Random tessellations: Stereological formulae in Euclidean space and Kendall's Problem in spherical space

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Contents

1. Introduction

Tessellations, or mosaics, have been a subject of interest for a very long time. Even ancient cultures, like the Sumerians or the Romans, used colored tiles to decorate floors and walls. Formal, mathematical definitions and deterministic tilings of the plane or higher dimensional spaces were considered much later. By a tessellation of $\mathbb{R}^d$, we understand a system of convex polytopes in $\mathbb{R}^d$ which cover the whole space and have pairwise no common interior points.

Random tessellations of Euclidean spaces are a classical topic in stochastic geometry. They are extensively studied in the literature, see e.g. [49, Chapter 10], [8, Chapter 9], [38] or [40] for an overview and results for general tessellations. There are also many articles discussing various properties of special models such as hyperplane tessellations, see e.g. [33], [36], [17], [31, Chapter 6], or the so called Voronoi tessellations, see e.g. [39], [41], [4].

As the title suggests, this thesis can be subdivided into two parts, investigating random tessellations in different spaces. In the first part, Chapter 2, we consider random tessellations of $\mathbb{R}^d$ and their sections with a fixed lower dimensional linear subspace. Related results can be found in [38, Chapter 6] or [35, Section 2.5]. A somehow dual framework is often investigated in stereology, where a deterministic object is intersected with a random linear or affine subspace. An example of such an approach is [24] or [3]. In these papers, a fixed object is intersected with a j-dimensional hyperplane, $j \in \{1, \dots, d\}$, and the rotational average of the intrinsic volumes of this section is investigated.

Let X be a stationary random tessellation in $\mathbb{R}^d$, that means, for any $y \in \mathbb{R}^d$, the shifted tessellation $X + y$ has the same distribution as X. We write $\mathcal{S}_d(X)$ for the system of cells of X. Then any element $F \in \mathcal{S}_d(X)$ is almost surely a convex polytope and can be decomposed into its lower dimensional faces. Let $\mathcal{S}_k(X)$ denote the system of all k-dimensional faces of X. If $x \in \mathbb{R}^d$ is in the relative interior of some $F \in \mathcal{S}_k(X)$, we define $F_k(x) := F$. Further, let S be a linear subspace of $\mathbb{R}^d$. Then $X_S := X \cap S$ is a tessellation in S. Let U_F denote the subspace parallel to a convex polytope F. We assume X and S to be in general position, which means that for all $k \in \{0, \dots, d\}$ the following holds almost surely: If $F \in \mathcal{S}_k(X)$, then $\dim(U_F \cap S) = 0$ or $U_F + S = \mathbb{R}^d$. Then the j-faces of X_S, $j \in \{0, \dots, l\}$, are given by intersecting the $(d-l+j)$-faces of X with S, i.e. we have $\mathcal{S}_j(X_S) = \{F \cap S : F \in \mathcal{S}_{d-l+j}(X),\ F \cap S \neq \emptyset\}$. Now we consider the stationary random measures

$$M_{d-l+j}(\cdot) := \sum_{F \in \mathcal{S}_{d-l+j}(X)} \mathcal{H}^{d-l+j}(F \cap \cdot) \quad \text{and} \quad M_{S,j}(\cdot) := \sum_{F \in \mathcal{S}_{d-l+j}(X)} \mathcal{H}^j(F \cap S \cap \cdot).$$

Here stationarity means that $M_{d-l+j}(\cdot + x)$ has the same distribution as $M_{d-l+j}(\cdot)$ for any $x \in \mathbb{R}^d$ and that $M_{S,j}(\cdot + y)$ has the same distribution as $M_{S,j}(\cdot)$ for any $y \in S$, since $M_{S,j}$ is concentrated on S. Under the Palm measure $\mathbb{P}_{M_{S,j}}$, the origin is almost surely contained in some j-dimensional face of X_S. Our main result of Chapter 2, Theorem 2.4.1, shows that the Palm measure $\mathbb{P}_{M_{S,j}}$ of $M_{S,j}$ has a density with respect to the Palm measure $\mathbb{P}_{M_{d-l+j}}$ of M_{d-l+j} and this density is given by the subspace determinant, or generalized sine function, $[S, F_{d-l+j}(0)]_j$, a number related to the relative position of S and $U_{F_{d-l+j}(0)}$. Thus, for any measurable f, we have

$$\mathbb{E}_{M_{S,j}}[f] = \mathbb{E}_{M_{d-l+j}}\left[[S, F_{d-l+j}(0)]_j \cdot f\right],$$

where the expectations mean integration with respect to the Palm measures. We provide two different proofs for this result. The first is somehow more straightforward, while the second is based on another general result, Theorem 2.4.4. We also show that Theorem 2.4.1 can be used to prove Theorem 2.4.4. An application is $f \equiv 1$, giving us a relation between the intensities of M_{d-l+j} and $M_{S,j}$

$$\lambda_{S,j} = \lambda_{d-l+j} \cdot \mathbb{E}^0_{M_{d-l+j}}[[S, F_{d-l+j}]_j],$$

where $\mathbb{E}^0_{M_{d-l+j}}$ denotes integration with respect to the *Palm distribution* of M_{d-l+j}, i.e. $\lambda^{-1}_{d-l+j}\mathbb{P}_{M_{d-l+j}}$ in case λ_{d-l+j} is positive and finite. See also [49, Theorem 4.4.6] where a similar result is obtained for stationary k-flat processes.

A tessellation is called isotropic, if its distribution is invariant under all rotations of SO_d. In this case, $\mathbb{E}^0_{M_{d-l+j}}[[S, F_{d-l+j}]_j]$ can be computed and we obtain Corollary 2.5.4 (see also [38, Theorem 6.3]):

$$\lambda_{S,j} = \lambda_{d-l+j} \cdot \frac{\Gamma\left(\frac{d-l+j+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}.$$

In Section 2.6, we state a weaker version of Theorem 2.4.1, for the *area-debiased* Palm measures $\mathbb{P}_{N_{d-l+j}}$ and $\mathbb{P}_{N_{S,j}}$.

Let A be a locally finite set in $\mathbb{R}^d$ with $\mathrm{conv}(A) = \mathbb{R}^d$. For $x \in A$, the Voronoi cell of x is given by

$$C(x, A) := \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - a\| \text{ for all } a \in A\},$$

and the collection of these cells is the Voronoi tessellation of A (see [49, Chapter 10.2]). If the Voronoi tessellation is generated by a Poisson process, it is called a Poisson–Voronoi tessellation. In Section 2.7, we apply Theorem 2.4.1 to some of the results in [4] for a stationary Poisson–Voronoi tessellation X generated by a Poisson process N and compute the joint distribution of the $l - j + 1$ neighbours of the $(d - l + j)$-face of X containing the typical point, randomly chosen on a j-face of X_S. These neighbours are the $l - j + 1$ points in N, having the same distance from this typical point. The affine hull of the $(d - l + j)$-face of X containing the typical point is uniquely defined by those neighbours (see [4, (2.1)].

In the second part, which consists of Chapter 3 and Chapter 4, we consider random tessellations of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. This setting is not as extensively studied in the literature as the Euclidean one is. The intersection of the unit sphere with a d-dimensional subspace is the unit sphere in the intersecting subspace and thus a great subsphere of $\mathbb{S}^d$ having unit radius. We call the intersection a great circle in case of $d = 2$. At the same time, d-dimensional subspaces partition the Euclidean space $\mathbb{R}^{d+1}$ into polyhedral cones. This relation plays an important role in spherical geometry, see e.g. [1], [9] [14, Chapter 2]. Random tessellations of the sphere generated by intersecting the unit sphere with

d-dimensional subspaces are studied in [9], [34, Section 6], [2] and the recent work [23] on conical tessellations. Voronoi tessellations in spherical space can be defined as in the Euclidean case, using the geodesic distance on $\mathbb{S}^d$. Random Voronoi tessellations on the sphere and their applications are investigated in [34, Section 7], [44], [51], [43, Section 3.7.6, Section 5.10] and [53].

We focus on the what became known as 'Kendall's Problem' or 'Kendall's Conjecture'. So far this line of investigation was only considered in the Euclidean setting. In our present work, we now formulate and investigate a spherical analogue. Consider a stationary and isotropic Poisson line process in the Euclidean plane and denote the almost surely unique cell containing the origin by Z_0. This cell is called the *zero cell* or *Crofton cell*. In the foreword of the first edition of [8], David G. Kendall stated the following conjecture: The conditional law for the shape of Z_0, given the area $A(Z_0)$, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape. This conjecture was strongly supported by a heuristic proof from R. Miles [37]. Two years later, a proof was given by Kovalenko in [26]. Kovalenko also provided a simplified proof in [28] and an extension to the typical cell of a Poisson–Voronoi tessellation in the plane in [27]. Further extensions to arbitrary dimensions and not necessarily isotropic Poisson hyperplane tessellations were made in [17], where the size of the Crofton cell was measured by the volume. In [18] the problem was extended and solved for typical cells of stationary Poisson–Voronoi tessellations in arbitrary dimensions and the size was measured by an intrinsic volume. In [20] a very general setting with a very general class of size functionals was considered, containing the aforementioned results as special cases. In [21], Kendall's Problem was extended to the typical k-faces of a Poisson hyperplane tessellation ($k \in 2, \dots, d-1\}$) and in [22] to the typical k-faces of a Poisson–Voronoi tessellation. In [19] typical cells of Poisson–Delaunay tessellations were considered. David Kendall died in 2007 in Cambridge at the age of 89 ([58]). Hence, he was still alive when his conjecture was proven and extended to arbitrary dimensions and different models.

We continue with some notation in order to present our results. On the unit sphere, there is no naturally distinguished point similar to the Euclidean origin, so we choose $\overline{0} := (1, 0, \dots, 0)^T$ as the *spherical origin*. Let d_s denote the geodesic metric on $\mathbb{S}^d$ and let $B_s(x, r)$ denote a spherical cap with radius r and centre x.

A *convex body* in $\mathbb{S}^d$ is the intersection of the unit sphere with some line-free closed convex cone in $\mathbb{R}^{d+1}$ which is not $\{0\}$. We denote the set of spherically convex bodies by $\mathcal{K}_s^d$. If we do not require the cone to be line-free but only require that the cone is not equal to some linear subspace of $\mathbb{R}^{d+1}$, the resulting set will be denoted by $\overline{\mathcal{K}}_s^d$. A *spherical polytope* is the intersection of $\mathbb{S}^d$ with a polyhedral cone. For more details on spherical geometry, we refer to [49, Section 6.5] and [14].

By a tessellation of $\mathbb{S}^d$, we understand a finite collection of spherical polytopes which have nonempty interiors, which cover $\mathbb{S}^d$ and have pairwise disjoint interiors. If the tessellation is isotropic, that means its distribution is invariant under the rotations in SO_{d+1}, there is almost surely a unique cell containing $\overline{0}$ in its interior. We call this cell the *spherical Crofton cell* or spherical zero cell and denote it by Z_0.

We denote the spherical Lebesgue measure on $\mathbb{S}^d$ by σ_d and the surface area of the unit sphere by $\omega_{d+1} := \sigma_d(\mathbb{S}^d)$. Let X be an isotropic Poisson process on $\mathbb{S}^d$. Then the intensity measure of X equals $\mathbb{E}[X(\cdot)] = \gamma_S \sigma_d(\cdot)$ for some $\gamma_S \geq 0$. We want to show that the Crofton cell of an isotropic spherical Poisson hyperplane tessellation, given a lower bound for its spherical volume, converges to a spherical cap for $\gamma_S \to \infty$. Therefore, we have to quantify the deviation of Z_0 from a spherical cap.

A functional $\vartheta : \overline{\mathcal{K}}_s^d \to [0, \infty)$ is called a deviation functional for the class of spherical caps, if it is continuous and $\vartheta(K) = 0$, for some $K \in \overline{\mathcal{K}}_s^d$ with $\sigma_d(K) > 0$, if and only if K is a spherical cap. An example for such a deviation functional is the difference between spherical circumradius and spherical inradius of K. Another example, denoted by Δ, is discussed in Theorem 3.3.4 and measures the deviation in the L^2-sense. In Section 3.1 and 3.2 the setting and the model are formally introduced. Section 3.3 contains geometrical results. On $\mathcal{K}_s^d$, the functional U_1 is given by

$$U_1(K) = \frac{1}{2\omega_{d+1}} \int_{\mathbb{S}^d} \mathbf{1}\{K \cap x^\perp \neq \emptyset\} \, \sigma_d(dx).$$

It can be interpreted as 1/2 of the measure of great subspheres meeting the spherical convex body K. Thus, it is a spherical analogue of the Euclidean functional V_1, which, for convex bodies, is proportional to the mean width.

In [12], the following inequality is shown. It can be interpreted as a spherical version of the Urysohn inequality. Let $K \in \mathcal{K}_s^d$ and let C be a spherical cap with $\sigma_d(C) = \sigma_d(K)$. Then

$$U_1(K) \geq U_1(C)$$

and equality holds if and only if K is a spherical cap. Theorem 3.3.2 and Theorem 3.3.4 are key ingredients for the following investigations, giving stability estimations for the result above. The first stability estimate is for general deviation functionals. It can quite easily be generalized to other size functionals besides σ_d. We denote by B_a some spherical cap with $\sigma_d(B_a) = a$.

Theorem 3.3.2. *For any $a \in (0, \omega_{d+1}/2)$ there is a function $f_a : [0, \infty) \to [0, 1]$, with $f_a(0) = 0$ and $f_a(t) > 0$ for $t > 0$, such that*

$$U_1(K) \geq (1 + f_a(\varepsilon)) U_1(B_a),$$

for any $\varepsilon > 0$ and $K \in \mathcal{K}_s^d$ with $\sigma_d(K) \geq a$, $\overline{0} \in K$ and $\vartheta(K) \geq \varepsilon$.

The second stability estimate is much more explicit. For a spherical cap C, denote its radius by α_C.

Theorem 3.3.4. *Let $K \subset \mathbb{S}^d$ be a spherically convex body and $C \subset \mathbb{S}^d$ a spherical cap with $\sigma_d(K) = \sigma_d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C \leq \pi/2 - \alpha_0$. Then there is a constant $\gamma = \gamma(d, \alpha_0)$ such that*

$$U_1(K) \geq \left(1 + \gamma \Delta(K)^2\right) U_1(C).$$

In Section 3.4, we prove the following theorems, describing not only the asymptotic shape of the spherical Crofton cell given a lower bound for its volume, which is a spherical cap, but also giving deviation inequalities for fixed intensities. For the first result, we fix a general deviation functional.

Theorem 3.2.1. *Let $\varepsilon > 0$ and let $0 < a < \omega_{d+1}/2$. Then there are constants $c_1, c_2 > 0$, such that*

$$\mathbb{P}(\vartheta(Z_0) \geq \varepsilon | \sigma_d(Z_0) \geq a) \leq c_1 \cdot \exp(-c_2 \cdot \gamma_S \cdot \omega_{d+1})$$

and the constants c_1, c_2 depend only on a, ε and d.

For the second result, we consider the deviation functional Δ.

Theorem 3.4.3. *Let $0 < a < \omega_{d+1}/2$ and $0 < \varepsilon < 1$. Then there are constants $\tilde{c}_1, \tilde{c}_2 > 0$, such that*

$$\mathbb{P}\left(\Delta(Z_0) > \varepsilon | \sigma_d(Z_0) \geq a\right) \leq \tilde{c}_1 \cdot \exp\left(-\tilde{c}_2 \cdot \varepsilon^{2(d+1)} \cdot \gamma_S \cdot \omega_{d+1}\right)$$

and the constant $\tilde{c}_1$ depends only on a, ε and d and the constant $\tilde{c}_2$ depends only on a and d.

In the next section, we use the techniques developed in Section 3.4 to show that for $0 < a < \omega_{d+1}/2$ the probability $\mathbb{P}(\sigma_d(Z_0) \geq a)$ decays exponentially if the intensity γ_S tends to infinity. We also provide the explicit rate of the exponential decay.

Section 3.6 contains analgous results for the tessellation induced by a fixed number $N \geq d+1$ of independent and uniformly distributed random great subspheres. Such a collection of great subspheres is also called a spherical binomial process of size N.

In Section 3.7, we measure the size of the Crofton cell with the spherical inball radius. Then the asymptotic shape of the Crofton cell, given a lower bound for the inradius, is still a spherical cap and we obtain similar deviation inequalities as in Section 3.4. This holds for both spherical Poisson hyperplane tessellations and the tessellations induced by spherical binomial processes of size $N \geq d+1$.

After investigating Crofton cells, a natural next step is to look at typical cells. Thus, in Chapter 4, we consider typical objects in spherical space. Since $\mathbb{S}^d$ is a homogeneous SO_{d+1}-space ([49, p. 584]), we use the framework of random measures on homogeneous spaces, see [29] and [46]. A particle process in Euclidean space is a point process on the space of nonempty compact subsets of $\mathbb{R}^{d+1}$. If Y is a stationary particle process with intensity γ_Y, there is a very intuitive representation for the distribution $\mathbb{Q}$ of the typical particle of Y (see [49, p. 106]):

$$\mathbb{Q}(\cdot) = \frac{1}{\gamma_Y} \mathbb{E} \sum_{K \in Y} \mathbf{1}\{K - c(K) \in \cdot\} \mathbf{1}\{c(K) \in [0,1]^d\},$$

where $c : \mathcal{K}^{d+1} \to \mathbb{R}^{d+1}$ is a centre function and $\mathbb{Q}$ is concentrated on sets having the Euclidean origin as centre. In spherical space, there are many $\varphi \in SO_{d+1}$ such that $\varphi \bar{0} = x$ for some fixed $x \in \mathbb{S}^d$. Let X' be an isotropic spherical particle process with intensity $\gamma_{X'}$ and denote its typical particle by Z. Then the distribution of Z is given by

$$\mathbb{P}(Z \in \cdot) = \frac{1}{\gamma_{X'} \omega_{d+1}} \mathbb{E} \left[\sum_{K \in X'} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1} K \in \cdot\} \, \kappa(c_s(K), d\varphi) \right],$$

where $c_s : \mathcal{K}_s^d \to \mathbb{S}^d$ is a suitable centre function and, for $x \in \mathbb{S}^d$, $\kappa(x, \cdot)$ is a probability measure on SO_{d+1} concentrated on the set $\{\varphi \in SO_{d+1} : \ \varphi \bar{0} = x\}$. We then prove a disintegration result for isotropic particle processes on $\mathbb{S}^d$ (a Euclidean analogon can be found in [49, Theorem 4.1.1]).

In Section 4.2, we interpret an isotropic tessellation X' of $\mathbb{S}^d$ as an isotropic particle process and use the aforementioned disintegration result to obtain the following relation between the Crofton cell Z_0 of X' and the typicall cell Z of X'.

Theorem 4.2.2. *Let $f : \mathcal{K}_s^d \to [0,\infty)$ be measurable and rotation invariant. Then*

$$\mathbb{E}\left[f(Z_0)\right] = \gamma_{X'}\mathbb{E}\left[f(Z)\cdot\sigma_d(Z)\right],$$

where $\gamma_{X'}$ is the intensity of the tessellation X'.

Using this result, we transfer Theorem 3.2.1 to the typical cell of a spherical Poisson hyperplane tessellation and we do the same for the typical cell of the tessellation induced by a binomial hyperplane process of size $N \geq d+1$.

In the second part of Chapter 4, we investigate the typical cell of a spherical Poisson–Voronoi tessellation. In Euclidean space, the Theorem of Slivnyak characterizes a stationary Poisson process using its Palm distribution (see [49, Theorem 3.3.5]). We prove a spherical version:

Theorem 4.3.1. *Let X be an isotropic point process on $\mathbb{S}^d$ having intensity measure $\mathbb{E}[X(\cdot)] = \lambda\cdot\sigma_d(\cdot)$ for some $0<\lambda<\infty$. Let $\mathbb{P}_X$ denote its Palm distribution. Then X is a Poisson point process if and only if*

$$\mathbb{P}_X(X\in A) = \mathbb{P}(X+\delta_{\overline{0}}\in A), \quad A\in\mathcal{B}(\mathcal{N}(\mathbb{S}^d)).$$

With the help of this theorem, in Section 4.4 we show that the distribution of the typical cell of the Poisson–Voronoi tessellation, induced by an isotropic Poisson process X, is equal to the distribution of the Crofton cell induced by a special spherical Poisson hyperplane process Y. The hyperplane process Y is the set of all great subspheres having equal distance to the spherical origin $\overline{0}$ and a point in X and thus clearly is not isotropic. This leads to a new functional $\tilde{U}$ on $\mathcal{K}_s^d$ defined by

$$\tilde{U}(K) := \int_{\mathbb{S}^d} \mathbf{1}\{f(x)^\perp\cap K\neq\emptyset\}\,\sigma_d(dx), \quad K\in\mathcal{K}_s^d,$$

where $f(x) := (x-\overline{0})\cdot(\|x-\overline{0}\|)^{-1}$ for $x\in\mathbb{S}^d\setminus\{\overline{0}\}$ and $f(\overline{0}) := -\overline{0}$. In this setting, we measure the size with the centred spherical inball radius

$$r_s(K) := \max\{r\geq 0:\ B_s(\overline{0},r)\subset K\}, \quad K\in\mathcal{K}_s^d.$$

Furthermore, let

$$R_s(K) := \min\{r\geq 0:\ K\subset B_s(\overline{0},r)\}, \quad K\in\mathcal{K}_s^d,$$

denote the centred spherical circumradius and define the deviation functional $\vartheta^{\overline{0}}$ (for the class of spherical caps with centre $\overline{0}$) by

$$\vartheta^{\overline{0}}(K) := R_s(K) - r_s(K), \quad K\in\mathcal{K}_s^d.$$

Section 4.5 is devoted to the following extremal and stability result for $\tilde{U}$.

Theorem 4.5.1. *Let $a\in(0,\pi/2)$, $K\in\mathcal{K}_s^d$ with $r_s(K)\geq a$ and $C := B_s(\overline{0},a)$. Then*

$$\tilde{U}(K)\geq\tilde{U}(C) = \sigma_d(B_s(\overline{0},2a))$$

with equality if and only if $K = C$. *Furthermore, let* $K \subseteq B_s(\overline{0}, \pi/2)$ *and* $\vartheta^{\overline{0}}(K) \geq \varepsilon > 0$. *Then*

$$\tilde{U}(K) \geq \tilde{U}(C) \cdot (1 + c_{20} \cdot \varepsilon^d),$$

where the constant $c_{20} = c_{20}(a, d)$ *only depends on* a *and the dimension* d.

In the last section, Section 4.6, we modify some tools from Section 3.3 and eventually prove an asymptotic result for the typical cell Z of a spherical Poisson–Voronoi tessellation.

Theorem 4.6.4. *Let* $0 < a < \pi/2$, $\varepsilon > 0$ *with* $a + \varepsilon \leq \pi/2$. *Let* X *be a Poisson process on* $\mathbb{S}^d$ *with intensity* $\gamma_S > 1/\omega_{d+1}$. *Then*

$$\mathbb{P}(R_s(Z) - r_s(Z) \geq \varepsilon | r_s(Z) > a) \leq c_{23} \cdot \exp\left(-\gamma_S \cdot c_{24} \cdot \varepsilon^d\right),$$

where the constant $c_{24} > 0$ *depends only on* a *and* d *and the constant* $c_{23} > 0$ *depends only on* a, d *and* ε.

2. Distributional formulae in stereology

2.1. Stationary random measures and Palm measures

In this chapter, all random elements are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A random measure M on $\mathbb{R}^d$ (see e.g. [25]) is a random variable, taking its values in the space $\mathbf{M}$ of all locally bounded measures on $\mathbb{R}^d$, equipped with the σ-field $\mathcal{M}$ generated by the mappings $\alpha \mapsto \alpha(B)$, $B \in \mathcal{B}^d$, $\alpha \in \mathbf{M}$. Here $\mathcal{B}^d$ denotes the Borel σ-field on $\mathbb{R}^d$. For $k \in \{0, \dots, d\}$ let $G(d,k)$ be the set of all k-dimensional linear subspaces of $\mathbb{R}^d$ and denote by $A(d,k)$ the set of all k-dimensional affine subspaces of $\mathbb{R}^d$. Let $\mathbf{N} \subset \mathbf{M}$ denote the measurable set of all measures that are integer-valued on bounded sets. A *point process* N (on $\mathbb{R}^d$) is an $\mathbf{N}$-valued random variable. Such a point process is called *simple* if $\mathbb{P}(N \in \mathbf{N}_s) = 1$, where $\mathbf{N}_s$ denotes the space of all measures $\varphi \in \mathbf{N}$ satisfying $\varphi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$. Any element $\varphi \in \mathbf{N}_s$ is identified with its *support* $\operatorname{supp}(\varphi) = \{x \in \mathbb{R}^d : \varphi(\{x\}) > 0\}$, a locally finite subset of $\mathbb{R}^d$.

We assume $(\Omega, \mathcal{A})$ to be equipped with a *measurable flow* $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$. This is a family of measurable mappings such that $(\omega, x) \mapsto \theta_x\omega$ is measurable, $\theta_0\omega = \omega$ for all $\omega \in \Omega$ and

$$\theta_x \circ \theta_y = \theta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

Since

$$\theta_x \circ \theta_{-x} = \theta_{x-x} = \theta_0 = \mathrm{id}, \quad x \in \mathbb{R}^d,$$

these mappings are bijective.

Example 2.1.1. Assume that $(\Omega, \mathcal{A}) = (\mathbf{N}_s, \mathcal{M} \cap \mathbf{N}_s)$ and define $\theta_x\varphi := \varphi - x$, for $\varphi \in \mathbf{N}_s$ and $x \in \mathbb{R}^d$. Taking $\mathbb{P}$ as the distribution of a stationary simple point process yields a model as used above.

A random measure M on $\mathbb{R}^d$ is called *adapted* to the flow (or *stationary*), if

$$M(\theta_x\omega, B - x) = M(\omega, B), \quad \omega \in \Omega, x \in \mathbb{R}^d, B \in \mathcal{B}^d, \tag{2.1.1}$$

where $B + x := \{y + x : y \in B\}$. The probability measure $\mathbb{P}$ is assumed to be *stationary*, in the sense that

$$\mathbb{P} \circ \theta_x = \mathbb{P}, \quad x \in \mathbb{R}^d,$$

$$\mathbb{P} \circ \theta_x(A) := \mathbb{P}(\theta_x A) = \mathbb{P}(\{\theta_x\omega : \ \omega \in A\}), \ A \in \mathcal{A}.$$

For $\alpha \in \mathbf{M}$ we define $\alpha + x \in \mathbf{M}$ by $(\alpha + x)(B) = \alpha(B - x)$, $B \in \mathcal{B}^d$. If M is a flow-adapted random measure then the distribution of the shifted process $M + x = M \circ \theta_{-x}$ is the same for any $x \in \mathbb{R}^d$. In fact, if $C \in \mathcal{M}$,

$$\mathbb{P}(\{\omega \in \Omega : \ (M + x)(\omega, \cdot) \in C\}) = \mathbb{P}(\{\omega \in \Omega : \ M(\theta_{-x}\omega, \cdot) \in C\})$$

$$
\begin{aligned}
&= \mathbb{P}(\{\theta_x\omega \in \Omega : \ M(\theta_{-x}\theta_x\omega, \cdot) \in C\}) \\
&= \mathbb{P}(\{\theta_x\omega \in \Omega : \ M(\omega, \cdot) \in C\}) \\
&= (\mathbb{P} \circ \theta_x)(\{\omega \in \Omega : \ M(\omega, \cdot) \in C\}) \\
&= \mathbb{P}(\{\omega \in \Omega : \ M(\omega, \cdot) \in C\}). \qquad (2.1.2)
\end{aligned}
$$

Therefore, we call M just *stationary*. Let

$$\Lambda(B) := \mathbb{E}[M(B)], \ B \in \mathcal{B}^d,$$

denote the *intensity measure* of M. If M is stationary and Λ is locally finite, it is given by

$$\Lambda(dx) = \lambda_M \, \mathcal{H}^d(dx),$$

where $\lambda_M := \mathbb{E}[M([0,1]^d)]$ is the *intensity* of M and $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure, which on $\mathbb{R}^d$ equals the Lebesgue measure on $\mathbb{R}^d$. To confirm this, we use (2.1.2) to obtain for all $B \in \mathcal{B}^d$ and $x \in \mathbb{R}^d$

$$
\begin{aligned}
\Lambda(B + x) &= \mathbb{E}[M(B + x)] \\
&= \mathbb{E}[M(B)] \\
&= \Lambda(B).
\end{aligned}
$$

Since the Lebesgue measure is (up to a constant) the only locally finite translation invariant measure on $\mathbb{R}^d$, the assumption follows.

The measure

$$\mathbb{P}_M(A) := \iint \mathbf{1}\{\theta_x\omega \in A, x \in [0,1]^d\} \ M(\omega, dx) \ \mathbb{P}(d\omega), \quad A \in \mathcal{A}, \qquad (2.1.3)$$

is called the *Palm measure* of M. If Λ is locally finite, $\mathbb{P}_M$ is finite and satisfies the *refined Campbell theorem*

$$\mathbb{E}\left[\int f(\theta_x, x) \, M(dx)\right] = \mathbb{E}_M\left[\int f(\theta_0, x) \, \mathcal{H}^d(dx)\right] \qquad (2.1.4)$$

for all measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$, where $\mathbb{E}_M$ denotes the integral with respect to $\mathbb{P}_M$.

Proof.

$$
\begin{aligned}
\mathbb{P}_M(\Omega) &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{\theta_x\omega \in \Omega, x \in [0,1]^d\} \ M(\omega, dx) \ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\} \ M(\omega, dx) \ \mathbb{P}(d\omega) \\
&= \mathbb{E}[M([0,1]^d)] = \Lambda([0,1]^d) < \infty.
\end{aligned}
$$

Now we consider the measure $\tilde{M}$ on $\Omega \times \mathbb{R}^d$, defined by

$$\tilde{M}(A \times B) := \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{\theta_x \omega \in A\} \mathbf{1}\{x \in B\} \; M(\omega, dx) \; \mathbb{P}(d\omega), \quad B \in \mathcal{B}^d, \; A \in \mathcal{A}.$$

If we fix $A \in \mathcal{A}$, $\tilde{M}(A \times \cdot)$ is a translation invariant measure on $\mathbb{R}^d$. To obtain this, let $B \in \mathcal{B}^d$, $y \in \mathbb{R}^d$. Then

$$\begin{aligned}
\tilde{M}(A \times (B + y)) &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x \in B + y\} \mathbf{1}\{\theta_x \omega \in A\} \; M(\omega, dx) \; \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x \in B + y\} \mathbf{1}\{\theta_x \omega \in A\} \; M(\omega, dx) \; \mathbb{P} \circ \theta_y(d\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x \in B + y\} \mathbf{1}\{\theta_x \theta_{-y} \omega \in A\} \; M(\theta_{-y}\omega, dx) \; \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x + y \in B + y\} \mathbf{1}\{\theta_{x+y} \theta_{-y} \omega \in A\} \; M(\omega, dx) \; \mathbb{P}(d\omega) \\
&= \tilde{M}(A \times B),
\end{aligned}$$

using the stationarity of $\mathbb{P}$ and M. Therefore, we can decompose $\tilde{M}$ as the product of Lebesgue measure on $\mathbb{R}^d$ and some measure $\mathbb{P}_M^{(1)}$ on Ω, that is

$$\tilde{M} = \mathbb{P}_M^{(1)} \otimes \mathcal{H}^d.$$

Hence,

$$\mathbb{P}_M(A) = \iint \mathbf{1}\{x \in [0,1]^d, \theta_x \omega \in A\} \; M(\omega, dx) \; \mathbb{P}(d\omega) = \tilde{M}(A \times [0,1]^d) = \mathbb{P}_M^{(1)}(A)$$

for all $A \in \mathcal{A}$. Instead of $[0,1]^d$ we could have used any $B \in \mathcal{B}^d$ with $\mathcal{H}^d(B) = 1$ to define $\mathbb{P}_M$. In order to show (2.1.4), we take some measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$ and apply the decomposition from above:

$$\begin{aligned}
\int_\Omega \int_{\mathbb{R}^d} f(\theta_x \omega, x) \; M(\omega, dx) \; \mathbb{P}(d\omega) &= \int_\Omega \int_{\mathbb{R}^d} f(\omega, x) \; \tilde{M}(d\omega, dx) \\
&= \int_\Omega \int_{\mathbb{R}^d} f(\omega, x) \; \mathcal{H}^d(dx) \; \mathbb{P}_M(d\omega).
\end{aligned}$$

□

If $0 < \lambda_M < \infty$, then we can define the *Palm probability measure* $\mathbb{P}_M^0 := \lambda_M^{-1} \mathbb{P}_M$ of M.

Theorem 2.1.2. *(see [32]) Dropping the assumption of a locally finite intensity measure, $\mathbb{P}_M$ is still σ-finite.*

We prove this result after the following two lemmata.

Lemma 2.1.3. *(see [32]) For* $f : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to [0, \infty]$ *we have*

$$\int_{\mathbb{R}^d} \int_{\Omega} \int_{\mathbb{R}^d} f(t, s, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt = \int_{\mathbb{R}^d} \int_{\Omega} \int_{\mathbb{R}^d} f(s, t, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt. \tag{2.1.5}$$

Proof. Using the stationarity of $\mathbb{P}$ and M we get

$$\begin{aligned}\iiint f(t, s, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt &= \iiint f(t, s, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P} \circ \theta_{-t}(d\omega)\ dt \\ &= \iiint f(t, s, \theta_{s+t} \omega)\ M(\theta_t \omega, ds)\ \mathbb{P}(d\omega)\ dt \\ &= \iiint f(t, s-t, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt. \qquad (2.1.6)\end{aligned}$$

Analogously we obtain

$$\iiint f(s, t, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt = \iiint f(s-t, t, \theta_s \omega)\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt. \tag{2.1.7}$$

A change of variables yields

$$\int_{\mathbb{R}^d} f(t, s-t, \theta_s \omega) dt = \int_{\mathbb{R}^d} f(s-t, t, \theta_s \omega) dt.$$

Therefore, we obtain

$$\begin{aligned}(2.1.6) &= \iiint f(t, s-t, \theta_s \omega)\ dt\ M(\omega, ds)\ \mathbb{P}(d\omega) \\ &= \iiint f(s-t, t, \theta_s \omega)\ dt\ M(\omega, ds)\ \mathbb{P}(d\omega) \\ &= (2.1.7),\end{aligned}$$

using Fubini's theorem. □

Lemma 2.1.4. *A measure* μ *on* $(\Omega, \mathcal{A})$ *is* σ*-finite if and only if there is a measurable function* $f : \Omega \to [0, \infty]$ *such that* $f(\omega) > 0,\ \omega \in \Omega$, *and* $\int_\Omega f(\omega) \mu(d\omega) < \infty$.

Proof. If μ is σ-finite, there is a partition $\{A_1,\ A_2, \ldots\}$ of Ω such that $\mu(A_i) < \infty$ and A_i is measurable for all $i \in \mathbb{N}$. Defining $\frac{1}{\mu(A_i)} := +\infty$ if $\mu(A_i) = 0$ and $0 \cdot \infty := 0$, the function

$$f : \begin{cases} \Omega \to [0, \infty], \\ \omega \mapsto f(\omega) := \sum_{i=1}^\infty w_i \cdot \mathbf{1}_{A_i}(\omega) \cdot \frac{1}{\mu(A_i)}, \end{cases}$$

where $w_i > 0,\ \sum_{i=1}^\infty w_i < \infty$, fulfills all requirements.
On the other hand, if $f : \Omega \to [0, \infty]$, $f(\omega) > 0,\ \omega \in \Omega$, $\int f(\omega) \mu(d\omega) < \infty$, the sets

$$A_1 := \{\omega \in \Omega :\ f(\omega) \geq 1\},$$

$$A_n := \{\omega \in \Omega :\ \frac{1}{n} \leq f(\omega) < \frac{1}{n-1}\},\ n \geq 2,$$

form a measurable partition of Ω such that $\mu(A_i) < \infty,\ i \in \mathbb{N}$. □

Proof of Theorem 2.1.2: Let $B_1, B_2, \ldots$ be a Borel-measurable partition of $\mathbb{R}^d$ such that all B_i are bounded. We define the measurable function $h : \mathbb{R}^d \times \mathbf{M} \to [0, \infty]$ by

$$h(t, \alpha) := \begin{cases} \sum_{i=1}^{\infty} 2^{-i} \frac{1}{\alpha(B_i)} \mathbf{1}_{B_i}(t), & \alpha(\mathbb{R}^d) \neq 0, \\ \qquad 1 \qquad\qquad , & \alpha(\mathbb{R}^d) = 0. \end{cases}$$

If $\alpha(B_i) = 0$ we put $\frac{1}{\alpha(B_i)} = \infty$ and again $0 \cdot \infty = 0$. Then $h(t, \alpha) > 0,\ t \in \mathbb{R}^d,\ \alpha \in \mathbf{M}$ and

$$\int h(t, \alpha)\ \alpha(dt) \leq \sum_{i=1}^{\infty} 2^{-i} = 1. \tag{2.1.8}$$

Since $h(t, M(\omega)) > 0$ for all t and ω (M is our stationary random measure), we have $h(t, M(\theta_{-t}\omega)) > 0$ for all t and ω and thus

$$\int h(t, M(\theta_{-t}\omega))\ dt > 0, \quad \omega \in \Omega.$$

Using the definition of $\mathbb{P}_M$, Fubini's theorem and Lemma 2.1.3, applied to the function $f(t, s, \omega) := \mathbf{1}\{s \in [0,1]^d\} h(t, M(\theta_{-t}\omega))$, we obtain

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} h(t, M(\theta_{-t}\omega))\ dt\ \mathbb{P}_M(d\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{s \in [0,1]^d\} h(t, M(\theta_{-t+s}\omega))\ dt\ M(\omega, ds)\ \mathbb{P}(d\omega) \\
&= \int_{\mathbb{R}^d} \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{s \in [0,1]^d\} h(t, M(\theta_{-t}\theta_s\omega))\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt \\
&= \int_{\mathbb{R}^d} \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{t \in [0,1]^d\} h(s, M(\theta_{-s}\theta_s\omega))\ M(\omega, ds)\ \mathbb{P}(d\omega)\ dt \\
&= \int_\Omega \int_{\mathbb{R}^d} h(s, M(\omega))\ M(\omega, ds)\ \mathbb{P}(d\omega) \\
&\leq \int_\Omega 1\ \mathbb{P}(d\omega) = 1 < \infty,
\end{aligned}$$

where we have used (2.1.8) for the estimation. Lemma 2.1.4 finishes the proof. □

If N is a stationary simple point process, we will always assume that $\mathbb{P}(N = \emptyset) = 0$. In consequence, it follows that the convex hull of N is almost surely given by $\mathbb{R}^d$ (see [49, Theorem 2.4.4]). We refer to [32] for the definition of $\mathbb{P}_M$ and $\mathbb{P}_M^0$ in a canonical framework, and to [42] for the $\{\theta_x\}$-framework for point processes.

2.2. Stationary tessellations

In introducing random tessellations, we largely follow Chapter 10 in [49]. A (deterministic) tessellation $\mathbf{m}$ of $\mathbb{R}^d$ is a countable system of compact and convex subsets of $\mathbb{R}^d$ (*cells*) such that the following properties hold. First, each cell has a non-empty interior. Second, the union of the cells is all of $\mathbb{R}^d$ and the interiors of different cells are disjoint. Third, any bounded subset of $\mathbb{R}^d$ is intersected by only finitely many of the cells. Given a tesselation $\mathbf{m}$ we write $\mathcal{S}_d(\mathbf{m})$ for the system of cells of $\mathbf{m}$.

If $\mathbf{m}$ is a tessellation, then any element F of $\mathcal{S}_d(\mathbf{m})$ is in fact a convex polytope (see Lemma 10.1.1 in [49]). Hence, its boundary can be decomposed into lower-dimensional polytopes called *faces* of F. A k-dimensional face ($k \in \{0, \dots, d\}$) of F is called *k-face* of F and of $\mathbf{m}$. The cells are also called d-faces. We denote by $\mathcal{S}_k(F)$ the system of all k-faces of F and define

$$\mathcal{S}_k(\mathbf{m}) := \bigcup_{F \in \mathcal{S}_d(\mathbf{m})} \mathcal{S}_k(F)$$

as the system of all k-faces of $\mathbf{m}$ and

$$\mathcal{S}(\mathbf{m}) := \bigcup_{k \in \{0,\dots,d\}} \mathcal{S}_k(\mathbf{m})$$

as the system of all faces of $\mathbf{m}$. A tessellation $\mathbf{m}$ is called *regular* if for any $F, F' \in \mathbf{m}$, the intersection $F \cap F'$ is either empty or a face of both F and F'. (A regular tessellation $\mathbf{m}$ is called *normal*, if any k-face is contained in exactly $d - k + 1$ cells.)

The system of closed subsets of some locally compact topological space E with a countable base will be denoted by $\mathcal{F}(E)$, the Borel-σ-algebra on $\mathcal{F}(E)$ will be denoted by $\mathcal{B}(\mathcal{F}(E))$. We equip the space $\mathbf{T}$ of all tessellations with the trace σ-algebra induced by the Fell-topology on $\mathcal{F}(\mathcal{F}(\mathbb{R}^d))$. The space of all regular tessellations will be denoted by $\mathbf{T}^*$. We write $\mathcal{F}' = \mathcal{F}\backslash\{\emptyset\}$. The necessary measurability statements are summarized in the following lemma (see [49, Lemma 10.1.2]).

Lemma 2.2.1. *The sets $\mathbf{T}$ and $\mathbf{T}^*$ are measurable sets in $\mathcal{F}(\mathcal{F}')$, the set of closed subsets of $\mathcal{F}'$. The map*

$$\varphi_k : \begin{cases} \mathbf{T}^* & \to \mathcal{F}(\mathcal{F}') \\ m & \mapsto \mathcal{S}_k(m) \end{cases}$$

is measurable for $k \in \{0, \dots, d-1\}$.

