Karlsruhe Institute of Technology

Wave phenomena analysis and numerics

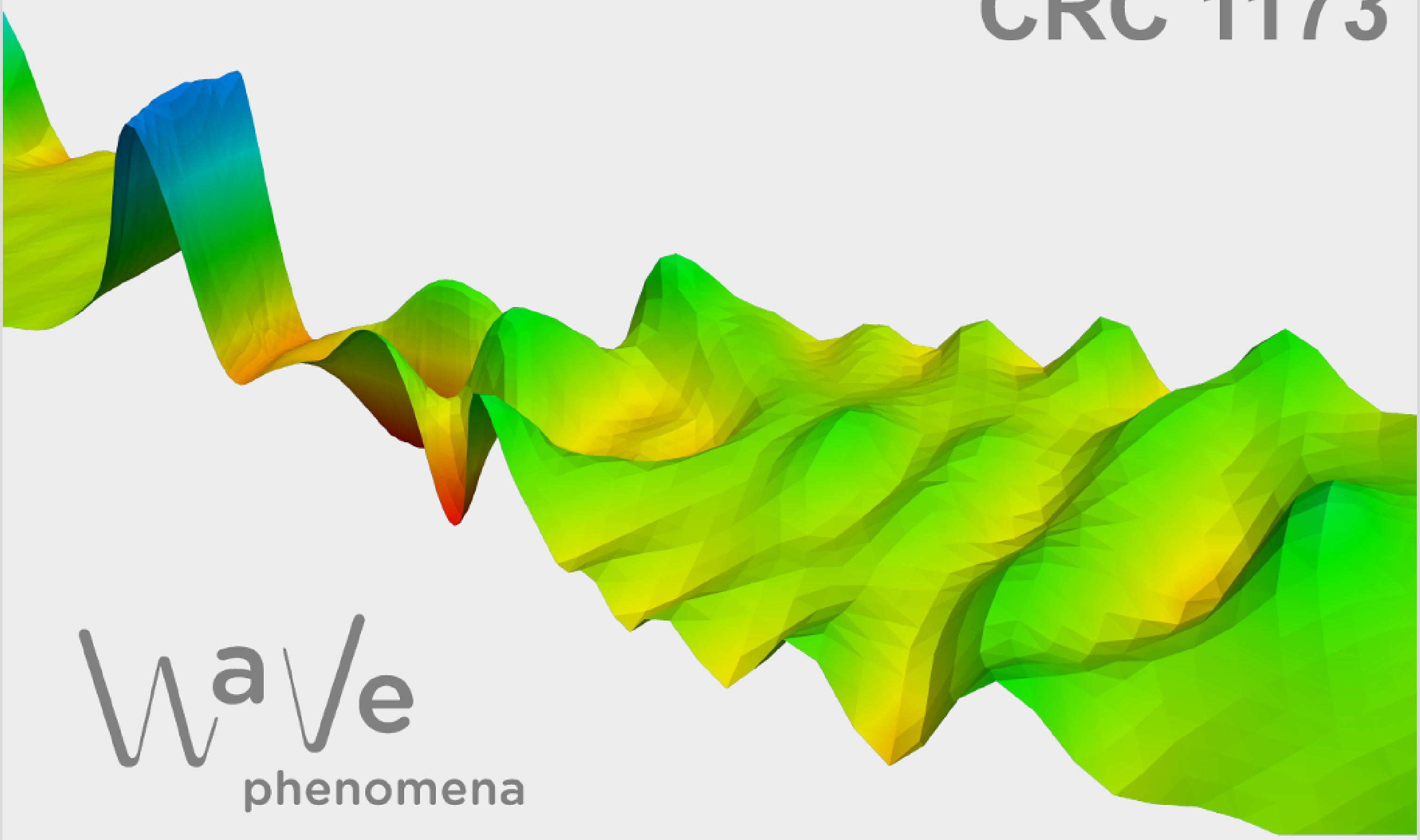
New criteria for the H^∞-calculus and the the Stokes operator on bounded Lipschitz domains

Peer Christian Kunstmann, Lutz Weis

CRC Preprint 2016/26, October 2016

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

Participating universities

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Funded by

DFG

ISSN 2365-662X

New criteria for the H^∞-calculus and the Stokes operator on bounded Lipschitz domains

Peer Christian Kunstmann and Lutz Weis*
Karlsruhe Institute of Technologie (KIT)
Institute for Analysis
Englerstr. 2, D – 76128 Karlsruhe, Germany
e-mail: peer.kunstmann@kit.edu, lutz.weis@kit.edu

Dedicted to Jan Prüss on the occasion of his 65th birthday

Abstract

We show that the Stokes operator A on the Helmholtz space $L^p_\sigma(\Omega)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, has a bounded H^∞-calculus if $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2d}$. Our proof uses a new comparison theorem for A and the Dirichlet Laplace $-\Delta$ on $L^p(\Omega)^d$, which is based on "off-diagonal" estimates of the Littlewood-Paley decompositions of A and $-\Delta$. This comparison theorem can be formulated for rather general sectorial operators and is well suited to extrapolate the H^∞-calculus from $L^2(U)$ to the $L^p(U)$-scale or part of it. It also gives some information on coincidence of domains of fractional powers.

AMS subject classification (MSC2010): 47 A 60, 47 D 06, 35 Q 30.

keywords: sectorial operators, bounded H^∞-calculus, Littlewood-Paley operators, domains of fractional powers, Stokes operator.

*Both authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

1 Introduction

For a sectorial operator A of angle ω on a Banach space X one can define the holomorphic functional calculus by a Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} f(\lambda) R(\lambda, A)\, d\lambda \tag{1}$$

for all bounded analytic funtions f on $\Sigma_\sigma := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \sigma\}$ with $\omega < \nu < \sigma < 2\pi$ and $\int_{\partial\Sigma_\sigma} |f(\lambda)|/|\lambda|\, |d\lambda| < \infty$. A has a bounded $H^\infty(\Sigma_\sigma)$-calculus if there is a uniform estimate

$$\|f(A)\|_{B(X)} \le C \|f\|_{H^\infty(\Sigma_\sigma)} \tag{2}$$

for all such f. This functional calculus has found a lot of interest in evolution equations because it often allows for optimal regularity estimates [4], [6], [12], [17], [20], [21]. By now there is a large literature establishing the boundedness of the H^∞-calculus for very large classes of partial differential operators A on $L^p(U)$-spaces with $1 < p < \infty$. However, even for the Laplace operator $A = -\Delta$ on $L^p(\mathbb{R}^d)$ the proof requires Fourier multiplier theorems (i.e. Littlewood-Paley theory), early results on elliptic operators estimated (1), (2) with the calculus of pseudo-differential operators and there are still some open questions, e.g., it seems that the boundedness for the Stokes operator on a Helmholtz space $L^p_\sigma(\Omega)$, Ω bounded, was only known for domains with a "smooth" boundary but not for a Lipschitz domain.

The purpose of this paper is to close this gap and also to present a new method for the H^∞-calculus, which is well adapted to the task of extrapolating a bounded H^∞-calculus for A on $L^2(U)$ to the whole L^p-scale, or part of it. Our approach is a refinement and a simplification of the comparison method of [11]. Assume that B has a bounded H^∞-calculus on a fixed $L^p(U)$-space. The idea is that if a second sectorial operator A is "close enough" to B it will inherit boundedness of the H^∞-calculus.

Recall that the boundedness of the H^∞-calculus can be characterized in terms of (certain) Littlewood-Paley estimates ([12, Remark 2(b)])

$$\|x\|_{L^p} \sim \Big\| \Big(\sum_{n\in\mathbb{Z}} |\varphi(2^n A)x|^2 \Big)^{1/2} \Big\|_{L^p}, \quad x \in L^p(U), \tag{3}$$

where the analytic function φ on Σ_σ decays polynomially at 0 and ∞. The idea is now that we "compare" the Littlewood-Paley decompositions of A and B and the "closeness" condition we use is motivated by the following simple calculation. Let ψ be a second function with the properties of φ such that $\sum_{n\in\mathbb{Z}} \psi^2(2^n\lambda) = 1$ for $\lambda \in \Sigma_\sigma$. Then, for $x \in L^p(U)$,

$$\begin{aligned}
& \Big\| \big(\sum_n |\varphi(2^n A)x|^2\big)^{1/2} \Big\|_{L^p} \\
= \; & \Big\| \Big(\sum_n \big|\varphi(2^n A)\big[\sum_m \psi^2(2^{n+m}B)x\big]\big|^2\Big)^{1/2} \Big\|_{L^p} \\
\le \; & \sum_m \Big\| \Big(\sum_n |\varphi(2^n A)\psi(2^{n+m}B)\psi(2^{n+m}B)x|^2\Big)^{1/2} \Big\|_{L^p} \\
\le \; & \Big(\sum_m a^{-|m|}\Big) \Big\| \Big(\sum_n |\psi(2^n B)x|^2\Big)^{1/2} \Big\|_{L^p} \le C\|x\|_{L^p},
\end{aligned} \tag{4}$$

if we assume the following "off-diagonal" estimates with respect to the Littlewood-Paley decomposition

$$\begin{aligned}
\Big\| \big(\sum_n |\varphi(2^n A)\psi(2^{n+m}B)x_n|^2\big)^{1/2} \Big\|_{L^p} & \le C a^{-|m|} \Big\| \big(\sum_n |x_n|^2\big)^{1/2} \Big\|_{L^p} \\
& \text{and} \\
\Big\| \big(\sum_n |\varphi(2^n A)'\psi(2^{n+m}B)'x_n'|^2\big)^{1/2} \Big\|_{L^{p'}} & \le C a^{-|m|} \Big\| \big(\sum_n |x_n'|^2\big)^{1/2} \Big\|_{L^{p'}}
\end{aligned} \tag{5}$$

for some $a > 1$. Note that the second condition gives the lower estimate of (3) by a similar dual argument. Hence, if B satisfies (3) and A satisfies (5) then by (4) also A has a bounded H^∞-calculus. As we shall show below (see Proposition 3), condition (5) holds, e.g., if $D(A^{\alpha_j}) = D(B^{\alpha_j})$ (with equivalent norms) for two indices with $\alpha_1 < 0 < \alpha_2$ and A is R-sectorial, i.e. for some $C > 0$ and all choices of $\lambda_1, \ldots, \lambda_n \notin \overline{\Sigma}_\sigma$, $x_1, \ldots, x_n \in L^p(U)$ we have

$$\Big\| \big(\sum_j |\lambda_j R(\lambda_j, A)x_j|^2\big)^{1/2} \Big\|_{L^p} \le C \Big\| \big(\sum_j |x_j|^2\big)^{1/2} \Big\|_{L^p}. \tag{6}$$

And this condition is clearly related to Littlewood-Paley theory. (6) is known to hold, e.g., if the semigroup e^{-tA} satisfies (generalized) Gaussian bounds (see, e.g. [17, Chapter 8]).

