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Abstract

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On Newton polytopes and growth properties of multivariate polynomials

by Dipl.-Math. Tomáš Bajbar

Many interesting properties of polynomials are closely related to the geometry of their Newton polytopes. In this dissertation thesis, we analyze the growth properties on $\mathbb{R}^n$ of multivariate polynomials $f \in \mathbb{R}[x]$ in terms of their so-called Newton polytopes at infinity. In fact, we introduce the broad class of so-called gem regular polynomials and characterize their coercivity via conditions solely containing information about the geometry of the vertex set of the Newton polytope at infinity, as well as sign conditions on the corresponding polynomial coefficients. For all other polynomials, the so-called gem irregular polynomials, we introduce sufficient conditions for coercivity based on those from the regular case. For some special cases of gem irregular polynomials, we establish necessary conditions for coercivity, too. We further introduce a stability concept for the coercivity of multivariate polynomials $f \in \mathbb{R}[x]$. In particular, we consider perturbations of f by polynomials up to the so-called degree of stable coercivity, and we analyze this stability concept in terms of the corresponding Newton polytopes at infinity. For coercive polynomials $f \in \mathbb{R}[x]$ we also introduce the order of coercivity as a measure expressing the order of growth of f, and we identify a broad class of multivariate polynomials $f \in \mathbb{R}[x]$ for which the order of coercivity and the degree of stable coercivity coincide. For these polynomials we give a geometric interpretation of this phenomenon in terms of their Newton polytopes at infinity, which we call the degree of convenience. Finally, we analyze the global diffeomorphism property of real polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ by studying the properties of the Newton polytopes at infinity corresponding to the sum of squares polynomials $\|F\|_2^2$. This allows us to identify a class of polynomial maps F for which their global diffeomorphism property on $\mathbb{R}^n$ is equivalent to their Jacobian determinant $\det JF$ vanishing nowhere on $\mathbb{R}^n$. In other words, we identify a class of polynomial maps for which the Real Jacobian Conjecture, which was proven to be false in general, still holds. We show some applications of our results, relate them to the existing literature and illustrate them with several examples.

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Dedicated to my family

Chapter 1

Introduction

It is an interesting question in polynomial optimization theory whether a given multivariate polynomial f attains its infimum on $\mathbb{R}^n$ or, more generally, on some non-compact basic semi-algebraic set $S \subseteq \mathbb{R}^n$. In fact, our subsequent studies, which culminated in the series of three articles [4–6] this dissertation thesis is based on, are motivated by the following statement from [65, Section 7] which is also cited in [76, 84]:

> 'This paper proposes a method for minimizing a multivariate polynomial $f(x)$ over its gradient variety. We assume that the infimum $f^\star$ is attained. This assumption is non-trivial, and we do not address the (important and difficult) question of how to verify that a given polynomial $f(x)$ has this property.'

Coercivity of a polynomial f on $\mathbb{R}^n$, that is, the property $f(x) \to +\infty$ holding whenever $\|x\| \to +\infty$ with some norm $\|\cdot\|$ defined on $\mathbb{R}^n$, is a sufficient condition for f having the above mentioned property. In fact, since a polynomial f is (lower semi-) continuous on $\mathbb{R}^n$, its coercivity implies the existence of a globally minimal point of f on $\mathbb{R}^n$ or over any nonempty closed subset of $\mathbb{R}^n$. It is, thus, an interesting problem how to verify or disprove that a given polynomial f is coercive on $\mathbb{R}^n$. As a further consequence of coercivity, f is bounded below on $\mathbb{R}^n$ by some $v \in \mathbb{R}$, so that the polynomial $f - v$ is positive semi-definite on $\mathbb{R}^n$. Since the coercivity of f is equivalent to the boundedness of its lower level sets $\{x \in \mathbb{R}^n |\ f(x) \leq \alpha\}$ for all $\alpha \in \mathbb{R}$, appropriate coercivity conditions are also useful as a tool for analyzing the boundedness property of basic semi-algebraic sets. Moreover, coercivity of a polynomial f implies the boundedness of trajectories of the polynomial gradient system $\dot{x} = -\nabla f(x)$, and properness of a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^m$ is equivalent to the coercivity of the sum of squares polynomial $\|F\|_2^2$. The latter is of crucial importance for obtaining results concerning the global diffeomorhism property of F which we shall present later in this thesis.

Before giving a brief overview of the present thesis, we first introduce some basic notation we shall use and we also define the Newton polytope (at infinty) - the crucial geometric object our results and analysis are based on.

For $f \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$, the ring of polynomials with real coefficients in n variables, we write $f(x) = \sum_{\alpha \in A(f)} f_\alpha x^\alpha$ with $A(f) \subseteq \mathbb{N}_0^n$, $f_\alpha \in \mathbb{R}$ for $\alpha \in A(f)$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$. We will assume that the set $A(f)$ is chosen minimally in the sense that $A(f) = \{\alpha \in \mathbb{N}_0^n | f_\alpha \neq 0\}$ holds. The degree of f is defined as $\deg(f) = \max_{\alpha \in A(f)} |\alpha|$ with $|\alpha| = \sum_{i=1}^n \alpha_i$.

In this thesis we will relate some growth properties of f with properties of the so-called Newton polytope at infinity (cf., e.g., [21])

$$\mathrm{New}_\infty(f) \ := \ \mathrm{conv}(A(f) \cup \{0\})$$

of f, that is, the convex hull of $A(f)$ and the origin. Note that the origin corresponds to the constant term of f which obviously is unrelated to coercivity. Hence, in the presence of a nonzero constant term the Newton polytope at infinity coincides with the usual Newton polytope $\mathrm{New}(f) = \mathrm{conv}\, A(f)$, and in absence of a nonzero constant term the latter could artificially be introduced to make the two concepts coincide, with no change in the coercivity behavior of f. In the following, however, we prefer to work with $\mathrm{New}_\infty(f)$ instead of $\mathrm{New}(f)$ to avoid the artificial assumption of the presence of a nonzero constant term. If no confusion is possible we will abbreviate the Newton polytope at infinity as $P := \mathrm{New}_\infty(f)$ and the set $A(f)$ as A.

Example 1.1. *The six-hump camel back function* $f(x) = x_1^2(4 - \frac{21}{10}x_1^2 + \frac{1}{3}x_1^4) + x_1x_2 + x_2^2(-4 + 4x_2^2)$ *yields* $A(f) = \{(2,0),(4,0),(6,0),(1,1),(0,2),(0,4)\}$ *with the corresponding sets* $\mathrm{New}(f)$ *and* $\mathrm{New}_\infty(f)$ *illustrated below.*

Various algebraic and analytic properties of polynomials are already well known to be encoded in the properties of their Newton polytopes. To name some of them, for example the number of isolated roots of n polynomial equations in n unknowns can be bounded by the (mixed) volumes of their Newton polytopes (cf., e.g., [43, 48, 78]), absolute irreducibility of a polynomial is implied by the indecomposability of its Newton polytope in the sense of Minkowski sums of polytopes [26], and there are also some results dealing with Newton polytopes in elimination theory [46]. There exists also an intimate connection between the properties of Newton polytopes and the so-called amoebas - objects, which can be viewed as images of zero sets of complex polynomials under a logarithm

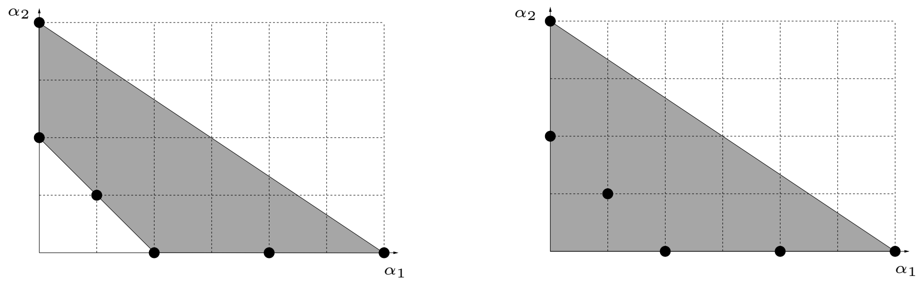

FIGURE 1.1: Illustration of Example 1.1. In the left picture the shaded area corresponds to the Newton polytope $\mathrm{New}(f)$ of the six-hump camel back function f. In the right picture the shaded area corresponds to the Newton polytope at infinity $\mathrm{New}_\infty(f)$ of the six-hump camel back function f. In both pictures the filled circles stand for the set $A(f)$ corresponding to f.

map [28, 58, 80]. These objects have recently been successfully used in various fields of mathematics [45, 80].

This thesis is structured as follows. In Chapter 2, which is based on the article [4], we analyse the coercivity property of multivariate polynomials $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ in terms of their Newton polytopes at infinity. Here we relate the coercivity property of multivariate polynomials to some properties of their Newton polytopes at infinity. In fact, we extract some useful and relevant information on the Newton polytope at infinity under study and we introduce the broad class of so-called gem regular polynomials and characterize their coercivity via three conditions (C1)–(C3) solely containing information about the geometry of the vertex set of the Newton polytope at infinity, as well as sign conditions on the corresponding polynomial coefficients. For all other polynomials, the so-called gem irregular polynomials, we introduce sufficient conditions for coercivity based on those from the regular case. For some special cases of gem irregular polynomials, we establish necessary conditions for coercivity and we also discuss the corresponding gap between the necessary and the sufficient conditions. Loosely speaking, considering our results, verifying or disproving the coercivity property of a given multivariate polynomial $f \in \mathbb{R}[x]$ can often be read off almost immediately from the coefficients corresponding to the vertices of the underlying Newton polytope at infinity $\mathrm{New}_\infty(f)$, which transforms the original problem into a more elegant, appealing one. We illustrate our results by various examples, which pave the way for their better understanding.

In the literature, there already exist results analyzing some other analytical properties of multivariate polynomials than coercivity via properties of the underlying Newton polytopes. For polynomials to be bounded from below, necessary conditions imposed on vertices of their Newton polytopes and on the corresponding coefficients are identified in [85]. These are in fact identical with our conditions (C1) and (C2) below (cf. Th. 2.8). This is not a coincidence due to the fact that every coercive polynomial is a polynomial bounded from below. Our additional condition (C3) can be viewed as a special condition for a polynomial being convenient (see, e.g., [21, 85] for the definition of convenient polynomials). In spite of these connections, in Chapter 2 we shall derive the conditions (C1)–(C3) with other proof techniques, mainly based on the application of theorems of the alternative, which allow us to develop also results in degenerate cases as well as sufficient conditions.

To mention some existing results on coercivity of polynomials, in [41, Section 3.2] the authors introduce a sufficient condition for coercivity on $\mathbb{R}^n$ of polynomials $f \in \mathbb{R}[x]$. On the one hand, this sufficient condition is computationally tractable, because it can be checked by solving a hierarchy of semi-definite programs. On the other hand, it is not satisfied by many coercive polynomials, as we shall show in Example 2.53. A simple reason for this effect is presented in Chapter 3 (see also [5]), where we prove that the sufficient condition from [41] actually characterizes the stronger property of so-called stable coercivity of gem regular polynomials, a concept which we first introduce later in Chapter 3.

The coercivity of polynomials in the convex setting is partially analyzed in [42], while the coercivity of a polynomial f defined on a basic semi-algebraic set S and its relation to the Fedoryuk and Malgrange conditions are examined in [84]. In [84, Th. 4.2] the authors prove that under the assumption of f being bounded from below on S, the Malgrange or the Fedoryuk conditions ([84, Defs. 4.2, 4.3]), characterize the coercivity of f on S.

Since our motivation for investigating the coercivity property of multivariate polynomials is driven by the fact that many numerical routines in polynomial optimization theory make the assumption that there exists a solution to a polynomial optimization problem under study (see, e.g., [65]), it is worth mentioning, that, meanwhile, there also exist probabilistic algorithms based on symbolic computation (see, e.g., [31, 32]) which allow to decide whether a given multivariate polynomial reaches its infimum over $\mathbb{R}^n$ without any assumptions and, in the latter case, the algorithm also computes a corresponding global minimal point. Under some additional regularity conditions the algorithm presented in [31] can even handle the case of polynomial equality constraints.

In Chapter 3, which is based on the article [5], we shall investigate the stability of the coercivity property of multivariate polynomials and its connection to their growth properties. To motivate this, we first consider the univariate case. A polynomial $f \in \mathbb{R}[x]$ then is called coercive on $\mathbb{R}$, if $f(x) \to +\infty$ holds for $|x| \to +\infty$. The latter is the case if and only if the leading coefficient of f is positive and the degree $\deg(f)$ of f is positive and even. This clearly is equivalent to the property $f(x)/|x|^q \to +\infty$ for all $|x| \to +\infty$ holding if and only if $q \in [0, \deg(f))$. Hence, for univariate coercive polynomials $f \in \mathbb{R}[x]$, the number $\deg(f)$ expresses how fast f grows on $\mathbb{R}$ and, thus, it can be viewed as a meaningful measure for the *order* of growth of f on $\mathbb{R}$. We call this number the *order of coercivity* of f.

It can be further observed that, in the univariate case, small perturbations of the coercive polynomial $f \in \mathbb{R}[x]$ by suitable univariate polynomials $g \in \mathbb{R}[x]$ preserve the coercivity property. In fact, considering a coercive polynomial $f \in \mathbb{R}[x]$ on $\mathbb{R}$, all perturbations of f to $f+g$ with polynomials $g \in \mathbb{R}[x]$ of degree not exceeding that of f, and with leading coefficient sufficiently close to zero, result in the coercivity of the univariate polynomial $f+g$ on $\mathbb{R}$. On the other hand, it is straightforward to see that small perturbations of a univariate coercive polynomial f to $f+g$ by polynomials g with degree exceeding that of f do not necessarily preserve the coercivity of $f+g$ on $\mathbb{R}$ and, thus, the number $\deg(f)$ can also be viewed as a measure expressing how *stable* the coercivity of f on $\mathbb{R}$ is. We call this number the *degree of stable coercivity* of f.

Hence, in the univariate case, the order of coercivity of a coercive polynomial $f \in \mathbb{R}[x]$ coincides with its degree of stable coercivity and both are equal to the number $\deg(f)$. The natural question arising in this context is whether the identity of these two numbers, after being properly defined in the multivariate setting, is also true for general multivariate coercive polynomials $f \in \mathbb{R}[x]$ and, if so, whether again these two numbers coincide with $\deg(f)$.

In Chapter 3, we will answer the first question affirmatively for a broad class of multivariate coercive polynomials, whereas we will provide a dissenting answer to the second question. More precisely, we will identify a broad class of multivariate coercive polynomials $f \in \mathbb{R}[x]$, for which the order of coercivity coincides with the degree of stable coercivity, and we will show that both are equal to a number which we shall call the *degree of convenience* of f and which, in general, differs from $\deg(f)$. This broad class of multivariate polynomials coincides with the class of all multivariate polynomials (both gem regular and irregular) for which our general sufficiency conditions for coercivity from Chapter 2 hold. It thus extends the results on coercive polynomials and their Newton polytopes at infinity obtained in Chapter 2. Interestingly, the degree of convenience of any multivariate polynomial $f \in \mathbb{R}[x]$ has a nice geometric interpretation with respect to the Newton polytope at infinity $\mathrm{New}_\infty(f)$ of f. Thus, analyzing the Newton polytope at infinity $\mathrm{New}_\infty(f)$ of a polynomial $f \in \mathbb{R}[x]$ enables one to explicitly determine the degree of convenience of f and, by the above considerations, also to directly compute the order of coercivity, or the degree of stable coercivity for a broad class of coercive polynomials. Finally, as applications of these results, we show that the gradient maps corresponding to a broad class of polynomials are always surjective, we establish Hölder type error bounds for such polynomials, and we link our results to the existence of solutions in the calculus of variations.

Chapter 4, which is based on the article [6], is motivated by an old and still interesting question of how to verify or disprove whether a given differentiable map $F : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible with a differentiable inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. In the present work, we shall call such maps global diffeomorphisms of $\mathbb{R}^n$ onto itself. The first well-known characterization of this global diffeomorphism property dates back to the work of Hadamard [33–35] and states that it is equivalent to the determinant $\det JF$ of the Jacobian matrix JF of F vanishing nowhere on $\mathbb{R}^n$, and to F being proper (cf. Th. 4.1 below). Here, F is called proper if preimages of compact sets under F always are compact.

In the case of complex polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$, the characterization of their global invertibility property directly refers to the celebrated Jacobian Conjecture from algebraic geometry, first formulated in [44] and asserting that if $\det JF$ is a nonzero constant function then F possesses a global polynomial inverse $F^{-1} : \mathbb{C}^n \to \mathbb{C}^n$. There exists a vast number of partial results on this conjecture where different approaches are used (see, e.g. [9, 10, 62, 71, 86, 89]). For more details on this open problem we refer to the survey papers [22, 82, 83, 88].

Following [52], in the setting of real polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$, the injectivity of F implies its surjectivity [10], and the global inverse F^{-1} of F is a polynomial if and only if $\det JF$ is a nonzero constant function [9]. If merely the existence, but not necessarily the polynomiality of the inverse map F^{-1} is sought for, one may conjecture that $\det JF$ vanishing nowhere on $\mathbb{R}^n$ implies the injectivity of F, and hence, also the existence of its global inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. This is the so-called "Real Jacobian Conjecture". It was, however, proven to be false by Pinchuk in [68], where a counterexample of a non-injective polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is constructed with $\det JF$ vanishing nowhere on $\mathbb{R}^2$.

Since, by Hadamard's above-mentioned theorem, the non-vanishing property of $\det JF$ actually is necessary for the global diffeomorphism property of F, it is thus an interesting question which additional conditions imposed on F, general enough, can assure that F is a global diffeomorphism of $\mathbb{R}^n$ onto itself. Answering this question, which is also posed by Bivià-Ausina in [11] and which is of significant importance in [15] as well, is the main motivation for Chapter 4.

From Hadamard's theorem it is clear that such additional conditions must be related to the properness of F. Since the latter properness may be characterized by the coercivity of the sum of squares polynomial $\|F\|_2^2$ as indicated above, sufficient conditions for the coercivity of polynomials will be the main tools used in Chapter 4. In fact, it will turn out that for a broad class of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ the coercivity of $\|F\|_2^2$ follows from $\det JF$ being nonzero on $\mathbb{R}^n$, so that at least for this class of polynomial maps the Real Jacobian Conjecture turns out to be true. This is the main result of Chapter 4.

We mention that, after Hadamard's fundamental contribution, further important results on the global diffeomorphism property were proved by Levy [53], Banach and Mazur [8], Caccioppoli [14], Plastock [69] and Rabier [70]. For a brief summary and further details see, e.g., [20, 30, 70]. It is worth mentioning that in mathematical economics, global invertibility properties of maps, as an object of interest, was originally highlighted in [74] and subsequently studied in [16, 25, 61]. The global invertibility of homogeneous maps, which are not smooth at the origin, is studied in [73].

In Chapter 5 we list some open problems related to our results and Appendix closes the present work.

Chapter 2

Coercive polynomials and their Newton polytopes

2.1 Chapter overview

This chapter is based on the article [4] and is structured as follows. In Section 2.2 we derive necessary conditions for coercivity of an arbitrary polynomial $f \in \mathbb{R}[x]$ which solely contain information about the geometry of the vertex set V of the Newton polytope at infinity P and sign conditions on the corresponding polynomial coefficients (Th. 2.8). Our technique of proof bases on the idea to evaluate f only along certain curves, which may be traced back at least to [72], see also [2, 64]. In Definition 2.18 we introduce the broad class of gem regular polynomials and show that the above analysis along curves cannot yield necessary conditions in addition to those stated in Theorem 2.8 in the gem regular case. For a special class of gem irregular polynomials, however, Theorem 2.29 states further necessary conditions for coercivity in terms of so-called circuit numbers (cf. [38]).

Section 2.3 deals with sufficient conditions for coercivity of f in terms of its Newton polytope at infinity. In Proposition 2.37 we prove that for gem regular polynomials the necessary conditions from Theorem 2.8 are in fact sufficient for coercivity. This leads to our main result, the Characterization Theorem 2.39 of coercivity for gem regular polynomials.

In Section 2.3.2 we formulate two sufficient conditions for coercivity for gem irregular polynomials (Ths. 2.41 and 2.44) in the spirit of those from the gem regular case and, again, containing information about the corresponding circuit numbers. Section 2.3.3 presents a simple connection between our sufficient conditions for coercivity and the

Fedoryuk and Malgrange conditions. In fact, in Corollary 2.51 we shall use some basic parts from the nontrivial results concerning characterization of coercivity of lower bounded polynomials via the Malgrange and Fedoryuk conditions from [84] to deduce that our Newton polytope type sufficient conditions for coercivity imply the Fedoryuk and Malgrange conditions. In Section 2.3.3 we also recall the concept of asymptotic and generalized critical values and we briefly mention their applications in polynomial optimization theory.

In Section 2.3.4 we show that, in contrast to our conditions, the sufficient condition for coercivity from [41, Section 3.2] cannot be verified for many coercive polynomials. For the explanation of the simple reason we relate the latter condition to the context of stable coercivity, which we analyze in more detail in Chapter 3. Throughout the whole Chapter 2, various illustrative examples are provided.

2.2 Necessary conditions for coercivity

2.2.1 Necessary sign conditions

Our derivation of necessary conditions for coercivity of f bases on a similar technique as presented in [2, 64, 72], that is, on evaluations of f along curves $\{x_{y,\beta}(t) |\, t \in \mathbb{R}\}$ with

$$x_{y,\beta}(t) \;:=\; (y_1 e^{\beta_1 t}, \ldots, y_n e^{\beta_n t})$$

and $y, \beta \in \mathbb{R}^n$ for $t \in \mathbb{R}$. We will often require that at least one entry of β is positive, that is, with $\mathbb{H} = \{h \in \mathbb{R} |\; h \geq 0\}$ we assume $\beta \in B := (-\mathbb{H}^n)^c$, where $(-\mathbb{H}^n)^c$ denotes the set $\mathbb{R}^n \setminus (-\mathbb{H}^n)$. As the vector β will act as a direction we could also restrict our attention to the case $\|\beta\| = 1$ but dispense with this for the ease of exposition. We abbreviate

$$I \;:=\; \{1, \ldots, n\}, \qquad Y \;:=\; \left\{ y \in \mathbb{R}^n \middle|\; \textstyle\prod_{i \in I} y_i \neq 0 \right\},$$

as well as

$$\Omega \;:=\; Y \times B.$$

Lemma 2.1. *Any* $(y, \beta) \in \Omega$ *satisfies* $\lim_{t \to \infty} \|x_{y,\beta}(t)\| = +\infty$.

Proof. In the case that $\|\cdot\|$ coincides with the ℓ_∞-norm $\|\cdot\|_\infty$ we obtain for any $(y, \beta) \in \Omega$

$$\lim_{t \to \infty} \|x_{y,\beta}(t)\|_\infty \;=\; \lim_{t \to \infty} \max_{i \in I} |y_i| e^{\beta_i t} \;=\; +\infty.$$

The equivalence of any norm $\|\cdot\|$ with $\|\cdot\|_\infty$ thus yields the assertion. $\square$

Next, for $f \in \mathbb{R}[x]$, $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t \in \mathbb{R}$ we define the one-dimensional restriction of f to the curve given by $x_{y,\beta}$,

$$\pi_f(y, \beta, t) \;:=\; f(x_{y,\beta}(t)) \;=\; \sum_{\alpha \in A} f_\alpha\, y^\alpha\, e^{\langle \alpha, \beta \rangle t},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, as well as

$$\Omega_f \;:=\; \{(y, \beta) \in \mathbb{R}^n \times \mathbb{R}^n |\; \lim_{t \to \infty} \pi_f(y, \beta, t) = +\infty\}.$$

Lemma 2.1 then immediately yields the following result.

Lemma 2.2. *The coercivity of $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ implies $\Omega \subseteq \Omega_f$.*

For any $\beta \in \mathbb{R}^n$ let us consider the optimization problem to maximize $\langle \alpha, \beta \rangle$ over the set $A_0 := A \cup \{0\}$, and denote the optimal value and the optimal point set of the latter problem by

$$d(\beta) \;:=\; \max_{\alpha \in A_0} \langle \alpha, \beta \rangle$$

and

$$A_0(\beta) \;:=\; \{\alpha \in A_0 |\; \langle \alpha, \beta \rangle = d(\beta)\},$$

respectively. Note that $d(\beta) \geq 0$ holds for all $\beta \in \mathbb{R}^n$ due to $0 \in A_0$ and that, as the all ones vector $\mathbb{1} \in \mathbb{R}^n$ satisfies $\langle \alpha, \mathbb{1} \rangle = |\alpha|$, we may write $\deg(f) = d(\mathbb{1})$.

For $f \in \mathbb{R}[x]$ and $\beta \in \mathbb{R}^n$ we define the auxiliary polynomial

$$f^\beta(x) \;:=\; \sum_{\alpha \in A_0(\beta)} f_\alpha\, x^\alpha$$

for $x \in \mathbb{R}^n$.

Proposition 2.3. *The inclusion $\Omega \subseteq \Omega_f$ implies the following assertions:*

a) For all $\beta \in B$ we have $d(\beta) > 0$.

b) For all $\beta \in B$ the polynomial f^β is positive semi-definite on $\mathbb{R}^n$.

Proof. For the proof of part a) assume that $d(\beta) \leq 0$ holds for some $\beta \in B$. Then all $\alpha \in A$ satisfy $\langle \alpha, \beta \rangle \leq d(\beta) \leq 0$ so that

$$\pi_f(\mathbb{1}, \beta, t) \;=\; \sum_{\alpha \in A} f_\alpha\, e^{\langle \alpha, \beta \rangle t}$$

is, as a function of t, bounded for $t \to \infty$. On the other hand, we have $(\mathbb{1}, \beta) \in \Omega$, so that the assumption $\Omega \subseteq \Omega_f$ implies $\lim_{t \to \infty} \pi_f(\mathbb{1}, \beta, t) = +\infty$, a contradiction.

For the proof of part b) choose any $(y, \beta) \in \Omega$. Since part a) implies $0 \notin A_0(\beta)$, the leading term of $\pi_f(y, \beta, \cdot)$ may be written as

$$\sum_{\alpha \in A_0(\beta)} f_\alpha \, y^\alpha \, e^{d(\beta)t} \;=\; e^{d(\beta)t} f^\beta(y).$$

As the assumption $\Omega \subseteq \Omega_f$ yields $\lim_{t\to\infty} \pi_f(y, \beta, t) = +\infty$, this leading term cannot tend to $-\infty$ for $t \to +\infty$. However, in view of part a), the latter would happen in the case $f^\beta(y) < 0$, so that $f^\beta(y) \geq 0$ has to hold for all $y \in Y$. As the topological closure of Y is $\mathbb{R}^n$, the continuity of f^β yields the assertion. □ □

2.2.2 Necessary conditions on the vertices of the Newton polytope

In the next step we will relate the assertions of Proposition 2.3 with statements about the Newton polytope at infinity $P = \mathrm{New}_\infty(f) = \mathrm{conv}\, A_0$. In fact, let

$$V_0 \;:=\; \mathrm{vert}\, P$$

denote the vertex set of P. Note that, due to $0 \in P \subseteq \mathbb{H}^n$, V_0 contains the origin, and that we have $V_0 \subseteq A_0$ by, for example, [90, Prop. 2.2(ii)]. In the following we shall call

$$V \;:=\; V_0 \setminus \{0\}$$

the *vertex set of P at infinity.* Our previous arguments imply the inclusion $V \subseteq A$.

With respect to the following lemma note that the above arguments entail that the element $\bar\alpha = 0$ of V_0 coincides with the singleton set $A_0(-\mathbb{1})$, where $-\mathbb{1}$ is *not* an element of B.

Lemma 2.4. *For all $\bar\alpha \in V$ the following assertions hold:*

a) There exists some $\beta \in B$ with $A_0(\beta) = \{\bar\alpha\}$.

b) In the case $\Omega \subseteq \Omega_f$ we have $f_{\bar\alpha} > 0$ and $\bar\alpha \in 2\mathbb{N}_0^n$.

Proof. Let $\bar\alpha \in V$. Then, due to $A_0 \subseteq P$, in particular the system

$$\sum_{\alpha \in A_0 \setminus \{\bar\alpha\}} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} \bar\alpha \\ 1 \end{pmatrix}, \qquad \lambda_\alpha \geq 0 \;\text{ for all } \alpha \in A_0 \setminus \{\bar\alpha\}$$

is inconsistent. By the Farkas lemma, the latter is equivalent to the existence of some $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ with

$$\langle \bar{\alpha}, \beta \rangle + \gamma > 0, \quad \langle \alpha, \beta \rangle + \gamma \leq 0, \ \alpha \in A_0 \setminus \{\bar{\alpha}\}. \tag{2.1}$$

Due to $0 \in A_0$ and $\bar{\alpha} \neq 0$ we have $0 \in A_0 \setminus \{\bar{\alpha}\}$ and conclude

$$\langle \bar{\alpha}, \beta \rangle \ > \ -\gamma \ \geq \ 0$$

from (2.1). For $\beta \in -\mathbb{H}^n$ this would contradict $\bar{\alpha} \in \mathbb{H}^n$, so that β must be an element of $(-\mathbb{H}^n)^c = B$. Moreover, (2.1) implies

$$\langle \bar{\alpha}, \beta \rangle \ > \ \langle \alpha, \beta \rangle, \ \alpha \in A_0 \setminus \{\bar{\alpha}\}$$

so that $d(\beta) = \langle \bar{\alpha}, \beta \rangle$ and $A_0(\beta) = \{\bar{\alpha}\}$ hold, that is, the assertion of part a).

To see part b), first use part a) to choose some $\beta \in B$ with $A_0(\beta) = \{\bar{\alpha}\}$. Proposition 2.3b) then implies $f_{\bar{\alpha}} \, x^{\bar{\alpha}} \geq 0$ for all $x \in \mathbb{R}^n$. The choice $x := \mathbb{1}$ and $f_{\bar{\alpha}} \neq 0$ yield the first assertion of part b). Moreover, for any $i \in I$ the choice $x := \mathbb{1} - 2e_i$ leads to $f_{\bar{\alpha}}(-1)^{\bar{\alpha}_i} \geq 0$, so that $f_{\bar{\alpha}} > 0$ implies $\bar{\alpha}_i \in 2\mathbb{N}_0$ and, thus, the second assertion of part b). □

In the next lemma, $\operatorname{cone} A$ denotes the convex cone generated by A.

Lemma 2.5. *The inclusion $\Omega \subseteq \Omega_f$ implies the following assertions:*

a) The set $\operatorname{cone} A$ *contains all unit vectors e_i, $i \in I$.*

b) For all $i \in I$ the set V contains a vector of the form $2k_i e_i$ with $k_i \in \mathbb{N}$.

Proof. To see the assertion of part a), let $i \in I$ and choose some $\beta \in \mathbb{R}^n$ with $\langle e_i, \beta \rangle > 0$. Then we have $\beta \in (-\mathbb{H}^n)^c = B$. By Proposition 2.3a) the value $d(\beta)$ thus is positive or, in other words, the system

$$\langle e_i, \beta \rangle > 0, \quad \langle \alpha, \beta \rangle \leq 0, \ \alpha \in A$$

is inconsistent. By the Farkas lemma, the latter is equivalent to $e_i \in \operatorname{cone} A$.

For the proof of part b), given any $i \in I$ we rewrite the fact $e_i \in \operatorname{cone} A$ from part a) as the existence of $K \subseteq A$ and $\lambda_\alpha > 0$, $\alpha \in K$, with $e_i = \sum_{\alpha \in K} \lambda_\alpha \, \alpha$. In particular, for any $j \in I \setminus \{i\}$ we have

$$0 \ = \ \sum_{\alpha \in K} \lambda_\alpha \, \alpha_j \, .$$

Due to $\alpha_j \geq 0$ for all $\alpha \in K$ this is only possible for $\alpha_j = 0$, that is, all elements of K must have the form $\alpha = k_i e_i$ with some $k_i \in \mathbb{N}$ and, in particular, there exists some element $\alpha \in A$ of this form.

Next, let $k_i^\star := \max\{k_i \in \mathbb{N} |\, k_i e_i \in A\}$ and $\alpha^i := k_i^\star e_i$. We will proceed to show $\alpha^i \in V$. Note that $\alpha^i \in P \setminus \{0\}$ is clear from $A \subseteq P$ and $k_i^\star \in \mathbb{N}$.

Assume that α^i is not a vertex of P. Then there exist $L \subseteq A_0 \setminus \{\alpha^i\}$ and $\lambda_\alpha > 0$, $\alpha \in L$, with $\sum_{\alpha \in L} \lambda_\alpha\, \alpha = \alpha^i$ and $\sum_{\alpha \in L} \lambda_\alpha = 1$. With the same reasoning as above, all elements of L must have the form $\alpha = k_i e_i$ with some $k_i \in \mathbb{N}_0$. In view of $\alpha^i \notin L$, this implies

$$k_i^\star \;=\; \alpha_i^i \;=\; \sum_{\alpha \in L} \lambda_\alpha\, \alpha_i \;=\; \sum_{\alpha \in L} \lambda_\alpha\, k_i \;<\; \sum_{\alpha \in L} \lambda_\alpha\, k_i^\star \;=\; k_i^\star,$$

a contradiction. Hence, we arrive at $k_i^\star e_i \in V_0 \setminus \{0\} = V$. Lemma 2.4b) finally entails that $k_i^\star$ necessarily must be even. □

Remark 2.6. Using $A \subseteq \mathbb{H}^n$, it is not hard to see that the assertion of Lemma 2.5a) is equivalent to the statement $\operatorname{cone} A = \mathbb{H}^n$. □

For later reference we observe that not only the condition $\Omega \subseteq \Omega_f$ (cf. Prop. 2.3a)) but still its necessary condition from Lemma 2.5b) implies $d(\beta) > 0$ for all $\beta \in B$:

Lemma 2.7. *For all $i \in I$ let the set V contain a vector of the form $2k_i e_i$ with $k_i \in \mathbb{N}$. Then $d(\beta) > 0$ holds for all $\beta \in B$.*

Proof. For any $\beta \in B$ there exists some $i \in I$ with $\beta_i > 0$, so that for the choice $\alpha = 2k_i e_i \in A_0$ we obtain

$$d(\beta) \;=\; \max_{a \in A_0} \langle \alpha, \beta \rangle \;\geq\; \langle 2k_i e_i, \beta \rangle \;=\; 2k_i \beta_i \;>\; 0.$$

□

The combination of Lemmata 2.2, 2.4b) and 2.5b) yields our main necessary conditions for coercivity of a polynomial involving the vertex set of P at infinity.

