# Equilibrium Dynamics in the Overlapping Generations Growth Model with Multi-Period-Lived Consumers

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# Abstract(English Version)

The topic of this dissertation are multi-period overlapping generations growth model. In the literature, these models are usually considered for two-period lived generations. If there is the consideration of equilibria in an economy with more lifetime periods, either prices are not endogenous or they are solved only numerically.

The present thesis presents the analytical approach for multi-period models with production. Here, first the model is introduced with a general structure and there is the description of a general approach to determine an equilibrium. This approach is applied in the following to the common economy with logarithmic utility and Cobb-Douglas production function. Here, we first consider particular labor distributions among the generations such that the equilibrium can be determined uniquely. For a lifetime of three periods the general approach combined with an assumption of a solution's structure allows to prove the existence of a unique equilibrium of that structure.
Even for particular distributions of labor supply there are phenomena that are not known in the two-period model. Hence, during the present work both the equilibrium dynamics and the results of the following analysis with respect to ling-run behavior, dynamic efficiency and optimality are compared to the well-known results of the two-period mode. For the main part the analysis takes place for the particular distributions of labor supply. Therefore, the applicability of efficiency and optimality criteria to the present model is shown in general.

Altogether this work paves the way for a theoretical existence result for many numerical studies and it is basic for future research in this area.

# Abstract (German Version)


Gegenstand dieser Dissertation sind mehrperiodige Wachstumsmodelle mit überlappenden Generationen. Diese Modellklasse wird in der Literatur vorwiegend für Generationen mit einer Lebenszeit von zwei Perioden betrachtet. Bei längeren Lebensdauern wird entweder auf endogen bestimmte Preise verzichtet oder eine rein numerische Vorgehensweise gewählt um eine Gleichgewichtslösung zu bestimmen.
In der vorliegenden Arbeit erfolgt die analytische Betrachtung der mehrperiodigen Modelle mit Produktion. Dazu wird zunächst das Modell in einer allgemeinen Struktur eingeführt und die allgemeine Herangehensweise zur Bestimmung einer Gleichgewichtslösung erläutert. Die Anwendung dieses Vorgehens findet im Folgenden an der gängigen Ökonomie mit logarithmischer Nutzenfunktion und Cobb-Douglas-Produktionsfunktion statt. Hier werden zunächst spezielle Verteilungen des Arbeitsangebots über verschiedene Generationen betrachtet, in denen eine Gleichgewichtslösung eindeutig bestimmt werden kann. Für die Lebenszeit von drei Perioden wird die allgemeine Herangehensweise mit der Annahme einer speziellen Lösungsstruktur verknüpft und so die Existenz eines eindeutigen Gleichgewichts dieser Struktur bewiesen.
Bereits für spezielle Verteilungen des Arbeitsangebotes lassen sich Phänomene erkennen, die aus dem zweiperiodigen Modell nicht bekannt sind. Daher werden im Rahmen der Arbeit sowohl die Gleichgewichtslösung als auch die Ergebnisse der anschließenden Analyse der Gleichgewichtsdynamik bezüglich Langzeitverhalten, dynamischer Effizienz und Optimalität mit den bekannten Resultaten des zweiperiodigen Modells verglichen. Die Analyse erfolgt in erster Linie für spezielle Verteilungen des Arbeitsangebots im Fall von drei Perioden lebenden Agenten. Dafür wird zunächst allgemein die Anwendbarkeit von Effizienz- und Optimalitätskriterien auf das vorliegende Modell gezeigt.
Diese Arbeit ist damit ein Türöffner zur Erschaffung einer theoretischen Existenzgrundlage für viele numerische Untersuchungen und stellt eine Basis für zukünftige Forschung in diesem Bereich dar.

# Contents

# Chapter 1

# Introduction

30 years are a quite large time span. If you think about your life 30 years ago many things will have changed significantly over time. Looking into history it is approximately the time span during which in the beginning of the 20th century two world wars took place or later the Berlin Wall divided Germany. During the last 30 years there were multiple crises as the Dot-com bubble or the financial crises in 2007. These are just some examples what happened during a longer time period in the last century.
Why do I think about a span of 30 years? This is approximately the length of one period in the general two-period overlapping generations model. In this model a generation is born at the beginning of one period and dies at the end of the following period. In the general model the action of a generation reduces to one decision in the first period. Assuming that agents enter the economy at about 20 years of age and die at the age of about 80, one period corresponds to a 30-year period. So there is the critique that making only one decision during such a long time period is not realistic (cf. Molnár and Simonovits [17]).
Reducing the length of one period requires an increase of the number of lifetime periods if a reduction of total lifetime is not desired. The theoretical analysis of the overlapping generations model with more than two periods, which is insufficiently considered in the literature yet, is the topic of the present work. We consider an overlapping generations economy with production where agents live for an arbitrary but fixed number of periods. The objective is to define and determine an inter-temporal competitive equilibrium. The definition takes place in a very general context while the analysis and determination is limited to a particular economy.

In the literature, overlapping generations models have been very popular ever since their introduction by Samuelson [21] more than 50 years ago. This paper focuses on the efficiency paradox, that competitive equilibria may be pareto inefficient. More recently overlapping generations models have often been used to explain bubbles and to analyze their impacts.

Bubbles in overlapping generations economies have been introduced by Tirole [25].
Several economists concentrated on the efficiency paradox (see Shell [22] or Cass and Yaari [9] for example). The consumption-loan model introduced by Samuelson has been extended in many ways. The question about existence of equilibria and their properties has been considered by Balasko and Shell [3] or Gale [14] just to mention two examples. Balasko et al. [5] analyze the overlapping generations economy as a pure exchange economy with fewer requirements on the utility functions and heterogeneous commodities. This paper provides an algorithm how an economy with long-lived agents can be transformed into a two-period economy with multiple heterogeneous commodities. As in the economy with production prices depend on each period's capital stock this transformation is not possible in the economy considered in this thesis. Thus, for the considered economy another approach is necessary. The idea of increasing the number of periods in general is not new as well. One of the first multi-period overlapping generations model was presented by Auerbach and Kotlikoff [2]. They provide a model where agents live for 55 periods. They do not only extend the number of periods, they also introduce a government sector and endogenous labor supply. While considering the decision problem of one household they state that there is no closed form solution for the decision variables, therefore they solve the model numerically. Prices are determined endogenously such that expectations and realized values coincide. In his work Colucci [10] provides a detailed overview over articles on overlapping generations. He considers a multi-period pure exchange overlapping generations model with government where he restricts the number of steady states. Molnár and Simonovits [17] consider a multi-period overlapping generations economy that they call 'overlapping cohort' economy because they argue that generation in its usual sense would only allow to consider up to four generations to live at the same time. Their focus is on different possibilities to generate expectations about the future. Again there is no production sector in their work. In Bullard [7] a n-period version of Samuelson's model is analyzed. In this model Bullard states that the aggregate savings function complicated. Although this paper considers a particular logarithmic utility function, he is able to show that in his economy there are at most two stationary equilibria, "the autarkic one and the monetary one" (Bullard [7]), that coincide with those in the two period model with money. Furthermore he gives some possible extensions that are worth to be discussed. One of them is introducing production.
Production in the two-period model was first introduced in Diamond [13]. His work focuses on the two-period economy and shows that introducing debt could solve the inefficiency problem. Since then there have been many publications on overlapping generation models that refine or extend the structure of the model. Galor and Ryder [15] provide the two-period overlapping generations model in a very clear and well-structured notation. They focus on the question of stability of steady states and in a later work the dynamic efficiency

is regarded as well (see Galor and Ryder [16]). Ríos-Rull [20] considers a very general model with labor-leisure choice, a neoclassical production function with productivity shock and uncertain lifetime limited to a maximal age. He gives a definition of a temporary equilibrium and analyzes numerically how an economy's different market structures affect the equilibrium allocation. An approach to determine an inter-temporal equilibrium is not given there.
As it is for example stated in the work of Auerbach and Kotlikoff [2] increasing the number of periods leads to a complicated framework. Therefore, today mostly these models are considered numerically. Simulating the model shifts the main focus from the existence and properties of equilibria to impacts of different features of the model. The structure of the overlapping generations model is used to evaluate social security systems themselves (see Wrede [26] for example) or impacts on those as it is done in de la Croix et al. [12]. In actual simulations often an uncertain lifetime is assumed. In general there is a benchmark solution that is the steady state solution, but to the best of my knowledge there is no existence result for an equilibrium.
This thesis presents a first step towards filling this gap. It provides a theoretical framework for the multi-period overlapping generation models with productive capital and defines the corresponding equilibrium conditions. Moreover, a particular economy is presented, that may serve as a reference model for future research.

The dissertation is organized as follows. Chapter 2 presents the general model's structure as well as the decision problems and the equilibrium concept. Chapters 3 and 4 consider the benchmark economy with particular utility and production functions. Chapter 3 derives properties of a dynamic equilibrium under the general assumption of a particular equilibrium structure. Moreover, it shows the existence and uniqueness of such an equilibrium in the three-period economy. Chapter 4 then analyzes the derived dynamic of the three-period economy with respect to long-run behavior and optimality. Furthermore, the results are compared to the well-known two-period dynamics. Finally, Chapter 5 gives a short summary and outlook.

# Chapter 2

# The General Model

The objective of this chapter is to give a detailed description of the model's general structure.
The outline of this chapter is as follows: Section 1 introduces the production sector and describes the firm's decision problem. Section 2 explains the structure of the population. Section 3 introduces the decision problem of an arbitrary consumer. Section 4 gives the definition of an equilibrium in the underlying economy. Finally, Section 5 provides an approach to determine an equilibrium.

## 2.1 Decision Problem of the Firm

The production side of the economy is given by a representative firm that uses capital and labor to produce the single consumption good, which is the numeraire. The input factors are denoted by $K$ for capital and $L$ for labor while $Y$ denotes output. The underlying production function is

$$F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, \quad (K, L) \mapsto F(K, L) = Y.$$

$F$ is assumed to be $C^2$ and homogeneous of degree one.
In each period $t \geq 0$ the firm pays a wage per unit of labor and return per unit of capital. Let $w_t$ denote the wage and $r_t$ the return. Profit of the firm in period $t \geq 0$ is given by

$$\Pi_t(K, L) = F(K, L) - w_t L - r_t K.$$

In each period $t \geq 0$ the firm decides about labor and capital demand, taken wage and return as given. So the decision problem of the firm reads

$$\max_{(K_t, L_t) > 0} \Pi_t(K_t, L_t).$$

Let $k_t = \frac{K_t}{L_t}$ be the capital-labor-ratio and define the gross production function in intensive form

$$f(k_t) := F(k_t, 1).$$

$f$ is assumed to satisfy the following properties:

$$f(k) \geq 0, \quad f'(k) > 0, \quad f''(k) < 0 \tag{2.1}$$

The first order conditions of the firm's decision problem lead to the following optimality conditions on $k_t$:

$$w_t = f(k_t) - k_t f'(k_t) \tag{2.2a}$$

$$r_t = f'(k_t) \tag{2.2b}$$

Thus, each sequence $\{(K_t, L_t)\}_{t\geq 0}$ which capital-labor-ratio satisfies equations (2.2) for all $t \geq 0$ is an optimal plan of capital and labor demand of the firm. Note that $w_t, r_t \geq 0$ by the properties of $f$ ($r_t > 0$ is obvious, $w_t \geq 0$ follows by the concavity).

## 2.2 Population Structure

The considered economy is a classical overlapping generations economy that is assumed to be perfectly competitive and all decisions and activities are done in infinite discrete time. In each period $t \geq 0$ a new generation is born which lives for $I+1$ consecutive periods ($I \in \mathbb{N}$), i.e., an agent born at the beginning of period $t$ dies at the end of period $t+I$. Lifetime $I+1$ is given exogenously. The number of agents born in each period $t \geq 0$ is assumed to be constant over time and denoted by $N$. Without loss of generality it is normalized to $N = 1$. Each generation is indexed by $i \in \{0, \ldots, I\} =: \mathbb{I}$, where $i$ denotes the remaining lifetime. That is in period $t$ the agent born in period $t$ is indexed by '$I$', the agent born in period $t-1$ with '$I-1$' and so forth. Thus, in each period there are $I+1$ agents alive. Note that for $I = 1$ the model describes the well-known model where agents live for two periods. Thus, the two-period model constitutes a benchmark for the multi-period model.
Figure 2.1 visualizes the population structure. The markings on the timeline illustrate the beginning of a new period and thus the end of the previous period.

## 2.3 Decision Problem of a Single Household

This section studies the behavior of a single consumer. Without loss of generality set $t = 0$ here. The consumer is a member of an arbitrary generation $i \in \mathbb{I}$. For purposes of simpler

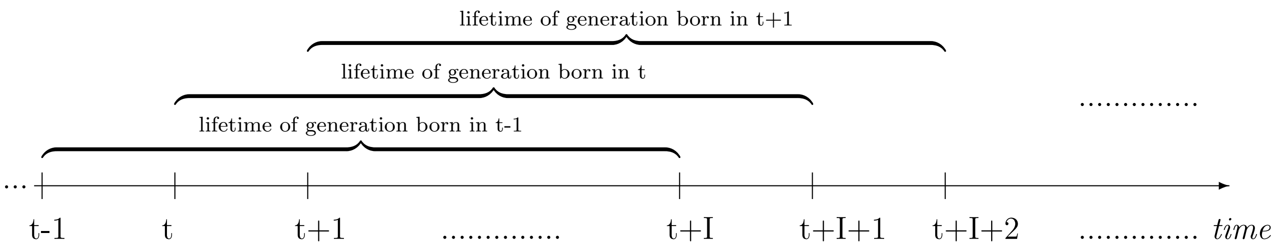



Figure 2.1: Generational structure of the model

indexation drop the generational index in this section.
In each period the consumer is endowed with wealth. Let $\omega_n$ denote the wealth in period $n = 0, ..., i$ which is defined by

$$\omega_n = \ell_n \hat{w}_n + \hat{r}_n k_n. \tag{2.3}$$

Wealth is a composition of labor and capital income. In each period $n \in \{0, ..., i\}$ the consumer supplies a specific amount of labor to the labor market, that is given exogenously and denoted by $\ell_n$. Supplied labor in period $n$ is refunded with wage $\hat{w}_n$. Analogously, capital $k_n \in \mathbb{R}$ is supplied to the capital market and refunded with return $\hat{r}_n$. The prices $\hat{w}_n$ and $\hat{r}_n$ are given or expected by the consumer.

The consumer has to decide how to divide up wealth between consumption and investment. Invested capital coincides with capital supply in the following period. Let $(c_n)_{n=0}^i$ denote the consumption plan and $(k_n)_{n=1}^i$ the investment plan of the consumer. Initial capital $k_0 \in \mathbb{R}$ is given and together with prices it determines the consumer's wealth $\omega_0$ in the initial period. Given prices, $\hat{w}^i = (\hat{w}_n)_{n=0}^i$ and $\hat{r}^i = (\hat{r}_n)_{n=0}^i$, and initial capital $k_0$ the agent decides about its consumption plan and an investment plan for the remaining lifetime periods. Consumption is required to be positive.
Note that in contrast to the two-period economy it is not sufficient to have expectations about the prices of the following period. Expectations are required up period $i$, the last period the agent is alive. So the consumer faces to the following budget constraints:

$$c_n + k_{n+1} \leq \omega_n \quad n = 0, ..., i, \; k_{i+1} := 0 \tag{2.4a}$$

$$c_n \geq 0 \qquad n = 0, ..., i \tag{2.4b}$$

Thus, the budget set is given in dependence of the prices and initial capital

$$\mathbb{B}(\hat{w}^i, \hat{r}^i, k_0) := \Big\{ \big((c_j)_{j=0}^i, (k_j)_{j=1}^i\big) \,|(2.4) \text{ are satisfied for all } j \Big\}.$$

In the two-period economy capital supply is required to be positive as well. In general investment may be negative as well, but negative investment reduces future wealth. By construction in the last period a consumer is alive, she consumes all her wealth and has no opportunity to make an investment. So wealth needs to be positive in that period and infinitely negative investment must be excluded by introducing a lower bound. This is presented in the following lemma.

**Lemma 2.3.1.** *Given price vectors $\hat{w}$ and $\hat{r}$ and initial capital $k_0$, an investment plan $(k_n)_{n=1}^{i}$ satisfying (2.4) also satisfies*

$$k_n + \sum_{j=n}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=n}^{j} \hat{r}_m} > 0 \qquad \forall n = 1, .., i.$$

In particular the constraint must hold for initial capital $k_0$. As it can be seen in the proof of Lemma 2.3.1, consumption in period $i$ would be negative if there was capital supply below the lower bound in one period. Obviously, this holds for $n = 0$ as well.

**Lemma 2.3.2.** *There is a no-bankruptcy condition for initial capital $k_0$:*

$$k_0 + \sum_{j=0}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=0}^{j} \hat{r}_m} > 0 \tag{2.5}$$

After the definition of boundaries for capital supply, consider the decision problem of the consumer. The objective of the consumer is to maximize lifetime utility. Consumer $i \in \mathbb{I}$ faces the following 'remaining lifetime' utility function

$$U\left((c_n)_{n=0}^{i}\right) = \sum_{n=0}^{i} \beta^n u(c_n)$$

with discount factor $\beta \in (0, 1)$ and period-wise utility function

$$u : \mathbb{R}_+ \to \mathbb{R}_+, \quad c \mapsto u(c).$$

The following assumption states the properties of $u$.

**Assumption 2.3.1.** *The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is $C^1$, strictly increasing and strictly concave. Moreover, it satisfies the Inada conditions*

$$\lim_{c \searrow 0} u'(c) = \infty$$
$$\lim_{c \to \infty} u'(c) = 0.$$

In the following assume that Assumption 2.3.1 holds. Now, the decision problem of a single household can be defined:

$$\max_{\left((c_n)_{n=0}^{i},(k_{n+1})_{n=0}^{i-1}\right)} \left\{U\left((c_n)_{n=0}^{i}\right) \mid \left((c_n)_{n=0}^{i}, (k_{n+1})_{n=0}^{i-1}\right) \in \mathbb{B}(\hat{w}^i, \hat{r}^i)\right\} \tag{2.6}$$

Taking into account that utility is strictly increasing in consumption constraint (2.4a) is binding and the upper and lower bound of investment are non-maximizing by Assumption 2.3.1. So the first order conditions describe the consumer's optimal decision:

$$\beta \hat{r}_{n+1} = \frac{u'(\ell_n \hat{w}_n + \hat{r}_n k_n - k_{n+1})}{u'(\ell_{n+1}\hat{w}_{n+1} + \hat{r}_{n+1}k_{n+1} - k_{n+2})} \left(= \frac{u'(c_n)}{u'(c_{n+1})}\right) \tag{2.7}$$

for all $n \in \{0, \ldots, i-1\}$ with $k_{i+1} := 0$ for purposes of a compact notation.
Even in the three-period economy the equations are much more complex than in the two-period economy, where the first order conditions reduce to one simple equation when there is no investment in the following period. Obviously, this simple equation is part of the equilibrium equations for each number of periods. But already in the three-period economy there is a second equation that depends on capital supply of three different periods. Thus, knowing current capital supply it is not possible to decide about next period's capital supply without an assumption about further decisions. However, the following lemma states a uniqueness result.

**Lemma 2.3.3.** *Suppose initial capital $k_0$ and prices $\hat{w}, \hat{r}$ satisfy the no-bankruptcy constraint*

$$k_0 > -\sum_{j=0}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=0}^{j} \hat{r}_m}.$$

*Then, there exists a unique utility-maximizing consumption-investment plan $(c_0^\star, .., c_i^\star, k_1^\star, .., k_i^\star)$ which is characterized by the equations*

$$\begin{aligned} c_n^\star + k_{n+1}^\star &= \ell^{i-n}\hat{w}_n + k_n^\star \hat{r}_n, \quad n = 0, \ldots, i, \quad k_{i+1}^\star := 0 \\ \hat{r}_n \beta u'(c_n^\star) &= u'(c_{n-1}^\star), \quad n = 1, \ldots, i. \end{aligned}$$

In spite of the additional complexity Lemma 2.3.3 guarantees that given prices the consumer's decision is uniquely defined by equations (2.7). Thus, the first order conditions are an optimality condition for each consumption-investment plan.

## 2.4 Equilibrium

The previous sections have shown that given initial capital supply for all generations, the economy, now denoted by $\mathcal{E}$, is described by the lifetime of the agents, the labor profile, the utility function, the agent's discount factor and the production function of the firm. That is $\mathcal{E} = \langle \mathbb{I}, \ell, u, \beta, f \rangle$ plus initial values.

So far, the single decisions have been considered. As the equilibrium links decisions over the generations and over time, in the following the generational index will be necessary as well as the general time index. The economy contains three markets: the labor market, the capital market and the consumption good market. It is sufficient to focus on the labor and the capital market as by Walras' law the third is cleared if these two are cleared.

Labor supply in period $t \geq 0$, is defined by the so-called labor profile. As mentioned in Section 2.3 the specific amount of labor a consumer supplies is given exogenously and constant over time, i.e., labor supply of generation $i$, $i \in \mathbb{I}$, is the same for all periods $t \geq 0$. The labor profile describes the fraction of total labor supply a generation supplies and is defined as

$$\ell : \mathbb{I} \to \mathbb{R}_+, i \mapsto \ell^i.$$

Total labor supply can be normalized to one without loss of generality, i.e., $\sum_{i \in \mathbb{I}} \ell^i = 1$.

Total capital supply in period $t \geq 0$ is defined as the sum of individual capital supply, $k_t^i$, of the different generations $i \in \mathbb{I}$.

Thus, total labor and capital are given by

$$1 = L_t \qquad \sum_{i \in \mathbb{I}} k_t^i = K_t$$

and by equations (2.2), the market clearing factor prices are

$$w_t = \mathcal{W}(k_t^A) = f(k_t^A) - k_t f'(k_t^A) \tag{2.8}$$

$$r_t = \mathcal{R}(k_t^A) = f'(k_t^A) \tag{2.9}$$

with $k_t^A = \frac{K_t}{L_t} = \sum_{i \in \mathbb{I}} k_t^i$ denoting aggregated capital.

Note that the definition of the decision problem of the firm requires $k^A$ to be positive. So far, a temporary equilibrium is described. The economy's set-up requires a dynamic structure of equilibrium. The assumption of rational consumers with perfect foresight fills this gap. So together with the optimality conditions (2.7) a general definition of equilibrium can be given:

**Definition 2.4.1.** *Let initial values for the number of agents born in period $t = 0$ and capital supply $\mathbf{k_0} = (k_0^i)_{i=0}^{I-1}$ satisfying $k_0^A > 0$ and the no-bankruptcy constraint for all $i$ be given. An* equilibrium *for $\mathcal{E}$ is a price sequence $\{w_t, r_t\}_{t \geq 0}$ together with an allocation $\{(\mathbf{c}_t, \mathbf{k}_t)\}_{t \geq 0}$ with $\mathbf{k}_t = (k_t^i)_{i=0}^{I-1}$ and $\mathbf{c}_t = (c_t^i)_{i \in \mathbb{I}} \gg 0$ satisfying the equilibrium equations, such that*

$$w_t = \mathcal{W}(k_t^A) \tag{2.10a}$$
$$r_t = \mathcal{R}(k_t^A) \tag{2.10b}$$
$$\beta u'(c_{t+1}^{i-1})\mathcal{R}(k_{t+1}^A) = u'(c_t^i), \quad i \in \mathbb{I}\backslash\{o\} \tag{2.10c}$$
$$c_t^i + k_{t+1}^{i-1} = \omega^i(k_t), \quad i \in \mathbb{I}. \tag{2.10d}$$

*For purposes of a compact notation, let* $k_t^i \equiv 0$ *for* $i = -1$.

Before continuing with the description how equilibria will be constructed, there is one labor profile that has to be excluded. There is one generation that does not participate in capital building: the old. If all labor would be supplied by this generation, the generations participating in this process, need to invest a negative amount of capital to generate positive consumption. Thus, in sum capital supply would be negative.

**Lemma 2.4.1.** *There is no equilibrium for the labor profile* $\ell^0 = 1,\ \ell^i = 0, i = 1, ..., I$.

**Assumption 2.4.1.** *The labor profile* $\ell : \mathbb{I} \to \mathbb{R}_+, i \mapsto \ell^i$, *satisfies* $\sum_{i=1}^{I} \ell^i > 0$.

As the objective is to define and analyze equilibria in the following it will be assumed that Assumption 2.4.1 holds.

## 2.5 Recursive Structure

The assumption of rational agents with perfect foresight together with the equilibrium conditions makes the construction of an equilibrium extensive. So in general computing an equilibrium amounts to solving an infinite number of coupled equations. This complexity is captured by the following functional approach. Focus on a special type of equilibria, Markov equilibria, where the following state only depends on the current state. In the two-period economy this holds, as the dynamical system defining capital supply evolution is one dimensional and of degree one: As it can be seen in Appendix A next period's capital stock is a function of the current capital stock, which defines all other variables in the model. A recursive structure is also very helpful if the model is simulated.

So following the two-period economy, capital supply is the state variable in the general multi-period economy as well. The state space is denoted by $\mathbb{K}$ and it is a subset of $\mathbb{R}^I$. The elements $\mathbf{k} \in \mathbb{K}$, with $\mathbf{k} = (k^i)_{i=0}^{I-1}$, describe the distribution of capital supply over the generations. The generation, that has just been born, is not included as it does not supply any capital by construction. Specifying the state space exactly will be one of the challenges during this thesis. The main problem here is to state the lower bound for generational capital supply. In Lemma 2.3.1 a lower bound is defined, but it depends on future prices. As the prices are not given, but are determined by future capital, this result cannot easily be transferred to the definition of the state space.

Wealth of generation $i \in \mathbb{I}$ is determined by a function of capital supply $\omega^i : \mathbb{K} \to \mathbb{R}$

$$\mathbf{k} \mapsto \omega^i(\mathbf{k}) := \ell^i \mathcal{W}(k^A) + k^i \mathcal{R}(k^A). \tag{2.11}$$

Now, the concept of Markov equilibrium can be defined formally:

**Definition 2.5.1.** *A* Markov equilibrium *for $\mathcal{E}$ is a mapping $\mathcal{K} : \mathbb{K} \longrightarrow \mathbb{K}$ such that $\left\{ \left\{ (c_t^i)_{i\in\mathbb{I}}, (k_t^i)_{i\in\mathbb{I}\setminus\{I\}} \right\}_{t\geq 0} \right\}$ defined as $\mathbf{k}_{t+1} = \mathcal{K}(\mathbf{k}_t)$, $t \geq 0$ and $c_t^i = \omega^i(\mathbf{k}_t) - k_{t+1}^{i-1}$ for $i \in \mathbb{I}$ together with the price sequence $\{w_t, r_t\}_{t\geq 0}$ defined as in (2.10) is an equilibrium.*

Note that any Markov Equilibrium $\mathcal{K} = (\mathcal{K}^i)_{i\in\mathbb{I}\setminus\{I\}}$ for $\mathcal{E}$ defines an induced function $\mathcal{C} = (\mathcal{C}^i)_{i\in\mathbb{I}} : \mathbb{K} \longrightarrow \mathbb{R}_{++}^{\mathbb{I}}$

$$\mathcal{C}^i(\mathbf{k}) := \omega^i(\mathbf{k}) - \mathcal{K}^{i-1}(\mathbf{k}), \tag{2.12}$$

such that given initial capital supply, $\mathbf{k_0} \in \mathbb{K}$, $\left\{ \left(\mathcal{K}(\mathbf{k}_0)\right)^t, \mathcal{C}\left(\left(\mathcal{K}(\mathbf{k}_0)\right)^t\right) \right\}_{t\geq 0}$ is an equilibrium allocation. Obviously, the wealth function is a key factor in the equilibrium equations. Hence, it is the topic of the following subsection.

## 2.5.1 Properties of the Wealth Function

The analysis of the wealth function's (2.11) properties mainly focuses the behavior of individual wealth with respect to capital supply of the different generations. First look at the monotonicity properties of the wealth function.
Before considering the derivatives of the wealth function of agent $i$, $\omega_t^i$, with respect to capital supply of different generations, that is $k_t^i$ and $k_t^j$ with $j \neq i$, analyze the properties of aggregated capital, $k^A = \sum_{i=0}^{I-1} k^i$, and prices (2.10) with respect to individual capital supply:

$$\frac{\partial k^A}{\partial k^i} = 1$$
$$\frac{\partial \mathcal{W}(k^A)}{\partial k^i} = -k^A f''(k^A)$$
$$\frac{\partial \mathcal{R}(k^A)}{\partial k^i} = f''(k^A)$$

Now, the monotonicity properties are analyzed:

$$\frac{\partial \omega^i(\mathbf{k})}{\partial k^i} = \ell^i(-(k^A f''(k^A)) + f'(k^A) + k^i f''(k^A) \tag{2.13}$$
$$\frac{\partial \omega^i(\mathbf{k})}{\partial k^j} = \ell^i(-k^A f''(k^A)) + k^i f''(k^A) \tag{2.14}$$

As the production function is assumed to be strictly increasing (cf. (2.1)), it is obvious that for all $k \in \mathbb{K}$ it is

$$\frac{\partial \omega^i(\mathbf{k})}{\partial k^i} > \frac{\partial \omega^i(\mathbf{k})}{\partial k^j} \;\forall\; i \neq j.$$

Thus, $\omega^i(\mathbf{k})$ is rather increasing in $k^i$ than in $k^j$, $j \neq i$. The reason is that an increase of $k^i$ has an increasing as well as a decreasing effect on capital income of generation $i$ while an increase of $k^j$ has only a decreasing effect on capital income of generation $i$ as return is decreasing in each $k^n, n \in \mathbb{I} \backslash \{I\}$. In addition, labor income only depends on aggregated capital. The effects of an increase of $k^i$ and $k^j$, $j \neq i$, on aggregated capital are the same.
Equation (2.14) shows that for $k^i < \ell^i k^A$ wealth $\omega^i(\mathbf{k})$ is strictly increasing in $k^j$ and otherwise it is decreasing as the production function is also assumed to be strictly concave (cf. (2.1)). The following example illustrates the fact that there is no general monotonicity result: Assume an economy where all labor is supplied by generation $I$, the young. Thus, $\omega^I$ only consists of labor income while $\omega^i$, $i < I$, only consist of capital income. As wage is increasing and return is decreasing in $k^i$, for all $i \in \mathbb{I} \backslash \{I\}$, generation $I$ benefits from any increase of capital supply independently of the generation. Otherwise, an increase of capital supply of any generation different from $i$ reduces capital income, and thus generational wealth, of generation $i$. Summarized, it can be said that the monotonicity properties of generational wealth with respect to other generations capital supply highly depend on the labor profile and the distribution of capital supply.
Considering the behavior of wealth with respect to the own capital supply (2.13) a stronger statement can be achieved:

$$\begin{aligned} f'(k^A) + f''(k^A)(k^i - \ell^i k^A) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \\ \Leftrightarrow k^i \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \left( \ell^i - \frac{f'(k^A)}{f''(k^A)k^A} \right) k^A \end{aligned} \tag{2.15}$$

Here again equation (2.15) shows that the labor profile plays a key role in the properties of the wealth function. But with an additional assumption on the production function monotonicity can be guaranteed.

**Assumption 2.5.1.** *Let $f$ satisfy*

$$\frac{f''(k^A)k^A}{f'(k^A)} \geq -1.$$

**Lemma 2.5.1.** *If Assumption 2.5.1 holds, $\omega^i(\mathbf{k})$ is strictly increasing in $k^i$, $i = 0, ..., I-1$.*

Besides the monotonicity behavior, now consider which values wealth takes. Obviously, there is no upper bound as individual capital supply can be infinitesimal high. As it has been shown in Section 2.3 there is a no-bankruptcy constraint for consumers capital supply, depending on future prices. This constraint implies a lower bound for wealth.

**Lemma 2.5.2.** *Given price vectors $w^I$ and $r^I$ for the next $I$ periods, in each period $t \geq 0$ and for each generation $i \in \mathbb{I}$ there is a critical value for wealth $(\omega_t^i)^{crit}$, defining a lower bound with*

$$(\omega_t^i)^{crit} := -\sum_{j=1}^{i} \frac{\ell^{i-j} w_{t+j}}{\prod_{m=1}^{j} r_{t+m}} \tag{2.16}$$

*Proof.* The result follows by setting the no-bankruptcy constraint of Lemma 2.3.1 in the definition of wealth. □

For $i = 0$ the critical value turns out to be constant and equal to 0. Moreover, it is obvious that $(\omega_t^i)^{crit}$ does not only depend on future capital supply but also on the labor profile $\ell$. So there exist special labor profiles, such that the dependance of future values vanishes and the critical value is zero as well:

**Lemma 2.5.3.** *Let $\ell^j = 0$ for $j \leq i,\ i \in \mathbb{I}$. Then, $\underline{\omega^j} := (\omega_t^j)^{crit} = 0$ for $j = 0, ..., i+1$.*

*Proof.* The result follows directly by computing the critical values with the labor profile $\ell^j = 0$ for $j \leq i,\ i \in \mathbb{I}$. □

The meaning of the labor profile described in Lemma 2.5.3 is that from the age of $I - j$ on there is no labor supply. Thus, this labor profile represents retirement.

### 2.5.2 The State Space

As it has already been indicated, the definition of the state space is not straightforward. The state space needs to represent all possible values of the capital evolution that satisfy some kind of feasibility. Following the two-period economy the intuition is to define $\mathbb{K} = \mathbb{R}^I_{++}$. If each generation supplied a positive amount of capital, there is no conflict with bankruptcy (2.5). But as Lemma 2.3.1 has shown that the lower bound of generational capital supply may be negative depending on labor profile and prices. Thus, the solution to the decision problem of the consumer might be negative for some generations. In this case it would be necessary to include zero investment and to deal with boundary solutions as there is a conflict with the requirement of representing optimal capital evolution. Moreover, the opportunity for inter-generational exchange of capital is a property that can lead to effects that are not usual in an overlapping generations economy yet. This central benefit of the multi-period economy should not be excluded by the definition of the state space. Introducing the option of negative capital supply requires further restrictions. By construction aggregated capital needs to be positive. But if only aggregated capital is required to be positive, the definition is too rough. By choosing one generation's capital supply infinitely large, the other's could be very small and violate the no-bankruptcy constraint (2.5). Hence, there is a conflict with feasibility. So obviously, it is optimal to define the state space as a subset of $\mathbb{R}^I$ such that aggregated capital is positive and the no-bankruptcy constraint (2.5) holds. In contrast to Section 2.3 prices are not given or expected exogenously and by the assumption of perfect foresight the lower bound depends on future prices and thus on future capital supply. Using Lemma 2.3.1 it is obvious that the definition of the lower bound of capital supply can be transferred into a lower bound of wealth. Hence, using Lemma 2.5.3 it is possible to define a superset of the state space:

$$\mathbb{K} \subset \left\{ k : \mathbb{I}\backslash\{I\} \to \mathbb{R} \,\middle|\, \sum_{i \in \mathbb{I}\backslash\{I\}} k^i > 0,\ \omega^0(k) > 0 \right\}$$

Note that this restriction is consistent with the assumptions in the model as the old generation consumes all its wealth and consumption is required to be positive. The definition of the state space turns out to be that complex, that even in the benchmark economy in Chapter 3 it will be done after the construction of optional Markov equilibria.

### 2.5.3 Constructing Markov Equilibria

This subsection describes a functional approach to obtain Markov equilibria as fixed points of an operator $T$ defined on a suitably chosen function space $\mathscr{K}$ similar to Barbie and Hillebrand [6]. In the following, there is guidance how to proceed in this approach.
First define a function space $\mathscr{K}$. Note that any Markov equilibrium, that is determined later, is an element of $\mathscr{K}$. Thus, depending on the generality of the set there may be further Markov equilibria that are not in $\mathscr{K}$.
The construction is as follows:
Suppose a mapping $\hat{\mathcal{K}} : \mathbb{K} \longrightarrow \mathbb{K}$, $\hat{\mathcal{K}} \in \mathscr{K}$ is given, that defines capital evolution in the following period. Given next period's capital supply $\mathbf{k_1} = (k_1^i)_{i=0}^{I-1} \in \mathbb{K}$, capital supply $\mathbf{k_2}$ is defined by $\mathbf{k_2} = \hat{\mathcal{K}}(\mathbf{k}_1)$. So given capital supply $\mathbf{k}$ of the current period, next period's capital supply is defined by the equilibrium equations (2.10) in an implicit function:

$$\beta u'(\omega^i(\mathbf{k_1}) - \hat{\mathcal{K}}^{i-1}(\mathbf{k_1}))\mathcal{R}(k_1^A) = u'(\omega^{i+1}(\mathbf{k}) - k_1^i), \quad i \in \mathbb{I}\backslash\{I\}. \tag{2.17}$$

Any solution for $\mathbf{k_1}$ implies a mapping $\mathcal{K}(\mathbf{k}) = \mathbf{k_1}$, that determines next period's capital stock in dependance of current capital stock. If the definition of the function space $\mathscr{K}$

guarantees that $\mathcal{K}(\mathbf{k}) \in \mathscr{K}$ and the definition of $\mathcal{K}(\mathbf{k})$ is unique, a well-defined operator $T : \mathscr{K} \longrightarrow \mathscr{K}$ can be defined such that $T(\hat{\mathcal{K}}) = \mathcal{K}$. By construction any fixed point of $T$ defines a Markov equilibrium. Note that the uniqueness of a solution of equations (2.17) is crucial and it may lead to further restrictions on the function space.

The complexity of multi-period overlapping generations models and the construction of their equilibria were pointed out in the current section. Moreover, it gave an instruction to derive equilibria. The next chapter treats a particular well-known economy, where the concepts are implemented.

## 2.6 Mathematical Appendix

***Proofs of Lemma 2.3.1 and Lemma 2.3.2*** . The result follows by the fact that consumption is required to be positive. The conjecture is that given prices $\hat{w}^i, \hat{r}^i$

$$k_n + \sum_{j=n}^{i} \frac{\ell_j \hat{w}_j}{\prod\limits_{m=0}^{j} \hat{r}_m} > 0 \quad n = 1, ..., i. \tag{2.18}$$

The argument is an induction argument. If the inequality holds for some $n \in \{1, ..., i\}$, it holds for $n - 1$ as well.
Recall the budget constraints (2.4), requiring positive consumption, i.e., $c_n > 0$, $\forall n = 0, ..., i$ and the sum of consumption and investment must not exceed wealth, i.e., $\omega_n \geq c_n + k_{n+1}$, $\forall n = 0, ...i$. For purposes of compact notation set $k_{i+1} := 0$.
First consider $n = i$:
By construction the consumer consumes all her wealth in that period, as she dies at the end of the period and has no opportunity for further investment, i.e., $c_i = \omega_i$. Together with the budget constraints it is

$$0 < c_i = \omega_i = \ell_i \hat{w}_i + k_i \hat{r}_i$$
$$\Leftrightarrow \quad k_i > -\frac{\ell_i \hat{w}_i}{\hat{r}_i} \left( = \sum_{j=i}^{i} \frac{\ell_j \hat{w}_j}{\prod\limits_{m=i}^{j} \hat{r}_m} \right)$$

Thus, (2.18) holds for $n = i$.
Assume that the inequality holds for $n + 1$ with $n \in \{1, ..., i - 1\}$, i.e.,

$$k_{n+1} + \sum_{j=n+1}^{i} \frac{\ell_j \hat{w}_j}{\prod\limits_{m=n+1}^{j} \hat{r}_m} > 0.$$

Now, consider index $n$:
By the budget constraints (2.4) the inequality can be concluded:

$$\begin{aligned} & 0 < c_n \leq \omega_n - k_{n+1} \\ \Leftrightarrow \quad & 0 < \ell_n \hat{w}_n + k_n \hat{r}_n - k_{n+1} \\ \Leftrightarrow \quad & k_n > \frac{k_{n+1}}{\hat{r}_n} - \frac{\ell_n \hat{w}_n}{\hat{r}_n} > -\frac{1}{\hat{r}_n} \sum_{j=n+1}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=n+1}^{j} \hat{r}_m} - \frac{\ell_n \hat{w}_n}{\hat{r}_n} = -\sum_{j=n}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=n}^{j} \hat{r}_m} \end{aligned}$$

By continuing the induction the conjecture must hold for the initial value $k_0$ as well. This proves Lemma 2.3.2.

□

***Proof of Lemma 2.3.3***. First note that given the optimal investment decisions $(k_1^\star, \ldots, k_i^\star)$, consumption is directly defined by

$$c_n^\star = \ell^{i-n} \hat{w}_n + k_n^\star \hat{r}_n - k_{n+1}^\star, \; n = 0, ..., i.$$

Again set $k_{i+1} := 0$ to simplify notation.

Therefore it is sufficient to show that the tupel $(k_1, \ldots, k_i)$ is uniquely defined by the first order conditions of the maximization problem of the consumer:

$$\hat{r}_n \beta u'(\ell_n \hat{w}_n + k_n \hat{r}_n - k_{n+1}) - u'(\ell_{n-1} \hat{w}_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n) = 0, \quad n = 1, \ldots, i.$$

Define an auxiliary function

$$\hat{r}_n \beta u'(\ell_n \hat{w}_n + k_n \hat{r}_n - k_{n+1}) - u'(\ell_{n-1} \hat{w}_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n) =: H_n(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i), \quad n = 1, \ldots, i.$$

Obviously, the zeros of $H_n, \; n = 1, \ldots, i$, define the optimal investment plan $(k_1^\star, \ldots, k_i^\star)$. Here again it can be seen very clearly that there are three investment decisions that need to be taken into account to solve the decision problem.
The idea of the proof is solving the problem backwards. As $k_{n+1} = 0$, $H_i$ only depends on two investment decisions. So first focus on that function:
For $n = i$, it is

$$H_i(k_{i-1}, k_i, 0, w^i, r^i) = \hat{r}_i \beta u'(\ell_i \hat{w}_i + k_i \hat{r}_i) - u'(\ell_{i-1} \hat{w}_{i-1} + k_{i-1} \hat{r}_{i-1} - k_i).$$

*Conjecture*:

Given $k_{i-1}$ there is $k_i \in (-\frac{\ell_i \hat{w}_i}{\hat{r}_i}, \ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1})$ that solves $H_i(k_{i-1}, k_i, 0, \hat{w}^i, \hat{r}^i) = 0$. In particular $k_{i-1}$ is defined uniquely.
So now, assume $k_{i-1}$ to be fixed.

Existence:
For the proof of existence consider the boundary behavior of $H_i$. As the utility function satisfies the Inada conditions (see Assumption 2.3.1) it is:

$$\begin{aligned}
&\lim_{k_i \searrow -\frac{\ell_i \hat{w}_i}{\hat{r}_i}} H_i(k_{i-1}, k_i, 0, \hat{w}^i, \hat{r}^i) \\
&= \lim_{k_i \searrow -\frac{\ell^0 \hat{w}_i}{\hat{r}_i}} \hat{r}_i \beta u'(\underbrace{\ell_i \hat{w}_i + k_i \hat{r}_i}_{\to 0}) - u'(\underbrace{\ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1} - k_i}_{\to\ const}) \\
&= \infty
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{k_i \nearrow \ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1}} H_i(k_{i-1}, k_i, 0, \hat{w}^i, \hat{r}^i) \\
&= \lim_{k_i \nearrow \ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1}} \hat{r}_i \beta u'(\underbrace{\ell_i \hat{w}_i + k_i \hat{r}_i}_{\to\ const}) - u'(\underbrace{\ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1} - k_i}_{\to 0}) \\
&= -\infty
\end{aligned}$$

Altogether, given $k_{i-1}$ the intermediate value theorem shows the existence of $k_i \in (-\frac{\ell_i \hat{w}_i}{\hat{r}_i}, \ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1})$ satisfying $H^i(k_{i-1}, k_i, 0, w^i, r^i) = 0$.

Uniqueness:
Uniqueness follows by the monotonicity properties of $H_i(-)$ with respect to $k_i$:

$$\frac{\partial H_i(-)}{\partial k_i} = \hat{r}_i \beta u''(\ell_i \hat{w}_i + k_i \hat{r}_i)\hat{r}_i + u''(\ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1} - k_i) < 0$$

The monotonicity of $H_i$ with respect to $k_i$ implies the existence of at most one zero of $H_i$. As existence has already been proven it follows that there exists a unique solution $k_i \in (-\frac{\ell_i w_i}{r_i}, \ell_{i-1}w_{i-1} + k_{i-1}r_{i-1})$ to $H_i(-) = 0$, depending only on $k_{i-1}$. Thus, $k_i$ can be written as a function: $k_i = K_i(k_{i-1})$. By the implicit function theorem it follows that

$$\frac{\partial K_i}{\partial k_{i-1}} = -\frac{\hat{r}_{i-1} u''(\ell_i \hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1} - k_i)}{\hat{r}_i \beta u''(\ell_i \hat{w}_i + k_i \hat{r}_i)\hat{r}_i + u''(\ell_{i-1}\hat{w}_{i-1} + k_{i-1}\hat{r}_{i-1} - k_i)} < 0.$$

Therefore, $H_{i-1}$ is given by $H_{i-1}(k_{i-2}, k_{i-1}, K_i(k_{i-1}), \hat{w}^i, \hat{r}^i)$.

For the following observation define

$$\underline{k}_n = -\sum_{j=n}^{i} \frac{\ell_j \hat{w}_j}{\prod_{m=0}^{j} \hat{r}_m}$$

as the lower bound for investment in period $n \in \{1, ..., i\}$. Note that by the requirement of positive consumption it is

$$k_n \to \underline{k}_n \qquad \Rightarrow \qquad k_{n+1} \to \underline{k}_{n+1}. \tag{2.19}$$

Choosing investment near its lower bound (see Lemma 2.3.1), investment in the following period is near its lower bound as well. The reason is that $\underline{k}_n$ corresponds to all future income taking into account future returns. If the agent consumes (almost) total income of the remaining lifetime in one period, she needs to go into debt in all future periods. In each following period she repays all labor income and investment again is the negative remaining lifetime income to avoid negative (or zero) consumption. Note that this is a key observation for the proof of existence of future capital supply.
Now, assume that for an arbitrary but fixed $n+1 \in \{0, ..., i-1\}$ there is a function $K_{n+1}(k_n)$ with $\frac{\partial K_{n+1}}{\partial k_n} < 0$ as it has been shown for $n = i$ and show that the property holds for $n$ as well:

$$H_n(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i)) = \hat{r}_n \beta u'(\ell_n \hat{w}_n + k_n \hat{r}_n - k_{n+1}) - u'(\ell_{n-1} w_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n)$$

The conjecture is that given $k_{n-1}$ there is a unique $k_n \in (\underline{k}_n, \ell_{n-1} w_{n-1} + k_{n-1} \hat{r}_{n-1})$ such that $H_n(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i) = 0$. Again existence follows by the Inada conditions and (2.19):

$$\begin{aligned}
&\lim_{k_n \searrow \underline{k}_n} H_i(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i) \\
&= \lim_{k_n \searrow \underline{k}_n} \hat{r}_n \beta u'(\ell_n \hat{w}_n + k_n \hat{r}_n - k_{n+1}) - u'(\ell_{n-1} \hat{w}_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n) \\
&\overset{(2.19)}{=} \lim_{k_{n+1} \searrow \underline{k}_{n+1}} \hat{r}_n \beta u'(\underbrace{\underline{k}_{n+1} - k_{n+1}}_{\to 0}) - u'(\ell_{n-1} \hat{w}_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n) \\
&= \infty
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{k_n \nearrow \ell_{n-1} w_{n-1} + k_{n-1} \hat{r}_{n-1}} H_n(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i) \\
&= \lim_{k_i \nearrow \ell_{n-1} w_{n-1} + k_{n-1} \hat{r}_{n-1}} \hat{r}_n \beta u'(\underbrace{\ell_n \hat{w}_n + k_n \hat{r}_n}_{\to\ const}) - u'(\underbrace{\ell_{n-1} \hat{w}_{n-1} + k_{n-1} \hat{r}_{n-1} - k_n}_{\to 0}) \\
&= -\infty
\end{aligned}$$

So again, the existence is ensured by the intermediate value theorem. For the proof of uniqueness use $k_{n+1} = K_{n+1}(k_n)$:

$$H_n(k_{n-1}, k_n, k_{n+1}, \hat{w}^i, \hat{r}^i) = H_n(k_{n-1}, k_n, K_{n+1}(k_n), \hat{w}^i, \hat{r}^i)$$

Similar to the first part of the proof uniqueness follows by the monotonicity of $H_n$:

$$\frac{\partial H_n(-)}{\partial k_n} = \hat{r}_n \beta u''\big(\ell^n \hat{w}_n + k_n \hat{r}_n\big)\left(\hat{r}_n \underbrace{- \frac{\partial K_{n+1}(k_n)}{\partial k_n}}_{>0}\right) + u''\big(\ell^1 \hat{w}_{n-1} + k^1_{n-1}\hat{r}_{n-1} - k_n\big) < 0$$

Thus, $k_{n-1}$ defines $k_n$ uniquely. The induced function $K_n(k_{n-1})$ is strictly decreasing:

$$\frac{\partial K_n(k_{n-1})}{\partial k_{n-1}} = -\frac{\hat{r}_{n-1} u''\big(\ell_{n-1}\hat{w}_{n-1} + k_{n-1}\hat{r}_{n-1} - k_n\big)}{\hat{r}_n \beta u''\big(\ell^n \hat{w}_n + k_n \hat{r}_n\big)\left(\hat{r}_n - \frac{\partial K_{n+1}(k_n)}{\partial k_n}\right) + u''\big(\ell^1 \hat{w}_{n-1} + k^1_{n-1}\hat{r}_{n-1} - k_n\big)} < 0$$

So $k_n$ has the same properties as $k_{n+1}$ and by induction summarized it can be said that knowing the initial value $k_0$ (satisfying the no-bankruptcy constraint), the future decisions $(k_1^\star, \ldots, k_i^\star)$ are uniquely defined. □

***Proof of Lemma 2.4.1***. Without loss of generality consider an arbitrary period $t = 0$. The labor profile leads to a contradiction to $k_2^A > 0$. Exemplary consider the three-period economy. Wealth is given by

$$\omega^2(\mathbf{k}) = 0 \qquad \omega^1(\mathbf{k}) = k^1 \mathcal{R}(k^A) \qquad \omega^0(\mathbf{k}) = \mathcal{W}(k^A) + k^0 \mathcal{R}(k^A)$$

Initial capital is irrelevant for the young generation. By the budget constraints (2.4) it follows for the young generation:

$$c_0^2 = \omega^2(\mathbf{k}_0) - k_1^1 > 0 \qquad \Rightarrow \qquad k_1^1 < 0 \qquad \Rightarrow \omega^1(\mathbf{k}_1) < 0$$

By the same argumentation in the following period it is

$$c_1^1 = \omega^1(\mathbf{k}_1) - k_2^0 > 0 \qquad \Rightarrow \qquad k_2^0 < 0 :$$

As this holds for all periods $t \geq 0$, $k_2^1 < 0$ holds as well and thus $k_2^A = k_2^1 + k_2^0 < 0$.

□

***Proof of Lemma 2.5.1***. If and only if $k^i < \left(\ell^i - \frac{f'(k^A)}{f''(k^A)k^A}\right) k^A$, the wealth function is strictly increasing in $k^i$. In addition it is

$$\frac{f''(k^A)k^A}{f'(k^A)} \geq -1 \Leftrightarrow 1 \leq -\frac{f'(k^A)}{f''(k^A)k^A} \text{ (as } f''(\cdot) < 0).$$

It follows, that

$$(1 + \ell^i)k^A \leq \left(\ell^i - \frac{f'(k^A)}{f''(k^A)k^A}\right) k^A$$

and therefore

$$k^i < (1 + \ell^i)k^A \Rightarrow k^i < \left(\ell^i - \frac{f'(k^A)}{f''(k^A)k^A}\right) k^A.$$

So it is sufficient to show that $k^i < (1 + \ell^i)k^A$.
Define $k^A_{-i} := \sum_{\substack{j \in \mathbb{I} \\ j \neq i}} k^j$. As $k \in \mathbb{K}$ it is clear that $k^i > -k^A_{-i}$. Having a labor profile $\ell$, all $\ell^i$ are defined to be greater or equal than 0. That is $-k^A_{-i} > -\frac{1+\ell^i}{\ell^i} k^A_{-i}$. Therefore it is

$$\begin{aligned} & k^i > -\frac{1+\ell^i}{\ell^i} k^A_{-i} \\ \Leftrightarrow \quad & -\ell^i k^i < (1 + \ell^i) k^A_{-i} \\ \Leftrightarrow \quad & k^i < (1 + \ell^i) k^A \end{aligned}$$

□

# Chapter 3

# The Benchmark Economy

In this chapter, the focus is on the special case where utility is the logarithmic function and the production function is of the Cobb-Douglas type. The corresponding two-period economy is well-known and well-analyzed. The objective of this chapter is to understand how equilibria qualitatively change when the number of lifetime periods is increased. Besides the form of equilibria, the chapter addresses the questions which input factors mainly affect the equilibrium and which effects occur that the two-period model does not capture. But as a first step it is necessary to characterize conditions for equilibria of the model with this special structure.

