattering by a medium covered with a very thin and highly conducting layer. Our main objective is to show that the Factorization Method, which is an inverse problem solution algorithm, can be applied to detect the position and shape of such objects from the measurements of the scattered waves at large distances. Such problems originate from applications such as land-mine detection, radar or seismic imaging. We consider two special cases of the problem which are derived as the TM- (transverse magnetic) and TE- (transverse electric) modes from the full Maxwell system. The studies of both modes are divided into the following parts:

(1) Instead of considering the full model with a thin highly conductive layer of given thickness δ, we first derive an approximate one. With the scaled asymptotic expansions technique [25], [12], one can show that for the layers of thickness δ and conductivity proportional to δ^{-1}, the model with the well-known conductive transmission conditions [45], [46] can be used as the first order approximation of the original model.

(2) Prior to considering the inverse problem we study the corresponding forward problem. We establish the well-posedness for both modes by a variational approach involving the Dirichlet-to-Neumann map on an auxiliary interface. This approach will be also used to solve the direct problems numerically.

(3) We show the applicability of the Factorization Method. It turns ou that for the TM-mode the FM works for partially coated obstacles (the results are published in [5]). For the TE-mode one has to make the assumption that the obstacle is fully coated. The study of the FM for the TE-mode has been recently submitted to Inverse Problems Journal [4].

(4) To test the factorization method numerically on a generic data, we developed solvers for the direct problems. Two approaches we used: the combined integral equation and finite element method as suggested in [37], [38] (implemented in MATLAB), and the finite element method (using the FreeFem++ solver [27]).

In Chapter 4 we study an interior transmission eigenvalue problem. Roughly speaking, interior eigenvalues are the wave numbers for which the far field operator lacks injectivity. In recent years the study of interior transmission eigenvalues became an important area in the inverse scattering research (see [13]). It has been shown that with the knowledge of the transmission eigenvalues it is possible to get information about the material properties of the scatterer [15], [22], [11], [18]. In a collaboration with I.Harris (Texas A&M) and A.Kleefeld (Brandenburg University of Technology) we showed that for the TE mode for a real valued boundary parameter interior eigenvalues exist. We also established monotonicity results which suggest that it is possible to retrieve information about the boundary parameter (if the refractive index is fixed) or about the refractive index (if the boundary parameter is fixed) from the knowledge of the interior eigenvalues. The monotonicity results were established by I. Harris and the computations of interior eigenvalues are thanks to A. Kleefeld. We present them here for the sake of completeness.

This work has been partly supported by German Research Foundation (DFG), grant KI906/14-1. The financial support is greatly acknowledged. The results of Chapter 4 were carried out during my research stay at the University of Delaware in Summer 2015. I thank Dr. Fioralba Cakoni for the hospitality during the stay and Karlsruhe House of Young Scientists (KHYS) for the financial support.

This work would not exists without the support of my colleagues. First of all, I would like to thank my advisor Prof. Dr. Andreas Kirsch for the formulation of the problem and for valuable discussions during recent years. I also thank PD Dr. Frank Hettlich for being the co-examiner of the thesis and for always being open to discuss any mathematical problem I would come with to his office. I am very thankful to Irene de Teresa Trueba for introducing me to the perturbation theory and suggesting to use FreeFem++ package for numerical simulations. Her suggestions enriched this work considerably. I am very grateful to Xiaodong Liu for being an initiator to collaborate which resulted in publishing [5] and [6]. Finally, I would like to thank my colleagues from the working group on Inverse Problem Thomas Rösch, Elena Cramer, Janina Stompe, Monika Behrens, Tilo Arens, Julian Ott and Uwe Zeltmann for providing a friendly atmosphere throughout the years.

1 Introduction

1.1 Aim of This Work and Previous Results

In this work we focus on the applicability of the *factorization method* for two scalar inverse scattering problems. The problems are derived as TM- and TE-modes from the time-harmonic Maxwell system where the scattering medium is coated by a thin highly conductive layer. The TM- and TE-modes are special cases when the electric or the magnetic field have only one non-zero component. For this modes we consider the scattering of the electromagnetic waves by an infinitely long cylinder with constant cross section [33]. The appearance of the thin highly conductive layer leads to conductive transmission conditions [2] which has been known for a long time in the study of electromagnetic induction in the Earth [45], [46].

In the following we use the abbreviations **(SP1)** and **(SP2)** for the scattering problems which correspond to the TM- and TE-modes, respectively. In both cases an incident wave of the form

$$u^i(x) = e^{ikx\cdot d}, \;\; x \in \mathbb{R}^2, \;\; k > 0,$$

with the direction of incidence $d \in S^1 = \{x \in \mathbb{R}^2 : \, |x| = 1\}$, is scattered by the medium, which results in the total field u given as the sum $u = u^i + u^s$, where u^s denotes the scattered field.

We assume that the scattering medium is embedded in a homogeneous background. Let $D \subset \mathbb{R}^2$ represent the support of the medium. Further, we assume that D has a smooth boundary ∂D and its complement $\mathbb{R}^2 \backslash \overline{D}$ is connected. The scatterer is characterized by the complex-valued index of refraction n such that $\operatorname{Re} n > 0$, $\operatorname{Im} n \geq 0$ on D and $\operatorname{Re} n \neq 1$ a.e. in D. The thin layer is represented by η, which is a real-valued function defined on ∂D, and it stays for the (scaled) surface conductivity.

In problem **(SP1)** the total field u satisfies the Helmholtz equation

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^2 \backslash \partial D, \tag{1.1}$$

with conductive transmission conditions of the form

$$u_+ - u_- = 0 \;\text{ on }\; \partial D \quad \text{and} \quad \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\eta u = 0 \;\text{ on }\; \partial D. \tag{1.2}$$

For now, $u_\pm$ and $\partial u_\pm / \partial \nu$ denote the limits of u and $\partial u / \partial \nu$ from the exterior $(+)$ and the interior $(-)$, respectively.

In problem **(SP2)** the total field satisfies the generalized Helmholtz equation

$$\nabla \cdot \left(\frac{1}{n} \nabla u \right) + k^2 u = 0 \;\text{ in }\; \mathbb{R}^2 \backslash \partial D \tag{1.3}$$

with the following transmission conditions:

$$\frac{\partial u_+}{\partial \nu} - \frac{1}{n_-} \frac{\partial u}{\partial \nu} = 0 \;\text{ on }\; \partial D \quad \text{and} \quad u_+ - u_- - i\eta \frac{\partial u_+}{\partial \nu} = 0 \;\text{ on }\; \partial D. \tag{1.4}$$

In order u^s to be outgoing we require it to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \;\; r = |x|, \tag{1.5}$$

uniformly for all directions $x/|x|$. The Sommerfeld radiation condition implies [16] the following asymptotic behavior of the scattered field

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|} \right) \right), \;\; |x| \to \infty,$$

uniformly with respect to $\hat{x} = x/|x| \in S^1$. The function u^∞ defined on S^1 is called the scattering amplitude, or the far field pattern. In the following we will also write $u^\infty = u^\infty(\cdot, d)$ to indicate that the far field corresponds to the scattered field due to the incident plane wave with direction $d \in S^1$.

We distinguish between the *direct* (or, also called *forward*) and *inverse problem*. In the *forward problem* the information about the scattering medium is given and we study the behavior of the scattered field. In this work, we address the questions of uniqueness, existence and stability, i.e., the continuous dependence, of the scattered field u^s with respect to the incident field u^i in an appropriate norm.

The *inverse problem* we will be studying consists of determining the location and the shape of the domain D from the knowledge of the far field patterns $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^1$ by the Factorization Method (FM). The Factorization Method belongs to the family of non-iterative methods and, unlike iterative methods, it does not require solving a sequence of forward problems. Computationally the method is fast. Another advantage of the FM is, that it works without prior knowledge on material properties or the number of components of the medium.

The rough idea of the Factorization Method is the following: for a given sampling point $z \in \mathbb{R}^2$ we determine whether or not the equation

$$\widetilde{F}g(\hat{x}) = e^{-ikz\cdot\hat{x}}, \quad \hat{x} \in S^1, \tag{1.6}$$

is solvable in $L^2(S^1)$, which in turn is equivalent to whether or not the given point z belongs to D. That is, we sample a region, which as we suppose contains the scatterer, and reconstruct its support based on the criteria above. The operator $\widetilde{F}$ in (1.6) is computed from the far field operator $F : L^2(S^1) \to L^2(S^1)$ which incorporates the far fields and is defined by

$$(Fg)(\hat{x}) = \int\limits_{S^1} u^\infty(\hat{x}, d) g(d) \, \mathrm{d}s(d) \quad \text{for } \hat{x} \in S^1. \tag{1.7}$$

The Factorization Method provides both necessary and sufficient conditions to determine if $z \in D$. Thus, an important by-product of the FM is an explicit proof of uniqueness of the inverse problem.

The FM has been introduced by Kirsch in 1998 for scattering by impenetrable sound soft or sound hard obstacles [29]. Since then, it has been applied to a variety of problems from acoustic and electromagnetic scattering and from problems arising in the electrical impedance tomography.

The monograph [32] captures just a few of them.

In the following we would like to mention the works which are closely related to the problems we will be studying and which were most helpful for our research.

In his work from 1998 in [28] F. Hettlich studied **(SP1)** and **(SP2)** with index of refraction n and the conductivity being a constant. The inverse problem was solved by a Newton-like method.

A.Kirsch proved the applicability of the FM for the problem of scattering by inhomogeneous media with transmission conditions, i.e., the version of **(SP1)** with $\eta = 0$:

$$\begin{aligned}
\Delta u + k^2 n u &= 0 \text{ in } \mathbb{R}^2 \backslash \partial D, \\
u_+ - u_- &= 0 \quad \text{on} \quad \partial D, \\
\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} &= 0 \quad \text{on } \partial D, \\
u^s = u - u^i & \text{ satisfies (1.5)},
\end{aligned}$$

with $n \in L^\infty(D)$ real-valued [30] and later for a complex-valued n [32]. In [32] was assumed that the contrast $q := n - 1$ is locally bounded from below and that the wave number k is not an interior transmission eigenvalue. In [41] A. Lechleiter weakened the assumptions on q and showed that for this case the Factorization Method works, regardless if k is an interior transmission eigenvalue.

In [34] A. Kirsch and A. Kleefeld applied the Factorization Method to **(SP1)** with refractive index $n = 1$ inside D and conductivity $\eta \geq 0$ on ∂D. The authors obtained the forward data by solving the direct problem numerically by a boundary element collocation method and presented reconstructions in 3D.

In [36] A.Kirsch and X.Liu proved the Factorization Method for the problem

$$\nabla \cdot (A\nabla u) + k^2 m u = 0 \ \text{ in } D, \tag{1.8}$$

$$\Delta u + k^2 u = 0 \ \text{ in } \mathbb{R}^2 \backslash \overline{D}, \tag{1.9}$$

$$u_+ - u_- = 0 \ \text{ on } \partial D, \tag{1.10}$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = 0 \ \text{ on } \partial D, \tag{1.11}$$

$$u^s = u - u^i \text{ satisfies (1.5)}, \tag{1.12}$$

with $A \in C^1(\overline{D}, \mathbb{C}^{2\times 2})$ and $m \in C^1(D)$ being complex-valued. With $A =$diag $(\frac{1}{n}, \frac{1}{n})$ this case corresponds to the **(SP2)** with $\eta = 0$. Detailed analysis of this work was crucial for us to prove the applicability of the Factorization Method for **(SP2)**.

F.Cakoni et. al. proved in [9] the applicability of Linear Sampling Method for a more general version of **(SP2)**. In their work the authors allow η to vanish on some parts of the boundary which leads to the mixed type transmission conditions. Although, the Linear Sampling Method is closely related to the Factorization Method and provides good numerical reconstructions, its mathematical theory is still incomplete.

In the next section we give a motivation for the conductive transmission conditions. In Chapter 2 we show that the FM works for **(SP1)** for partially coated case, i.e., η may vanish on a part of the boundary ∂D, and without any restriction on the wave number $k > 0$. In Chapter 3 we study an interior eigenvalue problem which appears in the context of problem **(SP1)**. In Chapter 4 we study the FM for (**SP2**) with $\eta = 0$. Finally, in Chapter 5 we prove the FM (**SP2**) for a completely coated obstacle.

1.2 Motivation

1.2.1 Derivation of the Full Model for the TE- and TM-Mode

We consider the problem of scattering of a time-harmonic electromagnetic wave by a penetrable inhomogeneous object covered by a thin highly conductive layer. We suppose that the covered object is embedded in a non-conductive homogeneous background.

Let $\mathcal{E}$ and $\mathcal{H}$ denote the electric and the magnetic field, respectively. The electromagnetic wave satisfies the Maxwell equations

$$\operatorname{curl}\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} = 0, \qquad \operatorname{curl}\mathcal{H} - \varepsilon\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E},$$

where ε, μ and σ are real-valued positive functions which stay for the electric permittivity, magnetic permeability and the conductivity, respectively. In the time-harmonic case we assume that the magnetic and the electric field can be decomposed into space dependent and time dependent parts as

$$\mathcal{E}(x,t) = E(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = H(x)e^{-i\omega t},$$

where $\omega > 0$ is the frequency. Then the (complex-valued) fields E and H satisfy

$$\operatorname{curl}E - i\omega\mu H = 0, \qquad \operatorname{curl}H + i\omega\varepsilon E = \sigma E.$$

Let E^{int}, H^{int} represent the electric and the magnetic fields, respectively, inside the scattering object, which we denote by D^-, E^δ, H^δ the electric and the magnetic fields inside the layer D^δ, and E^{ext}, H^{ext} the corresponding fields in the exterior $D^+ := \mathbb{R}^3\backslash\overline{(D \cup D^\delta)}$. Then the propagation of the electromagnetic wave is described by the following set of equations:

$$\left.\begin{array}{rcl}\operatorname{curl}E^{\text{int}} - i\omega\mu_0 H^{\text{int}} & = & 0\\ \operatorname{curl}H^{\text{int}} + i\omega\varepsilon E^{\text{int}} & = & \sigma\ E^{\text{int}}\end{array}\right\}\quad \text{in } D^-, \tag{1.13}$$

$$\left.\begin{array}{rcl}\operatorname{curl}E^{\delta} - i\omega\mu_0 H^{\delta} & = & 0\\ \operatorname{curl}H^{\delta} + i\omega\varepsilon_1 E^{\delta} & = & \sigma^\delta E^{\delta}\end{array}\right\}\quad \text{in } D^\delta, \tag{1.14}$$

and

$$\left.\begin{array}{rcl} \operatorname{curl} E^{\text{ext}} - i\omega\mu_0 H^{\text{ext}} & = & 0 \\ \operatorname{curl} H^{\text{ext}} + i\omega\varepsilon_0 E^{\text{ext}} & = & 0 \end{array}\right\} \text{ in } D^+. \tag{1.15}$$

We assume that μ_0, ε_0 and ε_1 are constants, whereas the layer's conductivity σ^δ, the conductivity σ and the electric permittivity ε inside D^- might depend on x. Further we only consider frequencies ω belonging to a "resonance region" [16], i.e., $\omega^2\mu_0\varepsilon_0 a$, where a is a typical dimension of the scatterer, is less than or comparable to 1. On the interfaces ∂D^- and ∂D^+ between the scatterer and the layer, and between the layer and the background medium, respectively, we have the continuity of the tangential component of both the electric and the magnetic fields:

$$\left.\begin{array}{l} \nu \times E^{\text{int}} - \nu \times E^\delta = 0 \\ \nu \times H^{\text{int}} - \nu \times H^\delta = 0 \end{array}\right\} \quad \text{on } \partial D^- \tag{1.16}$$

and

$$\left.\begin{array}{l} \nu \times E^{\text{ext}} - \nu \times E^\delta = 0 \\ \nu \times H^{\text{ext}} - \nu \times H^\delta = 0 \end{array}\right\} \quad \text{on } \partial D^+, \tag{1.17}$$

where ν is the unit normal vector to the tangential plane to the boundary ∂D^+ or ∂D^- directed into the exterior of D^-.

We consider the scattering of an incident time-harmonic electromagnetic wave $\mathcal{E}^i(x,t) = E^i(x)e^{-i\omega t}$, $\mathcal{H}^i(x,t) = H^i(x)e^{-i\omega t}$, $x \in \mathbb{R}^3, t \in \mathbb{R}_{>0}$ with E^i and H^i satisfying $\operatorname{curl} E^i - ikH^i = 0$ and $\operatorname{curl} H^i + ikE^i = 0$ in all of $\mathbb{R}^3$, where we set

$$k = \omega\sqrt{\epsilon_0\mu_0}.$$

Then the (exterior) total field consists of the sum of incident and scattered fields

$$E^{\text{ext}} = E^i + E^s,$$
$$H^{\text{ext}} = H^i + H^s,$$

where E^s, H^s is an outgoing wave which satisfies the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0, \; r = |x|, \tag{1.18}$$

uniformly with respect to all directions $\hat{x} = x/|x|$.

We will study a special case, where the scattering object is represented by an infinitely long cylinder with the axis coincident with the x_3-axis (for a point $x \in \mathbb{R}^3$ we write $x = (x_1, x_2, x_3)^\top$). Let now $D^- \subset \mathbb{R}^2$ represent its cross section and $D^+ = \mathbb{R}^2 \backslash \overline{(D^- \cup D^\delta)}$ the exterior (see Figure 1.1). We

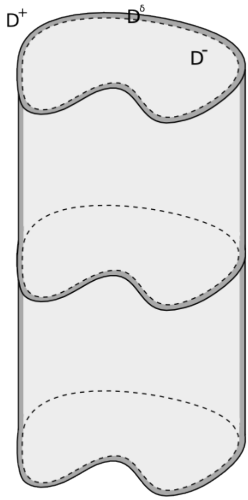

Figure 1.1: Notation for the domains

also assume, that σ^δ, σ and ε depend only on x_1 and x_2. In this special case we consider two special situations, or two different modes. Precisely, when the incident field E^i is given by $(0, 0, E_3^i)$, the so-called E-mode (also transverse magnetic, TM-mode), and when H^i is given by $(0, 0, H_3^i)$, which is called $H-$mode (or TE-mode).

We start with the E-mode. Assume that the incident field is of the form $E^i = (0, 0, E_3^i)^\top$, where E_3^i is independent of x_3. Then the first two components of E^{int}, E^s and E^δ are zero as well, and E_3^{int}, E_3^s and E_3^δ are functions of x_1 and x_2. From (1.15) we get

$$\operatorname{curl}\operatorname{curl} E^{\text{ext}} = \omega^2 \mu_0 \varepsilon_0 E^{\text{ext}} \text{ in } \mathbb{R}^3 \setminus \overline{(D \cup D^\delta)}$$

or (recall $k = \omega\sqrt{\varepsilon_0 \mu_0}$)

$$\Delta E_3^{\text{ext}} + k^2 E_3^{\text{ext}} = 0 \quad \text{in } D^+.$$

Analogously, (1.13) and (1.14) become

$$\Delta E_3^\delta + k_1^2 E_3^\delta = 0 \quad \text{in } D^\delta, \qquad \Delta E_3^{\text{int}} + k^2 n E_3^{\text{int}} = 0 \text{ in } D^-,$$

respectively, with $k_1^2 = \omega^2 \mu_0 \varepsilon_1 + i\omega\mu_0\sigma^\delta$ and $n = \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$.

Substituting $\frac{1}{i\omega\mu_0}\operatorname{curl} E$ for H into the equations with the transmission conditions (1.16)–(1.17) we get

$$\left.\begin{aligned} E_3^{\text{int}} - E_3^\delta &= 0 \\ \frac{\partial E_3^{\text{int}}}{\partial\nu} - \frac{\partial E_3^\delta}{\partial\nu} &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial D^- \tag{1.19}$$

and

$$\left.\begin{aligned} E_3^{\text{ext}} - E_3^\delta &= 0 \\ \frac{\partial E_3^{\text{ext}}}{\partial\nu} - \frac{\partial E_3^\delta}{\partial\nu} &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial D^+, \tag{1.20}$$

where we write $\partial E_3/\partial\nu$ for $\nu \cdot \nabla E$. The analogue of the Silver-Müller radiation condition in $\mathbb{R}^2$ is the Sommerfeld radiation condition:

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial E_3^s}{\partial r} - ikE_3^s\Big) = 0, \; r = |x|, \tag{1.21}$$

uniformly in $\hat{x} = \frac{x}{|x|}$.

For the $H-$mode we set $H^i = (0, 0, H_3^i)^\top$. Then $H^{\text{ext}} = (0, 0, H_3^{\text{ext}})^\top$ and H_3^{ext} also satisfies the Helmholtz equation

$$\Delta H_3^{\text{ext}} + k^2 H_3^{\text{ext}} = 0 \quad \text{in } D^+. \tag{1.22}$$

Inside the inhomogeneity D^- holds

$$\operatorname{curl} \frac{1}{i\omega\mu_0(-i\omega\varepsilon + \sigma)} \operatorname{curl} H_3^{\mathrm{int}} = H_3^{\mathrm{int}}$$

or

$$\operatorname{curl} \frac{1}{k^2\left(\frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon}\right)} \operatorname{curl} H_3^{\mathrm{int}} = H_3^{\mathrm{int}}$$

which yields

$$-\operatorname{div}\left(\frac{1}{k^2 n} \nabla H_3^{\mathrm{int}}\right) = H_3^{\mathrm{int}} \quad \text{in } D^-,$$

where n is again given by $n = \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$. In the same way we get that

$$-\operatorname{div}\left(\frac{1}{k_1^2} \nabla H_3^{\delta}\right) = H_3^{\delta} \quad \text{in } D^{\delta},$$

with $k_1^2 = \omega^2\mu_0\varepsilon_1 + i\omega\mu_0\sigma^\delta$. As in the $E-$mode, we substitute $\frac{1}{\sigma - i\omega\varepsilon}\operatorname{curl} H$ for E in (1.16)–(1.17), which gives the transmission conditions:

$$\left.\begin{aligned} H_3^{\mathrm{int}} - H_3^{\delta} &= 0 \\ \frac{1}{k^2 n_-}\frac{\partial H_3^{\mathrm{int}}}{\partial\nu} - \frac{1}{k_{1_-}^2}\frac{\partial H_3^{\delta}}{\partial\nu} &= 0 \end{aligned}\right\} \quad \text{on } \partial D^- \tag{1.23}$$

and

$$\left.\begin{aligned} H_3^{\mathrm{ext}} - H_3^{\delta} &= 0 \\ \frac{1}{k^2}\frac{\partial H_3^{\mathrm{ext}}}{\partial\nu} - \frac{1}{k_{1_+}^2}\frac{\partial H_3^{\delta}}{\partial\nu} &= 0 \end{aligned}\right\} \quad \text{on } \partial D^+. \tag{1.24}$$

Here, $k_{1_\pm}^2$ stays for the limit of k_1^2 approaching ∂D^- (for $-$) and ∂D^+ (for $+$) along the normal ν, and n_- is the limit of n approaching ∂D^- along ν. As in the previous case, the model is completed by the Sommerfeld radiation condition for H_3^s.

We summarize the derivations above. Let u^+, u^δ and u^- denote the total fields inside D^+, D^δ and D^-, respectively. For the $E-$mode we have

$$\Delta u^+ + k^2 u^+ = 0 \quad \text{in } D^+, \tag{1.25}$$

$$\Delta u^\delta + k_1^2 u^\delta = 0 \quad \text{in } D^\delta, \tag{1.26}$$

$$\Delta u^- + k^2 n u^- = 0 \quad \text{in } D^-, \tag{1.27}$$

$$u^+ = u^i + u^s \tag{1.28}$$

with the boundary conditions

$$u^+ - u^\delta = 0, \qquad \frac{\partial u^+}{\partial \nu} - \frac{\partial u^\delta}{\partial \nu} = 0 \quad \text{on } \partial D^+ \tag{1.29}$$

and

$$u^\delta - u^- = 0, \qquad \frac{\partial u^\delta}{\partial \nu} - \frac{\partial u^-}{\partial \nu} = 0 \quad \text{on } \partial D^-. \tag{1.30}$$

For the $H-$mode holds

$$\Delta u^+ + k^2 u^+ = 0 \quad \text{in } D^+, \tag{1.31}$$

$$\operatorname{div}\left(\frac{1}{(k_1^2/k^2)}\nabla u^\delta\right) + k^2 u^\delta = 0 \quad \text{in } D^\delta, \tag{1.32}$$

$$\operatorname{div}\left(\frac{1}{n}\nabla u^-\right) + k^2 u^- = 0 \quad \text{in } D^-, \tag{1.33}$$

$$u^+ = u^i + u^s \tag{1.34}$$

with the transmission conditions

$$u^+ - u^\delta = 0, \qquad \frac{\partial u^+}{\partial \nu} - \frac{1}{(k_{1_+}^2/k^2)}\frac{\partial u^\delta}{\partial \nu} = 0 \quad \text{on } \partial D^+ \tag{1.35}$$

and

$$u^\delta - u^- = 0, \qquad \frac{1}{(k_{1_-}^2/k^2)}\frac{\partial u^\delta}{\partial \nu} - \frac{1}{n_-}\frac{\partial u^-}{\partial \nu} = 0 \quad \text{on } \partial D^-. \tag{1.36}$$

In both cases, the scattered fields satisfy the Sommerfeld radiation condition (1.21).

We consider the special case where the layer D^δ is of constant thickness δ and that the conductivity σ^δ is of the order $1/\delta$. For each point $x \in D^\delta$, we suppose that σ remains constant along the normal ν (later we give precise assumptions on the thickness of the layer D^δ and the smoothness of the boundary ∂D^-).

In this work, instead of studying the full model involving a thin highly conductive layer of a given thickness δ, we will be working with an approximate one. In the following section, employing the scaled asymptotic expansions technique [25] we will show that the model with the well-known conductive transmission conditions [2] represents the first order approximation of the the full model involving the layer. For the surface materials having the properties described above, the first order approximation is good enough. However, for more complicated coatings approximations of higher order are used (see e.g. [7] where surface impedance involves a second order surface operator).

The use of approximate models is a common practice and has its theoretical and practical advantages. From the theoretical point of view, the analysis of the direct and the inverse problem is less technical. In practice, to obtain a numerical solution of a problem with a thin layer by standard numerical methods, for instance, finite elements, it is necessary to use a finer mesh. This increases the size of the discrete model and consequently the cost of computation.

1.2.2 Approximate Transmission Conditions of the First Order

As in the previous section, let D^δ represent the thin layer of (constant) thickness δ and let $D^\pm$ denote the exterior (+) and the interior (−) of inhomogeneity (see Figure 1.2 on the right). We also use $u^\pm$ and u^δ to distinguish between the total field inside $D^\pm$ and D^δ, respectively. We first derive the approximate transmission conditions for the TM-mode.

TM-Mode. We have

$$\Delta u^+ + k^2 u^+ = 0 \quad \text{in} \quad D^+, \tag{1.37}$$

$$\Delta u^\delta + k_1^2 u^\delta = 0 \quad \text{in} \quad D^\delta, \tag{1.38}$$

$$\Delta u^- + k^2 n u^- = 0 \quad \text{in} \quad D^-, \tag{1.39}$$

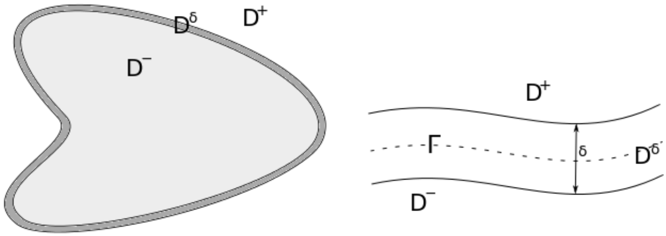

Figure 1.2: Notation for the domains

where $k_1^2 = \omega^2 \varepsilon_1 \mu_0 + i\omega\sigma_1\mu_0$. On the interfaces ∂D^- and ∂D^+ the transmission conditions hold

$$u^- - u^\delta = 0 \quad \text{on } \partial D^-, \qquad u^+ - u^\delta = 0 \quad \text{on } \partial D^+, \tag{1.40}$$

$$\frac{\partial u^-}{\partial \nu} - \frac{\partial u^\delta}{\partial \nu} = 0 \quad \text{on } \partial D^-, \qquad \frac{\partial u^+}{\partial \nu} - \frac{\partial u^\delta}{\partial \nu} = 0 \quad \text{on } \partial D^+. \tag{1.41}$$

Following the approach described in [25] and [12], we formally assume that for sufficiently small δ the field u^δ can be represented by the series

$$u^\delta(x) = u_0^\delta(x) + \delta u_1^\delta(x) + \delta^2 u_2^\delta(x) + \dots \quad \text{for } x \in D^\delta. \tag{1.42}$$

Furthermore, we extend $u^\pm$ analytically into D^δ and assume that the extensions are also given by

$$u^\pm = u_0^\pm + \delta u_1^\pm + \delta^2 u_2^\pm + \dots \quad \text{in } D^\delta. \tag{1.43}$$

To start with, let Γ represent a closed curve inside the layer 'in the middle between' ∂D^- and ∂D^+ (see Figure 1.2 (b)) and let $x_\Gamma(s) = \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}$, $s \in [0, L] \subset \mathbb{R}$ denote its counter-clockwise parametrization with respect to the arc length. We assume that Γ is C^2 smooth. Then, for sufficiently small δ (to be precise, for $\delta < \min_{s\in[0,L]}(2/c(s))$, where $c(s)$ is the curvature of

Γ at the point $x_\Gamma(s) \in \Gamma$), the thin layer can be parametrized in terms of new coordinates $(s,t) \in [0,L] \times \big(-\delta/2, \delta/2\big)$ through

$$x = x_\Gamma(s) + t\nu(s),$$

where $\nu(s)$ is the unit outward normal to Γ at $s \in [0,L]$. The boundaries ∂D^+ and ∂D^- can be written in curvilinear coordinates as

$$\partial D^+ = \{x_\Gamma(s) + (\delta/2)\nu(s),\ s \in [0,L]\}$$

and

$$\partial D^- = \{x_\Gamma(s) - (\delta/2)\nu(s),\ s \in [0,L]\}.$$

In each case we assume that u_j^δ and $u_j^\pm$, $j = 0,1,\ldots$ are bounded and do not depend on δ. Our aim is to compute the jumps $u^+ - u^-$ and $\frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu}$ across Γ. We truncate the series in (1.42) and (1.43) to only the first two terms and calculate the jumps

$$(u_0^+ - u_0^-) + \delta(u_1^+ - u_1^-) \quad \text{on} \quad \Gamma,$$

and

$$\left(\frac{\partial u_0^+}{\partial \nu} - \frac{\partial u_0^-}{\partial \nu}\right) + \delta\left(\frac{\partial u_1^+}{\partial \nu} - \frac{\partial u_1^-}{\partial \nu}\right) \quad \text{on} \quad \Gamma.$$

Let u be a function defined on D^δ. We define $\tilde{u} : \mathbb{R}_+ \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \to \mathbb{R}$ as

$$\tilde{u}(s,t) := u(x),$$

where $x = x_\Gamma(s) + t\nu(s)$ for $(s,t) \in [0,L] \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$. The Laplacian of u in the parametric coordinates (s,t) is given by

$$\Delta u = \frac{1}{(1+tc)}\frac{\partial}{\partial s}\left(\frac{1}{(1+tc)}\frac{\partial}{\partial s}\tilde{u}\right) + \frac{1}{(1+tc)}\frac{\partial}{\partial t}\left((1+tc)\frac{\partial}{\partial t}\tilde{u}\right),$$

where c is the curvature.

So, with respect to curvilinear coordinates the Helmholtz equation (1.38) and the continuity conditions (1.40)–(1.41) have the following form

$$\frac{1}{(1+tc)}\frac{\partial}{\partial s}\left(\frac{1}{(1+tc)}\frac{\partial}{\partial s}\tilde{u}^\delta\right) + \frac{1}{(1+tc)}\frac{\partial}{\partial t}(1+tc)\frac{\partial}{\partial t}\tilde{u}^\delta + k_1^2\tilde{u}^\delta = 0 \tag{1.44}$$

in $(0,L)\times\left(-\frac{\delta}{2},\frac{\delta}{2}\right)$,

$$u^+\left(x_\Gamma(\,\cdot\,)+\frac{\delta}{2}\nu\right)=\tilde{u}^\delta\left(\,\cdot\,,\frac{\delta}{2}\right),\qquad u^-\left(x_\Gamma(\,\cdot\,)-\frac{\delta}{2}\nu\right)=\tilde{u}^\delta\left(\,\cdot\,,-\frac{\delta}{2}\right),\tag{1.45}$$

$$\frac{\partial u^+}{\partial\nu}\left(x_\Gamma(\,\cdot\,)+\frac{\delta}{2}\nu\right)=\frac{\partial\tilde{u}^\delta}{\partial t}\left(\,\cdot\,,\frac{\delta}{2}\right),\frac{\partial u^-}{\partial\nu}\left(x_\Gamma(\,\cdot\,)-\frac{\delta}{2}\nu\right)=\frac{\partial\tilde{u}^\delta}{\partial t}\left(\,\cdot\,,-\frac{\delta}{2}\right)\tag{1.46}$$

in $[0,L)$.

Next we introduce a new variable

$$\xi=\frac{t}{\delta},\quad t\in\left[-\frac{\delta}{2},\frac{\delta}{2}\right],$$

and define the function $\tilde{u}^\delta(s,\xi):=\tilde{u}^\delta(s,t/\delta)$ (by abuse of notation we will continue writing $\tilde{u}^\delta$). After the rescaling the equations (1.44)–(1.46) become

$$\frac{1}{(1+\delta\xi c)}\frac{\partial}{\partial s}\frac{1}{(1+\delta\xi c)}\frac{\partial}{\partial s}\tilde{u}^\delta+\frac{1}{\delta^2}\frac{1}{(1+\delta\xi c)}\frac{\partial}{\partial\xi}(1+\delta\xi c)\frac{\partial}{\partial\xi}\tilde{u}^\delta$$
$$+k_1^2\tilde{u}^\delta=0\ \text{ in } (0,L)\times\left(-\frac{1}{2},\frac{1}{2}\right),\tag{1.47}$$

$$u^\pm(x_\Gamma(\,\cdot\,)\pm\frac{\delta}{2}\nu)=\tilde{u}^\delta\left(\,\cdot\,,\pm\frac{1}{2}\right),\tag{1.48}$$

$$\frac{\partial u^\pm}{\partial\nu}(x_\Gamma(\,\cdot\,)\pm\frac{\delta}{2}\nu)=\frac{1}{\delta}\frac{\partial\tilde{u}^\delta}{\partial\xi}\left(\,\cdot\,,\pm\frac{1}{2}\right)\tag{1.49}$$

in $[0,L)$. From (1.49) and the expansions (1.42) and (1.43), comparing the same powers of δ we conclude

$$\frac{1}{\delta}\frac{\partial\tilde{u}_0^\delta}{\partial\xi}\left(\,\cdot\,,\pm\frac{1}{2}\right)=0.\tag{1.50}$$

Now, multiplying (1.47) by $(1+\delta\xi c)^2$, using the asymptotics $\tilde{u}^\delta = \tilde{u}_0^\delta + \delta\tilde{u}_1^\delta + \delta^2\tilde{u}_2^\delta + \dots$ and equating the same powers of δ we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi^2}\tilde{u}_j^\delta + \left(3\xi c\frac{\partial^2}{\partial\xi^2} + c\frac{\partial}{\partial\xi}\right)\tilde{u}_{j-1}^\delta \\
&\quad + \left(3\xi^2c^2\frac{\partial^2}{\partial\xi^2} + 2\xi c^2\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial s^2} + k_1^2\right)\tilde{u}_{j-2}^\delta \\
&\quad + \left(\xi^3c^3\frac{\partial^2}{\partial\xi^2} + \xi^2c^3\frac{\partial}{\partial\xi} + \xi c\frac{\partial^2}{\partial s^2} - \xi c'\frac{\partial}{\partial s} + 3\xi k_1^2 c\right)\tilde{u}_{j-3}^\delta \\
&\quad + 3c\xi^2k_1^2c^2\tilde{u}_{j-4}^\delta + \xi^3k_1^2c^3\tilde{u}_{j-5}^\delta = 0 \qquad (1.51)
\end{aligned}$$

for $(s,\xi) \in (0,L)\times\left(-\frac{1}{2},\frac{1}{2}\right)$ and $j = 0,1,2,\dots$. By convention, we set $\tilde{u}_j^\delta = 0$ for negative j.