This argument is particularly well suited for extrapolation from $L^2(U)$ to the $L^p(U)$-scale. Often it is possible to check the equality of some fractional domains of A and B on $L^2(U)$. Then we have (5) on $L^2(U)$ for some $a > 1$ by the last remark. If A and B are R-sectorial in L^p then we have (5) in L^p for $a = 1$. Interpolating (5) in L^2 and (5) in L^p then gives (5) on all $L^q(U)$ with q between 2 and p (with a different $a > 1$, we refer to Theorem 5 below), i.e. if A satisfies (6) and B has a bounded H^∞-calculus on these $L^q(U)$-spaces, so does A. For an illustration concerning elliptic operators, see Section 4.

We mentioned that condition (5) can be obtained from the equality (with equivalent norms) of fractional domains of A and B. Conversely, condition (5) implies equality of certain fractional domains (see Theorem 1). Our extrapolation scheme makes it therefore possible also to extrapolate the coincidence of fractional domains of A and B from L^2 to the L^p-scale.

For the Stokes operator A on the Helmholtz space $L^p_\sigma(\Omega)$ we have the additional difficulty that we want to compare A with the Dirichlet Laplace operator $B = -\Delta$ on the larger space $L^p(\Omega)^d$. To this end we introduce a variant of condition (5) including a retraction of $L^p(\Omega)^d$ onto $L^p_\sigma(\Omega)$, which is defined by the Helmholtz projection (see Theorem 9). According to the latter argument the Stokes operator has a bounded H^∞-calculus on $L^p_\sigma(\Omega)$ if we can show that

- the Helmholtz projection P_p is bounded on $L^p(\Omega)^d$,
- we have $L^2_\sigma(\Omega)_{s/2,A} = H^s_\sigma(\Omega)$ and $(L^2(\Omega)^d)_{s/2,B} = H^s(\Omega)^d$ for $|s| < 1/2$ where $X_{s,A}$ denontes $(D(A^s), \|A^s \cdot\|)^\sim$ on X,
- A is R-sectorial on $L^p_\sigma(\Omega)$.

The latter we show by extending Shen's proof in [22] for sectoriality of A to a square function estimate as in (6) for $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{2d}$, $d \geq 3$.

See Section 6 for the precise statement of the theorem on the Stokes operator on bounded Lipschitz domains. In Section 3 we prove our comparison result based on (5). There we will use the random sum techniques of [11] to formulate our result in a Banach space setting. We recall essential definitions and statements from [12] and [11] in Section 2. Since $\mathbb{E}\|\sum \varepsilon_n x_n\|_{L^p} \sim \|\sum |x_n|^2)^{1/2}\|_{L^p}$ in an $L^p(U)$-space for a Rademacher sequence (ε_n) there is no essential difference between the L^p-case and the general setting.

2 Preliminaries

In this paper, X is always a complex Banach space. The space of bounded operators in X is denoted by $B(X)$. We recall the notation $\Sigma_\omega := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \omega\}$ for $\omega \in (0,\pi)$. By abuse of notation we set $\overline{\Sigma}_0 := [0,\infty)$.

We shall need the concept of R-boundedness. A set of operators $\mathscr{T}$ from $X \to Y$ is called *R-bounded* if there exists a constant C such that, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $T_1, \ldots, T_n \in \mathscr{T}$, we have

$$\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j T_j x_j\|_Y \leq C\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|_X.$$

The smallest constant C is denoted $\mathcal{R}(\mathscr{T})$. R-boundedness is stronger than uniform boundedness, and for sets in $B(X)$ it is equivalent to uniform boundedness if and only if X is a Hilbert space (this is an unpublished result due to Pisier, one has to combine [1, Prop. 1.13] with Kwapien's characterization of Hilbert spaces as those Banach spaces having type 2 and cotype 2, see also [17, N 2.12]). Obviously, singletons $\{T\}$ are always R-bounded, we shall use this fact later.

A *sectorial operator* of type $\omega \in [0, \pi)$ is an injective linear operator A in X with dense domain $D(A)$ and dense range $R(A)$, such that its spectrum $\sigma(A)$ is contained in the complex sector $\overline{\Sigma}_\omega$ and one has uniform boundedness of

$$\{\lambda(\lambda - A)^{-1} : \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\theta\} \tag{7}$$

for any $\theta \in (\omega, \pi)$. The infimum of all such $\omega \in [0, \pi)$ is denoted $\omega(A)$.

A sectorial operator A of type ω in X is called *R-sectorial* of type ω, if the sets in (7) are R-bounded in $B(X)$. The infimum of all such ω is denoted $\omega_R(A)$. A sectorial operator A of type ω in X is called *almost R-sectorial* of type ω, if the sets

$$\{\lambda A(\lambda - A)^{-2} : \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\theta\}, \quad \theta \in (\omega, \pi),$$

are R-bounded in $B(X)$. The infimum of all such ω is denoted $\omega_r(A)$.

If A is a sectorial operator in a Banach space X and $\alpha \in \mathbb{R}$ we denote by $\dot{X}_{\alpha,A}$ the completion of $D(A^\alpha)$ with respect to the norm $\|A^\alpha \cdot \|_X$. For the scale $(\dot{X}_{\alpha,A})$ of homogeneous fractional domain spaces and their properties we refer to [11, Sect. 2] and [17, Sect. 15.E].

For an angle $\omega \in (0, \pi)$ we denote by $H^\infty(\Sigma_\omega)$ the set of all bounded homolorphic functions on Σ_ω and by $H_0^\infty(\Sigma_\omega)$ the subset of those functions $f \in H^\infty(\Sigma_\omega)$ that satisfy, for some $\varepsilon > 0$, $|f(z)| = O(|z|^\varepsilon)$ as $z \to 0$ and $|f(z)| = O(|z|^{-\varepsilon})$ as $z \to \infty$. If A is a sectorial operator in X of type ω and $\theta \in (\omega, \pi)$ then for $f \in H_0^\infty(\Sigma_\theta)$ the absolutely convergent Dunford type integral

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_\nu} f(\lambda)(\lambda - A)^{-1} \, d\lambda$$

defines a bounded operator $f(A)$ on X, which is independent of $\nu \in (\omega, \theta)$. The operator A is said to have a bounded $H^\infty(\Sigma_\theta)$-calculus, if there is a constant C such that

$$\|f(A)\| \leq C\|f\|_{\infty,\Sigma_\theta}$$

for all $f \in H_0^\infty(\Sigma_\theta)$. In this case, the functional calculus $f \mapsto f(A)$ extends to a bounded algebra homomorphism $H^\infty(\Sigma_\theta) \to B(X)$. The infimum of all such angles θ is denoted $\omega_H(A)$. For more details on the construction of the H^∞-calculus and its properties we refer to [3, 12, 11, 10]).

We recall the characterization of [11, Theorem 4.1]: If A is an almost R-sectorial operator in X and $\psi \in H^\infty(\Sigma_\omega) \setminus \{0\}$ where $\omega \in (\omega_r(A), \pi)$ then each of the following two conditions is equivalent to A having a bounded $H^\infty(\Sigma_\sigma)$-calculus for each $\sigma \in (\omega_r(A), \pi)$:

(i) There are constants $C_{\psi,A}, C_{\psi,A'} > 0$ such that, for $x \in X$ and $x' \in X'$,

$$\begin{aligned}
\sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j \psi(t2^j A)x \Big\|_X &\le C_{\psi,A}\|x\|_X, \\
\sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j \psi(t2^j A')x' \Big\|_{X'} &\le C_{\psi,A'}\|x'\|_{X'}.
\end{aligned}$$

(ii) There are constants $C_{\psi,A}, C'_{\psi,A} > 0$ such that, for $x \in X$,

$$C'_{\psi,A}\|x\|_X \le \sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j \psi(t2^j A)x \Big\|_X \le C_{\psi,A}\|x\|_X$$

Here, X' denotes the dual space of X. For later use we remark that, as a consequence of the contraction principle (see [11, Prop. 2.5]), one has for finite subsets $F_1, F_2 \subseteq \mathbb{Z}$ the following monotonicity

$$\mathbb{E}\Big\| \sum_{j\in F_1} \varepsilon_j x_j \Big\| \le \mathbb{E}\Big\| \sum_{j\in F_2} \varepsilon_j x_j \Big\| \quad \text{if } F_1 \subseteq F_2. \tag{8}$$

At the end of this section we recall that any sectorial operator A that has a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (\omega(A), \pi)$ is almost R-sectorial and satisfies $\omega_H(A) = \omega_r(A)$ (see [11, Corollary 4.4]). So in the sequel we just say that A has a bounded H^∞-calculus.

