Risk-Sensitive Stopping Problems for Continuous-Time Markov Chains

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Referentin: Prof. Dr. Nicole Bäuerle
Korreferent: Prof. Dr. Ulrich Rieder

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1 Introduction

1.1 Motivation and Historical Overview

The theory of optimal stopping deals with the problem of finding an optimal time to take a given action based on observable stochastic processes in order to maximize an expected reward. Problems of this kind can arise for a variety of different situations. Typical examples can be found in operations research, where an owner of a company has to decide for the right moment to replace some machinery before it breaks down, or where a manager has to decide between sequentially arriving applicants for a job position. The latter is well-known as the "secretary problem". Other famous stopping problems can be found by names like "bandit problem", "job search problem" or "house selling problem". One can imagine, there are a lot of situations where a decision maker has to choose one of sequentially arriving offers, or in a continuous-time context, where he has to find the optimal time to execute an action.

Optimal stopping problems first arose in the context of statistics in Wald [1945], where the sequential analysis of statistical observations was studied. A few years later, Snell [1952] generalized the sequential analysis to pure mathematical stopping problems without the statistical context. It followed an increased interest in this new subject, as more and more research developed around the topic of optimal stopping. Especially the work of Chow & Robbins [1961, 1963, 1965] contributed greatly to the theory of optimal stopping and designed techniques to solve such problems. The theory developed so far was summarized in Chow et al. [1971] and further refined in various publications, like for example Gugerli [1986] or Davis [1993].

However, all these problems were formulated to maximize some expected reward gained by optimal stopping. They do not account for the individual preference relation or attitude towards risk of a given investor or decision maker. Such effects can be considered by studying utility functions, which were first introduced by von Neumann & Morgenstern [1944]. He showed that such effects, as the behavior of investors under risk, can be modelled by utility functions. Later, Pratt [1964] and Arrow [1965] refined this connection and introduced the Arrow-Pratt measure of absolute risk aversion, which allowed to quantify and compare the risk aversion of two investors using different utility functions. Among all possible utility functions, a particular one came in the focus of interest. The exponential utility was especially highlighted and investigated. As discussed in Pratt [1964] and Fishburn [1970], this utility function provides a lot of attractive properties that simplify many expected utility optimization problems. Due to this prominence, optimization probems for expected utility functions are very often referred to as *risk-sensitive*. Such optimization problems can be seen in various publications, like for example Howard & Matheson [1972] and Chung & Sobel [1987] for risk-sensitive control of Markov decision processes, Ghosh & Saha [2014] for risk-sensitive control of continuous-time Markov chains or Denardo & Rothblum [1979] and Hall et al. [1979] for risk-sensitive optimal stopping.

Literature on expected utility optimization or on optimal stopping problems for other utility functions is more scarce. And literature dealing with general utility without confining itself on a particular utility function is even harder to find. Some examples are Rieder [1991] for stochastic games, Kadota et al. [2001] for optimal stopping of discrete-time decision processes, Bäuerle & Rieder [2011] and Bäuerle & Rieder [2014] for optimal control of discrete-time Markov decision processes. Most of the publications concerning general utility functions deal with discrete-time settings. Especially for the context of optimal stopping under general utility functions, we can name Müller [2000], where the expected utility maximization of optimal stopping problems is studied for a discrete-time sequence of independent and identically distributed random variables, or Bäuerle & Rieder [2015] for partially observable risk-sensitive stopping problems in discrete time. Kadota et al. [1996] considered optimal stopping for denumerable Markov chains under general utility, but also in a discrete-time setting.

In this thesis, we want to consider optimal stopping problems for continuous-time Markov chains under general utility. In view of the terminology *risk-sensitive* for exponential utility, we will call our problem the *generalized risk-sensitive stopping problem for continuous-time Markov chains*. To the best of our knowledge, such optimal stopping problems for general utility and continuous-time stochastic processes were not considered in literature until now.

Continuous-time Markov chains are a special class of Markov processes, which are constant between random jump times. Only at these jump times the value of the chain will change into a new random state. The choice of a continuous-time Markov chain as underlying process will allow for a piecewise approach to the problem and for convenient interpretability of our results. Moreover, as discussed in Kushner & Dupuis [1992], a great class of diffusion processes can be approximated arbitrarily closely using continuous-time Markov chains.

As already mentioned, utility functions can be used to represent the preference relation and attitude towards risk of a decision maker or an investor who employs it. They can also be used to quantify the level of risk aversion and thus describe the behavior of an investor under risk. By permitting general utility functions and not confining ourselves to a specific one, like for expample the exponential utility, we allow for a variety of different risk preferences and can thus model the behavior under risk of a lot of various investors. But since the exponential utility is that prominent in a lot of applications, we will accompany every chapter and main result of this thesis by comparing the general case with the exponential one. We will see that the exponential utility leads indeed to a considerable simplification in contrast to the general case and yields results which are similar to stopping problems for discrete-time Markov chains.

1.2 Outline of the Thesis

This thesis is divided into nine chapters. In chapter 2, chapter 3 and chapter 4 we give a brief introduction to continuous-time Markov chains, filtrations, stopping times for Markov chains and utility functions. These chapters lay the foundation for the generalized risk-sensitive stopping problem for continuous-time Markov chains, which will be introduced in chapter 5. In chapter 6 we will express this stopping problem in terms of value functions. Chapter 7 forms the main part of this thesis and contains a discrete-time approach to the stopping problem. In chapter 8 conditions will be given which guarantee the optimality of special stopping times. Lastly, chapter 9 gives an alternative continuous-time approach to the stopping problem.

In the following, we outline the thesis and give a brief overview of the results in each chapter. A more detailed description will be given at the beginning of each chapter.

Chapter 2:
In section 2.1 continuous-time Markov chains will be introduced. Moreover, in section 2.2 known properties and results are summarized. Section 2.3 will discuss that continuous-time Markov chains can be fully described knowing two of its characteristic quantities: The jump times and the embedded discrete-time Markov chain.

Chapter 3:
In section 3.1 we will discuss suitable filtrations for a continuous-time Markov chain X and describe them using jump times and post-jump states of X. Lastly, we will give a useful decomposition characterization for stopping times τ with respect to these filtrations. It will allow for a piecewise representation of stopping times between each jump of X, which will be of great importance for subsequent chapters.

Chapter 4:
We aim to establish continuous-time stopping problems, which also account for individual preferences or attitudes towards risk of a given investor. Such preference relations can be modelled by using utility functions. In particular, we will distinguish between two classes of utility functions: The ones which are and those which are not classically defined on the whole real line. The latter have to be extended manually on the whole real line to become suitable for the stopping problem introduced in the subsequent chapter.

Chapter 5:
In section 5.1 the central optimization problem for this thesis, the generalized risk-sensitive stopping problem for continuous-time Markov chains

$$\mathbb{E}\left[U\left(-c\tau + g(X_\tau)\right)\right] \to \max_{\tau\in\Sigma}!$$

is introduced. It can be interpreted as follows: Suppose an investor can receive a reward $g(X_t)$, based on a reward function g and on the actual state X_t of a continuous-time Markov chain X at time $t \geq 0$. He can observe the evolution of the Markov chain and wait for it to attain more favorable states, which yield a higher reward. But by waiting up to time $t \geq 0$, he will be charged with a fee of $-ct$, based on some cost rate $c > 0$. Given a utility function U, his goal is thus to maximize the expected utility of the reward he can achieve, over the set Σ of all feasible stopping times τ. The prefix "generalized" stems from the fact that the considered utility function does not have to be a particular one but can be chosen arbitrarily.

Section 5.2 will cover the introduction of similar stopping problems, the generalized risk-sensitive n–step stopping problems. These can be seen as a modification of the above-mentioned problem by assuming that for $n \in \mathbb{N}_0$, an n–step stopping problem will stop at the latest after n jumps of the underlying Markov chain. These n–step stopping problems will be the key elements for tackling the unrestricted stopping problem using the discrete-time approach in chapter 7. In section 5.3 and section 5.4 we will impose conditions under which these stopping problems are well-posed and allow for an analytical connection, which will be discussed in the subsequent chapter. Lastly, the special case of exponential utility functions will be covered in section 5.5 and serves as a comparison between the general case and the exponential case, which is a very common and often used utility function in utility maximization theory.

Chapter 6:
Section 6.1 introduces the concept of value functions, which are closely connected to the stopping problems introduced in the previous chapter. We will define an unrestricted value function for the unrestricted stopping problem, as well as n–step value functions for the n–step stopping problems. Moreover, we will see that the unrestricted value function turns out to be the limit function of the n-step ones, if $n \to \infty$. Studying these functions will help us in our attempt to characterize or even solve the generalized risk-sensitive stopping problem for continuous-time Markov chains. The second part of this chapter, section 6.2, will treat again the special case of exponential utility. We will see that this particular choice of utility function will lead to a significant simplification of the stopping problem.

Chapter 7:
This chapter forms the main part of this thesis. Section 7.2 will cover the so-called reward iteration formula, which will allow us to calculate an n–step value function for a fixed stopping time τ, given that the preceding $(n-1)$–step value function for τ is known. This allows for a recursive approach over $n \in \mathbb{N}_0$. Section 7.3 will again compare the case of general utility with the case of exponential utility.

In section 7.4 we will again establish a iteration type formula, the so-called Bellman equation. This equation makes it possible to recursively calculate every n–step value function from the previous one, without having to fix a particular stopping time. Furthermore, we will see that calculating one step of this Bellman equation will involve the solving of a deterministic one-dimensional optimization problem. This solution on the other hand yields a piecewise instruction how the optimal stopping time for an n–step stopping problem can be constructed. The procedure of iteratively calculating an n–step value function from the preceding one can be interpreted as a discrete-time approach over all steps $n \in \mathbb{N}_0$. As for $n \to \infty$ the n–step value functions converge to the unrestricted value function, the Bellman equation yields a method to approximate a solution to the unrestricted stopping problem using the sequence of n–step value functions. Following this, section 7.5 will again treat the special case of exponential utility.

In section 7.6 we will go a step further and transform the aforementioned iteration type Bellman equation into a fixed-point equation. Instead of calculating every n–step value function recursively and considering the limit of these functions to gain the unrestricted value function, we will show that this value function can also be obtained as a solution of the fixed-point equation. Again, solving this equation will require to solve a deterministic optimization problem, which will provide a candidate for the optimal stopping time for the unrestricted stopping problem. We will see in section 7.7, that under certain conditions this candidate is indeed optimal and thus provides the solution to the generalized risk-sensitive stopping problem for continuous-time Markov chains. Finalizing this chapter, section 7.8 will discuss the particular choice of exponential utility as utility function. We will see that the fixed-point equation from the previous section will simplify tremendously. The optimal stopping time for this specific case will only be able to stop at discrete but random times, namely the jump times of the underlying Markov chain.

Chapter 8:
The discrete-time approach established in chapter 7 required the iterative solving of the Bellman equation to gain the n–step value functions and their corresponding optimal stopping times. Analogously, we need to solve a fixed-point equation to gain the unrestricted value function and its corresponding optimal stopping time. In general, solving these equations is a difficult and tedious task. In this chapter, we will give conditions which will guarantee the optimality of certain stopping times, without having to solve the aforementioned equations.

After a brief outline in section 8.1, we will discuss in section 8.2 under which conditions it is never optimal to stop in certain states of the underlying Markov chain. We will see that we have to treat the cases of classical and extended utility functions separately, as they lead to different results. For extended utility functions we will show that it is never optimal to wait arbitrarily long in a given state of the Markov chain. If the jump into a subsequent state does not occur before a certain threshold, a rational investor will always stop at this threshold. On the other hand, for classical utility functions this has not to be true. We will give an explicit condition that allows for an identification of states for which it is indeed never optimal to stop, as long as the Markov chain resides in this state.

Analogously, we will discuss in section 8.3 under which conditions it is optimal to stop immediately, as soon as the underlying Markov chain hits a certain state. Again, we will differentiate between classical and extended utility functions. Both will lead to similar, but not identical conditions for which the optimality of immediate stopping can be guaranteed.

In section 8.4, we will discuss our results in this chapter for the special case of exponential utility. The conditions given in section 8.2 and section 8.3 will lead in this case to the optimality of so-called one-step look ahead stopping times. These stopping times compare the actual utility an investor would gain by immediate stopping with the expected utility he would gain, if he waits exactly for one change of state of the underlying markov chain. We will show that comparing these two values is sufficient to derive an optimal stopping time for the unrestricted stopping problem under exponential utility.

Chapter 9:
In contrast to the discrete-time approach in chapter 7 we want to discuss a different approach to tackle the generalized risk-sensitive stopping problem for continuous-time Markov chains. After a brief introduction into this method in section 9.1 we will establish a partial differential equation in section 9.2, the Hamilton-Jacobi-Bellman equation. As customary for such dynamic programming approaches, the goal will be to prove that a solution of this Bellman equation is indeed the value function corresponding to the generalized risk-sensitive stopping problem for continuous-time Markov chains. This procedure is known as verification. Given that the requirements of this verification are fulfilled, the optimal stopping time for our stopping problem turns out to be a first hit time.

Having both the discrete-time and the continuous-time approach to tackle the generalized risk-sensitive stopping problem for continuous-time Markov chains, we can compare both methods and discuss their advantages and disadvantages. As it turns out, the discrete-time approach requires less assumptions on the regularity of the considered value functions, making it viable for stopping models that can not be tackled using the continuous-time approach via the Hamilton-Jacobi-Bellman equation and the verification technique.

At the end, we will discuss the continuous-time approach in section 9.3 for the special choice of exponential utility. We will see that the Hamilton-Jacobi-Bellman equation degenerates to a simpler fixed-point equation. In fact, we will show that for this particular utility function, both approaches will lead to an almost similar solution technique.

2 Continuous-Time Markov Chains

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with sample space Ω, σ–field $\mathcal{F}$ and probability measure $\mathbb{P}$. Furthermore, let S be a countable *state space*. We now want to consider a special class of continuous-time stochastic processes, namely the (homogeneous) continuous-time Markov chains:

2.1 Definition of Continuous-Time Markov Chains

Definition 2.1 (continuous-time Markov chain)

A continuous-time stochastic process $X := (X_t)_{t \geq 0}$ taking values in S is called a *continuous-time Markov chain*, if for every $n \in \mathbb{N}$, $h > 0$, each $0 \leq t_0 < t_1 < \ldots < t_n$ and every $x_0, \ldots, x_n, x_{n+1} \in S$, such that

$$\mathbb{P}\left(X_{t_k} = x_k,\, 0 \leq k \leq n\right) > 0,$$

the *Markov property*

$$\mathbb{P}\left(X_{t_n+h} = x_{n+1} | X_{t_k} = x_k,\, 0 \leq k \leq n\right) = \mathbb{P}\left(X_{t_n+h} = x_{n+1} | X_{t_n} = x_n\right) \tag{2.1}$$

holds.

Furthermore, the continuous-time Markov chain X is called *homogeneous*, if the additional condition

$$\mathbb{P}\left(X_{t_k+h} = x_{k+1} | X_{t_k} = x_k\right) = \mathbb{P}\left(X_{t+h} = x_{k+1} | X_t = x_k\right) \tag{2.2}$$

holds for every $0 \leq k \leq n$ and $t \geq 0$.

Remark 2.2 (notes on the name *Markov chain* and alternative definitions)

(a) Stochastic processes which exhibit the Markov property (2.1) are generally called *Markov processes*. In this work we study a special class of Markov processes, namely those with denumerable state space S. In this case of an S–valued Markov process, it is common to use the word “chain” rather than “process”. Thus, we call the class of considered stochastic processes “Markov chains”.

(b) Let X be an continuous-time S–valued stochastic process. Then the following statements are equivalent and thus serve as alternative definitions for a Markov chain:

(i) X is a Markov chain according to Definition 2.1.

(ii) For each $n \in \mathbb{N}$, $h > 0$, every $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ and each bounded $\left(\mathcal{P}(S) - \mathcal{B}(\mathbb{R})\right)$–measurable function $f \colon S \to \mathbb{R}$ the equation

$$\mathbb{E}\left[f(X_{t_n+h})|\sigma(X_{t_0}, \ldots, X_{t_n})\right] = \mathbb{E}\left[f(X_{t_n+h})|\sigma(X_{t_n})\right] \tag{2.3}$$

holds.

(iii) For each $t \geq 0$ and every bounded $\left(\sigma(X_s,\ s \geq t) - \mathcal{B}(\mathbb{R})\right)$–measurable random variable Y the equation

$$\mathbb{E}\left[Y|\sigma(X_s,\ 0 \leq s \leq t)\right] = \mathbb{E}\left[Y|\sigma(X_t)\right] \tag{2.4}$$

holds.

For further reference, these equivalences can be found in [Brémaud, 1981, Appendix A1, Theorem T40].

The Markov property (2.1) ensures that for every discrete set of points in time $0 \leq t_0 < t_1 < \ldots < t_n$, the distribution of X at time $t_n + h$ does only depend on the last "observation" at time t_n, but not on the whole history at $t_0, \ldots, t_n$. In this sense, the Markovian structure legitimates the name of the Markov chain. In addition, the homogeneity condition (2.2) ensures that the above-mentioned distributions do not depend on the exakt points in time, but rather on the duration since the last observation. Thus, the probability that a homogeneous Markov chain changes its state from some $x \in S$ to some $j \in S$ does only depend on the elapsed time $h > 0$, but not on any explicit point in time.

Definition 2.3 (transition probabilities of a continuous-time Markov chain)

(a) Given a continuous-time Markov chain X, the *transition probability* $p_{xj}(s,t)$ from state $x \in S$ at time $s \geq 0$ to state $j \in S$ at time $t \geq s$ is defined by

$$p_{xj}(s,t) := \mathbb{P}\left(X_t = j|X_s = x\right). \tag{2.5}$$

(b) In the case of a homogeneous continuous-time Markov chain, the definition of a transition probability simplifies to

$$p_{xj}(t) := \mathbb{P}\left(X_t = j|X_0 = x\right) \tag{2.6}$$

for every $x, j \in S$ and $t \geq 0$.

Note that in the homogeneous case the transition probability from state $x \in S$ at time $s \geq 0$ to state $j \in S$ at time $t \geq s$ can be expressed as

$$p_{xj}(s,t) = \mathbb{P}\left(X_t = j | X_s = x\right) = \mathbb{P}\left(X_{t-s} = j | X_0 = x\right) = p_{xj}(t-s),$$

since the only knowledge needed is the difference in time between t and s and thus only one argument is required. Hence, the definition for transition probabilities for homogeneous Markov chains is consistent with the general one.

Notation 2.4

To simplify notation, we will write $\mathbb{P}_x(\cdot)$ in lieu of $\mathbb{P}(\,\cdot\,|X_0 = x)$. Thus, the transition probability of a homogeneous continuous-time Markov chain X from state $x \in S$ to $j \in S$ after a duration of $t \geq 0$ can be expressed as

$$p_{xj}(t) = \mathbb{P}\left(X_t = j | X_0 = x\right) = \mathbb{P}_x\left(X_t = j\right).$$

2.2 Transition Functions and Intensity Rates

Definition 2.5 (transition function of a homogeneous continuous-time Markov chain)

Let X be a homogeneous continuous-time Markov chain.

(a) Consider the transition probabilities in Definition 2.3. For every $x, j \in S$, the mapping $p_{xj} \colon [0,\infty) \to [0,1]$ with

$$p_{xj}(t) = \mathbb{P}_x\left(X_t = j\right)$$

is called *transition function* of X, if $t \mapsto p_{xj}(t)$ is right-continuous in 0, i.e.

$$\lim_{t \downarrow 0} p_{xj}(t) = \delta_{xj}. \tag{2.7}$$

(b) If the corresponding transition functions exist, i.e. the mappings $t \mapsto p_{xj}(t)$ are right-continuous in 0 for all $x, j \in S$, then the mapping $P \colon [0,\infty) \to [0,1]^{S\times S}$ defined by

$$P(t) := \left(p_{xj}(t)\right)_{x,j\in S} \tag{2.8}$$

is called the *transition matrix function* of X.

Assumption 2.6

From now on, we will assume that for every homogeneous continuous-time Markov chain the corresponding transition functions exist. By writing *Markov chain X with transition matrix function P* we will always refer to a homogeneous continuous-time Markov chain with existing transition functions.

Proposition 2.7

Let X be a Markov chain with transition matrix function P. Then the following statements hold:

(a) P satisfies the *Chapman-Kolmogorov* equation

$$p_{xj}(t+s) = \sum_{k \in S} p_{xk}(s) p_{kj}(t) \quad \forall x, j \in S, s, t \geq 0, \tag{2.9}$$

respectively written as matrix-multiplication:

$$P(t+s) = P(s)P(t) \quad \forall s, t \geq 0. \tag{2.10}$$

(b) $p_{xj}(t) \geq 0$ and $p_{xx}(t) > 0$ for all $x, j \in S$ and $t \geq 0$.

(c) $P(t)$ is a stochastic matrix for every $t \geq 0$.

(d) $t \mapsto p_{xj}(t)$ is uniformly continuous for every $x, j \in S$ and the right-hand derivative exists for $t = 0$. More precisely, the following right-handed limits exist:

$$\begin{aligned} q_{xx} &:= \lim_{h \downarrow 0} \frac{p_{xx}(h) - 1}{h} \in [-\infty, 0] \quad \text{for all } x \in S, \\ q_{xj} &:= \lim_{h \downarrow 0} \frac{p_{xj}(h)}{h} \in [0, \infty) \quad \text{for all } x, j \in S, x \neq j. \end{aligned}$$

(e) For all $t \geq 0$ and $x \in S$ we get

$$p_{xx}(t) \geq e^{-q_{xx} t}. \tag{2.11}$$

Proof of Proposition 2.7

(a)-(c) The proofs of this statements can be found, for example, in [Liggett, 2010, Theorem 2.12 + Theorem 2.13].

(d) This statement is proven, for example, in [Brémaud, 1999, Theorem 2.1].

(e) A proof of this assertion is given in [Liggett, 2010, Theorem 2.14 (a)].

□

Definition 2.8 (intensity matrix, stability and conservation)

Let X be again a Markov chain with transition matrix function P.

(a) The right-hand derivatives q_{xj} from Proposition 2.7 (d) are called *intensity rates* of X.

(b) The matrix $Q := (q_{xj})_{x,j\in S}$ is called *intensity matrix* of X.

(c) P is called *conservative*, if

$$-q_{xx} = \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} < \infty \quad \text{for all } x \in S. \tag{2.12}$$

(d) P is called *stable*, if

$$0 < \lambda := \sup_{x\in S} \big\{ -q_{xx} \big\} < \infty. \tag{2.13}$$

Notation 2.9

Since the intensity rate q_{xx} is always non-positive, it is convenient to define

$$q_x := -q_{xx} \in [0, \infty]$$

for every $x \in S$. The notation of q_x will often appear in the remainder of this work and thus legitimates this small notational abbreviation.

Sometimes, instead of P, we will call the intensity matrix Q stable, respectively conservative, and will not distinguish between the two terminologies.

Remark 2.10 (interpretation of intensity rates)

(a) Although permitted by definition, the case of $q_x = \infty$ for some $x \in S$ is explicitly excluded for stable or conservative transition matrix functions P. An infinitely high intensity rate q_x would imply an infinitely high value of the right-hand derivative at point 0 and would cause the Markov chain X to instantaneously leave state x as soon as it is attained. By assuming stability or conservativity we can ensure a strictly positive sojourn time in every state $x \in S$.

(b) By demanding the intensity matrix to be stable we can guarantee the existence of an upper bound for all intensity rates q_x, $x \in S$. This will imply that for any given finite time interval, the Markov chain X will not exhibit an infinitely high number of jumps. This property is often referred to as *non-explosivity*-property and is adressed to in Proposition 2.21.

(c) The case of $q_x = 0$ for some $x \in S$ has also a clear interpretation. Knowing $p_{xx}(0) = 1$ and concluding from inequality (2.11) of Proposition 2.7 (e) that $p_{xx}(t) \geq e^{q_x t} = 1$, implies that $p_{xx}(t) = 1$ for all $t \geq 0$. The Markov chain will never leave such a state x. A state $x \in S$ with $q_x = 0$ is thus called *absorbing*.

Remark 2.11 (infinitesimal description)

The intensity matrix Q of a Markov chain X with transition matrix function P is sometimes called *infinitesimal generator* of X. The main reason for this denomination is one of the main features of Markov chains: Under certain conditions to the intensity matrix Q, it is possible to reconstruct the whole transition matrix function P using the so-called Kolmogorov's backward differential system as stated in the next theorem. Thus, we do not need to know the actual transition probabilities of a Markov chain for every transition duration $t \geq 0$. An infinitesimally small part of this information (namely the intensity rates, the right-handed derivatives of the transition functions in $t = 0$) is sufficient to generate all the transition rates, provided Kolmogorov's backward differential system can be solved.

Theorem 2.12 (Kolmogorov's backward differential system)

Let X be a Markov chain with a stable and conservative transition matrix function P. Then *Kolmogorov's backward differential system* is satisfied:

$$\begin{cases} \frac{d}{dt} P(t) = Q\, P(t), & \text{for all } t \geq 0, \\ P(0) = E. \end{cases} \tag{2.14}$$

with $E := (\delta_{xj})_{x,j \in S} \in \mathbb{R}^{S \times S}$ being the identity matrix. An alternative formulation uses the component-wise notation:

$$\begin{cases} \frac{d}{dt} p_{xj}(t) = -q_j\, p_{xj}(t) + \sum\limits_{\substack{k \in S, \\ k \neq x}} q_{xk}\, p_{kj}(t), & \text{for all } t \geq 0, \\ p_{xj}(0) = \delta_{xj}, \end{cases} \tag{2.15}$$

for every $x, j \in S$.

Proof of Theorem 2.12
For a complete proof, the reader may be referred to [Brémaud, 1999, Chapter 8, Theorem 3.1]. □

Note that in the case of a finite state space S, the solution of (2.14) always exists and is uniquely given by

$$P(t) = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!} =: e^{tQ}. \tag{2.16}$$

In general, the existence and uniqueness of a solution of (2.14) cannot be guaranteed without imposing additional conditions on Q. This case shall not be discussed any further in this work. For additional reference, see again in Brémaud [1999].

Assumption 2.13 (standing assumptions for the intensity matrix of a Markov chain)

From now on, we will assume that given a Markov chain X with transition matrix function P, the corresponding intensity matrix Q satisfies the stability and conservation conditions (2.13) and (2.12). We will call X a Markov chain with intensity matrix Q.

Example 2.14 (reconstruction of the transition matrix function P)

Let $S = \{0, 1\}$ be the state space of the system and X a continuous-time Markov chain with intensity matrix Q given by

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

for some $\alpha, \beta > 0$. Clearly Q is conservative and stable and thus fulfills Kolmogorov's backward differential system (2.14). Since the state space S is finite, the transition matrix function P is uniquely determined by (2.16)

$$P(t) = e^{\begin{pmatrix} -\alpha t & \alpha t \\ \beta t & -\beta t \end{pmatrix}}.$$

We omit the details for calculating P explicitly by referring to [Liggett, 2010, Example 2.6]. The corresponding transition functions are then given by

$$\begin{aligned}
p_{00}(t) &= \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}, \\
p_{01}(t) &= \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}, \\
p_{10}(t) &= \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t}, \\
p_{11}(t) &= \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t}.
\end{aligned}$$

As we have seen we were able to reconstruct the whole transition matrix function P using only the infinitesimal generator Q.

2.3 Jump Times and Embedded Discrete-Time Markov Chains

For a given Markov chain X with transition matrix function P, the transition probability $p_{xx}(0)$ is obviously 1. Due to the right-hand continuity for $t = 0$, we can conclude that $p_{xx}(t) > 0$ for every $x \in S$ and sufficiently small $t \geq 0$. Thus, being in a state $x \in S$ the Markov chain will $\mathbb{P}_x$–almost surely stay a positive period in this state before changing to another one. Hence, we can describe the behavior of the Markov chain X in the following way: Starting in an initial state $x_0 \in S$, the Markov chain remains in this state for a certain time before “jumping” into another state in which it will remain again for a certain duration bigger than zero. A natural question arising from this observation would be whether the Markov chain can be fully characterized using the knowledge of the time points at which X changes its states and the probability distribution of these transitions. To answer this question rigorously, we need to define the above-mentioned quantities.

Definition 2.15 (jump and sojourn times of Markov chains, embedded chains)

Let X be a Markov chain with intensity matrix Q. Then we define

(a) the *jump times* of X by

$$S_0 := 0, \quad S_n := \inf\{t > S_{n-1} | X_t \neq X_{S_{n-1}}\},\ n \in \mathbb{N}, \tag{2.17}$$

(b) the *sojourn times* of X by

$$T_n := S_n - S_{n-1},\ n \in \mathbb{N}, \tag{2.18}$$

(c) the *embedded Markov chain* $(Z_n)_{n \in \mathbb{N}_0}$ of X by

$$Z_n := X_{S_n},\ n \in \mathbb{N}_0. \tag{2.19}$$

The discrete-time stochastic process $(Z_n)_{n \in \mathbb{N}_0}$ defined above is indeed a discrete-time Markov chain, as the following theorem will show.

Theorem 2.16 (transition probabilities of the embedded chain and distribution of sojourn times)

Let X be a Markov chain with intensity matrix Q. Then the following statements are true:

(a) The embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ of X is a homogeneous discrete-time Markov chain with transition probabilities p_{xj}, s.t.

$$p_{xj} := \begin{cases} \delta_{xj} & , \text{ if } q_x = 0, \\ \begin{cases} \frac{q_{xj}}{q_x}, & \text{if } x \neq j \\ 0, & \text{if } x = j \end{cases} & , \text{ if } q_x > 0 \end{cases} \tag{2.20}$$

for every $x, j \in S$. We denote the corresponding discrete-time transition matrix $\tilde{P}$ by

$$\tilde{P} = \left(p_{xj}\right)_{x,j\in S}.$$

(b) Given $(Z_n)_{n\in\mathbb{N}_0}$, the sequence of sojourn times $(T_n)_{n\in\mathbb{N}}$ is independent and each T_n is exponentially distributed with

$$T_n \sim Exp(q_{Z_{n-1}}). \tag{2.21}$$

Proof of Theorem 2.16
A proof of this theorem can be found in [Brémaud, 1999, Chapter 8, Theorem 4.2]. □

Definition 2.17 (n–step transition probabilities of the embedded Markov chain)

The probability of a Markov chain X to change its state from $x \in S$ to some $j \in S$ after exactly $n \in \mathbb{N}_0$ jumps is given by

$$p_{xj}^{(n)} := \left(\tilde{P}^n\right)_{x,j\in S} \quad \text{for } n \in \mathbb{N} \tag{2.22}$$

and

$$p_{xj}^{(0)} := \delta_{xj}.$$

We call $p_{xj}^{(n)}$ the *n–step transition probability* from $x \in S$ to $j \in S$ of the embedded Markov chain.

Assumption 2.18 (irreducibility of Markov chains)

If not stated otherwise, we will assume for the remainder of this thesis that every considered Markov chain X is *irreducible*. This means that for every $x, j \in S$, there exists an integer $n \in \mathbb{N}$ such that

$$p_{xj}^{(n)} > 0.$$

As a consequence, we can conclude that

$$q_x > 0 \quad \text{for all } x \in S,$$

since we always find an $n \in \mathbb{N}$ such that the n–step transition probability of leaving any state $x \in S$ into another state is non-negative. Thus, every state of X is not absorbing, exhibiting a positive probability to reach and also leave this state.

Remark 2.19 (alternative characterization of the irreducibility property)

Instead of considering the n–step transition probabilities of the embedded Markov chain in order to check the irreducibility of a Markov chain, we can use the following characterization:

A Markov chain X is irreducible, if and only if the transition functions satisfy

$$p_{xj}(t) > 0$$

for all $t > 0$ and every $x, j \in S$ such that $x \neq j$.

Corollary 2.20 (on the distribution of sojourn times and joint densities)

Let X be a Markov chain with intensity matrix Q.

(a) Due to the memorylessness property of the exponential distribution we can easily formulate the following conclusion of Theorem 2.16 (b):

Suppose that the Markov chain X is in some state $x \in S$ at time $t \geq 0$, i.e. $X_t = x$. Then the sojourn time τ in this state until the next jump is also exponentially distributed with

$$\tau \sim Exp(q_x).$$

(b) Suppose again that the Markov chain X is in some state $x \in S$ at time $t \geq 0$ and assume $q_x > 0$. Let τ denote the sojourn time until the next jump, i.e. the Markov chain will change into a new state at $\tau + t$. Since the sojourn time τ is independent of the actual time t, the joint distribution of τ and $X_{\tau+t}$ conditioned by $X_t = x$ is given by

$$\begin{aligned}\mathbb{P}(\tau \leq s, X_{\tau+t} = j \mid X_t = x) &= \mathbb{P}(\tau \leq s, X_\tau = j \mid X_0 = x)\\ &= \mathbb{P}_x(\tau \leq s, X_\tau = j)\\ &= \mathbb{P}_x(\tau \leq s) \cdot \mathbb{P}_x(Z_1 = j)\\ &= \begin{cases} \Big(1 - \exp(-q_x \cdot s)\Big) \cdot p_{xj}, & \text{if } x \neq j,\\ 0, & \text{if } x = j.\end{cases}\end{aligned}$$

for every $s \geq 0$ and $j \in S$.

As a consequence, the joint density of τ and $X_{\tau+t}$ conditioned by $X_t = x$ is given by

$$\begin{aligned}f_{\tau, X_{\tau+t}}(s, j \mid X_t = x) &= f_{\tau, X_\tau}(s, j \mid X_0 = x)\\ &= f_\tau(s \mid X_0 = x) \cdot f_{X_\tau}(j \mid X_0 = x)\\ &= \begin{cases} q_x \cdot \exp(-q_x \cdot s) \cdot p_{xj}, & \text{if } x \neq j,\\ 0, & \text{if } x = j\end{cases}\\ &= \begin{cases} q_x \cdot \exp(-q_x \cdot s) \cdot \frac{q_{xj}}{q_x}, & \text{if } x \neq j,\\ 0, & \text{if } x = j\end{cases}\\ &= \begin{cases} \exp(-q_x \cdot s) \cdot q_{xj}, & \text{if } x \neq j,\\ 0, & \text{if } x = j\end{cases}\end{aligned} \tag{2.23}$$

for every $s \geq 0$ and $j \in S$.

(c) A special case of (2.23) for $t = 0$ and $\tau = T_1 = S_1$ will play an important role in subsequent chapters, stating

$$f_{S_1, Z_1}(s, j \mid X_0 = x) = \begin{cases} \exp(-q_x \cdot s) \cdot q_{xj}, & \text{if } x \neq j,\\ 0, & \text{if } x = j\end{cases} \tag{2.24}$$

for every $s \geq 0$, $j \in S$ and $x \in S$ such that $q_x > 0$.

An obvious property of the sequence of jump times $(S_n)_{n \in \mathbb{N}_0}$, emerging from Definition 2.15 and Theorem 2.16, is its $\mathbb{P}$–almost sure strict monotonicity

$$S_0 < S_1 < S_2 < S_3 < \dots, \quad \mathbb{P}\text{–a.s.}. \tag{2.25}$$

Considering a Markov chain X with intensity matrix Q (and thus assuming the validity of the stability condition due to Assumption 2.13), we can guarantee that $q_{Z_{n-1}} \leq \lambda < \infty$ for every $n \in \mathbb{N}$ (where λ is given as in (2.13)) and thus state the following proposition:

Proposition 2.21 (non-explosivity)

Let X be a Markov chain with intensity matrix Q. Due to the stability condition (2.13) the sequence of jump times $(S_n)_{n\in\mathbb{N}_0}$ converges $\mathbb{P}$–almost surely to infinity:

$$\lim_{n\to\infty} S_n = \infty \quad \mathbb{P}\text{–a.s.} \tag{2.26}$$

Proof of Proposition 2.21

For every $n \in \mathbb{N}$, we get $q_{Z_{n-1}} \leq \lambda < \infty$. By choosing a small but arbitrary $\varepsilon > 0$, Theorem 2.16 yields (given $(Z_n)_{n\in\mathbb{N}_0}$)

$$\mathbb{P}\left(T_n < \varepsilon\right) = 1 - \exp\left(-q_{Z_{n-1}} \cdot \varepsilon\right) \leq 1 - \exp\left(-\lambda\varepsilon\right) =: C < 1$$

for every $n \in \mathbb{N}$. Thus, using the independence of the sequence $(T_n)_{n\in\mathbb{N}}$ we get

$$\mathbb{P}\left(T_1 < \varepsilon, \ldots, T_n < \varepsilon\right) = \prod_{k=1}^{n} \mathbb{P}\left(T_k < \varepsilon\right) \leq C^n \stackrel{n\to\infty}{\longrightarrow} 0.$$

Hence, the sequence $(T_n)_{n\in\mathbb{N}}$ is $\mathbb{P}$–almost surely no null sequence. As a consequence, the series $\sum_{n\in\mathbb{N}} T_n$ diverges $\mathbb{P}$–almost surely and yield

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} T_k = \sum_{n\in\mathbb{N}} T_n = \infty \quad \mathbb{P}\text{–a.s.}$$

□

In this sense, the Markov chain does not "explode". For any given compact interval of time, the number of jumps of a Markov chain with with intensity matrix Q remains finite.

As we have seen, for every homogeneous Markov chain X, its sequence of jump times $(S_n)_{n\in\mathbb{N}_0}$ and its embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ can always be constructed and are well-defined. On the other hand, we can obviously use these jump times and the embedded chain to find a characterization for the Markov chain:

$$X_t = \sum_{n\in\mathbb{N}_0} Z_n \cdot \mathbb{1}_{\{S_n \leq t < S_{n+1}\}}. \tag{2.27}$$

Alternatively, using the identity $\mathbb{1}_{\{S_n \leq t < S_{n+1}\}} = \mathbb{1}_{\{S_n \leq t\}}\left(1 - \mathbb{1}_{\{S_{n+1} \leq t\}}\right)$ for every $t \geq 0$ and $n \in \mathbb{N}_0$, the Markov chain can also be characterized by

$$X_t = \sum_{n\in\mathbb{N}_0} Z_n \cdot \mathbb{1}_{\{S_n \leq t\}}\left(1 - \mathbb{1}_{\{S_{n+1} \leq t\}}\right). \tag{2.28}$$

3 Filtrations and Stopping Times for Markov Chains

In order to define the stopping problem for continuous-time Markov chains rigorously and derive further results, we need to introduce the class of admissible stopping times for the stopping problem and thus specify the filtrations for which the stopping times are defined.

3.1 Filtrations for Markov Chains

In this section we want to study the natural filtration generated by a Markov chain X with intensity matrix Q and try to characterize this filtration using the information given by the jump times $(S_n)_{n\in\mathbb{N}_0}$ and the embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ of X. Furthermore, we want to introduce a new family of filtrations using stopped processes of X and characterize them as well.

Definition 3.1 (natural filtration and filtrations of stopped processes)

Let X be a Markov chain with intensity matrix Q.

(a) The Filtration $(\mathcal{F}_t^X)_{t\geq 0}$ defined by

$$\mathcal{F}_t^X := \sigma\left(X_s,\ 0 \leq s \leq t\right), \quad t \geq 0 \tag{3.1}$$

is called the *natural filtration* of X.

(b) Fix an $n \in \mathbb{N}_0$ and consider the stopped process $\left(X_t^{S_n}\right)_{t\geq 0}$ defined by

$$X_t^{S_n} := X_{t\wedge S_n}. \tag{3.2}$$

Define the *n–step filtration* $\left(\mathcal{F}_t^{n,X}\right)_{t\geq 0}$ of X by

$$\mathcal{F}_t^{n,X} := \sigma\left(X_s^{S_n},\ 0 \leq s \leq t\right), \quad t \geq 0. \tag{3.3}$$

Remark 3.2 (interpretation of $(\mathcal{F}_t^X)_{t\geq 0}$ and $(\mathcal{F}_t^{n,X})_{t\geq 0}$)

The n–step filtration $\left(\mathcal{F}_t^{n,X}\right)_{t\geq 0}$ models the information of X up to the n–th jump of the Markov chain and thus contains obviously less information than the natural filtration $\left(\mathcal{F}_t^X\right)_t \geq 0$. Clearly, these n–step filtrations increase in $n \in \mathbb{N}$:

$$\mathcal{F}_t^{n,X} \subseteq \mathcal{F}_t^{n+1,X} \subseteq \mathcal{F}_t^X \quad \text{for all } t \geq 0, n \in \mathbb{N}_0. \tag{3.4}$$

Furthermore, since $S_n \to \infty$ for $n \to \infty$ according to Proposition 2.21, we can conclude that this increasing sequence of n–step filtrations converges to the natural filtration in the sense that

$$\bigvee_{n\in\mathbb{N}_0} \mathcal{F}_t^{n,X} = \mathcal{F}_t^X \quad \text{for all } t \geq 0. \tag{3.5}$$

Another possibility to interpret a Markov chain is by means of marked point processes, using the jump times $(S_n)_{n\in\mathbb{N}_0}$ and the embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ as marks. We will now define the corresponding filtrations using these quantities.

Definition 3.3 (filtrations generated by jump times and embedded Markov chain)

Let X be a Markov chain with intensity matrix Q, jump time sequence $(S_n)_{n\in\mathbb{N}_0}$ and embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$. We then define the filtrations $(\mathcal{D}_t)_{t\geq 0}$ and $(\mathcal{D}_t^n)_{t\geq 0}$ for every $n \in \mathbb{N}_0$ by

$$\mathcal{D}_t^n := \bigvee_{k=1}^{n} \left(\sigma\left(\mathbb{1}_{\{S_k\leq s\}},\ 0 \leq s \leq t\right) \vee \sigma\left(Z_k \cdot \mathbb{1}_{\{S_k\leq s\}},\ 0 \leq s \leq t\right)\right), \quad n \in \mathbb{N}_0, \tag{3.6}$$

$$\mathcal{D}_t := \bigvee_{k\in\mathbb{N}} \left(\sigma\left(\mathbb{1}_{\{S_k\leq s\}},\ 0 \leq s \leq t\right) \vee \sigma\left(Z_k \cdot \mathbb{1}_{\{S_k\leq s\}},\ 0 \leq s \leq t\right)\right). \tag{3.7}$$

For a more elaborate discussion of this topic, we may refer to Pham [2010], [Protter, 2005, Chapter 6] or [Last & Brandt, 1995, Chapter 2]

Remark 3.4 (interpretation of $(\mathcal{D}_t)_{t\geq 0}$ and $(\mathcal{D}_t^n)_{t\geq 0}$)

By construction, the following statements hold:

(a) For all $n \in \mathbb{N}$, $(\mathcal{D}_t^n)_{t\geq 0}$ is the smallest filtration, such that

  (i) $S_0, \ldots, S_n$ are $(\mathcal{D}_t^n)_{t\geq 0}$–stopping times,

  (ii) $Z_0, \ldots, Z_n$ are $\left(\mathcal{D}_{S_n}^n,\ \mathcal{P}(S)\right)$–measurable.

(b) $(\mathcal{D}_t)_{t\geq 0}$ is the smallest filtration, such that

(i) $S_0, S_1, \ldots$ are $(\mathcal{D}_t)_{t\geq 0}$–stopping times,

(ii) For all $n \in \mathbb{N}_0$, Z_n is $(\mathcal{D}_{S_n},\ \mathcal{P}(S))$–measurable.

(c) Analoguously to Remark 3.2, the sequence of filtrations $\left((\mathcal{D}_t^n)_{t\geq 0}\right)_{n\in\mathbb{N}_0}$ is again monotonically increasing in n

$$\mathcal{D}_t^n \subseteq \mathcal{D}_t^{n+1} \subseteq \mathcal{D}_t \quad \text{for all } t \geq 0, n \in \mathbb{N}_0. \tag{3.8}$$

(d) Furthermore, this sequence converges to $(\mathcal{D}_t)_{t\geq 0}$

$$\bigvee_{n\in\mathbb{N}_0} \mathcal{D}_t^n = \mathcal{D}_t \quad \text{for all } t \geq 0. \tag{3.9}$$

For further reference on part (a), this statement is treated more elaborately in Pham [2010].

The next proposition will show that the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ of X coincides with the filtration $(\mathcal{D}_t)_{t\geq 0}$, whereas the n–step filtrations $(\mathcal{F}_t^{n,X})_{t\geq 0}$ coincide with the filtrations $(\mathcal{D}_t^n)_{t\geq 0}$.

Proposition 3.5 (filtration equivalences)

For every $t \geq 0$

(a) $\mathcal{F}_t^X = \mathcal{D}_t$,

(b) $\mathcal{F}_t^{n,X} = \mathcal{D}_t^n$ for all $n \in \mathbb{N}_0$.

Proof of Proposition 3.5

Let $t \geq 0$. Then it holds:

(a) For all $s \in [0, t]$ and every $k \in \mathbb{N}_0$ the mappings

$$\omega \mapsto \mathbb{1}_{\{S_{k+1}(\omega)\leq s\}} \quad \text{and}$$
$$\omega \mapsto Z_k(\omega) \cdot \mathbb{1}_{\{S_k(\omega)\leq s\}}$$

are $\mathcal{D}_s$–measurable and thus $\mathcal{D}_t$–measurable.

As a consequence of the representation (2.28), the mapping

$$\omega \mapsto X_s(\omega)$$

is also $\mathcal{D}_s$–measurable and thus $\mathcal{D}_t$–measurable. Hence, we get

$$\left\{X_s^{-1}(A),\ 0 \leq s \leq t\right\} \in \mathcal{D}_t \quad \text{for every } A \in \mathcal{P}(S)$$

and thus

$$\mathcal{F}_t^X \subseteq \mathcal{D}_t.$$

On the other hand, the jump times S_n are for every $n \in \mathbb{N}_0$ defined as the first hitting times of the n–th jump of X (cf. Definition 2.15) and thus $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times. In addition, we know that for every $n \in \mathbb{N}_0$, the state of X after the n–th jump is given by $Z_n = X_{S_n}$ and thus $\mathcal{F}_{S_n}^X$–measurable. Since we know that $(\mathcal{D}_t)_{t\geq 0}$ is the smallest filtration fulfilling these two properties (cf. Remark 3.4), we can conclude

$$\mathcal{D}_t \subseteq \mathcal{F}_t^X$$

and altogether

$$\mathcal{F}_t^X = \mathcal{D}_t.$$

(b) Analoguously to the characterizations (2.27), respectively (2.28) for the Markov chain X, we can represent the stopped process $\left(X_t^{S_n}\right)_{t\geq 0}$ by

$$\begin{aligned} X_t^{S_n} &= \sum_{k=0}^{n-1} Z_k \cdot \mathbb{1}_{\{S_k \leq t < S_{k+1}\}} + Z_n \cdot \mathbb{1}_{\{S_n \leq t\}} \\ &= \sum_{k=0}^{n-1} Z_k \cdot \mathbb{1}_{\{S_k \leq t\}} \left(1 - \mathbb{1}_{\{S_{k+1} \leq t\}}\right) + Z_n \cdot \mathbb{1}_{\{S_n \leq t\}} \end{aligned} \tag{3.10}$$

for every $n \in \mathbb{N}_0$. Again, the mappings

$$\begin{aligned} \omega &\mapsto \mathbb{1}_{\{S_{k+1}(\omega) \leq s\}} && \text{for every } k \in \{0, 1, \ldots, n-1\} \text{ and} \\ \omega &\mapsto Z_k(\omega) \cdot \mathbb{1}_{\{S_k(\omega) \leq s\}} && \text{for every } k \in \{0, 1, \ldots, n\} \end{aligned}$$

are $\mathcal{D}_s^n$–measurable and thus $\mathcal{D}_t^n$–measurable for all $s \in [0, t]$. Hence, $X_s^{S_n}$ itself is $\mathcal{D}_t^n$–measurable for every $s \in [0, t]$. We get

$$\mathcal{F}_t^{n,X} \subseteq \mathcal{D}_t^n.$$

On the other hand, we know that for every $k \in \{0, 1, \ldots, n\}$ the jump times S_k are $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping times. Furthermore, the random variables

$$Z_k = X_{S_k} = X_{S_k}^{S_n}$$

are $\mathcal{F}_{S_k}^{n,X}$-measurable and thus $\mathcal{F}_{S_n}^{n,X}$-measurable for every $k \in \{0, 1, \ldots, n\}$. By Remark 3.4, we know that $(\mathcal{D}_t^n)_{t\geq 0}$ is the smallest filtration fulfilling these properties above. This leads to

$$\mathcal{D}_t^n \subseteq \mathcal{F}_t^{n,X}$$

and finally

$$\mathcal{F}_t^{n,X} = \mathcal{D}_t^n$$

for every $n \in \mathbb{N}_0$.

□

3.2 Stopping Times for Markov Chains

In this section we want to introduce the class of stopping times which will be treated in the subsequent sections and chapters. Considering the special structural properties of continuous-time Markov chains and the corresponding natrual filtrations, the question arises whether these properties lead to a simpler representation of this class of stopping times.

Definition 3.6 ($\mathbb{P}_x$–almost surely finite stopping times)

(a) A mapping $\tau\colon \Omega \to [0, \infty]$ is called a stopping time with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, if

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

We say τ is a $(\mathcal{F}_t)_{t\geq 0}$–stopping time.

(b) Let X be a Markov chain with intensity matrix Q and $(\mathcal{F}_t^X)_{t\geq 0}$ denote the corresponding natrual filtration. The set of all $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times shall by denoted by Σ.

(c) Let X be a Markov chain with intensity matrix Q and $(\mathcal{F}_t^X)_{t\geq 0}$ denote the corresponding natrual filtration. We define for every $x \in S$ the set of all $\mathbb{P}_x$*–almost surely finite* $(\mathcal{F}_t^X)_{t\geq 0}$*–stopping times* by Σ_x. Every $\tau \in \Sigma_x$ is a $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time such that $\mathbb{P}_x(\tau < \infty) = 1$, provided that the Markov chain X starts in an initial value $X_0 = x \in S$.

(d) Let X be a Markov chain with intensity matrix Q and $(\mathcal{F}_t^{n,X})_{t\geq 0}$ denote the corresponding n–step filtration for some $n \in \mathbb{N}_0$. The set of all $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping times shall by denoted by Σ_n.

Remark 3.7 (monotonicity of the stopping sets)

Let $x \in S$. According to (3.4), every ($\mathbb{P}_x$–almost surely finite) $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time is also a ($\mathbb{P}_x$–almost surely finite) $(\mathcal{F}_t^{n+1,X})_{t\geq 0}$–stopping time and also a ($\mathbb{P}_x$–almost surely finite) $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time for all $n \in \mathbb{N}_0$. Hence we get the monotonicity property

$$\Sigma_n \subseteq \Sigma_{n+1} \subseteq \Sigma \quad \text{for all } n \in \mathbb{N}_0. \tag{3.11}$$

Example 3.8 (jump times are stopping times)

It follows directly from Proposition 3.5 that the jump times S_n, $n \in \mathbb{N}_0$ of an continuous-time Markov chain X are stopping times. More precisely:

- S_n is an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time for every $n \in \mathbb{N}_0$ and
- for a fixed $n \in \mathbb{N}_0$, S_k is an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time for every $k \in \{0, 1, \ldots, n\}$.

3.2.1 Characterization of Stopping Times with respect to the n–Step Filtration

Let X be a Markov chain X with intensity matrix Q. The first step for characterizing stopping times will be a decomposition result for stopping times with respect to the n–step filtration $(\mathcal{F}_t^{n,X})_{t\geq 0}$. This result, as stated in the following theorem, can be found for instance in Bayraktar & Zhou [2014]. For further literature on the structure of stopping times, the reader may be referred to Pham [2010] or [Davis, 1993, Appendix A2].

Theorem 3.9 (decomposition of $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping times)

Let $n \in \mathbb{N}_0$. A mapping $\tau\colon \Omega \to [0,\infty]$ is an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time, if and only if it has the following decomposition:

$$\tau = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} + \tau^n \mathbb{1}_{\{S_n \leq \tau\}} \tag{3.12}$$

where for every $k \in \{0,1,\dots,n\}$:

(i) $\tau^k \geq S_k$,

(ii) τ^k is an $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time,

(iii) There exists a measurable mapping $h_k\colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau^k = h_k(S_1,\dots,S_k,Z_0,Z_1,\dots,Z_k) + S_k. \tag{3.13}$$

This decomposition (3.12) is unique in the sense that every term in the sum of (3.12) is uniquely determined on the set $\{S_k \leq \tau < S_{k+1}\}$ and independent of the structure of τ^k after the jump S_{k+1}.

Notation 3.10

Due to the equivalency in Theorem 3.9, every $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time τ is uniquely determined by the piecewise description using the $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping times τ^k, respectively the corresponding mappings h_k for $k \in \{0,1,\dots,n\}$. We will refer to this $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping times as *stopping rules* and use the notations

$$\tau = (\tau^0, \tau^1, \dots, \tau^n) \tag{3.14}$$

or

$$\tau = (h_0, h_1, \dots, h_n) \tag{3.15}$$

for the decomposition representation (3.12). Also note that for every $k \in \{0,1,\dots,n\}$ the stopping rules τ^k and the corresponding mappings h_k are connected by equation (3.13) and differ in the additive term S_k.

Remark 3.11 (alternative representation of decomposition (3.12))

Another representation of decomposition (3.12) is given by [Bayraktar & Zhou, 2014, Proposition 2.3]:

$$\tau = \tau^0 \mathbb{1}_{\{\tau^0 < S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{\tau^0 \geq S_1\} \cap \dots \cap \{\tau^{k-1} \geq S_k\} \cap \{\tau^k < S_{k+1}\}} + \tau^n \mathbb{1}_{\{\tau^0 \geq S_1\} \cap \dots \cap \{\tau^{n-1} \geq S_n\}}. \quad (3.16)$$

Proof of Theorem 3.9
By Proposition 3.5 we know that the filtrations $(\mathcal{F}_t^{n,X})_{t \geq 0}$ and $(\mathcal{D}_t^n)_{t \geq 0}$ are equivalent. According to [Bayraktar & Zhou, 2014, Theorem 2.1 and Lemma 3.2], a mapping τ is a $(\mathcal{D}_t^n)_{t \geq 0}$–stopping time and thus an $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time, if and only if it has the decomposition (3.16) with mappings τ^k, $k \in \{0, 1, \dots, n\}$ fulfilling the properties required in Theorem 3.9. Furthermore [Bayraktar & Zhou, 2014, Proposition 2.3] states the equivalence of the decompositions (3.16) and (3.12) and thus yield the desired assertion. □

Remark 3.12 (interpretation of decomposition (3.12))

(a) Considering decomposition (3.16), an interpretation for every $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time $\tau = (\tau^0, \tau^1, \dots, \tau^n)$ can be given in the following way: At first, the stopping time τ follows the stopping rule τ^0. If the stopping rule τ^0 does not trigger before the first jump time S_1, the stopping rule τ^0 is discarded and τ starts to follow the new stopping rule τ^1. Inductively, the stopping time τ will follow the stopping rule τ^k, provided that every single previous stopping rule $\tau^0, \dots, \tau^{k-1}$ did not trigger before the corresponding jump time $S_1, \dots, S_k$. Stopping rule τ^k stays in effect as long as it will not stop itself or the next jump S_{k+1} didn't occur so far. If the jump time S_{k+1} is reached, τ^k will be discarded again and the stopping time τ starts to follow the stopping rule τ^{k+1}. In the case that τ did not stop beforehand and all stopping rules $\tau^0, \dots, \tau^{n-1}$ are discarded, τ will follow stopping rule τ^n indefinitely.

(b) For every $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time τ, the stopping rule τ^0 is an $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time such that there exists a mapping $h_0 \colon S \to [0, \infty]$ with $\tau^0 = h_0(Z_0)$. Given an arbitrary initial value $X_0 = Z_0 = x \in S$ of our Markov chain X, the stopping rule τ^0 is given by $\tau^0 = h_0(x)$ and thus a deterministic constant, depending only on x.

(c) Given a Markov chain X with intensity matrix Q and an $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time τ, the event of stopping before the first jump of the Markov chain occurs at time S_1 is given by

$$\{\tau < S_1\} = \{\tau^0 < S_1\}.$$

Hence, the probability of stopping before this first jump time, given an initial value

$x \in S$, can be calculated by

$$\begin{aligned}\mathbb{P}_x(\tau < S_1) &= \mathbb{P}_x(\tau^0 < S_1) = 1 - \mathbb{P}_x\left(S_1 \le h_0(Z_0)\right) = 1 - \mathbb{P}\left(S_1 \le h_0(x)\right) \\ &= 1 - \mathbb{P}\left(T_1 \le h_0(x)\right) = \exp\left(-q_x \cdot h_0(x)\right). \qquad (3.17)\end{aligned}$$

Example 3.13 (example for a decomposition of n–step stopping times)

Let $n \in \mathbb{N}_0$. Continuing Example 3.8, an $(\mathcal{F}_t^{n,X})_{t\ge 0}$–stopping time $\tau = S_j$ for $j \in \{0, 1, \dots, n\}$ can be decomposed such that

$$\tau = \tau^0 \mathbb{1}_{\{\tau^0 < S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{\tau^0 \ge S_1\} \cap \dots \cap \{\tau^{k-1} \ge S_k\} \cap \{\tau^k < S_{k+1}\}} + \tau^n \mathbb{1}_{\{\tau^0 \ge S_1\} \cap \dots \cap \{\tau^{n-1} \ge S_n\}}$$

for $(\mathcal{F}_t^{k,X})_{t\ge 0}$–stopping times $\tau^k \ge S_k$, $k \in \{0, 1, \dots, n\}$. Since we know that the stopping time τ will trigger exactly at the j–th jump of the underlying Markov chain and neither before nor after it, we can conclude that the involved stopping rules are given by

$$\tau^k = \begin{cases} S_j, & \text{for } k = j, \\ \infty, & \text{for } k \in \{0, 1 \dots, j-1, j+1, \dots, n\}. \end{cases}$$

Every stopping rule τ^k is obviously an $(\mathcal{F}_t^{k,X})_{t\ge 0}$–stopping time such that $\tau^k \ge S_k$ and by defining the measurable mappings h_k as

$$h_k(S_1, \dots, S_k, Z_0, Z_1, \dots, Z_k) = \begin{cases} 0, & \text{for } k = j, \\ \infty, & \text{for } k \in \{0, 1 \dots, j-1, j+1, \dots, n\} \end{cases}$$

we get

$$\tau^k = h_k(S_1, \dots, S_k, Z_0, Z_1, \dots, Z_k) + S_k.$$

Note that the first j stopping rules $\tau^0, \dots, \tau^{j-1}$ are set to infinity to guarantee that the stopping time τ does not trigger before the j–th jump. On the other hand, setting the stopping rules $\tau^{j+1}, \dots, \tau^n$ to infinity is not mandatory. Since the stopping rules only have to be uniquely determined on the sets $\{S_k \le \tau < S_{k+1}\}$ and

$$\{S_k \le S_j < S_{k+1}\} = \emptyset \quad \mathbb{P}_x\text{–a.s. for } k \ne j,$$

the stopping rules τ^k for $k > j$ will never trigger and can be chosen arbitrarily.

Using equation (3.14) in Notation 3.10, the decomposition representation from (3.16) for S_j yields

$$S_j = (\infty, \dots, \infty, S_j, \infty, \dots, \infty).$$

Using equation (3.15) on the other hand, the decomposition representation can be written as

$$S_j = (\infty, \dots, \infty, 0, \infty, \dots, \infty).$$

Before we now address ourselves to the next step, namely the characterization of stopping times with respect to the natural filtration, we first restate a result from Bayraktar & Zhou [2014], which was already used and cited in the proof of Theorem 3.9, but which also will come handy on its own in later chapters:

Lemma 3.14 (charaterization of stopping rules)

For every $k \in \mathbb{N}_0$, a random variable τ^k is a $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time such that $\tau^k \geq S_k$ if and only if there exists a measurable mapping $h_k \colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau^k = h_k(S_1, \ldots, S_k, Z_0, Z_1, \ldots, Z_k) + S_k.$$

This Assertion is taken from [Bayraktar & Zhou, 2014, Lemma 3.2.], basicly stating that the properties of the stopping rules from decomposition (3.12) in Theorem 3.9 are not independent of one another, but rather that properties (i)+(ii) imply (iii) and vice versa.