From (1.51) we read the equations for $j = 0,1,2$:

$$\frac{\partial^2}{\partial\xi^2}\tilde{u}_0^\delta = 0, \qquad (1.52)$$

$$\frac{\partial^2}{\partial\xi^2}\tilde{u}_1^\delta + \left[3\xi c\frac{\partial^2}{\partial\xi^2} + c\frac{\partial}{\partial\xi}\right]\tilde{u}_0^\delta = 0, \qquad (1.53)$$

$$\frac{\partial^2}{\partial\xi^2}\tilde{u}_2^\delta + \left[3\xi c\frac{\partial^2}{\partial\xi^2} + c\frac{\partial}{\partial\xi}\right]\tilde{u}_1^\delta + \left[3\xi^2c^2\frac{\partial^2}{\partial\xi^2} + 2\xi c^2\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial s^2} + k_0^2\right]\tilde{u}_0^\delta = 0 \qquad (1.54)$$

in $(0,L)\times\left(\frac{1}{2},-\frac{1}{2}\right)$. Recall that our goal is to compute the jumps

$$u^+(x_\Gamma(\,\cdot\,)+0\nu) - u^-(x_\Gamma(\,\cdot\,)+0\nu)$$

and

$$\frac{\partial u^+}{\partial\nu}(x_\Gamma(\,\cdot\,)+0\nu) - \frac{\partial u^-}{\partial\nu}(x_\Gamma(\,\cdot\,)+0\nu) \text{ in } [0,L].$$

Let $s \in [0,L]$ be fixed. Using the Taylor series expansion, after equating the same powers of δ, we get

$$\begin{aligned}
&u^\pm\big(x_\Gamma(s) \pm \frac{\delta}{2}\nu\big) = u_0^\pm(x_\Gamma(s)+0\nu) \\
&\quad + \delta\left(\pm\frac{1}{2}\frac{\partial}{\partial\nu}u_0^\pm(x_\Gamma(s)+0\nu) + u_1^\pm(x_\Gamma(s)+0\nu)\right) + \delta^2(\,\dots\,) + \dots \qquad (1.55)
\end{aligned}$$

This holds for all $s \in [0, L]$. On the other hand, from the continuity condition (1.45) and from the ansatz (1.42) we have

$$u^{\pm}\big(x_\Gamma(\,\cdot\,) \pm \frac{\delta}{2}\big) = \tilde{u}^{\delta}\big(\,\cdot\,, \pm\frac{1}{2}\big) = \tilde{u}_0^{\delta}\big(\,\cdot\,, \pm\frac{1}{2}\big) + \delta\tilde{u}_1^{\delta}\big(\,\cdot\,, \pm\frac{1}{2}\big) + \ldots \text{ in } [0, L]. \tag{1.56}$$

We introduce the following notation: let $[\tilde{u}^{\delta}(\,\cdot\,, \pm\frac{1}{2})]$ denote the difference $\big(\tilde{u}^{\delta}(\,\cdot\,, \frac{1}{2}) - \tilde{u}^{\delta}(\,\cdot\,, -\frac{1}{2})\big)$ and $\langle u^{\pm}(x_\Gamma(\cdot) + 0\nu)\rangle$ the average $\frac{1}{2}\big(u^{+}(x_\Gamma(\cdot) + 0\nu) + u^{-}(x_\Gamma(\cdot) + 0\nu)\big)$. Then from (1.55) and (1.56) we obtain

$$\begin{aligned}
&[u^{\pm}(x_\Gamma(s)) + 0\nu] \\
&\quad = [u_0^{\pm}(x_\Gamma(s) + 0\nu)] + \delta[u_1^{\pm}(x_\Gamma(s) + 0\nu)] + \mathcal{O}(\delta^2) \qquad (1.57)\\
&\quad = \big[\tilde{u}_0^{\delta}\big(s, \pm\frac{1}{2}\big)\big] + \delta\Big(\big[\tilde{u}_1^{\delta}\big(s, \pm\frac{1}{2}\big)\big] - \Big\langle \frac{\partial}{\partial\nu} u_0^{\pm}(x_\Gamma(s) + 0\nu)\Big\rangle\Big) + \mathcal{O}(\delta^2)
\end{aligned}$$

for all $s \in [0, L]$. Analogously, for the jump in the normal derivative we have

$$\begin{aligned}
&\Big[\frac{\partial u^{\pm}}{\partial\nu}(x_\Gamma(s) + 0\nu)\Big] \\
&\qquad = \Big[\frac{\partial u_0^{\pm}}{\partial\nu}(x_\Gamma(s) + 0\nu)\Big] + \delta\Big[\frac{\partial u_1^{\pm}}{\partial\nu}(x_\Gamma(s) + 0\nu)\Big] + \mathcal{O}(\delta^2) \\
&\qquad = \frac{1}{\delta}\Big[\frac{\partial\tilde{u}_0^{\delta}}{\partial\xi}\big(s, \pm\frac{1}{2}\big)\Big] + \Big[\frac{\partial\tilde{u}_1^{\delta}}{\partial\xi}\big(s, \pm\frac{1}{2}\big)\Big] + \\
&\qquad\quad + \delta\Big(\Big[\frac{\partial\tilde{u}_2^{\delta}}{\partial\xi}\big(s, \pm\frac{1}{2}\big)\Big] - \Big\langle \frac{\partial^2}{\partial\nu^2} u_0^{\pm}(x_\Gamma(s) + 0\nu)\Big\rangle\Big) + \mathcal{O}(\delta^2)
\end{aligned} \tag{1.58}$$

for all $s \in [0, L]$.

We want to express the jumps $[\tilde{u}_j^{\delta}(\cdot, \pm\frac{1}{2})]$, $j = 0, 1$ in terms of the functions $[u_0^{\pm}(x(\cdot) + 0\nu)]$ and $[u_1^{\pm}(x(\cdot) + 0\nu)]$. First, from (1.52) we observe that $\tilde{u}_0^{\delta}(s, \,\cdot\,)$ is linear for all $s \in [0, L]$. Furthermore, (1.50) yields that $\tilde{u}_0^{\delta}(s, \,\cdot\,)$ is a constant, possibly a different one for different $s \in [0, L]$. Thus,

$$[\tilde{u}_0^{\delta}\big(\,\cdot\,, \pm\frac{1}{2}\big)] = 0 \text{ in } [0, L]. \tag{1.59}$$

Also, (1.53) reduces to

$$\frac{\partial^2}{\partial \xi^2}\tilde{u}_1^\delta = 0 \quad \text{in } (0,L) \times \big(-\frac{\delta}{2}, \frac{\delta}{2}\big) \tag{1.60}$$

and (1.54) to

$$\frac{\partial^2}{\partial \xi^2}\tilde{u}_2^\delta + c\frac{\partial}{\partial \xi}\tilde{u}_1^\delta + \Big(\frac{\partial^2}{\partial s^2} + k_1^2\Big)\tilde{u}_0^\delta = 0 \quad \text{in } (0,L) \times \big(-\frac{1}{2}, \frac{1}{2}\big). \tag{1.61}$$

Thus, by (1.60) $\frac{\partial}{\partial \xi}\tilde{u}_1^\delta(s, \cdot)$ is constant for each $s \in [0, L]$ and then

$$\Big[\frac{\partial \tilde{u}_1^\delta}{\partial \xi}\big(\cdot, \pm\frac{1}{2}\big)\Big] = 0. \tag{1.62}$$

The fundamental theorem of calculus, (1.62) and the continuity conditions (1.49) (again, one uses the series ansatz and equates the same powers of δ) yield

$$\begin{aligned}
[\tilde{u}_1^\delta\big(s, \pm\frac{1}{2}\big)] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial \tau}\tilde{u}_1^\delta(s,\tau)\,\mathrm{d}\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_0^-}{\partial \nu}\big(x_\Gamma(s) - \frac{1}{2}\nu\big)\,\mathrm{d}\tau \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_0^+}{\partial \nu}\big(x_\Gamma(s) + \frac{1}{2}\nu\big)\,\mathrm{d}\tau = \Big\langle \frac{\partial u_0^\pm}{\partial \nu}\big(x_\Gamma(s) \pm \frac{1}{2}\nu\big)\Big\rangle
\end{aligned} \tag{1.63}$$

for all $s \in [0, L]$. Thus, from (1.57), (1.59) and (1.63) we obtain

$$\begin{aligned}
&[u^\pm(x_\Gamma(s) + 0\nu)] \\
&= \delta\Big(\Big\langle \frac{\partial u_0^\pm}{\partial \nu}\big(x_\Gamma(s) \pm \frac{1}{2}\nu\big)\Big\rangle - \Big\langle \frac{\partial}{\partial \nu}u_0^\pm(x_\Gamma(s) + 0\nu)\Big\rangle\Big) + \mathcal{O}(\delta^2) \\
&= \mathcal{O}(\delta^2),
\end{aligned}$$

where in the last equality we applied the Taylor series expansion of $\frac{\partial}{\partial \nu}u_0^\pm(x_\Gamma(\cdot) \pm \frac{1}{2}\nu)$ along ν.

With (1.50) and (1.62), (1.58) reduces to

$$
\begin{aligned}
&\Big[\frac{\partial u^{\pm}}{\partial \nu}(x_\Gamma(s)+0\nu)\Big] \\
&= \delta\bigg(\Big[\frac{\partial \tilde u_2^\delta}{\partial \xi}\big(s,\pm\frac{1}{2}\big)\Big] - \big\langle \frac{\partial^2}{\partial \nu^2} u_0^{\pm}(x_\Gamma(s)+0\nu)\big\rangle\bigg) + \mathcal{O}(\delta^2)
\end{aligned}
$$

for all $s \in [0, L]$.

We compute the jump in $[\frac{\partial \tilde u_2^\delta}{\partial \xi}(\,\cdot\,,\pm\frac{1}{2})]$ using (1.61). For all $s \in [0, L]$ we have

$$
\begin{aligned}
\Big[\frac{\partial \tilde u_2^\delta}{\partial \xi}(s,\pm\frac{1}{2})\Big] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial^2 \tilde u_2^\delta}{\partial \tau^2}(s,\tau)\mathrm{d}\tau \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} c(s)\frac{\partial}{\partial \tau}\tilde u_1^\delta(s,\tau) + \big(\frac{\partial^2}{\partial s^2}+k_1^2\big)\tilde u_0^\delta(s,\tau)\mathrm{d}\tau \\
&= -c(s)\Big\langle \frac{\partial u_0^{\pm}}{\partial \nu}(x_\Gamma(s)\pm\frac{1}{2}\nu)\Big\rangle \\
&\qquad - \big(\frac{\partial^2}{\partial s^2}+k_1^2\big)\langle u_0^{\pm}(x_\Gamma(s)\pm\frac{1}{2}\nu)\rangle,\ s\in[0,L], \qquad (1.64)
\end{aligned}
$$

where in the last equality we used that $\frac{\partial}{\partial t}\tilde u_1^\delta(s,\cdot)$ and $\tilde u_0^\delta(s,\cdot)$ are constant for all $s \in \Gamma$, and the continuity conditions. For $u_j^{\pm}, j = 0, 1$ the derivative $\partial u_j^{\pm}/\partial s, j = 0, 1$ stays for the tangential derivative on Γ.

Thus,

$$
\begin{aligned}
\left[\frac{\partial u^{\pm}}{\partial \nu}(x_\Gamma(\cdot)+0\nu)\right] &= -\delta k_1^2 \langle u_0^{\pm}(x_\Gamma(\cdot)\pm \frac{1}{2}\nu)\rangle - \delta c(\cdot)\big\langle \frac{\partial u_0^{\pm}}{\partial \nu}(x_\Gamma(\cdot)\pm\frac{1}{2}\nu)\big\rangle \\
&\quad + \delta \frac{\partial^2}{\partial s^2}\langle u_0^{\pm}(x_\Gamma(\cdot)\pm\frac{1}{2}\nu)\rangle + \delta\big\langle \frac{\partial^2}{\partial \nu^2} u_0^{\pm}(x_\Gamma(\cdot)+0\nu)\big\rangle\big) + \mathcal{O}(\delta^2) \\
&= -\delta k_1^2\langle u_0^{\pm}(x_\Gamma(\cdot)+0\nu)\rangle - \delta c(\cdot)\big\langle \frac{\partial u_0^{\pm}}{\partial \nu}(x_\Gamma(\cdot)+0\nu)\big\rangle \\
&\quad + \delta \frac{\partial^2}{\partial s^2}\langle u_0^{\pm}(x_\Gamma(\cdot)+0\nu)\rangle + \delta\big\langle \frac{\partial^2}{\partial \nu^2} u_0^{\pm}(x_\Gamma(\cdot)+0\nu)\big\rangle\big) + \mathcal{O}(\delta^2).
\end{aligned}
\tag{1.65}
$$

Let σ_1 be given by

$$
\sigma_1^\delta(s,\xi) = \frac{\lambda(s)}{\delta} \quad \text{in } (0,L)\times\big(-\frac{\delta}{2},\frac{\delta}{2}\big),
$$

where λ is a bounded real valued function which does not depend on δ. Since $u_j^{\pm}, j = 0,1,2$ and their derivatives are uniformly bounded with respect to δ, we get

$$
[u^{\pm}(s,0)] = \mathcal{O}(\delta^2) \;\text{ and }\; \left[\frac{\partial u^{\pm}}{\partial \nu}(s,0)\right] = -i\lambda\omega\mu_0 u + \mathcal{O}(\delta) \tag{1.66}
$$

for all $s \in [0,L]$.

Neglecting the terms of order δ (in the the literature on approximate transmission conditions it is also called first order approximation) in the boundary conditions yields the following approximate model: Let $D \subset \mathbb{R}^2$ represent the inhomogeneity and let λ be a real valued and positive function defined on the boundary ∂D. The total field u satisfies

$$
\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{D}, \tag{1.67}
$$

$$
\Delta u + k^2 n u = 0 \quad \text{in } D, \tag{1.68}
$$

$$
u_+ - u_- = 0 \quad \text{on } \partial D, \tag{1.69}
$$

$$
\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D. \tag{1.70}
$$

It is exactly the model for the scattering problem (**SP1**).

TE-Mode. For the TE-mode we have

$$\Delta u^+ + k^2 u^+ = 0 \quad \text{in } D^+, \tag{1.71}$$

$$\operatorname{div} \frac{1}{(k_1^2/k^2)} \nabla u^\delta + k^2 u^\delta = 0 \quad \text{in } D^\delta, \tag{1.72}$$

$$\operatorname{div} \left(\frac{1}{n} \nabla u^- \right) + k^2 u^- = 0 \quad \text{in } D^-, \tag{1.73}$$

with the transmission conditions on the interfaces

$$u^+ - u^\delta = 0, \qquad \frac{\partial u^+}{\partial \nu} - \frac{1}{(k_{1_+}^2/k^2)} \frac{\partial u^\delta}{\partial \nu} = 0 \quad \text{on } \partial D^+ \tag{1.74}$$

and

$$u^\delta - u^- = 0, \qquad \frac{1}{(k_{1_-}^2/k^2)} \frac{\partial u^\delta}{\partial \nu} - \frac{1}{n_-} \frac{\partial u^-}{\partial \nu} = 0 \quad \text{on } \partial D^-. \tag{1.75}$$

We proceed as in the previous case. We assume that the fields $u^\delta, u^\pm$ can be represented by the series

$$u^\delta = u_0^\delta + \delta u_1^\delta + \delta^2 u_2^\delta + \dots, \quad u^\pm = u_0^\pm + \delta u_1^\pm + \delta^2 u_2^\pm + \dots \quad \text{in } D^\delta. \tag{1.76}$$

Further, write (1.72) in the curvilinear coordinates (s,t):

$$\frac{1}{(1+tc)} \frac{\partial}{\partial s} \frac{1}{k_1^2} \frac{1}{(1+tc)} \frac{\partial \tilde{u}^\delta}{\partial s} + \frac{1}{(1+tc)} \frac{\partial}{\partial t} \left(\frac{(1+tc)}{k_1^2} \frac{\partial \tilde{u}^\delta}{\partial t} \right) + \tilde{u}^\delta = 0 \tag{1.77}$$

in $(0,L) \times (-\frac{\delta}{2}, -\frac{\delta}{2})$, where $\tilde{u}(s,t) = u(x)$ with $x(s,t) = x_\Gamma(s) + t\nu(s)$, $(s,t) \in (0,L) \times (-\frac{\delta}{2}, -\frac{\delta}{2})$.

After the rescaling $\tilde{u}(s,\xi) = \tilde{u}(s, \frac{t}{\delta})$ (1.77) becomes

$$\begin{aligned} &\frac{1}{(1+\delta\xi c)} \frac{\partial}{\partial s} \frac{1}{k_1^2} \frac{1}{(1+\delta\xi c)} \frac{\partial \tilde{u}^\delta}{\partial s} \\ &\quad + \frac{1}{\delta^2} \frac{1}{(1+\delta\xi c)} \frac{\partial}{\partial \xi} \left(\frac{(1+\delta\xi c)}{k_1^2} \frac{\partial \tilde{u}^\delta}{\partial \xi} \right) + \tilde{u}^\delta = 0 \end{aligned} \tag{1.78}$$

in $(0, L) \times (-\frac{1}{2}, -\frac{1}{2})$, with the boundary conditions

$$u^{+}(x_{\Gamma}(\cdot) + \frac{1}{2}\nu) = \tilde{u}^{\delta}(\cdot, \frac{1}{2}), \quad \frac{\partial u^{+}}{\partial \nu}(x_{\Gamma}(\cdot) + \frac{1}{2}\nu) = \frac{1}{\delta}\frac{1}{(k_{1_{+}}^{2}/k^{2})}\frac{\partial \tilde{u}^{\delta}}{\partial \xi}(\cdot, \frac{1}{2}) \tag{1.79}$$

and

$$u^{-}(x_{\Gamma}(\cdot) - \frac{1}{2}\nu) = \tilde{u}^{\delta}(\cdot, -\frac{1}{2}), \quad \frac{\partial u^{-}}{\partial \nu}(x_{\Gamma}(\cdot) - \frac{1}{2}\nu) = \frac{1}{\delta}\frac{1}{(k_{1_{-}}^{2}/k^{2})}\frac{\partial \tilde{u}^{\delta}}{\partial \xi}(\cdot, -\frac{1}{2}) \tag{1.80}$$

in $[0, L]$.

Substituting the series ansatz (1.76) into (1.79)–(1.80) and making comparison of the coefficients we see that

$$\tilde{u}_{0}^{\delta}(\cdot, \frac{1}{2}) = \tilde{u}_{0}^{\delta}(\cdot, -\frac{1}{2}) = 0 \quad \text{in } [0, L]. \tag{1.81}$$

Multiplying (1.78) by $(1 + \delta\xi c)^{2}$ and equating the same powers of δ we get

$$\begin{aligned}
&\frac{\partial}{\partial \xi}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial \xi}\tilde{u}_{j}^{\delta}\right) + \left(3\xi c\frac{\partial}{\partial \xi}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial \xi}\right) + \frac{c}{k_{1}^{2}}\frac{\partial}{\partial \xi}\right)\tilde{u}_{j-1}^{\delta} \\
&+ \left(3\xi^{2}c^{2}\frac{\partial}{\partial \xi}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial \xi}\right) + 2\xi c^{2}\frac{1}{k_{1}^{2}}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial s}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial s}\right) + 1\right)\tilde{u}_{j-2}^{\delta} \\
&+ \left(\xi^{3}c^{3}\frac{\partial}{\partial \xi}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial \xi}\right) + \frac{\xi^{2}c^{3}}{k_{1}^{2}}\frac{\partial}{\partial \xi} + \xi c\frac{\partial}{\partial s}\left(\frac{1}{k_{1}^{2}}\frac{\partial}{\partial s}\right) - \frac{\xi c'}{k_{1}^{2}}\frac{\partial}{\partial s} + 3\xi c\right)\tilde{u}_{j-3}^{\delta} \\
&+ 3\xi^{2}c^{2}\tilde{u}_{j-4}^{\delta} + \xi^{3}c^{3}\tilde{u}_{j-5}^{\delta} = 0, j = 0, 1, \ldots,
\end{aligned} \tag{1.82}$$

with the convention $\tilde{u}_j = 0$ for negative j. The equations for $j = 0, 1, 2$ are:

$$\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\tilde{u}_0^\delta\right) = 0, \tag{1.83}$$

$$\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\tilde{u}_1^\delta\right) + \left(3\xi c\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\right) + \frac{c}{k_1^2}\frac{\partial}{\partial \xi}\right)\tilde{u}_0^\delta = 0, \tag{1.84}$$

$$\begin{aligned}&\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\tilde{u}_2^\delta\right) + \left(3\xi c\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\right) + \frac{c}{k_1^2}\frac{\partial}{\partial \xi}\right)\tilde{u}_1^\delta \\ &+ \left(3\xi^2 c^2\frac{\partial}{\partial \xi}\left(\frac{1}{k_1^2}\frac{\partial}{\partial \xi}\right) + 2\xi c^2\frac{1}{k_1^2}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial s}\left(\frac{1}{k_1^2}\frac{\partial}{\partial s}\right) + 1\right)\tilde{u}_0^\delta = 0.\end{aligned} \tag{1.85}$$

The jumps across Γ are given by (compare with (1.57) and (1.58)):

$$\begin{aligned}&[u^\pm(x_\Gamma(s)) + 0\nu] \\ &\quad = [\tilde{u}_0^\delta(s, \pm\frac{1}{2})] \\ &\qquad + \delta\left([\tilde{u}_1^\delta(s, \pm\frac{1}{2})] - \frac{1}{2}\left(\frac{\partial u_0^+}{\partial \nu^2}(x_\Gamma(s) + 0\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s) + 0\nu)\right)\right) \\ &\qquad + \mathcal{O}(\delta^2)\end{aligned} \tag{1.86}$$

and

$$\begin{aligned}&\frac{\partial u^+}{\partial \nu}(x_\Gamma(s) + 0\nu) - \frac{1}{n_-}\frac{\partial u^-}{\partial \nu}(x_\Gamma(s) + 0\nu) \\ &= \frac{1}{(k_1^2/k^2)}\left[\frac{\partial \tilde{u}_1^\delta}{\partial \xi}(s, \pm\frac{1}{2})\right] + \delta\left(\frac{1}{(k_1^2/k^2)}\left[\frac{\partial \tilde{u}_2^\delta}{\partial \xi}(s, \pm\frac{1}{2})\right]\right. \\ &\qquad \left. - \frac{1}{2}\left(\frac{\partial^2 u_0^+}{\partial \nu^2}(x_\Gamma(s) + 0\nu) + \frac{1}{n_-}\frac{\partial^2 u_0^-}{\partial \nu^2}(x_\Gamma(s) + 0\nu)\right)\right) + \mathcal{O}(\delta^2)\end{aligned} \tag{1.87}$$

for all $s \in [0, L]$.

We assume that the conductivity of the layer is of the form

$$\sigma^\delta(s, \xi) = \frac{\eta(s)}{\delta}, \quad (s, \xi) \in (0, L) \times (-1/2, 1/2), \tag{1.88}$$

where η does not depend on δ. Therefore, k_1^2 depends only on s. From (1.83) and (1.81) we conclude that $\tilde{u}_0^\delta$ is constant along the normal ν. This implies that, for each $s \in [0, L]$, $\frac{\partial}{\partial \xi} \tilde{u}_1^\delta(s, \cdot)$ is constant too (see (1.84)). As in (1.63), by the fundamental theorem of calculus and using the boundary conditions (1.79) and (1.80), we get for the jump $[\tilde{u}_1^\delta(\cdot, \pm\frac{1}{2})]$:

$$[\tilde{u}_1^\delta(\cdot, \pm\frac{1}{2})] = \frac{1}{2}(k_1^2/k^2)\left(\frac{\partial u_0^+}{\partial \nu}(x_\Gamma(\cdot) + \frac{1}{2}\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(\cdot) - \frac{1}{2}\nu)\right). \tag{1.89}$$

Thus, (1.86) becomes

$$\begin{aligned}
&[u^\pm(x_\Gamma(s)) + 0\nu] \\
&= \frac{\delta}{2}(k_1^2/k^2)\left(\frac{\partial u_0^+}{\partial \nu}(x_\Gamma(s) + \frac{1}{2}\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s) - \frac{1}{2}\nu)\right) \\
&\quad - \frac{\delta}{2}\left(\frac{\partial u_0^+}{\partial \nu^2}(x_\Gamma(s) + 0\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s) + 0\nu)\right) + \mathcal{O}(\delta^2) \\
&= \frac{i\omega\eta(s)\mu_0}{2k^2}\left(\frac{\partial u_0^+}{\partial \nu}(x_\Gamma(s) + 0\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s) + 0\nu)\right) + \mathcal{O}(\delta)
\end{aligned} \tag{1.90}$$

for all $s \in [0, L]$. It remains to compute $\left[\frac{\partial \tilde{u}_2^\delta}{\partial \xi}(\cdot, \pm\frac{1}{2})\right]$. From (1.85) and (1.79)–(1.80) we get

$$\begin{aligned}
\frac{1}{k_1^2}\left[\frac{\partial \tilde{u}_2^\delta}{\partial \xi}(s, \pm\frac{1}{2})\right] &= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{c}{k_1^2}\frac{\partial}{\partial \tau}\tilde{u}_1^\delta(s, \tau) + \left(\frac{\partial}{\partial s}\left(\frac{1}{k_1^2}\frac{\partial}{\partial s}\right) + 1\right)\tilde{u}_0^\delta(s, \tau)\,\mathrm{d}\tau \\
&= -\frac{c(s)}{2k^2}\left(\frac{\partial u_0^+}{\partial \nu}(x_\Gamma(s) + \frac{1}{2}\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s) - \frac{1}{2}\nu)\right) \\
&\quad - \left(\frac{\partial}{\partial s}\left(\frac{1}{k_1^2}\frac{\partial}{\partial s}\right) + 1\right)\langle u_0^\pm(x_\Gamma(s) \pm \frac{1}{2}\nu)\rangle.
\end{aligned}$$

for each $s \in [0, L]$.

Thus,

$$\begin{aligned}
&\frac{\partial u^+}{\partial \nu}(x_\Gamma(s)+0\nu) - \frac{1}{n_-}\frac{\partial u^-}{\partial \nu}(x_\Gamma(s)+0\nu) \\
&= \delta\Bigg(-\frac{c(s)}{2}\left(\frac{\partial u_0^+}{\partial \nu}(x_\Gamma(s)+0\nu) + \frac{1}{n_-}\frac{\partial u_0^-}{\partial \nu}(x_\Gamma(s)+0\nu)\right) \\
&\qquad - k^2\left(\frac{\partial}{\partial s}\left(\frac{1}{k_1^2}\frac{\partial}{\partial s}\right)+1\right)\langle u_0^\pm(x_\Gamma(s)+0\nu)\rangle \\
&\qquad - \frac{1}{2}\left(\frac{\partial^2 u_0^+}{\partial \nu^2}(x_\Gamma(s)+0\nu) + \frac{1}{n_-}\frac{\partial^2 u_0^-}{\partial \nu^2}(x_\Gamma(s)+0\nu)\right)\Bigg) + \mathcal{O}(\delta^2)
\end{aligned}$$

for all $s \in [0, L]$. Assuming that $\partial k_1^2/\partial s$ remains bounded we conclude

$$\frac{\partial u^+}{\partial \nu}(x_\Gamma(\cdot)+0\nu) - \frac{1}{n_-}\frac{\partial u^-}{\partial \nu}(x_\Gamma(\cdot)+0\nu) = \mathcal{O}(\delta). \tag{1.91}$$

Thus, the model with first order approximate transmission conditions for the $H-$mode has the following form:

$$\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2\backslash\overline{D}, \\
\operatorname{div}\left(\frac{1}{n}\right)\nabla u + k^2 u &= 0 \quad \text{in } D, \\
u_+ - u_- - i\eta\frac{\partial u^+}{\partial \nu} &= 0 \quad \text{on } \partial D, \\
\frac{\partial u^+}{\partial \nu} - \frac{1}{n_-}\frac{\partial u^-}{\partial \nu} &= 0 \quad \text{on } \partial D,
\end{aligned}$$

where η is real valued and positive.

1.2.3 Numerical Validation

In this section, by means of numerical experiments we show that the far fields of the full model (involving the layer of thickness δ) converge to the far fields of the approximate model, as δ goes to zero.

In the following examples our data set is represented by a $\mathbb{C}^{32\times 32}$ matrix F, where each entry is the far field pattern $u^\infty(\theta_j, \theta_l), j, l \in \{1, \dots 32\}$, with $\theta_j = 2\pi j/32$ and $\theta_l = 2\pi l/32$ denoting the corresponding incident direction of the plane wave and the observation point, respectively. The data is generated through a P^1 finite elements discretization using FreeFem++ [27]. The problem over $\mathbb{R}^2$ is reduced to a bounded domain with the help of Dirichlet-to-Neumann mapping [24].

We compute the far fields for the TM-mode for a kite-shaped object parametrized by $\gamma(t) = (\cos(t)+0.65\cos(2t)-0.65),\ 1.5\sin(t))^\top$, $t \in [0, 2\pi]$. Further, we set $n(x, y) = 0.2 + (x^2 + y^2)$, $k = 3$, $R = 5$ and $\eta = 0.5$, where R is the radius of the exterior disk (see also Figure 1.3).

Table 1 shows the relative error computed by

$$\frac{\|F - F^\delta\|_2}{\|F^\delta\|_2}, \tag{1.92}$$

where F^δ is the matrix containing the far fields $u^{\infty,\delta}$ corresponding to the full model with $\delta > 0$. As we can see the convergence is linear.

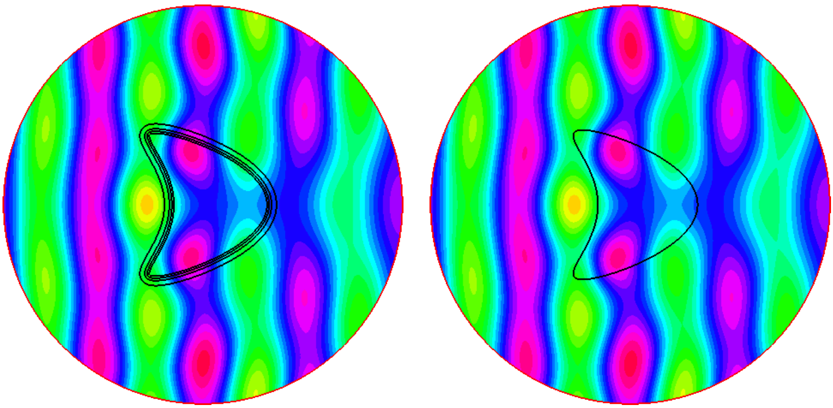

Figure 1.3: *Real part of the total field for a kite-shaped obstacle for the full model with $\delta = 0.05$ (on the left) and for the approximate model (on the right). The direction of incidence is $d = [1\ 0]^\top$.*

	Relative error
$\delta = 0.06$	0.0655
$\delta = 0.05$	0.0532
$\delta = 0.04$	0.0410
$\delta = 0.03$	0.0291
$\delta = 0.02$	0.0164

Table 1: Relative errors computed by (1.92) for a kite-shaped domain for the TM-mode.

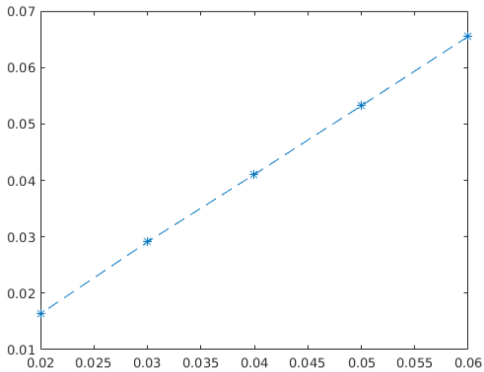

Figure 1.4: *Relative error of the far fields for a kite-shaped domain for the TM-mode.*

For the TE-mode we compute the far fields for a unit disk with the parameters $n(x) = 0.2+(x_1^2+x_2^2)$, $k = 3$, $\eta = 1.5$. Figure 1.5 shows the real part of the total field of the full model and of the approximation. As we can see, the total filed discontinuous across the boundary of the object. Figure 1.5 represents the plot of relative errors for $\delta = 0.08, 0.06, 0.04, 0.03, 0.02, 0.01$.

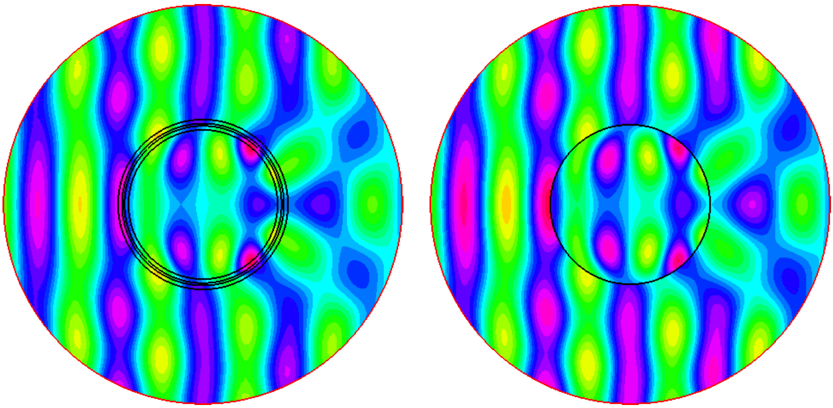

Figure 1.5: *Real part of the total field for the unit disk for the full model with* $\delta = 0.05$ *(on the left) and for the approximate model (on the right). The direction of incidence is* $d = [1\ 0]^\top$.

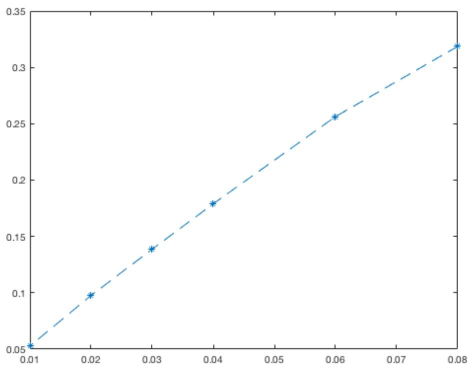

Figure 1.6: *Relative error of the far fields for the unit disk for the TE-mode.*

2 Direct and Inverse Problem for TM-mode

2.1 Mathematical Formulation of the Direct Scattering Problem

Throughout this chapter let D represent a finite union of bounded domains with C^2 boundary ∂D and connected exterior $\mathbb{R}^2\backslash\overline{D}$. Further, let ν denote the unit outward normal vector to ∂D.

Assume that $k > 0$, $n \in C(\overline{D})$ with $\mathrm{Im}(n) \geq 0$, $\mathrm{Re}(n) > 0$ and $n - 1 \neq 0$ in D, and $\lambda \in C(\partial D)$ with $\lambda(x) \geq 0$ for all $x \in \partial D$. We consider the following direct problem given u^i which satisfies the Helmholtz equation

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^2$$

find $u^s \in C^2(\mathbb{R}^2\backslash\overline{D}) \cap C^1(\mathbb{R}^2\backslash D)$ and $u \in C^2(D) \cap C^1(\overline{D})$ such that

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D}, \tag{2.1}$$

$$\Delta u + k^2 n u = 0 \quad \text{in} \quad D, \tag{2.2}$$

$$u^s - u = -u^i \quad \text{on} \quad \partial D, \tag{2.3}$$

$$\frac{\partial u^s}{\partial \nu} - \frac{\partial u}{\partial \nu} + i\lambda u = -\frac{\partial u^i}{\partial \nu} \quad \text{on} \quad \partial D, \tag{2.4}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|, \tag{2.5}$$

where the last equation holds uniformly in $x/|x|$. We will call a solution to the Helmholtz equation whose domain of definition contains the exterior of some disk *radiating* if it satisfies (2.5).