3 Criteria via Littlewood-Paley operators

Our first theorem gives our basic comparison criterion for the boundedness of the H^∞-calculus and the coincidence of fractional domains of two operators A and B on a Banach space X in terms of an off-diagonal estimate of their Littlewood-Paley decompositions. Variants of it in more concrete situations will be given in subsequent sections.

Theorem 1. *Let B have a bounded $H^\infty(\Sigma_\sigma)$-calculus on a Banach space X and let A be an almost R-sectorial operator in X. Assume that there are* *functions $\varphi, \psi \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ where $\nu > \sigma$ such that, for some $\beta_0, \beta_1 > 0$ and all $l \in \mathbb{Z}$,*

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+l}A)\psi(t2^jB) : j \in \mathbb{Z}\} \;\le\; C_0 2^{-\beta_0|l|}, \tag{9}$$

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+l}A)'\psi(t2^jB)' : j \in \mathbb{Z}\} \;\le\; C_1 2^{-\beta_1|l|}. \tag{10}$$

Then A has a bounded H^∞-calculus on X. Furthermore, if X is reflexive and for all $\alpha \in (-\beta_0, \beta_0)$ the functions $\varphi_\alpha(\lambda) = \lambda^\alpha\varphi(\lambda)$, $\psi_\alpha(\lambda) = \lambda^\alpha\psi(\lambda)$ still belong to $H_0^\infty(\Sigma_\nu)$ then for α with $|\alpha| < \beta_0$

$$D(B^\alpha) \subseteq D(A^\alpha), \qquad \|A^\alpha x\| \lesssim \|B^\alpha x\| \text{ for } x \in D(B^\alpha). \tag{11}$$

If $\varphi_\alpha, \psi_\alpha \in H_0^\infty(\Sigma_\nu)$ for $|\alpha| < \beta_1$ then

$$D(A^\alpha) \subseteq D(B^\alpha), \qquad \|B^\alpha x\| \lesssim \|A^\alpha x\| \text{ for } x \in D(A^\alpha). \tag{12}$$

Remark. Our proof shows that A has a bounded H^∞-calculus if we make only the weaker assumptions

$$\sup_{1\le s,t\le 2} \sum_{l\in\mathbb{Z}} \mathcal{R}\{\varphi(s2^{j+l}A)\psi(t2^jB) : j \in \mathbb{Z}\} < \infty, \tag{13}$$

$$\sup_{1\le s,t\le 2} \sum_{l\in\mathbb{Z}} \mathcal{R}\{\varphi(s2^{j+l}A)'\psi(t2^jB)' : j \in \mathbb{Z}\} < \infty. \tag{14}$$

Proof. We need regularizing operators. Setting

$$h_n(z) = n^2 z(1+nz)^{-1}(n+z)^{-1}$$

we let $U_n := h_n(A)^m$ where $m \in \mathbb{N}$ is $\ge \beta_0$. Then U_n maps into $D(A^m) \cap R(A^m)$ and $U_n x \to x$ for all $x \in X$. Moreover, $C_A := \sup_n \|U_n\| < \infty$ and we have

$$\|x\|_X \le \sup_n \|xU_n\|_X \le C_A\|x\|_X \quad \text{for all } x \in X.$$

We shall estimate square functions for the operator A via a reproducing type formula. To this end we choose $\widetilde{\psi} \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ as $\widetilde{\psi}(z) = [\int |\psi(t)|^2\, dt/t]^{-1}\overline{\psi(\overline{z})}$ so that $\int_0^\infty \psi(t)\widetilde{\psi}(t)\,\frac{dt}{t} = 1$ and put

$$\rho(z) := \int_1^2 \psi(tz)\widetilde{\psi}(tz)\,\frac{dt}{t}.$$

Then $\rho \in H_0^\infty(\Sigma_\nu)$ and $\sum_{j\in\mathbb{Z}} \rho(2^jz) = 1$ for all $z \in \Sigma_\nu$, which leads to

$$\sum_{j\in\mathbb{Z}} \rho(2^jB)x = x \quad \text{for all } x \in D(B) \cap R(B)$$

where the series is absolutely convergent in X (since the Dunford calculus even gives $\sum_j \|\rho(2^j B)h_1(B)\| < \infty$). For $x \in D(B^m) \cap R(B^m)$ and $|\alpha| < \beta_0$ we thus have

$$\varphi_{-\alpha}(s2^k A)A^\alpha U_n x = 2^{-k\alpha} s^{-\alpha} U_n \varphi(s2^k A)x,$$

and we thus obtain

$$\begin{aligned}
&\sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi_{-\alpha}(s2^k A)A^\alpha U_n x\Big\|_X \\
\le\;& C_A \sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \sum_j \varepsilon_k 2^{-k\alpha} s^{-\alpha} \varphi(s2^k A)\rho(2^{k-j}B)x\Big\|_X \\
\le\;& C_A 2^{|\alpha|} \sup_{s\in[1,2]} \sum_j 2^{-\alpha j} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi(s2^k A) 2^{-(k-j)\alpha} \rho(2^{k-j}B)x\Big\|_X \\
\le\;& C_A 2^{|\alpha|} \sup_{s\in[1,2]} \sum_j 2^{-\alpha j} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_{k-j} \varphi(s2^k A) 2^{-(k-j)\alpha} \rho(2^{k-j}B)x\Big\|_X,
\end{aligned}$$

where we observe that the sequence $(\varepsilon_{k-j})_k$ has the same distribution as $(\varepsilon_k)_k$. Now we shift the summation index k by j and use (8) to obtain

$$\le C_A 2^{|\alpha|} \sup_{s\in[1,2]} \sum_j 2^{-\alpha j} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi(s2^{k+j} A) 2^{-k\alpha} \rho(2^k B)x\Big\|_X.$$

We continue by writing $\rho(t2^k B)x$ as an integral and use

$$2^{-k\alpha}\widetilde{\psi}(t2^k B)x = t^\alpha \widetilde{\psi}_{-\alpha}(t2^k B)B^\alpha x$$

(recall that $x \in D(B^m) \cap D(R^m) \subseteq D(B^\alpha)$) to obtain

$$\begin{aligned}
&\sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_k \varepsilon_k \varphi_{-\alpha}(s2^k A)A^\alpha U_n x\Big\|_X \\
\le\;& C_A 2^{2|\alpha|} \sup_{s\in[1,2]} \int_1^2 \sum_j 2^{-\alpha j} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi(s2^{k+j}A)\psi(t2^k B)\widetilde{\psi}_{-\alpha}(t2^k B)B^\alpha x\Big\|_X \frac{dt}{t} \\
\le\;& C_A 2^{2|\alpha|} \sup_{s,t\in[1,2]} \sum_j 2^{-\alpha j} \mathcal{R}\{\varphi(s2^{k+j}A)\psi(t2^k B) : k \in \mathbb{Z}\} \times \\
&\qquad \sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \widetilde{\psi}_{-\alpha}(t2^k B)B^\alpha x\Big\|_X \\
\le\;& C_A 2^{2|\alpha|} C_0 \Big(\sum_j 2^{-(\beta_0-|\alpha|)|j|}\Big) C_{\widetilde{\psi}_{-\alpha},B} \|B^\alpha x\|_X
\end{aligned}$$

where the constant is finite due to $|\alpha| < \beta_0$. We obtain the estimate

$$\begin{aligned} &\sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi_{-\alpha}(s2^k A')(A^\alpha)' U_n' x' \Big\|_{X'} \\ \le\ & C_A 2^{2|\alpha|} C_1 \Big(\sum_j 2^{-(\beta_1-|\alpha|)|j|} \Big) C_{\widetilde{\psi}_{-\alpha},B'} \|(B^\alpha)' x'\|_{X'} \end{aligned}$$

for the dual square function in the same way with a finite constant, if $|\alpha| < \beta_1$.

For $\alpha = 0$ we have

$$\sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi(s2^k A) x \Big\|_X \le \sup_n \sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k \varphi(s2^k A) U_n x \Big\|_X$$

and our estimates show that A has a bounded H^∞-calculus. But then

$$\|A^\alpha U_n x\|_X \sim \sup_{s\in[1,2]} \mathbb{E}\Big\| \sum_k \varepsilon_k \varphi_{-\alpha}(s2^k A) A^\alpha U_n x \Big\|_X,$$

and we have shown

$$\sup_n \|A^\alpha U_n x\|_X \lesssim \|B^\alpha x\|_X$$

for all $|\alpha| < \beta_0$. By reflexivity of X we find a weakly convergent subsequence of $(A^\alpha U_n x)$, and weak closedness of A^α implies $x \in D(A^\alpha)$ for $x \in D(B^m) \cap R(B^m)$. Then (11) follows, since $D(B^m) \cap R(B^m)$ is a core for B^α.

The dual estimate gives

$$\|(A^\alpha)' x'\|_{X'} \lesssim \|(B^\alpha)' x'\|_{X'} \quad \text{for all } |\alpha| < \beta_1,$$

which implies, if X is reflexive, by [11, Corollary 5.6] or Proposition 11 that

$$\|B^\alpha x\|_X \lesssim \|A^\alpha x\|_X \quad \text{for all } |\alpha| < \beta_1,$$

and (12) follows.