3.2.2 Characterization of Stopping Times with respect to the Natural Filtration

In the previous section we discussed a decomposition result for $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping times, given a Markov chain X with a intensity matrix Q. Since we are primarily interested in $(\mathcal{F}_t^{X})_{t\geq 0}$–stopping times, a characterization by a similar decomposition result like (3.12) from Theorem 3.9 would be desirable. In fact, such a decomposition is possible for stopping times from the class Σ_x (for some $x \in S$), i.e. $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^{X})_{t\geq 0}$–stopping times, as stated in the next proposition.

Proposition 3.15 (decomposition of $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times)

Let $x \in S$ and $\tau\colon \Omega \to [0,\infty]$ be a mapping such that $\mathbb{P}_x(\tau < \infty) = 1$. Then τ is an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time, if and only if it has the following decomposition:

$$\tau = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{\infty} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.} \tag{3.18}$$

where for every $k \in \mathbb{N}_0$:

(i) $\tau^k \geq S_k$,

(ii) τ^k is an $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time,

(iii) There exists a measurable mapping $h_k\colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau^k = h_k(S_1, \dots, S_k, Z_0, Z_1, \dots, Z_k) + S_k.$$

This decomposition (3.18) is $\mathbb{P}_x$–almost surely unique in the sense that every term in the sum of (3.18) is – up to a $\mathbb{P}_x$–nullset – uniquely determined on the set $\{S_k \leq \tau < S_{k+1}\}$ and independent of the structure of τ^k after the jump S_{k+1}.

Proof of Proposition 3.15

Let $x \in S$ and $\tau\colon \Omega \to [0,\infty]$ be a mapping such that $\mathbb{P}_x(\tau < \infty) = 1$.

1. Assume that τ is an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time.

   (i) Define for every $n \in \mathbb{N}_0$ the mapping $\tau_n\colon \Omega \to [0,\infty]$ by

   $$\tau_n := \tau \wedge S_n.$$

   τ_n is then a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time such that

   $$\tau_n \nearrow \tau \quad \text{for } n \to \infty.$$

   Now Theorem 3.9 becomes applicable for τ_n and yields the decomposition

   $$\tau_n = \tau_n^0 \mathbb{1}_{\{\tau_n < S_1\}} + \sum_{k=1}^{n-1} \tau_n^k \mathbb{1}_{\{S_k \leq \tau_n < S_{k+1}\}} + \tau_n^n \mathbb{1}_{\{S_n \leq \tau_n\}} \tag{3.19}$$

   for mappings τ_n^k, $k \in \{0, 1, \dots, n\}$ such that

   - $\tau_n^k \geq S_k$,
   - τ_n^k is an $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time,

- There exists a measurable mapping $h_{k,n}\colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau_n^k = h_{k,n}(S_1,\dots,S_k,Z_0,Z_1,\dots,Z_k) + S_k.$$

(ii) Furthermore, for every $n \in \mathbb{N}_0$ we get

- $\{\tau_n < S_1\} = \{\tau \wedge S_n < S_1\} = \{\tau < S_1\}$,
- For all $k \in \{1,\dots,n-1\}$:
  $\{S_k \leq \tau_n < S_{k+1}\} = \{S_k \leq \tau \wedge S_n < S_{k+1}\} = \{S_k \leq \tau < S_{k+1}\}$ and
- $\{S_n \leq \tau_n\} = \{S_n \leq \tau \wedge S_n\} = \{S_n \leq \tau\} = \{S_n = \tau\}$.

Hence, decomposition (3.19) can be expressed as

$$\tau_n = \tau_n^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{n-1} \tau_n^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} + \tau_n^n \mathbb{1}_{\{S_n \leq \tau\}} \tag{3.20}$$

(iii) Theorem 3.9 states the uniqueness of the decomposition in the sense that for every $n \in \mathbb{N}_0$ the k–step stopping times τ_n^k are uniquely determined

- on $\{\tau_n < S_1\} = \{\tau < S_1\}$ for $k = 0$,
- on $\{S_k \leq \tau_n < S_{k+1}\} = \{S_k \leq \tau < S_{k+1}\}$ for $k \in \{1,\dots,n-1\}$,
- on $\{S_n \leq \tau_n\} = \{S_n \leq \tau\}$ for $k = n$.

Thus we get

$$\tau^k := \tau_n^k = \tau_{n+1}^k \qquad \text{for all } n \in \mathbb{N}_0,\ k \in \{0,1,\dots,n\} \tag{3.21}$$

on the corresponding sets $\{\tau < S_1\}$, $\{S_k \leq \tau < S_{k+1}\}$, $k \in \{1,\dots,n-1\}$, respectively $\{S_n \leq \tau\}$. This leads for every $n \in \mathbb{N}_0$ to an equivalent representation of (3.20):

$$\tau_n = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} + \tau^n \mathbb{1}_{\{S_n \leq \tau\}}, \tag{3.22}$$

where the mappings τ^k, $k \in \{0,1,\dots,n\}$ defined by (3.21) do not depend on n and fulfill

(i) $\tau^k \geq S_k$,

(ii) τ^k is an $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time,

(iii) There exists a measurable mapping $h_k\colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau^k = h_k(S_1,\dots,S_k,Z_0,Z_1,\dots,Z_k) + S_k.$$

(iv) We know that $S_n < S_{n+1}$ and thus

$$\{S_{n+1} \le \tau\} \subseteq \{S_n \le \tau\}$$

for every $n \in \mathbb{N}_0$. Additionally, we get $S_n \to \infty$ for $n \to \infty$ by Proposition 2.21 and hence

$$\bigcap_{n\in\mathbb{N}_0} \{S_n \le \tau\} = \{\tau = \infty\}.$$

Since $\mathbb{P}_x(\tau = \infty) = 0$ by assumption, we get

$$\mathbb{1}_{\{S_n\le\tau\}} \to 0 \quad \mathbb{P}_x\text{–a.s. for } n \to \infty.$$

(v) Using the steps $(i) - (iv)$ above, we can conclude that

$$\begin{aligned}
\tau &= \lim_{n\to\infty} \tau_n \\
&= \lim_{n\to\infty} \left(\tau^0 \mathbb{1}_{\{\tau<S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{S_k\le\tau<S_{k+1}\}} + \tau^n \mathbb{1}_{\{S_n\le\tau\}}\right) \\
&= \tau^0 \mathbb{1}_{\{\tau<S_1\}} + \sum_{k=1}^{\infty} \tau^k \mathbb{1}_{\{S_k\le\tau<S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.}
\end{aligned}$$

where τ^k fulfills the required properties for every $k \in \mathbb{N}_0$.

2. Assume now that decomposition (3.18) holds with τ^k fulfilling the required properties for every $k \in \mathbb{N}_0$.

(i) Let $n \in \mathbb{N}_0$ and define τ_n by

$$\tau_n := \tau^0 \mathbb{1}_{\{\tau<S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{S_k\le\tau<S_{k+1}\}} + \tau^n \mathbb{1}_{\{S_n\le\tau\}}. \tag{3.23}$$

Clearly, since $S_n \to \infty$ $\mathbb{P}_x$–almost surely for $n \to \infty$ and τ being $\mathbb{P}_x$–almost surely finite, we get

$$\tau_n \to \tau \quad \mathbb{P}_x\text{–a.s. for } n \to \infty.$$

Theorem 3.9 states that τ_n is an $(\mathcal{F}_t^{n,X})_{t\ge0}$–stopping time for every $n \in \mathbb{N}_0$. Since $\mathcal{F}_t^{n,X} \subseteq \mathcal{F}_t^X$ for every $n \in \mathbb{N}_0$ and $t \ge 0$, we can conclude that each τ_n is also an $(\mathcal{F}_t^X)_{t\ge0}$–stopping time.

(ii) According to the step above, we have a sequence $(\tau_n)_{n\in\mathbb{N}_0}$ of $(\mathcal{F}_t^X)_{t\ge0}$–stopping times with decomposition (3.23) for every $n \in \mathbb{N}_0$, which all use the same stopping rules τ^k. We will now show that the sequence $(\tau_n)_{n\in\mathbb{N}_0}$ is increasing. To this end we observe that for every $n \in \mathbb{N}_0$

$$\begin{aligned}
\tau_n \le \tau_{n+1} &\Leftrightarrow \tau^n \mathbb{1}_{\{S_n\le\tau\}} \le \tau^n \mathbb{1}_{\{S_n\le\tau<S_{n+1}\}} + \tau^{n+1} \mathbb{1}_{\{S_{n+1}\le\tau\}} \\
&\Leftrightarrow \tau^n \mathbb{1}_{\{S_{n+1}\le\tau\}} \le \tau^{n+1} \mathbb{1}_{\{S_{n+1}\le\tau\}} \\
&\Leftrightarrow \tau^n \le \tau^{n+1} \text{ on } \{S_{n+1} \le \tau\}.
\end{aligned}$$

We know by assumption that for every $n \in \mathbb{N}_0$, the stopping rule τ^n is $\mathbb{P}_x$–almost surely uniquely determined on the set $\{S_n \leq \tau < S_{n+1}\}$ and can be arbitrary otherwise. Thus, by setting

$$\tau^n := S_{n+1} \text{ on } \{S_{n+1} \leq \tau\}$$

and noting that $\tau^{n+1} \geq S_{n+1}$ we can conclude that

$$\tau^n \leq \tau^{n+1} \text{ on } \{S_{n+1} \leq \tau\}$$

and thus

$$\tau_n \leq \tau_{n+1}$$

for every $n \in \mathbb{N}_0$.

(iii) Since $\tau_n \to \tau$ $\mathbb{P}_x$–almost surely for $n \to \infty$ and $(\tau_n)_{n \in \mathbb{N}_0}$ being an increasing sequence, we get

$$\{\tau > t\} = \bigcup_{n \in \mathbb{N}_0} \{\tau_n > t\} \in \mathcal{F}_t^X$$

for every $t \geq 0$ and hence know that τ is an $(\mathcal{F}_t^X)_{t \geq 0}$–stopping time.

□

Notation 3.16

Analogously to Notation 3.10, every $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping time τ is uniquely determined by the piecewise description using the stopping rules τ^k, respectively the corresponding mappings h_k for $k \in \mathbb{N}_0$. We will use the notations

$$\tau = (\tau^0, \tau^1, \dots) \tag{3.24}$$

or

$$\tau = (h_0, h_1, \dots) \tag{3.25}$$

for the decomposition representation (3.18).

Remark 3.17 (interpretation of decomposition (3.18))

The statements made in Remark 3.12 remain valid for the decomposition in Proposition 3.15. Briefly outlined, the stopping time τ will follow the stopping rules τ^k, $k \in \mathbb{N}_0$ on the corresponding sets $\{S_k \leq \tau < S_{k+1}\}$. As long as a stopping rule τ^k does not stop before the $(k+1)$th jump, it will be discarded after S_{k+1} and the next stopping rule τ^{k+1} takes effect.

Some useful and imminent consequences concerning the relationship between $(\mathcal{F}_t^{n,X})_{t \geq 0}$–stopping times and $(\mathcal{F}_t^X)_{t \geq 0}$–stopping times that can be concluded from Proposition 3.15 and its proof, are stated in the following corollary.

Corollary 3.18

(a) Let $\tau = (\tau^0, \tau^1, \dots)$ be an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time. Then the random variable $\tau \wedge S_n$ is an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time for every $n \in \mathbb{N}_0$ such that

$$\tau \wedge S_n = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{n-1} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} + S_n \mathbb{1}_{\{S_n \leq \tau\}}$$

or simply

$$\tau \wedge S_n = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n)$$

for suitable $\tau^0, \dots, \tau^{n-1}$ from decomposition (3.12).

(b) On the other hand, let $n \in \mathbb{N}_0$ and $\tau = (\tau^0, \tau^1, \dots, \tau^n)$ be an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time. Then there exists an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time $\tilde{\tau}$ such that

$$\tilde{\tau} = (\tau^0, \tau^1, \dots, \tau^n, S_{n+1}, S_{n+2}, \dots).$$

(c) Let $n \in \mathbb{N}_0$ and consider the $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time $\tau = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n)$. This stopping time fulfills

$$\tau = \tau \wedge S_n$$

and there exists an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time $\tilde{\tau}$ such that

$$\tau = \tilde{\tau} \wedge S_n.$$

A possible choice for such a $\tilde{\tau}$ would be

$$\tilde{\tau} = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n, S_{n+1}, \dots).$$

Since stopping times of the form $\tau \wedge S_n$ will play an important role in chapter 7, it is convenient to define a suitable set for such stopping times.

Definition 3.19 (stopping times stopped before the n–th jump)

Let $n \in \mathbb{N}_0$ and $x \in S$. Define the set

$$\Sigma_{n,x} := \left\{ \tau \in \Sigma_x \;\middle|\; \exists \tilde{\tau} \in \Sigma_x, \text{ such that } \tau = \tilde{\tau} \wedge S_n \right\}. \tag{3.26}$$

By Corollary 3.18 (a), every $\tau \in \Sigma_{n,x}$ is an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time fulfilling the decomposition representation $\tau = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n)$. On the other hand, every $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time τ, such that $\tau = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n)$ permits by Corollary 3.18 (c) the representation $\tau = \tilde{\tau} \wedge S_n$ for a stopping time $\tilde{\tau} \in \Sigma_x$. Thus the set $\Sigma_{n,x}$ can alternatively be represented as

$$\Sigma_{n,x} = \left\{ \tau \in \Sigma_n \;\middle|\; \tau \text{ can be represented as } \tau = (\tau^0, \tau^1, \dots, \tau^{n-1}, S_n) \right\}. \tag{3.27}$$

Lemma 3.20 (monotonicity of the stopping sets)
For every $n \in \mathbb{N}_0$ and $x \in S$ the following monotonicity property is valid:

$$\Sigma_{n,x} \subseteq \Sigma_{n+1,x} \subseteq \Sigma_x. \tag{3.28}$$

Proof of Lemma 3.20
Let $n \in \mathbb{N}_0$, $x \in S$ and $\tau \in \Sigma_{n,x}$. By definition, there exists a stopping time $\tilde{\tau} \in \Sigma_x$ such that $\tau = \tilde{\tau} \wedge S_n$. Hence it holds $\tau = \tau \wedge S_n$ and $S_n < S_{n+1}$ yields $\tau = \tau \wedge S_n = \tau \wedge S_{n+1}$. Thus, by definition we get $\tau \in \Sigma_{n+1,x}$ and therefore

$$\Sigma_{n,x} \subseteq \Sigma_{n+1,x}.$$

The second part of the assertion follows directly from the definition of $\Sigma_{n,x}$ as

$$\Sigma_{n,x} \subseteq \Sigma_x$$

for every $n \in \mathbb{N}_0$. □

4 Utility Functions

In this chapter we will briefly introduce the *utility functions* we want to consider in this thesis. The concept of value functions is very common and wide spread in the field of optimization theory. Especially some prominent representives like exponential utility, power utility or logarithmic utility are quite popular choices for utility functions and are very often found in a lot of papers from various research fields, particularly in those with connections to finance, as the concept of utility functions has a clear economical interpretation.

Utility functions are strongly connected to preferences and preference relations. We assume that an investor, who has the decision between two different offers, is always able to determine which of these two alternatives he prefers more. These preferences are also captured in the concept of utility functions. At least under certain assumptions on the preference relations, one can establish an one-to-one connection between preference relations and utility functions. This relation was first established by John von Neumann and Oskar Morgenstern in 1944 in von Neumann & Morgenstern [1944].

We say that using a utility function U an investor prefers an offer A over an offer B, if and only if $U(A) > U(B)$. In this case offer A is said to yield a higher utility for the investor than offer B. In case of $U(A) = U(B)$ we say that the investor is indifferent between these two offers, as they yield the same utility. In general, an "offer" can be almost everything: A general good, like food or a car, a promise of a future reward, or especially common in finance: some cash flow. In the later case of a cash flow, the modelling of a preference relation is rather canonical. An economically rational investor will always prefer the greatest cash flow. Thus a utility function which maps this preference onto some utility has to exhibit some monotonicity property, meaning that a higher cash flow will result in a higher utility. This monotonicity is the most important property for utility functions. Another often desired property is concavity. Roughly spoken, the marginal utility gain is the smaller the higher the considered cash flow. An additional unit of cash yields a higher additional gain of utility, if the investor has a low endowment to begin with. The richer the investor gets the lower the surplus of utility gained by an additional unit. For a further reading on the topic of utility functions and the connection to preference relations we may refer to [Föllmer & Schied, 2004, chapter].

In the context of this thesis, we want to consider some revenue, consistent of some cost and reward functional, and evaluate it under some utility function. The details will be discussed elaborately in the subsequent chapter 5. Based on the reasoning above, we will now define the utility functions we will want to consider in the rest of this work:

Definition 4.1 (utility function)

Let $U\colon D \to \mathbb{R}$ and $D \subseteq \mathbb{R}$ be the maximal domain of the mapping. U is called *utility function* if it is strictly increasing and strictly concave on D and twice differentiable on the interior of D.

As usual in the theory of utility optimization the absolute values a utility function can attain do not have an economical interpretation. According to the strictly increasing monotonicity of a utility function, a higher utility is always preferred over a lower utility value. Thus if we are in the next chapter interested in maximizing some expected utility over some set of feasible stopping times, we are not truly interested in its value itself, but rather in finding – if existent – the maximizer of this set, namely the optimal stopping time for which the maximal expected utility is attained at. This maximizer gives an investor explicit rules for stopping the process optimally and thus maximizing some total reward in the sense of achieving the highest expected utility possible. Clearly, this optimal stopping time depends heavily on the chosen utility function. As mentioned before, the absolute values such a utility function can attain are rather uninteresting. As a result to this observation, two utility functions are often identified with each other and are said to be equivalent, if they are equal up to an affine transformation. A more significant property of a utility function is its *risk aversion*. A prominent measure therefore is the Arrow-Pratt measure of absolute risk aversion (ARA_U):

$$ARA_U(x) := -\frac{U''(x)}{U'(x)}, \quad x \in int(D), \tag{4.1}$$

which was developed by John W. Pratt and Kenneth J. Arrow and published in Pratt [1964], respectively Arrow [1965].

This measure is commonly used in literature to model the attitude of an investor towards risk. The higher the ARA_U–value, the more risk avoiding is an investor with utility function U. As one can simply see, the Arrow-Pratt measure doesn't use any absolute value if U itself, but instead it's first two derivatives, describing the monotonicity and curvature of U. Obviously, this measure is invariant under affine transformations. Thus, any utility functions, which are equal up to an affine transformation, have the same absolute risk aversion. Any such investors using these utility functions are said to have the same attitude towards risk and are treated equivalently.

Example 4.2 (examples for utility functions)

Prominent and commonly used utility functions are

- the exponential utility:

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x} \quad \text{for some } \gamma > 0,$$

  having a *constant absolute risk aversion* ($CARA$): $ARA_U(x) \equiv \gamma$,

- the power utility:

$$U\colon [0,\infty) \to \mathbb{R}, \quad U(x) := x^{\gamma} \quad \text{ for some } \gamma \in (0,1),$$

  having a *decreasing absolute risk aversion* $(DARA)$: $ARA_U(x) = (1-\gamma)\frac{1}{x}$,

- the logarithmic utility:

$$U\colon (0,\infty) \to \mathbb{R}, \quad U(x) := \ln(\gamma x) \quad \text{ for some } \gamma > 0.$$

  having a *decreasing absolute risk aversion* $(DARA)$: $ARA_U(x) = \frac{1}{x}$.

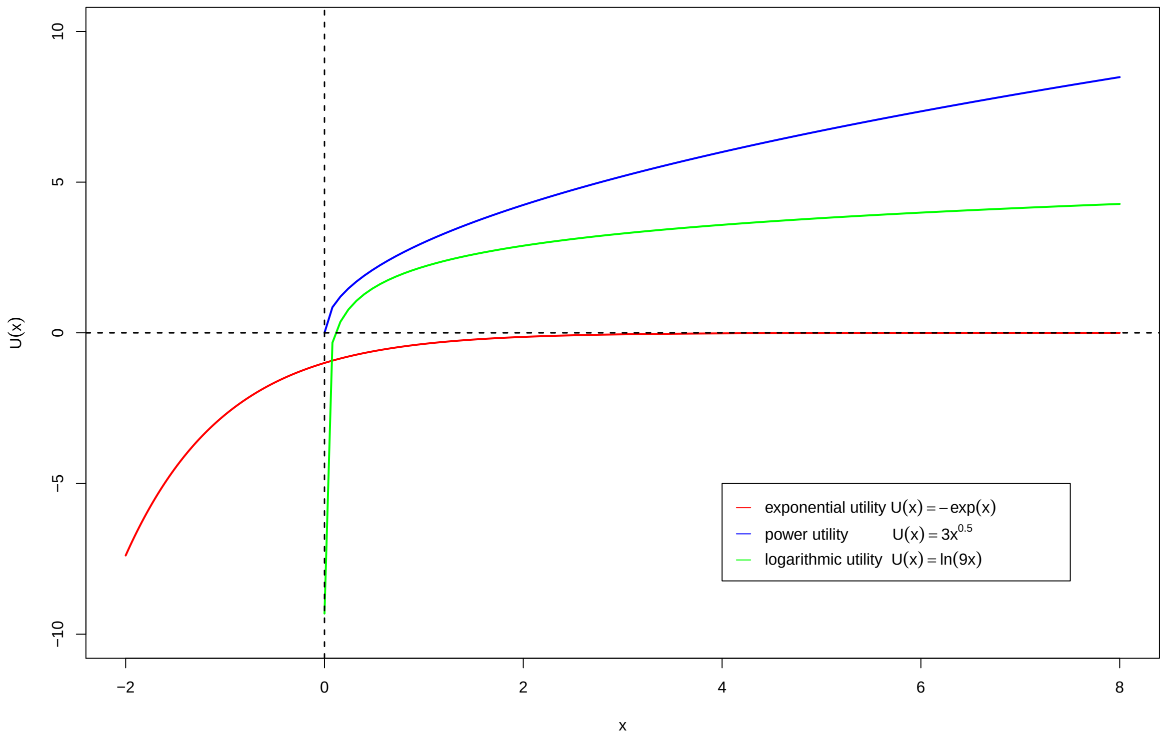

Figure 4.1: some examples for utility functions

As already mentioned at the beginning of this chapter, we want to use the definition of utility functions to evaluate some revenue, given as the sum of some cost and reward functional. Without supplementary assumptions we cannot control this revenue in the sense that it can't be bounded from below. As we will allow for the costs to become arbirarily high, the corresponding reward term may not be able to compensate for it, rendering the total revenue unbounded from below. Using a utility function defined according to Definition 4.1 on some domain $D \subsetneq \mathbb{R}$, like for example the power- or logarithmic utility, will not allow to account for arguments which are not bounded from below. Of course we would be able to confine ourselves to the case of utility functions with domain $\mathbb{R}$, but then we would lose some well-known and prominent cases shuch as the above-mentioned power or logarithmic utility functions. To this end we want to generalize the definition of a utility function by extending its domain to the whole real line:

Definition 4.3 (extended utility function)

Let $\tilde{U}$ be a utility function according to Definition 4.1 with domain $D \subsetneq \mathbb{R}$. We then define the extended utility function $U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$x \mapsto U(x) = \begin{cases} \tilde{U}(x), & \text{if } x \in D, \\ -\infty, & \text{if } x \notin D. \end{cases} \tag{4.2}$$

Basically for every argument $x \notin D$, we set the value of the utility function to $-\infty$. We thus have to extend the range of U by adding $-\infty$. Note that an extended utility function defined according to Definition 4.3 is still monotone and concave, but loses the strict monotonicity, the strict concavity as well as the differentiability in general. But restricted on the set D these properties obviously still hold. We summarize this in the following lemma:

Lemma 4.4 (properties of extended utility functions)

Let U be an extended utility function from Definition 4.3, derived from a classical utility function $\tilde{U}\colon D \to \mathbb{R}$. Then U is increasing and concave on $\mathbb{R}$ and strictly increasing, strictly concave on D, as well as twice differentiable on $int(D)$.

Note that this concept of extended utility functions is only needed if the maximal domain of a classical utility function according to Definition 4.1 does not coincide with the whole real line. If on the other hand we have a classical utility function with maximal domain $\mathbb{R}$, like for example the exponential utility, then Definition 4.1 and Definition 4.3 coincide. In this case the given utility function does never attain the value $-\infty$.

Remark 4.5 (interpretation of extended utility functions)

We can interpret extended utility functions in the following way: Suppose that an investor expresses his preferences using a classical utility function $\tilde{U}\colon D \to \mathbb{R}$ which is not defined on the whole real line. This maximal domain D of $\tilde{U}$ represents the solvency region of the investor. Going below this solvency region will result in bankruptcy, which is the worst case scenario for an investor. By extending this classical utility function to

$U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ according to Definition 4.3, we assign this case of bankruptcy the worst possible utility, namely $-\infty$. Note again that the actual values of utility functions do not have any economical meaning. The main purpose is to compare different cash flows in terms of provided utitily. Clearly, defaulting should never be preferable for any investor. Thus, setting the utility of such cases to $-\infty$ is economically meaningful.

Utility functions like power or logarithmic utility are for example only defined for non-negative, respectively positive domains. In such cases the investor is solvent as long as he owns non-negative, respectively positive amounts of money. He is not able to borrow money in order to go below these defaulting points.

The exponential utility on the other hand is defined on the whole real line and doesn't need to be extended. We can interpret this case in the way that an investor using an exponential utility does not have the risk of defaulting and can borrow any amount of money.

Example 4.6 (examples for extended utility functions)

With respect to Example 4.2, the following functions are extended utility functions:

- The extended power utility

$$U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}, \quad U(x) := \begin{cases} x^\gamma, & x \geq 0, \\ -\infty, & x < 0 \end{cases}$$

  for $\gamma \in (0,1)$.

- The extended logarithmic utility

$$U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}, \quad U(x) := \begin{cases} \ln(\gamma x), & x > 0, \\ -\infty, & x \leq 0 \end{cases}$$

  for $\gamma > 0$.

Notation 4.7 (designation of utility functions)

In the following chapters we will always consider utility functions defined on the whole real line. If the utility function in question is given by Definition 4.1 with maximal domain $D = \mathbb{R}$, we will call it *classical utility function* on $\mathbb{R}$. If on the other hand the considered utility function is given by Definition 4.3, we will call it *extended utility function*. In case there is no danger of confusion and no distinction necessary, we will just say *utility function*.

5 Generalized Risk-Sensitive Stopping Problems for Continuous-Time Markov Chains

In this chapter we want to introduce the main problem of this work, the generalized risk-sensitive stopping problem for homogeneous continuous-time Markov chains. A key feature in this work is the renunciation of a fixed predetermined utility function. Instead, the formulation of the stopping problem will be made under an arbitrary and general utility function. As discussed in the previous chapter, we do not want to restrict ourselves to the classical criterion of optimizing the expected reward under exponential utility. Thus, the term *risk-sensitive* refers to the general case of an arbitrary utility function in the sense of Definition 4.3.

5.1 Setup for the Stopping Problem for Continuous-Time Markov Chains

The generalized risk-sensitive stopping problem for continuous-time Markov chains forms the main object of interest in this thesis and will be defined in this section. Every theory we will develop serves the purpose to solve this problem and characterize its solutions as good as possible. To this end, we agree upon the following model setup for the generalized risk-sensitive stopping problem for continuous-time Markov chains, that will be used throughout the subsequent chapters and sections.

Definition 5.1 (generalized risk-sensitive stopping problem for continuous-time Markov chains)

Suppose that the model setting consists of the following elements:

- The Markov chain X with intensity matrix Q. Recalling Assumption 2.13, this means having a homogeneous continuous-time Markov chain taking values in S, with a transition matrix function which is right-continuous in zero and a corresponding intensity matrix which is stable and conservative.
- An initial value $x \in S$ such that $X_0 = x$.
- The set of all $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times Σ_x.
- An arbitrary utility function $U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ as specified in Definition 4.3.
- A constant cost rate $c > 0$.
- A lower bounded reward function $g\colon S \to \mathbb{R}$, specifying the reward gained upon stopping the problem.

Then the *generalized risk-sensitive stopping problem for continuous-time Markov chains* is given by

$$\mathbb{E}_x\left[U\left(-c\tau + g(X_\tau)\right)\right] \to \max_{\tau\in\Sigma_x}! \tag{5.1}$$

Given the continuous-time Markov chain X an investor or controller of this stopping problem can observe the evolution of the process at any given time $t \geq 0$ and thus knows the reward $g(X_t)$ he would gain from it by stopping the process at this point in time. Waiting and observing the process is penalized by a running cost rate $c > 0$. Hence, stopping the Markov chain at time $t \geq 0$ nets the investor a total reward of $-ct + g(X_t)$ monetary units. This total reward shall be evaluated under the individual utility function of the controller, given by his personal preferences and attidute towards risk. We shall suppose that this utility is chosen from a general class of utility functions as defined in Definition 4.3 and do not restrict ourselves to a certain type of utility function. As usual in the theory of utility optimization, we cannot hope for maximizing every possible random outcome of such utility. Instead, we restrict ourselves to the maximization of the expectation of the utility. The resulting stopping problem consists of maximizing this expected utility over all feasible stopping times, namely all $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times, and trying to find – if possible – the optimal stopping time for which this maximum is attained at.

Remark 5.2 (interpretation of the stopping problem for continuous-time Markov chains)

We can interpret the stopping problem for a continuous-time Markov chain X in the following way:

(a) X_t represents an offer the investor receives at time $t \geq 0$. He then has to decide whether he wants to accept this offer or not. Observing the process X and thus having the possibility to accept an offer is not free of charge. The investor has to pay the cummulative costs $-ct$ (according to the running cost rate $c > 0$) for observing the income of offers up to time $t \geq 0$. If he doesn't accept, the flow of offers just as the cummulative running costs continue. If he accepts an offer, the whole process stops and terminates the stopping problem. By stopping at some time $t \geq 0$, the investor then gains the reward $g(X_t)$ and has to pay a fee of $-ct$. This revenue is evaluated under the individual utility function U modelling his personal preferences and attidute towards risk.

(b) The stopping problem represents a special kind of control problem. For any time $t \geq 0$, the investor has to choose his actions to control the process. In this case his set of actions consists of stopping or not stopping the process by accepting or declining the current offer.

(c) For a general process Y the observed offers Y_t can change randomly for every moment $t \geq 0$. Even for a compact time intervall $[a, b] \subset [0, \infty)$ the investor could observe an uncountably infinity number of different offers. This would make the stopping problem extremely difficult to handle. As we restrict ourselfs to the case of continuous-time Markov chains, we get a more tractable situation. Using the notation of jump times and embedded Markov chains according to Definition 2.15, changes in the state of the Markov chain X occur only at designated jump times $(S_n)_{n\in\mathbb{N}_0}$. Between these jumps the chain remains in its last state. Furthermore, according to Proposition 2.21 the number of jumps within a compact interval of time $[a, b] \subset [0, \infty)$ remains $\mathbb{P}$–almost surely finite. Thus, a continuous-time Markov chain X can be interpreted as a discrete sequence of offers observable to the investor, represented by the states $(Z_n)_{n\in\mathbb{N}_0}$ of X immediately after a change of state. Since the sojourn times between two jumps are exponentially distributed, we can say that the investor receives an offer at some random jump time S_n, which remains valid and can be accepted for an exponentially distributed duration until the next jump occurs, replacing the current offer Z_n irrevocably by the next one Z_{n+1}.

(d) The optimization over the set Σ_x of all $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times is economically meaningful. Using a stopping time which is not necessarily ($\mathbb{P}_x$–almost surely) finite, will potentially cause an investor to observe the given stopping problem indefinitely, rejecting every single offer and never terminating the process. Such a behavior can of course not be realized in any real life application. Thus, the restriction on ($\mathbb{P}_x$–almost surely) finite stopping times is neccessary to guarantee econmical meaningfulness.

(e) One of the main goals is to find an optimal stopping time which maximizes the expected utility given in this context. An economically rational investor will try to apply this optimal stopping rule. Despite of optimizing over ($\mathbb{P}_x$–almost surely) finite stopping times, the desired supremum has not to be attained by such a ($\mathbb{P}_x$–almost surely) finite stopping time. To avoid such situations, we will impose additional conditions to guarantee for the optimal stopping time to be ($\mathbb{P}_x$–almost surely) finite.

5.2 Generalized Risk-Sensitive n–Step Stopping Problem

The risk-sensitive stopping problem for continuous-time Markov chains exhibits an infinite time horizont. The investor doesn't have to accept any incoming offer in any given finite interval of time. The only restriction on his choice of a stopping time he wants to apply is whether it lies in the set of feasible stopping times Σ_x (for a given initial value $X_0 = x$). Using the interpretation of continuous-time Markov chains as a countable sequence of offers with an exponentially distributed period of validity given in Remark 5.2, we can argue that the investor has the opportunity to observe a countably infinite number of incoming offers.

Of course it is sometimes reasonable to restrict this infinite number of incoming offers by an upper bound for the maximal numbers of possible offers. Since the offers appear randomly, namely at the moments of the jump times of the Markov chain, we cannot restrict the infinite time horizont of the stopping problem. Instead, we reformulate the general risk-sensitive stopping problem for continuous-time Markov chains (5.1) to a version which allows the investor to receive at most $n \in \mathbb{N}_0$ different offers. If he doesn't accept any of these offers and thus doesn't stop the problem himself, it will terminate itself by the time of the n-th jump, forcing the investor to take the n-th incoming offer. In terms of the continuous-time Markov chain X, we will now consider the stopped version of it by considering the new process $(X_t^{S_n})_{t\geq 0}$.

Definition 5.3 (generalized risk-sensitive n–step stopping problem)

Using the same notation as in section 5.1 and denoting the sequence of jump times of the Markov chain X by $(S_n)_{n\in\mathbb{N}_0}$ we define the *general risk-sensitive n-step stopping problem for continuous-time Markov chains* by

$$\mathbb{E}_x \left[U \left(-c(\tau \wedge S_n) + g(X_{\tau\wedge S_n}) \right) \right] \to \max_{\tau\in\Sigma_x}! \tag{5.2}$$

Remark 5.4 (alternative formulation of the n–step stopping problem (5.2))

As discussed before, the stopping problem (5.2) terminates at the latest after the n-th jump of the Markov chain. Thus by applying Corollary 3.18, for all $x \in S$ and every feasible stopping time $\tau \in \Sigma_x$ we can conclude that $\tau \wedge S_n$ is a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time for all $n \in \mathbb{N}_0$. Using the notation in Definition 3.19 we can write $\tau \wedge S_n \in \Sigma_{n,x}$. On the other hand, every stopping time in $\Sigma_{n,x}$ permits the representation $\tau \wedge S_n$ for a suitable stopping time $\tau \in \Sigma_x$. Thus, the general risk-sensitive n-step stopping problem for continuous-time Markov chains (5.2) can be expressed as

$$\mathbb{E}_x\left[U\left(-c\tau + g(X_\tau)\right)\right] \to \max_{\tau\in\Sigma_{n,x}}! \tag{5.3}$$

As we will see in chapter 7 studying the n–step stopping problem (5.2) will be essential in order to tackle the unrestricted version (5.1) for countably infinite incoming offers. These two problems are closely connected. Analysing problem (5.2) will give us the necessary tools to tackle problem (5.1) by means of limit results by letting n converge to infinity.

5.3 Well-Posedness of the Stopping Problems

Before we tackle the stopping problems (5.1), respectively (5.2) and try to find optimal stopping times for them, we first address to the question of well-posedness of this problems. We want to call a stopping problem *well-posed*, if the maximum of the expected utility over all feasible stopping times (or at least the supremum of it) can be attained by a value less than infinity. In that way we can guarantee that there exists no stopping time which leads to an arbitrarily high expected utility. Thus we can ensure that this stopping problem remains economically meaningful. A model, which allows an investor to achieve arbitrarily high amounts of utility does not reflect any realistic situation in any financial market.

To ensure the well-posedness of the stopping problems for every utility function, a possible way is to impose a certain integrability condition on the Markov chain X and the reward function g. We will formulate this condition as a standing assumption, which will hold throughout the remainder of this thesis.

Assumption 5.5 (integrability condition)

We will assume that every combination of Markov chain X, reward rate g and cost rate c studied in context of the stopping problem (5.1) fulfills the following integrability condition:

$$M_x := \sup_{\tau \in \Sigma_x} \mathbb{E}_x \Big[-c\tau + g^+(X_\tau) \Big] < \infty \quad \text{for every } x \in S, \tag{5.4}$$

where the mapping $g^+ \colon S \to [0, \infty)$ is defined as $g^+ := \max\{g, 0\}$.

This assumption guarantees the well-posedness of the stopping problems for any untility function as shown in the following Lemma.

Lemma 5.6 (well-posedness of the stopping problems)

Under the validity of Assumption 5.5 the stopping problems (5.1) and (5.2) for every $n \in \mathbb{N}_0$ are well-posed.

Proof of Lemma 5.6
Let $U \colon D \to \mathbb{R}$ be a utility function with domain $D \subseteq \mathbb{R}$. With U being strictly concave and strictly increasing, there exists an affine linear function $D \to \mathbb{R},\ x \mapsto mx + d$ for some constants $m > 0$ and $d \in \mathbb{R}$ such that

$$U(x) \leq mx + d \tag{5.5}$$

for every $x \in D$. Using (5.4), this yields

$$\sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-c\tau + g(X_\tau)\right)\right] \leq m \cdot \sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[-c\tau + g^+(X_\tau)\right] + d = mM_x + d < \infty$$

and thus the well-posedness of the stopping problem (5.1). Furthermore, the monotonicity of the feasible stopping time sets in equation (3.28) of Lemma 3.20 provides, together with the alternative version of the n–step stopping problem (5.3), the ineqality

$$\begin{aligned} \sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-c(\tau \wedge S_n) + g(X_{\tau \wedge S_n})\right)\right] &= \sup_{\tau \in \Sigma_{n,x}} \mathbb{E}_x \left[U\left(-c\tau + g(X_\tau)\right)\right] \\ &\leq \sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-c\tau + g(X_\tau)\right)\right] \\ &< \infty \end{aligned}$$

and hence the second part of the assertion. □

Remark 5.7 (sufficient conditions for the validity of Assumption 5.5)
The integrability condition (5.4) in Assumption 5.5 is obviously fulfilled if

$$g^+(X_\tau) = \max\{g(X_\tau), 0\}$$

is upper bounded by a constant independent of $x \in S$ and $\tau \in \Sigma_x$. There are mainly two situations which imply the above-mentioned boundedness from above and which are quite common in real life applications:

(a) The reward function $g\colon S \to \mathbb{R}$ is bounded and thus especially upper bounded. This reflects the situation where the reward an investor can receive (based on the actual state of the underlying Markov chain) is not allowed to be arbitrarily high. Instead the reward is chosen from a given finite interval within the bounds of the function g.

(b) The state space S of the underlying Markov chain is finite. In this case the upper bound can be chosen to be the maximal value the reward function $g\colon S \to \mathbb{R}$ can attain:

$$g^+(X_\tau) \leq \max\{g_{\max}, 0\},$$

where $g_{\max} := \max_{x \in S} g(x)$. In this situation there is only a finite number of reward values an investor can achieve, depending on the finite states the underlying Markov chain can attain.

5.4 Additional Assumptions Imposed on the Stopping Problems

In order to be able to compare the two stopping problems (5.1) and (5.2) analytically, we will make the following general assumption:

Assumption 5.8

We will assume that every choice of utilty function U, Markov chain X, reward rate g and cost rate c studied in context of the stopping problem (5.1) fulfills the following inequality:

$$\liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right)\right] \geq \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right)\right] \tag{5.6}$$

for all $x \in S$ and every $\tau \in \Sigma_x$.

This assumption is rather technical and hard to verify. Following the ideas given in [Bäuerle & Rieder, 2011, Remark 10.2.1] we will now give sufficient conditions under which Assumption 5.8 is valid.

Lemma 5.9 (sufficient conditions for Assumption 5.8)

(a) The inequality (5.6) in Assumption 5.8 is valid, if the following condition is satisfied:

$$\liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - cS_n + g(Z_n)\right) \mathbb{1}_{\{\tau > S_n\}} \right] \geq 0 \quad \text{for all } x \in S, \tau \in \Sigma_x. \tag{5.7}$$

(b) Inequality (5.7) and thus (5.6) are valid, if the following condition is satisfied:

$$\limsup_{n\to\infty} \mathbb{E}_x \left[U^-\left(-ct - cS_n + g(Z_n)\right) \mathbb{1}_{\{\tau > S_n\}} \right] = 0 \quad \text{for all } x \in S, \tau \in \Sigma_x, \tag{5.8}$$

where the mapping $U^-\colon \mathbb{R} \to \mathbb{R}$ is defined as $U^- := -\min\{U, 0\}$.

Proof of Lemma 5.9

(a) Suppose (5.7) holds. Then we get for every $x \in S$ and every $\tau \in \Sigma_x$ that

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right)\right] \\
=& \liminf_{n\to\infty} \Big(\mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right) \mathbb{1}_{\{\tau > S_n\}} \right] \\
&\qquad + \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] \Big) \\
\geq& \liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - cS_n + g(X_{S_n})\right) \mathbb{1}_{\{\tau > S_n\}} \right] \\
&+ \liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] \\
\overset{(\star)}{=}& \liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - cS_n + g(Z_n)\right) \mathbb{1}_{\{\tau > S_n\}} \right] + \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right)\right] \\
\overset{(5.7)}{\geq}& \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right)\right].
\end{aligned}$$

Thus (5.6) is valid. The equality in $(\star)$ holds due to the monotone convergence theorem by decomposing the utility into positive and negative part:

$$\begin{aligned}
U^+\left(-ct - c\tau + g(X_\tau)\right) &:= \max\left\{ U\left(-ct - c\tau + g(X_\tau)\right),\ 0\right\}, \\
U^-\left(-ct - c\tau + g(X_\tau)\right) &:= \max\left\{ -U\left(-ct - c\tau + g(X_\tau)\right),\ 0\right\}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] \\
=& \liminf_{n\to\infty} \mathbb{E}_x \left[U^+\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} - U^-\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] \\
=& \lim_{n\to\infty} \mathbb{E}_x \left[U^+\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] - \lim_{n\to\infty} \mathbb{E}_x \left[U^-\left(-ct - c\tau + g(X_\tau)\right) \mathbb{1}_{\{\tau \leq S_n\}} \right] \\
=& \mathbb{E}_x \left[U^+\left(-ct - c\tau + g(X_\tau)\right)\right] - \mathbb{E}_x \left[U^-\left(-ct - c\tau + g(X_\tau)\right)\right] \\
=& \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau)\right)\right],
\end{aligned}$$

since

$$\left(U^{+}\left(-ct-c\tau+g(X_{\tau})\right)\mathbb{1}_{\{\tau\leq S_n\}}\right)_{n\in\mathbb{N}_0}$$

and

$$\left(U^{-}\left(-ct-c\tau+g(X_{\tau})\right)\mathbb{1}_{\{\tau\leq S_n\}}\right)_{n\in\mathbb{N}_0}$$

are non-negative increasing sequences.

(b) Suppose (5.8) holds. Note that for every function U the decomposition $U = U^{+} - U^{-}$ into a positive part $U^{+} := \max\{U, 0\}$ and a negative part $U^{-} := -\min\{U, 0\}$ is valid. Then we get for every $x \in S$ and every $\tau \in \Sigma_x$ that

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}_x\left[U\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
=&\liminf_{n\to\infty} \mathbb{E}_x\left[U^{+}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}-U^{-}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
\geq&\liminf_{n\to\infty} \mathbb{E}_x\left[U^{+}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
&+\liminf_{n\to\infty} \mathbb{E}_x\left[-U^{-}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
=&\liminf_{n\to\infty} \mathbb{E}_x\left[U^{+}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
&-\limsup_{n\to\infty} \mathbb{E}_x\left[U^{-}\left(-ct-cS_n+g(Z_n)\right)\mathbb{1}_{\{\tau>S_n\}}\right]\\
\geq&0,
\end{aligned}$$

where the last inequality holds due to (5.8) and the positivity of the first summand. Thus (5.7) is valid.

□

5.5 Stopping Problems for Exponential Utility Functions

Among all utility functions the exponential utility is probably the most common. It arises in a lot of different kinds of optimization problems where some cost or reward functionals are evaluated under utility functions. Due to this special role, many authors call optimization problems with applied exponential utility *risk-sensitive*. In case of stopping problems wuch problems could be called *(classical) risk-sensitive* stopping problems.

As already mentioned in the beginning of this chapter, a key feature of this thesis is that we did not restrict ourselves to any special choice of a utility function. Therefore we called the problems we want to consider *generalized risk-sensitive* stopping problems. But as the name already suggests, we see our problems as generalization of the classicas risk-sensitive case. It is therefore reasonable to treat the well-known case of classical risk-sensitive stopping problems as a special case of the theory we want to establish. Hence we will consistenly compare the generalized case with the special choice of the exponential utiltiy as underlying utility function for our stopping problem and discuss the resulting simplifications with respect to the general theory.

We will begin with a reformulation of the stopping problem (5.1), respectively the n–step stopping problem (5.2) for the special choice of exponential utility as utility function. Hence we suppose that the underlying utility function U is a classical one regarding Definition 4.1 and given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

The (classical) risk-sensitive stopping problem for continuous-time Markov chains is then given by

$$\mathbb{E}_x\Big[- e^{c\gamma\tau - \gamma g(X_\tau)} \Big] \to \max_{\tau \in \Sigma_x}! \tag{5.9}$$

Analogously the classical risk-sensitive n–step stopping problem for continuous-time Markov chains can be expressed as

$$\mathbb{E}_x\Big[- e^{c\gamma(\tau\wedge S_n) - \gamma g(X_{\tau\wedge S_n})} \Big] \to \max_{\tau \in \Sigma_x}! \tag{5.10}$$

This does not look special at first glance, but we will see in the subsequent chapters that problems (5.9) and (5.10) are much easier to solve than their general versions (5.1) and (5.2).

6 Value Functions

In this chapter we will introduce the concept of so-called value functions which are closely connected to the stopping problems (5.1) and (5.2). We will see that studying these value functions will help us in our attempt to characterize or even solve the n–step stopping problem (5.2). On the other hand, the knowledge about (5.2) will help to tackle the unrestricted problem (5.1).

Section 6.1 of this chapter will cover the general theory for value functions under arbitrary utility functions. This will be the main part of this chapter and will lay the foundation for the theory in the subsequent chapters. In section 6.2 we will cover the special case where the stopping problem is formulated for the choice of exponential utility as utility function. We will see that this will lead to a considerable simplification of the theory made in the first section.

6.1 Value Functions for General Utility Functions

For the remainder of this section we will assume the setup made for the generalized risk-sensitive stopping problem for continuous-time Markov chains as in section 5.1.

Definition 6.1 (value functions)

Considering the generalized risk-sensitive stopping problem for continuous-time Markov chains (5.1), we define

(a) the *value function* $V(\,\cdot\,,\tau)\colon [0,\infty)\times S\to[-\infty,\infty)$ *for stopping time* $\tau\in\Sigma$ by

$$V(t,x,\tau) := \mathbb{E}_x\left[U\left(-ct-c\tau+g(X_\tau)\right)\right],$$

(b) the *value function* $V\colon [0,\infty)\times S\to[-\infty,\infty)$ by

$$V(t,x) := \sup_{\tau\in\Sigma_x} V(t,x,\tau).$$

For $(t,x)\in[0,\infty)\times S$ the value $V(t,x,\tau)$ of the value function $V(\cdot,\tau)$ expresses the expected utility of the "reward" $-ct-c\tau+g(X_\tau)$ the investor gains, if he applies stopping time $\tau\in\Sigma$. $V(t,x)$ therefore expresses the maximal expected utility the investor can achieve by optimizing over all feasible stopping times in Σ_x. Thus, for any initial value

$x \in S$ the value $V(0,x)$ of the value function $V(\cdot)$ represents the generalized risk-sensitive stopping problem for continuous-time Markov chains (5.1). $V(0,x)$ constitutes the optimal attainable expected utility of the problem (5.1), whereas the stopping time $\tau^\star \in \Sigma_x$, for which the supremum in $V(0,x)$ (if existent) is attained at, depicts the optimal stopping time and thus the solution to the stopping problem (5.1). For an arbitrary $t \geq 0$, the main difference of $V(t,x)$ to the stopping problem (5.1) is the additional cost term $-ct$. This can be interpreted in the way that our stopping problem is already running for a period of t time units without being stopped and has thus acummulated the additional costs of ct. After time t the investor "starts" to observe the problem and treats it as if it was started anew at time t. Hence, the initial value of the Markov chain X is set to $X_0 = x$. Stopping the problem at time $s \geq 0$ after "resetting" it yields a reward of $-ct - cs + g(X_s)$, where the costs are composed of the cumulated costs with respect to the elapsed times t and s before, respectively after resetting. In this sense $V(t,x)$ poses for $t > 0$ an slightly different stopping problem than (5.1), which is – in the above-mentioned sense – equivalent to $V(0,x)$. We will see that in order to calculate $V(0,x)$ we also have to know every other value $V(t,x)$ for all $t \geq 0$. Thus, the goal will be to find a useful characterization for the whole value function which will allow us to calculate its values for some $(t,x) \in [0,\infty) \times S$ in a more effective way than maximizing over the set of feasible stopping times.

The term *solving* $V(t,x)$ is thereby defined as the attempt to find the optimal stopping time $\tau^\star \in \Sigma_x$ for which the supremum in $V(t,x)$ is attained at. We will see that in order to solve (5.1), respectively $V(0,x)$ for any $x \in S$, we will need to solve $V(t,x)$ for every $(t,x) \in [0,\infty) \times S$.

Definition 6.2 (n–step value functions)

Let $n \in \mathbb{N}_0$. Considering the generalized risk-sensitive n–step stopping problem for continuous-time Markov chains (5.2), we define

(a) the *n-step value function $V_n(\,\cdot\,,\tau)\colon [0,\infty) \times S \to [-\infty,\infty)$ for stopping time $\tau \in \Sigma$* by

$$V_n(t,x,\tau) := \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau \wedge S_n}) \right) \right],$$

(b) the *n-step value function $V_n\colon [0,\infty) \times S \to [-\infty,\infty)$* by

$$V_n(t,x) := \sup_{\tau \in \Sigma_x} V_n(t,x,\tau).$$

Again, for every $x \in S$ and $n \in \mathbb{N}_0$ the value $V_n(0,x)$ of the n–step value function $V_n(\cdot)$ is equivalent to the generalized risk-sensitive n–step stopping problem (5.2). The interpretation above about the additional term $-ct$ remains analogously valid for n–step value functions. Additionally, for solving $V_n(0,x)$ and thus problem (5.2) for any $x \in S$ and $n \in \mathbb{N}_0$, we will need to solve $V_n(t,x)$ for every $(t,x) \in [0,\infty) \times S$. Thus we are again interested in the whole value function, rather than just some single values of it.

Remark 6.3 (notation and existence of value functions)

(a) With a slight abuse of notation, we will call both functions $(t,x) \mapsto V(t,x,\tau)$ for $\tau \in \Sigma$ and $(t,x) \mapsto V(t,x)$ value functions. Analogously, we call $(t,x) \mapsto V_n(t,x,\tau)$ for $\tau \in \Sigma$ and $(t,x) \mapsto V_n(t,x)$ n–step value functions for every $n \in \mathbb{N}_0$. It will be clear from the corresponding contexts which functions are meant.

(b) Under the integrability condition (5.4) of Assumption 5.5

$$M_x := \sup_{\tau \in \Sigma_x} \mathbb{E}_x\Big[- c\tau + g^+(X_\tau)\Big] < \infty \quad \text{for every } x \in S$$

Lemma 5.6 guarantees the well-posedness of the stopping problems. Due to

$$\sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau \wedge S_n}) \right) \right] \le \sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-c(\tau \wedge S_n) + g(X_{\tau \wedge S_n}) \right) \right],$$

$$\sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-ct - c\tau + g(X_\tau) \right) \right] \le \sup_{\tau \in \Sigma_x} \mathbb{E}_x \left[U\left(-c\tau + g(X_\tau) \right) \right] \tag{6.1}$$

and using (5.5), we can find constants $m > 0$ and $d \in \mathbb{R}$ such that

$$V_n(t,x) \le mM_x + d < \infty \quad \text{and} \tag{6.2}$$

$$V(t,x) \le mM_x + d < \infty. \tag{6.3}$$

Both value functions take finite values for every $(t,x) \in [0,\infty) \times S$ and thus are well-defined. In this sense, the existence of the value functions is always guaranteed under Assumption 5.5.

(c) The 0–step value functions are trivially given by

$$\begin{aligned} V_0(t,x,\tau) &= \mathbb{E}_x \left[U(-ct - c(\tau \wedge S_0) + g(X_{\tau \wedge S_0})) \right] = \mathbb{E}_x \left[U(-ct + g(X_0)) \right] \\ &= U(-ct + g(x)) \quad \text{and} \end{aligned} \tag{6.4}$$

$$V_0(t,x) = \sup_{\tau \in \Sigma_x} V_0(t,x,\tau) = U(-ct + g(x)) \tag{6.5}$$

for every $\tau \in \Sigma$, $t \ge 0$ and $x \in S$, since $S_0 = 0$.

(d) Note that we explicitly allow for a value function $V(\cdot\ ,\ \tau)$ or an n–step value function $V_n(\cdot\ ,\ \tau)$ for any $n \in \mathbb{N}$ and some $\tau \in \Sigma$ to take an arbitrarily small negative value. Thus the value range $[-\infty, \infty)$ for the value functions is intentional. It symbolizes the worst expected utility an investor can possibly achieve by applying the stopping time τ.

(e) For every $n \in \mathbb{N}_0$, $t \ge 0$ and $x \in S$ we get a lower estimate for $V_n(t,x)$ and $V(t,x)$ by using the special constant stopping time $\tau = 0$:

$$V_n(t,x) \ge V_n(t,x,0) = \mathbb{E}_x \left[U(-ct + g(X_0)) \right] = U(-ct + g(x)),$$

$$V(t,x) \ge V(t,x,0) = \mathbb{E}_x \left[U(-ct + g(X_0)) \right] = U(-ct + g(x)).$$

Hence, $V_n(t,x)$ and $V(t,x)$ are bounded from below by $U(-ct+g(x))$ and are only allowed to take arbitrarily small values if the associated utility function is set to $-\infty$ for these fixed $t \geq 0$ and $x \in S$. The difference to $V_n(t,x,\tau)$ or $V(t,x,\tau)$ lies in the fact that the expectation operator is not responsible for $V_n(t,x)$ or $V(t,x)$ to take the value $-\infty$.

(f) Note that U is a measurable function. Therefore, the mappings $t \mapsto V(t,x,\tau)$ and $t \mapsto V_n(t,x,\tau)$ are also measurable for every $x \in S$, $\tau \in \Sigma_x$ and $n \in \mathbb{N}_0$.

(g) Another property that emerges immediately from the definition of value functions is that for all $n \in \mathbb{N}_0$, $x \in S$ and $\tau \in \Sigma$

$$t \mapsto V_n(t,x,\tau), \quad t \mapsto V_n(t,x),$$

$$t \mapsto V(t,x,\tau), \quad t \mapsto V(t,x)$$

are decreasing functions.

Example 6.4 (example for explicitly calculable value functions)

Let $S = \{0,1\}$ and X a continuous-time Markov chain with intensity matrix Q given by

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

for some $\alpha, \beta > 0$ (cf. Example 2.14) and initial value $X_0 = x \in S$. Furthermore, let $U\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto -e^{-\gamma x}$ be an exponential utility function with $\gamma > 0$, $c > 0$ the cost rate and $g\colon S \to \mathbb{R}$ the reward function in this setting.

Consider now the stopping time $\tau := S_2 \in \Sigma$, where S_n denotes the n–th random jump time of X. Then the n–step value functions $V_n(t,x,\tau)$ for $n \in \mathbb{N}_0$ as well as the value function $V(t,x,\tau)$ are given by

$$V_0(t,x,\tau) = -e^{\gamma ct - \gamma g(x)},$$

$$V_1(t,x,\tau) = \begin{cases} -e^{\gamma ct - \gamma g(1)} \cdot \frac{\alpha}{\alpha - \gamma c}, & \text{for } x = 0 \text{ and } \alpha > \gamma c, \\ -e^{\gamma ct - \gamma g(0)} \cdot \frac{\beta}{\beta - \gamma c}, & \text{for } x = 1 \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise} \end{cases}$$

$$\begin{aligned} V_n(t,x,\tau) &= V(t,x,\tau) \\ &= \begin{cases} -e^{\gamma ct - \gamma g(x)} \cdot \frac{\alpha}{\alpha - \gamma c} \cdot \frac{\beta}{\beta - \gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise} \end{cases} \end{aligned}$$

for $n \geq 2$.

Proof
The corresponding embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ of X is given by the deterministic sequence

$$Z_n = \frac{1}{2}\Big(1 - (-1)^{n+x}\Big),$$

which alternates between the states 0 and 1, depending on the initial value $X_0 = Z_0 = x$. $(Z_n)_{n\in\mathbb{N}_0}$ is deterministic, since the transition probabilities for $(Z_n)_{n\in\mathbb{N}_0}$ are given by

$$p_{00} = p_{11} = 0, p_{01} = \frac{q_{01}}{-q_{00}} = 1 \text{ and } p_{10} = \frac{q_{10}}{-q_{11}} = 1.$$

The Markov chain thus changes continuously between the two states 0 and 1 using random exponentially distributed sojourn times between these jumps. To be more precise, the sojourn times $(T_n)_{n\in\mathbb{N}}$ are (given $(Z_n)_{n\in\mathbb{N}_0}$) independent and

$$T_n \sim Exp(q_{Z_{n-1}}).$$

This information can now be used to calculate the desired value functions:

The 0–step value function can be obtained very easy. It holds

$$V_0(t,x,\tau) = \mathbb{E}_x\left[U\left(-ct - c(\tau\wedge S_0) + g(X_{\tau\wedge S_0})\right)\right] = U\left(-ct + g(x)\right) = -e^{\gamma ct - \gamma g(x)}.$$

For the other value functions, we will differentiate between the two possible initial values $x = 0$ and $x = 1$.