We also will refer to this problem in the following equivalent form: given u^i with

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^2$$

find $u^s|_{\mathbb{R}^2\backslash D} \in C^2(\mathbb{R}^2\backslash\overline{D}) \cap C^1(\mathbb{R}^2\backslash D)$ and $u^s|_{\overline{D}} \in C^2(D) \cap C^1(\overline{D})$ such that

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D}, \tag{2.6}$$

$$\Delta u^s + k^2 n u^s = -k^2(n-1)u^i \quad \text{in} \quad D, \tag{2.7}$$

$$u^s_+ - u^s_- = 0 \quad \text{on} \quad \partial D, \tag{2.8}$$

$$\frac{\partial u^s_+}{\partial \nu} - \frac{\partial u^s_-}{\partial \nu} + i\lambda u^s_+ = -i\lambda u^i \quad \text{on} \quad \partial D, \tag{2.9}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|. \tag{2.10}$$

Here, $u^s_\pm$ and $\partial u^s_\pm/\partial\nu$ denote the limit of u^s and $\partial u^s/\partial\nu$ from the exterior $(+)$ and interior $(-)$, respectively.

In the following we also want to account for n and λ having discontinuities. We weaken our assumptions and require only

$$n \in L^\infty(D), \ \mathrm{Im}\,(n) \geq 0, \ \mathrm{Re}\,(n) > 0 \ \text{ and } \ n-1 \neq 0 \ \text{ a.e. in } \ D, \tag{2.11}$$

and

$$\lambda \in L^\infty(\partial D) \ \text{ with } \ \lambda \geq 0 \ \text{ a.e. on } \ \partial D. \tag{2.12}$$

We cannot assume anymore that u is smooth and have to specify in which sense the equations (2.6)–(2.7) and the boundary conditions (2.8)–(2.9) have to be understood.

Let $H^1(D)$ denote the Sobolev space and $H^1_{loc}(\mathbb{R}^2\backslash D)$ the local Sobolev space defined as

$$H^1(D) := \{u : u \in L^2(D), \nabla u \in L^2(D)\}, \quad \text{and}$$
$$H^1_{loc}(\mathbb{R}^2\backslash\overline{D}) := \{u : u \in H^1(B_R\backslash\overline{D}), \text{ for every } R, \text{ such that } D \subset B_R\},$$

where B_R is a ball of radius $R > 0$ centered at the origin $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$. Further, we denote by $H^{1/2}(\partial D)$ the trace space of $H^1(D)$ and by $H^{-1/2}(\partial D)$ its dual.

We assume $u^s|_D \in H^1(D)$ and $u^s|_{\mathbb{R}^2\setminus\overline{D}} \in H^1_{loc}(\mathbb{R}^2\setminus\overline{D})$ satisfying (2.1) and (2.2), respectively, in the distributional sense. In the next Lemma we show that this implies in particular that $\Delta u^s|_D \in L^2(D)$ and $\Delta u^s|_{\mathbb{R}^2\setminus\overline{D}} \in L^2_{loc}(\mathbb{R}^2\setminus D)$.

Lemma 2.1.1. *Let $u \in L^2(D)$ and $v \in L^2_{loc}(\mathbb{R}^2\setminus\overline{D})$ satisfy (2.6) and (2.7), respectively, in the distributional sense. Then $\Delta u \in L^2(D)$ and $\Delta v \in L^2_{loc}(\mathbb{R}^2\setminus\overline{D})$.*

Proof. We show the assertion for $u \in L^2(D)$. By definition, for distributional Laplacian we have:

$$\langle \Delta u, \varphi \rangle = \iint\limits_D u\, \Delta\varphi \,\mathrm{d}x, \qquad \text{for all} \qquad \varphi \in C_0^\infty(D).$$

We require u to satisfy (2.7) in the distributional sense which implies

$$\langle \Delta u, \varphi \rangle = \iint\limits_D (-k^2)(nu + (n-1)u^i)\,\varphi \,\mathrm{d}x, \;\; \text{for all} \;\; \varphi \in C_0^\infty(D). \quad (2.13)$$

By the Cauchy-Schwarz inequality holds

$$\begin{aligned} |\langle \Delta u, \varphi \rangle| &\le k^2 \|n\|_{L^\infty(D)} \iint\limits_D |u\,\varphi| \,\mathrm{d}x + k^2 \|(n-1)\|_{L^\infty(D)} \iint\limits_D |u^i \varphi| \,\mathrm{d}x \\ &\le k^2 (\|n\|_{L^\infty(D)} \|u\|_{L^2(D)} + \|n-1\|_{L^\infty(D)} \|u^i\|_{L^2(D)}) \|\varphi\|_{L^2(D)} \end{aligned}$$

for all $\varphi \in C_0^\infty(D)$.

Since $C_0^\infty(D)$ is dense in $L^2(D)$ we can extend $\langle \Delta u, \cdot \rangle$ by the right hand side of (2.13) for $\varphi \in L^2(D)$, i.e.,

$$\langle \Delta u, \varphi \rangle = \iint\limits_D (-k^2)(nu + (n-1)u^i)\,\varphi \,\mathrm{d}x, \qquad \text{for all} \qquad \varphi \in L^2(D).$$

That is, Δu defines a continuous linear functional on $L^2(D)$. Thus Δu is in the dual of $L^2(D)$, which again can be identified with $L^2(D)$. Therefore, the equation (2.7) holds in L^2-sense. The case for $v \in L^2_{loc}(\mathbb{R}^2\setminus D)$ is completely analogous.

□

By the trace theorem $u^s|_D$ and $u^s|_{\mathbb{R}^2\setminus\overline{D}}$ possess traces in $H^{1/2}(\partial D)$. Moreover, by Theorem 5.8 in [8] (and the remarks following it), for a function $u \in H^1(D)$ with $\Delta u \in L^2(D)$ the trace $\partial u/\partial \nu \in H^{-1/2}(\partial D)$ is well-defined by

$$\langle \frac{\partial u}{\partial \nu}, v \rangle = \iint\limits_D \Delta u \, \overline{v} + \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x \quad \text{for all} \quad v \in H^1(D), \tag{2.14}$$

where $\langle \cdot\,, \cdot \rangle$ is the dual form in the dual system $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D)\rangle$. Note that (2.14) is just the Green's theorem [16] in a wider space. For $u \in H^1_{loc}(\mathbb{R}^2\setminus\overline{D})$ with $\Delta u \in L^2_{loc}(\mathbb{R}^2\setminus\overline{D})$ the trace $\partial u/\partial \nu \in H^{-1/2}(\partial D)$ is defined by

$$\langle \frac{\partial u}{\partial \nu}, v \rangle = \iint\limits_{\mathbb{R}^2\setminus\overline{D}} \Delta u \, \overline{v} + \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x \quad \text{for all} \quad v \in H^1_{loc}(\mathbb{R}^2\setminus\overline{D}). \tag{2.15}$$

Now we define the direct scattering problem in Sobolev spaces: let D, $n \in L^\infty(D)$ and $\lambda \in L^\infty(\partial D)$ be given, and let n and λ satisfy the assumptions (2.11) and (2.12), respectively. For $f \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$ find $u \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2\setminus\overline{D}, \tag{2.16}$$

$$\Delta u + k^2 n u = f \quad \text{in} \quad D, \tag{2.17}$$

$$u_+ - u_- = 0 \quad \text{on} \quad \partial D, \tag{2.18}$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\lambda u_+ = h \quad \text{on} \quad \partial D, \tag{2.19}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|. \tag{2.20}$$

Note that setting $f = -k^2(n-1)u^i$ and $h = -i\lambda u^i$ the scattering problem (2.6)–(2.10) becomes a special case of (2.16)–(2.20). The equations (2.16)–(2.17) are understood in the distributional sense and the boundary conditions (2.18)–(2.19) are assumed in the sense of traces, where $u|_+, \partial u_+/\partial \nu$ and $u_-, \partial u_-/\partial \nu$ denote the traces taken from the interior and the exterior of D, respectively. From the regularity theory for elliptic differential equations [21] it is known that u is analytic in $\mathbb{R}^2\setminus\overline{D}$. In particular, the radiation condition (2.20) makes sense.

2.2 Existence and Uniqueness

The aim of this section is to show that a solution to (2.16)–(2.20) exists, is unique and depends continuously on the source terms f and h.

Theorem 2.2.1. *For any $f \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$, there exists at most one solution $u \in H^1_{loc}(\mathbb{R}^2)$ of (2.16)-(2.20).*

Proof. By the definition, the scattering problem (2.16)–(2.20) has the following equivalent variational formulation: For given $f \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$ find $u \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\iint\limits_{\mathbb{R}^2\setminus\overline{D}} [\nabla u \cdot \nabla\overline{\varphi} - k^2 u\overline{\varphi}]\,\mathrm{d}x + \iint\limits_{D} [\nabla u \cdot \nabla\overline{\varphi} - k^2 n u\overline{\varphi}]\,\mathrm{d}x$$
$$-i\int\limits_{\partial D} \lambda u\overline{\varphi}\,\mathrm{d}s = -\iint\limits_{D} f\overline{\varphi}\,\mathrm{d}x - \langle h, \varphi\rangle \tag{2.21}$$

for any test function $\varphi \in H^1(\mathbb{R}^2)$ with compact support. As before, $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for dual form in the dual system $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D)\rangle$. Further, we require u to satisfy the Sommerfeld radiation condition (2.20).

Let now v be the difference of two solutions. Then v solves (2.21) with $h = 0$ and $f = 0$. We show that v vanishes in all of $\mathbb{R}^2$.

Choose a ball B_R centered at the origin with $R > 0$ big enough such that $\overline{D} \subset B_R$. Let $\phi \in C^\infty(\mathbb{R}^2)$ be such that $\phi(x) = 1$ for $|x| \le R$ and $\phi(x) = 0$ for $|x| \ge R + 1$. We set $\varphi = \phi v$ and substitute it into (2.21):

$$\iint\limits_{R<|x|<R+1} \left[\nabla v \cdot \nabla\overline{\varphi} - k^2 v\overline{\varphi}\right]\mathrm{d}x + \iint\limits_{B_R\setminus\overline{D}} \left[|\nabla v|^2 - k^2|v|^2\right]\mathrm{d}x$$
$$+\iint\limits_{D} \left[|\nabla v|^2 - k^2 n|v|^2\right]\mathrm{d}x - i\int\limits_{\partial D} \lambda|v|^2\,\mathrm{d}s = 0. \tag{2.22}$$

By regularity results [21] v is analytic outside D. In particular, $\Delta v + k^2 v = 0$ holds in classical sense in $\mathbb{R}^2 \backslash \overline{B_R}$. We apply the Green's first theorem [16] to the first integral in (2.22):

$$-\int\limits_{|x|=R} \frac{\partial v}{\partial \nu} \overline{v}\,\mathrm{d}s + \iint\limits_{B_R \backslash \overline{D}} \left[|\nabla v|^2 - k^2 |v|^2 \right] \mathrm{d}x$$
$$+ \iint\limits_{D} \left[|\nabla v|^2 - k^2 n |v|^2 \right] \mathrm{d}x - i \int\limits_{\partial D} \lambda |v|^2\,\mathrm{d}s = 0.$$

From the assumptions on λ and n it follows that

$$\mathrm{Im} \int\limits_{\partial B_R} \overline{v} \frac{\partial v}{\partial \nu} \mathrm{d}s \leq 0. \tag{2.23}$$

Theorem 3.6 in [8] implies $v = 0$ in $\mathbb{R}^2 \backslash \overline{D}$. Thus, $\Delta v + k^2 n v = 0$ in D, or if extend n for example by 1 in the exterior of D, $\Delta v + k^2 n v = 0$ in $\mathbb{R}^2$ with $v = 0$ in $\mathbb{R}^2 \backslash \overline{D}$. Then the unique continuation principle [44], which holds for elliptic equations in 2D with coefficients in L^∞, applies giving that v is identically zero in all of $\mathbb{R}^2$.

□

To show the existence we will follow the approach introduced by P.Hähner in [24], the idea of which is to consider an equivalent form of (2.16)–(2.20) in a bounded domain B_R. Instead of the asymptotic Sommerfeld radiation condition a special boundary condition on the artificial boundary ∂B_R is imposed. We will also use this approach to solve the direct problem numerically.

We define the following Dirichlet-to-Neumann mapping $\Lambda_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ by

$$\Lambda_k : g \mapsto \frac{\partial \tilde{u}}{\partial \nu}, \tag{2.24}$$

where $\tilde{u} \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{B_R})$ is the solution of the exterior Dirichlet problem

$$\Delta \tilde{u} + k^2 \tilde{u} = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{B_R}, \tag{2.25}$$

$$\tilde{u} = g \quad \text{on } \partial B_R, \tag{2.26}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right) = 0, \quad r = |x|. \tag{2.27}$$

The exterior Dirichlet problem (2.25)–(2.27) is well-posed and thus, the Dirichlet-to-Neumann mapping Λ_k is well-defined and bounded.

To show the existence we will need the following important property of the operator Λ_k, which we formulate as a Lemma (see e.g. Theorem 5.22 in [8]).

Lemma 2.2.2. *The Dirichlet to Neumann operator Λ_k is a bounded linear operator from $H^{1/2}(\partial B_R)$ to $H^{-1/2}(\partial B_R)$. Furthermore, there exists a bounded operator $\Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ satisfying*

$$- \int_{\partial B_R} \Lambda_0 w \, \overline{w} \, \mathrm{d}s \geq c \|w\|^2_{H^{1/2}(\partial B_R)} \tag{2.28}$$

for some constant $c > 0$ such that $\Lambda_k - \Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is compact.

Next we show the equivalence between the scattering problem defined in a bounded domain B_R

$$\Delta u + k^2 u = 0 \quad \text{in} \quad B_R \backslash \overline{D}, \tag{2.29}$$

$$\Delta u + k^2 n u = f \quad \text{in} \quad D, \tag{2.30}$$

$$u_+ - u_- = 0 \quad \text{on} \quad \partial D, \tag{2.31}$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\lambda u_+ = h \quad \text{on} \quad \partial D, \tag{2.32}$$

$$\frac{\partial u}{\partial \nu} = \Lambda_k u \quad \text{on} \quad \partial B_R, \tag{2.33}$$

for $R > 0$ such that $\overline{D} \subset B_R$, and the problem given by (2.16)–(2.20).

An important ingredient of our proof will be the following representation theorem (Theorem 3.1 in [8]).

Theorem 2.2.3. ***(Representation Theorem).*** *Let $u^s \in C^2(\mathbb{R}^2\backslash\overline{D}) \cap C(\mathbb{R}^2\backslash D)$ be a solution to the Helmholtz equation in the exterior of D satisfying the Sommerfeld radiation condition and such that $\partial u/\partial\nu$ exists in the sense of uniform convergence as $x \to \partial D$. Then for $x \in \mathbb{R}^2\backslash\overline{D}$ we have that*

$$u^s(x) = \int\limits_{\partial D} \left(u^s(y)\frac{\partial}{\partial\nu(y)}\Phi_k(x,y) - \frac{\partial u^s}{\partial\nu}(y)\Phi_k(x,y)\right) \mathrm{d}s(y). \tag{2.34}$$

For a solution $u \in C^2(D) \cap C^1(\overline{D})$ of the Helmholtz equation in D holds

$$u(x) = \int\limits_{\partial D} \left(\frac{\partial u}{\partial\nu}(y)\Phi_k(x,y) - u(y)\frac{\partial}{\partial\nu(y)}\Phi_k(x,y)\right) \mathrm{d}s(y), \; x \in D. \tag{2.35}$$

The function Φ_k is called the *fundamental solution* to the Helmholtz equation and is given by

$$\Phi_k(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad x \neq y, \tag{2.36}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. For a fixed $y \in \mathbb{R}^2$ (that is, Φ_k represents a point source at y) the far field of Φ_k is given by (see Section 4.1 in [8])

$$\Phi_k^\infty(\hat{x},y) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot y}, \quad \hat{x} \in S^1. \tag{2.37}$$

By Remark 5.10 in [8] the Representation Theorem 2.2.3 holds also for H^1-solutions of the Helmholtz equation (in this case one has to interpret the boundary integrals as the dual forms).

Lemma 2.2.4. *Problems (2.29)–(2.33) and (2.16)–(2.20) are equivalent.*

Proof. We follow the arguments of Lemma 5.24 in [8]. Assume $u \in H^1_{loc}(\mathbb{R}^2)$ is a solution to (2.16)–(2.20). Then the restriction $u|_{B_R}$ is in $H^1(B_R)$ and solves (2.29)–(2.33).

Let now $u \in H^1(B_R)$ be a solution to (2.29)–(2.33). Then u can be extended to all of $\mathbb{R}^2$ such that u satisfies (2.16)–(2.20). Indeed, let $\tilde{u}$ be the solution to the exterior Dirichlet problem (2.25)–(2.27) with $\tilde{u} = u$ on ∂B_R. In particular, $\frac{\partial u}{\partial \nu} = \Lambda_k u = \frac{\partial \tilde{u}}{\partial \nu}$. By the representation formula (2.35) for u in the bounded domain $B_R \backslash \overline{D}$ we have

$$u(x) = \int\limits_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y) \\ - \int\limits_{|x|=R} \left(u(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial u}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y) \qquad (2.38)$$

for $x \in B_R \backslash \overline{D}$. For simplicity of notation, in (2.38) we use integrals instead of the the dual forms.

Let $x \in B_R$ be fixed. Then, since $\tilde{u}$ and $\Phi(x, \cdot)$ solve the Helmholtz equation in the exterior of B_R, by the Green's second identity we obtain

$$\int\limits_{|y|=R} \left(\tilde{u}(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y) \\ = \int\limits_{|y|=R_1} \left(\tilde{u}(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y) \qquad (2.39)$$

for any $R_1 > R$, or

$$\int\limits_{|y|=R} \left(\tilde{u}(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y) \\ = \lim_{R \to \infty} \int\limits_{|y|=R} \left(\tilde{u}(y) \frac{\partial \Phi(x,y)}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}(y) \Phi(x,y) \right) \mathrm{d}s(y). \qquad (2.40)$$

Both $\tilde{u}$ and $\Phi(x, \cdot)$ are radiating, and both $|\tilde{u}(y)|$ and $|\Phi(x,y)|$ are $\mathcal{O}\big(\frac{1}{\sqrt{|y|}}\big)$ as $|y| \to \infty$ (see e.g. in [16] for the *Sommerfeld's finiteness condition* for the 3D case, the 2D case is analogous). Applying the Cauchy-Schwarz inequality we get that the limit on the right hand side of (2.40) vanishes:

$$\lim_{R\to\infty} \int\limits_{|y|=R} \left(\tilde{u}(y)\frac{\partial\Phi(x,y)}{\partial\nu} - \frac{\partial\tilde{u}}{\partial\nu}(y)\Phi(x,y)\right)\mathrm{d}s(y) =$$

$$\lim_{R\to\infty} \int\limits_{|y|=R} \tilde{u}(y)\left(\frac{\partial}{\partial\nu}\Phi(x,y) - ik\Phi(x,y)\right)\mathrm{d}s(y)$$

$$- \lim_{R\to\infty} \int\limits_{|y|=R} \Phi(x,y)\left(\frac{\partial}{\partial\nu}\tilde{u}(y) - ik\tilde{u}(y)\right)\mathrm{d}s(y)$$

$$\leq \lim_{R\to\infty} \int\limits_{|y|=R} |\tilde{u}(y)|^2\,\mathrm{d}s(y) \int\limits_{|y|=R} \left|\frac{\partial}{\partial\nu}\Phi(x,y) - ik\Phi(x,y)\right|^2 \mathrm{d}s(y)$$

$$- \lim_{R\to\infty} \int\limits_{|y|=R} |\Phi(x,y)|^2\,\mathrm{d}s(y) \int\limits_{|y|=R} \left|\frac{\partial}{\partial\nu}\tilde{u}(y) - ik\tilde{u}(y)\right|^2 \mathrm{d}s(y) = 0.$$

Noting that $u = \tilde{u}$ and $\frac{\partial u}{\partial\nu} = \frac{\partial\tilde{u}}{\partial\nu}$ on ∂B_R with (2.40) we get

$$u(x) = \int\limits_{\partial D} \left(u(y)\frac{\partial\Phi(x,y)}{\partial\nu} - \frac{\partial u}{\partial\nu}(y)\Phi(x,y)\right)\mathrm{d}s(y) \tag{2.41}$$

for $x \in B_R$. Thus, u can be extended by the right hand side of (2.41) to all of $\mathbb{R}^2$ to a radiating solution.

□

Now we are ready to prove the well-posedness of the direct scattering problem.

Theorem 2.2.5. *Let $f \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$. Then the problem (2.29)–(2.33) has a unique solution $u \in H^1(B_R)$. Furthermore,*

$$\|u\|_{H^1(B_R)} \leq C_R(\|h\|_{H^{-1/2}(\partial D)} + \|f\|_{L^2(D)}) \tag{2.42}$$

with a positive constant C_R independent of f and h.

Proof. We first write (2.29)–(2.33) in the following equivalent variational form: For given $f \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$ find $u \in H^1(B_R)$ such that

$$\iint_{B_R\setminus\overline{D}} [\nabla u \cdot \nabla\overline{\varphi} - k^2 u\overline{\varphi}]\,\mathrm{d}x + \iint_D [\nabla u \cdot \nabla\overline{\varphi} - k^2 n u\overline{\varphi}]\,\mathrm{d}x$$
$$- \int_{|x|=R} \Lambda_k u\,\overline{\varphi}\mathrm{d}s - i\int_{\partial D} \lambda u\overline{\varphi}\,\mathrm{d}s = -\iint_D f\overline{\varphi}\,\mathrm{d}x - \int_{\partial D} h\overline{\varphi}\,\mathrm{d}s, \tag{2.43}$$

for all $\varphi \in H^1(B_R)$. The boundary integrals have to be interpreted as the dual forms. We write (2.43) as

$$a(u,\varphi) = b(\varphi) \quad \text{for all} \quad \varphi \in H^1(B_R), \tag{2.44}$$

with

$$a(u,\varphi) = \iint_{B_R\setminus\overline{D}} [\nabla u \cdot \nabla\overline{\varphi} - k^2 u\overline{\varphi}]\,\mathrm{d}x - \int_{|x|=R} \Lambda_k u\,\overline{\varphi}\,\mathrm{d}s$$
$$+ \iint_D [\nabla u \cdot \nabla\overline{\varphi} - k^2 n u\overline{\varphi}]\,\mathrm{d}x - i\int_{\partial D} \lambda u\,\overline{\varphi}\,\mathrm{d}s,$$

and

$$b(\varphi) = -\iint_D f\overline{\varphi}\,\mathrm{d}x - \int_{\partial D} h\overline{\varphi}\,\mathrm{d}s.$$

Further, we represent a as a sum $a = a_1 + a_2$, where

$$a_1(u,\varphi) = \iint_{B_R\setminus\overline{D}} [\nabla u \cdot \nabla\overline{\varphi} + u\overline{\varphi}]\,\mathrm{d}x - \int_{\partial B_R} \Lambda_0 u\overline{\varphi}\,\mathrm{d}s$$
$$+ \iint_D [\nabla u \cdot \nabla\overline{\varphi} + u\overline{\varphi}]\,\mathrm{d}x - i\int_{\partial D} \lambda u\overline{\varphi}\,\mathrm{d}s$$

and

$$a_2(u,\varphi) = -\iint\limits_{B_R\setminus\overline{D}} (1+k^2)u\overline{\varphi}\,\mathrm{d}x - \int\limits_{|x|=R} (\Lambda_k - \Lambda_0)u\overline{\varphi}\,\mathrm{d}s$$
$$- \iint\limits_{D} (1+k^2 n)u\overline{\varphi}\,\mathrm{d}x,$$

with Λ_0 being the operator defined in Lemma 2.2.2. By the boundedness of Λ_0 and the trace theorem, the sesquilinear form a_1 is bounded, i.e., $|a_1(u,\varphi)| \le c||u||_{H^1(B_R)}||\varphi||_{H^1(B_R)}$ for all $u, \varphi \in H^1(B_R)$. Riesz representation theorem yields that there exists a bounded linear operator $A_1 : H^1(B_R) \to H^1(B_R)$ such that

$$a_1(u,\varphi) = (A_1 u, \varphi)_{H^1(B_R)} \quad \text{for all } \varphi \in H^1(B_R). \tag{2.45}$$

By Lemma 2.2.2 and the assumptions (2.11) and (2.12) on n and λ, respectively, for all $u \in H^1(B_R)$ holds

$$\mathrm{Re}\ a_1(u,u) = \|u\|^2_{H^1(B_R)} - \int\limits_{|x|=R} \Lambda_0 u\,\overline{u}\,\mathrm{d}s \ge \|u\|^2_{H^1(B_R)}.$$

That is, a_1 is strictly coercive. The Lax-Milgram theorem (see Theorem 13.26 in [40]) implies that the operator $A_1 : H^1(B_R) \to H^1(B_R)$ has a bounded inverse.

The sesquilinear form a_2 is bounded as well, and by the Cauchy-Schwarz inequality we have that

$$|b(\varphi)| \le (\|f\|_{L^2(D)} + \|h\|_{H^{-1/2}(\partial D)})\|\varphi\|_{H^1(B_R)}. \tag{2.46}$$

Riesz representation theorem yields that there exists a bounded linear operator $A_2 : H^1(B_R) \to H^1(B_R)$ and an element $\widetilde{v} \in H^1(B_R)$ such that

$$a_2(u,\varphi) = (A_2 u, \varphi)_{H^1(B_R)} \quad \text{for all } \varphi \in H^1(B_R)$$

and

$$b(\varphi) = (\widetilde{v}, \varphi)_{H^1(B_R)} \quad \text{for all } \varphi \in H^1(B_R).$$

Moreover, $\|b\| = \|\tilde{v}\|_{H^1(B_R)}$. It holds also that $A_2 : H^1(B_R) \to H^1(B_R)$ is compact. From the Cauchy-Schwarz inequality and the trace theorem, for all $u \in H^1(B_R)$ we have

$$\begin{aligned}
\|A_2u\|^2_{H^1(B_R)} &= (A_2u, A_2u)_{H^1(B_R)} = |a_2(u, A_2u)| \\
&= |\iint\limits_{B_R\setminus\overline{D}} (1+k^2)u(\overline{A_2u})\,\mathrm{d}x| + |\int\limits_{|x|=R} (\Lambda_k - \Lambda_0)u(\overline{A_2u})\,\mathrm{d}s| \\
&\qquad + |\iint\limits_{D} (1+k^2n)u(\overline{A_2u})\,\mathrm{d}x| \\
&\le (1+\max\{1, \|n\|_{L^\infty(D)}\}k^2)\|u\|_{L^2(B_R)}\|A_2u\|_{L^2(B_R)} \\
&\qquad + \|(\Lambda_k - \Lambda_0)u\|_{H^{-1/2}(\partial D)}\|A_2u\|_{H^{1/2}(\partial D)} \\
&\le C(\|u\|_{L^2(B_R)} + \|(\Lambda_k - \Lambda_0)u\|_{H^{-1/2}(\partial D)})\|A_2u\|_{H^1(B_R)}
\end{aligned}$$

for some $C > 0$. Thus,

$$\|A_2u\|_{H^1(B_R)} \le C(\|u\|_{L^2(B_R)} + \|(\Lambda_k - \Lambda_0)u\|_{H^{-1/2}(\partial D)}).$$

Let $\{u_j\}_{j\in\mathbb{N}} \subset H^1(B_R)$ be such that $\|u_j\|_{H^1(B_R)} \le M$ for all $j \in \mathbb{N}$ and some $M > 0$. By the Rellich's embedding theorem, the embedding $\mathcal{I} : H^1(B_R) \to L^2(B_R)$ is compact. Therefore $\{u_j\}_{j\in\mathbb{N}}$ contains a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ which is strongly convergent in $L^2(D)$. Moreover, since $\Lambda_k - \Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is compact, $\{u_{j_k}\}_{k\in\mathbb{N}}$ contains a subsequence, still denoted by $\{u_{j_k}\}_{k\in\mathbb{N}}$ such that $\{(\Lambda_k - \Lambda_0)u_{j_k}\}_{k\in\mathbb{N}}$ converges strongly in $H^{-1/2}(\partial D)$. Since $\|A_2u\|$ is bounded by $\|u\|_{L^2(B_R)}$ and $\|(\Lambda_k - \Lambda_0)u\|_{H^{-1/2}(\partial D)}$ we conclude that $\{A_2u_{j_k}\}_{k\in\mathbb{N}} \subset H^1(B_R)$ is strongly convergent. That is, A_2 is compact.

The variational formulation (2.44) is equivalent to the problem:

$$\text{Find } u \in H^1(B_R) \text{ such that } A_1u + A_2u = \widetilde{v}, \tag{2.47}$$

where A_1 is bounded and strictly coercive and A_2 is compact. The Riesz-Fredholm theory and the uniqueness result (Theorem 2.2.1) imply that $A_1 + A_2$ is boundedly invertible on $H^1(B_R)$, i.e., the problem (2.43) or, equivalently, the problem (2.29)–(2.33) has a unique solution. The estimate (2.42) follows from (2.46).

□

2.3 Formulation of Inverse Problem. Far Field Operator

We consider the situation when the inhomogeneity D is illuminated by plane waves $u^i(x) = e^{ikx\cdot d}, x \in \mathbb{R}^2$, in all directions $d \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. As already mentioned in the introduction, the scattered field has the following asymptotic behavior:

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(\hat{x}) + \mathcal{O}\bigg(\frac{1}{|x|^{3/2}}\bigg), \quad r \to \infty, \quad \text{as} \quad |x| \to \infty,$$

uniformly in $\hat{x} = x/|x|$. We will also write $u^\infty(\cdot, d)$ to indicate that the far field corresponds to the incident plane wave with the direction of incidence $d \in S^1$.

In the inverse problem our data is given by the far fields $u^\infty(\hat{x}, d)$ for all observation points and all incidence directions $\hat{x}, d \in S^1$. Our goal is to determine the support D of the scatterer.

In fact, not all the measurements of u^∞ are needed due to symmetries in the far fields. Precisely, the following reciprocity relation holds.

Theorem 2.3.1. *Let $u^\infty(\hat{x}, d)$ be a far field pattern corresponding to the scattering problem (2.6)–(2.10) with the observation direction $\hat{x} \in S^1$ direction $d \in S^1$ of the incident plane wave. Then*

$$u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x}) \quad \textit{for all} \quad x, d \in S^1. \tag{2.48}$$

Proof. One can show that (see e.g. Theorem 4.2 in [8])

$$\sqrt{8\pi k} e^{-i\pi/4} \big(u^\infty(\hat{x}, d) - u^\infty(-d, -\hat{x})\big) \tag{2.49}$$
$$= \int_{\partial D} u_+(y, d) \frac{\partial}{\partial \nu} u_+(y, -\hat{x}) - u_+(y, -\hat{x}) \frac{\partial}{\partial \nu} u_+(y, d) \, \mathrm{d}s(y),$$

where $u_+(\cdot, d)$ and $\partial u_+(\cdot, d)/\partial \nu$ are the traces of the total field (again for simplicity of notation we keep writing integrals) corresponding to the incidence direction $d \in S^1$. Then by the boundary conditions (2.3)–(2.4) and the definition of the trace operator in $H^{-1/2}$ (2.14) we get

$$\begin{aligned}
&\sqrt{8\pi k} e^{-i\pi/4}\big(u^\infty(\hat{x}, d) - u^\infty(-d, -\hat{x})\big) \\
&\quad = \int_{\partial D} u_-(y, d)\left(\frac{\partial}{\partial \nu} u_-(y, -\hat{x}) - i\lambda u_-(y, -\hat{x})\right) \\
&\qquad\qquad - u_-(y, -\hat{x})\left(\frac{\partial}{\partial \nu} u_-(y, d) - i\lambda u_-(y, d)\right) \mathrm{d}s(y) \\
&\quad = \int_{\partial D} u_-(y, d)\frac{\partial}{\partial \nu} u_-(y, -\hat{x}) - u_-(y, -\hat{x})\frac{\partial}{\partial \nu} u_-(y, d) \, \mathrm{d}s(y) \\
&\quad = \iint_D u(y, d)(-k^2 n u(y, -\hat{x})) - u(y, -\hat{x})(-k^2 n u(y, d)) \, \mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

□

The reciprocity relation (2.48) is also one of the criteria to verify whether boundary conditions of an approximate model are reasonable (in the same way as in the proof of Theorem 2.3.1 one can see, that the reciprocity relation holds for the full model involving the thin layer).

Next we define the far field operator $F : L^2(S^1) \to L^2(S^1)$:

$$(Fg)(\hat{x}) = \int_{S^1} u^\infty(\hat{x}, d) g(d) \, \mathrm{d}s(d) \qquad \text{for} \quad \hat{x} \in S^1. \tag{2.50}$$

By the superposition principle Fg is the far field corresponding to the scattering problem (2.6)–(2.10) with the incident wave given by

$$v_g(x) = \int_{S^1} e^{ikx\cdot\hat{\theta}} g(\hat{\theta}) \, \mathrm{d}\hat{\theta}, \quad x \in \mathbb{R}^2. \tag{2.51}$$

The function v_g is an entire solution to the Helmholtz equation and it is called the *Herglotz wave function* with kernel g.

Using the reciprocity relation (2.48) it is possible to show (see e.g. Theorem 4.3 in [8]) that the far field operator F has dense range provided it is one-to-one. With respect to injectivity of F we have the following result.

Theorem 2.3.2. *Let $\Gamma \subseteq \partial D$ be relatively open, such that $\lambda \geq \lambda_0 > 0$ a.e. on Γ and $\lambda = 0$ a.e. on $\partial D \backslash \Gamma$ and let n be real valued. Assume that k^2 is not an eigenvalue of the following interior eigenvalue problem*

$$\Delta w + k^2 n w = 0 \quad in \quad D, \;\; \Delta v + k^2 v = 0 \quad in \quad D, \tag{2.52}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial D \tag{2.53}$$

$$w = v \quad on \quad \partial D, \tag{2.54}$$

$$v|_\Gamma = 0 \quad on \quad \Gamma, \tag{2.55}$$

i.e., the only solution $(w, v) \in H^1(D) \times H^1(D)$ of is the trivial one $(w, v) = (0, 0)$. Then the far field operator F is injective.

Proof. Let $g \in L^2(S^1)$ be such that $Fg = 0$ on S^1. By the superposition principle $Fg = u^\infty$, where u^∞ is the far field pattern of u^s satisfying (2.1)-(2.5) with the incident field given by the Herglotz function v_g:

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{2.56}$$

$$\Delta u + k^2 n u = 0 \quad \text{in} \quad D, \tag{2.57}$$

$$u^s - u = -v_g \quad \text{on} \quad \partial D, \tag{2.58}$$

$$\frac{\partial u^s}{\partial \nu} - \frac{\partial u}{\partial \nu} + i\lambda u = -\frac{\partial v_g}{\partial \nu} \quad \text{on} \quad \partial D, \tag{2.59}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|, \tag{2.60}$$

Since $u^\infty = 0$, Rellich's Lemma and the unique continuation principle [8] imply that u^s vanishes in $\mathbb{R}^2 \backslash \overline{D}$. Therefore, the pair $(v, w) := (v_g|_D, u)$ is a solution of the following problem:

$$\Delta w + k^2 n w = 0 \text{ in } D, \tag{2.61}$$

$$\Delta v + k^2 v = 0 \text{ in } D, \tag{2.62}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = i\lambda v \text{ on } \partial D, \tag{2.63}$$

$$w - v = 0 \text{ on } \partial D. \tag{2.64}$$

From (2.61)–(2.64), by the definition of trace operator we conclude

$$i\int\limits_{\Gamma} \lambda |v|^2 \mathrm{d}s = \left\langle \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu}, v \right\rangle$$

$$= \iint\limits_{D} |\nabla w|^2 - k^2 n |w|^2 - |\nabla v|^2 + k^2 |v|^2 \mathrm{d}x. \tag{2.65}$$

Taking the imaginary part of (2.65) yields $v = 0$ a.e. on Γ. Therefore, the problem (2.61)–(2.64) is equivalent to (2.52)–(2.55).

