Electromagnetic Wave Scattering at Biperiodic Surfaces

Variational Formulation, Boundary Integral Equations and High Order Solvers

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Acknowledgment

Already during my school education and academic studies of electrical engineering I felt strongly drawn to mathematics. However, afterwards in my professional career in industry I was confronted with problems where I whished to have a deeper understanding beyond the scope as an engineer. So, I provided myself with a textbook about functional analysis, began to read and quickly realized my desire to learn more seriously about this topic and to give mathematics a bigger part in my life. This was the starting point of my new career as a mathematician. Nevertheless, it was anything but an easy step for me to give up my save employment in industry and to take in again the status of a student. After a long and hard struggle, I terminated my job and went to the Karlsruher Institute of Technology where I found all the lectures I wanted to attend, such as Integral Equations or Inverse Problems beside many others. The first mentioned lectures were given by Prof. Dr. Andreas Kirsch and PD Dr. Tilo Arens and opened the door to my master thesis about an inverse heat conducting problem in the working group “Inverse Problems” led by Andreas Kirsch. Subsequently, I got an offer as a scientific assistant for funding my graduation there – an opportunity which is not often given in life and I feel special gratitude to it.

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1. Introduction

In this thesis we consider time-harmonic electromagnetic wave scattering at impenetrable biperiodic surfaces in a homogeneous medium. Besides their rigorous analysis in biperiodic Sobolev spaces, which aims at answering the questions about existence and uniqueness of solutions, we will derive a high order solver for its numerical approximation – a collocation method based on trigonometric polynomials.

The propagation of electromagnetic waves is described by Maxwell's equations. In its simplified form for time-harmonic waves, propagating in an isotropic and homogeneous medium without charges and external currents, this system reduces to

$$\begin{aligned} \operatorname{curl} E - \mathrm{i}\omega\mu H &= 0, \\ \operatorname{curl} H + (\mathrm{i}\omega\varepsilon - \sigma)E &= 0, \end{aligned} \tag{1.1a}$$

connecting the electric field E and the magnetic field H to each other. Here, the material parameter $\varepsilon, \mu > 0$ are the electric permittivity and magnetic permeability, respectively, where $\sigma \geq 0$ is the conductivity and $\omega > 0$ denotes the frequency.

In general, given a domain $\Omega \subseteq \mathbb{R}^3$ (the scatterer) as well as some incident waves E^{i} and H^{i}, which satisfy Maxwell's system in all of $\mathbb{R}^3$, scattering problems by a perfect conductor consist of the determination of certain fields E^{s} and H^{s} (the scattered fields) which satisfy Maxwell's system in $\mathbb{R}^3 \setminus \Omega$, together with the boundary condition

$$\mathrm{n} \times (E^{\mathrm{i}} + E^{\mathrm{s}}) = 0 \qquad \text{on } \partial\Omega \tag{1.1b}$$

and a suitable radiation condition. Here, n denotes the unit normal vector on $\partial\Omega$.

In our context, the scatterer is unbounded and possesses a biperiodic structure, i.e., it has a periodic shape in two spacial dimensions, say in x_1- and x_2-direction. Many of such structures are conceivable. We assume its boundary $\partial\Omega$ to be describable by the graph of a biperiodic and Lipschitz continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Later, in view of the numerical treatment, we require f to be smooth. Furthermore, we assume the incident fields to be biperiodic as well, up to a certain phase shift.

Such problems appear often in applications, for instance in micro- and nano-optics such as the design of thin solar cells, the design of photonic crystals with a certain band gap structure, the construction of holographic films and optical storage devices. Besides their practical relevance, they contribute(d) an interesting and fascinating research area in mathematics during the last 100 years.

1.1. State of the Art

The beginning of the last century can be seen as the starting point of investigations of scattering problems in a periodic setting, when in 1907 Lord Rayleigh published his famous work about the behaviour of sound scattered by a perfectly reflecting regularly grooved grating [46]. From this time up to now many scientists provided valuable contributions, not only for acoustic but also for electromagnetic scattering problems. Here, the literature can be mainly divided into two parts: one for the oneperiodic and the other one for, the already mentioned, biperiodic setting. In contrast, oneperiodic structures exhibit periodicity only in one spatial direction, say in x_1-direction, while they are constant in the other, say in x_2-, direction. In such structures the time-harmonic Maxwell's system reduces to scalar valued Helmholtz equations – a simplification which is not longer possible in the biperiodic setting. For oneperiodic structures, a good overview about the state of the art at the beginning of the 1980s is given by Petit in [44] and about the end of the 1990s by Bao, Cowsar and Masters in [13].