To justify the remark, note that for $\alpha = 0$ we only need summability of the R-bounds in the last estimate of the argument above. □

Remark 2. (a) If we only assume that B has an $H^\infty(\Sigma_\sigma)$-calculus and (9) then it follows already that $\{\varphi(tA) : t > 0\}$ is R-bounded. In particular, if $\varphi(\lambda) = \lambda(1+\lambda)^{-2}$ then the assumption of almost R-sectoriality of A can be omitted since it is automatically fulfilled. Sketch of proof: Replacing x in the k-th term by x_k we obtain for each $s \in [1,2]$ and $\alpha = 0$:

$$\mathbb{E}\Big\| \sum_k \varepsilon_k \varphi(s2^k A) x_k \Big\| \le C\mathbb{E}\Big\| \sum_k \varepsilon_k \widetilde{\psi}(s2^k B) x_k \Big\| \lesssim \mathbb{E}\Big\| \sum_k \varepsilon_k x_k \Big\|$$

by almost R-sectoriality of B. Now we use [17, Example 2.16].

(b) For many natural choices of φ and $\widetilde{\psi}$, e.g. $z^\alpha(1+z)^{-\beta}$ for $0<\beta<\alpha$ or $z^\alpha e^{-z}$, one can omit the sup over $s,t\in[1,2]$ and simply put $s=t=1$ in (9) and (10) (cp. [16]).

(c) It is clear from the proof that the functions φ,ψ in (9) and (10) need not be the same.

(d) Condition (10) in Theorem 1 can be replaced by

$$C_1' := \sup_{1\le s,t\le 2} \sum_{l\in\mathbb{Z}} \mathcal{R}\{\psi(t2^jB)\varphi(s2^{j+l}A) : j\in\mathbb{Z}\} < \infty. \tag{10'}$$

As before, this condition allows to estimate (with a suitably chosen $\widetilde{\varphi}\in H_0^\infty$)

$$\sup_{t\in[1,2]} \mathbb{E}\Big\| \sum_k \varepsilon_k \psi(t2^kB)x\Big\|_X \le C_1' \sup_{s\in[1,2]} \mathbb{E}\Big\| \sum_j \varepsilon_j \widetilde{\varphi}(s2^jA)x\Big\|_X.$$

Since B has a bounded H^∞-calculus, we have

$$\sup_{t\in[1,2]} \mathbb{E}\Big\| \sum_k \varepsilon_k \psi(t2^kB)x\Big\|_X \ge C'_{\psi,B}\|x\|_X,$$

and hence

$$\sup_{s\in[1,2]} \mathbb{E}\Big\| \sum_j \varepsilon_j \widetilde{\varphi}(s2^jA)x\Big\|_X \ge C_1'^{-1}C'_{\psi,B}\|x\|_X.$$

We conclude that A has a bounded H^∞-calculus.

Our next result gives a partial converse of the second part of Theorem 1: The equality of fractional domains is a convenient way to verify the conditions (9) and (10) of Theorem 1.

Proposition 3. *Let A and B be almost R-sectorial operators of angle ω in X. Suppose that, for some $\alpha_0,\alpha_1>0$, we have*

$$D(B^\alpha)\subseteq D(A^\alpha) \quad \textit{and} \quad \|A^\alpha x\| \lesssim \|B^\alpha x\| \tag{15}$$

for $\alpha=\pm\alpha_0$ and

$$D(A^\alpha)\subseteq D(B^\alpha) \quad \textit{and} \quad \|B^\alpha x\| \lesssim \|A^\alpha x\| \tag{16}$$

for $\alpha=\pm\alpha_1$. If $\varphi,\psi,\lambda^{\pm\alpha}\varphi(\lambda),\lambda^{\pm\alpha}\psi(\lambda)$ are in $H_0^\infty(\Sigma_\nu)$ (where $\nu>\omega$) for $\alpha=\alpha_0,\alpha_1$ then (9) *holds with $\beta_0=\alpha_0$ and* (10) *holds with $\beta_1=\alpha_1$.*

Proof. We write, for $\alpha = \pm\alpha_0$,

$$\begin{aligned} & \varphi(t2^{l+j}A)\psi(s2^jB) \\ = \; & \left(\frac{t}{s}\right)^\alpha 2^{l\alpha}(t2^{l+j})^{-\alpha}A^{-\alpha}\varphi(t2^{l+j}A)[A^\alpha B^{-\alpha}](s2^j)^\alpha B^\alpha\psi(s2^jB) \\ = \; & \left(\frac{t}{s}\right)^\alpha 2^{l\alpha}\,\widetilde{\varphi}(t2^{l+j}A)\,M\,\widetilde{\psi}(s2^jB), \end{aligned}$$

where $\widetilde{\varphi}(z) = z^{-\alpha}\varphi(z)$ and $\widetilde{\psi}(z) = z^\alpha\psi(z)$ are in $H_0^\infty(\Sigma_\nu)$ by assumption, and $M \in B(X)$ denotes the bounded extension of $A^{-\alpha}B^\alpha$ (here we use (15)). By almost R-sectoriality then the sets $\{\widetilde{\varphi}(tA) : t > 0\}$ and $\{\widetilde{\psi}(sB) : s > 0\}$ are R-bounded (see [11, Lemma 3.3]). Taking $\alpha = -\alpha_0 < 0$ for $l \geq 0$ and $\alpha = \alpha_0 > 0$ for $l < 0$ we see that (9) follows with $\beta_0 = \alpha_0$.

For the proof of (10) we note that, by Proposition 11 below (with $X = Y$, $R = I$), condition (16) implies

$$D((B')^{-\alpha}) \subseteq D((A')^{-\alpha}) \quad \text{and} \quad \|(A')^{-\alpha}x'\| \lesssim \|(B')^{-\alpha}x'\| \quad \text{for } \alpha = \pm\alpha_1.$$

Hence we can repeat the argument. $\square$

Combination with Theorem 1 yields a result which should be compared to [11, Theorem 5.1] with an additional restriction on the range of α (we also refer to [18, Theorem 1.1], but mention that in case $P = I$, also the assertion of [11, Theorem 7.9] is correct).

Corollary 4. *Let B have a bounded H^∞-calculus on X and A be almost R-sectorial in X. If* (15) *and* (16) *hold for $\alpha = \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ which are different, then A has a bounded H^∞-calculus on X.*

Proof. If $\alpha_1 < \alpha_2 < 0$ or $0 < \alpha_2 < \alpha_1$ we shift the scales of (homogeneous) fractional domain spaces (see [11, Proposition 2.1]) and obtain, by Theorem 1 and Proposition 3, that A has a bounded H^∞-calculus in $\dot{X}_{\alpha_2,B} = \dot{X}_{\alpha_2,A}$. But then A has a bounded H^∞-calculus in X. $\square$

4 Extrapolation in the L^p-scale

We now describe a quite general method which allows us to extend the boundedness of the H^∞-calculus of differential operators A on a space $L^2(\Omega)$, where (Ω, d, μ) could be, e.g., a metric measure space with the doubling property, to the part of A on $L^p(\Omega)$-spaces, $p \neq 2$. It will be clear from the proof that this argument also applies to other interpolation scales such as Sobolev- and Besov spaces or scales of fractional domains of a sectorial operator (if, e.g., the operators have BIP). The following statements refine [11, Corollary 8.3]. Recall that a family of sectorial operators A_p on a scale of spaces $L^p(U)$, p in a real interval I, is *consistent* if, for $\lambda < 0$ and $p, \widetilde{p} \in I$ we have $R(\lambda, A_p)x = R(\lambda, A_{\widetilde{p}})x$ for $x \in L^p(U) \cap L^{\widetilde{p}}(U)$.

Theorem 5. *Let $p_0 \in (1, \infty) \setminus \{2\}$, and suppose that we are given two consistent families of sectorial operators A_p and B_p on $L^p(U)$ for $p = 2, p_0$ and all p between 2 and p_0. Let B_p have a bounded H^∞-calculus in $L^p(U)$ for $p = 2, p_0$, and let A be almost R-sectorial in $L^{p_0}(U)$. Assume further that there are functions $\varphi, \psi \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ such that, for some constants $\delta, C > 0$, and all $l \in \mathbb{Z}$, $s, t \in [1, 2]$,*

$$\sup_{j \in \mathbb{Z}} \|\varphi(s2^{j+l}A)\psi(t2^j B)\|_{L^2 \to L^2} \leq C 2^{-\delta|l|}, \tag{17}$$

$$\sup_{j \in \mathbb{Z}} \|\psi(t2^j B)\varphi(s2^{j+l}A)\|_{L^2 \to L^2} \leq C 2^{-\delta|l|}, \tag{18}$$

$$\mathcal{R}\{\varphi(s2^{j+l}A)\psi(t2^j B) : j \in \mathbb{Z}\} \leq C \quad \text{in } L^{p_0}(U) \text{ and } L^{p_0'}(U), \tag{19}$$

$$\mathcal{R}\{\psi(t2^j B)\varphi(s2^{j+l}A) : j \in \mathbb{Z}\} \leq C \quad \text{in } L^{p_0}(U) \text{ and } L^{p_0'}(U). \tag{20}$$

Then A has a bounded H^∞-calculus in $L^p(U)$ for p between 2 and p_0. Furthermore, we have for $|\alpha| < \theta_p \delta$, where θ_p determined by $\frac{1}{p} = \frac{\theta_p}{2} + \frac{1-\theta_p}{p_0}$, that

$$D(A_p^\alpha) = D(B_p^\alpha), \qquad \|A_p^\alpha x\| \sim \|B_p^\alpha x\| \text{ for } x \in D(A_p^\alpha),$$

as long as the functions $\varphi_\alpha(\lambda) = \lambda^\alpha \varphi(\lambda)$ and $\psi_\alpha(\lambda) = \lambda^\alpha \psi(\lambda)$ are still in $H_0^\infty(\Sigma_\nu)$ for $|\alpha| < \theta_p \delta$.