We now consider a random tessellation X as a measurable mapping from the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the space $\mathbf{T}^*$ of all (regular) tessellations. To deal with exceptional cases it is convenient to include an extra (trivial) partition $\mathbf{m}_\infty = \{\mathbb{R}^d\}$ into $\mathbf{T}$ and to set $\mathcal{S}_k(\mathbf{m}_\infty) := \emptyset$ for $k \in \{0, \dots, d-1\}$. It is assumed then that $\mathbb{P}(X = \mathbf{m}_\infty) = 0$.

A random tessellation X is called *stationary*, if

$$X \circ \theta_x = \{C - x : C \in X\}. \tag{2.2.1}$$

This definition is consistent with the definition of stationarity for random measures, see (2.1.1). It follows that $\{X = \mathbf{m}_\infty\}$ is a shift-invariant set and that

$$\mathcal{S}_k(X \circ \theta_x) = \{F - x : F \in \mathcal{S}_k(X)\}, \quad k \in \{0, \dots, d\}. \tag{2.2.2}$$

Let X be a stationary tessellation and $k \in \{0, \dots, d\}$. Since $\mathcal{H}^k(F \cap \cdot)$, for $F \in \mathcal{S}_k(X)$, is almost surely a locally finite measure on $\mathbb{R}^d$ and the mapping $F \mapsto \mathcal{H}^k(F \cap B)$ is almost surely measurable for all $B \in \mathcal{B}(\mathbb{R}^d)$ (see [54, Corollary 2.1.4]), we have that

$$M_k := \sum_{F \in \mathcal{S}_k(X)} \mathcal{H}^k(F \cap \cdot) \tag{2.2.3}$$

is a random measure on $\mathbb{R}^d$.

Due to (2.2.2), M_k is stationary in the sense of (2.1.1):

$$\begin{aligned} M_k(\omega, B+x) &= \sum_{F\in\mathcal{S}_k(X(\omega))} \mathcal{H}^k(F\cap(B+x)) \\ &= \sum_{F\in\mathcal{S}_k(X(\omega))} \mathcal{H}^k((F-x)\cap B) \\ &= \sum_{F\in\mathcal{S}_k(X(\theta_x\omega))} \mathcal{H}^k(F\cap B) \\ &= M_k(\theta_x\omega, B). \end{aligned}$$

Moreover, it is supported by the *random closed set* (see [49, page 464])

$$Y_k := \bigcup_{F\in\mathcal{S}_k(X)} F. \tag{2.2.4}$$

Note that Y_k is stationary in the sense that $Y_k\circ\theta_x = Y_k - x$ for all $x\in\mathbb{R}^d$. Whenever $x\in\mathbb{R}^d$ is in the relative interior of some $F\in\mathcal{S}_k(X(\omega))$, we define $F_k(\omega,x) := F$. Otherwise we define $F_k(\omega,x) := \{x\}$. It follows that

$$F_k(\theta_x\omega, 0) = F_k(\omega,x) - x, \quad x\in\mathbb{R}^d. \tag{2.2.5}$$

Now we want to introduce a random measure N_k, whose Palm measure is closely related to $\mathbb{P}_{M_k}$. We follow the approach in [4] and [5]. For any non-empty compact and convex set $C\subset\mathbb{R}^d$, let $c(C)$ denote its Steiner point (see [49, (14.28), p. 613]). Then $c(C+x) = c(C)+x,\ x\in\mathbb{R}^d$, and $c(C)$ is in the (relative) interior of C. We define the stationary random measure N_k by

$$N_k := \sum_{F\in\mathcal{S}_k(X)} \delta_{c(F)}.$$

If its intensity is finite, we can consider the Palm probability measure $\mathbb{P}^0_{N_k}$. Under this measure, the origin is almost surely a centre of a k-face ($F_k(0)$), the *area-debiased typical k-face* of X (see [4], the remark after Formula (2.10)).

2.3. Sections through tessellations: Notation and preliminary results

Let X be a stationary tessellation. We fix $l\in\{1,\dots,d\}$ and an l-dimensional affine subspace $S\subset\mathbb{R}^d$. By U_S we denote the linear subspace parallel to S.

Lemma 2.3.1.

$$X_S := \{C\cap S : C\in X,\ C\cap S\neq\emptyset\}$$

can be considered as a tessellation in S. Moreover, X_S is stationary in the sense that

$$X_S\circ\theta_y = \{C-y : C\in X_S\}, \quad y\in U_S.$$

Proof. In case of $l = d$ there is nothing to show, so we assume $l \leq d-1$. The elements of X_S are trivially compact and convex. Since X is almost surely a countable system, so is X_S. In order to show that X_S is locally finite, we take some bounded Borel set $B \subset S$. Then B is a bounded subset of $\mathbb{R}^d$ and $\{F \in X : \; F \cap B \neq \emptyset\}$ contains almost surely only finitely many elements. Therefore, $\#\{F \in X_S : \; F \cap B \neq \emptyset\}$ is almost surely finite too. Furthermore,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{F \in X} \mathbf{1}\{F \cap S \neq \emptyset, \; \operatorname{int}(F) \cap S = \emptyset\}\right] \\
&= \; \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{F_d(x) \cap S \neq \emptyset, \; \operatorname{int}(F_d(x)) \cap S = \emptyset\} \; N_d(dx)\right] \\
&= \; \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{1}\{((F_d \circ \theta_x)(0) + x) \cap S \neq \emptyset, \; \operatorname{int}((F_d \circ \theta_x)(0) + x) \cap S = \emptyset\} \; N_d(dx)\right] \\
&= \; \mathbb{E}_{N_d}\left[\int_{\mathbb{R}^d} \mathbf{1}\{(F_d(0) + x) \cap S \neq \emptyset, \; \operatorname{int}(F_d(0) + x) \cap S = \emptyset\} \; \mathcal{H}^d(dx)\right].
\end{aligned}$$

Now we consider the set $A := \{x \in \mathbb{R}^d : \; (F_d(\omega,0)+x) \cap S \neq \emptyset, \; \operatorname{int}(F_d(\omega,0)+x) \cap S = \emptyset\}$. Since $F_d(0)$ is almost surely a convex polytope, the following holds almost surely:

$$\begin{aligned}
-A = \{-x \in \mathbb{R}^d : \; &\text{there exists } y \in F_d(0), \; s \in S \text{ s.t. } y + x = s \\
&\text{and for all } y \in \operatorname{int}(F_d(0)), \; s \in S \text{ we have } y + x \neq s\} \\
= \{x \in \mathbb{R}^d : \; &x \in F_d(\omega,0) - S, \; x \notin \operatorname{int}(F_d(\omega,0)) - S\} \\
= \{x \in \mathbb{R}^d : \; &x \in F_d(\omega,0) + S, \; x \notin \operatorname{int}(F_d(\omega,0)) + S\} \\
= \{x \in \mathbb{R}^d : \; &x \in F_d(\omega,0) + S, \; x \notin \operatorname{int}\left(F_d(\omega,0) + S\right)\}.
\end{aligned}$$

Therefore, $\mathcal{H}^d(A) = 0$ $\mathbb{P}_{N_d}$-almost surely and

$$\mathbb{P}(\text{there is an } F \in X : \; F \cap S \neq \emptyset, \; \operatorname{int}(F) \cap S = \emptyset) = 0.$$

It follows that $\dim(F \cap S) = l$ almost surely for every $F \in X$ with $F \cap S \neq \emptyset$. Additionally, for every x in the relative interior of some $C \in X_S$, we have x is in the interior of the cell $F \in X$, where $C = F \cap S$. With that we can conclude that different cells in X_S do have disjoint interiors. The stationarity is inherited from the stationarity of the tessellation X. For $y \in U_S$ and $C \in X$, we have $(C - y) \cap S \neq \emptyset \;\Leftrightarrow\; C \cap S \neq \emptyset$ and therefore

$$\begin{aligned}
X_S \circ \theta_y &= \{C \cap S : C \in X \circ \theta_y, \; C \cap S \neq \emptyset\} \\
&= \{(C - y) \cap S : C \in X, \; (C_y) \cap S \neq \emptyset\} \\
&= \{(C \cap S) - y : C \in X, \; C \cap S \neq \emptyset\} \\
&= \{C - y : C \in X_S\}.
\end{aligned}$$

□

Lemma 2.3.2. *Let P be a polytope and Q be a polyhedral set in $\mathbb{R}^d$ such that their relative interiors have nonempty intersection. Then $P \cap Q$ is a polytope and its faces are intersections of faces of P with faces of Q.*

Proof. Since P is a polytope, it is a bounded, polyhedral set (see [52, Theorem 3.2.5]). Thus $P \cap Q$ is a polyhedral set and since P is bounded, $P \cap Q$ is also bounded. From [52, Theorem 3.2.5] it follows that $P \cap Q$ is a polytope.

Let F_1 be a face of P and F_2 be a face of Q such that $F_1 \cap F_2 \neq \emptyset$. Furthermore let $x, y \in P \cap Q$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$ such that $\lambda x + \mu y \in F_1 \cap F_2$. Then $\lambda x + \mu y \in F_1$ and by the definition of a face (see e.g. [52, p. 79]) we have $x, y \in F_1$. Analogously we obtain $x, y \in F_2$ and therefore $x, y \in F_1 \cap F_2$. This means $F_1 \cap F_2$ is a face of $P \cap Q$.

Now let $F \neq \emptyset$ be a face of $P \cap Q$. Then F is convex and we choose some $z \in \operatorname{relint}(F)$. Then $z \in P$ and $z \in Q$ and by [52, Theorem 2.6.10 and Corollary 2.6.7], there is a unique face F_1 of P and a unique face F_2 of Q such that $z \in \operatorname{relint}(F_1)$ and $z \in \operatorname{relint}(F_2)$. As shown above, $F_1 \cap F_2$ is a face of $P \cap Q$. Since $z \in \operatorname{relint}(F_1) \cap \operatorname{relint}(F_2)$, [45, Theorem 6.5] gives $\operatorname{relint}(F_1) \cap \operatorname{relint}(F_2) = \operatorname{relint}(F_1 \cap F_2)$. Therefore $\operatorname{relint}(F) \cap \operatorname{relint}(F_1 \cap F_2) \neq \emptyset$ and [52, Corollary 2.6.7] gives $F = F_1 \cap F_2$. □

For the rest of this chapter we assume now that X and S are in *general position*. This means, that for all $k \in \{0, \dots, d\}$ the following holds $\mathbb{P}$-almost surely: If U_F denotes the subspace parallel to $F \in \mathcal{S}_k(X)$, then $\dim(U_F \cap U_S) = 0$ or $U_F + U_S = \mathbb{R}^d$.

Lemma 2.3.3. *Let $j \in \{0, \dots, l\}$, $l = \dim(S)$. Then*

$$\mathcal{S}_j(X_S) = \{F \cap S : F \in \mathcal{S}_{d-l+j}(X),\ F \cap S \neq \emptyset\} \quad \mathbb{P}\text{-}a.s.$$

Proof. Beforehand, we want to show that for $F \in \mathcal{S}_{d-l+j}(X)$ with $F \cap S \neq \emptyset$ we have $\operatorname{relint}(F) \cap S \neq \emptyset$ almost surely. We abbreviate $k := d - l + j$ and use the same line of arguments as used in the proof of Lemma 2.3.1. Consider the set

$$A^{(k)} := \{x \in \mathbb{R}^d :\ \operatorname{relint}(F_k(\omega, 0) + x) \cap S = \emptyset,\ (F_k(\omega, 0) + x) \cap S \neq \emptyset\}$$

and note that $F_k(0)$ is $\mathbb{P}_{N_k}$-almost surely a convex and compact polytope of dimension k. Then

$$\begin{aligned} -A^{(k)} &= \{-x \in \mathbb{R}^d :\ \text{there exists } y \in F_k(\omega, 0),\ s \in S \text{ s.t. } y + x = s \\ &\qquad\qquad \text{and for all } y \in \operatorname{relint}(F_k(\omega, 0)), s \in S \text{ we have } y + x \neq s\} \\ &= \{x \in \mathbb{R}^d :\ x \in F_k(\omega, 0) - S,\ x \notin \operatorname{relint}(F_k(\omega, 0)) - S\} \\ &= \{x \in \mathbb{R}^d :\ x \in F_k(\omega, 0) + S,\ x \notin \operatorname{relint}(F_k(\omega, 0)) + S\} \\ &= \{x \in F_k(\omega, 0) + S :\ x \notin \operatorname{relint}(F_k(\omega, 0) + S)\} \end{aligned}$$

and $F_k(\omega, 0) + S$ is $\mathbb{P}_{N_k}$-almost surely closed and convex. Therefore, $\mathcal{H}^d(A^{(k)}) = 0$ $\mathbb{P}_{N_k}$-almost surely and using (2.1.4) for $\mathbb{P}_{N_k}$ and (2.2.5) we obtain

$$\mathbb{E} \sum_{F \in \mathcal{S}_k(X)} \mathbf{1}\{\operatorname{relint}(F) \cap S = \emptyset,\ F \cap S \neq \emptyset\}$$

$$= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{\mathrm{relint}(F_k(\omega, x)) \cap S = \emptyset,\ F_k(\omega, x) \cap S \neq \emptyset\}\ N_k(\omega, dx)\ \mathbb{P}(d\omega)$$

$$= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{\mathrm{relint}(F_k(\theta_{-x}\omega, x)) \cap S = \emptyset,\ F_k(\theta_{-x}\omega, x) \cap S \neq \emptyset\}\ \mathcal{H}^k(dx)\ \mathbb{P}_{N_k}(d\omega)$$

$$= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{\mathrm{relint}(F_k(\omega, 0) + x) \cap S = \emptyset,\ (F_k(\omega, 0) + x) \cap S \neq \emptyset\}\ \mathcal{H}^k(dx)\ \mathbb{P}_{N_k}(d\omega)$$

$$= \int_\Omega \mathcal{H}^d(A_k)\ \mathbb{P}_{N_k}(d\omega) = 0.$$

This means

$$\mathbb{P}(\text{there exists } F \in \mathcal{S}_{d-l+j} :\ F \cap S \neq \emptyset,\ \mathrm{relint}(F) \cap S = \emptyset) = 0.$$

Then $\dim(F \cap S) = \dim(U_F \cap U_S)$ almost surely. The dimension formula gives

$$d \geq \dim(U_F + U_S) = d - l + j + l - \dim(U_F \cap U_S).$$

Due to the assumption of general position, there are two possibilities. If $\dim(U_F \cap U_S) = 0$, the inequality above reads $d \geq d + j$ and thus, $j = 0$. If $\dim(U_F + U_S) = d$, we get $d = d + j - \dim(U_F \cap U_S)$ and therefore $\dim(U_F \cap U_S) = j$. It remains to show that for every $F \in \mathcal{S}_j(X_S)$ there is almost surely an $\hat{F} \in \mathcal{S}_{d-l+j}(X)$ such that $F = \hat{F} \cap S$. However, this is an immediate consequence of Lemma 2.3.2 and the fact that for $\hat{F} \in \mathcal{S}_{d-l+j}(X)$ we have almost surely $\dim(\hat{F} \cap S) = j$. □

Let $S \in A(d, l)$ be an l-dimensional affine subspace of $\mathbb{R}^d$. As an immediate consequence, the measure $M_{S,j}$, defined by

$$M_{S,j} := \sum_{F \in \mathcal{S}_{d-l+j}(X)} \mathcal{H}^j(F \cap S \cap \cdot), \tag{2.3.1}$$

is $\mathbb{P}$-almost surely a locally finite measure on $\mathbb{R}^d$. To show this, we use Lemma 2.3.3 to obtain

$$M_{S,j} = \sum_{F \in \mathcal{S}_j(X_S)} \mathcal{H}^j(F \cap \cdot) \quad \mathbb{P}\text{-almost surely}$$

and $\mathcal{S}_j(X_S)$ is a locally finite system of j-dimensional polytopes. Since X is stationary, we can identify $M_{S,j}$ with a stationary random measure on S. Here stationarity clearly refers to the equations $M_{S,j}(\theta_y\omega, \cdot) = M_{S,j}(\omega, \cdot + y)$ for all $\omega \in \Omega$ and all $y \in U_S$. To confirm this we use the definition of $M_{S,j}$ and (2.2.2) to obtain

$$\begin{aligned} M_{S,j}(\theta_y\omega, \cdot) &= \sum_{F \in \mathcal{S}_{d-l+j}(X(\theta_y\omega))} \mathcal{H}^j(F \cap S \cap \cdot) = \sum_{F \in \mathcal{S}_{d-l+j}(X(\omega))} \mathcal{H}^j((F - y) \cap S \cap \cdot) \\ &= \sum_{F \in \mathcal{S}_{d-l+j}(X(\omega))} \mathcal{H}^j(((F \cap S) - y) \cap \cdot) = M_{S,j}(\omega, \cdot + y). \end{aligned}$$

Hence, the Palm measure of $M_{S,j}$ is given by

$$\mathbb{P}_{M_{S,j}}(A) = \iint \mathbf{1}\{\theta_x\omega \in A, x \in B \cap S\}\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega), \quad A \in \mathcal{A}, \tag{2.3.2}$$

where $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\mathcal{H}^l(B \cap S) = 1$. For convenience in later proofs, we assume B to be a cube with edge length 1, an l-face in S and a $(d-l)$-face in $(U_S)^\perp + y$ for some $y \in S$, using the usual construction for Palm measures in the Euclidean space S.

For $j \in \{0, \dots, l\}$, a $(d-l+j)$-dimensional linear space $U \subset \mathbb{R}^d$ and an l-dimensional subspace S we define a number $[S, U]_j$ as follows (see [49, p. 598]). Whenever $\dim(S \cap U) > j$ we put $[S, U]_j := 0$. Otherwise we choose an orthonormal basis in $S \cap U$ and extend it first to an orthonormal basis of S, then to an orthonormal basis of U. Then we define $[S, U]_j$ to be the d-dimensional volume of the parallelepiped spanned by the obtained vectors. This also shows that $[S, U]_j = [U, S]_j$. If F is a $(d-l+j)$-dimensional affine subspace, we define $[S, F]_j$ as $[S, U_F]_j$. If S is also an affine subspace, we define $[S, F]_j$ as $[U_S, U_F]_j$. For convenience, we define $[S, F]_j := 0$ whenever F is an affine subspace whose dimension does not equal $d - l + j$. Note that $[S, \mathbb{R}^d]_l = 1$ in case $j = l$.

2.4. Sections through tessellations: Distributional formulae

We are now in a position to formulate the main result of this chapter. This result is a general (distributional) version of a well-known principle of stereology, see e.g. [49, Theorem 4.4.7], which describes the directional distribution of the intersection of a stationary k-flat process ($k \in \{2, \dots, d-1\}$) with a fixed linear subspace $S \in G(d, d-k+j)$ as an integral with respect to the directional distribution of the original process. See also [49, Theorem 4.5.3], where the specific j-volume of intersections of stationary k-surface processes with linear subspaces of dimension $d - k + j$ is computed, using the specific k-volume of the original process and an integral with respect to its directional distribution.

Let X be a stationary tessellation of $\mathbb{R}^d$ and $S \in G(d, l)$, $l \in \{1, \dots, d\}$, a linear subspace. We assume X and S to be in general position.

Theorem 2.4.1. *For all $j \in \{0, \dots, l\}$ and all measurable $f : \Omega \to [0, \infty)$ we have that*

$$\mathbb{E}_{M_{S,j}}[f] = \mathbb{E}_{M_{d-l+j}}[[S, F_{d-l+j}(0)]_j \cdot f]. \tag{2.4.1}$$

An easy example is $f \equiv 1$, giving a relation between the intensities, see also (2.5.5) or Corollary 2.5.4.

Whenever $x \in S$ is in the relative interior of some $F \in \mathcal{S}_j(X_S(\omega))$, we define $F_j^S(\omega, x) := F$. Otherwise we define $F_j^S(\omega, x) := \{x\}$. As another example we choose f as the ith intrinsic volume, $i \in \{0, \dots, j\}$, of the volume weighted typical $(d-l+j)$-face of X intersected with S. Under $\mathbb{P}_{M_{S,j}}$, the volume weighted typical j-face $F_j^S(0)$ of X_S is $F_{d-l+j}(0) \cap S$, hence

$$\mathbb{E}_{M_{S,j}}[V_i(F_j^S(0))] = \mathbb{E}_{M_{d-l+j}}[[S, F_{d-l+j}(0)]_j \cdot V_i(F_{d-l+j}(0) \cap S)].$$

We provide two different proofs for this theorem, the first is somehow more straightforward, while the second is based on another general result, Theorem 2.4.4. We also show that Theorem 2.4.1 can be used to prove Theorem 2.4.4.

Let $S \in A(d, l)$. Note, that the measures $M_{S,j}$ are only adapted with respect to θ_y, $y \in U_S$. In particular, any of these measures are concentrated on S. Moreover, for $x \in U_S^\perp$

and any measurable function $g : S \to [0,\infty)$, we have

$$
\begin{aligned}
\int_S g(y)\ M_{S,j}(\theta_x\omega, dy) &= \sum_{F\in\mathcal{S}_{d-l+j}(X(\theta_x\omega))} \int_{\mathbb{R}^d} \mathbf{1}_{F\cap S}(y) g(y)\ \mathcal{H}^j(dy) \\
&= \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{\mathbb{R}^d} \mathbf{1}_{(F-x)\cap S}(y) g(y)\ \mathcal{H}^j(dy) \\
&= \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{\mathbb{R}^d} \mathbf{1}_{F\cap(S+x)}(y) g(y-x)\ \mathcal{H}^j(dy) \\
&= \int_{S+x} g(y-x)\ M_{S+x,j}(\omega, dy), \qquad (2.4.2)
\end{aligned}
$$

where we used the translation invariance of $\mathcal{H}^j$. Due to the stationarity of $\mathbb{P}$, we have $M_{S,j} \circ \theta_x \overset{d}{=} M_{S,j}$.

Lemma 2.4.2. *For measurable $f : \mathbb{R}^d \to [0,\infty)$ and S as in Theorem 2.4.1, we have $\mathbb{P}$-a.s.*

$$\int_{\mathbb{R}^d} [S, F_k(\omega,z)]_j f(z)\ M_k(\omega, dz) = \int_{S^\perp} \int_{S+y} f(x) M_{S+y,j}(\omega, dx)\ \mathcal{H}^{d-l}(dy), \qquad (2.4.3)$$

where $k := d-l+j$.

Proof. Due to the definition of M_k and M_{S+y}, the right-hand side of (2.4.3) equals

$$
\begin{aligned}
\int_{S^\perp} \int_{S+y} f(x)\ M_{S+y,j}(\omega, dx)\ \mathcal{H}^{d-l}(dy) &= \int_{S^\perp} \sum_{F\in\mathcal{S}_k(X(\omega))} \int_{F\cap(S+y)} f(x)\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(dy) \\
&= \sum_{F\in\mathcal{S}_k(X(\omega))} \int_{S^\perp} \int_{F\cap(S+y)} f(x)\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(dy)
\end{aligned}
$$

and the left-hand side of (2.4.3) equals

$$\int_{\mathbb{R}^d} [S, F_k(\omega,z)]_j f(z)\ M_k(\omega, dz) = \sum_{F\in\mathcal{S}_k(X(\omega))} \int_F [F,S]_j f(z)\ \mathcal{H}^k(dz).$$

Thus, it is sufficient to prove

$$[F,S]_j \int_F f(x) \mathcal{H}^k(dx) = \int_{S^\perp} \int_{F\cap(S+y)} f(x)\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(dy),$$

for an arbitrary but fixed k-dimensional polytope F, such that U_F and S are in general position.

We consider the following orthonormal bases

$$U_F = [a_1, \ldots, a_{d-l}, \ldots, a_{d-l+j}],$$

$$U_F \cap (U_F \cap S)^\perp = [a_1, \ldots, a_{d-l}],$$
$$S = [b_1, \ldots, b_l],$$
$$S^\perp = [c_1, \ldots, c_{d-l}],$$

where $b_1 = a_{d-l+1}, \ldots, b_j = a_{d-l+j}$. For the subspace determinant the following equalities hold

$$[S,F]_j = [F,S]_j = |\det(a_1, \ldots, a_{d-l}, b_1, \ldots, b_l)| = |\det(\langle a_i, c_n\rangle_{i,n=1}^{d-l})|.$$

The first equations can be found in [49] page 598. For the latter we use

$$a_i = \sum_{n=1}^{l} \langle a_i, b_n\rangle b_n + \sum_{n=1}^{d-l} \langle a_i, c_n\rangle c_n$$

to get

$$|\det(a_1, \ldots, a_{d-l}, b_1, \ldots, b_l)| = |\det(\langle a_i, c_n\rangle_{i,n=1}^{d-l})|.$$

Since F and S are in general position, the orthogonal projection

$$T : U_F \cap (U_F \cap S)^\perp \to \ S^\perp$$

is bijective and has Jacobian $[S,F]_j$.

In a first step, we assume $F \subset U_F$. A change of variables yields

$$\begin{aligned}
I &:= \int_{S^\perp} \int_{F \cap (S+y)} f(x)\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(dy) \\
&= [S,F]_j \int_{U_F \cap (U_F \cap S)^\perp} \int_{U_F \cap (S+T(\tilde{y}))} f(x)\mathbf{1}\{x \in F \cap (S + T(\tilde{y}))\}\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(d\tilde{y}).
\end{aligned}$$

Observing $S + T(\tilde{y}) = S + \tilde{y}$ and

$$U_F \cap (S + \tilde{y}) = U_F \cap ((U_F + \tilde{y}) \cap (S + \tilde{y})) = U_F \cap ((U_F \cap S) + \tilde{y})$$

for all $\tilde{y} \in U_F \cap (U_F \cap S)^\perp$, we get

$$\begin{aligned}
I &= [S,F]_j \int_{U_F \cap (U_F \cap S)^\perp} \int_{U_F \cap ((U_F \cap S)+\tilde{y})} f(x)\mathbf{1}\{x \in F \cap ((U_F \cap S) + \tilde{y})\}\ \mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(d\tilde{y}) \\
&= [S,F]_j \int_{U_F} f(z)\mathbf{1}\{z \in F\}\ \mathcal{H}^k(dz)
\end{aligned}$$

using Fubini's theorem.

For general F, we choose vectors $y_S \in S$ and $y_S^\perp \in S^\perp$ such that $F = F_0 + y_S + y_S^\perp$ and $F_0 \subset U_F$. Then $[S,F_0]_j = [S,F]_j$ and

$$\begin{aligned}
&\int_{S^\perp} \int_{F \cap (S+y)} f(x)\mathcal{H}^j(dx)\ \mathcal{H}^{d-l}(dy) \\
&= \int_{S^\perp} \int f(\tilde{x} + y_S + y_S^\perp)\mathbf{1}\{\tilde{x} \in F_0 \cap (S + y - y_S - y_S^\perp)\}\ \mathcal{H}^j(d\tilde{x})\ \mathcal{H}^{d-l}(dy)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S^\perp} \int f(\tilde{x} + y_S + y_S^\perp) \mathbf{1}\{\tilde{x} \in F_0 \cap (S + y - y_S^\perp)\} \; \mathcal{H}^j(d\tilde{x}) \; \mathcal{H}^{d-l}(dy) \\
&= \int_{S^\perp} \int \underbrace{f(\tilde{x} + y_S + y_S^\perp)}_{:=g(\tilde{x})} \mathbf{1}\{\tilde{x} \in F_0 \cap (S + \tilde{y})\} \; \mathcal{H}^j(d\tilde{x}) \; \mathcal{H}^{d-l}(d\tilde{y}) \\
&= [S, F_0]_j \int g(z) \mathbf{1}\{z \in F_0\} \; \mathcal{H}^k(dz) \\
&= [S, F]_j \int f(z + y_S + y_S^\perp) \mathbf{1}\{z \in F_0\} \; \mathcal{H}^k(dz) = [S, F]_j \int_F f(\tilde{z}) \; \mathcal{H}^k(d\tilde{z}).
\end{aligned}
$$

□

Proof of Theorem 2.4.1: Let B be as in (2.3.2). Using the stationarity of $\mathbb{P}$, Fubini, (2.3.2) and (2.4.2) we get

$$
\begin{aligned}
\mathbb{E}_{M_{S,j}}[f] &= \int_\Omega \int_S \mathbf{1}_{B \cap S}(x) f(\theta_x \omega) \; M_{S,j}(\omega, dx) \mathbb{P}(d\omega) \\
&= \int_{S^\perp} \int_\Omega \int_S \mathbf{1}_{B \cap S^\perp}(y) \mathbf{1}_{B \cap S}(x) f(\theta_x \omega) \; M_{S,j}(\omega, dx) \; \mathbb{P} \circ \theta_{-y}(d\omega) \; \mathcal{H}^{d-l}(dy) \\
&= \int_{S^\perp} \int_\Omega \int_S \mathbf{1}_{B \cap S^\perp}(y) \mathbf{1}_{B \cap S}(x) f(\theta_{x+y} \omega) \; M_{S,j}(\theta_y \omega, dx) \; \mathbb{P}(d\omega) \; \mathcal{H}^{d-l}(dy) \\
&= \int_\Omega \int_{S^\perp} \int_{S+y} \mathbf{1}_{B \cap S^\perp}(y) \mathbf{1}_{B \cap S}(x - y) f(\theta_x \omega) \; M_{S+y,j}(\omega, dx) \; \mathcal{H}^{d-l}(dy) \; \mathbb{P}(d\omega),
\end{aligned}
$$

since $B \cap S$ and $B \cap S^\perp$ are unit cubes in S and $S^\perp$, respectively. The equivalence

$$
x - y \in B \cap S, y \in B \cap S^\perp \;\Leftrightarrow\; x \in B, x - y \in S, y \in S^\perp,
$$

Lemma 2.4.2, (2.2.5) (a consequence of the stationarity of $\mathcal{S}_k(X)$) and the definition of M_k then give

$$
\begin{aligned}
\mathbb{E}_{M_{S,j}}[f] &= \int_\Omega \int_{S^\perp} \int_{S+y} \mathbf{1}_B(x) f(\theta_x \omega) \; M_{S+y,j}(\omega, dx) \; \mathcal{H}^{d-l}(dy) \; \mathbb{P}(d\omega) \\
&= \iint \mathbf{1}_B(z) [S, F_k(\omega, z)]_j f(\theta_z \omega) \; M_k(\omega, dz) \; \mathbb{P}(d\omega) \\
&= \iint \mathbf{1}_B(z) [S, F_k(\theta_z \omega, 0)]_j f(\theta_z \omega) \; M_k(\omega, dz) \; \mathbb{P}(d\omega) \\
&= \mathbb{E}_{M_k}[[S, F_k(0)]_j \cdot f].
\end{aligned}
$$

□

For the next theorem we need to introduce some notation. Let S be a linear subspace of dimension l, let B_S denote the unit ball in S and define

$$
d_S(x, y) := \inf\{r \geq 0 : y \in x + rB_S\}, \quad x, y \in \mathbb{R}^d.
$$

Note that $d_S(x, y) = \infty$ iff $x - y \notin S$. For a closed set $A \subset \mathbb{R}^d$ we put

$$d_S(A, x) := \inf\{d_S(x, y) : y \in A\}$$

and if $d_S(A, x) < \infty$, we define

$$P_S(A, x) := \{y \in A : d_S(y, x) = d_S(A, x)\}.$$

If $d_S(A, x) = \infty$, we put $P_S(A, x) := \{x\}$. Next we define

$$\mathrm{exo}_S(A) := \{x \in \mathbb{R}^d : \mathrm{card}(P_S(A, x)) \geq 2\}.$$

For $x \in \mathbb{R}^d \setminus \mathrm{exo}_S(A)$ and $d_S(A, x) < \infty$ we put $p_S(A, x) := y$, where $P_S(A, x) = \{y\}$. For all other $x \in \mathbb{R}^d$ we let $p_S(A, x) := x$.

For $k = d - l + j$ and $j \in \{0, \dots l\}$ we define the mapping $\pi_{S,k} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_{S,k}(\omega, x) := p_S(Y_k(\omega), x), \quad \omega \in \Omega,\ x \in \mathbb{R}^d, \tag{2.4.4}$$

where $Y_k(\omega) = \bigcup_{F \in \mathcal{S}_k(X(\omega))} F$ is the k-skeleton of X (see (2.2.4)).

Since $Y_k(\theta_x \omega) = Y_k(\omega) - x$ for all $x \in \mathbb{R}^d$, we have $d_S(Y_k(\theta_x\omega), 0) = d_S(Y_k(\omega), x)$ and it easily follows that

$$\pi_{S,k}(\omega, x) = \pi_{S,k}(\theta_x\omega, 0) + x, \quad x \in \mathbb{R}^d. \tag{2.4.5}$$

Next we define

$$H_{S,k}(\omega, x) := \{y \in S : y + x \notin \mathrm{exo}_S(Y_k(\omega)),\ x = \pi_{S,k}(\omega, x + y)\}, \quad x \in Y_k(\omega), \tag{2.4.6}$$

and $H_{S,k}(\omega, x) := \{0\}$ for $x \notin Y_k(\omega)$. Since for $x \in Y_k$ the unique metric projection of x onto Y_k is x, we always have $0 \in H_{S,k}(\omega, x)$. For $x \in Y_k(\omega)$, $H_{S,k}(\omega, x) + x$ is the set of all points in $S + x$ having x as the unique result of the metric projection (with respect to d_s) onto $Y_k(\omega)$. This also implies that $H_{S,k}(\omega, x)$ has to be a subset of $S \cap (S \cap U_{F_k(\omega,x)})^\perp$, and since X and S are assumed to be in general position, we almost surely have $\dim(\mathrm{aff}(H_{S,k}(x))) = l - j$. Additionally,

$$H_{S,k}(\omega, x) = H_{S,k}(\theta_x\omega, 0), \quad x \in \mathbb{R}^d. \tag{2.4.7}$$

Lemma 2.4.3. *Let S, l and j as in Theorem 2.4.1 and let $k := d - l + j$. Then $\mathcal{H}^l(\mathrm{exo}_S(Y_k) \cap S) = 0$ and $\mathcal{H}^d(\mathrm{exo}_S(Y_k)) = 0$.*

Proof. Since $Y_k(X)$ is a closed set, $Y_k \cap S$ is also closed. Furthermore, $\mathcal{H}^l(\mathrm{exo}_S(Y_k) \cap S) = \mathcal{H}^l(\mathrm{exo}_S((Y_k \cap S))$. Now we obtain $\mathcal{H}^l(\mathrm{exo}_S(Y_k \cap S)) = 0$ $\mathbb{P}$-almost surely by applying [16, Lemma 2.1] in the l-dimensional Euclidean space S.

Applying [16, Lemma 2.1] in $S + x$, we obtain

$$\mathcal{H}^l(\mathrm{exo}_S(Y_k) \cap (S + x)) = 0$$

for every $x \in S^\perp$. Using Fubini's theorem, this gives us

$$\mathcal{H}^d(\mathrm{exo}_S(Y_k)) = \int_{\mathbb{R}^d} \mathbf{1}\{y \in \mathrm{exo}_S(Y_k)\}\ \mathcal{H}^d(dy)$$

$$
\begin{aligned}
&= \int_{S^\perp} \int_S \mathbf{1}\{x + y \in \mathrm{exo}_S(Y_k)\}\ \mathcal{H}^l(dy)\ \mathcal{H}^{d-l}(dx) \\
&= \int_{S^\perp} \int_{S+x} \mathbf{1}\{y \in \mathrm{exo}_S(Y_k)\}\ \mathcal{H}^l(dy)\ \mathcal{H}^{d-l}(dx) \\
&= \int_{S^\perp} \mathcal{H}^l(\mathrm{exo}_S(Y_k) \cap (S + x))\ \mathcal{H}^{d-l}(dx) \\
&= 0.
\end{aligned}
$$

□

Theorem 2.4.4. *For all $j \in \{0, \dots, l\}$ and all measurable functions $f, g : \Omega \to [0, \infty)$ we have that*

$$
\mathbb{E}[f \cdot (g \circ \theta_{\pi_{S,d-l+j}(0)})] = \mathbb{E}_{M_{d-l+j}} \left[[S, F_{d-l+j}(0)]_j \cdot g \cdot \int_{H_{S,d-l+j}(0)} f \circ \theta_y\ \mathcal{H}^{l-j}(dy) \right]. \quad (2.4.8)
$$

For $f \equiv 1$ this can be understood as follows. The expectation of g after a random shift to the nearest neighbour of the origin in Y_{d-l+j}, with respect to the metric d_S, can be computed by taking the expectation of g with respect to the measure $\mathbb{P}_{M_{d-l+j}}$, multiplied with some density (namely $[S, F_{d-l+j}(0)]_j \cdot \mathcal{H}^{l-j}(H_{S,d-l+j}(0))$).

For $g \equiv 1$, this means the expectation of a measurable function f can be reconstructed from the measure $\mathbb{P}_{M_{d-l+j}}$, using a density ($[S, F_{d-l+j}(0)]_j \cdot \mathcal{H}^{l-j}(H_{S,d-l+j}(0))$ and taking the conditional expectation, given ω, of f after a random shift θ_y where y is uniformly distributed in $H_{S,d-l+j}(0)$. The fact that $0 < \mathcal{H}^{l-j}(H_{S,d-l+j}(0)) < \infty$ $\mathbb{P}_{M_{d-l+j}}$-almost surely can be found in Lemma 2.5.2.

Proof of Theorem 2.4.4: Let B be as in (2.3.2). Then $\mathcal{H}^l(B \cap S) = 1$. We consider the function

$$
\omega \mapsto g(\omega) \int_{H_{S,d-l+j}(0)} (f \circ \theta_y)(\omega) \mathcal{H}^{l-j}(dy).
$$

Since $(x, \omega) \mapsto \theta_x \omega$, f and g are measurable and $\mathcal{H}^{l-j}$ is σ-finite on the measurable set $H_{S,d-l+j}(\omega, 0)$, this function is measurable. Applying Theorem 2.4.1 to this function, (2.3.2) and (2.3.1) yield

$$
\begin{aligned}
&\mathbb{E}_{M_{d-l+j}} \left[[S, F_{d-l+j}(0)]_j \cdot g \cdot \int_{H_{S,d-l+j}(0)} f \circ \theta_y\ \mathcal{H}^{l-j}(dy) \right] \\
&= \mathbb{E}_{M_{S,j}} \left[g \cdot \int_{H_{S,d-l+j}(0)} f \circ \theta_y\ \mathcal{H}^{l-j}(dy) \right] \\
&= \int_\Omega \int_S \mathbf{1}_B(x) (g \circ \theta_x)(\omega) \int_{H_{S,d-l+j}(\theta_x \omega, 0)} (f \circ \theta_{y+x})(\omega)\ \mathcal{H}^{l-j}(dy)\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \sum_{F \in \mathcal{S}_{d-l+j}(X(\omega))} \int \mathbf{1}_{B \cap S \cap F}(x) g(\theta_x \omega) \int_{H_{S,d-l+j}(\omega, x)} (f \circ \theta_{y+x})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega)
\end{aligned}
$$

$$= \int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_F \int_{H_{S,d-l+j}(\omega,x)} \mathbf{1}_{B\cap S\cap F}(\pi_{S,d-l+j}(\omega,x+y))(g\circ\theta_{\pi_{S,d-l+j}(\omega,x+y)})(\omega)$$

$$\times (f\circ\theta_{x+y})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega).$$

Now $y \in H_{S,d-l+j}(\omega,x), x\in F$ imply $x = \pi_{S,d-l+j}(\omega,x+y)$. Then choosing some x_0 such that $F = F_0 + x_0$ where $F_0 \subset U_F, x_0 \in S\cap(S\cap U_F)^\perp$ and, also using the definition of $H_{S,d-l+j}(\omega,x)$, the integral above equals

$$\int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{U_F} \int_{H_{S,d-l+j}(\omega,x+x_0)} \mathbf{1}_{B\cap S}(\pi_{S,d-l+j}(\omega,x+x_0+y))(g\circ\theta_{\pi_{S,d-l+j}(\omega,x+x_0+y)})(\omega)$$

$$\times \mathbf{1}_{\{x+x_0\in F\}}(f\circ\theta_{x+x_0+y})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega)$$

$$= \int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{U_F} \int_{H_{S,d-l+j}(\omega,x+x_0)} \mathbf{1}_{B\cap S}(\pi_{S,d-l+j}(\omega,x+x_0+y))(g\circ\theta_{\pi_{S,d-l+j}(\omega,x+x_0+y)})(\omega)$$

$$\times \mathbf{1}_{\{\pi_{S,d-l+j}(\omega,x+x_0+y)\in F\}}(f\circ\theta_{x+x_0+y})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega)$$

$$= \int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{U_F\cap S} \int_{S\cap(S\cap U_F)^\perp} \mathbf{1}_{B}(\pi_{S,d-l+j}(\omega,x+x_0+y))(g\circ\theta_{\pi_{S,d-l+j}(\omega,x+x_0+y)})(\omega)$$

$$\times \mathbf{1}_{\{x+x_0+y\notin \mathrm{exo}_S(Y_{d-l+j}(\omega)),\ \pi_{S,d-l+j}(\omega,x+x_0+y)=x+x_0\in F\}}$$

$$\times (f\circ\theta_{x+x_0+y})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega).$$

Since, for $x\in U_F$, $x+x_0\in \mathrm{aff}(F)\cap S$ and $y\in S\cap(S\cap U_F)^\perp$ imply

$$d_S(x+x_0+y,\mathrm{aff}(F)) = \|y\|_2 = \|x+x_0+y-(x+x_0)\|_2 = d_S(x+x_0+y,x+x_0)$$

and $x+x_0+y\notin \mathrm{exo}_S(Y_{d-l+j}(\omega))$ is given, we have $\pi_{S,d-l+j}(\omega,x+x_0+y) = x+x_0$. Therefore, in this case, the condition $\pi_{S,d-l+j}(\omega,x+x_0+y) = x+x_0\in F$ is equivalent to $\pi_{S,d-l+j}(\omega,x+x_0+y)\in F$. Since also $x_0\in S\cap(S\cap U_F)^\perp$, we can rewrite the integral as

$$\int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_{U_F\cap S} \int_{S\cap(S\cap U_F)^\perp} \mathbf{1}_{B}(\pi_{S,d-l+j}(\omega,x+y))(g\circ\theta_{\pi_{S,d-l+j}(\omega,x+y)})(\omega)$$

$$\times \mathbf{1}_{\{x+y\notin \mathrm{exo}_S(Y_{d-l+j}(\omega))\}}\mathbf{1}_{\{\pi_{S,d-l+j}(\omega,x+y)\in F\}}(f\circ\theta_{x+y})(\omega)\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^j(dx)\mathbb{P}(d\omega). \quad (2.4.9)$$

Since $\mathcal{H}^l(\{z\in S : z\in \mathrm{exo}_S(Y_{d-l+j})\}) = 0$ $\mathbb{P}$-almost surely (Lemma 2.4.3), for $\mathcal{H}^l$-almost every $z\in S$ there is almost surely a unique $\pi_{S,d-l+j}(z)\in F$, for some $F\in S_{d-l+j}(X)$ if $Y_{d-l+j}(X)\cap S\neq\emptyset$. By Lemma 2.3.1, $X_S = \{C\cap S : C\in X,\ C\cap S\neq\emptyset\}$ is a tessellation in S and therefore the boundary of the cells of X_S can be decomposed into lower-dimensional polytopes. By Lemma 2.3.3, each element of $\mathcal{S}_j(X_S)$ is the intersection of S with some $F\in\mathcal{S}_{d-l+j}(X)$. Thus $Y_{d-l+j}(X)\cap S=\emptyset$ if and only if $X_S=\{S\}$, which implies that $X=\{\mathbb{R}^d\}$. This event was assumed to have probability 0 (see Section 2.2). Using Fubini's theorem and that for $\mathcal{H}^l$-almost every $z\in S$ there is almost surely a unique $\pi_{S,d-l+j}(z)\in F$, for some $F\in S_{d-l+j}(X)$, we obtain

$$(2.4.9) = \int_\Omega \sum_{F\in\mathcal{S}_{d-l+j}(X(\omega))} \int_S \mathbf{1}_B(\pi_{S,d-l+j}(\omega,z))(g\circ\theta_{\pi_{S,d-l+j}(\omega,z)})(\omega)$$

$$
\begin{aligned}
&\qquad\qquad \times \mathbf{1}_{\{z \notin \mathrm{exo}_S(Y_{d-l+j}(\omega))\}} \mathbf{1}_{\{\pi_{S,d-l+j}(\omega,z) \in F\}} (f \circ \theta_z)(\omega)\ \mathcal{H}^l(dz)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_S \sum_{F \in \mathcal{S}_{d-l+j}(X(\omega))} \mathbf{1}_B(\pi_{S,d-l+j}(\omega,z))(g \circ \theta_{\pi_{S,d-l+j}(\omega,z)})(\omega) \\
&\qquad\qquad \times \mathbf{1}_{\{\pi_{S,d-l+j}(\omega,z) \in F\}} (f \circ \theta_z)(\omega)\ \mathcal{H}^l(dz)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_S \mathbf{1}_B(\pi_{S,d-l+j}(\omega,z)(g \circ \theta_{\pi_{S,d-l+j}(\omega,z)})(\omega)(f \circ \theta_z)(\omega)\ \mathcal{H}^l(dz)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_S \mathbf{1}_B(\pi_{S,d-l+j}(\theta_z\omega,0) + z)(g \circ \theta_{\pi_{S,d-l+j}(\theta_z\omega,0)})(\theta_z\omega) f(\theta_z\omega)\ \mathcal{H}^l(dz)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_S \mathbf{1}_B(\pi_{S,d-l+j}(\omega,0) + z)\ \mathcal{H}^l(dz)\ (g \circ \theta_{\pi_{S,d-l+j}(\omega,0)}(\omega) f(\omega)\ \mathbb{P}(d\omega) \\
&= \mathbb{E}[f \cdot (g \circ \theta_{\pi_{S,d-l+j}(0)})]
\end{aligned}
$$

where we have also used (2.4.5), the stationarity of $\mathbb{P}$ and $\mathcal{H}^l(S \cap B) = 1$, see (2.3.2). □

2.5. An alternative approach

Now we present another proof of Theorem 2.4.4, which is not based on Theorem 2.4.1 and we then use Theorem 2.4.4 to prove Theorem 2.4.1.