Theorem 2.8. *Let $f \in \mathbb{R}[x]$ be coercive on $\mathbb{R}^n$. Then the following three conditions hold:*

$$V \subseteq 2\mathbb{N}_0^n. \tag{C1}$$

All $\alpha \in V$ satisfy $f_\alpha > 0$. (C2)

For all $i \in I$ the set V contains a vector of the form $2k_i e_i$ with $k_i \in \mathbb{N}$. (C3)

Remark 2.9. For later reference we remark that the assumption of a coercive polynomial f in Theorem 2.8 may be replaced by the assumption $\Omega \subseteq \Omega_f$. □

Example 2.10. *Assume that the function*

$$f(x) \;=\; f_{4,2}x_1^4x_2^2 + f_{3,3}x_1^3x_2^3 + f_{2,3}x_1^2x_2^3 + f_{1,3}x_1x_2^3 + f_{0,4}x_2^4 + f_{0,3}x_2^3 + f_{2,0}x_1^2$$

is coercive on $\mathbb{R}^2$. In the following we shall use Theorem 2.8 to derive necessary conditions on the coefficients f_α, $\alpha \in A$, in $f(x) = \sum_{\alpha\in A} f_\alpha\, x^\alpha$ with $A \subseteq \{(4,2),(3,3),(2,3),(1,3),(0,4),(0,3),(2,0)\}$ (see Fig. 3.1).

Due to (C1) *the point $(3,3)$ cannot be contained in any choice of A, as it would be a vertex of P, while $(3,3) \notin 2\mathbb{N}_0^2$. Hence, $f_{3,3}$ has to vanish.*

Due to (C3) *the point $(2,0)$ must be contained in any choice of A, and by* (C2) *we necessarily have $f_{2,0} > 0$.*

Due to (C3) *also the point $(0,4)$ must be contained in any choice of A, as the alternative point $(0,3)$ would violate the evenness condition of* (C3)*. By* (C2) *we also have $f_{0,4} > 0$.*

If the point $(4,2)$ is not contained in A, neither $(2,3)$ nor $(1,3)$ can be elements of A, since $(2,3)$ would be a vertex of P while $(2,3) \notin 2\mathbb{N}_0^2$ and, for the hence necessary case $(2,3) \notin A$ the point $(1,3)$ would be a vertex of P, in contradiction to (C1)*. In this case we arrive at $\{(0,4),(2,0) \subseteq A \subseteq \{(0,4),(0,3),(2,0)\}$ with $f_{0,4}, f_{2,0} > 0$ and $f_{0,3} \in \mathbb{R}$.*

If, on the other hand, $(4,2)$ is contained in A, then it is a vertex of P and we conclude $f_{4,2} > 0$ from (C2)*. We arrive at $\{(4,2),(0,4),(2,0)\} \subseteq A \subseteq \{(4,2),(2,3),(1,3),(0,4),(0,3),(2,0)\}$ with $f_{4,2}, f_{0,4}, f_{2,0} > 0$ and $f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R}$.*

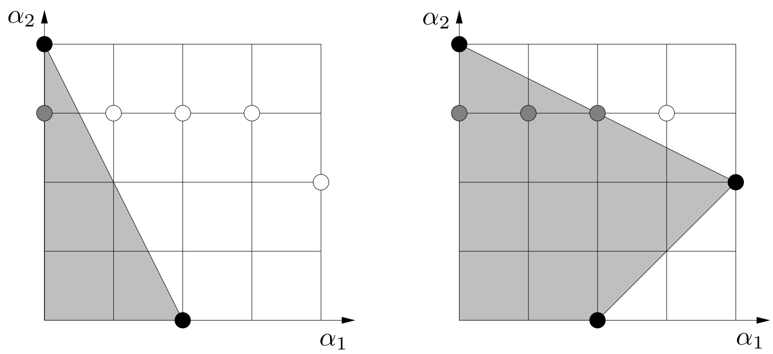

FIGURE 2.1: Illustration of Example 2.10. On the left: the exponent $(4,2)$ is not contained in A. On the right: the exponent $(4,2)$ is contained in A. In both pictures the shaded area corresponds to the Newton polytope at infinity P. The filled circles stand for the set V, while the shaded circles describe other possible exponents of f with arbitrary real coefficients. The void circles describe exponents of f with zero coefficients.

Example 2.11. *By Theorem 2.8 the so-called Motzkin form* $m(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2$ *is not coercive on* $\mathbb{R}^2$*, since it violates* (C3) *(while* (C1) *and* (C2) *are satisfied).*

2.2.3 A nondegeneracy notion for coercive polynomials

As a motivation for our further discussion note that the conditions (C1) and (C2) from Theorem 2.8 concern vertices of P and that these are, in view of Lemma 2.4a), singleton sets $A_0(\beta)$ for some $\beta \in B$. Proposition 2.3b), however, may provide additional necessary conditions in cases where $A_0(\beta)$ is not a singleton, especially if $A_0(\beta)$ contains some $\alpha \in V^c := A \setminus V$. In fact, for the special case $f(x) = x_1^4x_2^2 + x_1^2x_2^3 + x_1x_2^3 + x_2^4 + x_2^3 + x_1^2$ of the function from Example 2.10 we obtain $A_0((1,2)) = \{(4,2),(2,3),(0,4)\}$ with $(2,3) \in V^c$. On the other hand, the latter situation is degenerate in the sense that the elements of $A_0((1,2))$ are not in general position, where we say that finitely many points from $\mathbb{R}^n$ are in general position if for any $k \in \{2,\dots,n+1\}$ no k of them lie in a common affine subspace of dimension $k-2$.

Remark 2.12. We emphasize that a perturbation analysis under this notion of general position would not be straightforward, as the points in our application are elements of $\mathbb{N}_0^n$, rather than $\mathbb{R}^n$. $\square$

In the following we shall first identify an appropriate nondegeneracy condition for coercive polynomials (Def. 2.18), then see that we cannot derive necessary conditions in addition to those from Theorem 2.8 for the nondegenerate case with our techniques (Lem. 2.25) and, in Section 2.2.4, move on to treat a degenerate case.

To develop the nondegeneracy notion, in the following we shall take a closer look at the face structure of P and its relation to points in A. Recall that G is a nonempty (closed) face of P if and only if $G = \{\alpha \in P |\, \langle \alpha,\beta\rangle = \max_{\alpha\in P}\langle\alpha,\beta\rangle\}$ holds for some $\beta \in \mathbb{R}^n$.

Lemma 2.13. *For all* $\beta \in \mathbb{R}^n$ *we have* $\max_{\alpha\in P}\langle\alpha,\beta\rangle = d(\beta)$.

Proof. Let $\beta \in \mathbb{R}^n$. From $A_0 \subseteq P$ the relation

$$d(\beta) \;=\; \max_{\alpha\in A_0}\langle\alpha,\beta\rangle \;\le\; \max_{\alpha\in P}\langle\alpha,\beta\rangle$$

is clear. To see the reverse inequality, choose some arbitrary point $\bar\alpha \in P$. Then there exist $K \subseteq A_0$ and $\lambda_\alpha > 0$, $\alpha \in K$, with $\sum_{\alpha\in K}\lambda_\alpha\,\alpha = \bar\alpha$ and $\sum_{\alpha\in K}\lambda_\alpha = 1$. This implies

$$\langle\bar\alpha,\beta\rangle \;=\; \sum_{\alpha\in K}\lambda_\alpha\langle\alpha,\beta\rangle \;\le\; \sum_{\alpha\in K}\lambda_\alpha\, d(\beta) \;=\; d(\beta)$$

and, thus, $\max_{\bar\alpha\in P}\langle\bar\alpha,\beta\rangle \le d(\beta)$. $\square$

In view of Lemma 2.13, the nonempty faces of P are given by the sets

$$P(\beta) \ := \ \{\alpha \in P |\ \langle \alpha, \beta \rangle = d(\beta)\}$$

with $\beta \in \mathbb{R}^n$. Since we are primarily interested in vectors $\beta \in B$, the next result clarifies which faces of P are singled out by this choice, and how they are related to the sets $A_0(\beta)$. In fact, let us define

$$\mathcal{G} \ := \ \{G \subseteq \mathbb{R}^n |\ G \neq \emptyset \text{ is a face of } P \text{ with } 0 \notin G\}$$

as well as the *gem* of f,

$$\mathrm{Gem}(f) \ := \ \bigcup_{G \in \mathcal{G}} G.$$

Remark 2.14. The set $\mathrm{Gem}(f)$ has widely been used in the literature on Newton polytopes of polynomials under different names. For example, in [67, 79] it is called 'Newton boundary at infinity'. Our terminology is motivated by Definition 2.18 below. □

Lemma 2.15. *Under condition* (C3) *the following assertions hold:*

a) $G \in \mathcal{G}$ holds if and only if there exists some $\beta \in B$ with $G = P(\beta)$.

b) $A_G = A_0 \cap G$ holds with $G \in \mathcal{G}$ if and only if there exists some $\beta \in B$ with $A_G = A_0(\beta)$.

Proof. For the proof of part a) choose $G \in \mathcal{G}$. As G is a nonempty face of P, we have $G = P(\beta)$ with some $\beta \in \mathbb{R}^n$. Assume that this holds with $\beta \in -\mathbb{H}^n$. Then, due to $P \subseteq \mathbb{H}^n$, all $\alpha \in P$ satisfy $\langle \alpha, \beta \rangle \leq 0$, and the latter upper bound is attained for $0 \in P$. This implies $d(\beta) = 0$ and $0 \in P(\beta) = G$, a contradiction. Hence, we arrive at $G = P(\beta)$ with $\beta \in (-\mathbb{H}^n)^c = B$.

To see the reverse inclusion, let $P(\beta)$ with $\beta \in B$ be given. Then $P(\beta)$ is a nonempty face of P, and all $\alpha \in P(\beta)$ satisfy $\langle \alpha, \beta \rangle = d(\beta) > 0$ by (C3) and Lemma 2.7. This excludes that $P(\beta)$ contains the origin, that is, we have $P(\beta) \in \mathcal{G}$.

The assertion of part b) immediately follows from part a) and the identity $A_0 \cap P(\beta) = A_0(\beta)$ for any $\beta \in B$. □

In the following let V_G denote the vertex set $\mathrm{vert}\, G$ for any of the polytopes $G \in \mathcal{G}$. From, e.g., [90, Prop. 2.3(i)] we know the identity $V_G = V_0 \cap G$ which, in view of $0 \notin G$, implies $V_G = V \cap G$. The next result then is an immediate consequence of the relation $V \subseteq A$.

Lemma 2.16. *Each $G \in \mathcal{G}$ satisfies $V_G \subseteq A \cap G$.*

Remark 2.17. Lemma 2.16 sheds some additional light on the well-known fact that the degree $\deg(f)$ of $f \in \mathbb{R}[x]$ must be even if f is coercive on $\mathbb{R}^n$. In fact, recall that we may write $\deg(f) = d(\mathbb{1})$. Due to Theorem 2.8, Lemma 2.15a) and $\mathbb{1} \in B$, the face $G = P(\mathbb{1})$ lies in $\mathcal{G}$ and, by Lemma 2.16, it satisfies $V_G \subseteq A \cap G = A_0(\mathbb{1})$. Consequently, $A_0(\mathbb{1})$ contains some vertex $\bar{\alpha} \in V$, and we arrive at $\deg(f) = d(\mathbb{1}) = \langle \bar{\alpha}, \mathbb{1} \rangle$. As all entries of $\bar{\alpha}$ are even by condition (C1) in Theorem 2.8, $\deg(f)$ must also be even. $\square$

The announced nondegeneracy notion just states equality in the assertion of Lemma 2.16. Note that $V \subseteq A$ and the definition $V^c = A \setminus V$ entail

$$A \cap G \;=\; (V \,\dot{\cup}\, V^c) \cap G \;=\; V_G \,\dot{\cup}\, (V^c \cap G) \tag{2.2}$$

so that the identity $V_G = A \cap G$ is equivalent to $V^c \cap G = \emptyset$.

Definition 2.18 (Gem degenerate exponent vectors and gem regular polynomials)**.**

a) An exponent vector $\alpha \in A$ is called gem degenerate *if $\alpha \in V^c \cap G$ holds for some $G \in \mathcal{G}$. We denote the set of all gem degenerate points $\alpha \in A$ by D.*

b) The polynomial $f \in \mathbb{R}[x]$ is called gem regular *if the set D is empty, otherwise it is called* gem irregular*.*

Clearly, gem regularity of $f \in \mathbb{R}[x]$ is equivalent to $V^c \cap G = \emptyset$ for all $G \in \mathcal{G}$. Furthermore, the definition of D gives rise to a partitioning of V^c into D and a set of 'remaining exponents' $R := V^c \setminus D$, so that we may write

$$A \;=\; V \,\dot{\cup}\, D \,\dot{\cup}\, R. \tag{2.3}$$

Example 2.19. *For the polynomial $f(x) = x_1^4 x_2^2 + x_1 x_2^3 + x_2^4 + x_2^3 + x_1^2$ we obtain $V = \{(4,2),(0,4),(2,0)\}$, $D = \emptyset$, and $R = \{(1,3),(0,3)\}$, so that f is gem regular (see Fig. 4.1). Note that for the face $G = P((-1,0))$ we have $(0,3) \in V^c \cap G$, but that due to $G \notin \mathcal{G}$ this does* not *mean gem degeneracy of the exponent vector $(0,3)$.*

Example 2.20. *The polynomial $f(x) = x_1^4 x_2^2 + x_1^2 x_2^3 + x_1 x_2^3 + x_2^4 + x_2^3 + x_1^2$ satisfies $V = \{(4,2),(0,4),(2,0)\}$, $D = \{(2,3)\}$, and $R = \{(1,3),(0,3)\}$, so that f is gem irregular (see Fig. 4.1).*

Example 2.21. *The Motzkin form $m(x) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2$ is a gem irregular polynomial with $V = \{(4,2,0),(2,4,0),(0,0,6)\}$, $D = \{(2,2,2)\}$, and $R = \emptyset$.*

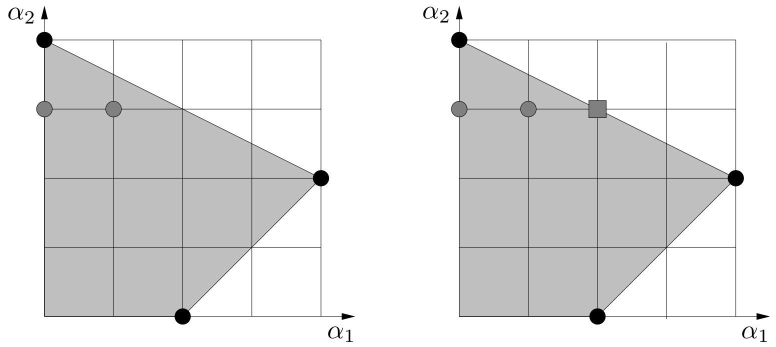

FIGURE 2.2: On the left: illustration of Example 2.19. On the right: illustration of Example 2.20. In both pictures the shaded area corresponds to the Newton polytope at infinity P. The filled circles stand for the set V, while the shaded circles describe the set R corresponding to f. The shaded square in the right picture describes the (singleton) set D corresponding to f.

To term the condition from Definition 2.18b) a *regularity* condition is justified by the fact that it is related to requiring general position of certain elements of A:

Lemma 2.22. *If for $f \in \mathbb{R}[x]$ and each $G \in \mathcal{G}$ the elements of $A \cap G$ are in general position, then f is gem regular.*

Proof. For each $G \in \mathcal{G}$ let the elements of $A \cap G$ be in general position and assume that $V^c \cap G \neq \emptyset$ holds for some $G \in \mathcal{G}$. Then, by Lemma 2.16 and (2.2), we have $|V_G| < |A \cap G|$. On the other hand, $\dim(G) + 1 \leq |V_G|$ holds as G is a polytope, where $\dim(G)$ denotes the dimension of the affine hull $\operatorname{aff}(G)$ of G. Hence, $A \cap G$ contains at least $\dim(G) + 2$ elements, while at the same time $A \cap G$ lies in the subspace $\operatorname{aff}(G)$ of dimension $\dim(G)$. This contradicts the assumption that the elements of $A \cap G$ are in general position. $\square$

Remark 2.23. The polynomial $f(x) = x_1^2 + x_2^2 + x_3^2 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2^2x_3^2$ shows that gem regularity is strictly weaker than the type of general position assumed in Lemma 2.22. In fact, $\mathrm{New}_\infty(f)$ is a cube and D is void, while for no facet $G \in \mathcal{G}$ the set $A \cap G$ is in general position. $\square$

The following characterization of the set D will be crucial in Section 2.3. It states that D contains exactly the exponent vectors in A which cannot be written as a convex combination of elements from V_0 with the origin entering with a positive weight. The proof is given below, prepared by the proof of a nonhomogeneous version of Motzkin's transposition theorem (Lemma A.1) in Appendix A.1.

Proposition 2.24. *Under condition* (C3) *the following are equivalent:*

a) $\alpha^\star \in D$,

b) $\alpha^\star \in V^c$, *and any choice of coefficients* λ_α, $\alpha \in V_0$, *with*

$$\alpha^\star \;=\; \sum_{\alpha \in V_0} \lambda_\alpha \, \alpha, \quad \sum_{\alpha \in V_0} \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \; \alpha \in V_0,$$

satisfies $\lambda_0 = 0$.

Proof. For any $\alpha^\star \in \mathbb{R}^n$, the fact that any choice of coefficients λ_α, $\alpha \in V_0$, with

$$\alpha^\star \;=\; \sum_{\alpha \in V_0} \lambda_\alpha \, \alpha, \quad \sum_{\alpha \in V_0} \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \; \alpha \in V_0,$$

satisfies $\lambda_0 = 0$ is equivalent to the inconsistency of the system

$$\sum_{\alpha \in V_0} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} \alpha^\star \\ 1 \end{pmatrix}, \quad \lambda_\alpha \geq 0, \; \alpha \in V, \quad \lambda_0 > 0. \tag{2.4}$$

For the application of Lemma A.1 we define

$$A := \begin{pmatrix} \cdots & \alpha & \cdots & 0 \\ \cdots & 1 & \cdots & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix},$$

where α runs through the set V_0 and where we use the convention that $0 \in V_0$ corresponds to its last entry, as well as

$$a := \begin{pmatrix} \alpha^\star \\ 1 \end{pmatrix}, \quad C := c^\intercal := (0, \ldots, 0, 1).$$

By Lemma A.1 the inconsistency of (2.4) is equivalent to the consistency of at least one of the systems

$$A^\intercal \rho + B^\intercal \sigma + \tau c = 0, \quad \langle a, \rho \rangle > 0, \quad \sigma, \tau \geq 0 \tag{2.5}$$

and

$$A^\intercal \rho + B^\intercal \sigma + \tau c = 0, \quad \langle a, \rho \rangle = 0, \quad \sigma \geq 0, \quad \tau > 0, \tag{2.6}$$

where we have used that τ is a scalar. Setting $\rho = (\beta, \gamma)$ with $\gamma \in \mathbb{R}$ yields that the consistency of (2.5) is equivalent to the consistency of

$$\langle \alpha, \beta \rangle \leq \tau, \; \alpha \in V, \quad \langle \alpha^\star, \beta \rangle > \tau, \quad \tau \geq 0, \tag{2.7}$$

and that the consistency of (2.6) is equivalent to the consistency of

$$\langle \alpha, \beta \rangle \leq \tau, \; \alpha \in V, \quad \langle \alpha^\star, \beta \rangle = \tau, \quad \tau > 0. \tag{2.8}$$

Note that in both systems the inequalities corresponding to the choice $\alpha = 0 \in V_0$ were dropped since they are always consistent in view of the nonnegativity of τ.

So far we have shown that part b) of the assertion can be reformulated as $\alpha^\star \in V^c$ and the consistency of at least one of the systems (2.7) and (2.8). Next we shall prove that for any $\alpha^\star \in A_0$ the system (2.7) must be inconsistent. In fact, as any $\alpha^\star \in A_0$ possesses some description

$$\alpha^\star \;=\; \sum_{\alpha \in V_0} \lambda_\alpha \, \alpha \quad \text{with} \quad \sum_{\alpha \in V_0} \lambda_\alpha = 1,\; \lambda_\alpha \geq 0,\; \alpha \in V_0,$$

the consistency of (2.7) implies the existence of some $\beta \in \mathbb{R}^n$ and $\tau \geq 0$ with

$$\langle \alpha^\star, \beta \rangle \;=\; \sum_{\alpha \in V_0} \lambda_\alpha \langle \alpha, \beta \rangle \;=\; \sum_{\alpha \in V} \lambda_\alpha \langle \alpha, \beta \rangle \;\leq\; \sum_{\alpha \in V} \lambda_\alpha \tau \;\leq\; \tau \sum_{\alpha \in V_0} \lambda_\alpha \;=\; \tau. \tag{2.9}$$

However, the consistency of (2.7) also implies $\langle \alpha^\star, \beta \rangle > \tau$, a contradiction. Hence, for any $\alpha^\star \in V^c \subseteq A_0$ the inconsistency of (2.4) is equivalent to the consistency of (2.8).

In the next step we will show that the consistency of (2.8) is equivalent to the existence of some $\beta \in B$ with $\alpha^\star \in A_0(\beta)$. In fact, an analogous argument as in the derivation of (2.9) from (2.7) shows that the consistency of (2.8) implies the optimality of the point $\alpha^\star$ for the maximization of $\langle \alpha, \beta \rangle$ over A_0, that is, $\alpha^\star \in A_0(\beta)$ with some $\beta \in \mathbb{R}^n$. More precisely, $\alpha^\star \geq 0$ and $\langle \alpha^\star, \beta \rangle = \tau > 0$ yield that the consistency of (2.8) entails $\beta \in (-\mathbb{H}^n)^c = B$. For the reverse implication, note that $\alpha^\star \in A_0(\beta)$ for some $\beta \in B$ means $\langle \alpha, \beta \rangle \leq \langle \alpha^\star, \beta \rangle$ for all $\alpha \in V$ and some $\beta \in B$. Moreover, by (C3) and Lemma 2.7 we have $d(\beta) = \langle \alpha^\star, \beta \rangle > 0$, so that the choice $\tau := d(\beta)$ proves the consistency of (2.8).

Altogether, this shows that part b) can be reformulated as $\alpha^\star \in V^c$ and the existence of some $\beta \in B$ with $\alpha^\star \in A_0(\beta)$. In view of (C3) and Lemma 2.15b), the latter is equivalent to $\alpha^\star \in V^c$ and the existence of some $G \in \mathcal{G}$ with $\alpha^\star \in A \cap G$, that is, to $\alpha^\star \in V^c \cap G$ for some $G \in \mathcal{G}$. This is, finally, just the definition for $\alpha^\star$ to lie in D, that is, part a) of the assertion. $\square$

The following lemma clarifies in which cases the assertion of Proposition 2.3b) may contain additional information on necessary conditions for coercivity, given the assertions of Theorem 2.8.

Lemma 2.25. *For $f \in \mathbb{R}[x]$ the following assertions hold:*

a) *If the conditions* (C1)–(C3) *from Theorem 2.8 hold and f is gem regular, then for all $\beta \in B$ the polynomial f^β is positive semi-definite on $\mathbb{R}^n$.*

b) If $\Omega \subseteq \Omega_f$ holds, then for all $G \in \mathcal{G}$ with $D \cap G \neq \emptyset$ we have

$$\sum_{\alpha \in V_G} f_\alpha x^\alpha \;\geq\; -\sum_{\alpha \in D\cap G} f_\alpha x^\alpha \tag{2.10}$$

for all $x \in \mathbb{R}^n$.

Proof. Let $\beta \in B$ and any $x \in \mathbb{R}^n$ be given. By (C3) and Lemma 2.15b) there is some $G \in \mathcal{G}$ with $A_0(\beta) = A \cap G$ so that

$$f^\beta(x) \;=\; \sum_{\alpha \in A\cap G} f_\alpha x^\alpha \tag{2.11}$$

holds. Under the assumption of part a) equations (2.2) and (2.11) yield

$$f^\beta(x) \;=\; \sum_{\alpha \in V_G} f_\alpha x^\alpha,$$

so that $V_G \subseteq V$, (C1) and (C2) imply the assertion of part a).

To see the assertion of part b), let $G \in \mathcal{G}$ with $D \cap G \neq \emptyset$ be given. By Lemma 2.5b) the inclusion $\Omega \subseteq \Omega_f$ implies (C3), so that Lemma 2.15b) guarantees the existence of some $\beta \in B$ with $A \cap G = A_0(\beta)$ and (2.11). Hence, the inclusion $\Omega \subseteq \Omega_f$, Proposition 2.3b) and (2.2) imply

$$0 \;\leq\; f^\beta(x) \;=\; \sum_{\alpha \in A\cap G} f_\alpha x^\alpha \;=\; \sum_{\alpha \in V_G} f_\alpha x^\alpha + \sum_{\alpha \in D\cap G} f_\alpha x^\alpha$$

for all $x \in \mathbb{R}^n$. This shows the assertion of part b). $\square$

Lemma 2.25a) expresses that Proposition 2.3b) and, thus, the approach used in Section 2.2, cannot provide necessary conditions for coercivity of gem regular polynomials f in addition to the conditions (C1)–(C3) stated in Theorem 2.8. In particular, although (C1)–(C3) where derived using only the special case of *singleton* sets $A_0(\beta)$ (cf., e.g., Lem. 2.4), the consideration of $\beta \in B$ with more general sets $A_0(\beta)$ in Proposition 2.3b) is superfluous.

For gem irregular polynomials f, however, further necessary conditions for coercivity may be derived from the assertion of Lemma 2.25b). The proof of the according statement directly follows from Lemma 2.2 and Lemma 2.25b).

Proposition 2.26. *Let $f \in \mathbb{R}[x]$ be coercive on $\mathbb{R}^n$. Then for all $G \in \mathcal{G}$ with $D \cap G \neq \emptyset$ the inequality*

$$\sum_{\alpha \in V_G} f_\alpha x^\alpha \;\geq\; -\sum_{\alpha \in D\cap G} f_\alpha x^\alpha$$

holds for all $x \in \mathbb{R}^n$.

For the following we observe that, under condition (C3), the unique correspondence between the sets $A_0(\beta)$, $\beta \in B$, and $A \cap G$, $G \in \mathcal{G}$, stated in Lemma 2.15, allows us to interchange the notation f^β with f^G so that, for example, equation (2.11) reads

$$f^G(x) \;=\; \sum_{\alpha \in A \cap G} f_\alpha \, x^\alpha.$$

In [85] the polynomials f^G are called *quasi-homogeneous components* of f.

2.2.4 Necessary conditions in a degenerate case

Lemma 2.2, Proposition 2.3b), Lemma 2.5b), and Lemma 2.15b) obviously allow to state a multitude of inequalities on the coefficients f_α, $\alpha \in A$, of a coercive polynomial, just by evaluating f^β at special vectors x for all $\beta \in B$ (or, equivalently, for all $G \in \mathcal{G}$). For example, the choice $x := \mathbb{1}$ yields

$$\sum_{\alpha \in A \cap G} f_\alpha \;\geq\; 0$$

for all $G \in \mathcal{G}$, and the choice $x = -\mathbb{1}$ leads to

$$\sum_{\substack{\alpha \in A \cap G \\ |\{i \in I \,|\, \alpha_i \in 2\mathbb{N}_0+1\}| \in 2\mathbb{N}_0}} f_\alpha \;\geq\; \sum_{\substack{\alpha \in A \cap G \\ |\{i \in I \,|\, \alpha_i \in 2\mathbb{N}_0+1\}| \in 2\mathbb{N}_0+1}} f_\alpha$$

for all $G \in \mathcal{G}$.

While, in view of Lemma 2.25a), many of these inequalities may not contain any information improving the conditions (C1)–(C3) from Theorem 2.8 due to $|D \cap G| = 0$, in the case of $G \in \mathcal{G}$ with $|D \cap G| > 0$ Proposition 2.26 provides a systematic way to gain further relations on the coefficients f_α, $\alpha \in A$. Our main result in the present section will state bounds on these coefficients in the case $|D \cap G| = 1$, under the additional assumption that G is a simplex, that is, the convex hull of affinely independent points. Note that in [38] the corresponding polynomial $f^G(x) = \sum_{A \cap G} f_\alpha \, x^\alpha$ is termed a *circuit polynomial.* The following examples illustrate this case.

Example 2.27. *Consider the polynomial*

$$f(x) \;=\; f_{4,2}x_1^4x_2^2 + f_{2,3}x_1^2x_2^3 + f_{1,3}x_1x_2^3 + f_{0,4}x_2^4 + f_{0,3}x_2^3 + f_{2,0}x_1^2$$

with $f_{4,2} \neq 0$*, whose coercivity on* $\mathbb{R}^2$ *implies* $f_{4,2}, f_{0,4}, f_{2,0} > 0$ *as well as* $f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R}$*, as we saw in Example 2.10. For* $f_{2,3} \neq 0$ *the face* $G = P((1,2))$ *lies in* $\mathcal{G}$*, is a simplex, and satisfies* $|D \cap G| = |\{(2,3)\}| = 1$*. In particular, the function* $f^G(x) = f_{4,2}x_1^4x_2^2 + f_{2,3}x_1^2x_2^3 + f_{0,4}x_2^4$ *is a circuit polynomial.*

Example 2.28. *The Newton polytope of the Motzkin form* $m(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2$ *from Example 2.11 is a simplex and satisfies* $|D \cap \mathrm{New}(m)| = |\{(2,2,2)\}| = 1$*. Thus,* m *is a circuit polynomial.*

Recall that, for any simplex G and $\alpha^\star \in G$, the coefficients λ_α, $\alpha \in V_G$, with

$$\sum_{\alpha \in V_G} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^\star \\ 1 \end{pmatrix}, \quad \lambda_\alpha \geq 0,\ \alpha \in V_G\,, \tag{2.12}$$

are unique. Using the natural convention $0^0 := 1$ in the polynomial setting (to cover the case of vanishing coefficients λ_α), we may define the *circuit number* (cf. [38])

$$\Theta(f, V_G, \alpha^\star) := \prod_{\alpha \in V_G} \left(\frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} \tag{2.13}$$

of $\alpha^\star$ with respect to f^G. Note that the arithmetic-geometric mean inequality immediately yields that for any $\alpha^\star \in G$ the circuit number $\Theta(f, V_G, \alpha^\star)$ bounds the sum of coefficients $\sum_{\alpha \in V_G} f_\alpha$ from below.

The following assertion has a similar structure as [38, Th. 1.1].

Theorem 2.29. *Let* $f \in \mathbb{R}[x]$ *be coercive on* $\mathbb{R}^n$*. Then the conditions* (C1)–(C3) *from Theorem 2.8 are satisfied, and for any* $\alpha^\star \in D$ *such that there exists a simplicial face* $G \in \mathcal{G}$ *with* $\alpha^\star \in G$ *and* $D \cap G = \{\alpha^\star\}$*, the following assertions hold.*

a) We have

$$f_{\alpha^\star} \geq -\Theta(f, V_G, \alpha^\star). \tag{2.14}$$

b) For $\alpha^\star \notin 2\mathbb{N}_0^n$ *we also have*

$$f_{\alpha^\star} \leq \Theta(f, V_G, \alpha^\star). \tag{2.15}$$

Proof. First, by Theorem 2.8, the conditions (C1)–(C3) are satisfied. Furthermore, under the stated assumptions Lemma 2.2 and Proposition 2.26 yield

$$\sum_{\alpha \in V_G} f_\alpha x^\alpha \geq -f_{\alpha^\star} x^{\alpha^\star} \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.16}$$

As a first step, we will rewrite this condition in terms of absolute values of x, where we put

$$|x|^\alpha \;:=\; \prod_{i\in I}|x_i|^{\alpha_i} \tag{2.17}$$

for any $\alpha \in \mathbb{N}_0^n$. Due to conditions (C1) and (C2), in the left hand side of (2.16) we may replace x^α by $|x|^\alpha$ for any $\alpha \in V_G$. In the right hand side we replace $x^{\alpha^\star}$ by $\mathrm{sign}(x^{\alpha^\star})|x^{\alpha^\star}| = \mathrm{sign}(x^{\alpha^\star})|x|^{\alpha^\star}$.