Proof. Complex interpolation gives the assumptions of Theorem 1 in $L^p(U)$ for p between 2 and p_0, however with the bound $C2^{-\theta_p \delta |l|}$. See also Remark 2 (d) and [11, Corollary 3.9]. □

Remark 6. Conditions (19) and (20) are satisfied if A is almost R-sectorial in $L^{p_0}(U)$ and $L^{p_0'}(U)$.

Corollary 7. *Suppose we are given two consistent families A_p and B_p on $L^p(U)$ for all p between 2 and some $p_0 \in (1, \infty) \setminus \{2\}$ such that*

a) B_{p_0} has a bounded $H^\infty(\Sigma_{\sigma_p})$-calculus on $L^{p_0}(U)$,

b) A_{p_0} is almost R-sectorial with angle ω_{p_0} on $L^{p_0}(U)$,

c) A_2 and B_2 have their numerical range in a sector Σ_{σ_2}, where $\sigma_2 < \pi/2$, and for <u>one</u> $\alpha \in \mathbb{R} \setminus \{0\}$ we have

$$D(A_2^\alpha) = D(B_2^\alpha), \qquad \|A_2^\alpha x\| \sim \|B_2^\alpha x\| \text{ for } x \in D(A_2^\alpha). \tag{21}$$

Then A_p has a bounded H^∞-calculus on $L^p(U)$ for all p between 2 and p_0 and

$$D(A_p^\beta) = D(B_p^\beta), \qquad \|A_p^\beta x\| \sim \|B_p^\beta x\| \text{ for } x \in D(A_p^\beta).$$

for $\beta = s\alpha$, $0 < s < \theta_p$, where $\theta_p \in (0,1)$ is given by $\frac{1}{p} = \frac{\theta_p}{2} + \frac{1-\theta_p}{p_0}$.

Remark. Condition c) is certainly fulfilled if A and B are self-adjoint or defined by a closed sectorial form and satisfy (21), e.g. for $\alpha = 1$ or $\alpha = 1/2$.

A version of this theorem where A_p is only defined on a (consistent) family of complemented subspaces X_p of $L^p(U)$, $p_0 \le p \le p_1$, will be applied in Section 6 in the context of the Stokes operator.

Proof. The operators A_2 and B_2 are accretive (even regularly accretive in the sense of [15]), and thus have a bounded H^∞-calculus (see, e.g., [17, Section 11]). Therefore $(L^2)^{\cdot}_{\beta,A} = (L^2)^{\cdot}_{\beta,B}$ for $\alpha \le \beta \le 0$ if $\alpha < 0$ and for $0 \le \beta \le \alpha$ for $\alpha > 0$. By a result of Kato ([15, Theorem 1.1]) and Proposition 11 it follows that in addition $(L^2)^{\cdot}_{\beta,A} = (L^2)^{\cdot}_{\beta,B}$ for $0 \le \beta < \min\{1/2, |\alpha|\}$ in the first case and for $-\min\{1/2, |\alpha|\} < \beta \le 0$ in the second. Now combine Proposition 3 (applied in L^2) and Theorem 5 to obtain a bounded H^∞-calculus for A_p with some angle. The optimal angle can then be obtained by [11, Corollary 3.9]. □

Furthermore we shall exploit the assertion on fractional domains in Theorem 5 in the following application. It is clear that this approach works in a large variety of situations.

Corollary 8. *Let A be an elliptic operator of order $2m$ with bounded measurable coefficients defined on $\mathbb{R}^d$ by a closed sectorial and coercive form with form domain $H_2^m(\mathbb{R}^d)$. Suppose that the semigroup (e^{-tA}) extends to a bounded C_0-semigroup on $L^{p_0}(\mathbb{R}^d)$ and $L^{p_1}(\mathbb{R}^d)$ where $1 \le p_0 < 2 < p_1 \le \infty$ (weak*-continuous for $p_1 = \infty$). Then, for $p_0 < p < p_1$, (e^{-tA}) extends to a C_0-semigroup in $L^p(\mathbb{R}^d)$ whose negative generator A_p has a bounded H^∞-calculus in $L^p(\mathbb{R}^d)$ and satisfies $D(A_p^\alpha) = H_p^{2m\alpha}(\mathbb{R}^d)$ for $0 < \alpha < \theta_p/2$ where $\theta_p \in (0,1)$ is given by $\frac{1}{p} = \frac{\theta_p}{2} + \frac{1-\theta_p}{p_0}$ in case $p_0 < p < 2$ and by $\frac{1}{p} = \frac{\theta_p}{2} + \frac{1-\theta_p}{p_1}$ in case $2 < p < p_1$.*

Proof. The assertion on the H^∞-calculus is already in [2], and this gives also R-sectoriality of A_p. Then we compare with the self-adjoint operator $B = (-\Delta)^m$ and use the proof of Corollary 7 and the arguments of Theorem 5. □

5 Criteria via Littlewood-Paley operators in complemented subspaces

Now we generalize the setting of Sections 3 and 4 and let A and B act in different Banach spaces X and Y, respectively, where X is isomorphic to a complemented subspace of Y.

Theorem 9. *Let X and Y be Banach spaces. Let $R : Y \to X$ and $S : X \to Y$ be bounded linear operators satisfying $RS = I_X$. Let B have a bounded $H^\infty(\Sigma_\sigma)$-calculus in Y and let A be almost R-sectorial in X. Assume that there are functions $\varphi, \psi \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ where $\nu > \sigma$ such that for some $\beta > 0$*

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+l}A)R\psi(t2^jB) : j \in \mathbb{Z}\} \le C_1 2^{-\beta|l|} \tag{22}$$

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\varphi(s2^{j+l}A)'S'\psi(t2^jB)' : j \in \mathbb{Z}\} \le C_2 2^{-\beta|l|}. \tag{23}$$

Then A has a bounded H^∞-calculus on X. If, in addition, X is reflexive and B-convex (e.g. an L^p-space with $1 < p < \infty$) and $\alpha \in \mathbb{R}$ is such that $|\alpha| < \beta$ and $\varphi_{-\alpha}(\lambda) = \lambda^{-\alpha}\varphi(\lambda)$, $\psi_{-\alpha}(\lambda) = \lambda^{-\alpha}\psi(\lambda)$ still belong to $H_0^\infty(\Sigma_\nu)$ then

$$D(A^\alpha) = \{x \in X : Sx \in D(B^\alpha)\} \quad \textit{with } \|A^\alpha x\|_X \sim \|B^\alpha Sx\|_Y.$$

Remark. If we only assume that

$$\sup_{1\le s,t\le 2} \sum_{l\in\mathbb{Z}} \mathcal{R}\{\varphi(s2^{j+l}A)R\psi(t2^jB) : j \in \mathbb{Z}\} < \infty, \tag{24}$$

$$\sup_{1\le s,t\le 2} \sum_{l\in\mathbb{Z}} \mathcal{R}\{\varphi(s2^{j+l}A)'S'\psi(t2^jB)' : j \in \mathbb{Z}\} < \infty, \tag{25}$$

in place of (22) and (23) then the proof below still gives a bounded H^∞-calculus for A in X.

Proof. We choose $\widetilde{\psi} \in H_0^\infty(\Sigma_\nu) \setminus \{0\}$ as in the proof of Theorem 1 so that $\int_0^\infty \psi(t)\widetilde{\psi}(t)\,\frac{dt}{t} = 1$ and put again

$$\rho(z) := \int_1^2 \psi(tz)\widetilde{\psi}(tz)\,\frac{dt}{t}$$

so that $\rho \in H_0^\infty(\Sigma_\nu)$ and $\sum_{j\in\mathbb{Z}} \rho(2^jz) = 1$ for all $z \in \Sigma_\nu$ and

$$\sum_{j\in\mathbb{Z}} \rho(2^jB)y = y \quad \text{for all } y \in D(B) \cap R(B)$$

where the series is absolutely convergent. In the following, we shall need regularizing operators and let $U_n := h_n(A)$ and $V_n(B) = h_n(B)^m$ where $m \in \mathbb{N}$ is $\geq \beta$ and

$$h_n(z) = n^2 z(1+nz)^{-1}(n+z)^{-1}.$$

Then the operators U_n and V_n are uniformly bounded in X and Y, respectively, and map into $D(A^m) \cap R(A^m)$ and $D(B^m) \cap R(B^m)$, respectively. Moreover, we have $U_n x \to x$ and $V_n y \to y$ as $n \to \infty$ for all $x \in X$ and all $y \in Y$, respectively. It follows that

$$\|Tx\|_X = \lim_{n\to\infty} \|TRV_nSx\|_X \leq \sup_n \|TRV_nSx\|_X$$

for any $x \in X$ and $T \in B(X)$. Thus we have for $x \in D(A^m) \cap R(A^m)$

$$\begin{aligned}
& \sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^k A)x \Big\|_X \\
\leq \quad & \sup_{s\in[1,2]} \sup_{N,n} \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^k A)RV_nSx \Big\|_X .
\end{aligned}$$