Proof of Theorem 2.4.4: We abbreviate $k := d - l + j$, $H(x) := H_{S,k}(x)$, and $\pi(x) := \pi_{S,k}(x)$. Let $f, g : \Omega \to [0, \infty)$ be measurable. By (2.4.5), we have

$$
(g \circ \theta_{\pi(0)})(\theta_x\omega) = g(\theta_{\pi(\theta_x\omega,0)}\theta_x\omega) = g(\theta_{\pi(\omega,x)-x}\theta_x\omega) = g(\theta_{\pi(\omega,x)}\omega).
$$

Taking a measurable set $B \subset \mathbb{R}^d$ of volume 1, applying Fubini's theorem, using that $\mathrm{exo}_S(Y_k)$ is almost surely a 0-set w.r.t. $\mathcal{H}^d$ and using that for almost all $x \notin \mathrm{exo}_S(Y_k)$ there is almost surely a unique $F \in \mathcal{S}_k(X)$ such that $x \in \mathrm{relint}(F)$, we get that the left-hand side of (2.4.8) equals

$$
\begin{aligned}
&\mathbb{E}\left[\int \mathbf{1}\{x \in B\}(f \circ \theta_x)(g \circ \theta_{\pi(x)})\ \mathcal{H}^d(dx)\right] \\
&= \mathbb{E}\left[\int \mathbf{1}\{x \in B\}(f \circ \theta_x)(g \circ \theta_{\pi(x)}) \mathbf{1}\{x \notin \mathrm{exo}_S(Y_k)\}\ \mathcal{H}^d(dx)\right] \\
&= \mathbb{E}\left[\sum_{F \in \mathcal{S}_k(X)} \int \mathbf{1}\{x \in B\}(f \circ \theta_x)(g \circ \theta_{\pi(x)}) \mathbf{1}\{\pi(x) \in \mathrm{relint}\, F\} \mathbf{1}\{x \notin \mathrm{exo}_S(Y_k)\}\ \mathcal{H}^d(dx)\right] \\
&= \mathbb{E}\left[\sum_{F \in \mathcal{S}_k(X)} \iint \mathbf{1}\{y + z \in B\}(f \circ \theta_{y+z})(g \circ \theta_{\pi(y+z)}) \mathbf{1}\{\pi(y+z) \in \mathrm{relint}\, F\}\right.
\end{aligned}
$$

$$\left.\times\ \mathbf{1}\{y+z\notin \mathrm{exo}_S(Y_k)\}\mathbf{1}\{y\in S_F, z\in S_F^\perp\}\ \mathcal{H}^k(dz)\ \mathcal{H}^{l-j}(dy)\right], \tag{2.5.1}$$

where $S_F = S\cap(S\cap U_F)^\perp$ denotes the orthogonal complement of $S\cap U_F$ in S and the last equality holds since X and S are almost surely in general position.

We now introduce for any k-dimensional polytope F a mapping $T_F:\mathbb{R}^d\to\mathbb{R}^d$ by

$$T_F(x) := z(F) + \Pi_{S_F^\perp}(x - z(F)),$$

where $z(F)$ is the centre of the ball circumscribing F. Note that

$$T_F(x) = \Pi_{S_F}(z(F)) + \Pi_{S_F^\perp}(x).$$

Whenever F and S are in general position, T_F, restricted to the affine hull $U_F + z(F)$ of F, is an injection with Jacobian $[F,S]_j$. In order to use this transformation, we note that $S_F^\perp + z(F) = S_F^\perp + \Pi_{S_F}(z(F))$ and we get

$$\begin{aligned}
&\iint \mathbf{1}\{y+z\in B\}\dots\mathbf{1}\{y\in S_F, z\in S_F^\perp\}\ \mathcal{H}^k(dz)\ \mathcal{H}^{l-j}(dy)\\
&= \iint \mathbf{1}\{y+z+\Pi_{S_F}(z(F))\in B\}\dots\mathbf{1}\{y\in S_F, z\in S_F^\perp\}\ \mathcal{H}^k(dz)\ \mathcal{H}^{l-j}(dy)\\
&= \iint \mathbf{1}\{y+z\in B\}\dots\mathbf{1}\{y\in S_F, z\in S_F^\perp+\Pi_{S_F}(z(F))\}\ \mathcal{H}^k(dz)\ \mathcal{H}^{l-j}(dy)\\
&= \iint \mathbf{1}\{y+z\in B\}\dots\mathbf{1}\{y\in S_F, z\in S_F^\perp+z(F)\}\ \mathcal{H}^k(dz)\ \mathcal{H}^{l-j}(dy).
\end{aligned}$$

Thus, (2.5.1) equals

$$\mathbb{E}\left[\sum_{F\in\mathcal{S}_k(X)}\iint \mathbf{1}\{y+T_F(x)\in B\}(f\circ\theta_{y+T_F(x)})(g\circ\theta_{\pi(y+T_F(x))})\mathbf{1}\{\pi(y+T_F(x))\in \operatorname{relint} F\}\right.$$

$$\left.\times\ [S,F]_j\mathbf{1}\{y+T_F(x)\notin \mathrm{exo}_S(Y_k)\}\mathbf{1}\{y\in S_F, x\in z(F)+U_F\}\ \mathcal{H}^k(dx)\ \mathcal{H}^{l-j}(dy)\right].$$

Since $T_F(x) = \Pi_{S_F}(z(F)-x)+x$, we have for all $y\in S_F$ and $x\in z(F)+U_F$ the following equivalence

$$\pi(y+T_F(x))\in\operatorname{relint} F,\ y+T_F(x)\notin \mathrm{exo}_S(Y_k) \iff y+\Pi_{S_F}(z(F)-x)\in H(x),\ x\in\operatorname{relint} F.$$

In either case $\pi(y+T_F(x)) = x$.

To show this, we first assume $\pi(y+T_F(x))\in\operatorname{relint}(F)$ and $y+T_F(x)\notin\mathrm{exo}_S(Y_k)$. From $T_F(x) = \Pi_{S_F}(z(F)-x)+x$ we obtain $\pi(y+\Pi_{S_F}(z(F)-x)+x\in\operatorname{relint}(F)$ and $y+\Pi_{S_F}(z(F)-x)+x\notin\mathrm{exo}_S(Y_k)$. Since $y\in S_F = S\cap(S\cap U_F)^\perp$, we get $y+\Pi_{S_F}(z(F)-x)\in S_F$ and thus $\pi(x+y+\Pi_{S_F}(z(F)-x)) = x\in\operatorname{relint} F$. Recalling the definition of $H(x) = \{y\in S: y+x\notin\mathrm{exo}_S(Y_k),\ \pi(y+x)=x\}$, we get $y+\Pi_{S_F}(z(F)-x)\in H(x)$.

For the second implication, we assume $y + \Pi_{S_F}(z(F) - x) \in H(x)$ and $x \in \operatorname{relint} F$. By definition of $H(x)$, we get $y+\Pi_{S_F}(z(F)-x)+x \notin \operatorname{exo}_S(Y_k)$ and $\pi(y+\Pi_{S_F}(z(F)-x)+x) = x \in \operatorname{relint} F$. Now the assertion follows from $\Pi_{S_F}(z(F) - x) + x = T_F(x)$.

Using the equivalence just shown and that $y + \Pi_{S_F}(z(F) - x) \in H(x)$ implies $y \in S_F$, it follows that

$$\begin{aligned}
\mathbb{E}[f \cdot (g \circ \theta_{\pi(0)})] = \mathbb{E}\Bigg[\sum_{F\in\mathcal{S}_k(X)} \iint & \mathbf{1}\{y + T_F(x) \in B\}(f \circ \theta_{y+T_F(x)})(g \circ \theta_x) \cdot [S, F]_j \\
& \times \mathbf{1}\{y + T_F(x) - x \in H(x)\}\mathbf{1}\{x \in \operatorname{relint}(F)\}\ \mathcal{H}^k(dx)\ \mathcal{H}^{l-j}(dy)\Bigg] \\
= \mathbb{E}\Bigg[\iint & \mathbf{1}\{y + T(x) \in B\}(f \circ \theta_{y+T(x)})(g \circ \theta_x)\mathbf{1}\{y + T(x) - x \in H(x)\} \\
& \times [S, F_k(x)]_j\ \mathcal{H}^{l-j}(dy)\ M_k(dx)\Bigg],
\end{aligned}$$

where $T(x) := T_F(x)$ if x is in the relative interior of some $F \in \mathcal{S}_k(X)$ and otherwise $T(x) := x$, and in the last step, we also used the definition of M_k and the fact that $\mathbb{P}$-almost surely $x \in \operatorname{relint} F$ holds for $\mathcal{H}^k$-almost all $x \in F$. Since X is stationary, (2.2.2) implies

$$T(x) = T(0) \circ \theta_x + x.$$

Since $F_k(\omega, x) = F_k(\theta_x\omega, 0) + x$ (see (2.2.5)), we have $[S, F_k(x)]_j = [S, F_k(0) \circ \theta_x]_j$ and using (2.4.7), we obtain

$$\begin{aligned}
\mathbb{E}[f \cdot (g \circ \theta_{\pi(0)})] = \mathbb{E}\Bigg[\iint & \mathbf{1}\{y + x + T(0) \circ \theta_x \in B\}(f \circ \theta_{y+T(0)\circ\theta_x} \circ \theta_x) \cdot (g \circ \theta_x) \\
& \times \mathbf{1}\{y + T(0) \circ \theta_x \in H(0) \circ \theta_x\}[S, F_k(0) \circ \theta_x]_j\ \mathcal{H}^{l-j}(dy)\ M_k(dx)\Bigg] \\
= \mathbb{E}_{M_k}\Bigg[\iint & \mathbf{1}\{y + x + T(0) \in B\}(f \circ \theta_{y+T(0)}) \cdot g \cdot \mathbf{1}\{y + T(0) \in H(0)\} \\
& \times [S, F_k(0)]_j\ \mathcal{H}^{l-j}(dy)\ \mathcal{H}^d(dx)\Bigg] \\
= \mathbb{E}_{M_k}\Bigg[\int & (f \circ \theta_{y+T(0)}) \cdot g(0)\mathbf{1}\{y + T(0) \in H(0)\}[S, F_k(0)]_j\ \mathcal{H}^{l-j}(dy)\Bigg],
\end{aligned}$$

where we also used (2.1.4) for $\mathbb{P}_{M_k}$, Fubini's theorem and $\mathcal{H}^d(B) = 1$. A final change of variables yields (2.4.8). $\square$

The case $S = \mathbb{R}^d$ is an interesting special case. Here we abbreviate $\pi_k := \pi_{\mathbb{R}^d,k}$ and $H_k := H_{\mathbb{R}^d,k}$.

Theorem 2.5.1. *Let $k \in \{0, \dots, d\}$. Then we have for any measurable $f, g : \Omega \to [0, \infty)$*

$$\mathbb{E}[f \cdot (g \circ \theta_{\pi_k(0)})] = \mathbb{E}_{M_k}\left[g \cdot \int_{H_k(0)} f \circ \theta_y\, \mathcal{H}^{d-k}(dy)\right].$$

Lemma 2.5.2. $0 < \mathcal{H}^{l-j}(H_{S,d-l+j}(0)) < \infty$ *with respect to both* $\mathbb{P}_{M_{S,j}}$ *and* $\mathbb{P}_{M_{d-l+j}}$.

PROOF: Let B as in (2.3.2) and abbreviate $k := d - l + j$. Then

$$\begin{aligned}
&\mathbb{P}_{M_{S,j}}\left(\mathcal{H}^{l-j}(H_{S,k}(0)) = \infty\right) \\
&= \iint \mathbf{1}\{x \in B\}\mathbf{1}\{\mathcal{H}^{l-j}(H_{S,k}(\theta_x\omega, 0)) = \infty\}\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega) \\
&= \iint \mathbf{1}\{x \in B\}\mathbf{1}\{\mathcal{H}^{l-j}(H_{S,k}(\omega, x)) = \infty\}\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega) \\
&\leq \iint \mathbf{1}\{x \in B\}\mathbf{1}\{\sum_{\substack{C\in\mathcal{S}_d(X(\omega))\\ C\cap\{x\}\neq\emptyset}} \mathcal{H}^{l-j}(C \cap S_{F_k(\omega,x)}) = \infty\}\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega). \qquad (2.5.2)
\end{aligned}$$

Since $\mathcal{S}_d(X)$ is $\mathbb{P}$-almost surely locally finite and $C \in \mathcal{S}_d(X)$ is almost surely a compact polytope, we have

$$\mathbf{1}\{\sum_{\substack{C\in\mathcal{S}_d(X(\omega))\\ C\cap\{x\}\neq\emptyset}} \mathcal{H}^{l-j}(C \cap S_{F_k(\omega,x)}) = \infty\} = 0$$

$\mathbb{P}$-almost surely. Therefore, the integrand on the right-hand side of (2.5.2) is zero. The same holds for M_k instead of $M_{S,j}$.

For $x \in \operatorname{relint}(F)$, $F \in \mathcal{S}_k(X)$ we have $d(x, Y_k \backslash F) > 0$ and thus $d_S(x, Y_k \backslash F) > 0$ $\mathbb{P}$-almost surely. It follows that for such x we have $\mathcal{H}^{l-j}(H_{S,k}(\omega, x)) > 0$ for $\mathbb{P}$-almost all ω. Using this fact, we get

$$\begin{aligned}
&\mathbb{P}_{M_{S,j}}(\mathcal{H}^{l-j}(H_{S,k}(0)) = 0) \\
&= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{x \in B\}\mathbf{1}\{\mathcal{H}^{l-j}(H_{S,k}(\omega, x)) = 0\}\ M_{S,j}(\omega, dx)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \sum_{F\in S_k(X(\omega))} \int_{F\cap S} \mathbf{1}\{x \in B\}\mathbf{1}\{\mathcal{H}^{l-j}(H_{S,k}(\omega, x)) = 0\}\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega) \\
&\leq \int_\Omega \sum_{F\in S_k(X(\omega))} \int_{F\cap S} \mathbf{1}\{x \in B\}\mathbf{1}\{x \notin \operatorname{relint}(F)\}\ \mathcal{H}^j(dx)\ \mathbb{P}(d\omega)
\end{aligned}$$

Since the relative boundary of F is the union of $(k-1)$-dimensional faces, the assumption of general position implies $\mathcal{H}^j((F \backslash \operatorname{relint}(F)) \cap S) = 0$ $\mathbb{P}$-almost surely. Thus, the probability above equals zero.

For $\mathbb{P}_{M_k}$ we get

$$\begin{aligned}
&\mathbb{P}_{M_k}(\mathcal{H}^{l-j}(H_{S,k}(0)) = 0) \\
&\leq \int_\Omega \sum_{F\in S_k(X(\omega))} \int_F \mathbf{1}\{x \in B\}\mathbf{1}\{x \notin \operatorname{relint}(F)\}\ \mathcal{H}^k(dx)\ \mathbb{P}(d\omega) \\
&= 0.
\end{aligned}$$

□

Proof of Theorem 2.4.1: Applying Theorem 2.5.1 with $(\mathbb{R}^d, k)$ replaced with (S, j), $f \equiv 1$ and considering the stationary tessellation $X_S := \{S \cap C : \ C \in X, \ C \cap S \neq \emptyset\}$ and the flow $\{\theta_x, \ x \in S\}$, yields

$$\mathbb{E}\left[g \circ \theta_{\pi_{S,j}(0)}\right] = \mathbb{E}_{M_j}\left[g \cdot \mathcal{H}^{l-j}(H_{S,j}(0))\right].$$

For almost all ω we have

- $\pi_{S,j}(\omega, 0)$ with respect to X_S equals $\pi_{S,d-l+j}(\omega, 0)$ with respect to X,
- $H_{S,j}(\omega, 0)$ with respect to X_S equals $H_{S,d-l+j}(\omega, 0)$ with respect to X and
- the measure M_j with respect to X_S equals $M_{S,j}$ with respect to X.

Therefore the equation above can be written as

$$\mathbb{E}[g \circ \theta_{\pi_{S,d-l+j}(0)})] = \mathbb{E}_{M_{S,j}}\left[g \cdot \mathcal{H}^{l-j}(H_{S,d-l+j}(0))\right].$$

Using (2.4.8) with $f \equiv 1$, we get

$$\mathbb{E}_{M_{S,j}}\left[g \cdot \mathcal{H}^{l-j}(H_{S,d-l+j}(0))\right] = \mathbb{E}_{M_{d-l+j}}\left[[S, F_{d-l+j}(0)]_j \cdot g \cdot \mathcal{H}^{l-j}(H_{S,d-l+j}(0))\right]. \quad (2.5.3)$$

Since $0 < \mathcal{H}^{l-j}(H_{S,d-l+j}(0)) < \infty$ with respect to both, $\mathbb{P}_{M_{S,j}}$ and $\mathbb{P}_{M_{d-l+j}}$, the assertion follows from the next lemma. □

Lemma 2.5.3. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures on Ω, $h, f : \Omega \to [0, \infty)$ and $f > 0$ $\mathbb{P}_1$-almost surely and $f > 0$ $\mathbb{P}_2$-almost surely. If*

$$\int g(\omega) \cdot f(\omega) \ \mathbb{P}_1(d\omega) = \int h(\omega) \cdot g(\omega) \cdot f(\omega) \ \mathbb{P}_2(d\omega) \quad (2.5.4)$$

for all measurable $g : \Omega \to [0, \infty)$, it follows that

$$\int g(\omega) \ \mathbb{P}_1(d\omega) = \int h(\omega) \cdot g(\omega) \ \mathbb{P}_2(d\omega).$$

Proof. Let $g : \Omega \to [0, \infty)$ be measurable and define $\tilde{g}(\omega) := g(\omega) \cdot (f(\omega))^{-1}$ if $f(\omega) > 0$ and $\tilde{g}(\omega) := 0$ else. Then $\tilde{g}$ is measurable and by (2.5.4), we obtain

$$\int g(\omega) \ \mathbb{P}_1(d\omega) = \int \tilde{g}(\omega) f(\omega) \ \mathbb{P}_1(d\omega) = \int h(\omega)\tilde{g}(\omega) f(\omega) \ \mathbb{P}_2(d\omega) = \int h(\omega) g(\omega) \ \mathbb{P}_2(d\omega).$$

□

Equation (2.4.1) implies in particular that, if the intensity λ_{d-l+j} of M_{d-l+j} is finite, the intensity $\lambda_{S,j} := \mathbb{E}[M_{S,j}([0,1]^d \cap S)]$ of $M_{S,j}$ is given by

$$\lambda_{S,j} = \lambda_{d-l+j} \mathbb{E}^0_{M_{d-l+j}}[[S, F_{d-l+j}(0)]_j]. \quad (2.5.5)$$

Following the language of Chapter 2 and Chapter 4 in [49], we call a tessellation X isotropic, if $X \overset{\mathcal{D}}{=} \vartheta X$ for all $\vartheta \in SO_d$. Here $\overset{\mathcal{D}}{=}$ denotes equality in distribution, $\vartheta X :=$

$\{\vartheta^{-1}K : K \in X\}$ and $\vartheta^{-1}K := \{\vartheta^{-1}x : x \in K\}$. As a consequence, we have $\mathcal{S}_k(\vartheta X) = \{\vartheta^{-1}F : F \in \mathcal{S}_k(X)\}$ for $\vartheta \in SO_d$ and $k \in \{0, \dots, d\}$. A random measure M is called isotropic, if $M(\vartheta B) \overset{\mathcal{D}}{=} M(B)$ for all $\vartheta \in SO_d$ and $B \in \mathcal{B}^d$.

We now additionally assume X to be isotropic. Then for $k \in \{0, \dots, d\}$, the random measure M_k is isotropic. To confirm this, let $B \in \mathcal{B}^d$, $\vartheta \in SO_d$ and use the definition of M_k (2.2.3), the rotation invariance of $\mathcal{H}^k$ and the isotropy of X to obtain

$$\begin{aligned} M_k(\vartheta B) &= \sum_{F \in \mathcal{S}_k(X)} \mathcal{H}^k(F \cap \vartheta B) = \sum_{F \in \mathcal{S}_k(X)} \mathcal{H}^k(\vartheta^{-1}F \cap B) \\ &= \sum_{F \in \mathcal{S}_k(\vartheta X)} \mathcal{H}^k(F \cap B) \overset{\mathcal{D}}{=} M_k(B). \end{aligned}$$

If X is isotropic, the expectation $\mathbb{E}^0_{M_{d-l+j}}[[S, F_{d-l+j}(0)]_j]$ can be computed explicitly.

Corollary 2.5.4. *Let X be isotropic. Then we have*

$$\lambda_{S,j} = \lambda_{d-l+j} \cdot \frac{\Gamma\left(\frac{d-l+j+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}. \tag{2.5.6}$$

Proof. We put $k := d - l + j$ and denote the invariant probability measure on the Grassmannian $G(d, q)$ by ν_q and the invariant probability measure on SO_d by ν. Let B be a ball with centred in the origin with $\mathcal{H}^d(B) = 1$. Using (2.1.4), the definition of M_k, (2.2.5), the isotropy of X, Fubini's theorem, the above mentioned identity for $\mathcal{S}_k(\vartheta X)$ and the invariance of $\mathcal{H}^k$, we obtain

$$\begin{aligned} \mathbb{E}^0_{M_k}\left[[S, F_k(0)]_j\right] &= \mathbb{E}^0_{M_k}\left[\int_{\mathbb{R}^d} [S, F_k(0)]_j \mathbf{1}\{x \in B\}\ \mathcal{H}^d(dx)\right] \\ &= \frac{1}{\lambda_k}\int_\Omega \int_{\mathbb{R}^d} [S, F_k(\theta_x\omega, 0)]_j \mathbf{1}\{x \in B\}\ M_k(\omega, dx)\ \mathbb{P}(d\omega) \\ &= \frac{1}{\lambda_k}\int_\Omega \int_{\mathbb{R}^d} [S, F_k(\omega, x)]_j \mathbf{1}\{x \in B\}\ M_k(\omega, dx)\ \mathbb{P}(d\omega) \\ &= \frac{1}{\lambda_k}\int_\Omega \sum_{F \in \mathcal{S}_k(X(\omega))} \int_F [S, F]_j \mathbf{1}\{x \in B\}\ \mathcal{H}^k(dx)\ \mathbb{P}(d\omega) \\ &= \frac{1}{\lambda_k}\int_\Omega \sum_{F \in \mathcal{S}_k(X(\omega))} [S, F]_j \mathcal{H}^k(F \cap B)\ \mathbb{P}(d\omega) \\ &= \frac{1}{\lambda_k}\mathbb{E}\left[\sum_{F \in \mathcal{S}_k(X)} [S, F]_j \mathcal{H}^k(F \cap B)\right] \\ &= \frac{1}{\lambda_k}\int_{SO_d} \mathbb{E}\left[\sum_{F \in \mathcal{S}_k(\vartheta^{-1}X)} [S, F]_j \mathcal{H}^k(F \cap B)\right]\ \nu(d\vartheta) \end{aligned}$$

$$= \frac{1}{\lambda_k}\mathbb{E}\left[\int_{SO_d}\sum_{F\in\mathcal{S}_k(X)}[S,\vartheta F]_j\mathcal{H}^k(\vartheta F\cap B)\ \nu(d\vartheta)\right]$$

$$= \frac{1}{\lambda_k}\mathbb{E}\left[\sum_{F\in\mathcal{S}_k(X)}\int_{SO_d}[S,\vartheta F]_j\mathcal{H}^k(F\cap\vartheta^{-1}B)\ \nu(d\vartheta)\right]$$

$$= \frac{1}{\lambda_k}\mathbb{E}\left[\sum_{F\in\mathcal{S}_k(X)}\mathcal{H}^k(F\cap B)\int_{SO_d}[S,\vartheta F]_j\ \nu(d\vartheta)\right]$$

where we also used that $\vartheta B = B$ for all $\vartheta \in SO_d$. Using $\dim(U_F) = k$, the fact that $[S,F]_j$ depends only on S and U_F, the subspace parallel to F, [49, (13.6)] and [49, Theorem 13.2.11], we get

$$\begin{aligned}
&\frac{1}{\lambda_k}\mathbb{E}\left[\sum_{F\in\mathcal{S}_k(X)}\mathcal{H}^k(F\cap B)\int_{SO_d}[S,\vartheta F]_j\ \nu(d\vartheta)\right]\\
&= \frac{1}{\lambda_k}\mathbb{E}\left[\sum_{F\in\mathcal{S}_k(X)}\mathcal{H}^k(F\cap B)\int_{G(d,k)}[S,L]_j\ \nu_k(dL)\right]\\
&= \int_{G(d,k)}[S,L]_j\ \nu_k(dL)\cdot\frac{1}{\lambda_k}\mathbb{E}\left[M_k(B)\right]\\
&= \int_{G(d,k)}[S,L]_j\ \nu_k(dL).
\end{aligned}$$

From [49, p. 133], we get

$$[S,L]_j = \frac{1}{\kappa_{l-j}}V^{l-j}(\Pi_{L^\perp}B_S),$$

where $\Pi_{L^\perp}$ is the orthogonal projection onto $L^\perp$ and B_S is the unit sphere in S. Note that, since S and L are in general position, the projection of B_S onto $L^\perp$ is indeed $(l-j)$-dimensional. Thus, we have

$$\begin{aligned}
\mathbb{E}^0_{M_k}\left[[S,F_k(0)]_j\right] &= \int_{G(d,k)}[S,L]_j\ \nu_k(dL)\\
&= \int_{G(d,k)}\frac{1}{\kappa_{l-j}}V^{l-j}(\Pi_{L^\perp}B_S)\ \nu_k(dL).
\end{aligned}$$

The mapping

$$h : \begin{cases} G(d,k) & \to G(d,d-k)\\ \quad L & \mapsto \quad L^\perp\end{cases}$$

transforms ν_k into $\nu_{d-k} = \nu_{l-j}$ and thus the above equals

$$\int_{G(d,l-j)}\frac{1}{\kappa_{l-j}}V_{l-j}(\Pi_L B_S)\ \nu_{l-j}(dL).$$

Applying [49, Theorem 6.2.2] and using $V_k(B^d) = \binom{d}{k}\frac{\kappa_d}{\kappa_{d-k}}$ (see [49, (14.8)]) as well as $\kappa_m = \frac{1}{m!}2^m\pi^{(m-1)/2}\Gamma((m+1)/2)$ we get

$$\begin{aligned}\mathbb{E}^0_{M_k}\left[[S, F_k(0)]_j\right] &= \frac{1}{\kappa_{l-j}} \cdot \frac{\Gamma\left(\frac{d-l+j+1}{2}\right)\Gamma\left(\frac{l-j+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}\binom{l}{l-j}\frac{\kappa_l}{\kappa_j}\\ &= \frac{\Gamma\left(\frac{d-l+j+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}.\end{aligned}$$

□

2.6. The typical cell of sections

We now adapt Proposition 2.1 from [4] to our setting. It describes a connection between M_k and N_k, where the latter was defined at the end of Section 2.3. Since we use a different centre function in the definition of N_k and work in the θ_x-framework, we give a proof of the following proposition.

Proposition 2.6.1. *For all measurable $g : \Omega \to [0, \infty)$ and $k \in \{0, \dots, d\}$, we have*

$$\mathbb{E}_{M_k}\left[g \circ \theta_{c(F_k(0))}\right] = \mathbb{E}_{N_k}\left[g \cdot \mathcal{H}^k(C_k(0))\right], \tag{2.6.1}$$

$$\mathbb{E}_{N_k}[g] = \mathbb{E}_{M_k}\left[(\mathcal{H}^k(F_k(0)))^{-1} g \circ \theta_{c(F_k(0))}\right] \tag{2.6.2}$$

where $C_k(\omega, x)$ is equal to $F \in \mathcal{S}_k(X(\omega))$ if and only if $x \in F$ for some $F \in \mathcal{S}_k(X(\omega))$, else we define $C_k(\omega, x) := \{x\}$.

Proof. Using the definition of N_k, Neveu's exchange formula [49, Theorem 3.4.5], (2.2.5) and the definition of M_k, we obtain

$$\begin{aligned}\mathbb{E}_{M_k}\left[g \circ \theta_{c(F_k(0))}\right] &= \int_\Omega g\left(\theta_{c(F_k(\omega,0))}\omega\right)\, \mathbb{P}_{M_k}(d\omega)\\ &= \int_\Omega \sum_{C \in \mathcal{S}_k(X(\omega))} \mathbf{1}\{c(F_k(\omega,0)) = c(C)\} g\left(\theta_{c(F_k(\omega,0))}\omega\right)\, \mathbb{P}_{M_k}(d\omega)\\ &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{c(F_k(\omega,0)) = y\} g(\theta_y\omega)\, N_k(\omega, dy)\, \mathbb{P}_{M_k}(d\omega)\\ &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{c(F_k(\theta_{-y}\theta_y\omega,0)) = y\} g(\theta_y\omega)\, N_k(\omega, dy)\, \mathbb{P}_{M_k}(d\omega)\\ &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{c(F_k(\theta_x\omega,0)) = -x\} g(\omega)\, M_k(\omega, dx)\, \mathbb{P}_{N_k}(d\omega)\\ &= \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}\{c(F_k(\omega,x)) = 0\} g(\omega)\, M_k(\omega, dx)\, \mathbb{P}_{N_k}(d\omega)\\ &= \int_\Omega g(\omega) \sum_{C \in \mathcal{S}_k(X(\omega))} \int_C \mathbf{1}\{c(F_k(\omega,x)) = 0\}\, \mathcal{H}^k(dx)\, \mathbb{P}_{N_k}(d\omega)\end{aligned}$$

$$= \int_{\Omega} g(\omega)\mathcal{H}^k\left(C_k(\omega,0)\right)\ \mathbb{P}_{N_k}(d\omega),$$

where in the last step, we have used the fact that 0 is $\mathbb{P}_{N_k}$-almost surely the centre of a unique k-face, which then lies inside this k-face.

In order to prove (2.6.2), we apply (2.6.1) with g replaced with $g\cdot(\mathcal{H}^k(C_k(0)))^{-1}$ and get

$$\mathbb{E}_{N_k}[g] = \mathbb{E}_{M_k}\left[g\circ\theta_{c(F_k(0))}(\mathcal{H}^k(C_k\circ\theta_{c(F_k(0))}(0)))^{-1}\right].$$

Now the assertion follows from

$$C_k(\theta_{c(F_k(\omega,0))}\omega,0) = F_k(\omega,0)-c(F_k(\omega,0)) \quad \mathbb{P}_{M_k}\text{-almost surely.}$$

□

Let $N_{S,j}$ denote the random measure N_j with respect to the tessellation X_S and let $C_{S,j}(\omega,x)$ be defined as $C_k(\omega,x)$ but with respect to the tessellation X_S.

Theorem 2.6.2. *For all measurable $g:\Omega\to[0,\infty)$, an l-dimensional linear subspace S, $l\in\{1,\dots,d\}$ and all $j\in\{0,\dots,l\}$ we have*

$$\mathbb{E}_{M_{S,j}}\left[g\circ\theta_{c(F_{d-l+j}(0))}\right] = \mathbb{E}_{N_{d-l+j}}\left[[S,C_{d-l+j}(0)]_j\cdot g\cdot\mathcal{H}^{d-l+j}(C_{d-l+j}(0))\right]. \tag{2.6.3}$$

If the intensity of M_{d-l+j} is finite, $g\equiv 1$ gives

$$\lambda_{S,j} = \lambda_{N_{d-l+j}}\mathbb{E}^0_{N_{d-l+j}}\left[[S,C_{d-l+j}(0)]_j\cdot\mathcal{H}^{d-l+j}(C_{d-l+j}(0))\right]. \tag{2.6.4}$$

For translation-invariant $f:\Omega\to[0,\infty)$ we have

$$\mathbb{E}_{N_{S,j}}\left[f\cdot\mathcal{H}^j(C_{S,j}(0))\right] = \mathbb{E}_{N_{d-l+j}}\left[[S,C_{d-l+j}(0)]_j\cdot f\cdot\mathcal{H}^{d-l+j}(C_{d-l+j}(0))\right]. \tag{2.6.5}$$

Proof: (2.6.3) follows directly from combining Proposition 2.6.1 with Theorem 2.4.1. In order to prove (2.6.5), we use the translation-invariance of f to obtain

$$\mathbb{E}_{M_{S,j}}\left[g\circ\theta_{c(F_{d-l+j}(0))}\right] = \mathbb{E}_{M_{S,j}}[g] = \mathbb{E}_{M_{S,j}}\left[g\circ\theta_{c(F_{S,j}(0))}\right].$$

Then the result follows from applying Proposition 2.6.1 to the left hand side of (2.6.3). □

Remark 2.6.3. *We have $C_{S,j}(0) = F_{d-l+j}(0)\cap S$ $\mathbb{P}_{N_{S,j}}$-almost surely, but $C_{S,j}(0)$ is not necessarily equal to $C_{d-l+j}(0)\cap S$.*

2.7. Sections through Poisson–Voronoi tessellations

In this section we fix a stationary Poisson process N of intensity γ. To work within the general setting of Section 2.2, we adapt the framework of Example 2.1.1 and define X as the stationary Voronoi tessellation based on N. This tessellation is stationary, isotropic and almost surely normal.

We fix $l\in\{1,\dots,d\}$ and a l-dimensional linear subspace $S\subset\mathbb{R}^d$ and consider the section of the Poisson–Voronoi tessellation (based on N) with S. According to [39, Proposition 3.4.1], the collection of convex sets

$$X_S := \{C\cap S : C\in X\}$$

is a normal tessellation in S, whenever $N \in \mathbf{N}_s$. Our aim is to describe the Palm measure of the stationary random measure

$$M_{S,j} := \sum_{F \in \mathcal{S}_{d+j-l}(X)} \mathcal{H}^j(F \cap S \cap \cdot),$$

see (2.3.1). The intensity

$$\lambda_{d-l+j} = \gamma^{(l-j)/d} \cdot \frac{2^{l-j+1}\pi^{(l-j)/2}}{d \cdot (l-j+1)!} \cdot \frac{\Gamma\left(l-j+\frac{d-l+j}{d}\right)}{\Gamma\left(\frac{d-l+j+1}{2}\right)} \\ \times \frac{\Gamma\left(\frac{(d-1)(l-j)+d+1}{2}\right)}{\Gamma\left(\frac{(d-1)(l-j)+d}{2}\right)} \cdot \frac{\left[\Gamma\left(\frac{d}{2}+1\right)\right]^{l-j+(d-l+j)/d}}{\left[\Gamma\left(\frac{d+1}{2}\right)\right]^{l-j}} \tag{2.7.1}$$

is well known (see e.g. [4, (1.2)]). It follows from (2.5.6) that the intensity $\lambda_{S,j}$ of $M_{S,j}$ is given by

$$\begin{aligned} \lambda_{S,j} &= \gamma^{(l-j)/d} \cdot \frac{2^{l-j+1} \cdot \pi^{(l-j)/2}}{d \cdot (l-j+1)!} \cdot \frac{\Gamma\left(l-j+\frac{d-l+j}{d}\right)}{\Gamma\left(\frac{d-l+j+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{(d-1)(l-j)+d+1}{2}\right)}{\Gamma\left(\frac{(d-1)(l-j)+d}{2}\right)} \\ &\quad \times \frac{\left[\Gamma\left(\frac{d}{2}+1\right)\right]^{l-j+(d-l+j)/d}}{\left[\Gamma\left(\frac{d+1}{2}\right)\right]^{l-j}} \cdot \frac{\Gamma\left(\frac{d-l+j+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \\ &= \gamma^{(l-j)/d} \cdot \frac{2^{l-j+1} \cdot \pi^{(l-j)/2}}{d \cdot (l-j+1)!} \cdot \Gamma\left(l-j+\frac{d-l+j}{d}\right) \cdot \frac{\Gamma\left(\frac{(d-1)(l-j)+d+1}{2}\right)}{\Gamma\left(\frac{(d-1)(l-j)+d}{2}\right)} \\ &\quad \times \frac{\left[\Gamma\left(\frac{d}{2}+1\right)\right]^{l-j+(d-l+j)/d}}{\left[\Gamma\left(\frac{d+1}{2}\right)\right]^{l-j+1}} \cdot \frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \end{aligned} \tag{2.7.2}$$

Following [49, Chapter 10.2], we mention some further properties of X. Let $k \in \{0, \dots, d\}$. For points $x_0, \dots, x_{d-k} \in \mathbb{R}^d$ in general position, we let $z(x_0, \dots, x_{d-k})$ denote the centre of the uniquely determined $(d-k)$-dimensional ball, having $x_0, \dots, x_{d-k}$ on its boundary. Furthermore, we let $F(x_0, \dots, x_{d-k})$ denote the k-dimensional affine subspace orthogonal to the above ball and containing $z(x_0, \dots, x_{d-k})$. Let $F \in \mathcal{S}_k(X(\omega))$. Then there are (lexicographically ordered) points $x_0, \dots, x_{d-k} \in N(\omega)$ such that

$$F = \{x \in F(x_0, \dots, x_{d-k}) : B^0(x, \|x - x_0\|) \cap N(\omega) = \emptyset\}. \tag{2.7.3}$$

Conversely, given different points $x_0, \dots, x_{d-k} \in N(\omega)$ such that the set F, defined as in (2.7.3), has nonempty relative interior, we have $F \in \mathcal{S}_k(X(\omega))$.

Now we introduce some further objects, following Section 2.2 and [4, Chapter 3]. Let x be in the relative interior of some $F \in \mathcal{S}_k(X(\omega))$ and choose $x_0, \dots, x_{d-k} \in N(\omega)$ as in (2.7.3). Since the points in N are almost surely in general position, the set $\{x_0, \dots, x_{d-k}\}$ is (almost surely) uniquely determined and we define

$$R_k(\omega, x) := \|x - x_0\| = \ldots = \|x - x_{d-k}\|,$$

$$X_{k,i}(\omega, x) := x_i, \quad i = 0, \ldots, d-k,$$
$$Z_k(\omega, x) := z(x_0, \ldots, x_{d-k}).$$

For points $x \in \mathbb{R}^d$ that are not in the relative interior of some k-face, we set $R_k(\omega, x) := 0$ and $X_{k,0}(\omega, x) = \ldots = X_{k,d-k}(\omega, x) = Z_k(\omega, x) := x$. For $k \leq d-1$, the number

$$R'_k(\omega, x) := \|X_{k,0}(\omega, x) - Z_k(\omega, x)\|$$

is positive and we can define the unit vectors

$$U_{k,i}(\omega, x) := \frac{X_{k,i}(\omega, x) - Z_k(\omega, x)}{R'_k(\omega, x)}, \quad i = 0, \ldots, d-k.$$

For convenience, we write

$$\Psi_k(\omega, x) := \{U_{k,0}(\omega, x), \ldots, U_{k,d-k}(\omega, x)\}.$$

If $k = d$, we define $R'_d(\omega, x) := 0$ and $U_{d,i}(\omega, x) := 0$. Furthermore, for $k \geq 1$, we define

$$R''_k(\omega, x) := \|x - Z_k(\omega, x)\|$$

and, given that $R''_k(\omega, x) > 0$, the unit vector

$$U_k(\omega, x) := \frac{Z_k(\omega, x) - x}{R''_k(\omega, x)}.$$

If $R''_k(\omega, x) = 0$, we choose $U_k(\omega, x)$ to equal some fixed unit vector. For $k = 0$ we choose $R''_0(\omega, x) = 0$ and $U_0(\omega, x) := 0$. For points x not in the relative interior of some k-face, we let $R'_k(\omega, x) = R''_k(\omega, x) \equiv 0$ and choose $U_{k,0}(\omega, x), \ldots, U_{k,d-k}, U_k(\omega, x)$ to be fixed unit vectors. For $k = d$ we define $U_{d,0}(\omega, x) := 0$.