In the following we focus on the case

$$x \in X \;:=\; \left\{x\in\mathbb{R}^n \,\middle|\, \prod\nolimits_{i\in I} x_i \neq 0\right\}$$

(see Rem. 2.30 for a discussion of the case $x \notin X$). Then we have $|x|^{\alpha^\star} > 0$, so that (2.16) implies

$$\sum_{\alpha\in V_G} f_\alpha\,|x|^{\alpha-\alpha^\star} \;\geq\; -f_{\alpha^\star}\;\mathrm{sign}(x^{\alpha^\star}) \;\text{ for all }\; x\in X. \tag{2.18}$$

With the two sets $X^\pm := \{x \in X|\ \mathrm{sign}(x^{\alpha^\star}) = \pm 1\}$ we arrive at the two separate conditions

$$\inf_{x\in X^+}\sum_{\alpha\in V_G} f_\alpha\,|x|^{\alpha-\alpha^\star} \;\geq\; -f_{\alpha^\star} \tag{2.19}$$

and

$$\inf_{x\in X^-}\sum_{\alpha\in V_G} f_\alpha\,|x|^{\alpha-\alpha^\star} \;\geq\; f_{\alpha^\star}\,. \tag{2.20}$$

Note that X^+ is nonempty for any $\alpha^\star \in \mathbb{N}_0^n$, whereas X^- is nonempty if and only if $\alpha^\star \notin 2\mathbb{N}_0^n$. This explains why the assertion of this theorem is split into parts a) and b).

In fact, let $X^\star$ either denote the set X^+ or a nonempty set X^-. We will show that the infimum appearing in conditions (2.19) and (2.20), that is, the infimum v_Q of the problem

$$Q: \quad \min_{x\in\mathbb{R}^n}\sum_{\alpha\in V_G} f_\alpha\,|x|^{\alpha-\alpha^\star} \quad \text{s.t.} \quad x\in X^\star$$

is bounded above by the infimum v_S of the problem

$$S: \quad \min_{s\in\mathbb{R}^{|V_G|}}\sum_{\alpha\in V_G} f_\alpha\,e^{s_\alpha} \quad \text{s.t.} \quad \sum_{\alpha\in V_F}\lambda_\alpha s_\alpha = 0,$$

where λ_α, $\alpha\in V_G$, denote the unique coefficients from (2.12) (in fact, both infima even coincide, see Rem. 2.31). As the objective function of Q is a posynomial, we will borrow some standard techniques from geometric programming for our further analysis.

We will use that for any $\bar{s}$ from the feasible set M_S of S the system of equations

$$\langle \alpha - \alpha^\star, z\rangle \;=\; \bar{s}_\alpha, \quad \alpha \in V_G\,, \tag{2.21}$$

possesses a solution $\bar{z}$. In fact, as the vectors $\alpha \in V_G$ are affinely independent as vertices of a simplex, the vectors $\begin{pmatrix}\alpha - \alpha^\star \\ 1\end{pmatrix}$, $\alpha \in V_G$, are linearly independent, and the system

$$\left\langle \begin{pmatrix}\alpha - \alpha^\star \\ 1\end{pmatrix}, \begin{pmatrix} z \\ \zeta \end{pmatrix} \right\rangle \;=\; \bar{s}_\alpha, \quad \alpha \in V_G\,,$$

possesses a solution $(\bar{z}, \bar{\zeta})$. Moreover, the feasibility of $\bar{s}$ implies

$$0 \;=\; \sum_{\alpha\in V_G} \lambda_\alpha \bar{s}_\alpha \;=\; \sum_{\alpha\in V_G} \lambda_\alpha\big(\langle \alpha - \alpha^\star, \bar{z}\rangle + \bar{\zeta}\big) \;=\; \left\langle \sum_{\alpha\in V_G} \lambda_\alpha(\alpha - \alpha^\star), \bar{z} \right\rangle + \bar{\zeta} \;=\; \bar{\zeta}$$

so that $\bar{z}$ solves (2.21).

Next, from any solution $\bar{z} \in \mathbb{R}^n$ of (2.21) we can construct an element of $X^\star$. In fact, for $X^\star = X^+$ the point x defined by $\bar{x}_i := e^{\bar{z}_i}$, $i \in I$, lies in X^+. On the other hand, if $X^\star = X^-$ holds with a nonempty set X^-, then $\alpha^\star$ possesses at least one odd entry $\alpha_j^\star$. The point x defined by $\bar{x}_j := -e^{\bar{z}_j}$ and $\bar{x}_i := e^{\bar{z}_i}$, $i \in I \setminus \{j\}$ then lies in X^-. Hence, in any of the two cases we arrive at $\bar{x} \in X^\star$ which implies

$$v_Q \;\le\; \sum_{\alpha\in V_G} f_\alpha |\bar{x}|^{\alpha-\alpha^\star}.$$

Furthermore, the latter right hand side satisfies

$$\sum_{\alpha\in V_G} f_\alpha |\bar{x}|^{\alpha-\alpha^\star} \;=\; \sum_{\alpha\in V_G} f_\alpha \prod_{i\in I} e^{(\alpha_i - \alpha_i^\star)\bar{z}_i} \;=\; \sum_{\alpha\in V_G} f_\alpha e^{\langle \alpha-\alpha^\star, \bar{z}\rangle} \;=\; \sum_{\alpha\in V_G} f_\alpha e^{\bar{s}_\alpha}$$

so that, as $\bar{s} \in M_S$ was chosen arbitrarily, we arrive at $v_Q \le v_S$.

Finally, let us explicitly compute v_S. Since S is a convex optimization problem with polyhedral feasible set, the globally minimal points of S coincide with its Karush-Kuhn-Tucker points. In fact, s is a Karush-Kuhn-Tucker point of S if there exists some $\mu \in \mathbb{R}$ with

$$f_\alpha e^{s_\alpha} \;=\; \mu\,\lambda_\alpha\,, \quad \alpha \in V_G. \tag{2.22}$$

The feasibility of s and (2.22) entail

$$1 \;=\; e^{\left(\sum\limits_{\alpha\in V_G} \lambda_\alpha s_\alpha\right)} \;=\; \prod_{\alpha\in V_G} (e^{s_\alpha})^{\lambda_\alpha} \;=\; \prod_{\alpha\in V_G} \left(\mu \frac{\lambda_\alpha}{f_\alpha}\right)^{\lambda_\alpha} \;=\; \mu \prod_{\alpha\in V_G} \left(\frac{\lambda_\alpha}{f_\alpha}\right)^{\lambda_\alpha}$$

so that μ as well as (by (2.22)) s are uniquely determined, and s coincides with the unique minimal point of S. The value v_S is the corresponding minimal value of S which, in view of (2.22), may be written as

$$\sum_{\alpha\in V_G} f_\alpha\, e^{s_\alpha} \;=\; \sum_{\alpha\in V_G} \mu\,\lambda_\alpha \;=\; \mu$$

and, thus, the infimum of S is

$$v_S \;=\; \mu \;=\; \prod_{\alpha\in V_G}\left(\frac{f_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} \;=\; \Theta(f, V_G, \alpha^\star). \tag{2.23}$$

As the infimum of Q is bounded above by v_S, the choice $X^\star = X^+$ in Q and (2.19) yields part a) of the assertion. Under the additional assumption of part b) the set X^- is nonempty, so that the choice $X^\star = X^-$ in Q together with (2.20) shows the assertion of part b). $\square$

Remark 2.30. In the above proof of Theorem 2.29, no additional conditions can be derived from (2.16) in the case $x \notin X$. To see this, let us define the index sets $I_0(x) = \{i \in I |\, x_i = 0\}$ and $I_0(\alpha^\star) = \{i \in I |\, \alpha^\star_i = 0\}$. Clearly, the condition $x \notin X$ is equivalent to $I_0(x) \neq \emptyset$. In the case $I_0(x) \not\subseteq I_0(\alpha^\star)$ there exists some $i \in I$ with $x_i = 0$ and $\alpha^\star_i > 0$ which implies $x_i^{\alpha^\star_i} = 0$ and $x^{\alpha^\star} = 0$. The condition resulting from (2.16) then contains no additional information as, in view of conditions (C1) and (C2), it holds anyway. Note that, in view of $I_0(x) \neq \emptyset$, this case includes the case $I_0(\alpha^\star) = \emptyset$, that is, $\alpha^\star \in \mathbb{N}^n$.

On the other hand, in the case $I_0(x) \subseteq I_0(\alpha^\star)$ all $i \in I_0(x)$ satisfy $x_i^{\alpha^\star_i} = 0^0 = 1$. Moreover, due to $\alpha^\star \in \operatorname{conv} V_G$ we necessarily have $\alpha_i = 0$ and, thus, $x_i^{\alpha_i} = 0^0 = 1$ for all $\alpha \in V_G$. Removing the variables x_i, $i \in I_0(x)$, and the exponent vector entries α_i, $i \in I_0(x)$, from condition (2.16) reduces it to a condition in a lower dimensional space of dimension $\tilde{n} = n - |I_0(x)|$ with $\tilde{n} \geq 1$ (as $I_0(x) = I$ is impossible due to $\alpha^\star \neq 0$). Since the lower dimensional variables $\tilde{x}$ possess no vanishing entries, the argument from the proof of Theorem 2.29 for the case $x \in X$ could be repeated, but as the resulting estimate of $f_{\alpha^\star}$ by the circuit number is independent of the dimension n of the decision variable of Q, we would not obtain new necessary conditions.

Summarizing, the condition (2.16) is not interesting for the case $x \notin X$. $\square$

Remark 2.31. The bounds on f_α stated in Theorem 2.29 actually are best possible in the sense that no better bounds can be derived from conditions (2.19) and (2.20). This is due to the fact that not only the estimate $v_Q \leq v_S$ holds, but even identity. In fact, the reverse inequality $v_Q \geq v_S$ readily follows from the arithmetic-geometric mean inequality: for any $\lambda_\alpha \geq 0$, $\alpha \in V_G$, with $\sum_{\alpha \in V_G} \lambda_\alpha = 1$ it yields for any $x \in \mathbb{R}^n$

$$\sum_{\alpha \in V_G} f_\alpha \, |x|^{\alpha - \alpha^\star} \;\geq\; \prod_{\alpha \in V_G} \left(\frac{f_\alpha \, |x|^{\alpha-\alpha^\star}}{\lambda_\alpha} \right)^{\lambda_\alpha} \;=\; |x|^{\sum_{\alpha \in V_G} \lambda_\alpha (\alpha - \alpha^\star)} \prod_{\alpha \in V_G} \left(\frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha}$$

where, again, the convention $0^0 = 1$ is used. If the λ_α, $\alpha \in V_G$, are additionally chosen such that $\alpha^\star = \sum_{\alpha \in V_G} \lambda_\alpha \, \alpha$, we obtain

$$\sum_{\alpha \in V_G} f_\alpha \, |x|^{\alpha - \alpha^\star} \;\geq\; \prod_{\alpha \in V_G} \left(\frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} \tag{2.24}$$

for all such λ as well as all $x \in \mathbb{R}^n$. While these inequalities give rise to the duality theory of geometric programming, we do not make use of it, as under the assumptions of Theorem 2.29 there only exists a single vector λ with the required specifications, and the right hand side in (2.24) may be replaced by the circuit number $\Theta(f, V_G, \alpha^\star)$. By (2.23) the circuit number coincides with v_S, so that the infimum of the left hand side in (2.24) taken over any set $X^\star \subseteq \mathbb{R}^n$ is bounded below by v_S. As v_Q is such an infimum, the relation $v_Q \geq v_S$ is shown. □

Example 2.32. *Consider the polynomial*

$$f(x) \;=\; f_{4,2} x_1^4 x_2^2 + f_{2,3} x_1^2 x_2^3 + f_{1,3} x_1 x_2^3 + f_{0,4} x_2^4 + f_{0,3} x_2^3 + f_{2,0} x_1^2$$

with $f_{4,2} \neq 0$, whose coercivity on $\mathbb{R}^2$ implies $f_{4,2}, f_{0,4}, f_{2,0} > 0$ as well as $f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R}$, as we saw in Example 2.10 and, for $f_{2,3} \neq 0$, the exponent $\alpha^\star = (2,3)$ lies in D and satisfies the assumptions of Theorem 2.29, as we saw in Example 2.27. In fact, we have $V_G = \{(4,2), (0,4)\}$ and $\lambda_{4,2} = \lambda_{0,4} = 1/2$. Hence, by Theorem 2.29a) and b) the coercivity of f implies

$$-2\sqrt{f_{4,2}\, f_{0,4}} \;\leq\; f_{2,3} \;\leq\; 2\sqrt{f_{4,2}\, f_{0,4}}\,. \qquad \square$$

Example 2.33. *Let us modify the Motzkin form from Example 2.11 such that the resulting polynomial does not violate the condition* (C3)*, for example to $\widetilde{m}(x) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 + \widetilde{m}_{2,2,2} x_1^2 x_2^2 x_3^2 + x_1^2 + x_2^2 + x_3^2$. For $\widetilde{m}_{2,2,2} \neq 0$ the exponent $\alpha^\star = (2,2,2)$ lies in D and satisfies the assumptions of Theorem 2.29 with the face $G = P(\mathbb{1})$, as we saw in Example 2.21. In fact, we have $V_G = \{(4,2,0), (2,4,0), (0,0,6)\}$ and $\lambda_{4,2,0} = \lambda_{2,4,0} = \lambda_{0,0,6} = 1/3$. By Theorem 2.29a) the coercivity of $\widetilde{m}$ hence implies $\widetilde{m}_{2,2,2} \geq -3$ which shows that the choice of the coefficient $\widetilde{m}_{2,2,2}$ in the original Motzkin form is, in this sense, a critical one.*

Example 2.34. *In [41, Ex. 3.2] the coercivity of $f(x) = x_1^6 + x_2^6 + f_{3,3} x_1^3 x_2^3 + x_1^4 - x_2 + 1$ on $\mathbb{R}^2$ is shown for the choice $f_{3,3} = -1$. The conditions* (C1)–(C3) *are clearly* *satisfied for any choice* $f_{3,3} \in \mathbb{R}$. *Moreover, the face* $G = P(\mathbb{1}) \in \mathcal{G}$ *is a simplex with* $|D \cap G| = |\{(3,3)\}| = 1$ *and, thus,* $\alpha^\star = (3,3)$ *satisfies the assumptions of Theorem 2.29 with* $V_G = \{(6,0),(0,6)\}$ *and* $\lambda_{6,0} = \lambda_{0,6} = 1/2$. *The coercivity of* f *hence implies* $f_{3,3} \in [-2,2]$.

Remark 2.35. The assumptions of Theorem 2.29 obviously exclude situations with $|D \cap G| > 1$ for $G \in \mathcal{G}$. While this makes our analysis incomplete, note that already the case $|D \cap G| > 0$ is degenerate in the sense that f then cannot be gem regular, and the elements of A then cannot be in general position. In this sense, cases with $|D \cap G| > 1$ are even more degenerate. □

Remark 2.36. The assumptions of Theorem 2.29 also exclude cases in which no face $G \in \mathcal{G}$ with $\alpha^\star \in G$ is a simplex. While such situations may be covered by our notion of gem regularity, they still are degenerate in the more restrictive sense that the vertices of each such G then cannot be in general position.

We believe, however, that it should be possible to generalize the assertion of Theorem 2.29 to non-simplicial faces of P by replacing the complete vertex set V_G of a face G corresponding to $\alpha^\star \in D$ by any affinely independent subset $V^\star \subseteq V_G$ with $\alpha^\star \in \operatorname{conv} V^\star$, and by using the according circuit number $\Theta(f, V^\star, \alpha^\star)$ in the estimates for $f_{\alpha^\star}$. Note that at least one such set $V^\star$ exists by Carathéodory's theorem, but as there may be several possible choices for $V^\star$, we would obtain several necessary inclusions for the coefficient $f_{\alpha^\star}$ by the technique from Theorem 2.29, and the tightest inclusions would form the necessary conditions. Unfortunately, we do not see how such results may be inferred from Proposition 2.26, as its assertion only covers complete sets $A \cap G$. Hence, we expect that these results cannot directly be deduced from our approach taken in Section 2.2, that is, the analysis along curves. □

2.3 Sufficient conditions for coercivity

We start by treating sufficient coercivity conditions for gem regular polynomials in Section 2.3.1 which actually lead to a coercivity *characterization*, before we move on to the degenerate case in Section 2.3.2.

2.3.1 A characterization of coercivity for gem regular polynomials

Proposition 2.37. *Let* f *be a gem regular polynomial satisfying the conditions* (C1)–(C3) *from Theorem 2.8. Then* f *is coercive on* $\mathbb{R}^n$.

Proof. Let $(x^k)_{k\in\mathbb{N}}$ be any sequence in $\mathbb{R}^n$ with $\lim_{k\to\infty}\|x^k\| = +\infty$. We have to show $\lim_{k\to\infty} f(x^k) = +\infty$.

With the definition $f^W(x) = \sum_{\alpha\in W} f_\alpha\, x^\alpha$ for $W \subseteq A$ and (2.3) we have $f = f^V + f^R$, as D is void by the assumption of gem regularity. The conditions (C1)–(C3) immediately imply the coercivity of f^V on $\mathbb{R}^n$, so that $\lim_{k\to\infty} f^V(x^k) = +\infty$ holds. In particular, we have $f^V(x^k) > 0$ for almost all $k \in \mathbb{N}$.

The proof will be complete if we can show the existence of some $\varepsilon > 0$ with

$$f^R(x^k) \;\geq\; (\varepsilon - 1) f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}, \tag{2.25}$$

as this implies

$$f(x^k) \;=\; f^V(x^k) + f^R(x^k) \;\geq\; \varepsilon\, f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}$$

and, thus, $\lim_{k\to\infty} f(x^k) = +\infty$.

In fact, by Proposition 2.24 for any $\alpha^\star \in R$ there exist coefficients λ_α, $\alpha \in V_0$, with

$$\alpha^\star \;=\; \sum_{\alpha\in V_0} \lambda_\alpha\, \alpha,\; \sum_{\alpha\in V_0} \lambda_\alpha = 1,\; \lambda_\alpha \geq 0,\; \alpha \in V,\; \lambda_0 > 0.$$

Hence, using (C1), the convention $0^0 = 1$ as well as (2.17) we may write

$$\begin{aligned}
f_{\alpha^\star}\,(x^k)^{\alpha^\star} \;&\geq\; -|f_{\alpha^\star}|\,|x^k|^{\alpha^\star} \;=\; -|f_{\alpha^\star}|\,|x^k|^{\sum_{\alpha\in V_0}\lambda_\alpha\,\alpha} \\
&=\; -|f_{\alpha^\star}|\,|x^k|^0 \prod_{\alpha\in V} ((x^k)^\alpha)^{\lambda_\alpha} \\
&\geq\; -|f_{\alpha^\star}| \prod_{\alpha\in V} \left(\max_{\alpha\in V}\,(x^k)^\alpha\right)^{\lambda_\alpha} \\
&=\; -|f_{\alpha^\star}| \left(\max_{\alpha\in V}\,(x^k)^\alpha\right)^{1-\lambda_0}.
\end{aligned}$$

In the following we denote, for $k \in \mathbb{N}$, by $\alpha(k)$ some $\alpha \in V$ with $(x^k)^{\alpha(k)} = \max_{\alpha\in V}(x^k)^\alpha$. (C1) and (C2) imply

$$f^V(x^k) \;=\; \sum_{\alpha\in V} f_\alpha\,(x^k)^\alpha \;\geq\; f_{\alpha(k)}\,(x^k)^{\alpha(k)} \;\geq\; \left(\min_{\alpha\in V} f_\alpha\right)(x^k)^{\alpha(k)},$$

so that, again by (C2),

$$\begin{aligned}
f_{\alpha^\star}\,(x^k)^{\alpha^\star} \;&\geq\; -|f_{\alpha^\star}| \left((x^k)^{\alpha(k)}\right)^{1-\lambda_0} \\
&\geq\; -\left(\min_{\alpha\in V} f_\alpha\right)^{-1} |f_{\alpha^\star}| \left((x^k)^{\alpha(k)}\right)^{-\lambda_0} f^V(x^k).
\end{aligned} \tag{2.26}$$

Next we shall show $\lim_{k\to\infty}(x^k)^{\alpha(k)} = +\infty$. On the contrary, assume that some subsequence $\left((x^{k_\ell})^{\alpha(k_\ell)}\right)_{\ell\in\mathbb{N}}$ is bounded above by some $M \in \mathbb{R}$. Then the definition of $\alpha(k_\ell)$ yields

$$f^V(x^{k_\ell}) \;=\; \sum_{\alpha\in V} f_\alpha\,(x^{k_\ell})^{\alpha} \;\le\; \sum_{\alpha\in V} f_\alpha\,(x^{k_\ell})^{\alpha(k_\ell)} \;\le\; M\sum_{\alpha\in V} f_\alpha$$

for all $\ell \in \mathbb{N}$. On the other hand, as a subsequence of $(x^k)_{k\in\mathbb{N}}$ the sequence $(x^{k_\ell})_{\ell\in\mathbb{N}}$ satisfies $\lim_{\ell\to\infty}\|x^{k_\ell}\| = +\infty$, so that the coercivity of f^V implies $\lim_{\ell\to\infty} f^V(x^{k_\ell}) = +\infty$, a contradiction.

The positivity of λ_0, thus, implies

$$\lim_{k\to\infty}\left((x^k)^{\alpha(k)}\right)^{-\lambda_0} \;=\; 0$$

and we arrive at $\lim_{k\to\infty}\gamma_k(\alpha^\star) = 0$ for the term

$$\gamma_k(\alpha^\star) \;:=\; \left(\min_{\alpha\in V} f_\alpha\right)^{-1}\left(|f_{\alpha^\star}|\left((x^k)^{\alpha(k)}\right)^{-\lambda_0}\right)$$

from (2.26). This implies

$$-\sum_{\alpha^\star\in R}\gamma_k(\alpha^\star) \;\ge\; -\tfrac{1}{2}$$

for almost all $k \in \mathbb{N}$, so that summing up the inequalities (2.26) over all $\alpha^\star \in R$ yields

$$\begin{aligned} f^R(x^k) \;&\ge\; \left(-\sum_{\alpha^\star\in R}\gamma_k(\alpha^\star)\right) f^V(x^k) \\ &\ge\; -\tfrac{1}{2}\,f^V(x^k) \end{aligned} \tag{2.27}$$

for almost all $k \in \mathbb{N}$, and (2.25) holds with $\varepsilon := \frac{1}{2}$. □

Example 2.38. *For the six-hump camel back function* $f(x) = x_1^2(4 - \frac{21}{10}x_1^2 + \frac{1}{3}x_1^4) + x_1x_2 + x_2^2(-4+4x_2^2)$, *direct inspection reveals that* $D(f) = \emptyset$, *and hence,* f *is gem regular (see Example 1.1 with the corresponding illustration below). Further, since* $V(f) = \{(6,0),(0,4)\}$ *with* $f_{(6,0)} = \frac{1}{3} > 0$ *and* $f_{(0,4)} = 4 > 0$ *holds,* f *fulfills the conditions* (C1)–(C3), *and thus, by latter Proposition 2.37,* f *is coercive on* $\mathbb{R}^2$.

Theorem 2.39 (Characterizations of Coercivity)**.**
For any gem regular polynomial $f \in \mathbb{R}[x]$ *the following three assertions are equivalent:*

a) f *is coercive on* $\mathbb{R}^n$.

b) $\Omega \subseteq \Omega_f$ *holds.*

c) The conditions (C1)–(C3) *from Theorem 2.8 hold.*

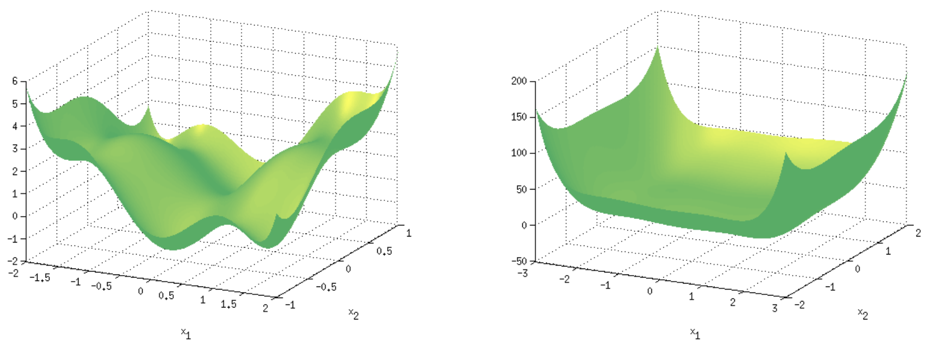

FIGURE 2.3: Illustration of Example 2.38. On the left: function f over the box $[-2, 2] \times [-1, 1]$. On the right: function f over the box $[-3, 3] \times [-1, 1]$, whose form confirms its coercivity.

Proof. Lemma 2.2 states that assertion a) implies b), in view of Remark 2.9 assertion b) implies c), and by Proposition 2.37 assertion c) implies a). □

Remark 2.40. While the equivalence of assertions a) and c) in Theorem 2.39 definitely is the important one from the application point of view, we emphasize that the equivalence of assertions a) and b) also is interesting in the following sense: it shows that the analysis of polynomials merely along certain curves is sufficiently strong to yield a characterization of an important property of polynomials, at least in the gem regular case.

More explicitly, the employed analysis along curves is not strong enough to yield necessary conditions on gem irregular coercive polynomials, where lower order monomials corresponding to the 'remaining exponents' set R may control the coercivity property (see Ex. 2.47 below). Note, however, that this analysis along curves is not used in the proof of the above sufficient condition, but that it is the estimate (2.25) which allows to ignore these lower order monomials in the gem regular case. □

2.3.2 Sufficient conditions in the degenerate case

By Carathéodory's theorem, for any degenerate multiplier $\alpha^\star \in D$ there exists a set of affinely independent points $V^\star \subseteq V$ with $\alpha^\star \in \operatorname{conv} V^\star$. In the case that a simplicial face $G \subseteq \mathcal{G}$ contains $\alpha^\star$, the set $V^\star$ can be chosen as the vertex set V_G of G. For non-simplicial faces G, however, there may exist several possibilities to choose $V^\star \subseteq V_G$.

For any set of affinely independent points $V^\star$ with $\alpha^\star \in \operatorname{conv} V^\star$, the solution λ of

$$\sum_{\alpha \in V^\star} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^\star \\ 1 \end{pmatrix}, \quad \lambda_\alpha \geq 0,\ \alpha \in V^\star \tag{2.28}$$

is unique, and again we may consider the circuit number

$$\Theta(f, V^\star, \alpha^\star) \;=\; \prod_{\alpha \in V^\star} \left(\frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha}.$$

If, in addition, $V^\star$ is chosen minimally in the sense that the presence of all points in $V^\star$ is necessary for $\alpha^\star \in \operatorname{conv} V^\star$ to hold, then we also have $\lambda_\alpha > 0$ for all $\alpha \in V^\star$.

While we were not able to use this approach in the derivation of necessary conditions in the degenerate case (cf. Rem. 2.36), it will be fruitful for the following.

Theorem 2.41. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the conditions* (C1)–(C3) *from Theorem 2.8. Furthermore for each $\alpha^\star \in D$ let $V^\star \subseteq V$ denote a minimal affinely independent set with $\alpha^\star \in \operatorname{conv} V^\star$, let $w(\alpha^\star) > 0$, $\alpha^\star \in D$, denote weights with $\sum_{\alpha^\star \in D} w(\alpha^\star) \le 1$, and let*

$$f_{\alpha^\star} \;>\; -w(\alpha^\star)\,\Theta(f, V^\star, \alpha^\star) \quad \text{if} \quad \alpha^\star \in 2\mathbb{N}_0^n$$

and

$$|f_{\alpha^\star}| \;<\; w(\alpha^\star)\,\Theta(f, V^\star, \alpha^\star) \quad \text{else.}$$

Then f is coercive on $\mathbb{R}^n$.

Proof. As in the proof of Proposition 2.37, let $(x^k)_{k \in \mathbb{N}}$ be any sequence in $\mathbb{R}^n$ with $\lim_{k\to\infty} x^k = +\infty$. In view of (2.3) we have $f = f^V + f^D + f^R$, where the conditions (C1)–(C3) imply $\lim_{k\to\infty} f^V(x^k) = +\infty$ and, thus, $f^V(x^k) > 0$ for almost all $k \in \mathbb{N}$. The proof will be complete if we can show the existence of some $\varepsilon > 0$ with

$$f^D(x^k) + f^R(x^k) \;\ge\; (\varepsilon - 1) f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}, \tag{2.29}$$

as this implies

$$f(x^k) \;=\; f^V(x^k) + f^D(x^k) + f^R(x^k) \;\ge\; \varepsilon\, f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}$$

and, thus, $\lim_{k\to\infty} f(x^k) = +\infty$.

In fact, the proof is based upon the estimate

$$f^{V^\star}(x^k) \;\ge\; \Theta(f, V^\star, \alpha^\star)\, |x^k|^{\alpha^\star} \tag{2.30}$$

for any $k \in \mathbb{N}$ and $\alpha^\star \in D$, where $\Theta(f, V^\star, \alpha^\star)$ is defined via the unique multipliers λ_α, $\alpha \in V^\star$, from (2.28).

To see (2.30), we distinguish similar cases as in Remark 2.30 and define the index sets

$$I_0(x^k) \;:=\; \{i \in I \,|\, x_i^k = 0\} \quad \text{and} \quad I_0(\alpha^\star) \;=\; \{i \in I \,|\, \alpha_i^\star = 0\}.$$

In the case $I_0(x^k) \not\subseteq I_0(\alpha^\star)$ there exists some $i \in I$ with $x_i^k = 0$ and $\alpha_i^\star \neq 0$, so that $(x_i^k)^{\alpha_i^\star} = 0$ and, thus, $|x^k|^{\alpha^\star} = 0$ holds. The relation (2.30) then collapses to the nonnegativity of $f^{V^\star}(x^k)$ which clearly holds in view of (C1) and (C2).

To study the second case, $I_0(x^k) \subseteq I_0(\alpha^\star)$, let us first discuss its special subcase $I_0(x^k) = \emptyset$. Then we have $|x^k|^\alpha > 0$ for any $\alpha \in V^\star$, so that the arithmetic-geometric mean inequality, together with (C1) and (C2), yields

$$\begin{aligned} f^{V^\star}(x^k) &= \sum_{\alpha \in V^\star} f_\alpha\,(x^k)^\alpha \;=\; \sum_{\alpha \in V^\star} f_\alpha\,|x^k|^\alpha \;\geq\; \prod_{\alpha \in V^\star} \left(\frac{f_\alpha\,|x^k|^\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} \\ &= \prod_{\alpha \in V^\star} \left(\frac{f_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} \prod_{\alpha \in V^\star} \left(|x^k|^\alpha\right)^{\lambda_\alpha} \;=\; \Theta(f, V^\star, \alpha^\star)\,|x^k|^{\alpha^\star}, \end{aligned}$$

that is, (2.30). Finally, for $\emptyset \neq I_0(x^k) \subseteq I_0(\alpha^\star)$ each $i \in I_0(x^k)$ satisfies $(x_i^k)^{\alpha_i^\star} = 0^0 = 1$ and, thus, $|x^k|^{\alpha^\star} = \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i^\star}$. Moreover, for each $i \in I_0(\alpha^\star)$ we find

$$0 \;=\; \alpha_i^\star \;=\; \sum_{\alpha \in V^\star} \lambda_\alpha\,\alpha_i\,,$$

so that the positivity of all λ_α, $\alpha \in V^\star$, implies $\alpha_i = 0$ for all $\alpha \in V^\star$. Hence, for any $\alpha \in V^\star$ and $i \in I_0(x^k) \subseteq I_0(\alpha^\star)$ we also have $(x_i^k)^{\alpha_i} = 0^0 = 1$ and, thus, $|x^k|^\alpha = \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i}$, so that we may write

$$f^{V^\star}(x^k) \;=\; \sum_{\alpha \in V^\star} f_{\alpha^\star} \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i}.$$

Since $|x_i^k| > 0$ holds for all $i \in I \setminus I_0(x^k)$, we may apply the arithmetic-geometric mean inequality to this term, as above in the case $I_0(x^k) = \emptyset$, and arrive at

$$f^{V^\star}(x^k) \;\geq\; \Theta(f, V^\star, \alpha^\star) \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i^\star} \;=\; \Theta(f, V^\star, \alpha^\star)\,|x^k|^{\alpha^\star}.$$

Hence, we have shown the estimate (2.30) in any case.

In view of

$$f_{\alpha^\star}\,(x^k)^{\alpha^\star} \begin{cases} = \; f_{\alpha^\star}\,|x^k|^{\alpha^\star} & \text{for } \; \alpha^\star \in 2\mathbb{N}_0^n \\ \geq \; -|f_{\alpha^\star}|\,|x^k|^{\alpha^\star} & \text{else,} \end{cases}$$

under the assumptions of the theorem there exists some $\delta(\alpha^\star) > 0$ with

$$\begin{aligned}
f_{\alpha^\star}\,(x^k)^{\alpha^\star} &\geq (\delta(\alpha^\star) - w(\alpha^\star)\,\Theta(f, V^\star, \alpha^\star))\;|x^k|^{\alpha^\star} \\
&= \big(\delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star) - w(\alpha^\star)\big)\;\Theta(f, V^\star, \alpha^\star)\,|x^k|^{\alpha^\star} \\
&\geq \big(\delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star) - w(\alpha^\star)\big)\;f^{V^\star}(x^k) \qquad (2.31) \\
&\geq \big(\delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star) - w(\alpha^\star)\big)\;f^{V}(x^k), \qquad (2.32)
\end{aligned}$$

where (2.31) holds due to (2.30) for a sufficiently small choice of $\delta(\alpha^\star)$, and (2.32) due to (C1) and (C2).