Considering this particular economy is quite common. Balasko and Shell [4] describe for the two-period economy that the logarithmic structure simplifies the equilibrium equations very much. Beyond that the economy often is used in numerical approaches (see for example Ascari and Rankin [1]). So the particular model structure presents a benchmark for future research in multi-period overlapping generations economies. As we will often refer to the two-period benchmark economy, this economy is described in Appendix A.

The chapter is organized as follows: Section 1 introduces the general benchmark economy where equilibria are determined for particular labor profiles. Section 2 focuses on the three-period economy where an equilibrium for an arbitrary labor profile can be determined. Section 3 compares the results of Section 2 to the two-period economy.

## 3.1 The $I+1$-Period Economy

Already in the previous chapter we recognized that the labor profile plays a crucial role in determining equilibria. Therefore, this section considers the general labor profile and particular labor profiles separately.

### 3.1.1 The general labor profile

An equilibrium is defined by a consumption and investment sequence such that the equilibrium equations (2.10) are satisfied as the capital stock directly determines the prices (see (2.10)). In this particular economy when $u(c) = \ln c$ and $f(k) = k^\alpha$ the equilibrium equations read:

$$\beta\alpha(k^A_{t+1})^{\alpha-1}c^i_t = c^{i-1}_{t+1}, \qquad i \in \mathbb{I}\backslash\{0\} \tag{3.1a}$$

$$c^i_t + k^{i-1}_{t+1} = \omega^i(\mathbf{k}_t), \qquad i \in \mathbb{I} \tag{3.1b}$$

with $\omega^i(\mathbf{k}) = \ell^i(1-\alpha)(k^A)^\alpha + k^i\alpha(k^A)^{\alpha-1}$. The equilibrium equations must be satisfied in all periods $t \geq 0$.

As the sequence $\left\{(c^i_t)_{i\in\mathbb{I}}\right\}_{t\geq 0}$ is induced by the sequence $\left\{(k^i_t)^{I-1}_{i=0}\right\}_{t\geq 0}$, see equation (2.12), the objective is to find a recursive structure of $\mathcal{K}(\mathbf{k}_t) = \mathbf{k}_{t+1}$, such that

$$\beta\alpha(k^A_{t+1})^{\alpha-1}(\omega^{i+1}(\mathbf{k}_t) - k^i_{t+1}) = \omega^i(\mathbf{k}_{t+1}) - \mathcal{K}^{i-1}(\mathbf{k}_{t+1}), \quad i \in \mathbb{I}\backslash\{I\}. \tag{3.2}$$

The construction of the recursive function $\mathcal{K}$ will be done as provided in Section 2.5.3. First have a look on the restriction on the state space and the function space. Section 2.5.2 has already determined a superset of $\mathbb{K}$:

$$\mathbb{K} \subset \left\{ k : \mathbb{I}\backslash\{I\} \to \mathbb{R} \;\middle|\; \sum_{i\in\mathbb{I}\backslash\{I\}} k^i > 0,\ \omega^0(\mathbf{k}) > 0 \right\}$$

Besides positive aggregated capital, elements of $\mathbb{K}$ guarantee, that the no-bankruptcy constraint (see Lemma 2.3.1) for each generation is satisfied. Obviously, each generation's wealth needs to be greater than a critical value, see Lemma 2.5.2. Besides positive aggregated capital, elements of $\mathbb{K}$ guarantee, that the no-bankruptcy constraint (see Lemma 2.3.1) for each generation is satisfied. Obviously, each generation's wealth needs to be greater than a critical value, see Lemma 2.5.2.

$$\mathbb{K} := \left\{ \mathbf{k} : \mathbb{I}\backslash\{I\} \to \mathbb{R} \;\middle|\; \sum_i k^i > 0, \omega^i(\mathbf{k}) > \underline{\omega}^i(\mathbf{k}), \underline{\omega}^0(\mathbf{k}) = 0 \right\}$$

Note that the lower bound is not necessarily constant. By construction it is $\underline{\omega}^i(\mathbf{k}) \leq 0$. If each generation had positive wealth in each period, there would not be any problem with future bankruptcy. But as Section 2.3 shows negative wealth may occur in general and can be compensated by sufficient capital supply by the remaining generations concerning total capital supply and by future income concerning the single consumer. That is, the requirement of non-negative wealth is sufficient but not necessary.

Thus, the set

$$\mathbb{K}_0 := \left\{ k : \mathbb{I} \backslash \{I\} \to \mathbb{R} \left| \sum_i k^i > 0, \omega^i(\mathbf{k}) > 0, i = 0, ..., I-1 \right. \right\} \tag{3.3}$$

is a subset of $\mathbb{K}$.

Note that positive wealth is a very different requirement than positive capital supply. As long as agents have labor income, capital supply certainly may be negative as long as capital income is not smaller than negative labor income. In addition note that there is no restriction on $\omega^I(\mathbf{k})$ as $i = I$ is excluded in $\mathbb{K}$. But by definition $\omega^I(\mathbf{k}) \geq 0$ holds, as aggregated capital needs to be positive, such that there will be no problem. Refining the definition of the state space will be done for particular economies.

The elements of the function space are the first candidates for the Markov equilibrium. For its definition follow the two-period economy where capital evolution is a function of capital, and in particular it is a constant fraction on current wealth. So the main assumption in this chapter is, that the equilibrium is a linear function of wealth. Thus, the function space $\mathscr{K}$ is subset of

$$\mathscr{K} \subset \tilde{\mathscr{K}} := \{\mathcal{K} : \mathbb{K} \to \mathbb{R} | \mathcal{K}(\mathbf{k}) = M\omega(\mathbf{k})\}$$

where $\omega(\mathbf{k}) := (\omega^i(\mathbf{k}))^\top_{i=0,\dots,I}$ denotes the vector containing all generations' wealth functions and $M \in \mathbb{R}^{I \times I+1}$ is the linear factor. Therefore, the evolution of capital supply of generation $i$ is given by

$$\mathcal{K}^i(\mathbf{k}) = \sum_{j=0}^{I} a_i^j \omega^j(\mathbf{k}) \quad i = 0, \dots, I-1.$$

The wealth functions for given capital $\mathbf{k} \in \mathbb{K}$ are given by

$$\begin{aligned} \omega^I(\mathbf{k}) &= \ell^I(1-\alpha)(k^A)^\alpha \\ \omega^i(\mathbf{k}) &= \ell^i(1-\alpha)(k^A)^\alpha + \alpha k^i (k^A)^{\alpha-1} \qquad\qquad i = 0, \dots, I-1. \end{aligned}$$

Note that the given production function satisfies Assumption 2.5.1 such that $\omega^i(\mathbf{k})$ is strictly increasing in $k^i$ by Lemma 2.5.1.

To proceed with the construction of an equilibrium, assume that $\hat{\mathcal{K}}(\mathbf{k}) \in \tilde{\mathscr{K}}$ defines the capital evolution in the following period. Thus, $\hat{\mathcal{K}}$ is a linear function of wealth, i.e., it is $\hat{\mathcal{K}}(\mathbf{k}) = \left(M\omega(\mathbf{k})^\top\right)$ with $M = \left(a_i^j\right)_{\substack{i=0,\dots,I-1 \\ j=0,\dots,I}} \in \mathbb{R}^{I \times I+1}$. Applying this assumption in the equilibrium equations $\mathbf{k}_1$ can be determined in dependance of $\mathbf{k}$ as the solution of the following system of equations:

$$\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+1}(\mathbf{k}) - k_1^i) = \omega^i(\mathbf{k}_1) - \hat{\mathcal{K}}^{i-1}(\mathbf{k}_1), \quad i = 0, ..., I-1 \tag{3.4}$$

Thus, the solution $\mathbf{k}_1$ defines the mapping $\mathcal{K}$ as described in Section 2.5.3. The operator $T$ (compare Section 2.5.3) maps $\hat{\mathcal{K}}$ on $\mathcal{K}$. The following lemma guarantees that, under the assumption on $\hat{\mathcal{K}}(\mathbf{k})$ to be an element of $\tilde{\mathscr{K}}$, the solution $\mathbf{k}_1$ also is an element of the function space $\tilde{\mathscr{K}}$.

**Lemma 3.1.1.** *The operator $T$ maps $\tilde{\mathscr{K}}$ into itself: $T(\tilde{\mathscr{K}}) \subset \tilde{\mathscr{K}}$.*

Hence, Lemma 3.1.1 implies that if $\hat{\mathcal{K}}(\mathbf{k}) = (M\omega(\mathbf{k}))$, the solution to the equilibrium equations can be written as $\mathbf{k}_1 = \tilde{M}\omega(k)$, $\tilde{M} \in \mathbb{R}^{I\times I+1}$, without losing a solution. Define an additional operator $\phi : \mathbb{R}^{I\times I+1} \to \mathbb{R}^{I\times I+1}$ such that $\phi(M) = \tilde{M}$. Note that $\phi$ defines $T$ uniquely. The equilibrium equations 3.4 define the components of $\tilde{M}$. First, determine the implicit function, that characterizes $\phi$. After having discussed the question about $\phi$ being well-defined, we look for fixed points. Fixed points of $\phi$ define what we will call *equilibrium candidates* in the following. The problem is that there is no guarantee that the mapping $\mathcal{K}$ generated by such a fixed point is a mapping on $\mathbb{K}$. An equilibrium will be a mapping such that this can be verified. So in the following equilibrium candidates may be determined without an explicit definition of the state space. Summarized, proceed as follows:
Given

$$M := \begin{pmatrix} a_0^0 & \dots & a_0^I \\ \vdots & \ddots & \vdots \\ a_{I-1}^0 & \dots & a_{I-1}^I \end{pmatrix} \in \mathbb{R}^{I\times I+1}$$

that defines capital evolution in the following period by $\hat{\mathcal{K}}(\mathbf{k}) = (M\omega(\mathbf{k})) \in \tilde{\mathscr{K}}$, the task is to find

$$\tilde{M} = \begin{pmatrix} \tilde{a}_0^0 & \dots & \tilde{a}_0^I \\ \vdots & \ddots & \vdots \\ \tilde{a}_{I-1}^0 & \dots & \tilde{a}_{I-1}^I \end{pmatrix} \in \mathbb{R}^{I\times I+1}$$

defining $\mathbf{k}_1 = \tilde{M}\omega(\mathbf{k})$ such that the equilibrium equations 3.4 are satisfied.

First derive the conditions for $\tilde{a}_i^j$ setting $\hat{\mathcal{K}}^i(\mathbf{k}) = \sum_{j=0}^{I} a_i^j\omega^j(\mathbf{k})$, $i = 0, ..., I-1$, and $\mathbf{k}_1 = \tilde{M}\omega(\mathbf{k})$ in the equilibrium equations (3.4). The equations defining $\tilde{M}$ are:

$$\beta\alpha\omega^1(\mathbf{k}) - \alpha(1+\beta)\sum_{j=0}^{I}\tilde{a}_0^j\omega^j(\mathbf{k}) - \ell^0(1-\alpha)\left(\tilde{M}\omega(\mathbf{k})\right)^A = 0$$

$$\beta\alpha\omega^{i+1}(\mathbf{k}) - \alpha(1+\beta)\sum_{j=0}^{I}\tilde{a}_i^j\omega^j(\mathbf{k}) - \ell^i(1-\alpha)\left(\tilde{M}\omega(\mathbf{k})\right)^A + \ell^I(1-\alpha)\left(\tilde{M}\omega(\mathbf{k})\right)^A a_{i-1}^I$$

$$+\sum_{n=0}^{I-1} a_{i-1}\left(\ell^n(1-\alpha)\left(\tilde{M}\omega(\mathbf{k})\right)^A + \alpha\sum_{j=0}^{I}\tilde{a}_n^j\omega^j(\mathbf{k})\right) = 0$$
$$i = 1, ..., I-1$$

Using $\left(\tilde{M}\omega(\mathbf{k})\right)^A = \sum_{j=0}^{I}\omega^j(\mathbf{k})\sum_{i=0}^{I-1}\tilde{a}_i^j$ leads to

$$\sum_{j=0}^{I}\omega^j(\mathbf{k})\left[-\alpha(1+\beta)\tilde{a}_0^j - \ell^0(1-\alpha)\sum_{n=0}^{I-1}\tilde{a}_n^j\right] + \alpha\beta\omega^1(\mathbf{k}) = 0 \tag{3.5a}$$

$$\sum_{j=0}^{I}\omega^j(\mathbf{k})\Bigg[-\alpha(1+\beta)\tilde{a}_i^j - \ell^i(1-\alpha)\sum_{n=0}^{I-1}\tilde{a}_n^j + \ell^I(1-\alpha)a_{i-1}^I\sum_{n=0}^{I-1}\tilde{a}_n^j$$
$$+\sum_{n=0}^{I-1}\tilde{a}_n^j(1-\alpha)\sum_{m=0}^{I-1}a_{i-1}^m\ell^m + \alpha\sum_{m=0}^{I-1}a_{i-1}^m\tilde{a}_m^j\Bigg] + \beta\alpha\omega^{i+1}(\mathbf{k}) = 0 \tag{3.5b}$$
$$i = 1, ..., I-1.$$

The system of equations is satisfied if the coefficients of $\omega^j(\mathbf{k})$ for all $j = 0, ..., I$ in (3.5a) and (3.5b) are equal to 0. Then, the components of $\tilde{M}$ are the solution to a linear system of equations. Denoting $\tilde{M} = \left(\tilde{a}^0, ..., \tilde{a}^I\right)$ where $\tilde{a}^j = \left(\tilde{a}_0^j, ..., \tilde{a}_{I-1}^j\right)^T \in \mathbb{R}^I$ and

$$\tilde{m} = \begin{pmatrix} \tilde{a}^0 \\ \vdots \\ \tilde{a}^I \end{pmatrix} \in \mathbb{R}^{I(I+1)},$$

$\tilde{m}$ is the solution to the linear system of equations $B_M\tilde{m} = b$ with $b = (b_n)_{n=0}^{I(I+1)} \in \mathbb{R}^{I(I+1)}$,

$$b = \begin{cases} -\beta\alpha & n = l(I+1) \quad (l = 1, ..., I) \\ 0 & otherwise \end{cases} \tag{3.6}$$

and

$$B_M = \begin{pmatrix} \tilde{B_M} & 0 & \dots & 0 \\ 0 & \tilde{B_M} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{B_M} \end{pmatrix} \in \mathbb{R}^{I(I+1)\times I(I+1)}$$

where $\tilde{B_M} = (b_{ln})_{l,n=0,...,I-1} \in \mathbb{R}^{I\times I}$ with

$$b_{ln} = \begin{cases} b_l + \alpha(a_{l-1}^l - 1 - \beta) & l = n \\ b_l + \alpha a_{l-1}^n & l \neq n \end{cases}$$
$$b_l := -\ell^l(1-\alpha) + (1-\alpha)\sum_{j=0}^{I}a_{l-1}^j\ell^j. \tag{3.7}$$

Additionally, $a_{-1} := 0$ is used for purposes of a compact notation. The implicit function $B_M \tilde{m} = b$ defines $\tilde{M}$. Whether the operator $\phi$, mapping $M$ on $\tilde{M}$, is well-defined, depends on the existence and uniqueness of the solution of the implicit function. In the following the properties of $\phi$, respectively the number of solutions depending on the properties of $B_M$, are considered.

**Lemma 3.1.2.** *The operator $\phi$ is well-defined if and only if* $\det(\tilde{B_M}) \neq 0$.

*Proof.* $\phi$ is well-defined if the system of equations $B_M \tilde{a} = b$ has a unique solution. That is equivalent to $\det(B_M) \neq 0$.
As $B_M$ is a block matrix with entries only in the diagonal, $\det(B_M) = (\det(\tilde{B_M}))^{I+1}$ holds. Therefore, $\det(B_M) \neq 0 \Leftrightarrow \det(\tilde{B_M}) \neq 0$. □

By Lemma 3.1.2 $\tilde{m}$ is uniquely defined if the determinant of $B_M$ is different from zero. So far, there is no restriction on the function space that guarantees that property. In particular note that even if $M$ satisfies the non-zero determinant of $B_M$, the image $\tilde{M}$ may not. Hence it is necessary to consider mappings that do not satisfy that property. If the determinant vanishes, $\det B_M = 0$, there are two possibilities: Either there is no solution or there are multiple. Multiple solutions would be a first indication that there are multiple equilibria. That is what concerns the following lemma.

**Lemma 3.1.3.** *The linear system of equations $B_M \tilde{a} = b$ has at most one solution. In particular there is no solution for* $\det(\tilde{B_M}) = 0$.

Obviously, this source of multiplicity of equilibria does not occur. In particular the result implies that each fixed point of $\phi$ necessarily leads to a Matrix $B$ with non-zero determinant. Under this condition the following corollary concludes the present findings.

**Assumption 3.1.1.** *Assume that $M$ is chosen such that* $\det(B_M) \neq 0$.

**Corollary 3.1.1.** *Any fixed point of the operator* $\phi : \left\{M \in \mathbb{R}^{I \times I+1} | \det(B_M) \neq 0\right\} \to \mathbb{R}^{I \times I+1}$ *satisfies Assumption 3.1.1.*

Note again that this corollary does not mean that the codomain is equal to the domain. A restriction of the domain or the codomain reduces the set that contains fixed point candidates ($\phi(M) = M$). In the following we focus on the restricted sets as the objective here is to find fixed points.
The following lemma answers the question if and how the the codomain of $\phi$ under assumption 3.1.1, where the image of $M$ under $\phi$ is uniquely defined, may be restricted.

**Lemma 3.1.4.** *Let Assumption 3.1.1 hold. The codomain of $\phi$ can be restricted to*

$$\left\{M \in \mathbb{R}^{I \times I+1} | a_i^0 = 0 \ \forall i = 0, ..., I-1\right\}.$$

For the following proof Cramer's rule will be used. It gives an instruction in the case of a non-vanishing determinant for computing the solution of a linear system of equations $Ax = b$ using the determinant. The l-th entry in the solution vector $(x)$ is given by the fraction of the determinant of the matrix $(A)$ where the l-th column is replaced by the "result" $(b)$ and the determinant of the matrix $(A)$ itself. For a better understanding you find Cramer's rule applied to the three-period economy in Section 3.4.1.

Before proving the lemma, consider matrix $B_M$ when one column is replaced by $b$. Obviously, the new matrix $(B_M)_l$ $(0 \leq l \leq I(I+1))$ is no longer a diagonal block matrix. But still the determinant can be determined by the product of the block matrices in the diagonal. Obviously, as $\det(B_M) \neq 0$ is equivalent to$\det(\tilde{B_M}) \neq 0$ $\tilde{B_I}$ has rank $I$. Adding columns to other columns does not change the value of the determinant, so each block, except that where the replacement took place, can be transformed to a diagonal matrix without changing the value of the determinant by Gaussian elimination. Finally, use the columns where the entries are different from 0 in the same row where $b$ has $-\beta\alpha$ as entry to eliminate these. The result again is a diagonal block matrix and the determinant is given by the product of the determinants of the entries.

For example, if the $l-th$ column is in block $i$ (totally there are I+1 blocks), the determinant of $(B_M)_l$ is given by $\det((B_M)_l) = (\det(\tilde{M_A}))^I \det(\tilde{B_{Mij}})$, where $(\tilde{B_M})_{ij}$ is matrix $\tilde{B_M}$ where the $j-th$ column is replaced by $-\beta\alpha e_i$ and again $e_i$ is the i-th unit vector.

With these explanations about Cramer's rule the following proof is straightforward.

*Proof.* The values $\tilde{a}_i^0$, $i = 0, ..., I-1$ are the first $I$ entries of the solution vector. So keeping the notation from above it is

$$\tilde{a}_i^0 = \frac{\det((B_M)_i)}{\det(B_M)} = \frac{(\det(\tilde{B_M}))^I \det((\tilde{B_M})_{0i})}{(\det(\tilde{B_M}))^{I+1}} = \frac{\det((\tilde{B_M})_{0i})}{\det(\tilde{B_M})}.$$

As the first $I$ entries of $b$ are equal to 0, $(\tilde{B_M})_{0i}$ has a 0-column in column $i$. Thus, $\det((\tilde{B_M})_{0i}) = 0$ and therefore $\tilde{a}_i^0 = 0$ for all $i = 0, ..., I-1$. □

Before considering particular scenarios, focus on the coefficients $a_i^j$. In the end the interest is in fixed points of $\phi$. As Lemma 3.1.4 restricts the codomain of $\phi$, obviously, it is sufficient to consider functions

$$\mathcal{K}(k) = A\omega_{-0}(k), \quad \omega_{-0}(k) = \left(\omega^i(\mathbf{k})\right)_{i=1}^I$$

with

$$A \in \mathbb{R}^{I \times I}, \quad A = \begin{pmatrix} a_0^1 & \dots & a_0^I \\ \vdots & \ddots & \vdots \\ a_{I-1}^1 & \dots & a_{I-1}^I \end{pmatrix}.$$

In addition, the definitions of $B_A$, $\tilde{B_A}$, $\tilde{a}$ and $\phi$ are suitably adapted as well.
Lemma 3.1.4 states that the coefficient of the wealth of generation '0', the oldest, is 0. That is, wealth of the oldest generation has no impact on future capital. The explanation is that $\omega^0(\mathbf{k})$ is consumed completely and generation '0' does not invest any capital and thus do not take part in building future capital.
Before characterizing the fixed points, address the question what can be expected about the coefficients in equilibrium. First clarify the meaning of coefficient $a_i^j$. It describes how an increase in $\omega^j(\mathbf{k})$ in the current period affects capital supply of generation $i$ in the following period. That is, the coefficients determine the influence of each generation's wealth on next period's individual capital supply. Note that for generation $i$ $\omega^j(\mathbf{k}),\ j \neq i$, has nothing to do with future or former income of generation '$i$'.
The interpretation suggests that in equilibrium $a_i^{i+1}$ should be positive. Any other result would be counterintuitive because it describes the 'own' capital supply in the following period. The reason is that one would assume that additional wealth in one period is divided up between consumption and investment analogously to wealth in general. Hence, if there is more wealth, that can be distributed, the chosen allocation intuitively is greater or equal the former decision in the sense that consumption as well as investment are greater or equal.
Concerning the other coefficients intuition is not that clear. As aggregated capital supply influences both next period's wage (positive) and return (negative) the question is which change is more valuable. That depends on different factors for example the labor profile $\ell$. Hence, one possible scenario is that $a_i^{i+1} > 0$ and $a_i^j \leq 0$ for all $j \neq i+1$. This scenario can be interpreted as follows: Take a look on an arbitrary generation. The higher its wealth is the higher is its investment. The other agents wealth influences the investment decision negatively, i.e., the higher the wealth of another generation the lower is the investment of the regarded generation. If wealth of another generation would be negative, investment increases compared to positive wealth of this other generation. That would make sense as the generation with negative wealth needs to go into debt to have positive consumption. As aggregated capital supply needs to be positive, the other generations compensate that by higher investment.

Obviously, the image of $A$ under $\phi$ strongly depends on the labor profile. Thus, there are labor profiles that admit statements about explicit structure of equilibria. In the next subsection the focus is on these particular labor profiles.

### 3.1.2 Special labor profiles

This subsection focuses on labor profiles with retirement. Retirement is defined as follows: there is $N \in \mathbb{I}$ such that $\ell^n = 0, \forall n < N$. That is, from a particular age on the generations do not supply labor any more. The definition implies that retired generations' wealth only consists of capital income. Retirement leads to strong statements about the structure of the equilibrium.

**Lemma 3.1.5.** *Let Assumption 3.1.1 hold. If there is $N \in \mathbb{I}$ such that $\ell^n = 0, \forall n < N$, the set containing all fixed points of $\phi$ can be restricted such that for each element $\tilde{A}$ of the codomain $\tilde{a}_i^j = 0$ for $j \neq i+1, i < N$.*

*In particular, for $i < N$ it is $\tilde{a}_i^{i+1} = \frac{\beta \sum_{m=0}^{i} \beta^m}{\sum_{m=0}^{i+1} \beta^m} < 1$.*

Thus, dividing the matrix $A$ into 4 block matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where $A_1 \in \mathbb{R}^{N \times N}$, $A_2 \in \mathbb{R}^{N \times I-N}$, $A_3 \in \mathbb{R}^{I-N \times N}$ and $A_4 \in \mathbb{R}^{I-N \times I-N}$, $A_1$ is a diagonal matrix with diagonal entries as determined in Lemma 3.1.5 and $A_2$ is the 0-matrix. None of that holds for $A_3$ and $A_4$. The extreme case is if $N = I$ holds. $N = I$ means that all labor is supplied by the young. Obviously, in this case the matrix $A$ is a diagonal matrix.

**Corollary 3.1.2.** *If labor is supplied only by the young, i.e., $\ell^I = 1$, the equilibrium is defined by*

$$\mathcal{K}(\mathbf{k}) = \left( \frac{\beta \sum_{m=0}^{i} \beta^m}{\sum_{m=0}^{i+1} \beta^m} \omega^{i+1}(\mathbf{k}) \right)_{i=0}^{I-1}.$$

In this particular economy each generation's next period's capital supply only depends on its current wealth. Wealth of the other generations does not affect the decision. This is a result that we know from the two-period economy. Moreover, by the construction of the economy it turns out that the restrictions on the function space are not necessary.

**Lemma 3.1.6.** *In the overlapping generations economy where all labor is supplied by the young, i.e., $\ell^I = 1$, the recursive equilibrium $\mathcal{K}(\mathbf{k}) = \bar{A}\omega_{-0}(\mathbf{k})$ is unique. In particular, the equilibrium is unique on a general function space that is not restricted to linear combinations of the wealth functions.*

Note the importance of that result. This particular economy has a unique equilibrium that is a Markov equilibrium. The following lemma defines the state space of this economy.

**Lemma 3.1.7.** *For $\ell^I = 1$ the state space is $\mathbb{R}_{++}^I$.*

*Proof.* The result follows easily with Lemma 2.3.1. □

The reason is that all future income consists only of capital income. It is necessary to save and there is no opportunity to borrow, as wealth from generation '$I-1$' on only consists of the capital income. Thus, savings must be big enough to compensate future consumption up to the last lifetime period. Knowing the dynamics, obviously, there is the question whether there are stationary points of the dynamics. In this economy, where all labor is supplied by the young, even a uniqueness result can be presented:

**Lemma 3.1.8.** *The dynamical system* $\mathcal{K}(\mathbf{k}) = (\mathcal{K}^i(\mathbf{k}))_{i=0}^{I-1}$, $\mathcal{K}^i(\mathbf{k}) = \left( \frac{\beta \sum_{j=0}^{i-1} \beta^j}{\sum_{j=0}^{i} \beta^j} \omega^{i+1}(\mathbf{k}) \right)_{i=0}^{I-1}$ *has a unique steady state.*

Next consider the economy where $N = I - 1$, i.e., $\ell^I + \ell^{I-1} = 1$. The economy is more general than that considered before, but it has still a particular structure such that it is possible to determine the equilibrium dynamics explicitly. The reason is that most entries of the equilibrium matrix $\bar{A}$ are determined by Lemma 3.1.5 and at the same time the coefficients generating next period's capital supply of generation '$I-1$' do not play a role in the equilibrium equations by construction.

It is clear that in this general context retirement after two time periods economically is not very interesting. The reason why it is not skipped here is that it is one of the models where an equilibrium can be determined explicitly and within the analysis of the three-period model in the following section, this context is a very useful extension of the two-period model. The three-period economy is one example that shows that in general, from $N = I - 2$ on the explicit computation of an equilibrium is no longer possible.

**Lemma 3.1.9.** *Let* $\ell^I + \ell^{I-1} = 1$. *Then,* $\phi$ *is well-defined and has a (unique) fixed point*

$$\bar{A} = \begin{pmatrix} \frac{\beta}{1+\beta} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \frac{\beta \sum_{m=0}^{I-2} \beta^m}{\sum_{m=0}^{I-1} \beta^m} & 0 \\ -\frac{\frac{\beta}{1+\beta}\ell^{I-1}(1-\alpha)}{\ell^{I-1}(1-\alpha)+\alpha \sum_{m=0}^{I} \beta^m} & \cdots & -\frac{\ell^{I-1}(1-\alpha)\frac{\sum_{m=0}^{I-2} \beta^m}{\sum_{m=0}^{I-1} \beta^m}}{\ell^{I-1}(1-\alpha)+\alpha \sum_{m=0}^{I} \beta^m} & \frac{\beta\alpha \sum_{m=0}^{I-1} \beta^m}{\ell^{I-1}(1-\alpha)+\alpha \sum_{m=0}^{I} \beta^m} \end{pmatrix}$$

Note that the equilibrium dynamics of that economy satisfies the intuition about the coefficients of the wealth function in the general part was right (see Section 3.1.1). Each generation's wealth has positive impact on their own next period's capital supply, the wealth of the others has a negative impact if it plays any role. Moreover, note that in this matrix

$\left|a_i^j\right| < 1$, for all $i = 0, ..., I-1;\ j = 1, ..., I$ holds.

The dynamics reflect that the economy is a generalization of the economy considered before. Setting $\ell^{I-1} = 0$ the dynamics coincide. The result demonstrates very well that introducing labor supply of the next generation leads to cross-wealth effects. The decision about capital supply in the following period now does not only depend on the own current wealth but also on the wealth of the other generations. In this particular case the decision of generations '1' to '$I-1$' about capital supply for the next period equals the former result, where all labor is supplied by the young. The decision of the young (generation '$I$') about capital supply in the next period differs. It depends on current wealth of all generations except the old. Note that the economy shows that cross-wealth effects are not excluded if all labor is supplied by one arbitrary generation. The only labor profile without these effects is if all labor is supplied by the young. By the argumentation done above the signs of the coefficients are very intuitive as next period's capital supply corresponds to the savings decision of the currently young generation. Having higher wealth the savings are higher as well. The savings also depend on next period's prices that are determined by all generation's decisions.
Moreover, it is obvious that negative capital supply of generation '$I-1$' can be realized. The assumption that increasing the number of periods in the overlapping generations economy allows borrowing, in contrast to the two-period model, is not only hypothetical. In an extreme case, when $\ell^I = 0$, it is clear that the young need to borrow to realize positive consumption as their wealth is zero.
Obviously, it is not necessary to consider the general model with an arbitrary labor profile to get effects that cannot occur in the two-period economy. Already in this quite simple modification of the model's structure, it is obvious that in the two-period model some effects are eliminated by construction.

The difficulties with the definition of the state space have been stated before. In this particular economy, $\ell^I + \ell^{I-1} = 1$, Lemma 2.5.3 defines it.

$$\mathbb{K} := \mathbb{K}_0,$$

where $\mathbb{K}_0$ is defined as in (3.3). Intuitively, the definition makes sense as well. Think of an arbitrary generation $i \in \mathbb{I} \setminus \{0, I\}$, i.e., not the youngest and not the oldest generation. If wealth was negative, next period's capital supply necessarily would be negative as well, as consumption needs to be positive. This continues for the following $i$ periods and when the generation is old capital supply is still negative as there is no labor income over all the periods and thus wealth when old would be negative as well. As wealth coincides with consumption when old, that states a contradiction. Note that here $\mathbb{K}_0 \neq \mathbb{R}^I_{++}$ as generation

'$I-1$' has both labor and capital income. Thus negative capital supply of that generation is compensated by labor income to a certain degree. In contrast to that in the economy where all labor is supplied by the young this set coincides with the set that requires positive capital supply as it has been defined above as for $\ell^i = 0$ it is

$$\omega^i(\mathbf{k}) > 0 \quad \Leftrightarrow \quad k^i > 0, \quad i = 0, ..., I-1.$$

With the exact definition of the state space it is possible to show that the dynamics define an equilibrium, i.e., they map $\mathbb{K}$ into itself.

**Lemma 3.1.10.** *Let $\ell^{I-1} + \ell^I = 1$. The fixed point derived in Lemma 3.1.9 determines equilibrium dynamics. That is, for $\mathbf{k} \in \mathbb{K}$ it is $\mathcal{K}(\mathbf{k}) := (\bar{A}\omega_{-0}(\mathbf{k})) \in \mathbb{K}$ holds as well.*

The implementation so far stresses that the two-period economy is very special. By construction almost all difficulties in the model vanish. With the introduction of more periods intuitively it became clear that one of the central steps, concerning the properties of the equilibrium dynamics, is from the two- to the three-period economy. Thus, the three-period economy is the topic of the following section.

## 3.2 The Three-Period Economy

The previous section presented the benchmark economy with an arbitrary number of lifetime periods. Now, the focus is on the three-period economy, where agents live for exactly three periods, i.e., $I = 2$, $\mathbb{I} = \{0, 1, 2\}$. In the following, the generations will often be called 'old' ($i = 0$), 'middle-aged' ($i = 1$) and 'young' ($i = 2$). As we have mentioned before the structural generalization of the two-period economy starts with three periods. This economy shows changes and allows interpreting them very nicely. Obviously, the dimension of the equilibrium dynamics changes in comparison to the two-period economy. But as Section 3.1.1 has shown there are some more effects in the multi-period economy that do not occur in the two-period economy. Do these arise already in the three-period economy?
We already have stated that the labor profile has a great impact on the equilibrium and thus, this section also investigates how the equilibrium properties change with respect to structural changes of the labor profile. In the three-period economy three cases are considered. First there are the labor profiles where the old are retired. As already mentioned in Section 3.1.2 here two possibilities are worth to consider separately: Labor is only supplied by the young as it is often the case in the two-period economy and labor supply is divided up between the young and the middle-aged. The third case is the general economy with an arbitrary labor profile.

### 3.2.1 Case I: Young Agents supply Total Labor

If only the young supply labor, the labor profile is $\ell = (0,0,1)$. Thus, the middle-aged and the old do not have labor income and therefore, their wealth only consists of capital income:

$$\begin{aligned}\omega^2(\mathbf{k}) =& (1-\alpha)(k^A)^\alpha \\ \omega^1(\mathbf{k}) =& \alpha k^1 (k^A)^{\alpha-1} \\ \omega^0(\mathbf{k}) =& \alpha k^0 (k^A)^{\alpha-1}\end{aligned}$$

By the results of Section 3.1.2, the model's dynamics is given by

$$\mathcal{K}(\mathbf{k}) = \left(\frac{\beta}{1+\beta}\omega^1(\mathbf{k}), \frac{\beta(1+\beta)}{(1+\beta+\beta^2)}\omega^2(\mathbf{k})\right). \tag{3.8}$$

Section 3.1.2 has already given an interpretation and description of the dynamics.
Knowing the dynamics, the next step is to derive its properties. Even the existence and uniqueness of a steady state could be derived for the arbitrary number of lifetime periods. Here, because of the reduced number of lifetime periods the unique steady state can be determined explicitly.

**Lemma 3.2.1.** *The dynamical system defined in (3.8) has a unique steady state $\bar{\mathbf{k}} \in \mathbb{K}$. In particular it is*

$$\bar{\mathbf{k}} = (\bar{k}^0, \bar{k}^1) = \left(\frac{\beta^2}{1+\beta+\beta^2}(1-\alpha)(\bar{k}^A)^\alpha \alpha (\bar{k}^A)^{\alpha-1}, \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha)(\bar{k}^A)^\alpha\right)$$

*with* $\bar{k}^A = \left(\beta\frac{(1+\beta)(1-\alpha)+\sqrt{(1-\alpha)((1+\beta)^2(1+2\alpha)+2\alpha(1+\beta^2))}}{2(1+\beta+\beta^2)}\right)^{\frac{1}{1-\alpha}}$.

In the two-period economy qualitatively there is no difference in the dynamics if there is old age labor supply or not (compare Appendix A). Already in the description of particular labor profiles in Section 3.1.2, it is obvious that this does not hold in the economy where agents live for more than two periods. So far there is no statement about steady states. First we will give a short overview over the economy where labor supply is divided up between the young and the middle-aged. Finally, the different results will be compared.

### 3.2.2 Case II: Retirement of the Old

This particular economy has an intuitive economic interpretation. In the two-period economy the length of one period is often assumed to be roughly 30 years. Increasing the number of periods naturally reduces their length if the total lifetime stays constant. In the three-period economy the interpretation is as follows: An agent enters the economy at the age of 20. She supplies labor for two periods, while the fraction of labor supply can change at the age of 40, until she is 60. In the last period she is retired and dies approximately at the age of 80. Thus, in that context the labor profile is given by $\ell = (0, \ell^1, 1-\ell^1)$. The results of Section 3.1.2 are shortly repeated here. Again, first have a look at the wealth functions. Given capital stock $\mathbf{k} \in \mathbb{K}$, $\mathbb{K}$ is defined in Section 3.1.1, they are given by

$$\begin{aligned}
\omega^2(\mathbf{k}) =& (1-\alpha)(1-\ell^1)(k^A)^\alpha \\
\omega^1(\mathbf{k}) =& \ell^1(1-\alpha)(k^A)^\alpha + \alpha k^1 (k^A)^{\alpha-1} \\
\omega^0(\mathbf{k}) =& \alpha k^0 (k^A)^{\alpha-1}.
\end{aligned}$$

Compared to the previous subsection, now, there is one generation, the middle-aged, that gets both capital income and labor income. Thus, the main difference to the previous subsection is that in the definition of the state space, negative capital supply for the middle-aged is possible as the only restriction is positive wealth.

Recall the results of Section 3.1.2, the dynamics is given by

$$\mathcal{K}(\mathbf{k}) = \left( \frac{\beta}{1+\beta}\omega^1(\mathbf{k}), \frac{\alpha\beta(1+\beta)\omega^2(\mathbf{k}) - \ell^1(1-\alpha)\frac{\beta}{1+\beta}\omega^1(\mathbf{k})}{\alpha(1+\beta+\beta^2) + \ell^1(1-\alpha)} \right). \tag{3.9}$$

In this, compared to Section 3.1.1, less complex economy the results discussed before are even more clear. As the old still have no labor income the decision of the middle-aged is not affected compared to the first case (Section 3.2.1). But the decision of the young changes significantly because of the additional labor income. The occurrence of cross-wealth effects shows this change. Unsurprisingly, the coefficient of wealth of the young in the equilibrium dynamics changes compared to the capital evolution in the first case. Moreover, the coefficients of both wealth functions, $\omega^1$ and $\omega^2$, depend on the labor profile. This property again stresses the importance of the labor profile. It not only affects wealth but also the linear factor. Increasing labor supply of the middle-aged decreases the investment decision of the young. As wealth of the young gets smaller and wealth of the middle-aged increases this property coincides with intuition.

Section 3.1.2 has already shown the feasibility for the economy with an arbitrary number of lifetime periods. Similar to the previous subsection, the dynamics has a unique steady state that can be defined explicitly. That is shown in the following lemma.

**Lemma 3.2.2.** *The dynamics defined in 3.9 has a unique steady state $\bar{k}$, namely:*

$$\bar{\mathbf{k}} = \left( \frac{\beta}{1+\beta}\omega^1(\bar{\mathbf{k}}), \frac{\alpha\beta(1+\beta)\omega^2(\bar{\mathbf{k}}) - \ell^1(1-\alpha)\frac{\beta}{1+\beta}\omega^1(\bar{\mathbf{k}})}{\alpha(1+\beta+\beta^2) + \ell^1(1-\alpha)} \right) \tag{3.10}$$

*with*

$$\omega^1(\bar{\mathbf{k}}) = \left(\bar{k}^A\right)^\alpha \left(\ell^1(1-\alpha) + \alpha\frac{\alpha\beta(1+\beta)(1-\ell^1)(1-\alpha)(\bar{k}^A)^{\alpha-1} - \ell^1(1-\alpha)}{\alpha(1+\beta+\beta^2)}\right)$$

$$\omega^2(\bar{\mathbf{k}}) = (1-\alpha)(1-\ell^1)\left(\bar{k}^A\right)^\alpha$$

$$\bar{k}^A = \left(\frac{\alpha\beta(1+\beta-\ell^1)(1-\alpha)}{2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)} + \sqrt{\left(\frac{\alpha\beta(1+\beta-\ell^1)(1-\alpha)}{2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)}\right)^2 + \frac{\alpha^2\beta^2(1-\ell^1)(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}}\right)^{\frac{1}{1-\alpha}}.$$

Obviously, the dynamics and the steady state are more complex here than in the previous section. Unsurprisingly, this evolution continues when there are no restrictions on the labor profile.

### 3.2.3 Case III: General Labor Profile

The previous two subsections have considered particular three-period economies where equilibria can be determined explicitly. Now, turn to the general three period model where labor supply is divided up between all three generations $\ell = (\ell^0, \ell^1, \ell^2)$. Here, an explicit computation of the equilibrium is no longer possible. Note that this is the smallest economy with this property by the results of Sections 3.2.1 and 3.2.2. Given $\mathbf{k} \in \mathbb{K}$, where $\mathbb{K} \subset \mathbb{R}^2$ needs to be specified later, the wealth functions are given by

$$\begin{aligned}
\omega^2(\mathbf{k}) &= \ell^2(1-\alpha)(k^A)^\alpha \\
\omega^1(\mathbf{k}) &= \ell^1(1-\alpha)(k^A)^\alpha + \alpha k^1(k^A)^{\alpha-1} \\
\omega^0(\mathbf{k}) &= \ell^0(1-\alpha)(k^A)^\alpha + \alpha k^0(k^A)^{\alpha-1}
\end{aligned}$$

as Section 3.1.1 has already shown. Note that here two generations have both labor and capital income and thus, negative capital supply is possible for both the middle-aged and the old generation.

Now, specify the conditions on the equilibrium dynamics that have been derived in Section 3.1.1. The operator $\phi$ with $\phi(A) = \tilde{A}$ is defined by the implicit function $B_A\tilde{a} = b$ with $b = (-\beta\alpha, 0, 0, -\beta\alpha)^T$ and $B_A \in \mathbb{R}^{4\times 4}$,

$$B_A := \begin{pmatrix} b_{00} & b_{01} & 0 & 0 \\ b_{10} & b_{11} & 0 & 0 \\ 0 & 0 & b_{00} & b_{01} \\ 0 & 0 & b_{10} & b_{11} \end{pmatrix}$$

where

$$\begin{aligned} b_{00} &= -(\alpha(1+\beta)+\ell^0(1-\alpha)) && (3.11a)\\ b_{01} &= -\ell^0(1-\alpha) && (3.11b)\\ b_{10} &= -\ell^1(1-\alpha)+\left((e_1^2)^\top A\ell\right)(1-\alpha) && (3.11c)\\ b_{11} &= -\ell^1(1-\alpha)+\left((e_1^2)^\top A\ell\right)(1-\alpha)+\alpha a_0^1-\alpha(1+\beta). && (3.11d)\end{aligned}$$

with $e_1^2 = (1,0)^\top$. Here we already make use of the fact that the wealth function of the old does not occur in the equilibrium dynamics (cf. Lemma 3.1.4). The solution of this system of equations defines $\tilde{A}$.

Recall the definition

$$\tilde{B_A} := \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

such that $B_A$ can be defined as $diag(\tilde{B_A}, \tilde{B_A})$. Applying Cramer's rule $\tilde{A}$ can be determined explicitly (see Section 3.4.1) and it is:

$$\begin{aligned} \tilde{a}_0^1 &= -\frac{\beta\alpha b_{11}}{b_{00}b_{11}-b_{01}b_{10}} \quad \wedge \quad \tilde{a}_1^1 = \frac{\beta\alpha b_{10}}{b_{00}b_{11}-b_{01}b_{10}} \\ \tilde{a}_0^2 &= \frac{\beta\alpha b_{01}}{b_{00}b_{11}-b_{01}b_{10}} \quad \wedge \quad \tilde{a}_1^2 = -\frac{\beta\alpha b_{00}}{b_{00}b_{11}-b_{01}b_{10}} \end{aligned} \tag{3.12}$$

It is worth to note that all entries of the image $\tilde{A}$ only depend on $a_0^1$ and $a_0^2$, because the entries of $B_A$ only depend on them. Thus, in search of a fixed point $\bar{A}$ of the operator $\phi$ it is sufficient to find $\bar{a}_0^1$ and $\bar{a}_0^2$. Then, the other entries of $\bar{A}$ of $\phi$ are uniquely determined by (3.12).

On closer inspection of the coefficients in (3.12) it is noticeable that $\tilde{a}_0^2$ and $\tilde{a}_1^2$ have different signs as long as $\tilde{a}_0^2 \neq 0$. Both $b_{00}$ and $b_{01}$ are independent of the matrix $A$ and negative. So if $\det \tilde{B}_A > 0$, it is $\tilde{a}_0^2 < 0$ and $\tilde{a}_1^2 > 0$ and if $\det \tilde{B}_A < 0$, it is the other way around.

To obtain a well-defined operator $\phi$ it is necessary to restrict $\mathbb{R}^{2\times 2}$ such that Assumption 3.1.1 holds. Define $\tilde{\mathbb{A}} := \{A \in \mathbb{R}^{2\times 2}\,|A \text{ satisfies Assumption 3.1.1}\}$. If $A \in \tilde{\mathbb{A}}$ holds, still the problem is that the image, $\phi(A)$, is not necessarily an element of $\tilde{\mathbb{A}}$ as it has already been stated prior to Assumption 3.1.1. As the objective is to find fixed points of $\phi$, it is sufficient to consider the restricted set as the superset of the fixed point set $\bar{\mathbb{A}} := \{A \in \mathbb{R}^{2\times 2}\,|\phi(A) = A\}$. The existence of a fixed point will be derived as follows: The idea is to restrict the function space by restricting $\tilde{\mathbb{A}}$ to a subset $\mathbb{A}$. The set $\mathbb{A} \subset \tilde{\mathbb{A}}$ needs to guarantee that there is a fixed point on $\mathbb{A}$ and that $\phi$ maps $\mathbb{A}$ into itself. The following lemma represents the result.

**Lemma 3.2.3.** *There is a fixed point on* $\mathbb{A} := \{A \in \mathbb{R}^{2\times 2} | a_0^1 \leq 1 \wedge a_0^2 \leq 0\}$.

**Corollary 3.2.1.** *Any fixed point $\bar{A}$ described in Lemma 3.2.3 satisfies $\bar{a}_1^1, \bar{a}_0^2 \in [-1, 0]$ and $\bar{a}_0^1, \bar{a}_1^2 \in [0, 1]$.*

Note that the set $\mathbb{A}$ defined in Lemma 3.2.3 contains the coefficient matrices of the equilibria in the two previous subsections. Moreover, it coincides with the guess in Section 3.1.1 about the sign of the coefficients. Together with Corollary 3.2.1 it becomes obvious that the coefficients determine a fraction of wealth that is invested. In particular for the positive coefficients this means that investment may not exceed wealth what coincides with the requirements of the budget constraints (2.4). The definition of $\mathbb{A}$ implies the exact definition of the function space $\mathscr{K}$:

$$\mathscr{K} = \{\mathcal{K} : \mathbb{K} \to \mathbb{K} \,|\, \mathcal{K}(\mathbf{k}) = A\omega_{-0}(\mathbf{k}),\ A \in \mathbb{A}\}$$

In the beginning it has been stated that fixed points of the operator are equilibrium candidates. It remains to show that the dynamics maps the state space into itself. The state space will be defined in dependance of the (general) coefficients $a_i^j$ in the function space $\mathscr{K}$, using the critical values for $\omega^j(\mathbf{k})$ defined in Lemma 2.5.2. So the state space is defined as:

$$\mathbb{K} := \left\{ \mathbf{k} = (k^0, k^1) \in \mathbb{R}^2 \,\middle|\, k^0 + k^1 > 0,\ \omega^0(\mathbf{k}) > 0,\ \omega^1(\mathbf{k}) > -\frac{\ell^0(1-\alpha)\beta}{-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)}\omega^2(\mathbf{k}) \right\}$$

Obviously, the basic state space $\mathbb{K}_0$, when the lower bound is equal to 0 for all $\omega^i(\mathbf{k})$, is a subset of $\mathbb{K}$ as required.

The definition of the state space is sufficient as the condition on $\omega^1(\mathbf{k})$ can be rewritten such that the lower bound is independent of $k$:

$$\begin{aligned} & \omega^1(\mathbf{k}) > -\frac{\ell^0(1-\alpha)\beta}{-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)} \\ \Leftrightarrow \quad & \frac{k^1}{k^A} > -\frac{\ell^0\ell^2(1-\alpha)^2\beta}{\alpha\left(-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)\right)} - \frac{\ell^1(1-\alpha)}{\alpha} \end{aligned}$$

**Definition 3.2.1.** *A dynamical system is called* feasible *if it maps the state space into itself.*

Following the definition of feasibility 3.2.1 it is obvious that a feasible equilibrium candidate is an equilibrium. In the following, we present a criterion for feasibility, such that some equilibrium candidates may be excluded as equilibrium.

**Lemma 3.2.4.** *A feasible dynamics $\mathcal{K} \in \tilde{\mathscr{K}}$ satisfies $\sum_{i=0}^{I} a_i^j > 0 \quad \forall j = 1, ..., I$.*

The result of Lemma 3.2.4 might be surprising but the problem is that the wealth of the old is not included in capital evolution (see Lemma 3.1.4). Thus, capital supply can always be chosen such that aggregated capital is positive by increasing $k^0$ without an impact on future capital supply. Hence, if the sum of coefficients would be negative for one $j = 1, ..., I$ it is possible to choose initial capital in the state space such that $\omega^j$ is quite large and $\omega^i$, $i \neq j$ are very small and the choice of $k^0$ guarantees that aggregated capital is greater than zero. Finally, it may happen, that aggregated capital supply in the upcoming periods gets negative. The positive sum of coefficients has another implication.

**Corollary 3.2.2.** *A feasible dynamics* $\mathcal{K} = A\omega_{-0}(\mathbf{k})$ *in the three-period economy satisfies* $\det A > 0$.

Now, the state space is defined and we derived feasibility criteria. Thus, it can be shown that the equilibrium candidate is an equilibrium.

**Lemma 3.2.5.** *Any fixed point of* $\phi$ *on the set* $\mathbb{A}$ *generates a feasible dynamics.*

Lemma 3.2.5 together with Lemma 3.2.3 states an existence result for the general three-period economy. By the structure or the manageable dimension of the three-period economy, respectively, it is possible to state a uniqueness result as well:

**Lemma 3.2.6.** *The fixed point described in Lemma 3.2.3 is the only equilibrium of the structure* $\mathcal{K}(k) = A\omega_{-0}(\mathbf{k})$. *In general, there are three equilibrium candidates.*

Finally, the results so far are summarized: In the three-period economy there is a unique Markov equilibrium that is a linear function of wealth. As long as the old generation is retired equilibria can be determined explicitly and constitute the unique equilibrium without an assumption on the equilibrium structure or the function space.
As in the previous sections a final result is the existence of a unique steady state in the general three-period economy.

**Lemma 3.2.7.** *The equilibrium dynamics* $\mathcal{K}(k) = \bar{A}\omega_{-0}(\mathbf{k})$ *with* $\bar{A} \in \mathbb{A}$ *has a unique steady state.*

The derivation of an equilibrium is now finished. The question how it differs structurally from the equilibrium of the corresponding two-period economy is the topic of the next section.

## 3.3 Comparison of Two-Period and Three-Period Scenario

This section compares the three-period economies considered in Section 3.2 to the two-period economy with old age labor supply. There, capital evolution is denoted by $\mathcal{K}_2(\mathbf{k})$ and given by

$$\mathcal{K}_2(\mathbf{k}) = \frac{\alpha\beta}{\alpha(1+\beta) + \ell^0(1-\alpha)}\omega^1(\mathbf{k})$$

as it is shown in Appendix A. The three-period economy's capital evolution is given by

$$\mathcal{K}_3(\mathbf{k}) = \left(a_0^1\omega^1(\mathbf{k}) + a_0^2\omega^2(\mathbf{k}), a_1^1\omega^1(\mathbf{k}) + a_1^2\omega^2(\mathbf{k})\right)$$

where the coefficients $a_i^j,\ i = 0, 1,\ j = 1, 2$ highly depend on the labor profile. Assuming $\ell = (0, 0, 1)$ capital evolution of the three period model structurally does not differ from that of the two-period model. The coefficients $a_0^2$ and $a_1^1$ are zero. Thus, in both cases future capital supply is a constant fraction of the own wealth, i.e., $0 < a_i^{i+1} < 1,\ i = 0, 1$.
Distributing labor supply among the young and the middle-aged in the three-period economy, it is obvious that capital evolution changes structurally. In contrast to the two-period economy introducing labor supply of one more generation has a great impact. In this economy it is $a_0^2 = 0,\ 0 < a_i^{i+1} < 1,\ i = 0, 1,\ -1 < a_1^1 < 0$. Thus, while the old again supply a constant fraction of wealth when middle-aged, cross wealth effects occur in capital supply of the middle-aged. By the property that wealth of the currently old has no influence on building capital in the two-period model there is no possibility to obtain effects like this even if the number of working generations is equal. Moreover, even in this very simple extension of the two-period economy where the equilibrium is determined explicitly negative savings occur.
Extending the economy to an arbitrary labor profile emphasizes the findings of the previous observations. Here cross-wealth effects occur in both capital supply functions. Structurally, again own wealth is evaluated positively and the other generations wealth negatively, i.e. $0 < a_i^{i+1} < 1,\ i = 0, 1, a_0^2, a_1^1 < 0$ , as it has been assumed.
The existence of a unique steady state is guaranteed in all economies. Thus, there is no difference so far. The next chapter considers the properties of the dynamics of these economies.