Theorem 2.7.1. *Assume that N is a stationary Poisson process of intensity $\gamma > 0$ and consider an l-dimensional linear subspace $S \subset \mathbb{R}^d$. Let $j \in \{0, \ldots, l\}$ and define $k := d - l + j$. Then the assertions (i)-(iv) of [4, Theorem 1.1] hold under the Palm probability measure $\mathbb{P}^0_{S,j}$ of $M_{S,j}$*

- *(i) The random variables $(\{x \in N : \|x\| > R_k(0)\}, R_k(0))$, $(R'_k(0))^2/R_k(0)^2$ and $(\Psi_k(0), U_k(0))$ are independent.*
- *(ii) $R^d_k(0)$ is gamma distributed with shape parameter $d - k + k/d$ and scale parameter $\gamma\kappa_d$.*
- *(iii) The conditional distribution of $\{x \in N : \|x\| > R_k(0)\}$ given $R_k(0) = r$ can be chosen to be the distribution of a homogeneous Poisson process on the complement of the ball $B(0, r)$, with intensity γ.*
- *(iv) For $k \in \{1, \ldots, d-1\}$, $(R'_k(0))^2/R_k(0)^2$ has a beta distribution with parameters $d(d-k)/2$ and $k/2$.*

Moreover, under $\mathbb{P}^0_{S,j}$ the distribution of $(\Psi_k(0), U_k(0))$ is given by

$$\mathbb{Q}_{S,j}(\cdot) := c_{S,j}^{-1} \int \cdots \int \mathbf{1}\{(\{\vartheta u_0, \dots, \vartheta u_{l-j}\}, \vartheta u) \in \cdot\} \Delta_{l-j}(u_0, \dots, u_{l-j})^{d-l+j+1}$$
$$\times [\vartheta L^\perp, S]_j \; \mathbb{S}_L(du_0) \dots \mathbb{S}_L(du_{l-j}) \; \mathbb{S}_{L^\perp}(du) \; \nu(d\vartheta), \qquad (2.7.4)$$

where L is a fixed $(l-j)$-dimensional subspace of $\mathbb{R}^d$, $\mathbb{S}_L$ denotes the uniform distribution on the unit sphere in L, $\Delta_{l-j}(u_0, \dots, u_{l-j})$ is the $(l-j)$-dimensional volume of the simplex spanned by $u_0, \dots, u_{l-j}$ and ν is the uniform distribution on the rotation group SO_d. The constant $c_{S,j}$ is given by

$$c_{S,j} := \frac{1}{((l-j)!)^{d-l+j+1}} \cdot \left(\frac{\Gamma\left(\frac{l-j}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \right)^{l-j} \cdot \frac{\Gamma\left(\frac{(d-1)(l-j)+d+1}{2}\right)}{\Gamma\left(\frac{d(l-j)}{2}\right)}$$
$$\times \frac{\Gamma\left(\frac{d-l+j+2}{2}\right) \cdots \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \cdots \Gamma\left(\frac{l-j-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{d-l+j+1}{2}\right) \Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{j+1}{2}\right)}, \qquad l-j \geq 2,$$
$$c_{S,j} := 2^{d-1} \frac{\Gamma(\frac{d}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{j+1}{2})\Gamma(\frac{d+1}{2})}, \qquad l-j=1,$$
$$c_{S,j} := 1, \qquad l=j.$$

Proof. Same as above, we abbreviate $\Psi_k(0) = \{U_{k,0}(0), \dots, U_{k,l-j}(0)\}$. By [4, Formula (2.1)] we almost surely have

$$[S, F_k(0)]_j = [S, \operatorname{span}(U_{k,0}(0), \dots, U_{k,l-j}(0))^\perp]_j.$$

Using Theorem 2.4.1 and [4, Theorem 1.1], we get

$$\begin{aligned}
\mathbb{Q}_{S,j}(\cdot) &= \mathbb{P}^0_{S,j}\left((\Psi_k(0), U_k(0)) \in \cdot\right) \\
&= \frac{1}{\lambda_{S,j}} \cdot \mathbb{E}_{M_{S,j}} \left[\mathbf{1}\{(\Psi_k(0), U_k(0)) \in \cdot\} \right] \\
&= \frac{1}{\lambda_{S,j}} \cdot \mathbb{E}_{M_k} \left[[S, F_{d-l+j}(0)]_j \mathbf{1}\{(\Psi_k(0), U_k(0)) \in \cdot\} \right] \\
&= \frac{\lambda_k}{\lambda_{S,j}} \cdot \mathbb{E}^0_{M_k} \left[[S, (\operatorname{span}(U_{k,0}(0) \dots, U_{k,l-j}(0)))^\perp]_j \mathbf{1}\{(\Psi_k(0), U_k(0)) \in \cdot\} \right] \\
&= \frac{\lambda_k c_k^{-1}}{\lambda_{S,j}} \int \dots \int \mathbf{1}\left\{(\{\vartheta u_0, \dots, \vartheta u_{l-j}\}, \vartheta u) \in \cdot\right\} \Delta_{l-j}(u_0, \dots, u_{l-j})^{k+1} \\
&\qquad \times [S, \vartheta L^\perp]_j \; \mathbb{S}_L(du_0) \dots \mathbb{S}_L(du_{l-j}) \; \mathbb{S}_{L^\perp}(du) \; \nu(d\vartheta). \qquad (2.7.5)
\end{aligned}$$

It remains to show that $c_{S,j} = c_k \cdot \frac{\lambda_{S,j}}{\lambda_k}$. Using Fubini's theorem, we get

$$c_{S,j} = \int \dots \int \Delta_{l-j}(u_0, \dots, u_{l-j})^{k+1} [S, \vartheta L^\perp]_j \; \mathbb{S}_L(du_0) \cdots \mathbb{S}_L(du_{l-j}) \; \mathbb{S}_{L^\perp}(du) \; \nu(d\vartheta)$$

$$= c_k \int_{SO_d} [S, \vartheta L^\perp]_j \ \nu(d\vartheta) \tag{2.7.6}$$

By [49, p. 133]), we have

$$\begin{aligned}(2.7.6) &= c_k \kappa_{l-j}^{-1} \int V^{l-j}(\Pi_{(\vartheta L^\perp)^\perp} B_S) \ \nu(d\vartheta)\\ &= c_k \kappa_{l-j}^{-1} \int V^{l-j}(\Pi_{\vartheta L} B_S) \ \nu(d\vartheta), \end{aligned}\tag{2.7.7}$$

where $\Pi_{\vartheta L} B_S$ denotes the orthogonal projection of the unit ball in S onto ϑL.
Let ν_q denote the unique, rotation invariant, probability measure on $G(d, q)$, the set of all q-dimensional linear subspaces of $\mathbb{R}^d$. Then

$$\begin{aligned}(2.7.7) &= c_k \kappa_{l-j}^{-1} \int_{G(d,l-j)} V_{l-j}(\Pi_F B_S) \ \nu_{l-j}(dF)\\ &= c_k \cdot \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{d+1}{2})\Gamma(\frac{j+1}{2})}, \end{aligned}\tag{2.7.8}$$

recalling the proof of Corollary 2.5.4. Using [4, Theorem 1.1], we obtain

$$\begin{aligned} c_k =& \frac{1}{((d-k)!)^{k+1}} \left[\frac{\Gamma((d-k)/2)}{\Gamma((d+1)/2)}\right]^{d-k} \frac{\Gamma((d^2-kd+k+1)/2)}{\Gamma((d^2-kd)/2)}\\ &\times \frac{\Gamma((k+2)/2)\cdots\Gamma(d/2)}{\Gamma(\frac{1}{2})\cdots\Gamma((d-k-1)/2)} \end{aligned}$$

for $k < d-1$, $c_{d-1} = 2^{d-1}$ and $c_d = 1$. This gives $c_{S,j}$ as required.

In order to prove assertion (ii), we recall that $[S, F_k(0)]_j$ is a.s. a function of $\Psi_k(0)$, say $f(\Psi_k(0))$. Then (i) and (ii) from [4, Theorem 1.1] and Theorem 2.4.1 give

$$\begin{aligned} \mathbb{P}^0_{S,j}(R^d_k(0) \in A) &= \frac{1}{\lambda_{S,j}} \int \mathbf{1}\{R^d_k(0) \in A\} \ d\mathbb{P}_{M_{S,j}}\\ &= \frac{1}{\lambda_{S,j}} \int \mathbf{1}\{R^d_k(0) \in A\}[S, F_k(0)]_j \ d\mathbb{P}_{M_k}\\ &= \frac{\lambda_k}{\lambda_{S,j}} \mathbb{P}^0_{M_k}(R^d_k(0) \in A) \cdot \mathbb{E}^0_{M_k}[S, (\mathrm{span}(U_{k,0}(0), \dots, U_{k,d-k}(0)))^\perp]_j\\ &= \mathbb{P}^0_{M_k}(R^d_k(0) \in A), \qquad A \subset \mathbb{R}, \end{aligned}$$

where we have used (2.7.6) in the last step. The proofs of (iii) and (iv) are essentially the same.

For assertion (i), it is sufficient to show

$$\begin{aligned} &\mathbb{P}^0_{S,j}((R'_k(0))^2/R_k(0)^2 \in A, (\Psi_k(0), U_k(0)) \in B)\\ &= \mathbb{P}^0_{S,j}((R'_k(0))^2/R_k(0)^2 \in A) \cdot \mathbb{P}^0_{S,j}((\Psi_k(0), U_k(0)) \in B) \end{aligned}$$

for all measurable A, B. The remaining identities can be shown analogously. Using [4, Theorem 1.1], we have

$$
\begin{aligned}
&\mathbb{P}^0_{S,j}((R'_k(0))^2/R_k(0)^2 \in A, (\Psi_k(0), U_k(0)) \in B) \\
&= \frac{\lambda_k}{\lambda_{S,j}} \mathbb{E}^0_{M_k} \left[\mathbf{1}\{(R'_k(0))^2/R_k(0)^2 \in A\} \mathbf{1}\{(\Psi_k(0), U_k(0)) \in B\} f(\Psi_k(0)) \right] \\
&= \mathbb{P}^0_{M_k}((R'_k(0))^2/R_k(0)^2 \in A) \cdot \frac{1}{\lambda_{S,j}} \mathbb{E}_{M_k} \left[\mathbf{1}\{(\Psi_k(0), U_k(0)) \in B\} f(\Psi_k(0)) \right] \qquad (2.7.9)
\end{aligned}
$$

Now the results already shown and (2.7.5) give

$$
(2.7.9) = \mathbb{P}^0_{S,j}((R'_k(0))^2/R_k(0)^2 \in A) \cdot \mathbb{P}^0_{S,j}((\Psi_k(0), U_k(0)) \in B)
$$

This completes the proof. $\square$

3. Kendall's Problem in spherical space: Crofton cells

We now consider random tessellations of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. This setting is not as extensively studied in the literature as the Euclidean one is. The intersection of the unit sphere with a d-dimensional subspace is the unit sphere in the intersecting subspace and thus a great subsphere of $\mathbb{S}^d$ having unit radius. We call the intersection a great circle in case of $d = 2$. At the same time, d-dimensional subspaces partition the Euclidean space $\mathbb{R}^{d+1}$ into polyhedral cones. This relation plays an important role in spherical geometry, see e.g. [1], [9] [14, Chapter 2]. Random tessellations of the sphere generated by intersecting the unit sphere with d-dimensional subspaces are studied in [9], [34, Section 6], [2] and the recent work [23] on conical tessellations. Voronoi tessellations in spherical space can be defined as in the Euclidean case, using the geodesic distance on $\mathbb{S}^d$. Random Voronoi tessellations on the sphere and applications are investigated in [34, Section 7], [44], [51], [43, Section 3.7.6, Section 5.10] and [53].

We focus on what became known as 'Kendall's Problem' or 'Kendall's Conjecture'. So far this line of investigation was only considered in the Euclidean setting. In our present work, we now formulate and investigate a spherical analogue. Consider a stationary and isotropic Poisson line process in the Euclidean plane and denote the almost surely unique cell containing the origin by Z_0. This cell is called the *zero cell* or *Crofton cell.* In the foreword of the first edition of [8], D.G. Kendall stated the following conjecture: The conditional law for the shape of Z_0, given the area $A(Z_0)$, converges weakly, as $A(Z_0) \to \infty$, to the degenerate law concentrated at the circular shape. A proof was given by Kovalenko in [26] and [28] and an extension to the typical cell of a Poisson–Voronoi tessellation in the plane in [27]. Further extensions to arbitrary dimensions and not necessarily isotropic Poisson hyperplane tessellations were made in [17], where the size of the Crofton cell was measured by the volume. In [18] the problem was extended and solved for typical cells of stationary Poisson–Voronoi tessellations in arbitrary dimensions and the size was measured by an intrinsic volume. In [20] a very general setting with a very general class of size functionals was considered, containing the aforementioned results as special cases. In [21], Kendall's Problem was extended to the typical k-faces of a Poisson hyperplane tessellation ($k \in 2, \dots, d-1\}$) and in [22] to the typical k-faces of a Poisson–Voronoi tessellation. In [19] typical cells of Poisson–Delaunay tessellations were considered.

3.1. Results from spherical geometry

In this chapter we will work in Euclidean space $\mathbb{R}^{d+1}$, $d \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ denote the unit sphere. Defining

$$d_s(x, y) := \arccos\langle x, y\rangle, \quad x, y \in \mathbb{S}^d, \tag{3.1.1}$$

we obtain the geodesic metric on $\mathbb{S}^d$. A geodesic ball (or spherical cap) on the unit sphere with radius $r \in [0, \pi]$ and centre $x \in \mathbb{S}^d$ is the set

$$B_s(x, r) := \{y \in \mathbb{S}^d : d_s(x, y) \leq r\}.$$

For a set $K \subset \mathbb{S}^d$, which is contained in an open hemisphere, we define the spherical circumball $B_s(K)$ as the smallest spherical cap on $\mathbb{S}^d$, containing K. It is shown e.g. in [6, Proposition 2.7] that this cap is unique. Its radius is denoted by $R_s(K)$. Relations to the (spherical) diameter are given in [11] and [47]. If $K \subset \mathbb{S}^d$ is closed but not necessarily contained in an open hemisphere, $R_s(K)$ is defined as the minimal radius r such that there is a spherical cap of radius r containing K. For $K \subset \mathbb{S}^d$ closed, the spherical inradius $r_s(K)$ is defined as $r_s(K) := \max\{r \in [0, \pi/2] : \text{there is } x \in \mathbb{S}^d \text{ such that } B_s(x, r) \subset K\}$. The centred spherical circumball $B_s(K, e)$, for a given point $e \in \mathbb{S}^d$, is the smallest spherical cap centred at e, which contains $K \subset \mathbb{S}^+_e := \{x \in \mathbb{S}^d : \langle x, e\rangle \geq 0\}$. The centred spherical inball $B_i(K, e)$, for a given point $e \in \mathbb{S}^d$, is the largest spherical cap centred at e, which is contained in K. The radii of these balls will be denoted by $R_s(K, e)$ and $r_s(K, e)$ (we shall omit the reference to the point e, if e is fixed and it is clear from the context that we are using the centred version).

In Euclidean space, the Crofton cell of a tessellation is the cell containing the origin. Since there are no distinguished points on a sphere, we choose $\overline{0} := (1, 0, \dots, 0)^\top$ as the spherical origin.

By a convex body in $\mathbb{S}^d$, we understand an intersection of a line-free closed convex cone $C \neq \emptyset$ in $\mathbb{R}^{d+1}$ with the unit sphere. Thus, a convex body is contained in some open hemisphere. We denote the set of spherically convex bodies by $\mathcal{K}^d_s$ and equip it with the spherical Hausdorff distance δ_s. If we do not require the cone to be line-free, the resulting intersection is only contained in some closed hemisphere. The set of all intersections of some closed convex cone $\widetilde{C} \neq \emptyset$ in $\mathbb{R}^{d+1}$, which is not equal to some linear subspace of $\mathbb{R}^{d+1}$, with $\mathbb{S}^d$ will be denoted by $\overline{\mathcal{K}}^d_s$. By excluding linear subspaces, we ensure that $\overline{\mathcal{K}}^d_s$ does not contain subspheres of $\mathbb{S}^d$. Note that $\overline{\mathcal{K}}^d_s$ is the closure of $\mathcal{K}^d_s$ with respect to the Hausdorff metric (see also [14]). A *spherical polytope* is the intersection of $\mathbb{S}^d$ with a polyhedral cone and if the polyhedral cone is line-free, its intersection with $\mathbb{S}^d$ is called a spherically convex polytope.

By a tessellation of $\mathbb{S}^d$, we understand a finite collection of spherical polytopes that have nonempty interiors, which cover $\mathbb{S}^d$ and have pairwise disjoint interiors.

For $A \subset \mathbb{S}^d$, the polar set is defined by $A^* := \{x \in \mathbb{S}^d : \langle x, a\rangle \leq 0 \text{ for all } a \in A\}$. If $K \in \mathcal{K}^d_s$, then $K^* \in \mathcal{K}^d_s$ and $(K^*)^* = K$ (see [49, p. 249]).

Following [12], we introduce a family U_j, $j \in \{0, \dots, d\}$, of geometric functionals on $\mathcal{K}^d_s$. For this, let $G(d+1, j)$, $j \in \{0, \dots, d+1\}$, be the space of j-dimensional linear subspaces of $\mathbb{R}^{d+1}$. Hence, for $L \in G(d+1, j+1)$ and $j \in \{0, \dots, d\}$, the intersection $L \cap \mathbb{S}^d$

is a j-dimensional great subsphere of $\mathbb{S}^d$. Further, let ν_j denote the rotation invariant probability measure on $G(d+1,j)$. For $K \in \overline{\mathcal{K}}_s^d$ and $j \in \{0,\dots,d\}$, we then define

$$U_j(K) := \frac{1}{2}\int_{G(d+1,d+1-j)} \mathbf{1}\{K \cap L \neq \emptyset\}\ \nu_{d+1-j}(dL). \tag{3.1.2}$$

In particular, $2 \cdot U_1(K)$ can be interpreted as the mass of all great subspheres hitting K, where

$$U_1(K) = \frac{1}{2}\int_{G(d+1,d)} \mathbf{1}\{K \cap L \neq \emptyset\}\ \nu_d(dL) = \frac{1}{2\omega_{d+1}}\int_{\mathbb{S}^d} \mathbf{1}\{K \cap x^\perp \neq \emptyset\}\ \sigma_d(dx), \tag{3.1.3}$$

and σ_d denotes spherical Lebesgue measure on $\mathbb{S}^d$ with total mass ω_{d+1}.

In [12], the authors deduce the following extremal property for U_1.

Theorem 3.1.1. *Let $K \in \overline{\mathcal{K}}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\sigma_d(K) = \sigma_d(C)$. Then*

$$U_1(K) \geq U_1(C). \tag{3.1.4}$$

Equality holds if and only if K is a spherical cap.

Remark 3.1.2. *Although in [12] Theorem 3.1.1 is only formulated for $K \in \mathcal{K}_s^d$, the first proof given in [12] remains valid for $K \in \overline{\mathcal{K}}_s^d$.*

In the Euclidean setting, the limit shape of the Crofton cell (suitably defined) of an isotropic and stationary Poisson hyperplane tessellation is a ball (see [17, Theorem 1]) and an inequality similar to (3.1.4) holds. Let $K \subset \mathbb{R}^d$ be a convex body and let B be a ball with $V_d(K) = V_d(B)$. Then

$$V_1(K) \geq V_1(B)$$

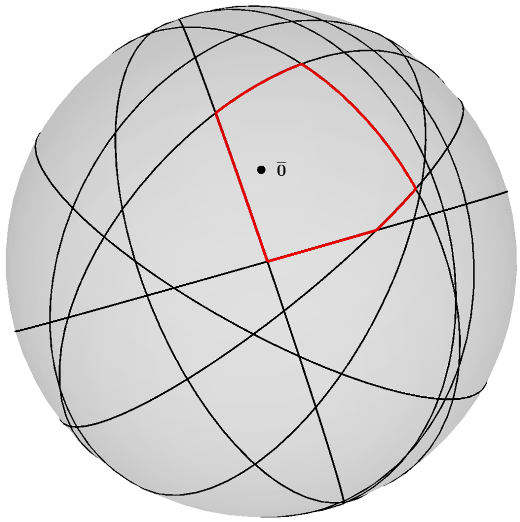
and equality holds if and only if K is a ball. This is a special case of a set of inequalities, which can be found, e.g., in [49, p. 613] and which state that

$$\left(\frac{\kappa_{d-j}}{\binom{d}{j}} V_j(K)\right)^k \geq \kappa_d^{k-j} \left(\frac{\kappa_{d-k}}{\binom{d}{k}} V_k(K)\right)^j,$$

for a convex body K, $0 < j < k \leq d$. Equality holds if and only if K is a ball. To obtain the special case, we put $j = 1$ and $k = d$. Based on Theorem 3.1.1 we have reasonable grounds on which to make the assumption that an analogous result on the sphere should be obtainable by similar arguments as in [20].

3.2. Spherical hyperplane tessellations and the Crofton cell

Let $X \neq 0$ be an isotropic Poisson process on $\mathbb{S}^d$. Since the spherical Lebesgue measure σ_d is (up to a constant) the only rotation invariant measure on $\mathbb{S}^d$ (see [49, chap. 13.2]), we have $\Theta(\cdot) := \mathbb{E}[X(\cdot)] = \gamma_S \cdot \sigma_d(\cdot)$ for some $\gamma_S > 0$. The number γ_S can be interpreted



Spherical Poisson hyperplane tessellation and Crofton cell with red boundary

(as in the stationary Euclidean case) as the intensity. The expected number of points on the sphere is

$$\mathbb{E}[X(\mathbb{S}^d)] = \gamma_S \cdot \omega_{d+1} = \gamma_S \cdot (d+1) \cdot \kappa_{d+1} = \gamma_S \cdot \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})},$$

where κ_n is the volume of the n-dimensional unit ball. Applying the measurable mapping

$$h : \begin{cases} \mathbb{S}^d & \to \; G(d+1,d) \cap \mathbb{S}^d, \\ x & \mapsto \; x^\perp \cap \mathbb{S}^d, \end{cases}$$

to every point in X, we obtain the spherical hyperplane process (or great subsphere process) $\tilde{X} := h(X)$.

A collection of spherical hyperplanes is said to be in general position if their normal vectors are in general position. By [49, Theorem 3.2.2. (b)] we have

$$\begin{aligned} &\mathbb{P}(\text{the points of } X \text{ are in general position}) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(\text{the points of } X \text{ are in general position} | X(\mathbb{S}^d) = k) \cdot \mathbb{P}(X(\mathbb{S}^d) = k) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(\xi_1, \dots, \xi_k \text{ are in general position}) \cdot \mathbb{P}(X(\mathbb{S}^d) = k), \end{aligned}$$

where $\xi_1, \xi_2, \dots$ are i.i.d. uniformly distributed on $\mathbb{S}^d$. Since i.i.d. uniformly distributed points on $\mathbb{S}^d$ are almost surely in general position, it follows that the spherical hyperplanes of $\tilde{X}$ are almost surely in general position.

Given there is at least one spherical hyperplane, they partition $\mathbb{S}^d$ almost surely into a collection of spherical polytopes, with pairwise disjoint interiors. Such a partition is called a spherical hyperplane tessellation of $\mathbb{S}^d$.

The *spherical Crofton cell* or *spherical zero cell* is the (almost surely uniquely determined) cell, which contains the spherical origin $\overline{0}$ in its (relative) interior. We will denote it by Z_0. Since Z_0 is the intersection of $\mathbb{S}^d$ with half-spaces, which are determined by linear subspaces (and the intersection of these half-spaces thus defines a polyhedral cone), it is always a spherical polytope. But only if the realisation of X contains at least $d+1$ points, it is also a spherical convex polytope, meaning its associated convex cone is line-free.

For $K \subset \mathbb{S}^d$ we define $\mathcal{H}_K := \{L \in G(d+1,d) \cap \mathbb{S}^d : L \cap K \neq \emptyset\}$. Then

$$\begin{aligned}\mathbb{E}\tilde{X}(\mathcal{H}_K) &= \gamma_S \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap K \neq \emptyset\}\ \sigma_d(dx) \\ &= 2 \cdot \gamma_S \cdot \omega_{d+1} \cdot U_1(K) =: \gamma_S \cdot \omega_{d+1} \cdot \Phi(K)\end{aligned}$$

and $\Phi(C_0) = 1$ for $C_0 := \{x \in \mathbb{S}^d : d_s(\overline{0}, x) \leq \pi/2\}$. Furthermore, we define

$$\mu(\cdot) := \frac{1}{\omega_{d+1}} \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap \mathbb{S}^d \in \cdot\}\ \sigma_d(dx).$$

Our aim is to show that the Crofton cell Z_0, given a lower bound for its spherical volume, converges to a spherical cap for $\gamma_S \to \infty$. This means that the conditional probability of Z_0 deviating from the shape of a spherical cap, given Z_0 has spherical volume at least a for some $a > 0$, converges to 0 for $\gamma_S \to \infty$. Therefore, we have to quantify the deviation of Z_0 from a spherical cap. In order to obtain a suitable quantification, we adapt a definition from [20] to our setting. A functional $\vartheta : \overline{\mathcal{K}}_s^d \to [0,\infty)$ is called a deviation functional for the class of spherical caps, if

(a) ϑ is continuous,

(b) $\vartheta(K) = 0$ for some $K \in \overline{\mathcal{K}}_s^d$ with $\sigma_d(K) > 0$ if and only if K is a spherical cap.

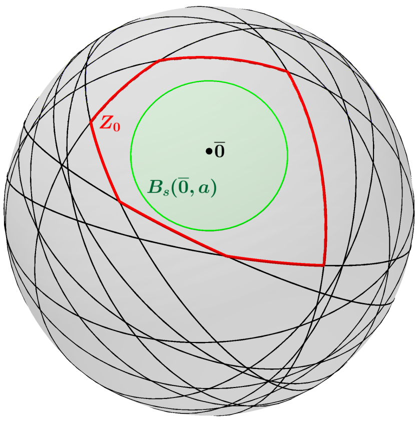
An example for such a deviation functional is the difference between spherical circumradius and spherical inradius of K. Another example measures the deviation of the shape in the L^2-sense and is discussed in Theorem 3.3.4. With these definitions, we can now formulate the main result of this chapter in the following theorem.

Theorem 3.2.1. *Let $a, \varepsilon > 0$ with $a < \omega_{d+1}/2$ and $\gamma_S \omega_{d+1} > 1$. Then there are constants $c_1, c_2 > 0$ such that*

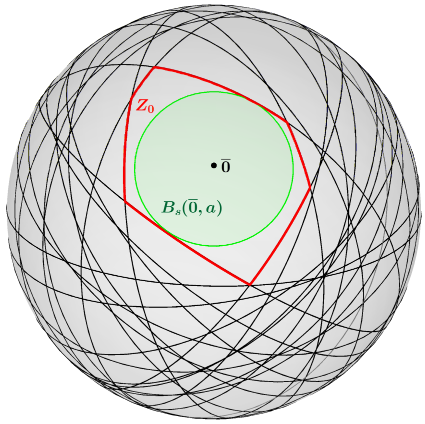
$$\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon | \sigma_d(Z_0) \geq a\right) \leq c_1 \cdot \exp\left(-c_2 \cdot \gamma_S \cdot \omega_{d+1}\right) \tag{3.2.1}$$

and the constants c_1, c_2 depend only on a, ε and d.

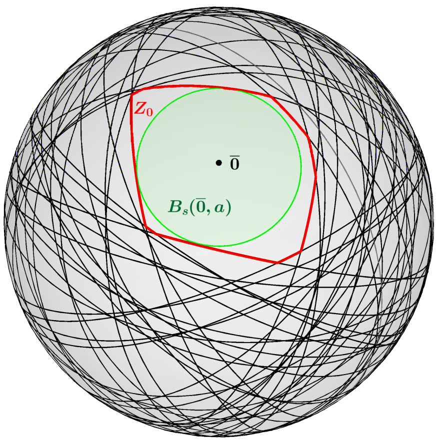
We visualize this result with some pictures obtained from simulations. For easier programming, we condition on a minimal spherical inball radius of $\frac{\pi}{8}$ centred at $\overline{0}$, instead of a minimal volume. The justification will come later in Section 3.7. The random points on $\mathbb{S}^2$ are simulated with R, the pictures are produced with GeoGebra (geogebra.org).



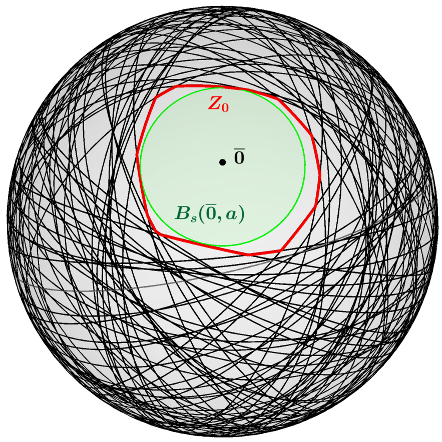
$\gamma_S = 1$, the realisation contains 17 great circles



$\gamma_S = 2$, the realisation contains 31 great circles



$\gamma_S = 4$, the realisation contains 61 great circles



$\gamma_S = 10$, the realisation contains 118 great circles

3.3. Geometric inequalities and stability results

In order to prove Theorem 3.2.1, we will first provide an approximation result for spherical polytopes, as well as two stability results for Theorem 3.1.1.

Lemma 3.3.1. *Let $K \in \overline{\mathcal{K}}_s^d$. Then there are constants k_1 and b_1, depending only on d, such that for all $k \geq k_1$ there is a spherical polytope Q with k vertices, without loss of generality on the boundary of K, satisfying*

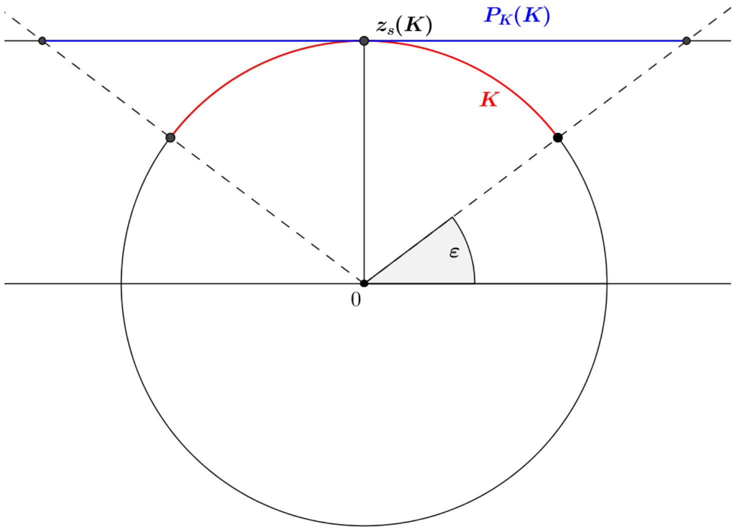
$$\delta_s(K, Q) \leq b_1 k^{-2/(d-1)}.$$

Proof. Let $\varepsilon > 0$ and assume $R_s(K) \leq \pi/2 - \varepsilon$. Later, this ε will depend on d. Let $z_s(K) \in \mathbb{S}^d$ denote the centre of the spherical circumball of K, $B_s^o(z_s(K))$ the open half sphere with centre $z_s(K)$ and E_x the tangent hyperplane to $\mathbb{S}^d$ in $x \in \mathbb{S}^d$. We consider the mapping

$$P_K : \begin{cases} B_s^o(z_s(K)) & \to E_{z_s(K)}, \\ x & \mapsto \langle x, z_s(K)\rangle^{-1} \cdot x, \end{cases}$$

meaning the image of $x \in \mathbb{S}^d$ is the intersection of $\operatorname{span}(x)$ and $E_{z_s(K)}$. From this we then obtain the following inequality for the Euclidean circumradius of the image of K

$$R := R(P_K(K)) \leq \frac{1}{\tan(\varepsilon)}.$$



Applying the main result from [7], we now get constants $k_0 = k_0(d)$ and $b_0 = b_0(d)$, such that the following is true. For $k \in \mathbb{N}$, $k \geq k_0$ there is a polytope $Q_0 \subset E_{z_s(K)}$ with k vertices, located on the boundary of $P_K(K)$, satisfying

$$\delta(R^{-1}P_K(K), R^{-1}Q_0) \leq b_0 k^{-2/(d-1)}.$$

Here δ denotes the Hausdorff-distance in $\mathbb{R}^{d+1}$. The polytopes $R^{-1}P_K(K)$ and $R^{-1}Q_0)$ lie in an affine subspace parallel to $z_s(K)^\perp$. Therefore $P_K(K) \subset Q_0 + R \cdot b_0 \cdot k^{-2/(d-1)} B_E$, where B_E is the unit ball in $z_s(K)^\perp$, and thus

$$\delta(P_K(K), Q_0) \leq \frac{1}{\tan(\varepsilon)} b_0 k^{-2/(d-1)}.$$

The mapping

$$\Pi_{\mathbb{S}^d} : \begin{cases} E_{z_s(K)} & \to \mathbb{S}^d, \\ x & \mapsto \frac{x}{\|x\|}, \end{cases}$$

is Lipschitz continuous with Lipschitz constant at most 2, which can be seen as follows. Using the triangle inequality, we get

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x-y}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|$$

$$
\begin{aligned}
&\leq \|x-y\| \cdot \frac{1}{\|x\|} + \|y\| \cdot \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \\
&= \frac{\|x-y\|}{\|x\|} + \left| \frac{\|y\|-\|x\|}{\|x\|} \right| \\
&= \frac{1}{\|x\|}\Big(\|x-y\| + \big|\|x\|-\|y\|\big| \Big) \\
&\leq \frac{1}{\|x\|}\Big(\|x-y\| + \|x-y\| \Big) \\
&\leq 2 \cdot \|x-y\|,
\end{aligned}
$$

since $\|x\| \geq 1$. We now need an estimate for the spherical distance of $x, y \in \mathbb{S}^d$. Since $t \mapsto \sin(t)$ is concave on $[0, \pi/2]$, we have for $\beta \in [0, \pi/2]$

$$
\frac{\sin(\beta)-\sin(0)}{\beta - 0} \geq \frac{\sin(\frac{\pi}{2}) - 0}{\frac{\pi}{2} - 0} \quad \Leftrightarrow \quad \frac{\sin(\beta)}{\beta} \geq \frac{2}{\pi} \quad \Leftrightarrow \quad \beta \geq \arcsin\left(\frac{2}{\pi}\cdot\beta\right),
$$

where we have used the monotonicity of arcsin. Applying this result with $\beta = \frac{\pi}{4}\|x-y\|$, we obtain

$$
\frac{\pi}{2} \cdot \|x-y\| \geq 2 \cdot \arcsin\left(\frac{\|x-y\|}{2}\right) = d_s(x,y).
$$

The last equality follows from

$$
\begin{aligned}
\cos\left(2\arcsin\left(\frac{\|x-y\|}{2}\right)\right) &= \left[\cos\left(\arcsin\left(\frac{\|x-y\|}{2}\right)\right)\right]^2 - \left[\sin\left(\arcsin\left(\frac{\|x-y\|}{2}\right)\right)\right]^2 \\
&= 1 - \frac{\|x-y\|^2}{4} - \frac{\|x-y\|^2}{4} = 1 - \frac{\langle x-y, x-y\rangle}{2} \\
&= 1 - \frac{2-2\langle x,y\rangle}{2} = \langle x,y\rangle,
\end{aligned}
$$

or directly from planar geometry. Thus the spherical polytope $Q := \Pi_{\mathbb{S}^d}(Q_0)$ satisfies $Q \subset \Pi_{\mathbb{S}^d}(P_K(K)) = K$ and

$$
\delta_s(Q,K) \leq \frac{1}{\tan(\varepsilon)} \cdot b_0 \cdot \pi \cdot k^{-2/(d-2)}, \quad k \geq k_0. \tag{3.3.1}
$$

For arbitrary $K \in \overline{\mathcal{K}}_s^d$, we divide K into 2^d pieces by intersecting with d hyperplanes, where each hyperplane is the linear span of $\overline{0}$ and $d-1$ of the remaining d standard basis vectors of $\mathbb{R}^{d+1}$ (or equivalently the orthogonal complement of one of those d remaining standard basis vectors). Then each piece of K is contained in a regular spherical d-simplex of edge-length $\pi/2$, which is the spherical convex hull of $d+1$ unit vectors. Its circumradius is $\arccos(1/\sqrt{d+1}) \in (\pi/4, \pi/2)$, see [10, Theorem 2]. Defining

$$
\varepsilon := \pi/2 - \arccos\left(\frac{1}{\sqrt{d+1}}\right),
$$

the individual pieces satisfy $R_s(K_i) \leq \frac{\pi}{2} - \varepsilon,\ i = 1, \dots, 2^d$. Applying the argumentation above to every piece, we obtain spherical polytopes $Q_i,\ i = 1, \dots, 2^d$, such that

$$\delta_s(Q_i, K_i) \leq \frac{1}{\tan(\varepsilon)} \cdot b_0 \cdot \pi \cdot k^{-2/(d-1)}, \quad i = 1, \dots, 2^d,\ k \geq k_0.$$

Defining $Q := \mathrm{conv}_s(\bigcup_{i=1}^{2^d} Q_i)$, we obtain a polytope with (at most) $2^d \cdot k$ vertices and

$$\delta_s(Q, K) \leq \frac{1}{\tan(\varepsilon)} \cdot b_0 \cdot \pi \cdot k^{-2/(d-1)}.$$

Since

$$\begin{aligned}\frac{1}{\tan(\varepsilon)} &= \frac{1}{\tan\left(\pi/2 - \arccos(1/\sqrt{d+1})\right)} \\ &= \frac{1}{\cot\left(\arccos(1/\sqrt{d+1})\right)} \\ &= \frac{\sin(\arccos(1/\sqrt{d+1}))}{1/\sqrt{d+1}} \\ &= \sqrt{d+1} \cdot \sqrt{1 - \frac{1}{d+1}} \\ &= \sqrt{d},\end{aligned}$$

the assertion follows with $k_1 = 2^d \cdot k_0$ and $b_1 = b_0 \cdot \pi \cdot \sqrt{d} \cdot 4^{d/(d-1)}$. □

We start with an abstract stability result for a general deviation functional ϑ. The proof is based on continuity and compactness arguments, which can be quite easily generalized for other size functionals besides the volume, see also Section 3.7.

Theorem 3.3.2. *For any $a \in (0, \omega_{d+1}/2)$ there is a function $f_a : [0, \infty) \to [0, 1]$, with $f_a(0) = 0$ and $f_a(t) > 0$ for $t > 0$, such that*

$$U_1(K) \geq (1 + f_a(\varepsilon))U_1(B_a), \tag{3.3.2}$$

for any $\varepsilon > 0$ and $K \in \overline{\mathcal{K}}_s^d$ with $\sigma_d(K) \geq a,\ \bar{0} \in K$ and $\vartheta(K) \geq \varepsilon$.

Remark 3.3.3. *In the Euclidean case the function f is independent of a (see [20, (7)]). Due to the missing homogenity, this is in general not true for the spherical setting.*

Proof. In [14] it is shown that the set $\overline{\mathcal{K}}_s^d$ is compact (see the remarks after Hilfssatz 2.3) and obviously $\mathcal{K}_s^d \subset \overline{\mathcal{K}}_s^d$. We consider the set

$$\mathcal{K}(a) := \left\{K \in \overline{\mathcal{K}}_s^d :\ \sigma_d(K) \geq a\right\}.$$

Since $\overline{\mathcal{K}}_s^d$ is compact and σ_d is continuous, $\mathcal{K}(a)$ is a closed subset and therefore compact. Let B_a be a spherical cap with $\sigma_d(B_a) = a$, then $B_a \in \mathcal{K}(a)$. The functional U_1 is a linear combination of the spherical intrinsic volumes, see [14, Korollar 5.2.5.], which are continuous and rotation invariant, see [14, p. 36] or [49, Theorem 6.5.2 and p. 256], and thus U_1 is continuous. In [14], this linear combination was used to define U_1 on $\overline{\mathcal{K}}_s^d$ and the representation (3.1.2), was shown in [14, Korollar 5.2.5.]. Therefore U_1 attains its minimum on $\mathcal{K}(a)$. The functional U_1 is also rotation invariant, which follows directly from the definition and the rotation invariance of ν_d. Using Theorem 3.1.1 and the rotation invariance, we obtain

$$\min\{U_1(K) : K \in \mathcal{K}(a)\} = U_1(B_a) =: \tau_a.$$

We now consider the set

$$\mathcal{K}_\varepsilon(a) := \{K \in \mathcal{K}(a) : \vartheta(K) \geq \varepsilon\}.$$

Since ϑ is continuous, $\mathcal{K}_\varepsilon(a)$ is also compact and U_1 attains its minimum on $\mathcal{K}_\varepsilon(a)$. Let $\tau_{a,\varepsilon}$ denote this minimum. From $\vartheta(K) \geq \varepsilon > 0$ for all $K \in \mathcal{K}_\varepsilon(a)$ and the equality case of Theorem 3.1.1, we get

$$\tau_{a,\varepsilon} > \tau_a, \text{ thus } \tau_{a,\varepsilon} =: (1 + g_a(\varepsilon))\tau_a,$$

where $g_a(\varepsilon) > 0$ for $\varepsilon > 0$ and $g_a(0) = 0$. Defining $f_a(t) := \min\{g_a(t), 1\}$ for $t \in [0, \infty)$, the assertion follows. □

This result can be made more explicit for a special deviation functional. For this, we use the notation and results from [12]. In particular, D, h and α, S_e, for $e \in \mathbb{S}^d$, are defined as seen in [12]. To be more specific, we use

$$D(x) := \int_0^x \sin^{d-1} t \, dt, \qquad x \in (0, \pi/2),$$

and

$$h(y) := \tan^d(D^{-1}(y)), \qquad y \in \operatorname{im}(D).$$

For $e \in \mathbb{S}^d$, we put $S_e := \mathbb{S}^d \cap e^\perp$ and $T_e := e + e^\perp$. Further, we define the open halfspace $H_e^+ := \{x \in \mathbb{R}^{d+1} : \langle x, e \rangle > 0\}$. Then the map $R_e : \mathbb{S}^d \cap H_e^+ \to T_e$ with $R_e(u) := \langle e, u \rangle^{-1} u$ is the radial projection to the tangent plane of $\mathbb{S}^d$ at e.