(i) Let $x = 0$. The embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ is then given by the sequence

$$(Z_0, Z_1, Z_2, Z_3, \dots) = (0, 1, 0, 1\dots)$$

and the first two sojourn times are independent and fulfill

$$T_1 \sim Exp(q_{Z_0}) = Exp(q_0) = Exp(\alpha) \quad \text{and} \quad T_2 \sim Exp(q_{Z_1}) = Exp(q_1) = Exp(\beta).$$

Calculating the distribution function for $e^{\gamma c T_1}$ yields

$$\mathbb{P}_0(e^{\gamma c T_1} \le t) = \begin{cases} 1 - t^{-\frac{\alpha}{\gamma c}}, & t \ge 1, \\ 0, & t < 1 \end{cases}$$

and leads to the expectation

$$\begin{aligned} \mathbb{E}_0\left(e^{\gamma c T_1}\right) &= \int_1^\infty \frac{\alpha}{\gamma c}\cdot t^{-\frac{\alpha}{\gamma c}-1}\cdot t\, dt = \frac{\alpha}{\gamma c}\cdot\int_1^\infty t^{-\frac{\alpha}{\gamma c}} dt \\ &= \begin{cases} -\frac{\alpha}{\gamma c}\cdot\frac{1}{-\frac{\alpha}{\gamma c}+1}, & \text{for } \alpha > \gamma c, \\ \infty, & \text{for } \alpha \le \gamma c \end{cases} \\ &= \begin{cases} \frac{\alpha}{\alpha - \gamma c}, & \text{for } \alpha > \gamma c, \\ \infty, & \text{for } \alpha \le \gamma c. \end{cases} \end{aligned}$$

Analogously we get

$$\mathbb{E}_0\left(e^{-cT_2}\right) = \begin{cases} \frac{\beta}{\beta-\gamma c}, & \text{for } \beta > \gamma c, \\ \infty, & \text{for } \beta \leq \gamma c. \end{cases}$$

Using the indepenence of T_1 and T_2 as well as the relations $S_1 = T_1$ and $S_2 = T_1 + T_2$ yields

$$\mathbb{E}_0\left(e^{\gamma c S_1}\right) = \begin{cases} \frac{\alpha}{\alpha-\gamma c}, & \text{for } \alpha > \gamma c, \\ \infty, & \text{for } \alpha \leq \gamma c \end{cases} \quad \text{and}$$

$$\begin{aligned} \mathbb{E}_0\left(e^{\gamma c S_2}\right) &= \mathbb{E}_0\left(e^{\gamma c T_1}\right) \cdot \mathbb{E}_0\left(e^{\gamma c T_2}\right) \\ &= \begin{cases} \frac{\alpha}{\alpha-\gamma c} \cdot \frac{\beta}{\beta-\gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

We can now calculate the 1–step value function:

$$\begin{aligned} V_1(t,0,\tau) &= \mathbb{E}_0\left[U\left(-ct - c(\tau \wedge S_1) + g(X_{\tau \wedge S_1})\right)\right] \\ &= \mathbb{E}_0\left[U\left(-ct - cS_1 + g(Z_1)\right)\right] \\ &= \mathbb{E}_0\left[-e^{-\gamma(-ct+g(1)-cS_1)}\right] \\ &= -e^{\gamma ct - \gamma g(1)} \cdot \mathbb{E}_0\left(e^{\gamma c S_1}\right) \\ &= \begin{cases} -e^{\gamma ct-\gamma g(1)} \cdot \frac{\alpha}{\alpha-\gamma c}, & \text{for } \alpha > \gamma c, \\ -\infty, & \text{for } \alpha \leq \gamma c. \end{cases} \end{aligned}$$

Since $\tau = S_2 = \tau \wedge S_2 = \tau \wedge S_n$ for all $n \geq 2$, all n–step value functions $V_n(t,0,\tau)$ for $n \geq 2$ and the value function $V(t,0,\tau)$ are equal. We get

$$\begin{aligned} V_2(t,0,\tau) &= V_n(t,0,\tau) = V(t,0,\tau) \\ &= \mathbb{E}_0\left[U\left(-ct - cS_2 + g(Z_2)\right)\right] \\ &= -e^{\gamma ct-\gamma g(0)} \cdot \mathbb{E}_0\left(e^{\gamma c S_2}\right) \\ &= \begin{cases} -e^{\gamma ct-\gamma g(0)} \cdot \frac{\alpha}{\alpha-\gamma c} \cdot \frac{\beta}{\beta-\gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise} \end{cases} \end{aligned}$$

for every $n \geq 2$.

(ii) Now let $x = 1$. The embedded Markov chain $(Z_n)_{n \in \mathbb{N}_0}$ is then given by the sequence

$$(Z_0, Z_1, Z_2, Z_3, \dots) = (1, 0, 1, 0 \dots)$$

and the first two sojourn times fulfill

$$T_1 \sim Exp(\beta) \quad \text{and} \quad T_2 \sim Exp(\alpha).$$

The calculations above can be used analogously and yield

$$\begin{aligned}
\mathbb{E}_1\left(e^{\gamma c S_1}\right) &= \begin{cases} \frac{\beta}{\beta-\gamma c}, & \text{for } \beta > \gamma c, \\ \infty, & \text{for } \beta \leq \gamma c \end{cases} \qquad \text{and} \\
\mathbb{E}_1\left(e^{\gamma c S_2}\right) &= \mathbb{E}_1\left(e^{\gamma c T_1}\right) \cdot \mathbb{E}_1\left(e^{\gamma c T_2}\right) \\
&= \begin{cases} \frac{\alpha}{\alpha-\gamma c} \cdot \frac{\beta}{\beta-\gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ \infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

Using $g(X_{S_1}) = g(Z_1) = g(0)$ and $g(X_{S_2}) = g(Z_2) = g(1)$ thus leads to

$$\begin{aligned}
V_1(t,1,\tau) &= \mathbb{E}_1\left[U\left(-ct - c(\tau \wedge S_1) + g(X_{\tau \wedge S_1})\right)\right] \\
&= \begin{cases} -e^{\gamma ct - \gamma g(0)} \cdot \frac{\beta}{\beta-\gamma c}, & \text{for } \beta > \gamma c, \\ -\infty, & \text{for } \beta \leq \gamma c \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
V_n(t,1,\tau) &= V(t,1,\tau) \\
&= \mathbb{E}_1\left[U\left(-ct - cS_2 + g(Z_2)\right)\right] \\
&= \begin{cases} -e^{\gamma ct - \gamma g(1)} \cdot \frac{\alpha}{\alpha-\gamma c} \cdot \frac{\beta}{\beta-\gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}$$

for every $n \geq 2$.

□

Proposition 6.5 (properties of value functions)

(a) By definition we get

$$\begin{aligned}
V(t,x,\tau) &\leq V(t,x) \quad \text{for all } t \geq 0, x \in S, \tau \in \Sigma_x, \\
V_n(t,x,\tau) &\leq V_n(t,x) \quad \text{for all } t \geq 0, x \in S, \tau \in \Sigma_x, n \in \mathbb{N}_0.
\end{aligned}$$

(b) Using the reasoning in Remark 5.4 and the alternative formulation (5.3) of the n–step stopping problem (5.1), the n–step value function V_n can be expressed as

$$V_n(t,x) = \sup_{\tau \in \Sigma_{n,x}} \mathbb{E}_x\left[U\left(-ct - c\tau + g(X_\tau)\right)\right] \tag{6.6}$$

(c) An immediate consequence of (6.1) is

$$V_n(t,x) \le V(t,x) \quad \text{for all } n \in \mathbb{N}_0, t \ge 0, x \in S. \tag{6.7}$$

Any n–step value function V_n is always dominated by the value function V. Thus, the optimal expected utility derived from the n–step stopping problem (5.2) is always smaller than the optimal expected utility derived from the stopping problem (5.1). This is intuitively clear, since the n–step stopping problem (5.2) terminates at the latest after n changes of the state of the Markov chain and does not permit the possibility to observe the system indefinitely.

(d) The monotonicity (3.28) of the set of feasible stopping times

$$\Sigma_{n,x} \subseteq \Sigma_{n+1,x} \quad \text{for all } n \in \mathbb{N}_0$$

coupled with the alternative representation (6.6) for the n–step value function yields the monotonicity property

$$V_n(t,x) \le V_{n+1}(t,x) \quad \text{for all } n \in \mathbb{N}_0, t \ge 0, x \in S. \tag{6.8}$$

Remark 6.6 (no monotonicity of value functions for fixed stopping times)

The monotonicity property (6.8) is generally not valid a fixed stopping time $\tau \in \Sigma$, as the following example illustrates:

Consider the situation in Example 6.4, where we were able to explicitly calculate the n–step value functions for the stopping time $\tau = S_2 \in \Sigma$. The 1–step and 2–step value functions are given by

$$V_1(t,x,\tau) = \begin{cases} -e^{\gamma ct - \gamma g(1)} \cdot \frac{\alpha}{\alpha - \gamma c}, & \text{for } x = 0 \text{ and } \alpha > \gamma c, \\ -e^{\gamma ct - \gamma g(0)} \cdot \frac{\beta}{\beta - \gamma c}, & \text{for } x = 1 \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise} \end{cases}$$

$$V_2(t,x,\tau) = \begin{cases} -e^{\gamma ct - \gamma g(x)} \cdot \frac{\alpha}{\alpha - \gamma c} \cdot \frac{\beta}{\beta - \gamma c}, & \text{for } \alpha > \gamma c \text{ and } \beta > \gamma c, \\ -\infty, & \text{otherwise.} \end{cases}$$

It is immediately evident that for $\alpha, \beta, \gamma > 0$ such that $\alpha > \gamma c \ge \beta$, $t \ge 0$ and $x = 0$ we get

$$V_1(t,0,\tau) = -e^{\gamma ct - \gamma g(1)} \cdot \frac{\alpha}{\alpha - \gamma c} > -\infty = V_2(t,0,\tau).$$

But even by excluding the cases of infinitely negative value functions the monotonicity property (6.8) could be violated:

Let now $t \geq 0$, $x = 0$ and $\alpha, \beta, \gamma > 0$ such that $\alpha > \gamma c$ and $\beta > \gamma c$. We get

$$\begin{aligned}
& V_1(t,0,\tau) > V_2(t,0,\tau) \\
\Longleftrightarrow & -e^{\gamma ct - \gamma g(1)} \cdot \frac{\alpha}{\alpha - \gamma c} > -e^{\gamma ct - \gamma g(0)} \cdot \frac{\alpha}{\alpha - \gamma c} \cdot \frac{\beta}{\beta - \gamma c} \\
& e^{-\gamma g(1)} < e^{-\gamma g(0)} \cdot \frac{\beta}{\beta - \gamma c} \\
\Longleftrightarrow & g(1) - g(0) > \frac{\ln\left(\frac{\beta}{\beta - \gamma c}\right)}{-\gamma} \\
\Longleftrightarrow & g(1) > g(0) - \frac{\ln\left(\frac{\beta}{\beta - \gamma c}\right)}{\gamma}.
\end{aligned}$$

$V_1(t,0,\tau)$ sybmolizes the expected utility an investor gains by applying stopping time $\tau = S_2$, if he is forced to stop before τ, namely immediately after the first jump of X from state $X_0 = 0$ into state 1. The reward which influences this expected utility is $g(1)$. One can just as well interpret $V_1(t,0,\tau)$ as the expected utility an investor gains by applying stopping time S_1 deliberately, since $V_1(t,0,S_2) = V(t,0,S_1)$ holds. On the other hand, we compare this value with $V_2(t,0,S_2) = V(t,0,S_2)$, the expected utility an investor gains by applying $\tau = S_2$. The inequality above states that the decision whether to stop after the first change of state or after the second (given that the initial value was $x = 0$), depends on the exact reward values $g(0)$ and $g(1)$ the investor could gain. If $g(1)$ is greater than $g(0)$ (minus some correction term dependent on the model data β, γ and c), then there is no incentive to wait for the second jump of the Markov chain back into state 0.

Proposition 6.7 (relation between value functions and n–step value functions, interchangeability of limits)

(a) For every $t \geq 0$, $x \in S$ and $\tau \in \Sigma$ the inequality

$$V(t,x,\tau) \leq \liminf_{n\to\infty} V_n(t,x,\tau) \tag{6.9}$$

holds.

(b) For every $t \geq 0$ and $x \in S$, the limit

$$V_\infty(t,x) := \lim_{n\to\infty} V_n(t,x) = \lim_{n\to\infty} \sup_{\tau\in\Sigma_x} \mathbb{E}_x \left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right)\right] \tag{6.10}$$

exists.

(c) The limit and the supremum in (6.10) are interchangeable and yield

$$V_\infty(t,x) = \lim_{n\to\infty} V_n(t,x) = V(t,x). \tag{6.11}$$

Proof of Proposition 6.7

(a) Let $t \geq 0$ and $x \in S$, $\tau \in \Sigma$. The desired inequality

$$V(t,x,\tau) \leq \liminf_{n\to\infty} V_n(t,x,\tau).$$

is just a reformulation of Assumption 5.8.

(b) For every $t \geq 0$ and $x \in S$, $(V_n(t,x))_{n\in\mathbb{N}_0}$ is an increasing sequence, dominated by $V(t,x) < \infty$ (according to (6.8) and (6.7) in Proposition 6.5). Hence, the limit $\lim_{n\to\infty} V_n(t,x)$ exists.

(c) Let $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$ be fixed. Using the alternative representation Equation (6.6) of n–step value functions in Proposition 6.5 and the monotonicity property $\Sigma_{n,x} \subseteq \Sigma_x$ in Lemma 3.20, we get

$$\begin{aligned} V_n(t,x) &= \sup_{\tau\in\Sigma_x} \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right)\right] \\ &= \sup_{\tau\in\Sigma_{n,x}} \mathbb{E}_x\left[U\left(-ct - c\tau + g(X_\tau)\right)\right] \\ &\leq \sup_{\tau\in\Sigma_x} \mathbb{E}_x\left[U\left(-ct - c\tau + g(X_\tau)\right)\right] \\ &= V(t,x) \end{aligned}$$

and thus

$$V_\infty(t,x) = \lim_{n\to\infty} V_n(t,x) \leq V(t,x).$$

On the other hand we know that

$$V_n(t,x) \geq V_n(t,x,\tau)$$

for every $\tau \in \Sigma$ and hence

$$V_\infty(t,x) = \lim_{n\to\infty} V_n(t,x) = \liminf_{n\to\infty} V_n(t,x) \geq \liminf_{n\to\infty} V_n(t,x,\tau) \overset{(a)}{\geq} V(t,x,\tau)$$

for every $\tau \in \Sigma_x$. This leads to

$$V_\infty(t,x) \geq \sup_{\tau\in\Sigma_x} V(t,x,\tau) = V(t,x)$$

and ultimately

$$V_\infty(t,x) = V(t,x).$$

□

Remark 6.8 (finiteness of optimal stopping times)

As mentioned in Remark 5.2 (e), the supremum of the value function $V(t,x,\tau)$ over all $\tau \in \Sigma_x$ does neither have to be a real maximum nor does it have to be attained by a $\mathbb{P}_x$–almost surely finite stopping time.

On the other hand, the case for n–step value functions is a different one. We will show that for every non-finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time for which the supremum in $V_n(t,x)$ is attained at, there exists another $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time which is $\mathbb{P}_x$–almost surely finite and yields the same value as the supremum. To this end, let $n \in \mathbb{N}_0$, $t \geq 0$, $x \in S$ and τ be a (possibly infinite) $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time such that

$$V_n(t,x,\tau) = V_n(t,x).$$

Since

$$V_n(t,x,\tau) = \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_n) + g(X_{\tau\wedge S_n})\right)\right]$$

utilizes the term $\tilde{\tau} := \tau \wedge S_n \in \Sigma_{n,x}$, we can replace τ by $\tilde{\tau}$ knowing that $\tau \wedge S_n = \tilde{\tau} \wedge S_n$ and thus

$$V_n(t,x,\tilde{\tau}) = V_n(t,x,\tau) = V_n(t,x).$$

Because $\tilde{\tau} \in \Sigma_{n,x}$ is a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time, the assertion is shown.

6.2 Value Functions for Exponential Utility Functions

We will now consider value functions for the special choice of exponential utility as utility function. Again, we suppose that the underlying utility function U is given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

We can easily see that for given $t \geq 0$, $x \in S$ and $\tau \in \Sigma_x$, the value functions $V(t,x,\tau)$ and $V(t,x)$ are given by

$$V(t,x,\tau) = \mathbb{E}_x\Big[-e^{c\gamma t + c\gamma\tau - \gamma g(X_\tau)}\Big] = e^{c\gamma t} \cdot \mathbb{E}_x\Big[-e^{c\gamma\tau - \gamma g(X_\tau)}\Big]$$

and

$$V(t,x) = \sup_{\tau\in\Sigma_x} \mathbb{E}_x\Big[-e^{c\gamma t + c\gamma\tau - \gamma g(X_\tau)}\Big] = e^{c\gamma t} \sup_{\tau\in\Sigma_x} \mathbb{E}_x\Big[-e^{c\gamma\tau - \gamma g(X_\tau)}\Big].$$

Due to the multiplicative structure of U, the cost term $-ct$ can be isolated from the remaining parameters, the expectation and even the supremum operator. Looking at the value function $V(t,x)$, we can see that in this particular case the time parameter $t \geq 0$ does not have any influence on the optimization over feasible stopping times. Thus we can suspect that the solution of this optimization (if existent) has to be independent of t. As an obvious consequence we can reduce the value functions to

$$\tilde{V}(x,\tau) := \mathbb{E}_x\Big[-e^{c\gamma\tau-\gamma g(X_\tau)}\Big] \tag{6.12}$$

and

$$\tilde{V}(x) := \sup_{\tau\in\Sigma_x} \mathbb{E}_x\Big[-e^{c\gamma\tau-\gamma g(X_\tau)}\Big]. \tag{6.13}$$

Note that in oder to get the original value functions, we need the relations

$$V(t,x,\tau) = e^{c\gamma t}\tilde{V}(x,\tau) \quad \text{and} \quad V(t,x) = e^{c\gamma t}\tilde{V}(x). \tag{6.14}$$

Analogously to the case of unrestricted value functions we can also simplify the n–step value functions $V_n(t,x,\tau)$ and $V_n(t,x)$ for every $n \in \mathbb{N}_0$, $t \geq 0$, $x \in S$ and $\tau \in \Sigma_x$ by defining

$$\tilde{V}_n(x,\tau) := \mathbb{E}_x\Big[-e^{c\gamma(\tau\wedge S_n)-\gamma g(X_{\tau\wedge S_n})}\Big], \tag{6.15}$$

respectively

$$\tilde{V}_n(x) := \sup_{\tau\in\Sigma_x} \mathbb{E}_x\Big[-e^{c\gamma(\tau\wedge S_n)-\gamma g(X_{\tau\wedge S_n})}\Big] \tag{6.16}$$

and noting the relation

$$V_n(t,x,\tau) = e^{c\gamma t}\tilde{V}_n(x,\tau) \quad \text{and} \quad V_n(t,x) = e^{c\gamma t}\tilde{V}_n(x). \tag{6.17}$$

As we have seen, the choice of exponential utility leads to a reduced value function in the sense that the optimization over all feasible stopping times $\tau \in \Sigma_x$ does not depend on the time parameter any more. The maximal expected utility gained by optimally terminating the stopping problem thus only depend on the initial value $x \in S$ of the underlying continuous-time Markov chain. This independence from the time parameter $t \geq 0$ is one of the main reasons why exponential utility is so popular for a lot of different optimization problems and especially stopping problems. Also note that only by neglecting an evaluation of the given reward under any utility function, which would be equivalent to taking U to be the identity, would allow us to get a similar situation where the optimization over all feasible stopping times is independent of the time parameter. But this mapping U would not fulfill the conditions of strict monotonicity and strict concavity as we required in Definition 4.1 and is therefore not a feasible utility function. Hence the exponential utility is the only classical utility function which exhibits such a property.

7 Discrete-Time Approach for the Generalized Risk-Sensitive Stopping Problem for Continuous-Time Markov Chains

7.1 Introduction

This chapter forms the main part of this thesis. Here, we want to establish a discrete-time approach in order to tackle the unrestricted stopping problem (5.1), respectively its corresponding unrestricted value function from Definition 6.1. To this end we want to utilize the n–step stopping problems introduced in (5.2), respectively their corresponding n–step value functions from Definition 6.2. We will proceed in several steps:

In section 7.2, we address to n–step value functions for a given stopping time $\tau \in \Sigma_x$ and initial value $x \in S$. We will be able to establish the so-called reward iteration formula, which will allow for an recursive approach to solve an n–step value function $V_n(t, x, \tau)$ for $n \in \mathbb{N}_0$, $t \geq 0$, $x \in S$ and $\tau \in \Sigma_x$ by applying the knowledge about the preceding $(n-1)$–step value function for the given $\tau \in \Sigma_x$. Section 7.3 will compare the case of general utility with the case of exponential utility.

In section 7.4 we will omit the need of a fixed stopping time $\tau \in \Sigma_x$ for initial value $x \in S$. This will also lead to an iteration type equation, the so-called Bellman equation. This makes it possible for $n \in \mathbb{N}$, $t \geq 0$ and $x \in S$ to recursively calculate every n–step value function $V_n(t, x)$ from the previous one, without having to fix a particular stopping time. Furthermore we will see that calculating one step of this Bellman equation will involve the solving of a deterministic one-dimensional optimization problem. This solution on the other hand will yield a mapping which fulfills the requirements of Theorem 3.9. Solving n steps of this Bellman equation will therefore yield a piecewise description of an $(\mathcal{F}_t^{n,X})_{t\geq 0}$–stopping time. As it turns out, this stopping time will be the solution to the n–step stopping problem and the corresponding n–step value function. As we already know from Proposition 6.7, knowing the sequence of n–step value functions allows for an arbitrarily good approximation of the unrestricted one. This legitimates the name *discrete-time approach* of this chapter, as we can interpret $V(t, x)$ to be the limit of a discrete sequence of functions, which can be attained in an interative way. Following this, section 7.5 will again treat the special case of exponential utility.

In section 7.6 we will transform the aforementioned Bellman equation into a fixed-point equation. Instead of calculating every n–step value function recursively and considering the limit of these functions to gain the unrestricted value function, we will show that this value function can also be obtained as a solution to the fixed-point equation. Again, solving this equation will require to solve a deterministic optimization problem. This solution will provide us with a mapping which can be used to construct an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time using Proposition 3.15. This yields a candidate for the optimal stopping time for the unrestricted stopping problem. We will see in section 7.7, that under certain conditions this candidate is indeed optimal and thus provides us with the solution to the generalized risk-sensitive stopping problem for continuous-time Markov chains. Finalizing this chapter, section 7.8 will discuss the particular choice of exponential utility. We will see that the fixed-point equation from the previous section will simplify in the sense that the corresponding deterministic optimization problem degenerates to the problem of choosing the greater of two values. The optimal stopping time for this specific utility function will only be able to stop at the jump times of the underlying Markov chain. More precisely, at every jump time, the optimal stopping time will either stop immediately after attaining the new state or will never stop as long as the Markov chain remains in this state.

7.2 Reward Iteration

In the previous chapter the concept of value functions and n–step value functions was introduced and their meaning in context of the stopping problems (5.1) and (5.2) was discussed. Furthermore a lot of connections between the various kinds of value functions were established. In particular, Proposition 6.7 stated the convergence

$$V_n(t,x) \to V(t,x) \quad \text{as } n \to \infty$$

for every $t \geq 0$ and $x \in S$. Additionally, the 0–step value function can always be trivially calculated by

$$V_0(t,x) = U(-ct + g(x))$$

for every $t \geq 0$ and $x \in S$, as shown in Remark 6.3 (c).

Thus, a feasible approach for obtaining the value function $V(t,x)$ would be to calculate all n–step value functions $V_n(t,x)$ and use them to gain $V(t,x)$ as the corresponding limit for $n \to \infty$. And in order to gain the n–step value functions $V_n(t,x)$, we could try to calculate $V_n(t,x,\tau)$ for every given $\tau \in \Sigma_x$. A possible course of action would be the attempt to establish a recursive formula in order to calculate the desired functions iteratively. We will show that this idea is viable and formulate such a formula, the so-called reward iteration. To this end we will excessively utilize the special structure of feasible $\mathbb{P}_x$–almost surely finite stopping times in Σ_x by using the characterization of $(\mathcal{F}_t^X)_{t\geq 0}$–stopping times given in section 3.2. As a remainder, Proposition 3.15 stated that for every $x \in S$ a mapping $\tau\colon \Omega \to [0,\infty]$ such that $\mathbb{P}_x(\tau < \infty) = 1$ is an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time, if and only if it has the decomposition

$$\tau = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{\infty} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.}$$

or in short

$$\tau = (\tau^0, \tau^1, \tau^2, \dots),$$

respectively

$$\tau = (h_0, h_1, h_2, \dots),$$

where for every $k \in \mathbb{N}_0$:

(i) $\tau^k \geq S_k$,

(ii) τ^k is an $(\mathcal{F}_t^{k,X})_{t\geq 0}$–stopping time,

(iii) There exists a measurable mapping $h_k\colon [0,\infty)^k \times S^{k+1} \to [0,\infty]$, such that $h_k \geq 0$ and

$$\tau^k = h_k(S_1, \dots, S_k, Z_0, Z_1, \dots, Z_k) + S_k.$$

An important fact to be reminded of, as stated in Remark 3.12 (b), is that for $x \in S$ and every $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time τ the corresponding stopping rule τ^0 is characterized by

$$\tau^0 = h_0(Z_0) = h_0(x)$$

for a measurable mapping $h_0\colon S \to [0,\infty]$ and is thus deterministically given.

In order to establish the desired reward iteration formula for n–step value functions as mentioned above, we will need a new concept of shifted stopping times, which will be introduced in the following definition. The basic underlying idea will be for $t \geq 0$ and a non-absorbing state $x \in S$ to look at an n–step value function $V_n(t, x, \tau)$ for an arbitrary $\tau \in \Sigma_x$ and at the corresponding expected utility an investor would gain by utilizing this particular stopping time. Now fix such a $\tau = (\tau^0, \tau^1, \dots) \in \Sigma_x$ and suppose that the investor decides not to stop before the first jump time S_1 of the Markov chain X and resolves to ignore τ (and thus in particular τ^0) until this first jump time. After the jump however, he wants to behave according to τ (using the remaining stopping rules $\tau^1, \tau^2, \dots$). This situation can be interpreted as restarting the Markov chain after S_1 anew. Under this condition of not stopping before the first jump, the investor will then know at which concrete time $S_1 = s \geq 0$ this first jump occured and into which state the Markov chain changed after the jump from the initial state x. This state will mark the new initial state $j \in S\backslash\{x\}$ of the Markov chain after the "restart". The remaining stopping rules will then form a new stopping time – the *shifted stopping time* – conditioned on the informations the investor gained up to time s. This shall be defined rigoroulsy in the subsequent definitions.

Definition 7.1 (restarted continuous-time Markov chain)

Let X be a continuous-time Markov chain with jump times $(S_n)_{n\in\mathbb{N}_0}$, embedded Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ and initial value $X_0 = x \in S$.

(a) Suppose that the time S_1 of the first jump of X as well as the state Z_1 of the Markov chain after the jump are known:

$$S_1 = s \geq 0 \quad \text{and} \quad Z_1 = j \in S.$$

Then we define the *restarted Markov chain* $(\tilde{X}_t)_{t\geq 0}$ by

$$\tilde{X}_t := X_{s+t} \quad \text{for all } t \geq 0 \quad \text{and } \tilde{X}_0 := X_s = j. \tag{7.1}$$

(b) Define the sequence of *shifted jump times* $(\tilde{S}_n)_{n\in\mathbb{N}_0}$ of $(S_n)_{n\in\mathbb{N}_0}$, respectively the *shifted embedded Markov chain* $(\tilde{Z}_n)_{n\in\mathbb{N}_0}$ of $(Z_n)_{n\in\mathbb{N}_0}$ by

$$\tilde{S}_n := S_{n+1}, \quad \text{respectively} \tag{7.2}$$

$$\tilde{Z}_n := Z_{n+1}. \tag{7.3}$$

Note that by Assumption 2.18, we defined the restarted Markov chain only for initial values $x \in S$ such that $q_x > 0$. In other words, these states x are not absorbing. In this case we can expect the next jump to occur in a finite time interval. A definition of restarted Markov chains for absorbing initial states are not meaningful, as the Markov chain never leaves such states.

Lemma 7.2 (properties of restarted Markov chains)

(a) Since we restricted ourselves to the case of homogeneous continuous-time Markov chains we can conclude (cf. Definition 2.1) that for every $s \geq 0$, $n \in \mathbb{N}$, $h > 0$, each $0 \leq t_0 < t_1 < \ldots < t_n$ and every $x_0, \ldots, x_n, x_{n+1} \in S$, such that

$$\mathbb{P}\left(\tilde{X}_{t_k} = x_k,\, 0 \leq k \leq n\right) > 0,$$

the Markov property (2.1)

$$\begin{aligned}
\mathbb{P}\left(\tilde{X}_{t_n+h} = x_{n+1} | \tilde{X}_{t_k} = x_k,\, 0 \leq k \leq n\right) &= \mathbb{P}\left(X_{s+t_n+h} = x_{n+1} | X_{s+t_k} = x_k,\, 0 \leq k \leq n\right) \\
&= \mathbb{P}\left(X_{t_n+h} = x_{n+1} | X_{t_k} = x_k,\, 0 \leq k \leq n\right) \\
&= \mathbb{P}\left(X_{t_n+h} = x_{n+1} | X_{t_n} = x_n\right) \\
&= \mathbb{P}\left(X_{s+t_n+h} = x_{n+1} | X_{s+t_n} = x_n\right) \\
&= \mathbb{P}\left(\tilde{X}_{t_n+h} = x_{n+1} | \tilde{X}_{t_n} = x_n\right)
\end{aligned}$$

holds. Thus, the restarted Markov chain $\tilde{X}$ is indeed a homogeneous continuous-time Markov chain and possesses (up to a different initial value) the same distribution as X.

(b) Let $n \in \mathbb{N}_0$, $s \geq 0$, $x \in S$ such that $q_x > 0$ and $j \in S \backslash \{x\}$. Using the restarting result above we then get

$$\begin{aligned}
\mathbb{P}_x(\tilde{S}_n \leq t | S_1 = s, X_{S_1} = j) &= \mathbb{P}_x(S_{n+1} \leq t | S_1 = s, X_{S_1} = j) \\
&= \mathbb{P}_x(S_{n+1} \leq t | S_1 = s, X_s = j) \\
&= \mathbb{P}_x(S_{n+1} \leq t | S_1 = s, \tilde{X}_0 = j) \\
&= \mathbb{P}(S_n + s \leq t | X_0 = j) \\
&= \mathbb{P}_j(S_n + s \leq t)
\end{aligned}$$

and thus

$$\tilde{S}_n\,{}_{|\tilde{S}_0 = s, \tilde{Z}_0 = j} = \tilde{S}_n\,{}_{|S_1 = s, Z_1 = j} \overset{\mathcal{D}}{=} S_n + s\,{}_{|Z_0 = j}. \tag{7.4}$$

(c) Similarly, for each $n \in \mathbb{N}_0$, $x \in S$ such that $q_x > 0$, $j \in S \backslash \{j\}$ and $l \in S$ we get

$$\begin{aligned}
\mathbb{P}_x(\tilde{Z}_n = l | \tilde{Z}_0 = j) &= \mathbb{P}_x(Z_{n+1} = l | Z_1 = j) \\
&= \mathbb{P}(Z_{n+1} = l | Z_1 = j) \\
&= \mathbb{P}(Z_n = l | Z_0 = j) \\
&= \mathbb{P}_j(Z_n = l)
\end{aligned}$$

and thus

$$\tilde{Z}_n\,{}_{|\tilde{Z}_0 = j} = \tilde{Z}_n\,{}_{|Z_1 = j} \overset{\mathcal{D}}{=} Z_n\,{}_{|Z_0 = j}. \tag{7.5}$$

Definition 7.3 (shifted stopping times)

Let $x \in S$, $s \geq 0$ and $\tau = (\tau^0, \tau^1, \tau^2, \dots) = (h_0, h_1, h_2, \dots) \in \Sigma_x$ such that

$$\tau^k = h_k(S_1, \dots, S_k, x, Z_1, \dots, Z_k) + S_k.$$

Then define the *shifted stopping time* $\overrightarrow{\tau}_{s,x}$ of τ by

$$\overrightarrow{\tau}_{s,x} := (\tilde{\tau}^0, \tilde{\tau}^1, \tilde{\tau}^2, \dots) = (\tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \dots), \tag{7.6}$$

where

$$\tilde{h}_k \colon [0,\infty)^k \times S^{k+1} \to [0,\infty],$$
$$\tilde{h}_k(s, x;\ \tilde{s}_1, \dots, \tilde{s}_k, \tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_k) := h_{k+1}(s, \tilde{s}_1, \dots, \tilde{s}_k, x, \tilde{z}_0, \tilde{z}_1, \dots, \tilde{z}_k) \tag{7.7}$$

and

$$\begin{aligned} \tilde{\tau}^k &:= \tilde{h}_k(s, x;\ \tilde{S}_1, \dots, \tilde{S}_k, \tilde{Z}_0, \tilde{Z}_1, \dots, \tilde{Z}_k) + \tilde{S}_k - s & (7.8)\\ &= h_{k+1}(s, \tilde{S}_1, \dots, \tilde{S}_k, x, \tilde{Z}_0, \tilde{Z}_1, \dots, \tilde{Z}_k) + \tilde{S}_k - s \\ &= h_{k+1}(s, S_2, \dots, S_{k+1}, x, Z_1, Z_2, \dots, Z_{k+1}) + S_{k+1} - s & (7.9) \end{aligned}$$

for every $k \in \mathbb{N}_0$.

Proposition 7.4 (shifted stopping times are well-defined)

Let $s \geq 0$, $x \in S$, $j \in S \backslash \{x\}$ and $\tau \in \Sigma_x$. Assume again that

$$S_1 = s \quad \text{and} \quad Z_1 = j.$$

Then the shifted stopping time $\overrightarrow{\tau}_{s,x}$ of τ is a $\mathbb{P}_j$–almost surely finite stopping time with respect to the natural filtration of the restarted Markov chain $(\tilde{X}_t)_{t\geq 0}$ as introduced in Definition 7.1, equation (7.1).

Remark 7.5 (interpretation of shifted stopping times)

Note that the mappings $\tilde{h}_k$ defined in (7.7) are again non-negative and that the shifted stopping rules $\tilde{\tau}^k$ defined in (7.8) yield $\tilde{\tau}^k \geq S_{k+1} - s$ for every $k \in \mathbb{N}_0$. Assuming that the first jump time of a Markov chain X occurs at $S_1 = s \geq 0$ we get $\tilde{\tau}^0 \geq 0$ and $\tilde{\tau}^k \geq S_{k+1} - S_1$ for every $k \in \mathbb{N}$.

Thus, under the assumption that X changes its state for the first time at $S_1 = s \geq 0$ from $Z_0 = x \in S$, such that $q_x > 0$, into $Z_1 = j \in S \backslash \{x\}$ and under the condition that a stopping time $\tau \in \Sigma_x$ did not stop before this first jump occured, $\overrightarrow{\tau}_{s,x}$ denotes the stopping time which behaves exactly like τ, beginning after the first jump time $S_1 = s$ and knowing that $Z_1 = j$.

This makes it possible to observe a stopping time and to "restart" it in the sense above, if it didn't stop until the first change of state of the underlying Markov chain X.

Also note that the concept of shifted stopping times was defined for $x \in S$ such that $q_x > 0$, according to Assumption 2.18. Analogously to the definition of restarted Markov chains we can guarantee that the shift occurs at a $\mathbb{P}_x$–almost surely finite time S_1.

Lemma 7.6 (properties of shifted stopping times)
Let $x \in S$, $j \in S\backslash\{x\}$, $\tau \in \Sigma_x$, $s \geq 0$ and $n \in \mathbb{N}_0$. Assume that $S_1 = s$, $Z_1 = j$ and that τ did not stop before S_1. Then the following statements about shifted stopping times can be made:

(a) According to (7.9) we get

$$\tilde{\tau}^k + s = h_{k+1}(s, S_2, \ldots, S_{k+1}, x, Z_1, Z_2, \ldots, Z_{k+1}) + S_{k+1} = \tau^{k+1}{}_{|S_1=s}.$$

As a consequence and as stated in Remark 7.5 about the interpretation of shifted stopping times, such a $\vec{\tau}_{s,x}$ represents the stopping time τ itself ("restarted" after the first jump time) under the additional condition that this first jump time and the subsequent state of the Markov chain are known and τ did not stop before this change occured. This event can be characterized by the set $\{S_1 < \tau\} = \{S_1 < \tau^0\}$ and leads to

$$\mathbb{P}_x\Big(\tau \cdot \mathbb{1}_{\{S_1 \leq \tau^0\}} \leq t \;\Big|\; S_1 = s, Z_1 = j\Big) = \mathbb{P}_j\Big((\vec{\tau}_{s,x} + s) \cdot \mathbb{1}_{\{s \leq \tau^0\}} \leq t\Big) \qquad (7.10)$$

for all $t \geq 0$.

(b) Combining (7.4) with (7.10) immediately yields

$$\begin{aligned} &\mathbb{P}_x\Big((\tau \wedge S_{n+1}) \cdot \mathbb{1}_{\{S_1 \leq \tau^0\}} \leq t \;\Big|\; S_1 = s, Z_1 = j\Big) \\ &= \mathbb{P}_j\Big((\vec{\tau}_{s,x} \wedge S_n + s) \cdot \mathbb{1}_{\{s \leq \tau^0\}} \leq t\Big) \end{aligned} \qquad (7.11)$$

for all $t \geq 0$.

(c) Using the right-continuity of X and the definition of the restarted Markov chain $\tilde{X}$ with $\tilde{X}_t = X_{s+t}$ for $S_1 = s$ and every $t \geq 0$ leads for all $y \in \mathbb{R}$ to

$$\begin{aligned} &\mathbb{P}_x\Big(g(X_{\tau \wedge S_{n+1}}) \cdot \mathbb{1}_{\{S_1 \leq \tau^0\}} = y \;\Big|\; S_1 = s, Z_1 = j\Big) \\ &= \mathbb{P}_j\Big(g(X_{(\vec{\tau}_{s,x} \wedge S_n)+s}) \cdot \mathbb{1}_{\{s \leq \tau^0\}} = y\Big) \\ &= \mathbb{P}_j\Big(g(\tilde{X}_{\vec{\tau}_{s,x} \wedge S_n}) \cdot \mathbb{1}_{\{s \leq \tau^0\}} = y\Big) \\ &= \mathbb{P}_j\Big(g(X_{\vec{\tau}_{s,x} \wedge S_n}) \cdot \mathbb{1}_{\{s \leq \tau^0\}} = y\Big). \end{aligned} \qquad (7.12)$$

Now we gathered the necessary tools to formulate the reward iteration formula mentioned at the beginning of this section, whose proof and interpretation will utilize the above-mentioned restarting techniques.

Theorem 7.7 (reward iteration)

Let $x \in S$, $t \geq 0$ and $\tau \in \Sigma_x$. Then the following *reward iteration* formula holds:

$$V_{n+1}(t,x,\tau) = U\big(-ct - c\tau^0 + g(x)\big)\cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j, \overrightarrow{\tau}_{s,x})\, ds \quad (7.13)$$

for every $n \in \mathbb{N}_0$, where the initial value function V_0 is given by

$$V_0(t,x,\tau) = U\big(-ct + g(x)\big). \quad (7.14)$$

Remark 7.8 (interpretation of Theorem 7.7)

Theorem 7.7 states that every consecutive n–step value function for given $x \in S$, $t \geq 0$ and stopping time $\tau \in \Sigma_x$ can be calculated using its predecessor in an appropriate way. Clearly, the 0–step value function corresponds to the case of immediate stopping, independent of the choice of the stopping time. The 0–step value function thus coincides with the utility gained by receiving a reward $g(x)$ based on the initial value of the underlying Markov chain and paying the inevitable costs $-ct$.

The two summands in the reward iteration formula itself, on the other hand, can be interpreted based on the observation whether the considered stopping time τ does stop before the first jump time S_1 of the Markov chain or not. This will also be the main idea in the proof of this theorem.

The first summand contributes to the value function $V_{n+1}(t,x,\tau)$ for stopping time τ with the utility an investor would gain, if the corresponding stopping problem stops before the first change of state of the underlying Markov chain, weighted by the probability of this event to happen. This event to occur means for the stopping time τ to behave according to the stopping rule τ^0, which is a deterministic stopping time in the sense that it only depends on the initial value of the Markov chain by a suitable mapping h_0 such that $\tau^0 = h_0(x)$.

The second summand contributes to $V_{n+1}(t,x,\tau)$ with the expected utility the investor would gain, conditioned on the event that the corresponding stopping problem does not terminate before the first jump time of the Markov chain. The following auxiliary lemma will show that this expectation is given in terms of the integral expression of the reward iteration formula. This integral requires the preceding n–step value function V_n, evaluated for the corresponding shifted stopping time $\overrightarrow{\tau}_{s,x}$ of τ for an appropriate choice $s \geq 0$. This originates from the idea that conditioned on the knowledge that τ did not stop before the first jump time S_1, the underlying Markov chain and the whole stopping problem can be restarted after S_1. Since τ did not stop beforehand, the actual jump time as well as the new state of the Markov chain can be observed. Knowing this, the stopping problem can be continued after the reset by applying $\vec{\tau}_{s,x}$, which exactly represents τ under this additional information, given that the jump time $S_1 = s$ is known. Since the $(n+1)$–step value function $V_{n+1}(t, x, \tau)$ corresponds to a stopping problem which allows at the most for $n+1$ jumps of the Markov chain, restarting the stopping problem exactly after the first jump results in at most n remaining jumps that still can be considered. Thus after restarting, the shifted stopping time has to be applied for an appropriate n–step value function V_n, where the term "appropriate" refers to the right choice of applied time and state for V_n. The integral and sum enveloping this n–step value function accommodate for these (weighted) choices of time and state, depending on the actual time at which the first change of state occured and which successive state the Markov chain took after the initial value x.

Now as stated in the remark above we will formulate the following auxiliary lemma in order to prove Theorem 7.7:

Lemma 7.9

Let $x \in S$, $t \geq 0$, $\tau \in \Sigma_x$ and $n \in \mathbb{N}_0$. Then the following statement holds:

$$\mathbb{E}_x\Big[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau \wedge S_{n+1}})\right) \mathbb{1}_{\{S_1 \leq \tau^0\}}\Big] = \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j, \vec{\tau}_{s,x})\, ds. \tag{7.15}$$

Proof of Lemma 7.9

Let $x \in S$, $\tau \in \Sigma_x$ and $n \in \mathbb{N}_0$ and recall decomposition (3.18)

$$\tau = \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \sum_{k=1}^{\infty} \tau^k \mathbb{1}_{\{S_k \leq \tau < S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.}$$

As stated in Remark 3.12 (b), the stopping rule $\tau^0 = h_0(Z_0) = h_0(x)$ is deterministically given by the initial value x of the Markov chain X. Furthermore, up to the first jump time S_1 of the Markov chain the stopping time τ is fully characterized by this deterministic stopping rule τ^0 and thus

$$\{\tau < S_1\} = \{\tau^0 < S_1\}.$$

Conditioned on the set $\{S_1 \leq \tau^0\} = \{S_1 \leq \tau\}$ we know that the stopping time τ will never trigger before S_1, guaranteeing at least one change of state of the underlying

Markov chain before τ stops. The probability for this event to happen is given by

$$\mathbb{P}_x(S_1 \le \tau) = \mathbb{P}_x(S_1 \le \tau^0) = 1 - \exp(-q_x \cdot \tau^0),$$

since $S_1 \sim Exp(q_x)$ according to Theorem 2.16.

Assume now that X changes its state for the first time at $S_1 = s$ for some $s \ge 0$ from $Z_0 = x$ into $Z_1 = j$ for some $j \in S \backslash \{x\}$ and assume additionally that τ does not stop before this first jump time. Appying Lemma 7.6 then leads to

$$\begin{aligned}
&\mathbb{E}_x \left[U\Big(-ct - c(\tau \wedge S_{n+1}) + g\left(X_{\tau \wedge S_{n+1}}\right) \Big) \mathbb{1}_{\{S_1 \le \tau^0\}} \Big| S_1 = s,\ Z_1 = j \right] \\
&= \mathbb{E}_j \left[U\Big(-ct - c(\vec{\tau}_{s,x} \wedge S_n + s) + g\left(X_{\vec{\tau}_{s,x} \wedge S_n}\right) \Big) \mathbb{1}_{\{s \le \tau^0\}} \right] \\
&= \mathbb{E}_j \left[U\Big(-c(t+s) - c(\vec{\tau}_{s,x} \wedge S_n) + g\left(X_{\vec{\tau}_{s,x} \wedge S_n}\right) \Big) \right] \mathbb{1}_{\{s \le \tau^0\}}.
\end{aligned}$$

Remembering the joint density of S_1 and Z_1

$$f_{S_1,Z_1}(s, j \mid X_0 = x) = \begin{cases} \exp(-q_x \cdot s) \cdot q_{xj}, & \text{if } x \ne j \text{ and } s \ge 0, \\ 0, & \text{otherwise} \end{cases}$$

as given in Corollary 2.20 (c) finally leads to

$$\begin{aligned}
&\mathbb{E}_x \Big[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau \wedge S_n}) \right) \mathbb{1}_{\{S_1 \le \tau^0\}} \Big] \\
&= \mathbb{E}_x \left[\mathbb{E}_x \left[U\Big(-ct - c(\tau \wedge S_{n+1}) + g\left(X_{\tau \wedge S_{n+1}}\right) \Big) \mathbb{1}_{\{S_1 \le \tau^0\}} \Big| S_1,\ Z_1 \right] \right] \\
&= \int_{-\infty}^{\infty} \sum_{j \in S} e^{-q_x \cdot s} \cdot q_{xj} \cdot \mathbb{1}_{\{j \ne q,\ s \ge 0\}} \\
&\qquad \cdot \mathbb{E}_x \left[U\Big(-ct - c(\tau \wedge S_{n+1}) + g\left(X_{\tau \wedge S_{n+1}}\right) \Big) \mathbb{1}_{\{S_1 \le \tau^0\}} \Big| S_1 = s,\ Z_1 = j \right] \, ds \\
&= \int_0^{\infty} e^{-q_x s} \sum_{\substack{j \in S, \\ j \ne x}} q_{xj} \cdot \mathbb{E}_j \left[U\Big(-c(t+s) - c(\vec{\tau}_{s,x} \wedge S_n) + g\left(X_{\vec{\tau}_{s,x} \wedge S_n}\right) \Big) \right] \mathbb{1}_{\{s \le \tau^0\}} \, ds \\
&= \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S, \\ j \ne x}} q_{xj} \cdot \mathbb{E}_j \left[U\Big(-c(t+s) - c(\vec{\tau}_{s,x} \wedge S_n) + g\left(X_{\vec{\tau}_{s,x} \wedge S_n}\right) \Big) \right] \, ds \\
&= \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S, \\ j \ne x}} q_{xj} \cdot V_n(t+s, j, \vec{\tau}_{s,x}) \, ds
\end{aligned}$$

and thus concludes the proof. □

We are now able to prove Theorem 7.7 rigorously, justifying the interpretation given in Remark 7.8.

Proof of Theorem 7.7
Let $x \in S$, $t \geq 0$, $\tau \in \Sigma_x$ and $n \in \mathbb{N}_0$. The initial equation for the 0–step value function stated in Theorem 7.7 are simply given by equation (6.4) from Remark 6.3 (c):

$$V_0(t,x,\tau) = U\big(-ct + g(x)\big).$$

As for the reward iteration formula itself, we will use the following reasoning:

The event that τ doesn't stop before the first jump of the underlying Markov chain is given by

$$\{S_1 > \tau\} = \{S_1 > \tau^0\}$$

for the deterministic stopping rule τ^0 of τ. The probability for this event to happen is given by

$$\mathbb{P}_x(S_1 > \tau) = \mathbb{P}_x(S_1 > \tau^0) = \exp(-q_x \cdot \tau^0),$$

since $S_1 \sim Exp(q_x)$ according to Theorem 2.16 and τ^0 being deterministic. Consequentially the event of τ stopping before the first jump time S_1 is

$$\{S_1 \leq \tau\} = \{S_1 \leq \tau^0\}.$$

The main idea to prove the desired reward iteration formula is to differentiate between these two events and to decide whether the stopping time did stop before S_1 or not. If τ did stop before S_1, the underlying Markov chain never changed its state and still remains in its initial state x. On the other hand, if τ did not stop before S_1, at least one change of state occured, enabling us to restart the Markov chain and to consider the corresponding shifted stopping time $\overrightarrow{\tau}_{S_1,x}$ after S_1. This leads to the following calculation, allowing the application of Lemma 7.9:

$$\begin{aligned}
V_{n+1}(t,x,\tau) &= \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau\wedge S_{n+1}})\right)\right] \\
&= \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau\wedge S_{n+1}})\right)\cdot \mathbb{1}_{\{S_1>\tau\}}\right] \\
&\quad + \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau\wedge S_{n+1}})\right)\cdot \mathbb{1}_{\{S_1\leq\tau\}}\right] \\
&= \mathbb{E}_x\left[U\left(-ct - c\tau^0 + g(X_{\tau^0})\right)\cdot \mathbb{1}_{\{S_1>\tau^0\}}\right] \\
&\quad + \mathbb{E}_x\left[U\left(-ct - c(\tau \wedge S_{n+1}) + g(X_{\tau\wedge S_{n+1}})\right)\cdot \mathbb{1}_{\{S_1\leq\tau^0\}}\right] \\
&= \mathbb{E}_x\left[U\left(-ct - c\tau^0 + g(X_0)\right)\cdot \mathbb{1}_{\{S_1>\tau^0\}}\right] \\
&\quad + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_n(t+s, j, \overrightarrow{\tau}_{s,x})\, ds \\
&= U\left(-ct - c\tau^0 + g(x)\right)\cdot \mathbb{P}_x\left(S_1 > \tau^0\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_n(t+s, j, \overrightarrow{\tau}_{s,x})\, ds \\
= \quad & U\left(-ct - c\tau^0 + g(x)\right) \cdot e^{-q_x \tau^0} \\
&+ \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_n(t+s, j, \overrightarrow{\tau}_{s,x})\, ds.
\end{aligned}
$$

This shows (7.13) and thus finalizes the proof. □

7.3 Reward Iteration for Exponential Utility Functions

We will now consider value functions for the special choice of exponential utility as utility function. Again, we suppose that the underlying utility function U is given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

As seen in section 6.2 of chapter 6, we can reduce the n–step value functions $V_n(t, x, \tau)$ for every $n \in t \geq 0$, $x \in s$ by ommiting the time parameter t and consider the new reduced n–step value functions $\tilde{V}_n(x, \tau)$ given in (6.15), which does not depend on time t anymore. The connection between $V_n(t, x, \tau)$ and $\tilde{V}(x, \tau)$ is given by (6.17):

$$V_n(t, x, \tau) = e^{c\gamma t} \tilde{V}_n(x, \tau).$$

This allows us to paraphrase Theorem 7.7 for this special choice of exponential utility:

Corollary 7.10 (reward iteration for exponential utility functions)

Let $x \in S$ and $\tau \in \Sigma_x$. Then the following *reward iteration* formula holds:

$$\tilde{V}_{n+1}(x, \tau) = -e^{(c\gamma - q_x)\tau^0 - \gamma g(x)} + \int_0^{\tau^0} e^{(c\gamma - q_x)s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot \tilde{V}_n(j, \overrightarrow{\tau}_{s,x})\, ds \tag{7.16}$$

for every $n \in \mathbb{N}_0$, where the initial value function $\tilde{V}_0$ is given by

$$\tilde{V}_0(x, \tau) = -e^{-\gamma g(x)}. \tag{7.17}$$

Proof of Corollary 7.10
Let $t \geq 0$, $x \in S$ and $\tau \in \Sigma_x$. The initial 0–step value function $V_0(t, x, \tau)$ is in case of exponential utility given by

$$V_0(t, x, \tau) = -e^{c\gamma t - \gamma g(x)} = -e^{-\gamma g(x)} \cdot e^{c\gamma t}.$$

Applying (6.17) therefore yields

$$\tilde{V}_0(x,\tau) = -e^{-\gamma g(x)}.$$

For the reward iteration formula (7.16) itself, we get for every $n \in \mathbb{N}_0$ by applying the original Theorem 7.7 and (6.17):

$$\begin{aligned}
&\tilde{V}_{n+1}(x,\tau) \\
&= e^{-c\gamma t} \cdot V_{n+1}(t,x,\tau) \\
&= e^{-c\gamma t} \cdot U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + e^{-c\gamma t} \cdot \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_n(t+s, j, \overrightarrow{\tau}_{s,x})\, ds \\
&= -e^{c\gamma\tau^0 - \gamma g(x)} \cdot e^{-q_x\cdot\tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot e^{c\gamma s} \cdot \tilde{V}_n(j, \overrightarrow{\tau}_{s,x})\, ds \\
&= -e^{(c\gamma - q_x)\tau^0 - \gamma g(x)} + \int_0^{\tau^0} e^{(c\gamma - q_x)s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot \tilde{V}_n(j, \overrightarrow{\tau}_{s,x})\, ds.
\end{aligned}$$

□

7.4 The Bellman Equation and Optimal Stopping Times for the n–Step Stopping Problem

Definition 7.11

Let $x \in S$, $d \in \mathbb{R}$ and $c > 0$. Then we can recursively define the sequence $\left(T_d^{(n)}(x)\right)_{n\in\mathbb{N}_0}$ by

$$T_d^{(0)}(x) := \min\left\{ \inf_{\substack{j\in S\setminus\{x\},\\ q_{xj}>0}} \frac{g(j)+d}{c}\,,\ \frac{g(x)+d}{c} \right\} \quad \text{and} \tag{7.18}$$

$$T_d^{(n)}(x) := \min\left\{ \inf_{\substack{j\in S\setminus\{x\},\\ q_{xj}>0}} T_d^{(n-1)}(j)\,,\ \frac{g(x)+d}{c} \right\}, \quad n \in \mathbb{N}. \tag{7.19}$$

Note that the reward function g is by definition lower bounded. Thus the infima in (7.18) and (7.19) exist.

Lemma 7.12

Let $x \in S$, $d \in \mathbb{R}$ and $c > 0$. Then the the sequence $\left(T_d^{(n)}(x)\right)_{n\in\mathbb{N}_0}$ can be expressed explicitly using n–step transition probabilities given in Definition 2.17:

$$T_d^{(n)}(x) = \inf_{j\in A^{(n)}(x)} \frac{g(j)+d}{c}, \quad n \in \mathbb{N}_0, \tag{7.20}$$

where

$$A^{(n)}(x) := \left\{ j \in S \,\middle|\, \exists k \in \{0, 1, \ldots, n+1\} : \; p_{xj}^{(k)} > 0 \right\}. \tag{7.21}$$

Proof of Lemma 7.12
Let $x \in S$, $d \in \mathbb{R}$ and $c > 0$. We will show Lemma 7.12 by induction. For $n = 0$ we note that since $q_x \neq 0$, we get $p_{xx} = p_{xx}^{(1)} = 0$. On the other hand, for $j \in S\backslash\{x\}$, the inequality $q_{xj} > 0$ is equivalent to $p_{xj} = p_{xj}^{(1)} > 0$. Additionally, the 0–step transition probability from x to $j \in S$ is given by $p_{xj}^{(0)} = \delta_{xj}$. This leads to

$$\begin{aligned}
T_d^{(0)}(x) &= \min\left\{ \inf_{\substack{j\in S\backslash\{x\},\\ q_{xj}>0}} \frac{g(j)+d}{c}, \frac{g(x)+d}{c} \right\} \\
&= \min\left\{ \inf_{\substack{j\in S,\\ p_{xj}^{(1)}>0}} \frac{g(j)+d}{c}, \inf_{\substack{j\in S,\\ p_{xj}^{(0)}>0}} \frac{g(j)+d}{c} \right\} \\
&= \inf_{\substack{j\in S,\\ p_{xj}^{(0)}>0 \text{ or } p_{xj}^{(1)}>0}} \frac{g(j)+d}{c} \\
&= \inf_{j\in A^{(0)}(x)} \frac{g(j)+d}{c}.
\end{aligned}$$

Now for the induction step, suppose that

$$T_d^{(n)}(x) = \inf_{j\in A^{(n)}(x)} \frac{g(j)+d}{c}$$

for some arbitrary but fixed $n \in \mathbb{N}_0$. Additionally note that for the n–step transition probabilities of the embedded Markov chain, the discrete-time version of the Chapman-

Kolmogorov equation holds and yields for every $x, l \in S$ and $k \in \mathbb{N}_0$:

$$\begin{aligned} p_{xl}^{(k+1)} &= \sum_{j \in S} p_{xj} p_{jl}^{(k)} \\ &= \sum_{j \in S \setminus \{x\}} p_{xj} p_{jl}^{(k)} \\ &= \sum_{\substack{j \in S \setminus \{x\}, \\ p_{xj} > 0, p_{jl}^{(k)} > 0}} p_{xj} p_{jl}^{(k)}. \end{aligned}$$

As a consequence we get $p_{xl}^{(k+1)} > 0$, if and only if there exists a state $j \in S \setminus \{x\}$ such that $p_{xj>0}$ and $p_{jl}^{(k)} > 0$. This leads together with the induction hypothesis to

$$\begin{aligned} &\inf_{\substack{j \in S \setminus \{x\}, \\ q_{xj} > 0}} T_d^{(n)}(j) \\ &= \inf_{\substack{j \in S \setminus \{x\}, \\ p_{xj} > 0}} \inf_{l \in A^{(n)}(j)} \frac{g(l) + d}{c} \\ &= \inf_{\substack{j \in S \setminus \{x\}, \\ p_{xj} > 0}} \inf_{\substack{l \in S, \\ \exists k \in \{0,1,\dots,n+1\} \\ \text{s.t. } p_{jl}^{(k)} > 0}} \frac{g(l) + d}{c} \\ &= \inf_{\substack{l \in S, \\ \exists k \in \{0,1,\dots,n+1\} \\ \text{s.t. } p_{xl}^{(k+1)} > 0}} \frac{g(l) + d}{c} \\ &= \inf_{\substack{l \in S, \\ \exists k \in \{1,\dots,n+2\} \\ \text{s.t. } p_{xl}^{(k)} > 0}} \frac{g(l) + d}{c}. \end{aligned}$$

Furthermore due to $p_{xl}^{(0)} = \delta_{xl}$, we have again

$$\frac{g(x) + d}{c} = \inf_{\substack{l \in S, \\ p_{xl}^{(0)} > 0}} \frac{g(l) + d}{c}.$$

Therefore we get

$$T_d^{(n+1)}(x) = \min \left\{ \inf_{\substack{j \in S \setminus \{x\}, \\ q_{xj} > 0}} T_d^{(n)}(j) \ , \ \frac{g(x) + d}{c} \right\}$$

$$= \min \left\{ \inf_{\substack{l \in S, \\ \exists k \in \{1,\dots,n+2\} \\ \text{s.t. } p_{xl}^{(k)} > 0}} \frac{g(l)+d}{c} \; , \; \inf_{\substack{j \in S, \\ p_{xj}^{(0)} > 0}} \frac{g(j)+d}{c} \right\}$$

$$= \inf_{\substack{l \in S, \\ \exists k \in \{0,1,\dots,n+2\} \\ \text{s.t. } p_{xl}^{(k)} > 0}} \frac{g(l)+d}{c}$$

$$= \inf_{l \in A^{(n+1)}(x)} \frac{g(l)+d}{c}.$$

This concludes the induction step and thus finalizes the proof. □

Lemma 7.13

Let $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$. Then it holds:

(a) The mapping $m_{n,t,x} \colon [0,\infty) \to \mathbb{R} \cup \{-\infty\}$, defined by

$$m_{n,t,x}(\vartheta) := U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \quad (7.22)$$

is bounded from above by a constant, which only depends on x.

(b) Suppose that the associated utility function U is a classical utility function on the whole real line and does not take the value $-\infty$. Then it holds:

(i) If the mapping $t \mapsto V_n(t,j)$ is measurable for every $j \in S$, then $m_{n,t,x}$ is continuous and almost everywhere differentiable on $[0,\infty)$ such that

$$m'_{n,t,x}(\vartheta) \quad (7.23)$$
$$= \Big(\sum_{\substack{j \in S, \\ j \neq x}} q_{xj} V_n(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(x)\big) - q_x U\big(-ct - c\vartheta + g(x)\big) \Big)$$
$$\cdot\, e^{-q_x \vartheta}$$

almost everywhere. Note that $t \mapsto V_0(t,j) = U(-ct + g(j))$ is always measurable by definition of U.

(ii) Suppose that $t \mapsto V_n(t,j)$ is measurable for every $j \in S$. Then the supremum

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

is either a maximum and is attained by a finite maximizer $\vartheta_n^\star$ or is an unattainable (finite) supremum, in which case the maximizer is set as $\vartheta_n^\star = \infty$.