Thus, with the notation from the proof of Proposition 2.37 for $\alpha^\star \in R$ and (2.27), we arrive at

$$\begin{aligned}
&f^D(x^k) + f^R(x^k) \\
&\geq \left(\sum_{\alpha^\star\in D} \big(\delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star) - w(\alpha^\star)\big) - \sum_{\alpha^\star\in R} \gamma_k(\alpha^\star)\right) f^V(x^k) \\
&\geq \left(\sum_{\alpha^\star\in D} \delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star) - \sum_{\alpha^\star\in R} \gamma_k(\alpha^\star) - 1\right) f^V(x^k)
\end{aligned}$$

and, due to

$$\lim_{k\to\infty} \sum_{\alpha^\star\in R} \gamma_k(\alpha^\star) \;=\; 0,$$

may choose

$$\varepsilon \;:=\; \frac{1}{2} \sum_{\alpha^\star\in D} \delta(\alpha^\star)\Theta^{-1}(f, V^\star, \alpha^\star)$$

in (2.29). □

Remark 2.42. We emphasize that, in contrast to our necessary condition for the degenerate case from Theorem 2.29, the sufficient condition from Theorem 2.41 holds for general polynomials $f \in \mathbb{R}[x]$, and does not make any assumptions on the structure of faces related to degenerate exponent vectors.

□

Remark 2.43. For the special case of a gem irregular polynomial $f \in \mathbb{R}[x]$ with a singleton set $D = \{\alpha^\star\}$ such that the minimal face $G \in \mathcal{G}$ with $\alpha^\star \in G$ is simplicial, the gap between the necessary condition from Theorem 2.29 and the sufficient condition from Theorem 2.41 reduces to the strictness of an inequality: the necessary condition states that (C1)–(C3) as well as

$$f_{\alpha^\star} \;\geq\; -\Theta(f, V_G, \alpha^\star) \;\text{ if }\; \alpha^\star \in 2\mathbb{N}_0^n$$

and

$$|f_{\alpha^\star}| \;\leq\; \Theta(f, V_G, \alpha^\star) \;\; \text{else}$$

hold, and the sufficient condition just replaces the nonstrict by strict inequalities in either case. Note that the choice $V^\star = V_G$ is mandatory for a minimal simplicial face G.

□

Other than in the special degenerate case from Remark 2.43, the gap between necessary and sufficient conditions is significantly larger, so that we expect that the necessary (cf. also Rem. 2.36) as well as the sufficient condition can be improved further. In fact, already for the case $D = \{(\alpha^\star)^1, (\alpha^\star)^2\}$ such that the minimal faces $G_i \in \mathcal{G}$ with $(\alpha^\star)^i \in G_i$ are simplicial and not identical, the need to choose weights $w((\alpha^\star)^1)$ and $w((\alpha^\star)^2)$ in Theorem 2.41 leads to a larger discrepancy to the necessary conditions from Theorem 2.29 than just the strictness of inequalities.

In the following we will show how Theorem 2.41 can be modified to improve the sufficient conditions in this respect. The price to pay is, unfortunately, that we need to require an extra condition on the polynomial $f \in \mathbb{R}[x]$ (cf. Rem. 2.42). For the statement of this condition, for any $\alpha^\star \in D$ choose a minimal affinely independent set $V^\star(\alpha^\star) \subseteq V$ with $\alpha^\star \in \operatorname{conv} V^\star(\alpha^\star)$ and define the set $\mathcal{V} := \{V^\star(\alpha^\star) |\, \alpha^\star \in D\}$. In particular, if two exponent vectors $(\alpha^\star)^1$ and $(\alpha^\star)^2$ satisfy $V^\star((\alpha^\star)^1) = V^\star((\alpha^\star)^2)$, then this set is only listed *once* in $\mathcal{V}$. We will need to require that the sets in $\mathcal{V}$ can be chosen to be mutually disjoint.

Theorem 2.44. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the conditions* (C1)–(C3) *from Theorem 2.8. Furthermore for each $\alpha^\star \in D$ let $V^\star(\alpha^\star) \subseteq V$ denote a minimal affinely independent set with $\alpha^\star \in \operatorname{conv} V^\star(\alpha^\star)$ such that the sets in $\mathcal{V} = \{V^\star(\alpha^\star) |\, \alpha^\star \in D\}$ are mutually disjoint, let $w(\alpha^\star) > 0$, $\alpha^\star \in D$, denote weights with $\sum_{\alpha^\star \in D \cap \operatorname{conv} V^\star} w(\alpha^\star) \leq 1$ for each $V^\star \in \mathcal{V}$, and let*

$$f_{\alpha^\star} \;>\; -w(\alpha^\star)\, \Theta(f, V^\star, \alpha^\star) \;\; \textit{if} \;\; \alpha^\star \in 2\mathbb{N}_0^n$$

and

$$|f_{\alpha^\star}| \;<\; w(\alpha^\star)\, \Theta(f, V^\star, \alpha^\star) \;\; \textit{else.}$$

Then f is coercive on $\mathbb{R}^n$.

Proof. This proof is identical to the proof of Theorem 2.41 until the estimate (2.31), from which we do *not* deduce the coarser estimate (2.32), but proceed as follows. To bound $f^D(x^k)$ from below, first we group the sum over all $\alpha^\star \in D$ which share the same set $V^\star \in \mathcal{V}$ and write

$$f^D(x^k) \;=\; \sum_{V^\star\in\mathcal{V}} \; \sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} f_{\alpha^\star}(x^k)^{\alpha^\star}.$$

For any $V^\star \in \mathcal{V}$ the inner sum satisfies

$$\begin{aligned}
\sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} f_{\alpha^\star}(x^k)^{\alpha^\star} \;&\geq\; \sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \Big(\big(\delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star) - w(\alpha^\star)\big)\, f^{V^\star}(x^k)\Big)\\
&\geq\; \left(\sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star) - 1\right) f^{V^\star}(x^k)\\
&\geq\; \min_{V^\star\in\mathcal{V}} \left(\sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star) - 1\right) f^{V^\star}(x^k).
\end{aligned}$$

As the sets $V^\star \in \mathcal{V}$ are mutually disjoint, for sufficiently small choices of $\delta(\alpha^\star)$, $\alpha^\star \in D$, the conditions (C1) and (C2) imply

$$\begin{aligned}
f^D(x^k) \;&=\; \sum_{V^\star\in\mathcal{V}} \; \sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} f_{\alpha^\star}(x^k)^{\alpha^\star}\\
&\geq\; \min_{V^\star\in\mathcal{V}} \left(\sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star) - 1\right) \sum_{V^\star\in\mathcal{V}} f^{V^\star}(x^k)\\
&\geq\; \left(\min_{V^\star\in\mathcal{V}} \sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star) - 1\right) f^{V}(x^k).
\end{aligned}$$

From here, the proof may be continued as the proof of Theorem 2.41, with the choice

$$\varepsilon \;:=\; \frac{1}{2}\min_{V^\star\in\mathcal{V}} \sum_{\alpha^\star\in D\,\cap\,\mathrm{conv}\,V^\star} \delta(\alpha^\star)\Theta^{-1}(f,V^\star,\alpha^\star). \qquad \square$$

As an application of Theorem 2.44 recall the above mentioned situation $D = \{(\alpha^\star)^1, (\alpha^\star)^2\}$ such that the minimal faces $G_i \in \mathcal{G}$ with $(\alpha^\star)^i \in G_i$ are simplicial and not identical. If, in addition, G_1 and G_2 are actually disjoint, then Theorem 2.44 may be applied, and the resulting sufficient conditions for coercivity differ from the necessary conditions of Theorem 2.29 again just by the strictness of inequalities.

Example 2.45. *Examples 2.32, 2.33, and 2.34 all satisfy the special condition discussed in Remark 2.43. In particular, the coercivity of the polynomial $f(x) = x_1^6+x_2^6+f_{3,3}x_1^3x_2^3+x_1^4-x_2+1$ on $\mathbb{R}^2$ may not only be guaranteed for $f_{3,3}=-1$, as stated in [41], but by Theorem 2.41 even for any $f_{3,3}\in(-2,2)$.*

Example 2.46. *Minimal examples for polynomials satisfying the special condition from Remark 2.43, but being critical in the sense that only the necessary conditions from*

Theorem 2.29 hold, but not the sufficient ones from Theorem 2.41, are $f^{\pm}(x) = x_1^2 \pm 2x_1x_2 + x_2^2$. *Direct inspection immediately reveals that neither* f^+ *nor* f^- *are coercive.*

Example 2.47. *A further effect is illustrated if the situation from Example 2.46 occurs for higher order terms, as in the polynomial* $f(x) = x_1^4 - 2x_1^2x_2^2 + x_2^4$, *which is critical and non-coercive for similar reasons as the polynomials from Example 2.46. Here, unlike in the gem regular case, 'remaining exponents' in the set* R *do have an influence on the coercivity of* f *since, for example,* $f(x) + x_1^2 + x_2^2$ *is coercive.*

Note that Theorem 2.41 presents our most general sufficient conditions for coercivity, while Theorems 2.39 and 2.44 refine them under more special assumptions. As any coercive and lower semi-continuous function on $\mathbb{R}^n$ attains its infimum, an obvious first application of Theorem 2.41 is that any polynomial $f \in \mathbb{R}[x]$ satisfying the assumptions of Theorem 2.41 attains its infimum v over $\mathbb{R}^n$. In particular, f is then bounded below, and $f - v$ is positive semi-definite on $\mathbb{R}^n$.

Moreover, as all lower level sets of any coercive function are bounded, a basic semi-algebraic set

$$S = \{x \in \mathbb{R}^n |\ g_1(x) = 0,\ \dots,\ g_l(x) = 0,\ h_1(x) \geq 0,\ \dots,\ h_m(x) \geq 0\}$$

with polynomials $g_1, \dots, g_l, h_1 \dots, h_m \in \mathbb{R}[x]$ is bounded if at least one of the functions g_i, $i = 1, \dots, l$, $-g_i$, $i = 1, \dots, l$, $-h_j$, $j = 1, \dots, m$, satisfies the assumptions of Theorem 2.41. In particular, the zero set of any polynomial $f \in \mathbb{R}[x]$ satisfying the assumptions of Theorem 2.41 is bounded.

A less obvious application is given in the next section.

2.3.3 The Malgrange and Fedoryuk conditions

In the following, using some immediate implications from the results presented in [84], we will show that the assumptions from Theorem 2.41 imposed on $f \in \mathbb{R}[x]$ directly imply that f fulfills the so-called Malgrange and Fedoryuk conditions on $\mathbb{R}^n$. To this end we use some nontrivial results concerning the characterization of coercivity of polynomials on closed semi-algebraic sets via the Malgrange or Fedoryuk conditions from [84]. Before doing so in Corollary 2.51, we shortly recall the definition of the Malgrange and Fedoryuk conditions and briefly mention the related concepts of asymptotic and generalized critical values of polynomials, which are also very interesting for theoretical and numerical aspects of global polynomial optimization theory (see, e.g., [75, 76]).

Definition 2.48 (Malgrange condition, see [1], [47])**.** *A polynomial $f \in \mathbb{R}[x]$ satisfies the* Malgrange condition *at a value $y \in \mathbb{R}$ if and only if there exists a constant $C > 0$ such that the inequality*

$$\|x\| \cdot \|\nabla f(x)\| \geq C$$

holds for all $x \in \mathbb{R}^n$ with $\|x\|$ sufficiently large and $f(x)$ sufficiently close to y.

Definition 2.49 (Fedoryuk condition, see [23], [84])**.** *A polynomial $f \in \mathbb{R}[x]$ satisfies the* Fedoryuk condition *on $\mathbb{R}^n$ if there exist positive constants δ and R such that*

$$\|\nabla f(x)\| \geq \delta \quad \text{for all } x \in \mathbb{R}^n \text{ with } \|x\| \geq R.$$

The Fedoryuk and Malgrange conditions arise in the context of analyzing the bifurcation sets and generalized critical values of polynomials $f : \mathbb{K}^n \to \mathbb{K}$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. For more details see, e.g., [1, 23, 39, 40, 47, 59, 84].

In the following the definition of asymptotic and generalized critical values of a polynomial $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ is recalled (see, e.g., [1, 47, 59, 76, 84]). Their connection with the Malgrange condition is studied in detail in [1].

Definition 2.50 (Asymptotic and generalized critical values)**.** *For $f \in \mathbb{R}[x]$ the set*

$$K_\infty(f, \mathbb{R}^n) := \{y \in \mathbb{R} |\ \exists \text{ sequence } (x^k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n, \|x^k\| \to \infty \text{ with } f(x^k) \to y$$
$$\text{and } (1 + \|x^k\|)\|\nabla f(x^k)\| \to 0\}$$

is called the set of asymptotic critical values of f on $\mathbb{R}^n$ and the set

$$K(f, \mathbb{R}^n) := \{y \in \mathbb{R} |\ \exists \text{ sequence } (x^k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n \text{ with } f(x^k) \to y$$
$$\text{and } (1 + \|x^k\|)\|\nabla f(x^k)\| \to 0\}$$

is called the set of generalized critical values of f on $\mathbb{R}^n$.

The concept of asymptotic critical values in a general functional analytic setting together with its far reaching application has been developed by Rabier in his seminal work [71]. In [76] it is already observed that the infimum of a lower bounded polynomial $f \in \mathbb{R}[x]$ is contained in the set $K(f, \mathbb{R}^n)$. This fact is further used in [75], where an efficient algorithm for computing the global infimum $\inf_{x\in\mathbb{R}^n} f(x)$ of $f \in \mathbb{R}[x]$ is developed based on computing the set $K(f, \mathbb{R}^n)$.

Corollary 2.51. *Let $f \in \mathbb{R}[x]$ satisfy the assumptions of Theorem 2.41. Then f also satisfies the Fedoryuk and Malgrange conditions on $\mathbb{R}^n$.*

Proof. By Theorem 2.41, the polynomial f is coercive and thus $\inf_{x\in\mathbb{R}^n} f(x) > -\infty$ holds. By setting $S := \mathbb{R}^n$ in [84, Th. 4.2] the assertion directly follows. □

2.3.4 A growth condition

While Example 2.45 shows that, in particular, the sufficient condition for coercivity from [41] can be improved with respect to possible values of polynomial coefficients, in the following we will show that our sufficient condition from Theorem 2.41 covers whole classes of polynomials which cannot be treated at all by the approach from [41].

To see this, we start by repeating the result from [41] explicitly (where the choice of the norm is, again, irrelevant).

Lemma 2.52 ([41, Lemma 3.1])**.**
Decompose $f \in \mathbb{R}[x]$ with $\deg(f) \in 2\mathbb{N}$ into a sum of polynomials, $f = f_0 + \cdots + f_{\deg(f)}$, where f_i is homogeneous of degree i for $i = 0, \dots, \deg(f)$. If the growth condition

$$\exists\,\delta > 0\;\forall\, x \in \mathbb{R}^n : \quad f_{\deg(f)}(x) \;\geq\; \delta\,\|x\|^{\deg(f)} \tag{G}$$

is satisfied, then f is coercive on $\mathbb{R}^n$.

The following example presents a polynomial which is coercive on $\mathbb{R}^n$ while violating the growth condition (G).

Example 2.53. *Consider the gem regular polynomial $f(x) := x_1^2+x_2^2+x_1^2x_2^2$ which clearly fulfills conditions* (C1)–(C3)*. By our Characterization Theorem 2.39 the polynomial f is coercive on $\mathbb{R}^2$, but this cannot be verified using the sufficiency criterion* (G)*. In fact, we have $\deg(f) = 4$, $f_4 = x_1^2x_2^2$, and choosing the Euclidean norm we obtain for every positive constant δ*

$$0 \;=\; f_4(0,1) \;<\; \delta\|(0,1)\|_2^4 \;=\; \delta.$$

The sufficiency criterion (G) *is, hence, violated although f is coercive. Many different examples having this property can be constructed easily in the same way. One only has to find a coercive polynomial $f \in \mathbb{R}[x]$ (e.g., using Ths. 2.39, 2.41 or 2.44) and a point $\bar{x} \neq 0$ such that $f_{\deg(f)}(\bar{x}) = 0$.*

In the subsequent Chapter 3 we show that, for gem regular polynomials of even degree, the growth condition (G) actually implies our sufficient conditions (C1)–(C3) for coercivity and is then, in view of Example 2.53, strictly stronger than our conditions. In fact, in Chapter 3 it turns out that, under the above assumptions, the growth condition (G) characterizes the stronger property of so-called stable coercivity of gem regular polynomials. The latter refers to the condition that coercivity prevails under certain sufficiently small perturbations of the polynomial coefficients. An alternative characterization of stable coercivity is possible by conditions (C1)–(C3) and an extra condition (C4) from Chapter 3, again in terms of the Newton polytope at infinity, so that in the gem regular case the even degree of the polynomial together with condition (G) may be characterized by (C1)–(C4).

Chapter 3

Coercive polynomials: Stability, order of growth, and Newton polytopes

3.1 Chapter overview

This chapter is based on the article [5] and it is structured as follows. Section 3.2 deals with the stability concept for coercivity of multivariate polynomials. In particular, for coercive polynomials $f \in \mathbb{R}[x]$, the notion of q-stable coercivity and the degree of stable coercivity $s(f)$ are introduced (see Defs. 3.1 and 3.2). Moreover, we introduce the degree of convenience $c(f)$ for general multivariate polynomials $f \in \mathbb{R}[x]$ (see Def. 3.6) and we show, as the main result of the section, that for a broad class of coercive polynomials $f \in \mathbb{R}[x]$ the degree of convenience $c(f)$ of f coincides with the corresponding degree of stable coercivity $s(f)$ of f (see Ths. 3.14 and 3.17).

In Section 3.3 we focus on those coercive polynomials $f \in \mathbb{R}[x]$ for which the general relation $s(f) \leq \deg(f)$ is tight, that is, the case of $\deg(f)$-stably coercive polynomials. The reason for investigating this special case is that in [41, Sec. 3.2] the authors introduce a sufficient condition for coercivity on $\mathbb{R}^n$ of polynomials $f \in \mathbb{R}[x]$ (see condition (G) in Lem. 2.52) which, on one hand, is computationally tractable because it can be verified by solving a hierarchy of semidefinite programs. On the other hand, as indicated in Example 2.53 (cf. [4, Ex. 3.16]), this condition is rather strong since many coercive polynomials violate it. As the reason for this effect we shall show that the condition actually characterizes the $\deg(f)$-stable coercivity of gem regular polynomials $f \in \mathbb{R}[x]$ (see Th. 3.23 below), which is a stronger property than general coercivity on $\mathbb{R}^n$.

Section 3.4 deals with the growth property of coercive polynomials $f \in \mathbb{R}[x]$ and with its connection to the stability concept introduced in Section 3.2. In particular, for coercive polynomials $f \in \mathbb{R}[x]$ the notion of q-coercivity and the order of coercivity $o(f)$ is introduced in order to measure how fast they grow on $\mathbb{R}^n$. We will prove that for a broad class of coercive polynomials $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ not only the aforementioned identity $s(f) = c(f)$ holds, but also $o(f) = c(f)$, which leads to the main results of the present chapter, that is, Theorem 3.32 and 3.37. Moreover, with Theorem 3.35 we sharpen the Characterization Theorem 2.39, the main result from [4].

In Section 3.5 we discuss some applications of our results on the growth properties of coercive polynomials in different areas. In particular, in Section 3.5.1, we show that a broad class of coercive polynomials $f \in \mathbb{R}[x]$ possesses surjective gradient maps $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$. In Section 3.5.2 we prove Hölder type error bounds for the inequality $f \leq 0$ with f from this broad class of coercive polynomials. In Section 3.5.3 we apply our main results concerning the growth of coercive polynomials to the problem of existence of solutions of a broad class of optimization problems arising in the calculus of variations, which closes the present chapter.

3.2 Stable coercivity

In this section we investigate for which polynomials coercivity is stable under small perturbations of the polynomial coefficients up to some given degree. The following definition is inspired by the definition of the stable compactness property of basic semi-algebraic sets from [60, Sec. 5].

Definition 3.1. *A polynomial $f \in \mathbb{R}[x]$ is called q-stably coercive on $\mathbb{R}^n$ for some $q \in \mathbb{N}_0$ if f is coercive and remains coercive on $\mathbb{R}^n$ for all sufficiently small perturbations by polynomials of degree at most q, that is, if there exists some $\varepsilon > 0$ such that for every $g \in \mathbb{R}[x]$ with $\deg(g) \leq q$ and $|g_\alpha| \leq \varepsilon$ for all $\alpha \in A(g)$, the polynomial $f + g$ is coercive on $\mathbb{R}^n$.*

Definition 3.2 (Degree of stable coercivity)**.** *For a coercive polynomial $f \in \mathbb{R}[x]$ we call the number*

$$s(f) := \max\{q \in \mathbb{N}_0 |\ f \textit{ is } q\textit{-stably coercive on } \mathbb{R}^n\}$$

the degree of stable coercivity of f on $\mathbb{R}^n$. A coercive polynomial $f \in \mathbb{R}[x]$ with degree of stable coercivity equal to $s(f)$ is called stably coercive on $\mathbb{R}^n$ of degree $s(f)$.

Lemma 3.3. *If $f \in \mathbb{R}[x]$ is coercive then the necessary conditions from Theorem 2.29 as well as the inequalities $0 \leq s(f) \leq \deg(f)$ hold.*

Proof. The first part of the assertion directly follows by Theorem 2.29. Since for a coercive polynomial $f \in \mathbb{R}[x]$ the value of its constant monomial $f_0 \in \mathbb{R}$ is irrelevant for its coercivity property, it holds that $f + c$ is coercive on $\mathbb{R}^n$ for all $c \in \mathbb{R}$. Thus, f is 0-stably coercive on $\mathbb{R}^n$ and $s(f) \geq 0$ follows. Further, for an arbitrary $i \in I$ and an arbitrary $\varepsilon > 0$, with $g^\varepsilon(x) := \varepsilon x_i^{\deg(f)+1}$ one clearly obtains $\deg(g^\varepsilon) = \deg(f) + 1$, $|g^\varepsilon_\alpha| \leq \varepsilon$ for all $\alpha \in A(g^\varepsilon) = \{(\deg(f) + 1)e_i\}$, and $(\deg(f) + 1)e_i \in V(f + g^\varepsilon)$. By Remark 2.17 it holds $\deg(f) \in 2\mathbb{N}$, and one obtains $(\deg(f) + 1)e_i \notin 2\mathbb{N}_0^n$, which means that $f + g^\varepsilon$ does not fulfill the condition (C1). Hence, Theorem 2.29 implies that $f + g^\varepsilon$ is not coercive on $\mathbb{R}^n$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this is a contradiction to f being $(\deg(f) + 1)$-stably coercive on $\mathbb{R}^n$ and, thus, one obtains $s(f) \leq \deg(f)$. ☐

Remark 3.4. Clearly, if $f \in \mathbb{R}[x]$ is q-stably coercive on $\mathbb{R}^n$ for some $q \in \mathbb{N}_0$, then f is $\tilde{q}$-stably coercive on $\mathbb{R}^n$ for all $\tilde{q} \in \mathbb{N}_0$ with $\tilde{q} < q$. ☐

The following example shows that the upper bound for $s(f)$ from Lemma 3.3 is not necessarily attained given a coercive polynomial $f \in \mathbb{R}[x]$.

Example 3.5. *Consider the gem regular polynomial* $f(x) = x_1^4 + x_2^2 + x_1^2 x_2^2$ *which, by Theorem 2.39, is coercive on* $\mathbb{R}^2$*. One has* $\deg(f) = 4$*, but direct inspection reveals that* $s(f) < \deg(f)$ *has to hold, as any perturbation of* f *by* $g^\varepsilon(x) := -\varepsilon x_2^4$ *with some* $\varepsilon > 0$ *results in* $f + g^\varepsilon$ *not being coercive on* $\mathbb{R}^2$*. In fact, one obtains* $\deg(f)e_2 \in V(f + g^\varepsilon)$ *with* $(f + g^\varepsilon)_{\deg(f)e_2} = -\varepsilon < 0$*, and thus,* $f + g^\varepsilon$ *does not fulfill the necessary condition for coercivity* (C2) *from Theorem 2.29 (see Fig.3.1).*

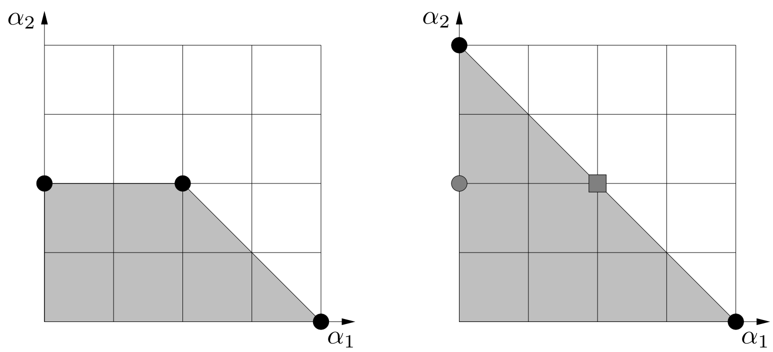

FIGURE 3.1: On the left: Illustration of $\text{New}_\infty(f)$ from Example 3.5. On the right: Illustration of $\text{New}_\infty(f + g^\varepsilon)$ from Example 3.5 for some $\varepsilon > 0$. In both pictures the shaded area corresponds to the Newton polytope at infinity of the corresponding function. The filled circles stand for the vertex set at infinity of the Newton polytope at infinity. In the right picture, the shaded circle stands for the singleton $R(f + g^\varepsilon)$ whereas the shaded square represents the singleton $D(f + g^\varepsilon)$.

Next we shall relate the degree of stable coercivity of f to a geometric property of the Newton polytope at infinity of f. For the following terminology note that the condition (C3) may be seen as a special condition for a polynomial being convenient (see, e.g., [21, 49, 85] for the definition of convenient polynomials). To the best of our knowledge, the notion of convenient polynomials (from the French "polynôme commode") for the first time is introduced in [49], where a deep connection between Milnor numbers and some geometric properties of Newton polytopes is revealed, whereas in [21] the setting of convenient polynomials is used for considerations on the existence of optimal solutions of constrained polynomial optimization problems. In [85] the convenient polynomials play a role for establishing a relation between Newton polytopes and bounded below polynomials.

Definition 3.6 (Degree of convenience)**.** *For a polynomial $f \in \mathbb{R}[x]$ we call the number*

$$c(f) := \min_{i \in I}\{\max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\}\}$$

the degree of convenience of f.

In the following, let Δ^n denote the lattice simplex $\operatorname{conv}\{0, e_i,\ i \in I\} \subseteq \mathbb{R}^n$ and for any $X \subseteq \mathbb{R}^n$ and $d \in \mathbb{R}$ we put $dX := \{dx \in \mathbb{R}^n |\ x \in X\}$. Note that the lattice polytope $d\Delta^n$ is maximal among all Newton polytopes at infinity of polynomials $f \in \mathbb{R}[x]$ with $\deg(f) \leq d$ in the sense that the inclusion $\operatorname{New}_\infty(f) \subseteq d\Delta^n$ holds for these f (and that it is attained as the Newton polytope at infinity of, for example, the polynomial $\sum_{i \in I} x_i^d$). The next proposition gives an alternative description of the degree of convenience from Definition 3.6.

Proposition 3.7. *For $f \in \mathbb{R}[x]$ one has*

$$c(f) = \max\{c \in \mathbb{N}_0 |\ c \cdot \Delta^n \subseteq \operatorname{New}_\infty(f)\}.$$

Proof. First we show $c(f) \leq \max\{c \in \mathbb{N}_0 |\ c \cdot \Delta^n \subseteq \operatorname{New}_\infty(f)\}$. By Definition 3.6 it holds

$$c(f) \leq c_i := \max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\} \quad \text{for all } i \in I, \tag{3.1}$$

and, hence

$$\begin{aligned} c(f)\Delta^n &= c(f)\operatorname{conv}\{0, e_i,\ i \in I\} = \operatorname{conv}\{0, c(f)e_i,\ i \in I\} \\ &\subseteq \operatorname{conv}\{0, c_i e_i,\ i \in I\} \subseteq \operatorname{conv} A_0(f) = \operatorname{New}_\infty(f), \end{aligned}$$

where the first inclusion follows from the property (3.1), and the second inclusion from the facts that $c_i e_i \in A_0(f)$ for all $i \in I$, and $0 \in A_0(f)$. This implies

$$c(f) \leq \max\{c \in \mathbb{N}_0 |\ c \cdot \Delta^n \subseteq \operatorname{New}_\infty(f)\}.$$

It remains to show $c(f) \geq \max\{c \in \mathbb{N}_0 |\ c \cdot \Delta^n \subseteq \mathrm{New}_\infty(f)\}$. Denoting $c^\star := \max\{c \in \mathbb{N}_0 |\ c \cdot \Delta^n \subseteq \mathrm{New}_\infty(f)\}$ one obtains

$$c^\star e_i \in \mathrm{New}_\infty(f) \quad \text{for all } i \in I.$$

The latter is only possible if for every $i \in I$ there exists $c_i^\star \in \mathbb{N}_0$ with $c_i^\star \geq c^\star$ and $c_i^\star e_i \in A_0(f)$. This implies

$$\max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\} \geq c^\star \quad \text{for every } i \in I,$$

which results in

$$\min_{i \in I}\{\max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\}\} \geq c^\star,$$

and the assertion follows. □

Example 3.8. *The geometric interpretation of the degree of convenience in Proposition 3.7 yields $c(f) = 2$ for the polynomial $f(x) = x_1^4 + x_2^2 + x_1^2 x_2^2$ from Example 3.5 (see Fig.4.1).*

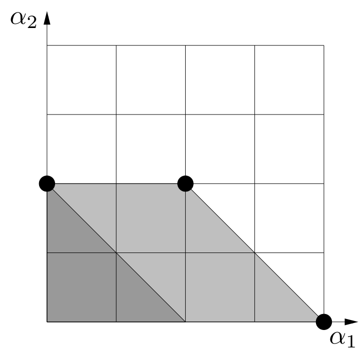

FIGURE 3.2: Illustration of $\mathrm{New}_\infty(f)$ together with $c(f)\Delta^2$ from Example 3.8. The light shaded area corresponds to the set $\mathrm{New}_\infty(f)$, and the filled circles stand for the vertex set at infinity $V(f)$. The dark shaded area corresponds to the set $c(f)\Delta^2$.

Lemma 3.9. *If $f \in \mathbb{R}[x]$ is coercive on $\mathbb{R}^n$, then we have $c(f) = 2\min_{i \in I} k_i \in 2\mathbb{N}$ with k_i, $i \in I$, from* (C3).

Proof. By Theorem 2.29, f fulfills the condition (C3), that is, for all $i \in I$ the set $V(f)$ contains a vector of the form $2k_i e_i$ with $k_i \in \mathbb{N}$. This implies $\max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\} = 2k_i$ for all $i \in I$, and with Definition 3.6 one obtains

$$c(f) = \min_{i \in I}\{\max\{c \in \mathbb{N}_0 |\ ce_i \in A_0(f)\}\} = 2 \min_{i \in I} k_i \in 2\mathbb{N}.$$

□

Example 3.10. *Due to* $k_1 = 2$ *and* $k_2 = 1$*, the formula for the degree of convenience in Lemma 3.9 yields* $c(f) = 2\min\{2,1\} = 2$ *for the polynomial* $f(x) = x_1^4 + x_2^2 + x_1^2 x_2^2$ *from Example 3.5.*

Lemma 3.11. *For all* $f \in \mathbb{R}[x]$ *it holds* $0 \leq c(f) \leq \deg(f)$*.*

Proof. The assertion follows from Proposition 3.7 since the inclusions $0 \cdot \Delta^n = \{0\} \subseteq \mathrm{New}_\infty(f) \subseteq \deg(f)\Delta^n$ hold for all $f \in \mathbb{R}[x]$. □

Next, for a coercive polynomial $f \in \mathbb{R}[x]$, we shall improve the upper bound on its degree of stable coercivity $s(f)$ from Lemma 3.3.

Lemma 3.12. *If* $f \in \mathbb{R}[x]$ *is coercive, then the necessary conditions from Theorem 2.29 as well as the inequalities* $0 \leq s(f) \leq c(f)$ *together with* $c(f) = 2\min_{i \in I} k_i \in 2\mathbb{N}$ *hold, where* k_i*,* $i \in I$*, come from* (C3)*.*

Proof. In view of Lemmata 3.3 and 3.9, we only have to show $s(f) \leq c(f)$. To this end, for some $i \in I$ with $c(f) = 2k_i$ and $\varepsilon > 0$ we put $g^\varepsilon(x) := \varepsilon x_i^{c(f)+1}$. Then g^ε fulfills $\deg(g^\varepsilon) = c(f) + 1$, $|g^\varepsilon_\alpha| \leq \varepsilon$ for all $\alpha \in A(g^\varepsilon) = \{(c(f)+1)e_i\}$ as well as $(c(f)+1)e_i \in V(f + g^\varepsilon)$. By Lemma 3.9 it holds $c(f) \in 2\mathbb{N}$, and one obtains $(c(f)+1)e_i \notin 2\mathbb{N}_0^n$, which means that $f + g^\varepsilon$ does not fulfill the condition (C1). Hence, Theorem 2.29 implies that $f + g^\varepsilon$ is not coercive on $\mathbb{R}^n$. Since $\varepsilon > 0$ can be chosen arbitrarily small, one obtains that f is not $(c(f)+1)$-stably coercive on $\mathbb{R}^n$ and, thus, $s(f) \leq c(f)$. □

Lemma 3.13. *If* $f \in \mathbb{R}[x]$ *is gem regular and satisfies the conditions* (C1)-(C3)*, then* f *is coercive on* $\mathbb{R}^n$ *with* $s(f) \geq c(f) = 2\min_{i \in I} k_i \in 2\mathbb{N}$*, where* k_i*,* $i \in I$*, come from* (C3)*.*

Proof. In view of Theorem 2.39 and Lemma 3.9, we only have to show $s(f) \geq c(f)$. By Definition 3.2, it suffices to prove that f is $c(f)$-stably coercive. To this end, we have to show the existence of some $\varepsilon > 0$ such that for all $g \in \mathbb{R}[x]$ with $\deg(g) \leq c(f)$ and $|g_\alpha| \leq \varepsilon$ for all $\alpha \in A(g)$, the polynomial $f + g$ is coercive on $\mathbb{R}^n$. We shall prove the latter by verifying the assumptions of Theorem 2.41 for $f + g$.