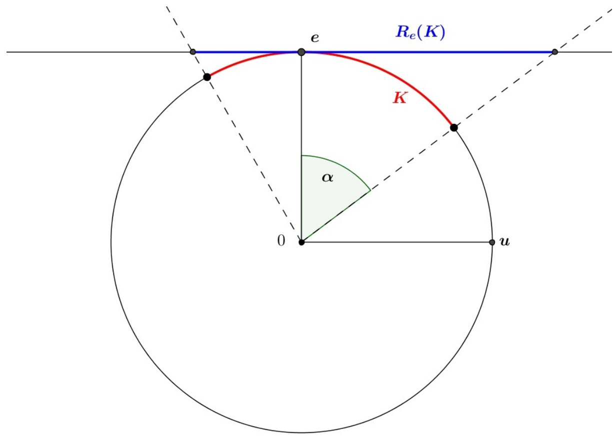
Let $K \subset \mathbb{S}^d$ be a spherically convex set with positive volume and $e \in -\operatorname{int}(K^*)$. The map $F_K : -\operatorname{int}(K^*) \to (0, \infty)$ with

$$F_K(e) := \int_K \langle e, u \rangle^{-(d+1)} \, \sigma_d(du)$$

assigns the volume of $R_e(K)$ in T_e to e. Let $\operatorname{Min}(F_K)$ denote the set of all $e \in -\operatorname{int}(K^*)$ such that F_K attains its minimum at e. It is shown in [12], that any $e \in \operatorname{Min}(F_K)$ is the centroid of $R_e(K)$ in T_e. We define $M(K) := \{e \in \operatorname{int}(K) : K \subset e^+\}$, where $e^+ := \{x \in \mathbb{R}^{d+1} : \langle x, e \rangle \geq 0\}$. Then $\operatorname{Min}(F_K) \subset M(K)$.

For $e \in M(K)$, the positive, continuous function $\alpha = \alpha_{K,e} : S_e \to (0, \pi/2)$ is defined by

$$\partial(R_e(K)) = \{e + \tan(\alpha(u))u : u \in S_e\}.$$



Using [49, Lemma 6.5.1], we can describe the volume of K in the form

$$\sigma_d(K) = \int_{S_e} \int_0^{\alpha(u)} \sin^{d-1} t \, dt \, \sigma_{d-1}(du),$$

and thus

$$\frac{\sigma_d(K)}{d\kappa_d} = \int_{S_e} D(\alpha(u)) \, \sigma^0_{d-1}(du),$$

where $\sigma^0_{d-1} := \sigma_{d-1}(\mathbb{S}^{d-1})^{-1}\sigma_{d-1}$ on great subspheres of $\mathbb{S}^d$. In particular, let $C \subset \mathbb{S}^d$ be a non-degenerate, line-free spherical cap. Then there is a constant $\alpha_C \in (0, \pi/2)$, which is independent of e, such that

$$\frac{\sigma_d(C)}{d\kappa_d} = \int_0^{\alpha_C} \sin^{d-1} t \, dt = D(\alpha_C), \qquad h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \tan^d(\alpha_C).$$

If $C^* \subset S^d$ is the polar of C, then $\alpha_{C^*} + \alpha_C = \pi/2$.

For a spherically convex set $K \subset S^d$, we define

$$\Delta(K)^2 := \inf\left\{ \int_{S_e} \left(D(\alpha(u)) - \int_{S_e} D(\alpha(u)) \, \sigma^0_{d-1}(du) \right)^2 \sigma^0_{d-1}(du) : e \in M(K) \right\},$$

which measures the deviation of the shape of K from a spherical cap in the L^2 sense. Clearly, $\Delta(K) = 0$ if and only if K is a spherical cap.

Theorem 3.3.4. *Let $K \subset \mathbb{S}^d$ be a spherically convex body and $C \subset \mathbb{S}^d$ a spherical cap with $\sigma_d(K) = \sigma_d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C \leq \pi/2 - \alpha_0$. Then there is a constant $\gamma = \gamma(d, \alpha_0)$ such that*

$$U_1(K) \geq \left(1 + \gamma \Delta(K)^2\right) U_1(C).$$

Proof. Let K be as stated in the theorem and let $e \in M(K)$. Since U_1 is rotation invariant, we can assume C to be centred at e. We continue to use the notation from [12]. Then

$$x_0 := \int_{S_e} D(\alpha(u)) \, \sigma^0_{d-1}(du) \in \operatorname{im}(D),$$

since $\alpha(u) \in (0, \pi/2)$, σ^0_{d-1} is a probability measure and D is monotone and continuous. For any $z \in \mathrm{im}(D)$, we have

$$h(z) - h(x_0) = h'(x_0)(z - x_0) + \frac{1}{2}h''(x_0 + \theta(z - x_0))(z - x_0)^2$$

for some $\theta = \theta(x_0, z) \in (0, 1)$. Since

$$h'(y) = \frac{d}{\cos^{d+1} D^{-1}(y)} \geq d$$

and

$$h''(y) = \frac{d(d+1)}{\cos^{d+2}(D^{-1}(y))} \frac{1}{\sin^{d-2}(D^{-1}(y))} \geq d(d+1),$$

for $y \in \mathrm{im}(D)$, we deduce that

$$h(z) - h(x_0) \geq h'(x_0)(z - x_0) + \binom{d+1}{2}(z - x_0)^2. \tag{3.3.3}$$

Substituting $z = D(\alpha(u))$, $u \in S_e$, in (3.3.3) and then integrating (3.3.3) with respect to σ^0_{d-1} over S_e, we obtain

$$\int_{S_e} h(D(\alpha(u)))\ \sigma^0_{d-1}(du) - h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) \geq 0 + \binom{d+1}{2}\Delta(K)^2.$$

Using that

$$h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) = h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \tan^d(\alpha_C) \leq \tan^d\left(\frac{\pi}{2} - \alpha_0\right) = \cot^d(\alpha_0)$$

and

$$\lambda^d(R_e(K)) = \frac{1}{d}\int_{S_e} \tan^d(\alpha(u))\ \sigma_{d-1}(du) = \frac{1}{d}\int_{S_e} h(D(\alpha(u)))\ \sigma_{d-1}(du)$$

(see [12, p. 14] or apply spherical coordinates in T_e), we conclude

$$\frac{\lambda^d(R_e(K))}{\kappa_d} = \int_{S_e} h(D(\alpha(u)))\ \sigma^0_{d-1}(du) \geq \left(1 + \gamma_1\Delta(K)^2\right) h\left(\frac{\sigma_d(K)}{d\kappa_d}\right), \tag{3.3.4}$$

where $\gamma_1 := \binom{d+1}{2}\tan^d(\alpha_0)$. Next, we recall some relations from [12]. The equality case of [12, (27)] gives

$$h\left(\frac{\sigma_d(K)}{d\kappa_d}\right) = h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \frac{\lambda^d(R_e(C))}{\kappa_d}$$

and the equality cases of [12, (26)] and [12, (30)] yield

$$h\left(\frac{\sigma_d(C)}{d\kappa_d}\right) = \frac{\kappa_d}{\lambda^d(R_e(C))}.$$

Now we use (26) and (30) from [12] for the first inequality, (3.3.4) for the second inequality and the identities above to obtain

$$h\left(\frac{\sigma_d(K^*)}{d\kappa_d}\right) \leq \frac{\kappa_d}{\lambda^d(R_e(K))}$$

$$
\begin{aligned}
&\le \frac{1}{1+\gamma_1\Delta(K)^2}\frac{1}{h\left(\frac{\sigma_d(K)}{d\kappa_d}\right)}\\
&= \frac{1}{1+\gamma_1\Delta(K)^2}\frac{\kappa_d}{\lambda^d(R_e(C))}\\
&= \frac{1}{1+\gamma_1\Delta(K)^2}h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)\\
&= \left(1-\frac{\gamma_1}{1+\gamma_1\Delta(K)^2}\Delta(K)^2\right)h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)\\
&\le \left(1-\gamma_2\Delta(K)^2\right)h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right), \qquad (3.3.5)
\end{aligned}
$$

where

$$\gamma_2 := \frac{\gamma_1}{1+\gamma_1(\pi/2)^2} \le \frac{\gamma_1}{1+\gamma_1\Delta(K)^2},$$

since $\Delta(K) \le \pi/2$. Next, we define

$$
\begin{aligned}
\gamma_3 : &= \min\left\{\gamma_2\frac{\tan^d(\alpha_0)}{D(\frac{\pi}{2}-\alpha_0)}\frac{\sin^{d+1}(\alpha_0)}{d}, \left(\frac{2}{\pi}\right)^2\right\}\\
&\le \gamma_2\frac{\tan^d(\alpha_{C^*})}{D(\alpha_{C^*})}\frac{\cos^{d+1}(\alpha_{C^*})}{d}\\
&= \gamma_2\frac{h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)}{\frac{\sigma_d(C^*)}{d\kappa_d}}h'\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)^{-1}.
\end{aligned}
$$

Note that the minimum in the definition of γ_3 is taken in order to ensure that $1-\gamma_3\Delta(K)^2 \ge 0$. Then the mean value theorem and the fact that h and h' are increasing imply that

$$
\begin{aligned}
&-h\left((1-\gamma_3\Delta(K)^2)\frac{\sigma_d(C^*)}{d\kappa_d}\right)+h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)\\
&\le h'\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right)\gamma_3\Delta(K)^2\frac{\sigma_d(C^*)}{d\kappa_d}\\
&\le \gamma_2\Delta(K)^2 h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right). \qquad (3.3.6)
\end{aligned}
$$

Combining (3.3.5) and (3.3.6), we get

$$h\left(\frac{\sigma_d(K^*)}{d\kappa_d}\right) \le h\left((1-\gamma_3\Delta(K)^2)\frac{\sigma_d(C^*)}{d\kappa_d}\right),$$

and hence

$$\frac{\sigma_d(K^*)}{\sigma_d(\mathbb{S}^d)} \le (1-\gamma_3\Delta(K)^2)\frac{\sigma_d(C^*)}{\sigma_d(\mathbb{S}^d)}.$$

Since
$$\frac{1}{2} - U_1(K) = \frac{\sigma_d(K^*)}{\sigma_d(\mathbb{S}^d)},$$
we deduce that
$$U_1(K) \geq \frac{1}{2}\gamma_3\Delta(K)^2 + (1-\gamma_3\Delta(K)^2)U_1(C).$$
Finally, we use the following
$$\frac{1}{2} - U_1(C) = \frac{\sigma_d(C^*)}{\sigma_d(\mathbb{S}^d)} = D(\alpha_{C^*}) \geq D(\alpha_0) \geq 2D(\alpha_0)U_1(C),$$
and therefore
$$\frac{1}{2} \geq (1+2D(\alpha_0))\, U_1(C),$$
to get
$$U_1(K) \geq \left[\gamma_3\Delta(K)^2 + 2D(\alpha_0)\gamma_3\Delta(K)^2 + 1 - \gamma_3\Delta(K)^2\right] U_1(C),$$
and thus
$$U_1(K) \geq \left(1 + 2D(\alpha_0)\gamma_3\Delta(K)^2\right) U_1(C),$$
which yields the assertion with $\gamma := 2D(\alpha_0)\gamma_3$. □

For our application, we need a result without a fixed upper bound on α_C. Making appropriate changes to the argumentation above, we obtain the following result.

Corollary 3.3.5. *Let $K \in \overline{\mathcal{K}}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $\sigma_d(K) = \sigma_d(C) > 0$. Let $\alpha_0 \in (0, \pi/2)$ be such that $\alpha_0 \leq \alpha_C$. Then*
$$U_1(K) \geq (1 + \tilde{\gamma}\Delta(K)^2)U_1(C),$$
where the constant
$$\tilde{\gamma} = 2 \cdot \min\left\{ \frac{\binom{d+1}{2}\sin^{d+1}(\alpha_0)\tan^{-2d}(\alpha_C)}{d + d\binom{d+1}{2}\left(\frac{\pi}{2}\right)^2\tan^{-d}(\alpha_C)}, \left(\frac{2}{\pi}\right)^2 D\left(\frac{\pi}{2} - \alpha_C\right)\right\}$$
depends on α_0, d and α_C.

Proof. We will not repeat all the arguments from the proof of Theorem 3.3.4, but only note where changes occur. Since we no longer can bound $\tan^d(\alpha_C)$ by $\cot^d(\alpha_0)$, the first change has to be made when deducing (3.3.4). The other arguments used to obtain (3.3.4) need not be changed and we obtain
$$\frac{\lambda^d(R_e(K))}{\kappa_d} \geq (1 + \tilde{\gamma}_1\Delta(K)^2)h\left(\frac{\sigma_d(K)}{d\kappa_d}\right), \tag{3.3.7}$$
where $\tilde{\gamma}_1 := \binom{d+1}{2}\tan^{-d}(\alpha_C)$. Now we continue analogously to the proof of Theorem 3.3.4 and obtain a modified version of (3.3.5)
$$h\left(\frac{\sigma_d(K^*)}{d\kappa_d}\right) \leq (1 - \tilde{\gamma}_2\Delta(K)^2)h\left(\frac{\sigma_d(C^*)}{d\kappa_d}\right), \tag{3.3.8}$$

where

$$\tilde{\gamma}_2 := \frac{\tilde{\gamma}_1}{1+\tilde{\gamma}_1(\pi/2)^2} \le \frac{\tilde{\gamma}_1}{1+\tilde{\gamma}_1\Delta(K)^2}.$$

Next, we define

$$\begin{aligned}\tilde{\gamma}_3 &:= \min\left\{\tilde{\gamma}_2 \frac{\tan^{-d}(\alpha_C)}{D\left(\frac{\pi}{2}-\alpha_C\right)}\frac{\sin^{d+1}(\alpha_0)}{d}, \left(\frac{2}{\pi}\right)^2\right\}\\ &\le \tilde{\gamma}_2 \frac{\tan^{d}(\alpha_{C^*})}{D(\alpha_{C^*})}\frac{\cos^{d+1}(\alpha_{C^*})}{d},\end{aligned}$$

since $\alpha_0 \le \alpha_C$, $\alpha_{C^*} = \pi/2 - \alpha_C$ and $1/\tan(x) = \tan(\pi/2 - x)$. Further following the previous proof, we obtain

$$U_1(K) \ge \frac{1}{2}\tilde{\gamma}_3\Delta(K)^2 + (1-\tilde{\gamma}_3\Delta(K)^2)U_1(C). \tag{3.3.9}$$

Using

$$\frac{1}{2} - U_1(C) = \frac{\sigma_d(C^*)}{\sigma_d(\mathbb{S}^d)} = D(\alpha_{C^*}) \ge 2D(\alpha_{C^*})U_1(C),$$

we get

$$\frac{1}{2} \ge \left(1 + 2D\left(\frac{\pi}{2}-\alpha_C\right)\right)U_1(C).$$

In combination with (3.3.9), we obtain

$$U_1(K) \ge \left(1 + 2D\left(\frac{\pi}{2}-\alpha_C\right)\tilde{\gamma}_3\Delta(K)^2\right)U_1(C).$$

Now the assertion follows with $\tilde{\gamma} = 2D(\pi/2-\alpha_C)\tilde{\gamma}_3$. □

Next, we compare the deviation functional Δ to another natural deviation functional. For this we assume that the assumptions of the theorem are satisfied. If $e \in M(K)$, then $\alpha = \alpha_{K,e}$ is well-defined and

$$\underline{\alpha}_e(K) := \min\{\alpha(u) : u \in S_e\}, \qquad \overline{\alpha}_e(K) := \max\{\alpha(u) : u \in S_e\}.$$

Then

$$\Delta_0(K) := \min\{\overline{\alpha}_e(K) - \underline{\alpha}_e(K) : e \in M(K)\}$$

measures the deviation of the shape of K from the shape of a spherical cap (centred at a point $e \in M(K)$). For any $e \in M(K)$ we have

$$\alpha(u) \in [\underline{\alpha}_e(K), \overline{\alpha}_e(K)], \qquad u \in S_e,$$

and therefore

$$\begin{aligned}\left|D(\alpha(u)) - \frac{\sigma_n(K)}{n\kappa_n}\right| &\le D(\overline{\alpha}_e(K)) - D(\underline{\alpha}_e(K))\\ &\le D'(\overline{\alpha}_e(K))(\overline{\alpha}_e(K) - \underline{\alpha}_e(K))\end{aligned}$$

$$= \sin^{d-1}(\overline{\alpha}_e(K))(\overline{\alpha}_e(K) - \underline{\alpha}_e(K))$$
$$\leq \overline{\alpha}_e(K) - \underline{\alpha}_e(K).$$

Thus, we have

$$\Delta(K) \leq \Delta_0(K). \tag{3.3.10}$$

Later, this relation will be used in the proof of Lemma 3.4.2.

Lemma 3.3.6. *Let $\alpha > 0$ and $0 < a < \omega_{d+1}/2$. Then there is an integer $\nu \in \mathbb{N}$, depending only on a, d and α, such that for every spherical polytope P with $\sigma_d(P) \geq a$ there is a spherical polytope $Q = Q(P)$ satisfying $\operatorname{ext}(Q) \subset \operatorname{ext}(P)$, $f_0(Q) \leq \nu$ and*

$$\Phi(Q) \geq (1 - \alpha)\Phi(P).$$

Furthermore, the mapping $P \mapsto Q(P)$ can be chosen to be measurable.

Proof. Recalling from the proof of Theorem 3.3.2, we first note that the functional U_1 is continuous on the compact set $\overline{\mathcal{K}}_s^d$. Together with the spherical Hausdorff distance $\overline{\mathcal{K}}_s^d$ forms a compact metric space. Thus, by [15, Corollary (7.18)], Φ is also uniformly continuous on $\overline{\mathcal{K}}_s^d$. Let α, P and a be as in the statement of the lemma and define $\varepsilon := \alpha \cdot \Phi(B_a)$, where B_a is a spherical cap with $\sigma_d(B_a) = a$. From the uniform continuity of $\Phi = 1/2 \cdot U_1$ we get $\delta = \delta(\varepsilon)$, independent of P, such that

$$|\Phi(P) - \Phi(K)| \leq \varepsilon = \alpha \cdot \Phi(B_a)$$

for all $K \in \overline{\mathcal{K}}_s^d$ with $\delta_s(K, P) \leq \delta(\varepsilon)$. From Lemma 3.3.1 we now get a spherical polytope $Q = Q(P)$ and a number $\nu = \nu(\varepsilon, d)$ satisfying

$$\operatorname{ext}(Q) \subset \operatorname{ext}(P) \text{ and } \delta_s(Q, P) \leq \delta(\varepsilon).$$

Since $Q \subset P$ and Φ is monotone, we get $\Phi(Q) \leq \Phi(P)$. Using Theorem 3.1.1 and $\sigma_d(P) \geq a$, we obtain

$$|\Phi(P) - \Phi(Q)| = \Phi(P) - \Phi(Q) \leq \varepsilon = \alpha \cdot \Phi(B_a) \leq \alpha \cdot \Phi(P).$$

Consequently, the first assertion follows from

$$\Phi(P) - \Phi(Q) \leq \alpha \cdot \Phi(P) \;\Leftrightarrow\; \Phi(Q) \geq (1 - \alpha)\Phi(P).$$

After identifying each spherical polytope with a Euclidean polytope which is the convex hull of the Euclidean origin and the vertices of the spherical polytope, the second assertion follows as in [17, Lemma 4.2]. $\square$

3.4. Probabilistic inequalities and proof of Theorem 3.2.1

After the geometric preparations of Section 3.3, we can proceed with estimating the conditional probability in Theorem 3.2.1. Since bounding the denominator is much simpler, we will provide an upper bound for the numerator as a separate result.

Lemma 3.4.1. *Let* $0 < a < \omega_{d+1}/2$ *and* $\varepsilon > 0$. *Furthermore, let*

$$\mathcal{K}_{a,\varepsilon} := \{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \in [a, \omega_{d+1}/2],\ \vartheta(K) \geq \varepsilon\} \tag{3.4.1}$$

and $\gamma_S \omega_{d+1} > 1$. *Then*

$$\mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon}) \leq c_3 \cdot (\gamma_S \omega_{d+1})^{d \cdot \nu} \cdot \exp\left(-\gamma_S \omega_{d+1} \left(1 + \frac{f_a(\varepsilon)}{3}\right) \Phi(B_a)\right), \tag{3.4.2}$$

where the constants c_3 *and* ν *depend only on* a, d *and* ε.

Proof. Let $N \in \mathbb{N}$. For $H_1, \ldots, H_N \in G(d+1, d) \cap \mathbb{S}^d$ such that $\overline{0} \notin H_i,\ i = 1, \ldots, N$, we define $H_{(N)} := (H_1, \ldots, H_N)$ and let $P(H_{(N)})$ denote the spherical Crofton cell of the tessellation induced by $H_1, \ldots, H_N$. In what follows, we consider $H_1, \ldots, H_N \in G(d+1, d) \cap \mathbb{S}^d$ such that $P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d$. This requires $N \geq d+1$. If $N \geq d+1$ and $H_1, \ldots, H_N$ are i.i.d. with a distribution which has a density with respect to the invariant measure, we have almost surely have $P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d$.

Define $\alpha := f_a(\varepsilon)/(2 + f_a(\varepsilon))$, then $(1-\alpha)(1+f_a(\varepsilon)) = 1 + \alpha$. Since $f_a(\varepsilon) \leq 1$, we have $\alpha \geq f_a(\varepsilon)/3$. Due to Lemma 3.3.6 and Theorem 3.3.2, there are at most $\nu = \nu(d, a, \varepsilon)$ vertices of $P(H_{(N)})$ such that the spherical convex hull $Q(H_{(N)})$ of these vertices satisfies

$$\begin{aligned} 1 \geq \Phi(Q(H_{(N)})) &\geq (1-\alpha)\Phi(P(H_{(N)})) \\ &\geq (1-\alpha)(1+f_a(\varepsilon))\Phi(B_a) \\ &= (1+\alpha)\Phi(B_a), \end{aligned}$$

where we used $\Phi(\cdot) = 2U_1(\cdot) \leq 1$. By Lemma 3.3.6, we can assume that the mapping

$$(H_1, \ldots, H_N) \mapsto Q(H_{(N)})$$

is measurable. Since μ is isotropic, every vertex of $Q(H_{(N)})$ lies μ^N-almost surely in exactly d of these great subspheres. The remaining subspheres do not hit $Q(H_{(N)})$. Hence, the number of great subspheres hitting $Q(H_{(N)})$ is $j \in \{d+1, \ldots, d \cdot \nu\}$. Without loss of generality we assume $H_1 \cap Q(H_{(N)}) \neq \emptyset, \ldots, H_j \cap Q(H_{(N)}) \neq \emptyset$. Then there are subsets $J_1, \ldots, J_{f_0(Q(H_{(N)}))}$ of $\{1, \ldots, j\}$, each of cardinality d, such that

$$\bigcap_{l \in J_i} H_l, \quad i = 1, \ldots, f_0(Q(H_{(N)})) \leq \nu,$$

give the vertices of $Q(H_{(N)})$. In the following, let $\sum_{(J_1, \ldots, J_\nu)}$ denote the sum over all ν-tuples of subsets of $\{1, \ldots, j\}$ with d elements. Let $K \subset C_0 = \{x \in \mathbb{S}^d : d_s(\overline{0}, x) \leq \pi/2\}$. Then

$$\int_{\mathcal{H}_{C_0}} \mathbf{1}\{H \cap K = \emptyset\}\ \mu(dH) = \Phi(C_0) - \Phi(K).$$

Using $\Phi(C_0) = 1$ and assuming $N \geq d+1$, we obtain

$$\mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d | \tilde{X}(\mathcal{H}_{C_0}) = N) = \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d | \tilde{X}(\mathcal{H}_{C_0}) = N) \cdot \Phi(C_0)^N$$

$$
\begin{aligned}
&= \int_{\mathcal{H}_{C_0}^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d\} \; \mu^N(d(H_1,\dots,H_N)) \\
&\le \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \int_{\mathcal{H}_{C_0}^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d\} \mathbf{1}\{H_i \cap Q(H_{(N)}) \neq \emptyset,\; i=1,\dots,j\} \\
&\qquad\qquad \times \mathbf{1}\{H_i \cap Q(H_{(N)}) = \emptyset,\; i=j+1,\dots,N\} \; \mu^N(d(H_1,\dots,H_N)) \\
&\le \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \sum_{(J_1,\dots,J_\nu)} \int_{\mathcal{H}_{C_0}^j} \int_{\mathcal{H}_{C_0}^{N-j}} \mathbf{1}\{\Phi(\mathrm{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{i\in J_r} H_i) \ge (1+\alpha)\Phi(B_a)\} \\
&\qquad\qquad\qquad \times \mathbf{1}\{H_l \cap \mathrm{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{i\in J_r} H_i = \emptyset,\; l=j+1,\dots,N\} \\
&\qquad\qquad\qquad \times \mu^{N-j}(d(H_{j+1},\dots,H_N)) \; \mu^j(d(H_1,\dots,H_j)) \\
&= \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \sum_{(J_1,\dots,J_\nu)} \int_{\mathcal{H}_{C_0}^j} \mathbf{1}\{\Phi(\mathrm{conv}_s \bigcup_{r=1}^{\nu} \bigcap_{i\in J_r} H_i) \ge (1+\alpha)\Phi(B_a)\} \\
&\qquad\qquad\qquad \times [\Phi(C_0) - \Phi(\mathrm{conv}_s \bigcup_{r=1}^{\nu} \bigcup_{i\in J_r} H_i)]^{N-j} \; \mu^j(d(H_1,\dots,j)) \\
&\le \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \binom{j}{d}^{\nu} [\Phi(C_0) - (1+\alpha)\Phi(B_a)]^{N-j} \cdot \Phi(C_0)^j \\
&= \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \binom{j}{d}^{\nu} [1 - (1+\alpha)\Phi(B_a)]^{N-j} . \qquad (3.4.3)
\end{aligned}
$$

Summation over N gives

$$
\begin{aligned}
\mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon}) &= \sum_{N=0}^{d} \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} | \tilde{X}(\mathcal{H}_{C_0}) = N) \mathbb{P}(\tilde{X}(\mathcal{H}_{C_0}) = N) \\
&\quad + \sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d | \tilde{X}(\mathcal{H}_{C_0}) = N) \mathbb{P}(\tilde{X}(\mathcal{H}_{C_0}) = N) \\
&\le \sum_{N=0}^{d} \mathbb{P}(\tilde{X}(\mathcal{H}_{C_0}) = N) \\
&\quad + \sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d | \tilde{X}(\mathcal{H}_{C_0}) = N) \mathbb{P}(\tilde{X}(\mathcal{H}_{C_0}) = N)
\end{aligned}
$$

For the second sum, we use (3.4.3), $\alpha \ge f_a(\varepsilon)/3$ and $\gamma_S \omega_{d+1} > 1$ to obtain

$$
\sum_{N=d+1}^{\infty} \mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon} \cap \mathcal{K}_s^d | \tilde{X}(\mathcal{H}_{C_0}) = N) \mathbb{P}(\tilde{X}(\mathcal{H}_{C_0}) = N)
$$

$$
\begin{aligned}
&\le \sum_{N=d+1}^{\infty}\sum_{j=d+1}^{d\cdot\nu}\binom{N}{j}\binom{j}{d}^{\nu}[1-(1+\alpha)\Phi(B_a)]^{N-j}\,\frac{(\gamma_S\omega_{d+1})^N}{N!}\exp(-\gamma_S\omega_{d+1})\\
&=\sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\exp(-\gamma_S\omega_{d+1})\frac{(\gamma_S\omega_{d+1})^j}{j!}\sum_{N=j}^{\infty}\frac{[1-(1+\alpha)\Phi(B_a)]^{N-j}}{(N-j)!}(\gamma_S\omega_{d+1})^{N-j}\\
&=\sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\exp(-\gamma_S\omega_{d+1})\frac{(\gamma_S\omega_{d+1})^j}{j!}\exp\left[\gamma_S\omega_{d+1}(1-(1+\alpha)\Phi(B_a))\right]\\
&=\exp(-\gamma_S\omega_{d+1}(1+\alpha)\Phi(B_a))\sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\frac{(\gamma_S\omega_{d+1})^j}{j!}\\
&\le \sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\frac{(\gamma_S\omega_{d+1})^j}{j!}\;\cdot\;\exp\left[-\gamma_S\omega_{d+1}\left(1+\frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right]\\
&\le \sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\frac{(\gamma_S\omega_{d+1})^{d\cdot\nu}}{j!}\;\cdot\;\exp\left[-\gamma_S\omega_{d+1}\left(1+\frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right].
\end{aligned}
$$

For the first sum, we obtain

$$
\begin{aligned}
\sum_{N=0}^{d}\mathbb{P}(\tilde X(\mathcal{H}_{C_0})=N) &= \sum_{N=0}^{d}\frac{(\gamma_S\omega_{d+1})^N}{N!}\exp(-\gamma_S\omega_{d+1})\\
&\le \sum_{N=0}^{d}\frac{(\gamma_S\omega_{d+1})^{d\cdot\nu}}{N!}\exp\left[-\gamma_S\omega_{d+1}\left(1+\frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right],
\end{aligned}
$$

since $1\ge(1+\alpha)\Phi(B_a)\ge(1+f_a(\varepsilon)/3)\Phi(B_a)$. Combining both estimates, we obtain

$$
\mathbb{P}(Z_0\in\mathcal{K}_{a,\varepsilon})\le c_3\cdot(\gamma_S\omega_{d+1})^{d\cdot\nu}\cdot\exp\left(-\gamma_S\omega_{d+1}\left(1+\frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right),
$$

where

$$
c_3=c_3(a,\varepsilon,d):=\sum_{j=d+1}^{d\cdot\nu}\binom{j}{d}^{\nu}\frac{1}{j!}+\sum_{N=0}^{d}\frac{1}{N!}.
$$

□

By choosing $\vartheta=\Delta$ and using Corollary 3.3.5 instead of Theorem 3.3.2, we are able to obtain the follwoing more explicit result.

Lemma 3.4.2. *Let* $0<a<\omega_{d+1}/2,\ 0<\varepsilon\le 1$ *and* $\gamma_S\omega_{d+1}>1$. *Furthermore, let*

$$
\tilde{\mathcal{K}}_{a,\varepsilon}=\{K\in\overline{\mathcal{K}}_s^d:\sigma_d(K)\in[a,\omega_{d+1}/2],\ \Delta(K)\ge\varepsilon\}. \tag{3.4.4}
$$

Then

$$
\mathbb{P}(Z_0\in\tilde{\mathcal{K}}_{a,\varepsilon})\le c_3\cdot(\gamma_S\omega_{d+1})^{d\cdot\nu}\cdot\exp\left(-\gamma_S\omega_{d+1}\left(1+\frac{\gamma'\varepsilon^{2(d+1)}}{3}\right)\Phi(B_a)\right), \tag{3.4.5}
$$

where the constant c_3 depends only on a, d and ε and the constant γ' depends only on a and d.

Proof. Let $Z_0 \in \tilde{\mathcal{K}}_{a,\varepsilon} \cap \mathcal{K}_s^d$. Let $e \in M(Z_0)$ be arbitrary. Suppose that all points of X are either in $B_s^o(e, \varepsilon)$ or $B_s^o(-e, \varepsilon)$, where $B_s^o(s, r)$ denotes the open spherical cap with centre $s \in \mathbb{S}^d$ and radius $r > 0$. Then we immediately get $\Delta_0(Z_0) < \varepsilon$. By (3.3.10), it follows that $\Delta(Z_0) < \varepsilon$, a contradiction to $Z_0 \in \tilde{\mathcal{K}}_{a,\varepsilon}$.

Therefore, there is a point $x \in X$ such that $x \in B_s(e, \pi/2)$ and $d_s(e, x) \geq \varepsilon$ or $x \in B_s(-e, \pi/2)$ and $d_s(-e, x) \geq \varepsilon$. Without loss of generality, we assume the latter to be true. Then

$$\sigma_d\left(B_s\left(x, \frac{\pi}{2}\right) \cap B_s\left(-e, \frac{\pi}{2}\right)\right) = \frac{d_s(-e, x)}{\pi}\frac{\omega_{d+1}}{2} \geq \varepsilon \cdot \frac{\omega_{d+1}}{2\pi}$$

and thus

$$\sigma_d(Z_0) \leq \frac{\omega_{d+1}}{2} - \frac{\omega_{d+1}}{2}\frac{\varepsilon}{\pi}.$$

Now let C be a spherical cap with $\sigma_d(C) \leq \omega_{d+1}/2 - (\varepsilon/\pi) \cdot (\omega_{d+1}/2)$ and denote its radius by α_C. Using [49, Lemma 6.5.1] and $\alpha_C < \pi/2$, we obtain

$$\begin{aligned}\sigma_d(C) = \omega_d \int_0^{\alpha_C} \sin^{d-1}(t)\ dt &\leq \frac{\omega_{d+1}}{2} - \frac{\varepsilon}{\pi}\frac{\omega_{d+1}}{2} \\ &= \omega_d \int_0^{\pi/2} \sin^{d-1}(t)\ dt - \frac{\omega_{d+1}\varepsilon}{2\pi}\end{aligned}$$

and thus

$$\frac{\varepsilon\omega_{d+1}}{2\pi\omega_d} \leq \int_{\alpha_C}^{\pi/2} \sin^{d-1}(t)\ dt \leq \frac{\pi}{2} - \alpha_C,$$

which gives us

$$\alpha_C \leq \frac{\pi}{2} - \frac{\varepsilon\omega_{d+1}}{2\pi\omega_d}.$$

Analogously to the proof of Lemma 3.4.1, we consider $N \in \mathbb{N}$ and $H_1, \ldots, H_N \in G(d+1, d) \cap \mathbb{S}^d$ such that the Crofton cell $P(H_{(N)})$ of the induced tessellation satisfies $P(H_{(N)}) \in \tilde{\mathcal{K}}_{a,\varepsilon} \cap \mathcal{K}_s^d$. Let C be a spherical cap satisfying $\sigma_d(C) = \sigma_d(P(H_{(N)}))$ and denote its radius by α_C. Using Corollary 3.3.5 instead of Theorem 3.3.2 and the monotonicity of Φ, we obtain

$$\Phi(P(H_{(N)})) \geq (1 + \tilde{\gamma}\varepsilon^2)\Phi(C) \geq (1 + \tilde{\gamma}\varepsilon^2)\Phi(B_a), \tag{3.4.6}$$

where

$$\tilde{\gamma} = 2\min\left\{\frac{\binom{d+1}{2}\sin^{d+1}(\alpha_0)\tan^{-2d}(\alpha_C)}{d + d\binom{d+1}{2}\left(\frac{\pi}{2}\right)^2\tan^{-d}(\alpha_C)}, \left(\frac{2}{\pi}\right)^2 D\left(\frac{\pi}{2} - \alpha_C\right)\right\}.$$

The argumenation above, applied to $P(H_{(N)})$ and the normals of $H_1, \ldots, H_N$ instead of Z_0 and the points of X, gives

$$\alpha_C \leq \frac{\pi}{2} - \varepsilon\frac{\omega_{d+1}}{2\pi\omega_d}.$$

Recalling

$$D(x) = \int_0^x \sin^{d-1}(t)\ dt \geq \int_0^x \left(\frac{2}{\pi}\right)^{d-1} t^{d-1}\ dt = \frac{2^{d-1}}{d\pi^{d-1}}x^d,$$

and with the fact that the tangent is increasing on $[0, \pi/2)$ and $\tan(x) \geq x$ for $x \in [0, \pi/2)$, we obtain

$$\begin{aligned}
\tilde{\gamma} &\geq 2 \min \left\{ \frac{\binom{d+1}{2} \sin^{d+1}(\alpha_0) \tan^{-2d}\left(\frac{\pi}{2} - \varepsilon \frac{\omega_{d+1}}{2\pi\omega_d}\right)}{d + d\binom{d+1}{2} \left(\frac{\pi}{2}\right)^2 \tan^{-d}(\alpha_0)}, \left(\frac{2}{\pi}\right)^2 D\left(\varepsilon \frac{\omega_{d+1}}{2\pi\omega_d}\right) \right\} \\
&\geq 2 \min \left\{ \frac{\binom{d+1}{2} \sin^{d+1}(\alpha_0) \tan^{2d}\left(\varepsilon \frac{\omega_{d+1}}{2\pi\omega_d}\right)}{d + d\binom{d+1}{2} \left(\frac{\pi}{2}\right)^2 \tan^{-d}(\alpha_0)}, \left(\frac{2}{\pi}\right)^{d+1} \frac{\omega_{d+1}^d}{d(2\pi\omega_d)^d} \varepsilon^d \right\} \\
&\geq 2 \min \left\{ \frac{\binom{d+1}{2} \sin^{d+1}(\alpha_0) \left(\frac{\omega_{d+1}}{2\pi\omega_d}\right)^{2d}}{d + d\binom{d+1}{2} \left(\frac{\pi}{2}\right)^2 \tan^{-d}(\alpha_0)} \varepsilon^{2d}, \left(\frac{2}{\pi}\right)^{d+1} \frac{\omega_{d+1}^d}{d(2\pi\omega_d)^d} \varepsilon^{2d} \right\} \\
&=: \gamma' \cdot \varepsilon^{2d},
\end{aligned}$$

where we made use of $\varepsilon \leq 1$ in the second to last line. Note that $\gamma' > 0$ depends only on a and d. Combining this with (3.4.6), we get

$$\Phi(P(H_{(N)})) \geq (1 + \gamma' \varepsilon^{2(d+1)}) \Phi(B_a).$$

Proceeding as in the proof of Lemma 3.4.1 gives the result. □

Now we are able to prove Theorem 3.2.1.

Proof. First we note that

$$\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon | \sigma_d(Z_0) \geq a\right) = \frac{\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon,\ \sigma_d(Z_0) \geq a\right)}{\mathbb{P}\left(\sigma_d(Z_0) \geq a\right)} = \frac{\mathbb{P}(Z_0 \in \mathcal{K}_{a,\varepsilon})}{\mathbb{P}\left(\sigma_d(Z_0) \geq a\right)}. \tag{3.4.7}$$

Let B_a be a spherical cap with $\sigma_d(B_a) = a$. Then

$$\begin{aligned}
\mathbb{P}(\sigma_d(Z_0) \geq a) &\geq \mathbb{P}(\tilde{X}(\mathcal{H}_{B_a}) = 0) \\
&= \exp(-\gamma_S \omega_{d+1} \Phi(B_a)).
\end{aligned} \tag{3.4.8}$$

Using (3.4.7), (3.4.8) and Lemma 3.4.1, we obtain

$$\begin{aligned}
\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon | \sigma_d(Z_0) \geq a\right) &\leq \frac{c_3 \cdot (\gamma_S \omega_{d+1})^{d \cdot \nu} \cdot \exp\left(-\gamma_S \omega_{d+1}\left(1 + \frac{f_a(\varepsilon)}{3}\right) \Phi(B_a)\right)}{\exp(-\gamma_S \omega_{d+1} \Phi(B_a))} \\
&= c_3 \cdot (\gamma_S \omega_{d+1})^{d \cdot \nu} \cdot \exp\left(-\gamma_S \omega_{d+1} \frac{f_a(\varepsilon)}{3} \Phi(B_a)\right) \\
&\leq c_1 \cdot \exp\left(-c_2 \cdot \gamma_S \omega_{d+1}\right),
\end{aligned}$$

where the constants c_1, c_3, ν and c_2 only depend on a, ε and d. □

Using Lemma 3.4.2 instead of 3.4.1 (and of course $\vartheta = \Delta$), we obtain a similar result but with a more explicit constant in the exponent.

Theorem 3.4.3. *Let $0 < a < \omega_{d+1}/2$, $0 < \varepsilon < 1$ and $\gamma_S \omega_{d+1} > 1$. Then there are constants $\tilde{c}_1, \tilde{c}_2 > 0$, such that*

$$\mathbb{P}\left(\Delta(Z_0) > \varepsilon | \sigma_d(Z_0) \geq a\right) \leq \tilde{c}_1 \cdot \exp\left(-\tilde{c}_2 \cdot \varepsilon^{2(d+1)} \cdot \gamma_S \cdot \omega_{d+1}\right) \tag{3.4.9}$$

and the constant $\tilde{c}_1$ depends only on a, ε and d and the constant $\tilde{c}_2$ depends only on a and d.

3.5. An asymptotic result for the size of the Crofton cell

Similar to [20, Theorem 2], we determine the asymptotic distribution function of $\sigma_d(Z_0)$, given the intensity γ_S tends to infinity. We use the techniques developed in the proof of Lemma 3.4.1 to obtain the following theorem.