## 3.4 Mathematical Appendix

### 3.4.1 Applying Cramer's rule in the Three-Period Economy

Cramer's rule tells that if the solution of a linear system of equations $Ax = b$ is unique the solution is given by $x_i = \frac{detA_i}{detA}$ where $A_i$ is equal to $A$ except the $i - th$ column that is replaced by $b$.
Thus, here the first column of $B$ is replaced by $b$ (3.6).

$$\tilde{a}_0^0 = \frac{1}{(x_1y_2 - x_2y_1)^3} \begin{vmatrix} 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 & 0 & 0 \\ -\beta\alpha & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & y_1 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 \\ -\beta\alpha & 0 & 0 & 0 & y_1 & y_2 \end{vmatrix} = 0$$

The determinant is equal to zero, as the first and the second row are multiples of each other. The determination of $\tilde{a}_1^0$ leads to the same result. Thus, next consider another example with a non-vanishing determinant:

$$\begin{aligned} \tilde{a}_0^1 =& \frac{1}{(x_1y_2 - x_2y_1)^3} \begin{vmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta\alpha & x_2 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 \\ -\beta\alpha & 0 & -\beta\alpha & 0 & y_1 & y_2 \end{vmatrix} \\ =& \frac{1}{(x_1y_2 - x_2y_1)^3} \left( \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)^2 \begin{vmatrix} -\beta\alpha & x_2 \\ 0 & y_2 \end{vmatrix} \\ =& \frac{-\beta\alpha y_2 (x_1y_2 - x_2y_1)^2}{(x_1y_2 - x_2y_1)^3} \\ =& -\frac{\beta\alpha y_2}{(x_1y_2 - x_2y_1)} \end{aligned}$$

The other components of $\tilde{A}$ can be computed analogously.

### 3.4.2 Proofs of Chapter 3

***Proof of Lemma 3.1.1.*** Let $\hat{\mathcal{K}(\mathbf{k})} = M\omega(\mathbf{k})$. The idea of the proof is as follows: Show that each $k_1^i$ defined by the equilibrium equations is a linear combination of $\omega^j(\mathbf{k}), j = 0, ..., I$ and $k_1^j,\ I - 1 \geq j > i$. Then, it can be concluded that $k_1^{I-1}$ is a linear combination of the wealth functions as there is no capital supply with greater upper index.
As $k_1^{I-2}$ is a linear combination of the wealth functions and $k_1^{I-1}$ it follows that it is a linear combination of the wealth functions as well. This can be continued up to $k_1^0$ and thus the lemma is proven. The equilibrium equations (3.4) under the assumption are

$$\beta\alpha(k_1^A)^{\alpha-1}(\omega^1(\mathbf{k}) - k_1^1) - \omega^0(\mathbf{k}_1) = 0$$
$$\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+1}(k) - k_1^i) - \omega^i(\mathbf{k}_1) + \sum_{j=0}^{I} a_i^j \omega^j(\mathbf{k}) = 0$$
$$i = 1, ..., I-1$$

$$\Leftrightarrow \quad \beta\alpha(\omega^1(\mathbf{k}) - k_1^1) - \ell^0(1-\alpha)k^A - \alpha k_1^0 = 0 \quad (3.13a)$$
$$\beta\alpha(\omega^{i+1}(\mathbf{k}) - k_1^i) - \ell^i(1-\alpha)k^A - \alpha k_1^i + \sum_{j=0}^{I} a_i^j(\ell^j(1-\alpha)k^A + \alpha k_1^j) = 0 \quad (3.13b)$$
$$i = 1, ..., I-1$$

Thus, from the first equation (3.13a) it follows

$$k_1^0 = \frac{\alpha\beta\omega^1(\mathbf{k}) - \ell^0(1-\alpha)\sum_{j=1}^{I-1} k_1^j}{\alpha(1+\beta) + \ell^0(1-\alpha)}$$

Obviously, $k_1^0$ is a linear combination of $\omega^1(\mathbf{k})$ and $k_1^j$, $j > 0$. The result follows by induction. Assume that the conjecture holds for all $k_1^j$ with $j < i$. From the i-th equilibrium equation (3.13b) it follows that

$$\begin{aligned} & k_1^i(-\alpha(1+\beta) - \ell^i(1-\alpha) + a_i^i(\ell^i(1-\alpha)+\alpha)) \\ &= \ell^i(1-\alpha)(\sum_{n<i} k_1^n + \sum_{n>i} k_1^n) - \sum_{n<i} k_1^n(\alpha + \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha))) \\ &\quad + \sum_{n>i} k_1^n(\alpha + \sum_{l=0}^{I}(a_n^l \ell^l(1-\alpha))) - \omega^{i+1}(\mathbf{k}) \end{aligned}$$

Substituting step by step all $k_1^j$ with $j < i$ in the end there is a linear combination $\omega^j(\mathbf{k})$, $j = 0, ..., I$ and $k_1^j$, $j > i$. The substitution is exemplary done only for $k_1^0$ here:

$$\begin{aligned} & k_1^i(-\alpha(1+\beta) - \ell^i(1-\alpha) + a_i^i(\ell^i(1-\alpha)+\alpha)) \\ &= \sum_{0<n<i} k_1^n(\ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha))) + \sum_{n>i} k_1^n(\ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha))) \\ &\quad - \frac{\alpha\beta\omega^1(\mathbf{k}) - \ell^0(1-\alpha)\sum_{j=1}^{I-1} k_1^j}{\alpha(1+\beta) + \ell^0(1-\alpha)}(\ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha))) - \omega^{i+1}(\mathbf{k}) \end{aligned}$$

$\Leftrightarrow$

$$k_1^i \left( -\alpha(1+\beta) - \ell^i(1-\alpha) + a_i^i(\ell^i(1-\alpha)+\alpha) + \frac{\ell^0(1-\alpha)(\ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha)))}{\ell^0(1-\alpha) + \alpha(1+\beta)} \right)$$

$$= \sum_{0<n<i} k_1^n \left( \ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha)) \right) + \sum_{n>i} k_1^n \left( \ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha)) \right)$$
$$- \frac{\alpha\beta\omega^1(\mathbf{k}) - \ell^0(1-\alpha)\left( \sum_{1<n<i} k_1^n + \sum_{n>i} k_1^n \right)}{\alpha(1+\beta) + \ell^0(1-\alpha)} (\ell^i(1-\alpha) - \alpha - \sum_{l=0}^{I}(a_i^n \ell^l(1-\alpha))) - \omega^{i+1}(\mathbf{k})$$

The other substitutions are completely analogous, such that in the end all $k_1^j,\ j = 0, ..., I-1$ are proven to be linear combinations of the wealth functions $\omega^i(\mathbf{k}), i = 0, ..., I$.

□

***Proof of Lemma 3.1.3***. As seen in Lemma 3.1.2 the system of equations has exactly one solution for $\det(\tilde{B_A}) \neq 0$.
Now, assume $\det(\tilde{B_A}) = 0$. It is a well-known result of analytical geometry that if the rows in a matrix are linear independent, the determinant is different from 0. Negating that result implies that the rows of $\tilde{B_A}$, noted as $\tilde{b}_i,\ i = 0, ..., I-1$ here, are linear dependent. That is, there are constants $x_i, i = 0, ..., I-1,\ x_i \neq 0$ for at least one $i$ such that $\sum_{i=0}^{I-1} x_i \tilde{b}_i = 0$ or alternatively spoken there is at least one row that can be written as a linear combination of the others.
Looking at the linear system of equations:

$$B_A \tilde{a} = b \quad \Leftrightarrow \quad \begin{array}{rcl} \tilde{B_A}\tilde{a}^0 & = & 0 \\ \tilde{B_A}\tilde{a}^1 & = & -\beta\alpha e_1 \\ & \vdots & \\ \tilde{B_A}\tilde{a}^I & = & -\beta\alpha e_I \end{array}$$

where $e_i \in \mathbb{R}^I$ denotes the $i-th$ unit vector.
Without loss of generality assume $\tilde{b}_0 = \sum_{n=1}^{I-1} \kappa_n \tilde{b}_n$. Look at the equations $\tilde{B_A}\tilde{a}^1 = -\beta\alpha e_1$.
Obviously, for $n = 1, ..., I-1$ $\tilde{b}_n \tilde{a}^1 = 0$ holds.
Then, for the first equation it is

$$\tilde{b}_0 \tilde{a}^1 = \sum_{n=1}^{I-1} \kappa_n \tilde{b}_n \tilde{a}^1 = 0 \neq -\beta\alpha.$$

That leads to a contradiction. Therefore there is no solution if $\det(\tilde{B_A}) = 0$. □

***Proof of Lemma 3.1.5***. Assume $N > 0$. Otherwise the statement is trivial.
First look at $\ell^0 = 0$. This implies $b_0 = -\ell^0(1-\alpha) = 0$, defined in (3.7) , as $a^i_{-1}$ is defined to be zero. Therefore,

$$b_{00} = -\alpha(1+\beta) \qquad b_{0j} = 0, j = 1, ..., I-1$$

The values $\tilde{a}^i_0, i = 1, ..., I-1$ can be determined by Cramer's rule again. By the argumentation about computing the determinant of the matrix where one column of $B_A$ is replaced by $b$ it follows:

$$\tilde{a}^i_0 = \frac{det((\tilde{B_A})_{1i})}{det(\tilde{B_A})}$$

Taking into account that $b_{0j} = 0, j = 1, ..., I-1$, $det((\tilde{B_A})_{1i}) = 0, i = 2, ..., I$ as the first row is the 0-vector. So the only value that is different from 0 is $\tilde{a}^1_0$. Its value can be computed directly as well:

$$\tilde{a}^1_0 = \frac{det((\tilde{B_A})_{11})}{det(\tilde{B_A})} = \frac{\begin{vmatrix} -\beta\alpha & 0 & \dots & 0 \\ 0 & b_{11} & \dots & b_{1(I-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{(I-1)1} & \dots & b_{(I-1)(I-1)} \end{vmatrix}}{\begin{vmatrix} -\alpha(1+\beta) & 0 & \dots & 0 \\ b_{01} & b_{11} & \dots & b_{1(I-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(I-1)1} & b_{(I-1)1} & \dots & b_{(I-1)(I-1)} \end{vmatrix}}$$

$$= \frac{-\beta\alpha \begin{vmatrix} b_{11} & \dots & b_{1(I-1)} \\ \vdots & \ddots & \vdots \\ b_{(I-1)1} & \dots & b_{(I-1)(I-1)} \end{vmatrix}}{-\alpha(1+\beta) \begin{vmatrix} b_{11} & \dots & b_{1(I-1)} \\ \vdots & \ddots & \vdots \\ b_{(I-1)1} & \dots & b_{(I-1)(I-1)} \end{vmatrix}}$$

$$= \frac{\beta}{1+\beta}$$

So for $\ell^0 = 0$ the restriction of the set of fixed point candidates is verified.
Proceed by induction. Assume that the conjecture holds for $m-1 \leq n < N$. Then, it is

$$b_m = -\ell^m(1-\alpha) + (1-\alpha)\sum_{j=0}^{I} a^j_{m-1}\ell^j = (1-\alpha)\sum_{j=N}^{I} a^j_{m-1}\ell^j \overset{j \geq N \geq n \geq m}{=} 0.$$

This implies

$$b_{mm} = \alpha(a^m_{m-1} - 1 - \beta) \qquad b_{mj} = 0, j \in \mathbb{I} \backslash \{m, I\}.$$

As before the entries of $\tilde{A}$ are given by

$$\tilde{a}^i_m = \frac{det((\tilde{B_A})_{(m+1)i})}{det(\tilde{B_A})}$$

and $det((\tilde{B_A})_{(m+1)i}) = 0$, with $i = 1, ..., m, m+2, ..., I$ by the same argumentation (0-row in row $m+1$) as above. Thus, it follows that

$$\tilde{a}^i_m = 0 \quad \forall i = 1, ..., m, m+2, ..., I.$$

If the conjecture holds for all values $0, ..., m-1$, again $\tilde{a}^{m+1}_m$ can be computed directly:

$$\tilde{a}^{m+1}_m = \frac{det((\tilde{B_A})_{(m+1)(m+1)})}{det(\tilde{B_A})}$$

$$= \frac{\begin{vmatrix} b_{00} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & b_{(m-1)(m-1)} & 0 & 0 & \dots & 0 \\ 0 & & \dots & 0 & -\beta\alpha & 0 & \dots & 0 \\ b_{(m+1)0} & & \dots & b_{(m+1)(m-1)} & 0 & b_{(m+1)(m+1)} & \dots & b_{(m+1)(I-1)} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{(I-1)0} & & \dots & b_{(I-1)(m-1)} & 0 & b_{(I-1)(m+1)} & \dots & b_{(I-1)(I-1)} \end{vmatrix}}{\begin{vmatrix} b_{00} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & b_{mm} & 0 & \dots & 0 \\ b_{(m+1)0} & \dots & & \dots & & \dots & b_{(m+1)(I-1)} \\ \vdots & & \ddots & & \ddots & & \vdots \\ b_{(I-1)0} & \dots & & \dots & & \dots & b_{(I-1)(I-1)} \end{vmatrix}}$$

$$= \frac{-\beta\alpha \prod_{l=0}^{m-1} b_{ll} \begin{vmatrix} b_{(m+1)(m+1)} & \cdots & b_{(m+1)(I-1)} \\ \vdots & \ddots & \vdots \\ b_{(I-1)m+1} & \cdots & b_{(I-1)(I-1)} \end{vmatrix}}{\prod_{l=0}^{m} b_{ll} \begin{vmatrix} b_{(m+1)(m+1)} & \cdots & b_{(m+1)(I-1)} \\ \vdots & \ddots & \vdots \\ b_{(I-1)(m+1)} & \cdots & b_{(I-1)(I-1)} \end{vmatrix}}$$

$$= -\frac{\beta\alpha}{b_{mm}} = -\frac{\beta}{a_{m-1}^{m} - 1 - \beta}$$

What is left to show is the exact value for $\tilde{a}_i^{i+1}$.
Obviously,

$$\tilde{a}_0^1 = \frac{\beta}{1+\beta} = \frac{\beta \sum_{m=0}^{0} \beta^m}{\sum_{m=0}^{1} \beta^m}$$

satisfies the formula. Again, we will proceed by induction. Assume the conjecture holds for $\tilde{a}_{i-1}^i$. Then, it is shown above that

$$\begin{aligned}
\tilde{a}_i^{i+1} &= -\frac{\beta}{a_{i-1}^{i} - 1 - \beta} \\
&= -\frac{\beta}{\frac{\beta \sum_{m=0}^{i-1} \beta^m}{\sum_{m=0}^{i} \beta^m} - 1 - \beta} \\
&= -\frac{\beta \sum_{m=0}^{i} \beta^m}{\underbrace{\beta \sum_{m=0}^{i-1} \beta^m - \sum_{m=0}^{i} \beta^m}_{=-1} - \beta \sum_{m=0}^{i} \beta^m} \\
&= \frac{\beta \sum_{m=0}^{i} \beta^m}{\sum_{m=0}^{i+1} \beta^m}
\end{aligned}$$

□

***Proof of Lemma 3.1.6.*** Recall the wealth functions and the equilibrium equations 3.4:

$$\begin{aligned}\omega^I(\mathbf{k}) =&(1-\alpha)(k^A)^\alpha \\ \omega^i(\mathbf{k}) =&\alpha k^1 (k^A)^{\alpha-1} \quad i=0,...,I-1.\end{aligned}$$

and

$$\beta\alpha(k_1^A)^{\alpha-1}(\omega^1(\mathbf{k})-k_1^0)-\omega^0(\mathbf{k}_1)=0 \tag{3.14a}$$

$$\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+1}(k)-k_1^i)-\Big(\omega^i(\mathbf{k}_1)-\hat{\mathcal{K}}^{i-1}(\mathbf{k}_1)\Big)=0 \quad i=1,...,I-1 \tag{3.14b}$$

From the first equilibrium equation (3.14a) it follows that $k_1^0=\frac{\beta}{1+\beta}\omega^1(\mathbf{k})$ and thus, as the solution must hold for any period, that $\mathcal{K}^0(\mathbf{k})=\frac{\beta}{1+\beta}\omega^1(\mathbf{k})$ .
Then, for $i=1$ in (3.14b) it is:

$$\begin{aligned}&\beta\alpha(k_1^A)^{\alpha-1}(\omega^2(\mathbf{k})-k_1^1)-\Big(\omega^1(\mathbf{k}_1)-\hat{K}^0(\mathbf{k}_1)\Big)=0\\ \Leftrightarrow\quad&\beta\alpha(k_1^A)^{\alpha-1}(\omega^2(\mathbf{k})-k_1^1)=\omega^1(\mathbf{k}_1)-\frac{\beta}{1+\beta}\omega^1(\mathbf{k}_1)\\ \Leftrightarrow\quad&\beta\alpha(k_1^A)^{\alpha-1}(\omega^2(\mathbf{k})-k_1^1)=\frac{1}{1+\beta}k_1^1\alpha(k_1^A)^{\alpha-1}\\ \Leftrightarrow\quad&k_1^1=\frac{\beta(1+\beta)}{1+\beta+\beta^2}\omega^2(\mathbf{k})\end{aligned}$$

Hence, by the same argumentation as above it follows that $\mathcal{K}^1(\mathbf{k})=\frac{\beta(1+\beta)}{1+\beta+\beta^2}\omega^2(\mathbf{k})$.

Now, proceed by induction. Assume that $\mathcal{K}^i(\mathbf{k})=\frac{\beta\sum_{j=0}^{i}\beta^j}{\sum_{j=0}^{i+1}\beta^j}\omega^{i+1}(\mathbf{k})$. Using the equilibrium equation (3.14b) for $i+1$, it is

$$\begin{aligned}&\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+2}(\mathbf{k})-k_1^{i+1})-\Big(\omega^{i+1}(\mathbf{k}_1)-\hat{K}^i(\mathbf{k}_1)\Big)=0\\ \Leftrightarrow\quad&\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+2}(\mathbf{k})-k_1^{i+1})=\omega^{i+1}(\mathbf{k}_1)-\frac{\beta\sum_{j=0}^{i}\beta^j}{\sum_{j=0}^{i+1}\beta^j}\omega^{i+1}(\mathbf{k}_1)\\ \Leftrightarrow\quad&\beta\alpha(k_1^A)^{\alpha-1}(\omega^{i+2}(\mathbf{k})-k_1^{i+1})=\frac{1}{\sum_{j=0}^{i+1}\beta^j}k_1^{i+1}\alpha(k_1^A)^{\alpha-1}\\ \Leftrightarrow\quad&\beta\omega^{i+2}(\mathbf{k})=\frac{1+\beta\sum_{j=0}^{i+1}\beta^j}{\sum_{j=0}^{i+1}\beta^j}k_1^{i+1}\\ \Leftrightarrow\quad&k_1^{i+1}=\frac{\beta\sum_{j=0}^{i+1}\beta^j}{\sum_{j=0}^{i+2}\beta^j}\omega^{i+2}(\mathbf{k})\end{aligned}$$

Thus, for each $i \in \mathbb{I}\setminus\{I\}$, it is $\mathcal{K}^i(\mathbf{k}) = \frac{\beta \sum_{j=0}^{i} \beta^j}{\sum_{j=0}^{i+1} \beta^j} \omega^{i+1}(\mathbf{k})$. In this proof, the assumption that the equilibrium dynamics is a linear function of wealth was not used. The equilibrium equations (3.14a) and (3.14b) could directly be solved for individual capital supply with some kind of backwards induction. Therefore, there is no other equilibrium.

□

***Proof of Lemma 3.1.8***. First, for purposes of a simple notation define $B_n^m := \sum_{j=n}^{m} \beta^j$. A steady state of $\mathcal{K}(\mathbf{k})$ must satisfy

$$\bar{k}^{I-1} = \frac{\beta B_0^{I-1}}{B_0^I}(1-\alpha)(\bar{k}^A)^\alpha$$
$$\bar{k}^i = \frac{\beta B_0^i}{B_0^{i+1}}\alpha(\bar{k}^A)^{\alpha-1}\bar{k}^{i+1} \quad i = 0, ..., I-2$$

by Lemma 3.1.6. Proceed by induction to show that each $\bar{k}^i,\ i \in \mathbb{I}\setminus\{I\}$ can be written in terms of $\bar{k}^A$.
Obviously, the conjecture is satisfied for $\bar{k}^{I-1}$ by definition. For $\bar{k}^{I-2}$ it is

$$\bar{k}^{I-2} = \alpha(\bar{k}^A)^{\alpha-1}\bar{k}^{I-1}\frac{\beta B_0^{I-2}}{B_0^{I-1}} = \alpha(\bar{k}^A)^{\alpha-1}\frac{\beta^2 B_0^{I-2}}{B_0^I}(1-\alpha)(\bar{k}^A)^\alpha.$$

Now, the conjecture is that

$$\bar{k}^{I-j} = \alpha^{j-1}\beta^j\frac{B_0^{I-j}}{B_0^I}(1-\alpha)(\bar{k}^A)^\alpha\left((\bar{k}^A)^{\alpha-1}\right)^{j-1},$$

which is satisfied for $j = 2$. Assume it is satisfied for $j$. Then, for $\bar{k}^{I-j-1}$ it is

$$\bar{k}^{I-j-1} = \alpha(\bar{k}^A)^{\alpha-1}\frac{\beta B_0^{I-j-1}}{B_0^{I-j}}\underbrace{\alpha^{j-1}\beta^j\frac{B_0^{I-j}}{B_0^I}(1-\alpha)(\bar{k}^A)^\alpha\left((\bar{k}^A)^{\alpha-1}\right)^{j-1}}_{=\bar{k}^{I-j}}$$
$$= \alpha^j\beta^{j+1}\frac{B_0^{I-j-1}}{B_0^I}(\bar{k}^A)^\alpha\left((\bar{k}^A)^{\alpha-1}\right)^j(1-\alpha).$$

As entries in the steady state vector only depend on the aggregated steady state value, it is sufficient to determine $\bar{k}^A$.

$$\bar{k}^A = \sum_{j=1}^{I}\bar{k}^{I-j} = (1-\alpha)\sum_{j=1}^{I}\alpha^{j-1}\beta^j\frac{B_0^{I-j}}{B_0^I}(\bar{k}^A)^\alpha\left((\bar{k}^A)^{\alpha-1}\right)^{j-1}$$
$$\Leftrightarrow \qquad \left(\bar{k}^A\right)^{1-\alpha} = \sum_{j=1}^{I}\bar{k}^{I-j} = (1-\alpha)\sum_{j=1}^{I}\alpha^{j-1}\beta^j\frac{B_0^{I-j}}{B_0^I}\left((\bar{k}^A)^{\alpha-1}\right)^{j-1}.$$

Substituting $x := \left(\bar{k}^A\right)^{1-\alpha}$ leads to

$$x = \sum_{j=1}^{I} \bar{k}^{I-j} = \sum_{j=1}^{I} \alpha^{j-1}\beta^j \frac{B_0^{I-j}}{B_0^I}(1-\alpha)x^{1-i}$$
$$\Leftrightarrow x^I - \sum_{j=1}^{I} \alpha^{j-1}\beta^j \frac{B_0^{I-j}}{B_0^I}(1-\alpha)x^{I-i} = 0.$$

The positive roots of the polynomial define the steady state values for $\bar{k}^A$. In the resubstitution it is necessary to extract a root. Thus, negative roots of the polynomial do not lead to a steady state value.
On closer inspection of the coefficients in the polynomial, it turns out that all coefficients are negative except the leading coefficient. By Descartes' rule of signs (cf. Struik [24]), it follows that if there is only one change of signs in the coefficients of a polynomial it has exactly one positive real root $x_0$.
Thus, there is a unique steady state determined by $\bar{k}^A = (x_0)^{\frac{1}{1-\alpha}}$. □

***Proof of Lemma 3.1.9***. First note that $\phi$ is well-defined on $\mathbb{A}$. Consider the structure of $\tilde{B_A} = (b_{mn})_{m,n=0,...,I-1}$. For $m = 0, ..., I-2$ it is

$$\begin{aligned} b_m &= -\ell^m(1-\alpha) + (1-\alpha)\sum_{j=0}^{I} a_{m-1}^j \ell^j \\ &= (1-\alpha)(\ell^{I-1}a_{m-1}^{I-1} + \ell^I a_{m-1}^I) \\ &= 0 \quad \text{ as } a_{m-1}^{I-1} = a_{m-1}^I = 0 \ \forall m = 0, ..., I-2 \end{aligned}$$

and thus

$$b_{mn} = \begin{cases} \alpha(a_{m-1}^m - 1 - \beta) & n = m \\ 0 & otherwise \end{cases}$$

$$\Rightarrow det(\tilde{B_A}) = \prod_{n=0}^{I-1} b_{nn} \neq 0.$$

Now, check if $\phi$ maps $\mathbb{A}$ into $\mathbb{A}$ By Lemma 3.1.5 given $A \in \mathbb{A}$ it is

$$\begin{aligned} &\tilde{a}_i^0 = 0 && && i = 0, ..., I-1 \\ &\tilde{a}_i^j = 0 && j \neq i+1 && i = 0, ..., I-2\ ; j = 0, ..., I \\ &\tilde{a}_i^{i+1} = \frac{\beta \sum_{m=0}^{i} \beta^m}{\sum_{m=0}^{i+1} \beta^m} && && i = 0, ..., I-2 \end{aligned}$$

Therefore the coefficients that are left to determine are

$$a_{I-1}^{j} \quad j = 1, ..., I.$$

Keeping the notation from above and using Cramer's rule it is

$$a_{I-1}^{j} = \frac{det(\tilde{B_A})_{Ij}}{det\tilde{B_A}},$$

where $det((\tilde{B_A})_{Ij})$ is given by

$$det((\tilde{B_A})_{Ij}) = \begin{cases} \beta\alpha \prod\limits_{\substack{n=0 \\ n\neq j-1}}^{I-2} b_{nn} \cdot b_{I-1j-1} & j = 1, ..., I-1 \\ -\beta\alpha \prod_{n=0}^{I-2} b_{nn} & j = I \end{cases}$$

The formula for $j = I$ is obvious. For $j < I$ it can be derived as follows. The last column of $\tilde{B_A}$ is replaced by the $j-th$ unit vector multiplied with $-\beta\alpha$. The first step is the Laplace expansion along this last column. Afterwards the last column and the $j-th$ row are crossed out, such that in column $j$ there remains only the entry in the last row $b_{I-1j-1}$ (it is $j-1$ as the rows are counted from 0 on):

$$det((\tilde{B_A})_{Ij}) = \begin{vmatrix} b_{00} & 0 & \cdots & & & & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ 0 & \ddots & b_{(j-2)(j-2)} & 0 & & & & 0 \\ \vdots & & \ddots & b_{(j-1)(j-1)} & \ddots & & & -\beta\alpha \\ \vdots & & & \ddots & b_{jj} & \ddots & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 & b_{(I-2)(I-2)} & 0 \\ b_{(I-1)0} & \cdots & & & & \cdots & b_{(I-1)(I-2)} & 0 \end{vmatrix}$$

$$= (-1)^{I+j}(-\beta\alpha) \begin{vmatrix} b_{00} & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & & & \vdots \\ 0 & \ddots & b_{(j-2)(j-2)} & 0 & & & 0 \\ \vdots & & \ddots & 0 & b_{jj} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 & b_{(I-2)(I-2)} \\ b_{(I-1)0} & \cdots & & b_{(I-1)(j-1)} & & \cdots & b_{(I-1)(I-2)} \end{vmatrix}$$

$$= (-1)^{I+j}(-\beta\alpha)(-1)^{I-1+j} b_{(I-1)(j-1)} \begin{vmatrix} b_{00} & 0 & \dots & & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & b_{(j-2)(j-2)} & 0 & & 0 \\ \vdots & & \ddots & b_{jj} & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & & \dots & 0 & b_{(I-2)(I-2)} \end{vmatrix}$$

$$= \beta\alpha b_{(I-1)(j-1)} \prod_{\substack{n=0 \\ n \neq j-1}}^{I-2} b_{nn}$$

What is left to determine to be able to compute $a_{I-1}^{j}$ are the remaining components of $\tilde{B}_A$, $b_{I-1\cdot}$:

$$\begin{aligned} b_{I-1} &= -\ell^{I-1}(1-\alpha) + (1-\alpha)\left(a_{I-2}^{I-1}\ell^{I-1} + a_{I-2}^{I}\ell^{I}\right) \\ &= -\ell^{I-1}(1-\alpha) + (1-\alpha)a_{I-2}^{I-1}\ell^{I-1} \\ &= (1-\alpha)\ell^{I-1}\left(\frac{\beta\sum\limits_{m=0}^{I-2}\beta^m}{\sum\limits_{m=0}^{I-1}\beta^m} - 1\right) \\ &= -\frac{\ell^{I-1}(1-\alpha)}{\sum\limits_{m=0}^{I-1}\beta^m} \\ b_{(I-1)(I-1)} &= -\frac{\ell^{I-1}(1-\alpha}{\sum\limits_{m=0}^{I-1}\beta^m} + \alpha\left(\frac{\beta\sum\limits_{m=0}^{I-2}\beta^m}{\sum\limits_{m=0}^{I-1}\beta^m} - 1 - \beta\right) \\ &= -\frac{\ell^{I-1}(1-\alpha) + \alpha\sum\limits_{m=0}^{I}\beta^m}{\sum\limits_{m=0}^{I-1}\beta^m} \\ b_{(I-1)(j-1)} &= b_{I-1} + \alpha a_{I-2}^{j-1} = b_{I-1} \end{aligned}$$

So for $\tilde{a}_{I-1}^{j}$ it is

$$\tilde{a}_{I-1}^{I} = \frac{-\beta\alpha}{b_{I-1I-1}} = \frac{\beta\alpha\sum\limits_{m=0}^{I-1}\beta^m}{\ell^{I-1}(1-\alpha) + \alpha\sum\limits_{m=0}^{I}\beta^m} < 1$$

$$\tilde{a}^j_{I-1} = \frac{-\beta\alpha\frac{\ell^{I-1}(1-\alpha)}{\sum_{m=0}^{I-1}\beta^m}}{\frac{\ell^{I-1}(1-\alpha)+\alpha\sum_{m=0}^{I}\beta^m}{\sum_{m=0}^{I-1}\beta^m}\cdot\alpha\cdot\frac{\sum_{m=0}^{j}\beta^m}{\sum_{m=0}^{j-1}\beta^m}}$$

$$= -\frac{\ell^{I-1}(1-\alpha)\frac{\sum_{m=0}^{j-1}\beta^m}{\sum_{m=0}^{j}\beta^m}}{\ell^{I-1}(1-\alpha)+\alpha\sum_{m=0}^{I}\beta^m} < 1$$

As this fixed point can be computed directly it is unique. But it is only unique in the general notation of $A$ as long as $\ell^I > 0$. Otherwise $\omega^I(k) = 0$ and its coefficients are arbitrary. In this case it is possible to reduce the general problem such that the values $a^I_i,\ i = 0, .., I-1$ are eliminated and therefore, $B_A \in \mathbb{R}^{I^2\times I^2}$ where exactly one block matrix is missing. In this problem again it is possible to find a unique fixed point. This shows the uniqueness, which was stated in brackets in the lemma.

□

***Proof of Lemma 3.1.10.*** In the following the different conditions on elements of $\mathbb{K}$ are verified. The first is that aggregated capital has to be positive:

$$(\mathcal{K}(\mathbf{k}))^A = \sum_{j=0}^{I-1}\omega^j(\mathbf{k})\frac{\beta\sum_{m=0}^{j-1}\beta^m}{\sum_{m=0}^{j}\beta^m}\left(1-\frac{\ell^{I-1}(1-\alpha)}{\ell^{I-1}(1-\alpha)+\alpha\sum_{m=0}^{I}\beta^m}\right)+\frac{\omega^I(\mathbf{k})\beta\alpha\sum_{m=0}^{I-1}\beta^m}{\ell^{I-1}(1-\alpha)+\alpha\sum_{m=0}^{I}\beta^m}$$

$$= \frac{\sum_{j=0}^{I-1}\omega^j(\mathbf{k})\frac{\beta\sum_{m=0}^{j-1}\beta^m}{\sum_{m=0}^{j}\beta^m}\alpha\sum_{m=0}^{I}\beta^m + \omega^I(\mathbf{k})\beta\alpha\sum_{m=0}^{I-1}\beta^m}{\ell^{I-1}(1-\alpha)+\alpha\sum_{m=0}^{I}\beta^m}$$

$$> 0$$

The second condition is that wealth needs to be positive. The proof is separated into two parts. First $\omega^{I-1}$ is proven to be positive and second the other wealth functions are considered.

Consider

$$\omega^{I-1}(\mathcal{K}(\mathbf{k})) = \left((\mathcal{K}(\mathbf{k}))^A\right)^\alpha\left(\ell^{I-1}(1-\alpha)+\alpha\frac{\mathcal{K}^{I-1}(\mathbf{k})}{(\mathcal{K}(\mathbf{k}))^A}\right).$$

Obviously, it is positive if and only if the term in brackets is positive:

$$
\begin{aligned}
&\ell^{I-1}(1-\alpha) + \alpha \frac{\mathcal{K}^{I-1}(\mathbf{k})}{(\mathcal{K}(\mathbf{k}))^A} \\
&= \ell^{I-1}(1-\alpha) + \alpha \frac{-\sum\limits_{j=0}^{I-1} \omega^j(\mathbf{k}) \frac{\beta \sum_{m=0}^{j-1} \beta^m}{\sum_{m=0}^{j} \beta^m} \ell^{I-1}(1-\alpha) + \omega^I(\mathbf{k})\beta\alpha \sum\limits_{m=0}^{I-1} \beta^m}{\sum\limits_{j=0}^{I-1} \omega^j(\mathbf{k}) \frac{\beta \sum_{m=0}^{j-1} \beta^m}{\sum_{m=0}^{j} \beta^m} \alpha \sum\limits_{m=0}^{I} \beta^m + \omega^I(\mathbf{k})\beta\alpha \sum\limits_{m=0}^{I-1} \beta^m} \\
&= \frac{\sum\limits_{j=0}^{I-1} \omega^j(\mathbf{k}) \frac{\beta \sum_{m=0}^{j-1} \beta^m}{\sum_{m=0}^{j} \beta^m} \ell^{I-1}(1-\alpha) \left( \sum\limits_{m=0}^{I} \beta^m - 1 \right) + \omega^I(\mathbf{k})\beta \sum\limits_{m=0}^{I-1} \beta^m (\alpha + \ell^{I-1}(1-\alpha))}{\sum\limits_{j=0}^{I-1} \omega^j(\mathbf{k}) \frac{\beta \sum_{m=0}^{j-1} \beta^m}{\sum_{m=0}^{j} \beta^m} \sum\limits_{m=0}^{I} \beta^m + \omega^I(\mathbf{k})\beta \sum\limits_{m=0}^{I-1} \beta^m} \\
&> 0
\end{aligned}
$$

For all the other wealth functions $j = 0, ..., I-2$ it is:

$$
\begin{aligned}
\omega^j(\mathcal{K}(\mathbf{k})) &= \left(\mathcal{K}(\mathbf{k}))^A\right)^\alpha \alpha \frac{\overbrace{\mathcal{K}^j(\mathbf{k})}^{>0}}{(\mathcal{K}(\mathbf{k}))^A} \\
&> 0
\end{aligned}
$$

as $\mathcal{K}^j(\mathbf{k})$ is a positive fraction of $\omega^{j+1}(\mathbf{k})$ which is positive for each feasible $\mathbf{k}$.

So in sum $\mathcal{K}$ maps into $\mathbb{K}$.

□

***Proof of Lemma 3.2.1****.* A steady state is a fixed point in each component of the dynamical system. So the following equations must be satisfied

$$
\begin{aligned}
\bar{k}^1 &= \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha)(\bar{k}^A)^\alpha \\
\bar{k}^0 &= \frac{\beta}{1+\beta}\bar{k}^1\alpha(\bar{k}^A)^{\alpha-1} \\
\Leftrightarrow \qquad \bar{k}^1 &= \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha)(\bar{k}^A)^\alpha \\
\bar{k}^0 &= \frac{\beta^2}{1+\beta+\beta^2}(1-\alpha)(\bar{k}^A)^\alpha\alpha(\bar{k}^A)^{\alpha-1}
\end{aligned}
$$

Obviously, both steady state variables are uniquely defined by the aggregated value. Therefore it is sufficient to consider the aggregated capital in the steady state:

$$\begin{aligned} \bar{k}^A = \bar{k}^1 + \bar{k}^0 = (\bar{k}^A)^\alpha \left( \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha) + \frac{\beta^2}{1+\beta+\beta^2}(1-\alpha)\alpha(\bar{k}^A)^{\alpha-1} \right) \\ \Leftrightarrow \quad (\bar{k}^A)^{1-\alpha} - \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha) - \frac{\beta^2}{1+\beta+\beta^2}(1-\alpha)\alpha(\bar{k}^A)^{\alpha-1} = 0 \end{aligned}$$

The last transformation can be done as for $\bar{k} \in \mathbb{K}$ the aggregated capital must be greater than zero.
Now, substitute $x := (\bar{k}^A)^{1-\alpha}$. So the equation reads:

$$x^2 - \frac{\beta(\beta+1)}{1+\beta+\beta^2}(1-\alpha)x - \frac{\beta^2}{1+\beta+\beta^2}(1-\alpha)\alpha = 0$$

So what needs to be done is to solve that simple quadratic equation:

$$\begin{aligned} x_{1/2} &= \frac{\beta(1+\beta)(1-\alpha)}{2(1+\beta+\beta^2)} \pm \frac{\beta}{2(1+\beta+\beta^2)}\sqrt{(1+\beta)^2(1-\alpha)^2 + 4\alpha(1+\beta+\beta^2)(1-\alpha)} \\ &= \beta\frac{(1+\beta)(1-\alpha) \pm \sqrt{(1-\alpha)((1+\beta)^2(1+2\alpha) + 2\alpha(1+\beta^2))}}{2(1+\beta+\beta^2)} \end{aligned}$$

Note that resubstituting only allows values that are greater than zero. As, obviously, the root is greater than the first summand and both summands are positive, there is a unique steady state value for aggregated capital

$$\bar{k}^A = \left( \beta\frac{(1+\beta)(1-\alpha) + \sqrt{(1-\alpha)((1+\beta)^2(1+2\alpha) + 2\alpha(1+\beta^2))}}{2(1+\beta+\beta^2)} \right)^{\frac{1}{1-\alpha}}$$

and thus, there is a unique steady state. □

***Proof of Lemma 3.2.2***. The proof is analogous to the proof of Lemma 3.1.8. The equations describing a steady state $\bar{k}$ are:

$$\bar{k}^0 = \frac{\beta}{1+\beta}\omega^1(\bar{\mathbf{k}}) \tag{3.15}$$

$$\bar{k}^1 = \frac{\alpha\beta(1+\beta)\omega^2(\bar{\mathbf{k}}) - \ell^1(1-\alpha)\frac{\beta}{1+\beta}\omega^1(\bar{\mathbf{k}})}{\alpha(1+\beta+\beta^2) + \ell^1(1-\alpha)} \tag{3.16}$$

$$\bar{k}^0 + \bar{k}^1 = \bar{k}^A \tag{3.17}$$

Using equation (3.15) in (3.16) leads to:

$$\begin{aligned}
&\bar{k}^1 = \frac{\alpha\beta(1+\beta)\omega^2(\bar{\mathbf{k}}) - \ell^1(1-\alpha)\bar{k}^0}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)} \\
\Leftrightarrow\quad &\alpha(1+\beta+\beta^2)\bar{k}^1 + \ell^1(1-\alpha)\bar{k}^1 = \alpha\beta(1+\beta)\omega^2(\bar{k}) - \ell^1(1-\alpha)\bar{k}^0 \\
\Leftrightarrow\quad &\alpha(1+\beta+\beta^2)\bar{k}^1 + \ell^1(1-\alpha)\bar{k}^A = \alpha\beta(1+\beta)\omega^2(\bar{k}) \\
\Leftrightarrow\quad &\bar{k}^1 = \frac{\alpha\beta(1+\beta)(1-\ell^1)(1-\alpha)(\bar{k}^A)^\alpha - \ell^1(1-\alpha)\bar{k}^A}{\alpha(1+\beta+\beta^2)}
\end{aligned}$$

Thus, $\bar{k}^1$ is uniquely determined by the steady state value for aggregated capital $\bar{k}^A$. As $\omega^1(\mathbf{k})$ is determined by aggregated capital and capital supply of the middle-aged it follows that $\bar{k}^0$ is also uniquely determined by $\bar{k}^A$. So it is sufficient to find a solution to the following equation:

$$\bar{k}^A = \bar{k}^0 + \bar{k}^1 = \alpha\frac{\beta(1+\beta)\omega^2(\bar{\mathbf{k}}) + (1+\beta+\beta^2)\frac{\beta}{1+\beta}\omega^1(k\bar{\mathbf{k}})}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}$$

with

$$\begin{aligned}
\omega^2(\bar{\mathbf{k}}) &= (1-\alpha)(1-\ell^1)(\bar{k}^A)^\alpha \\
\omega^1(\bar{\mathbf{k}}) &= (\bar{k}^A)^\alpha\left(\ell^1(1-\alpha) + \frac{\alpha\beta(1+\beta)(1-\ell^1)(1-\alpha)(\bar{k}^A)^{\alpha-1} - \ell^1(1-\alpha)}{1+\beta+\beta^2}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
(\bar{k}^A)^{1-\alpha}\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right) &= \alpha\Big(\beta(1+\beta)(1-\ell^1)(1-\alpha) \\
&+\frac{\beta(1+\beta+\beta^2)}{1+\beta}\ell^1(1-\alpha) + \frac{\beta}{1+\beta}\left(\alpha\beta(1+\beta)(1-\ell^1)(1-\alpha)(\bar{k}^A)^{\alpha-1} - \ell^1(1-\alpha)\right)\Big)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\quad (\bar{k}^A)^{1-\alpha}\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right) &= \alpha\beta\big((1+\beta)(1-\ell^1)(1-\alpha) \\
&+\beta\ell^1(1-\alpha) + \alpha\beta(1-\ell^1)(1-\alpha)(\bar{k}^A)^{\alpha-1}\big)
\end{aligned}$$

Now, substitute $x := (\bar{k}^A)^{1-\alpha}$. Then, it is

$$\begin{aligned}
&x\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right) - \alpha\beta(1-\alpha)(1+\beta-\ell^1) - \alpha^2\beta^2(1-\ell^1)(1-\alpha)\frac{1}{x} = 0 \\
\Leftrightarrow\quad &x^2 - \frac{\alpha\beta(1+\beta-\ell^1)(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}x - \frac{\alpha^2\beta^2(1-\ell^1)(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)} = 0
\end{aligned}$$

As for $x = 0$ the term on the left hand side is negative, there are two roots, one positive and one negative. As it is necessary to extract a root to perform the back substitution, the negative root does not define a steady state. So the unique steady state is defined by $\bar{k}^A = x^{\frac{1}{1-\alpha}}$ with $x > 0$. □

***Proof of Lemma 3.2.3.*** The proof makes use of the Schauder fixed point theorem. The mapping $\phi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, \phi(A) = \tilde{A}$ is defined by

$$\phi(A) = \begin{pmatrix} -b_{11} & b_{10} \\ b_{01} & -b_{00} \end{pmatrix} \frac{\beta\alpha}{\det \tilde{B}_A}$$

with $b_{ij}$ defined in 3.11. Obviously, $\phi$ is uniquely determined by $a_0^1, a_0^2$. Thus, it is sufficient to find a fixed point of a mapping $\tilde{\phi} : \mathbb{R}^2 \to \mathbb{R}^2,\ (a_0^1, a_0^2) \mapsto (-b_{11}, b_{10})\frac{\beta\alpha}{\det \tilde{B}_A}$.
Set $\tilde{\mathbb{A}} := \{(a^1, a^2) | a^1 \leq 1 \wedge a^2 \leq 0\}$. $\mathbb{R}^2$ satisfies all requirements of the Schauder fixed point theorem and obviously, $\tilde{\mathbb{A}}$ is compact and convex and $\tilde{\phi}$ is continuous. What is left to be shown is that $\tilde{\phi}$ maps $\tilde{\mathbb{A}}$ into itself.
For $a_0 \in \tilde{\mathbb{A}}$ it is

$$b_{10} = -(1 - a_0^1)\ell^1(1-\alpha) + a_0^2\ell^2(1-\alpha) \leq 0 \quad \Rightarrow \qquad b_{11} = b_{10} - \alpha(1 + \beta - a_0^1) < 0$$

and

$$\begin{aligned}\det \tilde{B}_A = b_{11}b_{00} - b_{01}b_{10} &= (b_{01} - \alpha(1+\beta))(b_{10} - \alpha(1+\beta - a_0^1)) - b_{01}b_{10} \\ &= \alpha\left(\alpha(1+\beta)(1+\beta-a_0^1) - (1+\beta)b_{10} - b_{01}(1+\beta-a_0^1)\right) > 0\end{aligned}$$

That implies that Assumption 3.1.1 holds and thus $\tilde{\phi}$ is well-defined. Now, $\tilde{\phi}(a_0) \in \tilde{\mathbb{A}}$ can be concluded:

$$\begin{aligned}\tilde{\phi}_1(a_0^1, a_0^2) &= \beta\frac{\alpha(1+\beta-a_0^1) - b_{01}}{(1+\beta)(\alpha(1+\beta-a_0^1) - b_{10}) - b_{01}(1+\beta-a_0^1)} \leq 1 \\ \tilde{\phi}_2(a_0^1, a_0^2) &= \beta\alpha\frac{b_{10}}{\det \tilde{B}_A} \leq 0\end{aligned}$$

Thus, there is a fixed point of $\tilde{\phi}$ on $\tilde{\mathbb{A}}$ and therefore, there is a fixed point of $\phi$ on $\mathbb{A}$ as well. □

***Proof of Lemma 3.2.4.*** Let $\omega_{-0}(\mathbf{k}) = (\omega^i(\mathbf{k}))_{i=1}^I$. The conjecture is that $\omega_{-0}(\mathbb{K}_0) = \mathbb{R}_{++}^{\mathbb{I}}$. If the conjecture holds and there is a tuple of coefficients $\left(a_i^j\right)_{i=0^I}$ such that $\sum_{i\in\mathbb{I}} a_i^j < 0$, it is possible to choose $\mathbf{k} \in \mathbb{K}_0$ such that $\omega^j(\mathbf{k})$ is very large and $\omega^n(\mathbf{k}), n \in \mathbb{I}\backslash\{0, j\}$ is infinitetisimal small. Thus, aggregated capital in the following period is or may be negative, which is excluded for feasible dynamics.
As $\mathbb{K}_0 \subset \mathbb{K}$ and the condition that aggregated capital needs to be positive is not relaxed in

the definition of the state space, this leads to a contradiction.
It remains to show that the conjecture holds. Let $x \in \mathbb{R}^I_{++}$ be arbitrary but fixed. Set

$$\begin{aligned} x_1 &= \omega^1(\mathbf{k}) \\ x_2 &= \omega^2(\mathbf{k}) \\ &\vdots \vdots \qquad\qquad \Leftrightarrow \qquad k^i = \frac{x_i - \ell^1(1-\alpha)(k^A)^\alpha}{\alpha(k^A)^{\alpha-1}} \quad i = 1, ..., I-1 \\ x_{I-1} &= \omega^{I-1}(\mathbf{k}) \\ x_I &= \omega^I(\mathbf{k}) \qquad\qquad \Leftrightarrow \qquad k^A = \left(\frac{x^I}{\ell^I(1-\alpha)}\right)^{\frac{1}{\alpha}} \end{aligned}$$

Thus, $k$ is uniquely defined by $x$. If $k \in \mathbb{K}_0$ holds the conjecture is proven.
The properties $k^A > 0$ and $\omega^i(\mathbf{k}) > 0$, respectively, hold by $x^I$ and $x^i$ with $i = 1, ..., I-1$ being positive. The only condition that has not been proven yet is that $\omega^0(\mathbf{k}) > 0$ holds:

$$k^0 = k^A - \sum_{i=1}^{I} k^i = \frac{\alpha(k^A)^\alpha - (1-\alpha)\sum_{i=1}^{I-1}\ell^i + \sum_{i=1}^{I-1} x^i}{\alpha(k^A)^{\alpha-1}}$$

Thus,

$$\begin{aligned} \omega^0(\mathbf{k}) &= \ell^0(1-\alpha)(k^A)^\alpha + \alpha(k^A)^\alpha - (1-\alpha)\sum_{i=1}^{I-1}\ell^i + \sum_{i=1}^{I-1} x^i \\ &= (k^A)^\alpha\left(\ell^0(1-\alpha) + \alpha - (1-\ell^0-\ell^I)(1-\alpha)\right) + f(k) - \omega^0(\mathbf{k}) - \omega^I(k) \\ &= (k^A)^\alpha\left(\ell^0(1-\alpha) + \alpha - (1-\alpha) + (\ell^0+\ell^I)(1-\alpha) - \ell^I(1-\alpha) + 1\right) - \omega^0(\mathbf{k}) \\ \Leftrightarrow \quad 2\omega^0(\mathbf{k}) &= (k^A)^\alpha\left(\ell^0(1-\alpha) + \alpha + \alpha + \ell^0(1-\alpha)\right) > 0 \end{aligned}$$

where $f(k) = \sum_{i=0}^{I}\omega^i(\mathbf{k})$ has been used. Dividing the last inequality by two leads to the required result that $\omega^0(\mathbf{k}) > 0$. Thus, $k \in \mathbb{K}_0$. □

***Proof of Lemma 3.2.5***. Before checking the conditions on aggregated capital and the wealth function, make a note of the properties on the coefficients of a fixed point on $\mathbb{A}$. As elements of $\mathbb{A}$ lead to a matrix $B_A$ with positive determinant, Cramer's rule can be applied and for a fixed point it is:

$$\begin{aligned} a_0^1 &= -\frac{\beta\alpha b_{11}}{b_{00}b_{11} - b_{01}b_{10}} \quad &\wedge& \quad a_1^1 = \frac{\beta\alpha b_{10}}{b_{00}b_{11} - b_{01}b_{10}} \\ a_0^2 &= \frac{\beta\alpha b_{01}}{b_{00}b_{11} - b_{01}b_{10}} \quad &\wedge& \quad a_1^2 = -\frac{\beta\alpha b_{00}}{b_{00}b_{11} - b_{01}b_{10}} \end{aligned}$$

with

$$\begin{aligned}
b_{00} &= -(\alpha(1+\beta)+\ell^0(1-\alpha)) \\
b_{01} &= -\ell^0(1-\alpha) \\
b_{10} &= -\ell^1(1-\alpha)+\left((e_1^2)^\top A\ell\right)(1-\alpha) \\
b_{11} &= -\ell^1(1-\alpha)+\left((e_1^2)^\top A\ell\right)(1-\alpha)+\alpha a_0^1-\alpha(1+\beta).
\end{aligned}$$

So now check the conditions. Let $k \in \mathbb{K}$ and $k_1 = \mathcal{K}(k)$.

$\underline{k_1^A > 0}$:

$$k_1^A = (a_0^1 + a_1^1)\omega^1(k_1) + (a_0^2 + a_1^2)\omega^2(\mathbf{k}) = \alpha\frac{(1+\beta-a_0^1)\omega^1(\mathbf{k})+(1+\beta)\omega^2(\mathbf{k})}{b_{00}b_{11}-b_{01}b_{10}}$$

which is positive if and only if
$\omega^1(\mathbf{k}) > -\frac{1+\beta}{1+\beta-a_0^1}\omega^2(\mathbf{k})$
which is satisfied by the definition of $\mathbb{K}$ as $-\frac{1+\beta}{1+\beta-a_0^1} > -\frac{\ell^0(1-\alpha)\beta}{-b_{11}+\ell^0(1-\alpha)(1+\beta-a_0^1)}$.

$\underline{\omega^0(\mathbf{k}_1)}$:

$$\begin{aligned}
&\omega^0(\mathbf{k}_1) > 0 \\
\Leftrightarrow&\frac{k_1^0}{k_1^A} > -\frac{\ell^0(1-\alpha)}{\alpha} \\
\Leftrightarrow&\frac{-b_{11}\omega^1(\mathbf{k})+b_{01}\omega^2(\mathbf{k})}{\alpha((1+\beta-a_0^1)\omega^1(\mathbf{k})+(1+\beta)\omega^2(\mathbf{k}))} > -\frac{\ell^0(1-\alpha)}{\alpha} \\
\Leftrightarrow&-b_{11}\omega^1(\mathbf{k})-\ell^0(1-\alpha)\omega^2(\mathbf{k}) > -\ell^0(1-\alpha)\left((1+\beta-a_0^1)\omega^1(\mathbf{k})-(1+\beta)\omega^2(\mathbf{k})\right) \\
\Leftrightarrow&\omega^1(\mathbf{k}) > -\frac{\ell^0(1-\alpha)\beta}{-b_{11}+\ell^0(1-\alpha)(1+\beta-a_0^1)}\omega^2(\mathbf{k})
\end{aligned}$$

That exactly is satisfied by the definition of $\mathbb{K}$.