As a first step we claim that, with

$$\varepsilon_1 \;:=\; \tfrac{1}{2}\min\{|f_\alpha|,\ \alpha \in V(f) \cap c(f)\Delta^n\}, \tag{3.2}$$

every polynomial $g \in \mathbb{R}[x]$ with $\deg(g) \leq c(f)$ and $|g_\alpha| \leq \varepsilon_1$, $\alpha \in A(g)$, satisfies $V(f+g) = V(f)$. In fact, one obtains

$$(f+g)_\alpha \;=\; f_\alpha + g_\alpha \;=\; f_\alpha \;>\; 0 \quad \text{for all } \alpha \in V(f) \setminus c(f)\Delta^n$$

due to $g_\alpha = 0$ for all $|\alpha| > c(f)$, and the definition of ε_1 further yields

$$(f+g)_\alpha \;=\; f_\alpha + g_\alpha \;\geq\; f_\alpha - |g_\alpha| \;\geq\; f_\alpha - \varepsilon_1 \;>\; 0 \;\; \text{for all } \alpha \in V(f) \cap c(f)\Delta^n.$$

This implies $V(f) \subseteq A(f+g)$ and, thus,

$$\mathrm{New}_\infty(f) \;=\; \mathrm{conv}\, V_0(f) \;\subseteq\; \mathrm{conv}\, A_0(f+g) \;=\; \mathrm{New}_\infty(f+g)$$

holds.

On the other hand, due to the inclusions $\mathrm{New}_\infty(g) \subseteq c(f)\Delta^n \subseteq \mathrm{New}_\infty(f)$, one obtains

$$\begin{aligned} \mathrm{New}_\infty(f+g) \;&\subseteq\; \mathrm{conv}\{\mathrm{New}_\infty(f) \cup \mathrm{New}_\infty(g)\} \\ &\subseteq\; \mathrm{conv}\{\mathrm{New}_\infty(f)\} = \mathrm{New}_\infty(f). \end{aligned}$$

We arrive at

$$\mathrm{New}_\infty(f+g) = \mathrm{New}_\infty(f) \tag{3.3}$$

and, hence, the asserted identity $V(f+g) = V(f)$. As a consequence, since f satisfies the conditions (C1)-(C3), so does $f+g$.

In the case that $f+g$ is gem regular, that is, the set $D(f+g)$ of gem degenerate exponents of $f+g$ is void, Theorem 2.39 yields the coercivity of $f+g$, and the proof is complete. However, gem regularity of $f+g$ is not guaranteed, so that the proof continues with the investigation of the case $D(f+g) \neq \emptyset$, where coercivity of $f+g$ may be checked via the further sufficient conditions from Theorem 2.41.

More explicitly, the proof will be complete if we can decrease ε_1 such that for all corresponding polynomials g, for all $\alpha^\star \in D(f+g)$, and for some minimal affinely independent set $V^\star \subseteq V(f+g)$ with $\alpha^\star \in \mathrm{conv}\, V^\star$ we can provide weights $w_{f+g}(\alpha^\star) > 0$, $\alpha^\star \in D(f+g)$, with $\sum_{\alpha^\star \in D(f+g)} w_{f+g}(\alpha^\star) \leq 1$ such that

$$(f+g)_{\alpha^\star} \;>\; -w_{f+g}(\alpha^\star)\, \Theta(f+g, V^\star, \alpha^\star) \;\; \text{for all } \alpha^\star \in 2\mathbb{N}_0^n$$

and

$$|(f+g)_{\alpha^\star}| \;<\; w_{f+g}(\alpha^\star)\, \Theta(f+g, V^\star, \alpha^\star) \;\; \text{for all } \alpha^\star \in (2\mathbb{N}_0^n)^c$$

hold.

First, define

$$\widehat{D} := \{\alpha^\star \in \mathbb{N}_0^n \,|\, \exists h \in \mathbb{R}[x] \text{ with } \mathrm{New}_\infty(h) = \mathrm{New}_\infty(f) \text{ and } \alpha^\star \in D(h)\},$$

the set of all possible gem degenerate exponent vectors of an arbitrary polynomial h with the same Newton polytope at infinity as the one corresponding to f. Due to (3.3) one has $D(f+g) \subseteq \widehat{D}$.

In order to define weights which are independent of g and, thus, ε, we put $w_{f+g}(\alpha^\star) := |\widehat{D}|^{-1}$ for all $\alpha^\star \in D(f+g)$. These constant weights obviously are positive and, due to $D(f+g) \subseteq \widehat{D}$, also satisfy

$$\sum_{\alpha^\star \in D(f+g)} w_{f+g}(\alpha^\star) \;=\; \frac{|D(f+g)|}{|\widehat{D}|} \;\leq\; 1.$$

Next, for every $\alpha^\star \in \widehat{D}$, there exists a minimal affinely independent set $V^\star \subseteq V(f) = V(f+g)$ with $\alpha^\star \in \operatorname{conv} V^\star$ and a constant $\varepsilon_2(\alpha^\star) > 0$ such that

$$0 \;<\; |\widehat{D}|^{-1} \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} - \varepsilon$$

holds for all $\varepsilon \in (0, \varepsilon_2(\alpha^\star)]$, as taking the limit $\varepsilon \downarrow 0$ in the right side of the latter inequality yields the positive number $|\widehat{D}|^{-1}\Theta(f, V^\star, \alpha^\star)$. We put

$$\varepsilon_2 \;:=\; \min_{\alpha^\star \in \widehat{D}} \varepsilon_2(\alpha^\star).$$

Choose any $g \in \mathbb{R}[x]$ with $\deg(g) \leq c(f)$ and $|g_\alpha| \leq \min\{\varepsilon_1, \varepsilon_2\}$, $\alpha \in A(g)$. Then, any $\alpha^\star \in D(f+g) \cap 2\mathbb{N}_0^n$ satisfies, due to $D(f+g) \subseteq \widehat{D}$, the property

$$\begin{aligned}
(f+g)_{\alpha^\star} \;&=\; g_{\alpha^\star} \;\geq\; -|g_{\alpha^\star}| \;\geq\; -\min\{\varepsilon_1, \varepsilon_2\} \;\geq\; -\varepsilon_2 \;\geq\; -\varepsilon_2(\alpha^\star) \\
&>\; -|\widehat{D}|^{-1} \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon_2(\alpha^\star)}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&\geq\; -|\widehat{D}|^{-1} \prod_{\alpha \in V^\star} \left(\frac{(f+g)_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&=\; -w_{f+g}(\alpha^\star)\Theta(f+g, V^\star, \alpha^\star).
\end{aligned}$$

In fact, the first equality holds by the gem regularity of f which entails $f_{\alpha^\star} = 0$, and the latter also implies $\alpha^\star \in A(g)$, explaining the second inequality. The last inequality holds because of $f_\alpha - \varepsilon_2(\alpha^\star) \leq f_\alpha - |g_\alpha| \leq (f+g)_\alpha$ for all $\alpha \in V^\star \cap A(g)$ and $f_\alpha - \varepsilon_2(\alpha^\star) \leq f_\alpha = (f+g)_\alpha$ for all $\alpha \in V^\star \cap A^c(g)$.

Analogously, for every $\alpha^\star \in D(f+g) \cap (2\mathbb{N}_0^n)^c$ one can see the relation

$$|(f+g)_{\alpha^\star}| \;<\; w_{f+g}(\alpha^\star)\Theta(f+g, V^\star, \alpha^\star)$$

so that, altogether, $f+g$ satisfies all sufficient conditions from Theorem 2.41 for the choice $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. □

Theorem 3.14. *If $f \in \mathbb{R}[x]$ is gem regular and satisfies the conditions* (C1)-(C3)*, then f is coercive on $\mathbb{R}^n$ with $s(f) = c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$, where k_i, $i \in I$, come from* (C3)*.*

Proof. By Theorem 2.39 the polynomial f is coercive on $\mathbb{R}^n$, and Lemma 3.12 implies $s(f) \le c(f)$. Simultaneously, Lemma 3.13 implies $s(f) \ge c(f)$ and we arrive at $s(f) = c(f)$. Finally, Lemma 3.9 implies $c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$, and the assertion follows. □

Example 3.15. *Example 3.10 and Theorem 3.14 yield $s(f) = 2$ for the gem regular polynomial $f(x) = x_1^4 + x_2^2 + x_1^2x_2^2$ with $\deg(f) = 4$ from Example 3.5.*

The following lemma shows that the regularity assumption from Lemma 3.13 may be weakened significantly.

Lemma 3.16. *If for $f \in \mathbb{R}[x]$ the sufficient conditions from Theorem 2.41 hold, then f is coercive on $\mathbb{R}^n$ with $s(f) \ge c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$.*

Proof. In view of Theorem 2.41 and Lemma 3.9 we only have to show $s(f) \ge c(f)$. The proof of Lemma 3.13 already covers the case of a gem regular polynomial f, that is, the case $D(f) = \emptyset$. In the following we will thus concentrate on the case $D(f) \neq \emptyset$, using the notation from the proof of Lemma 3.13.

Define $\varepsilon_3 := \frac{1}{2}\min\{|f_\alpha|,\ \alpha \in (V(f) \cup D(f)) \cap c(f)\Delta^n\}$. Clearly, since $\varepsilon_3 \le \varepsilon_1$, we obtain by the construction from the beginning of the proof of Lemma 3.13 that for every polynomial $g \in \mathbb{R}[x]$ with $\deg(g) \le c(f)$ and $|g_\alpha| \le \varepsilon_3$, $\alpha \in A(g)$, one has $\mathrm{New}_\infty(f+g) = \mathrm{New}_\infty(f)$ with $f+g$ satisfying conditions (C1)-(C3). Further, by definition of ε_3 one has $D(f) \subseteq D(f+g)$, because $(f+g)_{\alpha^\star} \neq 0$ holds for all $\alpha^\star \in D(f)$. In fact, for each $\alpha^\star \in D(f) \cap c(f)\Delta^n$, one obtains

$$|(f+g)_{\alpha^\star}| = |f_{\alpha^\star} + g_{\alpha^\star}| \ge |f_{\alpha^\star}| - |g_{\alpha^\star}| \ge |f_{\alpha^\star}| - \varepsilon_3 > 0,$$

and for all remaining exponent vectors $\alpha^\star \in D(f) \setminus c(f)\Delta^n$ the property

$$(f+g)_{\alpha^\star} = f_{\alpha^\star} + g_{\alpha^\star} = f_{\alpha^\star} \neq 0.$$

We note that the set $D(f+g) \setminus D(f)$ may or may not be nonempty, that is, the perturbation of f by g may or may not create new degenerate points $\alpha^\star$. In the following we shall treat these two possibilities as two subcases.

Subcase $D(f+g) \setminus D(f) = \emptyset$

By assumption the polynomial f fulfills the sufficient conditions from Theorem 2.41 with some weights $w_f(\alpha^\star)$, $\alpha^\star \in D(f)$. We claim that for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)$ there exists a constant $\varepsilon_4(\alpha^\star) > 0$ such that

$$0 < \left(w_f(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} + f_{\alpha^\star} \right) - \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_4(\alpha^\star)], \tag{3.4}$$

and for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)^c$ there exists a constant $\varepsilon_4(\alpha^\star) > 0$ such that

$$0 < \left(w_f(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} - |f_{\alpha^\star}| \right) - \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_4(\alpha^\star)]. \tag{3.5}$$

This is the case since taking the limit $\varepsilon \downarrow 0$ of the right side of (3.4), one obtains the number $w_f(\alpha^\star)\Theta(f, V^\star, \alpha^\star) + f_{\alpha^\star}$, which is positive due to the assumption that f fulfills the sufficiency conditions from Theorem 2.41 and, taking the limit $\varepsilon \downarrow 0$ of the right side of (3.5), one obtains the number $w_f(\alpha^\star)\Theta(f, V^\star, \alpha^\star) - |f_{\alpha^\star}|$, which is positive for the same reason.

Set $\varepsilon_4 := \frac{1}{2} \min\{\varepsilon_4(\alpha^\star) |\ \alpha^\star \in D(f)\} > 0$ and consider $g \in \mathbb{R}[x]$ with $\deg(g) \leq c(f)$ and $|g_\alpha| \leq \varepsilon < \min\{\varepsilon_3, \varepsilon_4\}$ for all $\alpha \in A(g)$. By $\varepsilon < \varepsilon_3 \leq \varepsilon_1$ (see (3.2)) and the arguments above, the polynomial $f + g$ fulfills the conditions (C1)-(C3). Further, using the fact $D(f) = D(f + g)$ and defining the weights w_{f+g} of the perturbed polynomial $f + g$ by

$$w_{f+g}(\alpha^\star) := w_f(\alpha^\star) \;\text{ for all }\; \alpha^\star \in D(f + g),$$

in view of $\varepsilon < \varepsilon_4$ one obtains the relations

$$\begin{aligned}
(f+g)_{\alpha^\star} &\geq f_{\alpha^\star} - |g_{\alpha^\star}| \geq f_{\alpha^\star} - \varepsilon > -w_f(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&\geq -w_f(\alpha^\star)\Theta(f + g, V^\star, \alpha^\star) \\
&= -w_{f+g}(\alpha^\star)\Theta(f + g, V^\star, \alpha^\star) \quad \text{for all } \alpha^\star \in D(f + g) \cap 2\mathbb{N}_0^n,
\end{aligned}$$

and

$$\begin{aligned}
|(f+g)_{\alpha^\star}| &\leq |f_{\alpha^\star}| + |g_{\alpha^\star}| \leq |f_{\alpha^\star}| + \varepsilon < w_f(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&\leq w_f(\alpha^\star)\Theta(f + g, V^\star, \alpha^\star) \\
&= w_{f+g}(\alpha^\star)\Theta(f + g, V^\star, \alpha^\star) \quad \text{for all } \alpha^\star \in D(f + g) \cap (2\mathbb{N}_0^n)^c,
\end{aligned}$$

where, in either case, the strict inequality holds due to (3.4) and (3.5), respectively, and the last inequality holds because of $f_\alpha - \varepsilon \leq f_\alpha - |g_\alpha| \leq (f + g)_\alpha$ for all $\alpha \in V^\star \cap A(g)$ and $f_\alpha - \varepsilon \leq f_\alpha = (f + g)_\alpha$ for all $\alpha \in V^\star \cap A^c(g)$.

The polynomial $f+g$ thus satisfies all assumptions from Theorem 2.41 and is, hence, coercive on $\mathbb{R}^n$.

Subcase $\boldsymbol{D(f+g) \setminus D(f) \neq \emptyset}$

By the sufficiency conditions from Theorem 2.41 and a continuity argument, for each $\alpha^\star \in D(f)$ there exists a constant $\delta_{\alpha^\star} > 0$ with $w_f(\alpha^\star) - \delta_{\alpha^\star} > 0$,

$$f_{\alpha^\star} > -(w_f(\alpha^\star) - \delta_{\alpha^\star})\Theta(f, V^\star, \alpha^\star) \quad \text{if } \alpha^\star \in D(f) \cap 2\mathbb{N}_0^n, \tag{3.6}$$

and

$$|f_{\alpha^\star}| < (w_f(\alpha^\star) - \delta_{\alpha^\star})\Theta(f, V^\star, \alpha^\star) \quad \text{if } \alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)^c. \tag{3.7}$$

Let the corresponding weights w_{f+g} for the polynomial $f+g$ be defined as follows:

$$w_{f+g}(\alpha^\star) := w_f(\alpha^\star) - \delta_{\alpha^\star} \quad \text{ for all } \alpha^\star \in D(f), \tag{3.8}$$

$$w_{f+g}(\alpha^\star) := |\widehat{D} \setminus D(f)|^{-1} \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \quad \text{ for all } \alpha^\star \in D(f+g) \setminus D(f). \tag{3.9}$$

Note that all weights $w_{f+g}(\alpha^\star)$ corresponding to the newly created degenerate exponent vectors $\alpha^\star \in D(f+g) \setminus D(f)$ are defined to be constant.

Clearly, we obtain $w_{f+g}(\alpha^\star) > 0$ for all $\alpha^\star \in D(f+g)$ and

$$\begin{aligned}
\sum_{\alpha^\star \in D(f+g)} w_{f+g}(\alpha^\star) &= \sum_{\alpha^\star \in D(f)} w_{f+g}(\alpha^\star) + \sum_{\alpha^\star \in D(f+g)\setminus D(f)} w_{f+g}(\alpha^\star) \\
&= \sum_{\alpha^\star \in D(f)} (w_f(\alpha^\star) - \delta_{\alpha^\star}) + \sum_{\alpha^\star \in D(f+g)\setminus D(f)} \left(|\widehat{D} \setminus D(f)|^{-1} \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \\
&= \sum_{\alpha^\star \in D(f)} w_f(\alpha^\star) - \sum_{\alpha^\star \in D(f)} \delta_{\alpha^\star} + \left(\sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \sum_{\alpha^\star \in D(f+g)\setminus D(f)} |\widehat{D} \setminus D(f)|^{-1} \\
&= \sum_{\alpha^\star \in D(f)} w_f(\alpha^\star) - \sum_{\alpha^\star \in D(f)} \delta_{\alpha^\star} + \left(\sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \frac{|D(f+g) \setminus D(f)|}{|\widehat{D} \setminus D(f)|} \\
&\leq \sum_{\alpha^\star \in D(f)} w_f(\alpha^\star) - \sum_{\alpha^\star \in D(f)} \delta_{\alpha^\star} + \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} = \sum_{\alpha^\star \in D(f)} w_f(\alpha^\star) \leq 1
\end{aligned}$$

with the penultimate inequality holding since $D(f) \subseteq D(f+g) \subseteq \widehat{D}$, and the last inequality holding by the ssumption that f fulfills the sufficiency conditions from Theorem 2.41.

Next, we claim that for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)$ there exists a constant $\varepsilon_5(\alpha^\star) > 0$ such that

$$0 < (w_f(\alpha^\star) - \delta_{\alpha^\star}) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} + f_{\alpha^\star} - \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_5(\alpha^\star)], \tag{3.10}$$

and for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)^c$ there exists a constant $\varepsilon_5(\alpha^\star) > 0$ such that

$$0 < (w_f(\alpha^\star) - \delta_{\alpha^\star}) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} - |f_{\alpha^\star}| - \varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_5(\alpha^\star)]. \tag{3.11}$$

This is the case since taking the limit $\varepsilon \downarrow 0$ of the right side of (3.10), one obtains the number $(w_f(\alpha^\star) - \delta_{\alpha^\star})\,\Theta(f, V^\star, \alpha^\star) + f_{\alpha^\star}$, and taking the limit $\varepsilon \downarrow 0$ of the right side of (3.11) yields the number $(w_f(\alpha^\star) - \delta_{\alpha^\star})\,\Theta(f, V^\star, \alpha^\star) - |f_{\alpha^\star}|$. In view of (3.6) and (3.7), both of these numbers are positive.

This implies that for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)$ one has for all $\varepsilon \in (0, \varepsilon_5(\alpha^\star)]$

$$\begin{aligned}
(f+g)_{\alpha^\star} \quad &\geq \quad f_{\alpha^\star} - \varepsilon > -(w_f(\alpha^\star) - \delta_{\alpha^\star}) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} \\
&= \quad -w_{f+g}(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} \geq -w_{f+g}(\alpha^\star)\Theta(f+g, V^\star, \alpha^\star)
\end{aligned}$$

and simultaneously for every $\alpha^\star \in D(f) \cap (2\mathbb{N}_0^n)^c$ one has for all $\varepsilon \in (0, \varepsilon_5(\alpha^\star)]$

$$\begin{aligned}
|(f+g)_{\alpha^\star}| \quad &\leq \quad |f_{\alpha^\star}| + \varepsilon < (w_f(\alpha^\star) - \delta_{\alpha^\star}) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} \\
&= \quad w_{f+g}(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha}\right)^{\lambda_\alpha} \leq w_{f+g}(\alpha^\star)\Theta(f+g, V^\star, \alpha^\star)
\end{aligned}$$

where, in either case, the strict inequality holds due to (3.10) and (3.11), respectively, and the last inequality holds because of $f_\alpha - \varepsilon \leq (f+g)_\alpha$ for all $\alpha \in V^\star$ as in the previous subcase.

The choice of $\varepsilon < \varepsilon_5 := \min\{\varepsilon_5(\alpha^\star),\ \alpha^\star \in D(f)\}$ results in all coefficients $(f+g)_{\alpha^\star}$, $\alpha^\star \in D(f)$, fulfilling the sufficient conditions for coercivity from Theorem 2.41. In order to finish the proof, we only have to guarantee this property for all remaining degenerate exponent vectors $D(f+g) \setminus D(f)$ of $f+g$, which we shall do in its following final part.

We claim that for every $\alpha^\star \in \widehat{D} \setminus D(f)$ there exists a constant $\varepsilon_6 > 0$ such that such that one has

$$0 < \left(|\widehat{D} \setminus D(f)|^{-1} \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} - \varepsilon \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_6].$$

This is the case since taking the limit $\varepsilon \downarrow 0$ of the right side of the latter inequality yields the positive number $\left(|\widehat{D} \setminus D(f)|^{-1} \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \Theta(f, V^\star, \alpha^\star)$. Due to $D(f+g) \subseteq \widehat{D}$, this implies that for every $\alpha^\star \in (D(f+g) \setminus D(f)) \cap 2\mathbb{N}_0^n$ and all $\varepsilon \in (0, \varepsilon_6]$ we obtain the property

$$\begin{aligned}
(f+g)_{\alpha^\star} &= g_{\alpha^\star} \geq -|g_{\alpha^\star}| \geq -\varepsilon \\
&> -\left(|\widehat{D} \setminus D(f)|^{-1} \sum_{\beta^\star \in D(f)} \delta_{\beta^\star} \right) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&= -w_{f+g}(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \geq -w_{f+g}(\alpha^\star) \Theta(f+g, V^\star, \alpha^\star)
\end{aligned}$$

with the first equality holding due to $f_{\alpha^\star} = 0$, and the last inequality holding because of $f_\alpha - \varepsilon \leq (f+g)_\alpha$ for all $\alpha \in V^\star$ as above.

Simultaneously, due to $D(f+g) \subseteq \widehat{D}$, for every $\alpha^\star \in D(f+g) \setminus D(f)$ with $\alpha^\star \notin 2\mathbb{N}_0^n$ and all $\varepsilon \in (0, \varepsilon_6]$ one obtains the property

$$\begin{aligned}
|(f+g)_{\alpha^\star}| &= |f_{\alpha^\star} + g_{\alpha^\star}| = |g_{\alpha^\star}| \leq \varepsilon \\
&< \left(|\widehat{D} \setminus D(f)|^{-1} \sum_{\alpha^\star \in D(f)} \delta_{\alpha^\star} \right) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \\
&= w_{f+g}(\alpha^\star) \prod_{\alpha \in V^\star} \left(\frac{f_\alpha - \varepsilon}{\lambda_\alpha} \right)^{\lambda_\alpha} \leq w_{f+g}(\alpha^\star) \Theta(f+g, V^\star, \alpha^\star),
\end{aligned}$$

where the second equality and the last inequality hold for the same reasons as above.

Finally, choosing $\varepsilon < \min\{\varepsilon_3, \varepsilon_5, \varepsilon_6\}$ we obtain that $f+g$ in the last subcase fulfills all sufficient conditions from Theorem 2.41 and $f+g$ is thus coercive on $\mathbb{R}^n$, which completes the proof. $\square$

The following theorem is shown along the same lines as Theorem 3.14, with Theorem 2.39 and Lemma 3.13 replaced by Theorem 2.41 and Lemma 3.16, respectively.

Theorem 3.17. *If for $f \in \mathbb{R}[x]$ the sufficient conditions from Theorem 2.41 hold, then f is coercive on $\mathbb{R}^n$ with $s(f) = c(f) = 2\min_{i \in I} k_i \in 2\mathbb{N}$, where k_i, $i \in I$, come from* (C3).

Remark 3.18. Theorems 3.14 and 3.17 imply, in particular, that the upper bound $c(f)$ for $s(f)$ from Lemma 3.12 is attained under mild assumptions, unlike the coarser upper bound $\deg(f)$ from Lemma 3.3 (cf. Ex. 3.15 and the subsequent Sec. 3.3). $\square$

3.3 Stable coercivity of maximum degree

As illustrated by Example 3.15, a large class of coercive polynomials f satisfies $s(f) < \deg(f)$. This motivates to study whether polynomials of maximal degree of stable coercivity, that is, with $s(f) = \deg(f)$, enjoy any special properties. The present section will provide a positive answer.

Lemma 3.19. *If $f \in \mathbb{R}[x]$ is $\deg(f)$-stably coercive, then the necessary conditions from Theorem 2.29 as well as*

$$\mathrm{New}_\infty(f) \;=\; \deg(f)\Delta^n \tag{C4}$$

hold.

Proof. Since any $\deg(f)$-stably coercive polynomial f is coercive on $\mathbb{R}^n$, the first part of the assertion holds by Theorem 2.29. With Definition 3.2 and Lemma 3.12 one obtains $\deg(f) \leq s(f) \leq c(f)$. Hence $\deg(f) \leq c(f)$ holds, and Proposition 3.7 implies $\deg(f)\Delta^n \subseteq \mathrm{New}_\infty(f)$. Since $\mathrm{New}_\infty(f) \subseteq \deg(f)\Delta^n$ is always true, one finally obtains $\mathrm{New}_\infty(f) = \deg(f)\Delta^n$. $\square$

Lemma 3.20. *Let $f \in \mathbb{R}[x]$ be gem regular. If the conditions* (C1)-(C4) *hold, then f is* $\deg(f)$*-stably coercive.*

Proof. By Theorem 3.14 it holds $s(f) = c(f)$ and since, by assumption, f fulfills (C4), it also holds $c(f) = \deg(f)$. Thus $s(f) = \deg(f)$ and, by Definition 3.2, f is $\deg(f)$-stably coercive on $\mathbb{R}^n$. $\square$

As, in the gem regular case, the necessary and sufficient conditions for coercivity for $f \in \mathbb{R}[x]$ coincide, from Lemmata 3.19 and 3.20 we may obtain a characterization theorem for stable coercivity for f of degree $\deg(f)$ in analogy to the Characterization Theorem 2.39 for coercivity. Before we state it (cf. Th. 3.23 below), we return to the relation of stable coercivity of degree $\deg(f)$ with the growth condition (G) from Lemma 2.52.

Lemma 3.21. *If $f \in \mathbb{R}[x]$ fulfills $\deg(f) \in 2\mathbb{N}$ and the growth condition* (G)*, then the conditions* (C1)-(C4) *hold.*

Proof. By Lemma 2.52, f is coercive on $\mathbb{R}^n$, so that Theorem 2.29 implies the conditions (C1)-(C3). Assume that condition (C4) is violated. Then we obtain the existence of some $i \in I$ with $f_{\deg(f)e_i} = 0$. The points $x(t) := te_i$ with $t > 0$ satisfy

$$f_{\deg(f)}(x(t)) \;=\; t^{\deg(f)} \sum_{|\alpha| = \deg(f)} f_\alpha \, e_i^\alpha,$$

where e_i^α vanishes whenever $\alpha_j > 0$ holds for some $j \neq i$. Hence, the summation must only be taken over exponents α with $|\alpha| = \deg(f)$ and $\alpha = ce_i$ with some constant $c > 0$. As the only possible choice is $c = \deg(f)$, we arrive at

$$f_{\deg(f)}(x(t)) \;=\; t^{\deg(f)} \, f_{\deg(f)e_i} \;\equiv\; 0 \;<\; \delta \, \|x(t)\|^{\deg(f)} \;=\; \delta \, t^{\deg(f)} \|e_i\|$$

for any $\delta > 0$ and $t > 0$. Consequently (G) is violated, in contradiction to the assumption. □

Lemma 3.22. *Let $f \in \mathbb{R}[x]$ be gem regular. If the conditions* (C1)-(C4) *hold, then we have* $\deg(f) \in 2\mathbb{N}$, *and the growth condition* (G) *is fulfilled.*

Proof. Conditions (C1)-(C3) and Theorem 2.39 guarantee the coercivity of f, so that Remark 2.17 yields $\deg(f) \in 2\mathbb{N}$.

Under condition (C4) and gem regularity, for all $x \in \mathbb{R}^n$ we have

$$f_{\deg(f)}(x) \;=\; \sum_{|\alpha| = \deg(f)} f_\alpha \, x^\alpha \;=\; \sum_{i \in I} f_{\deg(f)e_i} \, x_i^{\deg(f)}.$$

Due to $\deg(f) \in 2\mathbb{N}$ we may also replace the terms x_i by $|x_i|$ in the latter expression, so that

$$\|x\|_{\text{w-}\deg(f)} \;:=\; \left(f_{\deg(f)}(x)\right)^{1/\deg(f)}$$

turns out to be a weighted ℓ_p norm with $p = \deg(f)$ and weights $f_{\deg(f)e_i}$, $i \in I$, which are actually positive due to (C2). By the equivalence of norms, for any other norm $\|\cdot\|$ on $\mathbb{R}^n$ there exists some $M > 0$ with $\|x\|_{\text{w-}\deg(f)} \geq M\|x\|$ for all $x \in \mathbb{R}^n$, and we arrive at

$$f_{\deg(f)}(x) \;=\; \|x\|_{\text{w-}\deg(f)}^{\deg(f)} \;\geq\; M^{\deg(f)} \, \|x\|^{\deg(f)}.$$

The choice $\delta := M^{\deg(f)}$ shows the assertion. □

Our results allow to state the subsequent theorem, which presents several characterizations of stable coercivity of maximal degree.

Theorem 3.23. *For a gem regular polynomial $f \in \mathbb{R}[x]$ the following four assertions are equivalent:*

a) f *is* $\deg(f)$*-stably coercive.*

b) f *is coercive and* (C4) *holds.*

c) f *fulfills conditions* (C1)–(C4)*.*

d) f *fulfills* $\deg(f) \in 2\mathbb{N}$ *and the growth condition* (G)*.*

Proof. By Lemma 3.19, assertion a) implies b), Theorem 2.39 shows that assertion b) implies c) and, in view of Lemma 3.20, assertion c) implies a). This shows the equivalence of assertions a), b), and c). The equivalence of assertions c) and d) is an immediate consequence of Lemmata 3.21 and 3.22. □

We finish this section with the statement of Lemma 3.20 without the assumption of gem regularity. The proof runs along the same lines as the proof of Lemma 3.20, with Theorem 3.14 replaced by Theorem 3.17.

Proposition 3.24. *If* $f \in \mathbb{R}[x]$ *fulfills the sufficient conditions from Theorem 2.41 as well as the condition* (C4)*, then* f *is* $\deg(f)$*-stably coercive.*

3.4 The order of coercivity

This section investigates how the speed of growth of a coercive polynomial is related to stable coercivity. As before, $\|\cdot\|$ shall denote some arbitrary norm on $\mathbb{R}^n$.