Furthermore, there exists a measurable mapping $f_n^\star\colon [0,\infty)\times S \to [0,\infty]$ such that $\vartheta_n^\star = f_n^\star(t,x)$.

Moreover, the mapping

$$[0,\infty) \to \mathbb{R}\cup\{-\infty\}, \quad t \mapsto \sup_{\vartheta\geq 0} m_{n,t,x}(\vartheta)$$

is measurable for every $x\in S$.

(c) Suppose now that the associated utility function U is an extended utility function, derived from a classical utility function with maximal domain of the form $[-d,\infty) \subsetneq \mathbb{R}$. Then it holds:

(i) For every $t\geq 0$ and $x\in S$ the mapping $\vartheta \mapsto m_{0,t,x}(\vartheta)$ is continuous on $\big[0, T_d^{(0)}(j) - t\big]$, differentiable on $\big[0, T_d^{(0)}(x)-t\big)$ and

$$m_{0,t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in \big[0, T_d^{(0)}(x)-t\big], \\ = -\infty, & \text{if } \vartheta \in \big(T_d^{(0)}(x)-t, \infty\big). \end{cases} \tag{7.24}$$

On $\big[0, T_d^{(0)}(x)-t\big)$ the derivative $m'_{0,t,x}$ is given by (7.23). In case that $t > T_d^{(0)}(x)$ and thus $\big[0, T_d^{(0)}(x)-t\big] = \emptyset$, the mapping $\vartheta\mapsto m_{0,t,x}(\vartheta)$ is constantly $-\infty$.

(ii) Let $t\geq 0$, $x\in S$ and $n\in\mathbb{N}$. If the mapping $t\mapsto V_n(t,j)$ is continuous on $\big[0, T_d^{(n-1)}(j)\big]$ for every $j\in S$ and

$$V_n(t,j) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n-1)}(j)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n-1)}(j), \infty\big) \end{cases} \tag{7.25}$$

for every $j\in S$, then $\vartheta\mapsto m_{n,t,x}(\vartheta)$ is continuous on $\big[0, T_d^{(n)}(x)\big]$, differentiable on $\big[0, T_d^{(n)}(x)-t\big)$ and

$$m_{n,t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in \big[0, T_d^{(n)}(x)-t\big], \\ = -\infty, & \text{if } \vartheta \in \big(T_d^{(n)}(x)-t, \infty\big). \end{cases} \tag{7.26}$$

On $\big[0, T_d^{(n)}(x)-t\big)$ the derivative $m'_{n,t,x}$ is given by (7.23). In case that $t > T_d^{(n)}(x)$ and thus $\big[0, T_d^{(n)}(x)-t\big] = \emptyset$, the mapping $\vartheta\mapsto m_{n,t,x}(\vartheta)$ is constantly $-\infty$.

(iii) Let $t\geq 0$, $x\in S$ and $n\in\mathbb{N}_0$ and let assumption (7.25) be fulfilled. We then get

$$\sup_{\vartheta\geq 0} m_{n,t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n)}(x)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n)}(x), \infty\big). \end{cases} \tag{7.27}$$

In case that $T_d^{(n)}(x) < 0$ and therefore $\big[0, T_d^{(n)}(x)\big] = \emptyset$, we get

$$(t,x) \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) \equiv -\infty.$$

Furthermore we get for every $t \in \big[0, T_d^{(n)}(x)\big]$ that

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = \max_{\vartheta \in \left[0, T_d^{(n)}(x) - t\right]} m_{n,t,x}(\vartheta). \tag{7.28}$$

The supremum is always attained by a finite maximizer $\vartheta_n^\star \in \big[0, T_d^{(n)}(x)\big]$, which depends on the actual choice of $x \in S$ and $t \in \big[0, T_d^{(n)}(x)\big]$.

Note that for $t > T_d^{(n)}(x)$, $m_{n,t,x}$ is constantly given by $-\infty$. In this case every $\vartheta \geq 0$ is a maximizer of $\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$. We will choose the smallest one and set $\vartheta_n^\star := 0$.

(iv) Let $x \in S$, $n \in \mathbb{N}_0$ and let the assumption (7.25) of (ii) be fulfilled. Then the mapping

$$[0,\infty) \to \mathbb{R} \cup \{-\infty\}, \quad t \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

is continuous on $\big[0, T_d^{(n)}(x)\big]$.

Furthermore, there exists a measurable mapping $f_n^\star \colon [0,\infty) \times S \to \mathbb{R}$ such that $\vartheta_n^\star = f_n^\star(t,x)$.

If the maximizer $\vartheta_n^\star$ for $\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$ is unique for $t \in \big[0, T_d^{(n)}(x)\big]$, then $t \mapsto f_n^\star(t,x)$ is even continuous.

Proof of Lemma 7.13

(a) For all $n \in \mathbb{N}_0$ and $t \geq 0$ the value function $V_n(t,x)$ is bounded from above by some constant M_x depending only on $x \in S$, as stated in Remark 6.3 (b). Hence, the following calculation will show that $m_{n,t,x}$ is also bounded from above in ϑ, where the upper bound only depends on the initial state $x \in S$, but not on $n \in \mathbb{N}_0$ or $t \geq 0$. To this end, note that by Definition 6.2 there exists for every $n \in \mathbb{N}_0$, $t \geq 0$, $x \in S$ and $\varepsilon > 0$ a stopping time $\tau^\varepsilon \in \Sigma_x$, such that

$$V_n(t,x) \leq V_n(t,x,\tau^\varepsilon) + \varepsilon.$$

Now let $n \in \mathbb{N}_0$, $t \geq 0$, $x \in S$, $\vartheta \geq 0$ and $\varepsilon > 0$. By setting $\tau^0 :\equiv \vartheta$ and interpreting τ^ε as the shifted stopping time $\overrightarrow{\tau}_{s,x}$ (cf. Definition 7.3) of some stopping time $\tau \in \Sigma_x$, such that

$$\tau := \tau^0 \mathbb{1}_{\{\tau < S_1\}} + \big(\tau^\varepsilon + S_1\big)\mathbb{1}_{\{\tau \geq S_1\}},$$

we thus can apply the reward iteration formula Equation (7.13) from Theorem 7.7 and hence conclude that

$$\begin{aligned}
m_{n,t,x}(\vartheta) &= U\big(-ct - c\vartheta + g(x)\big)\cdot e^{-q_x\cdot\vartheta} + \int_0^\vartheta e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n(t+s,j)\,ds \\
&\leq U\big(-ct - c\tau^0 + g(x)\big)\cdot e^{-q_x\cdot\tau^0} + \int_0^{\tau^0} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\Big(V_n(t+s,j,\tau^\varepsilon)+\varepsilon\Big)\,ds \\
&= V_{n+1}(t,x,\tau) + \int_0^\vartheta e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot\varepsilon\,ds \\
&\leq M_x + \int_0^\vartheta e^{-q_x s}\,ds \sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot\varepsilon \\
&= M_x + \sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot\left(-\frac{1}{q_x}\right)\big(e^{-q_x\cdot\vartheta}-1\big)\cdot\varepsilon \\
&= M_x + \sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x}\cdot\big(1-e^{-q_x\cdot\vartheta}\big)\cdot\varepsilon \\
&\leq M_x + \varepsilon < \infty.
\end{aligned}$$

Since $\vartheta \geq 0$ was arbitrary, we get the desired upper boundedness for $m_{n,t,x}$.

Let $\vartheta \geq 0$ and define the mapping $f\colon [0,\vartheta] \to \mathbb{R}$,

$$f(s) := e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n(t+s,j).$$

We will now show the Lebesgue integrability of f on $[0,\vartheta]$. To this end we denote the positive part of a function f by $f^+ := \max\{f,0\} \geq 0$ and the negative part by $f^- := \max\{-f,0\} \geq 0$. Therefore it holds that

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^- = f + 2f^-.$$

Using the monotonicity of the utility function U leads to

$$U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x\vartheta} \leq U\big(g(x)\big) =: B_x$$

and thus

$$\int_0^{\vartheta} f(s)\, ds \leq |M_x| + |B_x| =: C_x < \infty$$

for all $\vartheta \geq 0$.

Note that $m_{n,t,x}$ has not to be bounded from below. But since $V_n(t,x) \geq V_n(t,x,0)$ for the special stopping time $\tau = 0$, we can conclude that

$$V_n(t+s,j) \geq V_n(t+s,j,0) = U\big(-c(t+s) + g(j)\big)$$

and thus

$$f(s) \geq e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot U\big(-c(t+s) + g(j)\big) =: h(s)$$

holds for every $s \geq 0$. We do not know whether f is a continuous mapping or not, but h on the other hand is continuous on $[0,\infty)$ and thus integrable on the compact interval $[0,\vartheta]$. This leads to

$$\int_0^{\vartheta} f^-(s)\, ds \leq \int_0^{\vartheta} h^-(s)\, ds < \infty.$$

In summary, we get

$$\int_0^{\vartheta} |f(s)|\, ds = \int_0^{\vartheta} f(s)\, ds + 2\int_0^{\vartheta} f^-(s)\, ds < \infty$$

and thus the Lebesgue integrability of f on $[0,\vartheta]$.

(b) Let U be a classical utility function on the whole real line.

(i) By assumption, $s \mapsto V_n(t+s,j)$ for $j \in S$ and thus $s \mapsto f(s)$ are measurable mappings. Since f is Lebesgue integrable on $[0,\vartheta]$, the Lebesgue differentiation theorem (cf. [Elstrodt, 1996, Theorem 4.14]) yields the absolute continuity and thus the almost everywhere differentiability of $\vartheta \mapsto \int_0^{\vartheta} f(s)\, ds$ such that

$$\frac{\partial}{\partial\vartheta} \int_0^{\vartheta} f(s)\, ds = f(\vartheta).$$

This yields together with the differentiability of the utility function U the continuity and almost everywhere differentiability of $m_{n,t,x}$ and

$$
\begin{aligned}
&m'_{n,t,x}(\vartheta) \\
&= \frac{\partial}{\partial \vartheta}\Big(U\big(-ct - c\vartheta + g(x)\big)\cdot e^{-q_x\vartheta}\Big) + f(\vartheta) \\
&= -cU'\big(-ct - c\vartheta + g(x)\big)\cdot e^{-q_x\vartheta} - q_x U\big(-ct - c\vartheta + g(x)\big)\cdot e^{-q_x\vartheta} \\
&\quad + e^{-q_x\vartheta}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n(t+\vartheta, j) \\
&= \Bigg(\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}V_n(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(x)\big) - q_x U\big(-ct - c\vartheta + g(x)\big)\Bigg) \\
&\quad \cdot e^{-q_x\vartheta}
\end{aligned}
$$

almost everywhere.

(ii) For every $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$ we know that $m_{n,t,x}$ is an upper bounded mapping on $[0,\infty)$ according to part (a). Hence, the supremum

$$\sup_{\vartheta\geq 0} m_{n,t,x}(\vartheta)$$

exists and takes a finite value for every $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$. Since $m_{n,t,x}$ is continuous on $[0,\infty)$, we can differentiate between two possible cases:

- The supremum is attained within a compact interval. It is thus a maximum and is attained at a finite value $0 \leq \vartheta_n^\star < \infty$. Note that this maximum does not have to be unique, but we can guarantee its existence.
- The supremum is not attained within a compact interval and can thus not be attained by any finite argument. We set the corresponding maximizer $\vartheta_n^\star = \infty$ and say that the supremum is attained at infinity, extending the positive real line to $[0,\infty]$.

In both cases this supremum depends explicitly on the choice of $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$. Thus we can interpret the maximizer $\vartheta_n^\star$ as the function value of some mapping $[0,\infty)\times S \to [0,\infty]$ such that $(t,x) \mapsto \vartheta_n^\star$. To see that this mapping can be chosen in a measurable way, we will distinguish between two different situations:

Define the set

$$I := \Big\{(t,x) \in [0,\infty)\times S \,\Big|\, \exists \vartheta_n^\star \in [0,\infty) \text{ such that } \sup_{\vartheta\geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(\vartheta_n^\star)\Big\}.$$

I contains all pairs $(t,x) \in [0,\infty) \times S$ such that the supremum

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

is attained at a finite value $0 \leq \vartheta_n^\star < \infty$. Then a measurable selection theorem from [Brown & Purves, 1973, Corollary 1] yields that I is a measurable set and that there exists a measurable mapping $\tilde{f}_n^\star \colon [0,\infty) \times S \to [0,\infty)$ such that $\tilde{f}_n^\star(t,x) = \vartheta_n^\star$ for $(t,x) \in I$ and

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(\tilde{f}_n^\star(t,x)) \quad \text{for all } (t,x) \in I.$$

Now denote the set of pairs $(t,x) \in [0,\infty) \times S$ such that the supremum

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

is not attained by any finite argument by

$$\begin{aligned} I^c &:= \big([0,\infty) \times S\big) \backslash I \\ &= \Big\{ (t,x) \in [0,\infty) \times S \Big| \nexists \vartheta_n^\star \in [0,\infty) \text{ such that } \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(\vartheta_n^\star) \Big\}. \end{aligned}$$

Clearly the set I^c is measurable, since I was also measurable. In this case of the supremum being unattainable, we have set the corresponding maximizer to $\vartheta_n^\star = \infty$.

Define now the mapping $f_n^\star \colon [0,\infty) \times S \to [0,\infty]$ by

$$f_n^\star(t,x) = \begin{cases} \tilde{f}_n^\star(t,x), & (t,x) \in I, \\ \infty, & (t,x) \in I^c. \end{cases}$$

By construction we know that

$$(f_n^\star)^{-1}(\{-\infty\}) = I^c$$

is a measurable set, yielding the measurability of $f_n^\star$ on $[0,\infty) \times S$. Moreover we get $f_n^\star(t,x) = \vartheta_n^\star$ for all $t \geq 0$ and $x \in S$ and

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(f_n^\star(t,x)) \quad \text{for all } (t,x) \in I.$$

In particular, we get the measurability of

$$[0,\infty) \to \mathbb{R} \cup \{-\infty\}, \quad t \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

for all $x \in S$.

(c) Now assume that U is an extended utility function, derived from a classical utility function with maximal domain of the form $[-d,\infty) \subsetneq \mathbb{R}$.

(i) Let $t \geq 0$ and $x \in S$. Consider now

$$\begin{aligned} & m_{0,t,x}(\vartheta) \\ &= U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_0(t+s,j)\, ds \\ &= \underbrace{U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta}}_{=:P^1(t,x,\vartheta)} + \underbrace{\int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot U\big(-ct - cs + g(j)\big)\, ds}_{=:P^2(t,x,\vartheta)} . \end{aligned}$$

Note that according to part (a), both terms $P^1(t,x,\cdot)$ and $P^2(t,x,\cdot)$ are bounded from above by a constant only depending on $x \in S$. We will now first discuss for which parameters $t \geq 0$ and $x \in S$ we get $m_{0,t,x}(\vartheta) = -\infty$ for all $\vartheta \geq 0$. To this end we observe that

$$\begin{aligned} U(-ct + g(x)) = -\infty &\Leftrightarrow -ct + g(x) < -d \\ &\Leftrightarrow t > \frac{g(x)+d}{c} \end{aligned}$$

and since $\vartheta \mapsto U(-ct - c\vartheta + g(x))$ is a decreasing mapping, we get $U(-ct - c\vartheta + g(x)) \equiv -\infty$, if and only if $t > \frac{g(x)+d}{c}$. Hence we know that

$$P^1(t,x,\cdot) \equiv -\infty \Leftrightarrow t > \frac{g(x)+d}{c}.$$

Now for P^2, we know that for any $\vartheta \geq 0$ the integral over the compact interval $[0,\vartheta]$ yields $-\infty$, if and only if the integrand equals $-\infty$ for any $s \in [0,\vartheta]$. This however is the case, if and only if there exists at least one state $j \in S \backslash \{x\}$, such that

$$U(-ct - cs + g(j)) = -\infty.$$

Since $s \mapsto U(-ct - cs + g(j))$ is a decreasing function, we need to require

$$U(-ct + g(j)) = -\infty$$

for at least one $j \in S \backslash \{x\}$ such that $q_{xj} \neq 0$, in order to guarantee that $P^2(t,x,\cdot) \equiv -\infty$. This however is the case, if $t > \frac{g(j)+d}{c}$ for at least one $j \in S \backslash \{x\}$ such that $q_{xj} \neq 0$.

In summary we get

$$m_{0,t,x}(\vartheta) = -\infty \text{ for all } \vartheta \geq 0$$

if and only if $t > \frac{g(x)+d}{c}$ or there exists a state $j \in S \backslash \{x\}$ with $q_{xj} \neq 0$ such that $t > \frac{g(j)+d}{c}$. On the other hand, it holds

$$\begin{aligned}
& m_{0,t,x}(\vartheta) \neq -\infty \text{ for some } \vartheta \geq 0 \\
\Leftrightarrow & \forall j \in S \backslash \{x\} \text{ such that } q_{xj} \neq 0: \ t \leq \frac{g(j)+d}{c} \text{ and } t \leq \frac{g(x)+d}{c} \\
\Leftrightarrow & t \leq \min \left\{ \inf_{\substack{j \in S \backslash \{x\}, \\ q_{xj} > 0}} \frac{g(j)+d}{c}, \ \frac{g(x)+d}{c} \right\} \\
\Leftrightarrow & t \leq T_d^{(0)}(x).
\end{aligned}$$

This shows the last assertion of (i).

Now for (7.24), assume that $t \in \left[0, T_d^{(0)}(x)\right]$. We can say that

$$\begin{aligned}
U(-ct - c\vartheta + g(x)) > -\infty & \Leftrightarrow -ct - c\vartheta + g(x) \geq -d \\
& \Leftrightarrow \vartheta \leq \frac{g(x)+d}{c} - t.
\end{aligned}$$

The same reasoning as above yields immediately that

$$P^1(t, x, \vartheta) > -\infty \Leftrightarrow \vartheta \leq \frac{g(x)+d}{c} - t.$$

In addition

$$P^2(t, x, \vartheta) > -\infty \Leftrightarrow \forall j \in S \backslash \{x\} \text{ such that } q_{xj} \neq 0: \ \vartheta \leq \frac{g(j)+d}{c} - t.$$

Thus, we get

$$\begin{aligned}
& m_{0,t,x}(\vartheta) > -\infty \\
\Leftrightarrow & \forall j \in S \backslash \{x\} \text{ such that } q_{xj} \neq 0: \ \vartheta \leq \frac{g(j)+d}{c} - t \text{ and } \vartheta \leq \frac{g(x)+d}{c} - t \\
\Leftrightarrow & \vartheta \leq \min \left\{ \inf_{\substack{j \in S \backslash \{x\}, \\ q_{xj} > 0}} \frac{g(j)+d}{c}, \ \frac{g(x)+d}{c} \right\} - t \\
\Leftrightarrow & \vartheta \leq T_d^{(0)}(x) - t
\end{aligned}$$

and therefore

$$m_{0,t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in \left[0, T_d^{(0)}(x) - t\right], \\ = -\infty, & \text{if } \vartheta \in \left(T_d^{(0)}(x) - t, \infty\right). \end{cases}$$

Hence (7.24) is shown.

For the remaining statements of part (i), note that U is continuous on $[-d, \infty)$ and differentiable on $(-d, \infty)$. As a consequence the mapping $(t, \vartheta) \mapsto P^1(t, x, \vartheta)$ is also continuous on the set D_0, where

$$\begin{aligned} D_0 &:= \big\{(t,\vartheta) \in [0,\infty)^2 \big|\, t \in \big[0, T_d^{(0)}(x)\big],\ \vartheta \in \big[0, T_d^{(0)}(x) - t\big]\big\} \qquad (7.29) \\ &= \big\{(t,\vartheta) \in [0,\infty)^2 \big|\, \vartheta \in \big[0, T_d^{(0)}(x)\big],\ t \in \big[0, T_d^{(0)}(x) - \vartheta\big]\big\} \end{aligned}$$

and differentiable on the set D_0°, where

$$D_0^\circ := D_0 \backslash \big\{(t,\vartheta) \in [0,\infty)^2 \big|\, t + \vartheta = T_d^{(0)}(x)\big\}.$$

For the integral part P^2 we remind ourselves that for $t \in \big[0, T_d^{(0)}(x)\big]$ and $\vartheta \in \big[0, T_d^{(0)}(x) - t\big]$ the mapping

$$s \mapsto U(-ct - cs + g(j))$$

is decreasing on $[0, \vartheta]$ for every $j \in S\backslash\{x\}$. It is continuous and bounded from below by $U(-ct - c\vartheta + g(j))$ and bounded from above according to part (a) of this proof. As a consequence

$$s \mapsto e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot U\big(-ct - cs + g(j)\big)$$

is integrable on the compact set $[0, \vartheta]$. This yields by the fundamental theorem of calculus the differentiability of $\vartheta \mapsto P^2(t, x, \vartheta)$ on $\big[0, T_d^{(0)}(x) - t\big)$.

Furthermore, we know that for every $\vartheta \in \big[0, T_d^{(0)}(x)\big]$ and $s \in [0, \vartheta]$ the mapping

$$t \mapsto e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot U\big(-ct - cs + g(j)\big)$$

is continuous on $\big[0, T_d^{(0)}(x) - \vartheta\big]$ and (by the same arguments as above) also bounded in t. A standard continuity argument for parameterized integrals (see for example [Klenke, 2013, Theorem 6.27]) yields the continuity of

$$t \mapsto \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot U\big(-ct - cs + g(j)\big)\, ds$$

on $\big[0, T_d^{(0)}(x) - \vartheta\big]$. In summary we know that $P^2(t, x, \cdot)$ is differentiable on $\big[0, T_d^{(0)}(x) - t\big)$ and $P^2(\cdot, x, \cdot)$ is continuous on D_0 as defined in (7.29).

Putting P^1 and P^2 together we ultimately get the desired differentiability of $m_{0,t,x}$ on $\big[0, T_d^{(0)}(x) - t\big)$ and additionally the continuity of

$$(t, \vartheta) \mapsto m_{0,t,x}(\vartheta)$$

on D_0.

On the set $\big[0, T_d^{(0)}(x) - t\big)$ the derivative $m'_{0,t,x}$ is clearly given by (7.23).

(ii) Let $t \geq 0$, $x \in S$, $n \in \mathbb{N}$ and assume that the mapping $t \mapsto V_n(t,j)$ is continuous on $\big[0, T_d^{(n-1)}(j)\big]$ for every $j \in S$ and

$$V_n(t,j) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n-1)}(j)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n-1)}(j), \infty\big) \end{cases}$$

for every $j \in S$, as required in (7.25). Consider now

$$m_{n,t,x}(\vartheta) = \underbrace{U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta}}_{=:P^1(t,x,\vartheta)} + \underbrace{\int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds}_{=:P^2(t,x,\vartheta)}.$$

For P^1 we can use the results in (i) to conclude

$$P^1(t, x, \cdot) \equiv -\infty \Leftrightarrow t > \frac{g(x)+d}{c}$$

and for fixed $t \in \big[0, \frac{g(x)+d}{c}\big]$

$$P^1(t, x, \vartheta) > -\infty \Leftrightarrow \vartheta \leq \frac{g(x)+d}{c} - t.$$

Since $\frac{g(x)+d}{c} \geq T_d^{(n)}(x)$, the continuity of $(t,\vartheta) \mapsto P^1(t,x,\vartheta)$ on the set D_n, where

$$\begin{aligned} D_n &:= \big\{(t,\vartheta) \in [0,\infty)^2 \big|\, t \in \big[0, T_d^{(n)}(x)\big],\ \vartheta \in \big[0, T_d^{(n)}(x) - t\big]\big\} \\ &= \big\{(t,\vartheta) \in [0,\infty)^2 \big|\, \vartheta \in \big[0, T_d^{(n)}(x)\big],\ t \in \big[0, T_d^{(n)}(x) - \vartheta\big]\big\} \end{aligned} \tag{7.30}$$

and its differentiability on the set D_n°, where

$$D_n^\circ := D_n \backslash \big\{(t,\vartheta) \in [0,\infty)^2 \,\big|\, t + \vartheta = T_d^{(n)}(x)\big\}$$

are analogeous to (i).

For P^2 we can state by assumption that if $t > T_d^{(n-1)}(j)$, then $V_n(t+s, j) = -\infty$ for every $s \geq 0$ and $j \in S\backslash\{x\}$ and thus

$$P^2(t, x, \cdot) \equiv -\infty \Leftrightarrow \exists j \in S\backslash\{x\} \text{ with } q_{xj} \neq 0, \text{ such that } t > T_d^{(n-1)}(j).$$

On the other hand if $t+s \leq T_d^{(n-1)}(j)$ for some $s \geq 0$, then we get $V_n(t+s, j) > -\infty$. Analogously, we can conclude that

$$P^2(t, x, \vartheta) > -\infty \Leftrightarrow \forall j \in S\backslash\{x\} \text{ such that } q_{xj} \neq 0 : \ \vartheta \leq T_d^{(n-1)}(j) - t.$$

By Definition 7.11 we get

$$T_d^{(n)}(x) \leq T_d^{(n-1)}(j)$$

for all $j \in S\backslash\{x\}$ such that $q_{xj} \neq 0$. Therefore, the assumed continuity of $V_n(\cdot, j)$ on $\big[0, T_d^{(n-1)}(j)\big]$ is also valid on $\big[0, T_d^{(n)}(x)\big]$ and yields analogously to (i) the continuity of

$$s \mapsto e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)$$

on the compact set $[0, \vartheta]$, where $t \in \big[0, T_d^{(n)}(x)\big]$ and $\vartheta \in \big[0, T_d^{(n)}(x) - t\big]$. Then again the fundamental theorem of calculus yields the differentiability of

$$\vartheta \mapsto P^2(t, x, \vartheta) = \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j) \, ds$$

on $\big[0, T_d^{(n)}(x) - t\big)$.

Furthermore, the same argument for parametrized integrals (cf. [Klenke, 2013, Theorem 6.27]) as in (i) also provides the continuity of

$$t \mapsto P^2(t, x, \vartheta) = \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j) \, ds$$

on $\big[0, T_d^{(n)}(x) - \vartheta\big]$. In summary we know that $\vartheta \mapsto P^2(t, x, \vartheta)$ is differentiable on $\big[0, T_d^{(n)}(x) - t\big)$ and $(t, x) \mapsto P^2(t, x, \vartheta)$ is continuous on D_n as defined in (7.30).

Now we can put P^1 and P^2 back together. For $m_{n,t,x}(\vartheta)$ to be greater than $-\infty$, we need to guarantee this finiteness for P^1 and P^2 simultaneously. Thus, we get

$$\begin{aligned}
&m_{n,t,x}(\vartheta) > -\infty \\
&\Leftrightarrow \forall j \in S\backslash\{x\} \text{ such that } q_{xj} \neq 0: \ \vartheta \leq T_d^{(n-1)}(j) - t \text{ and } \vartheta \leq \frac{g(x)+d}{c} - t \\
&\Leftrightarrow \vartheta \leq \min\left\{ \inf_{\substack{j \in S\backslash\{x\}, \\ q_{xj}>0}} T_d^{(n-1)}(j) \ , \ \frac{g(x)+d}{c} \right\} - t \\
&\Leftrightarrow \vartheta \leq T_d^{(n)}(x) - t.
\end{aligned}$$

Thus we get $m_{n,t,x}(\vartheta) > -\infty$, if and only if $t \in \left[0, T_d^{(n)}(x)\right]$ and $\vartheta \in \left[0, T_d^{(n)}(x) - t\right]$, respectively $(t, \vartheta) \in D_n$, where D_n is given according to (7.30). This shows (7.26) of part (ii).

For the remaining statements of part (ii) we can combine the differentiablity and continuity results of P^1 and P^2 to ultimately get the desired differentiability of $m_{n,t,x}$ on $\left[0, T_d^{(n)}(x) - t\right)$ and additionally the continuity of

$$(t, \vartheta) \mapsto m_{n,t,x}(\vartheta)$$

on D_n.

The derivative $m'_{n,t,x}$ is again obviously given by (7.23) on $\left[0, T_d^{(n)}(x) - t\right)$.

In fact, in (i) and (ii) we proved the continuity on the set D_n, $n \in \mathbb{N}_0$, instead of just for $\vartheta \in \left[0, T_d^{(n)}(x) - t\right]$. But we will need this stronger result for the following part (iii).

(iii) Suppose $t \geq 0$, $x \in S$ and $n \in \mathbb{N}_0$. If $n \geq 1$, let assumption (7.25) of part (ii) be fulfilled. Then the equations (7.24) and (7.26) from part (i), respectively part (ii), immediately yield

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = \sup_{\vartheta \in \left[0, T_d^{(n)}(x) - t\right]} m_{n,t,x}(\vartheta)$$

for $t \in \left[0, T_d^{(n)}(x)\right]$. Since $\vartheta \mapsto m_{n,t,x}(\vartheta)$ is additionally continuous on $\left[0, T_d^{(n)}(x) - t\right]$ the supremum can be attained by a maximum, providing equation (7.28)

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = \max_{\vartheta \in \left[0, T_d^{(n)}(x) - t\right]} m_{n,t,x}(\vartheta).$$

We call the value $\vartheta_n^\star$ at which the maximum is attained at a maximizer and get

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(\vartheta_n^\star).$$

This maximizer is obviously dependent on $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$.

In addition we know that $m_{n,t,x} \equiv -\infty$ for any $x \in S$, $n \in \mathbb{N}_0$, if and only if $t > T_d$. This immediately leads to

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n)}(x)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n)}(x), \infty\big). \end{cases}$$

Just as stated in Lemma 7.13 (c) (iii) itself, we set $\vartheta_n^\star = 0$ if $t > T_d^{(n)}(x)$ but every other $\vartheta \geq 0$ would also be a valid maximizer.

In case that $T_d^{(n)}(x) < 0$ and thus $\big[0, T_d^{(n)}(x)\big] = \emptyset$, we get $\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = -\infty$ for every $t \geq 0$.

(iv) We will now show the continuity of

$$[0, \infty) \to \mathbb{R} \cup \{-\infty\}, \quad t \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

on $\big[0, T_d^{(n)}(x)\big]$. To this end we will utilize a result from Bäuerle & Rieder [2011] to show that taking the supremum of $m_{n,t,x}(\vartheta)$ over all $\vartheta \geq 0$ (which in this case is equivalent to taking the maximum of $m_{n,t,x}(\vartheta)$ over all $\vartheta \in \big[0, T_d^{(n)}(x) - t\big]$) preserves the continuity of $m_{n,t,x}$ in t. At first we remind ourselfes that in (i) and (ii) we were able to show the continuity of

$$(t, \vartheta) \mapsto m_{n,t,x}(\vartheta)$$

on the compact set

$$D_n = \big\{(t, \vartheta) \in [0, \infty)^2 |\, t \in \big[0, T_d^{(n)}(x)\big],\ \vartheta \in \big[0, T_d^{(n)}(x) - t\big]\big\}$$

as defined in (7.30).

Define now

$$D_n(t) := \big\{\vartheta \geq 0 |\ (t, \vartheta) \in D_n\big\} = \big[0, T_d^{(n)}(x) - t\big].$$

Clearly $D_n(t)$ is a compact set for every $t \in \big[0, T_d^{(n)}(x)\big]$. Consider now the set-valued mapping

$$t \mapsto D_n(t).$$

This set-valued mapping assigns every $t \in \big[0, T_d^{(n)}(x)\big]$ to a non-empty subset of $\big[0, T_d^{(n)}(x)\big]$. Without going into detail, we refer to [Bäuerle & Rieder, 2011, Appendix A.2] for a more detailed study of set-valued mappings and use Definition A.2.1 and Lemma A.2.2 (d) of Bäuerle & Rieder [2011] to conclude that $t \mapsto D_n(t)$ is continuous. This allows the application of Proposition 2.4.8 Bäuerle and Rieder established to guarantee the continuity of

$$t \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

on $\big[0, T_d^{(n)}(x)\big]$. Furthermore, Proposition 2.4.8 also provides the existence of a measurable mapping $f_n^\star \colon \big[0, T_d^{(n)}(x)\big] \times S \to \mathbb{R}$ such that $\vartheta_n^\star = f_n^\star(t,x)$. This emerges basicly from a selection theorem (Theorem A.2.3) of Bäuerle & Rieder [2011] which originated from Kuratowski & Ryll-Nardzewski [1965].

In addition, Proposition 2.4.8 of Bäuerle & Rieder [2011] also yields the continuity of $t \mapsto f_n^\star(t,x)$ on $\big[0, T_d^{(n)}(x)\big]$, given that the maximizer $\vartheta_n^\star$ is unique for every $t \in \big[0, T_d^{(n)}(x)\big]$.

Now for $t > T_d^{(n)}(x)$, we have set $\vartheta_n^\star$ to zero, according to part (iii). Thus we can easily extend $f_n^\star$ in a measurable way to $f_n^\star \colon [0,\infty) \times S \to \mathbb{R}$ by setting $f_n^\star(t,x) = 0$ for $x \in S$ and $t > T_d^{(n)}(x)$. Of course we will never have uniqueness of $\vartheta_n^\star$ for $t > T_d^{(n)}(x)$ as the maximizing function is constantly set to $-\infty$. But by setting the maximizers to $\vartheta_n^\star = f_n^\star(t,x) = 0$ for $t > T_d$ we can extend the continuity of $t \mapsto f_n^\star(t,x)$ to $[0,\infty)$, since $f_n^\star(T_d^{(n)}(x), x) = 0$ by (7.28).

This concludes the proof of Lemma 7.13, as all assertions are shown.

□

Theorem 7.14 (Bellman equation)

Let $x \in S$, $t \geq 0$ and $n \in \mathbb{N}_0$. Then it holds:

(a) The *Bellman equation*

$$V_{n+1}(t,x) = \sup_{\vartheta \geq 0} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \Big\} \tag{7.31}$$

is valid. The initial value function V_0 is given by

$$V_0(t,x) = U\big(-ct + g(x)\big). \tag{7.32}$$

(b) If U is a classical utility function on the whole real line, then the supremum in (7.31) exists, depending on $x \in S$ and $t \geq 0$. It is either a maximum and is attained by a finite maximizer $\vartheta_n^\star$ or is an unattainable (finite) supremum, in which case the maximizer is set as $\vartheta_n^\star = \infty$.

Furthermore, there exists a measurable mapping $f_n^\star \colon [0, \infty) \times S \to [0, \infty]$ such that $f_n^\star(t, x) = \vartheta_n^\star$.

(c) If U is an extended utility function derived from a classical utility function with maximal domain of the form $[-d, \infty) \subsetneq \mathbb{R}$, then the supremum in (7.31) exists and is even a maximum, depending on $x \in S$ and $t \geq 0$. It is attained by a finite maximizer $\vartheta_n^\star \in \big[0, T_d^{(n)}(x) - t\big]$ with $T_d^{(n)}(x)$ given in Definition 7.11.

Moreover, there exists a measurable mapping $f_n^\star \colon [0, \infty) \times S \to \big[0, T_d^{(n)}(x)\big]$ such that $f_n^\star(t, x) = \vartheta_n^\star$. If the maximizer $\vartheta_n^\star$ is unique for every $t \in \big[0, T_d^{(n)}(x)\big]$, then $t \mapsto f_n^\star(t, x)$ is even continuous on $[0, \infty)$.

(d) For $n \in \mathbb{N}$ the optimal stopping time $\tau_n^\star \in \Sigma_x$ for $V_n(t, x) = \sup_{\tau \in \Sigma_x} V_n(t, x, \tau)$ such that $V_n(t, x) = V_n(t, x, \tau_n^\star)$ is given by

$$\tau_n^\star := \Big(f_{n-1}^\star(t, x), f_{n-2}^\star(S_1 + t, Z_1) + S_1, \dots, f_0^\star(S_{n-1} + t, Z_{n-1}) + S_{n-1}, S_n, S_{n+1}, \dots\Big). \tag{7.33}$$

For $n = 0$ the optimal stopping time $\tau_0^\star \in \Sigma_x$ for $V_0(t, x) = U\big(-ct + g(x)\big)$ is trivially given by

$$\tau_0^\star := 0.$$

(e) If U is a classical utility function like in (b), then the value functions $V_n(t, x)$ are measurable in $t \geq 0$ for every $n \in \mathbb{N}_0$ and $x \in S$.

(f) If U is an extended utility function like in (c), then the value function $V_0(t, x)$ is continuous in $t \in \big[0, \frac{g(x)+d}{c}\big]$ and

$$V_0(t, x) \begin{cases} > -\infty, & \text{if } t \in \big[0, \frac{g(x)+d}{c}\big], \\ = -\infty, & \text{if } t \in \big(\frac{g(x)+d}{c}, \infty\big). \end{cases} \tag{7.34}$$

For $n \in \mathbb{N}$ the value functions $V_n(t, x)$ are continuous in $t \in \big[0, T_d^{(n-1)}(x)\big]$. Furthermore

$$V_n(t, x) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n-1)}(x)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n-1)}(x), \infty\big). \end{cases} \tag{7.35}$$

Remark 7.15

(a) The optimal stopping time in (7.33) is given using notation (3.24) in Notation 3.16. By using the piecewise description via stopping rules, the optimal stopping time would read

$$\tau_n^\star = \left(\tau_n^{\star,0}, \tau_n^{\star,1}, \tau_n^{\star,2}, \ldots\right), \tag{7.36}$$

where the stopping rules $\tau_n^{\star,k}$ are given by

$$\tau_n^{\star,k} := \begin{cases} f_{n-1-k}^\star(S_k + t, Z_k) + S_k, & k < n, \\ S_k, & k \geq n. \end{cases} \tag{7.37}$$

(b) Note, that the optimal stopping time in equation (7.33) is not only a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_u^X)_{u\geq 0}$–stopping time but even a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_u^{X,n})_{u\geq 0}$–stopping time and can be represented by

$$\tau_n^\star = \left(f_{n-1}^\star(t,x), f_{n-2}^\star(S_1 + t, Z_1) + S_1, \ldots, f_0^\star(S_{n-1} + t, Z_{n-1}) + S_{n-1}, S_n\right), \tag{7.38}$$

respectively

$$\tau_n^\star = \left(\tau_n^{\star,0}, \tau_n^{\star,1}, \ldots, \tau_n^{\star,n-1}, S_n\right). \tag{7.39}$$

More precisely, the choice of the stopping rules $\tau_n^{\star,k}$ for $k \geq n$ is absolutely arbitrary, since we optimize a corresponding n–step stopping problem which terminates itself at the latest immediately after the n–th jump of the underlying Markov chain. Thus the only stopping rules $\tau_n^{\star,k}$ of $\tau_n^\star$ that influence $V_n(t,x,\tau_n^\star)$ are

$$\tau_n^{\star,0}, \tau_n^{\star,1}, \ldots, \tau_n^{\star,n-1},$$

respectively

$$f_{n-1}^\star(t,x), f_{n-2}^\star(S_1 + t, Z_1) + S_1, \ldots, f_0^\star(S_{n-1} + t, Z_{n-1}) + S_{n-1}.$$

(c) The optimal stopping time $\tau_n^\star$ for $V_n(t,x)$ depends explicitly on the choice of $t \geq 0$ and $x \in S$. To be absolutely precise, one could add these two parameters to the notation of the optimal stopping time, writing $\tau_{n,t,x}^\star$ instead of just $\tau_n^\star$. But in order to keep the notation simpler and more readable, we omitted these additional arguments. Of course, this dependency always has to be kept in mind nonetheless.

(d) For a general stopping time $\tau \in \Sigma_x$, the corresponding stopping rules τ^k for $k \in \mathbb{N}_0$ are given by

$$\tau^k = h_k(S_1, \ldots, S_k, Z_0, Z_1, \ldots, Z_k) + S_k$$

for measurable mappings h_k with $k \in \mathbb{N}_0$. They depend on the whole history of jump times $S_1, \ldots, S_k$ and of post-jump states $Z_0, Z_1, \ldots, Z_k$.

The optimal stopping time $\tau_n^\star$ for $V_n(t,x) = \sup_{\tau \in \Sigma_x} V_n(t,x,\tau)$ has an even simpler structure. The stopping rules

$$\tau_n^{\star,k} = \begin{cases} f^\star_{n-1-k}(S_k + t, Z_k) + S_k, & k < n, \\ S_k, & k \geq n. \end{cases}$$

do not depend on the whole history, but solely on the last jump time S_k and the last state Z_k which the Markov chain attained. In this sense, we say that the optimal stopping time exhibits a *Markovian structure*.

(e) Due to the fact that the considered problem terminates at the latest immediately after the n–th jump as well as the optimal stopping problem terminates not later than reaching the n–th jump time S_n, we can conclude that $\tau_n^\star$ is indeed a $\mathbb{P}_x$–almost surely finite stopping time (this will be rigorously proven in the proof of Theorem 7.14).

(f) Note that in the setting of Theorem 7.14 (b), the $\mathbb{P}_x$–almost sure finiteness is generally not true for the single stopping rules $\tau_n^{\star,k}$ of which $\tau_n^\star$ is composed. Since Theorem 7.14 (b) explicitly allows for the maximizer to be infinite for some $t \geq 0$ and $x \in S$, the corresponding mapping $f_n^\star$ can also attain infinity. This allows for some stopping rules $\tau_n^{\star,k}$ with $k < n$ to reach infinity with positive probability due to

$$\mathbb{P}_x(\tau_n^{\star,k} = \infty) = \mathbb{P}_x(f^\star_{n-1-k}(S_k + t, Z_k) = \infty) > 0.$$

(g) In the setting of Theorem 7.14 (b) we set the maximizer $\vartheta_n^\star = \infty$, if the supremum in (7.31) is not a maximum which can be attained by a finite value. In this case equation (7.31) reads

$$\begin{aligned} &V_{n+1}(t,x) \\ &= \sup_{\vartheta \geq 0} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \Big\} \\ &= \limsup_{\vartheta \to \infty} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \Big\}. \end{aligned}$$

Proof of Theorem 7.14

The proof will be divided into three parts. Step 1 will adress to the existence of the maximizer as well as the existence of a measurable mapping for both cases (b) and (c). Step 2 will treat with the part of (d) which claims $\tau_n^\star \in \Sigma_x$, whereas step 3 will cover the validity of the Bellman equation (7.31) in part (a), as well as the optimality of $\tau_n^\star$ in (d), the measurability of $V_n(t,x)$ in (e) and the continuity of $V_n(t,x)$ in addition to (7.34) and (7.35) in case of an extended utility function in (f):

1. For every $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$ the supremum in (7.31) can be expressed in terms of
$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta),$$
using the definition of $m_{n,t,x}$ in (7.22) of Lemma 7.13. As shown in part (a) of the same lemma, $m_{n,t,x}$ is an upper bounded mapping on $[0, \infty)$. Hence, the supremum exists and takes a finite value for every $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$.

   Suppose that U is a classical utility function on the whole real line. Then the assertions from (b) follow directly from Lemma 7.13 (b) (ii).

   Now assume that U is an extended utility function derived from a classical utility function with maximal domain of the form $[-d, \infty) \subsetneq \mathbb{R}$. We suppose for the moment that the value functions $V_n(t, x)$ are continuous in t on $\big[0, T_d^{(n-1)}(x)\big]$ for every $n \in \mathbb{N}_0$ and $x \in S$ (Analogously we assume that $V_0(t, x)$ is continuous in t on $\big[0, \frac{g(x)+d}{c}\big]$ for every $x \in S$). This continuity is part of assertion (e) and will be shown iteratively in step 3.

   By Lemma 7.13 (c) (iii) and especially equation (7.28) we know that for every $t \geq 0$ and $x \in S$, the supremum in (7.31) is attained by a finite maximizer
$$\vartheta_n^\star \in \big[0, T_d^{(n)}(x) - t\big].$$
The existence of a measurable mapping $f_n^\star \colon [0, \infty) \times S \to \big[0, T_d^{(n)}(x) - t\big]$ as well as its continuity in t stems directly from Lemma 7.13 (c)(iv).

2. As already stated in Remark 7.15, the representation (7.33) of the stopping time $\tau_n^\star$ is equivalent to (7.38)
$$\tau_n^\star = \Big(\tau_n^{\star,0}, \tau_n^{\star,1}, \dots, \tau_n^{\star,n-1}, S_n\Big),$$
where
$$\tau_n^{\star,k} = f_{n-1-k}^\star(S_k + t, Z_k) + S_k$$
for all $k < n$. Furthermore, set $\tau_n^{\star,k} := S_k$ for all $k \geq 0$ such that
$$\tau_n^\star = \tau_n^{\star,0} \mathbb{1}_{\{\tau_n^\star < S_1\}} + \sum_{k=1}^{\infty} \tau_n^{\star,k}\, \mathbb{1}_{\{S_k \leq \tau_n^\star < S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.} \tag{7.40}$$
and define the mappings $h_k \colon [0, \infty)^k \times S^{k+1} \to [0, \infty]$, $k \in \mathbb{N}_0$ such that
$$h_k(s_1, \dots, s_k, z_0, z_1, \dots, z_k) = \begin{cases} f_{n-1-k}^\star(s_k + t, z_k), & k < n, \\ 0, & k \geq n. \end{cases}$$

Clearly, all h_k are measurable mappings and $h_k \geq 0$ for all $k \in \mathbb{N}_0$, since $f^\star_{n-1-k} \geq 0$ are measurable for every $k \in \mathbb{N}_0$. Additionally, we obviously get

$$\tau_n^{\star,k} = h_k(S_1, \dots, S_k, Z_0, Z_1, \dots, Z_k) + S_k \geq S_k.$$

To show that $\tau_n^{\star,k}$ is indeed an $(\mathcal{F}_u^{X,k})_{u\geq 0}$–stopping time, we apply Lemma 3.14 which yields the desired result, since $\tau_n^{\star,k}$ can be represented by a measurable mapping $h_k \geq 0$, depending on $S_1, \dots, S_k, Z_0, \dots, Z_k$. Hence $\tau_n^{\star,k}$ is an $(\mathcal{F}_u^{X,k})_{u\geq 0}$–stopping time.

In addition, we get by construction in (7.40) that $\tau_n^\star$ stopps at the latest at $\tau_n^{\star,n} = S_n$. Due to $\mathbb{P}_x(S_n < \infty) = 1$, we can conclude that $\mathbb{P}_x(\tau_n^\star < \infty) = 1$.

Overall, the decomposition result for $(\mathcal{F}_u^X)_{u\geq 0}$–stopping times in Proposition 3.15 yields the desired assertion that $\tau_n^\star \in \Sigma_x$.

3. We will show the validity of the Bellman equation (7.31), the optimality of $\tau_n^\star \in \Sigma_x$ and part (e) of Theorem 7.14 by induction over $n \in \mathbb{N}_0$:

<u>Induction basis:</u>

To show the statements for $n = 0$, we recall Remark 6.3 (c), stating

$$V_0(t,x) = U\big(-ct + g(x)\big) = V_0(t,x,\tau)$$

for all $x \in S$, $t \geq 0$ and every $\tau \in \Sigma_x$. Thus by applying the reward iteration formula (7.13) from Theorem 7.7, the following calculation holds for every $x \in S$ and $t \geq 0$:

$$\begin{aligned} V_1(t,x) &= \sup_{\tau\in\Sigma_x} V_1(t,x,\tau) \\ &= \sup_{\tau\in\Sigma_x} \Big\{ U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x\cdot\tau^0} \\ &\qquad + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_0(t+s, j, \vec{\tau}_{s,x})\, ds \Big\} \\ &= \sup_{\tau\in\Sigma_x} \Big\{ U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x\cdot\tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} \cdot V_0(t+s, j)\, ds \Big\}. \end{aligned}$$

This implies that for the optimization over all $\tau \in \Sigma_x$ only the first stopping rule τ^0 of $\tau \in \Sigma_x$ does have any influence. Recalling that for every $x \in S$ and $\tau \in \Sigma_x$ the corresponding stopping rule τ^0 only depends deterministically on x, we can conclude that the maximization over all $\tau \in \Sigma_x$ is equivalent to the maximization over all deterministic $(\mathcal{F}_u^{0,X})_{u\geq 0}$–stopping times $\tau \geq 0$. This on the other hand is equivalent to a maximization over all non-negative real numbers $\vartheta \geq 0$. Hence we conclude

$$\begin{aligned}
V_1(t,x) &= \sup_{\tau \in \Sigma_x} \left\{ U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_0(t+s,j)\, ds \right\} \\
&= \sup_{\tau^0 \geq 0} \left\{ U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_0(t+s,j)\, ds \right\} \qquad (7.41) \\
&= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_0(t+s,j)\, ds \right\} \\
&= \sup_{\vartheta \geq 0} m_{0,t,x}(\vartheta) \qquad (7.42)
\end{aligned}$$

for all $x \in S$, $t \geq 0$ and $m_{0,t,x}$ as defined in (7.22). This yields the induction basis for the Bellman equation.

Of course we know that every stopping rule τ^0 depends explicitly on the choice of $x \in S$ and thus the maximizer $\vartheta_0^\star$ in (7.42) must also have this intrinsic dependency, regardless of $m_{0,t,x}$ itself. On the other hand, step 1 of this proof implies that due to $m_{0,t,x}$, the optimal choice of an $\vartheta_0^\star$ does also have to depend on the time $t \geq 0$.

For the optimality of the stopping time

$$\tau_0^\star = (f_0^\star(t,x), S_1, S_2, \dots)$$

for $V_1(t,x)$ we refer again to the reasoning above, stating that the optimization over all $\tau \in \Sigma_x$ is reduced to the optimization over all (deterministic) 0–step stopping rules (as seen in (7.41)), respectively all non-negative real numbers. Due to step 1 of this proof we know of the existence of such a maximizer $\vartheta_0^\star$, as well as the existence of a measurable mapping $f_0^\star$ such that the optimal 0–step stopping rule is given by

$$\tau_0^{\star,0} = \vartheta_0^\star = f_0^\star(t,x).$$

Since only the first stopping rule is relevant and the 1–step stopping problem terminates anyway at the latest at the first jump time S_1, we set $\tau_0^{\star,k} = S_k$ for all $k > 0$. This leads to the optimality of

$$\tau_0^\star = (f_0^\star(t,x), S_1, S_2, \dots) = (\tau_0^{\star,0}, S_1, S_2, \dots)$$

for $V_1(t,x) = \sup_{\tau \in \Sigma_x} V_1(t,x,\tau) = V_1(t,x,\tau_0^\star)$.

Moreover, in case of an extended utility function derived from a classical utility function with maximal domain of the form $[-d, \infty)$, we can apply Lemma 7.13 (c) (iv) to conclude that

$$t \mapsto V_1(t,x) = \sup_{\vartheta \geq 0} m_{0,t,x}(\vartheta)$$

is continuous on $\big[0, T_d^{(0)}(x)\big]$ for every $x \in S$ and that equation Equation (7.35) is valid for $n = 1$. This concludes the induction basis.

Induction hypothesis:

Assume that for some $n \in \mathbb{N}$ the Bellman equation (7.31)

$$V_n(t,x) = \sup_{\vartheta \geq 0} \Bigg\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_{n-1}(t+s, j)\, ds \Bigg\}$$

is valid for all $t \geq 0$ and $x \in S$ and the optimal stopping time $\tau^\star_{n,t,x} \in \Sigma_x$ for $V_n(t,x) = \sup_{\tau \in \Sigma_x} V_n(t,x,\tau)$ such that $V_n(t,x) = V_n(t,x,\tau^\star_{n,t,x})$ is given by

$$\tau^\star_{n,t,x} = \Big(f^\star_{n-1}(t,x), f^\star_{n-2}(S_1+t, Z_1)+S_1, \ldots, f^\star_0(S_{n-1}+t, Z_{n-1})+S_{n-1}, S_n, S_{n+1}, \ldots \Big)$$

respectively

$$\tau^\star_{n,t,x} = \Big(\tau^{\star,0}_{n,t,x}, \tau^{\star,1}_{n,t,x}, \tau^{\star,2}_{n,t,x}, \ldots \Big), \tag{7.43}$$

where

$$\tau^{\star,k}_{n,t,x} = \begin{cases} f^\star_{n-1-k}(S_k + t, Z_k) + S_k, & \text{for } k \in \{0, \ldots, n-1\}, \\ S_k, & \text{for } k \geq n \end{cases} \tag{7.44}$$

for all $t \geq 0$ and $x \in S$. As usual, we set $S_0 = 0$ and $Z_0 = x$. Also note that we will use the extended notation for the optimal stopping time $\tau^\star_{n,t,x}$ to clarify its dependence on the actual choice of t and x, as this will be needed in the remainder of this proof.

Furthermore in case of an extended utility function, we additionally assume that $t \mapsto V_n(t,x)$ is continuous on $\big[0, T_d^{(n-1)}(x)\big]$ for all $x \in S$ and fulfills equation (7.35).

Induction step:

Fix now some $t \geq 0$ and $x \in S$. We will show that the induction step for the validity of the Bellman equation (7.31) and the optimality of the stopping time in

(7.33) holds for $n+1$ and for the fixed but arbitrary parameters t and x. To this end, consider the stopping time

$$\tau^\star_{n+1,t,x} = \Big(\tau^{\star,0}_{n+1,t,x}, \tau^{\star,1}_{n+1,t,x}, \tau^{\star,2}_{n+1,t,x}, \dots\Big), \tag{7.45}$$

where

$$\tau^{\star,k}_{n+1,t,x} = \begin{cases} f^\star_{n-k}(S_k+t, Z_k) + S_k, & \text{for } k \in \{0,\dots,n\}, \\ S_k, & \text{for } k \geq n+1. \end{cases} \tag{7.46}$$

Note that $\tau^\star_{n+1,t,x} \in \Sigma_x$ according to step 2 of this proof and that $\tau^{\star,0}_{n+1,t,x} = f^\star_n(t,x)$ is a deterministic stopping rule.

For further convenience, we will additionally use the description of stopping rules as given in the decomposition result of Proposition 3.15:

$$\tau^{\star,k}_{n+1,t,x} = h_k(S_1, S_2, \dots, S_k, Z_0, Z_1, Z_2, \dots, Z_k) + S_k,$$

with h_k such that

$$h_k(s_1, s_2, \dots, s_k, z_0, z_1, z_2, \dots, z_k) := \begin{cases} f^\star_{n-k}(s_k+t, z_k), & k \in \{0,\dots,n\}, \\ 0, & k \geq n+1 \end{cases} \tag{7.47}$$

for all $k \in \mathbb{N}_0$. Note that $s_0 := 0$ and $z_0 := x$. Thus, $\tau^\star_{n+1}$ can be written in short as

$$\tau^\star_{n+1,t,x} = \Big(h_0, h_1, h_2, \dots\Big).$$

Now fix an $s \geq 0$ and a $j \in S\backslash\{x\}$. Consider the shifted stopping time $\overrightarrow{\tau}^\star_{n+1,t,s,x}$ of $\tau^\star_{n+1,t,x}$. According to Definition 7.3, $\overrightarrow{\tau}^\star_{n+1,t,s,x}$ is defined by

$$\overrightarrow{\tau}^\star_{n+1,t,s,x} = (\tilde\tau^0, \tilde\tau^1, \tilde\tau^2, \dots) = (\tilde h_0, \tilde h_1, \tilde h_2, \dots) \tag{7.48}$$

where

$$\begin{gathered} \tilde h_k \colon [0,\infty)^k \times S^{k+1} \to [0,\infty], \\ \tilde h_k(s, x;\ \tilde s_1, \dots, \tilde s_k, \tilde z_0, \tilde z_1, \dots, \tilde z_k) := h_{k+1}(s, \tilde s_1, \dots, \tilde s_k, x, \tilde z_0, \tilde z_1, \dots, \tilde z_k) \end{gathered}$$

and

$$\begin{aligned} \tilde\tau^k &= \tilde h_k(s, x;\ \tilde S_1, \dots, \tilde S_k, \tilde Z_0, \tilde Z_1, \dots, \tilde Z_k) + \tilde S_k - s \\ &= h_{k+1}(s, \tilde S_1, \dots, \tilde S_k, x, \tilde Z_0, \tilde Z_1, \dots, \tilde Z_k) + \tilde S_k - s \end{aligned}$$

for every $k \in \mathbb{N}_0$. Note that we use again the notation of shifted jump times $\tilde S_k = S_{k+1}$ and the shifted embedded Markov chain $(\tilde Z_k)_{k\in\mathbb{N}_0}$, $\tilde Z_k = Z_{k+1}$ for $k \in \mathbb{N}_0$, as introduced in Definition 7.1 (b).

Appliying (7.47), this leads to

$$\begin{aligned}\tilde{\tau}^0 &= h_1(s, x, \tilde{Z}_0) + \tilde{S}_0 - s \\ &= f^\star_{n-1}(s+t, \tilde{Z}_0) + \tilde{S}_0 - s \quad \text{and} \\ \tilde{\tau}^k &= h_{k+1}(s, \tilde{S}_1, \ldots, \tilde{S}_k, x, \tilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_k) + \tilde{S}_k - s \\ &= \begin{cases} f^\star_{n-1-k}(\tilde{S}_k + t, \tilde{Z}_k) + \tilde{S}_k - s, & \text{for all } k \in \{1, \ldots, n\} \\ \tilde{S}_k - s, & \text{for } k \geq n+1. \end{cases}\end{aligned}$$

Assume now that the first jump time and the subsequent state of the underlying Markov chain are known and given by $S_1 = s$, respectively $Z_1 = j$. Using now the properties of shifted jump times (7.4) and the shifted embedded Markov chain (7.5) from Lemma 7.2 (b) and (c), as well as utilizing representation (7.44) for the optimal n–step stopping rules from the induction hypothesis, we can conclude that

$$\begin{aligned}\tilde{\tau}^0_{|S_1=s, Z_1=j} &= f^\star_{n-1}(s+t, \tilde{Z}_0) + \tilde{S}_0 - s_{\ |S_1=s, Z_1=j} \\ &= f^\star_{n-1}(s+t, j) \\ &= \tau^{\star,0}_{n,t+s,j}\end{aligned}$$

and

$$\begin{aligned}\tilde{\tau}^k_{|S_1=s, Z_1=j} &= f^\star_{n-1-k}(\tilde{S}_k + t, \tilde{Z}_k) + \tilde{S}_k - s_{\ |S_1=s, Z_1=j} \\ &\overset{\mathcal{D}}{=} f^\star_{n-1-k}(S_k + s + t, Z_k) + S_k + s - s_{\ |Z_0=j} \\ &= f^\star_{n-1-k}(S_k + s + t, Z_k) + S_{k\ |Z_0=j} \\ &= \tau^{\star,k}_{n,t+s,j}\end{aligned}$$

for $k \in \{1, \ldots, n\}$, as well as

$$\begin{aligned}\tilde{\tau}^k_{|S_1=s, Z_1=j} &= \tilde{S}_k - s_{\ |S_1=s, Z_1=j} \\ &\overset{\mathcal{D}}{=} S_k + s - s_{\ |Z_0=j} \\ &= S_{k\ |Z_0=j} \\ &= \tau^{\star,k}_{n,t+s,j}\end{aligned}$$

for $k \geq n+1$.

In summary, we were able to establish a connection between the stopping rules of the shifted stopping time $\overrightarrow{\tau}^\star_{n+1,t,s,x}$ of $\tau^\star_{n+1,t,x}$ and the stopping rules of the optimal stopping time $\tau^\star_{n,t+s,j}$ from the induction hypothesis, given that the first jump time and the subsequent state of the underlying Markov chain are known. By construction in (7.48), respectively (7.43), this leads ultimately to the following relation of $\overrightarrow{\tau}^{\star}_{n+1,t,s,x}$ and $\tau^{\star}_{n,t+s,j}$:

$$\overrightarrow{\tau}^{\star}_{n+1,t,s,x}\,{}_{|S_1=s,Z_1=j} = \tau^{\star}_{n,t+s,j}.$$

Using the induction hypothesis about the optimality of the stopping time $\tau^{\star}_{n,t+s,j}$ for $V_n(t+s,j)$ thus yields

$$V_n(t+s,j) = V_n\big(t+s,j,\tau^{\star}_{n,t+s,j}\big) \overset{\mathcal{D}}{=} V_n\Big(t+s,j,\overrightarrow{\tau}^{\star}_{n+1,t,s,x}\Big). \tag{7.49}$$

Now we are able to prove the optimality of stopping time $\tau^{\star}_{n+1,t,x}$ for $V_{n+1}(t,x)$ as well as the validity of the Bellman equation (7.31) for the $(n+1)$–th step.