Existence and Uniqueness. In principle, there are two main approaches to tackle the question about existence and uniqueness of boundary value problems: the variational approach and the integral equation method.

They also provide the basis for numerical realizations in form of finite element methods and the numerical solution of boundary integral equations. Both approaches are also applicable for scattering problems in a periodic setting – and in the 1990s there were established a plenty of existence and uniqueness results in this context.

For the variational approach, a domain truncation process is characteristic: one truncates the domain to a unit cell, which covers one period, and considers the weak formulation of the scattering problem only therein. This process generates artificial boundary conditions, which take the radiation condition into account via Dirichlet-to-Neumann maps or the Calderon operator. The approach for the oneperiodic case is for instance shown by Elschner and Schmidt in [28]. The corresponding approach for the biperiodic case is treated by Abboud and Nédélec in [1], by Bao, Dobson and Cox in [10, 11, 12, 14, 26], by Bonnet-Bendhia and Starling in [17] and by Schmidt in [50]. It is worth mentioning the work of Arens in [7] for an application of this approach to the Helmholtz equation in three dimensions. All of them yield essentially one main result: unique solutions exist for all frequencies outside a certain discrete set.

The integral equation method (to be more precisely the indirect one) consists of a representation of the scattered field in form of a potential ansatz with an unknown density, which is intended to determine by exploiting certain jump relations to end up in a boundary integral equation for this density. This equation is used for both the proof of existence and for numerical computations. (In contrast, and for the sake of completeness, the direct integral equation method uses formulations which are directly based on the representation formulas of Green and Stratton-Chu for the solution.) Important contributions for this approach came already in the early 1990s from Nédélec and Starling ([43]) and Chen, Dobson and Friedman ([24],[27]). Current results in connection with non-self-intersecting multilayered structures can be found in Bugert's dissertation thesis [22]. In summary, again unique solvability can be ensured outside a certain discrete set regarding some material parameters or the frequency.

The Numerical Treatment. The variational approach is very popular among mathematicians, because it makes the application of the well-studied finite element method possible. Hence, many of the above mentioned articles contain already implementations or provide at least the basis, see for instance Bao in [10, 11]. For biperiodic structures an adaptive finite element method can be found in [15].

The integral equation method yields a boundary integral equation for which in general three approaches are available to obtain an approximate solution: Galerkin, collocation and Nyström methods. A standard reference for a good introduction into this subject is the monograph of Saranen and Vainikko ([48]) or in a more general context the textbook of Kress ([36]). An essential disadvantage of boundary integral equation methods in comparison with finite element method is the solution of big systems of linear equations with dense matrices.

Galerkin methods are projection methods and are encountered often in the literature. There exists a complete theory with existence, uniqueness, stability and convergence results also for the case of non-smooth boundaries. The numerical implementation in form of boundary element methods is well-established (see Sauter and Schwab in [49]).

Collocation methods belong to the projection methods as well and are often used in applications, since they are in principle easier to implement. An open question concerns stability in the case of non-smooth boundaries.

In combination with matrix compression techniques, both Galerkin and collocation methods exhibit a complexity which is comparable to the corresponding one of finite element methods. Here, low algebraic convergence rates with approximately linearly growing complexity are characteristic. Those techniques include the fast multipole method, panel clustering and adaptive corss approximation (see Rjasanow and Steinbach in [47]) as well as the $\mathcal{H}$-matrix calculus (see Hackbusch in [32]).

Nyström (or quadrature) methods approximate the integral by appropriate quadrature rules. For problems in two dimensions, Colton and Kress ([25]) or Meier, Arens, Chandler-Wilde and Kirsch ([41]) achieved for smooth boundary curves any algebraic convergence rate. Furthermore, methods with exponential convergence rate and a quadratic count of operations are known, see Kussmaul ([37]) and Martensen ([39]). For smooth surfaces the method is easy to implement, as it requires only the composite trapezoidal rule in combination with a rule that uses the same quadrature points and easy to determine weights to overcome the integration over the singularity.