Theorem 3.5.1. *Let $0 < a < \omega_{d+1}/2$ and let B_a be a spherical cap with $\sigma_d(B_a) = a$. Then*

$$\lim_{\gamma_S \to \infty} \gamma_S^{-1} \cdot \ln(\mathbb{P}(\sigma_d(Z_0) \geq a)) = -2 \cdot \omega_{d+1} \cdot U_1(B_a). \tag{3.5.1}$$

Proof. Let $\kappa \in (0, 1)$ and

$$\mathcal{K}_{a,0} := \{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \geq a,\ \vartheta(K) \geq 0\} = \{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \geq a\}.$$

Let $N \in \mathbb{N}$ and let $H_1, \ldots, H_N \in G(d+1, d) \cap \mathbb{S}^d$ such that $P(H_{(N)}) \in \mathcal{K}_{a,0}$. By (3.1.4) we have $U_1(P(H_{(N)})) \geq U_1(B_a)$ and thus $\Phi(P(H_{(N)})) \geq \Phi(B_a)$. By Lemma 3.3.6 we obtain a number $\nu = \nu(d, a, \kappa)$ and a spherical polytope $Q(P(H_{(N)})) =: Q$ with at most ν vertices and $\operatorname{ext}(Q) \subset \operatorname{ext}(P(H_{(N)}))$ such that

$$\Phi(Q) \geq \left(1 - \frac{\kappa}{2}\right) \Phi(P(H_{(N)})) \geq \left(1 - \frac{\kappa}{2}\right) \Phi(B_a). \tag{3.5.2}$$

Proceeding as in the proof of Lemma 3.4.1, we obtain

$$\mathbb{P}(Z_0 \in \mathcal{K}_{a,0} | \tilde{X}(\mathcal{H}_{C_0}) = N) \leq \sum_{j=d+1}^{d \cdot \nu} \binom{N}{j} \binom{j}{d}^{\nu} \left[1 - \left(1 - \frac{\kappa}{2}\right) \Phi(B_a)\right]^{N-j}$$

for $N \geq d+1$. After summation over N, where we deal with the cases $N \in \{0, \ldots, d\}$ as in the proof of Lemma 3.4.1, and assuming $\gamma_S \omega_{d+1} > 1$, we get

$$\begin{aligned} \mathbb{P}(Z_0 \in \mathcal{K}_{a,0}) &\leq c_4 \left(\gamma_S \omega_{d+1}\right)^{d \cdot \nu} \exp\left(-\left(1 - \frac{\kappa}{2}\right) \Phi(B_a) \gamma_S \omega_{d+1}\right) \\ &\leq c_5 \cdot \exp\left(-\left(1 - \kappa\right) \Phi(B_a) \gamma_S \omega_{d+1}\right), \end{aligned} \tag{3.5.3}$$

for suitable constants $c_4, c_5 > 0$, which depend only on a, d and κ. For the last inequality, we used that $x \mapsto x^{d \cdot \nu} \exp(-\kappa/2 \cdot x)$ is bounded.

Combining (3.5.3) with (3.4.8), we get

$$\exp\left(-\gamma_S \omega_{d+1} \Phi(B_a)\right) \leq \mathbb{P}(\sigma_d(Z_0) \geq a) \leq c_5 \cdot \exp\left(-(1 - \kappa) \gamma_S \omega_{d+1} \Phi(B_a)\right).$$

This yields

$$\liminf_{\gamma_S \to \infty} \gamma_S^{-1} \ln \mathbb{P}(\sigma_d(Z_0) \geq a) \geq -2 \cdot \omega_{d+1} \cdot U_1(B_a)$$

and

$$\limsup_{\gamma_S \to \infty} \gamma_S^{-1} \ln \mathbb{P}(\sigma_d(Z_0) \geq a) \leq -2 \cdot (1-\kappa) \cdot \omega_{d+1} \cdot U_1(B_a).$$

The left-hand side of the second estimate is independent of κ and therefore

$$\lim_{\gamma_S \to \infty} \gamma_S^{-1} \ln \mathbb{P}(\sigma_d(Z_0) \geq a) = -2 \cdot \omega_{d+1} \cdot U_1(B_a),$$

which completes this proof. □

3.6. Spherical binomial hyperplane tessellation

In this section, we consider a binomial process instead of the Poisson process X. Let $Y_i,\ i \in \mathbb{N}$, be a sequence of independent and identically distributed random variables on $\mathbb{S}^d$ and

$$Y_1 \sim \frac{\sigma_d(\cdot)}{\omega_{d+1}}.$$

Let $d+1 \leq N \in \mathbb{N}$, then

$$Y := \sum_{i=1}^{N} \delta_{Y_i}$$

is a binomial point process of size N on $\mathbb{S}^d$. Likewise to in the previous chapter, we apply the mapping h and consider the tessellation induced by

$$\tilde{Y} := h(Y),$$

with intensity measure

$$\mathbb{E}\tilde{Y}(\cdot) = N \cdot \mu(\cdot) = \frac{N}{\omega_{d+1}} \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap \mathbb{S}^d \in \cdot\}\ \sigma_d(dx).$$

For $N < d+1$ the spherical Crofton cell $Z_0(Y)$ induced by $\tilde{Y}$ is not contained in some open hemisphere.

Lemma 3.6.1. *Let $0 < a < \omega_{d+1}/2$ and $\varepsilon > 0$. Then*

$$\mathbb{P}(Z_0(Y) \in \mathcal{K}_{a,\varepsilon}) \leq c_6 \cdot N^{d \cdot \nu} \cdot \exp\left(\ln\left[1 - \left(1 + \frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right] \cdot N\right), \tag{3.6.1}$$

where

$$\mathcal{K}_{a,\varepsilon} = \left\{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \geq a,\ \vartheta(K) \geq \varepsilon\right\}$$

and c_6, ν depend only on a, ε and d.

Proof. Let $P(H_{(N)})$ denote the spherical Crofton cell induced by $H_1, \dots, H_N \in G(d+1, d) \cap \mathbb{S}^d$. Then

$$\mathbb{P}(Z_0(Y) \in \mathcal{K}_{a,\varepsilon}) = \int_{\mathcal{H}_{C_0}^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_{a,\varepsilon}\} \, \mu^N(d(H_1, \dots, H_N)).$$

We proceed similar to the first part of the proof of Lemma 3.4.1 and start by recalling some definitions and results. Let $\alpha := f_a(\varepsilon)/(2 + f_a(\varepsilon))$. Then $\alpha \geq f_a(\varepsilon)/3$ and there are at most $\nu = \nu(d, a, \varepsilon)$ vertices of $P(H_{(N)})$ such that the spherical convex Hull Q of these vertices satisfies

$$1 \geq \Phi(Q) \geq (1 + \alpha)\Phi(B_a).$$

We proceed similar to the first part of the proof of Lemma 3.4.1 and obtain

$$\begin{aligned}
\mathbb{P}(Z_0(Y) \in \mathcal{K}_{a,\varepsilon}) &\leq \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j} \binom{j}{d}^{\nu} [1 - (1+\alpha)\Phi(B_a)]^{N-j} \\
&\leq \sum_{j=d+1}^{d\cdot\nu} \frac{N^{d\cdot\nu}}{j!} \binom{j}{d}^{\nu} \left[1 - \left(1 + \frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right]^{N-j}. \\
&=: c_6 \cdot N^{d\cdot\nu} \left[1 - \left(1 + \frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)\right]^{N}.
\end{aligned}$$

□

Theorem 3.6.2. *Let $a, \varepsilon > 0$ with $a < \omega_{d+1}/2$. Then there are constants $c_7, c_8 > 0$ such that*

$$\mathbb{P}\left(\vartheta(Z_0(Y)) \geq \varepsilon | \sigma_d(Z_0(Y)) \geq a\right) \leq c_7 \cdot \exp\left(-c_8 \cdot N\right) \tag{3.6.2}$$

and the constants c_7, c_8 depend only on a, ε and d.

Proof. For the denominator we get

$$\mathbb{P}(\sigma_d(Z_0(Y)) \geq a) \geq \mathbb{P}(\tilde{Y}(\mathcal{H}_{B_a}) = 0) = (1 - \Phi(B_a))^N. \tag{3.6.3}$$

Using this and (3.6.1), we get

$$\begin{aligned}
\mathbb{P}\left(\vartheta(Z_0) \geq \varepsilon | \sigma_d(Z_0) \geq a\right) &\leq c_6 N^{d\cdot\nu} \left(\frac{1 - \left(1 + \frac{f_a(\varepsilon)}{3}\right)\Phi(B_a)}{1 - \Phi(B_a)}\right)^N \\
&\leq c_7 \cdot \exp\left(-c_8 \cdot N\right)
\end{aligned}$$

for suitable constants $0 < c_7, c_8$. In the last step, we used $f_a(\varepsilon) > 0$ for $\varepsilon > 0$. □

Additionally we can determine the asymptotic size of $Z_0(Y)$, using the same arguments as in the proof of Theorem 3.5.1.

Theorem 3.6.3. *Let $0 < a < \omega_{d+1}/2$ and let B_a be a spherical cap with $\sigma_d(B_a) = a$. Then*

$$\lim_{N\to\infty} N^{-1} \cdot \ln(\mathbb{P}(\sigma_d(Z_0(Y)) \geq a)) = \ln[1 - 2 \cdot U_1(B_a)]. \tag{3.6.4}$$

Proof. Let $\kappa \in (0,1)$ and

$$\mathcal{K}_{a,0} = \{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \geq a,\ \vartheta(K) \geq 0\} = \{K \in \overline{\mathcal{K}}_s^d : \sigma_d(K) \geq a\}$$

analogously to the proof of Theorem 3.5.1. Using the same arguments, we get $\nu = \nu(a,\kappa,d)$ such that

$$\begin{aligned}
\mathbb{P}(Z_0(Y) \in \mathcal{K}_{a,0}) &\leq \sum_{j=d+1}^{d\cdot\nu} \binom{N}{j}\binom{j}{d}^{\nu}\left[1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right]^{N-j} \\
&\leq \sum_{j=d+1}^{d\cdot\nu} \frac{N^{d\cdot\nu}}{j!}\binom{j}{d}^{\nu}\left[1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right]^{N-j} \\
&=: c_9 \cdot N^{d\cdot\nu} \cdot \left[1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right]^{N},
\end{aligned}$$

where $c_9 > 0$ depends only on a, d and κ. Thus, we obtain the upper bound

$$\begin{aligned}
N^{-1}\ln(\mathbb{P}(\sigma_d(Z_0(Y)) \geq a)) &\leq N^{-1} \cdot \ln\left[c_9 \cdot N^{d\cdot\nu} \cdot \left(1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right)^{N}\right] \\
&= \frac{\ln(c_9)}{N} + d\cdot\nu\cdot\frac{\ln(N)}{N} + \ln\left[1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right]
\end{aligned}$$

and therefore

$$\limsup_{N\to\infty} N^{-1}\cdot\ln(\mathbb{P}(\sigma_d(Z_0(Y)) \geq a)) \leq \ln\left[1-\left(1-\frac{\kappa}{2}\right)\Phi(B_a)\right].$$

Since this holds for any $\kappa \in (0,1)$ and the left-hand side does not depend on κ, we get

$$\limsup_{N\to\infty} N^{-1}\cdot\ln(\mathbb{P}(\sigma_d(Z_0(Y)) \geq a)) \leq \ln\left[1-\Phi(B_a)\right].$$

The lower bound is much easier to obtain. From

$$\mathbb{P}(\sigma_d(Z_0(Y)) \geq a) \geq \mathbb{P}(\tilde{Y}(\mathcal{H}_{B_a}) = 0) = (1-\Phi(B_a))^N,$$

we immediately get

$$\liminf_{N\to\infty} N^{-1}\cdot\mathbb{P}(\sigma_d(Z_0(Y)) \geq a) \geq \ln\left[1-\Phi(B_a)\right].$$

Combining these two bounds, the assertion follows from $\Phi(B_a) = 2\cdot U_1(B_a)$. □

3.7. The spherical inball radius

Now we use the spherical inball radius instead of the spherical volume to measure the size of a cell. In order to derive a similar result as Theorem 3.2.1, we note the following properties

(1) The spherical inball radius is continuous (with respect to the spherical Hausdorff distance),

(2) it is not identically 0,

(3) for $K \subset M \subset \mathbb{S}^d$ we have $r_s(K) \leq r_s(M)$,

(4) for $K \in \overline{\mathcal{K}}_s^d$ and a spherical cap C with $r_s(K) \geq r_s(C)$, we have $\sigma_d(K) \geq \sigma_d(C)$.

Due to the last property, Theorem 3.1.1 still holds if we replace the spherical volume with the spherical inball radius. In fact, these four properties also hold for the spherical inball radius centred at some fixed $e \in \mathbb{S}^d$, if we require the cap C in (4) to have the centre e.

Corollary 3.7.1. *Let $K \in \overline{\mathcal{K}}_s^d$ and let $C \subset \mathbb{S}^d$ be a spherical cap with $r_s(K) = r_s(C)$. Then*

$$U_1(K) \geq U_1(C). \tag{3.7.1}$$

Equality holds if and only if K is a spherical cap.

Proof. Let $K \in \overline{\mathcal{K}}_s^d$ and let C be a spherical cap with $r_s(K) = r_s(C)$. Due to property (4), we get $\sigma_d(K) \geq \sigma_d(C)$ and thus the inequality (3.7.1) follows immediately from Theorem 3.1.1. If K also is a spherical cap, we get $\sigma_d(K) = \sigma_d(C)$ and therefore $U_1(K) = U_1(C)$ by Theorem 3.1.1.

Let $K \in \overline{\mathcal{K}}_s^d$ and let C be a spherical cap with $r_s(K) = r_s(C)$. Let $U_1(K) = U_1(C)$. We now assume K is not a spherical cap. Since $r_s(K) = r_s(C)$, we obtain $\sigma_d(K) > \sigma_d(C)$. But then there is a spherical cap C_2 with $\sigma_d(C_2) = \sigma_d(K)$ and $C \subsetneq C_2$. The representation (3.1.3) now gives $U_1(C_2) > U_1(C)$. Applying Theorem 3.1.1 to K and C_2, we obtain

$$U_1(K) \geq U_1(C_2) > U_1(C),$$

which is a contradiction. ☐

Replacing all spherical caps in the proof above with spherical caps having the centre e, we note that the following version of Corollary 3.7.1 also holds for the spherical inball radius centred at e.

Corollary 3.7.2. *Let $e \in \mathbb{S}^d$ be fixed. Let $K \in \overline{\mathcal{K}}_s^d$ and let $C_e \subset \mathbb{S}^d$ be a spherical cap centred at e with $r_s(K, e) = r_s(C_e, e)$. Then*

$$U_1(K) \geq U_1(C_e).$$

Equality holds if and only if K is a spherical cap centred at e.

Due to this, the following results hold for both the spherical inball radius and the centred spherical inball radius. In the latter case, the limit shape will be a cap centred at e and the deviation functional has to be modified accordingly. A possible example is the difference between centred spherical circumradius and centred spherical inball radius, which plays an important role in Section 4.3. For ease of notation, we will only write down the results for the spherical inball radius and a general deviation functional.

Now we need to adapt the results from Section 3.3 to our new setting. Mostly, this is just a change in notation due to the similar properties of σ_d and r_s.

Lemma 3.7.3 (Adaption of Theorem 3.3.2)**.** *Let $a \in (0, \pi/2)$, $\varepsilon > 0$ and $K \in \overline{\mathcal{K}}_s^d$ with $r_s(K) \geq a$, $\overline{o} \in K$ and $\vartheta(K) \geq \varepsilon$. Then there is a function $\tilde{f}_a : \mathbb{R}^+ \to \mathbb{R}^+$ with $\tilde{f}_a(0) = 0$ and $\tilde{f}_a(t) > 0$, $t > 0$, such that*

$$\Phi(K) \geq (1 + \tilde{f}_a(\varepsilon))\Phi(B_s(\overline{0}, a)). \tag{3.7.2}$$

Proof. Recalling the proof of Theorem 3.3.2, we only need to apply Corollary 3.7.1 instead of Theorem 3.1.1, since only the continuity of σ_d is used. □

Lemma 3.7.4 (Adaption of Lemma 3.4.1)**.** *Let $0 < a < \pi/2$, $\varepsilon > 0$ and $\gamma_S \omega_{d+1} > 1$. Furthermore, let*

$$\mathcal{K}'_{a,\varepsilon} := \left\{ K \in \overline{\mathcal{K}}_s^d : r_s(K) \in \left[a, \frac{\pi}{2}\right],\ \vartheta(K) \geq \varepsilon \right\}. \tag{3.7.3}$$

Then

$$\mathbb{P}(Z_0 \in \mathcal{K}'_{a,\varepsilon}) \leq c_{10} \cdot (\gamma_S \omega_{d+1})^{d \cdot \nu} \cdot \exp\left(-\gamma_S \omega_{d+1} (1 + \frac{\tilde{f}_a(\varepsilon)}{3}) \Phi(B_s(\overline{0}, a)) \right), \tag{3.7.4}$$

where the constants c_{10} and ν depend only on a, d and ε.

Proof. Substituting the adapted lemmata and $B_s(\overline{0}, a)$ instead of B_a, the argumentation is exactly the same as in the proof of Lemma 3.4.1. □

Let Y be a binomial process of size $N \geq d+1$ and, as in the previous section, denote the induced Crofton cell by $Z_0(Y)$. In this case, the argumentation differs only very slightly, see the proof of Lemma 3.6.1, and thus we immediately obtain the following result.

Lemma 3.7.5. *Let $0 < a < \pi/2$ and $\varepsilon > 0$. Then*

$$\mathbb{P}(Z_0(Y) \in \mathcal{K}'_{a,\varepsilon}) \leq c_{11} \cdot N^{d \cdot \nu} \cdot \exp\left(\ln\left[1 - \left(1 + \frac{\tilde{f}_a(\varepsilon)}{3}\right) \Phi(B_s(\overline{0}, a)) \right] \cdot N \right), \tag{3.7.5}$$

where the constants c_{11} and ν depend only on a, d and ε.

Having done these crucial estimates, we are able to prove the following theorems for the asymptotic shape of Crofton cells having large spherical inradii.

Theorem 3.7.6. *Let $a, \varepsilon > 0$ with $a < \pi/2$ and let $\gamma_S \omega_{d+1} > 1$. Then there are constants $c_{12}, c_{13} > 0$, depending only on a, d and ε, such that*

$$\mathbb{P}(\vartheta(Z_0) \geq \varepsilon | r_s(Z_0) \geq a) \leq c_{13} \cdot \exp(-c_{13} \cdot \gamma_S \omega_{d+1}). \tag{3.7.6}$$

Proof. Combining

$$\begin{aligned} \mathbb{P}(r_s(Z_0) \geq a) &= \mathbb{P}(\tilde{X}(\mathcal{H}_{B_s(\overline{0},a)}) = 0) \\ &= \exp[-\gamma_S \omega_{d+1} \Phi(B_s(\overline{0}, a))] \end{aligned}$$

with Lemma 3.7.4 the result follows in similar way to the proof of Theorem 3.2.1. □

Theorem 3.7.7. *Let $a, \varepsilon > 0$ with $a < \pi/2$. Then there are constants $c_{14}, c_{15} > 0$, depending only on a, d and ε, such that*

$$\mathbb{P}(\vartheta(Z_0(Y)) \geq \varepsilon | r_s(Z_0(Y)) \geq a) \leq c_{14} \cdot \exp(-c_{15} \cdot N). \tag{3.7.7}$$

Proof. As before in the proof of Theorem 3.7.6, we only need to combine

$$\begin{aligned} \mathbb{P}(r_s(Z_0(Y)) \geq a) &= \mathbb{P}(\tilde{Y}(\mathcal{H}_{B_s(\overline{0},a)}) = 0) \\ &= \left(1 - \Phi(B_s(\overline{0}, a))\right)^N, \end{aligned}$$

with Lemma 3.7.5. □

In fact, the results in this section hold for any size-functional on the sphere, having the properties (1) to (4) mentioned at the beginning of this chapter.

4. Kendall's Problem in spherical space: typical cells

4.1. Typical objects in spherical space

After having studied the Crofton cell in the previous chapter, a natural next step is to look at typical cells, expecting to find similar results. In Euclidean space, there is a very intuitive representation for the distribution of the typical grain of a stationary particle process.

Let X' be a stationary particle process in $\mathbb{R}^{d+1}$ with intensity $\gamma_{X'}$. Denote the distribution of the associated typical particle by $\mathbb{Q}'$. Then the following holds for any measurable $B \subset \mathbb{R}^{d+1}$ satisfying $V_{d+1}(B) < \infty$ (see e.g. [49, p. 106] or [49, Theorem 4.1.3])

$$\mathbb{Q}'(\cdot) = \frac{1}{\gamma_{X'} \cdot V_{d+1}(B)} \mathbb{E}\left[\sum_{K \in X'} \mathbf{1}\{K - c(K) \in \cdot\}\mathbf{1}\{c(K) \in B\}\right],$$

where $c : \mathcal{K}^{d+1} \to \mathbb{R}^{d+1}$ is a suitable centre function. The distribution $\mathbb{Q}'$ is concentrated on sets having the Euclidean origin as centre. In spherical space, it is not immediately clear how to centre a particle, since for $x \in \mathbb{S}^d$ there are many $\varphi \in SO_{d+1}$ satisfying $\varphi\overline{0} = x$, where SO_{d+1} denotes the rotation group on $\mathbb{R}^{d+1}$. In order to deal with this situation, we will work in the framework of [29], where the author considers the general setting of random measures on a homogeneous space. We start by giving some additional definitions and specialising the setting of [29] to our situation.

In particular we consider the compact group SO_{d+1} and denote the unique, rotation invariant probability measure on SO_{d+1} by ν. The group SO_{d+1} operates continuously on $\mathbb{S}^d$ (see [49, Theorem 13.2.2]). Furthermore the operation is transitive, i.e. the projection

$$\pi_x : \ SO_{d+1} \to \mathbb{S}^d, \ \varphi \mapsto \varphi x,$$

is surjective for every $x \in \mathbb{S}^d$, and thus $\mathbb{S}^d$ is a homogeneous space (see [49, p. 582]). Defining

$$\sigma_d^0 := \nu \circ \pi_{\overline{0}}^{-1} = \frac{1}{\omega_{d+1}} \cdot \sigma_d,$$

we obtain a rotation invariant probability measure on $\mathbb{S}^d$. Next we consider

$$SO_{d+1}^{\overline{0}} := \{\varphi \in SO_{d+1} : \ \varphi\overline{0} = \overline{0}\}$$

and denote by $\kappa(\overline{0}, \cdot)$ the $SO_{d+1}^{\overline{0}}$-invariant probability measure on this (compact) subgroup. Putting

$$\kappa(\overline{0}, SO_{d+1} \setminus SO_{d+1}^{\overline{0}}) := 0, \tag{4.1.1}$$

we extend the measure to a probability measure on SO_{d+1}. For $x \in \mathbb{S}^d$ let

$$SO_{d+1}^{\overline{0},x} := \{\varphi \in SO_{d+1} : \ \varphi\overline{0} = x\}$$

and for $\varphi_x \in SO_{d+1}^{\overline{0},x}$ arbitrary but fixed, let

$$\kappa(x, B) := \int \mathbf{1}\{\varphi_x \circ \varphi \in B\} \ \kappa(\overline{0}, d\varphi), \quad B \in \mathcal{B}(SO_{d+1}). \tag{4.1.2}$$

This definition is independent of the choice of φ_x (see [29, (2.7)]) and satisfies

$$\int \mathbf{1}\{\varphi \in \cdot\} \ \kappa(\psi x, d\varphi) = \int \mathbf{1}\{\psi \circ \varphi \in \cdot\} \ \kappa(x, d\varphi), \quad \psi \in SO_{d+1}, \ x \in \mathbb{S}^d. \tag{4.1.3}$$

In addition, we have the following disintegration (equation (2.9) in [29])

$$\int_{SO_{d+1}} f(\varphi) \ \nu(d\varphi) = \int_{\mathbb{S}^d} \int_{SO_{d+1}} f(\varphi) \ \kappa(x, d\varphi) \ \sigma_d^0(dx). \tag{4.1.4}$$

In the following sections, we assume all random elements to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a measurable flow $\{\theta_\varphi : \ \varphi \in SO_{d+1}\}$. We further assume $\mathbb{P}$ to be invariant, i.e.

$$\mathbb{P} \circ \theta_\varphi = \mathbb{P}, \quad \varphi \in SO_{d+1}. \tag{4.1.5}$$

A random measure ξ on $\mathbb{S}^d$ is called adapted or isotropic, if

$$\xi(\theta_\varphi \omega, \varphi B) = \xi(\omega, B), \quad \omega \in \Omega, \ \varphi \in SO_{d+1}, \ B \in \mathcal{B}(\mathbb{S}^d), \tag{4.1.6}$$

where $\varphi B = \{\varphi x : x \in B\}$. For measurable $f : \mathbb{S}^d \to \mathbb{R}_+$ and isotropic ξ, we have

$$\int f(x) \ \xi(\theta_\varphi \omega, dx) = \int f(\varphi x) \ \xi(\omega, dx). \tag{4.1.7}$$

For each locally finite measure η on $\mathbb{S}^d$ and for each $\varphi \in SO_{d+1}$ let the rotated measure $\varphi\eta$ be given by

$$(\varphi\eta)(\varphi B) = \eta(B), \quad B \in \mathcal{B}(\mathbb{S}^d). \tag{4.1.8}$$

Thus we can rewrite (4.1.6) as

$$\xi(\theta_\varphi \omega, \cdot) = \varphi\xi(\omega, \cdot). \tag{4.1.9}$$

The Palm measure of an isotropic random measure ξ is a finite measure on Ω defined by

$$\mathbb{P}_\xi(A) := \int_\Omega \int_{\mathbb{S}^d} \int_{SO_{d+1}} \mathbf{1}\{\theta_\varphi^{-1}\omega \in A\} \cdot w(x) \ \kappa(x, d\varphi) \ \xi(\omega, dx) \ \mathbb{P}(d\omega), \ A \in \mathcal{F}, \tag{4.1.10}$$

where $w : \mathbb{S}^d \to \mathbb{R}_+$ is a measurable function satisfying $\int w(x) \ \sigma_d^0(dx) = 1$. This definition is independent of the choice of w (see [29, (3.8)]), allowing us to choose $w \equiv 1$. In what follows, $\mathbb{E}_\xi$ denotes integration with respect to $\mathbb{P}_\xi$. Furthermore the refined Campbell Theorem ([29, Theorem 3.7]) holds:

Theorem 4.1.1. *Let $f : \Omega \times SO_{d+1} \to \mathbb{R}_+$ be measurable. Then*

$$\mathbb{E} \int_{\mathbb{S}^d} \int_{SO_{d+1}} f(\theta_\varphi^{-1}, \varphi)\ \kappa(x, d\varphi)\ \xi(dx) = \mathbb{E}_\xi \int_{SO_{d+1}} f(\theta_{id}, \varphi)\ \nu(d\varphi). \tag{4.1.11}$$

In the canonical setting, this is a special case of [46, Theorem 1]. Note that $\mathbb{P}_\xi$ is not a probability measure but satisfies

$$\mathbb{P}_\xi(\Omega) = \mathbb{E}\left[\xi(\mathbb{S}^d)\right].$$

Let X' be an isotropic particle process on $\mathbb{S}^d$. This is a point process in $\mathcal{K}_s^d$ for which the following holds

$$X'(\theta_\varphi \omega) = \varphi X'(\omega) := \{\varphi K\ :\ K \in X'(\omega)\}, \quad \omega \in \Omega,\ \varphi \in SO_{d+1}. \tag{4.1.12}$$

The intensity of X' is defined by the expected number of particles, normalized by the surface area of $\mathbb{S}^d$

$$\gamma_{X'} := \frac{\mathbb{E}[X'(\mathcal{K}_s^d)]}{\omega_{d+1}}.$$

A rotation covariant spherical centre function is a mapping $c_s : \mathcal{K}_s^d \to \mathbb{S}^d$ satisfying

$$c_s(\varphi K) = \varphi c_s(K), \quad K \in \mathcal{K}_s^d,\ \varphi \in SO_{d+1}. \tag{4.1.13}$$

An example for such a centre function is the spherical circumcentre, i.e. the centre of the smallest spherical cap containing K. In order to define the typical particle of X', we fix a centre function and consider the marked random measure

$$\begin{aligned}\zeta'(\omega) &:= \sum_{K \in X'(\omega)} \int_{SO_{d+1}} \delta_{(c_s(K), \varphi^{-1}K)}\ \kappa(c_s(K), d\varphi) \\ &= \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \delta_{(c_s(K), \varphi^{-1}K)}\ \kappa(c_s(K), d\varphi)\ X'(\omega, dK).\end{aligned} \tag{4.1.14}$$

In what follows, we will also use the notation

$$\zeta'(\omega, \cdot) := \zeta'(\omega)(\cdot).$$

This random measure is invariant in the sense of [29, Remark 3.9]. To see this, let $\psi \in SO_{d+1}$ and $B \subset \mathbb{S}^d,\ A \subset \mathcal{K}_s^d$ be measurable. Then we have

$$\begin{aligned}\zeta'(\theta_\psi \omega, (\psi B) \times A) &= \sum_{K \in X'(\theta_\psi \omega)} \int_{SO_{d+1}} \delta_{(c_s(K), \varphi^{-1}K)}(\psi B \times A)\ \kappa(c_s(K), d\varphi) \\ &= \sum_{K \in X'(\omega)} \int_{SO_{d+1}} \delta_{(c_s(\psi K), \varphi^{-1}\psi K)}(\psi B \times A)\ \kappa(c_s(\psi K), d\varphi) \\ &= \sum_{K \in X'(\omega)} \int_{SO_{d+1}} \delta_{(\psi c_s(K), \varphi^{-1}\psi K)}(\psi B \times A)\ \kappa(\psi c_s(K), d\varphi)\end{aligned}$$

$$= \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \delta_{(c_s(K),\varphi^{-1}\psi K)}(B\times A)\ \kappa(\psi c_s(K), d\varphi)$$

$$= \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \delta_{(c_s(K),\varphi^{-1} K)}(B\times A)\ \kappa(c_s(K), d\varphi)$$

$$= \zeta'(\omega, B\times A),$$

where we made use of (4.1.12), (4.1.13) and (4.1.3) successively.

The distribution of the typical particle Z of X' is defined as the mark distribution of the random measure ζ'. The mark space is $\mathcal{K}^d_{\overline{0}} := \{K \in \mathcal{K}^d_s : c_s(K) = \overline{0}\}$, the set of all spherical convex sets with centre $\overline{0}$.

The Palm measure of ζ' is a measure on $\Omega \times \mathcal{K}^d_{\overline{0}}$ defined by

$$\mathbb{P}_{\zeta'}(\cdot) := \int_\Omega \int_{\mathbb{S}^d\times\mathcal{K}^d_{\overline{0}}} \int_{SO_{d+1}} \mathbf{1}\{(\theta^{-1}_\varphi\omega, K)\in\cdot\}\ \kappa(x,d\varphi)\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega). \tag{4.1.15}$$

If the intensity $\gamma_{X'}$ of X' is positive and finite, we have

$$\begin{aligned}\mathbb{P}_Z(\cdot) := \mathbb{P}(Z\in\cdot) &= \frac{1}{\gamma_{X'}\cdot\omega_{d+1}}\mathbb{P}_{\zeta'}(\Omega\times\cdot)\\ &= \frac{1}{\gamma_{X'}\omega_{d+1}}\int_\Omega \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1}K\in\cdot\}\ \kappa(c_s(K),d\varphi)\ \mathbb{P}(d\omega). \end{aligned}\tag{4.1.16}$$

In the Euclidean case, the typical particle of an isotropic and stationary particle process is still isotropic, but not stationary. In the spherical setting the typical particle cannot be isotropic, since its centre is almost surely $\overline{0}$, but its distribution still has some symmetry.

It is also possible to introduce the typical particle in terms of the Palm distribution $\tilde{\mathbb{P}}_0$ of $\zeta'(\cdot\times\mathcal{K}^d_{\overline{0}})$, the point process of centres. Define $P(\omega,x) = K$ if $x = c_s(K)$ for some $K \in X'(\omega)$ and $P(\omega,x) = \{x\}$ else. Using (4.1.13) and (4.1.12), we get $P(\theta_\varphi\omega,\varphi x) = \varphi P(\omega,x)$ for all $\varphi\in SO_{d+1}$. Together with (4.1.11), the definition of ζ' and (4.1.16) we obtain

$$\begin{aligned}\tilde{\mathbb{P}}_0(P(\theta_{id},\overline{0})\in\cdot) &= \frac{1}{\gamma_{X'}\omega_{d+1}}\int_\Omega \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \mathbf{1}\{P(\theta^{-1}_\varphi\omega,\overline{0})\in\cdot\}\ \kappa(c_s(K),d\varphi)\ \mathbb{P}(d\omega)\\ &= \frac{1}{\gamma_{X'}\omega_{d+1}}\int_\Omega \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1}P(\omega,c_s(K))\in\cdot\}\ \kappa(c_s(K),d\varphi)\ \mathbb{P}(d\omega)\\ &= \frac{1}{\gamma_{X'}\omega_{d+1}}\int_\Omega \sum_{K\in X'(\omega)} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1}K\in\cdot\}\ \kappa(c_s(K),d\varphi)\ \mathbb{P}(d\omega)\\ &= \mathbb{P}(Z\in\cdot).\end{aligned}$$

Therefore, the following Lemma as well as Theorem 4.1.3 are a consequence of more general results, see e.g. [29, Proposition 3.10]. We give the proofs for convenience.

Lemma 4.1.2. *Let X' be an isotropic particle process on $\mathbb{S}^d$ and let Z denote the typical particle of X'. Then the distribution of Z is invariant under rotations fixing $\overline{0}$, that means*

$$\mathbb{P}(\varphi Z\in\cdot) = \mathbb{P}(Z\in\cdot) \quad \varphi\in SO^{\overline{0}}_{d+1}.$$

Proof. Let $\varphi \in SO_{d+1}^{\overline{0}}$, let $A \subset \mathcal{K}_{\overline{0}}^d$ be measurable and $\varphi A := \{\varphi K : K \in A\}$. Since $SO_{d+1}^{\overline{0}}$ is a subgroup, we also have $\varphi^{-1} \in SO_{d+1}^{\overline{0}}$. Using (4.1.16), (4.1.15), $\kappa(x, SO_{d+1}) = 1$, for all $x \in \mathbb{S}^d$, and (4.1.14), we obtain

$$
\begin{aligned}
\mathbb{P}(\varphi Z \in A) &= \frac{1}{\gamma_{X'}\omega_{d+1}} \mathbb{P}_{\zeta'}(\Omega \times (\varphi^{-1}A)) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} \mathbf{1}\{(\theta_\psi^{-1}\omega, K) \in \Omega \times (\varphi^{-1}A)\}\ \kappa(x, d\psi)\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} \mathbf{1}\{\varphi K \in A\}\ \kappa(x, d\psi)\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_{\overline{0}}^d} \mathbf{1}\{\varphi K \in A\}\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{\varphi \circ \psi^{-1} K \in A\}\ \kappa(c_s(K), d\psi)\ X'(\omega, dK)\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{(\psi \circ \varphi^{-1})^{-1} K \in A\}\ \kappa(c_s(K), d\psi)\ X'(\omega, dK)\ \mathbb{P}(d\omega).
\end{aligned}
$$

Next we use the definition of $\kappa(x, \cdot)$, (4.1.2), the invariance of $\kappa(\overline{0}, \cdot)$ under $SO_{d+1}^{\overline{0}}$, (4.1.14), (4.1.15) and (4.1.16) to obtain

$$
\begin{aligned}
\mathbb{P}(\varphi Z \in A) &= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{(\varphi_{c_s(K)} \circ \psi \circ \varphi^{-1})^{-1} K \in A\}\ \kappa(\overline{0}, d\psi)\ X'(\omega, dK)\, \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{(\varphi_{c_s(K)} \circ \psi)^{-1} K \in A\}\ \kappa(\overline{0}, d\psi)\ X'(\omega, dK)\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{\psi^{-1} K \in A\}\ \kappa(c_s(K), d\psi)\ X'(\omega, dK)\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_{X'}\omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} \mathbf{1}\{K \in A\}\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \mathbb{P}(Z \in A).
\end{aligned}
$$

□

The refined Campbell Theorem holds for ζ' in the following form (see [29, Remark 3.9])

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} f(\theta_\varphi^{-1}\omega, \varphi, K)\ \kappa(x, d\varphi)\ \zeta'(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \int_{\Omega \times \mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} f(\omega, \varphi, K)\ \nu(d\varphi)\ \mathbb{P}_{\zeta'}(d(\omega, K)),
\end{aligned} \tag{4.1.17}
$$

for all measurable functions $f : \Omega \times SO_{d+1} \times \mathcal{K}_{\overline{0}}^d \to [0, \infty)$. In this setting, we can prove a disintegration result for the intensity measure of X', similar to the Euclidean case (see [49, Theorem 4.1.1]).

Theorem 4.1.3. *Let X' be an isotropic particle process on $\mathbb{S}^d$ with intensity $0 < \gamma_{X'} < \infty$ and let $f : \mathcal{K}_s^d \to [0, \infty)$ be measurable. Then the following holds*

$$\int_\Omega \int_{\mathcal{K}_s^d} f(K)\ X'(\omega, dK)\ \mathbb{P}(d\omega) = \gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_0^d} \int_{SO_{d+1}} f(\varphi K)\ \nu(d\varphi)\ \mathbb{P}_Z(dK). \tag{4.1.18}$$

Proof. Using the definition of $\mathbb{P}_Z$, (4.1.17) and (4.1.14), we obtain

$$\begin{aligned}
&\gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_0^d} \int_{SO_{d+1}} f(\varphi K)\ \nu(d\varphi)\ \mathbb{P}_Z(dK) \\
&= \int_{\Omega \times \mathcal{K}_0^d} \int_{SO_{d+1}} f(\varphi K)\ \nu(d\varphi)\ \mathbb{P}_{\zeta'}(d(\omega, K)) \\
&= \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_0^d} \int_{SO_{d+1}} f(\varphi K)\ \kappa(x, d\varphi)\ \zeta'(\omega, d(x, K))\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\varphi_1 \circ \varphi_2^{-1})K)\ \kappa(c_s(K), d\varphi_1)\ \kappa(c_s(K), d\varphi_2)\ X'(\omega, dK)\ \mathbb{P}(d\omega).
\end{aligned}$$

Since $\mathbb{P}$ is invariant and X' is adapted, we can use (4.1.13), (4.1.3), Fubini's theorem and the invariance of ν to obtain

$$\begin{aligned}
&\gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_0^d} \int_{SO_{d+1}} f(\varphi K)\ \nu(d\varphi)\ \mathbb{P}_Z(dK) \\
&= \int_{SO_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\varphi_1 \circ \varphi_2^{-1})\psi K)\ \kappa(c_s(\psi K), d\varphi_1)\ \kappa(c_s(\psi K), d\varphi_2) \\
&\qquad \times X'(\omega, dK)\ \mathbb{P}(d\omega)\ \nu(d\psi) \\
&= \int_{SO_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\varphi_1 \circ \varphi_2^{-1})\psi K)\ \kappa(\psi c_s(K), d\varphi_1)\ \kappa(\psi c_s(K), d\varphi_2) \\
&\qquad \times X'(\omega, dK)\ \mathbb{P}(d\omega)\ \nu(d\psi) \\
&= \int_{SO_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\psi \circ \varphi_1 \circ \varphi_2^{-1} \circ \psi^{-1} \circ \psi)K)\ \kappa(c_s(K), d\varphi_1) \\
&\qquad \times \kappa(c_s(K), d\varphi_2)\ X'(\omega, dK)\ \mathbb{P}(d\omega)\ \nu(d\psi) \\
&= \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\psi \circ \varphi_1 \circ \varphi_2^{-1} \circ \psi^{-1} \circ \psi)K)\ \nu(d\psi) \\
&\qquad \times \kappa(c_s(K), d\varphi_1)\ \kappa(c_s(K), d\varphi_2)\ X'(\omega, dK)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} \int_{SO_{d+1}} f((\psi \circ \varphi_1 \circ \varphi_2^{-1})K)\ \nu(d\psi)\ \kappa(c_s(K), d\varphi_1) \\
&\qquad \times \kappa(c_s(K), d\varphi_2)\ X'(\omega, dK)\ \mathbb{P}(d\omega) \\
&= \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} \int_{SO_{d+1}} \int_{SO_{d+1}} f(\psi K)\ \nu(d\psi)\ \kappa(c_s(K), d\varphi_1)\ \kappa(c_s(K), d\varphi_2)\ X'(\omega, dK)\ \mathbb{P}(d\omega)
\end{aligned}$$

$$= \int_\Omega \int_{\mathcal{K}_s^d} \int_{SO_{d+1}} f(\psi K)\ \nu(d\psi)\ X'(\omega, dK)\ \mathbb{P}(d\omega)$$

$$= \int_{SO_{d+1}} \int_\Omega \int_{\mathcal{K}_s^d} f(\psi K)\ X'(\omega, dK)\ \mathbb{P}(d\omega)\ \nu(d\psi)$$

$$= \int_\Omega \int_{\mathcal{K}_s^d} f(K)\ X'(\omega, dK)\ \mathbb{P}(d\omega),$$

where we again used the isotropy of X' to obtain the last equation. □

4.2. The typical cell of spherical Poisson hyperplane tessellations

As in [2], we can interpret the tessellation generated by the spherical hyperplane process $\tilde{X}$ as an isotropic particle process X'. The distribution of the typical cell Z of X' is given by (4.1.16). The following relation between the typical cell and the Crofton cell of an isotropic tessellation on $\mathbb{S}^d$ is a special case of a well-known relationship valid in all homogeneous tessellations, see e.g. [29, Corollary 8.4]. Its Euclidean counterpart can be found in [49, Theorem 10.4.1]. We give the proof for convenience and add an explicit expression for the intensity of X' if the tessellation is induced by an isotropic spherical Poisson hyperplane process. In advance, we give some properties of the functions

$$h_m : \begin{cases} [0,\infty) & \to \mathbb{R}, \\ \quad t & \mapsto (-1)^{m+1}e^{-t} + 2\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{t^{m-2i}}{(m-2i)!}, \end{cases}$$

$m \in \mathbb{N}_0$, which will occur in the afore-mentioned explicit expression of the intensity of X'.