If k^2 is not an eigenvalue of the interior eigenvalue problem then $(w, v) = (0, 0)$ is the only solution of (2.52)–(2.55). In particular, $v_g = 0$ in D and, by analyticity, in all of $\mathbb{R}^2$. This implies (see e.g. [8], Section 3.2) that $g = 0$.

□

Following the lines of the above proof we conclude that for complex-valued n with Im $n > 0$ on an open subset in $D_0 \subset D$ no eigenvalues exist: comparing the imaginary parts on the left and the right hand side of (2.65) would give $w = 0$ on D_0 and, by unique continuation [44], $w = 0$ in D. Then the boundary conditions (2.63)–(2.64) become $v = w = 0$ and $\partial v / \partial \nu = \partial w / \partial \nu = 0$ on ∂D. By the representation formula

$$v(x) = \int\limits_{\partial D} \left(\frac{\partial v}{\partial \nu}(y) \Phi(x, y) - v(y) \frac{\partial \Phi}{\partial \nu(y)}(x, y) \right) \mathrm{d}s(y) = 0 \quad \text{for} \quad x \in D.$$

Remark 2.3.3. *The interior eigenvalues form at most a discrete countable set with infinity as the only accumulation point.*

The case $\lambda = 0$, i.e., the problem (2.52)–(2.54) is well-known, and it has been shown that interior eigenvalues exist and form a discrete set [13], [14]. The situation when $\lambda > 0$ on some open subset $\Gamma \subseteq \partial D$ introduces the additional requirement (2.55) and, to the author's knowledge, it has not been studied yet if for general n and D the interior eigenvalues always exist. Inspired by the problem (2.61)–(2.64) in Chapter 3 we will show existence of interior eigenvalues where the boundary condition (2.63) is replaced by

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \lambda v \text{ on } \partial D,$$

with λ real-valued and positive.

2.4 The Factorization Method

To prove the applicability of the FM we proceed in the following three steps: we

(1) derive a factorization of the far field operator of the form $F = GT^*G^*$;

(2) characterize D by test functions;

(3) establish a link between the test functions and the data operator F.

In this section we put the following assumptions (cf. [41]) on the contrast

$$q := n - 1 \quad \text{in } D.$$

Assumption 2.4.1. *For $q \in L^\infty(D)$ holds $Im\, q \geq 0$, $q \neq 0$ a.e. in D. There exists c_0 with $1 + Re\, q \geq c_0$ a.e. in D. Further, there exists $t_0 \in (0, \pi)$ such that*

$$Re\,(e^{-it_0} q) \geq c|q| \text{ a.e. in } D \tag{2.66}$$

for some $c > 0$. One of the following assumptions is satisfied: Either

$$\text{for all } y \in D \text{ there is } \delta > 0 \text{ such that} \iint_{|x-y|<\delta} \frac{1}{|q(x)|}\,\mathrm{d}x < \infty, \tag{2.67}$$

or

$$\iint_{D_\varepsilon} \frac{1}{|q(x)|}\,\mathrm{d}x < \infty \tag{2.68}$$

where $D_\varepsilon = \{x \in D : \, dist(x, \partial D) < \varepsilon\}$.

Note, if $|q|$ is not bounded from below, then by (2.68) the contrast can vanish on ∂D provided it decays slowly in the neighborhood of ∂D such that $1/|q|$ is integrable.

To study the factorization method for the general case of partially coated obstacles, i.e., to take into consideration that λ might vanish on some part of the boundary, we introduce Sobolev spaces on an open arc. We use the definitions and the notation of Section 8.1 in [8].

Let $\Gamma \subset \partial D$ be an open subset of ∂D. We define the space of restrictions to Γ of functions in $H^{1/2}(\partial D)$ as

$$H^{1/2}(\Gamma) = \{u|_\Gamma : u \in H^{1/2}(\partial D)\}$$

with the norm

$$\|u\|_{H^{1/2}(\Gamma)} := \min\{\|U\|_{H^{1/2}(\partial D)} \text{ for } U \in H^{1/2}(\partial D) \text{ with } U|_\Gamma = u\}.$$

It can be shown (cf. Theorem A4 in [42]) that there exist a bounded extension operator $\tau : H^{1/2}(\Gamma) \to H^{1/2}(\partial D)$, i.e., for any $u \in H^{1/2}(\Gamma)$ there exists an extension $\tau u \in H^{1/2}(\partial D)$ such that

$$\|\tau u\|_{H^{1/2}(\partial D)} \leq C\|u\|_{H^{1/2}(\Gamma)},$$

where $C > 0$ is independent of u. Further, we define

$$\tilde{H}^{1/2}(\Gamma) := \{u \in H^{1/2}(\partial D) : \operatorname{supp} u \subseteq \overline{\Gamma}\},$$

where supp u is the largest relatively closed subset of ∂D such that $u = 0$ a.e. on $\partial D \backslash \operatorname{supp} u$. The space $\tilde{H}^{1/2}(\Gamma)$ can be identified with the trace space of $H^1_0(D, \partial D\backslash\overline{\Gamma})$ where

$$H^1_0(D, \partial D\backslash\overline{\Gamma}) = \{u \in H^1(D) : u|_{\partial D\backslash\overline{\Gamma}} = 0 \text{ in the trace sense}\}.$$

The extension by zero of $u \in \tilde{H}^{1/2}(\Gamma)$ to the whole boundary ∂D is in $H^{1/2}(\partial D)$ and that the associated zero extension operator is bounded. The spaces $\tilde{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ equipped with the restriction of the inner product of $H^{1/2}(\partial D)$ (an introduction of the space $H^{1/2}(\partial D)$ is provided e.g. in Sections 1.4 and 1.5 in [8]) are Hilbert spaces. Let

$$H^{-1/2}(\Gamma) \text{ denote the dual space of } \tilde{H}^{1/2}(\Gamma)$$

and

$$\tilde{H}^{-1/2}(\Gamma) \text{ be the dual space of } H^{1/2}(\Gamma)$$

with respect to the duality pairing defined as follows.

A bounded functional $F \in H^{-1/2}(\Gamma)$ can be seen as the restriction to Γ of some $\tilde{F} \in H^{-1/2}(\partial D)$ in the following sense: let $\tilde{u} \in H^{1/2}(\partial D)$ denote the extension by zero of $u \in \tilde{H}^{1/2}(\Gamma)$, then the restriction F is defined through

$$F(u) = \tilde{F}(\tilde{u}).$$

Thus, we define $H^{-1/2}(\Gamma)$ as

$$H^{-1/2}(\Gamma) := \{v|_\Gamma : v \in H^{-1/2}(\partial D)\}$$

with the dual form

$$\langle v|_\Gamma, u\rangle_{H^{-1/2}(\Gamma)\times\tilde{H}^{1/2}(\Gamma)} = \langle v, \tilde{u}\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)}.$$

We define the support $\operatorname{supp} F$ of a bounded linear functional $F \in H^{-1/2}(\partial D)$ as the largest relatively closed subset of ∂D such that the restriction of F to $\partial D\backslash\operatorname{supp} F$ is zero. With this we identify

$$\tilde{H}^{-1/2}(\Gamma) := \{v \in H^{-1/2}(\partial D) : \operatorname{supp} v \subseteq \overline{\Gamma}\}.$$

Thus, the extension by zero $\tilde{v} \in H^{-1/2}(\partial D)$ of $v \in H^{-1/2}(\Gamma)$ is well-defined. The dual form between $\tilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ is given by

$$\langle v, u|_\Gamma\rangle_{\tilde{H}^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = \langle \tilde{v}, u\rangle_{H^{-1/2}(\partial D),H^{1/2}(\partial D)},$$

where $u \in H^{1/2}(\partial D)$. Note that the embeddings

$$\tilde{H}^{1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow \tilde{H}^{-1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$$

are continuous. Moreover, Rellich's embedding theorem (see Theorem 1.32 in [8]) and Theorem 1.36 in [8] imply the embeddings

$$\tilde{H}^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$$

and

$$H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow \tilde{H}^{-1/2}(\Gamma)$$

are compact.

In the following we assume

$$\lambda \geq \lambda_0 > 0 \quad \text{a.e. on } \Gamma$$

and

$$\lambda = 0 \quad \text{a.e. on } \partial D \backslash \Gamma,$$

where $\Gamma \subset \partial D$ is relatively open in ∂D.

Theorem 2.4.2. *The far field operator F has a factorization of the form*

$$F = \frac{1}{\overline{\gamma}} G T^* G^*.$$

where $\gamma = exp(i\pi/4)/\sqrt{8\pi k}$ and $T : L^2(D) \times L^2(\Gamma) \to L^2(D) \times L^2(\Gamma)$ is given by

$$T \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \frac{\overline{q}\varphi_1}{k^2|q|} - \sqrt{|q|}w|_D \\ -i\lambda(\varphi_2 + w) \end{pmatrix}, \tag{2.69}$$

with $w \in H^1_{loc}(\mathbb{R}^2)$ being the radiating solution of the following problem

$$\Delta w + k^2 w = -\varphi_1 \sqrt{|q|} \quad \text{in } \mathbb{R}^2 \backslash \partial D, \tag{2.70}$$

$$w_+ = w_- \qquad \text{on } \partial D, \tag{2.71}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = i\lambda\varphi_2 \quad \text{on } \Gamma, \qquad \frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = 0 \quad \text{on } \partial D \backslash \Gamma. \tag{2.72}$$

Proof. We rewrite the problem (2.6)–(2.10) in the following way: Let $f \in L^2(D)$ and $h \in L^2(\Gamma)$ be given. Find $u \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{2.73}$$

$$\Delta u + k^2(1+q)u = -k^2 \frac{q}{\sqrt{|q|}} f \quad \text{in} \quad D, \tag{2.74}$$

$$u_+ - u_- = 0 \quad \text{on} \quad \partial D, \tag{2.75}$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\lambda u = -h \ \text{ on } \Gamma, \qquad \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = 0 \ \text{ on } \partial D \backslash \Gamma, \tag{2.76}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad r = |x|. \tag{2.77}$$

Next, we define the data-to-pattern operator $G : L^2(D) \times L^2(\Gamma) \to L^2(S^1)$ by

$$G : \begin{pmatrix} f \\ h \end{pmatrix} \mapsto u^\infty, \tag{2.78}$$

where u^∞ is the far field pattern of the solution to (2.73)–(2.77). By the well-posedness of the direct problem G is well-defined. Further, let $H : L^2(S^1) \to L^2(D) \times L^2(\Gamma)$ be given by $Hg = \begin{pmatrix} H_1 g \\ H_2 g \end{pmatrix}$, where $H_1 : L^2(S^1) \to L^2(D)$ and $H_2 : L^2(S^1) \to L^2(\Gamma)$ are defined as

$$(H_1\psi)(x) = \sqrt{|q(x)|} \int_{S^1} \psi(\theta) e^{ikx\cdot\theta} \, \mathrm{d}s(\theta), \ x \in D, \tag{2.79}$$

and

$$(H_2\varphi)(x) = i\lambda(x) \int_{S^1} \varphi(\theta) e^{ikx\cdot\theta} \, \mathrm{d}s(\theta), \ x \in \Gamma. \tag{2.80}$$

By the superposition principle follows $F = GH$. The adjoint $H^* : L^2(D) \times L^2(\Gamma) \to L^2(S^1)$ of H is given by

$$H^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} (\hat{x}) = \iint_D \varphi_1(y) \sqrt{|q(y)|} e^{-ik\hat{x}\cdot y} \, \mathrm{d}y - i \int_\Gamma \varphi_2(y) \lambda(y) e^{-ik\hat{x}\cdot y} \, \mathrm{d}s(y),$$

for $\hat{x} \in S^1$.

From the asymptotic behavior of the fundamental solution (note that the far field of $\Phi_k(\cdot, y)$ is given by

$$\Phi_k^\infty(\hat{x}, y) = \gamma e^{-ik\hat{x}\cdot y}, \quad \text{for} \quad y \in \mathbb{R}^2 \text{ fixed}),$$

it follows that $\gamma H^*(\varphi_1, \varphi_2)^\top$ is the far field w^∞ of the function w, which is the sum of the volume and the single layer potentials with the densities $\varphi_1 \sqrt{|q|} \in L^2(D)$ and $\tilde{\varphi}_2 \lambda \in L^2(\partial D)$ (where $\tilde{\varphi}_2 \in L^2(\partial D)$ denotes an extension of φ_2):

$$w(x) = \iint_D \varphi_1(y) \sqrt{|q(y)|} \Phi_k(x, y) \mathrm{d}y - i \int_{\partial D} \tilde{\varphi}_2(y) \lambda(y) \Phi_k(x, y) \, \mathrm{d}y,$$

for $x \in \mathbb{R}^2 \backslash \partial D$. By properties of the volume [31] and the single layer potentials [42], $w \in H^1_{loc}(\mathbb{R}^2)$, is radiating and satisfies

$$\Delta w + k^2 w = -\varphi_1 \sqrt{|q|} \quad \text{in } \mathbb{R}^2 \backslash \partial D, \tag{2.81}$$

$$w_+ = w_- \quad \text{on } \partial D, \tag{2.82}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = i\lambda\varphi_2 \quad \text{on } \Gamma, \quad \frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = 0 \quad \text{on } \partial D \backslash \Gamma, \tag{2.83}$$

where (2.81) is understood in L^2-sense and the boundary conditions (2.82)–(2.83) in the sense of traces. It can be shown, analogous to the case of (2.6)–(2.10), that the radiating solution to (2.81)–(2.83) exists, is unique and depends continuously on φ_1 and φ_2.

Writing (2.70)–(2.72) as

$$\Delta w + k^2(1+q)w = -k^2 \frac{q}{\sqrt{|q|}} \left(\frac{\overline{q}}{k^2|q|}\varphi_1 - \sqrt{|q|}w \right) \quad \text{in } \mathbb{R}^2 \backslash \partial D, \tag{2.84}$$

$$w_+ = w_- \quad \text{on } \partial D, \tag{2.85}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} + i\lambda w = i\lambda(\varphi_2 + w) \quad \text{on } \Gamma, \quad \frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = 0 \quad \text{on } \partial D \backslash \Gamma, \tag{2.86}$$

since (2.73)–(2.77) is well-posed, we immediately see that
$\gamma H^*(\varphi_1, \varphi_2)^\top = w^\infty = G\left(\frac{\overline{q}\varphi_1}{k^2|q|} - \sqrt{|q|}w, -i\lambda(\varphi_2 + w)\right)^\top$ for all $(\varphi_1, \varphi_2)^\top \in L^2(D) \times L^2(\Gamma)$.

Then $\gamma H^* = GT$, or $\overline{\gamma}H = T^*G^*$, where $T : L^2(D) \times L^2(\Gamma) \to L^2(D) \times L^2(\Gamma)$ is defined by (2.69). Thus, the far field operator F can be represented as $F = (1/\overline{\gamma})GT^*G^*$.

□

The next theorem provides a link between D and the range of the data-to-pattern operator G.

Theorem 2.4.3. *For any $z \in \mathbb{R}^2$ define ϕ_z by*

$$\phi_z(\hat{x}) := e^{-ik\hat{x}\cdot z}, \; \hat{x} \in S^1. \tag{2.87}$$

Then

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}(G), \tag{2.88}$$

where $G : L^2(D) \times L^2(\Gamma) \to L^2(S^1)$ *is the data-to-pattern operator defined in (2.78).*

Proof. Assume $z \in D$ and let $B[z, \varepsilon]$ be some closed ball centered at z with radius $\varepsilon > 0$ such that $B[z, \varepsilon] \subset D$. We choose a cut-off function $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 1$ for $|t| \geq \varepsilon$ and $\psi(t) = 0$ for $|t| \leq \varepsilon/2$, and define $v \in C^\infty(\mathbb{R}^2)$ by

$$v(x) := \psi(|x - z|)\Phi_k(x, z),\ x \in \mathbb{R}^2.$$

Since v and $\Phi_k(\cdot, z)$ coincide in the exterior of $B[z, \varepsilon]$, Rellich's Lemma implies $v^\infty = \Phi_k^\infty(\cdot, z) = \gamma\phi_z$. Also, v solves (2.73)–(2.77) with

$$f = -\frac{1}{k^2}\frac{\sqrt{|q|}}{q}(\Delta v + k^2(1 + q)v) \text{ in } D$$

(by Assumption 2.4.1 $f \in L^2(D)$) and $h = -i\lambda v|_\Gamma$. Thus, $G(f, h) = \gamma\phi_z$.

Let now $z \notin D$ and assume on the contrary that there exists $(f, h) \in L^2(D) \times L^2(\Gamma)$ such that $G(f, h) = \phi_z$. Let u be the solution of (2.73)–(2.77) determined by f and h, and $u^\infty = G(f, h)$ be its far field pattern. Since ϕ_z is the far field pattern of $\Phi(\cdot, z)/\gamma$, by Rellich's Lemma and analytic continuation we have $u(x) = \Phi(x, z)/\gamma$ for all $x \in \mathbb{R}^2 \backslash (\overline{D} \cup \{z\})$.

But $|\nabla\Phi_k(x, z)| = \frac{k}{4}|H_1^{(1)}(k|x - z|)|$ is in $\mathcal{O}(1/|x - z|)$ as $x \to z$, where $H_1^{(1)}$ is the Hankel function of the first kind of order one. Thus, for any disk $B(z, \varepsilon)$, $\varepsilon > 0$, containing z, we have $\Phi_k(\cdot, z) \notin H^1(B_z)$. This implies (regardless if $z \in \mathbb{R}^2 \backslash \overline{D}$ or $z \in \partial D$) that $\Phi(\cdot, z) \notin H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$. However, $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$. We arrive at a contradiction.

□

A crucial step in proving the applicability of the factorization method is to establish a relation between the range of the (not explicitly known) operator G and the range of on operator which incorporates the given data, that is, the far fields. We will use the most general range identity result, which was first formulated by A. Kirsch [29] and further refined by A. Lechleiter [41].

Theorem 2.4.4. *(Range Identity [41]). Let $X \subset U \subset X^*$ be a Gelfand triple with Hilbert space U and reflexive Banach space X such that the embedding is dense. Furthermore, let V be a second Hilbert space and $F : V \to V$, $H : V \to X$ and $T : X \to X^*$ be linear and bounded operators with*

$$F = H^* T H.$$

We make the following assumptions:

(a) H is compact and injective.

(b) $Re\,T$ has the form $Re\,T = T_0 + T_1$ with some coercive operator T_0 and some compact operator $T_1 : X \to X^*$.

(c) $Im\,T$ is non-negative X, i.e., $\langle Im\,T\phi, \phi\rangle \geq 0$ for all $\phi \in X$.

Further we assume that one of the following conditions is fulfilled

(d) T is injective.

(e) $Im\,T$ is positive on the finite dimensional null space of $Re\,T$, i.e., for all $\phi \neq 0$ such that $Re\,T\phi = 0$ it holds $\langle Im\,T\phi, \phi\rangle > 0$ for all $\phi \in X$.

Then the operator $F_\sharp := |Re\,F| + Im\,F$ is positive definite and the ranges of $H^* : X^* \to V$ and $F_\sharp^{1/2} : V \to V$ coincide.

Remark 2.4.5. *If the imaginary part of the middle operator $T : X \to X^*$ is non-positive, one sets (see Section 2.5.1 in [32])*

$$F_\sharp := |Re\,F| + |Im\,F|.$$

The real and the imaginary parts of an operator F on a Hilbert space are given by

$$\operatorname{Re} F = \frac{1}{2}(F + F^*) \quad \text{and} \quad \operatorname{Im} F = \frac{1}{2i}(F - F^*),$$

respectively. By the spectral theorem, a compact self-adjoint and positive definite operator $A : H \to H$ on a Hilbert space H possess a complete eigensystem $\{\lambda_j, \psi_j\}_{j\in\mathbb{N}}$ with strictly positive eigenvalues λ_j and corresponding normalized eigenfunctions $\psi_j \in H$. With this eigensystem A has the following diagonalization:

$$A\psi = \sum_j \lambda_j (\psi, \psi_j)_H \psi_j, \quad \text{for all} \quad \psi \in H.$$

We define the square root of A as

$$A^{1/2}\psi = \sum_j \sqrt{\lambda_j} (\psi, \psi_j)_H \psi_j, \quad \psi \in H. \tag{2.89}$$

In following we collect properties of G and T and show that the operators appearing in the factorization of F satisfy the assumptions (a)–(d) of the Theorem 2.4.4.

Theorem 2.4.6. *(a) G^* is compact and injective.*

(b) The real part of the middle operator $e^{it_0}T$, with $t_0 \in (0, \pi)$ chosen such that (2.66) is satisfied, has a decomposition of the form $Re\,T = T_0 + T_1$, where T_0 is coercive and T_1 is compact.

(c) For $Im\,T$ holds

$$\left\langle Im\,T \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle \le 0 \text{ for all } \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^2(D)\times L^2(\Gamma),$$

where $\langle\cdot,\cdot\rangle$ is the dual product between $L^2(D)\times L^2(\Gamma)$ and $L^2(D)\times L^2(\Gamma)$.

(d) T is one-to-one.

Proof. (a) By Schauder's theorem, G is compact if and only if its adjoint G^* is. To show the compactness of $G : L^2(D)\times L^2(\Gamma) \to L^2(S^1)$ we follow the arguments of Lemma 1.13 in [32]. Choose a ball B_R centered at the origin with radius $R > 0$ such that $\overline{D} \subset B_R$. We decompose G as $G = G_2G_1$, where $G_1 : L^2(D)\times L^2(\Gamma) \to C(\partial B_R)\times C(\partial B_R)$ is a bounded linear operator given by

$$G_1(f, h) = \big(w|_{\partial B_R}, \frac{\partial w}{\partial \nu}|_{B_R}\big)$$

with $w \in H^1_{loc}(\mathbb{R}^2)$ being the solution to (2.73)–(2.77). Note that w is analytic outside $\overline{D}$. From the Representation Theorem 2.2.3 and the asymptotic behavior of the fundamental solution Φ_k (Section 4.1 in [8]) we have for the far field of w:

$$w^\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial B_R} \left(w(y) \frac{\partial e^{ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial w}{\partial \nu}(y) e^{-ik\hat{x}\cdot y} \right) \mathrm{d}s(y) \tag{2.90}$$

We define $G_2 : C(\partial B_R) \times C(\partial B_R) \to L^2(S^1)$ by the right hand side of (2.90). The kernels in (2.90) are smooth and therefore G_2 is compact. Thus, G is compact as a composition of a bounded and a compact operator.

Now we compute the adjoint of G. Let v be the solution of the boundary value problem defined in (2.6)–(2.10) with the incident field u^i given by $\overline{v_g}$ where v_g is the Herglotz wave function

$$v_g(y) = \int\limits_{S^1} e^{ik\hat{x}\cdot y} g(\hat{x}) \,\mathrm{d}s(\hat{x}), \quad y \in \mathbb{R}^2.$$

Here and in the following, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We now define $w \in H^1_{loc}(\mathbb{R}^2 \backslash \partial D)$ by

$$w = \begin{cases} v + \overline{v_g} & \text{in D} \\ v & \text{in } \mathbb{R}^2 \backslash \overline{D} \end{cases} .$$

Thus, w satisfies

$$\Delta w + k^2 w = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{2.91}$$

$$\Delta w + k^2(1+q) w = 0 \quad \text{in} \quad D, \tag{2.92}$$

$$w_+ - w_- = -\overline{v_g} \quad \text{on} \quad \partial D, \tag{2.93}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} + i\lambda w_- = -\frac{\partial \overline{v_g}}{\partial \nu} \quad \text{on} \quad \partial D, \tag{2.94}$$

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial w}{\partial r} - ikw \right) = 0, \quad r = |x|. \tag{2.95}$$

We claim that the adjoint operator $G^* : L^2(S^1) \to L^2(D) \times L^2(\Gamma)$ is given by

$$G^* g = \begin{pmatrix} k^2 \frac{\overline{q}}{\sqrt{|q|}} \overline{w} \\ \overline{w}_-|_\Gamma \end{pmatrix}. \tag{2.96}$$

Indeed, let $f \in L^2(D)$, $h \in L^2(\Gamma)$ and $g \in L^2(S^1)$ be given and let $u \in H^1_{loc}(\mathbb{R}^2)$ be the solution to (2.73)–(2.77). Then

$$
\begin{aligned}
(G(f,h),g)_{L^2(S^1)} &= \int\limits_{S^1} u^\infty(\hat{x})\overline{g(\hat{x})}\,\mathrm{d}s(\hat{x}) \\
&= \int\limits_{S^1} \left(\int_{\partial D} u_+(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - e^{-ik\hat{x}\cdot y}\frac{\partial u_+(y)}{\partial \nu}\,\mathrm{d}s(y)\right)\overline{g(\hat{x})}\,\mathrm{d}s(d) \\
&= \int\limits_{\partial D} u_+(y)\frac{\partial \overline{v_g(y)}}{\partial \nu} - \overline{v_g(y)}\frac{\partial u_+(y)}{\partial \nu}\,\mathrm{d}s(y) \\
&= \int\limits_{\partial D} u_+(y)\Big(\frac{\partial w_-(y)}{\partial \nu} - \frac{\partial w_+(y)}{\partial \nu} - i\lambda(y)w_-(y)\Big) \\
&\qquad\qquad - \Big(w_-(y) - w_+(y)\Big)\frac{\partial u_+(y)}{\partial \nu}\,\mathrm{d}s(y) \\
&= \int\limits_{\partial D} u_+(y)\Big(\frac{\partial w_-(y)}{\partial \nu} - i\lambda(y)w_-(y)\Big) - w_-(y)\frac{\partial u_+(y)}{\partial \nu}\,\mathrm{d}s(y) \\
&= \int\limits_{\partial D} u_+(y)\Big(\frac{\partial w_-(y)}{\partial \nu} - i\lambda(y)w_-(y)\Big) \\
&\qquad - w_-(y)\left(\frac{\partial u_-(y)}{\partial \nu} - i\lambda(y)u_+(y)\right)\mathrm{d}s(y) + \int\limits_{\Gamma} w_-(y)h(y)\,\mathrm{d}s(y) \\
&= \int\limits_{\partial D} u_-(y)\frac{\partial w_-(y)}{\partial \nu} - w_-(y)\frac{\partial u_-(y)}{\partial \nu}\mathrm{d}s(y) + \int\limits_{\Gamma} w_-(y)h(y)\,\mathrm{d}s(y) \\
&= \iint\limits_{D} u(x)(-k^2(1+q(x))w(x)) + w(x)(k^2(1+q(x))u(x))\,\mathrm{d}x \\
&\qquad\qquad + k^2\iint\limits_{D} w(x)\frac{q}{\sqrt{|q|}}f(x)\,\mathrm{d}x + \int\limits_{\Gamma} w_-(y)h(y)\,\mathrm{d}s(y) \\
&= k^2\iint\limits_{D} w(x)\frac{q}{\sqrt{|q|}}f(x)\,\mathrm{d}x + \int\limits_{\Gamma} w_-(y)h(y)\,\mathrm{d}s(y),
\end{aligned}
$$

For convenience we write boundary integrals instead of dual forms. In the forth and the sixth equality we have used the conductive boundary conditions (2.93)– (2.94) and (2.75)–(2.76) for w and u, respectively. The fifth equality holds because both u and w are radiating solutions. In the eighth equality we have applied the Green's theorem, (2.92) and (2.74). Thus, $G^*g = \left(k^2\overline{w}\frac{\overline{q}}{\sqrt{|q|}}, \overline{w_-}|_\Gamma\right)^\top$ for all $g \in L^2(S^1)$.

We proceed by showing that the adjoint operator G^* is injective. Let $g \in L^2(S^1)$ be such that $G^*g = 0$, i.e., since $|q| \neq 0$ a.e. in D, $(w|_D, w_-|_\Gamma)^\top = (0,0)^\top$. From (2.91)–(2.95) we conclude that w satisfies

$$\Delta w + k^2 w = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D}, \tag{2.97}$$

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D, \tag{2.98}$$

$$w_+ - w_- = -\overline{v_g} \quad \text{on} \quad \partial D, \tag{2.99}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = -\frac{\partial \overline{v_g}}{\partial \nu} \quad \text{on} \quad \partial D, \tag{2.100}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \quad r = |x|. \tag{2.101}$$

As in the Section (2.2) we can show, that (2.97)–(2.101) has at most one solution. It is not hard to see that $w|_D = \overline{v_g}|$ and $w|_{\mathbb{R}^2\backslash\overline{D}} = 0$ solves the problem above. Therefore, $v_g = 0$ in D and, by unique continuation, in $\mathbb{R}^2$. Jacobi-Anger expansion (see e.g. Section 3.2 in [8]) implies $g = 0$.

(b) We decompose T into the sum $T = T_0 + T_1$ with $T_0 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \frac{\overline{q}}{k^2|q|}\varphi_1 \\ -i\lambda\varphi_2 \end{pmatrix}$ and $T_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = -\begin{pmatrix} \sqrt{|q|}w \\ i\lambda w \end{pmatrix}$.

By the well-posedness of (2.70)–(2.72) the mapping $(\varphi_1, \varphi_2)^\top \mapsto w|_D$ from $L^2(D) \times L^2(\Gamma)$ into $H^1(D)$ is bounded, and the trace theorem implies $w|_\Gamma \in H^{1/2}(\Gamma)$. Since the embeddings $H^1(D) \hookrightarrow L^2(D)$ and $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ are compact, it follows that $T_1 : L^2(D) \times L^2(\Gamma) \to L^2(D) \times L^2(\Gamma)$ is compact as well.

Let $\langle\cdot,\cdot\rangle$ denote the dual product between $L^2(D)\times L^2(\Gamma)$ and $L^2(D)\times L^2(\Gamma)$. By the assumption (2.66), we have:

$$\begin{aligned}
&\left\langle \operatorname{Re} e^{it_0} T_0 \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle = \operatorname{Re}\left\langle e^{it_0}\, T_0 \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle \\
&= \frac{1}{k^2}\iint\limits_D \operatorname{Re}\frac{e^{-it_0}q}{|q|}|\varphi_1|^2 \mathrm{d}x + \operatorname{Im}(e^{it_0})\int\limits_\Gamma \lambda|\varphi_2|^2\,\mathrm{d}s \\
&\geq \frac{c}{k^2}\|\varphi_1\|^2_{L^2(D)} + \lambda_0 \operatorname{Im}(e^{it_0})\|\varphi_2\|^2_{L^2(\Gamma)} \\
&\geq \frac{1}{2}\min\left\{\frac{c_0}{k^2}, \lambda_0 \operatorname{Im}(e^{it_0})\right\}\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\|^2_{L^2(D)\times L^2(\Gamma)}.
\end{aligned}$$

(c) Let $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^2(D)\times L^2(\Gamma)$. Then

$$\left\langle T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle = \iint\limits_D \left(\frac{\overline{q}}{k^2|q|}\varphi_1 - \sqrt{|q|}w\right)\overline{\varphi_1}\,\mathrm{d}x - \int\limits_\Gamma i\lambda(\varphi_2 + w)\overline{\varphi_2}\,\mathrm{d}s$$

$$\begin{aligned}
&= \iint\limits_D \frac{1}{k^2}\frac{\overline{q}}{|q|}|\varphi_1|^2 \mathrm{d}x - i\int\limits_\Gamma \lambda|\varphi_2|^2\,\mathrm{d}s \\
&\qquad - \iint\limits_D \sqrt{|q|}w\overline{\varphi_1}\,\mathrm{d}x - i\int\limits_\Gamma \lambda w\overline{\varphi_2}\,\mathrm{d}s,
\end{aligned}$$

We examine the last two terms. From (2.70)–(2.72) and the definition of the trace $\partial w/\partial\nu$ we see:

$$\begin{aligned}
\iint\limits_D |\nabla w|^2 - k^2|w|^2 - \overline{\varphi}_1\sqrt{|q|}w \mathrm{d}x &= \int\limits_{\partial D}\overline{\frac{\partial w_-}{\partial\nu}}w_-\,\mathrm{d}s \\
&= \int\limits_{\partial D}\overline{\frac{\partial w_+}{\partial\nu}}w_+\,\mathrm{d}s + i\int\limits_\Gamma \lambda w\overline{\varphi_2}\,\mathrm{d}s
\end{aligned}$$

(here we again use integrals instead of dual forms). Application of the Green's theorem in $B_R \backslash D$, the radiation condition and the asymptotic behavior of u (that is, $|u(x)| \in \mathcal{O}(1/\sqrt{|x|})$ as $|x| \to \infty$) yield

$$\begin{aligned}
\operatorname{Im} \int\limits_{\partial D} \overline{\frac{\partial w_+}{\partial \nu}} w_+ \,\mathrm{d}s &= -\operatorname{Im} \iint\limits_{B_R \backslash \overline{D}} |\nabla w| - k^2 |w|^2 \,\mathrm{d}x + \operatorname{Im} \int\limits_{|x|=R} \overline{\frac{\partial w}{\partial \nu}} w \,\mathrm{d}s \\
&= \operatorname{Im} \,(-ik \int\limits_{|x|=R} |w|^2 \,\mathrm{d}s) + \operatorname{Im} \int\limits_{|x|=R} \overline{\left(\frac{\partial w}{\partial \nu} - ikw \right)} w \,\mathrm{d}s \\
&\to -2\pi k \int\limits_{S^1} |w^\infty|^2 \,\mathrm{d}s \qquad \text{as } R \to \infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\operatorname{Im} \left\langle T \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\rangle \\
&\qquad = -\iint\limits_{D} \frac{1}{k^2} \frac{\operatorname{Im} q}{|q|} |\varphi_1|^2 \mathrm{d}x - \int\limits_{\Gamma} \lambda |\varphi_2|^2 \mathrm{d}s - 2\pi k \int\limits_{S^1} |w^\infty|^2 \le 0.
\end{aligned}$$

(d) Let $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^2(D) \times L^2(\Gamma)$ such that $T \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then the solution $w \in H^1_{loc}(\mathbb{R}^2)$ of (2.70)–(2.72) becomes

$$\begin{aligned}
&\Delta w + k^2 (1+q) w = 0 \quad \text{in } \mathbb{R}^2 \backslash \partial D, \\
&w_+ = w_- \quad \text{ on } \partial D, \\
&\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} + i\lambda w = 0 \text{ on } \partial D.
\end{aligned}$$

From the uniqueness of the solution to (2.6)–(2.10) we conclude $w = 0$ in all of $\mathbb{R}^2$. Thus, $\frac{k^2 \overline{q}}{|q|} \varphi_1 = 0$ in D and $\lambda \varphi_2 = 0$ on Γ. Since $|q| \neq 0$ a.e. in D and $\lambda \neq 0$ a.e. on Γ we conclude $(\varphi_1, \varphi_2) = (0, 0)$.

□

Now we can state the first main result of this section.

Theorem 2.4.7. *Let the Assumption 2.4.1 hold. For $z \in \mathbb{R}^2$ define $\phi_z \in L^2(S^2)$ by (2.87). Then*

$$z \in D \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|(\phi_z, \psi_j)_{L^2(S^1)}|^2}{\lambda_j} < \infty,$$

where $F_\sharp = |Re\, F| + |Im\, F|$ and (λ_j, ψ_j) is its eigensystem. In other words, the sign of the function

$$W(z) = \left[\sum_{j=1}^{\infty} \frac{|(\phi_z, \psi_j)_{L^2(S^1)}|^2}{\lambda_j} \right]^{-1} \tag{2.102}$$

is the characteristic function of D.