For problems in three dimensions the realization is more difficult because of the more complicated structure of the kernel of the integral operators – the singularity depends now on both the distance and the direction. In this situation it is hard to find a quadrature rule with a high order convergence rate. For the case of a globally parametrizable surface by means of a sphere, a first implementation was successfully realized by Wienert ([52]) in 1990, who removed the singularity by using rotations and spherical coordinates. A complete convergence analysis came more than 10 years later, see Ganesh, Graham and Sloan ([30, 31]).

Another approach were chosen by Bruno and Kunyanski ([19, 20]) in 2001. To make the idea clearer, we assume the surface $\partial\Omega$ to be globally parametrizable by a map $\Psi : Q := (-\pi, \pi)^2 \to \partial\Omega$, which gives us finally an integral equation of the second kind

$$\varphi(t) - \int_Q k(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau = \psi(t), \qquad t \in Q,$$

whose approximate solution we are looking for. Using a cut-off function χ with $\chi(t) = 0$, for $|t| \geq \varrho$, and $\chi \equiv 1$ in a neighborhood of zero, the singularity can be isolated,

$$\begin{aligned}\varphi(t) - \int_Q k(t,\tau)\,\chi(\tau - t)\,\varphi(\tau)\,\mathrm{d}\tau \\ - \int_Q k(t,\tau)\,(1 - \chi(\tau - t))\,\varphi(\tau)\,\mathrm{d}\tau = \psi(t),\end{aligned} \tag{1.2}$$

for $t \in Q$. Substituting τ in the first integral by polar coordinates centered at t and applying the transformation rule removes the singularity. As a consequence, the Nyström method realized by the composite trapezoidal rule is applicable, leading to high order convergence. Although the idea is pretty simple, the implementation is technically difficult and an analysis how the convergence rate is related to the overall complexity was not given. During the next 10 years some efforts have been made to fill this gap:

- Heinemeyer ([33]) interpreted this method as a method of locally corrected weights (as suggested in [23]) and proved pointwise convergence of the discrete operators with super-algebraic convergence rate, but did not give a convergence rate of the overall scheme.

- Arens interpreted in his habilitation thesis ([7]) the method as a collocation method based on trigonometric polynomials and he was the first who rigorously and completely showed stability and super-algebraic convergence rate with quadratic computational complexity for the semi-discrete scheme for a variant of this method. However, the complexity estimate for the fully discrete scheme leaves room for improvement.
- Bruno, Dominguez and Sayas published in [18] the most complete analysis, but they limit themselves to scattering problems.

In this thesis, the scheme of Arens in [7] is improved by reducing the overall complexity.

The Evaluation of Green's Function. The implementation of the integral equation method in a biperiodic setting requires the evaluation of the (quasi-) periodic Green's function for the Helmholtz equation (including their partial derivatives) several times. Since for this purpose its usual series representation is disadvantage, the availability of efficient evaluation methods is a crucial issue. The lack of those methods might be the reason for why Nédélec and Starling ([43]) as well as Dobson and Friedman ([27]) did not pursue or did not implement their ansatzes.

In his review article [38], Linton compared different expressions for Green's function and recommends Ewald's method, which is to split up the function into two exponentially convergent series, one of them containing the singularity. This method was successfully picked up by Arens in [7], who derived different representations which are best suited for numerics. Arens, Lechleiter, Sandfort and Schmitt performed in ([8]) evaluations of the Green's function (and their partial derivatives) based on the preparatory work of [7] and gave rigorous error estimates for the numerical approximation of the function.

There are also efforts to avoid the Green's function, at least in the oneperiodic and multilayered setting, see [16] and [54, 55].