$\underline{\omega^1(\mathbf{k}_1)}$:

Verify the following inequality:

$$\begin{aligned}
&\omega^1(\mathbf{k}_1) = \ell^1(1-\alpha)(k_1^A)^\alpha + \frac{k_1^1}{k_1^A}(k_1^A)^\alpha\alpha > -\frac{\ell^2\ell^0(1-\alpha)^2\beta}{-b_{11}+\ell^0(1-\alpha)(1+\beta-a_0^1)}(k_1^A)^\alpha \\
\Leftrightarrow&\ell^1(1-\alpha)+\frac{b_{10}\omega^1(\mathbf{k})-b_{00}\omega^2(\mathbf{k})}{(1+\beta-a_0^1)\omega^1(\mathbf{k})+(1+\beta)\omega^2(\mathbf{k})} > -\frac{\ell^0\ell^2(1-\alpha)^2\beta}{-b_{11}+\ell^0(1-\alpha)(1+\beta-a_0^1)} \\
\Leftrightarrow&(\ell^1\beta+a_0^2\ell^2)(1-\alpha)\omega^1(\mathbf{k})+(\ell^0(1-\alpha)+(1+\beta)(\ell^1(1-\alpha)+\alpha))\omega^2(\mathbf{k}) \\
&\qquad > -\frac{\ell^0\ell^2(1-\alpha)^2\beta}{-b_{11}+\ell^0(1-\alpha)(1+\beta-a_0^1)}\left((1+\beta-a_0^1)\omega^1(\mathbf{k})+(1+\beta)\omega^2(\mathbf{k})\right)
\end{aligned}$$

$$\begin{aligned}\Leftrightarrow &\left[(\ell^1\beta + a_0^2\ell^2)(1-\alpha)(-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)) + \ell^0\ell^2(1-\alpha)^2\beta(1+\beta-a_0^1)\right]\omega^1(\mathbf{k}) \\ &> -\left[\ell^0\ell^2(1-\alpha)^2\beta(1+\beta)\right. \\ &\quad \left. - \left(\ell^0(1-\alpha) + (\ell^1(1-\alpha)+\alpha)(1+\beta)\right)(-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1))\right]\omega^2(\mathbf{k})\end{aligned}$$

Now, use that $\mathbf{k} \in \mathbb{K}$:

$$\begin{aligned}&\left[(\ell^1\beta + a_0^2\ell^2)(1-\alpha)(-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)) + \ell^0\ell^2(1-\alpha)^2\beta(1+\beta-a_0^1)\right]\omega^1(\mathbf{k}) \\ &> -\left[(\ell^1\beta + a_0^2\ell^2)\ell^0(1-\alpha)^2\beta + \frac{\ell^0\ell^0\ell^2(1-\alpha)^3\beta^2(1+\beta-a_0^1)}{-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1)}\right]\omega^2(\mathbf{k}) \\ &> -\left[(\ell^1\beta + a_0^2\ell^2)\ell^0(1-\alpha)^2\beta + \ell^0\ell^2(1-\alpha)^2\beta^2\right]\omega^2(\mathbf{k})\end{aligned}$$

Hence, it can be concluded that:

$$\begin{aligned}&-\left[(\ell^1\beta + a_0^2\ell^2)\ell^0(1-\alpha)^2\beta + \ell^0\ell^2(1-\alpha)^2\beta^2\right]\omega^2(\mathbf{k}) \\ &> -\left[\ell^0\ell^2(1-\alpha)^2\beta(1+\beta)\right. \\ &\quad \left. - \left(\ell^0(1-\alpha) + (\ell^1(1-\alpha)+\alpha)(1+\beta)\right)(-b_{11} + \ell^0(1-\alpha)(1+\beta-a_0^1))\right]\omega^2(\mathbf{k}) \\ \Leftrightarrow &-(\ell^1\beta + (a_0^2-1)\ell^2)\ell^0(1-\alpha)^2\beta \\ &> -b_{11}\left(\ell^0(1-\alpha) + (\ell^1(1-\alpha)+\alpha)(1+\beta)\right) - (\ell^0(1-\alpha) + \alpha(1+\beta))\ell^0(1+\beta-a_0^1) \\ &\quad - \ell^0\ell^1(1-\alpha)^2(1+\beta)(1+\beta-a_0^1) \\ \Leftrightarrow &(1-a_0^2)\ell^2\ell^0(1-\alpha)^2\beta + \ell^0\ell^1(1-\alpha)^2(1+\beta+\beta^2-(1+\beta)a_0^1) \\ &> -b_{11}\left(\ell^0(1-\alpha) + (\ell^1(1-\alpha)+\alpha)(1+\beta)\right) - (\ell^0(1-\alpha) + \alpha(1+\beta))\ell^0(1+\beta-a_0^1)\end{aligned}$$

As the first term is greater than zero and the last term is smaller than zero the inequality is satisfied. □

***Proof of Lemma 3.2.6***. The existence and even uniqueness of such an equilibrium for $\ell^0 = 0$ is shown in Section 3.2.2. So here $\ell^0 \neq 0$ is assumed.
As the dimension of the economy is not that high all fixed points of $\phi$ can be described, which is done first in a straightforward computation.
A matrix $\bar{A}$ is required such that

$$B_{\bar{A}}\bar{a} = b.$$

Following Corollary 3.1.1 it can be assumed that $\det B_A \neq 0$ without losing any solution. The image $\tilde{A}$ of an arbitrary matrix $A$ under $\phi$ has been characterized in equation (3.12). As the entries of $B_A$ only depend on the first row of $A$, any image and in particular a fixed point of $\phi$ is characterized by $a_0^1$ and $a_0^2$. If these two entries coincide with the corresponding values of the image, there is one fixed point defined by the remaining entries of the image. So the existence of a fixed point is ensured if there are $\bar{a}_0^1$ and $\bar{a}_0^2$ such that:

$$\bar{a}_0^1 = -\frac{\beta\alpha\bar{b}_{11}}{\det B_{\bar{A}}} \qquad \wedge \qquad \bar{a}_1^1 = \frac{\beta\alpha\bar{b}_{10}}{\det B_{\bar{A}}}$$

where $\bar{b}_{ij}$ denote the entries of $B_{\bar{A}}$. As $\det B_{\bar{A}} \neq 0$ and $\bar{b}_{01} \neq 0$ as $\ell^0 \neq 0$, $\bar{a}_0^2$ is different from zero as well. So it can be concluded:

$$\begin{aligned} & detB_{\bar{A}} = \frac{\bar{b}_{01}\beta\alpha}{\bar{a}_0^2} \\ \Rightarrow\quad & \frac{\bar{b}_{01}\beta\alpha}{\bar{a}_0^2}\bar{a}_0^1 = -\alpha\beta\bar{b}_{11} \\ \Leftrightarrow\quad & \bar{a}_0^1\bar{b}_{01} = -\bar{a}_0^2\bar{b}_{11} \\ \Leftrightarrow\quad & -\bar{a}_0^1\ell^0(1-\alpha) = -\bar{a}_0^2\left(\ell^1(1-\alpha) - (\ell^1\bar{a}_0^1 + \ell^2\bar{a}_0^2)(1-\alpha) + \alpha(1+\beta-\bar{a}_0^1)\right) \\ \Leftrightarrow\quad & \bar{a}_0^1 = \frac{\ell^1(1-\alpha) + \alpha(1+\beta) - \ell^2(1-\alpha)\bar{a}_0^2}{(\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)}\bar{a}_0^2 \end{aligned}$$

So the determinant can be quoted in terms of $\bar{a}_0^2$. In general, it is

$$\det B_A = \alpha\left[(1+\beta-a_0^1)\left(\alpha(1+\beta)+\ell^0(1-\alpha)\right) + (1+\beta)(1-\alpha)\left((1-a_0^1)\ell^1 - a_0^2\ell^2\right)\right].$$

Before regarding the final equation consider the following:

$$\begin{aligned} & (1+\beta-\bar{a}_0^1) \\ = & \frac{(\ell^1(1-\alpha)+\alpha)(1+\beta)\bar{a}_0^2 - \ell^0(1+\beta)(1-\alpha) - (\ell^1(1-\alpha)+\alpha(1+\beta))\,\bar{a}_0^2 + \ell^2(1-\alpha)(\bar{a}_0^2)^2}{(\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)} \\ = & \frac{(\ell^1(1-\alpha)\beta + \ell^2(1-\alpha)\bar{a}_0^2)\,\bar{a}_0^2 - \ell^0(1-\alpha)(1+\beta)}{(\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)} \end{aligned}$$

Combining the previous results in the equation $detB_{\bar{A}} = \frac{\bar{b}_{01}\beta\alpha}{\bar{a}_0^2}$ leads to

$$\begin{aligned} & \bar{a}_0^2\left[(1+\beta)\ell^1(1-\alpha)\left((\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)\right)\right. \\ & \quad + \left(\alpha(1+\beta)+\ell^0(1-\alpha)\right)\left[\left(\ell^1(1-\alpha)\beta + \ell^2(1-\alpha)\bar{a}_0^2\right)\bar{a}_0^2 - \ell^0(1-\alpha)(1+\beta)\right] \\ & \quad - (1+\beta)\ell^1(1-\alpha)\left[\left(\alpha(1+\beta)+\ell^1(1-\alpha)\right)\bar{a}_0^2 - (\bar{a}_0^2)^2\ell^2(1-\alpha)\right] \\ & \quad \left. -\bar{a}_0^2\ell^2(1-\alpha)(1+\beta)\left((\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)\right)\right] \\ & \quad = -\beta\ell^0(1-\alpha)\left(\ell^1(1-\alpha)+\alpha)\bar{a}_0^2 - \ell^0(1-\alpha)\right). \end{aligned}$$

Thus, $\bar{a}_0^2$ is defined by the roots of

$$
\begin{aligned}
&(\bar{a}_0^2)^3\left(\left(\alpha(1+\beta)+\ell^0(1-\alpha)\right)\ell^2(1-\alpha)+(1+\beta)\ell^2\ell^1(1-\alpha)^2-(1+\beta)\ell^2(1-\alpha)(\ell^1(1-\alpha)+\alpha)\right)\\
&+(\bar{a}_0^2)^2\left(\left(\alpha(1+\beta)+\ell^0(1-\alpha)\right)\ell^1(1-\alpha)\beta-\alpha\beta(1+\beta)(1-\alpha)\ell^1+\ell^2(1-\alpha)(1+\beta)\right)\\
&+\bar{a}_0^2\left(-\left(\alpha(1+\beta)+\ell^0(1-\alpha)\right)\ell^0(1-\alpha)(1+\beta)-\ell^0(1-\alpha)^2(1+\beta)\ell^1+\beta\ell^0(1-\alpha)(\ell^1(1-\alpha)+\alpha)\right)\\
&-\ell^0\ell^0(1-\alpha)^2\beta=0
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&(\bar{a}_0^2)^3\ell^2(1-\alpha)^2\ell^0+(\bar{a}_0^2)^2\ell^0(1-\alpha)^2\left(\ell^1\beta+\ell^2(1+\beta)\right)\\
&+\bar{a}_0^2\ell^0(1-\alpha)\left(-\alpha(1+\beta+\beta^2)-\ell^0(1-\alpha)(1+\beta)-\ell^1(1-\alpha)\right)-\ell^0\ell^0(1-\alpha)^2\beta=0
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&(\bar{a}_0^2)^3\ell^2(1-\alpha)+(\bar{a}_0^2)^2(1-\alpha)\left(\ell^1\beta+\ell^2(1+\beta)\right)\\
&-\bar{a}_0^2\left(\alpha(1+\beta+\beta^2)+\ell^0(1-\alpha)(1+\beta)+\ell^1(1-\alpha)\right)-\ell^0(1-\alpha)\beta=0
\end{aligned}
$$

Note that in the last step $\ell^0(1-\alpha)$ can be crossed out as it is assumed to be different from 0. Otherwise the equation is fulfilled automatically as it reads $'0=0'$.
As this is a cubic equation the existence of a real root is ensured. Having a closer look at the coefficients the roots can be characterized a little bit:
Obviously, there is one positive root as the leading coefficient is greater than zero and the constant term is negative. But it can easily be shown that there are three real roots:

(i) As already mentioned the polynomial converges to infinity for $\bar{a}_0^2 \to \infty$

(ii) For $\bar{a}_0^2 = 0$ the polynomial is smaller than zero ($-\ell^0(1-\alpha)\beta$).

(iii) Setting $\bar{a}_0^2 = -1$ the polynomial is equal to

$$(1-\alpha)\left(\ell^1\beta+\ell^2\beta\right)+\alpha(1+\beta+\beta^2)+\ell^0(1-\alpha)+\ell^1(1-\alpha)>0.$$

(iv) For $\bar{a}_0^2 \to -\infty$ the polynomial converges to $-\infty$ as well.

Thus, the three roots lie in the intervals $(-\infty,-1),\ (-1,0),\ (0,\infty)$. These roots define the three equilibrium candidates. Obviously, the second root, $a_0^2 \in (-1,0)$, defines the equilibrium that has already been considered. It remains to show, that the other roots define an infeasible dynamics. Both candidates contradict Lemma 3.2.4.

First consider $a_0^2 > 0$:
$a_0^2 + a_1^2 = \frac{\alpha^2\beta(1+\beta)}{\det \tilde{B_A}}$. By the definition of $a_0^2$ by Cramer's rule $a_0^2 > 0 \Leftrightarrow \det \tilde{B_A} < 0$. Thus, the sum of the coefficients of $\omega^2(\mathbf{k})$ is negative and the fixed point defined by $a_0^2 > 0$ may not be feasible.

Now, consider $a_0^2 < -1$:
First the upper bound will be redefined. Show that for $(a_0^2)^\star = -\frac{\beta\ell^1+\sqrt{\beta^2\ell^1\ell^1-4\ell^0\ell^2(1+\beta)}}{2\ell^2}$ the polynomial above is greater than zero as well, thus the considered interval changes to $\left(-\infty, -\frac{\beta\ell^1+\sqrt{\beta^2\ell^1\ell^1-4\ell^0\ell^2(1+\beta)}}{2\ell^2}\right)$. In the following, set $\kappa := \beta^2\ell^1\ell^1 - 4\ell^0\ell^2(1+\beta)$.

$$\begin{aligned}
&((\bar{a}_0^2)^\star)^3\ell^2(1-\alpha) + ((\bar{a}_0^2)^\star)^2(1-\alpha)\left(\ell^1\beta + \ell^2(1+\beta)\right)\\
&\quad - (\bar{a}_0^2)^\star\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)\right) - \ell^0(1-\alpha)\beta\\
=&-\frac{\left(\beta\ell^1+\sqrt{\kappa}\right)^3}{8\ell^2\ell^2}(1-\alpha) + \frac{\left(\beta\ell^1+\sqrt{\kappa}\right)^2}{4\ell^2\ell^2}(1-\alpha)\left(\ell^1\beta + \ell^2(1+\beta)\right) - \ell^0(1-\alpha)\beta\\
&\quad + \frac{\left(\beta\ell^1+\sqrt{\kappa}\right)}{2\ell^2}\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)\right)\\
=&\frac{2\beta\ell^1 - \beta\ell^1 - \sqrt{\kappa}}{8\ell^2\ell^2}\left(\beta\ell^1+\sqrt{\kappa}\right)^2(1-\alpha) + \frac{2\beta^2\ell^1\ell^1 + 2\beta\ell^1\sqrt{\kappa} + 4\ell^0\ell^2(1+\beta)}{4\ell^2}(1+\beta)(1-\alpha)\\
&\quad + \frac{\left(\beta\ell^1+\sqrt{\kappa}\right)}{2\ell^2}\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)\right) - \ell^0(1-\alpha)\beta\\
=&\beta\ell^1(1-\alpha)\frac{\beta^2\ell^1\ell^1 + \beta\ell^1\sqrt{\kappa} + 2\ell^0\ell^2(1+\beta)}{4\ell^2\ell^2} - \sqrt{\kappa}(1-\alpha)\frac{\beta^2\ell^1\ell^1 + \beta\ell^1\sqrt{\kappa} + 2\ell^0\ell^2(1+\beta)}{4\ell^2\ell^2}\\
&\quad + \beta\ell^1(1+\beta)(1-\alpha)\frac{\beta\ell^1+\sqrt{\kappa}}{2\ell^2} + \ell^0(1+\beta)^2(1-\alpha) - \ell^0(1-\alpha)\\
&\quad + \frac{\beta\ell^1+\sqrt{\kappa}}{2\ell^2}\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)\right)\\
=&\beta^2\ell^1\ell^1\frac{\beta\ell^1+\sqrt{\kappa}}{4\ell^2\ell^2} + \frac{\ell^1\beta(1-\alpha)\ell^0(1+\beta)}{2\ell^2} - \frac{\beta^2\ell^1\ell^1\sqrt{\kappa}(1-\alpha)}{4\ell^2\ell^2}\\
&\quad - \beta\ell^1(1-\alpha)\frac{\beta^2\ell^1\ell^1 + 4\ell^0\ell^2(1+\beta)}{4\ell^2\ell^2} - \frac{\sqrt{\kappa}(1-\alpha)\ell^0(1+\beta)}{2\ell^2}\\
&\quad + \ell^0(1-\alpha)(1+\beta+\beta^2) + \frac{\beta\ell^1+\sqrt{\kappa}}{2\ell^2}\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)(1+\beta+\beta^2)\right)\\
=&\frac{\ell^1\beta(1-\alpha)\ell^0(1+\beta)}{2\ell^2} - \frac{\beta\ell^1\ell^0(1-\alpha)(1+\beta) + \sqrt{\kappa}(1-\alpha)(1+\beta)}{2\ell^2}\\
&\quad + \ell^0(1-\alpha)(1+\beta+\beta^2) + \frac{\beta\ell^1+\sqrt{\kappa}}{2\ell^2}\left(\alpha(1+\beta+\beta^2) + \ell^0(1-\alpha)(1+\beta) + \ell^1(1-\alpha)(1+\beta+\beta^2)\right)\\
=&\ell^0(1-\alpha)(1+\beta+\beta^2) + \frac{\beta\ell^1+\sqrt{\kappa}}{2\ell^2}(1+\beta+\beta^2)(\ell^1(1-\alpha)+\alpha)
\end{aligned}$$

The conjecture is that $a_0^1 + a_1^1 < 0$. Obviously, is $\det\tilde{B_A} > 0$, as $a_0^2 < 0$, by the same argumentation as above. Thus,

$$a_0^1 + a_1^1 = \frac{\alpha^2\beta(1+\beta-a_0^1)}{\det\tilde{B_A}} < 0 \Leftrightarrow (1+\beta-a_0^1) < 0$$

With $a_0^1$ defined above in dependence of $a_0^2$ it is

$$1+\beta-a_0^1=\frac{\beta\ell^1(1-\alpha)a_0^2-(1+\beta)\ell^0(1-\alpha)+\ell^2(1-\alpha)(a_0^2)^2}{(\ell^1(1-\alpha)+\alpha)a_0^2-\ell^0(1-\alpha)}<0$$
$$\overset{a_0^2<0}{\Leftrightarrow}\beta\ell^1(1-\alpha)a_0^2-(1+\beta)\ell^0(1-\alpha)+\ell^2(1-\alpha)(a_0^2)^2>0$$

Obviously, the left polynomial in the last inequality has a positive and a negative root. As $\beta\ell^1(1-\alpha)>0$ the inequality is satisfied for all $a_0^2$ that are smaller than the negative root. The negative root in fact is $(a_0^2)^\star$ and $a_0^2<(a_0^2)^\star$ by assumption. Thus, the coefficients of $\omega^1(\mathbf{k})$ contradict Lemma 3.2.4. □

***Proof of Lemma 3.2.7***. The equilibrium dynamics is given by $\mathcal{K}(\mathbf{k})=A\omega_{-0}(\mathbf{k})$. The objective is to find $\bar{\mathbf{k}}=\left(\bar{k}^0,\bar{k}^1\right)$ such that

$$\begin{aligned}\bar{k}^0&=a_0^1\omega^1(\bar{\mathbf{k}})+a_0^2\omega^2(\bar{\mathbf{k}})\\ \bar{k}^1&=a_1^1\omega^1(\bar{\mathbf{k}})+a_1^2\omega^2(\bar{\mathbf{k}})\\ \bar{k}^A&=\bar{k}^0+\bar{k}^1\end{aligned}$$

with

$$\begin{aligned}\omega^1(\bar{\mathbf{k}})&=(\bar{k}^A)^\alpha\left(\ell^1(1-\alpha)+\alpha\frac{\bar{k}^1}{\bar{k}^A}\right)\\ \omega^2(\bar{\mathbf{k}})&=\ell^2(1-\alpha)(\bar{k}^A)^\alpha\end{aligned}$$

Using the definition of $\bar{k}^1$ in $\omega^1(\bar{k})$ leads to:

$$\begin{aligned}&\omega^1(\bar{\mathbf{k}})=(\bar{k}^A)^\alpha\left(\ell^1(1-\alpha)+\alpha\frac{a_1^1\omega^1(\bar{k})+a_1^2\omega^2(\bar{k})}{\bar{k}^A}\right)\\ \Leftrightarrow\quad&\omega^1(\bar{\mathbf{k}})\left(1-a_1^1\alpha(\bar{k}^A)^{\alpha-1}\right)=(\bar{k}^A)^\alpha\left(\ell^1(1-\alpha)+\alpha\frac{a_1^2\ell^2(1-\alpha)(\bar{k}^A)^\alpha}{\bar{k}^A}\right)\\ \Leftrightarrow\quad&\omega^1(\bar{\mathbf{k}})=(\bar{k}^A)^\alpha\frac{\ell^1(1-\alpha)+\alpha\frac{a_1^2\ell^2(1-\alpha)(\bar{k}^A)^\alpha}{\bar{k}^A}}{1-a_1^1\alpha(\bar{k}^A)^{\alpha-1}}\end{aligned}$$

As both $\omega^1(\bar{\mathbf{k}})$ and $\omega^2(\bar{\mathbf{k}})$ can be written in dependance of $\bar{k}^A$, now $\bar{k}^A$ will be determined.

$$\begin{aligned}&\bar{k}^A=\bar{k}^1+\bar{k}^0=(a_0^1+a_1^1)\omega^1(\bar{k})+(a_0^2+a_1^2)\omega^2(\bar{k})\\ \Leftrightarrow\quad&\left(\bar{k}^A\right)^{1-\alpha}=(a_0^1+a_1^1)\frac{\ell^1(1-\alpha)+\alpha a_1^2\ell^2(1-\alpha)(\bar{k}^A)^{\alpha-1}}{1-a_1^1\alpha(\bar{k}^A)^{\alpha-1}}+(a_0^2+a_1^2)\ell^2(1-\alpha)\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & \left(\bar{k}^A\right)^{1-\alpha} - a_1^1\alpha = (a_0^1 + a_1^1)\ell^1(1-\alpha) + (a_0^1 + a_1^1)\alpha a_1^2\ell^2(1-\alpha)(\bar{k}^A)^{\alpha-1} \\
& \qquad + (a_0^2 + a_1^2)(1-\alpha)\ell^2 - a_1^1\alpha(a_0^2 + a_1^2)\ell^2(1-\alpha)(\bar{k}^A)^{\alpha-1} \\
\Leftrightarrow \quad & \left(\bar{k}^A\right)^{1-\alpha} - \left((1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1\right) \\
& \qquad - \alpha\ell^2(1-\alpha)\underbrace{(a_0^1 a_1^2 - a_1^1 a_0^2)}_{=\det A}\left(\bar{k}^A\right)^{\alpha-1} = 0
\end{aligned}$$

Now, substitute $\left(\bar{k}^A\right)^{1-\alpha} =: x$:

$$x^2 - \left((1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1\right) x - \alpha\ell^2(1-\alpha)\det A = 0$$

Obviously, again only the positive roots are interesting as $k^A = x^{\frac{1}{1-\alpha}}$ is a root of $x$. As $\det A > 0$ by Corollary 3.2.2 with Descartes' rule of signs (cf. Struik [24]) the uniqueness of this positive root is guaranteed.
In particular it is

$$\begin{aligned}
k^A = \Bigg( & \frac{(1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1}{2} \\
& + \sqrt{\left(\frac{(1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1}{2}\right)^2 + \alpha\ell^2(1-\alpha)\det A}\Bigg)^{\frac{1}{1-\alpha}}
\end{aligned}$$

□

# Chapter 4

# The Equilibrium Dynamics

The previous chapter analyzed the capital evolution of the benchmark economy. In the three-period economy a uniqueness result has been achieved. Except for the existence of a unique steady state there was no analysis of the dynamics' properties so far. This chapter will address the questions how the equilibrium dynamics behave in the long-run and if there is some kind of optimality. For this analysis, the definitions need to be transferred from the context of the two-period economy to the generalization as well as it needs to be checked if the criteria that are known are applicable or transferable, respectively.

The chapter is organized as follows: Section 1 considers the stability properties of the equilibrium dynamics. Section 2 treats dynamic efficiency and section 3 pareto optimality. These sections first introduce the corresponding concepts for the general multi-period economy and analyze the equilibrium derived in Chapter 3 with respect to these concepts.

## 4.1 Stability Properties and the Steady State

The objective of this section is the analysis of the stability-properties of the three-period benchmark economy. Again, the section distinguishes the cases of the general economy and the economy where the labor profile is restricted such that the old generation is retired.

### 4.1.1 Case I: The General Three-Period Economy

The three-period economy with an arbitrary labor profile has a unique equilibrium of the type

$$\mathcal{K}(k^0, k^1) = \begin{pmatrix} a_0^1 & a_0^2 \\ a_1^1 & a_1^2 \end{pmatrix} \begin{pmatrix} \omega^1(\mathbf{k}) \\ \omega^2(\mathbf{k}) \end{pmatrix}$$

(see 3.2.5). The previous chapter's appendix (see Proof of Lemma 3.2.7) has given the explicit characterization of the unique steady state $\bar{\mathbf{k}} = (\bar{k}^0, \bar{k}^1)$. The following lemma addresses the stability of the dynamics. The existence of a unique steady state raises the question of global and local long-run behavior. The criterion of local stability is a general criterion for dynamical systems. Thus, it is applicable without conflicts.

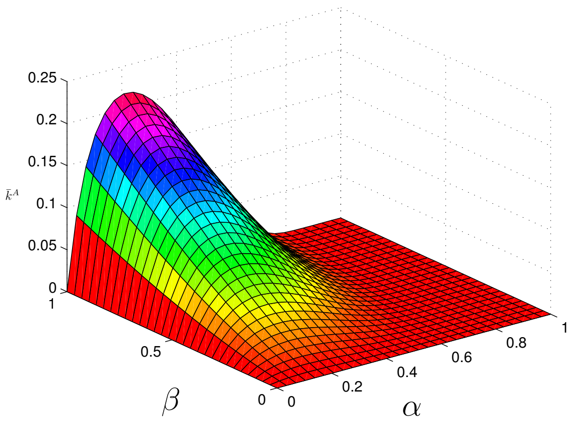
**Lemma 4.1.1.** *The steady state determined in Lemma 3.2.5 is either saddle path stable or asymptotically stable.*

Obviously, the convergence in the long-run is not proven in general. Lemma 4.1.1 states that if instability occurs, it is of a particular type. A specification would only be possible if the coefficients in the equilibrium dynamics were further restricted. Then, if instability occurs, it might be possible to define critical values for the parameters $\alpha$, $\beta$ and the labor profile such that the dynamics is stable or not. But saddle path stability at least guarantees that for each $k_0^0$ there is a $k_0^1$ such that $\mathcal{K}^t(\mathbf{k}_0)$ converges towards the steady state $\bar{\mathbf{k}}$ for $t \to \infty$.

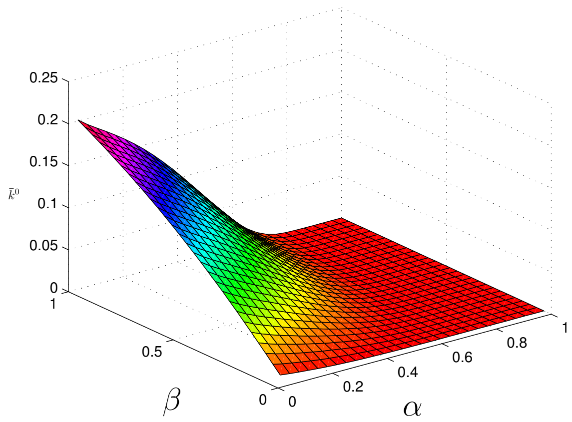
### 4.1.2 Case II: Retirement of the Old in the Three-Period Economy

This section considers the particular economy where labor is only supplied by the young and the middle-aged. Section 3.2.2 shows that there is a unique steady state (see equation (3.10)). Before analyzing the stability properties of the economy, focus on the behavior of the steady state values depending on the labor profile and the other parameters. Figure 4.1 shows the steady values for $\ell^1 = \frac{1}{2}$. Naturally, the most interesting value is $\bar{k}^1$, because here compared to the well-known two-period economy new effects may occur. The figure shows that the decision of positive or negative capital supply does not only depend on the labor profile. Furthermore, it is visible that negative savings are not only realized in the extreme example where $\ell^1 = 1$. Figure 4.1(c) stresses that there are positive and negative values for $\bar{k}^1$. Concerning the dependance on the parameters except the labor profile, Figure 4.1 emphasizes that at least the steady state value of aggregated capital supply, $\bar{k}^A$, is increasing in $\beta$. $\bar{k}^0$ seems to be increasing in $\beta$ as well. This result is intuitive as the higher $\beta$ the higher future utility is evaluated. Thus, future consumption is more valuable and it is more attractive to invest capital. Regarding the parameter $\alpha$ no similar observation is made. The value for $\ell^1$ is chosen exemplary here. In Appendix B there are similar considerations for the values $\ell^1 \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Figures B.1 until B.4 emphasize the observations for these values. It is worth to note that in all cases the steady state may be very close to zero. Thus, when plotting the dynamical behavior later, a numerical convergence to a value very close to zero does not necessarily represent impoverishment of the economy.

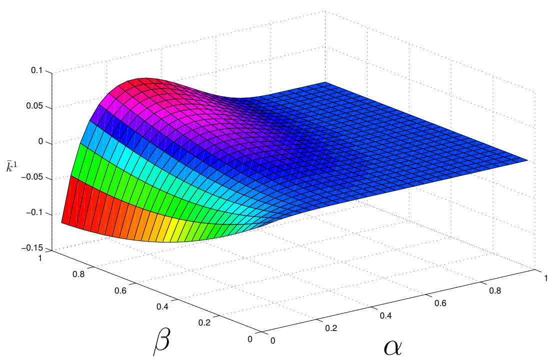
Now, after closer inspection of the steady state's properties regard the long-run behavior. As in this economy the equilibrium dynamics is known explicitly, the stability result of




(a) $\bar{k}^A$

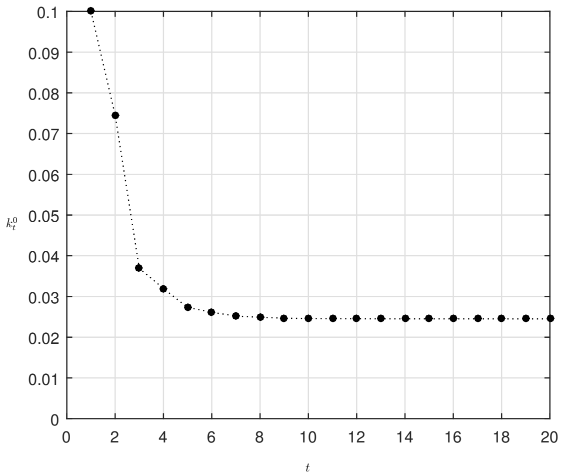



(b) $\bar{k}^0$

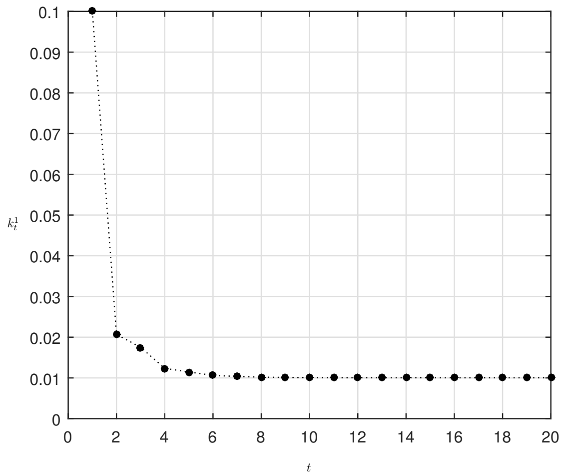



(c) $\bar{k}^1$

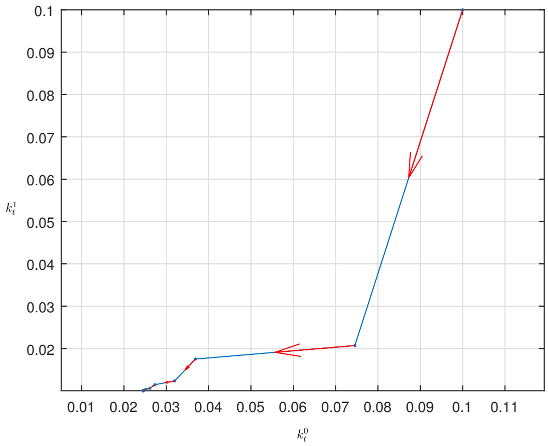
Figure 4.1: Steady state values for $\ell^1 = \frac{1}{2}$ in dependence of parameters $\alpha$ and $\beta$




(a) $k^0$




(b) $k^1$




(c) image

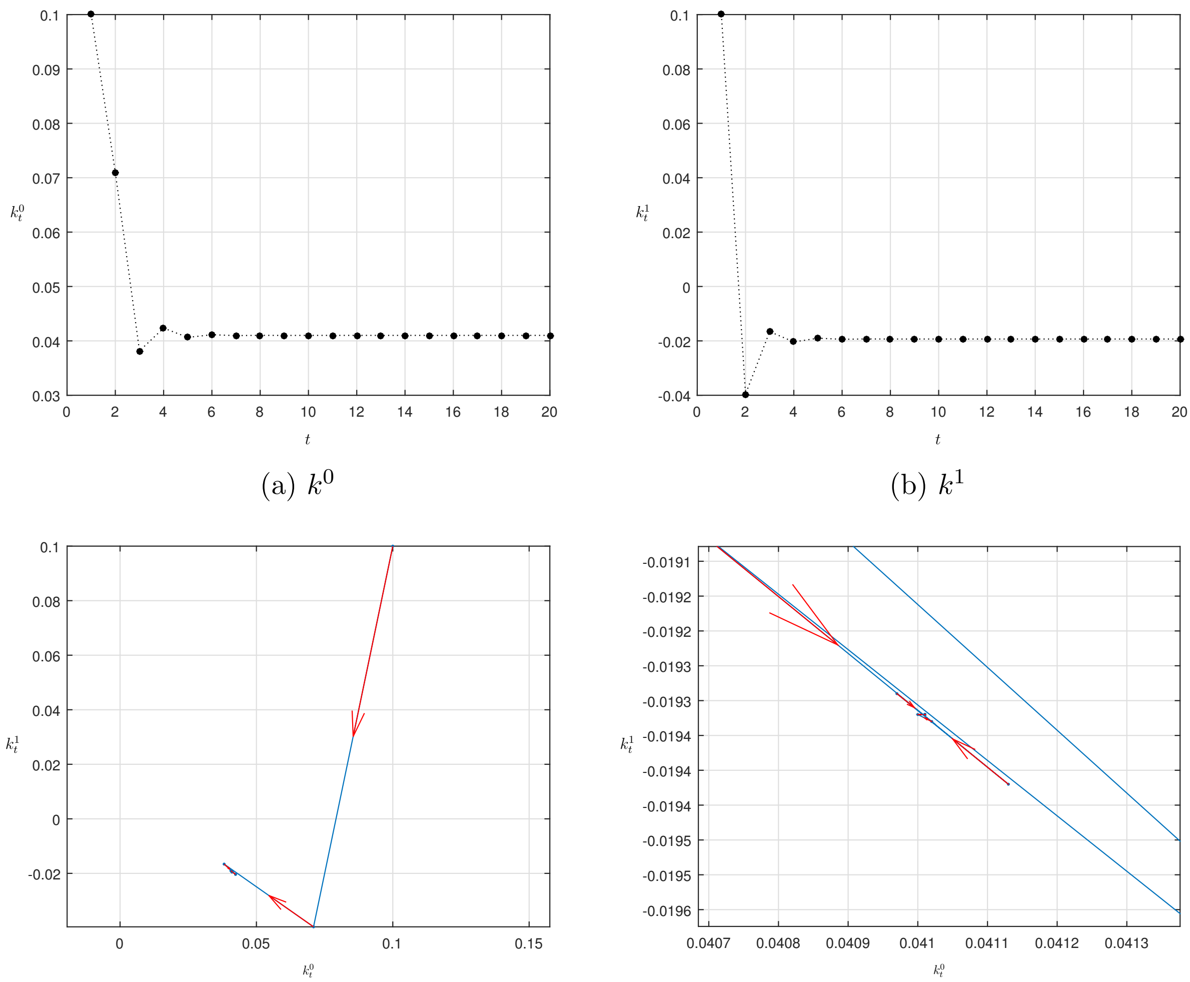
Figure 4.2: Dynamics with $\ell^1 = \frac{1}{2}$, $\alpha = 0.5$ $\beta = 0.5$, $\mathbf{k_1} = (0.1, 0.1)$

Lemma 4.1.1 may be specified.

**Lemma 4.1.2.** *The steady state determined in Lemma 3.2.2 is locally asymptotically stable.*

Lemma 4.1.2 states a very strong result. If the old generation is retired, the economy converges towards its steady state.

Figures 4.2 to 4.4 show the convergence for different parameter values for $\alpha$ and $\beta$. Note that for numerical reasons the initial value is denoted by $\mathbf{k}_1$ in the figures. As proven in Lemma 4.1.2 the system converges for all parameter values. But the figures show that the evolution differs structurally. While in Figure 4.2 the dynamics converges monotone to the steady state in both components, zooming in the evolution in total of Figure 4.3(c) shows that the convergence is not monotone at all (see Figure 4.3(d)). Here the convergence of $k_t^0$ as well as $k_t^1$ is not monotone. Some values are greater and some are smaller than the

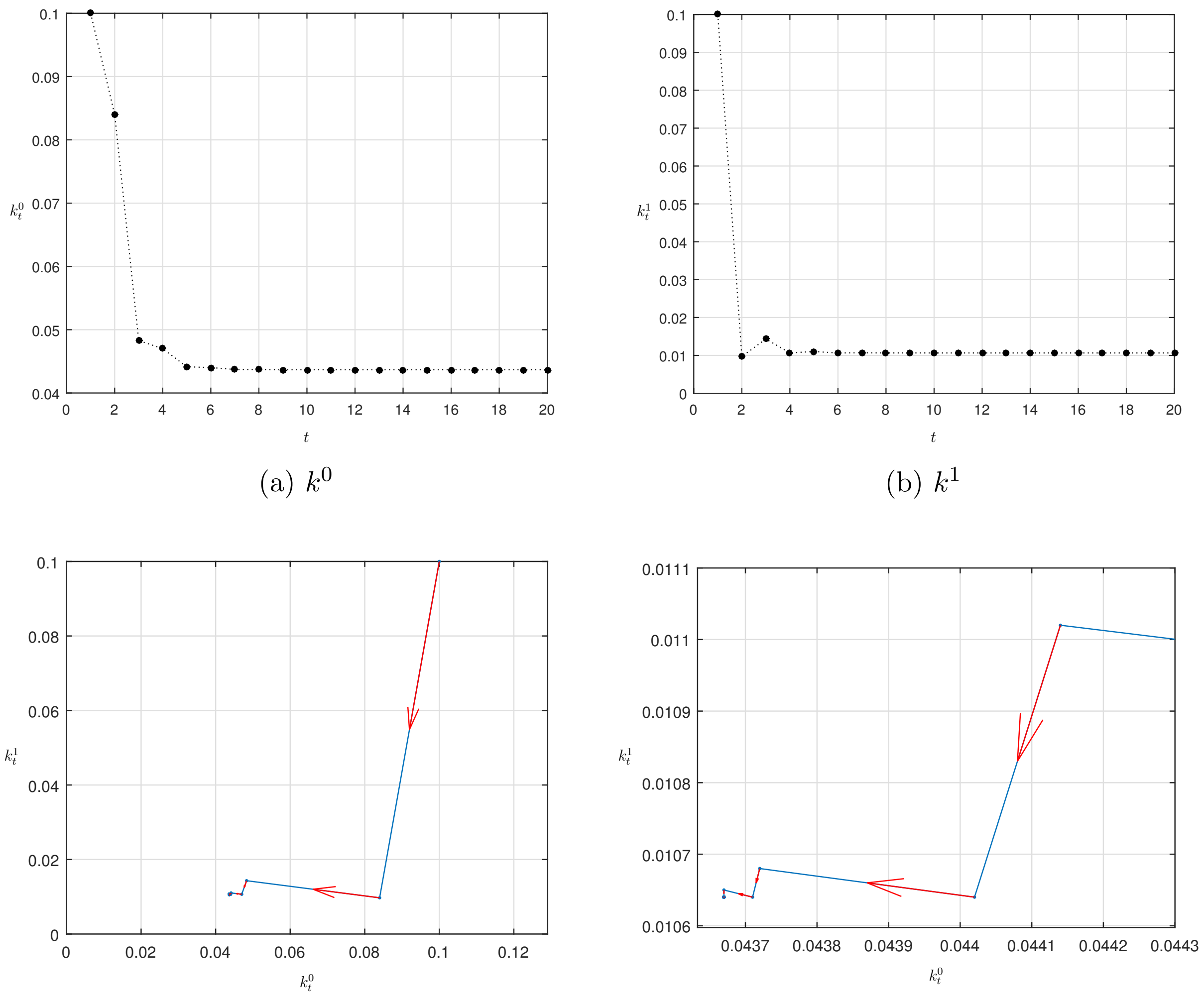



(a) $k^0$ (b) $k^1$

(c) Evolution of the image (d) Evolution near steady state

Figure 4.3: Dynamics with $\ell^1 = \frac{1}{2}$; $\alpha = 0.1$, $\beta = 0.2$; initial value $\mathbf{k}_1 = (0.1, 0.1)$

steady state value. In particular it is not alternating as Figure 4.3(a) for $t = 1, 2, 3$ shows. In Figure 4.4 the convergence in total is little oscillating. In particular there is a repeating pattern. The oscillating pattern is due to the evolution of $k_t^1$ (Figure 4.4(b)). $k_t^0$ seems to converge monotone again. Again the figures in this subsection have been chosen exemplary. In Appendix B figures B.5 to B.39, show the same figures for other values of $\ell^1 \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. For each parameter set the evolution of the dynamics has been visualized for at least three initial values: $(0.1, 0.1)$, $(10, 10)$ and one near the steady state. Figures 4.5 and 4.6 show that the pattern of dynamics depends on the initial value. Both initial values $\mathbf{k}_1$ are near the steady state ($\bar{k} \hat{=} (0.044, 0.011)$), but in the first figure the convergence is oscillating in both components ($k_t^0$ and $k_t^1$) and in the second it is monotone. All figures that show the convergence for any parameter set, point out that the convergence is very fast. In most economies it takes at most ten steps to be that near to the steady state value that in the




(a) $k^0$

(b) $k^1$

(c) Evolution of $\mathbf{k}$

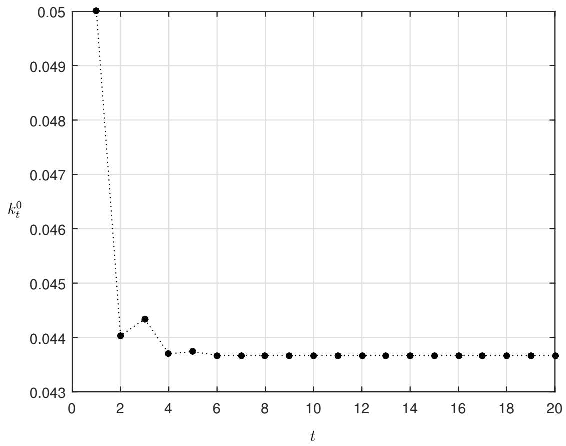
(d) Evolution near steady

Figure 4.4: Dynamics with $\ell^1 = \frac{1}{2}$, $\alpha = 0.33$ $\beta = 0.4$; initial value $\mathbf{k_1} = (0.1,\ 0.1)$

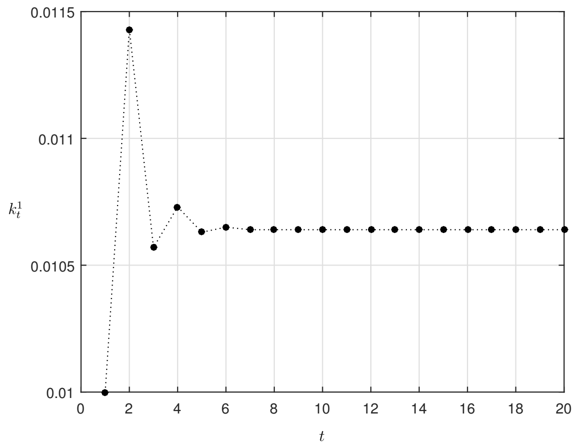
figure no difference in the values may be recognized. Figures 4.4 to 4.7 indicate for the particular economy that convergence is independent of initial values. Thus, the steady state probably is globally asymptotically stable as well. Computing the iterations for different parameters and initial values substantiate this observation and shows the fast convergence as well.

### 4.1.3 Comparison to the Two-Period Economy

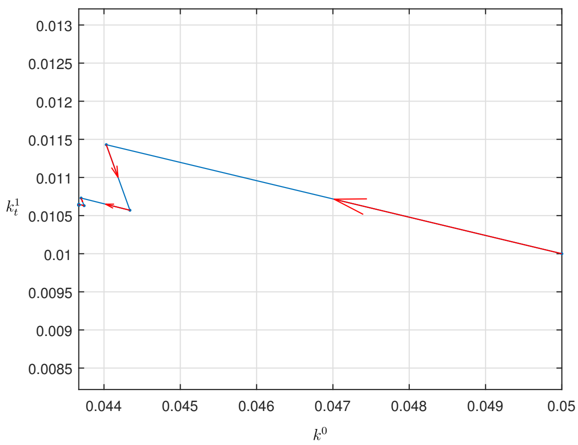
Analogous to the equilibrium structure (see Section 3.3) this section compares the stability properties of the two-period to the three-period economy. The previous Subsection 4.1.2 shows the properties of the three-period economy with retirement that are completed by the considerations in Appendix B. Appendix A contains the corresponding results and figures




(a) $k^0$ for $\mathbf{k_1} = (0.05,\ 0.01)$



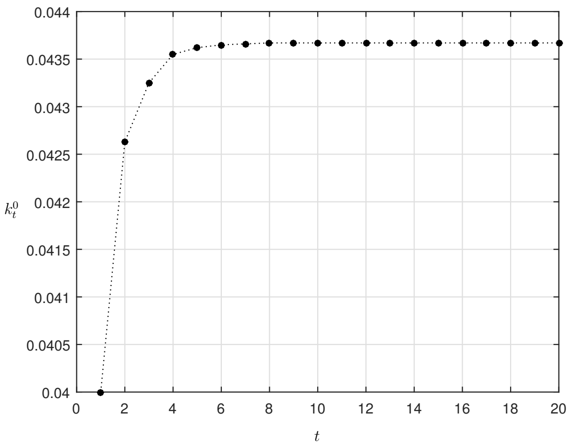

(b) $k^1$ for $\mathbf{k_1} = (0.05,\ 0.01)$



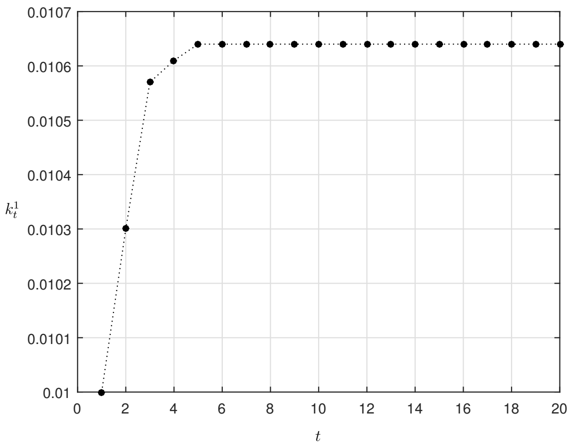

(c) $\mathbf{k_1} = (0.05,\ 0.01)$

Figure 4.5: Dynamics with $\ell^1 = \frac{1}{2}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k_1} = (0.05,\ 0.01)$

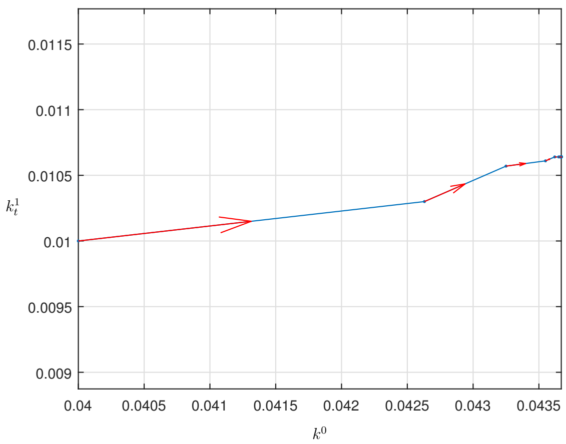
of the two-period economy with old age labor supply. Comparing the steady state value of aggregated capital $\bar{k}^A$ for different parameter values of labor supply of the old respectively the middle-aged there are some similarities: If all labor is supplied by the young $\bar{k}^A$ seems to be increasing in $\beta$ and decreasing in $\alpha$ in both economies. For other values of $\ell^0$ in the two-period respectively $\ell^1$ in the three-period economy it is at most increasing in $\beta$. The maximum value of $\bar{k}^A$ in total decreases in labor supply of the young in both economies. In the two-period economy this fact is very intuitive: The higher the income of the young the higher is their investment and thus capital supply. In the three-period economy one could imagine that the deficit of investment of the young is compensated by investment of the middle-aged when labor supply of the middle-aged increases. Obviously, though there is an evolution in this direction, especially when individual capital supply of the middle-aged is negative, but there is no compensation in total.