Proceeding, for fixed n, as in the proof of Theorem 1 we get

$$\begin{aligned}
& \sup_{s\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^k A)RV_nSx \Big\|_X \\
\leq \quad & \sup_{s\in[1,2]} \sum_j \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^k A)R\rho(2^{k-j}B)V_nSx \Big\|_X \\
= \quad & \sup_{s\in[1,2]} \sum_j \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^k A)R\rho(2^{k-j}B)V_nSx \Big\|_X \\
\leq \quad & \sup_{s\in[1,2]} \sum_j \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^{k+j} A)R\rho(2^{k}B)V_nSx \Big\|_X .
\end{aligned}$$

We continue by writing $\rho(t2^kB)V_nSx$ as an integral:

$$\begin{aligned}
\leq \quad & \sup_{s\in[1,2]} \int_1^2 \sum_j \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \varphi(s2^{k+j}A)R\psi(t2^kB)\widetilde{\psi}(t2^kB)V_nSx \Big\|_X \frac{dt}{t} \\
\leq \quad & \sup_{s,t\in[1,2]} \sum_j \mathcal{R}\{\varphi(s2^{k+j}A)R\psi(t2^kB) : k \in \mathbb{Z}\} \sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \widetilde{\psi}(t2^kB)V_nSx \Big\|_Y \\
\leq \quad & C_1\Big(\sum_j 2^{-\beta|j|)}\Big) \sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \widetilde{\psi}(t2^kB)V_nSx \Big\|_Y \leq C_1\Big(\sum_j 2^{-\beta|j|}\Big) C_{\widetilde{\psi},B} \, \|V_nSx\|_Y \\
\leq \quad & C_1' C_{\widetilde{\psi},B} \, \|V_n\| \|S\| \, \|x\|_X .
\end{aligned}$$

Since $S' : Y' \to X'$, $R' : X' \to Y'$ and $S'R' = I_{X'}$, we obtain the estimate

$$\sup_{s\in[1,2]} \mathbb{E}\Big\| \sum_k \varepsilon_k \varphi(s2^k A')x' \Big\|_{X'} \le C_2 C_{\widetilde{\psi},B'} \big(\sup_n \|U_n'\|\big) \|R'\| \|x'\|_{X'}$$

for the dual square function in the same way. From the first estimate and this we obtain that A has a bounded H^∞-calculus in X. Now we put $\widetilde{\varphi}(z) = [\int |\varphi(t)|^2\, dt/t]^{-1} \overline{\varphi(\overline{z})}$ so that $\int_0^\infty \varphi(t)\widetilde{\varphi}(t)\, \frac{dt}{t} = 1$ and

$$\eta(z) := \int_1^2 \varphi(tz)\widetilde{\varphi}(tz)\, \frac{dt}{t}.$$

satisfies $\eta \in H_0^\infty(\Sigma_\nu)$ and $\sum_{j\in\mathbb{Z}} \eta(2^j z) = 1$ for all $z \in \Sigma_\nu$. We conclude

$$\sum_{k\in\mathbb{Z}} \eta(s2^k A)x = x, \qquad x \in D(A) \cap R(A),$$

where the series converges absolutely in X. Since X is B-convex, we also have by [11, Proposition 3.5] that

$$\sup_{1\le s,t\le 2} \mathcal{R}\{\psi(s2^{k+l}B)S\varphi(t2^k A) : k \in \mathbb{Z}\} \le C_2 2^{-\beta|l|}.$$

Now we let $|\alpha| < \beta$ and estimate, similarly as we have done before, for $x \in D(A^m) \cap R(A^m)$,

$$\begin{aligned}
\|B^\alpha V_n Sx\|_Y &\lesssim \sup_{t\in[1,2]} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j \psi_{-\alpha}(t2^j B) B^\alpha V_n Sx \Big\|_Y \\
&\le \sup_{t\in[1,2]} \sup_N E\Big\| \sum_{|j|\le N} \varepsilon_j t^{-\alpha} 2^{-\alpha j} V_n \psi(t2^j B)Sx \Big\|_Y
\end{aligned}$$

$$\begin{aligned}
&\le 2^{|\alpha|} \sup_{t\in[1,2]} \sum_k \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j 2^{-\alpha j} V_n \psi(t2^j B) S\eta(2^{j-k}A)x \Big\|_Y \\
&\le 2^{|\alpha|} \sup_{t\in[1,2]} \sum_k 2^{-\alpha k} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j V_n \psi(t2^j B) S 2^{-\alpha(j-k)} \eta(s2^{j-k}A)x \Big\|_Y \\
&\le 2^{|\alpha|} \sup_{t\in[1,2]} \sum_k 2^{-\alpha k} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j V_n \psi(t2^{j+k} B) S 2^{-\alpha j} \eta(s2^j A)x \Big\|_Y \\
&\le 2^{|\alpha|} \sup_{t\in[1,2]} \int_1^2 \sum_k 2^{-\alpha k} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j 2^{-\alpha j} V_n \psi(t2^{j+k}B) S\varphi(s2^j A)\widetilde{\varphi}(s2^j A)x \Big\|_Y \frac{ds}{s} \\
&\le 2^{|\alpha|} \sup_{s,t\in[1,2]} \sum_k 2^{-\alpha k} \mathcal{R}\{V_n \psi(t2^{j+k}B) S\varphi(s2^j A) : j \in \mathbb{Z}\} \sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k 2^{-\alpha k} \widetilde{\varphi}(s2^k A)x \Big\|_X \\
&\le 2^{2|\alpha|} C_2 \sup_n \|V_n\| \Big(\sum_l 2^{-(\beta-|\alpha|)|l|} \Big) \mathbb{E}\Big\| \sum_k \varepsilon_k \widetilde{\varphi}_{-\alpha}(s2^k A) A^\alpha x \Big\|_X \\
&\lesssim \|A^\alpha x\|_X.
\end{aligned}$$

So we have shown

$$\sup_n \|B^\alpha V_n Sx\|_Y \lesssim \|A^\alpha x\|_X, \quad x \in D(A^m) \cap R(A^m). \tag{26}$$

If $x \in D(A^\alpha)$ then the argument we used in the proof of Theorem 1 shows that $Sx \in D(B^\alpha)$ and

$$\|B^\alpha Sx\|_Y \lesssim \|A^\alpha x\|_X.$$

In the same way, using $\rho, \psi, \widetilde{\psi}$ in place of $\eta, \varphi, \widetilde{\varphi}$, we obtain

$$\sup_n \|A^\alpha U_n Ry\|_X \lesssim \|B^\alpha y\|_Y, \quad y \in D(B^m) \cap R(B^m). \tag{27}$$

Again, the argument we used in the proof of Theorem 1 shows for $y \in D(B^\alpha)$ that $Ry \in D(A^\alpha)$ and

$$\|A^\alpha Ry\|_X \lesssim \|B^\alpha y\|_Y.$$

Finally, if $x \in X$ and $y = Sx \in D(B^\alpha)$ this implies $x = RSx = Ry \in D(A^\alpha)$ and

$$\|A^\alpha x\|_X = \|A^\alpha RSx\|_X \lesssim \|B^\alpha Sx\|_Y,$$

and the proof is finished. □

We give conditions that imply (22) and (23) in the style of Proposition 3.

Proposition 10. *Condition* (22) *holds if*

$$R(D(B^\alpha)) \subseteq D(A^\alpha), \qquad \|A^\alpha Ry\|_X \le C\|B^\alpha y\|_Y, \quad y \in D(B^\alpha),$$

for $\alpha = \alpha_1, \alpha_2$ *where* $\alpha_1 < 0 < \alpha_2$. *Similarly, condition* (23) *holds if*

$$S'(D((A')^\alpha)) \subseteq D((B')^\alpha), \qquad \|(A')^\alpha S'y'\|_{X'} \le C\|(B')^\alpha y'\|_{Y'}, \quad y' \in D((B')^\alpha),$$

for $\alpha = \alpha_1, \alpha_2$ *where* $\alpha_1 < 0 < \alpha_2$, *or if*

$$S(D(A^\alpha)) \subseteq D(B^\alpha), \qquad \|B^\alpha Sx\|_Y \le C\|A^\alpha x\|_X, \quad x \in D(A^\alpha)$$

for $\alpha = \alpha_1, \alpha_2$ *where* $\alpha_1 < 0 < \alpha_2$.

Proof. Similiar to the proof of Proposition 3. For the last statement we use the following Proposition 11. □

The following is a simplified version of [11, Proposition 5.5], sufficient for our purposes, which we prove here for convenience.

Proposition 11. *Let X and Y be Banach spaces. Let B and A be closed injective operators wih dense domain and range in Y and X, respectively. Let $R : Y \to X$ be a bounded linear operator and $C > 0$. Then $(i) \Rightarrow (ii)$ where*

- (i) *$R(D(B^{-1})) \subset D(A^{-1})$ and $\|A^{-1}Ry\|_X \leq C\|B^{-1}y\|_Y$ for all $y \in D(B^{-1})$.*
- (ii) *$R'(D(A')) \subseteq D(B')$ and $\|B'R'x'\|_{Y'} \leq C\|A'x'\|_{X'}$ for all $x' \in D(A')$.*

If X is reflexive then (i) and (ii) are equivalent.

Proof. Assume that (i) holds. Let $x' \in D(A')$ and $y \in D(B)$. Then $By \in D(B^{-1})$, $RBy \in D(A^{-1})$, and $x := A^{-1}RBy \in D(A)$. We thus have

$$\langle By, R'x'\rangle = \langle RBy, x'\rangle = \langle Ax, x'\rangle = \langle x, A'x'\rangle = \langle A^{-1}RBy, A'x'\rangle,$$

and, using (i),

$$|\langle By, R'x'\rangle| \leq \|A^{-1}RBy\|_X\|A'x'\|_{X'} \leq C\|y\|_Y\|A'x'\|_{X'}.$$

This means $R'x' \in D(B')$ and $\|B'R'y\|_{Y'} \leq C\|A'x'\|_{X'}$.