1.2. Results Presented in this Thesis

Before we start to describe the electromagnetic scattering problem, that we are interested in, in the necessary mathematical precision, afterwards continue with investigations concerning the question about existence and uniqueness of solutions and finally derive a high order solver for their numerical approximation, it is indispensable to provide in a first step the correct framework in form of the function spaces where solutions are sought for. This will be done in Chapter 2. Here, we have chosen the approach of [34] for arbitrary bounded Lipschitz domains (an elementary and comprehensive presentation looking for its equals in particular regarding Chapter 5 therein and which is managed without Sobolev-Slobodeckiĭ spaces) and transferred their ideas to the biperiodic setting. This proceeding differs from the usual procedure in the literature, such as in [7] or [22] (where, by means of a special partition of unity, many results can be obtained from corresponding ones for nonperiodic Sobolev spaces), and appears to be new. Since the key idea from the approach of [34] is to exploit results of periodic Sobolev spaces on cuboids, their methods seemed to be the "natural choice" – and therefore best suited for our purposes. We start with basis results for Sobolev spaces, where our main focus is on Sobolev spaces for functions on cuboids for reasons which were mentioned above. Theorem 2.30, Corollary 2.32, Lemma 2.37 (part *(i)*), Theorem 2.38 and 2.40, together with Theorem A.2, as well as the special choice of the partition of unity in Theorem 2.42 resulted from discussions with Andreas Kirsch. Some of those results refer to a multiplication operator in the trace spaces of vector fields which appear not to have been published in this form so far. Then we introduce the Q-periodic setting, in particular the notion of a cell set and give results which hold in this more general context before we turn to cell sets of Lipschitz layer type, the setting which will be the most important one in this thesis. Throughout this thesis we use the term "Q-periodic" as a synonym of "biperiodic". Admittedly, the development of all these results is pretty exhaustive, but allows a detailed analysis. In Subsection 2.1.4 we connect both approaches from the literature, at least for smooth surfaces. Last but not least, it is worth mentioning that the setting for the variational approach requires the consideration of subspaces where modified differential operators such as ∇_β or div_β satisfy certain conditions. The results in this context seem also to be new.

Chapter 3 aims at a precise formulation of the scattering problem (1.1) and at the investigation of its unique solvability by means of methods from a variational approach. For this we fix the geometrical setting and take a closer look at upward propagating waves. The latter one is the substitute of the Silver-Müller radiation condition from the nonperiodic setting and the method of choice in our situation. This radiation condition was firstly proposed by Lord Rayleigh in [46]. We carry over the ansatz from [7] to electromagnetic scattering, see also [22], and give a more detailed analysis adapted to our purposes, especially as a preparation for the Calderon operator introduced later. After those preliminary considerations we are able to give a precise weak formulation and show uniqueness in the standard way. Afterwards, we introduce the Calderon operator and proceed with a detailed analysis to obtain its most important properties for further investigations. Such an analysis could not be found in the literature. By means of this operator, we rewrite our scattering problem equivalently into its variational form and continue to prove existence of solutions. For this we were inspired by [35] and [42] (after personal communication with Andreas Kirsch), where the solution space is split up into a direct sum which allows a dissection of the problem in easier to analyse ones. Here, we adapted the idea to the Q-periodic setting and end up with Theorem 3.42, the main result of Chapter 3.

The topic of Chapter 4 is the integral equation method – yielding the boundary integral equation which will be later the basis for our numerical scheme. At first we recall the definition of the Q-(quasi-)periodic Green's function for the Helmholtz equation and its most important properties from [7]. By means of this fundamental solution we define vector potentials and follow thereby very closely the presentation in [34], with corresponding modifications for the Q-periodic framework. Here, special attention should be paid to a certain transmission problem, as it provides the important jump relations of the vector potentials and thus gives rise to the definition of the boundary integral operators $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$. While $\mathcal{L}_\alpha$ can be written as a compact perturbation of an isomorphism, a similar result in the case of Lipschitz surfaces is not known for the operator $\mathcal{M}_\alpha$. At this point we have to impose more regularity on the surface and rely on results from [21]. A special technique, as described in the proof of [7, Theorem 4.22] or the proof of [22, Lemma 4.15], then makes the results from [21] applicable. With the tools derived so far at hand, the derivation of the boundary integral equation (for the unknown density from the potential ansatz for the solution of our scattering problem) and the investigation of its solvability is now straightforward. Finally, some technical efforts are addressed to the verificaton of the assumptions on the kernels of the corresponding integral operator, in particular on the weak singularity which is supposed to be of a special kind, see Assumption 5.6.

In Chapter 5, the numerical scheme is presented. As mentioned above, it is a variant of the method from [19, 20] and constitutes an improvement of the scheme in [7], which consists in a reduction of the overall complexity by introducing another orthogonal projection. A key tool is the removal of the weak singularity by a transformation into polar coordinates for the first integral in (1.2). As a consequence, the corresponding integral operator takes on a non-standard form making the analysis of its mapping properties, as well as for its approximation, technically complicated. Therefore, the approach is demonstrated at first on single integral equations and later generalized to systems. The scheme is a collocation method and achieves super-algebraic convergence rate (provided the surface is smooth). As another novelty, the analysis yields an explicit dependence of the constants in the stability and convergence estimates on some number ϱ which couples the support of various cut-off functions to each other. The results were developed in collaboration with Tilo Arens and prepublished in [9]. Therein, the application to typical boundary value problems such as potential and scattering problems both for bounded obstacles and for biperiodic surfaces is emphasized and numerical examples are presented which demonstrate the expected convergence rates in practice.