Proof. Theorem 2.4.6 and the range identity (Theorem 2.4.4) yield $\mathcal{R}(F_\sharp^{1/2}) = \mathcal{R}((F_\sharp^*)^{1/2}) = \mathcal{R}(G)$ (note, $F_\sharp$ is selfadjoint). Picard's theorem [31] implies $\psi \in L^2(S^1)$ belongs to $\mathcal{R}(F_\sharp^{1/2})$ if and only if

$$\sum_{j=1}^{\infty} \frac{|(\psi, \psi_j)_{L^2(S^1)}|^2}{\lambda_j} < \infty.$$

Finally, Theorem 2.4.3 completes the proof.

□

2.5 Numerical Results

In this section, we study the applicability of our method through some numerical simulations in $\mathbb{R}^2$.
In the first example the forward data was generated for a kite-shaped object by coupling of the finite element and boundary integral equation method as suggested in [37], [38]. For the numerical treatment of the integral equations we applied the Nystrom method with 128 quadrature points, for the finite element method we used the MATLAB PDE toolbox.

The computed data set is represented by a $\mathbb{C}^{64\times 64}$ matrix F, where each entry is the far field pattern $u^\infty(\theta_j, \theta_l), j, l \in \{1, \dots 64\}$, with $\theta_j = 2\pi j/64$

and $\theta_l = 2\pi l/64$ denoting the corresponding incident direction of the plane wave and the observation point, respectively. Further, we compute the matrix $F_\sharp = |\mathrm{Re}\, F| + |\mathrm{Im}\, F|$ which represents a discretized version of the operator $F_\sharp$. The real and imaginary part of a matrix $A \in \mathbb{C}^{N\times N}$ is given by

$$\mathrm{Re}\,(A) = \frac{A + A^*}{2} \quad \text{and } \mathrm{Im}\,(A) = \frac{A - A^*}{2i},$$

respectively. We define the absolute value of a matrix $A \in \mathbb{C}^{N\times N}$ with a singular value decomposition $A = U\Lambda V^*$ as

$$|A| = U|\Lambda|V^*,$$

with $|\Lambda| = \mathrm{diag}|\lambda_j|, j = 1, ...N$. For our reconstructions we used a grid $\mathcal{G}$ of 200×200 equally spaced sampling points on the rectangle $[-4, 4] \times [-4, 4]$. Let $\{(\sigma_n, \psi_n) : n = 1, ..., 64\}$ represent the eigensystem of the matrix $F_\sharp$. Then the analogous W of the indicator function in (2.102) is given by

$$W(z) := \left[\sum_{j=1}^{64} \frac{|\phi_z^* \psi_n|^2}{|\sigma_n|}\right]^{-1}, \; z \in \mathcal{G},$$

where $\phi_z = (e^{-ik\theta_1 \cdot z}, e^{-ik\theta_2 \cdot z}, \ldots, e^{-ik\theta_{64} \cdot z})^\top \in \mathbb{C}^{64}$. Although, the sum is finite we expect the value of $W(z)$ to be much larger for the points belonging to D then for those lying outside of the domain.

Figure 2.1 (a) shows the real part of total field for a kite-shaped obstacle, corresponding to the plane $u^i(x) = e^{ikx\cdot d}, x \in \mathbb{R}^2$ with $k = 2$, $d = [\cos(\pi/3) \;\; \sin(\pi/3)]^\top$. The scatterer is given by a kite-shaped domain with the boundary ∂D parametrized by $\gamma(t) = (\cos(t) + 0.65\cos(2t) - 0.65), 1.5\sin(t))^\top$, $t \in [0, 2\pi]$. Refractive index is $n(x) = 1 + 10i|\sin(x_1)| + (x_1^2 + x_2^2)$, for $x \in D$, and $\eta(x) = |x_1| + x_2^2$ for $x \in \partial D$.

For the second example to compute the far field for the same objects with the parameters $n(x) = 1.2 + (x_1^2 + x_2^2)$, $k = 3$ and $\eta = 1.5$ with the help of FreeFem++ package [27] with P^1 finite elements. This time we take only 32 incidence and 32 observation directions. Let $F_{32} \in \mathbb{C}^{32\times 32}$ denote the data matrix. Figure 2.2 shows the real part of total field corresponding to the plane with the incidence direction $d = [1 \;\; 0]^\top$ and the reconstruction.

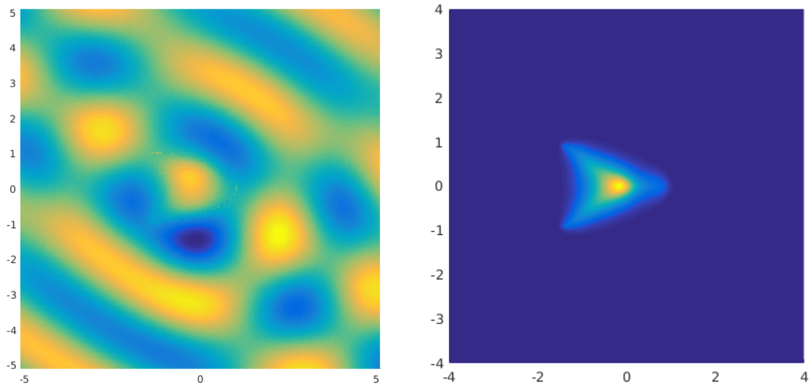

Figure 2.1: *On the left: Real part of the total field of a kite-shaped obstacle corresponding to the incidence direction* $d = [\cos(\pi/3), \sin(\pi/3)]^\top$. *On the right: reconstruction by the* $F_\sharp$ *Method.*

In the next example we keep the parameters n and k the same and compute the far fields for the kite shaped object coated by a highly conductive layer of thickness $\delta = 0.07$ with the conductivity $\sigma^\delta = 1.5/0.07$. Let $F_{32}^\delta \in \mathbb{C}^{32\times 32}$ denote the data matrix. Figure 2.3 shows the total field corresponding to the plane with the incidence direction $d = [1\ 0]^\top$ and the reconstruction.

The relative error of the approximation F_{32}^δ on F_{32} is approximately 10 percent:

$$\frac{\|F_{32} - F_{32}^\delta\|_2}{\|F^\delta_{32\|_2}} \approx 0.1039.$$

Still the FM gives a good reconstruction.

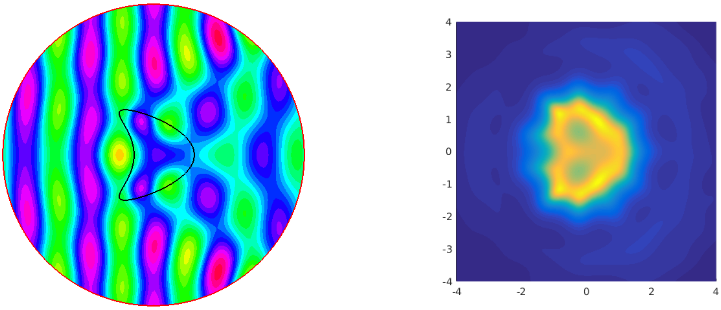

Figure 2.2: *On the left: Real part of the total field of a kite-shaped obstacle corresponding to the incidence direction* $d = [1\ 0]^\top$. *On the right: reconstruction by the* $F_\sharp$ *Method.*

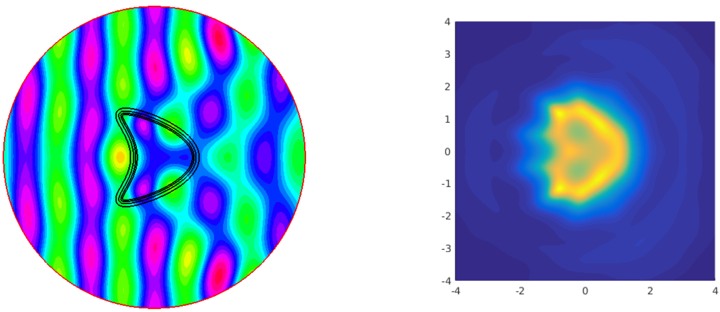

Figure 2.3: *On the left: Real part of the total field of a coated kite-shaped obstacle corresponding to the incidence direction* $d = [1\ 0]^\top$. *On the right: reconstruction by the* $F_\sharp$ *Method.*

3 Interior Eigenvalue Problem

3.1 Introduction

In Chapter 2 we have shown that the support of the scatterer can be determined by the Factorization Method. Having localized the scatterer it is desirable to retrieve information about its material properties. Recent studies on the interior transmission eigenvalues suggest the latter carry additional information about the scatterer. For example, [15] and [22] show that constant and piecewise constant refractive indices, respectively, can be reconstructed with the knowledge of the interior eigenvalues. In [11], [19], and [26] the interior eigenvalues are used to detect cavities (that is, the subregions in the scatterer where the contrast is zero). Furthermore, it has been shown that the interior eigenvalues can be determined from the far field data (see e.g. [10], [26], [35], and [48]). This suggests that the interior eigenvalues can have practical applications in engineering areas such as non-destructive testing.

In the previous chapter, while proving the injectivity of the far field operator F in Theorem 2.3.2 we encountered the following interior problem

$$\Delta w + k^2 n w = 0 \quad \text{in } D, \tag{3.1}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{3.2}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = i\lambda v \quad \text{on } \partial D, \tag{3.3}$$

$$w = v \quad \text{on } \partial D. \tag{3.4}$$

We showed that F is injective if and only if there does not exist a Herglotz wave function v_g such that (w, v_g) is a solution to (3.1)–(3.4) with $v = v_g$. There is at most a discrete set of values of k such that (3.1)–(3.4) has a non-trivial solution (see Remark 2.3.3). We call such k's interior eigenvalues. Due to the presence of an imaginary term in the transmission conditions (3.3) the well-established techniques cannot be used to prove the existence of interior eigenvalues and, to the author's knowledge, (3.1)–(3.4) is an open problem. However, in [18] D. Colton and Y-J. Leung studied (3.1)–(3.4) for the case where D is a unit ball in 3D and n is spherically stratified. In this work the authors showed that complex eigenvalues exist, accumulate on the real axis, and determine uniquely the index of refraction.

In this chapter we study (3.1)–(3.4), where the boundary parameter is real-valued, i.e., we replace (3.3) by

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \eta v \quad \text{on } \partial D,$$

with real-valued η. For this problem, which is rather of academic interest, we show that the interior eigenvalues exist and form a discrete set with $+\infty$ as an accumulation point. Further, we show that the first interior eigenvalue is a monotonic function of the refractive index n and the boundary parameter η. Later we obtain a uniqueness result for constant n and η (this result is thanks to I. Harris, who is one of the co-authors of [3]). Finally, we present some numerical examples which confirm the theory.

3.2 Problem Definition and Variational Formulation

Let $D \subset \mathbb{R}^m$, $m \in \{2, 3\}$, represent a bounded simply connected domain. We define the Sobolev space

$$H_0^1(D) = \big\{u \in L^2(D) : |\nabla u| \in L^2(D) \text{ and } u = 0 \text{ on } \partial D\big\}$$

and

$$\tilde{H}_0^2(D) = \big\{u \in H^2(D) : u \in H^2(D) \cap H_0^1(D)\big\}.$$

Since $\tilde{H}_0^2(D)$ is a subspace of $H^2(D)$ we equip with the $H^2(D)$ norm defined as

$$\|u\|_{H^2(D)} = \sum_{|\alpha| \leq 2} \|D^\alpha u\|_{L^2(D)}, \tag{3.5}$$

$$\alpha := (\alpha_1, \dots, \alpha_m),\ \alpha_j \in \mathbb{N}_0,\ j = 1, \dots, m,\ |\alpha| = \alpha_1 + \cdots + \alpha_m.$$

The interior transmission eigenvalue problem is defined as follows: for given functions $n \in L^\infty(D)$ and $\eta \in L^\infty(\partial D)$ find $k > 0$ and nontrivial $(w, v) \in L^2(D) \times L^2(D)$ such that $w - v \in \tilde{H}_0^2(D)$ and (w, v) satisfies

$$\Delta w + k^2 n w = 0 \text{ in } D, \tag{3.6}$$

$$\Delta v + k^2 v = 0 \text{ in } D, \tag{3.7}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \eta v \text{ on } \partial D, \tag{3.8}$$

$$w - v = 0 \text{ on } \partial D. \tag{3.9}$$

We put the following assumptions on n, η, and ∂D.

Assumption 3.2.1.

1. *The boundary ∂D is of class $\mathcal{C}^2$.*
2. *n is real-valued. It holds either $0 < n_{min} \le n < 1$ or $n > 1$ a.e. in D.*
3. *$\eta \in L^\infty(\partial D)$ is real-valued such that $\eta > 0$ a.e. on ∂D.*

The pair $(w, v) \in L^2(D) \times L^2(D)$ is assumed to satisfy (3.6)–(3.7) in the distributional sense. We now let $u \in \tilde{H}_0^2(D)$ denote the difference $w - v$. Then u satisfies

$$\Delta u + k^2 n u = -k^2 (n - 1) v \ \text{ in } D \tag{3.10}$$

or

$$(\Delta + k^2) \frac{1}{n - 1} (\Delta u + k^2 n u) = 0 \ \text{ in } D \tag{3.11}$$

in the distributional sense.

Since we only require v to be in $L^2(D)$, we need to specify, in which sense the boundary condition (3.8) has to be understood. We first note that by

(3.7) the Laplacian of v is also in $L^2(D)$ (see Lemma 2.1.1). To this end, let

$$L^2_\Delta(D) = \{w \in L^2(D) : \, \Delta w \in L^2(D)\}$$

be equipped with the graph norm $\|w\|_{L^2_\Delta(D)} = \|w\|_{L^2(D)} + \|\Delta w\|_{L^2(D)}$, and let

$$\tilde{H}^{1/2}(\partial D) = \left\{ \varphi \in H^{1/2}(\partial D) : \, \frac{\partial w}{\partial \nu} = \varphi \text{ for } w \in \tilde{H}^2_0(D) \right\}.$$

Further, we denote the dual space of $\tilde{H}^{1/2}(\partial D)$ by $\tilde{H}^{-1/2}(\partial D)$. Since $\tilde{H}^{1/2}(\partial D) \subset H^{1/2}(\partial D)$ we have that $H^{-1/2}(\partial D) \subset \tilde{H}^{-1/2}(\partial D)$. In the following theorem, we show that elements from $L^2_\Delta(D)$ possess a trace in the dual space $\tilde{H}^{-1/2}(\partial D)$.

Theorem 3.2.2. *The mapping $\gamma : u \mapsto u|_{\partial D}$ defined in $C^\infty(\overline{D})$ can be extended to a linear continuous mapping from $L^2_\Delta(D)$ to $\tilde{H}^{-1/2}(\partial D)$.*

Proof. Let $u \in C^\infty(\overline{D})$ and $\phi \in C^\infty(\overline{D})$ such that $\phi = 0$ on ∂D. By the Green's second theorem [16] we have

$$\int_{\partial D} u \frac{\partial \overline{\phi}}{\partial \nu} \, \mathrm{d}s = \int_D u \overline{\Delta \phi} - \overline{\phi} \Delta u \, \mathrm{d}x. \tag{3.12}$$

The Cauchy-Schwarz inequality yields

$$\left| \left\langle u, \frac{\partial \phi}{\partial \nu} \right\rangle \right| \leq C \|u\|_{L^2_\Delta(D)} \|\phi\|_{H^2(D)} \tag{3.13}$$

with some $C > 0$, for all $u \in C^\infty(\overline{D})$ and all $\phi \in C^\infty(\overline{D})$ with $\phi = 0$ on ∂D.

Since $C^\infty_0(\overline{D})$ is dense in $H^1_0(D)$, and, in particular, in $\tilde{H}^2_0(D)$, (3.12) can be extended for $\phi \in \tilde{H}^2_0(D)$:

$$\left\langle u, \frac{\partial \phi}{\partial \nu} \right\rangle = \int_D u \overline{\Delta \phi} - \overline{\phi} \Delta u \, \mathrm{d}x \qquad \text{for all} \qquad u \in C^\infty(\overline{D}), \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\tilde{H}^{-1/2}(\partial D)$ and $\tilde{H}^{1/2}(\partial D)$.

Now, let $f \in \tilde{H}^{1/2}(\partial D)$. Then there exists $\phi \in \tilde{H}^2_0(D)$ with $\frac{\partial \phi}{\partial \nu} = f$ such that (see e.g. Theorem 8.8 in [47]) $\|f\|_{H^{1/2}(\partial D)} \geq c\|\phi\|_{H^2(D)}$ for some $c > 0$. Thus, (3.13) implies that

$$|\langle u, f\rangle| \leq C\|u\|_{L^2_\Delta(D)}\|f\|_{H^{1/2}(\partial D)} \quad \text{for all } \; f \in \tilde{H}^{1/2}(\partial D)$$

and all $u \in C^\infty(\overline{D})$. Therefore, the mapping

$$f \mapsto \langle u, f\rangle$$

defines a continuous linear functional on $\tilde{H}^{1/2}(\partial D)$ and

$$\|u\|_{\tilde{H}^{-1/2}(\partial D)} = \sup_{\substack{f\in \tilde{H}^{1/2}(\partial D),\\ \|f\|_{H^{1/2}(\partial D)}=1}} |\langle u, f\rangle| \leq C\|u\|_{L^2_{\Delta(D)}}.$$

Thus, $\gamma : u \mapsto u|_{\partial D}$ defined on $C^\infty(\overline{D})$ is continuous with respect to the norm of $L^2_\Delta(D)$. Since $C^\infty(\overline{D})$ is dense in $L^2_\Delta(D)$ (see [23] page 54), γ can be extended by continuity to a bounded linear mapping from $L^2_\Delta(D)$ to $\tilde{H}^{-1/2}(\partial D)$.

□

By the previous theorem, equations (3.7) and (3.10) imply that

$$v = -\frac{1}{k^2(n-1)}(\Delta + k^2 n)u \, \in \, \tilde{H}^{-1/2}(\partial D).$$

We write the boundary condition (3.8) as

$$\frac{1}{\eta}\frac{\partial u}{\partial \nu} = -\frac{1}{k^2(n-1)}(\Delta + k^2 n)u \; \text{ on } \partial D. \tag{3.15}$$

Since $\frac{1}{\eta}\frac{\partial u}{\partial \nu} \in L^2(\partial D) \subset \tilde{H}^{-1/2}(\partial D)$, the equality (3.15) is understood in $\tilde{H}^{-1/2}(\partial D)$ sense. Combining (3.11), (3.14), and (3.15) we arrive at a variational formulation of (3.6)–(3.9), which reads as follows: find $u \in \tilde{H}^2_0(D)$ such that

$$\left\langle -\frac{1}{k^2(n-1)}(\Delta u + k^2 n u), \frac{\partial \varphi}{\partial \nu}\right\rangle = \left\langle \frac{1}{\eta}\frac{\partial u}{\partial \nu}, \frac{\partial \varphi}{\partial \nu}\right\rangle$$
$$= \int_D -\frac{1}{k^2(n-1)}(\Delta u + k^2 n u)\Delta\overline{\varphi} - \frac{1}{n-1}(\Delta u + k^2 n u)\overline{\varphi}\,\mathrm{d}x \tag{3.16}$$

for all $\varphi \in \tilde{H}_0^2(D)$. Again, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\tilde{H}^{1/2}(\partial D)$ and $\tilde{H}^{-1/2}(\partial D)$. Taking into account the regularity of u and φ, and multiplying both sides by k^2 the identity (3.16) becomes:

$$\int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} \, \mathrm{d}s + \int_D \frac{1}{n-1} (\Delta u + k^2 n u)(\Delta \overline{\varphi} + k^2 \overline{\varphi}) \, \mathrm{d}x = 0 \qquad (3.17)$$

for all $\varphi \in \tilde{H}_0^2(D)$.

The functions v and w are related to u through

$$v = -\frac{1}{k^2(n-1)}(\Delta u + k^2 n u) \quad \text{and} \quad w = -\frac{1}{k^2(n-1)}(\Delta u + k^2 u).$$

Definition 3.2.3. *Values of $k > 0$ for which the interior eigenvalue problem (3.6)–(3.9) has a nontrivial solution $v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v \in \tilde{H}_0^2(D)$ are called interior eigenvalues. If $k > 0$ is an interior eigenvalue, we call the solution $u \in \tilde{H}_0^2(D)$ of (3.17) the corresponding eigenfunction.*

3.3 Discreteness of the Interior Eigenvalues

In this section, we will prove that the set of interior eigenvalues is at most discrete. To this end, we will write the interior eigenvalue problem as a quadratic eigenvalues problem for k^2. From the variational formulation (3.17) can be written as

$$\mathbb{T}u + k^2 \mathbb{T}_1 u + k^4 \mathbb{T}_2 u = 0, \qquad (3.18)$$

where the operator $\mathbb{T} : \tilde{H}_0^2(D) \mapsto \tilde{H}_0^2(D)$ is the bounded, self-adjoint operator defined by means of the Riesz representation theorem such that

$$(\mathbb{T}u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \, \overline{\Delta \varphi} \, \mathrm{d}x \quad \text{for all} \quad \varphi \in \tilde{H}_0^2(D). \qquad (3.19)$$

By Theorem 8.13 in [21] (note $\partial D \in C^2$), there exists a constant $C > 0$ such that

$$\|u\|^2_{H^2(D)} \le C\Big(\|u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)}\Big) \quad \text{for all } u \in \tilde{H}^2_0(D).$$

Since the trace of u is zero we even have that $\|u\|_{L^2(D)} \le c\|\Delta u\|_{L^2(D)}$ for some $c > 0$. Indeed, by the definition of the trace operator

$$0 = \Big\langle \frac{\partial u}{\partial \nu}, u \Big\rangle_{H^{-1/2}(\partial D), H^{1/2}(\partial D)} = \iint\limits_D |\nabla u|^2 + \Delta u\,\overline{u}\mathrm{d}x$$

for all $u \in \tilde{H}^2_0(D)$. The Cauchy-Schwarz inequality yields

$$\|\nabla u\|^2_{L^2(D)} \le \|\Delta u\|_{L^2(D)}\|u\|_{L^2(D)} \qquad \text{for all } u \in \tilde{H}^2_0(D).$$

By the min-max principle [43]

$$\lambda_1(D) \le \frac{\|\nabla u\|^2_{L^2(D)}}{\|u\|^2_{L^2(D)}} \qquad \text{for all } u \in H^1_0(D),$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Therefore,

$$\lambda_1(D)\|u\|^2_{L^2(D)} \le \|\nabla u\|^2_{L^2(D)} \le \|\Delta u\|_{L^2(D)}\|u\|_{L^2(D)}$$

or

$$\|u\|_{L^2(D)} \le (1/\lambda_1(D))\|\Delta u\|_{L^2(D)}$$

for all $u \in \tilde{H}_0(D)$.

Thus, the operator $\mathbb{T}$, for $n - 1 > 0$, (or $-\mathbb{T}$, for $0 < n < 1$) is coercive on $\tilde{H}_0(D)$ and, by the Lax-Milgram Lemma [40], has a bounded inverse. Next, we define the operator $\mathbb{T}_1 : \tilde{H}^2_0(D) \mapsto \tilde{H}^2_0(D)$ by means of the Riesz representation theorem such that for all $\varphi \in \tilde{H}^2_0(D)$

$$(\mathbb{T}_1 u, \varphi)_{H^2(D)} = -\int\limits_D \frac{1}{n-1}(\overline{\varphi}\,\Delta u + u\,\overline{\Delta\varphi})\,\mathrm{d}x + \int\limits_D \nabla u \cdot \overline{\nabla\varphi}\,\mathrm{d}x$$
$$+ \int\limits_{\partial D} \frac{1}{\eta}\frac{\partial u}{\partial \nu}\overline{\frac{\partial \varphi}{\partial \nu}}\,\mathrm{d}s.$$

The operator $\mathbb{T}_1$ is self-adjoint and compact. Indeed, let us define the auxiliary operator $\mathbb{A} : \tilde{H}_0^2(D) \mapsto \tilde{H}_0^2(D)$ such that

$$(\mathbb{A}u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} u \, \overline{\Delta\varphi} \, \mathrm{d}x$$

and

$$(\mathbb{A}^* u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \overline{\varphi} \, \Delta u \, \mathrm{d}x.$$

It is easy to see that $\|\mathbb{A}u\|_{H^2(D)}$ is bounded by $\|u\|_{L^2(D)}$. By Rellich's embedding theorem, this implies that $\mathbb{A}$, and therefore, by Schauder's Theorem $\mathbb{A}^*$, are compact. The compactness of $\mathbb{T}_1$ follows from the compactness of $\mathbb{A}$ and $\mathbb{A}^*$ along with the fact that $H^{1/2}(\partial D)$ and $H^2(D)$ are compactly embedded in $L^2(\partial D)$ and $H^1(D)$, respectively. At last, we define $\mathbb{T}_2 : \tilde{H}_0^2(D) \mapsto \tilde{H}_0^2(D)$ by means of the Riesz representation theorem such that

$$(\mathbb{T}_2 u, \varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \, \overline{\varphi} \, \mathrm{d}x \quad \text{for all} \quad \varphi \in \tilde{H}_0^2(D).$$

$\mathbb{T}_2$ is compact and self-adjoint.

We are now ready to prove the discreteness of the set of interior eigenvalues.

Theorem 3.3.1. *Assume that $n > 1$ or $0 < n < 1$ a.e. in D and $\eta > 0$ a.e. on ∂D then the set of interior eigenvalues is at most discrete. Moreover, the only accumulation point for the set of interior eigenvalues is $+\infty$.*

Proof. Let $\sigma = 1$ when $n - 1 \geq \alpha > 0$ and $\sigma = -1$ when $1 - n \geq \alpha > 0$. We write (3.18) as

$$u + \sigma k^2 (\sigma \mathbb{T})^{-1} \mathbb{T}_1 u + \sigma k^4 (\sigma \mathbb{T})^{-1} \mathbb{T}_2 u = 0$$

or, equivalently (since $\sigma\mathbb{T}_2$ is a positive self-adjoint operator), as

$$\left(\mathbb{K} - \frac{1}{k^2} \mathbb{I} \right) U = 0 \tag{3.20}$$

with $U = \big(u, k^2(\sigma\mathbb{T}_2)^{1/2}u\big)^\top \in \tilde{H}_0^2(D)\times\tilde{H}_0^2(D)$ and $\mathbb{K} : \tilde{H}_0^2(D)\times\tilde{H}_0^2(D) \to \tilde{H}_0^2(D) \times \tilde{H}_0^2(D)$ given by

$$\mathbb{K} = \begin{pmatrix} \sigma(\sigma\mathbb{T})^{-1}\mathbb{T}_1 & (\sigma\mathbb{T})^{-1}\left(\sigma\mathbb{T}_2\right)^{1/2} \\ -\left(\sigma\mathbb{T}_2\right)^{1/2} & 0 \end{pmatrix}.$$

The square root $\left(\sigma\mathbb{T}_2\right)^{1/2}$ of the compact self-adjoint operator $\sigma\mathbb{T}_2$ is defined by $\left(\sigma\mathbb{T}_2\right)^{1/2} = \displaystyle\int_0^\infty \lambda^{1/2}\mathrm{d}E_\lambda$, where E_λ is the spectral measure associated with $\sigma\mathbb{T}_2$. The operator $\left(\sigma\mathbb{T}_2\right)^{1/2}$ is compact and self-adjoint.

Thus, (3.20) yields that the interior eigenvalues k are the inverse of the eigenvalues for the compact-matrix operator $\mathbb{K}$. Therefore, the interior eigenvalues form at most a discrete set with $+\infty$ as the only accumulation point. Moreover, by the First Riesz Theorem [40] the eigenspaces for each eigenvalue have finite multiplicity.

□

3.4 Existence of the Interior Eigenvalues

We prove the existence of infinitely many interior eigenvalues using the Theorem 2.3 in [13]. We recall this key result in the following lemma.

Lemma 3.4.1. *([13], Theorem 2.3) Let $k \mapsto A_k$ be a continuous mapping from $(0,\infty)$ to the set of self-adjoint positive definite bounded linear operators on the Hilbert space U and assume that B is a self-adjoint non-negative compact linear operator on U. We assume that there exist two positive constants k_0 and k_1 such that*

1. *$A_{k_0} - k_0^2 B$ is positive on U*
2. *$A_{k_1} - k_1^2 B$ is non-positive on a $m-$dimensional subspace of U*

then each of the equations $\lambda_j(k) - k^2 = 0$ for $j = 1, \dots, m$ has at least one solution in $[k_0, k_1]$ where $\lambda_j(k)$ is such that $A_k - \lambda_j(k)B$ has a non-trivial kernel.

Recall the variational formulation of the interior eigenvalue problem (3.17):

$$\int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} \, \mathrm{d}s + \int_D \frac{1}{n-1} (\Delta u + k^2 n u)(\overline{\Delta \varphi} + k^2 \overline{\varphi}) \, \mathrm{d}x = 0, \tag{3.21}$$

for all $\varphi \in \tilde{H}_0^2(D)$. We define the following bounded sesquilinear forms on $\tilde{H}_0^2(D)$:

$$\mathcal{A}_k(u, \varphi) = \int_D \frac{1}{n-1} (\Delta u + k^2 u)(\overline{\Delta \varphi} + k^2 \overline{\varphi}) + k^4 u \overline{\varphi} \, \mathrm{d}x + k^2 \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \overline{\frac{\partial \varphi}{\partial \nu}} \, \mathrm{d}s, \tag{3.22}$$

$$\widetilde{\mathcal{A}}_k(u, \varphi) = \int_D \frac{n}{1-n} (\Delta u + k^2 u)(\Delta \overline{\varphi} + k^2 \overline{\varphi}) + \Delta u \, \Delta \overline{\varphi} \, \mathrm{d}x, \tag{3.23}$$

$$\mathcal{B}(u, \varphi) = \int_D \nabla u \cdot \nabla \overline{\varphi} \, \mathrm{d}x, \quad \text{and} \tag{3.24}$$

$$\widetilde{\mathcal{B}}(u, \varphi) = \int_D \nabla u \cdot \nabla \overline{\varphi} \, \mathrm{d}x + \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} \, \mathrm{d}s. \tag{3.25}$$

Now, we write the interior eigenvalue problem either as

$$\mathcal{A}_k(u, \varphi) - k^2 \mathcal{B}(u, \varphi) = 0 \quad \text{for all} \quad \varphi \in \tilde{H}_0^2(D), \quad \text{for } n > 1, \tag{3.26}$$

or as

$$\widetilde{\mathcal{A}}_k(u, \varphi) - k^2 \widetilde{\mathcal{B}}(u, \varphi) = 0 \quad \text{for all} \quad \varphi \in \tilde{H}_0^2(D), \quad \text{for } 0 < n < 1. \tag{3.27}$$

Using the Riesz representation theorem we can define the bounded linear operators $\mathbb{A}_k$, $\widetilde{\mathbb{A}}_k$, $\mathbb{B}$, and $\widetilde{\mathbb{B}} : \tilde{H}_0^2(D) \mapsto \tilde{H}_0^2(D)$ such that

$$(\mathbb{A}_k u, \varphi)_{H^2(D)} = \mathcal{A}_k(u, \varphi), \quad (\widetilde{\mathbb{A}}_k u, \varphi)_{H^2(D)} = \widetilde{\mathcal{A}}_k(u, \varphi),$$

$$(\mathbb{B} u, \varphi)_{H^2(D)} = \mathcal{B}(u, \varphi) \quad \text{and} \quad (\widetilde{\mathbb{B}} u, \varphi)_{H^2(D)} = \widetilde{\mathcal{B}}(u, \varphi).$$

Since n and η are real valued the sesquilinear forms are Hermitian and therefore the operators are self-adjoint. Due to compact embeddings of $H^2(D)$ into $H^1(D)$ and $H^{1/2}(\partial D)$ into $L^2(\partial D)$ the operators $\mathbb{B}$ and $\tilde{\mathbb{B}}$ are compact. Also since $\eta > 0$ both operators $\mathbb{B}$ and $\tilde{\mathbb{B}}$ are positive (note that the trace of u on ∂D is zero).

For the case when $n > 1$ it has been shown in [14] that

$$\mathcal{A}_k(u,u) \geq C||\Delta u||^2_{L^2(D)} + k^2 \int\limits_{\partial D} \frac{1}{\eta} \left| \frac{\partial u}{\partial \nu} \right|^2 \mathrm{d}s \geq C||\Delta u||^2_{L^2(D)}$$

where $C > 0$ only depends on the refractive index n. Also for $\widetilde{\mathcal{A}}_k$, for the case $0 < n < 1$, we have

$$\widetilde{\mathcal{A}}_k(u,u) = \int\limits_D \frac{n}{1-n} |\Delta u + k^2 u|^2 + |\Delta u|^2 \,\mathrm{d}x \geq ||\Delta u||^2_{L^2(D)}.$$

Therefore, for both $\mathcal{A}_k$ and $\widetilde{\mathcal{A}}_k$ holds

$$\mathcal{A}_k(u,u) \geq C\|u\|_{H^2(D)} \quad \text{and} \quad \widetilde{\mathcal{A}}_k(u,u) \geq c\|u\|_{H^2(D)}$$

for all $k \geq 0$, where the constants C and c are positive and independent of $u \in \tilde{H}^2_0(D)$. In the next theorem we summarize the properties of the operators $\mathbb{A}_k$, $\widetilde{\mathbb{A}}_k$, $\mathbb{B}$, and $\widetilde{\mathbb{B}}$.

Theorem 3.4.2. *Assume that either $n > 1$ or $0 < n < 1$ a.e. in D and that $\eta > 0$ a.e. on ∂D then*

1. *the operators $\mathbb{B}$ and $\widetilde{\mathbb{B}}$ are positive, compact, and self-adjoint.*
2. *the operator $\mathbb{A}_k$ is a coercive self-adjoint operator provided that $n > 1$.*
3. *the operator $\widetilde{\mathbb{A}}_k$ is a coercive self-adjoint operator provided that $0 < n < 1$.*

Therefore, the operators $\mathbb{A}_k - k^2\mathbb{B}$ and $\widetilde{\mathbb{A}}_k - k^2\widetilde{\mathbb{B}}$ satisfy the Fredholm property.

Note that the interior eigenvalues are the solutions to $\lambda_j(k) - k^2 = 0$ where $\lambda_j(k) = \lambda_j(k; n, \eta)$ are the eigenvalues for the generalized eigenvalue problem

$$\mathbb{A}_k u = \lambda_j(k)\mathbb{B}u \quad \text{for} \quad 1 < n \qquad \text{or} \qquad \widetilde{\mathbb{A}}_k u = \lambda_j(k)\widetilde{\mathbb{B}}u \quad \text{for} \quad 0 < n < 1. \tag{3.28}$$

From the above discussion we have that $\mathbb{A}_k$, $\widetilde{\mathbb{A}}_k$, $\mathbb{B}_k$, and $\widetilde{\mathbb{B}}_k$ satisfy the assumptions of Theorem 2.3 of [13]. To prove existence it remains to show that the operators $\mathbb{A}_k - k^2\mathbb{B}$ and $\widetilde{\mathbb{A}}_k - k^2\widetilde{\mathbb{B}}$ are positive for some k_0 and non-positive for some k_1 on a finite dimensional subspace of $\tilde{H}_0^2(D)$.