Lemma 4.2.1. *The functions $h_m,\ m \in \mathbb{N}_0$, have the following properties:*

(1) $h'_m = h_{m-1},\ m \geq 1$,

(2) $h_0(t) = 2 - e^{-t} \geq 1,\ h_1(t) = e^{-t} + 2t \geq 1$,

(3) $h_m(0) = 1,\ m \geq 0$,

(4) h_m *is strictly increasing and* $h_m \geq 1$, $m \geq 0$,

(5) h_m *is convex for* $m \geq 1$,

(6) $0 \leq h_m(t) - \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^m}{m!}\right) \leq \frac{t^m}{m!},\ m \in \mathbb{N}_0$.

Proof. First note that for $m \in \mathbb{N}_0$

$$h_m(t) = (-1)^{m+1}e^{-t} + 2\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{t^{m-2i}}{(m-2i)!} = (-1)^{m+1}e^{-t} + \sum_{i=0}^{m} \left(1 + (-1)^{m-i}\right) \frac{t^i}{i!}.$$

The properties (1), (2) and (3) now follow directly from this identity. (4) and (5) are a direct consequence of the first three properties.

We will prove (6) by induction. For $m = 0$, we have $h_0(t) = 2 - e^{-t}$ and thus clearly

$$0 \leq 2 - e^{-t} - 1 = 1 - e^{-t} \leq 1.$$

Now assume that (6) holds for some $m \geq 1$. Then the first inequality of (6) is equivalent to

$$1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} \leq h_m(t).$$

Using this inequality and the properties (1) and (3), we obtain

$$\begin{aligned}
&h_{m+1}(t) - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&= \int_0^t h_m(s)\ ds + h_{m+1}(0) - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&\geq \int_0^t \left(1 + \frac{s}{1} + \ldots + \frac{s^m}{m!}\right)\ ds + 1 - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&= 0.
\end{aligned}$$

For the second inequality, we note that the second inequality of (6) is equivalent to

$$h_m(t) \leq 1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^m}{m!}.$$

Using this inequality and again the properties (1) and (3), we get

$$\begin{aligned}
&h_{m+1}(t) - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&= \int_0^t h_m(s)\ ds + h_{m+1}(0) - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&\leq \int_0^t \left(1 + \frac{s}{1} + \ldots + \frac{s^m}{m!} + \frac{s^m}{m!}\right)\ ds + 1 - \left(1 + \frac{t}{1} + \ldots + \frac{t^m}{m!} + \frac{t^{m+1}}{(m+1)!}\right) \\
&= \frac{t^{m+1}}{(m+1)!}.
\end{aligned}$$

Hence (6) also holds for $m + 1$, which concludes the proof. □

Theorem 4.2.2. *Let $f : \mathcal{K}_s^d \to [0, \infty)$ be measurable and rotation invariant. Let X' be an isotropic tessellation of $\mathbb{S}^d$ with intensity $\gamma_{X'}$, let Z_0 denote the spherical Crofton cell and Z the typical cell of X'. Then*

$$\mathbb{E}\left[f(Z_0)\right] = \gamma_{X'}\mathbb{E}\left[f(Z) \cdot \sigma_d(Z)\right]. \tag{4.2.1}$$

If X' is a spherical hyperplane tessellation induced by a spherical Poisson hyperplane process $\tilde{X}$ with intensity γ_S, we have

$$\gamma_{X'}\omega_{d+1} = h_d(\gamma_S\omega_{d+1}),$$

where

$$h_d(t) = (-1)^{d+1}e^{-t} + 2\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{t^{d-2i}}{(d-2i)!}.$$

Proof. From (4.1.18) and the rotation invariance of f we get

$$\begin{aligned}
\mathbb{E}[f(Z_0)] &= \mathbb{E} \sum_{K \in X'} f(K)\mathbf{1}\{\overline{0} \in \operatorname{int}(K)\} \\
&= \gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_{\overline{0}}^d} \int_{SO_{d+1}} f(\varphi K)\mathbf{1}\{\overline{0} \in \operatorname{int}(\varphi K)\}\ \nu(d\varphi)\ \mathbb{P}_Z(dK) \\
&= \gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_{\overline{0}}^d} f(K) \int_{SO_{d+1}} \mathbf{1}\{\varphi\overline{0} \in \operatorname{int}(K)\}\ \nu(d\varphi)\ \mathbb{P}_Z(dK) \\
&= \gamma_{X'}\omega_{d+1} \int_{\mathcal{K}_{\overline{0}}^d} f(K) \cdot \frac{\sigma_d(K)}{\omega_{d+1}}\ \mathbb{P}_Z(dK),
\end{aligned}$$

since the inner integral in the second to last line defines a rotation invariant probability measure on $\mathbb{S}^d$. This completes the first part of the proof.

For the second part, we use Schläfli's theorem (see [48, p. 209 - p. 212] or [9, (1.1)] in modern language), providing an explicit formula for the number of cells $N(k)$ generated by $k \geq 1$ great subspheres in general position,

$$N(k) = 2\sum_{i=0}^{d} \binom{k-1}{i}.$$

Recall that the spherical hyperplane process $\tilde{X}$, is defined by $\tilde{X} := h(X)$, where X is a spherical Poisson point process and

$$h : \begin{cases} \mathbb{S}^d & \to G(d+1,d) \cap \mathbb{S}^d, \\ x & \mapsto x^{\perp} \cap \mathbb{S}^d, \end{cases}$$

with $\mathbb{E}[X(\mathbb{S}^d)] = \gamma_S\omega_{d+1}$ beeing the expected number of points. If X contains no points, we consider the whole of $\mathbb{S}^d$ as one cell and thus define $N(0) := 1$. Then

$$\begin{aligned}
\gamma_{X'} &= \frac{1}{\omega_{d+1}} \cdot \mathbb{E}\left[N(X(\mathbb{S}^d))\right] \\
&= \frac{1}{\omega_{d+1}} \cdot \sum_{k=1}^{\infty} 2 \cdot \sum_{i=0}^{d} \binom{k-1}{i} \cdot \mathbb{P}(X(\mathbb{S}^d) = k) + \frac{1}{\omega_{d+1}} \cdot \mathbb{P}(X(\mathbb{S}^d) = 0) \\
&= \frac{2}{\omega_{d+1}} \cdot \sum_{k=1}^{\infty}\sum_{i=0}^{d} \binom{k-1}{i} \cdot e^{-\gamma_S\omega_{d+1}} \frac{(\gamma_S\omega_{d+1})^k}{k!} + \frac{1}{\omega_{d+1}} e^{-\gamma_S\omega_{d+1}} \\
&= \frac{2}{\omega_{d+1}} \cdot \sum_{i=0}^{d} \frac{1}{i!} \cdot \sum_{k=i+1}^{\infty} e^{-\gamma_S\omega_{d+1}} \frac{(\gamma_S\omega_{d+1})^k}{(k-i-1)!} \cdot \frac{1}{k} + \frac{1}{\omega_{d+1}} e^{-\gamma_S\omega_{d+1}}. \qquad (4.2.2)
\end{aligned}$$

Defining furthermore

$$f_i(x) := \sum_{k=i+1}^{\infty} \frac{1}{k} \cdot \frac{x^k}{(k-i-1)!},$$

we get

$$f_i(0) = 0 \text{ and } f_i'(x) = \sum_{k=i+1}^{\infty} \frac{x^{k-1}}{(k-i-1)!} = x^i \cdot e^x.$$

Applying [55, p. 174, Formula 419] iteratively yields

$$\begin{aligned} f_i(\gamma_S \omega_{d+1}) &= \int_0^{\gamma_S \omega_{d+1}} x^i \cdot e^x \, dx \\ &= \sum_{k=0}^{i} \left(e^{\gamma_S \omega_{d+1}} \cdot (-1)^k (\gamma_S \omega_{d+1})^{i-k} \cdot \frac{i!}{(i-k)!} \right) - (-1)^i \cdot i!. \end{aligned}$$

Combining this with (4.2.2) and using $\sum_{i=0}^{d}(-1)^{i+1} = -\frac{1}{2}(1+(-1)^d)$, we obtain

$$\begin{aligned} \gamma_{X'} &= \frac{2e^{-\gamma_S \omega_{d+1}}}{\omega_{d+1}} \sum_{i=0}^{d} (-1)^{i+1} + \frac{2}{\omega_{d+1}} \sum_{i=0}^{d} \sum_{k=0}^{i} \left[(-1)^k \cdot \frac{(\gamma_S \omega_{d+1})^{i-k}}{(i-k)!} \right] + \frac{e^{-\gamma_S \omega_{d+1}}}{\omega_{d+1}} \\ &= \frac{e^{-\gamma_S \omega_{d+1}}}{\omega_{d+1}} \left(-1 - (-1)^d + 1 \right) + \frac{2}{\omega_{d+1}} \sum_{i=0}^{d} \sum_{k=0}^{i} \left[(-1)^k \cdot \frac{(\gamma_S \omega_{d+1})^{i-k}}{(i-k)!} \right] \\ &= \frac{(-1)^{d+1} e^{-\gamma_S \omega_{d+1}}}{\omega_{d+1}} + \frac{2}{\omega_{d+1}} \sum_{i=0}^{d} \sum_{k=0}^{i} \left[(-1)^k \cdot \frac{(\gamma_S \omega_{d+1})^{i-k}}{(i-k)!} \right]. \end{aligned}$$

In order to complete the proof, it remains to show that

$$2 \sum_{i=0}^{d} \sum_{k=0}^{i} \left((-1)^k \frac{t^{i-k}}{(i-k)!} \right) = 2 \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{t^{d-2i}}{(d-2i)!}.$$

This can be seen as follows

$$\begin{aligned} 2 \sum_{i=0}^{d} \sum_{k=0}^{i} \left((-1)^k \frac{t^{i-k}}{(i-k)!} \right) &= 2 \sum_{k=0}^{d} \sum_{i=k}^{d} (-1)^k \frac{t^{i-k}}{(i-k)!} = 2 \sum_{k=0}^{d} \sum_{i=0}^{d-k} (-1)^k \frac{t^i}{i!} \\ &= 2 \sum_{i=0}^{d} \sum_{k=0}^{d-i} (-1)^k \frac{t^i}{i!} = 2 \sum_{i=0}^{d} \frac{t^i}{i!} \sum_{k=0}^{d-i} (-1)^k \\ &= \sum_{i=0}^{d} \left(1 + (-1)^{d-i} \right) \frac{t^i}{i!} = 2 \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{t^{d-2i}}{(d-2i)!}. \end{aligned}$$

□

Now we are able to extend our results for the asymptotic shape of the spherical Crofton cell to typical cells.

Theorem 4.2.3. *Let $a, \varepsilon > 0$ with $a < \omega_{d+1}/2$. Let X be an isotropic, spherical Poisson hyperplane process on $\mathbb{S}^d$ with intensity γ_S such that $\gamma_S \omega_{d+1} > 1$. Let Z and Z_0 denote the typical cell and the spherical Crofton cell of the induced tessellation. Then there are constants $c_{16},\ c_{17} > 0$, depending only on a, d and ε, such that*

$$\mathbb{P}(\vartheta(Z) \geq \varepsilon | \sigma_d(Z) \geq a) \leq c_{16} \cdot \exp\left(-c_{17} \cdot \gamma_S \omega_{d+1}\right). \tag{4.2.3}$$

If we substitute X by a binomial process Y of size $N \geq d+1$, then there are constants $c_{18},\ c_{19} > 0$, depending only on a, d and ε, such that

$$\mathbb{P}(\vartheta(Z(Y)) \geq \varepsilon | \sigma_d(Z(Y)) \geq a) \leq c_{18} \cdot \exp\left(-c_{19} \cdot N\right). \tag{4.2.4}$$

Proof. We first note the trivial upper bound $\sigma_d(Z_0) \leq \omega_{d+1}$. In order to estimate the denominator, we use (4.2.1) and (3.4.8) to obtain

$$\begin{aligned}
\mathbb{P}(\sigma_d(Z) \geq a) &= \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z) \geq a\}\right] \\
&= \frac{1}{\gamma_{X'}} \cdot \gamma_{X'} \cdot \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z) \geq a\} \cdot \sigma_d(Z) \cdot \frac{1}{\sigma_d(Z)}\right] \\
&= \frac{1}{\gamma_{X'}} \cdot \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z_0) \geq a\} \cdot \frac{1}{\sigma_d(Z_0)}\right] \\
&\geq \frac{1}{\gamma_{X'}} \cdot \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z_0) \geq a\} \cdot \frac{1}{\omega_{d+1}}\right] \\
&= \frac{1}{\gamma_{X'} \cdot \omega_{d+1}} \cdot \mathbb{P}\left(\sigma_d(Z_0) \geq a\right) \\
&\geq \frac{1}{\gamma_{X'} \cdot \omega_{d+1}} \cdot \exp\left(-\gamma_S \omega_{d+1} \cdot \Phi(B_a)\right).
\end{aligned}$$

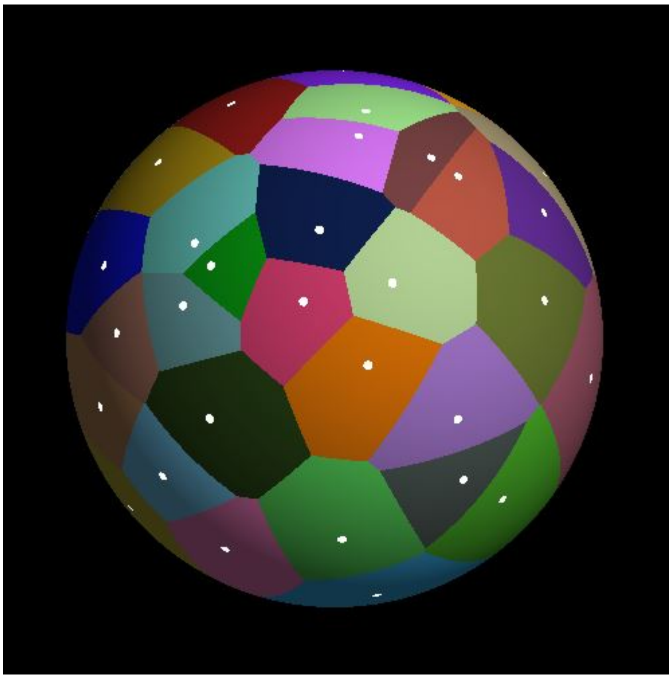
For the numerator, we use (4.2.1) and (3.4.2) and proceed as above

$$\begin{aligned}
\mathbb{P}(\sigma_d(Z) \geq a,\ \vartheta(Z) \geq \varepsilon) &= \frac{1}{\gamma_{X'}} \cdot \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z_0) \geq a,\ \vartheta(Z_0) \geq \varepsilon\} \cdot \frac{1}{\sigma_d(Z_0)}\right] \\
&\leq \frac{1}{\gamma_{X'}} \cdot \mathbb{E}\left[\mathbf{1}\{\sigma_d(Z_0) \geq a,\ \vartheta(Z_0) \geq \varepsilon\} \cdot \frac{1}{a}\right] \\
&\leq \frac{c_3}{a \cdot \gamma_{X'}} \cdot (\gamma_S \omega_{d+1})^{d \cdot \nu} \cdot \exp\left(-\gamma_S \omega_{d+1} \cdot \left(1 + \frac{f_a(\varepsilon)}{3}\right) \cdot \Phi(B_a)\right).
\end{aligned}$$

Combining these two estimates, we obtain the first result. In the binomial case, we use (3.6.3) instead of (3.4.8) and (3.6.1) instead of (3.4.2). $\square$

4.3. Spherical Poisson–Voronoi tessellations and Poisson processes on the sphere

After the consideration of the spherical Crofton cell and the typical cell of Poisson hyperplane tessellations on the sphere, it is a natural step to take a look at the Poisson–Voronoi



Spherical Voronoi tessellation, [56]

tessellations in spherical space, since in Euclidean space this tessellation is one of the classical models in stochastic geometry. Voronoi tessellations of a fixed number of points on $\mathbb{S}^2$ are briefly mentioned in [34].

Let $A \subset \mathbb{S}^d$ be locally finite. The Voronoi cell generated by $x \in A$ is (similar to the Euclidean case) given by

$$C(x, A) := \left\{ y \in \mathbb{S}^d : \ d_s(y, x) \leq d_s(y, z) \text{ for all } z \in A \right\}.$$

The set of all these cells forms the Voronoi tessellation generated by A. Note that for every $\varphi \in SO_{d+1}$, we have

$$C(\varphi x, \varphi A) = \varphi C(x, A). \tag{4.3.1}$$

Let X be an isotropic point process on $\mathbb{S}^d$ with intensity measure

$$\mathbb{E}[X(\cdot)] = \gamma_S \cdot \sigma_d(\cdot) = \gamma_S \cdot \omega_{d+1} \cdot \sigma_d^0(\cdot).$$

Additionally, we asume the intensity γ_S to be positive and finite, $0 < \gamma_S < \infty$. The Palm distribution of X is the normalized Palm measure of X, which is given by (4.1.10). From now on we will work exclusively with the Palm distribution and for brevity we will denote it by

$$\mathbb{P}_X(A) := \frac{1}{\gamma_S \omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d} \int_{SO_{d+1}} \mathbf{1}\{\theta_\varphi^{-1}\omega \in A\} \ \kappa(x, d\varphi) \ X(\omega, dx) \ \mathbb{P}(d\omega), \quad A \in \mathcal{F}, \tag{4.3.2}$$

without further indication of the normalization. The refined Campbell theorem (4.1.11) takes the form

$$\mathbb{E} \int_{\mathbb{S}^d} \int_{SO_{d+1}} f(\theta_\varphi^{-1}, \varphi) \ \kappa(x, d\varphi) \ X(dx) = \gamma_S \omega_{d+1} \cdot \mathbb{E}_X \int_{SO_{d+1}} f(\theta_{id}, \varphi) \ \nu(d\varphi), \tag{4.3.3}$$

or equivalently

$$\mathbb{E} \int_{\mathbb{S}^d} \int_{SO_{d+1}} f(\theta_{id}, \varphi)\ \kappa(x, d\varphi)\ X(dx) = \gamma_S \omega_{d+1} \cdot \mathbb{E}_X \int_{SO_{d+1}} f(\theta_\varphi, \varphi)\ \nu(d\varphi). \tag{4.3.4}$$

for all measurable functions $f : \Omega \times SO_{d+1} \to [0, \infty)$. Here $\mathbb{E}_X$ denotes the integration with respect to the probability measure $\mathbb{P}_X$.
In Euclidean space, the Theorem of Slivnyak characterizes a stationary Poisson process using its Palm distribution (see [49, Theorem 3.3.5]). The following version of the Mecke-Slivnyak Theorem is a special case of a very general result, [13, Theorem 4.21]. In the present special case, it can be proved (using the Mecke characterization of Poisson processes) in a few lines.

Theorem 4.3.1. *Let X be an isotropic point process on $\mathbb{S}^d$ with positive and finite intensity. Let $\mathbb{P}_X$ denote its Palm distribution. Then X is a Poisson point process, if and only if*

$$\mathbb{P}_X(X \in A) = \mathbb{P}(X + \delta_{\overline{0}} \in A), \quad A \in \mathcal{B}(\mathcal{N}(\mathbb{S}^d)). \tag{4.3.5}$$

4.4. The typical cell of a spherical isotropic Poisson–Voronoi tessellation

For Poisson–Voronoi tessellations, there is a very natural way to choose a centre function, namely the nucleus $x \in X$ of the cell $C(x, X)$. This leads to a slightly different approach to define the typical cell, involving the underlying Poisson point process and its Palm distribution. In order to define the distribution of the typical cell in this setting, we consider the random measure $\zeta : \Omega \to \mathcal{M}(\mathbb{S}^d \times \mathcal{K}_s^d)$, defined by

$$\zeta(\omega) := \sum_{x \in X(\omega)} \int_{SO_{d+1}} \delta_{(x, \varphi^{-1} C(x, X(\omega)))}\ \kappa(x, d\varphi). \tag{4.4.1}$$

This definition is very similar to definition (4.1.14) and ζ is also invariant in the sense of [29], Remark 3.9. Let $\psi \in SO_{d+1}$ and $B \subset \mathbb{S}^d,\ A \subset \mathcal{K}_s^d$ be measurable. Then we get

$$\begin{aligned}
\zeta(\theta_\psi \omega, (\psi B) \times A) &= \sum_{x \in X(\theta_\psi \omega)} \int_{SO_{d+1}} \delta_{(x, \varphi^{-1} C(x, X(\theta_\psi \omega)))}((\psi B) \times A)\ \kappa(x, d\varphi) \\
&= \sum_{x \in X(\omega)} \int_{SO_{d+1}} \delta_{(\psi x, \varphi^{-1} C(\psi x, X(\theta_\psi \omega)))}((\psi B) \times A)\ \kappa(\psi x, d\varphi) \\
&= \sum_{x \in X(\omega)} \int_{SO_{d+1}} \delta_{(\psi x, \varphi^{-1} \psi C(x, X(\omega)))}((\psi B) \times A)\ \kappa(\psi x, d\varphi) \\
&= \sum_{x \in X(\omega)} \int_{SO_{d+1}} \delta_{(x, \varphi^{-1} \psi C(x, X(\omega)))}(B \times A)\ \kappa(\psi x, d\varphi) \\
&= \sum_{x \in X(\omega)} \int_{SO_{d+1}} \delta_{(x, \varphi^{-1} C(x, X(\omega)))}(B \times A)\ \kappa(x, d\varphi)
\end{aligned}$$

$$= \zeta(\omega, B \times A),$$

where we made use of (4.1.9), (4.3.1) and (4.1.3). Recalling that X is an isotropic Poisson process with intensity measure $\mathbb{E}[X(\cdot)] = \gamma_S \cdot \omega_{d+1} \cdot \sigma_d^0(\cdot)$, the distribution of the typical cell Z is defined as the mark distribution of the random measure ζ and therefore

$$\begin{aligned}
\mathbb{P}(Z \in \cdot) &= \frac{1}{\gamma_S \omega_{d+1}} \mathbb{P}_\zeta(\Omega \times \cdot) \\
&= \frac{1}{\gamma_S \omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_s^d} \int_{SO_{d+1}} \mathbf{1}\{(\theta_\varphi^{-1}\omega, K) \in \Omega \times \cdot\}\ \kappa(x, d\varphi)\ \zeta(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_S \omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d \times \mathcal{K}_s^d} \mathbf{1}\{K \in \cdot\}\ \zeta(\omega, d(x,K))\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_S \omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1} C(x, X(\omega)) \in \cdot\}\ \kappa(x, d\varphi)\ X(\omega, dx)\ \mathbb{P}(d\omega).
\end{aligned} \tag{4.4.2}$$

Using (4.3.4), (4.1.9), (4.3.1) and Theorem 4.3.1, we obtain further

$$\begin{aligned}
\mathbb{P}(Z \in \cdot) &= \frac{1}{\gamma_S \omega_{d+1}} \int_\Omega \int_{\mathbb{S}^d} \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1} C(\varphi \cdot \overline{0}, X(\omega)) \in \cdot\}\ \kappa(x, d\varphi)\ X(\omega, dx)\ \mathbb{P}(d\omega) \\
&= \frac{1}{\gamma_S \omega_{d+1}} \gamma_S \omega_{d+1} \int_\Omega \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1} C(\varphi \cdot \overline{0}, X(\theta_\varphi \omega)) \in \cdot\}\ \nu(d\varphi)\ \mathbb{P}_X(d\omega) \\
&= \int_\Omega \int_{SO_{d+1}} \mathbf{1}\{\varphi^{-1} \varphi C(\overline{0}, X(\omega)) \in \cdot\}\ \nu(d\varphi)\ \mathbb{P}_X(d\omega) \\
&= \mathbb{P}_X(C(\overline{0}, X) \in \cdot) = \mathbb{P}(C(\overline{0}, X + \delta_{\overline{0}}) \in \cdot).
\end{aligned}$$

This relation allows us to interpret the typical cell of a spherical Poisson–Voronoi tessellation as the spherical Crofton cell of a special spherical Poisson hyperplane tessellation. This spherical hyperplane process Y is the set of all great subspheres, having equal spherical distance to the spherical origin $\overline{0}$ and to a point $x \in X$. This leads to a non isotropic process. The definition of Y can be made more explicit, using the map

$$f : \begin{cases} \mathbb{S}^d \setminus \{\overline{0}\} & \to \mathbb{S}^d, \\ x & \mapsto \frac{x - \overline{0}}{\|x - \overline{0}\|}. \end{cases}$$

Putting $f(\overline{0}) := -\overline{0}$, we can define f on the whole of $\mathbb{S}^d$ without influencing the results to come. Now let $x \in X$ and $z \in \mathbb{S}^d \cap f(x)^\perp$. Then

$$d_s(\overline{0}, z) = \arccos(\langle z, \overline{0} \rangle) = \arccos(\langle z, \overline{0} \rangle + \langle z, x - \overline{0} \rangle) = \arccos(\langle z, x \rangle) = d_s(x, z),$$

which means the spherical Poisson hyperplane process Y is given by

$$Y := \sum_{x \in X} \delta_{f(x)^\perp \cap \mathbb{S}^d}. \tag{4.4.3}$$

Using Lemma 6.5.1 from [49], we obtain for the intensity measure of Y

$$\begin{aligned}\mathbb{E}[Y(\cdot)] &= \mathbb{E}[\mathrm{card}\{x \in X : f(x)^\perp \cap \mathbb{S}^d \in \cdot\}] = \gamma_S \int_{\mathbb{S}^d} \mathbf{1}\{f(x)^\perp \cap \mathbb{S}^d \in \cdot\}\ \sigma_d(dx) \\ &= \gamma_S \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{\{\overline{0},-\overline{0}\}\vee u} \sin^{d-1}(d_s(\{\overline{0},-\overline{0}\},t)) \cdot \mathbf{1}\{f(t)^\perp \cap \mathbb{S}^d \in \cdot\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \\ &=: \gamma_S \cdot \omega_{d+1} \cdot \tilde{\mu}(\cdot). \end{aligned} \tag{4.4.4}$$

Here $\{\overline{0},-\overline{0}\} \vee u = \mathrm{pos}(\{\overline{0},-\overline{0},u\}) \cap \mathbb{S}^d$ is the spherical convex hull of the points $\overline{0}, -\overline{0}$ and u. This set is a two-dimensional half circle, containing u and bounded by $\overline{0}$ and $-\overline{0}$. In order to derive similar results as in the case of isotropic spherical Poisson hyperplane tessellations, we have to further examine the functional $\tilde{\mu}$ and eventually derive a stability result, similar to Theorem 3.3.4.

4.5. Another stability result

Let $K \in \overline{\mathcal{K}}_s^d$. Evaluating the functional $\tilde{\mu}$ especially for the set

$$\mathcal{H}_K := \{L \in G(d+1,d) \cap \mathbb{S}^d : L \cap K \neq \emptyset\},$$

we obtain a new functional on the space of spherically convex sets

$$\begin{aligned}\mathbb{E}[Y(\mathcal{H}_K)] &= \gamma_S \cdot \omega_{d+1} \cdot \tilde{\mu}(\mathcal{H}_K) \\ &= \gamma_S \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{\{\overline{0},-\overline{0}\}\vee u} \sin^{d-1}(d_s(\{\overline{0},-\overline{0}\},t)) \cdot \mathbf{1}\{f(t)^\perp \cap K \neq \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \\ &=: \gamma_S \cdot \tilde{U}(K). \end{aligned} \tag{4.5.1}$$

This functional will essentially play the role of $U_1(\cdot)$ in a stability estimate similar to Theorem 3.3.4. We will continue with recalling two definitions from the very beginning of this chapter and defining another deviation functional before stating our stability result in the next theorem.

Let $K \in \overline{\mathcal{K}}_s^d$ with $\overline{0} \in K$. The spherical inball radius centred in $\overline{0}$ is given by

$$r_s(K) := r_s(K,\overline{0}) = \max\{r \geq 0 :\ B_s(\overline{0},r) \subset K\}. \tag{4.5.2}$$

This will be our chosen method to measure the size of spherically convex sets. The spherical circumball radius centred in $\overline{0}$ is given by

$$R_s(K) := R_s(K,\overline{0}) = \min\{r \geq 0 :\ K \subset B_s(\overline{0},r)\}. \tag{4.5.3}$$

The difference between these two values is a natural deviation functional, which additionally has a very intuitive interpretation

$$\vartheta^{\overline{0}}(K) := R_s(K) - r_s(K). \tag{4.5.4}$$

Theorem 4.5.1. *Let $a \in (0, \pi/2)$, $K \in \overline{\mathcal{K}}_s^d$ with $r_s(K) \geq a$ and $C := B_s(\overline{0}, a)$. Then*

$$\tilde{U}(K) \geq \tilde{U}(C) = \sigma_d(B_s(\overline{0}, 2a)) \tag{4.5.5}$$

with equality if and only if $K = C$. Furthermore, let $K \subseteq B_s(\overline{0}, \pi/2)$ and $\vartheta^{\overline{0}}(K) \geq \varepsilon > 0$. Then

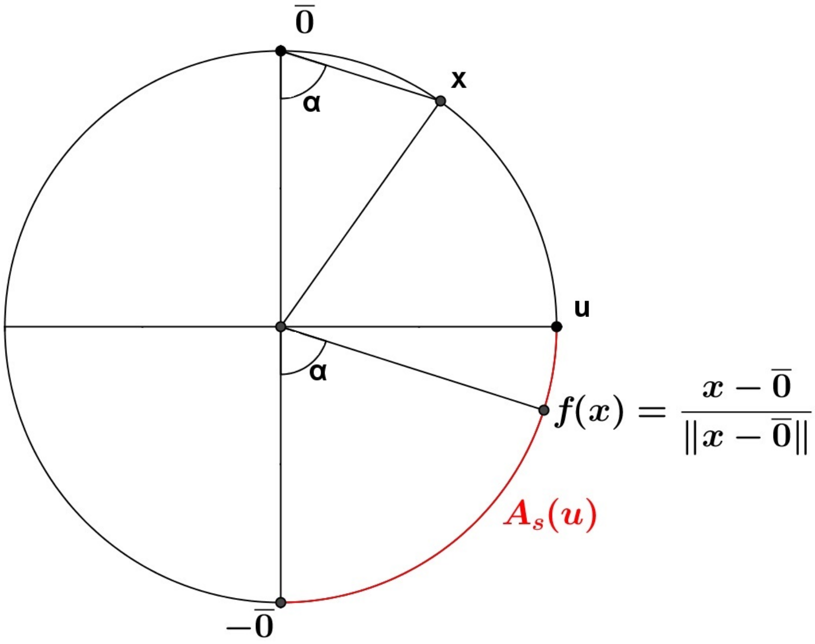
$$\tilde{U}(K) \geq \tilde{U}(C) \cdot (1 + c_{20} \cdot \varepsilon^d) \tag{4.5.6}$$

where the constant $c_{20} = c_{20}(a, d)$ only depends on a and the dimension d.

Proof. In order to examine the functional $\tilde{U}$, we have to examine the map f. Recall $f(x) = \frac{x - \overline{0}}{\|x - \overline{0}\|}$ and consider the isosceles triangle, formed by the (Euclidean) origin, $\overline{0}$ and $x \in \{\overline{0}, -\overline{0}\} \vee u$, where $x \neq \overline{0}$ and $x \neq -\overline{0}$. In this triangle, denote the angle in $\overline{0}$ by α. Then the following holds

$$\alpha = \frac{\pi - d_s(\overline{0}, x)}{2} < \frac{\pi}{2}$$

and α is equal to the angle between $-\overline{0}$ and $f(x) = \frac{x - \overline{0}}{\|x - \overline{0}\|}$ (see the next picture).



Furthermore $f(-\overline{0}) = -\overline{0}$ and we have for $u \in \overline{0}^{\perp} \cap \mathbb{S}^d$

$$f(\{\overline{0}, -\overline{0}\} \vee u) = \{y \in \{\overline{0}, -\overline{0}\} \vee u : \langle \overline{0}, y \rangle < 0\} =: A_s(u)$$

and

$$d_s(f(x), u) = \frac{\pi}{2} - \alpha = \frac{d_s(\overline{0}, x)}{2}. \tag{4.5.7}$$

Thus any circular arc A, bounded by u and some $y \in A_s(u)$, and the image measure $\sigma_1 \circ f^{-1}$ satisfy

$$(\sigma_1 \circ f^{-1})(A) = 2 \cdot \sigma_1(A). \tag{4.5.8}$$

Every circular arc $B \subset A_s(u)$ can be written as the difference of two circular arcs beginning in u, therefore this equation holds for every circular arc which is a subset of $A_s(u)$. Since measures on $A_s(u)$ are determined by their values on circular arcs, (4.5.8) holds for general, measurable subsets of $A_s(u)$.

$$\begin{aligned}\tilde{U}(K) &= \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, f^{-1}(t))) \cdot \mathbf{1}\{t^\perp \cap K \neq \emptyset\}\ (\sigma_1 \circ f^{-1})(dt)\ \sigma_{d-1}(du) \\ &= 2 \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, f^{-1}(t))) \cdot \mathbf{1}\{t^\perp \cap K \neq \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du).\end{aligned}$$

We define $\tilde{S}_u := \{-\overline{0}, u\}$. For $t \in A_s(u)$ we have $d_s(\{\overline{0}, -\overline{0}\}, f^{-1}(t)) = 2 \cdot d_s(\tilde{S}_u, t)$. To see this, we first consider the case $d_s(\tilde{S}_u, t) = d_s(u, t)$. Then the assertion follows directly from (4.5.7). If $d_s(\tilde{S}_u, t) = d_s(-\overline{0}, t)$ holds, using (4.5.7) we obtain

$$\pi/4 \leq \pi/2 - d_s(-\overline{0}, t) = d_s(u, t) = d_s(\overline{0}, f^{-1}(t))/2.$$

Using this, we get

$$\begin{aligned}d_s(\{\overline{0}, -\overline{0}\}, f^{-1}(t)) &= d_s(-\overline{0}, f^{-1}(t)) = \pi - d_s(\overline{0}, f^{-1}(t)) \\ &= 2\left(\frac{\pi}{2} - d_s(u, t)\right) = 2d_s(-\overline{0}, t),\end{aligned}$$

and therefore

$$\tilde{U}(K) = 2 \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2 \cdot d_s(\tilde{S}_u, t)) \cdot \mathbf{1}\{t^\perp \cap K \neq \emptyset\}\ \sigma_1(dt)\sigma_{d-1}(du). \tag{4.5.9}$$

From $r_s(K) \geq a$ we immediately get $C \subset K$. Let $u \in \overline{0}^\perp \cap \mathbb{S}^d$ and $t \in A_s(u)$. Then $d_s(\tilde{S}_u, t) \leq \pi/4$ holds and thus $\sin^{d-1}(2d_s(\tilde{S}_u, t)) \geq 0$. Using (4.5.9) we obtain our first claim

$$\tilde{U}(K) \geq \tilde{U}(C).$$

In order to calculate the value of $\tilde{U}(C)$, we consider the mapping

$$g_u : \left[0, \frac{\pi}{2}\right] \to A_s(u),$$

defined by $g_u(y) \in A_s(u)$ and $d_s(g_u(y), u) = y$. Using this transformation in (4.5.9), we get

$$\begin{aligned}\tilde{U}(C) &= 2 \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2d_s(\tilde{S}_u, t))\mathbf{1}\{t^\perp \cap C \neq \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \\ &= 2 \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_0^a \sin^{d-1}(2d_s(\tilde{S}_u, g_u(y)))\ dy\ \sigma_{d-1}(du). \end{aligned} \tag{4.5.10}$$

Additionally assuming $a \leq \pi/4$, we get $d_s(\tilde{S}_u, g_u(y)) = d_s(u, g_u(y)) = y$ for all $y \in [0, a]$ and it follows

$$(4.5.10) = 2 \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_0^a \sin^{d-1}(2y)\ dy\ \sigma_{d-1}(du)$$

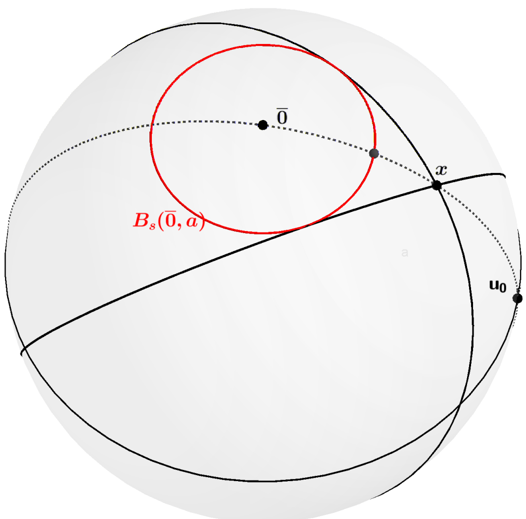
$$
\begin{aligned}
&= 2\omega_d \int_0^a \sin^{d-1}(2y)\ dy \\
&= \omega_d \int_0^{2a} \sin^{d-1}(x)\ dx \\
&= \sigma_d(B_s(\overline{0}, 2a)).
\end{aligned}
$$

In the case $a > \pi/4$, we have $d_s(\tilde{S}_u, g_u(y)) = d_s(-\overline{0}, g_u(y)) = \pi/2 - y$ for $y > \pi/4$ and we obtain

$$
\begin{aligned}
(4.5.10) &= 2\omega_d \left(\int_0^{\pi/4} \sin^{d-1}(2y)\ dy + \int_{\pi/4}^{a} \sin^{d-1}\left(2(\frac{\pi}{2} - y)\right)\ dy \right) \\
&= 2\omega_d \left(\int_0^{\pi/4} \sin^{d-1}(2y)\ dy + \int_{\pi/4}^{a} \sin^{d-1}(2y)\ dy \right) \\
&= 2\omega_d \int_0^{a} \sin^{d-1}(2y)\ dy. \\
&= \sigma_d(B_s(\overline{0}, 2a)).
\end{aligned}
$$

This concludes the proof of the first part of our theorem. The equality case follows from the stability result (4.5.6), which will be proven next.

Let $K \in \overline{\mathcal{K}}_s^d$ with $r_s(K) \geq a$, $K \subseteq B_s(\overline{0}, \pi/2)$ and $\vartheta^{\overline{0}}(K) \geq \varepsilon > 0$. Then there is a point $x \in K$ such that $d_s(\overline{0}, x) = a + \varepsilon$ and, since K is a spherically convex set with $r_s(K) \geq a$, $(B_s(\overline{0}, a) \vee x) \subset K$. Furthermore, we define $M(x, a) := (B_s(\overline{0}, a) \vee x) \setminus B_s(\overline{0}, a)$. Note that the condition $K \subseteq B_s(\overline{0}, \pi/2)$ implies $a + \varepsilon \leq \pi/2$.



Then the following holds

$$\tilde{U}(K) \geq \tilde{U}(B_s(\overline{0}, a) \vee x)$$

$$
\begin{aligned}
&= 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2 d_s(\tilde{S}_u, t)) \mathbf{1}\{t^{\perp} \cap (B_s(\overline{0}, a) \vee x) \neq \emptyset\} \, \sigma_1(dt) \, \sigma_{d-1}(du) \\
&= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2 d_s(\tilde{S}_u, t)) \mathbf{1}\{t^{\perp} \cap B_s(\overline{0}, a) = \emptyset\} \\
&\qquad\qquad \times \mathbf{1}\{t^{\perp} \cap M(x, a) \neq \emptyset\} \, \sigma_1(dt) \, \sigma_{d-1}(du) \\
&\geq \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2 d_s(\tilde{S}_u, t)) \\
&\qquad\qquad \times \mathbf{1}\{t^{\perp} \cap M(x, a) \cap (\{\overline{0}, -\overline{0}\} \vee u) \neq \emptyset\} \, \sigma_1(dt) \, \sigma_{d-1}(du) \\
&= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_0^{\pi/2} \sin^{d-1}(2 d_s(\tilde{S}_u, g_u(y))) \\
&\qquad\qquad \times \mathbf{1}\{g_u(y)^{\perp} \cap M(x, a) \cap (\{\overline{0}, -\overline{0}\} \vee u) \neq \emptyset\} \, dy \, \sigma_{d-1}(du) \\
&= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon, u)} \sin^{d-1}(2 d_s(\tilde{S}_u, g_u(y))) \, dy \, \sigma_{d-1}(du), \qquad (4.5.11)
\end{aligned}
$$

where $\delta(\varepsilon, u)$ denotes the spherical diameter of $M(x,a) \cap (\{\overline{0}, -\overline{0}\} \vee u)$. The inner integral can be simplified by considering different cases, similar as in the calculation of $\tilde{U}(C)$.

1. Let $a + \delta(\varepsilon, u) \leq \pi/4$. Then $d_s(\tilde{S}_u, g_u(y)) = y$ for all $y \in [a, a + \delta(\varepsilon, u)]$ and we get

$$
(4.5.11) = \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon, u)} \sin^{d-1}(2y) \, dy \, \sigma_{d-1}(du).
$$

2. Assume $a \leq \pi/4$ but $a + \delta(\varepsilon, u) > \pi/4$. Then $d_s(\tilde{S}_u, g_u(y)) = \pi/2 - y$ for $y > \pi/4$ and we obtain

$$
\begin{aligned}
(4.5.11) &= \tilde{U}(C) \\
&\quad + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \left(\int_a^{\pi/4} \sin^{d-1}(2y) \, dy + \int_{\pi/4}^{a+\delta(\varepsilon, u)} \sin^{d-1} \left(2 (\frac{\pi}{2} - y) \right) \, dy \right) \sigma_{d-1}(du) \\
&= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon, u)} \sin^{d-1}(2y) \, dy \, \sigma_{d-1}(du).
\end{aligned}
$$

3. Let $a \geq \pi/4$. Then we have $d_s(\tilde{S}_u, g_u(y)) = \pi/2 - y$ for all $y \in [a, a + \delta(\varepsilon, u)]$ and therefore

$$
\begin{aligned}
(4.5.11) &= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon, u)} \sin^{d-1}(\pi - 2y) \, dy \, \sigma_{d-1}(du) \\
&= \tilde{U}(C) + 2 \int_{\overline{0}^{\perp} \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon, u)} \sin^{d-1}(2y) \, dy \, \sigma_{d-1}(du).
\end{aligned}
$$

In all cases, we obtain the same expression.