To this end, we remind ourselves of the 0–step stopping rule $\tau^{\star,0}_{n+1,t,x} = f^{\star}_n(t,x)$ of $\tau^{\star}_{n+1,t,x}$ being deterministic and non-negative. Furthermore,

$$\vartheta^{\star}_n = f^{\star}_n(t,x) = \tau^{\star,0}_{n+1,t,x}$$

was the maximizer of

$$\sup_{\vartheta\geq 0}\left\{U\big(-ct-c\vartheta+g(x)\big)\cdot e^{-q_x\cdot\vartheta} + \int_0^{\vartheta} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n(t+s,j)\,ds\right\}$$

in the n–th step of the Bellman equation.

By application of the reward iteration formula from Theorem 7.7 and (7.49), this eventually leads to

$$\begin{aligned}
&V_{n+1}(t,x,\tau^{\star}_{n+1,t,x})\\
&= U\big(-ct-c\tau^{\star,0}_{n+1,t,x}+g(x)\big)\cdot e^{-q_x\cdot\tau^{\star,0}_{n+1,t,x}}\\
&\quad + \int_0^{\tau^{\star,0}_{n+1,t,x}} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n\big(t+s,j,\overrightarrow{\tau}^{\star}_{n+1,t,s,x}\big)\,ds\\
&= U\big(-ct-cf^{\star}_n(t,x)+g(x)\big)\cdot e^{-q_x\cdot f^{\star}_n(t,x)}\\
&\quad + \int_0^{f^{\star}_n(t,x)} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n\big(t+s,j,\overrightarrow{\tau}^{\star}_{n+1,t,s,x}\big)\,ds\\
&= U\big(-ct-cf^{\star}_n(t,x)+g(x)\big)\cdot e^{-q_x\cdot f^{\star}_n(t,x)} + \int_0^{f^{\star}_n(t,x)} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V_n(t+s,j)\,ds
\end{aligned}$$

$$= U\big(-ct - c\vartheta_n^\star + g(x)\big) \cdot e^{-q_x \cdot \vartheta_n^\star} + \int_0^{\vartheta_n^\star} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds$$

$$= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\}$$

and finally to

$$\begin{aligned} &V_{n+1}(t, x) \\ &\geq V_{n+1}(t, x, \tau_{n+1,t,x}^\star) \\ &= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\}. \end{aligned} \tag{7.50}$$

On the other hand, we get for every arbitrary choice of stopping times $\tau \in \Sigma_x$ that

$$\begin{aligned} &V_{n+1}(t, x, \tau) \\ &= U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} \underbrace{e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj}}_{\geq 0} \cdot \underbrace{V_n(t+s, j, \vec{\tau}_{s,x})}_{\leq V_n(t+s,j)}\, ds \\ &\leq U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \\ &\leq \sup_{\tau^0 \geq 0} \left\{ U\big(-ct - c\tau^0 + g(x)\big) \cdot e^{-q_x \cdot \tau^0} + \int_0^{\tau^0} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\} \\ &= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\} \end{aligned}$$

by application of the reward iteration formula and by using the fact that a maximization over all deterministic stopping rules $\tau^0 \geq 0$ is equivalent to the maximization over all non-negative real numbers $\vartheta \geq 0$.

Since the choice of τ was arbitrary, we get

$$V_{n+1}(t, x) \leq \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\}$$

and ultimately, together with (7.50)

$$
\begin{aligned}
V_{n+1}(t,x) &= V_{n+1}(t,x,\tau^{\star}_{n+1,t,x}) \\
&= \sup_{\vartheta\geq 0}\left\{U\big(-ct-c\vartheta+g(x)\big)\cdot e^{-q_x\cdot\vartheta}+\int_0^{\vartheta}e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}}q_{xj}\cdot V_n(t+s,j)\,ds\right\}.
\end{aligned}
$$

Thus, the Bellman equation (7.31) holds for the $(n+1)$–th step as well as $\tau^{\star}_{n+1,t,x}$ is optimal for $V_{n+1}(t,x)$.

Now assume that the underlying utility function is a classical one, defined on the whole real line. By Lemma 7.13 (b) (ii) we know that

$$
t\mapsto V_1(t,x)=\sup_{\vartheta\geq 0}m_{0,t,x}(\vartheta)
$$

is measurable. Applying the induction hypothesis about the measurability of $t\mapsto V_n(t,j)$ for all $j\in S$, Lemma 7.13 (b) (ii) will lead to the measurability of

$$
t\mapsto V_{n+1}(t,x)=\sup_{\vartheta\geq 0}m_{n,t,x}(\vartheta)
$$

on $[0,\infty)$ for every $x\in S$.

As last course of action, assume that the underlying utility function is an extended one derived from a utility function with maximal domain of the form $[-d,\infty)$. Applying the induction hypothesis about the continuity of $t\mapsto V_n(t,j)$ for all $j\in S$, coupled with Lemma 7.13 (c) (iv) yields the continuity of

$$
t\mapsto V_{n+1}(t,x)=\sup_{\vartheta\geq 0}m_{n,t,x}(\vartheta)
$$

on $\big[0,T_d^{(n)}(x)\big]$ for every $x\in S$, as well as the validity of equation (7.35).

By the principle of induction, we get the desired assertions for every $n\in\mathbb{N}_0$. Due to $t\geq 0$ and $x\in S$ being arbitrarily chosen, these assertions are also valid for every $t\geq 0$ and $x\in S$.

□

An immediate conclusion from Theorem 7.14 under knowledge of the existence of the maximizers $\vartheta_n^\star = f_n^\star(t,x)$ in (7.31) for $t \geq 0$ and $x \in S$ can be stated as follows:

Corollary 7.16 (Bellman equation with applied maximizers)
Let $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$. Then the Bellman equation (7.31) can be expressed as

$$V_{n+1}(t,x) = U\big(-ct - cf_n^\star(t,x) + g(x)\big) \cdot e^{-q_x \cdot f_n^\star(t,x)} + \int_0^{f_n^\star(t,x)} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds. \tag{7.51}$$

Note that in case of $f_n^\star(t,x) = \infty$ for given $t \geq 0$, $x \in S$ and $n \in \mathbb{N}_0$ equation (7.51) is just a symbolic notation for

$$V_{n+1}(t,x) = \limsup_{\vartheta \to \infty}\; U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds \tag{7.52}$$

according to Remark 7.15 (g).

Example 7.17 (example for the application of the Bellman equation)
Let $S = \{0,1\}$ and X a continuous-time Markov chain with intensity matrix Q given by

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

for some $\alpha, \beta > 0$ (cf. Example 2.14) and initial value $X_0 = x \in S$. Furthermore, let

$$U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}, \quad x \mapsto \begin{cases} \ln(x), & x > 0, \\ -\infty, & x \leq 0 \end{cases}$$

be an extended utility function, derived from a classical logarithmic utility. Moreover, let $c > 0$ be the cost rate and $g\colon S \to \mathbb{R}$ the reward function in this setting. Assume that

$$g(1) = K \cdot g(0) \tag{7.53}$$

for some $K > 1$. Then it holds:

(a) For every $n \in \mathbb{N}_0$ and every $x \in S$ we get

$$T_d^{(n)}(x) = \frac{g(0)}{c},$$

where $T_d^{(n)}(x)$ is defined according to (7.19) of Definition 7.11.

(b) For every $n \in \mathbb{N}_0$, $t \geq 0$ and initial value $x = 1$, the n–step value function $V_n(t,1)$ is given by

$$V_n(t,1) = U\big(-ct + g(1)\big) = \ln\big(-ct + g(1)\big).$$

The corresponding optimal stopping time $\tau_n^\star$ will stop immediately, thus

$$\tau_n^\star = 0.$$

(c) For every $n \in \mathbb{N}_0$, $t \in \big[0, \frac{g(0)}{c}\big]$ and initial value $x = 0$ the maximizer $\vartheta_n^\star$ is either a unique solution of the equation

$$e = \left[1 + \frac{\frac{\alpha}{c}\big(g(1) - g(0)\big)}{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)}\right]^{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)} \tag{7.54}$$

or set to zero. If

$$t \in \left[0, \frac{g(0)}{c} - \frac{1}{\alpha \ln(K)}\right), \tag{7.55}$$

then the maximizer is always a unique solution of (7.54) and takes values in $(0, \frac{g(0)}{c} - t)$. In this case the maximizer $\vartheta_n^\star$ is strictly greater than zero, yielding an optimal stopping rule which does not stop immediately, but before the first jump time S_1 of the underlying Markov chain, if this jump does not occur fast enough.

For $t \in \big[\frac{g(0)}{c} - \frac{1}{\alpha \ln(K)}, \frac{g(0)}{c}\big)$ the time parameter t and the respective cumulated costs $-ct$ may have become too large. In this case the maximizer is again set to $\vartheta_n^\star = 0$. The optimal stopping time would again stop immediately.

(d) Assume the specific case of

$$g(0) = 10, \quad g(1) = \frac{e^{10} + 99}{10} \approx 2212.55 \quad \text{and} \quad \alpha = \beta = c = 1.$$

For initial value $x = 1$, any $n \in \mathbb{N}_0$ and any $t \geq 0$, the optimal stopping time would stop immediately. The corresponding maximizer is thus given by

$$\vartheta_n^\star = f_n^\star(t,1) = 0.$$

For initial value $x = 0$ and any $n \in \mathbb{N}_0$ the maximizer $\vartheta_n^\star$ is set to zero, if $t \geq 9.9$ and

$$\vartheta_n^\star = f_n^\star(t,0) = 9.9 - t$$

for $t \in [0, 9.9)$.

In summary the optimal stopping time $\tau_n^\star$ for $V_n(t,x)$ is given by

$$\tau_n^\star = \begin{cases} 0, & x = 1 \quad \text{or} \quad x = 0 \text{ and } t \geq 9.9, \\ \min\{9.9 - t, S_1\}, & x = 0 \text{ and } t \in [0, 9.9). \end{cases}$$

This means that a rational investor will wait for the Markov chain to jump into the superior state 1 (provided the initial value of the underlying Markov chain is $x = 0$), as long as this change of state occurs before the break-point at 9.9. A longer waiting period is not optimal and does not yield a higher expected utility. Note that in this setting the default point is reached at $T_d^{(n)}(x) = \frac{g(0)}{c} = 10$. If the investor will wait for 10 time units and will thus cumulate costs of $-c \cdot 10 = -10$, he will achieve a utility value of $-\infty$. Therefore the 10 time units are the maximal period of time that the investor can possibly wait. The optimal value of 9.8902 is very close to this 10 time units. Hence we can say that the optimal behavior for an investor with initial value $x = 0$ is to wait for a change into the better state for nearly the maximal possible duration. If this change of state does not happen before time $t = 9.9$, he will stop shortly before the default time to prevent bankruptcy.

Proof

(a) Let $n \in \mathbb{N}_0$ and $x \in S$. Using Lemma 7.12, we can represent $T_d^{(n)}(x)$ as

$$T_d^{(n)}(x) = \inf_{j \in A^{(n)}(x)} \frac{g(j)}{c},$$

where

$$A^{(n)}(x) := \left\{ j \in S \,\middle|\, \exists k \in \{0, 1, \ldots, n+1\} : \; p_{xj}^{(k)} > 0 \right\}.$$

In this setting the discrete-time embedded Markov chain $(Z_n)_{n \in \mathbb{N}_0}$ has the transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Just as stated in Example 6.4, we can express the transition probabilities for jumping from state $x \in S$ into state $j \in S$ by

$$p_{xj} = 1 - \delta_{xj}.$$

Furthermore the n–step transition probabilities, as defined in Definition 2.17, are given by

$$p_{xj}^{(n)} = \begin{cases} \delta_{xj}, & \text{for } n \text{ even}, \\ 1 - \delta_{xj}, & \text{for } n \text{ odd}. \end{cases}$$

Now let $x \in S$ and $j \in S \backslash \{x\}$. Then we always get $p_{xx}^{(0)} = 1 > 0$ and $p_{xj}^{(1)} = 1 > 0$. Thus both possible states are contained in every single set $A^{(n)}(x)$ for all $n \in \mathbb{N}_0$. We therefore get

$$A^{(n)}(x) = S$$

and thus

$$T_d^{(n)}(x) = \inf_{j \in A^{(n)}(x)} \frac{g(j)}{c} = \inf_{j \in S} \frac{g(j)}{c} = \min\left\{\frac{g(0)}{c}, \frac{g(1)}{c}\right\} = \frac{g(0)}{c},$$

since $g(0) < g(1)$ and $c > 0$.

(b) In this example we only have two possible states the underlying Markov chain can attain. If the initial value of the chain is thus given by $x = 1$ and we know that this state yields the higher reward $g(1) > g(0)$, then it is obviously clear that the optimal stopping time for every n–step value function $V_n(t, 1)$ is given by $\tau_n^\star = 0$. It is always optimal to stop immediately, if the initial value is given by the superior state 1. Waiting in this state will only cumulate unnecessary costs. If the Markov chain jumps into the inferior state 0, the reward will be even smaller. This holds true for every single n–step value function $V_n(t, x)$. As a result we can calculate $V_n(t, x)$ by

$$V_n(t, 1) = V_n(t, 1, \tau_n^\star) = U\big(-ct + g(1)\big) = \ln\big(-c\gamma t + \gamma g(1)\big)$$

for every $n \in \mathbb{N}_0$.

(c) Let $n \in \mathbb{N}_0$ and $x = 0$. If $t \geq T_d^{(n)}(x) = \frac{g(0)}{c}$, then Lemma 7.13 (c) implies that the mapping $m_{n,t,0}$ from (7.22) is constantly given by $-\infty$. The corresponding maximizer in this case is therefore set to $\vartheta_n^\star = 0$. Now let $t \in \big[0, \frac{g(0)}{c}\big)$. As a consequence of Lemma 7.13 (c), as well as part (a) and (b) of this proof, $m_{n,t,0}$ is greater than $-\infty$ on the intervall $\big[0, \frac{g(0)}{c} - t\big)$ and given by

$$\begin{aligned}
&m_{n,t,0}(\vartheta) \\
&= U\big(-ct - c\vartheta + g(0)\big) \cdot e^{-q_0 \cdot \vartheta} + \int_0^\vartheta e^{-q_0 s} \sum_{\substack{j \in S,\\ j \neq 0}} q_{0j} \cdot V_n(t+s, j)\, ds \\
&= U\big(-ct - c\vartheta + g(0)\big) \cdot e^{-q_0 \cdot \vartheta} + \int_0^\vartheta e^{-q_0 s} q_{01} \cdot V_n(t+s, 1)\, ds \\
&= U\big(-ct - c\vartheta + g(0)\big) \cdot e^{-\alpha \cdot \vartheta} + \int_0^\vartheta e^{-\alpha s} q_{01} \cdot U\big(-ct - cs + g(1)\big)\, ds \\
&= \ln\big(-c\gamma(t+\vartheta) + \gamma g(0)\big) \cdot e^{-\alpha \cdot \vartheta} + \int_0^\vartheta e^{-\alpha s} q_{01} \cdot \ln\big(-c\gamma(t+s) + \gamma g(1)\big)\, ds
\end{aligned}$$

for $\vartheta \in \big[0, \frac{g(0)}{c} - t\big)$. Moreover, U is differentiable on $(0, \infty)$ and the derivative U' of U is

$$U(x) = \frac{1}{x}, \qquad \text{for all } x > 0.$$

Therefore, the derivative of $m'_{n,t,0}$ is given by (7.23) as

$$
\begin{aligned}
&m'_{n,t,0}(\vartheta) \\
&= e^{-q_0\vartheta} \cdot \Big(\sum_{\substack{j\in S,\\ j\neq 0}} q_{0j} V_n(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(0)\big) - q_0 U\big(-ct - c\vartheta + g(0)\big)\Big) \\
&= \left(q_{01} \ln\big(-c\gamma(t+\vartheta) + \gamma g(1)\big) - c \cdot \frac{1}{-c(t+\vartheta) + g(0)} - q_0 \ln\big(-c\gamma(t+\vartheta) + \gamma g(0)\big)\right) \\
&\quad \cdot e^{-\alpha\vartheta} \\
&= \left(\alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(1)\big) - c \cdot \frac{1}{-c(t+\vartheta) + g(0)} - \alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(0)\big)\right) \\
&\quad \cdot e^{-\alpha\vartheta} \\
&= e^{-\alpha\vartheta} \cdot \left(\alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(1)\big) - \alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(0)\big) - \frac{1}{\frac{g(0)}{c} - (t+\vartheta)}\right)
\end{aligned}
$$

for all $\vartheta \in \big[0, \frac{g(0)}{c} - t\big)$.

According to Theorem 7.14, the maximizer $\vartheta_n^\star$ for $\sup_{\vartheta\geq 0} m_{n,t,0}$ exists and is given by a finite value from $\big[0, \frac{g(0)}{c} - t\big]$. We can calculate it by identifying the roots of the derivative $m'_{n,t,0}$. Let $t < \frac{g(0)}{c}$ and $\vartheta < \frac{g(0)}{c} - t$. Then it holds:

$$
\begin{aligned}
&m'_{n,t,0}(\vartheta) = 0 \\
&\Leftrightarrow 0 = \alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(1)\big) - \alpha \ln\big(-c\gamma(t+\vartheta) + \gamma g(0)\big) - \frac{1}{\frac{g(0)}{c} - (t+\vartheta)} \\
&\Leftrightarrow \frac{1}{\frac{g(0)}{c} - (t+\vartheta)} = \alpha \ln\left(\frac{-c\gamma(t+\vartheta) + \gamma g(1)}{-c\gamma(t+\vartheta) + \gamma g(0)}\right) \\
&\Leftrightarrow 1 = \alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right) \ln\left(\frac{g(1) - c(t+\vartheta)}{g(0) - c(t+\vartheta)}\right) \\
&\Leftrightarrow 1 = \alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right) \ln\left(1 + \frac{g(1) - g(0)}{g(0) - c(t+\vartheta)}\right) \\
&\Leftrightarrow e = \left[1 + \frac{g(1) - g(0)}{g(0) - c(t+\vartheta)}\right]^{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)} \\
&\Leftrightarrow e = \left[1 + \frac{\frac{\alpha}{c}\big(g(1) - g(0)\big)}{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)}\right]^{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)} .
\end{aligned}
$$

This shows the validity of (7.54). In order to determine the maximizer $\vartheta_n^\star$ we thus have to solve equation (7.54), provided that this yields indeed a maximum of $m_{n,t,0}$.

For the existence and uniqueness of the solution we define the mapping

$$h\colon \left[0, \frac{g(0)}{c} - t\right) \to [0,\infty], \quad h(\vartheta) := \left[1 + \frac{\frac{\alpha}{c}\big(g(1)-g(0)\big)}{\alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right)}\right]^{\alpha\left(\frac{g(0)}{c}-(t+\vartheta)\right)}.$$

Note that h is obviously continuous. Moreover h is of the form

$$x \mapsto \left(1 + \frac{k}{x}\right)^x$$

for

$$k := \frac{\alpha}{c}\big(g(1)-g(0)\big) > 0 \quad \text{and} \quad x := \alpha\left(\frac{g(0)}{c} - (t+\vartheta)\right) > 0.$$

We know from basic calculus that

$$\left(1+\frac{k}{x}\right)^x < e^k$$

for all $x > 0$, $k > 0$ and

$$\lim_{x \searrow 0}\left(1+\frac{k}{x}\right)^x = 1$$

for all $k > 0$. Therefore we know that h is a strictly decreasing mapping such that

$$\lim_{\vartheta \searrow \frac{g(0)}{c}-t} h(\vartheta) = 1 < e.$$

Suppose now that $0 \le t < \frac{g(0)}{c} - \frac{1}{\alpha \ln(K)}$, as postulated in (7.55) of this example, where K was a growth constant for g given in (7.53). We will now show that $h(0) > e$:

$$\begin{aligned}
h(0) &= \left[1 + \frac{\frac{\alpha}{c}\big(g(1)-g(0)\big)}{\alpha\left(\frac{g(0)}{c} - t\right)}\right]^{\alpha\left(\frac{g(0)}{c}-t\right)} \\
&\ge \left[1 + \frac{\frac{\alpha}{c}\big(g(1)-g(0)\big)}{\alpha\left(\frac{g(0)}{c}\right)}\right]^{\alpha\left(\frac{g(0)}{c}-t\right)} \\
&= \left(\frac{g(1)}{g(0)}\right)^{\alpha\left(\frac{g(0)}{c}-t\right)} \\
&> \left(\frac{g(1)}{g(0)}\right)^{\alpha\left(\frac{1}{\alpha\ln(K)}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{g(1)}{g(0)}\right)^{\frac{1}{\ln(K)}} \\
&\geq K^{\frac{1}{\ln(K)}} \\
&= \left(e^{\ln(K)}\right)^{\frac{1}{\ln(K)}} \\
&= e.
\end{aligned}
$$

In summary we can find two arguments ϑ_1 and ϑ_2 such that $h(\vartheta_1) < e < h(\vartheta_2)$. Since h is continuous and strictly decreasing, an application of the intermediate value theorem yields a unique solution $\vartheta_n^\star$ to equation (7.54)

$$h(\vartheta) = e$$

on $\left(0, \frac{g(0)}{c} - t\right)$, if $t \in \left[0, \frac{g(0)}{c} - \frac{1}{\alpha \ln(K)}\right]$. Since the solution has no dependency on $n \in \mathbb{N}_0$ we can conclude that every maximizer $\vartheta_n^\star = f^\star(t, 0)$ is strictly greater than zero and smaller than $\frac{g(0)}{c} - t$.

(d) For $g(0) = 10$, $g(1) = \frac{e^{10}+99}{10} \approx 2212.55$ we can choose the growth factor $K > 1$ from (7.53) to be

$$K = \frac{e^{10} + 99}{100} \approx 221,26.$$

Furthermore, the default time $T_d^{(n)}(x)$ is given by

$$T_d^{(n)}(x) = \frac{g(10)}{c} = 10.$$

For $t \geq 10$ we know that the maximizer $\vartheta_n^\star$ is set to zero. Note that in this example, there is no dependency on $n \in \mathbb{N}_0$.

Now let $t < 10$. We know from part (c) of Example 7.17 that if t fulfills the inequality (7.55)

$$t < \frac{g(0)}{c} - \frac{1}{\alpha \ln(K)} = 10 - \frac{1}{\ln\left(\frac{e^{10}+99}{100}\right)} \approx 9.815,$$

then the corresponding maximizer $\vartheta_n^\star$ is uniquely determined as the solution of equation (7.54), which reads

$$e = \left(1 + \frac{\frac{e^{10}+99}{10} - 10}{10 - t - \vartheta}\right)^{10-t-\vartheta} = \left(1 + \frac{\frac{e^{10}-1}{10}}{10 - t - \vartheta}\right)^{10-t-\vartheta}$$

in this specific case. Substituting

$$u := t + \vartheta$$

yields

$$e = \left(1 + \frac{\frac{e^{10}-1}{10}}{10-u}\right)^{10-u}.$$

We now by part (c) and the intermediate value theorem that a unique solution $u^\star \in (0,10)$ exists. Indeed we can easily check that

$$u^\star = 9.9$$

is the solution of the equation above:

$$\begin{aligned}
&\left(1 + \frac{\frac{e^{10}-1}{10}}{10-u^\star}\right)^{10-u^\star} \\
&= \left(1 + \frac{\frac{e^{10}-1}{10}}{0.1}\right)^{0.1} \\
&= \left(1 + (e^{10}-1)\right)^{0.1} \\
&= e.
\end{aligned}$$

Therefore the solution of (7.54) is given by

$$\vartheta_n^\star = f_n^\star(t,0) = 9.9 - t$$

for every $n \in \mathbb{N}_0$. We can easily see that in this case the solution $\vartheta_n^\star$ exists and is greater than zero, if and only if $t < 9.9$. This is even more than the general interval $[0, 9.815]$, as given in (7.55). On the other hand, for $t \geq 9.9$ the maximizer is set to $\vartheta_n^\star = 0$. By Theorem 7.14 (d) the optimal stopping time for $V_n(t,x)$ is thus given by

$$\tau_n^\star := \left(f_{n-1}^\star(t,x), f_{n-2}^\star(S_1+t, Z_1)+S_1, \ldots, f_0^\star(S_{n-1}+t, Z_{n-1})+S_{n-1}, S_n, S_{n+1}, \ldots\right).$$

For initial value $x = 1$ we know that $f_{n-1}^\star(t,1) = 0$. Thus the stopping time always terminates immediately and we can write $\tau_n^\star = 0$. The same holds true for initial value $x = 0$ and $t \geq 9.9$.

For initial value $x = 0$ and $t < 9.9$ the situation is more interesting. We note that in this setting the next state of the underlying Markov chain after the first jump will always be the superior state 1 and hence $Z_1 = 1$. By studying the first two stopping rules of $\tau_n^\star$ we can see that

$$\begin{aligned}
\tau_n^{\star,0} &= f_{n-1}^\star(t,0) = 9.9 - t \text{ and} \\
\tau_n^{\star,1} &= f_{n-2}^\star(S_1+t, Z_1) + S_1 = f_{n-2}^\star(S_1+t, 1) + S_1 = S_1
\end{aligned}$$

for all $t < 9.9$. The optimal stopping time will thus follow the 0–step stopping rule $\tau_n^{\star,0}$ as long as the first jump does not occur. If the jump happens after $\tau_n^{\star,0} = 9.9 - t$, then the stopping time will terminate at time $9.9 - t$. On the other hand, if the jump happens before time $9.9 - t$, then $\tau_n^{\star,0}$ will be discarded and $\tau_n^{\star,1}$ will take effect. But $\tau_n^{\star,1} = S_1$ always terminates immediately. In summary, the optimal stopping time $\tau_n^\star$ can be expressed as

$$\tau_n^\star = \min\{9.9 - t, S_1\}.$$

□

As we have seen the Bellman equation from Theorem 7.14 provides us with an iterative approach to solve an n–step value function by reducing the problem into a deterministic maximization problem (7.31), provided that the preceding $(n-1)$–step value function is fully solved for every $x \in S$ and $t \geq 0$. Furthermore, every iteration step yields structural information about the stopping rules required to gain the optimal stopping time for the n–step value function. As we were originally interested in the unrestricted value function $V(t, x)$, we can use the Bellman equation to calculate every single n–step value function for every $x \in s$, $t \geq 0$ and use the fact that

$$\lim_{n \to \infty} V_n(t, x) = V(t, x)$$

according to Proposition 6.7 (c). This will at least allow for a useful approximation of $V(t, x)$, even if we did not calculate all n–step value functions, but only up to some sufficiently high step n.

But solving every iteration step and calculating the corresponding maximizers is a non-trivial task. In the next section we will discuss an approach for the unrestricted stopping problem and the corresponding unrestricted value function, which doesn't require to consider the limit of n–step value functions. Instead we will transform the already known Bellman equation from Theorem 7.14 into a fixed-point type equation.

But before we tackle the unrestricted value function, we will first discuss this section with regard to the special choice of an exponential utility as underlying utility function.

7.5 The Bellman Equation for Exponential Utility Functions

Again, we suppose that the underlying utility function U is given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

Similarly to Corollary 7.10, where we were restated the reward iteration formula from Theorem 7.7 for the case of exponential utility, we now want to discuss the Bellman equation from Theorem 7.14 for this particular utility function. The next theorem will cover the adaption of the Bellman equation to the particular n–step value functions $\tilde{V}_n(x)$, as defined in (6.16) of section 6.2 for exponential utility. We will see that the Bellman equation will simplify tremendously. More precisely, the deterministic maximization problem given in the general formulation (7.31) will degenerate into the simple task of choosing the greater value out of two possibilities. Furthermore, we will see that the corresponding maximizers for (7.31) can only take two values:

$$\vartheta_n^\star = 0 \quad \text{ or } \quad \vartheta_n^\star = \infty.$$

These two maximizers will correspond to the two possible arguments for the maximization problem. This leads to optimal stopping rules which behave in only two possible ways, depending on the current state of the underlying Markov chain. These optimal stopping rules will either stop immediately as soon as a certain state is reached, or will never terminate as long as the Markov chain resides in this state. In particular we will find out that the optimal stopping rules, respectively the optimal stopping time consisting of these rules, will never stop between two jumps of the Markov chain. This is of course not surprising, but rather expected. Note that we already mentioned that in the case of exponential utility the time parameter $t \geq 0$, describing the cumulated costs $-ct$ up to time t, does not have any influence on the value functions or the corresponding stopping problems. Since the choice of an optimal stopping time thus only depends on the initial value $x \in S$ of the underlying Markov chain, the decisions whether to stop or not, should only be made at the jump times of the Markov chain at which the changes of state occur.

Before we will confirm the above-mentioned claims for optimal stopping of n–step value functions in case of exponential utility, we first will address the Bellman equation (7.31) of Theorem 7.14 itself. Note that the function, over which the supremum in (7.31) is taken, can be expressed in terms of $m_{n,t,x}$, as defined in (7.22) of Lemma 7.13.

Note that in case of exponential utility, we can express $m_{n,t,x}$ for every $n \in \mathbb{N}_0$, $t \geq 0$

and $x \in S$ by applying (6.16)

$$\begin{aligned}
m_{n,t,x}(\vartheta) &= U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \\
&= -e^{c\gamma t + c\gamma\vartheta - \gamma g(x)} \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot e^{c\gamma t + s} \tilde{V}_n(j)\, ds \\
&= e^{c\gamma t} \left(-e^{(c\gamma - q_x)\vartheta - \gamma g(x)} + \int_0^\vartheta e^{(c\gamma - q_x)s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot \tilde{V}_n(j)\, ds \right) \\
&= e^{c\gamma t} \left(-e^{(c\gamma - q_x)\vartheta} \cdot e^{-\gamma g(x)} + \int_0^\vartheta e^{(c\gamma - q_x)s}\, ds \cdot \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot \tilde{V}_n(j) \right) \\
&= e^{c\gamma t} \left(-e^{-\gamma g(x)} \cdot e^{-(q_x - c\gamma)\vartheta} - \frac{1}{q_x - c\gamma} \left(e^{-(q_x - c\gamma)\vartheta} - 1 \right) \cdot \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot \tilde{V}_n(j) \right) \\
&= e^{c\gamma t} \left(-e^{-\gamma g(x)} \cdot e^{-(q_x - c\gamma)\vartheta} + \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \cdot \left(1 - e^{-(q_x - c\gamma)\vartheta} \right) \right).
\end{aligned}$$

Following the definition of the reduced n–step value function in (6.16), we define

$$\tilde{m}_{n,x}(\vartheta) := -e^{-\gamma g(x)} \cdot e^{-(q_x - c\gamma)\vartheta} + \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \cdot \left(1 - e^{-(q_x - c\gamma)\vartheta} \right). \tag{7.56}$$

Therefore, we can express $m_{n,t,x}$ as

$$m_{n,t,x}(\vartheta) = e^{c\gamma t} \cdot \tilde{m}_{n,x}(\vartheta). \tag{7.57}$$

Now for every $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$, the Bellman equation (7.31) in Theorem 7.14

$$V_{n+1}(t, x) = \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta)$$

is equivalent to

$$\tilde{V}_{n+1}(x) = \sup_{\vartheta \geq 0} \tilde{m}_{n,x}(\vartheta). \tag{7.58}$$

With this knowledge we can now formulate the Bellman theorem for exponential utility functions. Note that we will differentiate the initial states $x \in S$ between two cases

$$x \in S \text{ such that } q_x \leq c\gamma \quad \text{and} \quad x \in S \text{ such that } q_x > c\gamma.$$

As we will see each case will lead to a different result concerning the optimality of n–step value functions. The first case $q_x \leq c\gamma$ will represent the situation where the cost rate $c > 0$ is too high for the Markov chain (with initial value x) to compensate. As a result the optimal stopping rule will be to stop immediately in such a state. On the other hand if $q_x > c\gamma$, then the question of optimality is not trivial but rather depends on the bellman equation (7.60). Based on this equation the optimal course of action is to stop immediately or not to stop at all.

Theorem 7.18 (Bellman equation for exponential utility functions)

(a) Let $n \in \mathbb{N}_0$ and $x \in S$ such that $q_x \leq c\gamma$. Then it holds that

$$\tilde{V}_n(x) = -e^{-\gamma g(x)}. \tag{7.59}$$

(b) Let $n \in \mathbb{N}_0$ and $x \in S$ such that $q_x > c\gamma$. Then we get

$$\tilde{V}_{n+1}(x) = \max\left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \right\}. \tag{7.60}$$

The initial value function $\tilde{V}_0$ is given by

$$\tilde{V}_0(x) = -e^{-\gamma g(x)}. \tag{7.61}$$

(c) For $n \in \mathbb{N}$ the optimal stopping time $\tau_n^\star$ for $\tilde{V}_n(x)$ is given by

$$\tau_n^\star = \left(\tilde{f}_{n-1}^\star(x), \tilde{f}_{n-2}^\star(Z_1) + S_1, \ldots, \tilde{f}_0^\star(Z_{n-1}) + S_{n-1}, S_n, S_{n+1}, \ldots\right), \tag{7.62}$$

where for every $k \in \{0, 1, \ldots, n-1\}$

$$\tilde{f}_k^\star(x) = \begin{cases} 0, & \text{if } q_x \leq c\gamma \quad \text{or} \quad q_x > c\gamma \text{ and } \tilde{V}_k(x) = -e^{-\gamma g(x)}, \\ \infty, & \text{if } q_x > c\gamma \text{ and } \tilde{V}_k(x) = \sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_{k-1}(j). \end{cases} \tag{7.63}$$

For $n = 0$ the optimal stopping time $\tau_n^\star$ for $\tilde{V}_n(x)$ is trivially given by

$$\tau_0^\star = 0.$$

Proof of Theorem 7.18

(a) Let $n \in \mathbb{N}_0$ and $x \in S$ such that $q_x \leq c\gamma$. We thus know that $\vartheta \mapsto e^{-(q_x - c\gamma)\vartheta}$ is increasing on $[0, \infty)$, which implies that

$$\vartheta \mapsto -e^{-\gamma g(x)} \cdot e^{-(q_x - c\gamma)\vartheta}$$

is a decreasing mapping. Moreover we know that $\vartheta \mapsto 1 - e^{-(q_x - c\gamma)\vartheta}$ is decreasing on $[0, \infty)$. Since $\frac{q_{xj}}{q_x - c\gamma} \leq 0$ and

$$\tilde{V}_n(j) = \sup_{\tau \in \Sigma_j} \mathbb{E}_j \Big[- e^{c\gamma(\tau \wedge S_n) - \gamma g(X_{\tau \wedge S_n})} \Big] \leq 0$$

we can conclude that

$$\vartheta \mapsto \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \cdot \left(1 - e^{-(q_x - c\gamma)\vartheta}\right)$$

is also a decreasing mapping. In summary we know that by (7.56) the mapping $\tilde{m}_{n,x}$ is decreasing on $[0, \infty)$. By (7.58) this yields

$$\tilde{V}_{n+1}(x) = \sup_{\vartheta \geq 0} \tilde{m}_{n,x}(\vartheta) = \tilde{m}_{n,x}(0) = -e^{-\gamma g(x)}$$

for every $n \in \mathbb{N}_0$. For $n = 0$ we trivially get $\tilde{V}_0(x) = -e^{-\gamma g(x)}$.

We see that the maximizer is given for every $n \in \mathbb{N}_0$ by $\vartheta_n^\star = 0$. By Theorem 7.14 (b) we set

$$\tilde{f}_n^\star(x) = \vartheta_n^\star = 0.$$

(b) Now let $n \in \mathbb{N}_0$ and $x \in S$ such that $q_x > c\gamma$. This means that $p := e^{-(q_x - c\gamma)\vartheta} \in (0, 1]$ for $\vartheta \in [0, \infty)$. We thus can interpret $\tilde{m}_{n,x}(\vartheta)$ for every $\vartheta \geq 0$ as weighted average

$$\tilde{m}_{n,x}(\vartheta) = -e^{-\gamma g(x)} \cdot p + \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \cdot (1 - p)$$

of the two quantities

$$-e^{-\gamma g(x)} \qquad \text{and} \qquad \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j).$$

Therefore, the inequality

$$\tilde{m}_{n,x}(\vartheta) \leq \max \left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \right\}$$

is valid for every $\vartheta \geq 0$ and hence

$$\tilde{V}_{n+1}(x) = \sup_{\vartheta \geq 0} \tilde{m}_{n,x}(\vartheta) \leq \max \left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \right\}.$$

On the other hand we know that for $\vartheta = 0$ we get $\tilde{m}_{n,x}(0) = -e^{-\gamma g(x)}$ and that

$$\lim_{\vartheta \to \infty} \tilde{m}_{n,x}(\vartheta) = \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j)$$

due to $e^{-(q_x - c\gamma)\vartheta} \to 0$ as $\vartheta \to \infty$.

This yields

$$\tilde{V}_{n+1}(x) = \sup_{\vartheta \geq 0} \tilde{m}_{n,x}(\vartheta) \geq \max \left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \right\}$$

and altogether the Bellman equation (7.60)

$$\tilde{V}_{n+1}(x) = \max \left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_n(j) \right\}.$$

As we have seen the maximization problem $\sup_{\vartheta \geq 0} \tilde{m}_{n,x}(\vartheta)$ reduces to a maximization over two arguments, where these arguments originated from

$$\tilde{m}_{n,x}(0) \text{ for } \vartheta = 0 \quad \text{or} \quad \lim_{\vartheta \to \infty} \tilde{m}_{n,x}(\vartheta) \text{ for } \vartheta \to \infty.$$

Hence the corresponding maximizers $\vartheta_n^\star$ from Theorem 7.14 (b) can only take two values:

$$\vartheta_n^\star = 0 \quad \text{or} \quad \vartheta_n^\star = \infty.$$

More precisely, we get

$$\tilde{f}_n^\star(x) = \vartheta_n^\star = \begin{cases} 0, & \text{if } \tilde{V}_n(x) = -e^{-\gamma g(x)}, \\ \infty, & \text{if } \tilde{V}_n(x) = \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_{n-1}(j). \end{cases}$$

(c) Let $n \in \mathbb{N}$. By combining the cases $q_x \leq c\gamma$ from part (a) and $q_x > c\gamma$ from part (b) we gain

$$\tilde{f}_k^\star(x) := \begin{cases} 0, & \text{if } q_x \leq c\gamma \quad \text{or} \quad q_x > c\gamma \text{ and } \tilde{V}_k(x) = -e^{-\gamma g(x)}, \\ \infty, & \text{if } q_x > c\gamma \text{ and } \tilde{V}_k(x) = \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}_{k-1}(j) \end{cases}$$

for every $k \in \{0, 1, \dots, n-1\}$. An application of Theorem 7.14 (d) thus yields the optimal stopping time $\tau_n^\star$ for $\tilde{V}_n(x)$ by

$$\tau_n^\star = \Big(\tilde{f}_{n-1}^\star(x), \tilde{f}_{n-2}^\star(Z_1) + S_1, \dots, \tilde{f}_0^\star(Z_{n-1}) + S_{n-1}, S_n, S_{n+1}, \dots\Big).$$

The case for $n = 0$ is again trivial and also yields $\tilde{V}_0(x) = -e^{-\gamma g(x)}$ and $\tau_0^\star = 0$.

□

Remark 7.19 (implications of the Bellman equation for exponential utility)

(a) As we have seen, the case of exponential utility leads to a significantly easier Bellman equation. Depending on the current state $x \in S$, we can decide whether the costs are too high (fulfilling $q_x \leq c\gamma$) or low enough (yielding $q_x > c\gamma$). In the first case of high costs the question of optimal stopping is trivially answered by immediate stopping. In the case of low costs we only need to compare two values and take the greater one. Depending on the outcome the corresponding optimal stopping rule stipulates immediate stopping or to never stop in this state. Any other stopping rule is not optimal. In particular it will never be profitable to stop between two jumps, contrary to the case of logarithmic utility in Example 7.17.

(b) Although we have chosen a specific utility function, we still can not give a general closed form for the n–step value functions. These still depend on the specific choice of the underlying Markov chain, its intensity rates and the reward function g in the stopping model. Only for the case of an initial state $x \in S$ such that $q_x \leq c\gamma$ we can infer the explicit closed form of $\tilde{V}_n(x)$, since it is trivially given by the initial utility gained by immediate stopping.

(c) Just as in the general case, the Bellman equation for exponential utility (7.60) allows us to calculate every n–step value function $\tilde{V}_n(x)$ in an iterative way. Each iteration step also provides one stopping rule for the optimal stopping time $\tau_n^\star$. On the other hand, we can not construct these stopping rules without knowledge of the preceding n–step value functions. Hence, even if we are solely interested in the optimal stopping time $\tau_n^\star$ of the n–step value function, we still need to solve every single k–step value function for $k \in \{0, 1, \dots, n-1\}$ in order to get access to $\tau_n^\star$. We will show in chapter 8 that this problem can be circumvented under certain conditions and that under these conditions the decision whether or not we should stop in a certain state can be made without any knowlede on the n–step value functions themselves.

7.6 The Fixed-Point Equation

Definition 7.20

Let $d \in \mathbb{R}$ and $c > 0$. We then define T_d by

$$T_d := \frac{g_{\inf} + d}{c}, \tag{7.64}$$

where

$$g_{\inf} := \inf_{j \in S} g(j).$$

Lemma 7.21

Let $x \in S$, $d \in \mathbb{R}$ and $c > 0$. Then the sequence $\left(T_d^{(n)}(x)\right)_{n \in \mathbb{N}_0}$ from Definition 7.11 is decreasing and

$$T_d^{(n)}(x) \to T_d \quad \text{as} \quad n \to \infty. \tag{7.65}$$

Proof of Lemma 7.21

Let $x \in S$, $d \in \mathbb{R}$ and $c > 0$. By Lemma 7.12 $T_d^{(n)}(x)$ is given for every $n \in \mathbb{N}_0$ as

$$T_d^{(n)}(x) = \inf_{j \in A^{(n)}(x)} \frac{g(j) + d}{c}, \quad n \in \mathbb{N}_0,$$

where

$$A^{(n)}(x) = \left\{ j \in S \middle| \, \exists k \in \{0, 1, \ldots, n+1\} : \, p_{xj}^{(k)} > 0 \right\}.$$

Clearly, we get $A^{(n)}(x) \subseteq A^{(n+1)}(x) \subseteq S$ for every $n \in \mathbb{N}_0$ and thus

$$T_d^{(n)}(x) \geq T_d^{(n+1)}(x) \geq T_d.$$

Therefore, $\left(T_d^{(n)}(x)\right)_{n \in \mathbb{N}_0}$ is an decreasing sequence with lower bound T_d.

For the convergence, we note that the increasing sequence of sets $\left(A^{(n)}(x)\right)_{n \in \mathbb{N}_0}$ converges to

$$A^{\infty}(x) := \bigcup_{n \in \mathbb{N}_0} A^{(n)}(x) = \left\{ j \in S | \, \exists k \in \mathbb{N}_0 : \, p_{xj}^{(k)} > 0 \right\}.$$

But with respect to the irreducibility assumption 2.18, this immediately yields

$$A^{\infty}(x) = S.$$

Therefore we get for every $x \in S$ and $n \to \infty$:

$$\begin{aligned} T_d^{(n)}(x) &= \inf_{j \in A^{(n)}(x)} \frac{g(j)+d}{c} \\ &\to \inf_{j \in A^{\infty}(x)} \frac{g(j)+d}{c} \\ &= \inf_{j \in S} \frac{g(j)+d}{c} \\ &= \frac{g_{\inf}+d}{c} \\ &= T_d. \end{aligned}$$

□

Similarly to section 7.4, we will first state an auxiliary lemma, which will be useful for the subsequent Theorem 7.23 and for the following chapter.

Lemma 7.22

Let $t \geq 0$ and $x \in S$. Then it holds

(a) The mapping $m_{t,x} \colon [0,\infty) \to \mathbb{R}$, defined by

$$m_{t,x}(\vartheta) := U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s, j)\, ds \quad (7.66)$$

is bounded from above by a constant, which only depends on x.

(b) If in addition U is a classical utility function on the whole real line and does not take the value $-\infty$, then $m_{t,x}$ is continuous and almost everywhere differentiable and

$$\begin{aligned} &m'_{t,x}(\vartheta) \qquad (7.67) \\ &= e^{-q_x\vartheta}\Bigg(\sum_{\substack{j \in S, \\ j \neq x}} q_{xj} V(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(x)\big) - q_x U\big(-ct - c\vartheta + g(x)\big)\Bigg) \end{aligned}$$

almost everywhere.

Furthermore, the supremum

$$\sup_{\vartheta \geq 0} m_{t,x}(\vartheta)$$

exists. It is either a maximum and is attained by a finite maximizer $\vartheta^{\star}$ or is an unattainable (finite) supremum, in which case the maximizer is set as $\vartheta^{\star} = \infty$.

In addition there exists a measurable mapping $f^\star\colon [0,\infty) \times S \to [0,\infty]$ such that $f^\star(t,x) = \vartheta^\star$.

Moreover, the mapping

$$[0,\infty) \to \mathbb{R} \cup \{-\infty\}, \quad t \mapsto \sup_{\vartheta \geq 0} m_{t,x}(\vartheta)$$

is measurable for every $x \in S$.

(c) If U is an extended utility function derived from a classical utility function with maximal domain of the form $[-d,\infty)$, the following assertions hold, where T_d is defined according to equation (7.64) of Definition 7.20 as

$$T_d = \frac{g_{\inf} + d}{c}.$$

(i) We get for all $x \in S$ and $t \geq 0$:

$$V(t,x) \begin{cases} > -\infty, & \text{if } t \in [0,T_d], \\ = -\infty, & \text{if } t \in (T_d,\infty). \end{cases} \tag{7.68}$$

(ii) $\vartheta \mapsto m_{t,x}(\vartheta)$ is continuous on $[0,T_d - t]$ and almost everywhere differentiable on $[0,T_d - t)$ and

$$m_{t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in [0,T_d - t], \\ = -\infty, & \text{if } \vartheta \in (T_d - t,\infty). \end{cases} \tag{7.69}$$

On $[0,T_d - t)$ the derivative $m'_{t,x}$ is almost everywhere given by (7.67) and set to zero otherwise. In case that $t > T_d$ and thus $[0,T_d - t] = \emptyset$, the mapping $\vartheta \mapsto m_{t,x}(\vartheta)$ is constantly $-\infty$.

(iii) We get

$$\sup_{\vartheta \geq 0} m_{t,x}(\vartheta) \begin{cases} > -\infty, & \text{if } t \in [0,T_d], \\ = -\infty, & \text{if } t \in (T_d,\infty). \end{cases} \tag{7.70}$$

In case that $T_d < 0$ and thus $[0,T_d] = \emptyset$, we get $(t,x) \mapsto \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) \equiv -\infty$.

Furthermore we get for every $t \in [0,T_d]$ that

$$\sup_{\vartheta \geq 0} m_{t,x}(\vartheta) = \max_{\vartheta \in [0,T_d - t]} m_{t,x}(\vartheta). \tag{7.71}$$

The supremum is always attained by a finite maximizer $\vartheta^\star \in [0,T_d]$, which depends on the actual choice of $x \in S$ and $t \in [0,T_d]$.

Note that for $t > T_d$, $m_{t,x}$ is constantly given by $-\infty$. In this case every $\vartheta \geq 0$ is a maximizer of $\sup_{\vartheta \geq 0} m_{t,x}(\vartheta)$. We will choose the smallest one and set $\vartheta^\star := 0$.

(iv) The mapping

$$[0,\infty) \to \mathbb{R} \cup \{-\infty\}, \quad t \mapsto \sup_{\vartheta \geq 0} m_{t,x}(\vartheta)$$

is continuous on $[0, T_d]$.

Furthermore, there exists a measurable mapping $f^\star \colon [0,\infty) \times S \to \mathbb{R}$ such that $\vartheta^\star = f^\star(t,x)$.

If the maximizer $\vartheta^\star$ for $\sup_{\vartheta\geq 0} m_{t,x}(\vartheta)$ is unique for $t \in [0, T_d]$, then $t \mapsto f^\star(t,x)$ is even continuous.

These statements are basically the unrestricted analogues to Lemma 7.13, replacing $V_n(t,x)$ with $V(t,x)$, respectively $m_{n,t,x}$ with $m_{t,x}$.

Proof of Lemma 7.22
Let $t \geq 0$ and $x \in S$. The proof of parts (a) and (b) is analogeous to the proof of Lemma 7.13 (a) and (b) and is omitted to avoid repetition. For part (c) assume now that U is an extended utility function with maximal domain of the form $[-d,\infty)$.

(i) According to Proposition 6.7, respectively Proposition 6.5 we know that

$$V(t,x) = \lim_{n\to\infty} V_n(t,x)$$

as well as

$$V_n(t,x) \leq V(t,x)$$

for all $n \in \mathbb{N}_0$, $x \in S$ and $t \geq 0$. Furthermore Theorem 7.14 (f) stated

$$V_n(t,x) \begin{cases} > -\infty, & \text{if } t \in \big[0, T_d^{(n-1)}(x)\big], \\ = -\infty, & \text{if } t \in \big(T_d^{(n-1)}(x), \infty\big). \end{cases}$$

for all $n \in \mathbb{N}$, $x \in S$ and $t \geq 0$.

Fix now an $n \in \mathbb{N}$, $x \in S$ and let $t \in \big[0, T_d^{(n-1)}(x)\big]$. Since $T_d \leq T_d^{(n-1)}(x)$ for all $x \in S$ and $n \in \mathbb{N}$ due to Lemma 7.21, we can conclude that $t \in [0, T_d]$. Then it holds that

$$V(t,x) \geq V_n(t,x) > -\infty.$$

On the other hand, if $t > T_d$, then Lemma 7.21 allows us to find an $N \in \mathbb{N}$ such that $t > T_d^{(n-1)}(x)$ for all $n \geq N$. Here (7.35) implies

$$V_n(t,x) = -\infty$$

for all $x \in S$, $n \geq N$ and $t > T_d$ and hence

$$V(t,x) = \lim_{n\to\infty} V_n(t,x) = -\infty$$

for $x \in s$ and $t > T_d$.

Im summary we get

$$V(t,x) \begin{cases} > -\infty, & \text{if } t \in [0, T_d], \\ = -\infty, & \text{if } t \in (T_d, \infty). \end{cases}$$

(ii) Analogously to the proof of Lemma 7.13, we will divide $m_{t,x}$ into two parts:

$$m_{n,t,x}(\vartheta) = \underbrace{U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta}}_{=:P^1(t,x,\vartheta)} + \underbrace{\int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s, j)\, ds}_{=:P^2(t,x,\vartheta)}.$$

Part P^1 is absolutely analogeous to the proof of Lemma 7.13 (c) (ii). We know that

$$\vartheta \mapsto P^1(t,x,\vartheta) \equiv -\infty \Leftrightarrow t > \frac{g(x)+d}{c} \geq T_d$$

and for $t \in \big[0, \frac{g(x)+d}{c}\big]$ we get

$$P^1(t,x,\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in [0, \frac{g(x)+d}{c} - t], \\ = -\infty, & \text{if } \vartheta \in (\frac{g(x)+d}{c} - t, \infty). \end{cases}$$

Furthermore we know that $\vartheta \mapsto P^1(t,x,\vartheta)$ is differentiable on $[0, \frac{g(x)+d}{c} - t)$.

For P^2 we will differentiate two cases:
If $t > T_d$ then $V(t+s,j) = -\infty$ holds for all $j \in S$ and all $s \geq 0$. As a consequence, we get

$$\vartheta \mapsto P^2(t,x,\vartheta) \equiv -\infty \Leftrightarrow t > T_d.$$

If $t \in [0, T_d]$ then

$$V(t+s,j) > -\infty \Leftrightarrow s \in [0, T_d - t].$$

This yields

$$P^2(t,x,\vartheta) \begin{cases} > -\infty, & \text{if } \vartheta \in [0, T_d - t], \\ = -\infty, & \text{if } \vartheta \in (T_d - t, \infty) \end{cases}$$

and thus together with P^1 the validity of (7.69), since $\frac{g(x)+d}{c} \geq T_d$ by definition.

For the continuity and almost everywhere differentiability of $m_{t,x}$ we will use the same reasoning as in Lemma 7.13 (b) (i) to state that the mapping

$$s \mapsto f(s) := e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s, j)$$

is integrable on $[0, \vartheta]$. The Lebesgue differentiation theorem (cf. [Elstrodt, 1996, Theorem 4.14]) now yields the continuity of $\vartheta \mapsto P^2(t, x, \vartheta)$ on $[0, T_d - t]$ as well as the almost everywhere differentiability on $[0, T_d - t)$. The same is thus also valid for $m_{t,x}$ itself and the derivative $m'_{t,x}$ is almost everywhere given by (7.67).

The assertions (iii) - (iv) are again analogeous to the proof of Lemma 7.13 (c) (iii) - (iv). □

Theorem 7.23 (fixed-point equation)

Let $x \in S$ and $t \geq 0$. Then it holds:

(a) The *fixed-point equation* is valid:

$$V(t,x) = \sup_{\vartheta \geq 0} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s, j)\, ds \Big\} \tag{7.72}$$

(b) If U is a classical utility function on the whole real line, then the supremum in (7.72) exists, depending on $x \in S$ and $t \geq 0$. It is either a maximum and is attained by a finite maximizer $\vartheta^\star$ or an unattainable (finite) supremum, in which case the maximizer is set as $\vartheta^\star = \infty$.

Furthermore, there exists a measurable mapping $f^\star\colon [0,\infty) \times S \to [0,\infty]$ such that $f^\star(t,x) = \vartheta^\star$. In addition the value function $V(t,x)$ is measurable in $t \geq 0$ for every $x \in S$.

(c) If U is an extended utility function derived from a classical utility function with maximal domain of the form $[-d, \infty) \subsetneq \mathbb{R}$, then the supremum in (7.72) exists and is even a maximum, depending on $x \in S$ and $t \geq 0$. It is attained by a finite maximizer $\vartheta^\star \in [0, T_d - t]$ with T_d given by (7.64) in Definition 7.20.

Moreover, there exists a measurable mapping $f^\star\colon [0,\infty) \times S \to [0, T_d]$ such that $f^\star(t,x) = \vartheta^\star$. If the maximizer $\vartheta^\star$ is unique for every $t \in [0, T_d]$, then $t \mapsto f^\star(t,x)$ is even continuous on $[0,\infty)$ for every $x \in S$.

In addition the value function $V(t,x)$ is continuous in $t \in [0, T_d]$ for every $x \in S$ and

$$V(t,x) \begin{cases} > -\infty, & \text{if } t \in [0, T_d], \\ = -\infty, & \text{if } t \in (T_d, \infty). \end{cases} \tag{7.73}$$

Remark 7.24

Analogously to Remark 7.15 (g), we set the maximizer $\vartheta^\star = \infty$, if the supremum in (7.72) is not a maximum which can be attained by a finite value. In this case (7.72) reads

$$V(t,x) = \sup_{\vartheta \geq 0} m_{t,x}(\vartheta) = \limsup_{\vartheta \to \infty} m_{t,x}(\vartheta).$$

Now we can prove Theorem 7.23 rigorously.

Proof of Theorem 7.23
Due to the upper boundedness of $m_{t,x}$ by Lemma 7.22 (a) we can guarantee the existence of the supremum in (7.72). By following the reasonings in step 1 of the proof of Theorem 7.14 in which the existence of the maximizers $\vartheta_n^\star$ as well as the existence of the corresponding measurable mappings $f_n^\star$ from Theorem 7.14 (b) and (*c*) are shown, we can reapply the exact same arguments to ensure the existence of a maximizer $\vartheta^\star \in [0,\infty]$ in (b), respectively $\vartheta^\star \in [0, T_d]$ in (c) for the optimization in (7.72). The existence of a measurable mapping $f^\star\colon [0,\infty) \times S \to [0,\infty]$ for (b), respectively $f^\star\colon [0,\infty) \times S \to [0,\infty]$ for (c) follows as well by the same arguments as given in the proof of Theorem 7.14. This leads to the validity of parts (b) and (c) of Theorem 7.23.

For the fixed-point equation (7.72) itself, let $n \in \mathbb{N}_0$ be arbitrary and apply Theorem 7.14 to achieve

$$\begin{aligned} V_{n+1}(t,x) &= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s, j)\, ds \right\} \\ &\leq \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V(t+s, j)\, ds \right\}, \end{aligned}$$

since $V_n(t,x) \leq V(t,x)$ for all $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$ according to (6.7) in Proposition 6.5 (c).

Furthermore, we can now exploit relation (6.11) between value functions and n–step value functions, stating

$$V(t,x) = \lim_{n\to\infty} V_n(t,x),$$

which was established in Proposition 6.7 (c). As a result of n being arbitrary, we thus gain the inequality

$$V(t,x) \leq \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V(t+s, j)\, ds \right\}$$

for all $t \geq 0$ and $x \in S$.

To show the opposite inequality, let $n \in \mathbb{N}_0$ again be arbitrary. A reapplication of Theorem 7.14 yields

$$\begin{aligned} V_{n+1}(t,x) &= \sup_{\vartheta \geq 0} \left\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds \right\} \\ &\geq U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds \end{aligned}$$

for every $t \geq 0$, $x \in S$ and $\vartheta \geq 0$. Taking the limit $n \to \infty$ and utilizing the relation $V(t,x) = \lim_{n\to\infty} V_{n+1}(t,x)$ leads to

$$\begin{aligned} V(t,x) &= \lim_{n\to\infty} V_{n+1}(t,x) \\ &\geq \lim_{n\to\infty} \left(U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds \right) \\ &= U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \lim_{n\to\infty} \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds \\ &= U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot \lim_{n\to\infty} V_n(t+s,j)\, ds \\ &= U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} \cdot V(t+s,j)\, ds, \end{aligned}$$

for every $t \geq 0$, $x \in S$ and $\vartheta \geq 0$. Here, the interchangeability of limit and integral as well as the interchangeability of limit and sum is justified by the monotone convergence theorem, since

$$V_n(t+s,j) \nearrow V(t+s,j) \qquad \text{as } n \to \infty$$

for all $t \geq 0$, $s \in [0,\vartheta]$ and $j \in S$ according to Proposition 6.7 (c) and (d). Also note that $V_n(t+s,j)$ is lower bounded in $n \in \mathbb{N}_0$, $s \in [0,\vartheta]$ and $j \in S$ for every $t \geq 0$, meaning

$$V_n(t+s,j) \geq V_0(t+s,j) = U\big(-c(t+s) + g(j)\big) \geq C \qquad \text{for } (s,j) \in [0,\vartheta] \times S$$

for some constant $C \in \mathbb{R}$, since $s \mapsto U\big(-c(t+s) + g(j)\big)$ is continuous on the compact interval $[0,\vartheta]$ (thus exhibiting a minimum) and $j \mapsto U\big(-c(t+s) + g(j)\big)$ having a minimum, because the reward function g was assumed to be bounded from below, according to its definition in section 5.1.

Continuing the reasoning above and the validity of the last calculation for every $\vartheta \geq 0$ finally leads to

$$V(t,x) \geq \sup_{\vartheta \geq 0} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s,j) \, ds \Big\}$$

for all $t \geq 0$ and $x \in S$.