1.3. Notational Conventions

Numbers, Sets and Operations. The symbol $\mathbb{N}$ denotes the set of natural numbers with the exception of the zero element 0. We define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, we introduce for $s \in \mathbb{R}$, with $s \geq 0$, the set $\mathbb{N}_{\geq s} := \{n \in \mathbb{N}_0 \mid n \geq s\}$. As usual, we denote for $s \in \mathbb{R}$ by $\lfloor s \rfloor$ the largest integer smaller than or equal to s, while $\lceil s \rceil$ denotes the smallest integer greater than or equal to s.

To simplify expressions in some formulas, we make the convenient agreement that $0^0 := 1$.

If M is a set and $A \subseteq M$, then A^c denotes the complement of A with respect to M, i.e., $A^c := \{m \in M \mid m \notin A\}$.

As usual, for $d \in \mathbb{N}$ and $\mu, \nu \in \mathbb{Z}^d$ the symbol $\delta_{\mu,\nu}$ designates the *Kronecker delta*, i.e.,

$$\delta_{\mu,\nu} := \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases}$$

For ease of notation, we define for $M \in \mathbb{N}^d$

$$\underline{M} := \min\{M_1, \ldots, M_d\} \qquad \text{and} \qquad \overline{M} := \max\{M_1, \ldots, M_d\}. \tag{1.3}$$

Given $z \in \mathbb{C}$ we denote by $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\arg(z) \in (-\pi, \pi]$ and $\overline{z}$ the *real part*, the *imaginary part*, the *absolute value*, the *argument* and the *complex conjugate* of z, respectively.

For $x = (x_1, x_2, x_3)^\top \in \mathbb{C}^3$, the vector $\tilde{x}$ in $\mathbb{C}^2$ and the vector x^* in $\mathbb{C}^3$ are given by

$$\tilde{x} := (x_1, x_2)^\top \qquad \text{and} \qquad x^* := (x_1, x_2, -x_3)^\top. \tag{1.4}$$

Let $a, b \in \mathbb{C}^d$. Then the (real) *dot product* $a \cdot b$ is defined by

$$a \cdot b := \sum_{j=1}^{d} a_j b_j. \tag{1.5}$$

The *cross product* $a \times b$ for $a, b \in \mathbb{C}^3$ is defined as usual. If $a, b \in \mathbb{C}^2$ we make the arrangement that

$$a \times b := a_1 b_2 - a_2 b_1. \tag{1.6}$$

Moreover, for $a = (a_1, a_2)^\top \in \mathbb{C}^2$ we set

$$a^\perp := \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}. \tag{1.7}$$

Given a *multi-index* $\alpha \in \mathbb{N}_0^d$, we define for $x \in \mathbb{C}^d$

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}. \tag{1.8}$$

Furthermore, we introduce for multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ the *binomial coefficient* $\binom{\alpha}{\beta}$ given by

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} \tag{1.9}$$

and mean by $\beta \leq \alpha$ that $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, d$.

The Size of some Mathematical Objects. Let $d \in \mathbb{N}$. For $x \in \mathbb{R}^d$, $|x|$ denotes the *Euclidean norm* and $|x|_\infty$ the *maximum norm*, while for a multi-index $\alpha \in \mathbb{N}_0^d$, $|\alpha|$ denotes its *order*, given by $|\alpha| := \alpha_1 + \cdots + \alpha_d$. Moreover, for a Lebesgue measurable subset Ω of $\mathbb{R}^d$, $|\Omega|$ means the *Lebesgue measure* of Ω. The context should exclude any confusion.