(a) $k^0$ for $\mathbf{k_1} = (0.04,\ 0.01)$

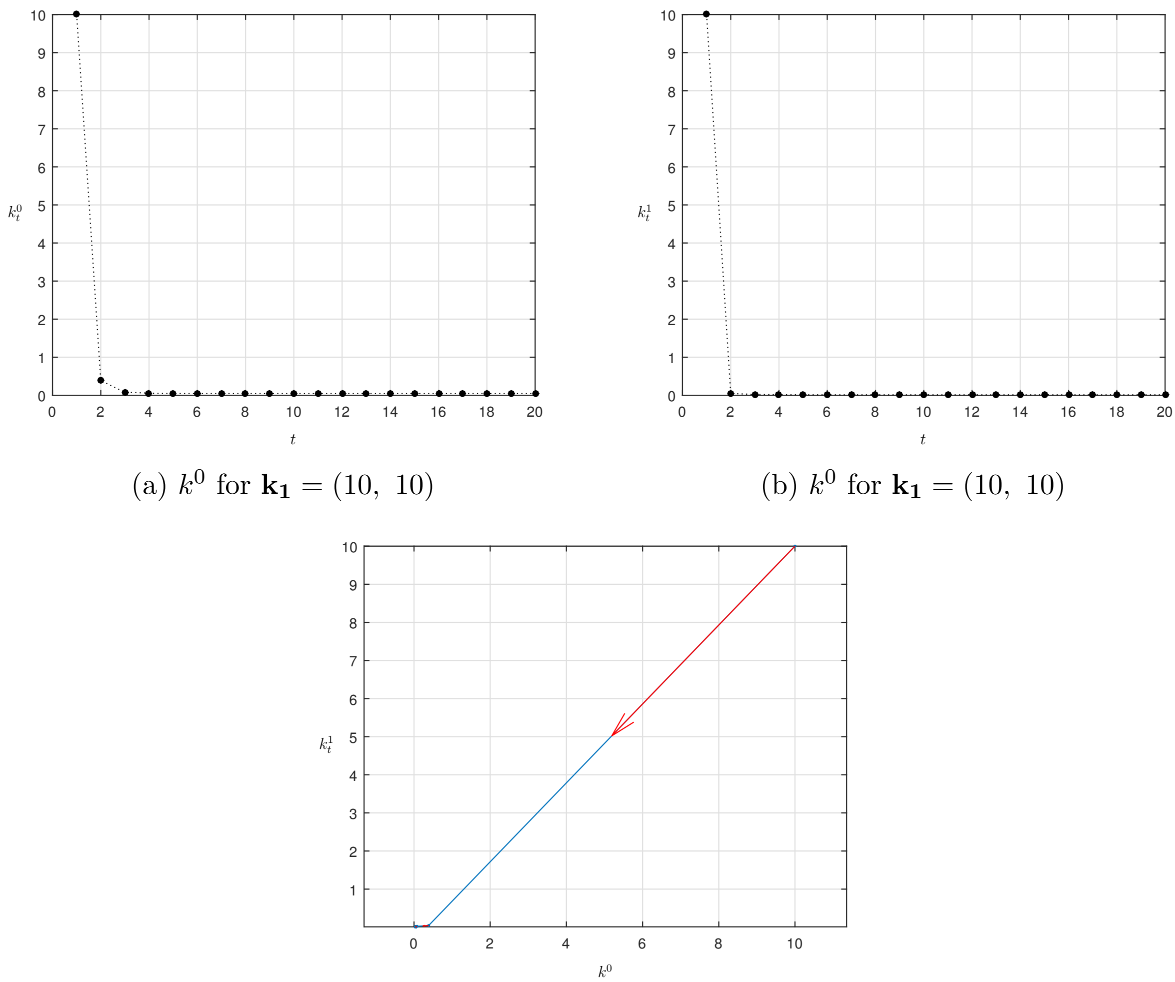



(b) $k^1$




(c) $\mathbf{k}$

Figure 4.6: Dynamics with $\ell^1 = \frac{1}{2}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k_1} = (0.04,\ 0.01)$



(a) $k^0$ for $\mathbf{k_1} = (10,\ 10)$

(b) $k^0$ for $\mathbf{k_1} = (10,\ 10)$

(c) $\mathbf{k_1} = (10,\ 10)$

Figure 4.7: Dynamics with $\ell^1 = \frac{1}{2}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k_1} = (10,\ 10)$

In the two-period economy aggregated capital is equal to capital supply of the old respectively the second generation (started with the young). Naturally, a one to one comparison is not possible because of the different dimensions. The most likely way is to compare aggregated capital supply in the two-period economy to individual capital supply of the middle aged in the three-period economy. Here the observation is that the visualization of the steady state of the two-period economy (Figure A.1) and the three-period economy (Figures 4.1 and B.1 to B.4) are similar for small values of labor supply of the old respectively the middle-aged. In case of high labor supply of the old respectively middle-aged the evolution of capital supply differs very much. One problem in this comparison is that capital supply of the middle-aged may be negative while this is forbidden in the two-period economy. Thus, the differences for a shift of labor supply to the second generation is intuitive.

The stability properties of the two economies are similar as well. Both of them converge towards their unique steady state. With the graphical and numerical foundation, respectively, we assume global stability for both economies. But while the two-period economy converges monotonously a general description of the evolution in the three-period economy is not possible. There are economies where the figures suggest monotone convergence but there are oscillating and other structures as well. Sometimes convergence seems to be monotone in at least one component.

## 4.2 Dynamic Efficiency

As it is well-known there is a lack of efficiency in the two-period overlapping generations economy. Regarding economies with more than two periods the question of efficiency naturally raises. In the first part of this section there are some preparations for the analysis as it needs to be checked which criterion is applicable in the multi-period context. The second part applies the criterion to the tree-period economy with retirement and finally the results are compared to the results of the two-period economy.,

### 4.2.1 Dynamic efficiency in the Multi-Period context

Dynamic efficiency in an overlapping generations economy is defined by the question if it is possible to increase aggregated consumption in one period without having a reduction in any other period. Efficiency is defined for the two-period economy for example in de la Croix and Michel [11]. First of all the definition must be extended to the multi-period context. Before considering dynamic efficiency itself there are some necessary preparations.

**Definition 4.2.1.** *Given* $k_0^A > 0$ *an* aggregated capital allocation *is a sequence* $\{k_t^A\}_{t\geq 0}$ *with* $k_t^A \in \mathbb{R}_{++},\ t \geq 0$. *An aggregated capital allocation* $\{k_t^A\}_{t\geq 0}$ *is called* feasible, *if*

$$f(k_t^A) - k_{t+1}^A \geq 0$$

*for all* $t \geq 0$.
*The set of feasible allocations is denoted by* $\mathbb{A}(k_0^A)$.

Obviously, each Markov equilibrium defines an aggregated capital allocation. Analogous to aggregated capital supply, aggregated consumption in an arbitrary period $t \geq 0$ is defined as

$$c^A := \sum_{i=0}^{I} c^i$$

Summed over the generations consumption and investment must not exceed output in each period $t \geq 0$. This condition is called aggregate resource constraint:

$$c_t^A \leq f(k_t^A) - k_{t+1}^A \quad \forall t \geq 0$$

In particular in each equilibrium equality holds because of the monotonicity properties of the utility function (see Assumption 2.3.1). In equilibrium aggregated consumption is defined in terms of aggregated capital and feasibility implies that aggregated consumption is positive. By the requirement of consumption to be positive the aggregated capital allocation defined by a Markov equilibrium is feasible. Now, all preparations are in place to define a dynamically efficient allocation.

**Definition 4.2.2.** *Given $k_0^A > 0$. A feasible aggregated capital allocation $\left\{k_t^A\right\}_{t\geq 0}$ is called* dynamically efficient *if there is no feasible allocation $\left\{\tilde{k}_t^A\right\}_{t\geq 0}$ such that*

$$f(k_t^A) - k_{t+1}^A \leq f(\tilde{k}_t^A) - \tilde{k}_{t+1}^A \quad \forall t \geq 0$$

*where the inequality holds strictly for at least one $t_0$.*
*Otherwise the allocation is called dynamically inefficient.*

The definition represents the introducing question. Note that efficiency is independent of preferences. In fact by definition dynamic efficiency only uses aggregated values of capital and the capital distribution is irrelevant. This observation is crucial. As the objective of this section is an analysis of the derived equilibrium with respect to dynamic efficiency the interest is in an efficiency criterion. As it is known for the two-period economy there are such criteria due to Cass [8] (Theorem 3 and Example 1). He exposes overaccumulation of capital as the source of inefficiency. The requirements and proofs only depend on aggregated capital.

**Corollary 4.2.1.** *The criteria for efficiency are the same as in the two-period economy where the capital evolution is replaced by the evolution of aggregated capital.*

For completeness Theorems 4.2.1 and 4.2.2 repeat the criteria in the current context.

**Theorem 4.2.1** (Efficiency for asymptotically stationary allocations)**.** *Let $\left\{k_t^A\right\}_{t\geq 0}$ be a feasible aggregated capital allocation that converges to a constant value $\bar{k} > 0$. Then it is dynamically efficient if*

$$f'(\bar{k}^A) > 1$$

*and dynamically inefficient if*

$$f'(\bar{k}^A) < 1.$$

**Theorem 4.2.2.** *Let $\left\{k_t^A\right\}_{t\geq 0}$ be a feasible aggregated capital allocation. It is dynamically efficient if and only if*

$$\sum_{t=0}^{\infty} \frac{1}{P_t} < \infty$$

*where* $P_0 = 1$ *and*

$$P_{t+1} = \frac{1}{f'\left(k_{t+1}^A\right)} P_t \quad t \geq 0.$$

Note that the consideration here is independent of the economy type. Hence, the criteria may be applied to any equilibrium allocation.

### 4.2.2 Efficiency in the Retirement Economy

This subsection applies the efficiency criterion derived in the previous subsection to the three-period benchmark economy where the old generation is retired. The main question is, how labor supply of the middle-aged influences efficiency. As the steady state of the equilibrium dynamics is shown to be globally asymptotically stable by Lemma 4.1.2 and the simulation, the Cass-Criterion for stable systems is applied. In the following the aggregated capital allocation induced by the equilibrium dynamics will be called *equilibrium allocation*.

**Lemma 4.2.1.** *There is a critical value* $\tilde{\ell}^1 < 1$ *such that the equilibrium allocation is dynamically efficient for all* $\ell^1 > \tilde{\ell}^1$. *In particular, it is*

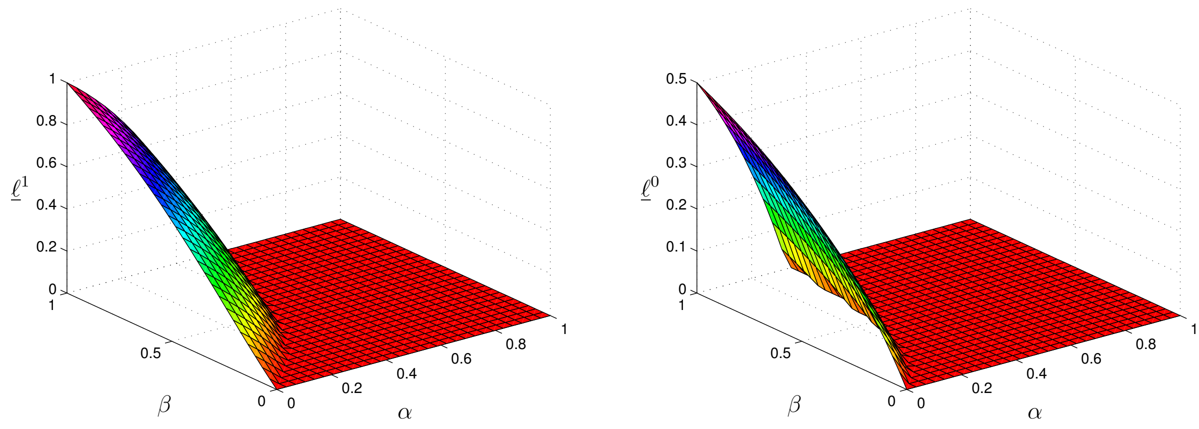
$$\tilde{\ell}^1 = \frac{\beta(1+2\beta)}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}.$$

Obviously, $\tilde{\ell}^1$ determined in Lemma 4.2.1 may be negative. As individual labor supply is required to be positive, let $\underline{\ell}^1 := \max\left\{0, \tilde{\ell}^1\right\}$ denote the critical value in total.
Lemma 4.2.1 states that the more labor is supplied by the middle-aged the higher is the probability that the equilibrium allocation is dynamically efficient. The reason is that dynamic inefficiency is caused by capital overaccumulation. That is, each reduction of investment in total makes dynamic efficiency more probable. Compared to the economy where all labor is supplied by the young the introduction of positive labor supply of the middle-aged corresponds to a reduction of investment in total. The reason is that labor income is split between two periods and lifetime consumption is not only financed by wealth when young, that is divided up between the lifetime periods by investments.

**Lemma 4.2.2.** *The critical value* $\underline{\ell}^1$ *determined above is smaller than* 1, *but there is no* $\underline{\ell}^* < 1$ *such that* $\underline{\ell}^1 < \underline{\ell}^*$ *for all* $\alpha, \beta \in (0,1)$.

**Corollary 4.2.2.** *If all labor is supplied by the middle-aged,* $\ell^1 = 1$, *the equilibrium allocation is dynamically efficient.*

Figure 4.8 shows how the critical value evolves in dependance on $\alpha$ and $\beta$. It validates Corollary 4.2.2.
By the structure of $\underline{\ell}^1$ and Figure 4.8(a) it is obvious that the boundary condition $\underline{\ell}^1 \geq 0$ is binding for many $\alpha$ and $\beta$. This observation leads to a critical value for $\alpha$ as well.




(a) The three-period economy (b) The two-period economy

Figure 4.8: Critical values of labor supply

**Corollary 4.2.3.** *The equilibrium is dynamically efficient if* $\alpha < \frac{\beta(1+2\beta)}{\beta(1+2\beta)+1+\beta+\beta^2}$.

Obviously, this value is smaller than one for all $\beta \in (0,1)$. Thus, a continuation of this principle, determining a critical value for $\beta$ as well, is not possible.

### 4.2.3 Comparison of the Two-Period and the Three-Period Economy

As the same efficiency criterion is applied for both the two- and the three-period economy the results may be compared very well. In both economies the introduction of labor supply of more than one (the young) generation makes dynamic efficiency more probable. In each economy there is a critical value for labor supply of the old respectively the middle-aged such that the equilibrium allocation is dynamically efficient for labor supply above the critical value. Figure 4.8 shows the critical values for the three- and the two-period economy. The figures (a) and (b) look similar. In total without the boundary condition the critical values $\tilde{\ell}^1$ and the corresponding $\tilde{\ell}^0$ in the two-period economy would converge to $-\infty$ for $\alpha \to 1$, but as labor supply is between 0 and 1 figures 4.8(a) and 4.8(b) show that for many $\alpha$ and $\beta$ this condition is binding. Moreover, the image of the critical value in the two-period economy seems to be more 'flat' than in the three-period economy and the maximal value is higher in Figure 4.8(a). In the two-period economy the critical value is smaller than $\frac{1}{2}$ (see Lemma A.0.1) while it is 1 in the three-period economy (see Lemma 4.2.2). Thus, there is always at least one labor profile such that the equilibrium allocation is dynamically efficient. The following lemma validates the observation that the values of the critical value in the two-period economy are smaller than in the three-period economy. Let

$\mathcal{E}_3 = \langle \{0,1,2\}, (0,\ell,1-\ell), ln(c), \beta, k^\alpha \rangle$ and $\mathcal{E}_2 = \langle \{0,1\}, (\ell,1-\ell), ln(c), \beta, k^\alpha \rangle$ define the three- and the two-period economy with the same parameter set, while the old in the three-period economy are retired. Thus besides the parameters $\alpha$ and $\beta$ the division of labor between the two-working generations is the same.

**Lemma 4.2.3.** *Given feasible initial values* $\mathbf{k}$ *for both economies, the following holds: If* $\mathcal{E}_3$ *is dynamically efficient* $\mathcal{E}_2$ *is dynamically efficient as well.*

The result is in some ways intuitive. Think of the special case where all labor is supplied by the young and consider the lifetime behavior of one young agent. In the three-period model investment, when the agent is young, must be large enough to handle consumption when middle-aged and when old as well. The reason is that income during these two periods only consists of capital income. In contrast to that in the two-period model only the consumption of one following period must be taken into account. Thus, capital overaccumulation is more likely in this three-period economy. Note that this observation only holds in the three-period economy with retirement and not for the three-period economy in general.
Finally, there are a few words about the general benchmark economy. The observations in the two-period and three-period economy with retirement suggest that in the general three-period economy dynamic efficiency is even more probable. With the same argumentation as above the intuition is that if the third generation supplies labor as well, overaccumulation occurs less. In an economy with an arbitrary number of lifetime periods one would conjecture that the less generations are retired, the more probable is dynamic efficiency. It is obvious that not only the number of working generations but also the distribution of individual labor supply among the generations influences dynamic efficiency.

## 4.3 Pareto Optimality

In contrast to dynamic efficiency pareto optimality describes optimality with respect to individual utility. Is it possible to make one generation better off without reducing any other agent's utility? Remember that an equilibrium in an overlapping generations economy is not necessarily pareto optimal as the first welfare theorem does not need to hold in this context (cf. Samuelson [21]). The problem is that over time there is an infinite number of agents and goods. Analogous to the previous section the definition of pareto optimality is extended from the well-known two-period case (e.g. de la Croix and Michel [11]) to the multi-period context in the first part before the application is possible. The second part presents a general pareto optimal solution for the benchmark economy with an arbitrary number of lifetime periods.

### 4.3.1 Pareto Optimality in the Multi-Period context

Already the introducing question stresses that now at least the distribution of the consumption is crucial for the inspection if an economy is pareto optimal or not. Hence, in contrast to the previous section, now an allocation is defined that includes this information.

**Definition 4.3.1.** *Given* $(\mathbf{c}_0, k_0^A) \in \mathbb{R}_{++}^{I+2}$*, a* feasible consumption capital allocation *is a sequence* $\left\{(c_t^i)_{i\in\mathbb{I}}, k_t^A\right\}_{t\geq 0} \in \mathbb{R}_{++}^{I+2}$ *satisfying*

$$f(k_t^A) = k_{t+1}^A + c_t^A$$

*for all* $t \geq 0$*. The set of feasible consumption capital allocations is denoted by* $\mathbb{A}^\star\big((\mathbf{c}_0, k_0^A)\big)$*.*

Again feasibility is defined in the context of aggregated values and analogous to the previous section a Markov equilibrium implies a feasible consumption capital allocation. But as the introduction hypothesizes the definition of pareto optimality is in terms of the preferences.

**Definition 4.3.2.** *Given* $k_0 \in \mathbb{K}$*, an allocation* $a \in \mathbb{A}^\star((\mathbf{c}_0, k_0^A))$ *is said to* pareto dominate $\tilde{a} \in \mathbb{A}^\star((\mathbf{c}_0, k_0^A))$ *if*

*(i)* $U_i\left((c_{t+n}^{i-n})_{n=0}^{i}\right) \geq U_i\left((\tilde{c}_{t+n}^{i-n})_{n=0}^{i}\right)$ *for all* $i = 0, ..., I-1$

*(ii)* $U\left((c_{t+i}^{I-i})_{i=0}^{I}\right) \geq U\left((\tilde{c}_{t+i}^{I-i})_{i=0}^{I}\right)$ *for all* $t \geq 0$

*and for at least one* $\hat{t} \geq 0$ *or one* $i \in \mathbb{I}$*, respectively, one of the inequalities holds strictly. A feasible allocation that is not dominated by any other feasible allocation is called* pareto optimal.

Obviously, a pareto optimal allocation is dynamically efficient. Otherwise aggregated consumption could be increased in one period without a reduction in any other period. An increase of aggregated consumption is equivalent to an increase of at least one individual consumption. This increases lifetime utility by Assumption 2.3.1.

**Lemma 4.3.1.** *A pareto optimal allocation* $a \in \mathbb{A}^\star((\mathbf{c}_0, k_0^A))$ *is dynamically efficient and satisfies*

$$u'(c_{t+i}^{I-i}) = \beta u'\left(c_{t+i+1}^{I-i-1}\right) f'(k_{t+i+1}^A), \quad \forall i = 0, ..., I-1 \tag{4.1}$$

Note that the equations (4.1) correspond to the first order conditions of the consumer. Thus, each equilibrium satisfies these equations. In the case of a converging two-period economy this states a criterion for pareto optimality as well: An equilibrium that is dynamically efficient is pareto optimal (compare de la Croix and Michel [11]). This result can be transferred to the multi-period economy.

**Lemma 4.3.2.** *Let* $\left\{\mathbf{c}_t^\star, (k_t^A)^\star\right\}_{t\geq 0}$ *be an consumption capital allocation induced by the equilibrium dynamics of an economy that converges to* $(\bar{\mathbf{c}}, \bar{k^A})$*. Then this equilibrium is pareto optimal if there is underaccumulation of capital and inefficient if there is overaccumulation.*

As over- and underaccumulation of capital are also the criterion for dynamic efficiency in a converging economy, the results how labor supply of the middle-aged influences efficiency in Section 4.2.2 are the same for pareto optimality. Thus, a separate comparison of the two- and the three-period economy for pareto optimality is not necessary.

### 4.3.2 A Pareto Optimal Solution

This subsection considers the general benchmark economy (cf. Section 3.1.1). In particular here a pareto optimal allocation of this economy will be determined. Therefore, usually the so called Social Planning Problem is used: How would a benevolent social planner decide about consumption and investment, taking into account the resource constraint, that is, which feasible allocation would he choose.
First the decision problem will be derived in general: Subject to the feasibility constraint the social planner's objective is to maximize a welfare function that takes into account utility over all generations. Depending on a discount factor $\delta \in (0,1)$ the welfare function $W(a,\delta) : \mathbb{A}(k_0) \times (0,1) \to \mathbb{R}$ is given by

$$\begin{aligned} W(a,\delta) &= \sum_{i=0}^{I-1} \delta^{I-i} U_i\left(\left(c_n^{i-n}\right)_{n=0}^{i}\right) + \sum_{t=0}^{\infty} \delta^t U\left(\left(c_{t+i}^{I-i}\right)_{i=0}^{I}\right) \\ &= \sum_{i=0}^{I-1} \delta^{I-i} \sum_{j=0}^{i} \beta^{j+I-i} u\left(c_j^{i-j}\right) + \sum_{t=0}^{\infty} \delta^t \sum_{j=0}^{I} \beta^j u\left(c_{t+j}^{I-j}\right) \\ &= \sum_{t=0}^{\infty} \delta^t \sum_{i=0}^{I} \left(\left(\frac{\beta}{\delta}\right)^{I-i} u(c_t^i)\right). \end{aligned}$$

Defining $v\left(\left(c^i\right)_{i=0}^{I}\right) := \sum_{i=0}^{I}\left(\left(\frac{\beta}{\delta}\right)^{I-i} u(c_t^i)\right)$ it is

$$W(a,\delta) = \sum_{t=0}^{\infty} \delta^t v\left(\left(c_t^i\right)_{i=0}^{I}\right). \tag{4.2}$$

and the Social Planning Problem reads:

$$\max_a \left\{W(a,\delta), a \in \mathbb{A}(k_0)\right\}$$

Now, consider the benchmark economy again.

**Lemma 4.3.3.** *Let* $k_0^A > 0$ *be given,* $u(c) = ln(c)$ *and* $f(k) = k^\alpha$, $\alpha \in (0,1)$. *For each discount factor* $\delta \in (0,1)$ *there exists a solution* $a^*$ *of the Social Planning Problem that only depends on aggregated capital. In particular it is*

$$c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1-\alpha\delta}{\sum_{i=0}^{I}\left(\frac{\delta}{\beta}\right)^i} f(k^A) \qquad (k')^A = \alpha\delta f(k^A)$$

*where* $c^j$ *denotes consumption and* $k'$ *future capital.*

Note that there are no structural assumptions, so it holds for all log utility economies. Thus given capital supply $\mathbf{k} \in \mathbb{K}$ there is a redistribution of capital supply such that for a suitable labor profile $\ell$ the solution is pareto optimal. In general, the pareto optimal solution is a mapping on $\mathbb{K}$ as the restrictions on the state space are based on the condition that consumption is positive. This is guaranteed in this solution.

## 4.4 Mathematical Appendix

***Proof of Lemma 4.1.1.*** First of all derive the characteristic polynomial of the Jacobian of $\mathcal{K}(\mathbf{k}) = (a_0^1\omega^1(\mathbf{k}) + a_0^2\omega^2(\mathbf{k}), a_1^1\omega^1(\mathbf{k}) + a_1^2\omega^2(\mathbf{k}))$:

$$\mathcal{DK}(\mathbf{k}) = \begin{pmatrix} a_0^1\frac{\partial\omega^1(\mathbf{k})}{\partial k^0} + a_0^2\frac{\partial\omega^2(\mathbf{k})}{\partial k^0} & a_0^1\frac{\partial\omega^1(\mathbf{k})}{\partial k^1} + a_0^2\frac{\partial\omega^2(\mathbf{k})}{\partial k^1} \\ \\ a_1^1\frac{\partial\omega^1(\mathbf{k})}{\partial k^0} + a_1^2\frac{\partial\omega^2(\mathbf{k})}{\partial k^0} & a_1^1\frac{\partial\omega^1(\mathbf{k})}{\partial k^1} + a_1^2\frac{\partial\omega^2(\mathbf{k})}{\partial k^1} \end{pmatrix}$$

Using the following properties of the wealth functions

$$\begin{aligned} \frac{\partial\omega^2(\mathbf{k})}{\partial k^0} &= \frac{\partial\omega^2(\mathbf{k})}{\partial k^0} = \ell^2(1-\alpha)\alpha(k^A)^{\alpha-1} \\ \frac{\partial\omega^1(\mathbf{k})}{\partial k^0} &= \ell^1(1-\alpha)\alpha(k^A)^{\alpha-1} + \frac{k^1}{k^A}\alpha(\alpha-1)(k^A)^{\alpha-1} \\ \frac{\partial\omega^1(\mathbf{k})}{\partial k^1} &= \frac{\partial\omega^1(\mathbf{k})}{\partial k^0} + \alpha(k^A)^{\alpha-1} \end{aligned}$$

the characteristic polynomial $C(\lambda, \mathbf{k})$ is given by:

$$C(\lambda, \mathbf{k}) = \lambda^2 - \lambda\ \mathrm{tr}(\mathcal{DK}(\mathbf{k})) + \det(\mathcal{DK}(\mathbf{k}))$$

with

$$
\begin{aligned}
\operatorname{tr}(\mathcal{DK}(\mathbf{k})) =& a_0^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + \frac{\partial \omega^1(\mathbf{k})}{\partial k^1} a_1^1 + (a_0^2 + a_1^2) \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \\
=& (a_0^1 + a_1^1) \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^1 \alpha (k^A)^{\alpha-1} + (a_0^2 + a_1^2) \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \\
=& \alpha (k^A)^{\alpha-1} \left( (a_0^1 + a_1^1) \left( \ell^1 (1-\alpha) - (1-\alpha) \frac{k^1}{k^A} \right) + a_1^1 + (a_0^2 + a_1^2)(1-\alpha)\ell^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
\det(\mathcal{DK}(\mathbf{k})) =& \left( a_0^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_0^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \right) \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^1} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \right) \\
& - \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \right) \left( a_0^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^1} + a_0^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \right) \\
=& a_0^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^1 \alpha (k^A)^{\alpha-1} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \right) \\
& + a_0^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^1 \alpha (k^A)^{\alpha-1} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^1} \right) \\
& - a_0^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \right) \\
& - a_0^1 \alpha (k^A)^{\alpha-1} \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \right) - a_0^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \left( a_1^1 \frac{\partial \omega^1(\mathbf{k})}{\partial k^0} + a_1^2 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} \right) \\
=& a_0^2 a_1^1 \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} (k^A)^{\alpha-1} - a_0^1 a_1^2 \alpha \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} (k^A)^{\alpha-1} \\
=& -\alpha \frac{\partial \omega^2(\mathbf{k})}{\partial k^0} (k^A)^{\alpha-1} \det(A) \\
=& -\alpha^2 \left( (k^A)^{\alpha-1} \right)^2 \ell^2 (1-\alpha) \det(A) \\
<& 0
\end{aligned}
$$

In sum it is

$$
\begin{aligned}
C(\lambda, \mathbf{k}) =& \lambda^2 - \lambda \alpha (k^A)^{\alpha-1} \left( (a_0^1 + a_1^1) \left( \ell^1 (1-\alpha) - (1-\alpha) \frac{k^1}{k^A} \right) + a_1^1 + (a_0^2 + a_1^2)(1-\alpha)\ell^2 \right) \\
& - \alpha^2 \left( (k^A)^{\alpha-1} \right)^2 \ell^2 (1-\alpha) \det(A).
\end{aligned}
$$

By the structure of the characteristic polynomial it is obvious that there is one positive eigenvalue $\lambda_+$ and a negative one $\lambda_-$. Thus, it is of the following structure:

$$
C(\lambda, \mathbf{k}) = \lambda^2 + p(\mathbf{k})\lambda - q(\mathbf{k})
$$

with $q(\mathbf{k}) > 0$ and the following estimation holds:

Case 1: $p(\mathbf{k}) \geq 0$

$$
\lambda_+ = -\frac{p(\mathbf{k})}{2} + \sqrt{\frac{p(\mathbf{k})^2}{4} + q(\mathbf{k})} \leq -\frac{p(\mathbf{k})}{2} + \frac{p(\mathbf{k})}{2} + \sqrt{q(\mathbf{k})} = \sqrt{q(\mathbf{k})}
$$

Case 2: $p(\mathbf{k}) < 0$

$$\lambda_- = -\frac{p(\mathbf{k})}{2} - \sqrt{\frac{p(\mathbf{k})^2}{4} + q(\mathbf{k})} \geq -\frac{p(\mathbf{k})}{2} - \frac{-p(\mathbf{k})}{2} - \sqrt{q(\mathbf{k})} = -\sqrt{q(\mathbf{k})}$$

Obviously, for $\sqrt{q(\mathbf{k})} < 1$ one eigenvalue is necessarily smaller than one according to amount. Note that in the stability criterion the eigenvalues in the steady state are evaluated. Thus, the interest is in $\sqrt{q(\bar{\mathbf{k}})}$. Set $\kappa := (1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1$ to check the root of the constant term:

$$\begin{aligned}
\sqrt{q(\bar{\mathbf{k}})} &= \alpha(\bar{k}^A)^{\alpha-1}\sqrt{\ell^2(1-\alpha)\det(A)} \\
&= \frac{2\alpha\sqrt{\ell^2(1-\alpha)\det(A)}}{\kappa + \sqrt{(\kappa)^2 + 4\alpha\ell^2(1-\alpha)\det(A)}} \\
&< 1
\end{aligned}$$

as in the equilibrium dynamics $\kappa = (1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1 > 0$ holds:

$$\begin{aligned}
&(1-\alpha)((a_0^1 + a_1^1)\ell^1 + (a_0^2 + a_1^2)\ell^2) + \alpha a_1^1 \\
=&\frac{\beta\alpha}{\det(\tilde{B}_A)}\left((1-\alpha)\left(\alpha(1+\beta-a_0^1)\ell^1 + \alpha(1+\beta)\ell^2\right) + \alpha(1-\alpha)(\ell^1(a_0^1 - 1) + a_0^2\ell^2)\right) \\
=&\frac{\beta\alpha^2(1-\alpha)}{\det(\tilde{B}_A)}\left(\beta\ell^1 + (1+\beta+a_0^2)\ell^2\right) \\
>&0
\end{aligned}$$

by Corollary 3.2.1. □

***Proof of Lemma 4.1.2.*** The dynamical system is asymptotically stable if all eigenvalues of the Jacobian matrix in the steady state are smaller than one according to amount (see e.g. Sorger [23]). The characteristic polynomial of the Jacobian in the general three-period economy has been derived in the proof of Lemma 4.1.1:

$$\begin{aligned}
C(\lambda,\mathbf{k}) =&\lambda^2 - \lambda\alpha(k^A)^{\alpha-1}\left((a_0^1 + a_1^1)\left(\ell^1(1-\alpha) - (1-\alpha)\frac{k^1}{k^A}\right) + a_1^1 + (a_0^2 + a_1^2)(1-\alpha)\ell^2\right) \\
&- \alpha^2\left((k^A)^{\alpha-1}\right)^2\ell^2(1-\alpha)\det(A).
\end{aligned}$$

Here the coefficients are

$$\begin{aligned}
a_0^1 &= \frac{\beta}{1+\beta} & a_1^1 &= -\frac{\frac{\beta}{1+\beta}\ell^1(1-\alpha)}{\alpha(1+\beta+\beta^2) + \ell^1(1-\alpha)} \\
a_0^2 &= 0 & a_1^2 &= \frac{\alpha\beta(1+\beta)}{\alpha(1+\beta+\beta^2) + \ell^1(1-\alpha)}.
\end{aligned}$$

Thus, it is

$$
\begin{aligned}
a_0^1 + a_1^1 &= \frac{\alpha\beta(1+\beta+\beta^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\frac{1}{1+\beta}\\
a_0^2 + a_1^2 &= \frac{\alpha\beta(1+\beta)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\\
\det A = a_0^1a_1^2 - a_1^1a_0^2 &= \frac{\alpha\beta^2}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\\
\ell^2 &= 1-\ell^1.
\end{aligned}
$$

With these terms the trace and the determinant of the Jacobian are

$$
\begin{aligned}
\operatorname{tr}(\mathcal{DK}(\mathbf{k})) &= \alpha(k^A)^{\alpha-1}\left[(a_0^1+a_1^1)\left(\ell^1(1-\alpha)-(1-\alpha)\frac{k^1}{k^A}\right)+a_1^1+(a_0^2+a_1^2)(1-\alpha)\ell^2\right]\\
&= \frac{\alpha\beta(k^A)^{\alpha-1}}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left[\frac{\alpha(1+\beta+\beta^2)}{1+\beta}\left(\ell^1(1-\alpha)-(1-\alpha)\frac{k^1}{k^A}\right)\right.\\
&\qquad\left.-\ell^1(1-\alpha)\frac{1}{1+\beta}+\alpha(1+\beta)(1-\alpha(1-\ell^1)\right]\\
&= \frac{\alpha\beta(k^A)^{\alpha-1}(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left[\frac{\ell^1}{1+\beta}\left(\alpha(1+\beta+\beta^2)-1-\alpha(1+\beta)^2\right)\right.\\
&\qquad\left.-\frac{\alpha(1+\beta+\beta^2)}{1+\beta}+\alpha(1+\beta)\right]\\
&= \frac{\alpha\beta(k^A)^{\alpha-1}(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left[\alpha(1+\beta)-\frac{\ell^1(1+\alpha\beta)}{1+\beta}-\frac{\alpha(1+\beta+\beta^2)}{1+\beta}\frac{k^1}{k^A}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\det(\mathcal{DK}(\mathbf{k})) &= -\alpha^2\left((k^A)^{\alpha-1}\right)^2\ell^2(1-\alpha)\det(A)\\
&= -\frac{\alpha^3\beta^2\left((k^A)^{\alpha-1}\right)^2(1-\ell^1)(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}
\end{aligned}
$$

Altogether the characteristic polynomial is given by

$$
\begin{aligned}
C(\lambda,\bar{\mathbf{k}}) =& \lambda^2 - \frac{\alpha^3\beta^2\left((\bar{k}^A)^{\alpha-1}\right)^2(1-\alpha)(1-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\\
&-\lambda\frac{\alpha\beta(\bar{k}^A)^{\alpha-1}(1-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left[\alpha(1+\beta)-\frac{\ell^1(1+\alpha\beta)}{1+\beta}-\frac{\alpha(1+\beta+\beta^2)}{1+\beta}\frac{\bar{k}^1}{\bar{k}^A}\right]
\end{aligned}
$$

In the steady state it is

$$\begin{aligned}
\frac{\bar{k}^1}{\bar{k}^A} &= \frac{\alpha\beta(1+\beta)(1-\alpha)(1-\ell^1)(\bar{k}^A)^{\alpha-1} - \ell^1(1-\alpha)}{\alpha(1+\beta+\beta^2)} \\
(\bar{k}^A)^{\alpha-1} &= \frac{2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)}{\alpha\beta\left((1-\alpha)(1+\beta-\ell^1)+\sqrt{\kappa}\right)} \\
\sqrt{\kappa} &:= \sqrt{(1-\alpha)^2(1+\beta-\ell^1)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}
\end{aligned}$$

Thus, $C(\lambda, \bar{\mathbf{k}})$ can be written in terms of $(\bar{k}^A)^{\alpha-1}$:

$$\begin{aligned}
C(\lambda,\bar{\mathbf{k}}) =& \lambda^2 - \frac{\alpha^3\beta^2\left((\bar{k}^A)^{\alpha-1}\right)^2(1-\alpha)(1-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)} \\
&- \lambda\frac{\alpha\beta(1-\alpha)(\bar{k}^A)^{\alpha-1}}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\Bigg(\alpha(1+\beta)-\ell^1\frac{1+\alpha\beta}{1+\beta} \\
&\qquad - \frac{\alpha\beta(1+\beta)(1-\alpha)(1-\ell^1)(\bar{k}^A)^{\alpha-1}-\ell^1(1-\alpha)}{1+\beta}\Bigg) \\
=& \lambda^2 - \frac{\alpha^3\beta^2\left((\bar{k}^A)^{\alpha-1}\right)^2(1-\alpha)(1-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)} \\
&- \lambda\frac{\alpha^2\beta(1-\alpha)(\bar{k}^A)^{\alpha-1}}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left(1+\beta-\ell^1-\beta(1-\alpha)(1-\ell^1)(\bar{k}^A)^{\alpha-1}\right)
\end{aligned}$$

The characteristic polynomial is a quadratic polynomial with positive leading coefficient. As the roots define the eigenvalues, all eigenvalues would be smaller than one if $C(1,\bar{\mathbf{k}}) > 0$. Thus, have a look at $C(1,\bar{\mathbf{k}})$:

$$C(1,\bar{\mathbf{k}}) = 1 - \frac{\alpha^2\beta(1-\alpha)(1+\beta-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}(\bar{k}^A)^{\alpha-1} + \frac{\alpha^2\beta^2(1-\alpha)(1-\ell^1)(1-\alpha-\alpha)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left((\bar{k}^A)^{\alpha-1}\right)^2$$

Set $\kappa := \frac{2}{\left((1-\alpha)(1+\beta-\ell^1)+\sqrt{(1-\alpha)^2(1+\beta-\ell^1)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\right)^2} > 0$. Thus,

$$\begin{aligned}
C(1,\bar{\mathbf{k}}) =& \kappa\Bigg[(1-\alpha)^2(1+\beta-\ell^1)^2+(1-\alpha)(1+\beta-\ell^1)\sqrt{(\kappa)} \\
&+ 2(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha) \\
&- \alpha(1-\alpha)^2(1+\beta-\ell^1)^2-\alpha(1-\alpha)(1+\beta-\ell^1)\sqrt{(\kappa)} \\
&+ 2(1-\alpha)(1-\ell^1)(1-2\alpha)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\Bigg] \\
=& \kappa\Bigg[(1-\alpha)^3(1+\beta-\ell^1)^2+\sqrt{(\kappa)}\,(1-\alpha)^2(1+\beta-\ell^1) \\
&+ 2(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))(1+1-2\alpha)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
=&\kappa\Bigg[(1-\alpha)^3(1+\beta-\ell^1)^2+\sqrt{(\kappa)}\,(1-\alpha)^2(1+\beta-\ell^1)\\
&+4(1-\ell^1)(1-\alpha)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\Bigg]\\
>&0
\end{aligned}
$$

Proceed analogously with $C(-1,\bar{\mathbf{k}})>0$. From the following transformation it can be concluded that each eigenvalue is greater than $-1$:

$$
C(-1,\bar{\mathbf{k}})=1+\frac{\alpha^2\beta(1-\alpha)(1+\beta-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}(\bar{k}^A)^{\alpha-1}-\frac{\alpha^2\beta^2(1-\alpha)(1-\ell^1)}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\left((\bar{k}^A)^{\alpha-1}\right)^2
$$

With $\bar{k}^A$ and $\kappa$ it is:

$$
\begin{aligned}
C(-1)=&\kappa\Bigg[(1-\alpha)^2(1+\beta-\ell^1)^2+(1-\alpha)(1+\beta-\ell^1)\sqrt{(\kappa)}\\
&+2(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\\
&+\alpha(1-\alpha)^2(1+\beta-\ell^1)^2+\alpha(1-\alpha)(1+\beta-\ell^1)\sqrt{(\kappa)}\\
&-2(1-\alpha)(1-\ell^1)(1-2\alpha)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\Bigg]\\
=&\kappa\Bigg[(1+\alpha)\left((1-\alpha)^2(1+\beta-\ell^1)^2+(1-\alpha)(1+\beta-\ell^1)\sqrt{(\kappa)}\right)\Bigg]\\
>&0
\end{aligned}
$$

Thus, each eigenvalue is greater than $-1$ and smaller than 1 and therefore the steady state of the equilibrium dynamics is asymptotically stable. □

***Proof of Lemma 4.2.1***. In a first step show that $f'(\bar{k}^A)$ is strictly increasing in $\ell^1$ for $\ell^1\in[0,1]$:

$$
\begin{aligned}
&f'(\bar{k}^A)=\alpha\left(\bar{k}^A\right)^{\alpha-1}\\
=&\frac{2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)}{\beta\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{(1+\beta-\ell^1)^2(1-\alpha)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))}\right)}
\end{aligned}
$$

This term is deviated with respect to $\ell^1$. For purposes of a compact notation define

$$
\kappa:=(1+\beta-\ell^1)^2(1-\alpha)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)).
$$

Then, the derivative of $f'$ with respect to $\ell^1$ is given by:

$$\begin{aligned}\frac{\partial f'(\bar{k}^A)}{\partial \ell^1} =& \frac{2}{\beta}\left(\frac{(1-\alpha)\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{\kappa}\right)}{\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{\kappa}\right)^2} + \frac{(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))(1-\alpha)}{\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{\kappa}\right)^2}\right.\\
&\left. - \frac{(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\frac{-2(1-\alpha)^2(1+\beta-\ell^1)+4(1-\ell^1)(1-\alpha)^2-4(1-\alpha)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))}{2\sqrt{\kappa}}}{\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{\kappa}\right)^2}\right)\\
&= \frac{2(1-\alpha)}{\beta\sqrt{\kappa}\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{\kappa}\right)^2}\Big[(1-\alpha)(1+\beta-\ell^1)\sqrt{\kappa}+(1-\alpha)^2(1+\beta-\ell^1)^2\\
&+4(1-\ell^1)(1-\alpha)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\\
&+(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\left[\sqrt{\kappa}+(1-\alpha)(1+\beta-\ell^1)-2(1-\ell^1)(1-\alpha)\right.\\
&\quad\left.+2(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\right]\Big]\\
&= \frac{2(1-\alpha)}{\beta\sqrt{\kappa}\left((1-\alpha)(1+\beta-\ell^1)+\sqrt{\kappa}\right)^2}\Big[\sqrt{\kappa}\left((1+\beta)(1-\alpha)+\alpha(1+\beta+\beta^2)\right)\\
&+(1-\alpha)^2(1+\beta-\ell^1)^2\\
&+\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)\\
&\quad\cdot\left(2(1-\alpha)(1-\ell^1)+(1-\alpha)(1+\beta-\ell^1)+2(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))\right)\Big]\\
&> 0\end{aligned}$$

By the monotonicity of $f'(\bar{k}^A)$ it follows that it is maximal for $\ell^1 = 1$. Setting $\ell^1 = 1$ it reduces to

$$f'(\bar{k}^A) = \frac{(\alpha(1+\beta+\beta^2)+1-\alpha)}{\beta^2(1-\alpha)} > \frac{(1-\alpha)}{\beta^2(1-\alpha)} = \frac{1}{\beta^2} > 1$$

Thus, if all labor is supplied by the middle-aged, the economy is dynamically efficient. Together with the monotonicity it follows that there is a value $\tilde{\ell^1} \in [0,1)$ such that the economy is dynamically efficient for all $\ell^1 > \tilde{\ell^1}$. Now, specify the critical value. The economy is efficient if

$$\frac{2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)}{\beta\left((1+\beta-\ell^1)(1-\alpha)+\sqrt{(1+\beta-\ell^1)^2(1-\alpha)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))}\right)} > 1$$

$\Leftrightarrow$

$$\begin{aligned}&2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)-\beta(1-\alpha)(1+\beta-\ell^1)\\
&\quad > \beta\sqrt{(1+\beta-\ell^1)^2(1-\alpha)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))}\end{aligned} \tag{4.3}$$

Now, distinguish two cases.

Case 1: $\ell^1 < \frac{\beta(1+\beta)(1-\alpha)-2\alpha(1+\beta+\beta^2)}{(1-\alpha)(2+\beta)}$

Note that

$$\begin{aligned} & \ell^1 < \frac{\beta(1+\beta)(1-\alpha)-2\alpha(1+\beta+\beta^2)}{(1-\alpha)(2+\beta)} \\ \Leftrightarrow \quad & 2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)-\beta(1-\alpha)(1+\beta-\ell^1)<0 \end{aligned}$$

Thus, the inequality (4.3) is violated as

$$\beta\sqrt{(1+\beta-\ell^1)^2(1-\alpha)^2+4(1-\alpha)(1-\ell^1)(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha))}>0.$$

In this case the economy is dynamically inefficient.

Case 2: $\ell^1 \geq \frac{\beta(1+\beta)(1-\alpha)-2\alpha(1+\beta+\beta^2)}{(1-\alpha)(2+\beta)}$

Analogous to case one the condition implies that

$$2\left(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\right)-\beta(1-\alpha)(1+\beta-\ell^1)>0.$$

Thus, inequality (4.3) can be transformed as follows:

$$\begin{aligned} & (4.3) \\ \Leftrightarrow \quad & 4\big(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\big)^2-4\big(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\big)\beta(1+\beta-\ell^1)(1-\alpha) \\ & \quad +\beta^2(1+\beta-\ell^1)^2(1-\alpha)^2 \\ & \qquad >\beta^2(1-\alpha)^2(1+\beta-\ell^1)^2+4\beta^2(1-\alpha)(1-\ell^1)\big(\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)\big) \\ \Leftrightarrow \quad & \alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)-\beta(1+\beta-\ell^1)(1-\alpha)-\beta^2(1-\alpha)(1-\ell^1)>0 \\ \Leftrightarrow \quad & \ell^1>\frac{(1-\alpha)\beta(1+2\beta)-\alpha(1+\beta+\beta^2)}{(1+\beta+\beta^2)(1-\alpha)}=\underline{\ell} \end{aligned}$$

What is left to show is that there is no contradiction with the two lower bounds. Obviously,

$$\frac{(1-\alpha)\beta(1+2\beta)-\alpha(1+\beta+\beta^2)}{(1+\beta+\beta^2)(1-\alpha)}>\frac{\beta(1+\beta)(1-\alpha)-2\alpha(1+\beta+\beta^2)}{(1-\alpha)(2+\beta)}$$

holds.

Altogether, the equilibrium allocation is dynamically efficient if and only if $\ell^1 \in \left(\frac{(1-\alpha)\beta(1+2\beta)-\alpha(1+\beta+\beta^2)}{(1+\beta+\beta^2)(1-\alpha)},1\right]$.

□

***Proof of Lemma 4.2.2.*** It has already been seen that the equilibrium is dynamically efficient if $\ell^1 = 1$. The critical value $\tilde{\ell^1}$ is strictly decreasing in $\alpha$, which can easily be verified with the derivation:

$$\begin{aligned}\frac{\partial \underline{\ell}}{\partial \alpha} &= \frac{-(+1+2\beta+3\beta^2)(1-\alpha)(1+\beta+\beta^2)+(1+\beta+\beta^2)((1-\alpha)(1+2\beta)\beta-\alpha(1+\beta+\beta^2)}{(1-\alpha)^2(1+\beta+\beta^2)^2}\\ &= -\frac{1}{(1-\alpha)^2}<0\end{aligned}$$

□

***Proof of Lemma 4.2.3.*** Recall that the equilibria of the economies are given by

$$\begin{aligned}\mathcal{K}_2(k) &= \frac{\beta\alpha(1-\ell)}{\alpha(1+\beta)+\ell(1-\alpha)}(1-\alpha)k^{\alpha},\ \ k\in\mathbb{R}_+\\ \mathcal{K}_3(\mathbf{k}) &= \left(\frac{\beta}{1+\beta}\omega^1(\mathbf{k}),\ \frac{\alpha\beta(1+\beta)\omega^2(\mathbf{k})-\ell^1(1-\alpha)\frac{\beta}{1+\beta}\omega^1(\mathbf{k})}{\alpha(1+\beta+\beta^2)+\ell^1(1-\alpha)}\right)\end{aligned}$$

By Lemma 4.1.2 and Appendix A it is known that the two-period economy converges globally towards its unique steady state

$$\bar{k}_2 = \frac{\beta}{1+\beta}(1-\alpha)k^{\frac{1}{1-\alpha}}$$

and together with the simulation it has been concluded that the three-period economy converges globally as well. Recall that for efficiency only aggregated values are relevant. The aggregated capital in the steady state is

$$\bar{k}_3^A = \left(\beta\frac{(1+\beta)(1-\alpha)+\sqrt{(1-\alpha)((1+\beta)^2(1+\alpha)+2\alpha(1+\beta^2))}}{2(1+\beta+\beta^2)}\right)^{\frac{1}{1-\alpha}}.$$

First consider the two-period economy. It is dynamically efficient if and only if

$$\begin{aligned}f'(\bar{k}_2) &= \frac{\alpha(1+\beta)+\ell(1-\alpha)}{(1-\alpha)\beta(1-\ell)}>1\\ \Leftrightarrow\quad \ell &> \frac{(1-\alpha)\beta-\alpha(1+\beta)}{(1-\alpha)(1+\beta}=:\underline{\ell}_2(<1)\end{aligned}$$

Now, show that $\underline{\ell}_2 < \underline{\ell}^1$ defined in Lemma 4.2.1:

$$\begin{aligned}&\frac{(1-\alpha)\beta(1+2\beta)-\alpha(1+\beta+\beta^2)}{(1-\alpha)(1+\beta+\beta^2)}>\frac{(1-\alpha)\beta-\alpha(1+\beta)}{(1-\alpha)(1+\beta)}\\ \Leftrightarrow\ &(1-\alpha)\beta(1+\beta)(1+2\beta)-\alpha(1+\beta+\beta^2)(1+\beta)\\ &\quad>(1-\alpha)\beta(1+\beta+\beta^2)-\alpha(1+\beta)(1+\beta+\beta^2)\\ \Leftrightarrow\ &(1+\beta)(1+2\beta)>1+\beta+\beta^2 \qquad \checkmark\end{aligned}$$

□

***Proof of Lemma 4.3.1.*** The proof is analogous to that of the two-period economy (see de la Croix and Michel [11]). The efficiency property has been motivated and explained in the preparation of this lemma. So it remains to show that equations 4.1 are a necessary condition for pareto optimality. Take consumption for all generations, except generation $t$, as given, as well as aggregated capital supply $k_t^A$ and $k_{t+I+1}^A$. There are $I$ resource constraints:

$$\begin{aligned} f(k_t^A) &= k_{t+1}^A + c_t^A \\ f(k_{t+1}^A) &= k_{t+2}^A + c_{t+1}^A \\ &\vdots \\ f(k_{t+I}^A) &= k_{t+I+1}^A + c_{t+I}^A \end{aligned}$$

The derivative of lifetime utility with respect to $k_{t+i}^A, \ i = 1, ..., I$ is

$$\begin{aligned} \frac{\partial U\left(\left(c_{t+j}^{I-j}\right)_{j=0}^{I-1}\right)}{\partial k_{t+i}^A} &= \sum_{j=0}^{I-1} \beta^j u'(c_{t+j}^{I-j}) \frac{\partial c_{t+j}^{I-j}}{\partial k_{t+i}^A} \\ &= -u'\left(c_{t+i-1}^{I-j+1}\right) + \beta u'\left(c_{t+i}^{I-i}\right) \beta^i \end{aligned}$$

as $\frac{\partial c_{t+j}^{I-j}}{\partial k_{t+i}^A} = 0$ for $j \neq i-1, i$. The first order condition sets the equations equal to zero. This leads to equations 4.1 □

***Proof of Lemma 4.3.2.*** The following proof is in the style of the proof of Theorem 2 in Okuno and Zilcha [18] combined with the proof of the equivalent result for two-period economies, Lemma 2.6, in de la Croix and Michel [11].

First note that in the multi-period overlapping generations economy considered here it is

$$\sum_{i \in \mathbb{I}} \omega^i(\mathbf{k}_t) = f(k_t^A) > 0 \qquad\qquad \left\| f(k_t^A) \right\| < N \tag{4.4}$$

for a constant $N > 0$. The first inequality holds by definition, the second follows from the fact that the economy converges. As $f$ is an increasing function by (2.1), it holds if there is a maximal respectively a supremum value for aggregated capital. If the convergence is such that aggregated capital always is smaller than aggregated capital in the limit, the limit $\bar{k}^A$ is this value. Otherwise for each $\varepsilon > 0$ there is a $t_0 \geq 0$ such that the equilibrium aggregated capital supply is at least smaller than $\bar{k}^A + \varepsilon$. As the set $\left\{k_t^A\right\}_{t=0}^{t_0}$ is finite there is a maximum $\tilde{k}^A$. Now, define $k_{max}^A := max\left\{\bar{k}^A + \varepsilon, \tilde{k}^A\right\}$ as the required value.