Theorem 3.4.3. *Assume that either $n > 1$ or $0 < n < 1$ a.e. in D and $\eta > 0$ a.e. on ∂D then for k sufficiently small for all $u \in \tilde{H}_0^2(D)$ there exists $\delta > 0$ such that*

$$\mathcal{A}_k(u,u) - k^2\mathcal{B}(u,u) \geq \delta\|\Delta u\|^2_{L^2(D)}$$

or

$$\widetilde{\mathcal{A}}_k(u,u) - k^2\widetilde{\mathcal{B}}(u,u) \geq \delta\|\Delta u\|^2_{L^2(D)}.$$

Proof. We first consider the case where $0 < n < 1$ and since $\eta > 0$ we have that

$$\begin{aligned}\widetilde{\mathcal{A}}_k(u,u) - k^2\widetilde{\mathcal{B}}(u,u) &\geq ||\Delta u||^2_{L^2(D)} - k^2\left(||\nabla u||^2_{L^2(D)} + \int\limits_{\partial D} \frac{1}{\eta}\left|\frac{\partial u}{\partial \nu}\right|^2 \,\mathrm{d}s\right)\\ &\geq ||\Delta u||^2_{L^2(D)} - k^2\left(||u||^2_{H^2(D)} + \int\limits_{\partial D} \frac{1}{\eta}\left|\frac{\partial u}{\partial \nu}\right|^2 \,\mathrm{d}s\right).\end{aligned}$$

Recall that the for all $u \in \tilde{H}_0^2(D)$ we have that there exists $C_1 > 0$ such that

$$||u||^2_{H^2(D)} \leq C_1||\Delta u||^2_{L^2(D)}.$$

Now let $\inf\limits_{x\in\partial D} \eta = \eta_{min} > 0$, then we have that $\frac{1}{\eta} \leq \frac{1}{\eta_{min}}$ for almost all $x \in \partial D$. Using these estimates yields that

$$\begin{aligned}&\widetilde{\mathcal{A}}_k(u,u) - k^2\widetilde{\mathcal{B}}(u,u)\\ &\quad\geq ||\Delta u||^2_{L^2(D)} - k^2\left(C_1||\Delta u||^2_{L^2(D)} + \frac{1}{\eta_{min}}\|\partial u/\partial \nu\|^2_{L^2(\partial D)}\right).\end{aligned}$$

By the trace theorem we obtain

$$\left\| \frac{\partial u}{\partial \nu} \right\|^2_{L^2(\partial D)} \leq C_2 ||u||^2_{H^2(D)}.$$

Combining this with the previous estimates we conclude

$$\widetilde{\mathcal{A}}_k(u,u) - k^2 \widetilde{\mathcal{B}}(u,u) \geq \left[1 - C_1 k^2 \left(1 + \frac{C_2}{\eta_{min}} \right) \right] ||\Delta u||^2_{L^2(D)}.$$

Since $\eta_{min} > 0$ we have that $\widetilde{\mathcal{A}}_k(u,u) - k^2 \widetilde{\mathcal{B}}(u,u) \geq \delta \|\Delta u\|^2_{L^2(D)}$ for all $k > 0$ sufficiently small.

For $n > 1$, since $\eta > 0$ a.e. on ∂D, we have

$$\begin{aligned}
&\mathcal{A}_k(u,u) - k^2 \mathcal{B}(u,u) \\
&= \int_D \frac{1}{n-1} |\Delta u + k^2 u|^2 + k^4 |u|^2 \, \mathrm{d}x - k^2 \int_D |\nabla u|^2 \, \mathrm{d}x + k^2 \int_{\partial D} \frac{1}{\eta} \left| \frac{\partial u}{\partial \nu} \right|^2 \mathrm{d}s \\
&\qquad \geq C \|\Delta u\|^2_{L^2(D)} - k^2 \|\nabla u\|^2_{L^2(D)} \\
&\qquad \geq C \|\Delta u\|^2_{L^2(D)} - k^2 \|u\|^2_{H^2(D)} \\
&\qquad \geq (C - k^2 C_1) \|\Delta u\|^2_{L^2(D)},
\end{aligned}$$

where again C_1 is the constant such that $||u||^2_{H^2(D)} \leq C_1 ||\Delta u||^2_{L^2(D)}$ for all $u \in \tilde{H}^2_0(D)$ and C is the constant where

$$\int_D \frac{1}{n-1} |\Delta u + k^2 u|^2 + k^4 |u|^2 \, \mathrm{d}x \geq C ||\Delta u||^2_{L^2(D)} \quad \text{for all} \quad u \in \tilde{H}^2_0(D).$$

Hence, for all k^2 sufficiently small we have that $\mathcal{A}_k(u,u) - k^2 \mathcal{B}(u,u) \geq \delta \|\Delta u\|^2_{L^2(D)}$, proving the claim.

□

We are now ready to prove the main result of this chapter.

Theorem 3.4.4. *Assume that either $n > 1$ or $0 < n < 1$ a.e. in D, then there exists infinitely many real interior eigenvalues.*

Proof. We will prove the result for the case of $n > 1$ and the other case is similar. Let $B_j = B(x_j, \varepsilon) := \{x \in \mathbb{R}^m : |x - x_j| < \varepsilon\}$ where $x_j \in D$ and $\varepsilon > 0$. Define $M(\varepsilon)$ as the number of disjoint balls B_j, i.e., $\overline{B_i} \cap \overline{B_j} = \emptyset$, with ε small enough such that $\overline{B_j} \subset D$. It can be shown by using separation of variables [17] that there exists infinitely many transmission eigenvalues to

$$\Delta w_j + k^2 n_{min} w_j = 0 \quad \text{and} \quad \Delta v_j + k^2 v_j = 0 \quad \text{in } B_j, \tag{3.29}$$

$$w_j - v_j = 0 \quad \text{and} \quad \frac{\partial w_j}{\partial \nu} - \frac{\partial v_j}{\partial \nu} = 0 \quad \text{on } \partial B_j. \tag{3.30}$$

where $n_{min} = \inf n(x)$ for $x \in D$. Let u_j denote the difference $u_j = v_j - w_j \in H_0^2(B_j)$ and let $\tilde{u}_j$ be the extension of u_j by zero to D. We note that $\tilde{u}_j \in H_0^2(D) \subset \tilde{H}_0^2(D)$. Since the supports of $\tilde{u}_j$ are disjoint we have that $\tilde{u}_j$ is orthogonal to $\tilde{u}_i$ for all $i \neq j$ in $\tilde{H}_0^2(D)$. This implies that $W_{M(\varepsilon)} = \text{span}\{\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_{M(\varepsilon)}\}$ forms an $M(\varepsilon)$ dimensional subspace of $\tilde{H}_0^2(D)$. Further, for any transmission eigenvalue k of (3.29)–(3.30) we have

$$\begin{aligned} 0 &= \int_D \frac{1}{n_{min} - 1} (\Delta \tilde{u}_j + k^2 \tilde{u}_j)(\Delta \overline{\tilde{u}_j} + k^2 n_{min} \overline{\tilde{u}_j}) \, \mathrm{d}x \\ &= \int_{B_j} \frac{1}{n_{min} - 1} (\Delta \tilde{u}_j + k^2 \tilde{u}_j)(\Delta \overline{\tilde{u}_j} + k^2 n_{min} \overline{\tilde{u}_j}) \, \mathrm{d}x \\ &= \int_{B_j} \frac{1}{n_{min} - 1} |\Delta \tilde{u}_j + k^2 \tilde{u}_j|^2 + k^4 |\tilde{u}_j|^2 \, \mathrm{d}x - k^2 \int_{B_j} |\nabla \tilde{u}_j|^2 \, \mathrm{d}x. \end{aligned}$$

Now, let k_ε be the first transmission eigenvalue of (3.29)–(3.30) in some ball B_j with the eigenfunction u_j. Then, for the extension $\tilde{u}_j$ we have

$$\mathcal{A}_{k_\varepsilon}(\tilde{u}_j, \tilde{u}_j) - k_\varepsilon^2 \mathcal{B}(\tilde{u}_j, \tilde{u}_j) = \int_D \frac{1}{n-1} |\Delta \tilde{u}_j + k_\varepsilon^2 \tilde{u}_j|^2 + k_\varepsilon^4 |\tilde{u}_j|^2 \, dx$$

$$- k_\varepsilon^2 \int_D |\nabla \tilde{u}_j|^2 \, \mathrm{d}x + k_\varepsilon^2 \int_{\partial D} \frac{1}{\eta} \left| \frac{\partial \tilde{u}_j}{\partial \nu} \right|^2 \, \mathrm{d}s$$

$$= \int_D \frac{1}{n-1} |\Delta \tilde{u}_j + k_\varepsilon^2 \tilde{u}_j|^2 + k_\varepsilon^4 |\tilde{u}_j|^2 \, \mathrm{d}x - k_\varepsilon^2 \int_D |\nabla \tilde{u}_j|^2 \, \mathrm{d}x$$

$$\leq \int\limits_{B_j} \frac{1}{n_{min}-1}|\Delta \tilde{u}_j + k_\varepsilon^2 \tilde{u}_j|^2 + k_\varepsilon^4 |\tilde{u}_j|^2 \,\mathrm{d}x - k_\varepsilon^2 \int\limits_{B_j} |\nabla \tilde{u}_j|^2 \,\mathrm{d}x = 0.$$

Thus, for all $u \in W_{M(\varepsilon)}$, we have $\mathcal{A}_{k_\varepsilon}(u,u) - k_\varepsilon^2 \mathcal{B}(u,u) \leq 0$. By Lemma 3.4.1 this gives that there are $M(\varepsilon)$ transmission eigenvalues in the interval $(0, k_\varepsilon]$. Now, note that as $\varepsilon \to 0$ that $M(\varepsilon) \to \infty$. Since the multiplicity of each eigenvalue is finite we conclude that there are infinitely many transmission eigenvalues.

□

From the proof of Theorem 3.4.4 we have the following upper bound on the first transmission eigenvalue of (3.6)–(3.9), which we denote by $k_1(n, \eta, D)$.

Corollary 3.4.5. *Let* $\sup_{x \in D} n(x) = n_{max}$ *and* $\inf_{x \in D} n(x) = n_{min}$*. Let* B_R *be a ball of radius* $R > 0$ *sufficiently small such that* $B_R \subseteq D$*. Then*

1. *if* $n > 1$ *for almost every* $x \in D$*, then*

$$k_1(n, \eta, D) \leq k_1(n_{min}, B_R),$$

where $k_1(n_{min}, B_R)$ *is the first transmission eigenvalue of* (3.29)–(3.30) *for the ball* B_R*.*

2. *if* $0 < n < 1$ *for almost every* $x \in D$*, then*

$$k_1(n, \eta, D) \leq k_1(n_{max}, B_R),$$

where $k_1(n_{max}, B_R)$ *is the first transmission eigenvalue of* (3.29)–(3.30) *for the ball* B_R *with* n_{min} *replaced by* n_{max}*.*

The bound in Corollary 3.4.5 becomes better if B_R is taken to be the largest ball such that $B_R \subseteq D$.

3.5 Monotonicity of the transmission eigenvalues

For this section we turn our attention to proving that the first transmission eigenvalue can be used to determine information about the material parameters n and η. To this end, we will show that the first transmission eigenvalue is a monotonic function with respect to the functions n and η. From the monotonicity we will obtain a uniqueness result for a homogeneous refractive index and homogeneous conductive boundary parameter. Recall that the transmission eigenvalues satisfy

$$\lambda_j(k; n, \eta) - k^2(n, \eta) = 0 \tag{3.31}$$

and the first transmission eigenvalue is the smallest root of (3.31) for $\lambda_1(k; n, \eta)$. Note that $\lambda_1(k; n, \eta)$ satisfies for $u \neq 0$

$$\lambda_1(k; n, \eta) = \min_{u \in \tilde{H}_0^2(D)} \frac{\mathcal{A}_k(u,u)}{\mathcal{B}(u,u)} \quad \text{for} \quad n > 1 \tag{3.32}$$

or

$$\lambda_1(k; n, \eta) = \min_{u \in \tilde{H}_0^2(D)} \frac{\widetilde{\mathcal{A}}_k(u,u)}{\widetilde{\mathcal{B}}(u,u)} \quad \text{for} \quad 0 < n < 1, \tag{3.33}$$

where the sesquilinear forms on $\tilde{H}_0^2(D)$ are defined by (3.22)–(3.25). It is clear that $\lambda_1(k; n, \eta)$ is a continuous function of $k \in (0, \infty)$. Note that the minimizers of (3.32) and (3.33) are the eigenfunctions corresponding to $\lambda_1(k; n, \eta)$. We will denote the first transmission eigenvalue by $k_1(n, \eta)$.

Theorem 3.5.1. *Assume that* $0 < n_1 \leq n_2$ *and* $0 < \eta_1 \leq \eta_2$*, then we have that*

1. *if* $n_1 > 1$*, then* $k_1(n_2, \eta_2) \leq k_1(n_1, \eta_1)$.
2. *if* $n_2 < 1$*, then* $k_1(n_1, \eta_1) \leq k_1(n_2, \eta_2)$.

Moreover, if the inequalities for the parameters n *and* η *are strict, then the first interior eigenvalue is strictly monotone with respect to* n *and* η.

Proof. We start with the case $n > 1$. Let $k_1 = k_1(n_1, \eta_1)$ and $k_2 = k_1(n_2, \eta_2)$. Therefore, for all $u \in \tilde{H}_0^2(D)$ such that $||\nabla u||_{L^2(D)} = 1$ the assumptions $n_1 \leq n_2$ and $\eta_1 \leq \eta_2$ yield

$$\begin{aligned}\lambda_1(k_1; n_2, \eta_2) &\leq \int_D \frac{1}{n_2 - 1} |\Delta u + k_1^2 u|^2 + k_1^4 |u|^2 \, \mathrm{d}x + k_1^2 \int_{\partial D} \frac{1}{\eta_2} \left| \frac{\partial u}{\partial \nu} \right|^2 \mathrm{d}s \\ &\leq \int_D \frac{1}{n_1 - 1} |\Delta u + k_1^2 u|^2 + k_1^4 |u|^2 \, \mathrm{d}x + k_1^2 \int_{\partial D} \frac{1}{\eta_1} \left| \frac{\partial u}{\partial \nu} \right|^2 \mathrm{d}s.\end{aligned}$$

Choose $u = u_1$ where u_1 is the normalized eigenfunction such that $||\nabla u_1||_{L^2(D)} = 1$ corresponding to the eigenvalue k_1. From (3.32) we have

$$\lambda_1(k_1; n_1, \eta_1) = \int\limits_D \frac{1}{n_1 - 1}|\Delta u_1 + k_1^2 u_1|^2 + k_1^4 |u_1|^2 \, \mathrm{d}x$$

$$+ k_1^2 \int\limits_{\partial D} \frac{1}{\eta_1} \left| \frac{\partial u_1}{\partial \nu} \right|^2 \mathrm{d}s,$$

since u_1 is the minimizer of (3.32) for $n = n_1$ and $\eta = \eta_1$. Thus $\lambda_1(k_1; n_2, \eta_2) \leq \lambda_1(k_1; n_1, \eta_1) = k_1^2$, i.e., $\lambda_1(k_1; n_2, \eta_2) - k_1^2 \leq 0$. In Theorem 3.4.3 we have shown that for all k^2 sufficiently small holds $\mathcal{A}_k(u, u) - k^2 \mathcal{B}(u, u) > 0$. This implies that there is a $\delta > 0$ such that for any $k^2 < \delta$ holds $\lambda_1(k; n_2, \eta_2) - k^2 > 0$. By the continuity we have that $\lambda_1(k; n_2, \eta_2) - k^2$ has at least one root in the interval $\left[\sqrt{\delta}, k_1\right]$. Since k_2 is the smallest root of $\lambda_1(k; n_2, \eta_2) - k^2$ we conclude that $k_2 \leq k_1$ proving the claim for this case.

For the case where $n_2 < 1$ we let $k_1 = k_1(n_1, \eta_1)$ and $k_2 = k_1(n_2, \eta_2)$ and the corresponding sesquilinear forms

$$\widetilde{\mathcal{A}}_k(u, \varphi) = \int\limits_D \frac{n}{1 - n}(\Delta u + k^2 u)(\Delta \overline{\varphi} + k^2 \overline{\varphi}) + \Delta u \, \Delta \overline{\varphi} \, \mathrm{d}x,$$

$$\widetilde{\mathcal{B}}(u, \varphi) = \int\limits_D \nabla u \cdot \nabla \overline{\varphi} \, \mathrm{d}x + \int\limits_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} \, \mathrm{d}s.$$

Recall that

$$\lambda_1(k; n_1, \eta_1) = \min_{u \in \tilde{H}_0^2(D)} \frac{\widetilde{\mathcal{A}}_k(u, u)\big|_{n = n_1}}{\widetilde{\mathcal{B}}(u, u)\big|_{\eta = \eta_1}},$$

where we have assumed that $n_1 \leq n_2$ and $\eta_1 \leq \eta_2$. For any value k and for all $u \in \tilde{H}_0^2(D)$ holds

$$\int_D \frac{n_1}{1-n_1}|\Delta u + k^2 u|^2 + |\Delta u|^2 \,\mathrm{d}x \leq \int_D \frac{n_2}{1-n_2}|\Delta u + k^2 u|^2 + |\Delta u|^2 \,\mathrm{d}x,$$

$$\int_D |\nabla u|^2 \,\mathrm{d}x + \int_{\partial D} \frac{1}{\eta_2}\left|\frac{\partial u}{\partial \nu}\right|^2 \,\mathrm{d}s \leq \int_D |\nabla u|^2 \,\mathrm{d}x + \int_{\partial D} \frac{1}{\eta_1}\left|\frac{\partial u}{\partial \nu}\right|^2 \,\mathrm{d}s.$$

Thus $\widetilde{\mathcal{A}}_k(u,u)\big|_{n=n_1} \leq \widetilde{\mathcal{A}}_k(u,u)\big|_{n=n_2}$ and $\widetilde{\mathcal{B}}(u,u)\big|_{\eta=\eta_2} \leq \widetilde{\mathcal{B}}(u,u)\big|_{\eta=\eta_1}$ for all $u \in \tilde{H}_0^2(D)$. Let now $u = u_2$, where u_2 is the eigenfunction corresponding with interior eigenvalue k_2. Then

$$\lambda_1(k_2; n_1, \eta_1) \leq \frac{\widetilde{\mathcal{A}}_{k_2}(u_2,u_2)\big|_{n=n_1}}{\widetilde{\mathcal{B}}(u_2,u_2)\big|_{\eta=\eta_1}} \leq \frac{\widetilde{\mathcal{A}}_{k_2}(u_2,u_2)\big|_{n=n_2}}{\widetilde{\mathcal{B}}(u_2,u_2)\big|_{\eta=\eta_2}} = k_2^2,$$

i.e. $\lambda_1(k_2; n_1, \eta_1) - k_2^2 \leq 0$. Similar arguments as in the previous case yield $k_1 \leq k_2$.

□

By the proof of the previous result we have the following uniqueness result for a homogeneous media and homogeneous boundary parameter η from the strict monotonicity of the first transmission eigenvalue.

Corollary 3.5.2.

1. *If it is known that $n > 1$ or $0 < n < 1$ is a constant and η is known and fixed, then n is uniquely determined by the first transmission eigenvalue.*
2. *If $n > 1$ or $0 < n < 1$ is known and fixed and η is a constant, then the first transmission eigenvalue uniquely determines η.*

It is known (see [13]) that for a every fixed $k \in (0, \infty)$ there exists an increasing sequence $\lambda_j(k; n, \eta)$ of positive generalized eigenvalues of (3.28) that satisfy

$$\lambda_j(k; n, \eta) = \min_{U \in \mathcal{U}_j} \max_{u \in U \setminus \{0\}} \frac{\mathcal{A}_k(u,u)}{\mathcal{B}(u,u)} \quad \text{for} \quad n > 1,$$

or

$$\lambda_j(k;n,\eta) = \min_{U\in\mathcal{U}_j} \max_{u\in U\setminus\{0\}} \frac{\widetilde{\mathcal{A}}_k(u,u)}{\widetilde{\mathcal{B}}(u,u)} \quad \text{for} \quad 0 < n < 1,$$

where $\mathcal{U}_j$ is the set of all j-dimensional subspaces U of $\tilde{H}_0^2(D)$. It is clear from the proof of Theorem 3.5.1 that if k_j is a transmission eigenvalue such that $\lambda_j(k;n,\eta) - k^2 = 0$, then $k_j(n,\eta)$ satisfies the monotonicity properties given in Theorem 3.5.1.

Corollary 3.5.3. *Assume that* $0 < n_1 \leq n_2$ *and* $0 < \eta_1 \leq \eta_2$ *and that* k_j *is a transmission eigenvalue such that* $\lambda_j(k) - k^2 = 0$, *where* $\lambda_j(k)$ *is a positive generalized eigenvalues of* (3.28)*, then we have:*

1. *if* $n_1 > 1$*, then we have that* $k_j(n_2,\eta_2) \leq k_j(n_1,\eta_1)$*.*
2. *if* $n_2 < 1$*, then we have that* $k_j(n_1,\eta_1) \leq k_j(n_2,\eta_2)$*.*

3.6 Numerical Results

In this section we present numerical examples which confirm the monotonicity results of the previous section. All computations are done for the 3D case. These results are thanks to A. Kleefeld, who is a co-author of [3]. The chosen objects for which the interior eigenvalues are computed are represented by a unit sphere centered at the origin, a peanut-shaped object, and a cushion-shaped object (see Figure 3.1). In all cases we assume that the refractive index n and the boundary parameter η are constants.

Numerical calculation of the interior eigenvalues for a sphere of radius $R > 0$ is done with a series expansion. One can show that the interior eigenvalues correspond to the values of k such that the determinant of the following matrix is zero [3]:

$$\begin{pmatrix} -j_p(kR) & j_p(k\sqrt{n}R) \\ -kj_p'(kR) - \eta j_p(kR) & k\sqrt{n}j_p'(k\sqrt{n}R) \end{pmatrix}, \tag{3.34}$$

where $j_p, p \geq 0$, denotes the spherical Bessel function of the first kind of order p.

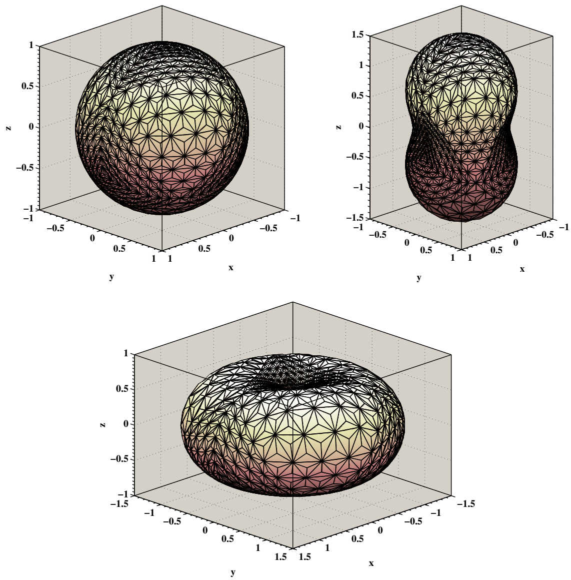

Figure 3.1: Left to right: Unit sphere centered at the origin, peanut-shaped obstacle, and cushion-shaped obstacle (at the bottom).

In Table 3.1, we list the first five interior eigenvalues for a unit sphere using the index of refraction $n = 4$ and various choices of η.

As we can see, for the limiting case $\eta = 0$ the interior eigenvalues are close to the 'classic' eigenvalues 3.141593, 3.692445, 4.261683 (see for example Table 12 in[39]). The limiting case for $\eta \to \infty$ gives the union of the interior Dirichlet eigenvalues for a unit sphere and a sphere of radius two which can easily be seen by considering the limit $\eta \to \infty$ in (3.34). The values are given by the zeros of $j_p(k)$ and $j_p(2k)$, respectively. The first four interior Dirichlet eigenvalues for a unit sphere are 3.141593, 4.493408, 5.236630, and 5.763441 (see also [39, Table 11]). The first four interior Dirichlet eigenvalues for a sphere of radius two are 1.570796, 2.246705, 2.881730, 3.493966.

η	1.	2.	3.	4.	5.
0.01	3.136 675	3.140 531	3.141 593	3.691 542	4.260 901
0.1	3.109 444	3.130 912	3.141 593	3.683 405	4.253 868
0.25	3.059 806	3.114 638	3.141 593	3.669 807	4.242 177
0.5	2.974 096	3.086 914	3.141 593	3.647 091	4.222 806
1	2.798 386	3.029 807	3.141 593	3.601 813	4.184 685
2	2.458 714	2.914 716	3.141 593	3.514 484	4.112 257
3	2.204 525	2.809 294	3.141 593	3.435 429	4.046 733
10	1.743 402	2.467 800	3.138 749	3.141 593	3.779 199
100	1.586 662	2.269 209	2.910 355	3.141 593	3.528 384
1000	1.572 369	2.248 952	2.884 610	3.141 593	3.497 455
10000	1.570 953	2.246 929	2.882 018	3.141 593	3.494 315

Table 3.1: The first five interior transmission eigenvalues for a unit sphere using the index of refraction $n = 4$ and various choices of η.

The interior eigenvalues for the peanut- and cushion-shaped objects are computed numerically from a boundary integral formulation of the interior eigenvalue problem (cf. Cossonnière and Haddar [20]). Table 3.2 lists the first five interior eigenvalues for a peanut-shaped object for $n = 1/2$ and $n = 4$ and different choses of η (recall, in Section 3.5 we distinguish between the cases $n < 1$ and $n > 1$). The peanut-shaped object is parametrically given by the spherical coordinates $x = \varrho \sin(\phi)\cos(\theta)$, $y = \varrho \sin(\phi)\sin(\theta)$, and $z = \varrho \cos(\phi)$ with azimuthal angle $\phi \in [0, \pi]$ and polar angle $\theta \in [0, 2\pi]$.

(n, η)	1.	2.	3.	4.	5.
(1/2,1/2)	1.481 359	1.754 289	2.080 586	2.106 238	2.245 421
(1/2,1)	1.889 608	2.245 548	2.713 844	2.727 860	2.934 707
(1/2,3)	2.482 082	2.947 498	3.640 550	3.695 166	3.997 475
(4,1/2)	2.754 035	2.987 131	3.460 241	3.517 669	3.583 455
(4,1)	2.678 956	2.930 558	3.404 815	3.456 156	3.534 554
(4,3)	2.391 812	2.723 728	3.196 562	3.198 664	3.291 749

Table 3.2: The interior eigenvalues for a peanut-shaped obstacle using the index of refractions $n = 1/2$ and $n = 4$ for $\eta = 1/2$, $\eta = 1$, and $\eta = 3$.

Figure 3.2 shows that for fixed $n = 4$ the first interior monotonically decreases for increasing η.

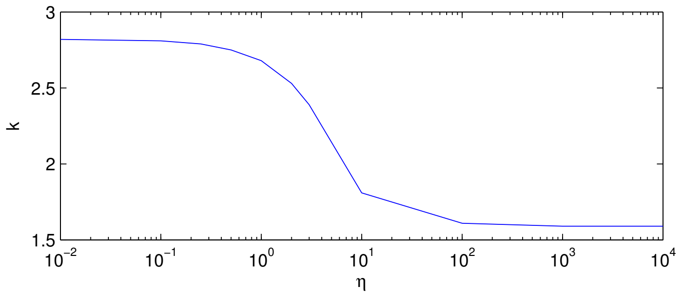

Figure 3.2: The monotonicity of the first interior eigenvalues for the peanut-shaped obstacle using $n = 4$ for increasing η.

As shown in Figure 3.3 for $n = 1/2$ and increasing η the first interior eigenvalue increases as well.

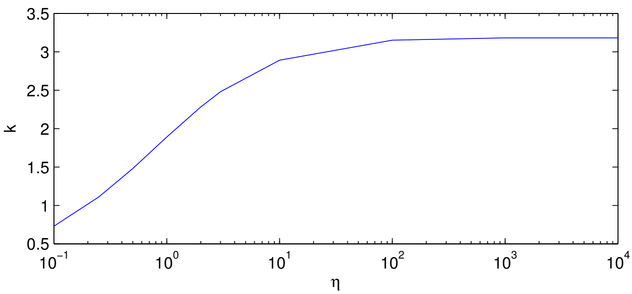

Figure 3.3: The monotonicity of the first interior eigenvalues for the peanut-shaped obstacle using $n = 1/2$ for increasing η.

In Table 3.3 we list the first five interior transmission eigenvalues for a cushion-shaped object that is given parametrically by spherical coordinates with $\varrho = 1 - \cos(2\phi)/2$. We consider the same parameters as in the previous case. As we see, the monotonic behavior of the transmission eigenvalues is the same as in the case with the peanut-shaped object.

(n,η)	1.	2.	3.	4.	5.
(1/2,1/2)	1.359 283	1.694 494	2.012 440	2.087 716	2.110 396
(1/2,1)	1.730 859	2.164 577	2.595 767	2.732 528	2.979 526
(1/2,3)	2.273 696	2.834 967	3.439 393	3.651 267	3.766 782
(4,1/2)	2.863 595	2.878 783	3.144 915	3.159 434	3.469 001
(4,1)	2.762 018	2.818 074	3.087 199	3.099 157	3.431 516
(4,3)	2.384 383	2.611 343	2.841 059	2.945 477	3.305 505

Table 3.3: The interior transmission eigenvalues for a cushion-shaped obstacle using the index of refractions $n = 1/2$ and $n = 4$ for $\eta = 1/2$, $\eta = 1$, and $\eta = 3$.

4 Factorization Method for TE-mode for $\eta = 0$

4.1 Introduction. Problem Definition

In this chapter we consider the case, where the scattering object is not coated, i.e., $\eta = 0$. The problem reads as follows: given an incident field u^i with

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^2$$

find $v \in H^1_{loc}(D)$ and $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ such that

$$\nabla \cdot A \nabla v + k^2 v = 0 \quad \text{in} \quad D, \tag{4.1}$$

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{4.2}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = h \quad \text{on} \quad \partial D, \tag{4.3}$$

$$u - v = f \quad \text{on} \quad \partial D, \tag{4.4}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad r = |x| \tag{4.5}$$

where (4.5) holds uniformly in $\hat{x} = x/|x|$. In (4.4) and (4.3) we set $f = -u^i$ and $h = -\partial u^i / \partial \nu$, respectively. Further, $D \subset \mathbb{R}^2$ is a union of bounded domains with C^2 boundary and connected exterior $\mathbb{R}^2 \backslash \overline{D}$, $A = (a_{ij})$ is a 2×2 matrix defined on D with complex-valued entries $a_{ij} \in L^\infty(D)$ and $k > 0$ is the wave number. As in Section 2.1 the equations (4.1)–(4.2) are understood in distributional sense and (4.3)–(4.4) in the sense of traces. Following the arguments of Lemma 2.1.1 one can show that (4.1) and (4.2) hold in L^2 sense. The trace of the conormal derivative $\partial v / \partial \nu_A$ for $H^1(D)$

functions such that $\nabla \cdot A\nabla v \in L^2(D)$ is in $H^{-1/2}(\partial D)$ and is well defined by

$$\langle \frac{\partial v}{\partial \nu_A}, \varphi \rangle = \iint\limits_D \overline{\nabla \varphi} \cdot A\nabla v + \overline{\varphi}\, \nabla \cdot A\nabla v \, \mathrm{d}x$$

for all $\varphi \in H^1(D)$. Throughout this chapter we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.

In [36] A.Kirsch and X.Liu showed that, under physically relevant assumptions on complex-valued A, the Factorization Method works for this case. However, the proofs in [36] are rather technical such that taking the same approach for the case with transmission conditions (1.4) would result in even more involved arguments. In this chapter we restrict ourselves to the case when A is real-valued and prove the factorization method in a more simple way. In Chapter 5 we follow the same approach to study the FM for the problem with conductive transmission conditions. By the physics of the problem, we require A to satisfy the following assumption.

Assumption 4.1.1. *The matrix-valued function $A : D \to \mathbb{R}^{2\times 2}$ with the entries $(a_{ij}) \in L^\infty(D)$ is symmetric. For almost all $x \in D$ holds*

$$\overline{\xi} \cdot A(x)\xi \geq c|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^2,$$

where c is a positive constant.

The well-posedness of (4.1)–(4.5) was established in [24] for A with entries in $C^1(\overline{D})$ for 3D case. In two dimensions, with the unique continuation result from [1] it is enough to assume that the coefficients in (4.1) are in $L^\infty(D)$.

Theorem 4.1.2. *Let $D \subset \mathbb{R}^2$ and $A : D \to \mathbb{R}^{2\times 2}$ satisfy the assumptions above and let $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$ be given. Let B_R denote a disk of radius $R > 0$ centered at zero such that $\overline{D} \subset B_R$. The transmission problem (4.1)–(4.5) has a unique solution $v \in H^1(D)$ and $u \in (B_R \backslash \overline{D})$ which satisfy*

$$\|v\|_{H^1(D)} + \|u\|_{H^1(B_R\backslash\overline{D})} \leq C_R(\|f\|_{H^{1/2}(\partial D)} + \|h\|_{H^{-1/2}(\partial D)}),$$

with $C_R > 0$ independent of f and h.

4.2 Factorization Method

Throughout this section we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D and not a Dirichlet eigenvalue of $-\nabla\cdot A\nabla$ in D. Next we define the interior Dirichlet-to-Neumann operator $\Lambda_k^- : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$:

$$\Lambda_k^- : f \mapsto \frac{\partial v}{\partial \nu},$$

with v being a solution of the Helmholtz equation in D: $\Delta v + k^2 v = 0$ in D with $v = f$ on ∂D. Further, let $\Lambda_{A,k}^- : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ denote the interior Dirichlet-to-Neumann operator:

$$\Lambda_{A,k}^- : f \mapsto \frac{\partial v}{\partial \nu_A},$$

where v is a solution $\nabla \cdot A\nabla v + k^2 v = 0$ in D with $v = f$ on ∂D.

Under the assumptions on k, the interior Dirichlet problems are well-posed which yields that the operators Λ_k^- and $\Lambda_{A,k}^-$ are well-defined and bounded.

Further, we define the exterior Dirichlet-to-Neumann operator
$\Lambda_k^+ : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$,

$$\Lambda_k^+ : f \mapsto \frac{\partial v}{\partial \nu},$$

where v is the radiating solution of the exterior Dirichlet problem

$$\begin{aligned} \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2 \backslash \overline{D}. \\ v = f \text{ on } \partial D. \end{aligned}$$

We also will refer to this problem in the following equivalent form (the equivalence can be shown with the same reasoning as Lemma 2.2.4)

$$\begin{aligned} \Delta v + k^2 v = 0 \text{ in } B_R \backslash \overline{D}. \\ v = f \text{ on } \partial D, \\ \frac{\partial v}{\partial \nu} = \Lambda_k v \text{ on } |x| = R, \end{aligned}$$

where B_R is a disk of radius $R > 0$ centered at the origin such that $\overline{D} \subset B_R$ and Λ_k is the Dirichlet-to-Neumann operator (2.24).

The exterior Dirichlet problem is well-posed and therefore the operator Λ_k^+ is well-defined and bounded.

The subscript k in the definitions stays for the wave number k. In the following we will also use operators $\Lambda_i^{\pm}$ and $\Lambda_{A,i}^-$, which correspond to the wave number $k = i$.

We define the far field operator $F : L^2(S^1) \to L^2(S^1)$ by

$$Fg(\hat{x}) = \int_{S^1} g(\hat{\theta}) u^{\infty}(\hat{x}, \hat{\theta}) \, \mathrm{d}s(\hat{\theta}), \tag{4.6}$$

where $u^{\infty}(\hat{x}, \hat{\theta})$ is the far field pattern of the solution to (4.1)–(4.5) corresponding to the incidence direction $\hat{\theta} \in S^1$ and the observation direction $\hat{x} \in S^1$. In the next theorem we show that F has a factorization of the form $F = H^* T H$.

Theorem 4.2.1. *The far field operator $F : L^2(S^1) \to L^2(S^1)$ defined by (4.6) has a factorization of the form $F = \gamma H^* T H$, where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $H : L^2(S^1) \to H^{1/2}(\partial D)$ is the Herglotz operator*

$$Hg(x) = \int_{S^1} e^{ikx\cdot d} g(d) \, \mathrm{d}s(d), \quad x \in \partial D$$

and $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is given by

$$Tf = (\Lambda_k^- - \Lambda_{A,k}^-) v_-,$$

where v_- is the trace of the radiating solution $v|_D \in H^1(D)$, $v|_{\mathbb{R}^2 \setminus \overline{D}} \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ to

$$\nabla \cdot A \nabla v + k^2 v = 0 \quad in \quad D, \tag{4.7}$$

$$\Delta v + k^2 v = 0 \quad in \quad \mathbb{R}^2 \setminus \overline{D}, \tag{4.8}$$

$$\frac{\partial v_+}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- f \quad on \quad \partial D, \tag{4.9}$$

$$v_+ - v_- = f \quad on \quad \partial D. \tag{4.10}$$

Proof. We define the data-to-pattern operator $G : H^{1/2}(\partial D) \to L^2(S^1)$ by $Gf = v^\infty$, where v^∞ is the far field pattern of the solution to (4.7)–(4.10).