For $z \in \mathbb{R}^d$ and $r > 0$ we set

$$\begin{aligned} \mathbb{B}_d(z, r) &:= \{x \in \mathbb{R}^d \mid |x - z| < r\}, \\ \mathbb{B}_d[z, r] &:= \{x \in \mathbb{R}^d \mid |x - z| \leq r\} \end{aligned} \quad \text{and} \quad \mathbb{S}^{d-1} := \partial\, \mathbb{B}_d(0, 1).$$

Mappings. Let A and B be non-empty sets. Sometimes we will denote the *set of all mappings* from A to B by B^A. In this sense, for instance the set $\mathbb{C}^\mathbb{N}$ consists of all sequences $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$.

Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ such that $\Omega \subsetneq \Omega'$. Furthermore, let $d' \in \mathbb{N}$. As usual, for a function $u : \Omega' \to \mathbb{C}^{d'}$, we denote by $u|_\Omega$ the *restriction* of u to Ω. Similarly in a converse manner, given a function $u : \Omega \to \mathbb{C}^{d'}$, we define $u|_0^{\Omega'} : \Omega' \to \mathbb{C}^{d'}$ to be the *extension* of u *by zero* to Ω'.

To simplify notation regarding expressions for functions, we make the following agreement: if the symbol "$\bullet$" appears in an expression, then this expression is to interpret as a function where "$\bullet$" stands for the independent variable, which domain of definition should be clear from the context. For example, if $a \in \mathbb{C}^d$, then by $a \cdot \bullet$ we mean the function $\mathbb{C}^d \ni z \mapsto a \cdot z \in \mathbb{C}$.

Normed Spaces and Linear Operators. Let X and Y be vector spaces over the field $\mathbb{F}$. For a linear mapping $T : X \to Y$, the set $\ker(T)$ denotes the *kernel* of T, that is

$$\ker(T) := \{x \in X \mid Tx = 0\}.$$

The mapping $T : X \to Y$ is called *antilinear* if $\mathbb{F} = \mathbb{C}$ and $T(\alpha_1 x_1 + \alpha_2 x_2) = \overline{\alpha_1}\, T(x_1) + \overline{\alpha_2}\, T(x_2)$, for all $\alpha_j \in \mathbb{C}$, $x_j \in X$, $j = 1, 2$.

Let $A, B \subseteq X$. Then $A + B$ indicates the set given by

$$A + B := \big\{ a + b \mid a \in A,\ b \in B \big\}.$$

Let now $(X, \|\cdot\|_X)$ be a normed vector space and $A, B \subseteq X$. Then $\operatorname{dist}(A, B)$ denotes the *distance* between A and B, i.e.,

$$\operatorname{dist}(A, B) := \inf \big\{ \|x - y\|_X \ \big|\ x \in A,\ y \in B \big\}.$$

Moreover, we denote by $\overline{A}$ and $\mathring{A}$ the *closure* and the *interior* of A, respectively.

Let U and V be closed subspaces of X such that $X = U + V$ and $U \cap V = \{0\}$. Then X is called *direct sum* of U and V, in sign

$$X = U \oplus V. \tag{1.10}$$

Let $(Y, \|\cdot\|_Y)$ be another normed vector space. The set $\mathcal{L}(X, Y)$ consists of all *linear* and *bounded* operators from X to Y. If Y coincides with X, then for simplicity we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. An operator $P \in \mathcal{L}(X)$ is called a *projection* if $P^2 = P$. If $T \in \mathcal{L}(X, Y)$ is bijective such that $T^{-1} \in \mathcal{L}(Y, X)$, then we call T an *isomorphism* and set

$$\mathcal{L}_{\mathrm{is}}(X, Y) := \big\{ T : X \to Y \mid T \text{ is a isomorphism} \big\}.$$

If for $T \in \mathcal{L}(X, Y)$ there holds $\|Tx\|_Y = \|x\|_X$ for all $x \in X$, then T is called *isometric*. If there exists an (isometric) isomorphism between the spaces X and Y, then X and Y are called (*isometrically*) *isomorphic*, in sign $X \simeq Y$ (or $X \cong Y$ respectively).

With $X \hookrightarrow Y$ we denote an *embedding* from X to Y, that is, a linear, bounded and injective mapping from X to Y. If X is a subspace of Y, sometimes we would like to emphasize that this embedding is given by the *identity* operator $\mathrm{id} : X \to Y$. In this case we will write $X \overset{\mathrm{id}}{\hookrightarrow} Y$ instead of $X \hookrightarrow Y$.