Definition 3.25. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called q-coercive on* $\mathbb{R}^n$ *for some nonnegative* $q \in \mathbb{R}$ *if* $f(x)/\|x\|^q \to +\infty$ *holds for* $\|x\| \to +\infty$*.*

Remark 3.26. Clearly, if $f \in \mathbb{R}[x]$ is q-coercive on $\mathbb{R}^n$ for some $q \geq 0$, then f is $\tilde{q}$-coercive on $\mathbb{R}^n$ also for all $\tilde{q} \in [0, q)$. □

Remark 3.27. By Definition 3.25 above, the 0-coercivity of some $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ coincides with the notion of coercivity of f on $\mathbb{R}^n$ we used so far. Therefore, in the following, instead of saying that f is 0-coercive on $\mathbb{R}^n$, we shall say that f is coercive on $\mathbb{R}^n$. □

Definition 3.28 (Order of coercivity)**.** *For a coercive function* $f : \mathbb{R}^n \to \mathbb{R}$ *on* $\mathbb{R}^n$ *we call the number*

$$o(f) := \sup\{q \geq 0 \,|\, f \text{ is } q\text{-coercive on } \mathbb{R}^n\}$$

the order of coercivity of f*. A coercive function* $f : \mathbb{R}^n \to \mathbb{R}$ *with order of coercivity equal to* $o(f)$ *is called coercive on* $\mathbb{R}^n$ *of order* $o(f)$*. For later purposes we introduce the set*

$$Q(f) := \{q \geq 0 \,|\, f \text{ is } q\text{-coercive on } \mathbb{R}^n\}.$$

Remark 3.29. A coercive function f on $\mathbb{R}^n$ with a finite order of coercivity $o(f)$ is q-coercive on $\mathbb{R}^n$ for all $0 \leq q < o(f)$, but f is not necessarily $o(f)$-coercive on $\mathbb{R}^n$. As an example for $n = 1$, take $f(x) = x^2$. One obtains $o(f) = 2$, f is q-coercive on $\mathbb{R}$ for all $0 \leq q < 2$, but f is not 2-coercive on $\mathbb{R}$. □

Lemma 3.30. *If $f \in \mathbb{R}[x]$ is coercive on $\mathbb{R}^n$ then the necessary conditions from Theorem 2.29 as well as $Q(f) \subseteq [0, c(f))$ and $o(f) \leq c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$ hold.*

Proof. The first part of the proof directly follows from Theorem 2.29. Further, using Lemma 3.9, one obtains $c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$. Now, on the contrary, assume $Q(f) \not\subseteq [0, c(f))$. Thus, there exists some $q \in Q(f)$ with $q \geq c(f)$. Then, for any $i^\star \in I$ with $c(f) = 2\min_{i\in I} k_i = 2k_{i^\star}$, one obtains $q \geq 2k_{i^\star}$. With $x^\nu := \nu e_{i^\star}$, $\nu \in \mathbb{N}$, we obtain a sequence $(x^\nu)_{\nu\in\mathbb{N}}$ with $\|x^\nu\|_2 \to +\infty$ for $\nu \to +\infty$. More precisely, from $x_j^\nu = 0$, $j \in I \setminus \{i^\star\}$, for all $\nu \in \mathbb{N}$, we conclude $\|x^\nu\|_2^q = |x_{i^\star}^\nu|^q$ as well as

$$f(x^\nu) = \sum_{\ell \leq 2k_{i^\star} : \, \ell e_{i^\star} \in A(f)} f_{\ell e_{i^\star}} (x_{i^\star}^\nu)^\ell.$$

Due to $\ell \leq 2k_{i^\star} \leq q$ for all ℓ in this summation, we arrive at

$$\begin{aligned} \lim_{\nu\to+\infty} \frac{f(x^\nu)}{\|x^\nu\|_2^q} &= \lim_{\nu\to+\infty} \sum_{\ell \leq 2k_{i^\star} : \, \ell e_{i^\star} \in A(f)} \frac{f_{\ell e_{i^\star}} (x_{i^\star}^\nu)^\ell}{|x_{i^\star}^\nu|^q} \\ &= \sum_{\ell \leq 2k_{i^\star} : \, \ell e_{i^\star} \in A(f)} \lim_{\nu\to+\infty} f_{\ell e_{i^\star}} \nu^{\ell - q} < +\infty. \end{aligned}$$

This contradicts the assumption of f being q-coercive on $\mathbb{R}^n$. Thus $Q(f) \subseteq [0, c(f))$ holds and, by Definition 3.28, also $o(f) \leq c(f)$. □

Lemma 3.31. *Let $f \in \mathbb{R}[x]$ be a gem regular polynomial satisfying the conditions* (C1)-(C3). *Then f is coercive on $\mathbb{R}^n$ with $Q(f) \supseteq [0, c(f)) \neq \emptyset$ and $o(f) \geq c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$.*

Proof. By Theorem 2.39, f is coercive on $\mathbb{R}^n$ and, using Lemma 3.9, one obtains $c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$ resulting in $[0, c(f)) \neq \emptyset$. Since f is gem regular, one also has $D(f) = \emptyset$, and, one may write $f = f^{V(f)} + f^{R(f)}$. Let q with $0 \leq q < c(f)$ be arbitrarily chosen. To show that f is q-coercive on $\mathbb{R}^n$, by equivalence of the norms on $\mathbb{R}^n$ it suffices to prove that $f^{V(f)}(x)/\|x\|_\infty^q \to \infty$ holds for $\|x\|_\infty \to \infty$. Then, as by the proof of Proposition 2.37 (see also Proposition 3.1 in [4]) there exists some $\varepsilon > 0$ such that for every sequence $(x^\nu)_{\nu\in\mathbb{N}}$ with $\lim_{\nu\to\infty} \|x^\nu\|_\infty = +\infty$ one has

$$f^{R(f)}(x^\nu) \geq (\varepsilon - 1) f^{V(f)}(x^\nu) \quad \text{for almost all } \nu \in \mathbb{N},$$

we obtain

$$\frac{f(x^\nu)}{\|x^\nu\|_\infty^q} = \frac{f^{V(f)}(x^\nu) + f^{R(f)}(x^\nu)}{\|x^\nu\|_\infty^q} \geq \varepsilon \frac{f^{V(f)}(x^\nu)}{\|x^\nu\|_\infty^q} \quad \text{for almost all } \nu \in \mathbb{N} \tag{3.12}$$

and, thus, the assertion.

In fact, for any $\nu \in \mathbb{N}$, with the numbers k_i, $i \in I$, from (C3) we obtain

$$f^{V(f)}(x^\nu) - \sum_{i\in I} f_{2k_ie_i}(x^\nu)^{2k_ie_i} = \sum_{\alpha \in V(f)\setminus\{2k_ie_i,\, i\in I\}} f_\alpha x^\alpha \geq 0,$$

where the nonnegativity follows from conditions (C1) and (C2). This implies

$$\frac{f^{V(f)}(x^\nu)}{\|x^\nu\|_\infty^q} \geq \sum_{i\in I} f_{2k_ie_i}\frac{(x_i^\nu)^{2k_i}}{\|x^\nu\|_\infty^q} = \sum_{i\in I} f_{2k_ie_i}\frac{|x_i^\nu|^{2k_i}}{\|x^\nu\|_\infty^q}.$$

Let $j(\nu) \in I$ denote some index with $|x^\nu_{j(\nu)}| = \|x^\nu\|_\infty$. Then we may continue to write

$$\sum_{i\in I} f_{2k_ie_i}\frac{|x_i^\nu|^{2k_i}}{\|x^\nu\|_\infty^q} \geq f_{2k_{j(\nu)}e_{j(\nu)}}\frac{|x^\nu_{j(\nu)}|^{2k_{j(\nu)}}}{\|x^\nu\|_\infty^q} \;\geq\; \left(\min_{i\in I} f_{2k_ie_i}\right)\|x^\nu\|_\infty^{2k_{j(\nu)}-q},$$

where the first inequality is true due to $f_{2k_ie_i} > 0$ for all $i \in I$ by conditions (C2) and (C3), which also implies the positivity of the term $\min_{i\in I} f_{2k_ie_i}$. Furthermore, by Lemma 3.9 one obtains

$$2k_{j(\nu)} - q \geq 2\min_{i\in I} k_i - q = c(f) - q,$$

so that for almost all $\nu \in \mathbb{N}$ we arrive at

$$\|x^\nu\|_\infty^{2k_{j(\nu)}-q} \geq \|x^\nu\|_\infty^{c(f)-q}.$$

In view of $q < c(f)$ our estimates imply

$$\lim_{\nu\to+\infty}\frac{f^{V(f)}(x^\nu)}{\|x^\nu\|_\infty^q} \geq \min_{i\in I} f_{2k_ie_i}\lim_{\nu\to+\infty}\|x^\nu\|_\infty^{c(f)-q} \;=\; +\infty.$$

Thus, f is q-coercive on $\mathbb{R}^n$, that is, $Q(f) \supseteq [0, c(f))$ holds and, by Definition 3.28, also $o(f) \geq c(f)$. □

The following result is a direct consequence of Theorem 3.14 and Lemmata 3.30 and 3.31.

Theorem 3.32. *Let $f \in \mathbb{R}[x]$ be a gem regular polynomial satisfying the conditions* (C1)-(C3). *Then f is coercive on $\mathbb{R}^n$ with*

$$s(f) = c(f) = o(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$$

and $Q(f) = [0, c(f))$.

Example 3.33. *Example 3.10 and Theorem 3.32 yield* $o(f) = 2$ *and* $Q(f) = [0, 2)$ *for the gem regular polynomial* $f(x) = x_1^4 + x_2^2 + x_1^2 x_2^2$ *from Example 3.5.*

The following result characterizes q-coercivity of arbitrary continuous functions in terms of a growth condition.

Lemma 3.34. *A continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* q*-coercive on* $\mathbb{R}^n$ *for some* $q \geq 0$ *if and only if*

$$\forall c_1 > 0 \quad \exists c_2 \geq 0 \quad \textit{with} \quad f(x) \geq c_1 \|x\|^q - c_2 \quad \forall x \in \mathbb{R}^n.$$

Proof. Let f be q-coercive on $\mathbb{R}^n$. Then

$$\forall c_1 > 0 \quad \exists M \geq 0 \quad \text{with} \quad f(x) \geq c_1 \|x\|^q \quad \forall \|x\| > M. \tag{3.13}$$

For an arbitrary value $c_1 > 0$, with

$$c_2 := \max\{0, \max_{\|x\| \leq M} c_1 \|x\|^q - f(x)\}, \tag{3.14}$$

it holds $f(x) \geq c_1 \|x\|^q - c_2$ for all $\|x\| \leq M$. Since $c_2 \geq 0$ holds due to (3.14), using (3.13) one simultaneously obtains

$$f(x) \geq c_1 \|x\|^q \geq c_1 \|x\|^q - c_2 \quad \forall \|x\| > M,$$

which finally results in

$$\forall c_1 > 0 \quad \exists c_2 \geq 0 \quad \text{with} \quad f(x) \geq c_1 \|x\|^q - c_2 \quad \forall x \in \mathbb{R}^n. \tag{3.15}$$

For the proof of the other direction assume that (3.15) is true. This implies

$$\forall c_1 > 0 \quad \exists c_2 \geq 0 \quad \text{with} \quad \frac{f(x)}{\|x\|^q} \geq c_1 - \frac{c_2}{\|x\|^q} \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{3.16}$$

Taking the limit for $\|x\| \to +\infty$ of both sides of (3.16) yields

$$\forall c_1 > 0 : \quad \lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|^q} \geq c_1$$

or, in other words,

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|^q} = +\infty$$

and the q-coercivity of f on $\mathbb{R}^n$ follows. $\square$

Theorem 3.35. *For any gem regular polynomial $f \in \mathbb{R}[x]$ the following assertions are equivalent:*

a) f is coercive on $\mathbb{R}^n$.

b) f fulfills conditions (C1)-(C3).

c) $Q(f) = [0, c(f)) \neq \emptyset$.

d) $c(f) > 0$, and the property

$$\forall c_1 > 0 \ \exists c_2 \geq 0 \text{ with } f(x) \geq c_1 \|x\|^q - c_2 \quad \text{for all } x \in \mathbb{R}^n$$

holds with some $q \geq 0$ if and only if $q \in [0, c(f))$.

Proof. For the equivalence of a) and b) see Theorem 2.39. For the proof that c) implies a) see Remarks 3.26 and 3.27. Further, b) implies c) by Theorem 3.32. For the equivalence of c) and d), see Lemma 3.34. □

Finally, we present a lower bound for the order of coercivity for polynomials which, unlike in Lemma 3.31, are not necessarily gem regular.

Lemma 3.36. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41. Then f is coercive on $\mathbb{R}^n$ with $Q(f) \supseteq [0, c(f)) \neq \emptyset$ and $o(f) \geq c(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$.*

Proof. The proof runs along the same lines as the proof of Lemma 3.31. In fact, by the proof of Theorem 2.41 (see [4, Th. 3.4]), there exists some $\varepsilon > 0$ such that for every sequence $(x^\nu)_{\nu\in\mathbb{N}}$ with $\lim_{\nu\to\infty} \|x^\nu\|_\infty = +\infty$ we have

$$f^{D(f)}(x^\nu) + f^{R(f)}(x^\nu) \geq (\varepsilon - 1) f^{V(f)}(x^\nu) \quad \text{for almost all } \nu \in \mathbb{N}. \tag{3.17}$$

Regarding (3.17) the same argumentation as in the proof of Lemma 3.31 can be used. □

Theorem 3.37. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41. Then f is coercive on $\mathbb{R}^n$ with*

$$s(f) = c(f) = o(f) = 2\min_{i\in I} k_i \in 2\mathbb{N}$$

as well as $Q(f) = [0, c(f)) \neq \emptyset$.

Proof. The assertion directly follows by Theorem 3.17 as well as Lemmata 3.30 and 3.36. □

For a polynomial $f \in \mathbb{R}[x]$ fulfilling the sufficiency conditions from Theorem 2.41, Lemma 3.34 gives a growth type characterization of its q-coercivity for exponent values $q \in Q(f) = [0, c(f))$. In the following, we shall show that such f fulfill a similar growth type condition as that from Lemma 3.34 even for the choice of the exponent value $q = c(f)$. We shall use this fact in Section 3.5 below to prove the existence of certain Hölder type error bounds, or to prove growth properties for Lagrangians in some problems arising in the calculus of variations.

Theorem 3.38. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41. Then for every $q \in \overline{Q(f)} = [0, c(f)]$ there exist constants $c_1 > 0$, $c_2 \geq 0$ with*

$$f(x) \geq c_1 \|x\|^q - c_2 \quad \textit{for all } x \in \mathbb{R}^n$$

Proof. As by Theorem 3.37, the polynomial f is q-coercive on $\mathbb{R}^n$ for all $q \in Q(f) = [0, c(f))$, the application of Lemma 3.34 yields the assertion for each $q \in Q(f)$. It hence only remains to prove the assertion for the special case $q = c(f)$. By the proof of Theorem 2.41 (see [4, Th. 3.4]) there exists a constant $\varepsilon > 0$ such that for every sequence $(x^\nu)_{\nu\in\mathbb{N}}$ with $\lim_{\nu\to\infty} \|x^\nu\|_\infty = +\infty$ we have

$$f^{D(f)}(x^\nu) + f^{R(f)}(x^\nu) \geq (\varepsilon - 1) f^{V(f)}(x^\nu) \quad \text{for almost all } \nu \in \mathbb{N}.$$

With analogous arguments as in the proof of Lemma 3.31 this results in

$$\frac{f(x^\nu)}{\|x^\nu\|_\infty^{c(f)}} \geq \varepsilon \frac{f^{V(f)}(x^\nu)}{\|x^\nu\|_\infty^{c(f)}} \geq \varepsilon \min_{i\in I} f_{2k_ie_i}$$

for almost all $\nu \in \mathbb{N}$ and, thus, in

$$\liminf_{\|x\|_\infty\to\infty} \frac{f(x^\nu)}{\|x^\nu\|_\infty^{c(f)}} \geq \varepsilon \min_{i\in I} f_{2k_ie_i} > 0.$$

Thus, there exists some $M > 0$ with

$$\frac{f(x)}{\|x\|_\infty^{c(f)}} \geq \frac{\varepsilon \min_{i\in I} f_{2k_ie_i}}{2} \quad \text{for all } \|x\|_\infty > M. \tag{3.18}$$

Defining

$$c_1 := \frac{\varepsilon \min_{i\in I} f_{2k_ie_i}}{2} \quad \text{and} \quad c_2 := \max\{0, \max_{\|x\|_\infty\leq M} c_1\|x\|_\infty^{c(f)} - f(x)\},$$

one obtains $f(x) \geq c_1\|x\|_\infty^{c(f)} - c_2$ for all $\|x\|_\infty \leq M$ and, due to $c_2 \geq 0$ and (3.18), the same inequality for all $\|x\|_\infty > M$. The assertion now follows from the equivalence of norms on $\mathbb{R}^n$. □

It is worth mentioning that the growth of polynomials is also analyzed in [63], where the authors use various algebraic tools for answering the question of how fast (not necessarily coercive) polynomials grow on semialgebraic sets.

3.5 Some applications

In this section we present three applications of our above main results. All of them are related to the speed of growth of coercive polynomials which, in view of the above results, may be characterized in terms of their degree of convenience.

3.5.1 Surjectivity of polynomial gradient maps

The following result is a well-known variational argument and may be found in, e.g., [12] for the univariate case. We provide its short proof for completeness.

Lemma 3.39. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and 1-coercive on $\mathbb{R}^n$. Then the gradient map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is surjective, that is, we have*

$$\{\nabla f(x) |\ x \in \mathbb{R}^n\} \ = \ \mathbb{R}^n.$$

Proof. By the 1-coercivity of f, for any $a \in \mathbb{R}^n$ the function $g_a(x) := f(x) - a^T x$ satisfies

$$\begin{aligned} \lim_{\|x\|\to\infty} \frac{g_a(x)}{\|x\|} \quad &= \quad \lim_{\|x\|\to\infty} \frac{f(x) - a^T x}{\|x\|} = \lim_{\|x\|\to\infty} \left(\frac{f(x)}{\|x\|} - a^T \frac{x}{\|x\|} \right) \\ &\geq \quad \lim_{\|x\|\to\infty} \left(\frac{f(x)}{\|x\|} - \max_{\|y\|=1} a^T y \right) = +\infty \end{aligned}$$

and is, thus, also 1-coercive on $\mathbb{R}^n$. Like in Remark 3.26, this implies the coercivity of g_a on $\mathbb{R}^n$ and, hence, there exists a global minimal point $x_a \in \mathbb{R}^n$ of g_a over $\mathbb{R}^n$. Fermat's rule implies $0 = \nabla g_a(x_a) = \nabla f(x_a) - a$, so that for any $a \in \mathbb{R}^n$ there exists some $x_a \in \mathbb{R}^n$ with $\nabla f(x_a) = a$, as asserted. □

Theorem 3.40. *Let $f \in \mathbb{R}[x]$ be a gem regular polynomial satisfying the conditions* (C1)-(C3)*. Then the polynomial gradient map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.*

Proof. By Theorem 3.32 one has $Q(f) = [0, o(f))$ with $o(f) \in 2\mathbb{N}$. This implies $o(f) \geq 2$, and by Remark 3.26 f is 1-coercive. The assertion, thus, follows from Lemma 3.39. □

Replacing Theorem 3.32 by Theorem 3.37 in the previous proof immediately yields the following result.

Theorem 3.41. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41. Then the polynomial gradient map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.*

We mention that checking surjectivity of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ is, in general, NP-hard (see, e.g., [7]). By Theorems 3.40 and 3.41, for some polynomial gradient maps $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ with $f \in \mathbb{R}[x]$, the surjectivity of ∇f can be guaranteed by the sufficient conditions for coercivity of f from Theorems 2.39 or 2.41. These, however, require to identify the set of vertices of Newton polytopes at infinity $V(f)$ together with the corresponding faces of $\mathrm{New}_\infty(f)$ not containing the origin $\mathcal{G}(f)$. This may be realized by, for example, vertex- or facet enumeration algorithms (for more details see, e.g., [3, 13]).

3.5.2 Hölder type error bounds

Error bounds for systems of inequalities possess important applications, for example in sensitivity analysis or in the formulation of termination criteria for optimization methods (for more details and references see, e.g., [54]). Some results on the global error bound property for systems of polynomials are already known. In [57] a Hölder type global error bound for a general polynomial system is proven, but the corresponding Hölder exponent remains unspecified (see [57, Th. 2.2]). In the special case of a convex quadratic inequality system satisfying the Slater condition, in [57] a global error bound result is proven with the explicit (Hölder) exponent one (see [57, Th. 3.1]), which can been seen as an analogon of the well-known Hoffman global error bound for linear inequality systems (see [36]). In [87], for a convex quadratic inequality system a generalization of the result from [57] is achieved, where the Slater condition is not needed, with the corresponding Hölder exponent not exceeding one and explicitly depending on the so-called degree of singularity of the system. Further generalizations of Hölder error bound results in the setting of piecewise convex quadratic systems, general piecewise convex polynomials or parametric polynomial systems can be found in [54–56]. In [54] and [55] an explicit Hölder exponent depending on the dimension n and the degree of the corresponding polynomial d is given.

In this section, we provide Hölder type error bounds for a broad class of (not necessarily convex) coercive polynomials, and we link the corresponding Hölder exponents to the degree of convenience.

For an arbitrary non-empty set $M \subseteq \mathbb{R}^n$, the distance function $\mathrm{dist}(\cdot, M) : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathrm{dist}(x, M) := \inf_{z \in M} \|x - z\|,$$

where $\|\cdot\|$ denotes some norm on $\mathbb{R}^n$, and, for any function $f:\mathbb{R}^n \to \mathbb{R}$, we define the residual function $[f(x)]_+ = \max\{0, f(x)\}$, $x \in \mathbb{R}^n$, and the lower level set $f^0_\leq = \{x \in \mathbb{R}^n |\, f(x) \leq 0\}$.

Lemma 3.42. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41 with $f^0_\leq \neq \emptyset$. Then there exist a constant $\gamma > 0$ and some $r > 0$ with*

$$\operatorname{dist}(x, f^0_\leq) \leq \gamma\, [f(x)]_+^{1/c(f)} \quad \textit{for all } \|x\| > r.$$

Proof. For the contrary, let us assume that for each $\gamma > 0$, $r > 0$ there exists some $x \in \mathbb{R}^n$ with $\|x\| > r$ and

$$\operatorname{dist}(x, f^0_\leq) > \gamma\, [f(x)]_+^{1/c(f)}.$$

In particular, the latter implies the existence of sequences $(\gamma^\nu) \subseteq \mathbb{R}$ with $\gamma^\nu \to +\infty$ for $\nu \to \infty$ and $(x^\nu) \subseteq \mathbb{R}^n$ with $\|x^\nu\| \to \infty$ for $\nu \to \infty$ such that

$$\left(\operatorname{dist}(x^\nu, f^0_\leq)\right)^{c(f)} > \gamma^\nu f(x^\nu) \tag{3.19}$$

holds for almost all $\nu \in \mathbb{N}$, where the coercivity of f is employed. By Theorem 3.38 there exist constants $c_1 > 0$ and $c_2 \geq 0$ yielding

$$f(x^\nu) \geq c_1 \|x^\nu\|^{c(f)} - c_2 > 0 \tag{3.20}$$

for almost all $\nu \in \mathbb{N}$. Due to the coercivity of f on $\mathbb{R}^n$, the nonempty set $f^0_\leq \subseteq \mathbb{R}^n$ is compact and, using the Weierstrass theorem, one obtains

$$\operatorname{dist}(x, f^0_\leq) = \inf_{z \in f^0_\leq} \|x - z\| \leq \inf_{z \in f^0_\leq} (\|x\| + \|z\|) = \|x\| + \min_{z \in f^0_\leq} \|z\| \tag{3.21}$$

for all $x \in \mathbb{R}^n$.

The combination of properties (3.19), (3.20) and (3.21) results in

$$\frac{\left(\|x^\nu\| + \min_{z \in f^0_\leq} \|z\|\right)^{c(f)}}{c_1 \|x^\nu\|^{c(f)} - c_2} \geq \frac{\left(\operatorname{dist}(x^\nu, f^0_\leq)\right)^{c(f)}}{[f(x^\nu)]_+} > \gamma^\nu \tag{3.22}$$

holding for almost all $\nu \in \mathbb{N}$. Taking the limes superior for $\nu \to +\infty$ in (3.22), yields

$$\frac{1}{c_1} = \limsup_{\nu \to +\infty} \frac{\left(\|x^\nu\| + \min_{z \in f^c_\leq} \|z\|\right)^{c(f)}}{c_1 \|x^\nu\|^{c(f)} - c_2} \geq \limsup_{\nu \to +\infty} \gamma^\nu = +\infty,$$

a contradiction. $\square$

Remark 3.43. Applying Theorem 3.37 in Lemma 3.42 reveals the upper bound $1/c(f) \leq 1/2$ on the Hölder exponent due to $c(f) \geq 2$. □

Before we state global Hölder type error bounds, we shortly recall the following version of a local error bound result for a single polynomial inequality from [57] (for more detail, see [57, Cor. 2.3]). The main result in [57] (Theorem 2.2) uses an error bound result for polynomial equality systems proven in [37, Lem. 2].

Lemma 3.44. *Let $f \in \mathbb{R}[x]$ with $f^0_\leq \neq \emptyset$ be given. Then, for all $\tilde{r} > 0$ with $f^0_\leq \cap \{x \in \mathbb{R}^n |\ \|x\| \leq \tilde{r}\} \neq \emptyset$, there exist some $\tilde{\gamma} > 0$ and $\tilde{q} > 0$ such that*

$$\operatorname{dist}(x, f^0_\leq) \leq \tilde{\gamma}\, [f(x)]_+^{1/\tilde{q}} \quad \textit{for all } \|x\| \leq \tilde{r}.$$

Now we are ready to formulate the main result of the present section, a global Hölder type error bound for coercive polynomials.

Theorem 3.45. *Let $f \in \mathbb{R}[x]$ be a polynomial satisfying the assumptions of Theorem 2.41 with $f^0_\leq \neq \emptyset$. Then there exist some $\bar{\gamma} > 0$ and $\tilde{q} > 0$ such that*

$$\operatorname{dist}(x, f^0_\leq) \leq \bar{\gamma} \max\{[f(x)]_+^{1/c(f)}, [f(x)]_+^{1/\tilde{q}}\} \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. By Lemma 3.42, there exist some $\gamma > 0$ and $r > 0$ such that

$$\operatorname{dist}(x, f^0_\leq) \leq \gamma\, [f(x)]_+^{1/c(f)} \quad \text{for all } \|x\| > r. \tag{3.23}$$

Without loss of generality, one can assume that $r > 0$ fulfills

$$f^0_\leq \cap \{x \in \mathbb{R}^n |\ \|x\| \leq r\} \neq \emptyset.$$

Then, Lemma 3.44 yields the existence of some $\tilde{\gamma} > 0$ and $\tilde{q} > 0$ with

$$\operatorname{dist}(x, f^0_\leq) \leq \tilde{\gamma}\, [f(x)]_+^{1/\tilde{q}} \quad \text{for all } \|x\| \leq r. \tag{3.24}$$

Setting $\bar{\gamma} := \max\{\gamma, \tilde{\gamma}\}$, the assertion directly follows by (3.23) and (3.24). □

3.5.3 Existence and uniqueness of solutions in the calculus of variations

The following general existence result for the fundamental problem in the calculus of variations in Sobolev spaces possesses a long history starting with Tonelli [81] as one of its first contributors. For more details, we refer to [18, 19]. For proofs see, for example, [18, Sec. 3.3] or [19, Sec. 3.4.1].

Theorem 3.46. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let $f \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $f = f(x, u, \xi)$, satisfy*

$$f(x, u, \cdot) \text{ is convex for every } (x, u) \in \overline{\Omega} \times \mathbb{R}, \tag{T1}$$

together with

$$\text{there exist } q > p \geq 1 \text{ and } c_1 > 0, c_2, c_3 \in \mathbb{R} \text{ such that}$$

$$f(x, u, \xi) \geq c_1 \|\xi\|_2^q + c_2 |u|^p + c_3 \text{ for all } (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n. \tag{T2}$$

Let the problem

$$(P) \qquad \inf_{u \in W^{1,q}(\Omega)} I(u) := \int_\Omega f(x, u(x), \nabla u(x)) \, dx \quad s.t. \quad u = u_0 \text{ on } \partial\Omega$$

be given with some $u_0 \in W^{1,q}(\Omega)$ such that $I(u_0) < \infty$. Then the problem (P) possesses a minimizer $\overline{u} \in W^{1,q}(\Omega)$. Furthermore, if $f(x, \cdot, \cdot)$ is convex for every $x \in \overline{\Omega}$ and either $f(x, \cdot, \xi)$ is strictly convex for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ or $f(x, u, \cdot)$ is strictly convex for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$ then the minimizer of (P) is unique.

A natural question arising in the context of Theorem 3.46 is how to verify the growth property (T2) with respect to the gradient information ∇u for some given Lagrangian f and, if at all, for which choices of exponent values q condition (T2) holds.

Before we state Theorem 3.48 as the main result of this section, we briefly show that for coercive polynomials strict convexity is not a stronger assumption than convexity. For an alternative proof, see Section A.2 in Appendix.

Lemma 3.47. *Let $f \in \mathbb{R}[x]$ be coercive and convex on $\mathbb{R}^n$. Then f is strictly convex on $\mathbb{R}^n$.*

Proof. Assume that f is convex, but not strictly convex on $\mathbb{R}^n$. Then f must be linear one some line segment of positive length in $\mathbb{R}^n$ (see, e.g., [77, Lem. 2]). This means that for some $x, y \in \mathbb{R}^n$, $x \neq y$, the function $F(t) := f(x + t(y - x))$ is linear on the interval $[0, 1]$. Since F inherits the polynomiality of f, it must be linear even on all of $\mathbb{R}$. On the other hand, F also inherits the coercivity of f, which contradicts its linearity . $\square$

For the case when the Lagrangian f from Theorem 3.46 is separable in the variable groups ξ and (x, u), as well as polynomial in x, an application of Theorem 3.38 yields the following result, where the degree of convenience arises as a natural upper bound for the choices of exponent values q.

Theorem 3.48. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let $f_1 \in \mathbb{R}[\xi]$ be convex on $\mathbb{R}^n$ and satisfy the conditions from Theorem 2.41. Let further $f_2 \in C^0(\overline{\Omega} \times \mathbb{R})$ be such that there exist $p \in [1, c(f_1))$ and $c_2, c_3 \in \mathbb{R}$ with*

$$f_2(x,u) \geq c_2|u|^p + c_3 \text{ for all } (x,u) \in \overline{\Omega} \times \mathbb{R}. \tag{T2*}$$

Let

$$(P) \qquad \inf_{u \in W^{1,q}(\Omega)} I(u) := \int_\Omega f_1(\nabla u(x)) + f_2(x, u(x))\, dx \quad s.t. \quad u = u_0 \text{ on } \partial\Omega$$

with some $u_0 \in W^{1,q}(\Omega)$ and some $q \in (p, c(f_1)]$ be given such that $I(u_0) < \infty$. Then the problem (P) possesses a minimizer $\overline{u} \in W^{1,q}(\Omega)$. Furthermore, if $f_2(x,\cdot)$ is convex on $\mathbb{R}$ for every $x \in \overline{\Omega}$, then the minimizer of (P) is unique.

Proof. The proof proceeds by verifying all assumptions of Theorem 3.46. For the Lagrangian $f(x,u,\xi) := f_1(\xi) + f_2(x,u)$ one obtains $f \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ due to $f_1 \in C^0(\mathbb{R}^n)$ and $f_2 \in C^0(\overline{\Omega} \times \mathbb{R})$. Further, convexity of f_1 on $\mathbb{R}^n$ yields the convexity of $f(x,u,\cdot)$ for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$ and thus the property (T1). The application of Theorem 3.38 to f_1 provides for each $q \in \overline{Q(f)} = [0, c(f)]$ the existence of some constants $c_1 > 0$ and $\tilde{c}_3 \geq 0$ with

$$f_1(\xi) \geq c_1\|\xi\|^q - \tilde{c}_3 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3.25}$$

Hence, for all choices $q \in \overline{Q(f)} = [0, c(f)]$ with $q > p$ the combination of the properties (T2*) and (3.25) yields

$$f(x,u,\xi) \geq c_1\|\xi\|^q + c_2|u|^p + c_4 \text{ for all } (x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n,$$

with exponents $q > p \geq 1$ and some constants $c_1 > 0$, $c_2 \in \mathbb{R}$ and $c_4 := c_3 - \tilde{c}_3 \in \mathbb{R}$. Property (T2) from Theorem 3.46 is thus also fulfilled and the existence assertion follows. For uniqueness, Lemma 3.47 yields strict convexity of f_1 on $\mathbb{R}^n$ which, together with the convexity of $f_2(x,\cdot)$ for all $x \in \overline{\Omega}$, provides that $f(x,\cdot,\cdot)$ is convex for all $x \in \overline{\Omega}$ and $f(x,u,\cdot)$ is strictly convex for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$. Finally, an application of the uniqueness part of Theorem 3.46 for f finishes the proof. □

Remark 3.49. Although the functional $I(u)$ from Theorem 3.48 is of a rather special form, it contains the following interesting special case. The choice $f_1(\xi) := \frac{1}{2}\|\xi\|_2^2 \in \mathbb{R}[\xi]$, $f_2(x, u(x)) := g(x)u$ with some sufficiently smooth function g yields a functional $I(u)$ which possesses as its corresponding *Euler-Lagrange equation* the well-known *Poisson equation*

$$\Delta u(x) = g(x) \quad \text{for all } x \in \Omega,$$

an important object of interest in theoretical physics or mechanical engineering. For more details see, e.g., [24, 29]. □

The following result is a straightforward implication of Theorem 3.48 for problems in the calculus of variations with polynomial Lagrangians depending only on the gradient information.

Corollary 3.50. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let $f \in \mathbb{R}[\xi]$ be convex on $\mathbb{R}^n$ and satisfy the conditions from Theorem 2.41. Let the problem*

$$(P) \qquad \inf_{u \in W^{1,q}(\Omega)} I(u) := \int_\Omega f(\nabla u(x))\, dx \quad s.t. \quad u = u_0 \ on\ \partial\Omega$$

with some $u_0 \in W^{1,q}(\Omega)$ and some $q \in (1, c(f)]$ be given such that $I(u_0) < \infty$. Then the problem (P) possesses a unique minimizer $\overline{u} \in W^{1,q}(\Omega)$.

Proof. The assertion follows with Theorem 3.48 by setting $f_1 := f$, $f_2 \equiv 0$, $c_2 = c_3 = 0$ and $p = 1$. □

An interesting illustration of the latter corollary is the following well-known existence result to the famous Dirichlet problem. We restate it here briefly in the light of the gem regularity of the corresponding Lagrangian and its degree of convenience.

Example 3.51 (Dirichlet's Energy Integral). *For an open bounded set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, consider Dirichlet's energy integral*

$$I(u) := \int_\Omega \|\nabla u\|_2^2\, dx.$$

Since the corresponding polynomial Lagrangian $f \in \mathbb{R}[\xi]$ with $f(\xi) = \sum_{i \in I} \xi_i^2$ is convex on $\mathbb{R}^n$, gem regular and it satisfies the conditions from Theorem 2.41, choosing $q := c(f) = 2$, Corollary 3.50 yields that the problem

$$(P) \qquad \inf_{u \in W^{1,2}(\Omega)} I(u) \quad s.t. \quad u = u_0 \ on\ \partial\Omega,$$

possesses a unique minimizer $\overline{u} \in W^{1,2}(\Omega)$ for all $u_0 \in W^{1,2}(\Omega)$.

It is worth mentioning that, for the case $n = 1$, the setting of polynomial Lagrangians is also used for some regularity considerations concerning solutions of variational problems (see, e.g. [17]).

Chapter 4

Global Polynomial Diffeomorphisms

4.1 Chapter overview

This chapter is based on the article [6] and it is structured as follows. In Section 4.2, we show that every sum of squares polynomial $\|F\|_2^2$ corresponding to some polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ fulfills conditions (C1) and (C2) and, using a determinant formula for Jacobians JF (see Lem. 4.8 below), we prove that polynomials $\|F\|_2^2$ corresponding to polynomial maps F with nonvanishing Jacobian determinants $\det JF$ fulfill also the condition (C3) (see Props. 4.9 and 4.10 below). Finally, a combination of Hadamard's theorem (see Th. 4.1 below) and the coercivity results from Chapter 2 enables us to identify a class of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ whose global diffeomorphism property on $\mathbb{R}^n$ is equivalent to their Jacobian determinant $\det JF$ vanishing nowhere on $\mathbb{R}^n$, which is the main result of the present chapter (see Ths. 4.11 and 4.12 below).