Now let X be reflexive and assume (ii). Then $A^{-1} = (A^{-1})'' = ((A')^{-1})'$. Let $y = Bz \in D(B^{-1})$. Let $x' = A'w' \in D((A')^{-1})$ where $w' = D(A')$. Then $R'w' \in D(B')$ and

$$\langle Ry, (A^{-1})'x'\rangle = \langle Ry, w'\rangle = \langle RBz, w'\rangle = \langle Bz, R'w'\rangle = \langle z, B'R'w'\rangle.$$

This yields

$$|\langle Ry, (A^{-1})'x'\rangle| \leq \|z\|_Y\|B'R'w'\|_{Y'} \leq \|z\|_Y C\|A'w'\|_{X'} = C\|B^{-1}y\|_Y\|x'\|_{X'}.$$

Hence $Ry \in D(A^{-1})$ and $\|A^{-1}Ry\|_X \leq C\|B^{-1}y\|_Y$. □

6 H^∞-calculus for the Stokes operator on Lipschitz domains

In this section we apply our results to the Stokes operator on bounded Lipschitz domains. In order to verify the assumptions we need R-sectoriality of the Stokes operator in L^q, we need the Helmholtz decomposition in H^s for $|s|$ small, and we need information on the fractional domains of the Stokes operator in L^2. We shall use arguments from [22] and [8] and results from [19]. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain where $d \geq 3$. As usual, we denote by $C^\infty_{c,\sigma}(\Omega)$ the space of all divergence-free vector fields in $C^\infty_c(\Omega)^d$ and by $L^q_\sigma(\Omega)$, where $q \in (1,\infty)$, the closure of $C^\infty_{c,\sigma}(\Omega)$ in $L^q(\Omega)^d$. We recall the definition of the Stokes operator A_q in $L^q_\sigma(\Omega)$ from [22, (1.7),(1.8)]:

$$A_q u := -\Delta u + \nabla\phi,$$

where $\phi \in L^q(\Omega)$ is such that $-\Delta u + \nabla\phi \in L^q_\sigma(\Omega)$ and $D(A_q)$ is the space of all $u \in W^{1,q}_0(\Omega)^d$ with $\operatorname{div}(u) = 0$ in Ω for which such a ϕ exists. We assume that $\operatorname{diam}(\Omega) = 1$. The following has been shown in [22, Theorem 1.1, Remark 6.4].

Proposition 12. *For any $\theta \in (\pi/2, \pi)$ there exists $\varepsilon > 0$, only depending on d, θ and the Lipschitz character of Ω, such that for*

$$\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2d} + \varepsilon \tag{28}$$

there is a constant $C_{q,\theta}$ satisfying

$$\|(\lambda + A_q)^{-1} f\|_{L^q} \le \frac{C_{q,\theta}}{|\lambda| + 1}\|f\|_{L^q}, \quad \lambda \in \Sigma_\theta, f \in L^q_\sigma(\Omega), \tag{29}$$

where $C_{q,\theta}$ only depends on d, q, θ and the Lipschitz character of Ω.

Consequently, for q satisfying (28), the Stokes operator A_q is sectorial in $L^q_\sigma(\Omega)$ and generates a bounded analytic semigroup. We check here how the proof given in [22] yields also R-sectoriality of A_q in $L^q_\sigma(\Omega)$ for q satisfying (28).

Proposition 13. *Under the assumptions of Proposition 12 there exists a constant $\tilde{C}_{q,\theta}$, only depending on d, q, θ and the Lipschitz character of Ω, such that*

$$\mathcal{R}\{(|\lambda| + 1)(\lambda + A_q)^{-1} : \lambda \in \Sigma_\theta\} \le \tilde{C}_{q,\theta}, \tag{30}$$

where the R-bound is taken for operators $L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$.

Proof. We follow the lines of [22, Proof of Theorem 1.1, p.421] and check that the arguments extend to square functions. For $\lambda \in \Sigma_\theta$, $f \in L^2(\Omega)^d$ we consider the problem

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= f \\ \operatorname{div} u &= 0 \end{aligned} \tag{31}$$

in Ω. There is a unique $u \in H^1_0(\Omega)^d$ and a function $\phi \in L^2(\Omega)$, unique up to constants, that satisfy (31). One has ([22, (6.9)])

$$(|\lambda| + 1)\|u\|_{L^2} \le C_0\|f\|_{L^2},$$

which implies the square function estimate

$$\Big\|\Big(\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{1/2}\Big\|_{L^2}\le C_0\Big\|\Big(\sum_j|f_j|^2\Big)^{1/2}\Big\|_{L^2}.$$

for finite collections $(f_j)_j$ in $L^2(\Omega)^d$ and $\Lambda := (\lambda_j)_j \in \Sigma_\theta$, where u_j is the solution of (31) with λ_j, f_j in place of λ, f. In this situation we put

$$T_\Lambda((f_j)_j) := \Big(\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{1/2},$$

which defines a sublinear and L^2-bounded operator. In order to use [22, Lemma 6.3] and thus obtain L^q-boundedness of T_Λ for q satisfying (28) we show the square function version of [22, (6.10)], i.e.

$$\Big(\fint_{\Omega\cap B}T_\Lambda((f_j)_j)^q\Big)^{1/q}\le C\Big(\fint_{\Omega\cap 2B}T_\Lambda((f_j)_j)^2\Big)^{1/2}$$

which means

$$\Big(\fint_{\Omega\cap B}\Big(\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{q/2}\Big)^{1/q}\le C\Big(\fint_{\Omega\cap 2B}\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{1/2}.$$

Here $B = B(x_0, r)$ denotes a ball with $x_0 \in \overline{\Omega}$ and $0 < r < c < 1$ and the $f_j \in L^2(\Omega)^d$ have $\operatorname{supp} f_j \subseteq \Omega \setminus 3B$. This is a square function version of [22, Lemma 6.2]. As in the proof given there it suffices to consider the cases $3B \subseteq \Omega$, which reduces to an interior estimate, and $x_0 \in \partial\Omega$. For the latter case we need a square function version of [22, Lemma 6.1], which will follow from a square function version of [22, Theorem 5.6] for the domain $U = \Omega \cap B$, i.e. from

$$\Big(\int_U\Big(\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{q/2}\Big)^{1/q}\le C\Big(\int_{\partial U}\sum_j(|\lambda_j|+1)^2|u_j|^2\Big)^{1/2}\qquad(32)$$

(cp. with [22, (5.18)]) where $q = \frac{2d}{d-1}$. As in [22] we use the estimate

$$\|(u_j)^*\|_{L^2(\partial U)}\le C\|u_j\|_{L^2(\partial U)}$$

where $(u_j)^*$ denotes the non-tangential maximal function of u_j with respect to the domain U. We also use the observation

$$|u_j(x)|\le C\int_{\partial U}\frac{(u_j)^*(y)}{|x-y|^{d-1}}\,d\sigma(y)\quad\text{for any } x\in U,$$

and the boundedness

$$\|I_1(F)\|_{L^2(\partial U)} \leq C\|F\|_{L^{q'}(U)}$$

of the operator

$$I_1(F)(y) = \int_U \frac{F(x)}{|x-y|^{d-1}}\,dx$$

where $\frac{1}{q}+\frac{1}{q'}=1$ (see [7, Lemma 6.1]). By a classical result of Marcinkiewicz and Zygmund (see [9, Ch. V, Sect. 2, Theorem 2.7]) the single operator I_1 has a vector-valued extension

$$\Big\|\Big(\sum_j |I_1(F_j)|^2\Big)^{1/2}\Big\|_{L^2(\partial U)} \leq \tilde{C}\Big\|\Big(\sum_j |F_j|^2\Big)^{1/2}\Big\|_{L^{q'}(U)}$$

(i.e. the set $\{I_1\}$ is R-bounded). Now we carry out the duality argument omitted in [22], but in our square function setting:

$$\begin{aligned}
&\Big|\int_U \sum_j (|\lambda_j|+1)u_j F_j\,dx\Big| \\
\leq\ & \sum_j \int_U\int_{\partial U} \frac{(|\lambda_j|+1)(u_j)^*(y)}{|x-y|^{d-1}}\,d\sigma(y)\,|F_j(x)|\,dx \\
=\ & \sum_j \int_{\partial U} (|\lambda_j|+1)(u_j)^*(y) I_1(F_j)(y)\,d\sigma(y) \\
\leq\ & \Big(\int_{\partial U}\sum_j |(|\lambda_j|+1)(u_j)^*|^2\,d\sigma(y)\Big)^{1/2}\Big(\int_{\partial U}\sum_j |I_1(|F_j|(y))|^2\,d\sigma(y)\Big)^{1/2} \\
\leq\ & C\tilde{C}\Big(\int_{\partial U}\sum_j |(|\lambda_j|+1)(u_j)^*|^2\,d\sigma(y)\Big)^{1/2}\Big\|\Big(\sum_j |F_j|^2\Big)^{1/2}\Big\|_{L^{q'}(U)}.
\end{aligned}$$

Taking the supremum over $(F_j)_j$ with $L^{q'}$-norm of the square function ≤ 1, we arrive at (32). □

We turn to the Helmholtz decomposition in Ω. We denote by $\mathbb{P} := \mathbb{P}_2$ the orthogonal projection in $L^2(\Omega)^d$ onto $L^2_\sigma(\Omega)$. It has been shown in [8, Theorem 11.1] that there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that, for $\frac{3}{2}-\varepsilon < q < 3+\varepsilon$, the operator $\mathbb{P}$ extends to a bounded projection $\mathbb{P}_q$ in $L^q(\Omega)^d$ onto $L^q_\sigma(\Omega)$ and that one has the Helmholtz decomposition

$$L^q(\Omega)^d = L^q_\sigma(\Omega) \oplus \nabla W^{1,q}(\Omega)$$

as a topological direct sum. Taking $R = \mathbb{P}_q : L^q(\Omega)^d \to L^q_\sigma(\Omega)$ and $S : L^q_\sigma(\Omega) \to L^q(\Omega)^d$ the inclusion, we see that we need for $q=2$ a Helmholtz decomposition of $H^s(\Omega)^d$ for $|s|$ small. The following is part of [19, Proposition 2.16]. For convenience and as details had been omitted in [19], we check here that the arguments given in [8] apply.