Hence, we were able to show both inequalities and thus gain the desired fixed-point equation

$$V(t,x) = \sup_{\vartheta \geq 0} \Big\{ U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s,j) \, ds \Big\}$$

for every $t \geq 0$ and $x \in S$. This concludes the proof part (a) of Theorem 7.23. □

An immediate conclusion from Theorem 7.23 under knowledge of the existence of the maximizer $\vartheta^\star = f^\star(t,x)$ in (7.72) for $t \geq 0$ and $x \in S$ can be stated as follows:

Corollary 7.25 (fixed-point equation with applied maximizer)

Let $x \in S$ and $t \geq 0$. Then the fixed-point equation (7.72) can be expressed as

$$V(t,x) = U\big(-ct - cf^\star(t,x) + g(x)\big) \cdot e^{-q_x \cdot f^\star(t,x)} + \int_0^{f^\star(t,x)} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s,j) \, ds. \quad (7.74)$$

Note that in case of $f^\star(t,x) = \infty$ for given $t \geq 0$ and $x \in S$ equation (7.74) is just a symbolic notation for

$$V(t,x) = \limsup_{\vartheta \to \infty} \; U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^{\vartheta} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(t+s,j) \, ds \quad (7.75)$$

according to Remark 7.24.

As we have seen in the previous section, the problem of solving the value function $V(t,x)$ for $t \geq 0$ and $x \in S$ could be tackled by solving all n–step value functions $V_n(t,x)$ iteratively using for example Theorem 7.14 and letting $V_n(t,x)$ converge to the desired unrestricted value function $V(t,x)$. But the tasks of iteratively finding a maximizer of (7.31) and using it to express the next $(n+1)$–step value function in a hopefully closed and manageable form for every iteration step of Theorem 7.14, are not easy in general.

Now having Theorem 7.23 at ones disposal opens up another possiblity for obtaining $V(t,x)$. If we would be able to identify the maximizer $\vartheta^\star$ for the optimization in (7.72), the problem of solving $V(t,x)$ would be reduced to the problem of finding a fixed-point of Equation (7.74) from Corollary 7.25.

In Theorem 7.14, we were additionally able to specify an optimal stopping time for an n–step value function and derive some structural properties that have to be valid. Theorem 7.23 on the other hand lacks this assertion about optimal stopping times for the unrestricted value function $V(t,x)$. This missing study of optimality will be covered in the following section.

7.7 Optimal Stopping Times for the Unrestricted Stopping Problem

By comparing the Bellman equation in Theorem 7.14 and the fixed-point equation in Theorem 7.23 one can easily see that they are structurally quite similar. Roughly spoken, the iteration type Bellman equation (7.31) converges to the fixed-point type equation (7.72) as n tends to infinity. Due to this structural similarity, one could suspect that an optimal stopping time for $V(t,x)$ could be analgeously derived from (7.33) of Theorem 7.14 by using the maximizer $\vartheta^\star = f^\star(t,x)$ from Theorem 7.23. This leads to the following definition:

Definition 7.26

Let $t \geq 0$, $x \in S$ and $f^\star\colon [0,\infty) \times S \to [0,\infty]$ be the maximizing function from Theorem 7.23. Then define

$$\tau^\star_{t,x} := \Big(f^\star(t,x),\ f^\star(S_1+t,Z_1)+S_1,\ f^\star(S_2+t,Z_2)+S_2,\ f^\star(S_3+t,Z_3)+S_3,\dots\Big). \quad (7.76)$$

$\tau^\star_{t,x}$ depends explicitly on the choice of t and x. If it is clear from the context, these two arguments are omited, writing

$$\tau^\star_{t,x} = \tau^\star.$$

This $\tau^\star$ would indeed be the best availible candidate for an optimal stopping time for the unrestricted value function $V(t,x)$. Unfortunately, this $\tau^\star$ is not optimal in general. However, under certain additional conditions we will be able to show the optimality of the very same stopping time.

At first we want to compare $\tau^\star$ with the optimal stopping time $\tau^\star_n \in \Sigma_x$ for $V_n(t,x)$. As stated in Remark 7.15 (e) and (f), we can guarantee that $\tau^\star_n$ is a $\mathbb{P}_x$–almost surely finite stopping time for every $n \in \mathbb{N}_0$. This property always holds despite the fact that for no single stopping rule $\tau^{\star,k}_n$ of $\tau^\star_n$, $k \in \{0,\dots,n-1\}$, the $\mathbb{P}_x$–almost sure finiteness has to be valid. This is owed to the fact that the stopping rules are set to $\tau^{\star,k}_n = S_k$ for every $k \geq n$

and thus forcing the stopping time $\tau_n^\star$ to stop at latest after n jumps of the underlying Markov chain.

As for the unrestricted case, we cannot guarantee the $\mathbb{P}_x$–almost sure finiteness of $\tau^\star$. Every stopping rule $\tau^{\star,k}$ is given by

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k \tag{7.77}$$

for $k \in \mathbb{N}_0$, thus applying the very same function $f^\star$ for every single stopping rule. As $f^\star\colon [0,\infty) \times S \to [0,\infty]$ originates from the maximization problem in (7.72) of Theorem 7.23, it is explicitly allowed to attain infinitely high values for some (or even all) arguments. Hence, there can be a positive probability for the stopping rules $\tau^{\star,k}$ to be infinite.

Opposed to the optimal stopping time $\tau_n^\star$ for $V_n(t,x)$, $\tau^\star$ exhibits no property which forces it to stop after a finite number of jumps of the underlying Markov chain. Thus, the $\mathbb{P}_x$–almost sure finiteness is generally not fulfilled. And even worse, due to the lacking $\mathbb{P}_x$–almost sure finiteness, we cannot apply the decomposition result from Proposition 3.15 to conclude that this piecewise description of $\tau^\star$ using the stopping rules $\tau^{\star,k}$, $k \in \mathbb{N}_0$, leads to $\tau^\star$ being an $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time. Hence we get in general that $\tau^\star \notin \Sigma_x$. On the other hand, by assuming $\tau^\star$ to be $\mathbb{P}_x$–almost surely finite we can conclude that $\tau^\star \in \Sigma_x$. Of course, this assumption has to be manually verified in every application.

We will summarize this observation in the following lemma:

Lemma 7.27

Let $t \geq 0$, $x \in S$ and $\tau^\star$ defined as in (7.76) of Definition 7.26. Then it holds:

$$\mathbb{P}_x\left(\tau^\star < \infty\right) = 1 \quad \Leftrightarrow \quad \tau^\star \in \Sigma_x.$$

Even if we can show that $\tau^\star \in \Sigma_x$, we still need to show its optimality for the unrestricted value function $V(t,x)$. To this end the following proposition will prove to be useful:

Proposition 7.28 (value splitting)

Let $t \geq 0$ and $x \in S$. Assume furthermore that $\tau^\star \in \Sigma_x$ for $\tau^\star$ from Definition 7.26. Then it holds

$$V(t,x) = \mathbb{E}_x\left[U\left(-ct - c\tau^\star + g\big(X_{\tau^\star}\big)\right) \cdot \mathbb{1}_{\{\tau^\star < S_n\}} + V(S_n + t, Z_n) \cdot \mathbb{1}_{\{\tau^\star \geq S_n\}}\right] \tag{7.78}$$

for every $n \in \mathbb{N}$.

Under the assumptions of Proposition 7.28 we can guarantee that $\tau^\star$ is indeed a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\geq 0}$–stopping time. Equation (7.78) now states that for every $t \geq 0$ and $x \in S$ the value function $V(t,x)$ can be splitted into the expectation of two parts, depending whether the stopping time $\tau^\star$ stops before or after the n–th jump of the underlying Markov chain. If $\tau^\star$ stops before S_n we gain the expected utility by applying stopping time $\tau^\star$. On the other hand, if $\tau^\star$ stops after S_n we can "restart" the underlying Markov chain after S_n and take the expectation of the reapplied value function at $S_n + t$ and Z_n.

Note that for $n = 1$ we get (as shown in the proof of Proposition 7.28)

$$\begin{aligned} V(t,x) &= \mathbb{E}_x\left[U\left(-ct - c\tau^\star + g\big(X_{\tau^\star}\big)\right)\cdot \mathbb{1}_{\{\tau^\star < S_1\}} + V(S_1+t, Z_1)\cdot \mathbb{1}_{\{\tau^\star \geq S_1\}}\right] \\ &= U\big(-ct - cf^\star(t,x) + g(x)\big)\cdot e^{-q_x\cdot f^\star(t,x)} + \int_0^{f^\star(t,x)} e^{-q_x s}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\cdot V(t+s,j)\, ds \end{aligned}$$

from Corollary 7.25. Thus, the special case $n = 1$ leads to the already established fixed-point equation with applied maximizers. As we have seen just now, the fixed-point equation can be interpreted in terms of the decision whether or not $\tau^\star$ has stopped before the first jump time S_1. Hence, Equation (7.78) in Proposition 7.28 can be seen as a generalization of the fixed-point equation by interpreting the value function in terms of the decision whether or not $\tau^\star$ has stopped before an arbitrary S_n for $n \in \mathbb{N}$.

Proof of Proposition 7.28

Let $t \geq 0$ and $x \in S$. Using the notation with dependency of $\tau^\star$ from t and x we will prove

$$V(t,x) = \mathbb{E}_x\left[U\left(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\right)\cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_n\}} + V(S_n+t, Z_n)\cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}}\right]$$

for all $n \in \mathbb{N}$ by induction over n. Furthermore, we will denote

$$\tau^\star_{t,x} = \left(\tau^{\star,0}_{t,x}, \tau^{\star,1}_{t,x}, \tau^{\star,2}_{t,x}, \dots\right),$$

where

$$\tau^{\star,k}_{t,x} = f^\star(S_k + t, Z_k) + S_k$$

for $k \in \mathbb{N}_0$. Note that $S_0 = 0$ and $Z_0 = x$.

Induction basis: Let $n = 1$. Using the the decomposition result in Proposition 3.15 and the fact that $\tau^{\star,0}_{t,x} = f^\star(t,x)$ is deterministic, we get

$$\begin{aligned} \{\tau^\star_{t,x} < S_1\} &= \{\tau^{\star,0}_{t,x} < S_1\} = \{f^\star(t,x) < S_1\} \quad \text{and} \\ \{\tau^\star_{t,x} \geq S_1\} &= \{\tau^{\star,0}_{t,x} \geq S_1\} = \{f^\star(t,x) \geq S_1\} \end{aligned}$$

and thus

$$\mathbb{P}_x\big(\tau^\star_{t,x} < S_1\big) = e^{-q_x f^\star(t,x)}$$

according to Corollary 2.20 (a). In addition, the joint density of (S_1, Z_1) given $X_0 = Z_0 = x$ equals

$$\sigma_{S_1,Z_1}(s,j \mid X_0 = x) = \begin{cases} \exp(-q_x \cdot s) \cdot q_{xj}, & \text{if } x \neq j, \\ 0, & \text{if } x = j \end{cases}$$

for every $s \geq 0$ and $j \in S$, due to Corollary 2.20 (c). Hence, we can conclude that

$$\begin{aligned} &\mathbb{E}_x \left[U\left(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_1\}} + V(S_1 + t, Z_1) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_1\}} \right] \\ &= \mathbb{E}_x \left[U\left(-ct - cf^\star(t,x) + g\big(X_0\big)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_1\}} \right] \\ &\quad + \int_0^\infty e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(s+t, j) \cdot \mathbb{1}_{\{f^\star(t,x) \geq s\}} \, ds \\ &= U\big(-ct - cf^\star(t,x) + g(x)\big) \cdot \mathbb{P}_x\big(\tau^\star_{t,x} < S_1\big) + \int_0^{f^\star(t,x)} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(s+t, j) \, ds \\ &= U\big(-ct - cf^\star(t,x) + g(x)\big) \cdot e^{-q_x f^\star(t,x)} + \int_0^{f^\star(t,x)} e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V(s+t, j) \, ds \\ &= V(t,x) \end{aligned}$$

where the last step ensues from Corollary 7.25.

Induction hypothesis: Assume that for some $n \in \mathbb{N}$ equation (7.78)

$$V(t,x) = \mathbb{E}_x \left[U\left(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} + V(S_{n-1} + t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \right]$$

is valid for all $t \geq 0$ and $x \in S$.

Induction step: For the induction step, we will show for all $t \geq 0$ and $x \in S$ that

$$\begin{aligned} &\mathbb{E}_x \left[U\left(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_n\}} + V(S_n + t, Z_n) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}} \right] \\ &= \mathbb{E}_x \left[U\left(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} + V(S_{n-1} + t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \right] \quad (7.79) \\ &= V(t,x), \end{aligned}$$

where the last step holds by induction hypothesis, thus concluding the validity of Equation (7.78) for all $n \in \mathbb{N}$ by induction.

The main difficulty will be to show Equation (7.79). To this end we will first gather the required properties of jump times, embedded Markov chains and stopping times that will be needed in the subsequent steps:

(i) Suppose S_{n-1} and Z_{n-1} are given and the Markov chain X is restarted after S_{n-1}. The first jump time after the reset shall be denoted by $\tilde{S}_1$, whereas the first new state attained by X after Z_{n-1} shall be denoted by $\tilde{Z}_1$. We thus get

$$\tilde{S}_1 \sim Exp(q_{Z_{n-1}})$$

and the joint density of $\tilde{S}_1$ and $\tilde{Z}_1$ given Z_{n-1} is thus given by

$$\sigma(s, j \mid Z_{n-1}) := \sigma_{\tilde{S}_1,\tilde{Z}_1}(s, j \mid Z_{n-1}) := \begin{cases} \exp(-q_{Z_{n-1}} \cdot s) \cdot q_{Z_{n-1}j}, & \text{if } Z_{n-1} \neq j, \\ 0, & \text{if } Z_{n-1} = j \end{cases}$$

according to Corollary 2.20 (c).

(ii) Given S_{n-1} and Z_{n-1} we can define the stopping time $\tau^{\star}_{S_{n-1}+t,Z_{n-1}}$ by

$$\tau^{\star}_{S_{n-1}+t,Z_{n-1}} = \left(\tau^{\star,0}_{S_{n-1}+t,Z_{n-1}}, \tau^{\star,1}_{S_{n-1}+t,Z_{n-1}}, \tau^{\star,2}_{S_{n-1}+t,Z_{n-1}}, \dots\right),$$

where

$$\tau^{\star,0}_{S_{n-1}+t,Z_{n-1}} = f^{\star}(S_{n-1} + t, Z_{n-1})$$

and

$$\tau^{\star,k}_{S_{n-1}+t,Z_{n-1}} = f^{\star}(\tilde{S}_k + S_{n-1} + t, \tilde{Z}_k)$$

for $k \in \mathbb{N}$. Analogously to (i), we assume the Markov chain to be restarted at S_{n-1} and denote the k–th jump time and the k–th state taken after the restart by $\tilde{S}_k$, respectively $\tilde{Z}_k$.

Most importantly, $\tau^{\star,0}_{S_{n-1}+t,Z_{n-1}} = f^{\star}(S_{n-1} + t, Z_{n-1})$ is again deterministic, given S_{n-1} and Z_{n-1}. This leads to

$$\{\tau^{\star}_{S_{n-1}+t,Z_{n-1}} < \tilde{S}_1\} = \{\tau^{\star,0}_{S_{n-1}+t,Z_{n-1}} < \tilde{S}_1\} = \{f^{\star}(S_{n-1} + t, Z_{n-1}) < \tilde{S}_1\} \quad (7.80)$$

given S_{n-1} and Z_{n-1}.

(iii) According to Definition 2.15 and Theorem 2.16 (b) we know that given the history $Z_0, Z_1, \dots, Z_{n-1}$, the sojourn time $T_n = S_n - S_{n-1}$ is independent of $S_{n-1} = \sum_{k=1}^{n-1} T_k$ and

$$T_n \sim Exp(q_{Z_{n-1}}).$$

Furthermore, the state Z_n of the Markov chain X after n jumps does only depend on the previous state and is characterized by the distribution given by the transition probabilities $\left(p_{Z_{n-1},j}\right)_{j \in S}$, as stated in Theorem 2.16 (a). Altogether, we can reapply Corollary 2.20 to formulate the joint density of T_n and Z_n, given Z_{n-1}:

$$\sigma_{T_n,Z_n}(s, j \mid Z_{n-1}) := \begin{cases} \exp(-q_{Z_{n-1}} \cdot s) \cdot q_{Z_{n-1}j}, & \text{if } Z_{n-1} \neq j, \\ 0, & \text{if } Z_{n-1} = j. \end{cases}$$

Hence we get $\sigma(\cdot|Z_{n-1}) := \sigma_{T_n,Z_n}(\cdot|Z_{n-1}) = \sigma_{\tilde{S}_1,\tilde{Z}_1}(\cdot|Z_{n-1})$.

(iv) By assumption we know that $\tau^\star_{t,x} \in \Sigma_x$ and thus fulfills the decomposition representation from Proposition 3.15. Hence, we get

$$\tau^\star_{t,x} = \tau^{\star,0}_{t,x} \mathbb{1}_{\{\tau^\star_{t,x} < S_1\}} + \sum_{k=1}^{\infty} \tau^{\star,k}_{t,x} \mathbb{1}_{\{S_k \le \tau^\star_{t,x} < S_{k+1}\}} \quad \mathbb{P}_x\text{–a.s.}$$

and thus

$$\tau^\star_{t,x} = \tau^{\star,k}_{t,x} \quad \text{on} \quad \{S_k \le \tau^\star_{t,x} < S_{k+1}\}.$$

(v) We can utilize the relation from (iv) to conclude

$$\begin{aligned}\{S_{n-1} \le \tau^\star_{t,x} < S_n\} &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{\tau^\star_{t,x} < S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{\tau^{\star,n-1}_{t,x} < S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) + S_{n-1} < S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\},\end{aligned}$$

which implies

$$\tau^\star_{t,x} = \tau^{\star,n-1}_{t,x} \quad \text{on } \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\}. \tag{7.81}$$

(vi) Using (v), we get

$$\begin{aligned}\{\tau^\star_{t,x} < S_n\} &= \{\tau^\star_{t,x} < S_{n-1}\} \,\dot\cup\, \{S_{n-1} \le \tau^\star_{t,x} < S_n\} \\ &= \{\tau^\star_{t,x} < S_{n-1}\} \,\dot\cup\, \big(\{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\}\big),\end{aligned}$$

where $A \,\dot\cup\, B$ represents the union of two disjoint sets A and B. This leads to

$$\mathbb{1}_{\{\tau^\star_{t,x} < S_n\}} = \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} + \mathbb{1}_{\{\tau^\star_{t,x} \ge S_{n-1}\}} \cdot \mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\}} \tag{7.82}$$

(vii) Using the decomposition representation in Proposition 3.15 we can conclude that

$$\begin{aligned}\{\tau^\star_{t,x} \ge S_n\} &= \{\tau^{\star,0}_{t,x} \ge S_1\} \cap \cdots \cap \{\tau^{\star,n-2}_{t,x} \ge S_{n-1}\} \cap \{\tau^{\star,n-1}_{t,x} \ge S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{\tau^{\star,n-1}_{t,x} \ge S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) + S_{n-1} \ge S_n\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) \ge S_n - S_{n-1}\} \\ &= \{\tau^\star_{t,x} \ge S_{n-1}\} \cap \{f^\star(S_{n-1}+t, Z_{n-1}) \ge T_n\}.\end{aligned}$$

This yields

$$\mathbb{1}_{\{\tau^\star_{t,x} \ge S_n\}} = \mathbb{1}_{\{\tau^\star_{t,x} \ge S_{n-1}\}} \cdot \mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) \ge T_n\}} \tag{7.83}$$

(viii) We know that

$$\begin{aligned}\{\tau^\star_{t,x} \geq S_n\} &= \{\tau^{\star,0}_{t,x} \geq S_1\} \cap \{\tau^{\star,1}_{t,x} \geq S_2\} \cap \cdots \cap \{\tau^{\star,n-1}_{t,x} \geq S_n\} \\ &= \{f^\star(t,x) \geq S_1\} \cap \{f^\star(S_1+t, Z_1) + S_1 \geq S_2\} \\ &\quad \cap \cdots \cap \{f^\star(S_{n-1}+t, Z_{n-1}) + S_{n-1} \geq S_n\}\end{aligned}$$

Here we can see, that on the set $\{\tau^\star_{t,x} \geq S_n\}$ the stopping time $\tau^\star_{t,x}$ only depends on the history $S_1, \ldots, S_n$ and $Z_0, Z_1, \ldots Z_{n-1}$, but not on any variables S_k and Z_k with higher index $k > n$, respectively $k \geq n$.

We will show (7.79) now in 5 steps:

1. Using (7.82) in (vi) yields

$$\begin{aligned}&\mathbb{E}_x\left[U\left(-ct - c\tau^\star_{t,x} + g\left(X_{\tau^\star_{t,x}}\right)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_n\}}\right] \\ &= \mathbb{E}_x\left[U\left(-ct - c\tau^\star_{t,x} + g\left(X_{\tau^\star_{t,x}}\right)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}}\right] \\ &\quad + \mathbb{E}_x\left[U\left(-ct - c\tau^\star_{t,x} + g\left(X_{\tau^\star_{t,x}}\right)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \cdot \mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\}}\right] \quad (7.84)\end{aligned}$$

2. Due to (7.80) in (ii) we have

$$X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}} = X_{\tau^{\star,0}_{S_{n-1}+t, Z_{n-1}}} = Z_{n-1}$$

on the set $\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}$ and thus

$$\begin{aligned}&\mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct - cS_{n-1} - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\left(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\right)\right)\right.\right. \\ &\qquad\qquad \left.\left. \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}} \middle| S_{n-1}\right] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\right] \\ &= \mathbb{E}_x\Big[\mathbb{E}_{Z_{n-1}}\Big[U\Big(- ct - cS_{n-1} - cf^\star(S_{n-1}+t, Z_{n-1}) + g\left(Z_{n-1}\right)\Big) \\ &\qquad\qquad \cdot \mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) < \tilde{S}_1\}} \Big| S_{n-1}\Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \\ &= \mathbb{E}_x\Big[U\Big(- ct - cS_{n-1} - cf^\star(S_{n-1}+t, Z_{n-1}) + g\left(Z_{n-1}\right)\Big) \\ &\qquad \cdot \mathbb{E}_{Z_{n-1}}\left[\mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) < \tilde{S}_1\}} \middle| S_{n-1}\right] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big].\end{aligned}$$

Assuming that S_{n-1} and Z_{n-1} are given and the Markov chain was restarted at S_{n-1} we have defined $\tilde{S}_1$ in (i) as the first jump time after the reset. According to (iii), this is equivalent to the perspective of X not being restarted and interpreting $\tilde{S}_1$ as the sojourn time T_n between the given S_{n-1} and S_n. This leads to

$$\begin{aligned}\mathbb{E}_{Z_{n-1}}\left[\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<\tilde{S}_1\}}\middle|S_{n-1}\right] &= \mathbb{E}\left[\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<\tilde{S}_1\}}\middle|S_{n-1},\ \tilde{X}_0 = Z_{n-1}\right]\\ &= \int_0^\infty \mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<u\}} e^{-q_{Z_{n-1}}u}\,du\\ &= \mathbb{E}_x\left[\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\middle|S_{n-1},\ Z_{n-1}\right]\end{aligned}$$

and thus

$$\begin{aligned}&\mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct-cS_{n-1}-c\tau^\star_{S_{n-1}+t,Z_{n-1}}+g\left(X_{\tau^\star_{S_{n-1}+t,Z_{n-1}}}\right)\right)\right.\right.\\ &\qquad\qquad \left.\left.\cdot\,\mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}}<\tilde{S}_1\}}\middle|S_{n-1}\right]\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\right]\\ &=\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^\star(S_{n-1}+t,Z_{n-1})+g\left(Z_{n-1}\right)\Big)\\ &\qquad\cdot\,\mathbb{E}_x\left[\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\middle|S_{n-1},\ Z_{n-1}\right]\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\Big]\\ &=\mathbb{E}_x\Big[\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^\star(S_{n-1}+t,Z_{n-1})+g\left(Z_{n-1}\right)\Big)\\ &\qquad\cdot\,\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\Big|S_{n-1},\ Z_{n-1}\Big]\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\Big]\\ &=\mathbb{E}_x\Big[\mathbb{E}_x\Big[\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^\star(S_{n-1}+t,Z_{n-1})+g\left(Z_{n-1}\right)\Big)\\ &\qquad\cdot\,\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\Big|S_{n-1},\ Z_{n-1}\Big]\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\Big|S_1,\dots,S_{n-1},Z_1,\dots,Z_{n-1}\Big]\Big],\end{aligned}$$

where the last equality holds due to the tower property. Now given the history $S_1,\dots,S_{n-1},Z_0=x,Z_1,\dots,Z_{n-1}$, we know by (viii) that $\tau^\star_{t,x}$ is fully described on the set $\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}$. This and a reapplication of the tower property leads to

$$\begin{aligned}&\mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct-cS_{n-1}-c\tau^\star_{S_{n-1}+t,Z_{n-1}}+g\left(X_{\tau^\star_{S_{n-1}+t,Z_{n-1}}}\right)\right)\right.\right.\\ &\qquad\qquad \left.\left.\cdot\,\mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}}<\tilde{S}_1\}}\middle|S_{n-1}\right]\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\right]\\ &=\mathbb{E}_x\Big[\mathbb{E}_x\Big[\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^\star(S_{n-1}+t,Z_{n-1})+g\left(Z_{n-1}\right)\Big)\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\\ &\qquad\cdot\,\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\Big|S_{n-1},\ Z_{n-1}\Big]\Big|S_1,\dots,S_{n-1},Z_1,\dots,Z_{n-1}\Big]\Big]\\ &=\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^\star(S_{n-1}+t,Z_{n-1})+g\left(Z_{n-1}\right)\Big)\\ &\qquad\cdot\,\mathbb{1}_{\{f^\star(S_{n-1}+t,Z_{n-1})<T_n\}}\cdot\mathbb{1}_{\{\tau^\star_{t,x}\geq S_{n-1}\}}\Big].\end{aligned}$$

On the other hand, we can utilize (7.81) from (v) to get

$$\begin{aligned}
&\mathbb{E}_x\left[U\left(-ct-c\tau^{\star}_{t,x}+g\left(X_{\tau^{\star}_{t,x}}\right)\right)\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})<T_n\}}\right]\\
&=\mathbb{E}_x\left[U\left(-ct-c\tau^{\star,n-1}_{t,x}+g\left(X_{\tau^{\star,n-1}_{t,x}}\right)\right)\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})<T_n\}}\right]\\
&=\mathbb{E}_x\Big[U\Big(-ct-c\big(f^{\star}(S_{n-1}+t,Z_{n-1})+S_{n-1}\big)+g(X_{S_{n-1}})\Big)\\
&\qquad\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})<T_n\}}\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\Big]\\
&=\mathbb{E}_x\Big[U\Big(-ct-cS_{n-1}-cf^{\star}(S_{n-1}+t,Z_{n-1})+g(Z_{n-1})\Big)\\
&\qquad\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})<T_n\}}\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\Big].
\end{aligned}$$

Finally we have

$$\begin{aligned}
&\mathbb{E}_x\left[U\left(-ct-c\tau^{\star}_{t,x}+g\left(X_{\tau^{\star}_{t,x}}\right)\right)\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})<T_n\}}\right]\\
&=\mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct-cS_{n-1}-c\tau^{\star}_{S_{n-1}+t,Z_{n-1}}+g\left(X_{\tau^{\star}_{S_{n-1}+t,Z_{n-1}}}\right)\right)\right.\right.\\
&\qquad\left.\left.\cdot \mathbb{1}_{\{\tau^{\star}_{S_{n-1}+t,Z_{n-1}}<\tilde{S}_1\}}\middle| S_{n-1}\right]\cdot \mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\right].
\end{aligned}\tag{7.85}$$

3. Suppose again that the history up to the $(n-1)$–th jump is known. More precisely, let $S_1=s_1,\dots,S_{n-1}=s_{n-1}$ and $Z_0=x, Z_1=z_1,\dots,Z_{n-1}=z_{n-1}$. We know by (viii) that $\tau^{\star}_{t,x}$ is fully described on the set $\mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}$. Moreover, given this history the joint distribution of T_n and Z_n is given by $\sigma(\cdot|z_{n-1})$ according to (iii). By additionally using (7.83) from (vii) we get

$$\begin{aligned}
&\mathbb{E}_x\left[V(S_n+t,Z_n)\mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_n\}}\middle|\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}\right]\\
&=\mathbb{E}_x\left[V(S_{n-1}+T_n+t,Z_n)\mathbb{1}_{\{\tau^{\star}_{t,x}\geq S_{n-1}\}}\cdot \mathbb{1}_{\{f^{\star}(S_{n-1}+t,Z_{n-1})\geq T_n\}}\middle|\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}\right]\\
&=\mathbb{E}_x\left[V(s_{n-1}+T_n+t,Z_n)\mathbb{1}_{\{\tau^{\star}_{t,x}\geq s_{n-1}\}}\cdot \mathbb{1}_{\{f^{\star}(s_{n-1}+t,z_{n-1})\geq T_n\}}\middle|\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}\right]\\
&=\mathbb{1}_{\{\tau^{\star}_{t,x}\geq s_{n-1}\}}\cdot\mathbb{E}_x\left[V(s_{n-1}+T_n+t,Z_n)\mathbb{1}_{\{f^{\star}(s_{n-1}+t,z_{n-1})\geq T_n\}}\middle|\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}\right]\\
&=\mathbb{1}_{\{\tau^{\star}_{t,x}\geq s_{n-1}\}}\cdot\int_0^\infty\sum_{j\in S}V(s_{n-1}+u+t,j)\cdot \mathbb{1}_{\{f^{\star}(s_{n-1}+t,z_{n-1})\geq u\}}\cdot\sigma(u,j|z_{n-1})\,du.
\end{aligned}$$

Furthermore, given the history $S_1 = s_1, \dots, S_{n-1} = s_{n-1}$ and $Z_0 = x, Z_1 = z_1, \dots, Z_{n-1} = z_{n-1}$ and knowing the joint distribution of $\tilde{S}_1$ and $\tilde{Z}_1$ is given by $\sigma(\cdot|z_{n-1})$ according to (i), as well as applying the results in (ii) for the set $\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}$, we can use the calculation above to conclude

$$\begin{aligned}
&\mathbb{E}_x\Bigg[\mathbb{E}_{Z_{n-1}}\Big[V(\tilde{S}_1 + S_{n-1} + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}}\Big| S_{n-1}\Big] \\
&\qquad \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Bigg|_{\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}}\Bigg] \\
&= \mathbb{E}_x\Bigg[\mathbb{E}_{z_{n-1}}\Big[V(\tilde{S}_1 + s_{n-1} + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{f^\star(s_{n-1}+t,z_{n-1}) \geq \tilde{S}_1\}}\Big] \\
&\qquad \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq s_{n-1}\}}\Bigg|_{\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}}\Bigg] \\
&= \mathbb{E}_x\Bigg[\int_0^\infty \sum_{j\in S} V(s_{n-1} + u + t, j) \cdot \mathbb{1}_{\{f^\star(s_{n-1}+t,z_{n-1}) \geq u\}} \cdot \sigma(u, j|z_{n-1})\, du \\
&\qquad \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq s_{n-1}\}}\Bigg|_{\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}}\Bigg] \\
&= \mathbb{E}_x\Bigg[\mathbb{E}_x\Big[V(S_n + t, Z_n)\mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}}\Big|_{\substack{S_1,\dots,S_{n-1},\\ Z_1,\dots,Z_{n-1}}}\Big]\Bigg|_{\substack{S_1=s_1,\dots,S_{n-1}=s_{n-1},\\ Z_1=z_1,\dots,Z_{n-1}=z_{n-1}}}\Bigg].
\end{aligned}$$

By applying the tower property we can ultimately establish the last formula in this step:

$$\begin{aligned}
&\mathbb{E}_x\Big[\mathbb{E}_{Z_{n-1}}\Big[V(\tilde{S}_1 + S_{n-1} + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}}\Big| S_{n-1}\Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \\
&= \mathbb{E}_x\Bigg[\mathbb{E}_x\Bigg[\mathbb{E}_{Z_{n-1}}\Big[V(\tilde{S}_1 + S_{n-1} + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}}\Big| S_{n-1}\Big] \\
&\qquad \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Bigg|_{\substack{S_1,\dots,S_{n-1},\\ Z_1,\dots,Z_{n-1}}}\Bigg]\Bigg] \\
&= \mathbb{E}_x\Bigg[\mathbb{E}_x\Bigg[\mathbb{E}_x\Big[V(S_n + t, Z_n)\mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}}\Big|_{\substack{S_1,\dots,S_{n-1},\\ Z_1,\dots,Z_{n-1}}}\Big]\Bigg|_{\substack{S_1,\dots,S_{n-1},\\ Z_1,\dots,Z_{n-1}}}\Bigg]\Bigg] \\
&= \mathbb{E}_x\Big[V(S_n + t, Z_n)\mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}}\Big]. \qquad (7.86)
\end{aligned}$$

4. We have already shown the desired equation (7.78) for $n = 1$ and all $t \geq 0$ and $x \in S$ in the induction basis. Using this formula on $V(S_{n-1} + t, Z_{n-1})$ for given S_{n-1} and Z_{n-1} yields

$$\begin{aligned}
&V(S_{n-1}+t, Z_{n-1}) \\
&= \mathbb{E}_{Z_{n-1}}\Big[U\Big(-c\big(S_{n-1}+t\big) - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\big(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}} \\
&\qquad + V(\tilde{S}_1 + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} \geq \tilde{S}_1\}}\Big].
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\mathbb{E}_x\Big[V(S_{n-1}+t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \qquad (7.87)\\
&= E_x\Big[\mathbb{E}_{Z_{n-1}}\Big[U\Big(-ct - cS_{n-1} - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\big(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}} \\
&\qquad + V(\tilde{S}_1 + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} \geq \tilde{S}_1\}}\Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big].
\end{aligned}$$

5. In this last step we will combine equations (7.84) – (7.87) of all four previous steps to show the desired equation (7.79) from the beginning of the induction step. To this end, let $t \geq 0$ and $x \in S$. Then it holds that

$$\begin{aligned}
&\mathbb{E}_x\Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} + V(S_{n-1}+t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \\
&\overset{(7.84)}{=} \mathbb{E}_x\Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}}\Big] \\
&\quad - \mathbb{E}_x\left[U\left(-ct - c\tau^\star_{t,x} + g\left(X_{\tau^\star_{t,x}}\right)\right) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \cdot \mathbb{1}_{\{f^\star(S_{n-1}+t, Z_{n-1}) < T_n\}}\right] \\
&\quad + \mathbb{E}_x\Big[V(S_{n-1}+t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \\
&\overset{(7.85)}{=} \mathbb{E}_x\Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}}\Big] \\
&\quad - \mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct - cS_{n-1} - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\left(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\right)\right)\right.\right. \\
&\qquad\qquad \left.\left. \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}} \middle| S_{n-1}\right] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\right] \\
&\quad + \mathbb{E}_x\Big[V(S_{n-1}+t, Z_{n-1}) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\Big] \\
&\overset{(7.87)}{=} \mathbb{E}_x\Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}}\Big] \\
&\quad - \mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct - cS_{n-1} - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\left(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\right)\right)\right.\right. \\
&\qquad\qquad \left.\left. \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t, Z_{n-1}} < \tilde{S}_1\}} \middle| S_{n-1}\right] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}}\right] \\
&\quad + \mathbb{E}_x\left[\mathbb{E}_{Z_{n-1}}\left[U\left(-ct - cS_{n-1} - c\tau^\star_{S_{n-1}+t, Z_{n-1}} + g\left(X_{\tau^\star_{S_{n-1}+t, Z_{n-1}}}\right)\right)\right.\right.
\end{aligned}$$

$$\cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} < \tilde{S}_1\}} \Big| S_{n-1} \Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \Big]$$

$$+ E_x \Big[\mathbb{E}_{Z_{n-1}} \Big[V(\tilde{S}_1 + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}} \Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \Big]$$

$$= \mathbb{E}_x \Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} \Big]$$

$$+ E_x \Big[\mathbb{E}_{Z_{n-1}} \Big[V(\tilde{S}_1 + t, \tilde{Z}_1) \cdot \mathbb{1}_{\{\tau^\star_{S_{n-1}+t,Z_{n-1}} \geq \tilde{S}_1\}} \Big] \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_{n-1}\}} \Big]$$

$$\overset{(7.86)}{=} \mathbb{E}_x \Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_{n-1}\}} \Big] + \mathbb{E}_x \Big[V(S_n + t, Z_n) \mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}} \Big]$$

$$= \mathbb{E}_x \Big[U\Big(-ct - c\tau^\star_{t,x} + g\big(X_{\tau^\star_{t,x}}\big)\Big) \cdot \mathbb{1}_{\{\tau^\star_{t,x} < S_n\}} + V(S_n + t, Z_n) \cdot \mathbb{1}_{\{\tau^\star_{t,x} \geq S_n\}} \Big].$$

This proves the validity of (7.79) and thus concludes the proof.

□

We are now able to postulate the main theorem of this section, stating the conditions needed for $\tau^\star$ to be the optimal stopping time for the unrestricted value function $V(t,x)$.

Theorem 7.29 (optimal stopping time for the unrestricted value function)

Let $t \geq 0$ and $x \in S$. Assume that

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1 + t, Z_1) + S_1,\ f^\star(S_2 + t, Z_2) + S_2,\ f^\star(S_3 + t, Z_3) + S_3, \dots \Big)$$

from Definition 7.26 fulfills $\tau^\star \in \Sigma_x$ and

$$\lim_{n\to\infty} \mathbb{E}_x \Big[V(S_n + t, Z_n) \cdot \mathbb{1}_{\{\tau^\star \geq S_n\}} \Big] = 0. \tag{7.88}$$

Then $\tau^\star$ is the optimal stopping time for $V(t,x)$.

Proof of Theorem 7.29
Let $t \geq 0$, $x \in S$ and $\tau^\star \in \Sigma_x$ such that (7.88) is satisfied. Since $S_n \to \infty$ $\mathbb{P}_x$–almost surely by Proposition 2.21, we know that

$$\mathbb{1}_{\{\tau^\star < S_n\}} \nearrow 1 \quad \mathbb{P}_x\text{–a.s.}$$

An application of the monotone convergence theorem as well as condition (7.88) yields

$$\begin{aligned} V(t,x) &= \lim_{n\to\infty} V(t,x) \\ &= \lim_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c\tau^\star + g\big(X_{\tau^\star}\big)\right) \cdot \mathbb{1}_{\{\tau^\star < S_n\}} + V(S_n + t, Z_n) \cdot \mathbb{1}_{\{\tau^\star \geq S_n\}} \right] \\ &= \lim_{n\to\infty} \mathbb{E}_x \left[U\left(-ct - c\tau^\star + g\big(X_{\tau^\star}\big)\right) \mathbb{1}_{\{\tau^\star < S_n\}} \right] + \lim_{n\to\infty} \mathbb{E}_x \left[V(S_n + t, Z_n) \mathbb{1}_{\{\tau^\star \geq S_n\}} \right] \\ &= \mathbb{E}_x \left[U\left(-ct - c\tau^\star + g\big(X_{\tau^\star}\big)\right) \right]. \end{aligned}$$

Hence, $\tau^\star \in \Sigma_x$ fulfills

$$V(t,x) = \sup_{\tau \in \Sigma_x} V(t,x,\tau) = V(t,x,\tau^\star)$$

and thus is the optimal stopping time for $V(t,x)$. □

Remark 7.30 (structure of the optimal stopping time)

(a) Theorem 7.29 gives us the optimal stopping time $\tau^\star$ for the unrestricted value function $V(t,x)$, provided that condition (7.88) is satisfied and that we were able to guarantee the finiteness of $\tau^\star$. Therefore $\tau^\star$ is a feasible stopping time from the class Σ_x and maximizes the value function.

(b) Provided that the requirements of Theorem 7.29 are satisfied and $\tau^\star$ is the optimal stopping time for $V(t,x)$, we can know that the main ingredient needed for $\tau^\star$ is the maximizing mapping $f^\star\colon [0,\infty) \times S \to [0,\infty]$, which stems from the maximization problem of fixed-point equation (7.72) of Theorem 7.23. As we can clearly see the stopping time $\tau^\star$ consists of stopping rules $\tau^{\star,k}$, which all utilize the same function $f^\star$. The only difference lies in the current jump time and the current state of the underlying Markov chain. Note again that provided $f^\star$ is known, a specific stopping rule $\tau^{\star,k} = f^\star(t+S_k, Z_k) + S_k$ for some $k \in \mathbb{N}_0$ is fully determined by the knowledge of S_k and Z_k. In particular, we do not need to know any information about preceding jump times or states before the k–th jump. An investor who applies this stopping time is able to observe the last jump time as well as the last state the Markov chain was in and therefore has every required information to calculate the value of $\tau^{\star,k}$. All he has to do is to terminate the stopping problem, if the next jump does not occur before this calculated value. On the other hand, if this jump happens before, he will discard $\tau^{\star,k}$ and will apply the next stopping rule $\tau^{\star,k+1}$.

(c) Being able to solve the fixed-point equation (7.72) will provide us with both the explicit form of the value function $V(t,x)$ and a candidate for the optimal stopping time for $V(t,x)$. Even if we are not interested in the value function itself but only in the optimal stopping time $\tau^\star$, we have in general no other means to get $\tau^\star$ without the knowledge of $V(t,x)$. In chapter 8 we will derive conditions under which we can at least identify the optimality of some special stopping rules without having to know $V(t,x)$ or solving the fixed-point equation (7.72).

7.8 The Fixed-Point Equation for Exponential Utility Functions

Again, we suppose that the underlying utility function U is given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

Analogously to section 7.5 we want to study Theorem 7.23 and Theorem 7.29 for the special case of exponential utility and analyze if we can again achieve a significant simplification of the fixex-point equation. In fact we will see that the situation is analogeous to Theorem 7.18. The iterative Bellman equation (7.60) will indeed transform into a fixed-point equation of similar structure.

To this end we note that according to (6.14) and (6.13) we get the reduced unrestricted value function $\tilde{V}(x)$ by

$$\tilde{V}(x) = \sup_{\tau \in \Sigma_x} \mathbb{E}_x\Big[- e^{c\gamma\tau - \gamma g(X_\tau)}\Big],$$

which fulfills for every $t \geq 0$ and $x \in S$ that

$$V(t,x) = e^{c\gamma t}\tilde{V}(x).$$

Using the definition of the mapping $m_{t,x}$ in (7.66) we can rewrite the fixed-point equation (7.72) from Theorem 7.23 in terms of

$$V(t,x) = \sup_{\vartheta \geq 0} m_{t,x}(\vartheta).$$

Analogous to (7.56) and (7.57) we can define the mapping

$$\tilde{m}_x(\vartheta) := -e^{-\gamma g(x)} \cdot e^{-(q_x - c\gamma)\vartheta} + \sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma}\tilde{V}(j) \cdot \big(1 - e^{-(q_x - c\gamma)\vartheta}\big) \tag{7.89}$$

and state the relation

$$m_{t,x}(\vartheta) = e^{c\gamma t} \cdot \tilde{m}_x(\vartheta). \tag{7.90}$$

Now for every $t \geq 0$ and $x \in S$, the fixed-point equation (7.72) is equivalent to

$$\tilde{V}(x) = \sup_{\vartheta \geq 0} \tilde{m}_x(\vartheta). \tag{7.91}$$

This allows us to formulate the fixed-point equation for the case of exponential utility:

Theorem 7.31 (fixed-point equation for exponential utility functions)

(a) Let $x \in S$ such that $q_x \leq c\gamma$. Then it holds that

$$\tilde{V}(x) = -e^{-\gamma g(x)}. \tag{7.92}$$

(b) Let $x \in S$ such that $q_x > c\gamma$. Then we get

$$\tilde{V}(x) = \max\left\{ -e^{-\gamma g(x)}, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}(j) \right\}. \tag{7.93}$$

(c) Define the mapping $\tilde{f}^\star \colon S \to \{0, \infty\}$ by

$$\tilde{f}^\star(x) := \begin{cases} 0, & \text{if } q_x \leq c\gamma \quad \text{or} \quad q_x > c\gamma \text{ and } \tilde{V}(x) = -e^{-\gamma g(x)}, \\ \infty, & \text{if } q_x > c\gamma \text{ and } \tilde{V}(x) = \sum\limits_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}(j) \end{cases} \tag{7.94}$$

and the stopping time $\tau^\star$ by

$$\tau^\star := \left(\tilde{f}^\star(x), \tilde{f}^\star(Z_1) + S_1, \tilde{f}^\star(Z_2) + S_2, \dots \right). \tag{7.95}$$

If $\tau^\star \in \Sigma_x$ and

$$\lim_{n \to \infty} \mathbb{E}_x \left[\tilde{V}(Z_n) \cdot \mathbb{1}_{\{\tau^\star \geq S_n\}} \right] = 0, \tag{7.96}$$

then $\tau^\star$ is the optimal stopping time for $\tilde{V}(x)$.

Proof of Theorem 7.31
Parts (a) and (b) are absolutely analogous to the proof of Theorem 7.18, replacing $\tilde{V}_n(x)$ with $\tilde{V}(x)$, $\tilde{m}_{n,x}$ with $\tilde{m}_x$ and applying Theorem 7.23 instead of Theorem 7.14. This will also yield the form of the corresponding maximizers and thus (7.94).

Given (7.94), part (c) of Theorem 7.31 is a direct application of Theorem 7.29. □

Remark 7.32 (discussion of Theorem 7.31)

(a) The fixed-point equation (7.93) is the unrestricted analogon to the iteration type Bellman equation (7.60).

(b) As we can clearly see from (7.94), we absolutely need the explicit solution of the unrestricted value function $\tilde{V}(x)$ in order to calculate $\tilde{f}^\star(x)$, if $q_x > c\gamma$. Without the knowledge of $\tilde{V}(x)$ we can not compute the stopping rules of the candidate $\tau^\star$

for the optimal stopping time. In the next chapter we will impose certain conditions which allow us to identify the values of $\tilde{f}^\star$ without knowing $\tilde{V}(x)$.

(c) Just as in the general case the additional condition (7.96) can not be dropped. We know that

$$\mathbb{1}_{\{\tau^\star \geq S_n\}} \to 0 \;\; \mathbb{P}_x\text{–a.s.} \quad \text{as} \quad n \to \infty$$

provided that $\tau^\star \in \Sigma_x$. Therefore we can see (7.96) as a condition on the embedded discrete time Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ as well as the reward function g of the stopping model, such that $\tilde{V}(Z_n)$ does not "grow to fast" in expectation.

(d) We know that we can express the intensity rates q_{xj} in terms of the transition probabilities p_{xj} of the embedded discrete-time Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ by

$$p_{xj} = \frac{q_{xj}}{q_x}, \text{ for } x, j \in S, x \neq j \quad \text{and} \quad p_{xx} = 0.$$

We therefore get

$$\begin{aligned}
\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma}\tilde{V}(j) &= \frac{1}{q_x - c\gamma}\sum_{\substack{j\in S,\\ j\neq x}} q_{xj}\tilde{V}(j) \\
&= \frac{q_x}{q_x - c\gamma}\sum_{\substack{j\in S,\\ j\neq x}} p_{xj}\tilde{V}(j) \\
&= \frac{q_x}{q_x - c\gamma}\sum_{j\in S} p_{xj}\tilde{V}(j) \\
&= \frac{q_x}{q_x - c\gamma}\mathbb{E}_x\big[\tilde{V}(Z_1)\big].
\end{aligned}$$

This allows for a different representation of the fixed-point equation (7.93) in terms of the expected value function with initial value Z_1:

$$\tilde{V}(x) = \max\left\{-e^{-\gamma g(x)},\ \frac{q_x}{q_x - c\gamma}\mathbb{E}_x\big[\tilde{V}(Z_1)\big]\right\}. \tag{7.97}$$

We can see in this representation that the maximum-operator in the fixed-point equation compares the actual initial utility $-e^{-\gamma g(x)}$ gained by immediate stopping with the expected value function $\mathbb{E}_x\big[\tilde{V}(Z_1)\big]$ weighted by the factor $\frac{q_x}{q_x - c\gamma}$, where Z_1 indicates the next state in which the underlying Markov chain will jump after leaving the initial state x.

Example 7.33 (example for the fixed-point equation for exponential utility functions)
Let $S = \{0, 1\}$ and X a continuous-time Markov chain with intensity matrix Q given by

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

for some $\alpha, \beta > 0$ (cf. Example 2.14). Furthermore, consider the exponential utility function

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$. Moreover, let $c > 0$ be the cost rate and $g\colon S \to \mathbb{R}$ the reward function in this setting. Assume that

$$\alpha \le c\gamma < \beta. \tag{7.98}$$

Then it holds:

(a) The value function $\tilde{V}(x)$ is given by

$$\tilde{V}(0) = -e^{-\gamma g(0)}$$

and

$$\tilde{V}(1) = \begin{cases} -e^{-\gamma g(1)}, & \text{if } g(0) \le g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right), \\ -\frac{\beta}{\beta - \gamma c} e^{\gamma c t - \gamma g(0)}, & \text{if } g(0) > g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right). \end{cases}$$

(b) The optimal stopping time $\tau^\star$ for $\tilde{V}(x)$ is given by

$$\tau^\star = \begin{cases} 0, & \text{if } x = 0 \quad \text{or} \quad x = 1 \text{ and } g(0) \le g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right), \\ S_1, & \text{if } x = 1 \text{ and } g(0) > g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right). \end{cases}$$

(c) Depending on x and on the reward values $g(0)$ and $g(1)$, we get

$$\tilde{V}(x) = \tilde{V}(x, 0) \text{ for stopping time } \tau = 0 \quad \text{or}$$
$$\tilde{V}(x) = \tilde{V}(x, S_1) \text{ for stopping time } \tau = S_1.$$

We have already calculated these value functions in Example 6.4 and can now verify them again.

(d) Suppose now that $\gamma = 1$, $c = 2$, $\alpha = 1$, $\beta = 3$ and $g(0) = 10$. If $g(1) = 8$, then the optimal stopping time is given by $\tau^\star = S_1$ for initial value 1 and $\tau^\star = 0$ for $x = 0$. If on the other hand $g(1) = 9$, then it is always optimal to stop immediately.

Proof

(a) Let $x = 0$. Theorem 7.31 (a) yields

$$\tilde{V}(0) = -e^{-\gamma g(0)}$$

for $\alpha = q_0 \leq c\gamma$.

Now let $x = 1$. Since $\beta = q_1 > c\gamma$, we use (7.93) of Theorem 7.31 (b) and compare

$$\begin{aligned}
&- e^{-\gamma g(1)} \geq \sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{1j}}{q_1 - c\gamma} \tilde{V}(j) \\
\Leftrightarrow &- e^{-\gamma g(1)} \geq \frac{q_{10}}{q_1 - c\gamma} \tilde{V}(0) \\
\Leftrightarrow &- e^{-\gamma g(1)} \geq -\frac{\beta}{\beta - c\gamma} e^{-\gamma g(0)} \\
\Leftrightarrow & e^{-\gamma(g(1)-g(0))} \leq \frac{\beta}{\beta - c\gamma} \\
\Leftrightarrow & g(1) - g(0) \geq -\frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right) \\
\Leftrightarrow & g(0) \leq g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right).
\end{aligned}$$

This yields

$$\tilde{V}(1) = \begin{cases} -e^{-\gamma g(1)}, & \text{if } g(0) \leq g(1) + \frac{1}{\gamma}\ln\left(\frac{\beta}{\beta - c\gamma}\right), \\ -\frac{\beta}{\beta - \gamma c} e^{\gamma c t - \gamma g(0)}, & \text{if } g(0) > g(1) + \frac{1}{\gamma}\ln\left(\frac{\beta}{\beta - c\gamma}\right). \end{cases}$$

(b) Using (7.94) and (7.95) of Theorem 7.31 (c) provides us with

$$\tilde{f}^\star(x) = \begin{cases} 0, & \text{if } x = 0 \quad \text{or} \quad x = 1 \text{ and } g(0) \leq g(1) + \frac{1}{\gamma}\ln\left(\frac{\beta}{\beta - c\gamma}\right), \\ \infty, & \text{if } x = 1 \text{ and } g(0) > g(1) + \frac{1}{\gamma}\ln\left(\frac{\beta}{\beta - c\gamma}\right) \end{cases}$$

and

$$\tau^\star = \left(\tilde{f}^\star(x), \tilde{f}^\star(Z_1) + S_1, \tilde{f}^\star(Z_2) + S_2, \dots\right).$$

For $x = 0$ or $x = 1$ and $g(0) \leq g(1) + \frac{1}{\gamma}\ln\left(\frac{\beta}{\beta - c\gamma}\right)$ we know that $\tilde{f}^\star(x) = 0$ and thus

$$\tau^\star = 0.$$

This stopping time is trivially $\mathbb{P}_x$–almost surely finite and fulfills (7.96).

For $x = 1$ and $g(0) > g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right)$ we know that $\tilde{f}^\star(x) = \infty$. Therefore, the stopping time $\tau^\star$ will not stop before the first jump time S_1. After the jump, the new state is deterministically given by $Z_1 = 0$, which leads to $\tilde{f}^\star(Z_1) = 0$. Hence we get

$$\tau^\star = S_1.$$

This stopping time is also $\mathbb{P}_x$–almost surely finite. It is very easy to see that (7.96) is fulfilled.

Therefore

$$\tau^\star = \begin{cases} 0, & \text{if } x = 0 \quad \text{or} \quad x = 1 \text{ and } g(0) \leq g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right), \\ S_1, & \text{if } x = 1 \text{ and } g(0) > g(1) + \frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right) \end{cases}$$

is the optimal stopping time for $\tilde{V}(x)$.

(c) In Example 6.4, we calculated

$$V(t, x, 0) = -e^{\gamma ct - \gamma g(x)}$$

and

$$V(t, x, S_1) = V_1(t, x, S_2) = -e^{\gamma ct - \gamma g(0)} \cdot \frac{\beta}{\beta - \gamma c}, \quad \text{for } x = 1 \text{ and } \beta > \gamma c.$$

These two value functions coincide with the ones we calculated, if we reapply the time parameter $t \geq 0$ via

$$V(t, x) = e^{c\gamma t} \tilde{V}(x).$$

(d) Let $g(0) = 10$. We calculate

$$\frac{1}{\gamma} \ln\left(\frac{\beta}{\beta - c\gamma}\right) = \ln(3) \approx 1.098.$$

If $g(1) = 8$, then $g(0) > \ln(3) + g(1)$. If $g(1) = 9$, then $g(0) \leq \ln(3) + g(1)$. Part (b) of the example yields the desired assertion.

□

8 Optimality of Special Stopping Times

In our setting of generalized risk-sensitive stopping problems for continuous-time Markov chains we derived several assertions for characterizations of optimal stopping times. In this chapter we want to discuss the optimality of stopping times which exhibit special structures and which have a clear interpretation. These special stopping times will mainly consist of stopping rules which imply the optimality of immediate stopping or never stopping in certain situations. Furthermore we will identify conditions under which the optimality of such stopping rules is guaranteed.

Note that in the continuous-time setting the two cases of immediate stopping or never stopping, if the Markov chain hits certain states, do not cover all possible stopping rules for our continuous-time stopping problem. We will see that there are indeed situations in which the question of optimality depends additionally on the actual sojourn time in a given state of the Markov chain. Thus we will be able to construct situations for which an optimal stopping rule between two jumps could read like this: If a certain state $x \in S$ is reached, do not stop at first but wait and hope for a short sojourn time until the next change of state occurs. If this happens "fast enough" then apply the new appropriate stopping rule for the next period. If this doesn't happen "fast enough", then stop after a certain period of time. This accommodates for situations where an optimal policy would consist at first of waiting for a "better" evolution of the underlying Markov chain. If this does not happen fast enough the cumulated costs of additional waiting up to some point outweigh the expected gain from future developments, making stopping and terminating the problem between two jumps optimal.

In some sense this chapter can be seen as a continuous-time generalization of a paper from Kadota et al. [1996], where optimal stopping under general utility, but discrete-time Markov chains was considered. As a consequence of this discrete-time setting the dependency of the value function on the time parameter is revoked. The question of optimal stopping is reduced to the search of a discrete time point at which such jumps can occur, but never between. In addition, Kadota, Kurano and Yasuda established conditions under which special stopping times, the so-called *one-step look ahead* (OLA) stopping times, are optimal. We will see that the conditions for immediate stopping or never stopping, as given in the subsequent sections, can be seen as generalizations of these OLA stopping times. In some special cases like the choice of an exponential utility as utility function, our setting can be simplified to match the criteria of Kadota et al. [1996].

8.1 General Concepts

In chapter 7 we were able to establish the Bellman equation ((7.31) in Theorem 7.14), which allowed us to calculate every n–step value function by iteratively solving all preceding k–step value functions for $k \in \{0, 1, \ldots, n\}$, instead of solving the n–step stopping problem (5.2) directly by maximizing over all feasible stopping times in Σ_x. In order to calculate all these value functions, it was required to solve a deterministic maximization problem over all non-negative numbers $\vartheta \geq 0$ in (7.31). Using the notation in (7.22) we can briefly recapitulate this deterministic problem by

$$m_{k,t,x}(\vartheta) \to \sup_{\vartheta \geq 0}! \tag{8.1}$$

The main feature was that the maximizers $\vartheta_k^\star = f_k^\star(t, x)$ solving these maximization problems for $k \in \{0, 1, \cdots, n\}$ were the key to construct the optimal stopping time

$$\tau_n^\star = \Big(f_{n-1}^\star(t, x), f_{n-2}^\star(S_1 + t, Z_1) + S_1, \ldots, f_0^\star(S_{n-1} + t, Z_{n-1}) + S_{n-1}, S_n, S_{n+1}, \ldots \Big)$$

for the n–step value function $V_n(t, x)$, as given in (7.33).

Similarily, we were able to extend this theory for the unrestricted value function by establishing the fixed-point equation ((7.72) in Theorem 7.23), which translated the task of solving the unrestricted stopping problem (5.1) by maximizing over all feasible stopping times in Σ_x into the task of solving a deterministic fixed-point equation, coupled with a deterministic maximization problem over all non-negative numbers $\vartheta \geq 0$ in (7.72). Using (7.66), this problem can be expressed as

$$m_{t,x}(\vartheta) \to \sup_{\vartheta \geq 0}! \tag{8.2}$$

We have also shown in Theorem 7.29 that the maximizer $\vartheta^\star = f^\star(t, x)$ in (7.72) can again be utilized to construct

$$\tau^\star = \Big(f^\star(t, x),\ f^\star(S_1 + t, Z_1) + S_1,\ f^\star(S_2 + t, Z_2) + S_2,\ f^\star(S_3 + t, Z_3) + S_3, \ldots \Big)$$

as formulated in (7.77). At least under the conditions of Theorem 7.29 we have shown that this $\tau^\star$ is an optimal stopping time for the unrestricted value function $V(t, x)$.

Both methods require us to be able to solve the corresponding maximization problems (8.1), respectively (8.2). In general it is quite difficult to solve these problems explicitly. Thus the question arises, whether one would be able to derive at least certain optimal stopping rules in a more computable way, without the need to solve the above-mentioned problems (8.1) or (8.2) explicitly. By "certain optimal stopping rules" we mean stopping rules, which stipulate an immediate stopping of the stopping problems (5.2) or (5.1) as soon as a certain time or state is reached, or stopping rules, which imply that the corresponding stopping problems will never terminate as long as the underlying Markov chain remains in a certain state.