First, show the conjecture that if the equilibrium allocation is not pareto optimal it is $\liminf_{t\to\infty} P_t > 0$ with $P_{t+1} = \frac{1}{f'((k_{t+1}^A)^\star} P_t, \quad P_0 = 1.$

As the equilibrium allocation $\left\{\mathbf{c}_t^\star, (k_t^A)^\star\right\}_{t\geq 0}$ is not pareto optimal, there is a feasible allocation $\left\{\hat{\mathbf{c}}_t, \hat{k}_t^A\right\}_{t\geq 0}$ that is a pareto improvement. Thus

$$\begin{aligned} U\Big(\left(\hat{c}_{t+j}^{I-j}\right)_{j=0}^{I}\Big) &\geq U\Big(\left(\left(c_{t+j}^{I-j}\right)^\star\right)_{j=0}^{I}\Big) \\ U\Big(\left(\hat{c}_{t+j}^{i-j}\right)_{j=0}^{i}\Big) &\geq U\Big(\left(\left(c_{t+j}^{i-j}\right)^\star\right)_{j=0}^{i}\Big) \qquad i = 0, .., I-1 \end{aligned} \tag{4.5}$$

where the strict inequality holds in at least one equation.
Now, define $A_t := \sum_{i=0}^{I} P_{t+i}\Big(\hat{c}_{t+i}^{I-i} - \left(c_{t+i}^{I-i}\right)^\star\Big)$ for $t \geq 0$ and $A_{-i} := \sum_{j=0}^{i} P_j\Big(\hat{c}_j^{i-j} - \left(c_j^{i-j}\right)^\star\Big)$ for $i = 0, ..., I-1$. Next, show that $A_t \geq 0$. The conjecture is without loss of generality shown for $t \geq 0$, the proof for $A_{-i}$ is analogous when the budget set at the end is taken as remaining lifetime budget set.
Assume that $A_t < 0$. Thus

$$\begin{aligned} & \sum_{i=0}^{I} P_{t+i}\hat{c}_{t+i}^{I-i} < \sum_{i=0}^{I} P_{t+i}\left(c_{t+i}^{I-i}\right)^\star \\ \Leftrightarrow \quad & \sum_{i=0}^{I} \frac{1}{\prod_{j=1}^{i} f'\left(\left(k_{t+j}^A\right)^\star\right)} \hat{c}_{t+i}^{I-i} < \sum_{i=0}^{I} \frac{1}{\prod_{j=1}^{i} f'\left(\left(k_{t+j}^A\right)^\star\right)} \left(c_{t+i}^{I-i}\right)^\star = \sum_{i=0}^{I} \frac{\ell^{I-i}\mathcal{W}(\mathbf{k}_{t+i}^\star)}{\prod_{j=1}^{i} f'\left(\left(k_{t+j}^A\right)^\star\right)} \end{aligned} \tag{4.6}$$

The equality in equation (4.6) holds as lifetime consumption discounted on its value in period $t$ must be equal to lifetime labor income discounted to the same period. Obviously, here exists a value $\nu_t$ such that $\left(\hat{c}_t^I + \nu_t, \hat{c}_{t+1}^{I-1}, ..., \hat{c}_{t+I}^I\right)$ belongs to the budget set

$$\mathcal{B}_t\left(\left(k_{\tilde{t}}^A\right)^\star\right)_{\tilde{t}\geq 0}\Big) := \left\{ \left(c_{t+i}^{I-i}\right)_{i=0}^{I} \in \mathbb{R}_{++} \,\middle|\, \sum_{i=0}^{I} \frac{1}{\prod_{j=1}^{i} f'\left(\left(k_{t+j}^A\right)^\star\right)} \left(c_{t+i}^{I-i}\right) = \sum_{i=0}^{I} \frac{\ell^{I-i}\mathcal{W}\left(\left(k_{t+i}^A\right)^\star\right)}{\prod_{j=1}^{i} f'\left(\left(k_{t+j}^A\right)^\star\right)} \right\}$$

This implies

$$\begin{aligned} U\Big(\left(\left(c_{t+j}^{I-j}\right)^\star\right)_{j=0}^{I}\Big) &= \max_{\left(c_{t+i}^{I-i}\right)_{i=0}^{I} \in \mathcal{B}_t\left(\left(k_{\tilde{t}}^A\right)^\star\right)_{\tilde{t}\geq 0}\Big)} U\left(\left(c_{t+i}^{I-i}\right)_{i=0}^{I}\right) \geq U\left(\hat{c}_t^I + \nu_t, \hat{c}_{t+1}^{I-1}, ..., \hat{c}_{t+I}^I\right) \\ &> U\left(\left(\hat{c}_{t+i}^{I-i}\right)_{i=0}^{I}\right). \end{aligned}$$

This states a contradiction. Thus $A_t \geq 0$ and there is at least one $t_0$ where the strict inequality holds by equation (4.5).
The second step is exactly the same as in de la Croix and Michel [11]. It is included here for completeness. Define

$$B_t := P_{t+1}\left[f\left(\left(k_{t+1}^A\right)^\star\right) - f\left(\hat{k}_{t+1}^A\right)\right] - P_t(\hat{k}_{t+1}^A - (k_{t+1}^A)^\star)$$

$B_t \geq 0$ holds as $g(k^A) := f(k^A) - f\left(\left(k_{t+1}^A\right)^\star\right) k$ is a concave function that has it maximum in $\left(k_{t+1}^A\right)^\star$. Thus, it is

$$f\left(\left(k_{t+1}^A\right)^\star\right) - f'\left(\left(k_{t+1}^A\right)^\star\right)\left(k_{t+1}^A\right)^\star \geq g(\hat{k}_{t+1}^A).$$

As $P_t > 0$ by definition, $B_t$ is positive.
In the last step consider for $T > t_0$:

$$\begin{aligned}
0 <& \sum_{t=0}^{T-1} A_t + B_t + \sum_{j=0}^{I-1} A_{-j} \\
=& P_0 \left[ \sum_{i=0}^{I} \hat{c}_0^i - (c_0^i)^\star - \hat{k}_1^A + (k_1^A)^\star \right] \\
&+ \sum_{t=1}^{T-1} P_t \left[ \sum_{i=0}^{I} \hat{c}_t^i - (c_t^i)^\star - \hat{k}_{t+1}^A + (k_{t+1}^A)^\star + f\left(\left(k_{t+1}^A\right)^\star\right) - f\left(\hat{k}_{t+1}^A\right) \right] \\
&+ P_T \left[ \sum_{i=0}^{I-1} \hat{c}_T^i - (c_T^i)^\star + f\left(\left(k_T^A\right)^\star\right) - f\left(\hat{k}_T^A\right) \right] \\
&+ \sum_{t=T+1}^{T+I} P_t \sum_{i=0}^{T+I-t-1} \hat{c}_t^i - (c_t^i)^\star \\
=& P_T \left[ \sum_{i=0}^{I-1} \hat{c}_T^i - (c_T^i)^\star + f\left(\left(k_T^A\right)^\star\right) - f\left(\hat{k}_T^A\right) \right] \\
&+ \sum_{t=T+1}^{T+I} P_t \sum_{i=0}^{T+I-t-1} \hat{c}_t^i - (c_t^i)^\star
\end{aligned}$$

As the first coefficients of $P_t$ are zero by the aggregate resource constraint (compare de la Croix and Michel [11]).

Finally, assume that $\liminf_{t\to\infty} P_t = 0$. This implies that $\sum_{t=0}^{T-1} A_t + B_t + \sum_{j=0}^{I-1} A_{-j}$ approaches to zero for $T \to \infty$ as $\sum_{i=0}^{I-1} \hat{c}_T^i - (c_T^i)^\star$ and $f\left(\left(k_T^A\right)^\star\right) - f\left(\hat{k}_T^A\right)$ are uniformly bounded by (4.4). This states a contradiction.

Hence, the conjecture is proven and the final result can be shown. If the equilibrium is dynamically inefficient, obviously pareto optimality fails by Lemma 4.3.1. The equilibrium is efficient if $f'\left(\left(\bar{k}_T^A\right)^\star\right) > 1$. Thus, the limit of the sequence $P_t$ is zero as the limit of $\frac{1}{f'\left(\left(k_T^A\right)^\star\right)}$ is less than one. Thus, by the conjecture proven before, the equilibrium is pareto optimal. □

***Proof of Lemma 4.3.3.*** Define

$$V(k^A) := \sup_a \{W(a,\delta) | a \in \mathbb{A}(k)\}. \tag{4.7}$$

First show that V is well-defined:
Define $k^A_{max} := \max\left\{k^A_0, \bar{k}^A\right\}$ with $f(\bar{k}) = \bar{k}^A$. By the monotonicity and the concavity of $f$ an upper bound for $f(k_t)$ can be defined: Starting with an initial value where the aggregation is greater than the fixed point, the upper bound for next period's aggregated capital supply gets smaller. Otherwise it increases up to the fixed point. Thus, for all $t \geq 0$ it is:

$$f(k^A_t) \leq f(k^A_{max}) =: y_{max}$$

Using the aggregate resource constraint and taking into account that both aggregated capital and aggregated consumption are assumed to be positive the upper bound holds for these values as well. Moreover, it is even an upper bound for individual consumption as this is positive as well. Thus, the welfare function can be estimated as follows:

$$\begin{aligned}
\Rightarrow \quad W(a,\delta) &= \sum_{t=0}^{\infty} \delta^t \sum_{i=0}^{I} \left( \left(\frac{\beta}{\delta}\right)^{I-i} u((c^i_t) \right) \\
&\leq \left(\frac{\beta}{\delta}\right)^{I} \sum_{t=0}^{\infty} \delta^t \sum_{i=0}^{I} \left( \left(\frac{\beta}{\delta}\right)^{-i} u(y_{max}) \right) \\
&= \left(\frac{\beta}{\delta}\right)^{I} u(y_{max}) \frac{1-\left(\frac{\delta}{\beta}\right)^{I+1}}{1-\frac{\delta}{\beta}} \frac{1}{1-\delta}
\end{aligned}$$

As the welfare function is bounded the maximum exists, and thus $V$ is well-defined.

V is solution to

$$V(k^A) = \sup_{\left((c^j)_{j=0}^I\right)} \left\{ v\left(\left(c^j\right)_{j=0}^I\right) + \delta V\left(f(k^A) - \sum_{j=0}^{I} c^j\right)\right\}$$

Initial guess: $V(k^A) = A \ln k^A + B$

$$\Rightarrow \quad V(k^A) = \sup_{\left((c^j)_{j=0}^I\right)} \left\{ v\left(\left(c^j\right)_{j=0}^I\right) + \delta\left(A \ln\left(f(k^A) - \sum_{j=0}^{I} c^j\right) + B\right)\right\}$$

So the first order conditions read:

$$\frac{1}{c^j}\left(\frac{\beta}{\delta}\right)^{I-j} - \frac{\delta A}{f(k^A) - \sum\limits_{i=0}^{I} c^i} = 0 \quad \Leftrightarrow \quad \left(\frac{\beta}{\delta}\right)^{I-j}\left(f(k^A) - \sum_{i=0}^{I} c^i\right) = \delta A c^j \quad \forall j = 0, \ldots, I \tag{4.8}$$

Therefore, it is

$$c^0 = \frac{f(k^A) - \sum_{i=1}^{I} c^i}{\left(\frac{\delta}{\beta}\right)^I \delta A + 1}$$

and using that value of $c^0$ leads to

$$f(k^A) - \sum_{i=0}^{I} c^i = \frac{\left(\frac{\delta}{\beta}\right)^I \delta A}{\left(\frac{\delta}{\beta}\right)^I \delta A + 1} \cdot \left(f(k^A) - \sum_{i=1}^{I} c^i\right)$$

$$\overset{in\ (4.8)}{\Rightarrow} \quad \delta A c^j = \left(\frac{\delta}{\beta}\right)^{I-j} \frac{\left(\frac{\delta}{\beta}\right)^I \delta A}{\left(\frac{\delta}{\beta}\right)^I \delta A + 1} \left(f(k^A) - \sum_{i=1}^{I} c^i\right) \quad \forall j = 1, \dots, I$$

$$\Leftrightarrow \quad c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + 1} \left(f(k^A) - \sum_{i=1}^{I} c^i\right) \quad \forall j = 1, \dots, I$$

Together it is

$$c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + 1} \left(f(k^A) - \sum_{i=1}^{I} c^i\right) \quad \forall j = 0, \dots, I$$

We continue with a kind of inductionary argument. Assume that for some $l \in \{0, \dots, I-1\}$ the following holds:

$$c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i} \left(f(k^A) - \sum_{i=l+1}^{I} c^i\right) \quad \forall j = 0, \dots, I$$

In the previous step it has been shown, that the assumption holds for $l = 0$.
In particular for $c^{l+1}$ is:

$$c^{l+1} = \left(\frac{\delta}{\beta}\right)^{l+1} \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i} \left(f(k^A) - \sum_{i=l+1}^{I} c^i\right)$$

Solving for $c^{l+1}$ leads to

$$c^{l+1} \left(\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i + \left(\frac{\delta}{\beta}\right)^{l+1}\right) = \left(\frac{\delta}{\beta}\right)^{l+1} \left(f(k^A) - \sum_{i=l+2}^{I} c^i\right)$$

$$\Leftrightarrow \quad c^{l+1} = \left(\frac{\delta}{\beta}\right)^{l+1} \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l+1} \left(\frac{\delta}{\beta}\right)^i} \left(f(k^A) - \sum_{i=l+2}^{I} c^i\right)$$

$$\Rightarrow \quad \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i} \left( f(k^A) - \sum_{i=l+1}^{I} c^i \right)$$

$$= \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i} \left( \frac{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l+1} \left(\frac{\delta}{\beta}\right)^i - \left(\frac{\delta}{\beta}\right)^{l+1}}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l+1} \left(\frac{\delta}{\beta}\right)^i} \right) \left( f(k^A) - \sum_{i=l+2}^{I} c^i \right)$$

$$= \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l+1} \left(\frac{\delta}{\beta}\right)^i} \left( f(k^A) - \sum_{i=l+2}^{I} c^i \right)$$

$$\Rightarrow \quad c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{l} \left(\frac{\delta}{\beta}\right)^i} \left( f(k^A) - \sum_{i=l+2}^{I} c^i \right) \quad \forall j = 0, \ldots, I$$

That is, for all $j = 0, \ldots, I$ the boundary case is

$$c^j = \left(\frac{\delta}{\beta}\right)^j \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{I} \left(\frac{\delta}{\beta}\right)^i} f(k^A)$$

So now $V(k)$ can be determined.

$$v\left(\{c^j\}_{j=0}^I\right) = \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-i} \ln c^j$$

$$= \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \left( \ln \left(\frac{\delta}{\beta}\right)^j + \ln f(k^A) - \ln \left( \left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{i=0}^{I} \left(\frac{\delta}{\beta}\right)^i \right) \right)$$

$$= \alpha \ln k^A \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} + (\ln \delta - \ln \beta) \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \cdot j - \ln \left( \left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \right)$$

$$\Rightarrow \quad V(k^A) = \alpha \ln k^A \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} + (\ln \delta - \ln \beta) \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \cdot j - \ln \left( \left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \right)$$

$$+ \delta A \alpha \ln k^A + \delta A \ln \frac{\left(\frac{\delta}{\beta}\right)^I \delta A}{\left(\frac{\delta}{\beta}\right)^I \delta A + \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j}} + \delta B$$

$$\stackrel{!}{=} A \ln k^A + B$$

$$\Rightarrow \quad A = \alpha \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} + \delta A \alpha \quad \text{and } B \text{ is a function of } A$$

$$\Leftrightarrow \quad A = \frac{\alpha}{1 - \alpha\delta} \sum_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} \qquad B = B(A)$$

$$
\begin{aligned}
\Rightarrow \quad c^i &= \left(\frac{\delta}{\beta}\right)^i \frac{1}{\left(\frac{\delta}{\beta}\right)^I \delta \frac{\alpha}{1-\alpha\delta} \sum\limits_{j=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-j} + \sum\limits_{j=0}^{I} \left(\frac{\delta}{\beta}\right)^j} f(k^A) \\
&= \left(\frac{\delta}{\beta}\right)^j \frac{1-\alpha\delta}{\left(\frac{\delta}{\beta}\right)^I \delta\alpha \sum\limits_{k=0}^{I} \left(\frac{\beta}{\delta}\right)^{I-k} + \sum\limits_{i=0}^{I} \left(\frac{\delta}{\beta}\right)^i (1-\alpha\delta)} f(k^A) \\
&= \left(\frac{\delta}{\beta}\right)^j \frac{1-\alpha\delta}{\alpha\delta \sum\limits_{k=0}^{I} \left(\frac{\delta}{\beta}\right)^k + \sum\limits_{i=0}^{I} \left(\frac{\delta}{\beta}\right)^i (1-\alpha\delta)} f(k^A) \\
\Leftrightarrow \quad c^i &= \left(\frac{\delta}{\beta}\right)^i \frac{1-\alpha\delta}{\sum\limits_{j=0}^{I} \left(\frac{\delta}{\beta}\right)^j} f(k^A)
\end{aligned}
$$

And for $(k')^A$ it follows:

$$
\sum_{i=0}^{I} c^i = (1-\alpha\delta) f(k^A) \quad \Rightarrow \quad (k')^A = f(k^A) - \sum_{i=0}^{I} c^i = \alpha\delta f(k^A) \tag{4.9}
$$

□

# Chapter 5

# Conclusions and Outlook

The multi-period overlapping generations model introduced in this thesis provides a theoretical base for equilibrium analysis. The general and comprehensive description of the model leaves room for specifications as well as extensions. It describes an economy with an arbitrary number of lifetime periods where optimal decisions of all involved parties are explained. Together with the assumption of perfect foresight an equilibrium is defined. The introduction of the concept of a forward-recursive equilibrium, a Markov equilibrium, provides an approach how such an equilibrium may be determined. Already in the general approach the challenges while proving the existence of such an equilibrium have been presented. Following the result of Balasko and Shell [4] that logarithmic utility simplifies the structure of the equilibrium equations the approach has been used for a particular utility and production function . This economy states a benchmark for all future research on multi-period overlapping generations economies. Here it has become obvious that the labor profile is a crucial property of an economy, because for particular labor profiles it turned out to be possible to determine equilibria explicitly. Regarding the two-period economy as a special case of the presented model, we recognized that the two-period economy is captured by these particular labor profiles in any case. From three periods on there are labor profiles such that this is no longer possible. Moreover, three is the smallest number of lifetime periods such that in the equilibrium dynamics cross wealth effects occur. For that reason the three-period economy is analyzed in detail with the result that there is an equilibrium dynamics that can be written as a linear combination of the wealth functions. As the resulting fixed point problem is still manageable even a uniqueness result has been achieved. But already regarding the three-period economy it becomes obvious that the presented approach is an existence approach. In general uniqueness will stay another challenge.

After the derivation of the equilibrium its properties are analyzed. For the three-period economy with retirement local stability has been proven and global stability has been suggested on the base of a numerical simulation, while for the general economy saddle-path stability has been proven. Local or even global stability depend on the coefficients in the equilibrium capital evolution. In a second step the focus was on dynamical efficiency. As the criteria that are commonly used for the two-period economy, only depend on aggregated values, it is possible to transfer them to the present multi-period model without further effort. The only change is that capital supply is replaced by aggregated capital supply. As these two values coincide in the two-period economy, the generalization holds for all considered overlapping generations economies. In the three-period economy with retirement the allocation induced by the Markov equilibrium is the more efficient the more labor is supplied in the second period. This is intuitive as overaccumulation of capital is the source of inefficiency and replacing capital income by labor income in later period reduces investment. By the same reason the two-period economy is more efficient than the three-period economy if the parameters are equal. As a consequence the intuition for the economy without retirement is that efficiency gets even more probable. The consideration of pareto optimality completes the analysis of the equilibrium dynamics. Here again the criteria could be transferred to the general context even if it has been more challenging. As an equilibrium allocation is pareto optimal if it is dynamically efficient, the analysis depending on the labor profile has not been done again. Finally, we presented a pareto optimal allocation for the general benchmark economy with an arbitrary number of lifetime periods that only depends on aggregated capital. In the analysis of the equilibrium dynamics the two- and the three-period economies are compared to see if or how the properties change.

The results of the three-period economy encourage further research on these economies. Besides extending the promising results of the three-period economy proving that there is an equilibrium of the benchmark economy independent of the labor profile and the number of lifetime periods, further utility functions may be looked at. For example, in the more general case of a CES utility function the question would be if the Markov equilibrium still is a function of the wealth functions. Moreover, from the two-period economy several extensions are known. One popular extension of basic overlapping generations model to capture the problem of inefficiency is to introduce a social security system. Note that this extension is already incorporated in our general economy. In the beginning labor income is also called labor and transfer income. That is because a transformation of the labor profile is able to capture any redistribution of wage. The labor profile in its original sense describes how total labor supply is distributed among the generations. A possible modification of the model would be to introduce a social security system that is financed by a proportional tax on labor income and the revenues of the tax are divided up between particular generations. This scenario is equivalent to an economy with a different labor profile: Each generation that pays the tax accounts for a smaller share of total labor supply, such that the labor income is equal to the net income in the model with taxes, and each receiving generation accounts for a higher share, such that labor income equals the transfer income. For example look at a three period economy with labor profile $(0, \frac{1}{2}, \frac{1}{2})$, i.e., the old generation is retired and has no labor income and total labor is divided up between the young and the middle-aged in equal parts. Assume there is a proportional tax on labor income of $\frac{1}{3}$, that finances a transfer payment to the retired. Thus, both the young and middle-aged consumer pay $\frac{1}{6}$ of the current wage as taxes and the old receive $\frac{1}{3}$ of the wage as transfer income. That is equivalent to the scenario when the labor profile is defined as $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

This property of the economy also seems to suggest that for the general three-period economy there are several labor profiles such that the economy is dynamically efficient for all other parameters as it is the case if all labor is supplied by the middle-aged. Indeed, efficiency has been shown for the economy when all labor is supplied by the middle-aged, but introducing old agent's labor supply admits to hope for an equivalent result with labor supply of the young for at least some labor profiles.

Obviously, the common extensions of the two-period economy are worth to consider as well. For example here population is assumed to be constant. Introducing population growth first raises the question if growth is constant or given by a growth function. In the general notation of the model perhaps a growth function is more realistic to capture the different length of periods when changing the number of lifetime periods. Another way to counteract overaccumulation of capital is to introduce governmental consumption. It is always worth to ask if the impact of an extension is smaller or larger in the two-period or multi-period economy. Following the financial crises in 2007 interest in bubbles has increased very much. As it can be seen for example in Barbie and Hillebrand [6], overlapping generations models with stochastic may create bubbly Markov equilibria. A theoretical framework for the analysis how these results evolve with an increase of the number of lifetime periods will be a benefit with its first steps in this dissertation.

# Appendix A

# The Two-Period Economy

As in the thesis there are many comparisons to the two-period economy, take a short look on this economy with old-age labor supply where utility is logarithmic and the production function is of the Cobb-Douglas structure as in Chapter 3. The production side is exactly the same as described in Section 2.1. As the model introduced in this thesis is a generalization of the two-period economy this holds as well for the consumption side. But in the two-period economy there is the possibility of solving the consumer's decision problem directly as it depends only on next period's capital stock. The labor profile reads $\ell = (\ell^0, 1-\ell^0)$. Capital supply is given by $k_t \geq 0$, which defines the prices

$$w_t = \mathcal{W}(k_t) = (1-\alpha)k_t^\alpha \qquad \wedge \qquad r_t = \mathcal{R}(k_t) = \alpha k_t^{1-\alpha}$$

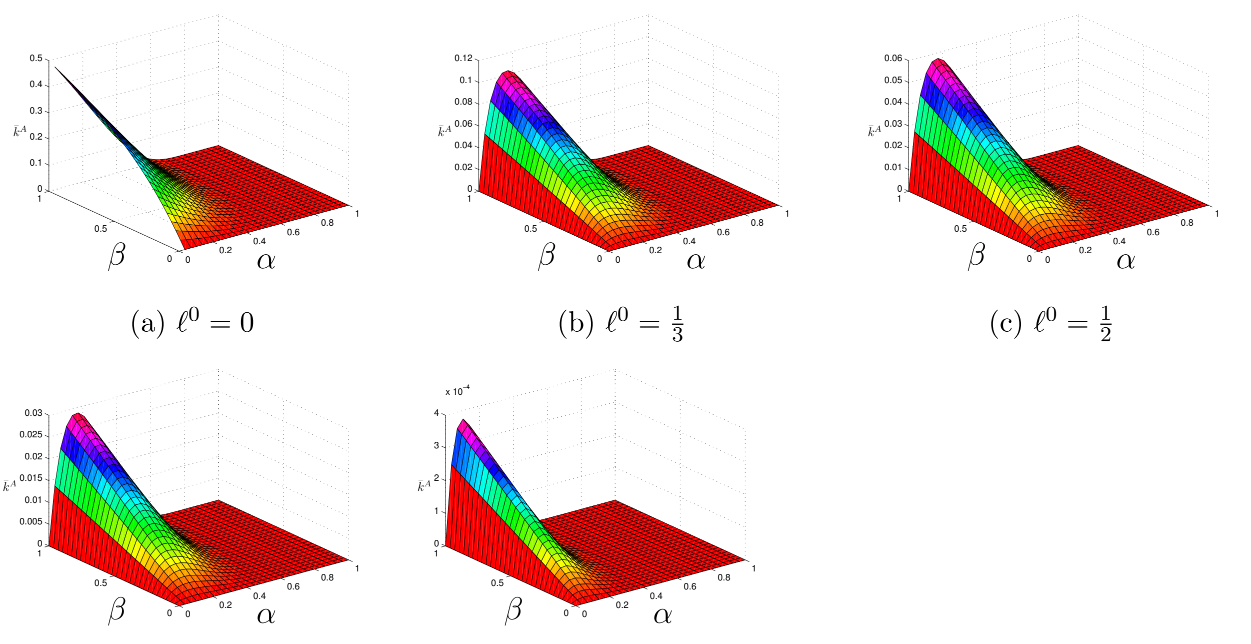
Taking into account expectations consistency the equilibrium equation reads:

$$\begin{aligned}
& \beta\mathcal{R}(k_{t+1}) = \frac{\omega^0(k_{t+1})}{\omega^1(k_t) - k_{t+1}} \\
\Leftrightarrow \qquad & \beta\mathcal{R}(k_{t+1}) = \frac{\ell^0\mathcal{W}(k_{t+1}) + k_{t+1}\mathcal{R}(k_{t+1})}{(1-\ell^0)\mathcal{W}(k_t) - k_{t+1}} \\
\Leftrightarrow \qquad & \beta\alpha = \frac{\ell^0(1-\alpha)k_{t+1} + \alpha k_{t+1}}{(1-\ell^0)\mathcal{W}(k_t) - k_{t+1}} \\
\Leftrightarrow \qquad & k_{t+1} = \frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}\mathcal{W}(k_t) = \frac{\beta\alpha}{\alpha(1+\beta)+\ell^0(1-\alpha)}\omega^1(k_t)
\end{aligned}$$

$k_{t+1}$ will be chosen positive. Otherwise the economy would break down in the next period. So the dynamics is given by $k_{t+1} = \frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}(1-\alpha)(k_t)^\alpha =: \psi(k_t)$. The dynamics has a unique non-trivial steady state

$$\bar{k} = \left(\frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}(1-\alpha)\right)^{\frac{1}{1-\alpha}}.$$

Figure A.1 shows how the steady state evolves for different values of $\ell^0$ depending on $\alpha$ and $\beta$. Local stability can be proven with the following criterion. If $\left|\psi'(\bar{k})\right| < 1$ the dynamics is



(a) $\ell^0 = 0$ (b) $\ell^0 = \frac{1}{3}$ (c) $\ell^0 = \frac{1}{2}$

(d) $\ell^0 = \frac{2}{3}$ (e) $\ell^0 = \frac{99}{100}$

Figure A.1: Steady state values for different $\ell^0$ in dependence of parameters $\alpha$ and $\beta$

locally stable:

$$\psi'(k) = \frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}(1-\alpha)\alpha k^{\alpha-1}$$

$$\Rightarrow \quad \left|\psi'(\bar{k})\right| = \frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}(1-\alpha)\alpha\left(\frac{\beta\alpha(1-\ell^0)}{\alpha(1+\beta)+\ell^0(1-\alpha)}(1-\alpha)\right)^{-1} = \alpha < 1$$

Global stability is demonstrated in the following figure: Obviously, as more labor is supplied by the old, less capital is accumulated. As the economy converges towards its steady state the Cass criterion for stationary allocations can be applied. The equilibrium is dynamically efficient if

$$f'\left(\bar{k}\right) > 1$$

$$\Leftrightarrow \quad \alpha\bar{k}^{\alpha-1} > 1$$

$$\Leftrightarrow \quad \frac{\alpha}{1-\alpha}\frac{\alpha(1+\beta)+\ell^0(1-\alpha)}{\beta\alpha(1-\ell^0)} > 1 \Leftrightarrow \qquad \ell^0 > \frac{\beta(1-\alpha)-\alpha(1+\beta)}{(1-\alpha)(1+\beta)} =: \tilde{\ell}^0$$

Thus, the equilibrium is first dynamically efficient in the presence of old agents labor supply than when all labor is supplied by the young. In particular there is a critical value such that the economy is dynamically efficient:

**Lemma A.0.1.** *If labor supply of the old is greater than one half, $\ell^0 \geq \frac{1}{2}$, the equilibrium is dynamically efficient.*

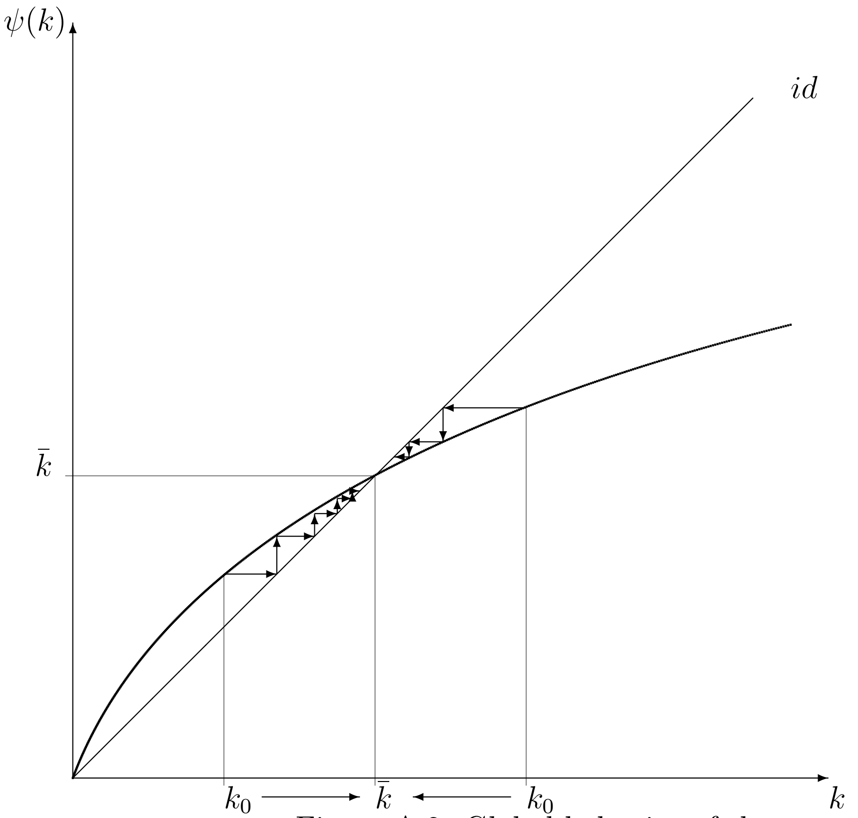



Figure A.2: Global behavior of the two-period model

*Proof.* The critical value for labor supply of the old, $\underline{\ell} = \frac{\beta(1-\alpha)-\alpha(1+\beta)}{(1-\alpha)(1+\beta)}$ is strictly decreasing in $\alpha$:

$$\frac{\partial \underline{\ell}}{\partial \alpha} = \frac{-(1+2\beta)(1-\alpha)(1+\beta)+(1+\beta)(\beta(1-\alpha)-\alpha(1+\beta))}{(1-\alpha)^2(1+\beta)^2} < 0$$

Thus, the maximal value for $\underline{\ell}$ with respect to $\alpha$ is for $\alpha = 0$:

$$\underline{\ell} < \frac{\beta}{1+\beta}$$

$\frac{\beta}{1+\beta}$ is strictly increasing in $\beta$ such that it is maximal for $\beta = 1$. Thus,

$$\underline{\ell} < \frac{1}{2}$$

□

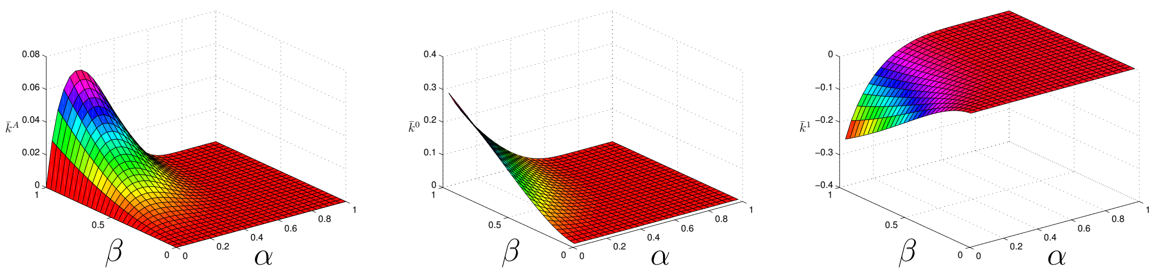
# Appendix B

# Simulation of the Dynamics of the Three-Period Economy

This part of the thesis completes the analysis of Chapter 4. Chapter 4 shows the properties of the steady state of the three-period economy with logarithmic utility and Cobb-Douglas production function with retirement and analyzes the behavior of the dynamics for one labor profile $\ell = \left(0, \frac{1}{2}, \frac{1}{2}\right)$. The present appendix validates the results for further suitably chosen labor profiles, namely $\ell \in \left\{(0,0,1), \left(0, \frac{1}{3}, \frac{2}{3}\right), \left(0, \frac{2}{3}, \frac{1}{3}\right), (0,1,0)\right\}$. The first section shows the steady states' properties, the second the dynamics for different parameter values.

## B.1 Steady States for different Labor Profiles

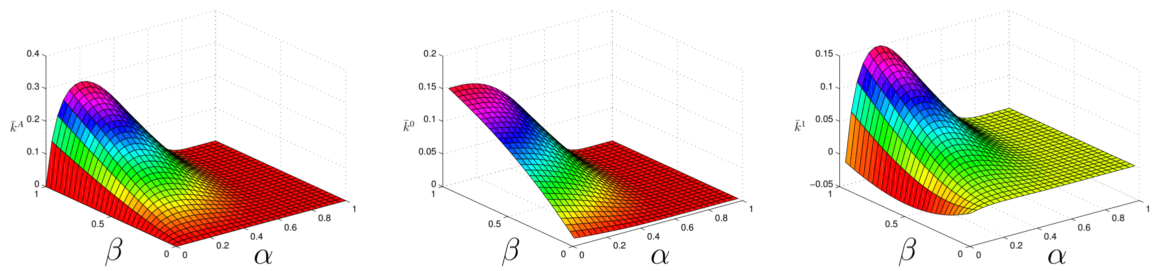
This section shows the steady states in dependence of the parameters $\alpha$ and $\beta$.




(a) $\bar{k}^A$ (b) $\bar{k}^0$ (c) $\bar{k}^1$

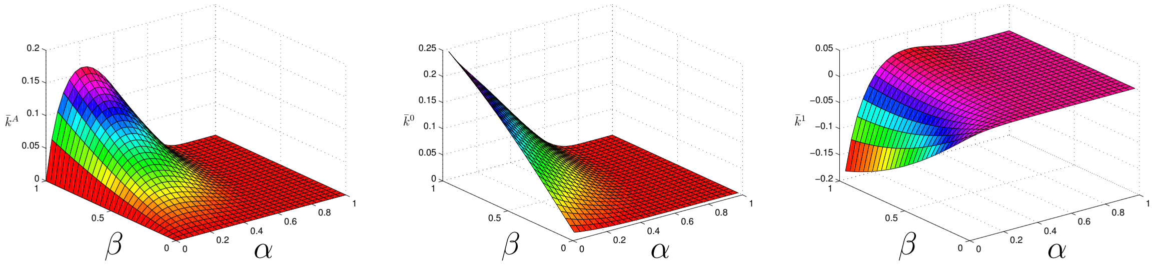
Figure B.1: Steady state values for $\ell^1 = 1$ in dependence of parameters $\alpha$ and $\beta$

Figure B.1 shows the steady state for the labor profile $(0, 1, 0)$. As it needs to be the case by construction capital supply of the middle-aged (Figure B.1 (c)) is negative in any case. The reason is that wealth of the young vanishes and there is the need to go into debt to




(a) $\bar{k}^A$ (b) $\bar{k}^0$ (c) $\bar{k}^1$

Figure B.2: Steady state values for $\ell^1 = \frac{1}{3}$ in dependence of parameters $\alpha$ and $\beta$

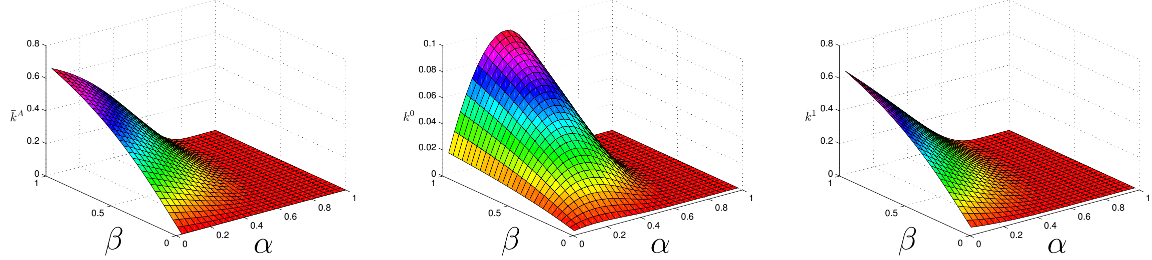



(a) $\bar{k}^A$ (b) $\bar{k}^0$ (c) $\bar{k}^1$

Figure B.3: Steady state values for $\ell^1 = \frac{2}{3}$ in dependence of parameters $\alpha$ and $\beta$

realize positive consumption. In contrast, Figure B.4 shows the other extreme labor profile $(0, 0, 1)$ where all labor is supplied by the young. Here, as well, the figure validates the result that capital supply needs to be positive for both generations (Figure B.4 (b) and (c)). In total, all Figures B.1 to B.4 show that aggregated capital is positive as well as old agents capital supply. Unsurprisingly, the absolute values depend on the labor profiles as well as the surface of the figures.

Moreover, note the evolution of the steady states with respect to $\beta$. As long as capital



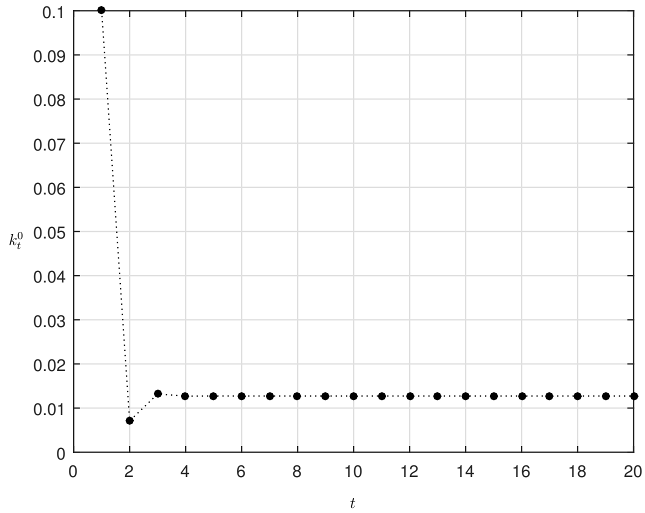

(a) $\bar{k}^A$ (b) $\bar{k}^0$ (c) $\bar{k}^1$

Figure B.4: Steady state values for $\ell^1 = 0$ in dependence of parameters $\alpha$ and $\beta$

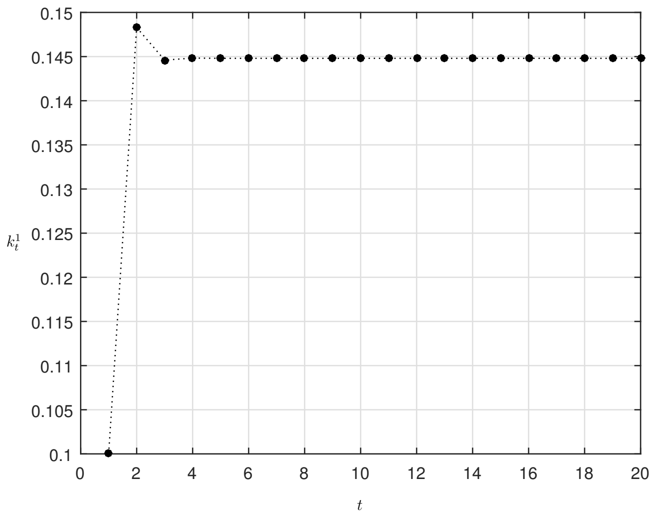
supply is positive it seems to be increasing in $\beta$. The higher future utility is evaluated the higher is investment. As higher investment implies higher wealth, which can be used for higher consumption, this result is intuitive (see Figures B.1 to B.4 (b)). But if there are negative values for capital supply, the behavior changes. In Figure B.1 (c) $\bar{k}^1$ seems to be decreasing in $\beta$. Thus, the higher future utility is evaluated the higher the young go into debt. Figure B.3 shows that if there are positive and negative values of capital supply the behavior with respect to $\beta$ is not clear.

## B.2 Dynamics for different Labor Profiles

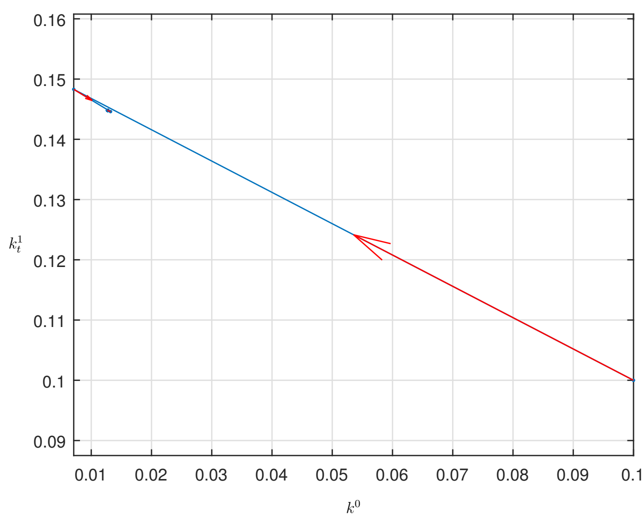
This appendix shows the dynamic behavior of the economies that correspond to those in Chapter 4 with different labor profiles. For each parameter set we choose three initial values $\mathbf{k}_1$: (0.1, 0.1), (10, 10) and one near the steady state. Again, the initial value is denoted by $\mathbf{k}_1$ for numerical reasons.
Chapter 4 already showed that dynamical behavior does not coincide over parameters and initial values. The observations in this appendix will validate these results. Figures B.5 to B.7 show the dynamical behavior of the three-period economy where all labor is supplied by the young and parameters $\alpha = 0.1$ and $\beta = 0.2$ for the different initial values. Even in this simple economy convergence is not necessarily monotone as it is the case in the two-period economy. Figure B.5(a) and (b) show that there is at least one outlier that contradicts monotonicity.
In general, note that all figures in this section show that convergence is very fast. A quantitative analysis is not possible because of the changing scaling. For example, Figure B.9(a) suggests that from step four on $k^0$ equals zero. There are two mistakes: First, the scaling is that rough that it cannot be distinguished between convergence to zero and to a value near zero, what happens here. Second, little variations in later values vanish as well because of the rough scaling. Hence, it is not surprising that non-monotone convergence mostly seems to be shown by figures that represent an initial value near the steady state. But in total all figures converge. This is verified by the figures' underlying computational results.
Finally, there is a short statement about the presented parameter values. The labor profiles coincide with those considered in the section about steady states and they are chosen exemplary over the interval $\ell^1 \in [0, 1]$. The discount factor $\beta$ usually is chosen near one, e.g., 0.99, for models with infinitely lived consumers (see Prescott [19] for example) which in general are calibrated with quarterly data. Transferring 0.99 to a 20 year period leads approximately to 0.45, 0.98 to 0.2. Thus, we choose $\beta \in \{0.2, 0.4, 0.5\}$. $1 - \alpha$ often is interpreted as the labor share and thus, $\alpha$ usually is chosen smaller than 0.5. Hence, we take the common value $\alpha = 0.33$ (cf. Prescott [19]) plus a great and a small value for $\alpha$.
Naturally, we regarded more than the presented parameter combinations and chose a representative set this appendix presents.




(a) $k^0$

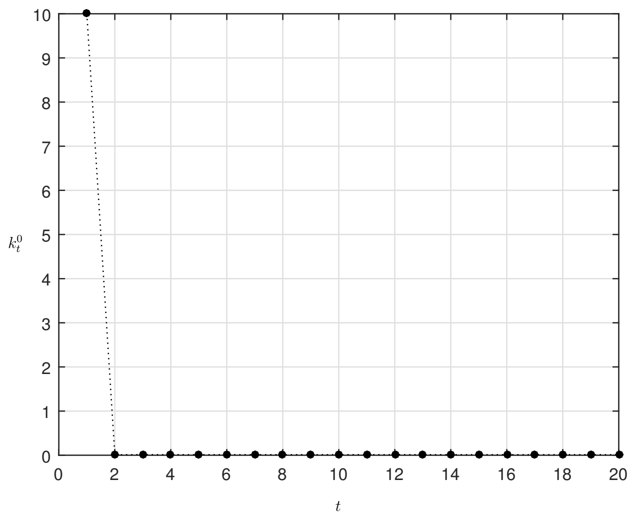



(b) $k^1$

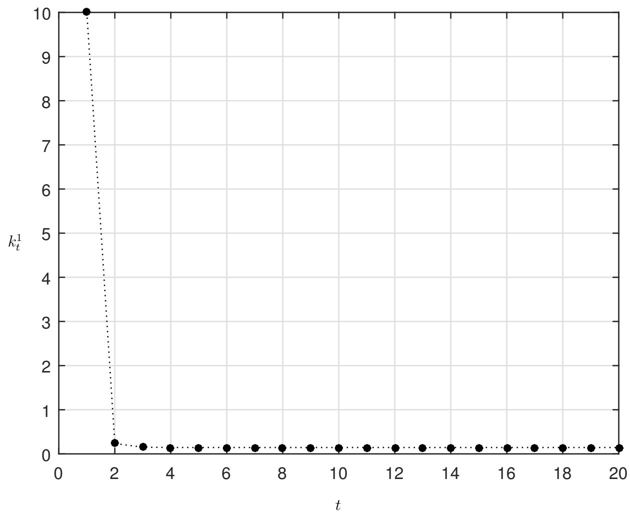



(c) $\mathbf{k}$

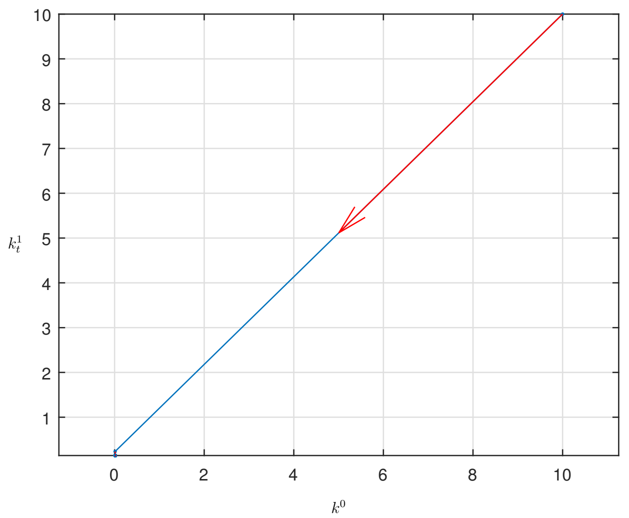
Figure B.5: Dynamics with $\ell^1 = 0$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

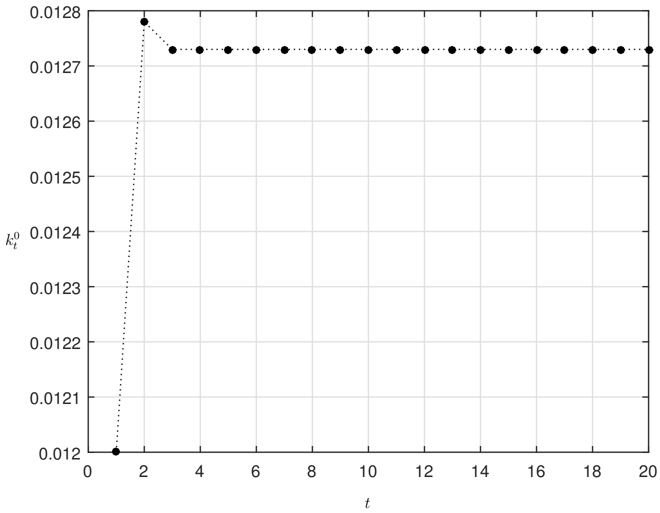



(b) $k^1$

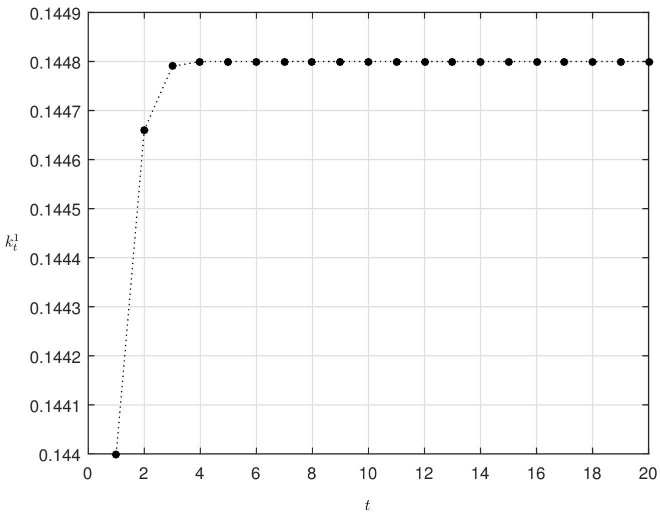



(c) $\mathbf{k}$

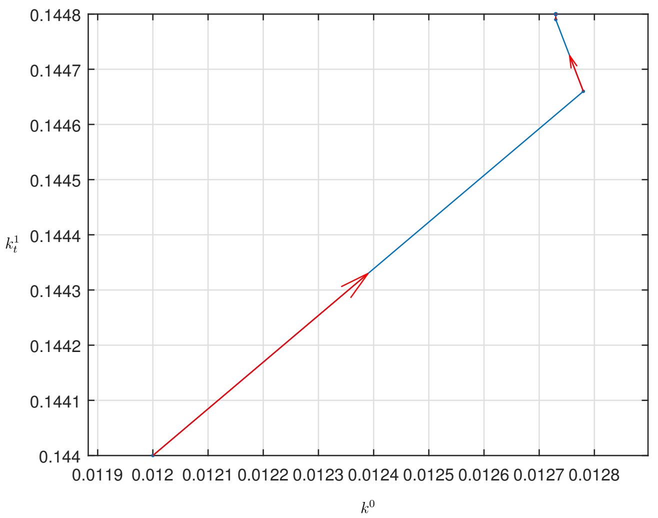
Figure B.6: Dynamics with $\ell^1 = 0$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

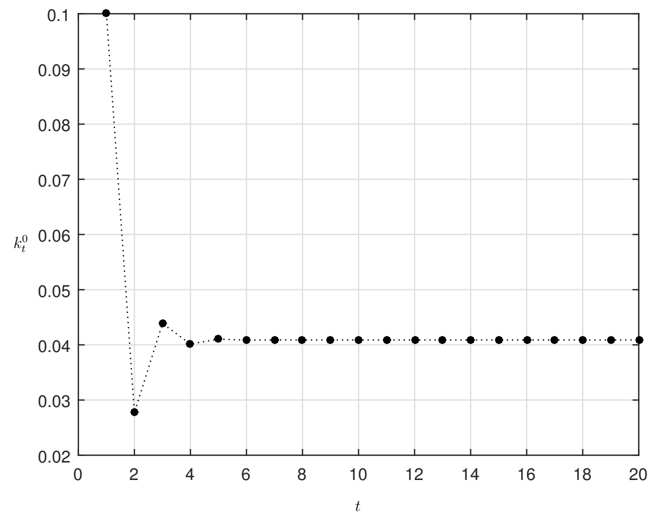



(b) $k^1$

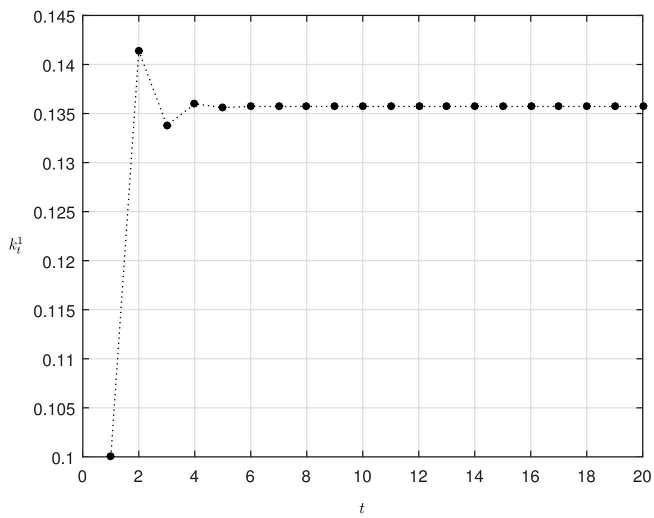



(c) $\mathbf{k}$

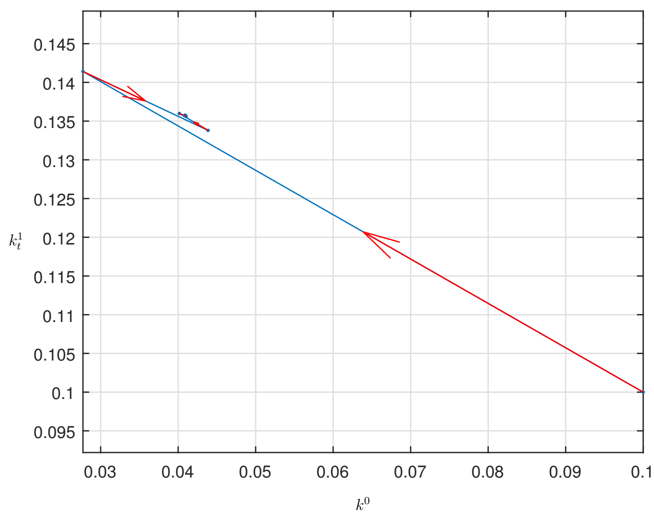
Figure B.7: Dynamics with $\ell^1 = 0$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.012,\ 0.144)$