Proposition 14. *For $|s| < \frac{1}{2}$, the Helmholtz projection $\mathbb{P}$ acts as a bounded linear projection P_s in $H^s(\Omega)^d$ and yields the decomposition*

$$H^s(\Omega)^d = H^s_\sigma(\Omega) \oplus \nabla H^{s+1}(\Omega)$$

as a topological direct sum where

$$H^s_\sigma(\Omega) := \{u \in H^s(\Omega)^d : \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\}.$$

Observe that $H^s_\sigma(\Omega) = H^s(\Omega)^d \cap L^2_\sigma(\Omega)$ in the proposition.

Proof. We argue as in [8, Proof of Theorem 11.1], denote the outer unit normal on $\partial\Omega$ by ν, and use the representation

$$\mathbb{P}u = u - \nabla \operatorname{div} \Pi_\Omega(u) - \nabla\psi$$

where Π_Ω denotes the Newton potential and ψ solves the Neumann problem

$$\Delta\psi = 0, \qquad \frac{\partial\psi}{\partial\nu} = \nu \cdot (u - \nabla \operatorname{div} \Pi_\Omega(u)).$$

Here $u \in H^s(\Omega)^d$, so $\Pi_\Omega(u) \in H^{s+2}(\Omega)^d$ and $\nabla \operatorname{div} \Pi_\Omega(u) \in H^s(\Omega)^d$ for $|s| < \frac{1}{2}$. We also observe $\operatorname{div}(u - \nabla \operatorname{div} \Pi_\Omega(u)) = 0$, hence (see [8, Section 9]) $u - \nabla \operatorname{div} \Pi_\Omega(u)$ has a normal component on $\partial\Omega$ and $\nu \cdot (u - \nabla \operatorname{div} \Pi_\Omega(u)) \in H^{s-1/2}(\partial\Omega)$. By [8, Theorem 9.2] in combination with [8, Remark, p.360], the Neumann problem above has a solution $\psi \in H^{s+1}(\Omega)$, unique up to constants, and $\nabla\psi \in H^s(\Omega)^d$. Thus $\mathbb{P}$ extends to a bounded operator P_s on $H^s(\Omega)^d$ for $|s| < \frac{1}{2}$ which is again a projection. We also see that $I - \mathbb{P}$ acts boundedly on $H^s(\Omega)^d$, and that

$$P_s(H^s(\Omega)^d) = \{u \in H^s(\Omega)^d : \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\} = H^s_\sigma(\Omega),$$

which finishes the proof. □

Let A_q denote the Stokes operator in $L^q_\sigma(\Omega)$ for q satisfying (28) and let $B_q = -\Delta$ with Dirichlet boundary conditions in $L^q(\Omega)^d$. Clearly, A_2 and B_2 are self adjoint and $D(A^{1/2}) = H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $D(B^{1/2}) = H^1_0(\Omega)^d$. But we have to relate these fractional domain spaces to the fractional Sobolev spaces of Proposition 14. The first assertion of the following proposition is part of [19, Theorem 5.1], we repeat the argument given there for convenience. The main ingredient is [19, Theorem 2.12]. The second assertion is well known.

Proposition 15. *For $|s| < \frac{1}{2}$, we have*

$$(L^2_\sigma(\Omega))^{\cdot}_{s/2,A_2} = H^s_\sigma(\Omega), \qquad (L^2(\Omega)^d)^{\cdot}_{s/2,B_2} = H^s(\Omega)^d.$$

Sketch of proof. Since the operators are boundedly invertible we can omit the dots, and by self adjointness and duality we can restrict to $0 < s < 1/2$. Since the operators are self adjoint we can use complex interpolation and it remains to show

$$[L^2_\sigma(\Omega), H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)]_s = H^s_\sigma(\Omega), \qquad [L^2(\Omega)^d, H^1_0(\Omega)^d]_s = H^s(\Omega)^d.$$

The latter is well known, and the former relies on [19, Theorem 2.12] which states that the scale of fractional divergence-free Sobolev spaces is a complex interpolation scale. □

Now we can prove our result for the Stokes operator.

Theorem 16. *Let $\Omega \subseteq \mathbb{R}^d$ where $d \geq 3$ be a bounded Lipschitz domain with* $\mathrm{diam}(\Omega) = 1$. *For q satisfying* (28) *the Stokes operator A_q has a bounded H^∞-calculus in $L^q_\sigma(\Omega)$. For these q we have $D(A_q^\alpha) = D((-\Delta_q)^\alpha) \cap L^q_\sigma(\Omega)$ for*

$$|\alpha| < \frac{1}{2} - \Big(\frac{1}{d} + 2\varepsilon\Big)^{-1}\Big|\frac{1}{2} - \frac{1}{q}\Big|.$$

Proof. The first assertion is proved by the arguments above. The second is also an application of Theorem 9. □

References

[1] W. Arendt, S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math. Z. 240, 311-343 (2002).

[2] S. Blunck, P.C. Kunstmann, Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus, Revista Mat. Iberoamericana 19 (3), 919-942 (2003).

[3] M.G. Cowling, I. Doust, A. McIntosh, A. Yagi, Banach space operators with a bounded H^∞ functional calculus, J. Aust. Math. Soc. A 60(1), 51-89 (1996).

[4] R. Denk, M. Hieber, J. Prüss, R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Memoirs AMS, Number 788, 2003.

[5] R. Denk, G. Dore, M. Hieber, J. Prüss, A. Venni, New thoughts on old results of R.T. Seeley, Math. Ann. 328 (4), 545-583 (2004).

[6] R. Denk, M. Kaip, General parabolic mixed order systems in L^p and applications, Operator Theory: Advances and Applications, Vol. 239, Birkhäuser, 2013.

[7] M. Dindos, M. Mitrea, Semilinear Poisson problems in Sobolev-Besov spaces on Lipschitz domains, Publ. Mat. 46, 353-403 (2002).

[8] E. Fabes, O. Mendez, M. Mitrea, Boundary layers on Soblev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains, J. Funct. Anal. 159, 323-368 (1998).

[9] J. Garcìa-Cuerva, J.L. Rubio di Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies, vol. 116, North-Holland, Amsterdam (1985). Notas de Matemática [Mathematical Notes], vol. 104.

[10] M. Haase, The Functional Calculus for Sectorial Operators, Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel (2006).

[11] N.J. Kalton, P.C. Kunstmann, L. Weis, Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators, Math. Ann. 336(4), 747-801 (2006).

[12] N.J. Kalton, L. Weis, The H^∞-calculus and sums of closed operators. Math. Ann. 321(2), 319-345 (2001).

[13] N.J. Kalton, L. Weis, The H^∞-functional calculus and square function estimates, in Nigel J. Kalton Selecta (F. Gesztesy, G. Godefroy, L. Grafakos, and L. Verbitsky, editors) Comtemporary Mathematicians, Birkhäuser-Springer, 2015.

[14] N.J. Kalton, L. Weis, Euclidean structures, manuscript, 2004.

[15] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Japan 13 (3), 246-274 (1961).

[16] P.C. Kunstmann, A. Ullmann, R_s-sectorial operators and generalized Triebel-Lizorkin Spaces, J. Fourier Anal. Appl. 20 (1) 135-185 (2014).

[17] P.C. Kunstmann, L. Weis, Maximal L^p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in: Functional Analytic Methods for Evolution Equations, Lecture Notes in Math., vol. 1855, pp. 65-311, Springer, Berlin (2004).

[18] P.C. Kunstmann, L. Weis, Erratum to: Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators, Math. Ann. 357, 801-804 (2013).

[19] M. Mitrea, S. Monniaux, The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains, J. Funct. Anal. 254, 1522-1574 (2008).

[20] A. Noll, J. Saal, H^∞-calculus for the Stokes operator on L^q-spaces, Math. Z. 244 (3), 651-688 (2003).

[21] J. Prüss, G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monographs in Mathematics, vol. 105, Birkhäuser, Basel, 2016.

[22] Z. Shen, Resolvent estimates in L^p for the Stokes operator in Lipschitz domains, Arch. Rational Mech. Anal. 205, 395-424 (2012).

[23] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland Mathematical Library, vol. 18. North-Holland, Amsterdam (1978).

[24] L. Weis, A new approach to maximal L^p-regularity, in: Evolution Equations and Their Applications in Physical and Life Sciences, Bad Herrenalb, 1998. Lecture Notes in Pure and Appl. Math., vol. 215, pp. 195-214. Dekker, New York (2001).