Let $H : L^2(S^1) \to H^{1/2}(\partial D)$ denote the Herglotz operator

$$Hg(x) = \int_{S^1} e^{ikx\cdot d} g(d)\,\mathrm{d}s(d), \quad x \in \partial D.$$

Since $Hg = v_g|_{\partial D}$ and $\Lambda_k^- Hg = \partial v_g/\partial\nu|_{\partial D}$, where v_g is the Herglotz wave function, by the superposition principle it follows that $F = -GH$.

The adjoint operator $H^* : H^{-1/2}(\partial D) \to L^2(S^1)$ is given by

$$H^*\psi(\hat{x}) = \int_{\partial D} \psi(y) e^{-ik\hat{x}\cdot y} \mathrm{d}s(y), \quad \hat{x} \in S^1, \tag{4.11}$$

where for simplicity of notation we use the integral instead of the dual form.

From the asymptotic behavior of the fundamental solution (2.36) to the Helmholtz equation it follows that $\gamma H^*\psi$ is the far field of the single layer potential

$$(S_L\psi)(x) = \int_{\partial D} \psi(y)\Phi_k(x,y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^2\backslash\partial D.$$

It is well-known [42] that the single layer potential can be continuously extended to the boundary ∂D, i.e.,

$$S_L\psi|_\pm = S\psi \quad \text{on } \partial D,$$

where $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is given by

$$S\psi(x) = \int_{\partial D} \psi(y)\Phi(x,y)\,\mathrm{d}s(y), \quad x \in \partial D.$$

Furthermore, the following jump condition hold on ∂D [42]:

$$\Lambda_k^+ S\psi - \Lambda_k^- S\psi = -\psi.$$

The single layer potential $S_L\psi$ with density $\psi \in H^{-1/2}(\partial D)$, solves the Helmholtz equation in $\mathbb{R}^2\backslash\partial D$ and satisfies the radiation condition. Therefore, we can view γH^* as the data-to-pattern operator $\gamma H^* : \psi \mapsto w^\infty$, where w^∞ is the far field operator of the solution of the following transmission problem:

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D, \tag{4.12}$$

$$\Delta w + k^2 w = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{D}, \tag{4.13}$$

$$\Lambda_k^+ w_+ - \Lambda_k^- w_- = -\psi \quad \text{on} \quad \partial D, \tag{4.14}$$

$$w_+ - w_- = 0 \quad \text{on} \quad \partial D, \tag{4.15}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \quad r = |x|. \tag{4.16}$$

Now, let $f \in H^{1/2}(\partial D)$ be given and let $v^\infty = Gf$ be the far-field pattern of the solution v to (4.7)–(4.10) with the source f. We define $w \in H^1_{loc}(\mathbb{R}^2)$ by

$$w = \begin{cases} \tilde{u} & \text{in } D, \\ v|_{\mathbb{R}^2\backslash\overline{D}} & \text{in } \mathbb{R}^2\backslash\overline{D}, \end{cases}$$

where $\tilde{u} \in H^1(D)$ is the solution to

$$\Delta\tilde{u} + k^2\tilde{u} = 0 \qquad \text{in} \quad D,$$

$$\tilde{u} = v_+ \quad \text{on} \quad \partial D.$$

Then w solves (4.12)–(4.16) with $\psi = -(\Lambda_k^- - \Lambda_k^+)v_+$. Applying Λ_k^- to (4.10) and subtracting (4.9) yields

$$(\Lambda_k^- - \Lambda_k^+)v_+ = (\Lambda_k^- - \Lambda_{A,k}^-)v_-. \tag{4.17}$$

Thus $\psi = -(\Lambda_k^- - \Lambda_{A,k}^-)v_-$.

Since $v = w$ in $\mathbb{R}^2\backslash\overline{D}$, by the Rellich's Lemma, the far fields of v and w coincide, i.e., $Gf = v^\infty = w^\infty = \gamma H^*(\Lambda_k^- - \Lambda_{A,k}^-)v_-$. This holds for all $f \in H^{1/2}(\partial D)$. Thus, $Gf = \gamma H^* T f$ for all $f \in H^{1/2}(\partial D)$ with $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ given by

$$T : f \mapsto (\Lambda_k^- - \Lambda_{A,k}^-)v_-, \tag{4.18}$$

where v_- is the trace of the solution to (4.7)–(4.10).

□

Next we show that the scattering domain D can be characterized by the range of H^*.

Theorem 4.2.2. *For any $z \in \mathbb{R}^2$, let $\phi_z \in L^2(S^1)$ be defined as*

$$\phi_z(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}, \quad \hat{x} \in S^1. \tag{4.19}$$

Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}(H^*)$.

Proof. We first show that for $z \in D$ holds $\phi_z \in \mathcal{R}(H^*)$. Let $f \in H^{1/2}(\partial D)$ be given by $f = \Phi_k(\cdot, z)$ on ∂D, where Φ_k is the fundamental solution (2.36). Let $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ be the radiating solution of the exterior Dirichlet problem

$$\begin{aligned} \Delta u + k^2 u = 0 \quad &\text{in } \mathbb{R}^2 \backslash \overline{D}, \\ u = f \quad &\text{on } \partial D. \end{aligned}$$

The exterior Dirichlet problem is uniquely solvable. Therefore $u = \Phi_k(\cdot, z)$ in $\mathbb{R}^2 \backslash D$. In particular, the far fields u^∞ and $\Phi_k^\infty(\cdot, z)$ coincide, i.e., $u^\infty(\hat{x}) = \Phi_k^\infty(\hat{x}, z) = \phi_z(\hat{x})$ for all $\hat{x} \in S^1$.

We define $w \in H^1_{loc}(\mathbb{R}^2)$ by

$$w = \begin{cases} \tilde{u} & \text{in } \; D, \\ u|_{\mathbb{R}^2 \backslash \overline{D}} & \text{in } \; \mathbb{R}^2 \backslash \overline{D}, \end{cases}$$

where $\tilde{u} \in H^1(D)$ is the solution of the interior Dirichlet problem:

$$\begin{aligned} \Delta \tilde{u} + k^2 \tilde{u} = 0 \quad &\text{in} \quad D, \\ \tilde{u} = f \quad &\text{on} \quad \partial D. \end{aligned}$$

Then w is the radiating solution of the following transmission problem:

$$\begin{aligned} \Delta w + k^2 w = 0 \quad &\text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \\ \Delta w + k^2 w = 0 \quad &\text{in} \quad D, \\ w_+ - w_- = 0 \quad &\text{on} \quad \partial D \\ \Lambda_k^+ w_+ - \Lambda_k^- w_- = -(\Lambda_k^- - \Lambda_k^+) f \quad &\text{on} \quad \partial D, \end{aligned}$$

with the far field pattern $w^\infty = u^\infty = \phi_z$. Since $w^\infty = \gamma H^*(\Lambda_k^- - \Lambda_k^+)f$, we get $\phi_z \in \mathcal{R}(H^*)$.

We prove the other direction by contradiction. Let $z \in \mathbb{R}^2 \backslash D$, and assume there is $\psi \in H^{-1/2}(\partial D)$ such that $H^*\psi = \phi_z$. That is, we assume that the far field of the solution $w \in H^1_{loc}(\mathbb{R}^2)$ to (4.12)–(4.16) with boundary data ψ coincide with the far field of the fundamental solution $\Phi_k(\cdot, z)$. By Rellich's Lemma and the unique continuation principle, w and $\Phi_k(\cdot, z)$ coincide in $\mathbb{R}^2 \backslash (D \cup \{z\})$. But for any disk B_z containing z in its interior, by assumption, $w \in H^1(B_z)$. At the same time, for any disk B_z containing z for the fundamental solution Φ_k we have $\Phi_k(\cdot, z) \notin H^1(B_z)$. We arrive at a contradiction.

□

In the next Lemma we collect some properties for the auxiliary operators Λ_k^-,$\Lambda_{A,k}^-$ and Λ_k^+.

Lemma 4.2.3. *(a) The difference of the operators* $\Lambda_k^- - \Lambda_i^-$, $\Lambda_{A,k}^- - \Lambda_{A,i}^-$ *and* $\Lambda_k^+ - \Lambda_i^+$ *is compact from* $H^{1/2}(\partial D)$ *to* $H^{-1/2}(\partial D)$.

(b) For all $f \in H^{1/2}(\partial D)$ *holds*

$$\langle \Lambda_{A,i}^- f, f\rangle \geq 0, \qquad \langle \Lambda_i^- f, f\rangle \geq 0,$$

and

$$-\langle \Lambda_i^+ f, f\rangle \geq 0.$$

(c) Assume that there is a constant $c > 0$ *such that*

$$\overline{\xi} \cdot (I - A(x))\xi \geq c|\xi|^2 \quad \textit{for all} \quad \xi \in \mathbb{C}^2 \quad \textit{and for almost all} \quad x \in D,$$

where $I : \mathbb{R}^2 \to \mathbb{R}^2$ *is the identity matrix. Then*

$$Re\,\langle(\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle \geq \hat{c}\|f\|^2_{H^{1/2}(\partial D)}$$

where $\hat{c} > 0$ *is a constant. If*

$$\overline{\xi} \cdot (A(x) - I)\xi \geq c|\xi|^2 \quad \textit{for all} \quad \xi \in \mathbb{C}^2 \quad \textit{and for almost all} \quad x \in D.$$

then

$$-Re\,\langle(\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle \geq \hat{c}\|f\|^2_{H^{1/2}(\partial D)}.$$

Proof. (a) We consider the case $\Lambda_k^- - \Lambda_i^- : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$. Let $f \in H^{1/2}(\partial D)$ be given and let $u_k, u_i \in H^1(D)$ denote the solutions of the Helmholtz equation in D with the Dirichlet data given by f and wave numbers k and i, respectively. Then $(\Lambda_k^- - \Lambda_i^-)f = \frac{\partial \tilde{u}}{\partial \nu}$ where $\tilde{u} \in H^1(D)$ solves

$$\Delta \tilde{u} + k^2 \tilde{u} = -(k^2+1)u_i \quad \text{in} \quad D, \qquad \tilde{u} = 0 \quad \text{on} \quad \partial D. \tag{4.20}$$

By an application of the Lax-Milgram Lemma it is easy to show that (4.20) is well posed for the right hand side in $L^2(D)$. The compactness of $\Lambda_k^- - \Lambda_i^-$ follows from the boundedness of the mapping $f \mapsto u_i$ from $H^{1/2}(\partial D)$ into $H^1(D)$, the compact embedding $H^1(D) \hookrightarrow L^2(D)$, the boundedness of $u_i \mapsto \tilde{u}$ from $L^2(D)$ into $H^1(D)$ and the trace theorem. The case $\Lambda_{A,k}^- - \Lambda_{A,i}^-$ is completely analogous.
The compactness of $\Lambda_k^+ - \Lambda_i^+$ can be shown in a similar way. Let $f \in H^{1/2}(\partial D)$ and $k > 0$ be given and let $u_k \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ be the solution to the exterior Dirichlet problem with the wave number k:

$$\begin{aligned} \Delta u_k + k^2 u_k = 0 \quad &\text{in} \quad B_R \backslash \overline{D}, \\ u_k = f \quad &\text{on} \quad \partial D \\ \frac{\partial u_k}{\partial \nu} = \Lambda_k u_k \quad &\text{on} \quad |x| = R. \end{aligned}$$

Further, let $u_i \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ denote the solution to the exterior Dirichlet problem with wave number i:

$$\begin{aligned} \Delta u_i - u_i = 0 \quad &\text{in} \quad B_R \backslash \overline{D}, \\ u_i = f \quad &\text{on} \quad \partial D \\ \frac{\partial u_i}{\partial \nu} = \Lambda_i u_i \quad &\text{on} \quad |x| = R. \end{aligned}$$

Here, Λ_k and Λ_i are the Dirichlet-to-Neumann operators (2.24) and B_R is a disk of radius $R > 0$ centered at zero such that $\overline{D} \subset B_R$. Let $w \in H^1(B_R \backslash \overline{D})$ denote the difference $u_k - u_i$. Then w satisfies

$$\Delta w + k^2 w = -(k^2+1)u_i \quad \text{in} \quad B_R \backslash \overline{D}, \tag{4.21}$$

$$w = 0 \quad \text{on} \quad \partial D \tag{4.22}$$

$$\frac{\partial w}{\partial \nu} = \Lambda_k w + (\Lambda_k - \Lambda_i)u_i \quad \text{on} \quad |x| = R. \tag{4.23}$$

This problem is well posed for arbitrary right-hand-side $L^2(B_R\backslash\overline{D})$ and $H^{-1/2}(\partial B_R)$ in (4.21) and (4.23), respectively. By compact embedding $H^1(B_R\backslash\overline{D}) \hookrightarrow L^2(B_R\backslash\overline{D})$ and, by compactness of $\Lambda_k - \Lambda_i = (\Lambda_k - \Lambda_0) + (\Lambda_0 - \Lambda_i)$ from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ (see Lemma 2.2.2) it follows that $f \mapsto \partial w/\partial\nu$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$, i.e., $\Lambda_k^+ - \Lambda_i^+ : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is compact.

(b) Since A is positive definite and by the definition of the trace operator it follows $\langle \Lambda_{A,i}^- f, f\rangle \geq 0$ for all $f \in H^{1/2}(\partial D)$. The case with Λ_i^- is analogous.
We show the assertion for $-\Lambda_i^+$. Let $f \in H^{1/2}(\partial D)$ and let $u \in H^1(B_R\backslash\overline{D})$ satisfy

$$\begin{aligned} \Delta u - u = 0 \quad &\text{in} \quad \mathbb{R}^2\backslash\overline{D}, \\ u = f \quad &\text{on} \quad \partial D, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} + u\right) = 0, \quad &r = |x|, \end{aligned}$$

uniformly in $\hat{x} = x/|x|$. Note that the radiation condition with $k = i$ implies exponential decay of $|u(x)|$ as $|x| \to \infty$. Then by the Green's Theorem (precisely by (2.15)) we have

$$\begin{aligned} -\langle \Lambda_i^+ f, f\rangle = -\left\langle \frac{\partial u}{\partial\nu}, u\right\rangle &= \iint\limits_{B_R\backslash\overline{D}} |\nabla u|^2 + |u|^2 \,\mathrm{d}x - \int\limits_{|x|=R} \frac{\partial u}{\partial\nu}\overline{u}\,\mathrm{d}s \\ &= \iint\limits_{B_R\backslash\overline{D}} |\nabla u|^2 + |u|^2 \,\mathrm{d}x + \int\limits_{|x|=R} |u|^2 \,\mathrm{d}s + o(1) \quad \text{as} \quad R \to \infty. \end{aligned}$$

Thus, $-\langle \Lambda_i^+ f, f\rangle = \|u\|_{H^1(\mathbb{R}^2\backslash\overline{D})} \geq 0$.

(c) Let $u \in H^1(D)$ denote the solution to

$$\Delta u - u = 0 \quad \text{in} \quad D, \qquad u = f \quad \text{on} \quad \partial D, \tag{4.24}$$

and let $w \in H^1(D)$ be the solution to

$$\nabla \cdot A\nabla w - w = 0 \quad \text{in} \quad D, \qquad w = f \quad \text{on} \quad \partial D. \tag{4.25}$$

We assume first that for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$ holds $\overline{\xi} \cdot (I - A(x))\xi \geq c|\xi|^2$. Thus,

$$\begin{aligned}
\langle(\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle &= \iint_D \overline{\nabla u} \cdot \nabla u + \overline{u}u - \overline{\nabla u} \cdot A\nabla w - \overline{u}w \, \mathrm{d}x \\
&= \iint_D \overline{\nabla u} \cdot (I - A)\nabla u + \overline{\nabla u} \cdot A\nabla u - \overline{\nabla u} \cdot A\nabla w \, \mathrm{d}x \\
&\quad + \iint_D \overline{u}u - 2\overline{u}w + \overline{w}w \, \mathrm{d}x + \iint_D \overline{u}w - \overline{w}w \, \mathrm{d}x \\
&= \iint_D \overline{\nabla u} \cdot (I - A)\nabla u \, \mathrm{d}x + \iint_D \overline{(\nabla u - \nabla w)} \cdot A(\nabla u - \nabla w) \, \mathrm{d}x \\
&\quad - \iint_D \overline{\nabla w} \cdot A\nabla w + \overline{w}w \, \mathrm{d}x + \iint_D \overline{\nabla w} \cdot A\nabla u + \overline{u}w \, \mathrm{d}x \\
&\quad + \iint_D \overline{u}u - 2\overline{u}w + \overline{w}w \, \mathrm{d}x.
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{Re}\langle(\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle &= \iint_D \overline{\nabla u} \cdot (I - A)\nabla u \, \mathrm{d}x \\
&\quad + \iint_D \overline{(\nabla u - \nabla w)} \cdot A(\nabla u - \nabla w) \, \mathrm{d}x + \|u - w\|_{L^2(D)}^2 \\
&\quad - \operatorname{Re} \iint_D \overline{\nabla w} \cdot A\nabla w + \overline{w}w \, \mathrm{d}x + \operatorname{Re} \overline{\iint_D \overline{A\nabla w} \cdot \nabla u + \overline{w}u \, \mathrm{d}x} \\
&\geq c\|\nabla u\|_{L^2(D)}^2 + \operatorname{Re}\left(-\langle\Lambda_{A,i}f, f\rangle + \langle\Lambda_{A,i}f, f\rangle\right) = c\|\nabla u\|_{L^2(D)}^2 \\
&> 0 \quad \text{for all } f \neq 0.
\end{aligned}$$

The last inequality is shown as follows. Assume $\operatorname{Re}\langle(\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle = 0$ and therefore $\nabla u = 0$. Then from (4.24) we have

$$\iint_D \nabla u \cdot \nabla\overline{\varphi} + u\overline{\varphi} \, \mathrm{d}x = \iint_D u\overline{\varphi} \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_0^\infty(\overline{D}).$$

By a density argument $\|u\|_{L^2(D)} = 0$ and consequently $\|u\|_{H^1(D)} = 0$. The trace theorem yields $f = 0$.

To show that there exists $\hat{c} > 0$ such that

$$\langle (\Lambda_i^- - \Lambda_{A,i}^-) f, f \rangle \geq \hat{c} \|f\|^2_{H^{1/2}(\partial D)} \quad \text{for all } f \in H^{1/2}(\partial D)$$

we use a contradiction. Assume that there is no such $\hat{c} > 0$. Then there is a sequence $\{f_j\}_{j\in\mathbb{N}}$ with $\|f_j\|_{H^{1/2}(\partial D)} = 1$ for all $j \in \mathbb{N}$ such that

$$\langle (\Lambda_i^- - \Lambda_{A,i}^-) f_j, f_j \rangle \to 0 \quad \text{as} \quad j \to \infty.$$

This implies that for corresponding sequence of the solutions $\{u_j\}_{j\in\mathbb{N}}$ to (4.24) holds $\|\nabla u_j\|_{L^2(D)} \to 0$ as $j \to \infty$. Moreover, by the well-posedness of (4.24), since $\|f_j\|_{H^{1/2}(\partial D)} = 1$ for all $j \in \mathbb{N}$, there is $M < 0$ such that $\|u_j\|_{H^1(D)} \leq M$ for all $j \in \mathbb{N}$. Further, for all $j \in \mathbb{N}$ holds

$$\iint_D \nabla u_j \overline{\nabla \varphi} + u_j \overline{\varphi} \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_0^\infty(\overline{D}).$$

By compact embedding of $H^1(D)$ into $L^2(D)$ it follows that $\{u_j\}_{j\in\mathbb{N}}$ has a strongly convergent subsequence in $L^2(D)$. We denote the subsequence again by $\{u_j\}_{j\in\mathbb{N}}$ and assume $u_j \to h$ in L^2 sense for some $h \in L^2(D)$. By the continuity of the inner product in L^2 we have

$$\lim_{j\to\infty} \iint_D \nabla u_j \overline{\nabla \varphi} + u_j \overline{\varphi} \, \mathrm{d}x = \iint_D h \overline{\varphi} \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_0^\infty(\overline{D}).$$

A density argument yields $h = 0$. This gives that $\|u_j\|_{H^1(D)} \to 0$ as $j \to \infty$. By the trace theorem, the latter implies $f_j \to 0$ for $j \to \infty$. But by assumption $\|f_j\|_{H^{1/2}(\partial D)} = 0$. We arrive at a contradiction.

Now we assume that there is $c > 0$ such that $\overline{\xi} \cdot (A(x) - I)\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$.
Then

$$\begin{aligned}
\langle (\Lambda_{A,i}^- - \Lambda_i^-)f, f\rangle &= \iint_D \overline{\nabla w} \cdot A\nabla w + |w|^2 - \overline{\nabla u} \cdot \nabla u - |u|^2 \,\mathrm{d}x \\
&= \iint_D \overline{\nabla w} \cdot (A - I)\nabla w + |\nabla w|^2 - 2\overline{\nabla u} \cdot \nabla w + |\nabla u|^2 \,\mathrm{d}x \\
&+ \iint_D |w|^2 - 2\overline{u}w + |u|^2 \,\mathrm{d}x \\
&+ 2\iint_D \overline{\nabla u} \cdot \nabla w + \overline{u}w \,\mathrm{d}x - 2\iint_D \overline{\nabla u} \cdot \nabla u + \overline{u}u \,\mathrm{d}x
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Re}\,\langle (\Lambda_{A,i}^- - \Lambda_i^-)f, f\rangle &= \iint_D \overline{\nabla w} \cdot (A - I)\nabla w \,\mathrm{d}x + \|\nabla w - \nabla u\|_{L^2(D)} \\
&+ \|u - w\|_{L^2(D)} + 2\mathrm{Re}\,(\overline{\langle \Lambda_i^- f, f\rangle} - \langle \Lambda_i^- f, f\rangle) \\
&\geq c\|\nabla w\|_{L^2(D)}.
\end{aligned}$$

By the same argument as in the previous case we conclude that there is $\hat{c} > 0$ such that

$$-\mathrm{Re}\,\langle (\Lambda_i^- - \Lambda_{A,i}^-)f, f\rangle \geq \hat{c}\|f\|^2_{H^{1/2}(\partial D)}$$

for all $f \in H^{1/2}(\partial D)$.

□

In the following we introduce the notion of the interior eigenvalues for the problem (4.1)–(4.5). We call $k > 0$ an *interior eigenvalue* corresponding to (4.1)–(4.5) if there exists a non-trivial solution $(u, v) \in H^1(D) \times H^1(D)$ to

$$\nabla \cdot A\nabla u + k^2 u = 0 \quad \text{in } D, \qquad \Delta v + k^2 v = 0 \quad \text{in D}, \tag{4.26}$$

$$u = v \quad \text{on } \partial D, \qquad \frac{\partial u}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{4.27}$$

With this Lemma we can show that the operators H and T appearing in the factorization of F satisfy the assumption of the Range Identity Theorem 2.4.4.

Theorem 4.2.4. *Assume k^2 is not an interior eigenvalue corresponding to (4.1)–(4.5), not a Dirichlet eigenvalue of $-\Delta$ in D and not a Dirichlet eigenvalue of $-\nabla \cdot A\nabla$. Then*

(a) H is compact and injective.

(b) $Re(-T)$ has the form $Re(-T) = T_0 + T_1$, where T_0 is coercive and T_1 is compact, provided $\overline{\xi} \cdot (I - A(x))\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$. If $\overline{\xi} \cdot (A(x) - I)\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$ then $Re\,T$ has a representation $Re\,T = \tilde{T}_0 + \tilde{T}_1$, with coercive $\tilde{T}_0$ and compact $\tilde{T}_1$. By coercivity we mean that there exists a constant $c > 0$ such that

$$\langle T_0 f, f\rangle \geq c||f||^2_{H^{1/2}(\partial D)} \quad \text{for all } f \in H^{1/2}(\partial D).$$

(c) $\langle Im\ -Tf, f\rangle > 0$ for all $f \in H^{1/2}(\partial D), f \neq 0$.

Proof. (a) The injectivity of the Herglotz operator (note k^2 is not an Dirichlet eigenvalue of $-\Delta$ in D) for three dimensional case is shown e.g. in Theorem 5.21 in [16]. The same arguments apply for the two dimensional case. Further from the regularity of the kernel it is easy to see that $g \mapsto v_g|_{B_R}$, where v_g is the Herglotz wave function (2.51) and B_R is a ball of radius $R > 0$ such that $D \subset B_R$, is a compact mapping from $L^2(S^1)$ into $H^1(B_R)$. From this and the trace theorem we conclude that $H : L^2(S^1) \to H^{1/2}(\partial D)$ is compact as a composition of a bounded and a compact operator.

(b) Consider the case when $\overline{\xi} \cdot (I - A(x))\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$. We will use the following identity

$$(\Lambda_k^- - \Lambda_{A,k}^-)f = (\Lambda_k^+ - \Lambda_{A,k}^-)v_+,$$

which can be derived by applying $\Lambda^-_{A,k}$ to (4.10) and then subtracting (4.9) from it. Let $f \in H^{1/2}(\partial D)$. By self-adjointness of Λ^-_k and $\Lambda^-_{A,k}$ (note A is real-valued) we have

$$\begin{aligned}\langle Tf, f\rangle &= \langle(\Lambda^-_k - \Lambda^-_{A,k})v_-, f\rangle = \langle(\Lambda^-_k - \Lambda^-_{A,k})(v_+ - f), f\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})f, f\rangle + \langle(\Lambda^-_k - \Lambda^-_{A,k})v_+, f\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})f, f\rangle + \langle v_+, (\Lambda^-_k - \Lambda^-_{A,k})f\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})f, f\rangle + \langle v_+, (\Lambda^+_k - \Lambda^-_{A,k})v_+\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})f, f\rangle + \langle(\Lambda^+_k - \Lambda^-_{A,k})^* v_+, v_+\rangle \qquad (4.28)\end{aligned}$$

By well-posedness of the problem (4.7)–(4.10) the mapping $B : f \mapsto v_+$ is bounded from $H^{1/2}(\partial D)$ to $H^{1/2}(\partial D)$. From (4.28) we see that T can be written as $T = (\Lambda^-_k - \Lambda^-_{A,k}) + B^*(\Lambda^+_k - \Lambda^-_{A,k})^* B$, where B^* denotes the adjoint of B. We write now T as a sum $T = T_0 + T_1$ where

$$T_0 = -(\Lambda^-_i - \Lambda^-_{A,i}) + B^*(\Lambda^+_i - \Lambda^-_{A,i})B$$

and

$$T_1 = -(\Lambda^-_k - \Lambda^-_i) + (\Lambda^-_{A,k} - \Lambda^-_{A,i}) + B^*\big((\Lambda^+_k - \Lambda^+_i)^* - (\Lambda^-_{A,k} - \Lambda^-_{A,i}\big)B$$

Lemma 4.2.3 (a) yields that T_1 is compact. Furthermore, by part (b) and (c) of the Lemma we have

$$\begin{aligned}-\mathrm{Re}\,\langle T_0 f, f\rangle &= \mathrm{Re}\,\langle(\Lambda^-_i - \Lambda^-_{A,i})f, f\rangle + \langle(\Lambda^-_{A,i} - \Lambda^+_i)v_+, v_+\rangle \\ &\geq \hat{c}\|f\|^2_{H^{1/2}(\partial D)}.\end{aligned}$$

for some $\hat{c} > 0$. Assume now $\overline{\xi} \cdot (A(x) - I)\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$. For this case we will use the identity (4.17):

$$(\Lambda^-_{A,k} - \Lambda^-_k)v_- = (\Lambda^+_k - \Lambda^-_k)v_+.$$

If we apply Λ^-_k to (4.10) and subtract (4.9) from it we get the equation above.

Let $f \in H^{1/2}(\partial D)$. Then

$$\begin{aligned}\langle Tf, f\rangle &= \langle(\Lambda^-_k - \Lambda^-_{A,k})v_-, v_+ - v_-\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})v_-, v_-\rangle + \langle(\Lambda^-_k - \Lambda^-_{A,k})v_-, v_+\rangle \\ &= -\langle(\Lambda^-_k - \Lambda^-_{A,k})v_-, v_-\rangle + \langle(\Lambda^-_k - \Lambda^+_k)v_+, v_+\rangle \qquad (4.29)\end{aligned}$$

Let $A : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ denote the mapping $f \mapsto v_-$, which is well defined and bounded by well-posedness of (4.7)–(4.10). Then T can be written as $T = \tilde{T}_0 + \tilde{T}_1$ with

$$\tilde{T}_0 = A^*(\Lambda_{A,i}^- - \Lambda_i^-)A + B^*(\Lambda_i^- - \Lambda_i^+)B,$$

and

$$\tilde{T}_1 = A^*((\Lambda_{A,k}^- - \Lambda_{A,i}^-) + (\Lambda_i^- - \Lambda_k^-))A + B^*((\Lambda_k^- - \Lambda_i^-) - (\Lambda_k^+ - \Lambda_i^+))B.$$

Again, Lemma 4.2.3 yields that $\tilde{T}_1$ is compact and

$$\begin{aligned} \operatorname{Re}\langle \tilde{T}_0 f, f\rangle &= \operatorname{Re}\langle (\Lambda_{A,i}^- - \Lambda_i^-)v_-, v_-\rangle + \langle (\Lambda_i^- - \Lambda_-^+)v_+, v_+\rangle \\ &\geq \hat{c}\|v_-\|^2_{H^{1/2}(\partial D)} \end{aligned}$$

for some $\hat{c} > 0$. The boundary conditions (4.9)–(4.10) imply

$$(\Lambda_k^+ - \Lambda_{A,k}^-)v_- = (\Lambda_k^- - \Lambda_k^+)f.$$

The operator $(\Lambda_k^- - \Lambda_k^+) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is an isomorphism, which can be deduced by examining the extension of the single layer potential to ∂D (see Theorem 7.3 in [8]). Thus

$$\begin{aligned} \|f\|_{H^{1/2}(\partial D)} &= \|(\Lambda_k^- - \Lambda_k^+)^{-1}(\Lambda_k^+ - \Lambda_{A,k}^-)v_-\|_{H^{1/2}(\partial D)} \\ &\leq \|(\Lambda_k^- - \Lambda_k^+)^{-1}\|\|(\Lambda_k^+ - \Lambda_{A,k}^-)\|\|v_-\|_{H^{1/2}(\partial D)}, \end{aligned}$$

which gives that $\operatorname{Re}\tilde{T}_0$ is coercive.

(c) Let $f \in H^{1/2}(\partial D)$. Then

$$\begin{aligned} \langle Tf, f\rangle &= \langle (\Lambda_k^- - \Lambda_{A,k}^-)v_-, f\rangle = \langle v_-, \Lambda_k^- f\rangle - \langle \Lambda_{A,k}^- v_-, f\rangle \\ &= \langle v_+, \Lambda_k^- f\rangle - \langle f, \Lambda_k^- f\rangle - \left\langle \frac{\partial v_+}{\partial \nu}, f\right\rangle + \langle f, \Lambda_k^- f\rangle \\ &= \langle v_+, \Lambda_k^- f\rangle - \left\langle \frac{\partial v_+}{\partial \nu}, f\right\rangle. \end{aligned}$$

Thus,

$$-\overline{\langle Tf, f\rangle} = \left\langle f, \frac{\partial v_+}{\partial \nu}\right\rangle - \langle \Lambda_k^- f, v_+\rangle. \tag{4.30}$$

Also,

$$\begin{aligned}\langle Tf,f\rangle &= \langle(\Lambda_k^- - \Lambda_{A,k}^-)v_-,f\rangle = \langle v_-,\Lambda_k^- f\rangle - \langle\Lambda_{A,k}^- v_-,f\rangle\\ &= \langle v_-,\frac{\partial v_+}{\partial\nu}\rangle - \langle v_-,\Lambda_{A,k}^- v_-\rangle - \langle\Lambda_{A,k}^- v_-,v_+\rangle + \langle\Lambda_{A,k}^- v_-,v_-\rangle\\ &= \left\langle v_-,\frac{\partial v_+}{\partial\nu}\right\rangle - \langle\Lambda_{A,k}^- v_-,v_+\rangle. \end{aligned}\tag{4.31}$$

Adding (4.31) to (4.30) yields

$$\begin{aligned}2i\,\mathrm{Im}\,\langle Tf,f\rangle &= \left\langle v_- + f,\frac{\partial v_+}{\partial\nu}\right\rangle - \langle\Lambda_{A,k}^- v_- + \Lambda_k^- f,v_+\rangle\\ &= \left\langle v_+,\frac{\partial v_+}{\partial\nu}\right\rangle - \left\langle\frac{\partial v_+}{\partial\nu},v_+\right\rangle\\ &= -2i\,\mathrm{Im}\left\langle\frac{\partial v_+}{\partial\nu},v_+\right\rangle.\end{aligned}$$

Therefore,

$$\mathrm{Im}\,\langle Tf,f\rangle = -\,\mathrm{Im}\left\langle\frac{\partial v_+}{\partial\nu},v_+\right\rangle.$$

Let B_R be a ball centered at the origin with radius $R > 0$ such that $\overline{D} \subset B_R$. Then by the definition of the trace in $H^{-1/2}(\partial D)$ and the Sommerfeld radiation condition we have

$$\begin{aligned}\mathrm{Im}\left\langle\frac{\partial v_+}{\partial\nu},v_+\right\rangle = &\ \mathrm{Im}\left(-\iint\limits_{B_R\setminus\overline{D}} |\nabla v|^2 - k^2|v|^2\,\mathrm{d}x\right)\\ &\qquad + \mathrm{Im}\left(\int\limits_{|x|=R}\frac{\partial v}{\partial\nu}\overline{v}\,\mathrm{d}s\right)\\ &= \mathrm{Im}\left(ik\int\limits_{|x|=R}|v|^2\mathrm{d}s + o(1)\right)\quad \text{as } R\to\infty.\end{aligned}$$

Thus, Im $\langle(-T)f,f\rangle \geq 0$. Assume there exists $f \in H^{1/2}(\partial D)$ such that Im $\langle(-T)f,f\rangle = 0$. Then $\lim_{R\to\infty}\int_{|x|=R}|u|^2\,\mathrm{d}s = 0$. Rellich's Lemma and the unique continuation principle imply $u = 0$ in $\mathbb{R}^2\setminus\overline{D}$. Thus, $v_+ = 0$ and $\partial v_+/\partial\nu = 0$. Since k^2 is not an interior eigenvalue, i.e., the only solution to (4.26)–(4.27) is the trivial one, we conclude $f = 0$. Thus, $\operatorname{Im}\langle(-T)f, f\rangle > 0$ for all $f \neq 0$.

□

The Range Identity Theorem 2.4.4 and Theorem (4.2.4) yield $\mathcal{R}(F_\sharp^{1/2}) = \mathcal{R}(H^*)$, where $F_\sharp = |\operatorname{Re} F| + |\operatorname{Im} F|$. Applying Theorem 4.2.2 we now can state the main result of this chapter.

Theorem 4.2.5. *$\phi_z \in L^2(S^2)$ by (2.87). Then*

$$z \in D \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|(\phi_z, \psi_j)_{L^2(S^1)}|^2}{\lambda_j} < \infty,$$

where (λ_j, ψ_j) is the eigensystem of $F_\sharp$.