(a) $k^0$

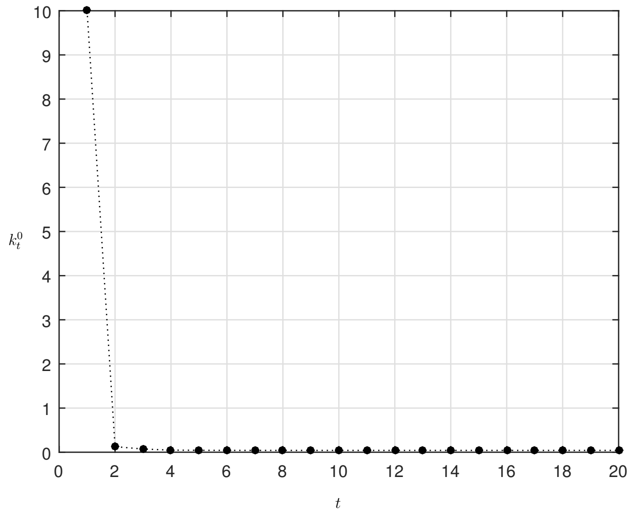



(b) $k^1$

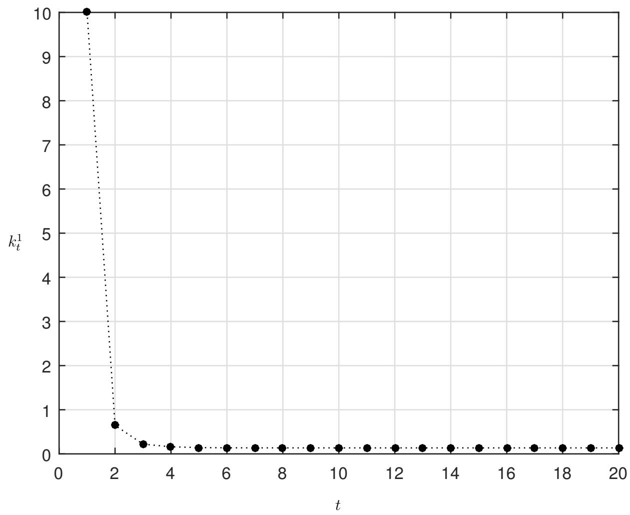



(c) $\mathbf{k}$

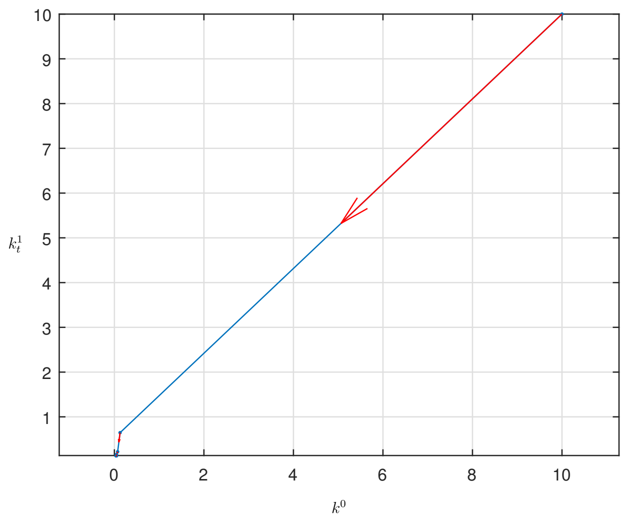
Figure B.8: Dynamics with $\ell^1 = 0$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

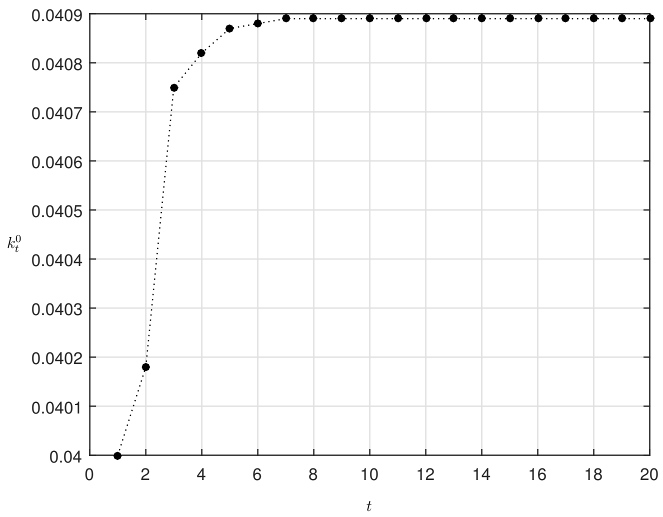



(b) $k^1$

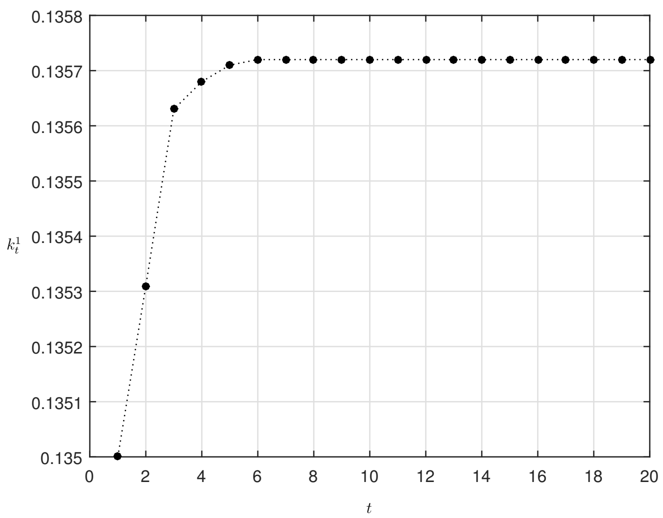



(c) $\mathbf{k}$

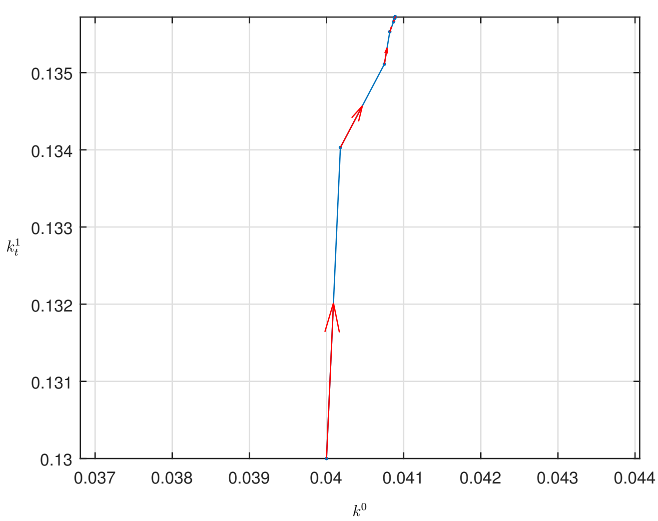
Figure B.9: Dynamics with $\ell^1 = 0$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

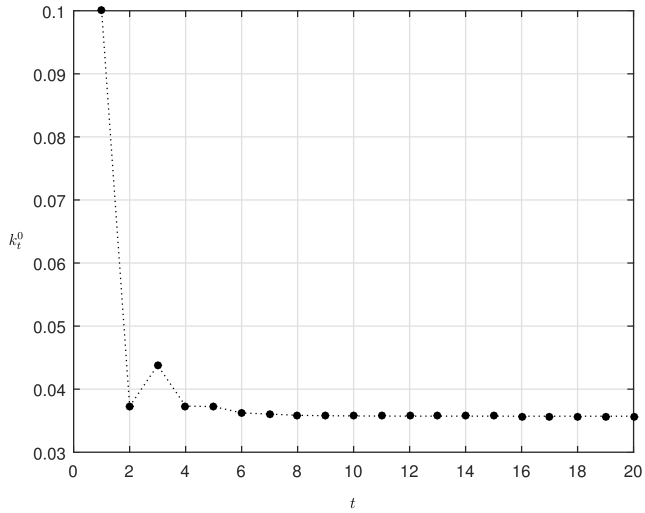



(b) $k^1$

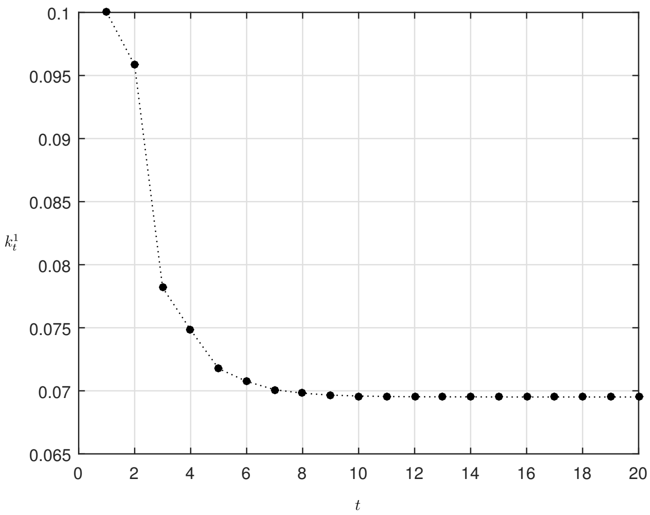



(c) $\mathbf{k}$

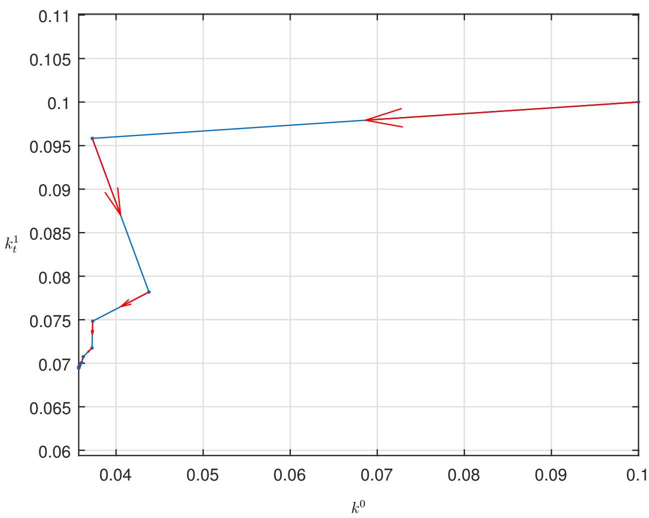
Figure B.10: Dynamics with $\ell^1 = 0$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.04,\ 0.135)$




(a) $k^0$

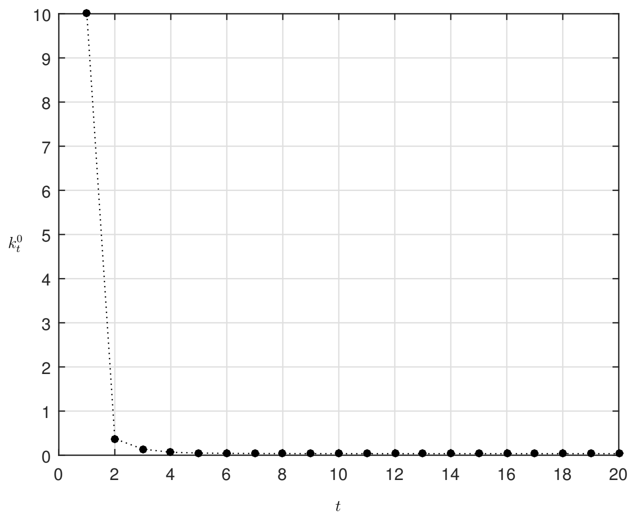



(b) $k^1$

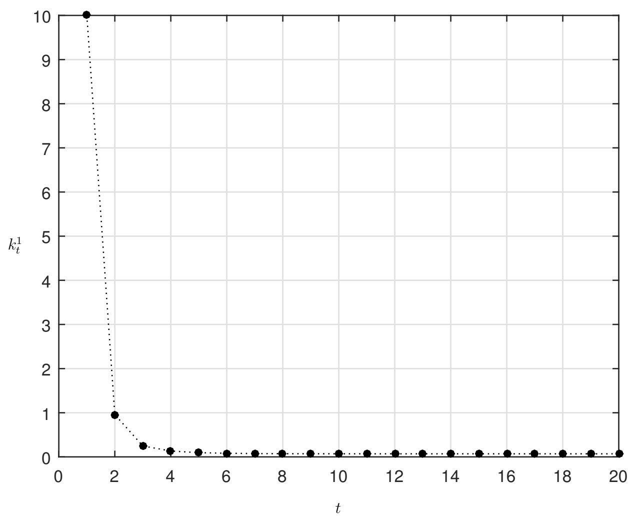



(c) $\mathbf{k}$

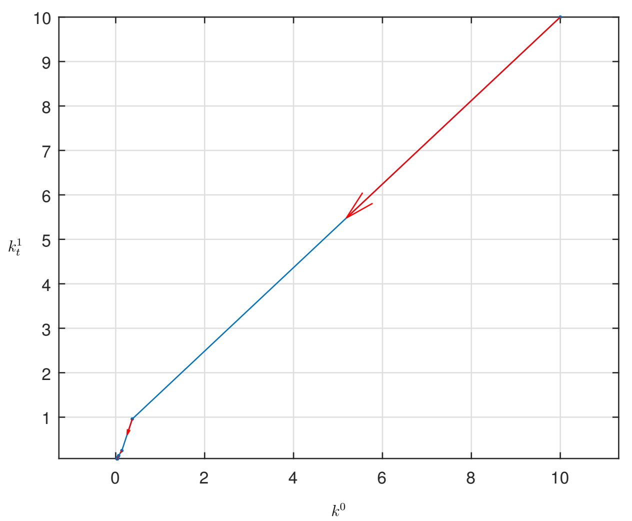
Figure B.11: Dynamics with $\ell^1 = 0$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

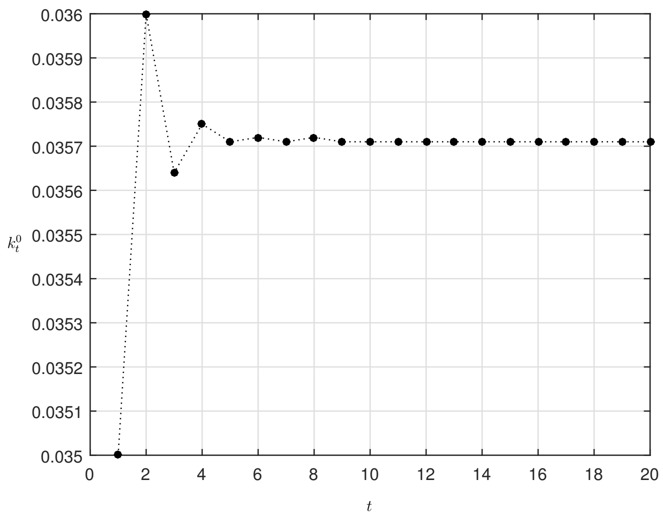



(b) $k^1$

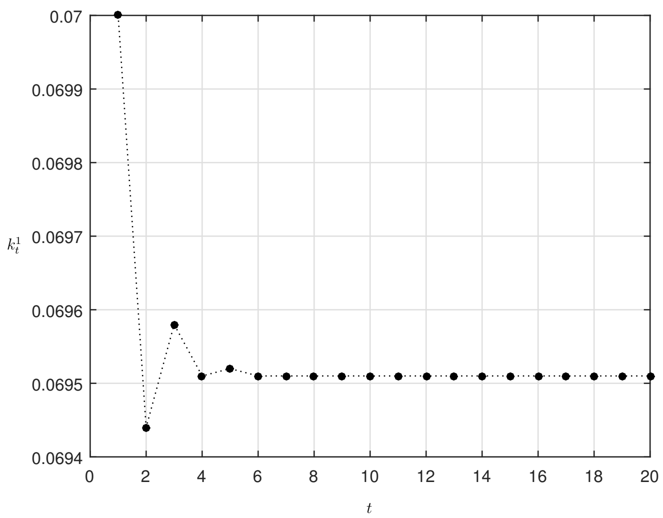



(c) $\mathbf{k}$

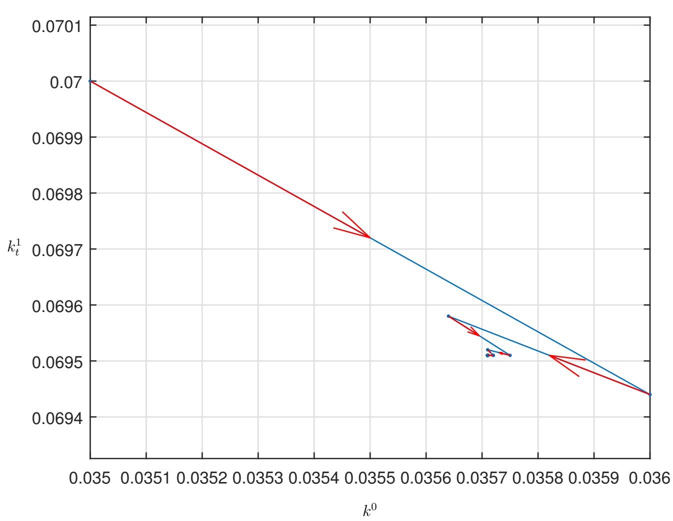
Figure B.12: Dynamics with $\ell^1 = 0$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

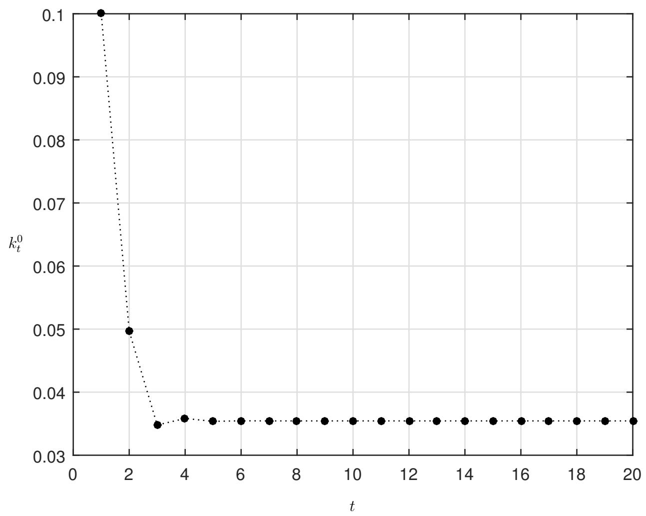



(b) $k^1$

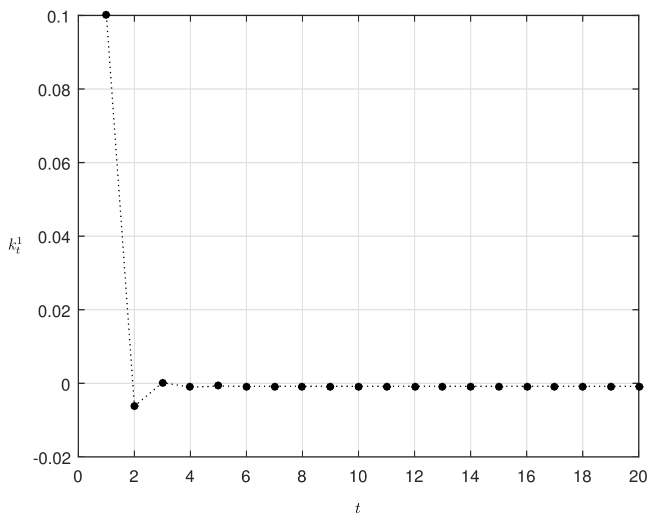



(c) $\mathbf{k}$

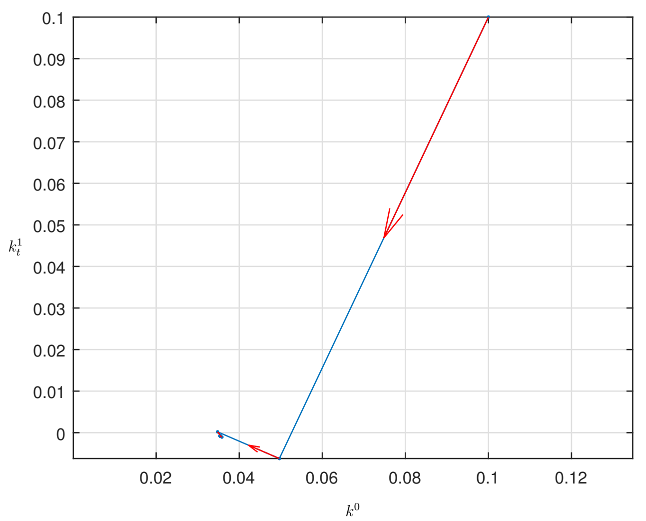
Figure B.13: Dynamics with $\ell^1 = 0$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.035,\ 0.07)$




(a) $k^0$

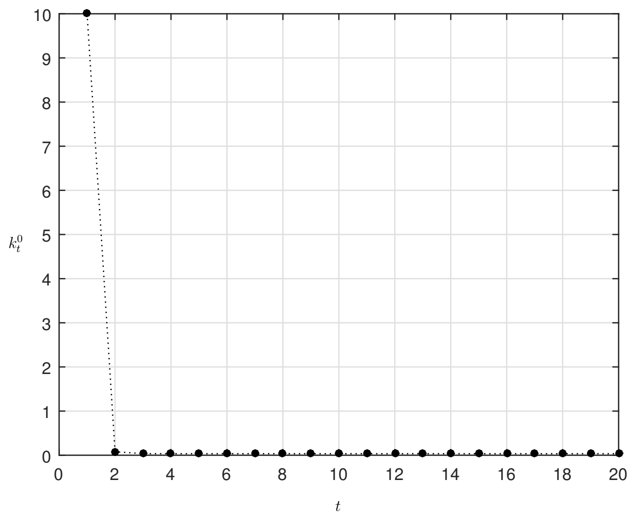



(b) $k^1$

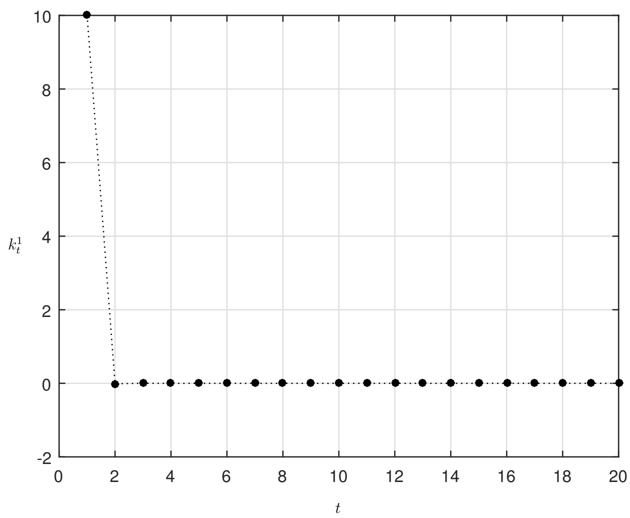



(c) $\mathbf{k}$

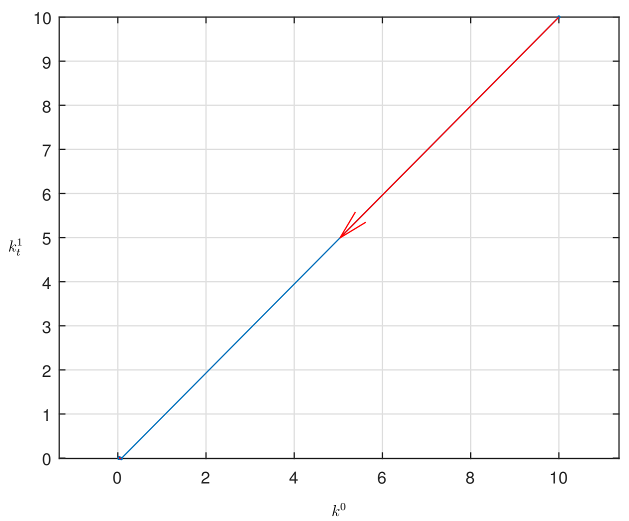
Figure B.14: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

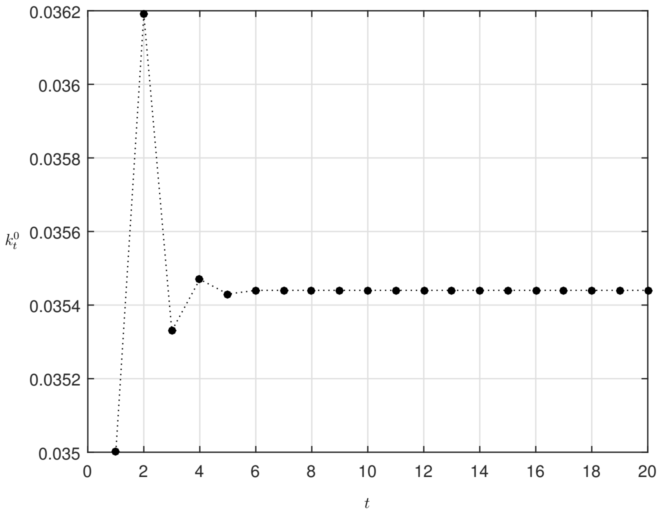



(b) $k^1$

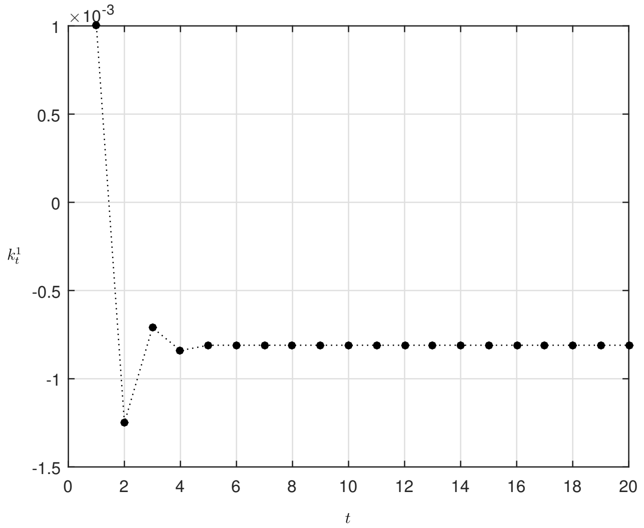



(c) $\mathbf{k}$

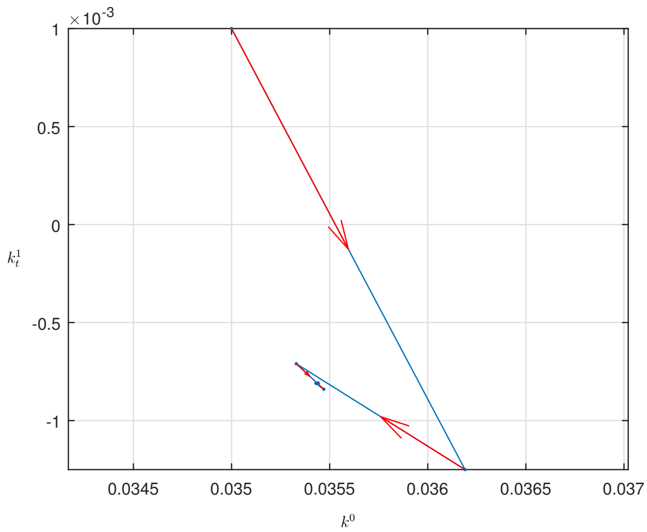
Figure B.15: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

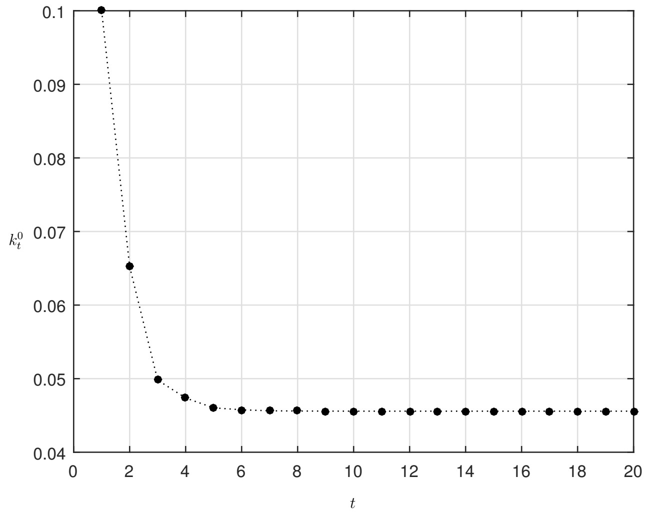



(b) $k^1$

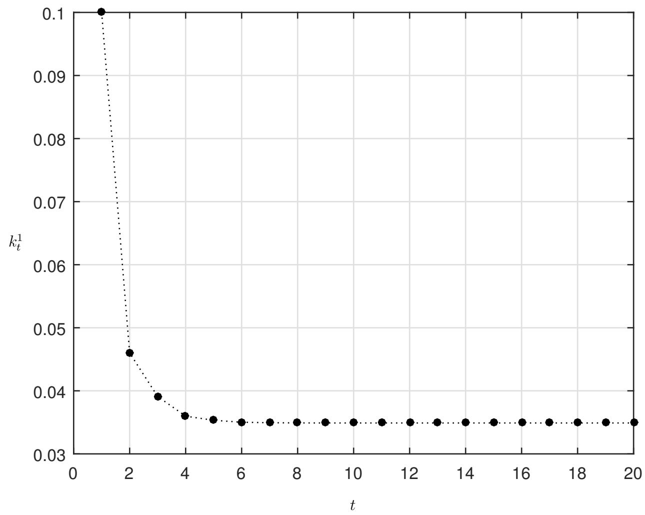



(c) $\mathbf{k}$

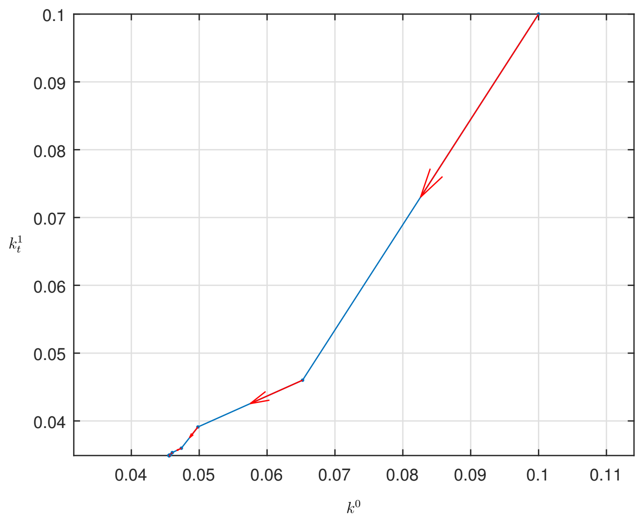
Figure B.16: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.035,\ 0.001)$




(a) $k^0$

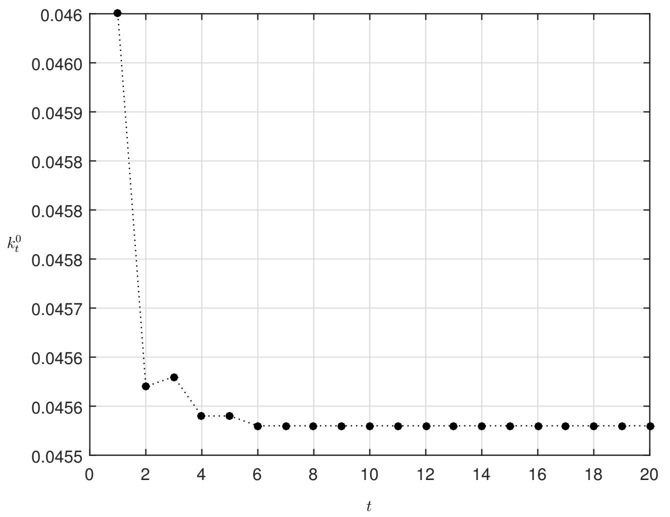



(b) $k^1$

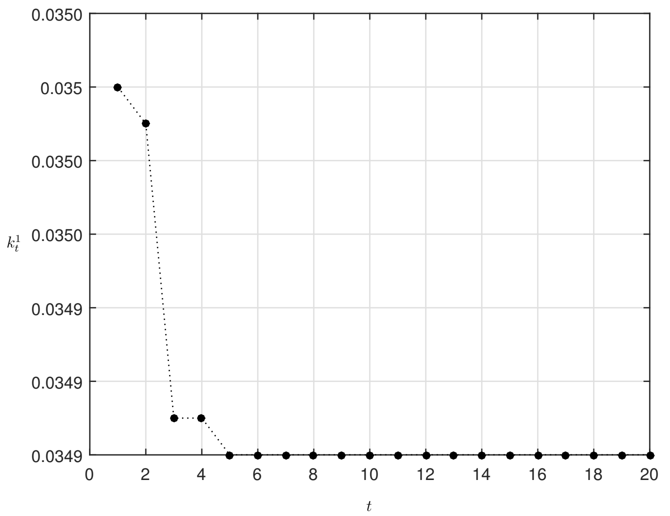



(c) $\mathbf{k}$

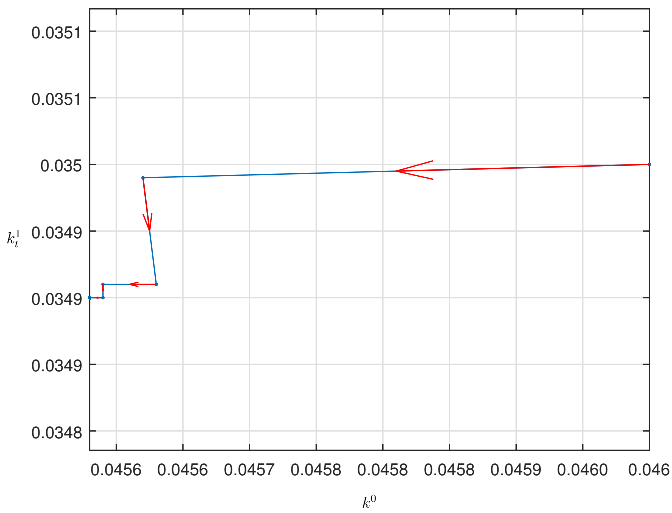
Figure B.17: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

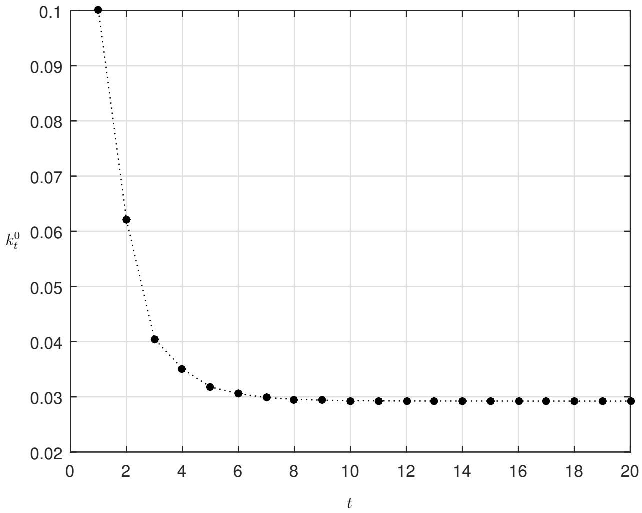



(b) $k^1$

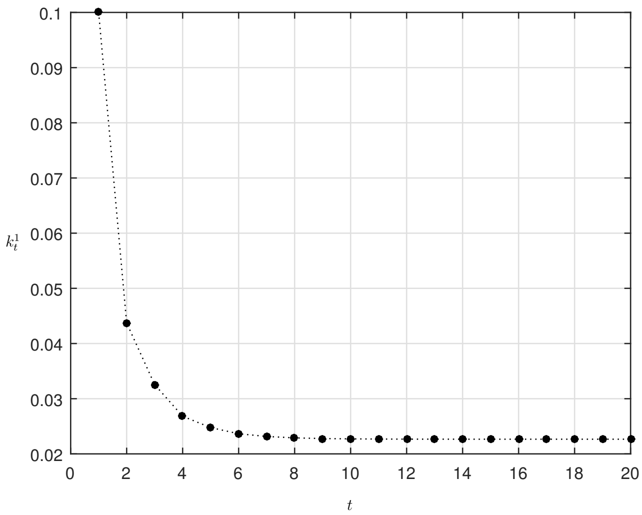



(c) $\mathbf{k}$

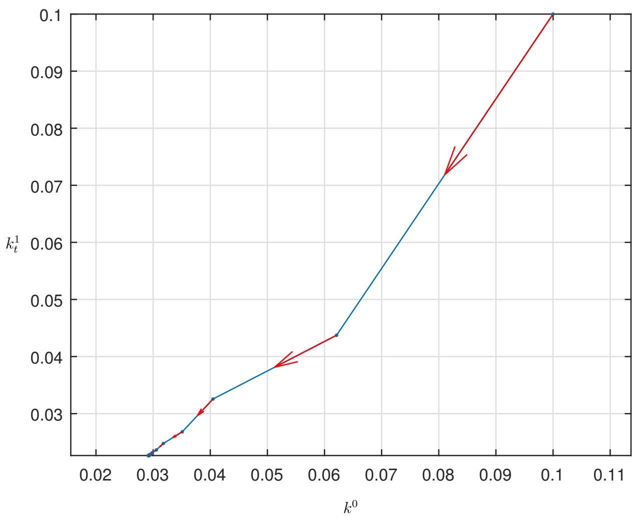
Figure B.18: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.046,\ 0.035)$




(a) $k^0$

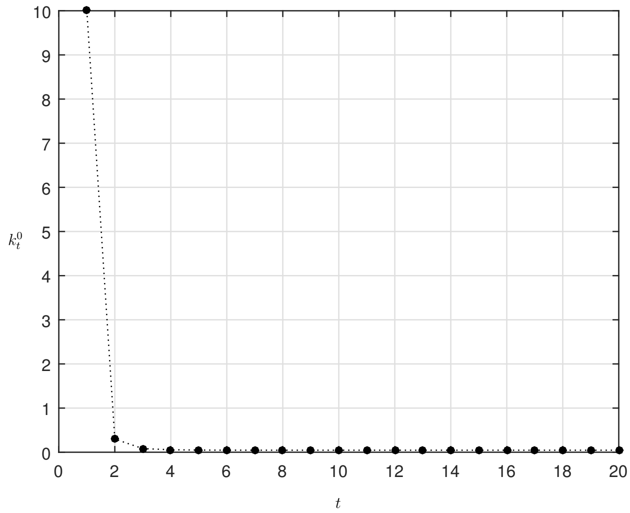



(b) $k^1$

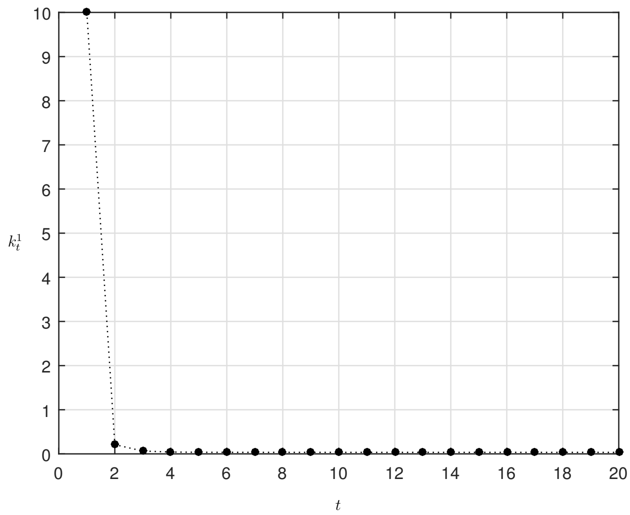



(c) $\mathbf{k}$

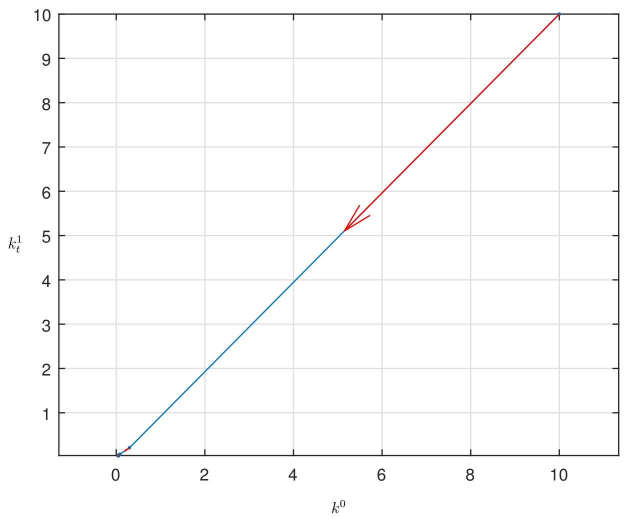
Figure B.19: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

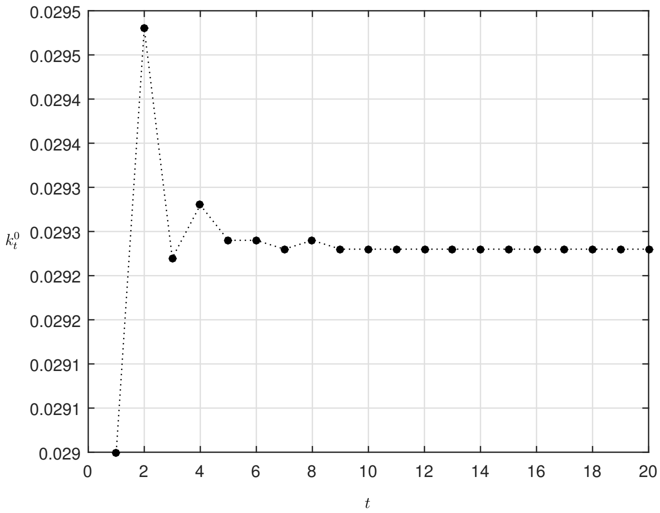



(b) $k^1$

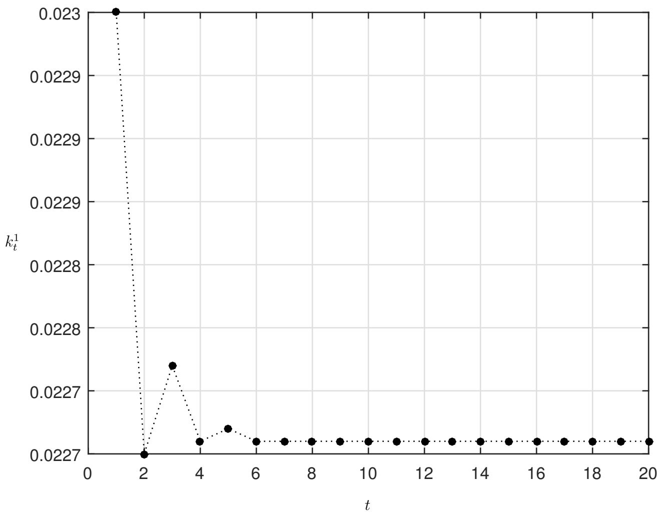



(c) $\mathbf{k}$

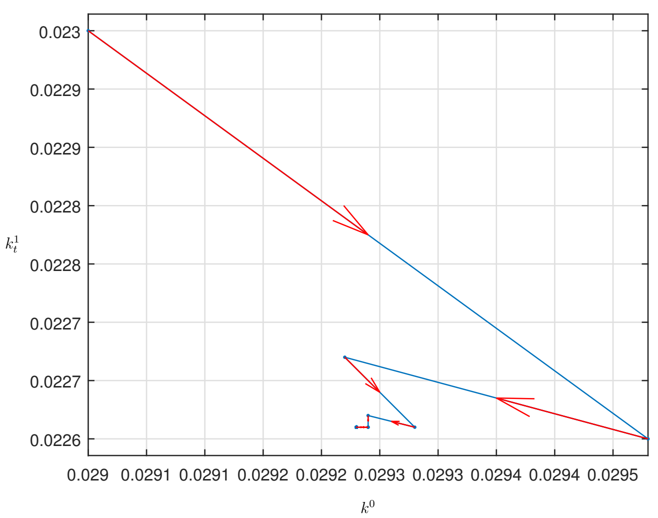
Figure B.20: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

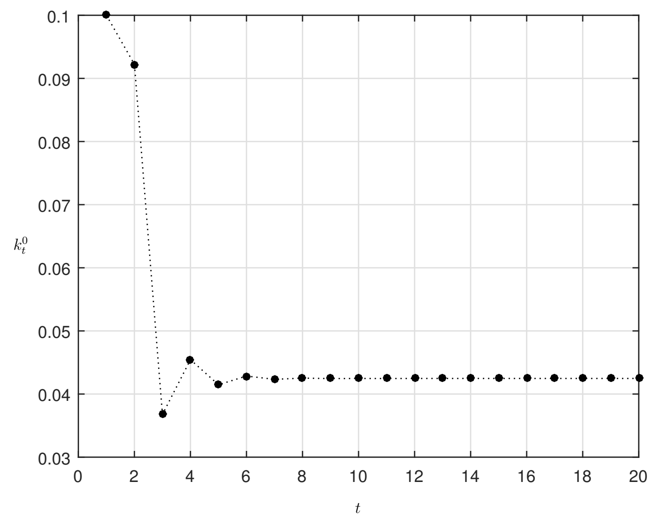



(b) $k^1$

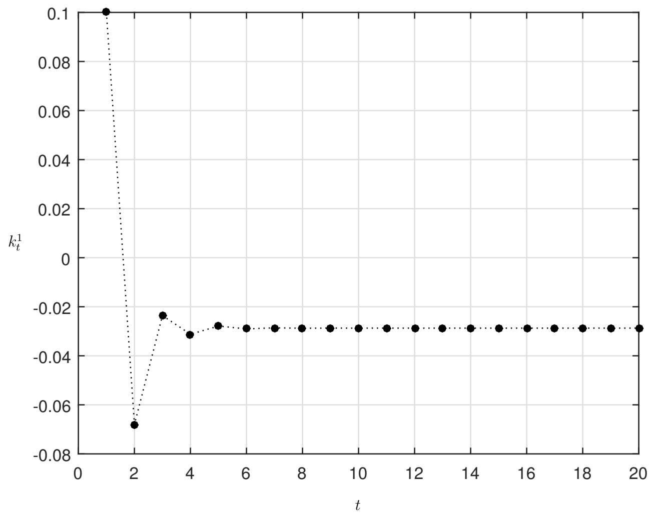



(c) $\mathbf{k}$

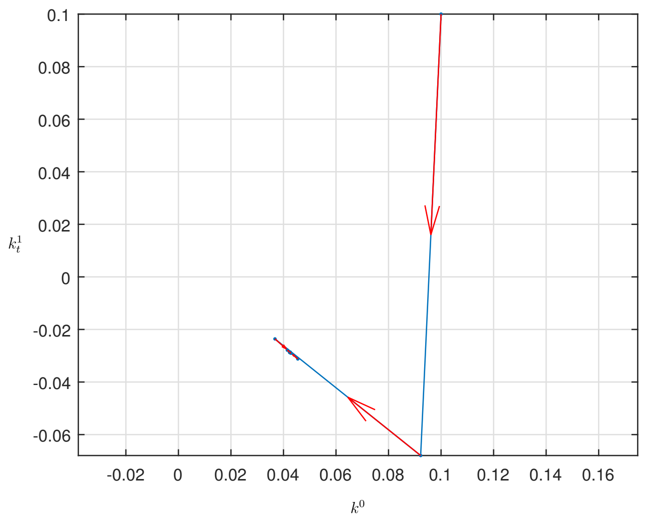
Figure B.21: Dynamics with $\ell^1 = \frac{1}{3}$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.029,\ 0.023)$




(a) $k^0$

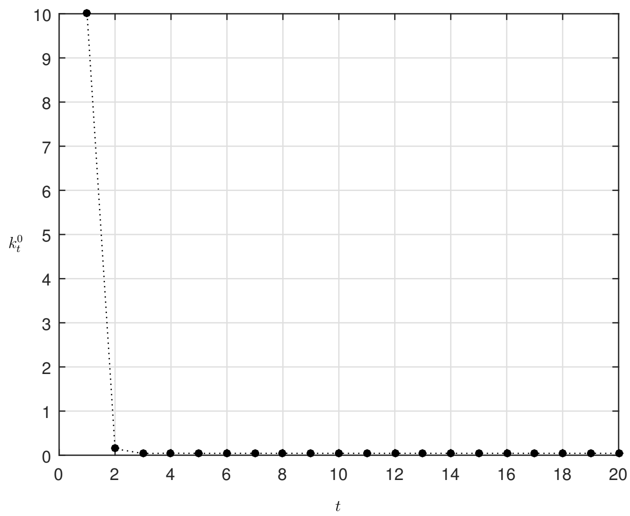



(b) $k^1$

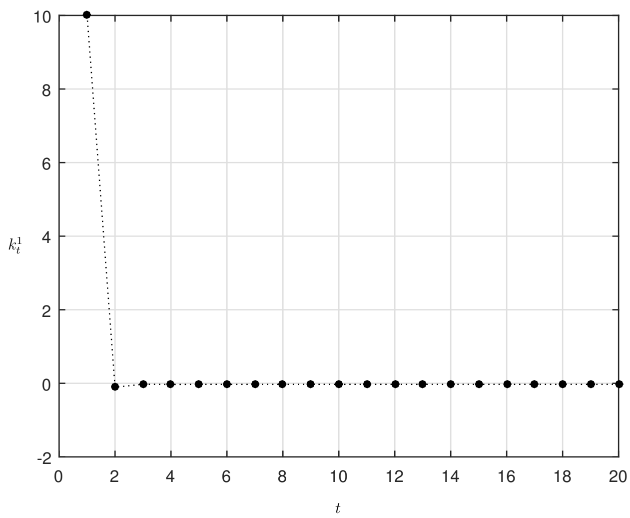



(c) $\mathbf{k}$

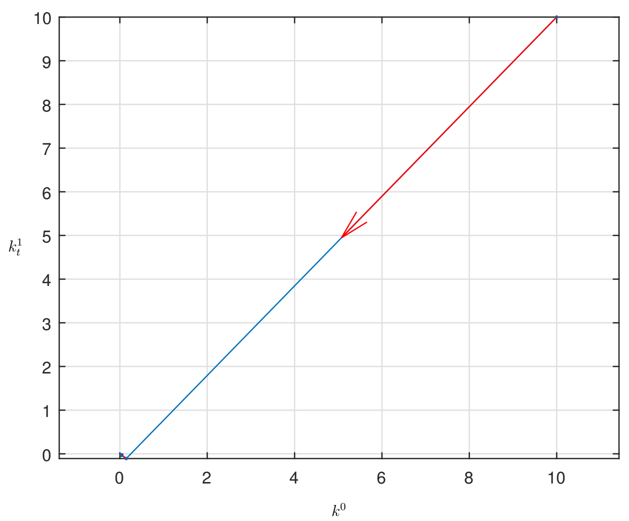
Figure B.22: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

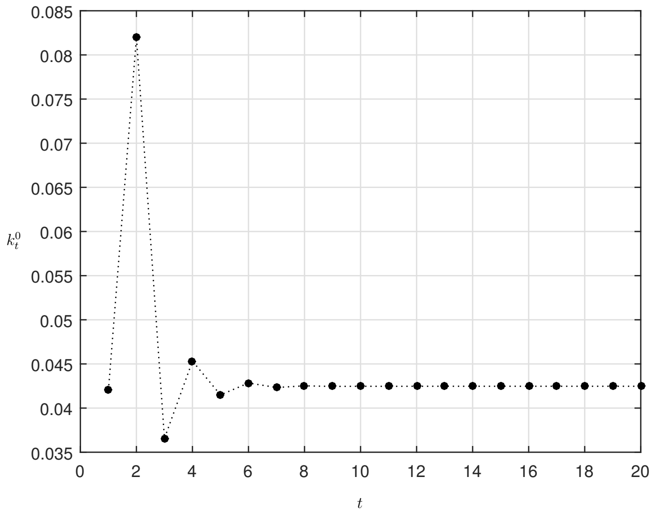



(b) $k^1$

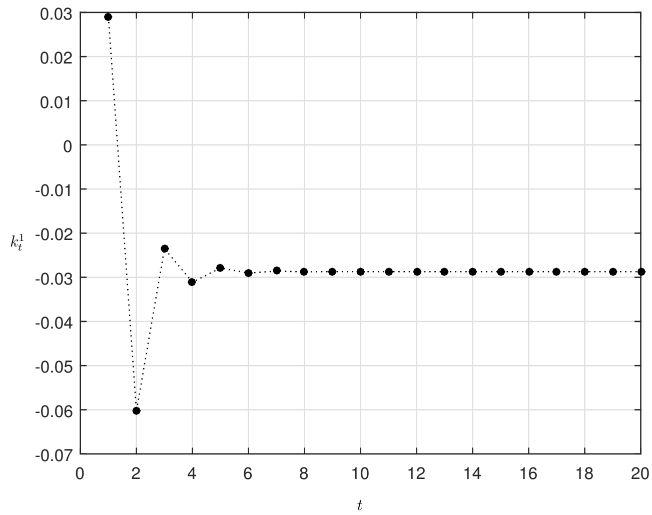



(c) $\mathbf{k}$

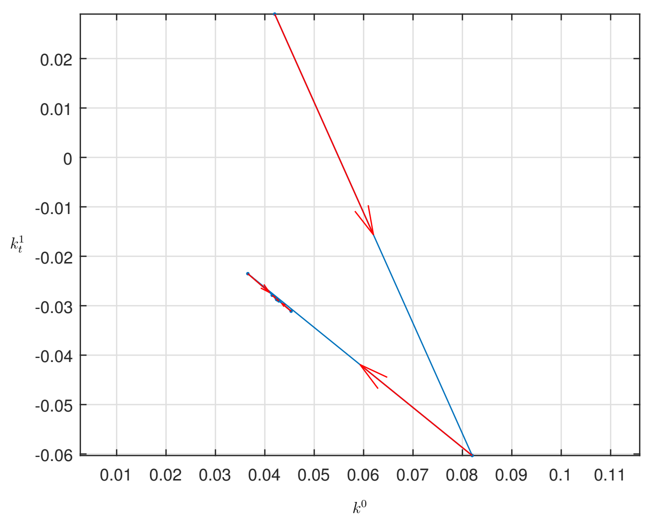
Figure B.23: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