Let us at first consider the unrestricted value function $V(t,x)$. For the remainder of this chapter we will assume that the requirements of Theorem 7.29 are fulfilled and the optimal stopping time for $V(t,x)$ is given by

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1+t,Z_1)+S_1,\ f^\star(S_2+t,Z_2)+S_2,\ f^\star(S_3+t,Z_3)+S_3,\dots \Big) \in \Sigma_x.$$

As we can see from Proposition 3.15, the stopping time $\tau^\star$ is uniquely determined between two jumps S_k and S_{k+1} by its corresponding stopping rule

$$\tau^{\star,k} = f^\star(S_k+t,Z_k)+S_k.$$

Moreover, this stopping rule is applied at the moment the k–th jump occures, given that $\tau^\star$ did not stop beforehand. But given that this case occured, the k–th jump time S_k, as well as the k–th state Z_k are known and determine $\tau^{\star,k}$ completely on the whole interval $[S_k,S_{k+1})$. As discussed in Remark 7.15 (d), these two quantities are the only ones which control the actual value of $\tau^{\star,k}$. Every other jump time $S_1,\dots,S_{k-1}$ or post-jump state $Z_0,Z_1,\dots,Z_{k-1}$ doesn't have any influence on the stopping rule. Thus, $\tau^{\star,k}$ is independent of every preceding stopping rule $\tau^{\star,0},\dots,\tau^{\star,k-1}$. Once the k–th jump time and the k–th state of the underlying Markov chain are observed, the optimal stopping time is completely determined up to the next jump.

Suppose now that the k–th jump time S_k, as well as the k–th state Z_k are known and the optimal stopping time $\tau^\star$ did not stop before S_k. Since the stopping rule $\tau^{\star,k}$ is given by knowing the exact value of $f^\star$ evaluated at S_k+t and Z_k, this maximizer function is the object of interest to decide the evolution of the optimal stopping time. We are especially interested in the following two cases:

If $f^\star(S_k+t,Z_k)=\infty$, the corresponding stopping rule is set to $\tau^{\star,k}=\infty$. Hence on the interval $[S_k,S_k+1)$, the stopping time $\tau^\star$ is constantly given by infinity and will thus never be able to stop before the next jump time S_{k+1}. As a consequence we can say that if $f^\star(S_k+t,Z_k)=\infty$ for given S_k, Z_k and $t\geq 0$, then the optimal stopping policy would be to wait until the next change of state of the underlying Markov chain occurs.

If $f^\star(S_k+t,Z_k)=0$, the corresponding stopping rule is set to $\tau^{\star,k}=S_k$. Hence on the interval $[S_k,S_k+1)$, the stopping time $\tau^\star$ is given by S_k and will thus by definition trigger immediately the moment the Markov chain jumps into state Z_k at time S_k. As a consequence we can say that if $f^\star(S_k+t,Z_k)=0$ for given S_k, Z_k and $t\geq 0$, then the optimal stopping policy would be to stop immediately once X reached Z_k at time S_k.

This concept remains valid for the study n–step value functions. The special conditions for the optimal stopping times for $V_n(t,x)$ are quite similar. Given the optimal stopping time

$$\tau_n^\star = \Big(f_{n-1}^\star(t,x), f_{n-2}^\star(S_1+t,Z_1)+S_1,\dots,f_0^\star(S_{n-1}+t,Z_{n-1})+S_{n-1},S_n,S_{n+1},\dots \Big)$$

for the n–step value function $V_n(t,x)$ and stopping rules

$$\tau_n^{\star,k} = \begin{cases} f^\star_{n-1-k}(S_k + t, Z_k) + S_k, & k < n, \\ S_k, & k \geq n, \end{cases}$$

we can again differentiate the two special cases:

If $f^\star_{n-1-k}(S_k + t, Z_k) = \infty$ for some $t \geq 0$, $k \in \{0, 1, \ldots, n-1\}$ and S_k, Z_k are known, then the corresponding stopping rule, and thus the stopping time itself is given by $\tau_n^\star = \tau_n^{\star,k} = \infty$ on $[S_k, S_{k+1})$. Analogously to the reasoning above, this prevents $\tau_n^\star$ from stopping before S_{k+1} and leads to the optimal stopping policy to never stop as long as the Markov chain remains in state Z_k.

If on the other hand $f^\star_{n-1-k}(S_k + t, Z_k) = 0$ for some $t \geq 0$, $k \in \{0, 1, \ldots, n-1\}$ and given S_k and Z_k, the stopping time $\tau^\star$ is given by S_k and thus stops directly after the Markov chain reached state Z_k at S_k.

Note that for $k = n$ we have $\tau_n^{\star,n} = S_n$, regardless of any functions $f_0^\star, \ldots, f_{n-1}^\star$. The stopping time will therefore stop immediately after reaching the n–th jump time, if it didn't stop beforehand. This accommodates for the concept of n–step value functions, respectively the n–step stopping problem (5.2), which terminates at latest after n jumps of the underlying Markov chain.

Another characteristic of optimal stopping times for n–step value functions is the fact that every single stopping rule $\tau_n^{\star,k}$ is characterized by a different function $f^\star_{n-1-k}$ and not by a single one like for optimal stopping times for unrestricted value functions. As a consequence, the decisions whether a stopping rule stipulates immediate stopping, no stopping or something different does not only depend on the last jump time and the last attained state, but also on the actual number of preceding jumps.

In any case the decision, whether or not one of the two special stopping rules applies, requires exact knowledge about the maximizer functions $f_k^\star$, respectively $f^\star$. In the following sections we will investigate under which conditions we can conclude the optimality of those special stopping rules without having to calculate these maximizers, respectively the corresponding maximizing problems (8.1) or (8.2).

8.2 Conditions for the Optimality of Not Stopping in Certain States

We will investigate the conditions of interest mainly for optimal stopping times for the unrestricted value function $V(t,x)$. Note that every result given is also valid for optimal stopping times for n–step value functions. The main difference lies in the additional dependency of the current step of the maximizers in (7.31) of Theorem 7.14. Note that if not stated otherwise or explicitly mentioned, one can replace every statement for $V(t,x)$,

$m_{t,x}$, the corresponding unrestricted stopping problem and its optimal stopping time, with the appropriate analogues $V_n(t,x)$, $m_{n,t,x}$, the n–step stopping problem and its optimal n–step stopping times for any $n \in \mathbb{N}_0$.

Optimality of not Stopping under Extended Utility Functions

Assume that the underlying utility function U for the unrestricted value function $V(t,x)$ is given by Definition 4.3, but does not fulfill the requirements of Definition 4.1 on the whole real line. This means that U is derived from a classical utility function $\tilde{U}$, which has a smaller maximal domain than $\mathbb{R}$. Using Definition 4.3 $\tilde{U}$ is thus extended to $\mathbb{R}$ by setting U to $-\infty$ where $\tilde{U}$ is not defined. Thus by considering mappings of the form

$$\vartheta \mapsto U(-c\vartheta + \text{const.})$$

we can deduce that there exists a $\vartheta_0 \geq 0$ such that $U(-c\vartheta + \text{const.}) = -\infty$ for all $\vartheta < \vartheta_0$. We can interpret this in the way that waiting for too long and cumulating too large costs will inevitably result in the worst possible utility for an investor, namely $-\infty$. According to Remark 4.5, this can be seen as a defaulting case, where an investor cumulated such high amounts of costs that he went bankrupt.

From intuition it should be evident that the optimality for $V(t,x)$ to wait in certain states as long as the underlying Markov chain resides in this state cannot be reasonable, since there is always the danger of defaulting to account for. On the contrary, given any jump time S_n and state Z_n and knowing that the optimal stopping time did not stop before S_n, the optimal stopping rule $\tau^{\star,n}$ is given by a fixed deterministic and *finite* value. If the next jump of the underlying Markov chain does not occure before $\tau^{\star,n}$, it is optimal to stop. As $S_{n+1} \sim Exp(q_{Z_n})$, given Z_n, there is always a positive probability that $\tau^\star$ stops before the next change of state. We can say that up to this $\tau^{\star,n}$, it is optimal to wait and hope for a change of X into a state which yields a higher reward. If this change does not occur before $\tau^{\star,n}$, the cumulated costs grow too high, forcing a rational investor to stop in order to avoid bankruptcy.

The same reasoning is also true for the n–step value functions $V_n(t,x)$ and their optimal stopping times, respectively optimal stopping rules. One only has to account for the dependency of the maximizers of (8.1), respectively (7.31) of Theorem 7.14. Also note that due to the nature of n–step value functions, the corresponding optimal stopping times $\tau_n^\star$ given by (7.33) stop at the latest after the n–th jump of the underlying Markov chain. Thus the only stopping rules of interest are the ones which are applied before this n–th change of state.

The following proposition summarizes the reasonings above and proves them rigorously. In case of the mainly discussed unrestricted stopping problem and its unrestricted value function $V(t,x)$, we will also be able to show the P_x–almost sure finiteness of the potential candidate $\tau^\star$ from (7.77) for the optimal stopping time for $V(t,x)$. This means that in the setting of Theorem 7.29 the requirement of $\tau^\star$ being $\mathbb{P}_x$–almost surely is automatically fulfilled and does not have to be checked manually.

Proposition 8.1 (optimality of finite stopping rules)

Let the underlying utility function $U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ of $V(t,x)$, respectively $V_n(t,x)$ be an extended utility function according to Definition 4.3, derived from a classical utility function with maximal domain $[-d,\infty)$. Moreover let T_d and $T_d^{(n)}(x)$ for $x \in S$ and $n \in \mathbb{N}_0$ be defined by (7.64), respectively (7.19). Then it holds

(a) The mapping $f^\star$ is bounded from above by T_d. For all $t \geq 0$ and $x \in S$, every stopping rule

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k, \quad k \in \mathbb{N}_0,$$

is $\mathbb{P}_x$–almost surely finite. As a consequence, it is never optimal to wait arbitrarily long in any given state.

Furthermore, $\tau^\star$ defined by

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1+t,Z_1)+S_1,\ f^\star(S_2+t,Z_2)+S_2,\ f^\star(S_3+t,Z_3)+S_3,\dots\Big)$$

is $\mathbb{P}_x$–almost surely finite and thus $\tau^\star \in \Sigma_x$ is automatically fulfilled.

(b) The mappings $f_0^\star, \dots, f_{n-1}$ are bounded from above. More precisely, we get

$$f_k^\star(t,x) \leq T_d^{(k)}(x) \qquad \text{for all } t \geq 0, x \in S, k \in \{0,1,\dots,n-1\}.$$

For all $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$, every stopping rule

$$\tau_n^{\star,k} = f_{n-1-k}^\star(S_k + t, Z_k) + S_k, \quad k \in \{0,1,\dots,n-1\},$$

is $\mathbb{P}_x$–almost surely finite. For $k \geq n$ the optimal stopping rules are given by $\tau_n^{\star,k} = S_k$. As a consequence, it is never optimal to wait arbitrarily long in any given state.

Proof of Proposition 8.1
We will prove part (a) of Proposition 8.1:

Suppose the domain on which U is bigger than $-\infty$ is given by $[-d,\infty)$ and let $t \geq 0$, $x \in S$ and $k \in \mathbb{N}_0$. Note that if $t > T_d$ where T_d is defined as usual by (7.64)

$$T_d = \frac{g_{\inf} + d}{c},$$

then $V(t,x) = -\infty$ and $m_{t,x} \equiv -\infty$ according to Lemma 7.22 (c)(i) and (ii). This represents the case where the cumulated costs are already too high to begin with, resulting in the bancruptcy of an investor independent of any choice he could make. As any applied stopping time will yield the same expected utility of $-\infty$, we will choose $\tau^\star = 0$ as “optimal” stopping time by convention. This is also reflected in Lemma 7.22

(c)(iii), where we set $\vartheta^\star = f^\star(t,x) = 0$. Thus, every $\tau^{\star,k}$ with

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k$$

fulfills $\tau^{\star,k} = S_k$, implying that every stopping rule will terminate as soon as it is in effect, rendering the optimal stopping time to be equal to zero.

Now let $t \in [0, T_d]$. Then an application of Theorem 7.23 (c) yields

$$\vartheta^\star = f^\star(t,x) \leq T_d - t \leq T_d.$$

As a consequence we get $f^\star(S_k + t, Z_k) \leq T_d$ for every $k \in \mathbb{N}_0$. Note that in case of $S_k + t > T_d$ we get $f^\star(S_k + t, Z_k) = 0 < T_d$. Thus every stopping rule

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k \leq T_d + S_k$$

is $\mathbb{P}_x$–almost surely finite.

As long as the next jump time S_{k+1} is not reached, stopping rule $\tau^{\star,k}$ is in effect. It will always terminate at the latest after a duration of T_d measured from S_k. A longer waiting period for the Markov chain to change its state is never optimal.

To prove the $\mathbb{P}_x$–almost surely finiteness of $\tau^\star$ we fix an $n \in \mathbb{N}$ and consider

$$\begin{aligned}
\{\tau^\star \geq S_n\} &= \{\tau^{\star,0} \geq S_1\} \cap \{\tau^{\star,1} \geq S_2\} \cap \cdots \cap \{\tau^{\star,n-1} \geq S_n\} \\
&= \bigcap_{k=1}^{n} \{\tau^{\star,k-1} \geq S_k\} \\
&= \bigcap_{k=1}^{n} \{f^\star(S_{k-1} + t, Z_{k-1}) + S_{k-1} \geq S_k\} \\
&= \bigcap_{k=1}^{n} \{f^\star(S_{k-1} + t, Z_{k-1}) \geq S_k - S_{k-1}\} \\
&= \bigcap_{k=1}^{n} \{f^\star(S_{k-1} + t, Z_{k-1}) \geq T_k\}.
\end{aligned}$$

Given Z_{k-1} we get $T_k \sim Exp(q_{Z_{k-1}})$ for the k–th sojourn time. Using the stability condition of the underlying Markov chain, which was assumed in Assumption 2.13, we know of the existence of a $\lambda \in (0, \infty)$ such that $q_j \leq \lambda$ for all $j \in S$. This leads to

$$\begin{aligned}
\mathbb{P}_x(T_k \leq t) &= \sum_{j \in S} \mathbb{P}_x(T_k \leq t | Z_{k-1} = j) \cdot \mathbb{P}_x(Z_{k-1} = j) \\
&= \sum_{j \in S} (1 - \exp(-q_j t)) \cdot \mathbb{P}_x(Z_{k-1} = j) \\
&\leq (1 - \exp(-\lambda t)) \sum_{j \in S} \mathbb{P}_x(Z_{k-1} = j) \\
&= 1 - \exp(-\lambda t).
\end{aligned}$$

Using the independence of the events $\big\{f^\star(S_{k-1}+t, Z_{k-1}) \geq T_k\big\}$, $k \in \big\{1,\dots,n\big\}$, we can cunclude that

$$\begin{aligned}
\mathbb{P}_x(\tau^\star \geq S_n) &= \mathbb{P}_x\left(\bigcap_{k=1}^{n}\{f^\star(S_{k-1}+t, Z_{k-1}) \geq T_k\}\right) \\
&= \prod_{k=1}^{n} \mathbb{P}_x\left(\{f^\star(S_{k-1}+t, Z_{k-1}) \geq T_k\}\right) \\
&\leq \prod_{k=1}^{n} \mathbb{P}_x\left(T_k \leq T_d\right) \\
&\leq \prod_{k=1}^{n} 1-\exp(-\lambda\, T_d) \\
&= [1-\exp(-\lambda\, T_d)]^n \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Since $\{\tau^\star \geq S_n\} \searrow \{\tau^\star = \infty\}$, we finally get

$$\mathbb{P}_x(\tau^\star = \infty) = \lim_{n\to\infty} \mathbb{P}_x(\tau^\star \geq S_n) = 0$$

and thus

$$\mathbb{P}_x(\tau^\star < \infty) = 1.$$

The stopping time $\tau^\star$ is hence $\mathbb{P}_x$–almost surely finite and therefore fulfills $\tau^\star \in \Sigma_x$. This finalizes the proof of part (a) of Proposition 8.1. For part (b) we note that every given argument above also applies for the case in Proposition 8.1 (b). Just replace T_d by $T_d^{(n)}(x)$, apply the results in Lemma 7.13 (c) as well as Theorem 7.14 (c) and (d) and repeat the reasoning above to conclude the proof. □

Optimality of not Stopping under Classical Utility Functions

Now suppose that the underlying utility function U for the unrestricted value function $V(t,x)$ is a classical one on the whole real line and does not take the value $-\infty$. In contrast to the upper case of an extended utility function, we do not have a bancruptcy threshold. Regardless of the amount of cumulated costs caused by arbirarily high waiting times, the corresponding utility given by U does not reach $-\infty$. Even higher losses result in even lower, but finite values of corresponding utility. In such cases it is a priori unclear, if there is a threshold after which an additional waiting in a certain state is still optimal or inefficient. As there is no bancruptcy treshold, a prolonged waiting in a certain state of the underlying Markov chain could possibly be optimal, depending on the expected state in which the Markov chain could jump as well as the expected time at which this jump could occur. We will now establish a condition under which we can guarantee the optimality of waiting in a given state, regardless of its duration. To this end we remind ourselves of the discussion at the beginning of this chapter, stating that indefinite waiting in a certain state is optimal, if the optimization problem (8.2), respectively its original problem (7.72) from Theorem 7.23 is solved for a maximizer set to infinity. As we are in the setting of Theorem 7.23 (b) and know that $\vartheta \mapsto m_{t,x}(\vartheta)$ is differentiable almost everywhere, we can demand for $m_{t,x}$ to fullfill

$$m'_{t,x}(\vartheta) > 0$$

almost everywhere. This, together with the knowledge of $m_{t,x}$ being continuous on $[0, \infty)$ according to Lemma 7.22 (b), will guarantee that

$$\sup_{\vartheta \geq 0} m_{t,x}(\vartheta) = \lim_{\vartheta \to \infty} m_{t,x}(\vartheta)$$

and the maximizer being $\vartheta^\star = \infty$. Of course we still have the same problem that was discussed at the beginning of this chapter. In order to verify conditions like $m'_{t,x} \geq 0$ explicitly we need to know the value function $V(t, x)$ itself. To avoid this difficulty, we will now define a new set which will help us to find states for which it is optimal to wait arbitrarily long for the next jump of the underlying Markov chain to occur.

Definition 8.2 (the S_t^∞–set)

Let $t \geq 0$. Define the set S_t^∞ by

$$S_t^\infty := \Big\{ x \in S \;\Big|\; \sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) > U\big(-c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-c\vartheta + g(x)\big) \quad \text{for all } \vartheta \geq t \Big\}. \tag{8.3}$$

Remark 8.3 (properties of the S_t^∞–set)

Clearly, S_t^∞ is a subset of the state space S for all $t \geq 0$. Furthermore, if $t \leq t'$ for some $t, t' \geq 0$, then we immediately get

$$S_t^\infty \subseteq S_{t'}^\infty \subseteq S. \tag{8.4}$$

Thus if some state $x \in S$ fulfills $x \in S_t^\infty$ for some $t \geq 0$, it will never "leave" the set again in the sense that $x \in S_{t'}^\infty$ for every $t' \geq t$.

This S_t^∞–set will give us a sufficient condition to decide whether or not it is optimal to wait arbitrarily long for the Markov chain to jump into a new state, given that the stopping problem already cumulated costs for a period of $t \geq 0$. The following theorem is valid for both the unrestricted value function and any n–step value function for $n \in \mathbb{N}_0$.

Theorem 8.4 (optimality condition for never stopping)

Let $t \geq 0$ and $x \in S$. Then it holds:

(a) If $x \in S_t^\infty$, then $m'_{t,x} > 0$ for every $\vartheta \geq 0$. As a consequence, the maximizer of (8.2), respectively (7.72) is given by

$$\vartheta^\star = f^\star(t', x) = \infty$$

for every $t' \geq t$.

If the underlying Markov chain will ever jump into state x after time t, then the optimal stopping rules for the unrestricted stopping problem will never terminate before the next jump time.

(b) Let $n \in \mathbb{N}$. If $x \in S_t^\infty$, then $m'_{k,t,x} > 0$ for every $\vartheta \geq 0$ and every $k \in \{0, 1, \ldots, n-1\}$. As a consequence, the maximizers of (8.1), respectively (7.31) are given by

$$\vartheta_k^\star = f_{n-1-k}^\star(t', x) = \infty$$

for any $k \in \{0, 1, \ldots, n-1\}$ and every $t' \geq t$.

If the underlying Markov chain will ever jump into state x after time t, then the optimal stopping rules for the n–step stopping problem will never terminate before the next jump time, as long as the n–th jump did not occur.

Proof of Theorem 8.4
We will use the inequalities

$$\begin{aligned} V_n(t,x) &\geq U(-ct + g(x)) \quad \text{and} \\ V(t,x) &\geq U(-ct + g(x)) \end{aligned}$$

given in Remark 6.3 (e) for all $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in S$.

Now fix a $t \geq 0$ and $x \in S$ and suppose $x \in S_t^\infty$. Hence we know that

$$\sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) > U\big(-c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-c\vartheta + g(x)\big)$$

is fulfilled for every $\vartheta \geq t$. Therefore we get

$$\sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(-ct - c\vartheta + g(j)\big) > U\big(-ct - c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-ct - c\vartheta + g(x)\big)$$

for every $\vartheta \geq 0$. Applying the above-mentioned inequalities thus yields the validity of

$$\sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{xj}}{q_x} V(t+\vartheta, j) > U\big(-ct - c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-ct - c\vartheta + g(x)\big)$$

$$\Leftrightarrow \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} V(t+\vartheta, j) > q_x U\big(-ct - c\vartheta + g(x)\big) + cU'\big(-ct - c\vartheta + g(x)\big)$$

$$\Leftrightarrow \sum_{\substack{j \in S,\\ j \neq x}} q_{xj} V(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(x)\big) - q_x U\big(-ct - c\vartheta + g(x)\big) > 0$$

for all $\vartheta \geq 0$ and therefore

$$m'_{t,x}(\vartheta) \geq 0 \quad \text{for all } \vartheta > 0.$$

The same argument also yields

$$m'_{k,t,x}(\vartheta) \geq 0 \quad \text{for all } \vartheta > 0$$

for every $k \in \{0, 1, \ldots, n-1\}$ for a given $n \in \mathbb{N}$.

As a consequence we immediately get

$$\sup_{\vartheta \geq 0} m_{t,x}(\vartheta) = \lim_{\vartheta \to \infty} m_{t,x}(\vartheta)$$

as well as

$$\sup_{\vartheta \geq 0} m_{k,t,x}(\vartheta) = \lim_{\vartheta \to \infty} m_{k,t,x}(\vartheta)$$

for all $k \in \{0, 1, \ldots, n-1\}$. The corresponding maximizers $\vartheta^\star$, respectively $\vartheta^\star_k$, $k \in \{0, 1, \ldots, n-1\}$ are therefore set to infinity.

An optimal stopping time $\tau^\star$ for $V(t, x)$ with stopping rules

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k, \quad k \in \mathbb{N}_0$$

will thus satisfy $\tau^{\star,k} > S_{k+1}$ and therefore never stop before the next change of state, if for given S_k and Z_k

$$Z_k \in S^\infty_{S_k + t}$$

is true. As a consequence, it is never optimal to stop in such a state.

Analogously, we get the same result for the stopping rules

$$\tau^{\star,k}_n = f^\star_{n-1-k}(S_k + t, Z_k) + S_k, \quad k \in \{0, 1, \ldots, n-1\}$$

for the optimal stopping time $\tau^\star_n$ of the n–step value function $V_n(t, x)$, $n \in \mathbb{N}_0$. Note that this is only true for the first n stopping rules $\tau^{\star,0}_n, \ldots, \tau^{\star,n-1}_n$, as the n–step stopping problem is terminated at the latest after the n–th jump of the underlying Markov chain. □

Remark 8.5 (interpretation of S_t^∞–sets)

With respect to Theorem 8.4 we can interpret a state $x \in S_t^\infty$ for $t \geq 0$ the following way: Since stopping the unrestricted stopping problem, respectively an n–step stopping problem, is never optimal as long as the underlying Markov chain resides in this state x, an investor can classify such a state as "bad state". The reward $g(x)$ gained from x has to be so unprofitable in comparison to other states which can be successors of x, that despite of the sojourn time in x, as long as it may take, it is always more lucrative to wait and hope for a "better" state with higher reward.

The monotonicity of S_t^∞ in $t \geq 0$ stated in Remark 8.3 is another interesting feature. It states that if a state $x \in S$ becomes "bad" for some $t \geq 0$ in the sense that $x \in S_t^\infty$, then it will never be able to become a more profitable state again. Once the cumulated costs $-ct$ are high enough the investor will be more inclined to wait for subsequent changes of state to compensate for the aggregated costs. This reflects in some way the attitude of an investor towards risk as well as his preferences. As discussed in chapter 4 such effects are indeed modelled utility functions. As we can see in Equation (8.3), the S_t^∞–set from Definition 8.2 is heavily dependent on the choice of a particular utility function.

Note that if $x \in S_0^\infty$, then $x \in S_t^\infty$ for all $t \geq 0$. Therefore the moment when the "bad" state x is reached doesn't play any role. Independent of $t \geq 0$ and the cumulated costs so far, it is never optimal to stop in state x. On the other hand, if there exists a $t' > 0$ such that $x \notin S_t^\infty$ for some $t < t'$ and $x \in S_t^\infty$ for $t \geq t'$, then the state x doesn't have to be "bad" from the beginning. As Theorem 8.4 only yields a sufficient, but not a necessary condition on the optimality of never stopping, there still exists the possibility that stopping in such a state x before the jump of the underlying Markov chain into another state can still be optimal, if the investor didn't already wait too long in the sense that the cumulated costs do not exceed $-ct'$. After time t' the state x becomes definitely "bad" and stopping in x will never again be optimal after this moment.

Note that the statements in Theorem 8.4 refer to fixed states $x \in S$ and fixed time values $t \geq 0$. We will discuss the situation again for the unrestricted case, but the case of n–step stopping problem is as always analogeous. According to Theorem 7.29, the optimal stopping time is given by

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1+t,Z_1)+S_1,\ f^\star(S_2+t,Z_2)+S_2,\ f^\star(S_3+t,Z_3)+S_3, \dots \Big),$$

where

$$\tau^{\star,k} = f^\star(S_k+t,Z_k)+S_k, \quad k \in \mathbb{N}_0.$$

Thus before the k–th jump time S_k the stopping rules $\tau^{\star,k}$ are random and do not permit a clear a priori decision whether or not it is never optimal to stop in state Z_k, if the Markov chain attained this state at time S_k. But due to the piecewise description of stopping times, an investor is very well able to observe the current state of the underlying

Markov chain as well as the time at which this state was attained at. Based on this information the investor can easily check whether the current state is part of the S_t^∞–set, or not. Therefore he knows immediately whether or not the optimal policy he has to apply up to the next jump is just to wait, independent of the possible duration of this waiting period.

Also note that in order to make such decision the investor solely needs the information required for S_t^∞, namely the model components for the stopping problem introduced in section 5.1. Of course the set S_t^∞ depends heavily on the choice of the underlying utility function U, which represents the preference relation and attidute of a given investor towards risk. In particular, an investor does not need to know the value function $V(t,x)$ itself, neither does he need to be able to solve the maximization problem (8.2), respectively (7.72) explicitly.

8.3 Conditions for the Optimality of Immediate Stopping in Certain States

Now we will investigate the conditions for which the optimal stopping policy of an investor requires an immediate stopping, as soon as a certain state and time is reached. To this end, we will again mainly focus on optimal stopping times for the unrestricted value function $V(t,x)$. Note that every result given is again also valid for optimal stopping times for n–step value functions $V_n(t,x)$. As we will see in the following sections, a simply calculable condition for the optimality of immediate stopping is not as easy as for the optimality of not stopping in a certain state. We will see that we need to impose an additional assumption on the structural connection of underlying Markov chain and utility function to deduce the validity of this condition. This assumption can be seen as the continuous-time generalization of an assumption made by Kadota et al. [1996], where optimal stopping times where also studied for arbitrary utility functions, but only in a time-discrete setting with time-discrete Markov chains.

Optimality of Immediate Stopping under Classical Utility Functions

Suppose that the underlying utility function U for the unrestricted value function $V(t,x)$ is a classical one on the whole real line and does not take the value $-\infty$. We will discuss this case at first because of the beneficial property of U being differentiable on the whole real line. We will also assume that the assumptions of Theorem 7.29 are fulfilled and that the optimal stopping time has therefore the structure

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1+t,Z_1)+S_1,\ f^\star(S_2+t,Z_2)+S_2,\ f^\star(S_3+t,Z_3)+S_3,\dots\Big)$$

given in (7.76).

Just as in the last section we know of the continuity and almost everywhere differentiability of $m_{t,x}$ (and $m_{n,t,x}$ for any $n \in \mathbb{N}_0$). Therefore, an obvious approach to find a condition under which the optimality of immediate stopping in certain states at certain times would be to demand the validity of

$$m'_{t,x}(\vartheta) \leq 0$$

for all $\vartheta \geq 0$. This would immediately imply that the maximization problem (8.2), respectively (7.72), is solved by the maximizer

$$\vartheta^\star = f^\star(t,x) = 0.$$

Given that the stopping problem was observed up to the k–th jump time S_k and knowing the current state S_k in which the Markov chain resides will lead to the stopping rule

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k = S_k$$

of the optimal stopping time $\tau^\star$, if $f^\star(S_k + t, Z_k) = 0$. Therefore the optimal stopping rule will terminate immediately as soon as the time $S_k + t$ has passed without stopping and the state Z_k was adopted. In this case it is never optimal to wait any additional period of time in hope of a new jump of the underlying Markov chain into a state with better reward. In some sense such a state $x \in S$, for which immediate stopping is optimal as soon as it is reached, is an ideal or "good" state in comparison to every successive state that could be attained at some point after x.

From intuition it should be plausible that if stopping in such a "good" state $x \in S$ is optimal at some time $t \geq 0$, then it should also be optimal if this state is reached at any later time point $t' \geq t$. In this case an investor has to deal with the extra costs of $-c(t'-t)$, but still has no hope for a higher expected utility gained by waiting any additional period of time.

Hence, for immediate stopping to be optimal, it has to be more rewarding than waiting for *any* successive state to follow. In this sense immediate stopping has to yield a higher actual utility than the expectation of every possible future utility that can be achieved due to the evolution of the underlying Markov chain. Sadly, this is a rather strong requirement that is not easy to check. Therefore we cannot simply repeat the approach given in Theorem 8.4 for never stopping in a certain state. We will need to impose an additional assumption on the structure of the underlying Markov chain in order to construct a condition which is easy enough to check and which does not require the knowledge of the value function $V(t,x)$ itself, respectively the explicit solution of the corresponding maximization problem (8.2). To this end we will define a new set similar to the S_t^∞–set from Definition 8.2.

Definition 8.6 (the S_t^0–set)

Let $t \geq 0$. Define the set S_t^0 by

$$S_t^0 := \Big\{ x \in S \,\Big|\, \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) \leq U\big(-c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-c\vartheta + g(x)\big) \quad \text{for all } \vartheta \geq t \Big\}. \tag{8.5}$$

Remark 8.7 (properties of the S_t^0–set)

Clearly, S_t^0 is a subset of the state space S for all $t \geq 0$. Furthermore, if $t \leq t'$ for some $t, t' \geq 0$, then we immediately get

$$S_t^0 \subseteq S_{t'}^0 \subseteq S. \tag{8.6}$$

Thus if some state $x \in S$ fulfills $x \in S_t^0$ for some $t \geq 0$, it will never "leave" the set again in the sense that $x \in S_{t'}^0$ for every $t' \geq t$.

As mentioned before, we would like to ensure the validity of statements like

$$x \in S_t^0 \Rightarrow f^\star(t, x) = 0$$

for some $t \geq 0$ and $x \in S$, similar to Theorem 8.4. For this purpose we will impose the subsequent assumption which establishes a connection between the structural properties of the underlying Markov chain and other model components of the stopping problem given in section 5.1, in particular the utility function U which reflects the preferences and attitude of an investor towards risk. Note that if the above-mentioned implication is indeed valid, then the monotonicity $S_t^0 \subseteq S_{t'}^0$ for $t' \geq t$ given in (8.6) yields the desired fact that once immediate stopping is optimal for some state $x \in S$ attained at some $t \geq 0$, it remains optimal if this state x is attained at any later moment $t' \geq t$.

Assumption 8.8 (closure assumption)

Suppose that for all $t \geq 0$, $x \in S_t^0$ and $j \in S \backslash \{x\}$ the implication

$$q_{xj} \neq 0 \Longrightarrow j \in S_t^0 \tag{8.7}$$

is valid.

Remark 8.9 (variants and interpretation of the closure assumption)

(a) The monotonicity $S_t^0 \subseteq S_{t'}^0$ for $t' \geq t$ given in (8.6) indicates that Assumption 8.8 even holds, if the weaker assumption

for all $t \geq 0$, $x \in S_t^0$ and $j \in S \backslash \{x\}$

$$q_{xj} \neq 0 \Longrightarrow j \in S_t^0 \tag{8.8}$$

is valid.

(b) The closure assumption can be reformulated using the contradiction of implication (8.7):

For all $t \geq 0$, $x \in S_t^0$ and $j \in S \backslash \{x\}$

$$\exists\, t' \geq t \text{ such that } j \notin S_{t'}^0 \Longrightarrow q_{xj} = 0. \tag{8.9}$$

(c) In order to give a suitable interpretation of the closure condition, we remind ourselves that using Theorem 2.16, an intensity rate $q_{xj} > 0$ for some $x \in S$ and $j \in S \backslash \{x\}$ implies that the transition probability p_{xj} of the corresponding embedded discrete-time Markov chain is strictly positive. In other words, the probability that the underlying Markov chain will change its state from x to some other state j is strictly greater than zero. For all $x \in S$, all $j \in S \backslash \{x\}$ with $q_{xj} > 0$ are thus direct possible successors of x.

Therefore if $x \in S_t^0$ for some $t \geq 0$, then assumption (8.7) claims that every possible direct successor of x has also be in S_t^0 and every $S_{t'}^0$ with $t' \geq t$. In other words, once the Markov chain X hits the set S_t^0 for some $t \geq 0$, it will never be able to leave it any more. In this sense, the set S_t^0 is closed with respect to the evolution of the process X.

(d) As we will see in the following theorem, the closure condition will account for the fact that immediate stopping in a certain state can only be optimal, if waiting in the same state is not profitable and if in addition waiting for any successive state $j \in S \backslash \{x\}$ after x (and its successors themselves) does not provide a higher expected utility.

Theorem 8.10 (optimality condition for immediate stopping)

Let $t \geq 0$, $x \in S$ and Assumption 8.8 be valid. Then it holds:

(a) If $x \in S_t^0$, then $m'_{t,x} \leq 0$ for every $\vartheta \geq 0$. As a consequence, the maximizer of (8.2), respectively (7.72) is given by

$$\vartheta^\star = f^\star(t', x) = 0$$

for every $t' \geq t$.

If the underlying Markov chain will ever jump into state $x \in S_t^0$ after time $t \geq 0$, then the optimal stopping rules for the unrestricted stopping problem will terminate immediately after this state was attained.

If $t \geq 0$ and $x \in S_t^0$ are the initial parameters of V, then

$$V(t, x) = V(t, x, 0) = U(-ct + g(x))$$

and the optimal stopping time is given by

$$\tau^\star = 0.$$

(b) Let $n \in \mathbb{N}$. If $x \in S_t^0$, then $m'_{k,t,x} \leq 0$ for every $\vartheta \geq 0$ and every $k \in \{0, 1, \ldots, n-1\}$. As a consequence, the maximizers of (8.1), respectively (7.31) are given by

$$\vartheta_k^\star = f_{n-1-k}^\star(t', x) = 0$$

for any $k \in \{0, 1, \ldots, n-1\}$ and every $t' \geq t$.

If the underlying Markov chain will ever jump into state $x \in S_t^0$ after time $t \geq 0$, then the optimal stopping rules for the n–step stopping problem will terminate immediately after this state was attained. As the n–step stopping problem terminated at the latest after the n–th jump, this property of immediate stopping does not only apply for stopping rules $\tau_n^{\star,0}, \ldots, \tau_n^{\star,n-1}$, but even for every stopping rule $\tau_n^{\star,k}$ with $k \geq n$.

If $t \geq 0$ and $x \in S_t^0$ are the initial parameters of V_n, then

$$V_n(t, x) = V_n(t, x, 0) = U(-ct + g(x))$$

and the optimal stopping time is given by

$$\tau_n^\star = 0.$$

Proof of Theorem 8.10
Let $t \geq 0$ and $x \in S_t^0$. Due to Proposition 6.5 (c) and (d) we know that

$$V(s,j) \geq V_n(s,j) \geq U(-cs + g(j))$$

for every $n \in \mathbb{N}_0$, $s \geq 0$ and $j \in S$.

Furthermore the Bellman equation (7.31) from Theorem 7.14 (a) reads

$$V_{n+1}(t,x) = \sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta),$$

where

$$m_{n,t,x}(\vartheta) = U\big(-ct - c\vartheta + g(x)\big) \cdot e^{-q_x \cdot \vartheta} + \int_0^\vartheta e^{-q_x s} \sum_{\substack{j \in S, \\ j \neq x}} q_{xj} \cdot V_n(t+s,j)\, ds$$

according to (7.22).

In addition, we remind ourselves that due to Lemma 7.13 (b) in the case of a classical utility function on the whole real line the mapping $m_{n,t,x}$ is differentiable almost everywhere and

$$m'_{n,t,x}(\vartheta)$$
$$= \Bigg(\sum_{\substack{j \in S, \\ j \neq x}} q_{xj} V_n(t+\vartheta, j) - cU'\big(-ct - c\vartheta + g(x)\big) - q_x U\big(-ct - c\vartheta + g(x)\big) \Bigg) \cdot e^{-q_x \vartheta}.$$

Thus we know that

$$m'_{n,t,x}(\vartheta) \leq 0$$
$$\Longleftrightarrow \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} V_n(t+\vartheta, j) \leq U\big(-ct - c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-ct - c\vartheta + g(x)\big)$$

for all $\vartheta \geq 0$.

We will now prove by induction over $n \in \mathbb{N}$, that for any $t \geq 0$ and $x \in S$ such that $x \in S_t^0$, the equality

$$V_n(t,x) = V_n(t,x,0) = U(-ct + g(x))$$

holds and the optimal stopping time $\tau_n^\star$ for the n–step value function is given by

$$\tau_n^\star = 0.$$

To this end, fix $t \geq 0$, $x \in S_t^0$ and suppose that $n = 1$. In this case we know that

$$V_0(t+\vartheta, j) = U(-ct - c\vartheta + g(j))$$

for all $\vartheta \geq 0$ and $j \in S$. By Definition 8.6, we thus get

$$x \in S_t^0 \Leftrightarrow \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(- c\vartheta + g(j)\big) \leq U\big(- c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(- c\vartheta + g(x)\big) \quad \forall \vartheta \geq t$$

$$\Leftrightarrow \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(- ct - c\vartheta + g(j)\big) \leq U\big(- ct - c\vartheta + g(x)\big)$$

$$+ \frac{c}{q_x} U'\big(- ct - c\vartheta + g(x)\big) \quad \forall \vartheta \geq 0$$

$$\Leftrightarrow m'_{0,t,x}(\vartheta) \leq 0 \quad \forall \vartheta \geq 0.$$

Thus the maximization problem (8.1) for $n = 0$ yields

$$\sup_{\vartheta \geq 0} m_{0,t,x}(\vartheta) = m_{0,t,x}(0)$$

and the corresponding maximizer is given by

$$\vartheta_0^\star = f_0^\star(t, x) = 0.$$

By Theorem 7.14 (d), the optimal stopping time $\tau_0^\star$ for the 1-step value function $V_1(t, x)$ is given by

$$\tau_1^\star = \big(f_0^\star(t, x), S_1, S_2, \dots\big) = 0$$

and

$$V_1(t, x) = V_1(t, x, 0) = U(-ct + g(x)).$$

Now suppose for the induction hypothesis that for a fixed but arbitrary $n \in \mathbb{N}$ we get:

Every $s \geq 0$ and $j \in S$ such that $j \in S_t^0$ imply

$$V_n(s, j) = V_n(s, j, 0) = U(-cs + g(j))$$

and $\tau_n^\star = 0$.

<u>induction step:</u>

Fix $t \geq 0$, $x \in S$ such that $x \in S_t^0$ and $\vartheta \geq 0$. Then we can decompose the state space S into

$$S = S_{t+\vartheta}^0 + \big(S_{t+\vartheta}^0\big)^c.$$

Therefore we get

$$\sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(- ct - c\vartheta + g(j)\big)$$

$$= \sum_{\substack{j \in S_{t+\vartheta}^0, \\ j \neq x}} \frac{q_{xj}}{q_x} U\big(- ct - c\vartheta + g(j)\big) + \sum_{j \notin S_{t+\vartheta}^0} \frac{q_{xj}}{q_x} U\big(- ct - c\vartheta + g(j)\big).$$

We know for every summand of the first sum, that

$$U\big(-ct-c\vartheta+g(j)\big)=V_n(t+\vartheta,j)$$

by induction hypothesis, since $j\in S^0_{t+\vartheta}$. On the other hand, an analysis of the summands of the second sum yields

$$q_{xj}=0$$

by Assumption 8.8, respectively the alternative version (8.9), since $x\in S^0_t$ and thus $x\in S^0_{t+\vartheta}$, but $j\notin S^0_{t+\vartheta}$. As a consequence we get

$$\frac{q_{xj}}{q_x}U\big(-ct-c\vartheta+g(j)\big)=0=\frac{q_{xj}}{q_x}V_n(t+\vartheta,j)$$

for every $j\notin S^0_{t+\vartheta}$.

This leads to

$$\begin{aligned}
&\sum_{\substack{j\in S,\\ j\neq x}}\frac{q_{xj}}{q_x}U\big(-ct-c\vartheta+g(j)\big)\\
&=\sum_{\substack{j\in S^0_{t+\vartheta},\\ j\neq x}}\frac{q_{xj}}{q_x}U\big(-ct-c\vartheta+g(j)\big)+\sum_{j\notin S^0_{t+\vartheta}}\frac{q_{xj}}{q_x}U\big(-ct-c\vartheta+g(j)\big)\\
&=\sum_{\substack{j\in S^0_{t+\vartheta},\\ j\neq x}}\frac{q_{xj}}{q_x}V_n(t+\vartheta,j)+\sum_{j\notin S^0_{t+\vartheta}}\frac{q_{xj}}{q_x}V_n(t+\vartheta,j)\\
&=\sum_{\substack{j\in S,\\ j\neq x}}\frac{q_{xj}}{q_x}V_n(t+\vartheta,j)
\end{aligned}$$

for every $\vartheta\geq 0$.

Now we can finally conclude our calculations by

$$\begin{aligned}
x\in S^0_t \Leftrightarrow &\sum_{\substack{j\in S,\\ j\neq x}}\frac{q_{xj}}{q_x}U\big(-c\vartheta+g(j)\big)\leq U\big(-c\vartheta+g(x)\big)+\frac{c}{q_x}U'\big(-c\vartheta+g(x)\big)\quad\forall\vartheta\geq t\\
\Leftrightarrow &\sum_{\substack{j\in S,\\ j\neq x}}\frac{q_{xj}}{q_x}U\big(-ct-c\vartheta+g(j)\big)\leq U\big(-ct-c\vartheta+g(x)\big)\\
&\qquad+\frac{c}{q_x}U'\big(-ct-c\vartheta+g(x)\big)\quad\forall\vartheta\geq 0\\
\Leftrightarrow &\sum_{\substack{j\in S,\\ j\neq x}}\frac{q_{xj}}{q_x}V_n(t+\vartheta,j)\leq U\big(-ct-c\vartheta+g(x)\big)\\
&\qquad+\frac{c}{q_x}U'\big(-ct-c\vartheta+g(x)\big)\quad\forall\vartheta\geq 0\\
\Leftrightarrow &\, m'_{n,t,x}(\vartheta)\leq 0\quad\forall\vartheta\geq 0.
\end{aligned}$$

The maximization problem (8.1) for the n–th step yields

$$\sup_{\vartheta \geq 0} m_{n,t,x}(\vartheta) = m_{n,t,x}(0)$$

and the corresponding maximizer is given by

$$\vartheta_n^\star = f_n^\star(t,x) = 0.$$

By Theorem 7.14 (d), the optimal stopping $\tau_n^\star$ for the $(n+1)$-step value function $V_{n+1}(t,x)$ is given by

$$\begin{aligned} \tau_{n+1}^\star &= \big(f_n^\star(t,x), f_{n-1}^\star(S_1+t,Z_1)+S_1,\ldots,f_0^\star(S_n+t,Z_n)+S_n,S_{n+1},\ldots\big) \\ &= \big(0, f_{n-1}^\star(S_1+t,Z_1)+S_1,\ldots,f_0^\star(S_n+t,Z_n)+S_n,S_{n+1},\ldots\big) \\ &= 0 \end{aligned}$$

and

$$V_{n+1}(t,x) = V_{n+1}(t,x,0) = U(-ct+g(x)).$$

This concludes part (b) of Theorem 8.10, as $x \in S_t^0$ also implies $x \in S_{t'}^0$ for every $t' \geq t$.

For part (a) we will utilize Proposition 6.7 (c), stating

$$V(t,x) = \lim_{n\to\infty} V_n(t,x).$$

Since $V_n(t,x) = U(-ct+g(x))$ for all $t \geq 0$, $x \in S$ such that $x \in S_t^0$ and $n \in \mathbb{N}_0$, we immediately get

$$V(t,x) = U(-ct+g(x))$$

for every $t \geq 0$ and $x \in S_t^0$. Looking at the fixed-point equation (7.72) of Theorem 7.23 (a), this implies that the corresponding maximization problem of Equation (7.72), respectively (8.2), reads

$$\sup_{\vartheta\geq 0} m_{t,x}(\vartheta) = m_{t,x}(0)$$

and is attained by

$$\vartheta^\star = f^\star(t,x) = 0.$$

As $x \in S_t^0$ implies $x \in S_{t'}^0$ for every $t' \geq t$, we can also conclude

$$f^\star(t',x) = 0$$

for every $t' \geq t$.

The optimal stopping time $\tau^\star$ according to Theorem 7.29, as given in Definition 7.26 assuming that the requirements of Theorem 7.29 are fulfilled, is therefore determined by

$$\begin{aligned} \tau^\star &= \big(f^\star(t,x), f^\star(S_1+t,Z_1)+S_1,\ldots\big) \\ &= \big(0, f^\star(S_1+t,Z_1)+S_1,\ldots\big) \\ &= 0. \end{aligned}$$

However if $t \geq 0$ and $x \in S_t^0$ are not the initial parameters of the stopping problem, but indicate a state x which the underlying Markov chain attains at some time t, then the optimal stopping rule will still be to stop immediately. □

Remark 8.11 (relation of the sets S_t^0 and S^∞)

(a) By comparing the S_t^∞–set (given by (8.3) in Definition 8.2) with the S_t^0–set (given by (8.5) in Definition 8.6), we can easily see that for every $t \geq 0$

$$S_t^0 \cap S_t^\infty = \emptyset \tag{8.10}$$

and in general

$$S_t^0 \cup S_t^\infty \subsetneq S. \tag{8.11}$$

This means that a given state $x \in S$ can be in the S_t^0–set, implying optimality of immediate stopping (at least under validity Assumption 8.8) in state x, or in the S_t^∞–set, implying optimality of never stopping in state x, or in neither of them. This accounts for the possibility of situations for which the optimal stopping rule will terminate between two jumps of the underlying Markov chain.

(b) According to Remark 8.3 and Remark 8.7, both S_t^∞ and S_t^0 are increasing in $t \geq 0$. As the time parameter t and thus the cumulated costs $-ct$ rise, a greater number of states $x \in S$ are inclined to be in one of these two sets. This means that an increasing time parameter and thus increasing cumulated costs will imply that the optimal stopping rules for a growing number of states will attain one of the two extreme cases: immediate stopping or never stopping. We can interpret this in the following way. Suppose an investor did already cumulate a high amount of costs and thus already exhibits a very unprofitable utility value. Furthermore, suppose that the current state of the underlying Markov chain is mediocre. There are better states than the current one, but also worse ones. The investor can now judge his situation in two ways:

First, he can argue that the reward gained by stopping in this state – less the cumulated costs he has to pay – yields no ample utility. The investor thus tends to wait for a change of state, hoping that this will bring him a better reward and thus utility value. Therefore he will tend to risk additional costs by waiting for a (hopefully) better state.

On the other hand, we could argue that the investor suspects, based on the transition probabilities of the underlying Markov chain, that there is a high probability that no significantly better state will be attained within a reasonable period of time. He thus fears to cumulate additional costs without gaining too much profit. The investor will hence tend to terminate the stopping problem immediately to avoid these additional costs.

(c) We will see in section 8.4 that in case of exponential utility as underlying utility function, the sets S_t^∞ and S_t^0 will again simplify. In this case we will show that

$$S_t^0 \cup S_t^\infty = S$$

for every $t \geq 0$. In fact we will even see that these sets do not depend on t at all. This coincides with our results in section 7.5, respectively section 7.8, that the optimal stopping time for n–step value functions as well as the unrestricted value function only decide whether to stop immediately in a given state, or not to stop as long as the underlying Markov chain resides in this state. Moreover, this decision is independent of $t \geq 0$.

Optimality of Immediate Stopping under Extended Utility Functions

Suppose now that the underlying utility function U for the unrestricted value function $V(t, x)$ is an extended one, derived from a classical utility function with maximal domain of the form $[-d, \infty]$. We will again assume that the assumptions of Theorem 7.29 are fulfilled and that the optimal stopping time has therefore the structure

$$\tau^\star = \Big(f^\star(t,x),\ f^\star(S_1+t, Z_1) + S_1,\ f^\star(S_2+t, Z_2) + S_2,\ f^\star(S_3+t, Z_3) + S_3, \ldots \Big)$$

given in (7.76).

By Lemma 7.22 (c) we know that $m_{t,x}$ is continuous on $[0, T_d]$ as well as differentiable on $[0, T_d)$, where T_d is as always defined according to (7.64) of Definition 7.20.

Similar to the case of classical utility functions, an obvious approach to find conditions under which immediate stopping is optimal in certain states at certain times would be to demand the validity of

$$m'_{t,x}(\vartheta) \leq 0$$

for all $\vartheta \in [0, T_d]$. This would immediately imply that the maximization problem (8.2), respectively (7.72) is solved by the maximizer

$$\vartheta^\star = f^\star(t, x) = 0.$$

Given that the stopping problem was observed up to the k–th jump time S_k and knowing the current state S_k in which the Markov chain resides will lead to the stopping rule

$$\tau^{\star,k} = f^\star(S_k + t, Z_k) + S_k = S_k$$

of the optimal stopping time $\tau^\star$, if $f^\star(S_k + t, Z_k) = 0$. Therefore the optimal stopping rule will terminate immediately as soon as the time $S_k + t$ has passed without stopping and the state Z_k was adopted. Analogeous to the case of classical utility functions, it is never optimal to wait any additional period of time in hope of a new jump of the underlying Markov chain into a state with better reward.

Note that the sole difference between the case of classical utility functions and the case of extended utility functions is the domain of the mapping $m_{t,x}$, for which it attains values greater than $-\infty$ and for which the derivative $m'_{t,x}$ of $m_{t,x}$ is defined. Demanding

$$m'_{t,x}(\vartheta) \leq 0$$

for all $\vartheta \in [0, T_d]$ is a weaker requirement than demanding the validity of this inequality for all $\vartheta \geq 0$, as it was the case for classical utility. We can therefore reapply the theory for immediate stopping under classical utility.

To avoid too much repetition, we will give only a brief outline of the situation for extended utility functions.

Definition 8.12 (the $S^0_{t,d}$–set for extended utility functions)

Let $t \geq 0$. Then the set $S^0_{t,d}$ is defined by

$$S^0_{t,d} := \Big\{ x \in S \,\Big|\, \sum_{\substack{j \in S,\\ j \neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) \leq U\big(-c\vartheta + g(x)\big) + \frac{c}{q_x} U'\big(-c\vartheta + g(x)\big) \quad \text{for all } \vartheta \in [t, T_d] \Big\}. \tag{8.12}$$

Remark 8.13 (properties of the set $S^0_{t,d}$)

(a) $S^0_{t,d}$ is again a subset of the state space S for all $t \geq 0$. Furthermore, if $t \leq t'$ for some $t, t' \geq 0$, then we immediately get

$$S^0_{t,d} \subseteq S^0_{t',d} \subseteq S. \tag{8.13}$$

(b) Note that for $t > T_d$ we get $[t, T_d] = \emptyset$ and therefore

$$S^0_{t,d} = S. \tag{8.14}$$

In case of extended utility functions the closure assumption can be formulated by

Assumption 8.14 (closure assumption for extended utility functions)

For all $t \in [0, T_d]$, $x \in S^0_{t,d}$ and $j \in S \backslash \{x\}$ the implication

$$q_{xj} \neq 0 \Longrightarrow j \in S^0_{t',d} \quad \text{for all} \quad t' \in [t, T_d] \tag{8.15}$$

is valid.

Theorem 8.15 (optimality condition for immediate stopping under extended utility)

(a) Let $x \in S$ and $t > T_d$. Then the optimal stopping time is always given by $\tau^\star = 0$ and
$$V(t, x) = V(t, x, 0) = U(-ct + g(x)).$$

(b) Let $t \in [0, T_d]$. If $x \in S^0_{t,d}$, then $m'_{t,x} \leq 0$ for every $\vartheta \in [0, T_d - t]$. As a consequence, the maximizer of (8.2), respectively (7.72) is given by
$$\vartheta^\star = f^\star(t', x) = 0$$
for every $t' \in [t, T_d]$.

If the underlying Markov chain will ever jump into state $x \in S^0_{t,d}$ after time $t \in [0, T_d]$, then the optimal stopping rules for the unrestricted stopping problem will terminate immediately after this state was attained.

If $t \in [0, T_d]$ and $x \in S^0_{t,d}$ are the initial parameters of V, then
$$V(t, x) = V(t, x, 0) = U(-ct + g(x))$$
and the optimal stopping time is given by
$$\tau^\star = 0.$$

Remark 8.16 (interpretation of Theorem 8.15)

(a) Note that the proof of Theorem 8.15 (b) is analogeous to the proof of Theorem 8.10. Part (a) follows directly from Theorem 7.23, since the mapping $m_{t,x}$ is constantly given by $-\infty$ in this case. This yields that every $\vartheta \geq 0$ is a valid maximizer for $\sup_{\vartheta \geq 0} m_{t,x}(\vartheta)$. In this case we agreed upon setting $\vartheta^\star = 0$ for every $x \in S$. The corresponding optimal stopping time $\tau^\star$ is therefore set to zero. Due to the high value of the time parameter $t > T_d$, the investor stops immediately in this case to avoid the possibility of bancruptcy, which can happen if $-ct + g(x) < -d$.

(b) For $t > T_d$ the derivative $m'_{t,x}$ of $m_{t,x}$ does not exist anywhere. Thus the frameworks of part (a) and (b) are slightly different. On the other hand we know that for $t > T_d$ we get $S^0_{t,d} = S$ according to Remark 8.13 (b). Therefore we still can interpret the set $S^0_{t,d}$ as the set of states for which immediate stopping is optimal. If $t > T_d$ we consistently get that every state leads to optimal immediate stopping.

In summary we have seen that in case of classical utility functions, we can identify certain states in S^0_t or S^∞_t, which lead to the optimality of immediate stopping or never stopping. But in general, there is always the possibility that for some states $x \in S$, the optimal stopping rule will follow a completely different policy and can stop between two states. In the case of extended utility functions we have seen that it is never optimal to wait arbitrarily long in a certain state. The case of optimality of never stopping will never occur. For immediate stopping we have seen that we can again identify some states in form of the set $S^0_{t,d}$, for which it is optimal to stop instantly.

8.4 Optimality of One-Step Look Ahead Stopping Times for Exponential Utility

We suppose in this section that the underlying utility function U is given by

$$U \colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

By Definition 4.1, this utility function is a classical one, defined on the whole real line. We want now to study some conditions under which the optimal stopping time for the unrestricted value function $V(t, x)$ would stop immediately in certain states or never stop in a given state at all. Following the outline of section 7.8, we know that the value function $V(t, x)$ can be reduced by considering (6.14) and (6.13), yielding

$$V(t, x) = e^{c\gamma t}\tilde{V}(x)$$

for every $t \geq 0$, $x \in S$, where

$$\tilde{V}(x) = \sup_{\tau \in \Sigma_x} \mathbb{E}_x\Big[- e^{c\gamma\tau - \gamma g(X_\tau)}\Big]$$

for every $x \in S$.

By applying Theorem 7.31 we know that the value function $\tilde{V}(x)$ fulfills

$$\tilde{V}(x) = -e^{-\gamma g(x)},$$

if $x \in S$ such that $q_x \leq c\gamma$ and

$$\tilde{V}(x) = \max\left\{ -e^{-\gamma g(x)}, \sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma}\tilde{V}(j) \right\},$$

if $x \in S$ such that $q_x > c\gamma$. The optimal stopping time (provided that the assumptions of Theorem 7.18 are satisfied) is given by (7.95)

$$\tau^\star = \Big(\tilde{f}^\star(x), \tilde{f}^\star(Z_1) + S_1, \tilde{f}^\star(Z_2) + S_2, \dots\Big),$$

where $f^\star$ is given by (7.94) as

$$\tilde{f}^\star(x) := \begin{cases} 0, & \text{if } q_x \leq c\gamma \quad \text{or} \quad q_x > c\gamma \text{ and } \tilde{V}(x) = -e^{-\gamma g(x)}, \\ \infty, & \text{if } q_x > c\gamma \text{ and } \tilde{V}(x) = \sum\limits_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma}\tilde{V}(j) \end{cases}$$

for every $x \in S$.

We already know from section 7.5 and section 7.8, that in case of exponential utility the optimal stopping time does not depend on the time parameter $t \geq 0$. As a consequence the only dependency lies in the initial value $x \in S$, respectively the model setting for the underlying continuous-time Markov chain. The optimal stopping time is thus fully characterized, if the Markov chain and the maximizing function $\tilde{f}^\star$ are known. We can directly see from (7.94) that $\tilde{f}^\star$ can only attain the values 0 and ∞, leading to immediate stopping in state $x \in S$, if $\tilde{f}^\star(x) = 0$, and not stopping in state $x \in S$, if $\tilde{f}^\star(x) = \infty$. For $x \in S$ such that $q_x \leq c\gamma$ we know that $\tilde{f}^\star(x) = 0$ is valid. But if $x \in S$ such that $q_x > c\gamma$ the situation is not that simple. In order to decide which of the two possible values $f^\star(x)$ will attain, we need to know the value function and thus need to solve the fixed-point equation (7.93).

Following the outline of section 8.2 and section 8.3 we will establish conditions under which we can circumvent the need to know the explicit form of $\tilde{V}(x)$. To this end we remember the S_t^0–set, as introduced in (8.5) of Definition 8.6. Adapting this set for the special case of exponential utility leads for every $t \geq 0$ to

$$\begin{aligned} S_t^0 = \Big\{x \in S \,\Big|\, &\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) \leq U\big(-c\vartheta + g(x)\big) + \\ &\frac{c}{q_x}U'\big(-c\vartheta + g(x)\big) \quad \text{for all } \vartheta \geq t\Big\} \\ = \Big\{x \in S \,\Big|\, &-\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x} e^{c\gamma\vartheta - \gamma g(j)} \leq -e^{c\gamma\vartheta-\gamma g(x)} + \frac{c\gamma}{q_x}e^{c\gamma\vartheta-\gamma g(x)} \quad \text{for all } \vartheta \geq t\Big\} \end{aligned}$$

$$= \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x} e^{-\gamma g(j)} \leq -e^{-\gamma g(x)} + \frac{c\gamma}{q_x} e^{-\gamma g(x)} \quad \text{for all } \vartheta \geq t \Big\}$$

$$= \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x} e^{-\gamma g(j)} \leq -\frac{q_x - c\gamma}{q_x} e^{-\gamma g(x)} \Big\}$$

$$= \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} \leq -(q_x - c\gamma) e^{-\gamma g(x)} \Big\} \tag{8.16}$$

Note that the set S_t^0 does not depend on the parameter $t \geq 0$ in this setting. Furthermore, the left-hand side of the inequality in (8.16) is always smaller than zero, whereas the right-hand side is greater or equal to zero, if $q_x \leq c\gamma$. Therefore we know that all $x \in S$ such that $q_x \leq c\gamma$ satisfy the inequality and are thus part of this set. We can hence rewrite (8.16) into

$$S_t^0 = \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} \leq -(q_x - c\gamma) e^{-\gamma g(x)} \Big\}$$

$$= \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} \leq -(q_x - c\gamma) e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \Big\}$$

$$= \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma} e^{-\gamma g(j)} \leq -e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \Big\}$$

We will give this set a new name and will see very soon in Remark 8.22 that this name is indeed meaningful and has a clear interpretation, which we can apply for our theory.