Let $\tilde{u}_0 := \Pi_{\overline{0}^\perp}(x)$ denote the orthogonal projection of x onto $\overline{0}^\perp$ and $u_0 := \tilde{u}_0/\|\tilde{u}_0\|$. Then u_0 is an element of $\overline{0}^\perp \cap \mathbb{S}^d$ and clearly there is a neighbourhood of u_0 in $\overline{0}^\perp \cap \mathbb{S}^d$, having a σ_{d-1}-content greater than 0, such that $\delta(\varepsilon, u) > 0$ on this neighbourhood. This shows that there is a stability result

$$\tilde{U}(K) \geq \tilde{U}(C)\left(1 + \frac{2}{\tilde{U}(C)} \int_{\overline{0}^\perp \cap \mathbb{S}^d} \int_a^{a+\delta(\varepsilon,u)} \sin^{d-1}(2y)\, dy\, \sigma_{d-1}(du)\right). \tag{4.5.12}$$

In order to determine an explicit result, we have to provide a lower bound for the double integral above. We consider the set

$$M(\varepsilon, a) := \{u \in \mathbb{S}^d \cap \overline{0}^\perp : \delta(\varepsilon, u) \geq \varepsilon/2\}. \tag{4.5.13}$$

Due to the symmetry of the situation, this set is clearly a spherical cap in $\mathbb{S}^d \cap \overline{0}^\perp \equiv \mathbb{S}^{d-1}$ with centre u_0. Thus it is sufficient to bound the diameter of $M(\varepsilon, a)$, if we want to bound the $(d-1)$-dimensional spherical volume of this set. This allows us to restrict ourselves, without loss of generality, to $d = 2$. We return to the set

$$M(x, a) = (B_s(\overline{0}, a) \vee x) \setminus B_s(\overline{0}, a).$$

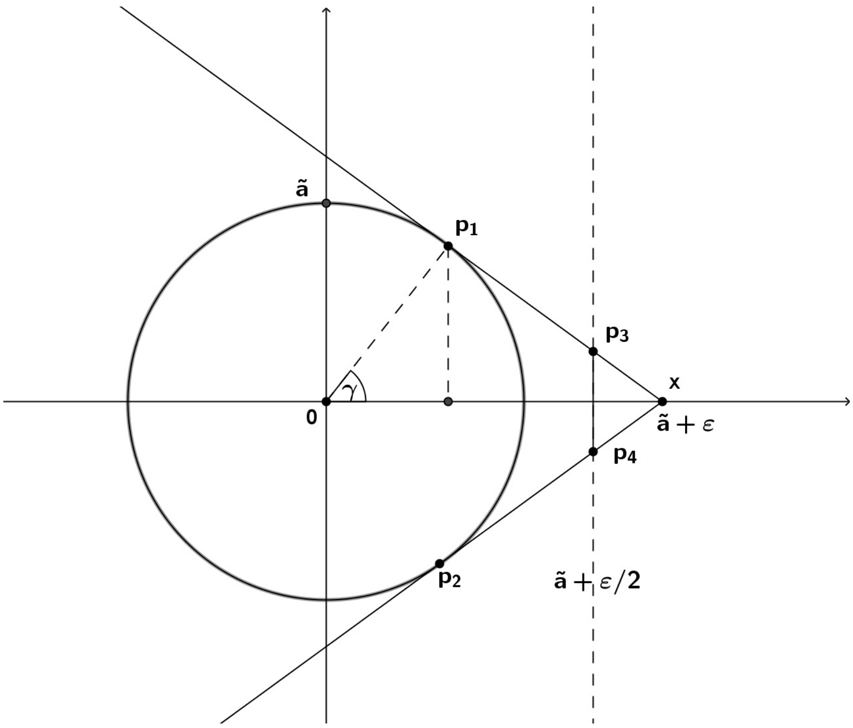
Its boundary consists of a spherical arc on the boundary of the spherical cap $B_s(\overline{0}, a)$ and two arcs of the spherical great circles containing x and touching $B_s(\overline{0}, a)$. These two touching points will be denoted by p_1 and p_2. Additionally, we denote the intersection of the boundary of $B_s(\overline{0}, a + \varepsilon/2)$ and $p_1 \vee x$ by p_3 and the intersection of this boundary with $p_2 \vee x$ by p_4. Then $\operatorname{diam}(M(\varepsilon, a))$ is at least as big as the length of the circular arc on $B_s(\overline{0}, a + \varepsilon/2)$ between p_3 and p_4. In particular we have $\operatorname{diam}(M(\varepsilon, a)) \geq \|p_3 - p_4\|$. Let $c_{21} = c_{21}(a, d, \varepsilon)$ be non negative, such that

$$\operatorname{diam}(M(\varepsilon, a)) \geq \varepsilon \cdot c_{21}(a, d, \varepsilon). \tag{4.5.14}$$

We want to show that $\inf_{\varepsilon \in (0, \pi/2 - a]} c_{21}(a, d, \varepsilon) > 0$ holds. For increasing ε, the boundary length of $B_s(\overline{0}, a)$ between p_1 and p_2 is nondecreasing, although this is possible for $\operatorname{diam}(M(\varepsilon, a))$. We fix $\tilde{p}_1$ and $\tilde{p}_2$ with respect to some arbitrary but fixed $\varepsilon_0 \in (0, \pi/2 - a)$ and consider the extremal case $\varepsilon = \pi/2 - a$. This means the additional point x lies in $\mathbb{S}^d \cap \overline{0}^\perp$. Then the spherical distance between the points p_3 and p_4, belonging to $\varepsilon = \pi/2 - a$, is at least as big as the spherical distance between the points $\tilde{p}_3$ and $\tilde{p}_4$, calculated from x, $\tilde{p}_1$ and $\tilde{p}_2$. Since the latter is clearly still greater than 0 and ε cannot grow further, we have

$$\liminf_{\varepsilon \to \frac{\pi}{2} - a} c_{21}(a, d, \varepsilon) > 0.$$

Now we consider the case $\varepsilon \to 0$. Since $a < \pi/2$, $a + \varepsilon < \pi/2$ and the projection onto the tangent plane in $\overline{0}$ is a bi-lipschitz mapping, we can, without loss of generality, consider the following situation.



Here $\tilde{a}$ is the radius of the projection of the circle that forms the border of the spherical cap $B_s(\overline{0}, a)$. We consider the planar situation sketched above. Let γ denote the angle between the horizontal axis and p_1 and let (x_1, y_1) denote the coordinates of p_1. We consider the two rectangular triangles formed by the points 0, p_1, x and 0, p_1, $(x_1, 0)$. Then

$$\frac{\tilde{a}}{\tilde{a}+\varepsilon} = \cos(\gamma) = \frac{x_1}{\tilde{a}}, \text{ thus } x_1 = \frac{\tilde{a}^2}{\tilde{a}+\varepsilon}.$$

Since p_1 lies on a circle with radius $\tilde{a}$, we immediately obtain

$$y_1 = \sqrt{\tilde{a}^2 - x_1^2} = \sqrt{\tilde{a}^2 - \frac{\tilde{a}^4}{(\tilde{a}+\varepsilon)^2}}.$$

In order to determine the slope $\tilde{\alpha}$ and the y-intercept $\tilde{\beta}$ of the tangent in p_1, we solve the following system of equations

$$\sqrt{\tilde{a}^2 - \frac{\tilde{a}^4}{(\tilde{a}+\varepsilon)^2}} = \tilde{\alpha}\left(\frac{\tilde{a}^2}{\tilde{a}+\varepsilon}\right) + \tilde{\beta}$$

$$\tilde{\beta} = -\tilde{\alpha}\,(\tilde{a}+\varepsilon)\,.$$

It follows that

$$\begin{aligned} \frac{\sqrt{\tilde{a}^2(\tilde{a}+\varepsilon)^2 - \tilde{a}^4}}{\tilde{a}+\varepsilon} &= \tilde{\alpha}\left(\frac{\tilde{a}^2}{\tilde{a}+\varepsilon}\right) - \tilde{\alpha}(\tilde{a}+\varepsilon) \\ &= \tilde{\alpha}\left(\frac{\tilde{a}^2 - (\tilde{a}+\varepsilon)^2}{\tilde{a}+\varepsilon}\right) \end{aligned}$$

and therefore

$$\tilde{\alpha} = \frac{\sqrt{\tilde{a}^2(\tilde{a}+\varepsilon)^2 - \tilde{a}^4}}{\tilde{a}^2 - (\tilde{a}+\varepsilon)^2}$$

and

$$\tilde{\beta} = (\tilde{a}+\varepsilon)\frac{\sqrt{\tilde{a}^2(\tilde{a}+\varepsilon)^2 - \tilde{a}^4}}{(\tilde{a}+\varepsilon)^2 - \tilde{a}^2}.$$

The distance between p_3 and p_4 is twice the value of the tangent at $x = \tilde{a} + \varepsilon/2$

$$\begin{aligned}\|p_3 - p_4\| &= 2\left(\tilde{\alpha}\cdot(\tilde{a}+\varepsilon/2) + \tilde{\beta}\right)\\ &= 2\left(\tilde{\alpha}\cdot(\tilde{a}+\varepsilon/2) - \tilde{\alpha}(\tilde{a}+\varepsilon)\right)\\ &= 2(-\tilde{\alpha}\cdot\varepsilon/2)\\ &= -\tilde{\alpha}\cdot\varepsilon.\end{aligned}$$

Since we have $\operatorname{diam}(M(\varepsilon, a)) \geq \|p_3 - p_4\|$, it is sufficient to show that

$$\liminf_{\varepsilon\to 0}\frac{\|p_3 - p_4\|}{\varepsilon} > 0.$$

With the above, we immediately get

$$\liminf_{\varepsilon\to 0} c_{21}(\varepsilon, a, d) > 0.$$

Plugging in our result for $\tilde{\alpha}$, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{\|p_3 - p_4\|}{\varepsilon} &= \liminf_{\varepsilon\to 0} -\tilde{\alpha} = \liminf_{\varepsilon\to 0} -\frac{\sqrt{\tilde{a}^2(\tilde{a}+\varepsilon)^2 - \tilde{a}^4}}{\tilde{a}^2 - (\tilde{a}+\varepsilon)^2}\\ &= \liminf_{\varepsilon\to 0} -\frac{\sqrt{\tilde{a}^2(\tilde{a}^2 + 2\tilde{a}\varepsilon + \varepsilon^2) - \tilde{a}^4}}{\tilde{a}^2 - (\tilde{a}^2 + 2\tilde{a}\varepsilon + \varepsilon^2)}\\ &= \liminf_{\varepsilon\to 0}\frac{\sqrt{2\tilde{a}^3\varepsilon + \tilde{a}^2\varepsilon^2}}{2\tilde{a}\varepsilon + \varepsilon^2}\\ &= \liminf_{\varepsilon\to 0}\frac{\sqrt{2\frac{\tilde{a}^3}{\varepsilon} + \tilde{a}^2}}{2\tilde{a} + \varepsilon} = \infty.\end{aligned}$$

We set

$$c_{22}(a, d) := \inf_{\varepsilon\in(0,\pi/2-a]} c_{21}(a, d, \varepsilon) > 0 \tag{4.5.15}$$

and from (4.5.14) we get

$$\operatorname{diam}(M(\varepsilon, a)) \geq \varepsilon\cdot c_{22}(a, d). \tag{4.5.16}$$

Thus

$$\sigma_{d-1}(M(\varepsilon, a)) \geq \sigma_{d-1}\left(B_s^{d-1}\left(\frac{c_{22}(a,d)\cdot\varepsilon}{2}\right)\right)$$

$$
\begin{aligned}
&= \omega_{d-1} \int_0^{c_{22}(a,d)\cdot\varepsilon/2} \sin^{d-2}(x)\ dx \\
&\geq \omega_{d-1} \int_0^{c_{22}(a,d)\cdot\varepsilon/2} \left(\frac{2}{\pi}\cdot x\right)^{d-1}\ dx \\
&= \omega_{d-1}\cdot\left(\frac{2}{\pi}\right)^{d-1}\cdot\frac{c_{22}(a,d)^{d-1}}{d-1}\cdot\varepsilon^{d-1},
\end{aligned}
$$

where $B_s^{d-1}(r)$ denotes a spherical cap with radius r in $\mathbb{S}^{d-1}$. Using this, we can further gauge the double integral in (4.5.12)

$$
\begin{aligned}
&\frac{2}{\tilde{U}(C)}\int_{\bar{0}^{\perp}\cap\mathbb{S}^d}\int_a^{a+\delta(\varepsilon,u)} \sin^{d-1}(2y)\ dy\ \sigma_{d-1}(du) \\
\geq&\frac{2}{\tilde{U}(C)}\int_{M(\varepsilon,a)}\int_a^{a+\varepsilon/2} \sin^{d-1}(2y)\ dy\ \sigma_{d-1}(du) \\
\geq&\frac{2}{\tilde{U}(C)}\int_{M(\varepsilon,a)}\int_a^{a+\varepsilon/2} \left(\min\{\sin(2a),\sin(2a+\varepsilon)\}\right)^{d-1}\ dy\ \sigma_{d-1}(du) \\
\geq&\frac{2}{\tilde{U}(C)}\int_{M(\varepsilon,a)}\int_a^{a+\varepsilon/2} \left(\min\{\sin(2a),\sin(a+\frac{\pi}{2})\}\right)^{d-1}\ dy\ \sigma_{d-1}(du) \\
\geq&\frac{2}{\tilde{U}(C)}\cdot\omega_{d-1}\cdot\left(\frac{2}{\pi}\right)^{d-1}\cdot\frac{c_{22}(a,d)^{d-1}}{d-1}\cdot\varepsilon^{d-1}\cdot\left(\min\{\sin(2a),\sin(a+\frac{\pi}{2})\}\right)^{d-1}\cdot\frac{\varepsilon}{2} \\
=:&\tilde{c}_{20}(a,d)\cdot\varepsilon^d,
\end{aligned}
$$

where we used $a+\varepsilon\leq\pi/2$. Defining

$$
c_{20}(a,d) := \min\left\{\tilde{c}_{20}(a,d),\left(\frac{2}{\pi}\right)^d\right\},
$$

we get

$$
\tilde{U}(K)\geq\tilde{U}(C)\cdot(1+c_{20}(a,d)\cdot\varepsilon^d),
$$

where the constant $c_{20}(a,d)>0$ depends only on a and the dimension d. The minimum was taken in order to ensure $c_{20}\cdot\varepsilon^d\leq 1$. □

4.6. The asymptotic shape of the typical cell of a spherical Poisson–Voronoi Tessellation

Before we can apply the techniques from Lemma 3.4.1 and prove a result similar to Theorem 3.2.1, we need to show the continuity of $\tilde{U}$ and an approximation result for spherical polytopes, similar to Lemma 3.3.6

Lemma 4.6.1. *The functional $\tilde{U}$ is continuous on $\overline{\mathcal{K}}_s^d$ with respect to the spherical Hausdorff distance.*

Proof. Let $K, L \in \overline{\mathcal{K}}_s^d$ such that $\delta_s(K,L) \le \varepsilon$. Without loss of generality, we assume $\tilde{U}(K) \ge \tilde{U}(L)$. Then we obtain

$$\begin{aligned}
|\tilde{U}(K) - \tilde{U}(L)| = 2 \cdot \Bigg(&\int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2d_s(\tilde{S}_u, t)) \cdot \mathbf{1}\{t^{\perp} \cap K \neq \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \\
&- \int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \int_{A_s(u)} \sin^{d-1}(2d_s(\tilde{S}_u, t)) \cdot \mathbf{1}\{t^{\perp} \cap L \neq \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \Bigg) \\
&= 2 \cdot \int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \int_{A_s(u)} \big[\mathbf{1}\{t^{\perp} \cap K \neq \emptyset\} - \mathbf{1}\{t^{\perp} \cap L \neq \emptyset\}\big] \\
&\qquad \times \sin^{d-1}(2d_s(\tilde{S}_u, t))\ \sigma_1(dt)\ \sigma_{d-1}(du) \\
&\le 2 \cdot \int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \int_{A_s(u)} \mathbf{1}\{t^{\perp} \cap K \neq \emptyset\} \cdot \mathbf{1}\{t^{\perp} \cap L = \emptyset\} \\
&\qquad \times \sin^{d-1}(2d_s(\tilde{S}_u, t))\ \sigma_1(dt)\ \sigma_{d-1}(du) \\
&\le 2 \cdot \int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \int_{A_s(u)} \mathbf{1}\{t^{\perp} \cap K \neq \emptyset\}\mathbf{1}\{t^{\perp} \cap L = \emptyset\}\ \sigma_1(dt)\ \sigma_{d-1}(du) \\
&\le 2 \int_{\overline{0}^{\perp}\cap\mathbb{S}^d} \varepsilon\ \sigma_{d-1}(du) = 2 \cdot \varepsilon \cdot \omega_d,
\end{aligned}$$

where we used $\delta_s(K,L) \le \varepsilon$ in the last line. □

Lemma 4.6.2. *Let $\alpha > 0$ and $0 < a < \pi/2$. Then there is an integer $\nu = \nu(\alpha, a, d) \in \mathbb{N}$, such that for every spherical polytope P with $r_s(P) \ge a$ there is a spherical polytope $Q = Q(P)$ satisfying* $\operatorname{ext}(Q) \subset \operatorname{ext}(P)$ *and $f_0(Q) \le \nu$ as well as*

$$\tilde{U}(Q) \ge (1-\alpha)\tilde{U}(P).$$

Furthermore, the mapping $P \mapsto Q(P)$ can be chosen to be measurable.

Proof. Since $\tilde{U}$ is continuous with respect to the spherical Hausdorff distance (Lemma 4.6.1), the assertions follow analogously to the proof of Lemma 3.3.6. □

Lemma 4.6.3. *Let $0 < a < \pi/2$ and let $\varepsilon > 0$ be such that $a + \varepsilon \le \pi/2$. Let X be an isotropic Poisson process with intensity γ_S such that $\gamma_S \omega_{d+1} > 1$ and*

$$\tilde{\mathcal{K}}_{a,\varepsilon} := \{K \in \overline{\mathcal{K}}_s^d : r_s(K) \in [a, \pi/2],\ \vartheta^{\overline{0}}(K) \ge \varepsilon\}.$$

Then

$$\mathbb{P}(Z \in \tilde{\mathcal{K}}_{a,\varepsilon}) \le c_{23} \cdot (\gamma_S \omega_{d+1})^{d\cdot\nu} \cdot \exp\left(-\gamma_S \left(1 + \frac{c_{20} \cdot \varepsilon^d}{3}\right) \cdot \sigma_d(B_s(\overline{0}, 2a))\right), \qquad (4.6.1)$$

where the constants ν und c_{20} depend only on a and d and the constant c_{23} depends only on a, d and ε.

Proof. We proceed as in the proof of Lemma 3.4.1. Let $\tilde{\Phi}(\cdot) := \frac{1}{\omega_{d+1}}\tilde{U}(\cdot)$ and $N \in \mathbb{N}$.

For $H_1, \dots, H_N \in G(d+1,d) \cap \mathbb{S}^d$, we define $H_{(N)} := (H_1, \dots, H_N)$ and let $P(H_{(N)})$ denote the spherical Crofton cell of the tessellation induced by $H_1, \dots, H_N$. Let $H_1, \dots, H_N$ be such that $P(H_{(N)}) \in \tilde{\mathcal{K}}_{a,\varepsilon}$ and define $\alpha := c_{20} \cdot \varepsilon^d/(2 + c_{20} \cdot \varepsilon^d)$. Then $\alpha \geq c_{20} \cdot \varepsilon^d/3$ and $(1-\alpha)(1 + c_{20} \cdot \varepsilon^d) = 1 + \alpha$. Due to Lemma 4.6.2 and (4.5.6), there are at most $\nu = \nu(a,d,\varepsilon)$ vertices of $P(H_{(N)})$ such that the spherically convex hull $Q(H_{(N)})$ of these vertices satisfies

$$
\begin{aligned}
\tilde{\Phi}(Q(H_{(N)})) &\geq (1-\alpha)\tilde{\Phi}(P(H_{(N)})) \\
&\geq (1-\alpha)(1 + c_{20} \cdot \varepsilon^d)\tilde{\Phi}(B_s(\overline{0}, a)) \\
&= \frac{1+\alpha}{\omega_{d+1}} \sigma_d(B_s(\overline{0}, 2a)).
\end{aligned}
$$

In order to complete the proof as we did for Lemma 3.4.1, we need to show that N $(d-1)$-dimensional great subspheres are $\tilde{\mu}^N$-almost surely in general position. Recalling

$$
\tilde{\mu}(\cdot) = \frac{1}{\omega_{d+1}} \int_{\mathbb{S}^d} \mathbf{1}\{f(x)^\perp \cap \mathbb{S}^d \in \cdot\} \, \sigma_d(dx)
$$

we will show that $\tilde{\mu}$ is absolutely continuous with respect to the isotropic measure

$$
\mu(\cdot) = \frac{1}{\omega_{d+1}} \int_{\mathbb{S}^d} \mathbf{1}\{x^\perp \cap \mathbb{S}^d \in \cdot\} \, \sigma_d(dx),
$$

which was defined in Section 3.2 and played a similar part in the proof of Lemma 3.4.1. Using the same arguments as we used to obtain (4.5.9), we get

$$
\begin{aligned}
\tilde{\mu}(\cdot) &= \frac{2}{\omega_{d+1}} \int_{\mathbb{S}^d \cap \overline{0}^\perp} \int_{A_s(u)} \sin^{d-1}(2d_s(\tilde{S}_u, t)) \cdot \mathbf{1}\{t^\perp \in \cdot\} \, \sigma_1(dt) \, \sigma_{d-1}(du) \\
&= \frac{2}{\omega_{d+1}} \int_{\mathbb{S}^d \cap \overline{0}^\perp} \int_{\{\overline{0}, -\overline{0}\} \vee u} \sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, t)) \cdot \frac{\sin^{d-1}(2d_s(\tilde{S}_u, t))}{\sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, t))} \\
&\qquad \times \mathbf{1}\{t \in A_s(u)\}\mathbf{1}\{t^\perp \in \cdot\} \, \sigma_1(dt) \, \sigma_{d-1}(du) \qquad (4.6.2)
\end{aligned}
$$

For $t \in A_s(u)$, we have $d_s(\{\overline{0}, -\overline{0}\}, t) = d_s(-\overline{0}, t) \in [0, \pi/2]$. Assuming $d_s(-\overline{0}, t) \in [\pi/4, \pi/2]$, we obtain the following upper bound

$$
\begin{aligned}
\max_t \frac{\sin^{d-1}(2d_s(\tilde{S}_u, t))}{\sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, t))} &= \max_{x \in [\frac{\pi}{4}, \frac{\pi}{2}]} \frac{\sin^{d-1}(2(\pi/2 - x))}{\sin^{d-1}(x)} \\
&= \max_{x \in [\frac{\pi}{4}, \frac{\pi}{2}]} \left(\frac{\sin(\pi - 2x)}{\sin(x)}\right)^{d-1} \\
&\leq \left(\frac{1}{\sin(\frac{\pi}{4})}\right)^{d-1} = \left(\sqrt{2}\right)^{d-1}.
\end{aligned}
$$

For $t \in A_s(u)$ and $d_s(-\overline{0}, t) \in (0, \pi/4]$, we have

$$
\sup_t \frac{\sin^{d-1}(2d_s(\tilde{S}_u, t))}{\sin^{d-1}(d_s(\{\overline{0}, -\overline{0}\}, t))} = \sup_{x \in (0, \frac{\pi}{4}]} \left(\frac{\sin(2x)}{\sin(x)}\right)^{d-1}
$$

$$\leq \sup_{x\in(0,\frac{\pi}{4}]} \left(\frac{2x}{1/\sqrt{2}x}\right)^{d-1}$$
$$\leq (2\sqrt{2})^{d-1}.$$

Using these upper bounds in (4.6.2), we conclude

$$\begin{aligned}\tilde{\mu}(\cdot) &\leq \frac{2}{\omega_{d+1}} \int_{\mathbb{S}^d\cap\overline{0}^\perp} \int_{\{\overline{0},-\overline{0}\}\vee u} \sin^{d-1}(d_s(\{\overline{0},-\overline{0}\},t)) \cdot (2\sqrt{2})^{d-1} \cdot \mathbf{1}\{t^\perp \in \cdot\}\ \sigma_1(dt)\ \sigma_{d-1}(du)\\ &= 2(2\sqrt{2})^{d-1}\cdot\mu(\cdot).\end{aligned}$$

This provides us with the required absolute continuity. Proceeding as in Lemma 3.4.1 and using $\gamma_S\omega_{d+1} > 1$, we obtain

$$\mathbb{P}(Z \in \tilde{\mathcal{K}}_{a,\varepsilon}) \leq c_{23}(a,\varepsilon,d)\cdot(\gamma_S\omega_{d+1})^{d\cdot\nu}\cdot\exp\left(-\gamma_S\omega_{d+1}\left(1+\frac{c_{20}\cdot\varepsilon^d}{3}\right)\cdot\tilde{\Phi}(B_s(\overline{0},a))\right).$$

Since

$$\tilde{\Phi}(B_s(\overline{0},a)) = 1/\omega_{d+1}\cdot\tilde{U}(B_s(\overline{0},a)) = \frac{\sigma_d(B_s(\overline{0},2a))}{\omega_{d+1}},$$

this completes the proof. □

Since our size functional is the spherical inball radius, centred at $\overline{0}$, we deduce, for $0 < a < \pi/2$,

$$\begin{aligned}\mathbb{P}(r_s(Z)\geq a) &= \exp\left(-\gamma_S\omega_{d+1}\tilde{\mu}(\mathcal{H}_{B_s(\overline{0},a)})\right)\\ &= \exp\left(-\gamma_S\omega_{d+1}\cdot\frac{\tilde{U}(B_s(\overline{0},a))}{\omega_{d+1}}\right)\\ &= \exp\left(-\gamma_S\cdot\sigma_d(B_s(\overline{0},2a))\right).\end{aligned}$$

Combining this result with Lemma 4.6.3, we obtain the following theorem for the asymptotic shape of the typical cell of a sphericall Poisson–Voronoi tessellation.

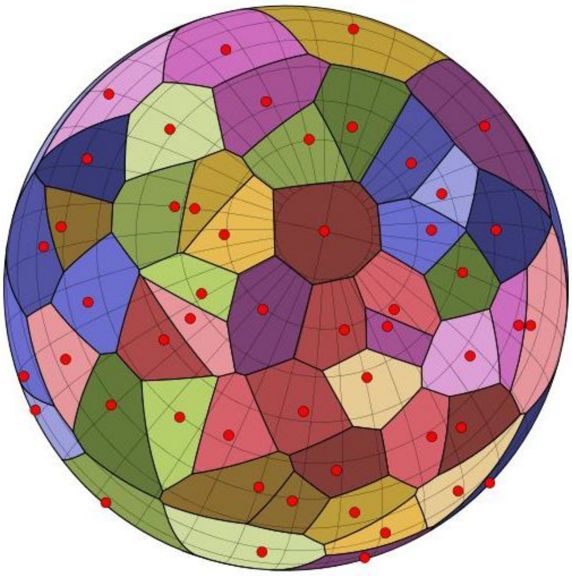
Theorem 4.6.4. *Let $0 < a < \pi/2$, $\varepsilon > 0$ with $a+\varepsilon \leq \pi/2$. Let X be a Poisson process on $\mathbb{S}^d$ with intensity γ_S such that $\gamma_S\omega_{d+1} > 1$. Then the typical cell Z of its spherical Poisson–Voronoi tessellation satisfies*

$$\mathbb{P}(R_s(Z) - r_s(Z) \geq \varepsilon | r_s(Z) > a) \leq c_{23}\cdot\exp\left(-\gamma_S\cdot c_{24}\cdot\varepsilon^d\right), \tag{4.6.3}$$

where the constant $c_{24} > 0$ depends only on a and d and the constant $c_{23} > 0$ depends only on a, d and ε.

The assumption $a+\varepsilon \leq \pi/2$ in the previous theorem can be replaced by the assumption $\varepsilon \leq \pi$. To see this, we first note that there is a constant $0 < c_{25} = c_{25}(a) < 1$ such that for $\tilde{\varepsilon} = \varepsilon\cdot c_{25}$ we have

$$a + \tilde{\varepsilon} \leq \frac{\pi}{2}.$$



Spherical Voronoi with added point at $\overline{0}$ and no points in $B_s(\overline{0}, 2a)$ [57]

Then, using Theorem 4.6.4, we get

$$\begin{aligned}\mathbb{P}(R_s(Z) - r_s(Z) \geq \varepsilon | r_s(Z) > a) &\leq \mathbb{P}(R_s(Z) - r_s(Z) \geq \tilde{\varepsilon} | r_s(Z) > a) \\ &\leq c_{23} \cdot \exp\left(-\gamma_S \cdot c_{24} \cdot (\tilde{\varepsilon})^d\right) \\ &= c_{23} \cdot \exp\left(-\gamma_S \cdot c_{24} \cdot c_{25}^d \cdot \varepsilon^d\right).\end{aligned}$$

Since $c_{25} < 1$, this estimate is in fact worse than the estimate in Theorem 4.6.4. The assumption $\varepsilon \leq \pi$ is more natural in the sense that for $R_s(K) > \pi$, the set $K \subset \mathbb{S}^d$ cannot be contained in a hemisphere and thus K cannot be a spherically convex set.

A. Simulation code in R

Methods for generating random points on the unit sphere can be found e.g. in [30]. In order to simulate uniformly distributed great circles on $\mathbb{S}^2$, we simulate the normal vectors which are uniformly distributed points on $\mathbb{S}^2$. The following code was used to simulate Figure 3.1 at the beginning of Section 3.2

```
### x,y,z standard-normal distributed, then normalise the vector (x,y,z)
### lambda=gamma_s*omega_{d+1}=gamma*4*pi
gamma=1
n=rpois(1,gamma*4*pi)
M = matrix(0, nrow=n, ncol=3)
k=0
print(n)
while(k<n)
{
x=rnorm(1,0,1)
y=rnorm(1,0,1)
z=rnorm(1,0,1)

s2=x^2+y^2+z^2
s=sqrt(s2)

x=x/s
y=y/s
z=z/s

print(x^2+y^2+z^2)

M[k+1,1]=x
M[k+1,2]=y
M[k+1,3]=z
k=k+1
}
```

The second block of code was used to simulate Figure 3.2 to Figure 3.5. Let $x \in \mathbb{S}^d$ and $0 < a < \pi/2$. Note that $x^\perp$ does not hit $B_s(\overline{0}, a)$ if and only if

$$d_s(\overline{0}^\perp, x) \geq a \;\Leftrightarrow\; x_3 \geq \sin(a).$$

```
### x,y,z standard-normal distributed, then normalise the vector (x,y,z)
##  lambda=gamma_s*omega_{d+1}=gamma*4*pi
```

```
##  a=minimal inballradius  (0,pi/2)
a=pi/8
gamma=6
n=rpois(1,gamma*4*pi)
M = matrix(0, nrow=n, ncol=3)
k=0
print(n)
while(k<n)
{
x=rnorm(1,0,1)
y=rnorm(1,0,1)
z=rnorm(1,0,1)

s2=x^2+y^2+z^2
s=sqrt(s2)

x=x/s
y=y/s
z=z/s

if(z < -sin(a))
{
M[k+1,1]=x
M[k+1,2]=y
M[k+1,3]=z

print(x^2+y^2+z^2)
print(k)

k=k+1
}
if(z > sin(a))
{
M[k+1,1]=x
M[k+1,2]=y
M[k+1,3]=z

print(x^2+y^2+z^2)
print(k)

k=k+1
}
}
```

This algorithm can be justified as follows. Let $N \in \mathbb{N}$, $a > 0$ and let $Z_1, Z_2, \ldots$ be independent and identically distributed with $Z_1 \sim \sigma_d^0(\cdot)$. Let X be an isotropic Poisson process on $\mathbb{S}^d$ with positive and finite intensity. For $a > 0$ denote

$$M(a) := \{x \in \mathbb{S}^d : d_s(x, \overline{0}^{\perp}) \leq a\}.$$

Using [49, Theorem 3.2.2 (b)], we obtain

$$\begin{aligned}
\mathbb{P}(X \in \cdot \,|X(M(a)) = 0, X(\mathbb{S}^d) = N) &= \frac{\mathbb{P}(X \in \cdot\,, X(M(a)) = 0, X(\mathbb{S}^d) = N)}{\mathbb{P}(X(M(a)) = 0, X(\mathbb{S}^d) = N)} \\
&= \frac{\mathbb{P}(X \in \cdot\,, X(M(a)) = 0 | X(\mathbb{S}^d) = N)}{\mathbb{P}(X(M(a)) = 0 | X(\mathbb{S}^d) = N)} \\
&= \frac{\mathbb{P}(\sum_{i=1}^N \delta_{Z_i} \in \cdot\,, Z_1 \notin M(a), \ldots, Z_N \notin M(a))}{\mathbb{P}(Z_1 \notin M(a), \ldots, Z_N \notin M(a))} \\
&= \mathbb{P}\left(\sum_{i=1}^N \delta_{Z_i} \in \cdot \,\middle|\, Z_1 \notin M(a), \ldots, Z_N \notin M(a) \right).
\end{aligned}$$

Bibliography

[1] Amelunxen, D. (2015). *Measures on polyhedral cones: characterizations and kinematic formulas.* http://arxiv.org/abs/1412.1569v2.

[2] Arbeiter, E., Zähle, M. (1992). *Geometric measures for random mosaics in spherical spaces.* Stochastics and Stochastics Reports, Vol. 46, 63–77.

[3] Auneau, J., Vedel Jensen, E., Rataj, J. (2012). *Closed form of the rotational Crofton formula.* Math. Nachr. 285, No. 23, 164–180.

[4] Baumstark, V., Last, G. (2007). *Some distributional results for Poisson–Voronoi Tessellations.* Advances in Applied Probability, 39, 16–40.

[5] Baumstark, V., Last, G. (2009) *Gamma distributions for stationary Poisson flat processes.* Advances in Applied Probability, 41, 911–939.

[6] Bridson, M.R., Haefliger, A. (1999). *Metric Spaces of Non-Positive Curvature.* Springer, Berlin

[7] Bronshteyn, E.M., Ivanov, L.D. (1975). *The approximation of convex sets by polyhedra.* Siberian Math. J. 16: 852–853.

[8] Chiu, S.N., Stoyan, D., Kendall, W.S. and Mecke, J. (2013). *Stochastic Geometry and its Applications.* Third Edition, Wiley, Chichester.

[9] Cover, T. M., Efron, B. (1967). *Geometrical probability and random points on a hypersphere.* Annals of Mathematical Statistics, 38: 213–220.

[10] Dekster, B.V., Wilker, J.B. (1991). *Simplexes in spaces of constant curvature.* Geometriae Dedicata, Vol. 38, Issue 1, 1–12

[11] Dekster, B.V. (1995). *The Jung Theorem for spherical and hyperbolic spaces.* Acta Mathematica Hungarica, Vol. 67, Issue 4, 315–331.

[12] Gao, F., Hug, D., Schneider, R. (2003). *Intrinsic volumes and polar sets in spherical space.* Math. Notae 41 (2001/02), 159–176.

[13] Gentner, D. S. (2011). *Palm theory, mass transports and ergodic theory for group-stationary processes.* KIT Scientific Publishing, Karlsruhe.

[14] Glasauer, S. (1995). *Integralgeometrie konvexer Körper im sphärischen Raum.* Doctoral dissertation, Freiburg i. Br.

[15] Hewitt, E., Stromberg, K. (1965). *Real and Abstract Analysis.* Springer-Verlag New York.

[16] Hug, D. , Last, G., Weil, W. (2006). *Polynomial parallel volume, convexity and contact distributions of random sets.* Probab. Theory and Relat. Fields 135, 169–200.

[17] Hug, D., Reitzner, M., Schneider, R. (2004). *The limit shape of the zero cell in a stationary Poisson hyperplane tessellation.* The Annals of Probability, 32, No 1B, : 1140–1167.

[18] Hug, D., Reitzner, M., Schneider, R. (2004). *Large Poisson–Voronoi cells and Crofton cells.* Adv. Appl. Prob. 36, 667–690.

[19] Hug, D., Schneider, R. (2005). *Large Typical Cells in Poisson-Delaunay Mosaics.* Rev. Roumaine Math. Pures Appl. 50, 657–670.

[20] Hug, D., Schneider, R. (2007). *Asymptotic Shapes of Large Cells in Random Tessellations.* Geom. Funct. Anal. 17: 156–191.

[21] Hug, D., Schneider, R. (2010). *Large faces in Poisson hyperplane mosaics.* The Annals of Probability, Vol. 38, No. 3, 1320–1344.

[22] Hug, D., Schneider, R. (2011). *Faces of Poisson–Voronoi mosaics.* Probability Theory and Related Fields, Vol. 151, 125–151.

[23] Hug, D., Schneider, R. (2016). *Random conical tessellations.* Discrete & Computational Geometry, 1–32.

[24] Vedel Jensen, E., Rataj, J. (2008). *A rotational integral formula for intrinsic volumes.* Advances in Applied Mathematics, 41, 530–560.

[25] Kallenberg, O. (2002). *Foundations of Modern Probability.* Second Edition, Springer, New York.

[26] Kovalenko, I.N. (1997). *A proof of a conjecture of David Kendall on the shape of a random polygons of large area* (Russian). Kibernetika i Sistemnyi Analiz, No. 4, pp. 3-10; Engl. transl. Cybernetics and Systems Analysis, Vol. 33, Issue 4, 461–467.

[27] Kovalenko, I.N. (1998). *An extension of a conjecture of D.G. Kendall concerning shapes of random polygons to Poisson–Voronoi cells.* in Voronois Impact on Modern Science (Engel, P. et al., eds.), Book I. Transl. from the Ukrainian, Kyiv: Institute of Mathematics. Proc. Inst. Math. Natl. Acad. Sci. Ukr., Math. Appl., Vol. 212, 266-274.

[28] Kovalenko, I.N. (1999). *A simplified proof of a conjecture of D.G. Kendall concerning shapes of random polygons.* Journal of Applied Mathematics and Stochastic Analysis, Vol. 12, 301–310.

[29] Last, G. (2010). *Stationary random measures on homogeneous spaces.* Journal of Theoretical Probability 23, 478–497.

[30] Marsaglia, G. (1972). *Choosing a point from the surface of a sphere.* The Annals of Mathematical Statistics, Vol. 43, No. 2, 645–646.

[31] Matheron, G. (1975). *Random Sets and Integral Geometry.* Wiley, New York.

[32] Mecke, J. (1967). *Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen.* Z. Wahrsch. verw. Gebiete, 9, 36–58.

[33] Miles, R. (1961). *Random polytopes: the generalisation to n dimensions of the intervals of a Poisson process.* Ph.D. Thesis, Cambridge University

[34] Miles, R.E. (1971). *Random points, sets and tessellations on the surface of a sphere.* Sankhyā: The Indian Journal of Statistics, Series A, Vol 33, No. 2, 145–174.

[35] Miles, R. (1972). *The random division of space.* Advances in Applied Probability, Vol. 4, Suppl., 243–266.

[36] Miles, R. (1974). *A synopsis of 'Poisson flats in Euclidean spaces'.* In *Stochastic Geometry.* ed. E. F. Harding and D. G. Kendall, Wiley, New York.

[37] Miles, R. (1995). *A heuristic proof of a long-standing conjecture of D.G. Kendall concerning the shapes of certain large random polytopes.* Advances in Applied Probability, Vol. 27, 397–417.

[38] Møller, J. (1989). *Random Tessellations in* $\mathbb{R}^d$. Advances in Applied Probability, 21, 37–73.

[39] Møller, J. (1994). *Lectures on Random Voronoi Tessellations.* Lecture Notes in Statistics, 87, Springer, New-York.

[40] Møller, J., Stoyan, D. (2007). *Stochastic geometry and random tessellations.* Research Report R-2007-28, Departement of Mathematical Sciences, Aalborg University

[41] Muche, L. (2005). *The Poisson–Voronoi tessellation: relationships for edges.* Advances in Applied Probability, 37, 279–296.

[42] Neveu, J. (1977). *Processus ponctuels.* École d'Eté de Probabilités de Saint-Flour VI. Lecture Notes in Mathematics, 598, 249–445, Springer, Berlin.

[43] Okabe, A., Boots, B., Sugihara, K., Nok Chiu, S. (2000). *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* Second Edition, Wiley, New York.

[44] Renka, R. (1997). *Algorithm 772: STRIPACK: Delaunay triangulation and Voronoi diagram on the surface of a sphere.* ACM Transactions on Mathematical Software, Vol. 23, Issue 3, 416–434.

[45] Rockafellar, R. T. (1997). *Convex analysis.* 10. print. and 1. paperb. print., Princeton, NJ : Princeton Univ. Press.

[46] Rother, W., Zähle, M. (1990). *Palm distributions in homogeneous spaces.* Math. Nachr, 149, 255–263.

[47] Santaló, L.A. (1946). *Convex Regions on the n-Dimensional Spherical Surface.* Annals of Mathematics, 2nd Series, Vol. 47 No. 3, 448–459.

[48] Schläfli, L. (1950). *Gesammelte Mathematische Abhandlungen I.* Birkhäuser, Basel.

[49] Schneider, R., Weil, W. (2008). *Stochastic and Integral Geometry.* Springer, Berlin.

[50] Sugihara, K. (2002). *Laguerre Voronoi diagram on the sphere.* Journal for Geometry and Graphics, Vol. 6, No. 1, 69–81.

[51] Tanemura, M. (1998). *Random packing and tessellation network on the sphere.* Forma, Vol. 13, No. 2, 99–121.

[52] Webster, R. (1994). *Convexity.* Oxford University Press.

[53] Yoshino, T., Matsuoka, A., Kurihara, T., Ishida, N., Kishimoto, N., Kimoto, K., Matsuura, S. (2012). *Application of Voronoi tessellation of spherical surface to geometrical models of skeleton forms of spherical radiolaria.* Forma, Vol. 27, 45–53.

[54] Zähle, M. (1982). *Random processes of Hausdorff rectifiable closed sets.* Mathematische Nachrichten, 108, 49–72.

[55] Zeidler, E. (ed.) (2013). *Springer-Taschenbuch der Mathematik, 3rd revised and extended Edition.* Springer Spektrum, Wiesbaden

[56] Bei, Z. http://www.senchalabs.org/philogl/PhiloGL/examples/voronoi/

[57] Davies, J. https://www.jasondavies.com/maps/voronoi/

[58] Bingham, N. J. (1 November 2007). http://www.independent.co.uk/news/obituaries/professor-david-kendall-398453.html