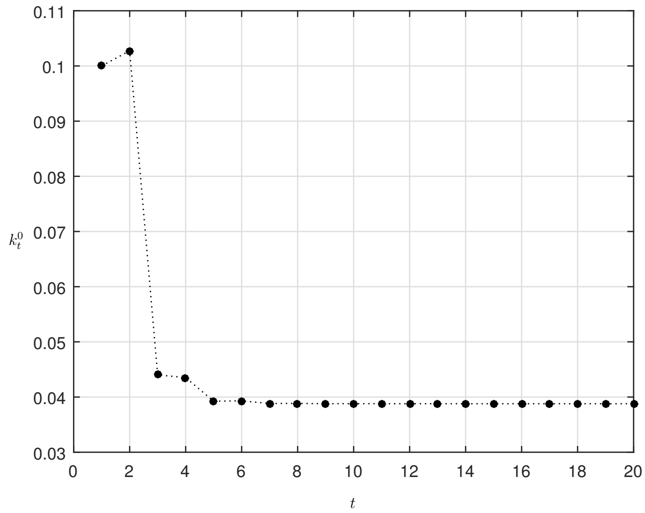



(b) $k^1$

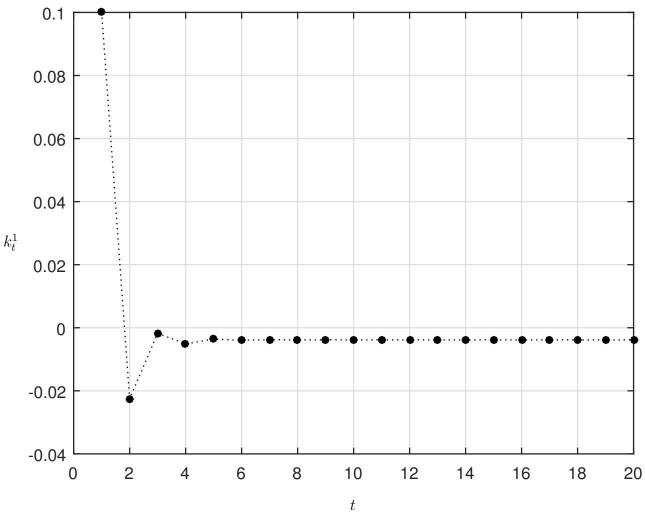



(c) $\mathbf{k}$

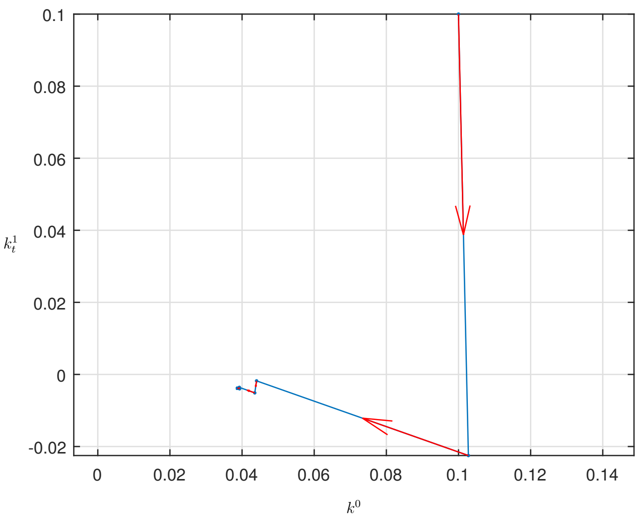
Figure B.24: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.042,\ -0.029)$




(a) $k^0$

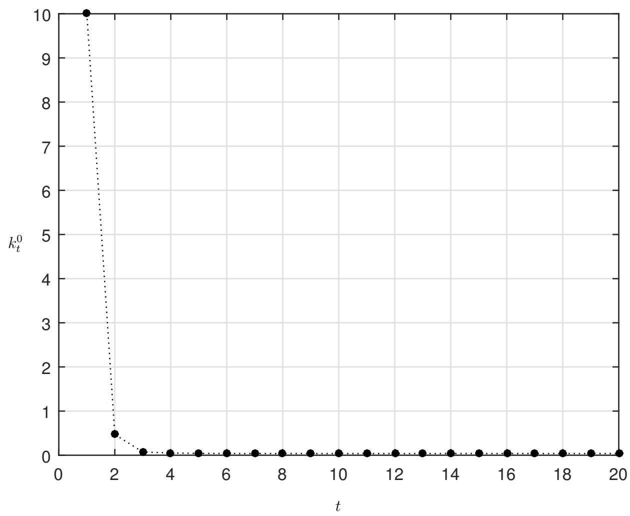



(b) $k^1$

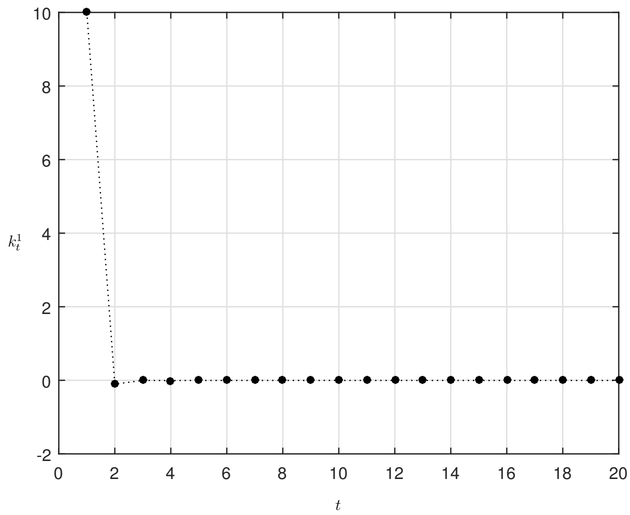



(c) $\mathbf{k}$

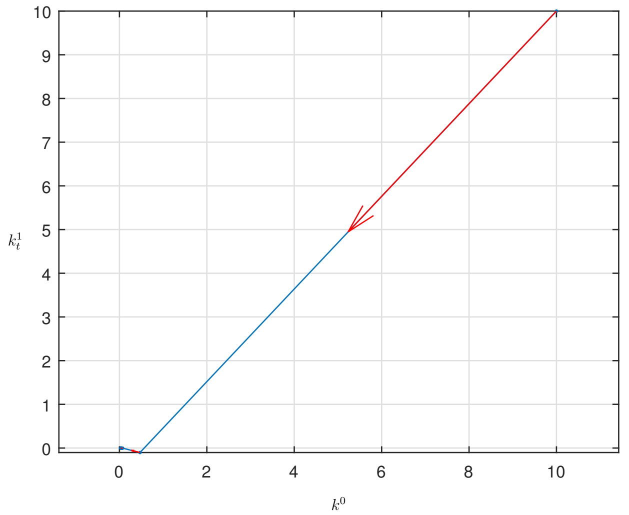
Figure B.25: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

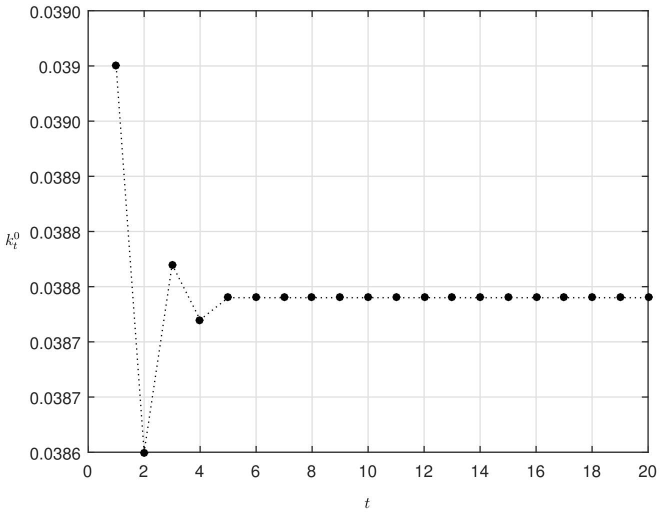



(b) $k^1$

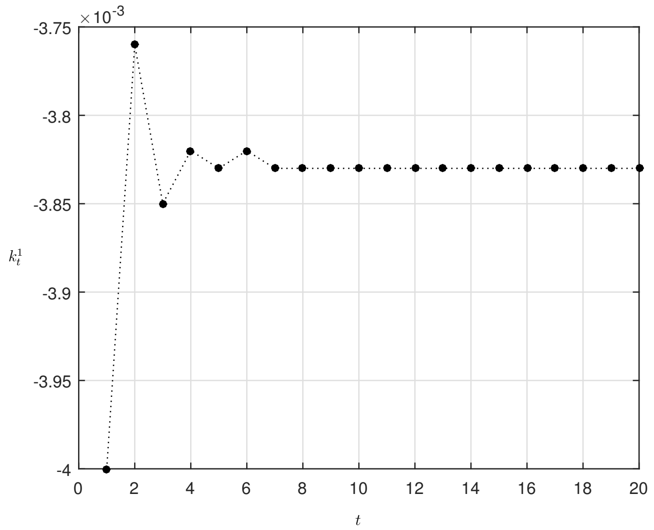



(c) $\mathbf{k}$

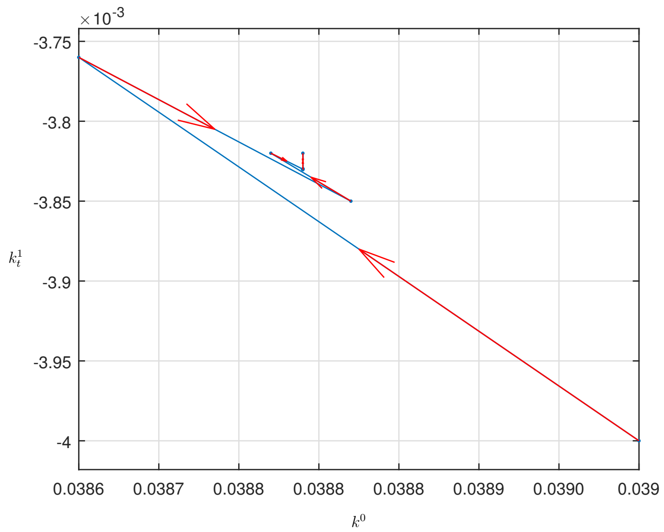
Figure B.26: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

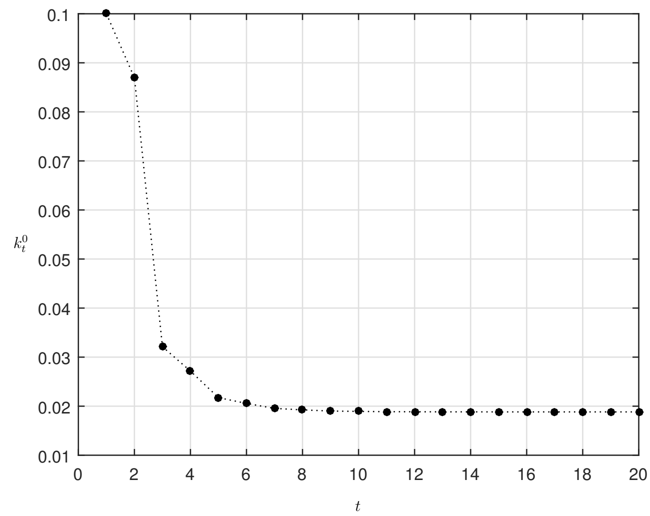



(b) $k^1$

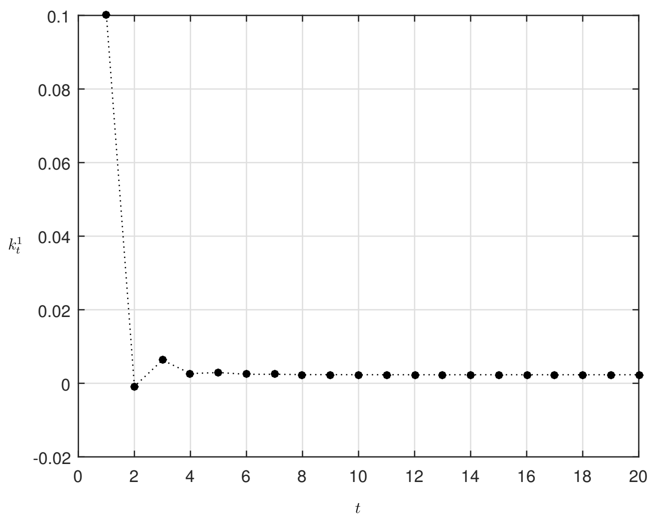



(c) $\mathbf{k}$

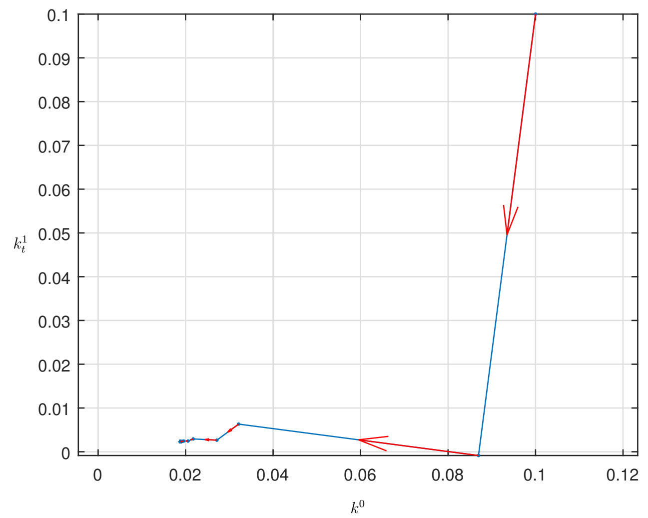
Figure B.27: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.039,\ -0.004)$




(a) $k^0$

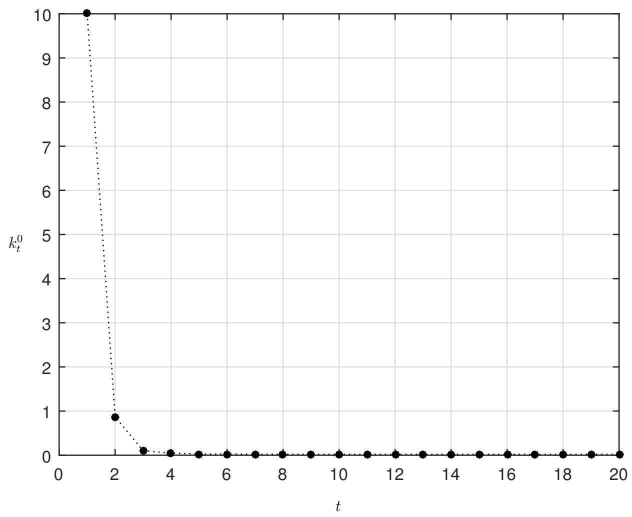



(b) $k^1$

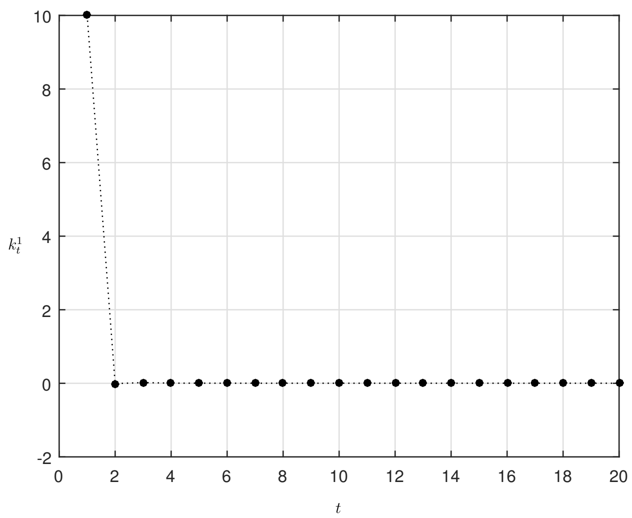



(c) $\mathbf{k}$

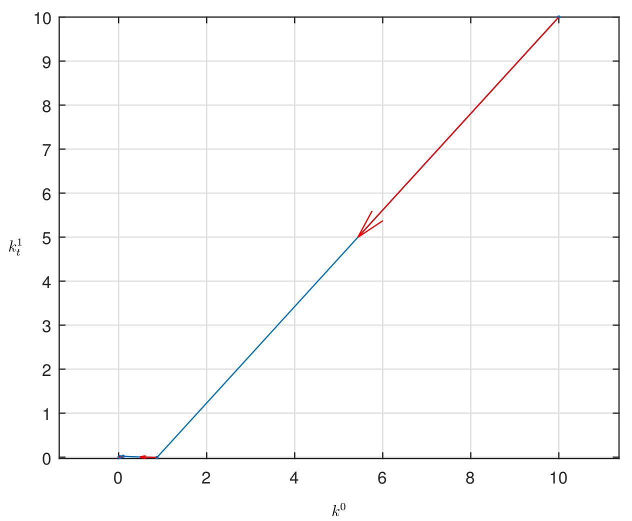
Figure B.28: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

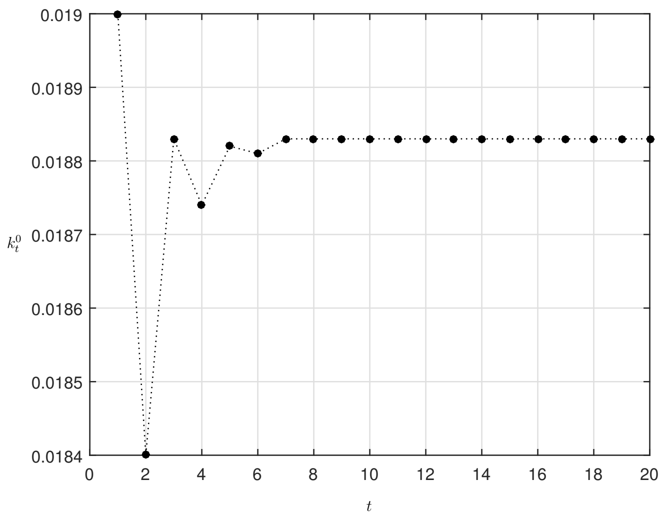



(b) $k^1$

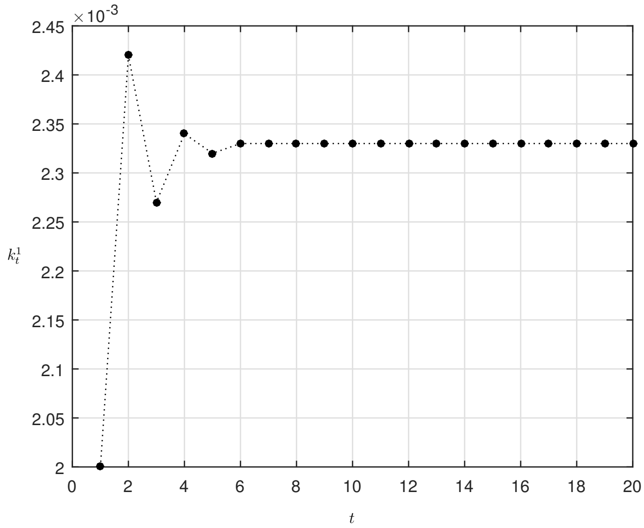



(c) $\mathbf{k}$

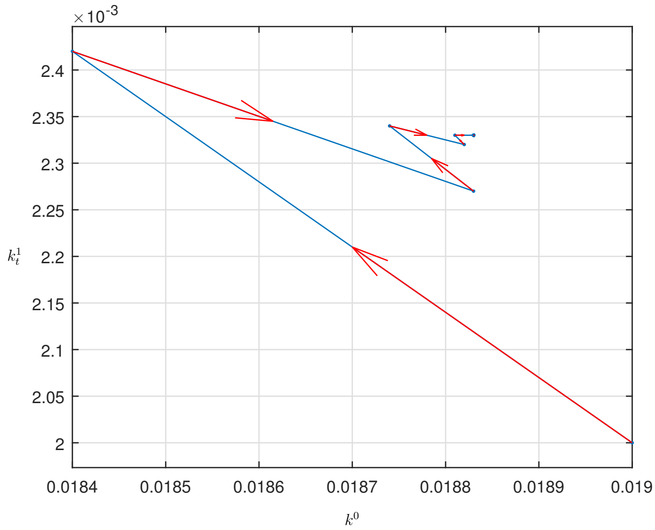
Figure B.29: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

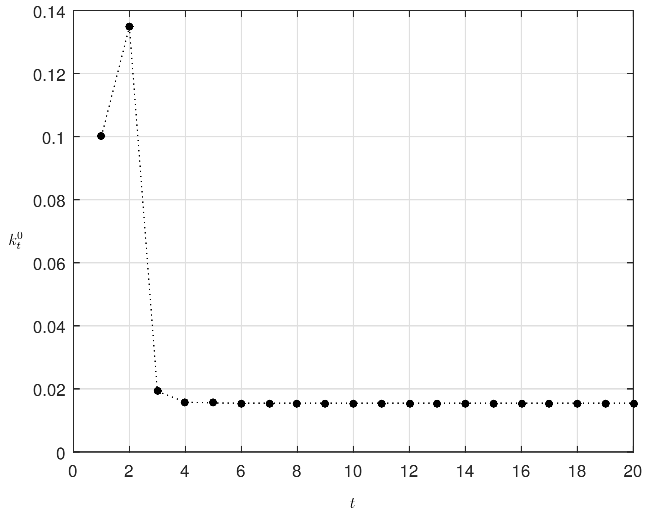



(b) $k^1$

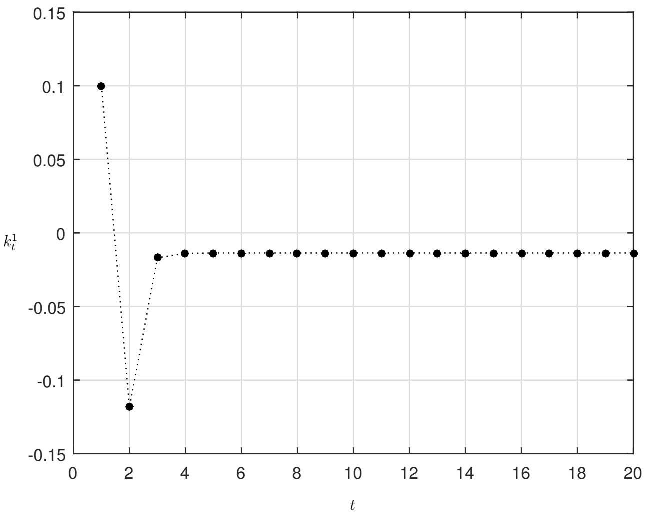



(c) $\mathbf{k}$

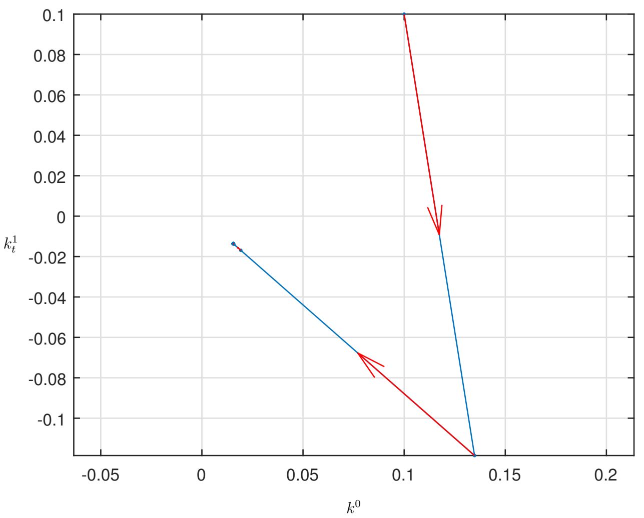
Figure B.30: Dynamics with $\ell^1 = \frac{2}{3}$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.019,\ 0.002)$




(a) $k^0$

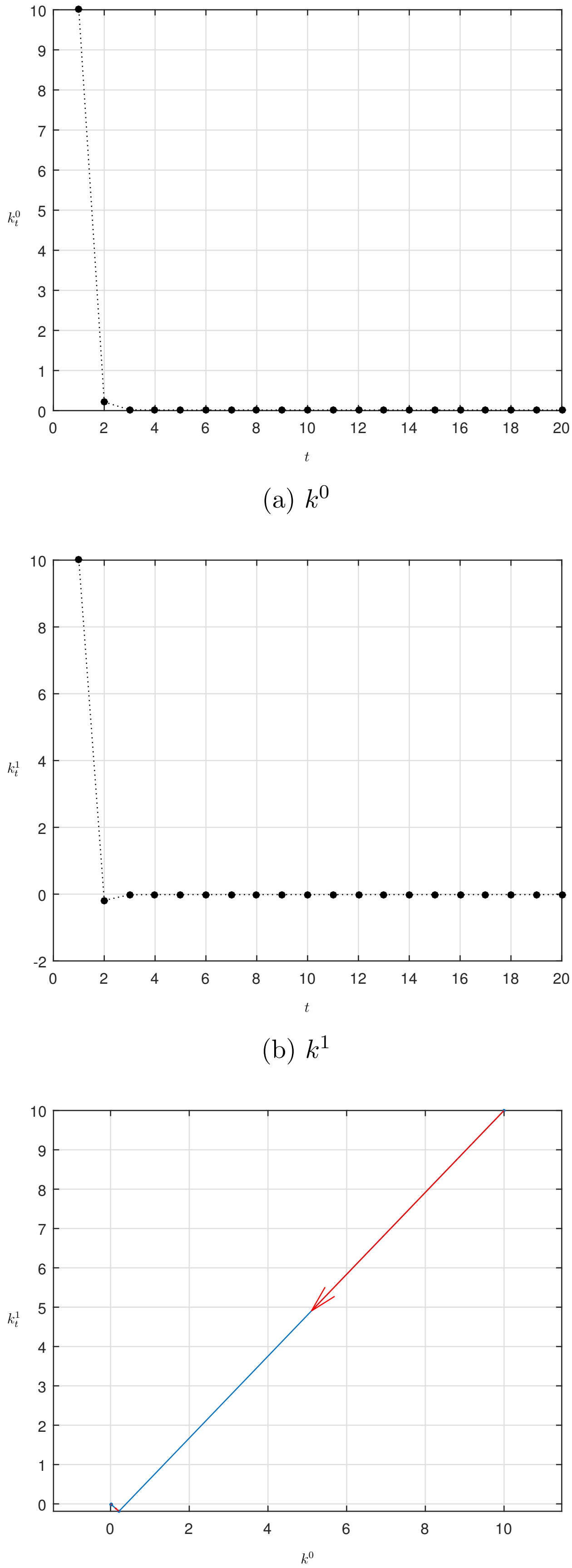



(b) $k^1$




(c) $\mathbf{k}$

Figure B.31: Dynamics with $\ell^1 = 1$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.1,\ 0.1)$

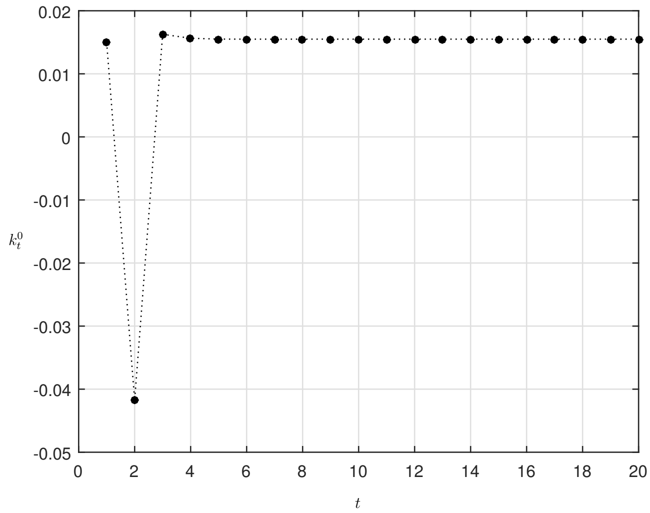



(a) $k^0$

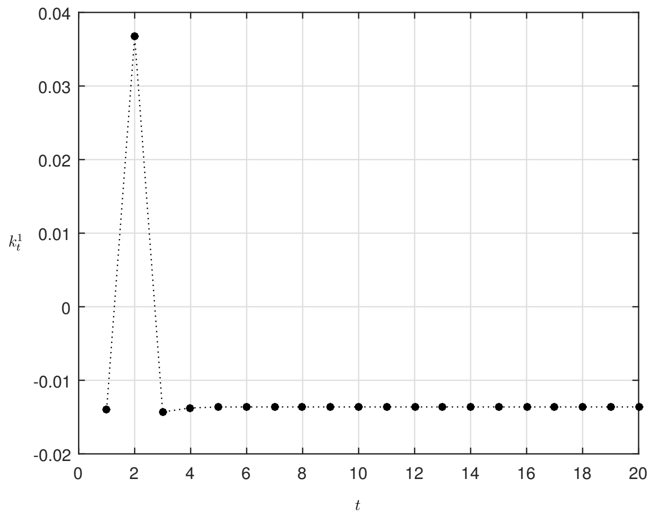
(b) $k^1$

(c) $\mathbf{k}$

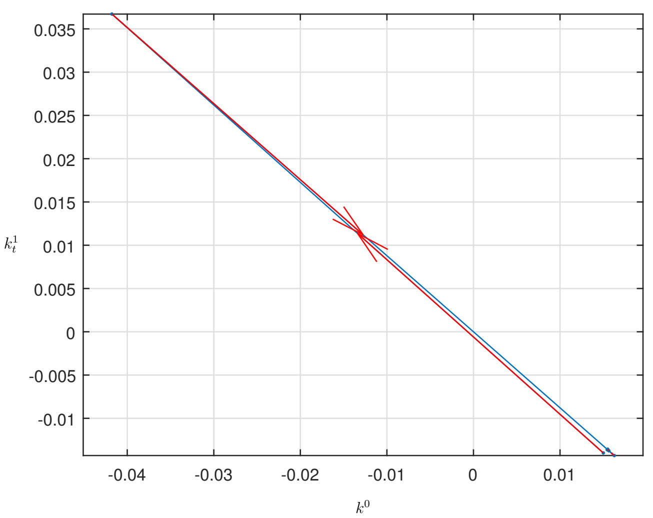
Figure B.32: Dynamics with $\ell^1 = 1$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

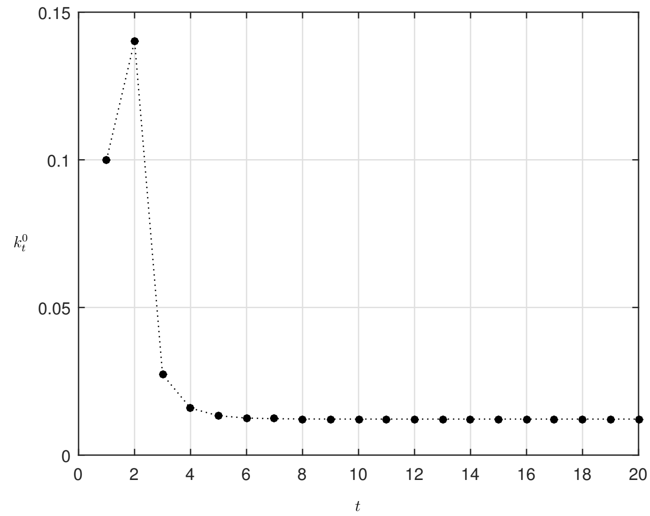



(b) $k^1$

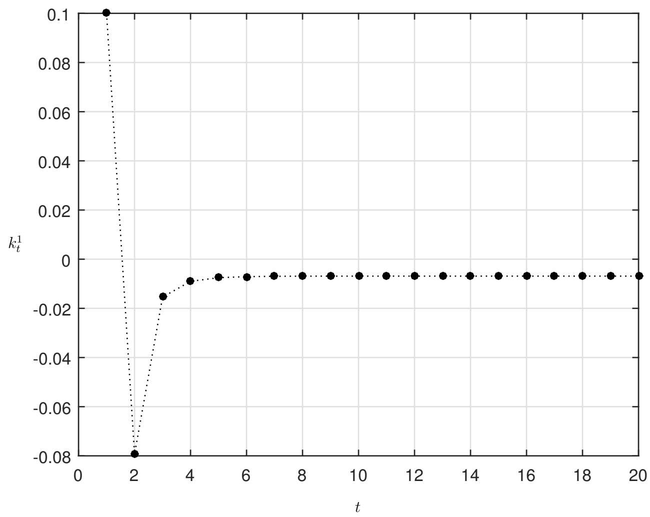



(c) $\mathbf{k}$

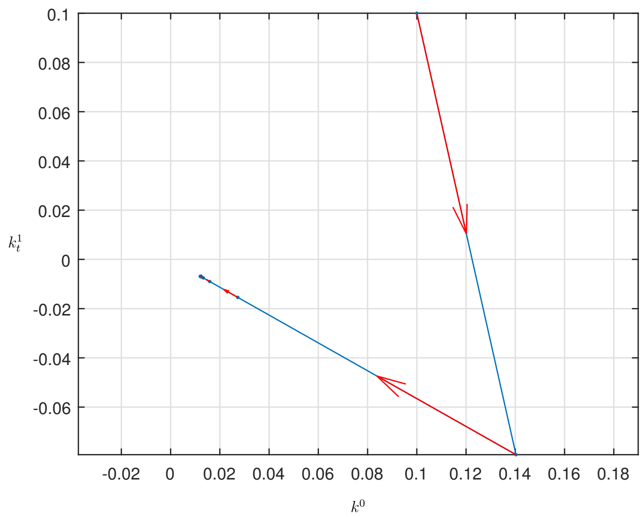
Figure B.33: Dynamics with $\ell^1 = 1$, $\alpha = 0.1$, $\beta = 0.2$, $\mathbf{k}_1 = (0.015,\ -0.014)$




(a) $k^0$

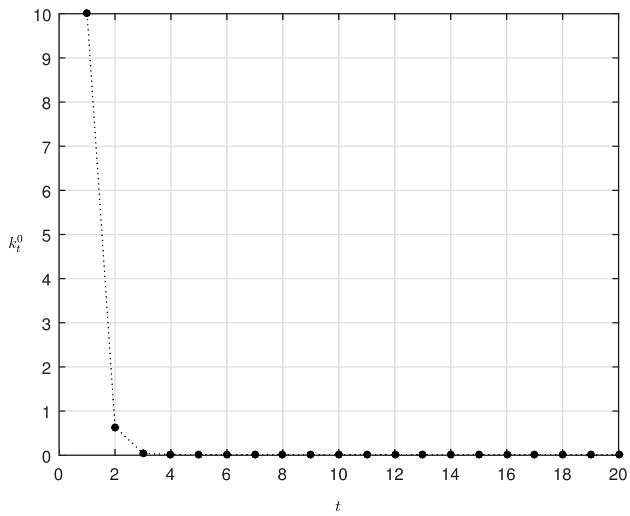



(b) $k^1$

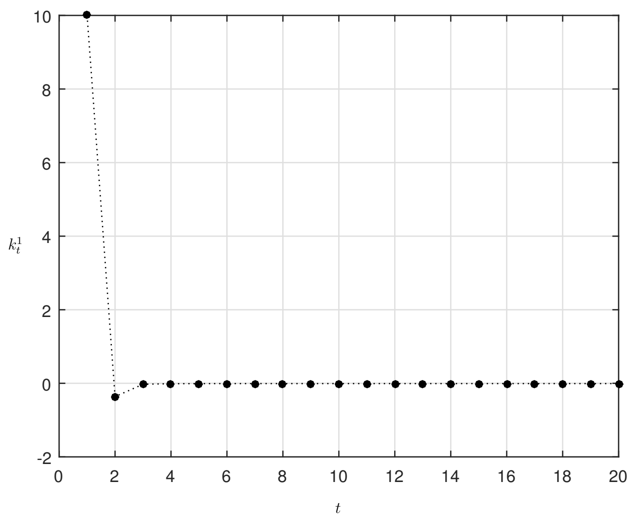



(c) $\mathbf{k}$

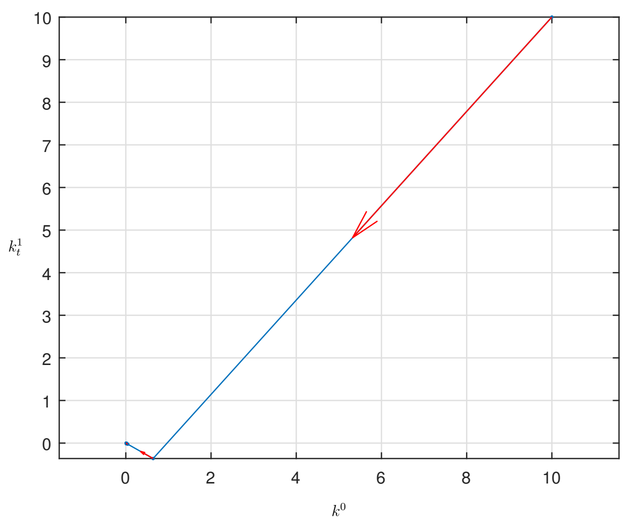
Figure B.34: Dynamics with $\ell^1 = 1$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

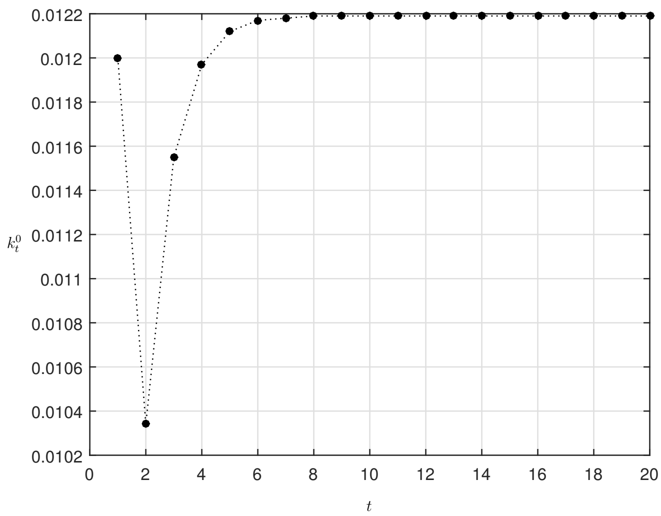



(b) $k^1$

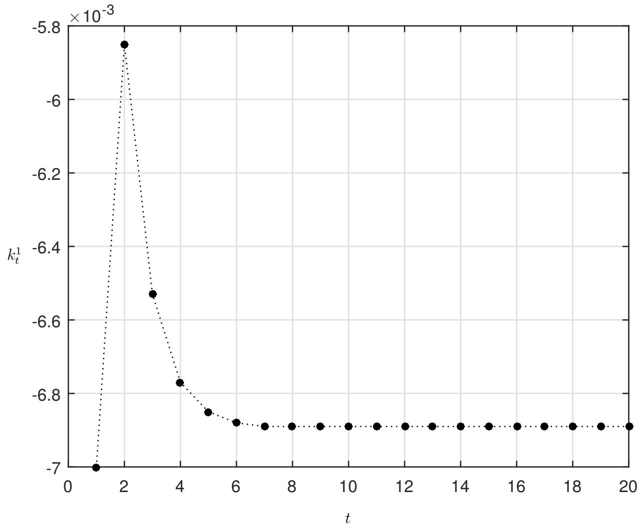



(c) $\mathbf{k}$

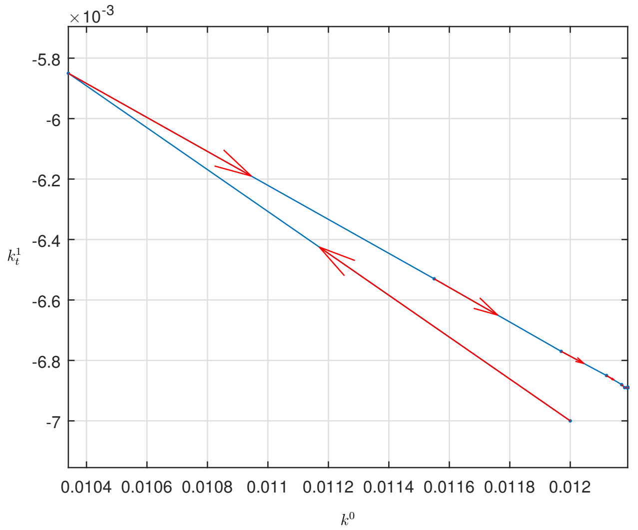
Figure B.35: Dynamics with $\ell^1 = 1$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$

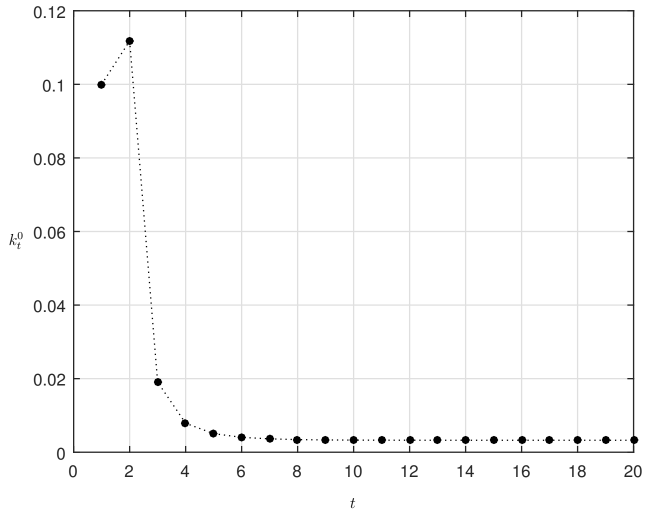



(b) $k^1$

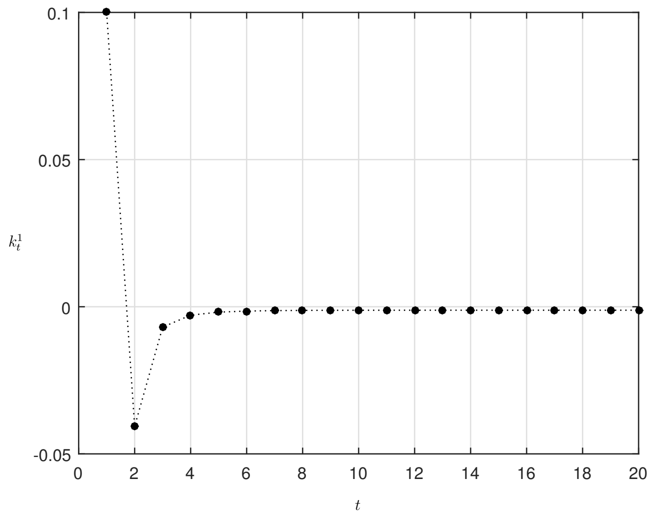



(c) $\mathbf{k}$

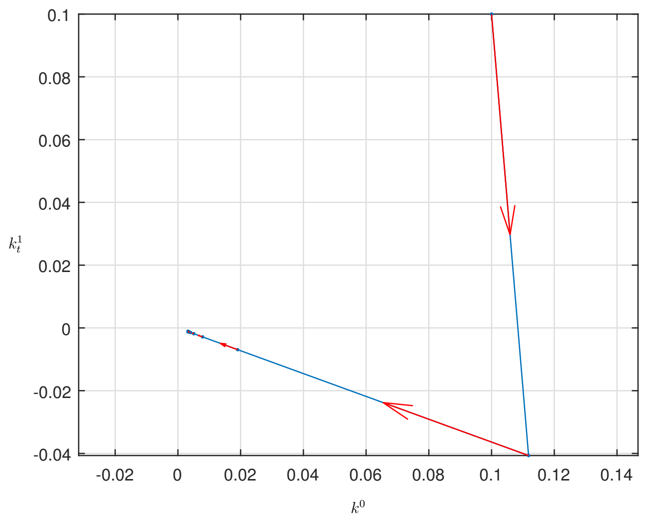
Figure B.36: Dynamics with $\ell^1 = 1$, $\alpha = 0.33$, $\beta = 0.4$, $\mathbf{k}_1 = (0.012,\ -0.007)$




(a) $k^0$

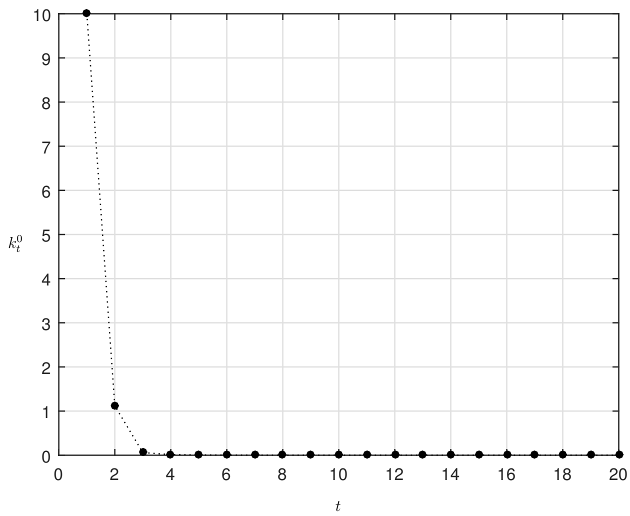



(b) $k^1$

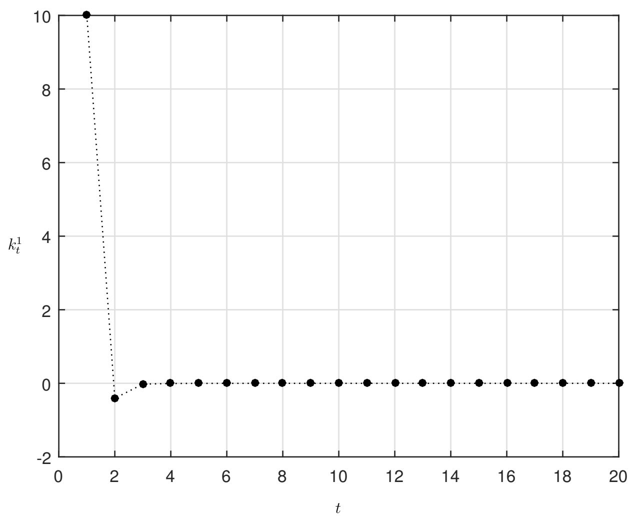



(c) $\mathbf{k}$

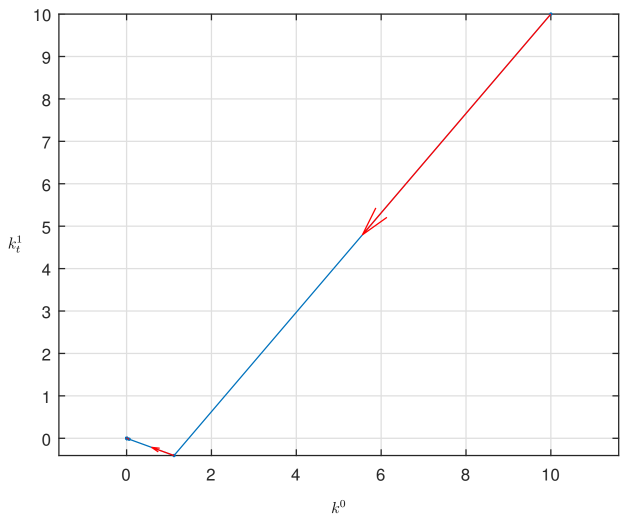
Figure B.37: Dynamics with $\ell^1 = 1$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.1,\ 0.1)$




(a) $k^0$

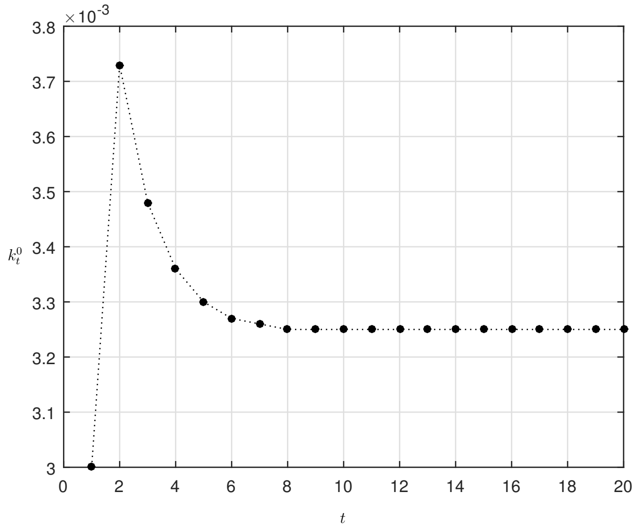



(b) $k^1$

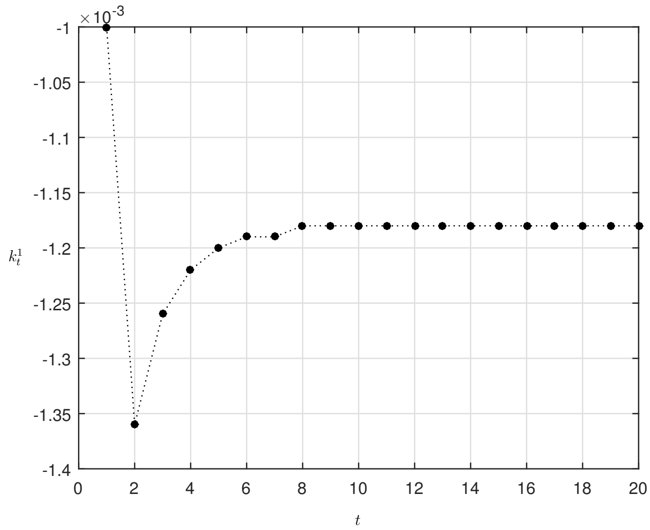



(c) $\mathbf{k}$

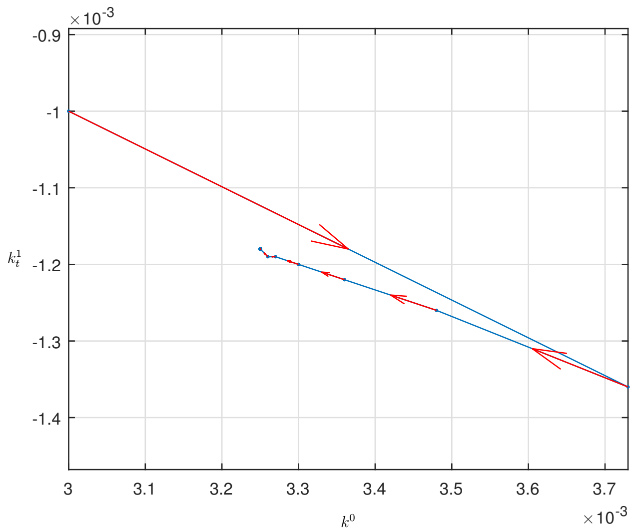
Figure B.38: Dynamics with $\ell^1 = 1$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (10,\ 10)$




(a) $k^0$




(b) $k^1$




(c) $\mathbf{k}$

Figure B.39: Dynamics with $\ell^1 = 1$, $\alpha = 0.5$, $\beta = 0.5$, $\mathbf{k}_1 = (0.003,\ -0.001)$

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# Erklärung

gemäß §4, Abs. 4 der Promotionsordnung vom 15.August 2006

Ich versichere wahrheitsgemäß, die Dissertation bis auf die in der Abhandlung angegebene Hilfe selbstständig angefertigt, alle benutzten Hilfsmittel vollständig und genau angegeben und genau kenntlich gemacht zu haben, was aus Arbeiten anderer und aus eigenen Veröffentlichungen unverändert oder mit Abänderungen entnommen wurde.

Hiermit erkläre ich, dass ich bisher an keiner anderen Hochschule ein Promotionsgesuch eingereicht habe.

Karlsruhe, 07.01.2016