This class of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ is described in terms of so-called Newton polytopes at infinity $\mathrm{New}_\infty(\|F\|_2^2)$ corresponding to $\|F\|_2^2$. More precisely, for a given polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$, in order to verify whether F belongs to the latter class, one has to identify the vertex set at infinity $V(\|F\|_2^2)$, the set of so-called gem-degenerate exponent vectors $D(\|F\|_2^2)$, and for the latter one also has to compute the corresponding circuit numbers (for definitions, see Chapter 2). The first may be realized by, for example, vertex- or facet enumeration algorithms (for more details see, e.g., [3, 13]).

We illustrate our main results in Example 4.13, where a one-parametric family of polynomial diffeomorphisms of $\mathbb{R}^2$ onto itself is analyzed by using our techniques. Since for some singular parameter value these techniques are not directly applicable, in Section 4.3 we also prove the invariance of the coercivity property under linear coordinate transformations, and show that our main results may be generalized by replacing the assumptions on $\|F\|_2^2$ by assumptions on $\|F \circ A^{-1}\|_2^2$ for some regular matrix $A \in \mathbb{R}^{n\times n}$ (see Cors. 4.18 and 4.19 below). In Example 4.20, we use such a transformation to apply our techniques to treat the case of the singular parameter from Example 4.13.

4.2 Global diffeomorphism property

Due to [30], the following theorem, which is of crucial importance for the present work, goes back at least to Jacques S. Hadamard [33–35]. For its proof see, e.g., [30], [51, Sec. 6.2], or [66, Cor. 4.3].

Theorem 4.1 (Hadamard)**.** *A map $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is a C^1-diffeomorphism of $\mathbb{R}^n$ onto itself if and only if the map F is proper and* $\det JF$ *vanishes nowhere on $\mathbb{R}^n$.*

Since for a continuous map $F : \mathbb{R}^n \to \mathbb{R}^n$ its properness is equivalent to the property $\|F(x)\|_2^2 \to +\infty$ whenever $\|x\| \to +\infty$ (see, e.g., [27, Prop. 3.1.15]), one can reformulate Theorem 4.1 in the setting of polynomial maps as follows.

Theorem 4.2. *A map $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \dots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$ is a C^1-diffeomorphism of $\mathbb{R}^n$ onto itself if and only if*

$$\det JF(x) \neq 0 \textit{ for all } x \in \mathbb{R}^n \tag{H1}$$

and

$$\|F(x)\|_2^2 \in \mathbb{R}[x] \textit{ is coercive on } \mathbb{R}^n. \tag{H2}$$

In the following we will identify conditions under which (H1) implies (H2), so that the diffeomorphism property of F in Theorem 4.2 may be characterized by condition (H1) alone, that is, the Real Jacobian Conjecture is true under these conditions. To this end, we shall first show that the function $f := \|F\|_2^2$ always satisfies the conditions (C1) and (C2).

For any two sets $X_1, X_2 \subseteq \mathbb{R}^n$ we denote by $X_1 + X_2 := \{x \in \mathbb{R}^n | \ \exists x_1 \in X_1, \exists x_2 \in X_2 : x = x_1 + x_2\} \subseteq \mathbb{R}^n$ their Minkowski sum and we define $dX_1 := \{x \in \mathbb{R}^n | \ \exists x_1 \in X_1 : x = dx_1\}$ for any $d \in \mathbb{R}$. We further denote by $\mathrm{vert}(P)$ the set of all vertices of some polytope $P \subseteq \mathbb{R}^n$. The proof of the following auxiliary result is given in Section A.3 in Appendix.

Lemma 4.3. *For any polytope $P \subseteq \mathbb{R}^n$ it holds $v \in \operatorname{vert}(P+P)$ if and only if $v = 2w$ with some $w \in \operatorname{vert}(P)$.*

The subsequent Lemma 4.4 will provide some useful properties regarding the Newton polytopes at infinity of squared polynomials, while Lemma 4.5 shall treat the case of sum of squares polynomials.

Lemma 4.4. *For any $f \in \mathbb{R}[x]$ the following properties hold.*

i) $\operatorname{New}_\infty(f^2) = \operatorname{New}_\infty(f) + \operatorname{New}_\infty(f)$

ii) $V(f^2) = 2V(f)$

iii) for each $\alpha \in V(f^2)$ it holds $(f^2)_\alpha = (f_{\frac{1}{2}\alpha})^2 > 0$.

Proof. Observe that due to

$$f^2(x) = \left(\sum_{\alpha \in A(f)} f_\alpha x^\alpha\right)^2 = \sum_{\alpha,\beta \in A(f)} f_\alpha f_\beta x^{\alpha+\beta}$$

$$= \sum_{\gamma \in A(f)+A(f)} \left(\sum_{\substack{\alpha,\beta \in A(f)\\ \alpha+\beta=\gamma}} f_\alpha f_\beta\right) x^\gamma, \tag{4.1}$$

the inclusion

$$A(f^2) \subseteq A(f) + A(f) \tag{4.2}$$

holds, which results in

$$\begin{aligned}
\operatorname{New}_\infty(f^2) &= \operatorname{conv}\left(\{0\} \cup A(f^2)\right) \subseteq \operatorname{conv}\left(\{0\} \cup (A(f)+A(f))\right)\\
&\subseteq \operatorname{conv}\left((\{0\} \cup A(f)) + (\{0\} \cup A(f))\right) = \operatorname{conv}\left(A_0(f) + A_0(f)\right)\\
&= \operatorname{conv}\left(A_0(f)\right) + \operatorname{conv}\left(A_0(f)\right) = \operatorname{New}_\infty(f) + \operatorname{New}_\infty(f),
\end{aligned} \tag{4.3}$$

where the first inclusion follows from (4.2) and the penultimate equality holds since the convex hull of the Minkowski sum of some given sets is the Minkowski sum of the convex hulls of the sets (see, e.g., [50, Prop. 4.12]).

Next, we shall show the inclusion

$$\operatorname{New}_\infty(f) + \operatorname{New}_\infty(f) \subseteq \operatorname{New}_\infty(f^2). \tag{4.4}$$

To this end it suffices to show $\operatorname{vert}(\mathrm{New}_\infty(f)+\mathrm{New}_\infty(f)) \subseteq \mathrm{New}_\infty(f^2)$. By Lemma 4.3 we have $\gamma \in \operatorname{vert}(\mathrm{New}_\infty(f)+\mathrm{New}_\infty(f))$ if and only if $\gamma = 2\delta$ holds with some (unique) $\delta \in \operatorname{vert}(\mathrm{New}_\infty(f)) = V_0(f) \subseteq A_0(f)$. If $\delta = 0$, then $\gamma = 0 \in \mathrm{New}_\infty(f^2)$ by definition. If $\delta \neq 0$, then with (4.1) one obtains for the coefficient $(f^2)_\gamma \in \mathbb{R}$ of f^2 corresponding to the vertex γ that

$$(f^2)_\gamma = \sum_{\substack{\alpha,\beta\in A(f)\\ \alpha+\beta=\gamma}} f_\alpha f_\beta = (f_\delta)^2 > 0, \tag{4.5}$$

where the last equality holds due to Lemma 4.3 and the inequality due to $f_\delta \neq 0$ following from $0 \neq \delta \in A(f)$. This implies $\gamma \in A(f^2)$ and hence $\gamma \in \mathrm{New}_\infty(f^2)$. Since $\gamma \in \operatorname{vert}(\mathrm{New}_\infty(f)+\mathrm{New}_\infty(f))$ was chosen arbitrarily, the inclusion

$$\operatorname{vert}(\mathrm{New}_\infty(f)+\mathrm{New}_\infty(f)) \subseteq \mathrm{New}_\infty(f^2)$$

follows.

The assertion i) follows from (4.3) and (4.4). The assertion ii) follows directly from the assertion i) by using Lemma 4.3. The assertion iii) follows directly from (4.5) above. □

Lemma 4.5. *For $f(x) = \sum_{i\in I} F_i^2(x)$ with $F_i \in \mathbb{R}[x]$, $i \in I$, the following properties hold.*

i) $\mathrm{New}_\infty(f) = \operatorname{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right)$

ii) $V_0(f) \subseteq \bigcup_{i\in I} 2V_0(F_i)$

iii) each $\alpha \in V(f)$ satisfies $f_\alpha > 0$.

Proof. Observe that due to

$$f(x) = \sum_{i\in I} F_i^2(x) = \sum_{i\in I}\sum_{\gamma\in A(F_i^2)} (F_i^2)_\gamma x^\gamma$$

$$= \sum_{\gamma\in\bigcup_{i\in I} A(F_i^2)} \left(\sum_{i\in I:\ \gamma\in A(F_i^2)} (F_i^2)_\gamma\right) x^\gamma, \tag{4.6}$$

the inclusion

$$A(f) \subseteq \bigcup_{i\in I} A(F_i^2), \tag{4.7}$$

and thus, also the inclusion

$$A_0(f) \subseteq \bigcup_{i\in I} A_0(F_i^2) \tag{4.8}$$

hold.

Part i)

First, one obtains

$$\begin{aligned}\mathrm{New}_\infty(f) &= \mathrm{conv}\,(A_0(f)) \subseteq \mathrm{conv}\left(\bigcup_{i\in I} A_0(F_i^2)\right) = \mathrm{conv}\left(\bigcup_{i\in I}\mathrm{conv}(A_0(F_i^2))\right)\\ &= \mathrm{conv}\left(\bigcup_{i\in I}\mathrm{New}_\infty(F_i^2)\right) = \mathrm{conv}\left(\bigcup_{i\in I}\mathrm{conv}(V_0(F_i^2))\right) = \mathrm{conv}\left(\bigcup_{i\in I} V_0(F_i^2)\right)\\ &= \mathrm{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right), \end{aligned} \tag{4.9}$$

where the inclusion holds due to (4.8) and the last equality due to Lemma 4.4 ii). In order to show the other inclusion

$$\mathrm{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right) \subseteq \mathrm{New}_\infty(f), \tag{4.10}$$

it suffices to prove that the vertex set of the polytope $\mathrm{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right)$ is contained in the set $\mathrm{New}_\infty(f)$. To this end let $\alpha \in 2\mathbb{N}_0^n$ be an arbitrary vertex of the polytope $\mathrm{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right)$. Then α is necessarily a vertex of each polytope $\mathrm{conv}\,(2V_0(F_i))$ containing α. Hence, by Lemma 4.4 ii), α is a vertex of each Newton polytope at infinity $\mathrm{New}_\infty(F_i^2)$ containing α, that is, $\alpha \in V_0(F_i^2)$ for each $i \in I$ with $\alpha \in \mathrm{New}_\infty(F_i^2)$. If $\alpha = 0$ then obviously $\alpha \in \mathrm{New}_\infty(f)$ by definition. Next we shall consider only the case $\alpha \neq 0$. Here it holds $\alpha \in V(F_i^2)$ for each $i \in I$ with $\alpha \in \mathrm{New}_\infty(F_i^2)$ and using (4.6) together with Lemma 4.4 ii) and iii) one obtains

$$\begin{aligned} f_\alpha &= \sum_{i\in I:\ \alpha\in A(F_i^2)} \left(F_i^2\right)_\alpha = \sum_{i\in I:\ \alpha\in V(F_i^2)} \left(F_i^2\right)_\alpha\\ &= \sum_{i\in I:\ \frac{\alpha}{2}\in V(F_i)} \left((F_i)_{\frac{\alpha}{2}}\right)^2 > 0. \end{aligned} \tag{4.11}$$

This implies $\alpha \in A(f)$, and hence, $\alpha \in \mathrm{New}_\infty(f)$.

The assertion i) follows from (4.9) and (4.10).

Part ii)

Due to i) it holds

$$V_0(f) = \mathrm{vert}\left(\mathrm{conv}\left(\bigcup_{i\in I} 2V_0(F_i)\right)\right) \subseteq \bigcup_{i\in I} 2V_0(F_i),$$

which proves the assertion ii).

Part iii)

Due to ii) it holds $V(f) \subseteq \bigcup_{i\in I} 2V(F_i)$. Thus for any $\alpha \in V(f)$ one has $\alpha \in \bigcup_{i\in I} 2V(F_i)$ and with (4.11) one obtains

$$f_\alpha = \sum_{i\in I\colon \frac{\alpha}{2}\in V(F_i)} \left((F_i)_{\frac{\alpha}{2}}\right)^2 > 0,$$

which proves the assertion iii). □

The last lemma yields for any sum of squares polynomial $f \in \mathbb{R}[x]$ the following property.

Proposition 4.6. *Every polynomial $f \in \mathbb{R}[x]$ with $f(x) = \sum_{i\in I} F_i^2(x)$, $F_i \in \mathbb{R}[x]$, $i \in I$, fulfills the conditions* (C1) *and* (C2).

Proof. By Lemma 4.5 ii) one obtains $V_0(f) \subseteq \bigcup_{i\in I} 2V_0(F_i)$, which results in

$$V(f) \subseteq V_0(f) \subseteq \bigcup_{i\in I} 2V_0(F_i) \subseteq 2\mathbb{N}_0^n,$$

and thus, f fulfills the condition (C1).

By Lemma 4.5 iii) one obtains $f_\alpha > 0$ for each $\alpha \in V(f)$, that is, f also fulfills the condition (C2). □

In order to analyze whether the sum of squares polynomial $f = \|F\|_2^2$ corresponding to some polynomial map F also fulfills the condition (C3), we shall use the following auxiliary result.

Lemma 4.7. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \dots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$, be given. If for each $j \in I$ there exist some $i \in I$ and $k \in \mathbb{N}$ with $ke_j \in A(F_i)$, then the polynomial $f = \|F\|_2^2$ satisfies condition* (C3).

Proof. For every $j \in I$ let there exist some $i \in I$ and some $k \in \mathbb{N}$ such that $ke_j \in A(F_i)$ holds. Define for each $j \in I$ the non-empty set

$$I(j) := \{i \in I|\ \exists k \in \mathbb{N} \text{ with } ke_j \in A(F_i)\}$$

and

$$m(j) := \max\{k \in \mathbb{N}|\ ke_j \in A(F_i),\, i \in I(j)\}$$

together with the set $\bar{I}(j) \subseteq I(j)$ of indices at which the maximal value $m(j)$ is attained. For each $j \in I$ it holds $m(j)e_j \in V(F_i)$ for all $i \in \bar{I}(j)$. Using Lemma 4.4 ii) one obtains for each $j \in I$

$$2m(j)e_j \in V(F_i^2) \quad \text{for all } i \in \bar{I}(j)$$

and by Lemma 4.4 iii) also

$$(F_i^2)_{2m(j)e_j} = \left((F_i)_{m(j)e_j}\right)^2 > 0. \tag{4.12}$$

With (4.6) and (4.12) one obtains for each $j \in I$

$$\begin{aligned} f_{2m(j)e_j} &= \sum_{i\in I\,:\,2m(j)e_j \in A(F_i^2)} (F_i^2)_{2m(j)e_j} = \sum_{i\in \bar{I}(j)} (F_i^2)_{2m(j)e_j} \\ &= \sum_{i\in \bar{I}(j)} \left((F_i)_{m(j)e_j}\right)^2 > 0, \end{aligned}$$

which implies $2m(j)e_j \in A(f)$. Since by definition of $m(j)$ it holds $ke_j \notin A(f)$ for all $k > m(j)$, one even obtains that for each $j \in I$ the vector $2m(j)e_j \in A(f)$ is a vertex of $\mathrm{New}_\infty(f)$. Thus, we arrive at $2m(j)e_j \in V(f)$ with some $m(j) \in \mathbb{N}$ for every $j \in I$. Thus, f fulfills the condition (C3), and the assertion follows. $\square$

In the following, for some vectors $a, b \in \mathbb{R}^n$, we use the notation $a \geq b$ if $a_i \geq b_i$ holds for all $i \in I$, and $\mathbb{1}$ denotes the all-ones vector $(1, \dots, 1)^T \in \mathbb{R}^n$.

The next result provides an explicit representation of the Jacobian determinant $\det JF$ of a polynomial map F, which will enable us to link the nowhere vanishing property of $\det JF$ to the condition (C3) of the polynomial $\|F\|_2^2$, as formulated in Proposition 4.9 below.

Lemma 4.8 (Determinant formula).
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \dots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$. Then all $x \in \mathbb{R}^n$ satisfy

$$\det JF(x) = \sum_{\substack{\alpha^i \in A(F_i),\, i\in I \\ \sum_{i\in I} \alpha^i \geq \mathbb{1}}} \left(\det(\alpha^1, \dots, \alpha^n) \prod_{i\in I} (F_i)_{\alpha^i} \right) x^{(\sum_{i\in I} \alpha^i) - \mathbb{1}}. \tag{4.13}$$

Proof. Let S_n denote the symmetric group on n elements, let $\mathrm{sign}(\sigma)$ denote the permutation sign of $\sigma \in S_n$, and for some arbitrarily given $x \in \mathbb{R}^n$ let the entries of $JF(x)$ be denoted by a_{ij}, $i, j \in I$. Then the Leibniz formula for determinants yields

$$\det JF(x) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i\in I} a_{i,\sigma(i)}$$

with

$$a_{i,\sigma(i)} = \frac{\partial}{\partial x_{\sigma(i)}} F_i(x) = \sum_{\alpha^i \in A(F_i)} (F_i)_{\alpha^i} \frac{\partial}{\partial x_{\sigma(i)}} x^{\alpha^i}$$

for all $\sigma \in S_n$ and $i \in I$. Interchanging multiplication and addition, and splitting the appearing products, further leads to

$$\prod_{i\in I} a_{i,\sigma(i)} \;=\; \sum_{(\alpha^1,\ldots,\alpha^n)\in A(F_1)\times\cdots\times A(F_n)} \left[\prod_{i\in I}(F_i)_{\alpha^i}\right]\cdot\left[\prod_{i\in I}\frac{\partial}{\partial x_{\sigma(i)}}x^{\alpha^i}\right]$$

for all $\sigma \in S_n$. In fact, in the above summation for any $i \in I$ it is sufficient to choose $\alpha^i \in A(F_i)$ with $\alpha^i_{\sigma(i)} \geq 1$, since the existence of some $j \in I$ with $\alpha^j_{\sigma(j)} = 0$ means that the monomial x^{α^j} does not depend on the variable $x_{\sigma(j)}$, resulting in

$$\frac{\partial}{\partial x_{\sigma(j)}}x^{\alpha^j} \;=\; 0 \quad\text{and}\quad \prod_{i\in I}\frac{\partial}{\partial x_{\sigma(i)}}x^{\alpha^i} = 0.$$

This shows

$$\prod_{i\in I} a_{i,\sigma(i)} \;=\; \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \alpha^i_{\sigma(i)}\geq 1,\, i\in I}} \left[\prod_{i\in I}(F_i)_{\alpha^i}\right]\cdot\left[\prod_{i\in I}\frac{\partial}{\partial x_{\sigma(i)}}x^{\alpha^i}\right]$$

for all $\sigma \in S_n$.

Next, for any $(\alpha^1,\ldots,\alpha^n)$ in the above summation and any $i \in I$ we have

$$\frac{\partial}{\partial x_{\sigma(i)}}x^{\alpha^i} \;=\; \alpha^i_{\sigma(i)} x_{\sigma_i}^{\alpha^i_{\sigma(i)}-1}\prod_{j\neq i} x_{\sigma(j)}^{\alpha^j_{\sigma(j)}}$$

and, since σ is a permutation,

$$\prod_{i\in I}\frac{\partial}{\partial x_{\sigma(i)}}x^{\alpha^i} \;=\; \left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot x^{\sum_{i\in I}\alpha^i-\mathbb{1}}.$$

We arrive at

$$\begin{aligned}\prod_{i\in I} a_{i,\sigma(i)} \;&=\; \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \alpha^i_{\sigma(i)}\geq 1,\, i\in I}} \left[\prod_{i\in I}(F_i)_{\alpha^i}\right]\cdot\left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot x^{\sum_{i\in I}\alpha^i-\mathbb{1}}\\ &=\; \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \alpha^i_{\sigma(i)}\geq 1,\, i\in I}} \left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot m(\alpha^1,\ldots,\alpha^n,x)\end{aligned}$$

for all $\sigma \in S_n$, where the monomial

$$m(\alpha^1,\ldots,\alpha^n,x) \;:=\; \left[\prod_{i\in I}(F_i)_{\alpha^i}\right]\cdot x^{\sum_{i\in I}\alpha^i-\mathbb{1}}$$

does not depend on σ. Hence, we may write

$$\begin{aligned}\det JF(x) &= \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \alpha^i_{\sigma(i)}\geq 1,\, i\in I}} \left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot m(\alpha^1,\dots,\alpha^n,x)\\ &= \sum_{\alpha^i\in A(F_i),\, i\in I} \sum_{\substack{\sigma\in S_n\\ \alpha^i_{\sigma(i)}\geq 1,\, i\in I}} \operatorname{sign}(\sigma)\cdot\left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot m(\alpha^1,\dots,\alpha^n,x).\end{aligned}$$

In the latter outer summation it suffices to consider $\alpha^i \in A(F_i)$, $i \in I$, with $\sum_{i\in I}\alpha^i \geq \mathbb{1}$, since otherwise there would exist some $j \in I$ with $\alpha^i_j = 0$ for all $i \in I$, resulting in $\alpha_j^{\sigma^{-1}(j)} = 0$ for any $\sigma \in S_n$. However, then the inner summation would be taken over the empty set.

After introducing this restriction on the outer summation, we may drop the constraint $\alpha^i_{\sigma(i)} \geq 1$, $i \in I$, in the inner summation since, for given $\sigma \in S_n$, its violation leads to a vanishing product $\prod_{i\in I}\alpha^i_{\sigma(i)}$. Thus we have shown the assertion

$$\begin{aligned}\det JF(x) &= \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \sum_{i\in I}\alpha^i\geq\mathbb{1}}} \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\cdot\left[\prod_{i\in I}\alpha^i_{\sigma(i)}\right]\cdot m(\alpha^1,\dots,\alpha^n,x)\\ &= \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \sum_{i\in I}\alpha^i\geq\mathbb{1}}} \det(\alpha^1,\dots,\alpha^n)\cdot m(\alpha^1,\dots,\alpha^n,x),\end{aligned}$$

where the final identity is due to the Leibniz formula for determinants. □

Proposition 4.9. *Let $F:\mathbb{R}^n\to\mathbb{R}^n$ with $F=(F_1,\dots,F_n)$, $F_i\in\mathbb{R}[x]$, $i\in I$, be given such that*

$$\det JF(0) \neq 0$$

holds. Then the polynomial $f = \|F\|_2^2$ satisfies condition (C3).

Proof. Assume that $f = \|F\|_2^2$ does not fulfill condition (C3). Then by Lemma 4.7 there exists an index $j^\star \in I$ such that for every $i \in I$ and every $k\in\mathbb{N}$ one has $ke_{j^\star}\notin A(F_i)$ and, thus, choosing $k = 1$ one especially obtains that for all $i \in I$

$$e_{j^\star} \notin A(F_i) \tag{4.14}$$

holds. Consider an arbitrary choice of exponent vectors $\alpha^i \in A(F_i)$, $i \in I$, with

$$\sum_{i\in I}\alpha^i = \mathbb{1}. \tag{4.15}$$

Since $\alpha^i \in \mathbb{N}_0^n$ for each $i \in I$, the system of equations (4.15) implies

$$\alpha_j^i \in \{0,1\} \text{ for all } i,j \in I. \tag{4.16}$$

Regarding (4.15), one also has

$$\sum_{i\in I} \alpha_{j^\star}^i = 1 \tag{4.17}$$

and thus, due to (4.16), there exists some (unique) $i^\star \in I$ such that $\alpha_{j^\star}^{i^\star} = 1$. By (4.14) there also exists some $j^{\star\star} \in I \setminus \{j^\star\}$ with $\alpha_{j^{\star\star}}^{i^\star} \neq 0$ and, consequently,

$$\|\alpha^{i^\star}\|_1 > 1.$$

Thus, the binary vector $\alpha^{i^\star}$ possesses at least two nonzero entries and, with (4.15), one obtains

$$\|\mathbb{1} - \alpha^{i^\star}\|_1 = \|\sum_{i\in I\setminus\{i^\star\}} \alpha^i\|_1 < n-1. \tag{4.18}$$

By (4.18) the remaining $n-1$ binary vectors α^i, $i \in I \setminus \{i^\star\}$, can possess at most $n-2$ non-zero entries in total. Thus, by the pigeonhole principle, there exists some $i^{\star\star} \in I \setminus \{i^\star\}$ with $\alpha^{i^{\star\star}} = 0$, which results in

$$\det(\alpha^1, \ldots, \alpha^n) = 0. \tag{4.19}$$

Since the choice of vectors $\alpha^i \in A(F_i)$, $i \in I$, with (4.15) was arbitrary, using Lemma 4.8 and (4.19) one finally obtains

$$\det JF(0) = \sum_{\substack{\alpha^i \in A(F_i),\, i\in I \\ \sum_{i\in I}\alpha^i = \mathbb{1}}} \left(\det(\alpha^1,\ldots,\alpha^n) \prod_{i=1}^n (F_i)_{\alpha^i} \right) = 0,$$

and the assertion follows.

□

The combination of Propositions 4.6 and 4.9 provides the following result.

Proposition 4.10. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \ldots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$, be given such that*

$$\det JF(0) \neq 0$$

holds. Then the polynomial $f = \|F\|_2^2$ fulfills the conditions (C1)-(C3).

The following two theorems contain the main results of this chapter. The first one assumes the gem-regularity of the polynomial $\|F\|_2^2$, while the second one treats also the case of gem-irregular polynomials $\|F\|_2^2$ under some further conditions imposed on the coefficients corresponding to the gem-degenerate exponent vectors of $\|F\|_2^2$ which also include the circuit number information.

Theorem 4.11. *For $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \ldots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$, let the polynomial $f = \|F\|_2^2$ be gem regular. Then the following two assertions are equivalent.*

a) F is a C^1-diffeomorphism of $\mathbb{R}^n$ onto itself.

b) $\det JF(x) \neq 0$ holds for all $x \in \mathbb{R}^n$.

Proof. Assertion a) implies b) by direct application of Theorem 4.1. For the proof of the reverse direction observe that assertion b) and Proposition 4.10 imply that $\|F\|_2^2$ fulfills the conditions (C1)-(C3) which, by Theorem 2.39, characterize the coercivity on $\mathbb{R}^n$ of the gem regular polynomial $\|F\|_2^2$. The map F thus fulfills the conditions (H1) and (H2), and Theorem 4.2 finally implies that F is a C^1-diffeomorphism of $\mathbb{R}^n$ onto itself. □

Theorem 4.12. *For $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \ldots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$, let $f = \|F\|_2^2$. For each $\alpha^\star \in D(f)$ let $V^\star \subseteq V(f)$ denote a minimal affinely independent set with $\alpha^\star \in \operatorname{conv} V^\star$ and the corresponding unique positive convex coefficients λ_α, $\alpha \in V^\star$, of $\alpha^\star$, let $w(\alpha^\star) > 0$, $\alpha^\star \in D(f)$, denote weights with $\sum_{\alpha^\star \in D(f)} w(\alpha^\star) \leq 1$, and let further*

$$f_{\alpha^\star} \;>\; -w(\alpha^\star)\,\Theta(f, V^\star, \alpha^\star) \quad \textit{if} \quad \alpha^\star \in 2\mathbb{N}_0^n$$

as well as

$$|f_{\alpha^\star}| \;<\; w(\alpha^\star)\,\Theta(f, V^\star, \alpha^\star) \quad \textit{else.}$$

Then the following two assertions are equivalent.

a) F is a C^1-diffeomorphism of $\mathbb{R}^n$ onto itself.

b) $\det JF(x) \neq 0$ holds for all $x \in \mathbb{R}^n$.

Proof. The proof runs along the same lines as the proof of Theorem 4.11, where Theorem 2.39 is replaced by Theorem 2.41. □

Example 4.13. *Consider the polynomial map $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ with $F_{t,1}(x) = x_1 + x_1^3 - t x_2^3$ and $F_{t,2}(x) = x_2 + x_1^3 + x_2^3$ for some parameter value $t \in \mathbb{R}$. We shall show that the map F_t is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself for all parameter values $t > -1$, and that F_t does not possess this diffeomorphism property for any $t < -1$.*

We define $f_t(x) := \|F_t(x)\|_2^2$ *for all* $x \in \mathbb{R}^2$*. First, let us consider the case* $t = 1$*. Observe that*

$$\det JF_1(x) = 1 + 3x_1^2 + 3x_2^2 + 18x_1^2x_2^2 > 0 \quad \text{for all } x \in \mathbb{R}^2$$

holds and hence $\det JF_1(x) \neq 0$ *for all* $x \in \mathbb{R}^2$ *is fulfilled. One further obtains*

$$f_1(x) = 2x_1^6 + 2x_2^6 + 2x_1^4 + 2x_2^4 + 2x_1^3x_2 - 2x_1x_2^3 + x_1^2 + x_2^2$$

with the corresponding gem

$$\mathcal{G}(f_1) = \operatorname{conv}((6,0),(0,6)),$$

which implies gem regularity of f_1*, since* $D(f_1) = V^c(f_1) \cap \mathcal{G}(f_1) = \emptyset$ *(see Fig. 4.1). According to Theorem 4.11 the map* F_1 *thus is a* C^1*-diffeomorphism of* $\mathbb{R}^2$ *onto itself.*

Next we shall consider only parameter values $t \neq 1$*. First, observe that the condition*

$$\det JF_t(x) = 1 + 3x_1^2 + 3x_2^2 + (9+9t)x_1^2x_2^2 \neq 0 \text{ for all } x \in \mathbb{R}^2 \tag{4.20}$$

is violated for any $t < -1$*, since the choice* $x(s) = (s,s)$ *with* $s \in \mathbb{R}$ *leads to the function* $\det JF_t(x(s)) = 1 + 6s^2 + (9+9t)s^4$ *which possesses real zeros. By Theorem 4.1,* F_t *can thus not be a* C^1*-diffeomorphism of* $\mathbb{R}^2$ *onto itself for any* $t < -1$*.*

On the other hand, (4.20) *holds for all* $t \geq -1$*, since the Jacobian determinant then is strictly positive. One further obtains*

$$f_t(x) = 2x_1^6 + (1+t^2)x_2^6 + 2(1-t)x_1^3x_2^3 + 2(x_1^4 + x_2^4) + 2(x_1^3x_2 - tx_1x_2^3) + x_1^2 + x_2^2$$

with

$$\mathcal{G}(f_t) = \operatorname{conv}((6,0),(0,6)),$$

which, for parameter values $t \neq 1$*, implies gem irregularity of* f_t*, since* $D(f_t) = V^c(f) \cap \mathcal{G}(f_t) = \{(3,3)\}$ *due to* $f_{t,(3,3)} = 2(1-t) \neq 0$ *(see Fig. 4.1). For* $\alpha^\star = (3,3) \in D(f_t)$ *the unique minimal affinely independent subset* $V^\star \subseteq V(f_t)$ *with* $\alpha^\star \in \operatorname{conv} V^\star$ *is given by the vertex set at infinity of* $\operatorname{New}_\infty(f_t)$ *itself, that is,* $V^\star := V(f_t) = \{(6,0),(0,6)\}$*. From the convex representation* $\alpha^\star = (3,3) = \frac{1}{2}(6,0) + \frac{1}{2}(0,6)$ *with the unique positive convex coefficients* $\lambda_{(6,0)} = \lambda_{(0,6)} = \frac{1}{2}$*, computing the corresponding circuit number yields*

$$\begin{aligned}\Theta(f_t, \alpha^\star, V(f_t)) = \left(\frac{f_{t,(6,0)}}{\lambda_{(6,0)}}\right)^{\lambda_{(6,0)}} \left(\frac{f_{t,(0,6)}}{\lambda_{(0,6)}}\right)^{\lambda_{(0,6)}} &= \sqrt{\frac{2}{\frac{1}{2}}}\sqrt{\frac{1+t^2}{\frac{1}{2}}} \\ &= 2\sqrt{2}\sqrt{(1+t^2)}.\end{aligned} \tag{4.21}$$

Further, choosing the weight $w((3,3)) := 1$*, the inequality*

$$|f_{t,(3,3)}| < \Theta(f_t, \alpha^\star, V(f_t))$$

holds if and only if $t \neq -1$*, because due to* $f_{t,(3,3)} = 2(1-t)$ *and* (4.21)*, the inequality*

$$|2(1-t)| < 2\sqrt{2}\sqrt{(1+t^2)}$$

holds if and only if $t \neq -1$*. According to Theorem 4.12, the latter fact together with* (4.20) *imply that the map* F_t *is a* C^1*-diffeomorphism of* $\mathbb{R}^2$ *onto itself for all parameter values* $t > -1$*, and the assertion follows.*

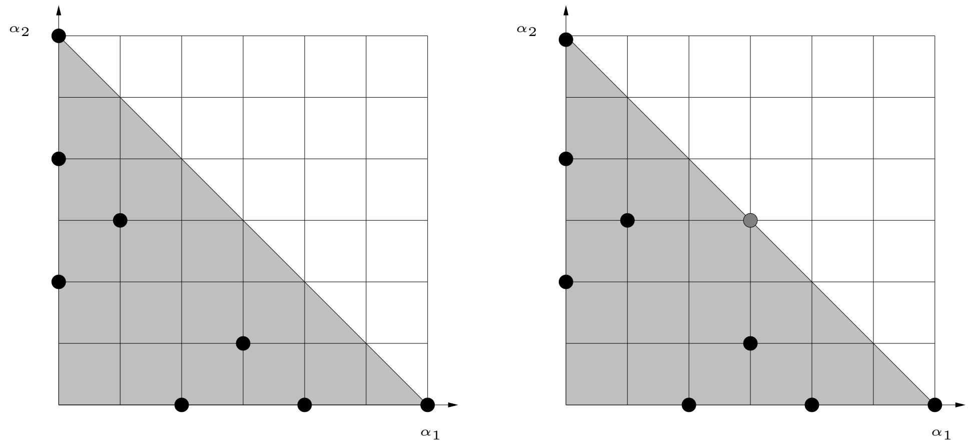

FIGURE 4.1: Illustration of Example 4.13. In the left picture, the shaded area corresponds to the Newton polytope at infinity $\mathrm{New}_\infty(f_1)$, and the black circles stand for the set $A(f_1)$. For the case $t \neq 1$, in the right picture, the shaded area corresponds to the Newton polytope at infinity $\mathrm{New}_\infty(f_t)$, the black circles stand for the set $A(f_t) \setminus D(f_t)$, and the shaded circle describes the (singleton) set $D(f_t)$.