Definition 8.17 (the one-step look ahead set)

Define the *one-step look ahead set* $S^\star$ by

$$S^\star := \Big\{ x \in S \Big| -\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma} e^{-\gamma g(j)} \leq -e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \Big\}. \tag{8.17}$$

Lemma 8.18 (alternative representation and interpretation of the $S^\star$–set)

The set $S^\star$ from Definition 8.17 can be written as

$$S^\star = \Big\{ x \in S \Big| \mathbb{E}_x \left[-e^{c\gamma S_1 - \gamma g(Z_1)} \right] \leq -e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \Big\}. \tag{8.18}$$

Proof of Lemma 8.18
We will show that for $x \in S$ such that $q_x > c\gamma$ the following assertion is true:

$$\sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma} e^{-\gamma g(j)} = \mathbb{E}_x\left[-e^{c\gamma S_1 - \gamma g(Z_1)}\right].$$

To this end we remind ourselves that the joint density f_{S_1,Z_1} of the random variables S_1 and Z_1, conditioned by $X_0 = Z_0 = x$, is according to Corollary 2.20 (c) given by

$$f_{S_1,Z_1}(s, j \mid X_0 = x) = \begin{cases} \exp(-q_x s) \cdot q_{xj}, & \text{if } x \neq j, \\ 0, & \text{if } x = j \end{cases}$$

for every $s \geq 0$, $j \in S$. This yields for $x \in S$ such that $q_x > c\gamma$

$$\begin{aligned}
\mathbb{E}_x\left[-e^{c\gamma S_1 - \gamma g(Z_1)}\right] &= -\int_0^\infty e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{c\gamma s - \gamma g(j)}\, ds \\
&= -\int_0^\infty e^{-q_x s} \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{c\gamma s} e^{-\gamma g(j)}\, ds \\
&= -\int_0^\infty e^{-(q_x - c\gamma)s}\, ds \cdot \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} \\
&= \frac{1}{q_x - c\gamma} \cdot \sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} \\
&= \sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x - c\gamma} e^{-\gamma g(j)}.
\end{aligned}$$

This implies the validity of (8.18) and thus concludes the proof. □

Analogously, we can now consider the S_t^∞–set from (8.3) of Definition 8.2. Analogeous to (8.16) we can simplify S_t^∞ to

$$\begin{aligned}
S_t^\infty = \Big\{ x \in S \,\Big|\, & \sum_{\substack{j\in S,\\ j\neq x}} \frac{q_{xj}}{q_x} U\big(-c\vartheta + g(j)\big) > U\big(-c\vartheta + g(x)\big) + \\
& \frac{c}{q_x} U'\big(-c\vartheta + g(x)\big) \quad \text{for all } \vartheta > t \Big\} \\
= \Big\{ x \in S \,\Big|\, & -\sum_{\substack{j\in S,\\ j\neq x}} q_{xj} e^{-\gamma g(j)} > -(q_x - c\gamma) e^{-\gamma g(x)} \Big\}.
\end{aligned}$$

We see again that there is no dependency on $t \geq 0$ in this context. Furthermore we can easily check that a state $x \in S$ such that $q_x \leq c\gamma$ can never fulfill the inequality in the

S_t^∞–set, since this would imply a negative left-hand side, but a positive right-hand side, which is a contradiction. This allows us to see that S_t^∞ is exactly the complementary set of S_t^0. We will summarize this observation in the following corollary:

Corollary 8.19

By definitions (8.5), (8.3) and (8.17), it holds:

(a) $S^\star = S_t^0$ for all $t \geq 0$,

(b) $S \backslash S^\star = S_t^\infty$ for all $t \geq 0$,

(c) $S_t^0 \cup S_t^\infty = S$ for all $t \geq 0$.

Every state $x \in S$ thus lies either in the set $S^\star = S_t^0$ or in the set $S \backslash S^\star = S_t^\infty$. Now the question remains, whether we can use this one-step look ahead set $S^\star$ in order to determine states, for which the optimal stopping time would stop immediately or never. We have seen in section 8.2, that this is indeed the case for states in $S_t^\infty = S \backslash S^\star$. For states in $S_t^0 = S^\star$, we have shown in section 8.3, that we need an additional closure condition in order to guarantee the optimality of immediate stopping. For the case of exponential utility, this condition is again independent of the time parameter $t \geq 0$ and can be formulated as follows:

Assumption 8.20 (closure assumption for exponential utility)

Suppose that for all $x \in S^\star$ and $j \in S \backslash \{x\}$ the implication

$$q_{xj} \neq 0 \Longrightarrow j \in S^\star \tag{8.19}$$

is valid.

Now we can apply Theorem 8.4 and Theorem 8.10 for the case of exponential utility and merge them into the following theorem:

Theorem 8.21 (optimality of one-step look ahead stopping times)

Let $x \in S$ and Assumption 8.20 be valid. Then it holds:

(a) If $x \in S^\star$, then the fixed-point equation (7.93) attains its maximum in the first argument and

$$\tilde{V}(x) = -e^{-\gamma g(x)}.$$

The corresponding maximizer is given by

$$\tilde{f}^\star(x) = 0.$$

If the underlying Markov chain will ever reach state x, the optimal stopping rule is given by immediate stopping.

(b) If $x \notin S^\star$, then the fixed-point equation (7.93) attains its maximum in the second argument and

$$\tilde{V}(x) = \sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} \tilde{V}(j).$$

The corresponding maximizer is given by

$$\tilde{f}^\star(x) = \infty.$$

If the underlying Markov chain will ever reach state x, the optimal stopping rule will never stop in x.

(c) Define

$$\tau^\star := \big(\tilde{f}^\star(x), \tilde{f}^\star(Z_1) + S_1, \tilde{f}^\star(Z_2) + S_2, \dots\big), \tag{8.20}$$

where

$$\tilde{f}^\star(x) = \begin{cases} 0, & \text{if } x \in S^\star, \\ \infty, & \text{if } x \notin S^\star. \end{cases} \tag{8.21}$$

If $\tau^\star$ is $\mathbb{P}_x$–almost surely finite and

$$\lim_{n \to \infty} \mathbb{E}_x\Big[\tilde{V}(Z_n) \cdot \mathbb{1}_{\{\tau^\star \geq S_n\}}\Big] = 0, \tag{8.22}$$

then the *one-step look ahead stopping time* $\tau^\star$ is the optimal stopping time for $\tilde{V}(x)$ and can be expressed in terms of a first hit time:

$$\tau^\star = \inf\{t \geq 0 | X_t \in S^\star\}. \tag{8.23}$$

Proof

The assertions follow directly from Theorem 7.23, Theorem 8.4 and Theorem 8.10. For the representation (8.23) of the optimal stopping time $\tau^\star$ as first hit time, note that by definition of $\tau^\star$ in (8.20) the optimal stopping rules $\tau^{\star,k}$ for $k \in \mathbb{N}_0$ are either given by

S_k, if $Z_k \in S^\star$, or by ∞, if $Z_k \notin S^\star$. If $\tau^{\star,k} = S_k$ for some $k \in \mathbb{N}_0$, then the stopping rule will stop as soon as state Z_k is reached (immediate stopping). This will terminate the whole stopping problem. If $\tau^{\star,k} = \infty$ for some $k \in \mathbb{N}_0$, then the stopping rule will never stop in state Z_k. In other words, the optimal stopping time terminates as soon as the embedded discrete-time Markov chain $(Z_n)_{n\in\mathbb{N}}$ hits set $S^\star$. This is the case if and only if the Markov chain itself hits the set $S^\star$. This yields representation (8.23). □

Remark 8.22 (interpretation of the one-step look ahead set and one-step look ahead stopping times)

In order to interpret the one-step look ahead set $S^\star$, we will use the representation (8.18) given in Lemma 8.18:

$$S^\star = \left\{ x \in S \,\middle|\, \mathbb{E}_x\left[-e^{c\gamma S_1 - \gamma g(Z_1)}\right] \leq -e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \right\}.$$

Also note that in terms of Theorem 8.21, the $S^\star$–set shall characterize the states $x \in S$ for which the optimal stopping rules are given by immediate stopping. There are exactly two reasons for a state $x \in S$ to be in the set $S^\star$:

<u>It holds that $q_x \leq c\gamma$:</u>

This case can be interpreted as the "high cost case". The intensity rate q_x, which influences the sojourn time in state x, is too low in comparison to the cost rate $c > 0$, weighted by the parameter $\gamma > 0$ which stems from the definition of the exponential utility. Note that by Corollary 2.20 (a), the expected sojourn time in state is given by $\frac{1}{q_x}$. This indicates that the expected waiting time for the next change of state to happen is relatively high. During this time an investor will cumulate relatively high costs. The condition $q_x \leq c\gamma$ thus gives a condition for which it is no longer profitable to wait for the next jump of the underling Markov chain, tolerating the cumulated costs and hoping for a state which yields a higher reward.

Note that in case of exponential utility, the absolute risk aversion ARA_U, which was introduced in (4.1) of chapter 4 is given by

$$ARA_U(x) \equiv \gamma,$$

according to Example 4.2 (a). We have discussed in chapter 4 that the absolute risk aversion models the attitude of an investor towards risk. The higher $\gamma > 0$, the more risk averse an investor becomes. On the other hand, increasing γ will also increase the number of states $x \in S$ which fulfill $q_x \leq c\gamma$ and thus are in the set $S^\star$. This coincides with our intuition: An investor with a high absolute risk aversion level will more likely refrain to wait in certain states for the next jump to happen, as he will more likely fear the additional costs he will cumulate in comparison with the expected time he has to wait in hope for a better state.

It holds that $q_x > c\gamma$ and $\mathbb{E}_x\left[-e^{c\gamma S_1-\gamma g(Z_1)}\right] \leq -e^{-\gamma g(x)}$:

This case can be interpreted as the "low cost case". Analogously to the case above the condition $q_x > c\gamma$ for a state $x \in S$ implies that the expected sojourn time in state x is low enough in comparison to the cost rate $c > 0$ and the risk aversion parameter $\gamma > 0$. An investor is thus more inclined to risk the waiting period up to the next jump of the Markov chain and to accept the additional costs he will cumulate doing this. The decision whether he will do it or not depends on the second inequality

$$\mathbb{E}_x\left[-e^{c\gamma S_1-\gamma g(Z_1)}\right] \leq -e^{-\gamma g(x)}.$$

Here, the investor compares two values: The utility $-e^{-\gamma g(x)}$ he would gain by immediate stopping and the expected utility $\mathbb{E}_x\left[-e^{c\gamma S_1-\gamma g(Z_1)}\right]$ he would gain by waiting exactly "one step", i.e the next change of state of the underlying Markov chain. This explains the name *one-step look ahead*. The investor compares the actual utility with the expected utility by looking one step ahead, thus the expected utility under the stopping time $\tau = S_1$. In terms of value functions, an investor thus compares

$$\tilde{V}(x,0) \text{ for stopping time } \tau = 0 \quad \text{and} \quad \tilde{V}(x,S_1) \text{ for stopping time } \tau = S_1.$$

Clearly, if he expects a higher utility by waiting for the first jump time S_1 than by stopping immediately, he will not stop in state x. This leads to $x \notin S^\star$ and in terms of Theorem 8.21 to the optimality of never stopping in state x.

On the other hand, if the acutal utility gained by immediate stopping is higher than the expected utility for stopping at time S_1, we can generally not expect that the investor will stop immediately. By definition of the value function

$$\tilde{V}(x) = \sup_{\tau\in\Sigma_x} \tilde{V}(x,\tau)$$

the investor has to compare all feasible stopping times.

But if the closure assumption 8.20 is valid, then Theorem 8.21 shows us that it is indeed sufficient to compare the utility gained by immediate stopping with the expected utility by stopping at S_1. In particular we do not need to analyse any other feasible stopping times. This legitimates the name of the one-step look ahead stopping time τ in (8.23), which is by Theorem 8.21 the optimal stopping time for the unrestricted stopping problem in case of exponential utility.

Example 8.23 (example for the optimality of one-step look ahead stopping times)

(a) Let $S = \mathbb{N}_0$ and X a homogeneous Poisson process with intensity rate $\lambda > 0$ and initial value $X_0 = x \in \mathbb{N}_0$. The intensity matrix $Q = (q_{xj})_{x,j \in \mathbb{N}_0}$ of X is given by

$$q_{xj} := \begin{cases} -\lambda, & \text{for } j = x, \\ \lambda, & \text{for } j = x+1, \\ 0, & \text{else.} \end{cases}$$

The Poisson process is a simple example for a continuous-time Markov chain with discrete, but not finite state space S. Note that at every single jump time S_n, the value of the process increases by exactly one unit. The corresponding embedded discrete-time Markov chain is therefore trivially given by $(Z_n)_{n \in \mathbb{N}_0}$, where $Z_n = n + x$.

Furthermore, consider the exponential utility function

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$. Moreover, let $c > 0$ be the cost rate and $g\colon S \to \mathbb{R}$ the reward function in this setting.

We want now to apply Theorem 8.21 in order to find the optimal stopping time $\tau^\star$ for the value function $\tilde{V}(x)$. For Theorem 8.21 to be applicable we need to guarantee the validity of Assumption 8.20. That is, for all $x \in S^\star$ and $j \in \mathbb{N}_0 \setminus \{x\}$ such that $q_{xj} \neq 0$ we need to ensure that $j \in S^\star$. In this specific example with a homogeneous Poisson process as continuous-time Markov chain, for every $x \in \mathbb{N}_0$ the only intensity rate q_{xj} not equal to zero and $j \neq x$ is given by $q_{x,x+1} = \lambda > 0$. In other words, for every $x \in S^\star$ the subsequent state $x+1$ has also to be in $S^\star$. Therefore, Assumption 8.20 simplifies in case of a homogeneous Poisson process to

$$x + 1 \in S^\star \quad \text{for all } x \in S^\star. \tag{8.24}$$

For (8.24) to be valid (and supposing $S^\star \neq \emptyset$), we thus need the existence of a threshold $\tilde{x} \in \mathbb{N}_0$, such that

$$x \in S^\star \quad \text{for all } x \geq \tilde{x}. \tag{8.25}$$

In this case the one-step look ahead set $S^\star$ has to be of the form

$$S^\star = \{\tilde{x}, \tilde{x}+1, \tilde{x}+2, \dots\}. \tag{8.26}$$

The question remains, whether such a threshold $\tilde{x} \in \mathbb{N}_0$ exists, such that Assumption 8.20 is satisfied. We will therefore try to calculate the one-step look ahead set $S^\star$ explicitly. As a remainder, $S^\star$ is given by (8.17) as

$$S^\star = \left\{ x \in \mathbb{N}_0 \;\middle|\; -\sum_{\substack{j \in S, \\ j \neq x}} \frac{q_{xj}}{q_x - c\gamma} e^{-\gamma g(j)} \leq -e^{-\gamma g(x)} \text{ and } q_x > c\gamma \quad \text{or} \quad q_x \leq c\gamma \right\}.$$

In this example the intensity rates q_x are all homogeneous and given by $q_x = \lambda$ for every $x \in \mathbb{N}_0$. Hence we get

$$S^\star = \left\{ x \in \mathbb{N}_0 \,\middle|\, -\frac{\lambda}{\lambda - c\gamma} e^{-\gamma g(x+1)} \leq -e^{-\gamma g(x)} \text{ and } \lambda > c\gamma \quad \text{or} \quad \lambda \leq c\gamma \right\}.$$

This allows us to differentiate between two global cases:

$\underline{\lambda \leq c\gamma}$:

The intensity rate $\lambda > 0$ is not higher than the cost rate $c > 0$, weighted by the risk aversion parameter $\gamma > 0$. This leads to the one-step look ahead set

$$S^\star = \mathbb{N}_0.$$

Obviously the closure condition is satisfied and the threshold for (8.26) is set to $\tilde{x} = 0$. Theorem 8.21 becomes applicable and yields immediately that

$$\tilde{V}(x) = -e^{-\gamma g(x)}$$

for every $x \in \mathbb{N}_0$. The optimal stopping time is given by

$$\tau^\star = 0.$$

We can again interpret this case in terms of expected sojourn times. As λ is small (relative to $c\gamma$), we expect a relatively long sojourn time of $\frac{1}{\lambda}$ in every state. Thus waiting for a higher state will accumulate too much additional costs in order to be worthwhile. Thus the optimal strategy is to stop immediately to avoid additional costs.

$\underline{\lambda > c\gamma}$:

The intensity rate $\lambda > 0$ is higher than the cost rate $c > 0$, weighted by the risk aversion parameter $\gamma > 0$. This leads to the one-step look ahead set

$$S^\star = \left\{ x \in \mathbb{N}_0 \,\middle|\, -\frac{\lambda}{\lambda - c\gamma} e^{-\gamma g(x+1)} \leq -e^{-\gamma g(x)} \right\}.$$

We get

$$\begin{aligned} x \in S^\star &\Leftrightarrow -\frac{\lambda}{\lambda - c\gamma} e^{-\gamma g(x+1)} \leq -e^{-\gamma g(x)} \\ &\Leftrightarrow e^{-\gamma(g(x+1) - g(x))} \geq \frac{\lambda - c\gamma}{\lambda} \\ &\Leftrightarrow g(x+1) - g(x) \leq \frac{1}{\gamma} \ln\left(\frac{\lambda}{\lambda - c\gamma}\right). \end{aligned}$$

Therefore the one-step look ahead set is given by

$$S^\star = \left\{ x \in \mathbb{N}_0 \,\middle|\, g(x+1) - g(x) \leq \frac{1}{\gamma} \ln\left(\frac{\lambda}{\lambda - c\gamma}\right) \right\}. \tag{8.27}$$

For the closure condition to be satisfied and thus exhibiting the form (8.26), we hence need to assume that there exists a threshold $\tilde{x} \in \mathbb{N}_0$ such that

$$\begin{cases} g(x+1) > g(x) + \frac{1}{\gamma}\ln\left(\frac{\lambda}{\lambda - c\gamma}\right) & \text{for all } x < \tilde{x}, \\ g(x+1) \leq g(x) + \frac{1}{\gamma}\ln\left(\frac{\lambda}{\lambda - c\gamma}\right) & \text{for all } x \geq \tilde{x}. \end{cases} \tag{8.28}$$

Assuming that (8.28) is valid for a threshold $\tilde{x} \in \mathbb{N}_0$, Theorem 8.21 becomes applicable and yields that the optimal stopping time solving $\widetilde{V}(x)$ is given by (8.23)

$$\begin{aligned} \tau^\star &= \inf\{t \geq 0 | X_t \in S^\star\} \\ &= \inf\{t \geq 0 | X_t \geq \tilde{x}\}. \end{aligned}$$

The optimal stopping time will trigger, as soon as the value $\tilde{x}$ is reached. In this case, the optimal stopping time is of reservation type. That means that the stopping time terminates as soon as a certain threshold, or reservation level is reached.

We can interpret this case in the following way: As the intensity rate λ is greater than $c\gamma$, we can heuristically argue that, due to the relatively high λ, the expected sojourn times between two jumps of the underlying Markov chain are comparatively small. As long as $x < \tilde{x}$ and thus $x \notin S^\star$, the marginal gains by jumping from state x with reward $g(x)$ to state $x+1$ with reward $g(x+1)$ are high. Thus waiting for the next jump, which we expect to happen relatively soon, is profitable for an investor, even if he has to pay additional cumulated costs. As soon as the threshold $\tilde{x}$ is reached, the marginal gain from successive states diminishes. Therefore the optimal stopping strategy is to stop as soon as the threshold is reached.

(b) Consider the special case of $\lambda = 2$, $c = 3$, $\gamma = \frac{1}{2}$ and initial state $x = 0$. Cleary, $\lambda > c\gamma$ holds and we thus are in the case where we need to check if condition (8.28) is satisfied. As reward function $g\colon \mathbb{N}_0 \to \mathbb{R}$ we choose

$$g(x) := 14\sqrt{x}.$$

In this scenario we can calculate

$$\frac{1}{\gamma}\ln\left(\frac{\lambda}{\lambda - c\gamma}\right) = 2\ln(4) \approx 2.77.$$

Knowing that g is a concave function with diminishing marginal gains, we easily check for which argument x the increments are for the first time smaller than $2\ln(4)$.

x	0	1	2	3	4	5	6	7	8
$g(x+1)-g(x)$	14.00	5.80	4.45	3.75	3.0	2.99	2.75	2.56	2.40

We can conclude that for the threshold $\tilde{x} = 6$ condition (8.28) is satisfied. Thus, the optimal stopping time is given by

$$\tau^\star = \inf\{t \geq 0 | X_t \geq 6\}.$$

An investor will therefore wait for 6 jumps of the Poisson process, before he will terminate the stopping problem.

Assume now that we increase the risk aversion parameter from $\gamma = \frac{1}{2}$ to $\gamma = \frac{3}{5}$ and let all other parameters untouched. This leads to

$$\frac{1}{\gamma} \ln \left(\frac{\lambda}{\lambda - c\gamma} \right) = \frac{5}{3} \ln (10) \approx 3.84.$$

Looking in the table above, we can immediately indentify the new threshold $\tilde{x} = 4$. The optimal stopping time is therefore given by

$$\tau^\star = \inf\{t \geq 0 | X_t \geq 4\}.$$

An investor with a higher risk aversion will thus terminate the stopping problem earlier. He perceives the reward margins from jump 4 to 6 as to small to tolerate the risk of high additional costs, if these jumps will not happen as soon as expected.

9 Continuous-Time Approach for the Generalized Risk-Sensitive Stopping Problem for Continuous-Time Markov Chains

9.1 Introduction

In this chapter we want to present an alternative approach to solve the unrestricted generalized risk-sensitive stopping problem

$$\mathbb{E}_x \left[U \left(-c\tau + g(X_\tau) \right) \right] \to \max_{\tau \in \Sigma_x}!$$

for continuous-time Markov chains, as introduced in (5.1) of chapter 5. Just as in the discrete-time approach of chapter 7 we will again mainly utilize the concept of value functions in order to get access to a theory which allows us to find a solution to (5.1). Note that the value functions for (5.1) are given by

$$\begin{cases} V(\,\cdot\,,\tau) \colon [0,\infty) \times S \to [-\infty,\infty), \\ (t,x) \mapsto V(t,x,\tau) = \mathbb{E}_x \big[U \left(-ct - c\tau + g(X_\tau) \right) \big] \end{cases}$$

for a given stopping time $\tau \in \Sigma$, respectively

$$\begin{cases} V \colon [0,\infty) \times S \to [-\infty,\infty), \\ (t,x) \mapsto V(t,x) = \sup\limits_{\tau \in \Sigma_x} V(t,x,\tau). \end{cases}$$

But in contrast to chapter 7 we will have no need to tackle the unrestricted value function $V(t,x)$ by introducing n–step value functions and interpret $V(t,x)$ as a limit

$$V(t,x) = \lim_{n \to \infty} V_n(t,x)$$

of the discrete sequence of n–step value functions $\big(V_n(t,x)\big)_{n \in \mathbb{N}_0}$ for $t \geq 0$ and $x \in S$. Instead we will use the popular dynamic programming approach in the field of optimal control. Basically it states that under certain conditions, the value function of a given stochastic optimization problem can be seen as a solution of a deterministic partial differential equation, the so-called *Hamilton-Jacobi-Bellman (HJB) equation*. This HJB-equation can be seen as a continuous-time analogon to the discrete-time Bellman equation, which was introduced 1957 by Richard Bellman in his work Bellman [1957]. The Bellman equation established in Theorem 7.14 serves as example. The HJB equation can also be seen as an extension of the Hamilton-Jacobi equation, a first-order partial differential equation which was established earlier by William Rowan Hamilton and Carl Gustav Jacob Jacobi, thus explaining the name "HJB".

One major downside of this theory is that in order for the value function to be a solution of a partial differential equation, it has to exhibit the required differentiability in the sense that every partial derivative stated in the HJB equation does have to exist. On the other hand, by finding a solution of the HJB equation we need to verify if it is indeed the wanted value function we sought. In fact, a classical smooth solution to an HJB equation does not have to exist in general. In the past decades a rich theory was developed to weaken the classical notion of solutions in order to generalize the dynamic programming approach. One of the most prominent concepts would be the so-called viscosity solutions to partial differential equations, introduced by Pierre-Louis Lions and Michael Crandall in 1983 for non-linear first order partial differential equations (cf. Crandall & Lions [1983]) and later extended by Robert Jensen (cf. Jensen [1988])and Hitoshi Ishii in 1989 (cf. Hitoshi [1989]) to non-linear second-order partial differential equations. At this point we also want to mention Fleming & Soner [2006], where Wendell Helms Fleming and Halil Mete Soner applied the concept of viscosity solutions to the optimal control of Markov processes.

As the main emphasis of this work lies in the study of generalized risk-sensitive stopping problems by using the discrete-time approach in chapter 7, we do not want to introduce the generalized concept of viscosity solutions in detail, but rather confine ourselfes to the existence of a classical smooth solution to the corresponding HJB equation, which will be stated in this chapter.

In the following we will establish the HJB equation for unrestricted generalized risk-sensitive stopping problem (5.1) and state a corresponding verification theorem to guarantee that a classical solution to the HJB equation is indeed the sought-after value function $V(t,x)$ we are interested in. Furthermore we can express the optimal stopping time to the unrestricted stopping problem in terms of a first hit time.

9.2 The Verification Theorem for Generalized Risk-Sensitive Stopping Problems

As mentioned above in the introduction the main task in this section will be to establish the HJB equation as well as formulate an appropriate verification theorem in order to guarantee a solution of this HJB equation being the desired value function of the corresponding stopping problem. To this end we first have to establish certain auxiliary propositions which will be needed in order to prove the verification theorem rigorously. This approach basicaly follows the standard procedure for verification of HJB equations for jump processes. For further literature on this topic the reader may be referred to [Bäuerle & Rieder, 2004, Section II.A.].

One aspect that has to be dealt with beforehand, is the existence of jumps and thus the discontinuity of the sample paths of the underlying continuous-time Markov chain. As we will see in the proof of Theorem 9.6, the basic approach to such verification theorems is an appropriate application of Itô's formula. In our case the underlying Markov process exhibits discontinuous jumps. Therefore, we need a version of Itô's formula which explicitly allows for such jumps:

Proposition 9.1 (Itô's formula for continuous-time Markov chains)

Let X be a continuous-time Markov chain as defined in chapter 2 and $G\colon [0,\infty)\times S\to\mathbb{R}$ such that $G\in C^{1,0}([0,\infty)\times S)$. Then we get for every $t,t'\geq 0$ such that $t'\geq t$ *Itô's formula for continuous-time Markov chains*:

$$
\begin{aligned}
G(t',X_{t'}) &= G(t,X_t) + \int_t^{t'} G_t(s,X_{s^-})ds + \sum_{t<s\leq t'}\Big[G(s,X_s)-G(s,X_{s^-})\Big] \\
&= G(t,X_t) + \int_t^{t'} G_t(s,X_{s^-})ds + \sum_{\substack{n\in\mathbb{N}_0,\\ t<S_n\leq t'}}\Big[G(S_n,Z_n)-G(S_n,Z_{n-1})\Big]. \qquad (9.1)
\end{aligned}
$$

Note that $(S_n)_{n\in\mathbb{N}_0}$ denotes the usual sequence of jump times of X and $(Z_n)_{n\in\mathbb{N}_0}$ represents the corresponding embedded discrete-time Markov chain for X. As the sample paths of X are càdlàg (right-continuous with existing left-hand limits), X_{s^-} denotes such a left-hand limit of X at time $s\geq 0$.

Proof of Proposition 9.1
Due to its construction the Markov chain X and thus the process $(t,X_t)_{t\geq 0}$ is a semi-martingale. Therefore Itô's formula for jump processes as given for example in [Protter, 2005, Theorem 33] or [Klebaner, 2012, Chapter 9.3, Equation (9.4)] becomes applicable:

$$
G(t',X_{t'}) = G(t,X_t) + \int_t^{t'} G_t(s,X_{s^-})ds + \sum_{t<s\leq t'}\Big[G(s,X_s)-G(s,X_{s^-})\Big]
$$

for $t'\geq t$. As a continuous-time Marcov chain is constant up to discontinuities at jump times, the last sum is not uncountably infinite, but indeed well-defined as its summands differ only from zero if s is a jump time S_n in the compact intervall $[t,t']$. Moreover as we assumed for the Markov chain to be stable and conservative, Proposition 2.21 implies almost surely a finite number of jumps in $[t,t']$. The sum above is thus even finite and

$$
\sum_{t<s\leq t'}\Big[G(s,X_s)-G(s,X_{s^-})\Big] = \sum_{\substack{n\in\mathbb{N}_0,\\ t<S_n\leq t'}}\Big[G(S_n,Z_n)-G(S_n,Z_{n-1})\Big].
$$

□

In regard of Proposition 9.1, the sum in (9.1) interpreted as a function in t' is discontinuous. To compensate this discontinuity we will additionally introduce two additional processes which are in some sense related to the Markov chain: the counting process for X and the corresponding compensator.

Definition 9.2 (counting process for continuous-time Markov chains)

(a) Consider the mapping $P\colon [0,\infty) \times \mathcal{P}(S) \times \Omega \to \mathbb{N}_0$,

$$P(t, M)(\omega) := \sum_{n\in\mathbb{N}_0} \mathbb{1}_{\{S_n \le t\}}(\omega)\, \mathbb{1}_{\{Z_n \in M\}}(\omega). \tag{9.2}$$

For every $M \subseteq S$ we call $\big(P(t, M)\big)_{t\ge 0}$ the *counting process* (on M) corresponding to X. $P(t, M)$ counts the random number of times a Markov chain X hits some set $M \subseteq S$ up to time $t \ge 0$.

(b) This counting process can be identified in a natural way with a *random counting measure* $\nu\colon \mathcal{B}\big([0,\infty)\big) \times \mathcal{P}(S) \to \mathbb{N}_0$:

$$\nu\big(A \times M\big) := \sum_{n\in\mathbb{N}_0} \delta_{(S_n, Z_n)}\big(A \times M\big), \tag{9.3}$$

where $\delta_{(S_n,Z_n)}$ is a (random) Dirac-measure on $\mathcal{B}\big([0,\infty)\big) \times \mathcal{P}(S)$.

Clearly, these two definitions are equivalent in the sense that for every $t \ge 0$ and $M \in \mathcal{P}(S)$ we get

$$P(t, M) = \nu\big([0, t] \times M\big).$$

For every $\Big(\mathcal{B}([0,\infty) \times \mathcal{P}(S)\Big)$–measurable function $f\colon [0,\infty) \times S \to \mathbb{R}$ and $t \ge 0$, $M \in \mathcal{P}(S)$ we get the integral representation

$$\begin{aligned}\int_{[0,t]\times M} f(s, j) P(ds, dj) &= \int_{[0,t]\times M} f(s, j)\nu(ds, dj) \\ &= \sum_{n\in\mathbb{N}_0} \int_{[0,t]\times M} f(s, j)\delta_{(S_n,Z_n)}(ds, dj) \\ &= \sum_{n\in\mathbb{N}_0} f(S_n, Z_n) \cdot \mathbb{1}_{\{S_n\le t, Z_n\in M\}} \\ &= \sum_{\substack{n\in\mathbb{N}_0,\\ 0\le S_n\le t}} f(S_n, Z_n) \cdot \mathbb{1}_{\{Z_n\in M\}}.\end{aligned}$$

For the special case $M = S$ this equation simplifies to

$$\int_{[0,t]\times S} f(s,j)P(ds,dj) = \sum_{j\in S}\int_0^t f(s,j)P(ds,\{j\}) = \sum_{\substack{n\in\mathbb{N}_0,\\ 0\leq S_n\leq t}} f(S_n,Z_n). \tag{9.4}$$

Definition 9.3 (compensator for counting processes of continuous-time Markov chains)

(a) We define the mapping $\tilde{P}\colon [0,\infty)\times\mathcal{P}(S)\times\Omega\to[0,\infty)$ by

$$\tilde{P}(t,M)(\omega) := \sum_{j\in M}\int_0^t \sum_{\substack{x\in S,\\ x\neq j}} q_{xj}\cdot \mathbb{1}_{\{X_{s^-}=x\}}(\omega)ds \tag{9.5}$$

and call the stochastic process $\big(\tilde{P}(t,M)\big)_{t\geq 0}$ (for every $M\in\mathcal{P}(S)$) the *compensator* of the counting process $\big(P(t,M)\big)_{t\geq 0}$.

(b) Analogously to the previous definition we also define the random counting measure $\tilde{\nu}\colon \mathcal{B}\big([0,\infty)\big)\times\mathcal{P}(S)\to\mathbb{N}_0$ by

$$\tilde{\nu}((s,t]\times M) := \tilde{P}(t,M) - \tilde{P}(s,M) \tag{9.6}$$

for every $s\leq t$ and $M\in\mathcal{S}$.

Again, we get $\tilde{P}(t,M) = \tilde{\nu}\big((0,t]\times M\big)$ for $t\geq 0$ and $M\in\mathcal{P}(S)$.

The name "compensator" is justified by the following theorem, which states that the discrete valued counting process $\big(P(t,M)\big)_{t\geq 0}$ can be adjusted by using the corresponding compensator to become a real-valued locale martingale:

Proposition 9.4 (compensated counting processes)

The process $\big(Q(t,M)\big)_{t\geq 0}$, defined by

$$Q(t,M) := P(t,M) - \tilde{P}(t,M)$$

is a locale $(\mathcal{F}_t^X)_{t\geq 0}$–martingale for every $M\in\mathcal{P}(S)$.

Proof of Proposition 9.4
For a proof of this proposition the reader may refer to [Davis, 1993, Proposition (26.7)]. □

One last aspect needed for stating and proving the subsequent verification theorem is the concept of restarting continuous-time Markov chains. In Definition 7.1 we already introduced such restarted Markov chains for the special case that the restart happened exactly at the first jump time S_1. We will now generalize this by allowing for an arbitrary restarting time.

Proposition 9.5 (restarted continuous-time Markov chain)

Let X be a homogeneuous continuous-time Markov chain with initial value $X_0 = x \in S$. Then it holds:

(a) For every restarting time $u \geq 0$ the process

$$\left(X_{t+u}\right)_{t\geq 0}$$

is again a homogeneous continuous-time Markov chain, given that the initial value X_u is known.

(b) For every $j \in S$, $u \geq 0$ and $t \geq 0$ we get

$$\mathbb{P}(X_{t+u} = j | X_u = x) = \mathbb{P}(X_t | X_0 = x).$$

$\left(X_{t+u}\right)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ thus share the same distribution, given that the corresponding initial values coincide.

Proof
Let X be a homogeneuous continuous-time Markov chain with initial value $X_0 = x \in S$ and let $u \geq 0$. Due to Definition 2.1 and the homogeneity property (2.2) we can immediately conclude that $\left(X_{t+u}\right)_{t\geq 0}$ is indeed a homogeneous continuous-time Markov chain, given that the initial value X_u is known. The same homogeneity property also implies the second part of Proposition 9.5. □

Now we are able to state the HJB equation for the unrestricted generalized risk-sensitive stopping problem for continuous-time Markov chains and formulate a correponding verfication theorem to guarantee that the solution of the HJB equation is the wanted value function $V(t, x)$ for the stopping problem.

Theorem 9.6 (verification theorem for the generalized risk-sensitive stopping problem)
Let $G \in C^{1,0}([0,\infty) \times S)$ be a solution of the Hamilton-Jacobi-Bellman equation (HJB)

$$0 = \max\left\{ G_t(t,x) + \sum_{j \in S} \Big(G(t,j) - G(t,x) \Big) q_{xj} \ , \ U(-ct + g(x)) - G(t,x) \right\} \tag{9.7}$$

and additionally fulfill the following growth-condition for all $x \in S$ and $t \geq 0$:

$$\mathbb{E}_x \left[\sum_{\substack{n \in \mathbb{N}_0, \\ 0 \leq S_n \leq t}} \Big| G(S_n, Z_n) \Big| \right] < \infty. \tag{9.8}$$

Then it holds:

(a) For all $t \geq 0$ and $x \in S$ the value function V fulfills

$$V(t,x) \leq G(t,x).$$

(b) If in addition the first hit time

$$\tau^\star := \inf \Big\{ s \geq 0 \ \Big| \ G(t+s, X_{t+s}) = U\big(-ct - cs + g(X_{t+s}) \big) \Big\} \tag{9.9}$$

is $\mathbb{P}_x$–almost surely finite, then $\tau^\star$ is the optimal stopping time for $V(t,x)$ and

$$V(t,x) = G(t,x).$$

Proof
Let $G \in C^{1,0}([0,\infty) \times S)$ be a solution of the HJB equation (9.7) which fulfills the additional condition (9.8).

(a) An application of Itô's formula for Markov chains as given in Proposition 9.1 on G and (t, X_t) for fixed time points $t, t' \geq 0$ such that $t' \geq 0$ yields

$$G(t', X_{t'}) = G(t, X_t) + \int_t^{t'} G_t(s, X_{s^-}) ds + \sum_{\substack{n \in \mathbb{N}_0, \\ t < S_n \leq t'}} \Big[G(S_n, Z_n) - G(S_n, Z_{n-1}) \Big].$$

A more detailed analysis of the last sum shows that it can be written in terms of the counting process P and the random counting measure ν, as introduced in Definition 9.2, equation (9.2) and (9.3), respectively. Thus, using equation (9.4), we get

$$
\begin{aligned}
&\sum_{\substack{n\in\mathbb{N}_0,\\ t<S_n\le t'}} \Big[G(S_n,Z_n)-G(S_n,Z_{n-1})\Big]\\
&=\sum_{t<s\le t'}\Big[G(s,X_s)-G(s,X_{s^-})\Big]\\
&=\int_{[t,t']\times S}\Big[G(s,j)-G(s,X_{s^-})\Big]P(ds,dj)\\
&=\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]P(ds,\{j\})\\
&=\underbrace{\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\tilde{P}(ds,\{j\})}_{(I)}\\
&\quad+\underbrace{\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\left(P-\tilde{P}\right)(ds,\{j\})}_{(II)},
\end{aligned}
$$

where $\tilde{P}$ denotes the compensator of the counting process P, according to Definition 9.3. The two parts (I) and (II) will now be treated separately:

(I) With respect to equation (9.5), for every $t\ge 0$ and $j\in S$ the compensator $\tilde{P}$ is given by

$$
\tilde{P}(t,\{j\})=\int_0^t\sum_{\substack{x\in S,\\ x\neq j}} q_{xj}\cdot \mathbb{1}_{\{X_{s^-}=x\}}\,ds
$$

and thus by using the stability assumption Assumption 2.13:

$$
\begin{aligned}
&\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\tilde{P}(ds,\{j\})\\
=&\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\cdot\sum_{\substack{x\in S,\\ x\neq j}} q_{xj}\cdot \mathbb{1}_{\{X_{s^-}=x\}}\,ds\\
=&\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\cdot q_{X_{s^-},j}\cdot \mathbb{1}_{\{X_{s^-}\neq j\}}\,ds\\
=&\sum_{j\in S}\int_t^{t'}\Big[G(s,j)-G(s,X_{s^-})\Big]\cdot q_{X_{s^-},j}\,ds.
\end{aligned}
$$

(II) Defining the process $(M_t)_{t\geq 0}$ by

$$M_t := \sum_{j\in S}\int_0^t \Big[G(s,j) - G(s,X_{s^-})\Big]\Big(P-\tilde{P}\Big)(ds,\{j\}) = \sum_{j\in S}\int_0^t \Big[G(s,j) - G(s,X_{s^-})\Big](\nu-\tilde{\nu})(ds,\{j\})$$

we can rewrite the expression in (II) to

$$\sum_{j\in S}\int_t^{t'} \Big[G(s,j) - G(s,X_{s^-})\Big]\Big(P-\tilde{P}\Big)(ds,\{j\}) = M_{t'} - M_t.$$

Note that according to Proposition 9.4, the compensated counting process

$$\Big(P(t,\{j\}) - \tilde{P}(t,\{j\})\Big)_{t\geq 0}$$

is a locale $(\mathcal{F}_t^X)_{t\geq 0}$–martingale for every $j \in S$. Thus, having a locale martingale as integrator, this martingale property can be transferred to the whole process $(M_t)_{t\geq 0}$. Moreover, we can even get a stronger assertion due to [Davis, 1993, Thm. 26.12]:

Under the validity of condition (9.8) from the requirements of this verification theorem, namely

$$\mathbb{E}_x\left[\sum_{\substack{n\in\mathbb{N}_0,\\ 0\leq S_n\leq t}} \Big|G(S_n,Z_n)\Big|\right] < \infty$$

for all $x \in S$ and $t \geq 0$, the process $(M_t)_{t\geq 0}$ becomes an $(\mathcal{F}_t^X)_{t\geq 0}$–martingale. Therefore, taking the expectation for every $t \geq 0$ and $x \in S$ yields

$$\mathbb{E}_x\left[\sum_{j\in S}\int_t^{t'} \Big[G(s,j) - G(s,X_{s^-})\Big]\Big(P-\tilde{P}\Big)(ds,\{j\})\right] = \mathbb{E}_x\Big[M_{t'} - M_t\Big] = 0. \tag{9.10}$$

Going back to Itô's formula and applying parts (I) and (II), we thus get

$$\begin{aligned}
&G(t', X_{t'}) \\
&= G(t, X_t) + \int_t^{t'} G_t(s, X_{s^-})ds + \sum_{\substack{n\in\mathbb{N}_0,\\ t<S_n\le t'}} \Big[G(S_n, Z_n) - G(S_n, Z_{n-1})\Big] \\
&= G(t, X_t) + \int_t^{t'} G_t(s, X_{s^-})ds + \sum_{j\in S}\int_t^{t'} \Big[G(s, j) - G(s, X_{s^-})\Big] \cdot q_{X_{s^-},j}\, ds + M_{t'} - M_t \\
&= G(t, X_t) + \int_t^{t'} G_t(s, X_{s^-}) + \sum_{j\in S} \Big[G(s, j) - G(s, X_{s^-})\Big] \cdot q_{X_{s^-},j}\, ds + M_{t'} - M_t \qquad (9.11)\\
&\le G(t, X_t) + 0 + M_{t'} - M_t.
\end{aligned}$$

The last inequality holds due to the fact that G is assumed to be a solution of the HJB equation (9.7). Therefore, the expression within the integral is always bounded from above by zero for every $t' \ge t$.

On the other hand we know that G, being a solution to the HJB equation (9.7), fulfills

$$U(-ct + g(x)) - G(t, x) \le 0$$

for all $x \in S$ and $t \ge 0$.

Now fix $x \in S$, $t \ge 0$ and suppose we have a stopping time $\tilde{\tau} \in \Sigma_x$ such that $\tilde{\tau} \ge t$. This yields

$$U(-c\tilde{\tau} + g(X_{\tilde{\tau}})) \le G(\tilde{\tau}, X_{\tilde{\tau}}) \le G(t, X_t) + M_{\tilde{\tau}} - M_t.$$

Taking the conditional expectation $\mathbb{E}[\cdot|X_t = x]$ leads to

$$E\big[U(-c\tilde{\tau}+g(X_{\tilde{\tau}}))\big|X_t = x\big] \le E\big[G(t, X_t)\big|X_t = x\big]+E\big[M_{\tilde{\tau}}-M_t\big|X_t = x\big] = G(t, x),$$

where the last equality holds, since the optional sampling theorem (cf. for example [Klenke, 2013, Satz 10.11]) for the $(\mathcal{F}_t^X)_{t\ge0}$–martingale $(M_t)_{t\ge0}$ implies

$$E\big[M_{\tilde{\tau}} - M_t\big|X_t = x\big] = 0.$$

Now let $\tau \in \Sigma_x$ be arbitrary and define $\tilde{\tau} := \tau + t$. Note that $\tilde{\tau}$ is again a $\mathbb{P}_x$–almost surely finite $(\mathcal{F}_t^X)_{t\ge0}$–stopping time such that $\tilde{\tau} \ge t$. Using Proposition 9.5 will now lead to

$$\begin{aligned} V(t,x,\tau) &= \mathbb{E}_x\big[U(-ct - c\tau + g(X_\tau))\big] \\ &= \mathbb{E}_x\big[U(-ct - c(\tilde{\tau} - t) + g(X_{\tilde{\tau}-t}))\big] \\ &= \mathbb{E}\big[U(-c\tilde{\tau} + g(X_{\tilde{\tau}-t}))\,\big|X_0 = x\big] \\ &= \mathbb{E}\big[U(-c\tilde{\tau} + g(X_{\tilde{\tau}}))\,\big|X_t = x\big] \\ &\le G(t,x). \end{aligned} \tag{9.12}$$

Since $\tau \in \Sigma_x$ was arbitrarily chosen, we can conclude that

$$V(t,x) = \sup_{\tau \in \Sigma_x} V(t,x,\tau) \le G(t,x).$$

(b) Let $x \in S$ and $t \ge 0$. Now consider the special stopping time

$$\tau^\star = \inf\Big\{s \ge 0 \,\Big|\, G(t+s, X_{t+s}) = U(-ct - cs + g(X_t + s))\Big\}.$$

Clearly, $\tau^\star$ is a first hit time and thus an $(\mathcal{F}_t^X)_{t\ge0}$–stopping time. By assumption we know that $\tau^\star$ is $\mathbb{P}_x$–almost surely finite and thus

$$\tau^\star \in \Sigma_x.$$

We will now show that this stopping time is indeed optimal for the value function $V(t,x)$ and that $V(t,x) = G(t,x)$.

Note that as a solution of the HJB equation (9.7), G always fulfills

$$G(s,j) \ge U(-cs + g(j))$$

for all $s \ge 0$ and $j \in S$.

By definition of the first hit time $\tau^\star$ we know that any $s < \tau^\star$ yields

$$G(t+s, X_{t+s}) > U(-ct - cs + g(X_{t+s})).$$

Therefore, as a solution of the HJB equation (9.7), G has to fulfill

$$G_t(t+s, X_{(t+s)^-}) + \sum_{j\in S}\Big(G(t+s,j) - G(t+s, X_{(t+s)^-})\Big) q_{X_{(t+s)^-},j} = 0$$

for any $s < \tau^\star$.

On the other hand, the continuity of U and G, as well as the right-continuity of the sample paths of X require

$$G(t+\tau^\star, X_{t+\tau^\star}) = U(-ct - c\tau^\star + g(X_{t+\tau^\star})).$$

Hence, applying Itô's formula and especially (9.11) yields for $\tau^\star + t \geq t$:

$$\begin{aligned}
&U(-ct - c\tau^\star + g(X_{t+\tau^\star})) \\
&= G(t+\tau^\star, X_{t+\tau^\star}) \\
&= G(t, X_t) + \int_t^{t+\tau^\star} G_t(s, X_{s^-}) + \sum_{j \in S} \Big[G(s,j) - G(s, X_{s^-}) \Big] \cdot q_{X_{s^-},j} \, ds + M_{t+\tau^\star} - M_t \\
&= G(t, X_t) + \int_0^{\tau^\star} G_t(t+s, X_{(t+s)^-}) + \sum_{j \in S} \Big[G(t+s,j) - G(t+s, X_{(t+s)^-}) \Big] \cdot q_{X_{(t+s)^-},j} \, ds \\
&\quad + M_{t+\tau^\star} - M_t \\
&= G(t, X_t) + M_{t+\tau^\star} - M_t.
\end{aligned}$$

Taking again the conditional expectation $\mathbb{E}[\cdot|X_t = x]$ thus yields

$$\begin{aligned}
\mathbb{E}\left[U(-ct - c\tau^\star + g(X_{t+\tau^\star})) \middle| X_t = x\right] &= \mathbb{E}\left[G(t, X_t) \middle| X_t = x\right] + \mathbb{E}\left[M_{t+\tau^\star} - M_t \middle| X_t = x\right] \\
&= G(t, x)
\end{aligned}$$

and hence by using the same shifting arguments as in (9.12) we get

$$\begin{aligned}
V(t, x, \tau^\star) &= \mathbb{E}_x\left[U(-ct - c\tau^\star + g(X_{\tau^\star}))\right] \\
&= \mathbb{E}\left[U(-ct - c\tau^\star + g(X_{t+\tau^\star})) \middle| X_t = x\right] \\
&= G(t, x).
\end{aligned}$$

Finally, we can conclude that

$$V(t, x) = \sup_{\tau \in \Sigma_x} V(t, x, \tau) \geq V(t, x, \tau^\star) = G(t, x)$$

and therefore together with part (a)

$$V(t, x) = G(t, x) = V(t, x, \tau^\star).$$

□

Remark 9.7 (comparison of Theorem 9.6 with Theorem 7.23)

We want now to compare the fixed-point equation stated in Theorem 7.31 of chapter 7 with the verification theorem in Corollary 9.8 of chapter 9.

A major difference between both approaches are definitely the assumptions made in order to establish both theories. The continuous-time approach in this chapter using the verification technique requires the differentiability of the value function in its time parameter. As a solution of the HJB equation Equation (9.7), the value function $V(t, x)$

has to possess its partial derivative in t. The fixed-point theorem in the discrete-time approach does not need this differentiability assumption. In particular, we didn't even have to assume continuity for our value functions. Without requiring it we were able to show in some cases, that the value functions we were looking for are indeed continuous. This makes the discrete-time approach more viable to a variety of applications, since we can even tackle stopping problems whose value fucntion is not differentiable at all.

Putting this difference aside, we can note that both approaches require similar additional conditions for the optimality of the stopping time they propose. This candidate needs to be $\mathbb{P}_x$–almost surely finite. This property needs to be manually verified in both approaches. Moreover, both require an additional growth condition to be valid in order to guarantee the optimality of the proposed stopping times. For the verification approach, this would be condition (9.8), whereas for the fixed-point theorem condition (7.88) needs to be shown.

Given the assumptions of both approaches, they are both viable to tackle the unrestricted stopping problem and the corresponding value function $\tilde{V}(x)$. Moreover, both provide us with a candidate for the optimal stopping time in explicit form. Given the nature of both approaches, this explicit form for the optimal stopping time $\tau^\star$ differs a little bit. The discrete-time approach allows for a piecewise description of stopping times using stopping rules between two jumps of the underlying Markov chain. An investor who follows the stopping policy given by $\tau^\star$ gains at every single jump time of the underlying Markov chain all information needed in order to know the optimal behavior up to the next jump. After a jump, he immediately gains the knowledge needed for the next period up to the subsequent jump. The continuous-time approach lacks this piecewise description. Here the optimal stopping time is simply given in terms of a first hit time. An investor following such a stopping time just waits for the first moment the hitting condition is satisfied. Therefore we can conclude as summary, that both representations of the optimal stopping time $\tau^\star$ allow for a simple application and execution of this stopping time.

We will see in the next section, that in the case of exponential utility, both approaches – as different as they may look – are in some sense equivalent. We will ascertain that both the fixed-point equation (7.72) as well as the HJB equation Equation (9.7) are describing the same situation and contain the same information about the value function we are looking for. Furthermore we will see that in case of exponential utility, the solution of the HJB equation does not need to fulfill any differentiability assumptions.

9.3 Continuous-Time Approach for the Classical Risk-Sensitive Stopping Problem under Exponential Utility

We will now consider value functions for the special choice of an exponential utility function. Again, we suppose that the underlying utility function U is given by

$$U\colon \mathbb{R} \to \mathbb{R}, \quad U(x) := -e^{-\gamma x}$$

for some $\gamma > 0$.

As discussed in section 6.2, the value function $V(t,x)$ can in this case be reduced by (6.14) and (6.13) to

$$V(t,x) = e^{c\gamma t}\tilde{V}(x),$$

where

$$\tilde{V}(x) := \sup_{\tau \in \Sigma_x} \mathbb{E}_x\Big[- e^{c\gamma\tau - \gamma g(X_\tau)}\Big].$$

We want now to adapt the HJB equation (9.7) for the special case of exponential utility and reformulate Theorem 9.6 for this situation. Since we are in search of a solution $G(t,x)$ of the HJB equation, which is – under validity of the assumptions of Theorem 9.6 – the value function $V(t,x)$ itself, we choose the obvious ansatz

$$G(t,x) = e^{c\gamma t}\tilde{G}(x) \tag{9.13}$$

for a suitable mapping $\tilde{G}\colon S \to \mathbb{R}$. Hence, for $G \in C^{1,0}([0,\infty) \times S)$ we can calculate its partial derivative by

$$G_t(t,x) = c\gamma e^{c\gamma t}\tilde{G}(x).$$

Therefore the HJB equation (9.7) can be expressed as

$$0 = e^{c\gamma t} \cdot \max\left\{ c\gamma\tilde{G}(x) + \sum_{j\in S}\Big(\tilde{G}(j) - \tilde{G}(x)\Big)q_{xj}\ ,\ -e^{-\gamma g(x)} - \tilde{G}(x)\right\}.$$

Furthermore, we know that we can express the intensity rates q_{xj} in terms of the transition probabilities p_{xj} of the embedded discrete-time Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ by

$$p_{xj} = \frac{q_{xj}}{q_x}, \text{ for } x,j \in S, x \neq j \quad \text{ and } \quad p_{xx} = 0.$$

We therefore get

$$\begin{aligned}\sum_{j\in S}\Big(\tilde{G}(j) - \tilde{G}(x)\Big)q_{xj} &= q_x \sum_{j\in S}\Big(\tilde{G}(j) - \tilde{G}(x)\Big)p_{xj} \\ &= q_x\Big(\sum_{j\in S}\tilde{G}(j)p_{xj} - \sum_{j\in S}\tilde{G}(x)p_{xj}\Big) \\ &= q_x\Big(\mathbb{E}_x\big[\tilde{G}(Z_1)\big] - \tilde{G}(x)\Big).\end{aligned}$$

This leads to the HJB equation

$$0 = \max\left\{(c\gamma - q_x)\tilde{G}(x) + q_x\mathbb{E}_x\big[\tilde{G}(Z_1)\big]\ ,\ -e^{-\gamma g(x)} - \tilde{G}(x)\right\}.$$

Note that by definition, we get

$$\tilde{V}(x) = \sup_{\tau\in\Sigma_x} \mathbb{E}_x\Big[-e^{c\gamma\tau-\gamma g(X_\tau)}\Big] < 0$$

for all $x \in S$. As a consequence we can search for solutions $\tilde{G}\colon S \to \mathbb{R}$ of the HJB equation (9.16), which also fulfill

$$\tilde{G}(x) < 0$$

for all $x \in S$.

Note that a solution $\tilde{G}$ of (9.16) has to satisfy both conditions

$$\begin{aligned} & -e^{-\gamma g(x)} - \tilde{G}(x) \le 0 \\ \Leftrightarrow\ & \tilde{G}(x) \ge -e^{-\gamma g(x)} \end{aligned} \tag{9.14}$$

and

$$\begin{aligned} & (c\gamma - q_x)\tilde{G}(x) + q_x\mathbb{E}_x\big[\tilde{G}(Z_1)\big] \le 0 \\ \Leftrightarrow\ & \mathbb{E}_x\big[\tilde{G}(Z_1)\big] \le \frac{q_x - c\gamma}{q_x}\tilde{G}(x). \end{aligned} \tag{9.15}$$

Moreover, for every single $x \in S$ one of these two conditions has to be fulfilled with "=".

We will now differentiate two cases:

- Let $x \in S$ such that $q_x \le c\gamma$. Since $q_x > 0$ and we assumed that $\tilde{G}(x) < 0$ for all $x \in S$, we can conclude that

$$\mathbb{E}_x\big[\tilde{G}(Z_1)\big] < 0 \le \frac{q_x - c\gamma}{q_x}\tilde{G}(x).$$

  In other words, for $x \in S$ such that $q_x \le c\gamma$ condition (9.15) is never satisfied with an "=". Therefore

$$\tilde{G}(x) = -e^{-\gamma g(x)}$$

  is valid.

- Let $x \in S$ such that $q_x > c\gamma$. In this case

$$\frac{q_x - c\gamma}{q_x} > 0$$

is true. We can therefore write (9.15) as

$$\tilde{G}(x) \geq \frac{q_x}{q_x - c\gamma} \mathbb{E}_x\big[\tilde{G}(Z_1)\big].$$

This yields togehter with (9.14) a new form for the HJB equation:

$$\tilde{G}(x) = \max\left\{-e^{-\gamma g(x)}\ ,\ \frac{q_x}{q_x - c\gamma} \mathbb{E}_x\big[\tilde{G}(Z_1)\big]\right\}.$$

Now we are able to postulate the verification theorem for exponential utility:

Corollary 9.8 (verification theorem for exponential utility)

Let $\tilde{G}\colon S \to (-\infty, 0)$ be a solution of the Hamilton-Jacobi-Bellman equation for exponential utility

$$\tilde{G}(x) = \max\left\{-e^{-\gamma g(x)}\ ,\ \frac{q_x}{q_x - c\gamma} \mathbb{E}_x\big[\tilde{G}(Z_1)\big]\right\} \tag{9.16}$$

and additionally fulfill the following growth-condition for all $x \in S$ and $b > 0$:

$$\mathbb{E}_x\left[\sum_{\substack{n\in\mathbb{N}_0,\\ 0\leq S_n\leq b}} e^{c\gamma S_n} \cdot \left|\tilde{G}(Z_n)\right|\right] < \infty. \tag{9.17}$$

Then it holds:

(a) For all $x \in S$ the value function $\tilde{V}$ fulfills

$$\tilde{V}(x) \leq \tilde{G}(x).$$

(b) If in addition the first hit time

$$\tau^\star := \inf\left\{s \geq 0 \mid \tilde{G}(X_{t+s}) = -e^{-\gamma g(X_t+s)}\right\} \tag{9.18}$$

is $\mathbb{P}_x$–almost surely finite, then $\tau^\star$ is the optimal stopping time for $\tilde{V}(x)$ and

$$\tilde{V}(x) = \tilde{G}(x).$$

Remark 9.9 (comparison of Corollary 9.8 with Theorem 7.31)

By comparing the fixed-point equation for exponential utility stated in Theorem 7.31 with the verification theorem for exponential utility in Corollary 9.8, we can clearly see that these are in some sense equivalent.

The HJB equation (9.16) is equivalent to the fixed-point equation (7.97). The task of finding a solution of the HJB equation is thus the same as to find a fixed-point for the above-mentioned equations. Note that in case of exponential utility we do not need to impose any additional differentiability assumption on the solution $\tilde{G}$. In both cases we require to check whether the found candidate for the optimal stopping time is $\mathbb{P}_x$–almost surely finite. Moreover, we need to verify in both cases certain growth conditions for the value functions. Using the verification theorem of the continuous-time approach this would be (9.17), whereas by using the fixed-point theorem in the discrete-time approach the growth condition to check would be (7.96).

Here we see that both approaches are viable to tackle the unrestricted stopping problem and the corresponding value function $\tilde{V}(x)$ and lead to the same result. Given the theory of both approaches the explicit form of the optimal stopping time differs a little bit. The discrete-time approach allows for a piecewise description of stopping times using stopping rules between two jumps of the underlying Markov chain. The continuous-time approach lacks this possibility. Here the optimal stopping time is compactly given in terms of a first hit time. Note that both representations of the optimal stopping time $\tau^\star$ allow for a simple application of this stopping time.

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