5 Direct and Inverse Problem for TE-mode

5.1 Direct Problem

Now we turn to the model for the coated scatterer. Let $D \subset \mathbb{R}^2$ be a finite union of bounded domains with C^2 boundary such that the exterior $\mathbb{R}^2 \backslash \overline{D}$ is connected. Let η represent the real-valued (scaled) surface conductivity on ∂D and A be a matrix-valued function defined on D. The direct problem reads as follows: given $k > 0$ and an incident field u^i with

$$\Delta u^i + k^2 u^i = 0 \quad \text{in} \quad \mathbb{R}^2$$

find $v \in H^1_{loc}(D)$ and $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ such that

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in } D, \tag{5.1}$$

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = h \qquad \text{on } \partial D, \tag{5.3}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = f \qquad \text{on } \partial D, \tag{5.4}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \qquad r = |x|. \tag{5.5}$$

In (5.4) and (5.3) we assume $f = -u^i$ and $h = -\partial u^i / \partial \nu$, respectively. We assume $A \in L^\infty(\overline{D}, \mathbb{C}^{2\times 2})$ and denote by $\operatorname{Re} A$ and $\operatorname{Im} A$ the matrices with the real and the imaginary parts of the entries of A, respectively. By the physics of the problem it holds that $\operatorname{Re} A$ and $\operatorname{Im} A$ are symmetric, and $\operatorname{Re}(\overline{\xi} \cdot A(x)\xi) \geq c|\xi|^2$ and $\operatorname{Im}(\overline{\xi} \cdot A(x)\xi) \leq 0$ for all $\xi \in \mathbb{C}^2$ and for almost all $x \in D$, where c is a positive constant. Due to the symmetry of A it follows $\mathrm{Im}(\overline{\xi} \cdot A\xi) = \overline{\xi} \cdot \mathrm{Im}(A)\xi$ and $\mathrm{Re}(\overline{\xi} \cdot A\xi) = \overline{\xi} \cdot \mathrm{Re}(A)\xi$. Further we allow η to have discontinuities and assume $\eta \in L^\infty(\partial D)$ with $\eta \geq \eta_0 > 0$ a.e. on ∂D.

We understand the equations (5.1) and (5.2) in the distributional sense and the boundary conditions (5.3) and (5.4) in the sense of the trace operator. Regularity theory for elliptic differential equations [21] implies u is analytic in $\mathbb{R}^2 \backslash \overline{D}$ and therefore the radiation condition (5.5) makes sense.

In the following we show that the problem (5.1)–(5.5) is well posed. The uniqueness result shown in Lemma 3.1 in [9] can be extended for A with $L^\infty(D)$ coefficients by the unique continuation principle stated in [1]. To prove the existence we again follow the approach of [24]. First we formulate (5.1)–(5.5) in a bounded domain.

Let $R > 0$ be big enough such that $\overline{D} \subset B_R$, where B_R is a disc of radius $R > 0$ centered at zero. Then (5.1)–(5.5) is equivalent (the justification for the equivalence is the same as in Lemma 2.2.4) to the following problem in B_R:

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in} \quad D, \tag{5.6}$$

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{5.7}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = h \qquad \text{on} \quad \partial D, \tag{5.8}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = f \qquad \text{on} \quad \partial D, \tag{5.9}$$

$$\frac{\partial u}{\partial \nu} = \Lambda_k u \quad \text{on} \quad \partial B_R, \tag{5.10}$$

with $h \in H^{-1/2}(\partial D)$ and $f \in H^{1/2}(\partial D)$, and $\Lambda_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ being the Dirichlet-to-Neumann mapping (2.24).

We also will consider (5.6)–(5.10) in the following equivalent variational form: find $w \in H^1(B_R \backslash \partial D)$ such that

$$\iint\limits_{D} \overline{\nabla\varphi} \cdot A\nabla w - k^2 \overline{\varphi} w \, \mathrm{d}x + \iint\limits_{B_R\backslash D} \overline{\nabla\varphi} \cdot \nabla w - k^2 \overline{\varphi} w \, \mathrm{d}x - \int\limits_{\partial D} \frac{i}{\eta} [\overline{\varphi}][w] \, \mathrm{d}s$$
$$- \langle \Lambda_k w, \varphi \rangle = -i \int\limits_{\partial D} \frac{1}{\eta} f [\overline{\varphi}] \, \mathrm{d}s - \langle h, \varphi_+ \rangle. \tag{5.11}$$

for all $\varphi \in H^1(B_R \backslash \partial D)$, where $[\varphi]$ and $[w]$ denote the jumps $\varphi_+ - \varphi_-$ or $w_+ - w_-$, respectively, across ∂D. Here and in the following, we denote by $\langle \cdot , \cdot \rangle$ the dual form in the dual system $\langle H^{-1/2}(\partial U), H^{1/2}(\partial U)\rangle$ with $U = D$ or $U = B_R$, depending on the context.

One readily sees that, if v and u solve (5.6)–(5.10) then $w|_D := v$ and $w|_{B_R\backslash\overline{D}} := u|_{B_R\backslash\overline{D}}$ satisfy (5.11). And vice versa, if w is a solution of (5.11) then $v := w|_D$ and $u|_{B_R\backslash\overline{D}} := w|_{B_R\backslash\overline{D}}$ satisfy (5.6)–(5.9) and $\partial u/\partial\nu = \Lambda_k u$ on ∂B_R.

Theorem 5.1.1. *For every $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$ the conductive transmission problem (5.1)–(5.5), or, equivalently, (5.11) is uniquely solvable. Moreover, the solution $w \in H^1(B_R\backslash\partial D)$ depends continuously on the boundary data, i.e., there exists a constant $C_R > 0$, independent of h and f, such that*

$$||w||_{H^1(B_R\backslash\partial D)} \le C_R\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big).$$

Proof. We define the following continuous sesquilinear forms on $H^1(B_R\backslash\partial D) \times H^1(B_R\backslash\partial D)$:

$$a_1(w,\varphi) = \iint\limits_{D} \overline{\nabla\varphi} \cdot Aw + \overline{\varphi} w \, \mathrm{d}x + \iint\limits_{B_R\backslash D} \overline{\nabla\varphi} \cdot \nabla w + \overline{\varphi} w \, \mathrm{d}x$$
$$- \int\limits_{\partial D} \frac{i}{\eta} [\overline{\varphi}][w] \, \mathrm{d}s - \langle \Lambda_0 w, \varphi \rangle$$

and

$$a_2(w,\varphi) = -(k^2+1) \iint\limits_{B_R} \overline{\varphi} w \, \mathrm{d}x - \langle (\Lambda_k - \Lambda_0) w, \varphi \rangle,$$

where Λ_0 fulfills the property (2.28) form Lemma (2.2.2). The right-hand-side of (5.11) defines a bounded conjugate linear functional L on $H^1(B_R\backslash\partial D)$:

$$L(\varphi) = -i \int\limits_{\partial D} \frac{1}{\eta} f[\overline{\varphi}]\,\mathrm{d}s - \langle h,\, \varphi_+\rangle.$$

Let $\eta_* = \operatorname{ess\,inf}_{\partial D}\eta$. By the Cauchy-Schwarz inequality and the trace theorem there exist positive constant c and a positive constant C, dependent on η, such that

$$\begin{aligned}|L\varphi| &\le \frac{c}{\eta_*}||f||_{H^{1/2}(\partial D)}||\varphi||_{H^1(B_R\backslash\partial D)} + c||h||_{H^{-1/2}(\partial D)}||\varphi||_{H^1(B_R\backslash\partial D)}\\ &\le C\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big)||\varphi||_{H^1(B_R\backslash\partial D)}\end{aligned}$$

for all $\varphi \in H^1(B_R\backslash\partial D)$. Thus,

$$||L|| \le C(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}).$$

We write (5.11) as the problem of determining $w \in H^1(B_R\backslash\partial D)$ such that

$$a_1(w,\varphi) + a_2(w,\varphi) = L(\varphi) \quad \text{for all } \varphi \in H^1(B_R\backslash\partial D). \tag{5.12}$$

By assumption, for the matrix A we have $\operatorname{Re}\overline{\xi}\cdot A(x)\xi \ge c|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and almost all $x \in D$ and some $c > 0$. Thus,

$$\begin{aligned}\operatorname{Re}\, a_1(w,w) &= \operatorname{Re}\iint\limits_D \overline{\nabla w}\cdot A\nabla w + |w|^2\,\mathrm{d}x + \iint\limits_{B_R\backslash D} |\nabla w|^2 + |w|^2\,\mathrm{d}x\\ &\qquad - \langle \Lambda_0 w, \varphi\rangle\\ &\ge \operatorname{Re}\iint\limits_D \overline{\nabla w}\cdot A\nabla w + |w|^2\,\mathrm{d}x + \iint\limits_{B_R\backslash D} |\nabla w|^2 + |w|^2\,\mathrm{d}x\\ &\ge \min\{1,c\}||w||^2_{H^1(D)} + ||w||^2_{H^1(B_R\backslash\overline{D})}\\ &\ge \min\{1,c\}||w||^2_{H^1(B_R\backslash\partial D)}.\end{aligned}$$

By the Riesz representation theorem we define the bounded linear operators $\mathcal{A}_1 : H^1(B_R\backslash\partial D) \to H^1(B_R\backslash\partial D)$ and $\mathcal{A}_2 : H^1(B_R\backslash\partial D) \to H^1(B_R\backslash\partial D)$ by

$$(\mathcal{A}_1 w,\varphi)_{H^1(B_R\backslash\partial D)} = a_1(w,\varphi) \quad \text{and} \quad (\mathcal{A}_2 w,\varphi)_{H^1(B_R\backslash\partial D)} = a_2(w,\varphi).$$

Then, in terms of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ (5.12) can be written as

$$\mathcal{A}_1 w + \mathcal{A}_2 w = F \tag{5.13}$$

with $F \in H^1(B_R \backslash \partial D)$ also defined by the Riesz representation theorem through $(F, \varphi)_{H^1(B_R \backslash \partial D)} = L(\varphi)$. In particular, $\|F\|_{H^1(B_R \backslash \partial D)} = \|L\| \leq C(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)})$.

Since Re $a_1(w, w) \geq c\|w\|^2_{H^1(B_R \backslash \partial D)}$, by the Lax-Milgram Lemma [40], the operator $\mathcal{A}_1$ is boundedly invertible on $H^1(B_R \backslash \partial D)$. By compactness of $\Lambda_k - \Lambda_0$ and the compact embedding of $H^1(B_R \backslash \partial D)$ into $L^2(B_R)$ we conclude that $\mathcal{A}_2$ is compact. Riesz-Fredholm theory yields that for all $F \in H^1(B_R \backslash)$ the solution of (5.13) exists, provided $\mathcal{A}_1 + \mathcal{A}_2$ is injective.

Assume, $\mathcal{A}_1 w + \mathcal{A}_2 w = 0$. This is equivalent to

$$a_1(w, \varphi) + a_2(w, \varphi) = 0 \quad \text{ for all } \varphi \in H^1(B_R \backslash \partial D),$$

or to (5.1)–(5.5) with $h = 0$ and $f = 0$. By Lemma 3.1 in [9] the problem (5.1)–(5.5) has at most one solution, and therefore $w = 0$. Thus, (5.13) is uniquely solvable and for the solution w holds

$$||w||_{H^1(B_R \backslash \partial D)} \leq ||\mathcal{A}_1 + \mathcal{A}_2||^{-1} C\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big), \tag{5.14}$$

or

$$||w||_{H^1(B_R \backslash \partial D)} \leq C_R\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big), \tag{5.15}$$

where $C_R > 0$ depends on η, R, D and the matrix A, and does not depend on f and h.

□

Remark 5.1.2. *Since $\eta \in L^\infty(\partial D)$ and $f, u, v \in H^{1/2}(\partial D) \subset L^2(\partial D)$ the boundary condition (5.4)*

$$\frac{\partial v}{\partial \nu_A} = \frac{i}{\eta}(f + v - u)$$

implies that $\partial v / \partial \nu_A \in L^2(\partial D)$. From the trace theorem and Theorem 5.1.1 we have the following estimate on the norm of $\partial v / \partial \nu_A$:

$$\left\| \frac{\partial v}{\partial \nu_A} \right\|_{L^2(\partial D)} \leq c(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}),$$

with $c > 0$ independent of f and h. As we will see in the next section the regularity of $\partial v/\partial \nu_A$ will play an important role in proving the factorization method.

5.2 Far Field Operator. Interior Eigenvalue Problem

Let $u^\infty(\hat{x}, \hat{\theta})$ denote the far field pattern of the solution to (5.1)–(5.5) corresponding to the incident plane wave u^i with the incidence direction $\hat{\theta} \in S^1$ and the observation direction $\hat{x} \in S^1$. As in the previous chapter we define the far field operator $F : L^2(S^1) \to L^2(S^1)$ by

$$Fg(\hat{x}) = \int_{S^1} g(\hat{\theta}) u^\infty(\hat{x}, \hat{\theta}) \, \mathrm{d}s(\hat{\theta}).$$

Note that also for this case, the far-fields u^∞ satisfy the reciprocity relation (2.48) which can be shown by substituting the boundary conditions (5.3)–(5.3) into (2.49). With respect to injectivity of F we have the following result.

Theorem 5.2.1. *Assume that k^2 is not an eigenvalue of the following interior eigenvalue problem*

$$\Delta w + k^2 w = 0 \ \textit{in}\ D, \qquad \nabla \cdot (A \nabla v) + k^2 v = 0 \ \textit{in}\ D, \tag{5.16}$$

$$\frac{\partial w}{\partial \nu} = 0 \ \textit{on}\ \partial D, \qquad \frac{\partial v}{\partial \nu_A} = 0 \ \textit{on}\ \partial D, \tag{5.17}$$

$$w = v \ \textit{on}\ \partial D, \tag{5.18}$$

i.e., the only solution $(w, v) \in H^1(D) \times H^1(D)$ of is the trivial one $(w, v) = (0, 0)$. Then the far field operator F is injective.

Proof. Let $g \in L^2(S^1)$ be such that $Fg = 0$ on S^1. By the superposition principle $Fg = u^\infty$, where u^∞ is the far field pattern corresponding to the incident field given by the Herglotz function

$$v_g(x) = \int_{S^1} e^{ikx\cdot d} g(d) \, \mathrm{d}s(d), \ \ x \in \mathbb{R}^2. \tag{5.19}$$

Thus, u^∞ is the far field pattern of the function u which satisfies:

$$\begin{aligned} \nabla \cdot A\nabla u + k^2 u &= 0 \quad \text{in } D, \\ \Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ \frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu_A} &= -\frac{\partial v_g}{\partial \nu} \quad \text{on } \partial D, \\ u_+ - u_- - i\eta \frac{\partial u}{\partial \nu_A} &= -v_g \quad \text{on } \partial D, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) &= 0, \qquad r = |x|. \end{aligned}$$

Here, u_+ and u_- denote the traces of u taken from the exterior and interior of the domain D, respectively. By assumption, $u^\infty = 0$. Rellich's Lemma and the unique continuation principle imply that u vanishes in $\mathbb{R}^2 \backslash \overline{D}$. Therefore, the pair $(w, v) := (v_g|_D, u|_D)$ is a solution of the following problem:

$$\nabla \cdot A\nabla v + k^2 v = 0 \text{ in } D, \tag{5.20}$$

$$\Delta w + k^2 w = 0 \text{ in } D, \tag{5.21}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = 0 \text{ on } \partial D, \tag{5.22}$$

$$w - v = i\eta \frac{\partial v}{\partial \nu_A} \text{ on } \partial D. \tag{5.23}$$

We show that for a $(w, v) \in H^1(D) \times H^1(D)$ which solves (5.20)–(5.23), the traces of w and v on ∂D coincide. Indeed, let $(w, v) \in H^1(D) \times H^1(D)$ be a solution of (5.20)–(5.23). By Green's first theorem we have

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle = \iint_D A\nabla v \cdot \overline{\nabla \varphi} - k^2 v \overline{\varphi} \, \mathrm{d}x$$

for all $\varphi \in H^1(D)$. We set $\varphi := w - v$. Furthermore, using the boundary conditions (5.22)–(5.23) and the Green's first theorem we get

$$\begin{aligned}
\int\limits_{\partial D} \frac{1}{i\eta}(w-v)\overline{(w-v)}\mathrm{d}s &= \left\langle \frac{\partial v}{\partial \nu_A}, w - v \right\rangle \\
&= \left\langle \frac{\partial v}{\partial \nu_A}, w \right\rangle - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2|v|^2 \,\mathrm{d}x \\
&= \left\langle \frac{\partial w}{\partial \nu}, w \right\rangle - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2|v|^2 \,\mathrm{d}x \\
&= \iint\limits_D |\nabla w|^2 - k^2|w|^2 \,\mathrm{d}x - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2|v|^2 \mathrm{d}x.
\end{aligned}$$

This implies that

$$\mathrm{Im} \int\limits_{\partial D} \frac{1}{i\eta}|w-v|^2 \,\mathrm{d}s = -\mathrm{Im} \iint\limits_D \overline{\nabla v} \cdot A\nabla v \,\mathrm{d}x \tag{5.24}$$

Since $\mathrm{Im}\ \overline{\xi} \cdot A(x)\xi = \overline{\xi} \cdot \mathrm{Im}\ (A(x))\xi \leq 0$ for all $\xi \in \mathbb{C}$ and all $x \in D$, the equality (5.24) is possible only if $\int_{\partial D} |w-v|^2 \mathrm{d}s = 0$. That is, the traces of u and w coincide on ∂D. The boundary conditions (5.22)–(5.23) imply $\partial v/\partial \nu_A = \partial w/\partial \nu = 0$ on ∂D. Thus, (5.16)–(5.18) is an equivalent formulation of (5.20)–(5.23).
If k^2 is not an eigenvalue of the interior eigenvalue problem then $(w, v) = (0, 0)$ is the only solution of (5.16)–(5.17). In particular, $v_g = 0$ in D and, by analyticity, in all of $\mathbb{R}^2$. This implies (see e.g. [8], Section 3.2) that $g = 0$.

□

Remark 5.2.2. *The interior eigenvalues form at most a discrete countable set with infinity as the only accumulation point.*

By the definition of the problem (5.20)–(5.23), the interior eigenvalues belong to a subset of the intersection of Neumann eigenvalues of $-\nabla \cdot A\nabla$ and $-\Delta$ in D. It can be shown that if $\overline{\xi} \cdot \mathrm{Im}(A(x_0))\xi < 0$ for all $\xi \in \mathbb{C}\backslash\{0\}$ at a point $x_0 \in D$ then there are no eigenvalues of $-\nabla \cdot A\nabla$, and, therefore, no interior eigenvalues. However, as we show below, if Im $A = 0$, the interior eigenvalues can exist.

Assume $D = B_1$ is a unit disk. Let $A = \operatorname{diag}(\frac{1}{a}, \frac{1}{a})$ be a real-valued diagonal matrix with $a \in \mathbb{R}_{>0}$. Then the problem (5.16)–(5.18) reads as

$$\Delta w + k^2 w = 0 \text{ in } B_1, \qquad \frac{1}{a}\Delta v + k^2 v = 0 \text{ in } B_1,$$
$$\frac{\partial w}{\partial \nu} = 0 \text{ on } \partial B_1, \qquad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B_1,$$
$$w = v \text{ on } \partial B_1,$$

Let w be given in polar coordinates as $w(r, \varphi) := J_n(kr)e^{in\varphi}$ for $r \in [0, 1]$, $\varphi \in [0, 2\pi)$ and some $n \in \mathbb{Z}$, with J_n being the $n-$th Bessel function. Then w solves the Helmholtz equation in B_1. We choose $k \in \mathbb{R}_{>0}$ such that $J_n'(kr)|_{r=1} = 0$. In this way, w is a Neumann eigenfunction of $-\Delta$ in B_1 corresponding to the eigenvalue k^2. Let $k_D = k\sqrt{a}$ and let $v(r, \varphi) = \frac{J_n(k)}{J_n(k_D)} J_n(k_D r)e^{in\varphi}$. We choose a so, that $J_n'(k_D r)|_{r=1} = 0$. Then v is a Neumann eigenfunction of $-\Delta$ in B_1 corresponding to the eigenvalue k_D^2. Moreover, on the boundary $r = 1$ holds:

$$w(1, \varphi) = J_n(k)e^{in\varphi} = \frac{J_n(k)}{J_n(k_D)} J_n(k_D)e^{in\varphi} = v(1, \varphi) \quad \text{for all} \quad \varphi \in [0, 2\pi).$$

Thus, k^2 is an interior eigenvalue.

5.3 Factorization Method

To derive the factorization of the far field operator F we follow the approach of Section 4.2.

Theorem 5.3.1. *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the far field operator $F : L^2(S^1) \to L^2(S^1)$ has a factorization of the form $F = \gamma H^* T H$, where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $H : L^2(S^1) \to H^{1/2}(\partial D)$ is the Herglotz operator*

$$Hg(x) = \int_{S^1} e^{ikx\cdot d} g(d)\, \mathrm{d}s(d), \quad x \in \partial D$$

and $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ *is given by*

$$Tf = (\Lambda_k^- - \Lambda_k^+)u,$$

where, Λ_k^- *and* Λ_k^+ *correspond to interior and exterior, respectively, Dirichlet-to-Neumann operators defined in Section 4.2,* u *is the trace of the solution* $v \in H^1(D)$, $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ *to*

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in } D, \tag{5.25}$$

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}, \tag{5.26}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi \quad \text{on } \partial D, \tag{5.27}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = \varphi \qquad \text{on } \partial D, \tag{5.28}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial \nu} - iku \right) = 0, \qquad r = |x|. \tag{5.29}$$

Proof. The proof follows exactly the lines of the proof of Theorem 4.2.1 with the difference that we do not use (4.17).

□

In the following theorem we show that the middle operator T in the factorization of F satisfies the assumptions of the Range Identity Theorem 2.4.4.

Theorem 5.3.2. *Assume* k^2 *is not an interior eigenvalue and not Dirichlet eigenvalue of* $-\Delta$ *in* D. *Then*

(a) $(-T)$ *has the form* $(-T) = T_0 + T_1$, *where* T_0 *is a coercive self-adjoint operator and* $T_1 : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ *is compact. By coercivity we mean that there exists a constant* $c > 0$ *such that*

$$\langle T_0 \varphi, \varphi \rangle \geq c ||\varphi||^2_{H^{1/2}(\partial D)} \qquad \text{for all } \varphi \in H^{1/2}(\partial D).$$

(b) $\langle Im\ (-T)\varphi, \varphi \rangle > 0$ *for all* $\varphi \in H^{1/2}(\partial D), \varphi \neq 0$.

Proof. (a) First, we write $T\varphi = (\Lambda_k^- - \Lambda_k^+)u$ as

$$T\varphi = (\Lambda_k^- - \Lambda_i^-)u + (\Lambda_i^+ - \Lambda_k^+)u + (\Lambda_i^- - \Lambda_i^+)u.$$

The differences $(\Lambda_k^- - \Lambda_i^-)$ and $(\Lambda_i^+ - \Lambda_k^+) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ are compact, which is shown in the Lemma 4.2.3. Let $A : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ define the mapping $\varphi \mapsto u$. By well-posedness of the direct problem and the trace theorem A is bounded. Thus, T can be written as $T = \tilde{T}_1 + (\Lambda_i^- - \Lambda_i^+)A$ with compact operator $\tilde{T}_1 := (\Lambda_k^- - \Lambda_i^-)A + (\Lambda_i^+ - \Lambda_k^+)A$.

We write the equation (5.27)

$$\Lambda_k^+ u - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi$$

as

$$\Lambda_i^+ u = (\Lambda_i^+ - \Lambda_k^+)u + \frac{\partial v}{\partial \nu_A} + (\Lambda_k^- - \Lambda_i^-)\varphi + \Lambda_i^- \varphi.$$

Thus

$$u = (\Lambda_i^+)^{-1}(\Lambda_i^+ - \Lambda_k^+)u + (\Lambda_i^+)^{-1}\frac{\partial v}{\partial \nu_A} + (\Lambda_i^+)^{-1}(\Lambda_k^- - \Lambda_i^-)\varphi \qquad (5.30)$$

$$+ (\Lambda_i^+)^{-1}\Lambda_i^- \varphi. \qquad (5.31)$$

Let $B : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ represent the mapping $\varphi \mapsto \partial v/\partial \nu_A$. By the well-posedness of the problem we get that $\varphi \mapsto (u - v - \varphi)$ is bounded from $H^{1/2}(\partial D)$ to $H^{1/2}(\partial D)$. The boundary condition (5.28)

$$\frac{\partial v}{\partial \nu_A} = \frac{1}{i\eta}(u - v - \varphi)$$

yields that $\varphi \mapsto \partial v/\partial \nu_A$ is bounded as a mapping from $H^{1/2}(\partial D)$ to $L^2(\partial D)$. Parameterizing ∂D and using Rellich's embedding theorem [8] we conclude that $L^2(\partial D)$ is compactly embedded in $H^{-1/2}(\partial D)$. Thus, the operator B is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. Now, using (5.30) we can write $(-T)$ as the sum $(-T) = T_0 + T_1$, where

$$T_1 := -\tilde{T}_1 - (\Lambda_i^- - \Lambda_i^+)(\Lambda_i^+)^{-1}\Big((\Lambda_i^+ - \Lambda_k^+)A + B + (\Lambda_k^- - \Lambda_i^-)\Big)$$

is compact, and

$$T_0 := (\Lambda_i^+ - \Lambda_i^-)(\Lambda_i^+)^{-1}\Lambda_i^-.$$

Since Λ_i^- is self-adjoint and coercive and $-(\Lambda_i^+)^{-1}$ is positive [36], the coercivity of $T_0 : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ follows immediately:

$$\begin{aligned}\langle T_0\varphi, \varphi\rangle &= \langle -\Lambda_i^-(\Lambda_i^+)^{-1}\Lambda_i^-\varphi, \varphi\rangle + \langle \Lambda_i^-\varphi, \varphi\rangle \\ &\geq \langle -(\Lambda_i^+)^{-1}\Lambda_i^-\varphi, \Lambda_i^-\varphi\rangle + c||\varphi||^2_{H^{1/2}(\partial D)} \\ &\geq c||\varphi||^2_{H^{1/2}(\partial D)}\end{aligned}$$

for all $\varphi \in H^{\frac{1}{2}}(\partial D)$.

(b)To show that $\operatorname{Im}\langle(-T)\varphi, \varphi\rangle > 0$ for all $\varphi \in H^{1/2}(\partial D), \varphi \neq 0$ we will use of the boundary condition (5.27), which in terms of Dirichlet-to-Neumann operators $\Lambda_k^\pm$ has the form

$$\Lambda_k^+ u - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi.$$

We write $\langle(-T)\varphi, \varphi\rangle$ as

$$\begin{aligned}\langle(-T)\varphi, \varphi\rangle &= \langle(\Lambda_k^+ - \Lambda_k^-)u, \varphi\rangle = \langle \Lambda_k^+ u, \varphi\rangle - \langle u, \Lambda_k^-\varphi\rangle \\ &= \Big\langle \frac{\partial v}{\partial \nu_A}, \varphi\rangle + \langle \Lambda_k^-\varphi, \varphi\Big\rangle - \langle u, \Lambda_k^+ u\rangle + \langle u, \frac{\partial v}{\partial \nu_A}\rangle.\end{aligned}$$

Then

$$\begin{aligned}2i \operatorname{Im}\langle(-T)\varphi, \varphi\rangle &= \langle(-T)\varphi, \varphi\rangle - \overline{\langle(-T)\varphi, \varphi\rangle} \\ &= \Big\langle \frac{\partial v}{\partial \nu_A}, \varphi\rangle - \langle u, \Lambda_k^+ u\rangle + \langle u, \frac{\partial v}{\partial \nu_A}\rangle \\ &\quad - \Big(\langle \varphi, \frac{\partial v}{\partial \nu_A}\rangle - \langle \Lambda_k^+ u, u\rangle + \langle \frac{\partial v}{\partial \nu_A}, u\rangle\Big) \\ &= 2i \operatorname{Im}\langle \Lambda_k^+ u, u\rangle + \Big\langle \frac{\partial v}{\partial \nu_A}, \varphi - u\Big\rangle - \Big\langle \varphi - u, \frac{\partial v}{\partial \nu_A}\Big\rangle\end{aligned}$$

$$= 2i \ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + 2i \ \mathrm{Im}\Big\langle \frac{\partial v}{\partial \nu_A}, \varphi - u\Big\rangle$$

$$= 2i \ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + 2i \ \mathrm{Im}\left(\frac{\partial v}{\partial \nu_A}, -i\eta\frac{\partial v}{\partial \nu_A}\right)_{L^2(\partial D)}$$

$$- 2i \ \mathrm{Im}\Big\langle \frac{\partial v}{\partial \nu_A}, v\Big\rangle.$$

In the last step we the boundary condition (5.28). Thus,

$$\mathrm{Im}\langle (-T)\varphi, \varphi\rangle = \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + \left(\frac{\partial v}{\partial \nu_A}, \eta\frac{\partial v}{\partial \nu_A}\right)_{L^2(\partial D)} - \mathrm{Im}\Big\langle \frac{\partial v}{\partial \nu_A}, v\Big\rangle$$

$$\geq \eta_0 \left\| \frac{\partial v}{\partial \nu_A}\right\|_{L^2(\partial D)} + \mathrm{Im}\langle \Lambda_k^+ u, u\rangle - \mathrm{Im}\Big\langle \frac{\partial v}{\partial \nu_A}, v\Big\rangle,$$

with $\eta_0 =$ess $\inf_{\partial D} \eta$. We compute the imaginary parts of $\langle \Lambda_k^+ u, u\rangle$ and $\big\langle \frac{\partial v}{\partial \nu_A}, v\big\rangle$. The Green's first theorem and the Sommerfeld radiation condition yield

$$\mathrm{Im}\left\langle \Lambda_k^+ u, u\right\rangle = \mathrm{Im}\left\langle \frac{\partial u}{\partial \nu}, u\right\rangle = \mathrm{Im}\left(-\iint\limits_{B_R\setminus\overline{D}} |\nabla u|^2 - k^2|u|^2 \,\mathrm{d}x\right)$$

$$+ \mathrm{Im}\left(\int\limits_{|x|=R} \frac{\partial u}{\partial \nu}\overline{u}\,\mathrm{d}s\right)$$

$$= \mathrm{Im}\left(ik \int\limits_{|x|=R} |u|^2 \mathrm{d}s + o(1)\right) \quad \text{as } R \to \infty,$$

and, by assumption on A,

$$\mathrm{Im}\left\langle \frac{\partial v}{\partial \nu_A}, v\right\rangle = \mathrm{Im} \iint\limits_D A\nabla v\cdot\overline{\nabla v} - k^2|v|^2\,\mathrm{d}x$$

$$= \iint\limits_D (\mathrm{Im}\ A)\nabla v\cdot\overline{\nabla v}\,\mathrm{d}x \leq 0.$$

Thus, Im $\langle (-T)\varphi, \varphi\rangle \geq 0$. Assume there exists $\varphi \in H^{1/2}(\partial D)$ such that Im $\langle (-T)\varphi, \varphi\rangle = 0$. Then $\|\partial v/\partial \nu_A\|_{L^2(\partial D)} = 0$ and $\lim_{R\to\infty}\int_{|x|=R}|u|^2\,\mathrm{d}s = 0$. Rellich's Lemma and the unique continuation principle imply $u = 0$ in

$\mathbb{R}^2\backslash\overline{D}$. Thus, $\partial u/\partial\nu = 0$. The boundary condition (5.27) yields $\Lambda_k^- \varphi = 0$. Since k^2 is not an interior eigenvalue, we conclude $\varphi = 0$. Thus, Im $\langle(-T)\varphi, \varphi\rangle > 0$ for all $\varphi \neq 0$.

□

With the previous theorem combined with Theorem 4.2.4, Theorem 4.2.2 and the Range Identity Theorem 2.4.4 we get the main result of this chapter.

Theorem 5.3.3. *Assume that k^2 is not a Dirichlet eigenvalue and not an eigenvalue of the interior eigenvalue problem (5.16)–(5.18).*

For $z \in \mathbb{R}^2$ we define $\phi_z \in L^2(S^1)$ by (2.87). Then

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}(F_\sharp^{1/2}), \tag{5.32}$$

and consequently

$$z \in D \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|(\phi_z, \psi_j)_{L^2(S^2)}|^2}{|\lambda_j|} < \infty.$$

where (λ_j, ψ_j) is an eigensystem of the operator $F_\sharp : L^2(S^2) \to L^2(S^2)$ given by

$$F_\sharp = |Re\ F| + |Im\ F|. \tag{5.33}$$

The sign of the function

$$W(z) = \left[\sum_{j=1}^{\infty} \frac{|\langle\phi_z, \psi_j\rangle_{L^2(S^2)}|^2}{|\lambda_j|}\right]^{-1} \tag{5.34}$$

is the characteristic function of D.

5.4 Numerical Results

In this section we present a numerical example to demonstrate the applicability of the factorization method. We compute the forward problem for a peanut-shaped scatterer with ∂D parametrized by $\gamma(t) =$

$(-1.5\sqrt{\cos^2(t) + .25\sin^2(t)}\sin(t), 1.8\sqrt{\cos^2(t) + .25\sin^2(t)}\cos(t))$ $t \in [0, 2\pi]$. Further, $k = 3, \eta = 0.5$ and $n = 0.5$. For the solution we used P^1 finite elements discretization. For the matrix A we set $A(x) = \text{diag}(x_1^2 + x_2^2 + 1.2, x_1^2 + x_2^2 + 1.2)$ for $x \in D$, the wave number is $k = 3$ and the conductivity $\eta = 3.5$. We reduce the scattering problem over $\mathbb{R}^2$ to a problem over a bounded domain with the help of Neumann-to-Dirichlet mapping [24] (see Section 2) and solve the forward problem using a P^1 finite elements discretization with the help of FreeFem++ package [27]. Figure 5.1 on the left represents the real part of the total field corresponding to the incident field with incident direction $d = [1\,0]^\top$.

Our data set is represented by a matrix $F \in \mathbb{C}^{32\times 32}$, where $F_{jl} = u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 32\}$, and $u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 32\}$ are the far fields corresponding to the incident direction of the plane wave $\theta_j = 2\pi j/32$ and the observation point $\theta_l = 2\pi l/32$.
In Figure 5.2 we plot the real part of the total field for the full model with $\delta = 0.6$ with the corresponding reconstruction by the Factorization Method (to the right). Despite the large error ($\|F - F^\delta\|/\|F^\delta\| \approx 0.23$, where F^δ is the matrix 32×32 matrix with containing the far fields of the full model) the reconstructions of the full and the approximate models are very similar.

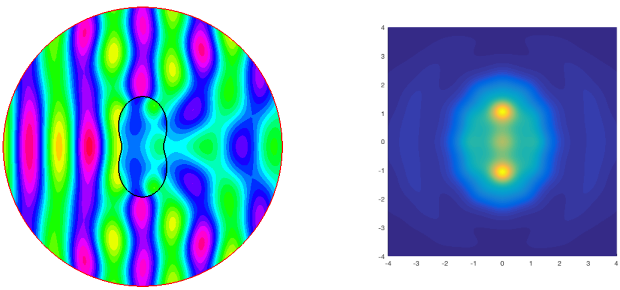

Figure 5.1: *From left to right: Total field of a peanut-shaped obstacle with the transmission conditions corresponding to the incident direction* $d = [1\ 0]^\top$. *Reconstruction by the Factorization Method.*

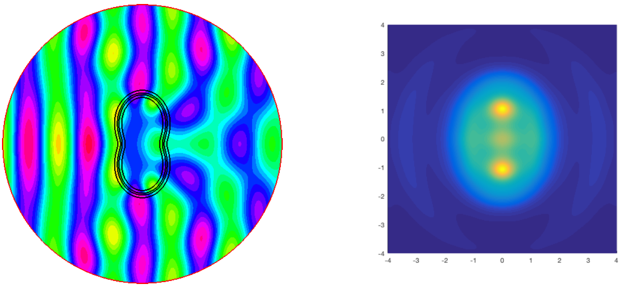

Figure 5.2: *From left to right: Total field of a peanut-shaped obstacle for the full model with $\delta = 0.6$ corresponding to the incident direction $d = [1\ 0]^\top$. Reconstruction by the Factorization Method.*

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