Remark 4.14. Example 4.13 also shows that, despite the assertion of Proposition 4.9, under the assumptions of Theorems 4.11 or 4.12 the diffeomorphism property of a polynomial map F may not solely be characterized by the condition $\det JF(0) \neq 0$. In fact, in the example we have $\det JF_t(0) \neq 0$ for any $t \in \mathbb{R}$, but F_t is not a C^1-diffeomorphism for $t < -1$. □

Remark 4.15. In [5, Lem. 2.22] we showed that gem regularity of a polynomial f is a weak condition in the sense that it follows from a general position property of the multiplier vectors $\alpha \in A(f)$. Unfortunately, the polynomials $f = \|F\|_2^2$ considered in the present chapter possess a special structure, so that gem regularity of such functions is not necessarily a mild assumption.

In fact, Example 4.13 provides a parametric family of such polynomials for which gem regularity and Theorem 4.11 may only be employed at a single choice of the parameter ($t = 1$). On the other hand, Theorem 4.12 covers the gem irregular case well enough to treat all members of the parametric family except for a singular choice of the parameter ($t = -1$), which will be considered separately in Example 4.20 below. $\square$

4.3 Coercivity under linear transformations

In this section we shall show how linear transformations can help to study the global diffeomorphism property of a polynomial map when the assumptions of Theorems 4.11 and 4.12 are violated, like for the singular parameter value in Example 4.13.

Proposition 4.16. *For any regular matrix $A \in \mathbb{R}^{n\times n}$ a function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive on $\mathbb{R}^n$ if and only if the function $f \circ A^{-1}$ is coercive on $\mathbb{R}^n$.*

Proof. Let A be a regular matrix, let f be coercive on $\mathbb{R}^n$, and consider a sequence $(y^\nu) \subseteq \mathbb{R}^n$ with $\lim_{\nu\to\infty} \|y^\nu\| = +\infty$. Then we have $\|y^\nu\| \leq \|A\|\|A^{-1}y^\nu\|$ for all $\nu \in \mathbb{N}$, where $\|A\|$ denotes the matrix norm of A induced by $\|\cdot\|$. This implies $\lim_{\nu\to\infty} \|A^{-1}y^\nu\| = +\infty$ and, by the coercivity of f, $\lim_{\nu\to\infty} f(A^{-1}y^\nu) = +\infty$, so that the coercivity of $f \circ A^{-1}$ is shown. The reverse direction may be shown along the same lines, using the identity $f = (f \circ A^{-1}) \circ A$. $\square$

We may also improve the formulation of Proposition 4.10 as follows:

Proposition 4.17. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \dots, F_n)$, $F_i \in \mathbb{R}[x]$, $i \in I$, be given such that*

$$\det JF(0) \neq 0$$

holds. Then for any regular matrix $A \in \mathbb{R}^{n\times n}$ the polynomial $\|F \circ A^{-1}\|_2^2$ fulfills the conditions (C1)-(C3)*.*

Proof. Since $F \circ A^{-1}$ is a polynomial map, the assertion follows from Proposition 4.10 and $\det J(F \circ A^{-1})(0) = \det JF(0) \cdot \det A^{-1} \neq 0$. $\square$

Corollary 4.18. *The assertion of Theorem 4.11 remains true, if the assumption of gem regularity of the polynomial $\|F\|_2^2$ is replaced by the assumption of gem regularity of the polynomial $\|F \circ A^{-1}\|_2^2$ for some regular matrix $A \in \mathbb{R}^{n\times n}$.*

Proof. We only have to modify the proof that assertion b) implies assertion a). In fact, for the given matrix $A \in \mathbb{R}^{n\times n}$ assertion b) and Proposition 4.17 imply that $\|F \circ A^{-1}\|_2^2$ fulfills the conditions (C1)-(C3) which, by Theorem 2.39, characterize the coercivity on $\mathbb{R}^n$ of the gem regular polynomial $\|F \circ A^{-1}\|_2^2$. Consequently, by Proposition 4.16 also the polynomial $\|F\|_2^2$ is coercive, so that the map F thus fulfills the conditions (H1) and (H2), and Theorem 4.2 implies the assertion. $\square$

The following result is shown analogously.

Corollary 4.19. *The assertion of Theorem 4.12 remains true, if the assumptions on the polynomial $\|F\|_2^2$ are replaced by the same assumptions on the polynomial $\|F \circ A^{-1}\|_2^2$ for some regular matrix $A \in \mathbb{R}^{n\times n}$.*

We illustrate Corollary 4.19 by sharpening the result given in Example 4.13.

Example 4.20. *Consider the polynomial map $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 4.13 with $F_{t,1}(x) = x_1 + x_1^3 - tx_2^3$ and $F_{t,2}(x) = x_2 + x_1^3 + x_2^3$ for some parameter value $t \in \mathbb{R}$. Then F_t is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself if and only if $t \geq -1$.*

In fact, by Example 4.13, it suffices to show that F_t is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself for the singular parameter value $t = -1$. In fact, using the linear coordinate transformation $x = A^{-1}y$ with the matrix

$$A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

one obtains the gem-irregular polynomial

$$f_{-1}(A^{-1}y) \quad := \quad \|F_{-1}(A^{-1}y)\|_2^2 =$$

$$= 2y_1^2 + 2y_2^2 + 8y_1^4 + 24y_1^2y_2^2 + 8y_1^6 + 48y_1^4y_2^2 + 72y_1^2y_2^4$$

with $D(f_{-1} \circ A^{-1}) = \{(4,2)\}$ and $V(f_{-1} \circ A^{-1}) = \{(6,0),(2,4),(0,2)\}$. From the positivity of the circuit number $\Theta(f_{-1} \circ A^{-1},(4,2),V(f_{-1} \circ A^{-1}))$ corresponding to the unique gem-degenerate exponent vector $\alpha^\star = (4,2)$ of $f_{-1} \circ A^{-1}$, one obtains the inequality

$$48 = \left(f_{-1} \circ A^{-1}\right)_{(4,2)} > -\Theta(f_{-1} \circ A^{-1},(4,2),V(f_{-1} \circ A^{-1})).$$

Since $\det JF_{-1}(x_1,x_2) = 1 + 3x_1^2 + 3x_2^2 > 0$ *holds for all $x \in \mathbb{R}^2$, Corollary 4.19 yields the assertion.*

Remark 4.21. In Pinchuk's counterexample to the Real Jacobian Conjecture (cf. [68]), the Jacobian determinant of F vanishes nowhere on $\mathbb{R}^2$ so that, by Proposition 4.17, the sum of squares polynomial $\|F \circ A^{-1}\|_2^2$ does satisfy the conditions (C1)-(C3) for any regular matrix $A \in \mathbb{R}^{2\times 2}$. Since, however, F is not a global C^1-diffeomorphism, $\|F \circ A^{-1}\|_2^2$ can neither be gem regular nor satisfy the additional sufficient conditions from Theorem 4.12 for any regular matrix $A \in \mathbb{R}^{2\times 2}$. □

Chapter 5

Conclusions and open problems

In this dissertation thesis, we analyzed growth properties on $\mathbb{R}^n$ of multivariate polynomials $f \in \mathbb{R}[x]$ in terms of their so-called Newton polytopes at infinity. In fact, in Chapter 2 we introduced the broad class of so-called gem regular polynomials and characterized their coercivity via conditions solely containing information about the geometry of the vertex set of the Newton polytope at infinity, as well as sign conditions on the corresponding polynomial coefficients. For all other polynomials, the so-called gem irregular polynomials, we introduced sufficient conditions for coercivity based on those from the regular case. For some special cases of gem irregular polynomials, we established necessary conditions for coercivity, too. In Chapter 3 we further introduced a stability concept for the coercivity of multivariate polynomials $f \in \mathbb{R}[x]$. In particular, we considered perturbations of f by polynomials up to the so-called degree of stable coercivity, and we analyzed this stability concept in terms of the corresponding Newton polytopes at infinity. For coercive polynomials $f \in \mathbb{R}[x]$ we also introduced the order of coercivity as a measure expressing the order of growth of f, and we identified a broad class of multivariate polynomials $f \in \mathbb{R}[x]$ for which the order of coercivity and the degree of stable coercivity coincide. For these polynomials we gave a geometric interpretation of this phenomenon in terms of their Newton polytopes at infinity, which we call the degree of convenience. Finally, in Chapter 4 we analyzed the global diffeomorphism property of real polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ by studying the properties of the Newton polytopes at infinity corresponding to the sum of squares polynomials $\|F\|_2^2$. This allowed us to identify a class of polynomial maps F for which their global diffeomorphism property on $\mathbb{R}^n$ is equivalent to their Jacobian determinant $\det JF$ vanishing nowhere on $\mathbb{R}^n$. In other words, we identified a class of polynomial maps for which the Real Jacobian Conjecture, which was proven to be false in general, still holds.

Concerning the results presented in Chapter 2, in the univariate case, that is, for $n = 1$ our results collapse to trivial statements. In fact, then we have $\mathrm{New}_\infty(f) = [0, \deg(f)]$ for any polynomial f so that, in particular, each polynomial f is gem regular. The characterization of coercivity from Theorem 2.39 by conditions (C1)–(C3) then simply states that the leading term of f has even degree and a positive coefficient.

For $n > 1$ a natural and more interesting question arising throughout this thesis is whether gem regularity, the conditions (C1)–(C3), and the remaining conditions introduced in Theorems 2.29, 2.41, and 2.44 can be verified algorithmically. To this end, in particular one needs to compute all vertices and faces of the polytope $\mathrm{New}_\infty(f)$. This could be done, for example, by using vertex and facet enumeration algorithms (cf., e.g., [3, 13]), but is beyond the scope of the present thesis.

In some applications, even stronger notions of coercivity are needed, like locally uniform coercivity of a parametric function $f : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ which is satisfied at $\bar{t} \in \mathbb{R}^r$ when $f(t, x) \to +\infty$ holds for $t \to \bar{t}$ and $\|x\| \to +\infty$. The application of our Newton polytope-type techniques to the latter concept in the case of multivariate polynomial functions $f \in \mathbb{R}[x]$ is subject of future research.

Our results from Chapter 3 show that, for a broad class of coercive polynomials, the degree of stable coercivity and the order of coercivity coincide with the degree of convenience. It is thus an interesting question whether this property holds true for all coercive polynomials. Answering this question positively would not only reveal an intimate connection between the stability and the order of growth concepts of coercive polynomials on the one hand, and a geometric property of the corresponding Newton polytopes at infinity expressed by the degree of convenience on the other hand, but in particular it would also show that the degree of stable coercivity as well as the order of coercivity always are even numbers.

Our results from Chapter 4 show that the global diffeomorphism property of a real polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ can sometimes be studied by analyzing the coercivity property of the sum of squares polynomial $\|F\|_2^2$ via its Newton polytope at infinity $\mathrm{New}_\infty(\|F\|_2^2)$. However, due to the special structure of the polynomial $\|F\|_2^2$, the assumptions of known sufficiency theorems for coercivity are not necessarily mild and may be expected to be violated. On the other hand, while preserving the coercivity property, suitable linear coordinate transformations may help to transform such a degenerated polynomial into another one, for which the known techniques for verifying coercivity can be applied.

In order to better understand the coercivity property of multivariate polynomials over $\mathbb{R}^n$, it is thus an interesting question whether for each coercive polynomial f there exists some linear coordinate transformation such that, in new coordinates, f fulfills the conditions from Theorem 2.39 or from Theorem 2.41, and how such a linear coordinate transformation may be constructed. We leave these questions for future research.

Appendix A

Appendix

A.1 A nonhomogeneous Motzkin transposition theorem

In the proof of Proposition 2.24 we use the following nonhomogeneous version of Motzkin's transposition theorem.

Lemma A.1. *For matrices and vectors of appropriate dimensions, the system*

$$Ax = a, \quad Bx \geq 0, \quad Cx > 0 \tag{A.1}$$

is inconsistent if and only if at least one of the systems

$$A^\mathsf{T}\rho + B^\mathsf{T}\sigma + C^\mathsf{T}\tau = 0, \quad \langle a, \rho \rangle > 0, \quad \sigma, \tau \geq 0 \tag{A.2}$$

and

$$A^\mathsf{T}\rho + B^\mathsf{T}\sigma + C^\mathsf{T}\tau = 0, \quad \langle a, \rho \rangle = 0, \quad \sigma, \tau \geq 0, \quad \tau \neq 0 \tag{A.3}$$

is consistent.

Proof. The system (A.1) is inconsistent if and only if the homogeneous system

$$(A, -a)\begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad (B, 0)\begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \quad \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} > 0 \tag{A.4}$$

is inconsistent, as for any solution x of (A.1) the vector $(x, 1)$ solves (A.4), and for any solution (x, y) of (A.4) we have $y > 0$, and x/y solves (A.1). By Motzkin's (homogeneous) transposition theorem, the system (A.4) is inconsistent if and only if the system

$$\begin{pmatrix} A^\mathsf{T} \\ -a^\mathsf{T} \end{pmatrix} \rho + \begin{pmatrix} B^\mathsf{T} \\ 0^\mathsf{T} \end{pmatrix} \sigma + \begin{pmatrix} C^\mathsf{T} & 0 \\ 0^\mathsf{T} & 1 \end{pmatrix} \begin{pmatrix} \tau \\ \mu \end{pmatrix} = 0, \quad \sigma, \tau, \mu \geq 0, \quad (\tau, \mu) \neq 0$$

is consistent. Rewriting this fact for the two cases $\mu > 0$ and $\mu = 0$ yields the assertion.

□

A.2 Alternative proof of Lemma 3.47

For the contrary, assume that f is not strictly convex on $\mathbb{R}^n$. Then, there exist some $x, y \in \mathbb{R}^n$ with $x \neq y$ and some $\bar{\lambda} \in (0,1)$ with

$$f((1-\bar{\lambda})x + \bar{\lambda}y) \geq (1-\bar{\lambda})f(x) + \bar{\lambda}f(y)$$

and, since f is convex on $\mathbb{R}^n$, the latter property yields

$$f((1-\bar{\lambda})x + \bar{\lambda}y) = (1-\bar{\lambda})f(x) + \bar{\lambda}f(y). \tag{A.5}$$

Defining the univariate polynomial $F \in \mathbb{R}[\lambda]$ by $F(\lambda) := f(x + \lambda(y - x))$ for all $\lambda \in \mathbb{R}$, the property (A.5) reduces to $F(\bar{\lambda}) = f(x) + \bar{\lambda}(f(y - f(x))$. Clearly, F inherits the convexity property on $\mathbb{R}$ from that of f on $\mathbb{R}^n$. In the next step, we shall show that F is linear over the interval $(0,1)$, which, due to the polynomial property of F, implies that F is in fact linear over the whole space $\mathbb{R}$.

Let us thus for the contrary assume that F is not linear on the interval $(0,1)$. Then, there exists some $\lambda^\star \in (0,1)$ with $F(\lambda^\star) \neq f(x) + \lambda^\star(f(y) - f(x))$. By the definition of F and its convexity, the latter property implies $F(\lambda^\star) < f(x) + \lambda^\star(f(y) - f(x))$, and thus, in view of (A.5), also $\bar{\lambda} \neq \lambda^\star$ holds. In the following we shall consider two cases.

If $\lambda^\star > \bar{\lambda}$, then with the convex coefficients $\lambda_1 := 1 - \bar{\lambda}/\lambda^\star$ and $\lambda_2 := \bar{\lambda}/\lambda^\star$ corresponding to the representation of the point $\bar{\lambda}$ as a convex combination $\bar{\lambda} = \lambda_1 \cdot 0 + \lambda_2 \cdot \lambda^\star$ of points 0 and $\lambda^\star$, one obtains

$$\begin{aligned} \lambda_1 F(0) + \lambda_2 F(\lambda^\star) &< \lambda_1 f(x) + \lambda_2(f(x) + \lambda^\star(f(y) - f(x))) \\ &= f(x) + \bar{\lambda}(f(y) - f(x)) = F(\bar{\lambda}), \end{aligned}$$

a contradiction to the convexity of F on $\mathbb{R}$.

If $\lambda^\star < \bar{\lambda}$, then with the convex coefficients $\lambda_1 := (1-\bar{\lambda})/(1-\lambda^\star)$ and $\lambda_2 := (\bar{\lambda} - \lambda^\star)/(1-\lambda^\star)$ corresponding to the representation of the point $\bar{\lambda}$ as a convex combination

$\bar{\lambda} = \lambda_1 \cdot \lambda^\star + \lambda_2 \cdot 1$ of points $\lambda^\star$ and 1, one obtains

$$\lambda_1 F(\lambda^\star) + \lambda_2 F(1) < \lambda_1(f(x) + \lambda^\star(f(y) - f(x))) + \lambda_2 f(y)$$

$$= f(x) + \bar{\lambda}(f(y) - f(x)) = F(\bar{\lambda}),$$

a contradiction to the convexity of F on $\mathbb{R}$.

Hence F is linear on $\mathbb{R}$ and one obtains a contradiction to the coercivity of F on $\mathbb{R}$, which the univariate polynomial F clearly inherits by its definition from the coercivity of f on $\mathbb{R}^n$. The assertion follows. □

A.3 Proof of Lemma 4.3

For any $\bar{v} \in \mathrm{vert}(P+P)$ there exists a vector $a \in \mathbb{R}^n \setminus \{0\}$ such that $\bar{v}$ is the unique optimal point of the problem

$$(\mathrm{LP1}) \qquad \max_{v \in P+P} a^T v.$$

Let $\bar{w} \in \mathrm{vert}(P)$ be an optimal point of the problem

$$(\mathrm{LP2}) \qquad \max_{w \in P} a^T w.$$

Since

$$\begin{aligned} \max_{v \in P+P} a^T v = \max_{(x,y) \in P \times P} a^T(x+y) &= \max_{x \in P} a^T x + \max_{y \in P} a^T y \\ &= 2 \max_{x \in P} a^T x = 2a^T \bar{w} = a^T 2\bar{w} \end{aligned} \tag{A.6}$$

holds, the point $2\bar{w} \in 2\,\mathrm{vert}(P)$ is an optimal point of (LP1). A a was chosen such that $\bar{v}$ is the unique optimal point of (LP1), one obtains $\bar{v} = 2\bar{w}$ with $\bar{w} \in \mathrm{vert}(P)$.

On the other hand, choose $\bar{w} \in \mathrm{vert}(P)$ and put $\bar{v} = 2\bar{w}$. To show is $\bar{v} \in \mathrm{vert}(P+P)$. Observe that there exists some $a \in \mathbb{R}^n \setminus \{0\}$ such that $\bar{w}$ is the unique optimal point of the problem (LP2). Using (A.6), the point $\bar{v}$ is thus an optimal point of (LP1). Assume that $\bar{v} \notin \mathrm{vert}(P+P)$ holds. Since (LP1) must possess a vertex solution, there exists an optimal point $\bar{z} := \bar{x} + \bar{y} \in P+P$ of (LP1) with $\bar{v} \neq \bar{z}$. For the point $\bar{u} := \frac{1}{2}(\bar{x} + \bar{y}) \in P$ we obtain the identity

$$a^T \bar{u} = \frac{1}{2} a^T (\bar{x} + \bar{y}) = \frac{1}{2} a^T \bar{z} = a^T \bar{w},$$

where the last equation holds since both $\bar{z}$ and $\bar{v} = 2\bar{w}$ are optimal for (LP1). The point $\bar{u} \in P$ is thus an optimal point of the problem (LP2), and the uniqueness of $\bar{w}$ implies

$\bar{w} = \bar{u}$. This leads to the contradiction $\bar{v} = \bar{z}$, and thus the assertion $\bar{v} \in \mathrm{vert}(P + P)$ follows. $\square$

A.4 Alternative proof of Lemma 4.8

The proof is done by induction using the Laplace expansion rule for matrix determinants.

Case $n = 1$

For an arbitrary polynomial $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \sum_{\alpha \in A(F)} F_\alpha x^\alpha$ one obtains due to $JF(x) = F'(x)$ the following equality

$$\det\, JF(x) = \det\, F'(x) = F'(x) = \sum_{\substack{\alpha \in A(F) \\ \alpha \geq 1}} \alpha F_\alpha x^{\alpha - 1},$$

which proves the correctness of the formula (4.13).

Induction step:

Assume that the determinant formula (4.13) is true for dimension $n-1$. Expanding the determinant of the Jacobian matrix $JF(x)$ of an arbitrary polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ along its n-th column by using the Laplace expansion rule yields

$$\det\, JF(x) = \sum_{j=1}^{n} (-1)^{j+n} \cdot (JF(x))_{(j,n)} \cdot \det\, J_{(j,n)}(x), \tag{A.7}$$

where

$$(JF(x))_{(j,n)} = \frac{\partial}{\partial x_n} F_j(x) \tag{A.8}$$

denotes the entry of the Jacobian matrix $JF(x) \in \mathbb{R}^{n \times n}$ corresponding to its j-th row and n-th column and the matrix $J_{(j,n)}(x) \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the matrix obtained from the Jacobian matrix $JF(x) \in \mathbb{R}^{n \times n}$ by omitting its j-th row and n-th column.

For the later purposes we first define projection maps $\pi_{i,n} : A(F_i) \to \mathbb{N}_0^{n-1}$ with $\pi_{i,n}(\alpha_1, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_{n-1})$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in A(F_i)$, $i \in I$ and a projection map $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ via $\pi_n(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ for all $x \in \mathbb{R}^n$. Denote further $I := \{1, \dots, n\}$ the index set of all coordinates in $\mathbb{R}^n$ and $I^j := I \setminus \{j\}$ for $j \in I$. For an arbitrary $i \in I$ and $\beta \in \mathbb{N}_0^{n-1}$ the set $\pi_{i,n}^{-1}(\beta) \subseteq A(F_i)$ denotes the inverse image of β under the projection map $\pi_{i,n}$.

For further working with the Laplace expansion formula (A.7) it is convenient to express all the information about the original Jacobian matrix $JF(x)$ containing n variables $x_1, \ldots, x_n$ only in terms of the first $n-1$ variables $x_1, \ldots, x_{n-1}$ while considering the variable x_n to be a parameter.

First, it holds

$$\begin{aligned} F_j(x) = \sum_{\alpha^j \in A(F_j)} (F_j)_{\alpha^j} x^{\alpha^j} &= \sum_{\alpha^j \in A(F_j)} (F_j)_{\alpha^j} x_n^{\alpha_n^j} \pi_n(x)^{\pi_{j,n}(\alpha^j)} \\ &= \sum_{\beta^j \in \pi_{j,n}(A(F_j))} \left(\sum_{\alpha \in \pi_{j,n}^{-1}(\beta^j)} (F_j)_\alpha x_n^{\alpha_n} \right) \pi_n(x)^{\beta^j} \end{aligned} \tag{A.9}$$

and due to (A.8) one obtains by differentiating (A.9) with respect to x_n

$$(JF(x))_{(j,n)} = \sum_{\beta^j \in \pi_{j,n}(A(F_j))} \left(\sum_{\substack{\alpha \in \pi_{j,n}^{-1}(\beta^j) \\ \alpha_n \geq 1}} (F_j)_\alpha \alpha_n x_n^{\alpha_n - 1} \right) \pi_n(x)^{\beta^j}. \tag{A.10}$$

Denoting

$$Q(\beta^j) := \left(\sum_{\substack{\alpha \in \pi_{j,n}^{-1}(\beta^j) \\ \alpha_n \geq 1}} (F_j)_\alpha \alpha_n x_n^{\alpha_n - 1} \right) \tag{A.11}$$

one can rewrite (A.10) as

$$(JF(x))_{(j,n)} = \sum_{\beta^j \in \pi_{j,n}(A(F_j))} Q(\beta^j) \cdot \pi_n(x)^{\beta^j}. \tag{A.12}$$

Since by the induction hypothesis the determinant formula (4.13) is true for the dimension $n-1$, its direct application to the matrix $J_{(j,n)}(x) \in \mathbb{R}^{n-1 \times n-1}$ results in

$$\begin{aligned} &\det J_{(j,n)}(x) = \\ &\sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i \in I^j \\ \sum_{i \in I^j} \beta^i \geq \mathbb{1}_{n-1}^T}} W(\beta^i,\ i \in I^j) \cdot \pi_n(x)^{\sum_{i \in I^j} \beta^i - \mathbb{1}_{n-1}^T} \end{aligned} \tag{A.13}$$

where

$$W(\beta^i,\ i \in I^j) := \det \left[\beta^i,\, i \in I^j\right] \cdot \prod_{i \in I^j} \left(\sum_{\alpha \in \pi_{i,n}^{-1}(\beta^i)} (F_i)_\alpha x_n^{\alpha_n} \right) \tag{A.14}$$

and

$$\left[\beta^i,\, i \in I^j\right] \in \mathbb{R}^{(n-1)\times(n-1)}$$

denotes the $(n-1, n-1)$-matrix consisting of projected exponent vectors $\beta^i \in \pi_{i,n}(A(F_i))$, $i \in I^j$ as row vectors.

Inserting (A.12) together with (A.13) into (A.7) and using

$$\pi_n(x)^{\beta^j} \cdot \pi_n(x)^{\sum_{i\in I^j} \beta^i - \mathbb{1}_{n-1}^T} = \pi_n(x)^{\sum_{i\in I} \beta^i - \mathbb{1}_{n-1}^T},$$

one directly obtains

$$\det\, JF(x) = \sum_{j\in I} \sum_{\beta^j \in \pi_{j,n}(A(F_j))} \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I^j \\ \sum_{i\in I^j} \beta^i \geq \mathbb{1}_{n-1}^T}} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)\, \pi_n(x)^{\sum_{i\in I} \beta^i - \mathbb{1}_{n-1}^T}. \tag{A.15}$$

The second and the third sum in (A.15) can be merged together resulting in

$$\det\, JF(x) = \sum_{j\in I} \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I^j} \beta^i \geq \mathbb{1}_{n-1}^T}} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)\, \pi_n(x)^{\sum_{i\in I} \beta^i - \mathbb{1}_{n-1}^T}. \tag{A.16}$$

The inner sum from (A.16) can be written as a difference of the following two sums:

$$\sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I^j} \beta^i \geq \mathbb{1}_{n-1}^T}} = \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I} \beta^i \geq \mathbb{1}_{n}^T}} - \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I^j} \beta^i \ngeq \mathbb{1}_{n-1}^T \\ \sum_{i\in I} \beta^i \geq \mathbb{1}_{n-1}^T}} \tag{A.17}$$

where for the second sum on the right side in (A.17) it holds

$$\sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I^j} \beta^i \ngeq \mathbb{1}_{n-1}^T \\ \sum_{i\in I} \beta^i \geq \mathbb{1}_{n-1}^T}} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)\, \pi_n(x)^{\sum_{i\in I} \beta^i - \mathbb{1}_{n-1}^T} = 0 \tag{A.18}$$

due to the fact that for an arbitrary choice of n vectors $\beta^i \in \pi_{i,n}(A(F_i)) \subseteq \mathbb{N}_0^{n-1}$, $i \in I$ with $\sum_{i\in I^j} \beta^i \ngeq \mathbb{1}_{n-1}^T$ it always holds

$$\det\ \left[\beta^i,\, i \in I^j\right] = 0,$$

and consequently,

$$W(\beta^i,\ i \in I^j) = 0.$$

as well.

The properties (A.17) and (A.18) imply that the inner sum in (A.16) is, in fact, independent on the index j

$$\det\ JF(x) = \sum_{j\in I} \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I}\beta^i \geq \mathbb{1}_{n-1}^T}} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)\, \pi_n(x)^{\sum_{i\in I}\beta^i - \mathbb{1}_{n-1}^T}.$$

Changing the order of summation in the latter identity results in

$$\begin{aligned}
\det\ JF(x) &= \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I}\beta^i \geq \mathbb{1}_{n-1}^T}} \sum_{j\in I} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)\, \pi_n(x)^{\sum_{i\in I}\beta^i - \mathbb{1}_{n-1}^T} \\
&= \sum_{\substack{\beta^i \in \pi_{i,n}(A(F_i)),\, i\in I \\ \sum_{i\in I}\beta^i \geq \mathbb{1}_{n-1}^T}} \pi_n(x)^{\sum_{i\in I}\beta^i - \mathbb{1}_{n-1}^T} \sum_{j\in I} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j) \qquad \text{(A.19)}
\end{aligned}$$

With definitions (A.11) and (A.14) one obtains

$$Q(\beta^j)\, W(\beta^i,\ i \in I^j) =$$

$$\det\ \left[\beta^i,\, i \in I^j\right] \left(\sum_{\substack{\alpha \in \pi_{j,n}^{-1}(\beta^j) \\ \alpha_n \geq 1}} (F_j)_\alpha \alpha_n x_n^{\alpha_n - 1} \right) \cdot \prod_{i \in I^j} \left(\sum_{\alpha \in \pi_{i,n}^{-1}(\beta^i)} (F_i)_\alpha x_n^{\alpha_n} \right) =$$

$$\det\ \left[\beta^i,\, i \in I^j\right] \sum_{\substack{\alpha^i \in \pi_{i,n}^{-1}(\beta^i),\, i\in I \\ \alpha_n^j \geq 1}} \prod_{i\in I} (F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i - 1} \alpha_n^j. \qquad \text{(A.20)}$$

Inserting the relation (A.20) in (A.19) results in

$$\sum_{j\in I} (-1)^{j+n}\, Q(\beta^j)\, W(\beta^i,\ i \in I^j)$$

$$= \sum_{j\in I} \left((-1)^{j+n} \det\ \left[\beta^i,\, i \in I^j\right] \sum_{\substack{\alpha^i \in \pi_{i,n}^{-1}(\beta^i),\, i\in I \\ \alpha_n^j \geq 1}} \prod_{i\in I} (F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i - 1} \alpha_n^j \right)$$

$$= \sum_{j\in I}\left(\sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}} (-1)^{j+n}\det\ \left[\beta^i,\, i\in I^j\right]\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\alpha_n^j\right)$$

$$= \sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\sum_{j\in I}\left((-1)^{j+n}\det\ \left[\beta^i,\, i\in I^j\right]\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\alpha_n^j\right)$$

$$= \sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\sum_{j\in I}\left((-1)^{j+n}\det\ \left[\beta^i,\, i\in I^j\right]\alpha_n^j\right)$$

$$= \sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\det\ \left[\alpha^i,\, i\in I\right] \tag{A.21}$$

Finally, inserting (A.21) into (A.19) one obtains

$$\det\ JF(x) =$$

$$\sum_{\substack{\beta^i\in\pi_{i,n}(A(F_i)),\, i\in I\\ \sum_{i\in I}\beta^i\geq 1\!\!1_{n-1}^T}}\pi_n(x)^{\sum_{i\in I}\beta^i-1\!\!1_{n-1}^T}\sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\det\ \left[\alpha^i,\, i\in I\right]$$

$$= \sum_{\substack{\beta^i\in\pi_{i,n}(A(F_i)),\, i\in I\\ \sum_{i\in I}\beta^i\geq 1\!\!1_{n-1}^T}}\sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\pi_n(x)^{\sum_{i\in I}\beta^i-1\!\!1_{n-1}^T}\det\ \left[\alpha^i,\, i\in I\right]$$

$$= \sum_{\substack{\beta^i\in\pi_{i,n}(A(F_i)),\, i\in I\\ \sum_{i\in I}\beta^i\geq 1\!\!1_{n-1}^T}}\sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\ x_n^{\sum_{i\in I}\alpha_n^i-1}\pi_n(x)^{\sum_{i\in I}\beta^i-1\!\!1_{n-1}^T}\det\ \left[\alpha^i,\, i\in I\right]$$

$$= \sum_{\substack{\beta^i\in\pi_{i,n}(A(F_i)),\, i\in I\\ \sum_{i\in I}\beta^i\geq 1\!\!1_{n-1}^T}}\sum_{\substack{\alpha^i\in\pi_{i,n}^{-1}(\beta^i),\, i\in I\\ \alpha_n^j\geq 1}}\prod_{i\in I}(F_i)_{\alpha^i}\, x^{\sum_{i\in I}\alpha^i-1\!\!1_n^T}\det\ \left[\alpha^i,\, i\in I\right]$$

$$= \sum_{\substack{\alpha^i\in A(F_i),\, i\in I\\ \sum_{i\in I}\alpha^i\geq 1\!\!1_n^T}}\prod_{i\in I}(F_i)_{\alpha^i}\, x^{\sum_{i\in I}\alpha^i-1\!\!1_n^T}\det\ \left[\alpha^i,\, i\in I\right]. \tag{A.22}$$

□

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