We call $\mathcal{L}(X,\mathbb{F})$ the *dual space* of X and denote this space by X^*. As usual, the evaluation of $\ell \in X^*$ at $x \in X$ is expressed by the *duality pairing* $\langle \ell, x\rangle$ and X^* is equipped with the operator norm, i.e.,

$$\|\ell\|_{X^*} := \sup_{x\in X\setminus\{0\}} \frac{|\langle \ell, x\rangle|}{\|x\|_X}, \qquad \ell \in X^*.$$

As an easy to verify consequence, we have the implication

$$\ell, \ell_n \in X^*,\ n \in \mathbb{N}, \ \text{ and } \ \ell_n \to \ell \quad \Rightarrow \quad \forall x \in X : \langle \ell_n, x\rangle \to \langle \ell, x\rangle. \tag{1.11}$$

Classical Function Spaces. Let $d \in \mathbb{N}$. For $\alpha \in \mathbb{N}_0^d$ the *partial differential operator* ∂^α of order $|\alpha|$ is defined by

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}\,,$$

and we consider ∂^α also for vector valued functions, see for instance [5]. By definition, ∂^α is the identity operator if $\alpha = 0$. Sometimes we will use the symbol ∂_j to denote the partial derivative of first order with respect to x_j.

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $d' \in \mathbb{N}$. If $u : \Omega \to \mathbb{C}^{d'}$ is differentiable, then we denote by $u'(x)$ the *Jacobian* of u in $x \in \Omega$.

Furthermore, let $m \in \mathbb{N}_0$. The spaces of continuously differentiable functions on Ω, on its closure $\overline{\Omega}$ and with compact support, respectively, are defined as usual by

$$\begin{aligned} C^m(\Omega,\mathbb{C}^{d'}) &:= \Big\{ u : \Omega \to \mathbb{C}^{d'} \;\Big|\; \forall\, 0 \le |\alpha| \le m : \partial^\alpha u : \Omega \to \mathbb{C}^{d'} \\ &\qquad\qquad \text{exists and is continuous} \Big\}, \\ C^m(\overline{\Omega},\mathbb{C}^{d'}) &:= \Big\{ u \in C^m(\Omega,\mathbb{C}^{d'}) \;\Big|\; \forall\, 0 \le |\alpha| \le m : \partial^\alpha u : \Omega \to \mathbb{C}^{d'} \\ &\qquad\qquad \text{can be continuously extended to } \overline{\Omega} \Big\}, \\ C_0^m(\Omega,\mathbb{C}^{d'}) &:= \Big\{ u \in C^m(\Omega,\mathbb{C}^{d'}) \;\Big|\; \operatorname{supp}(u) \text{ is compact and} \\ &\qquad\qquad \operatorname{supp}(u) \subseteq \Omega \Big\}. \end{aligned}$$

We recall that the *support* of a function $u : \Omega \to \mathbb{C}^{d'}$ is given by

$$\operatorname{supp}(u) := \overline{\{x \in \Omega \mid u(x) \neq 0\}}.$$

If Ω is additionally bounded, then as norms in $C^m(\overline{\Omega}, \mathbb{C}^{d'})$ and $C_0^m(\Omega, \mathbb{C}^{d'})$, we take

$$\|u\|_{C_0^m(\Omega,\mathbb{C}^{d'})} := \|u\|_{C^m(\overline{\Omega},\mathbb{C}^{d'})} := \max_{|\alpha|\leq m} \|\partial^\alpha u\|_\infty, \tag{1.12}$$

where $\|\cdot\|_\infty$ denotes the *supremum norm*. The corresponding spaces of *smooth* functions are

$$C^\infty(\Omega, \mathbb{C}^{d'}) = \bigcap_{k=0}^{\infty} C^k(\Omega, \mathbb{C}^{d'}),$$

$$C^\infty(\overline{\Omega}, \mathbb{C}^{d'}) = \bigcap_{k=0}^{\infty} C^k(\overline{\Omega}, \mathbb{C}^{d'}), \qquad C_0^\infty(\Omega, \mathbb{C}^{d'}) = \bigcap_{k=0}^{\infty} C_0^k(\Omega, \mathbb{C}^{d'}).$$

Let $\Omega' \subseteq \mathbb{R}^d$ be another open subset of $\mathbb{R}^d$ and $m \in \mathbb{N}_0 \cup \{\infty\}$. We call $u : \Omega \to \Omega'$ a C^m-*diffeomorphism* from Ω to Ω', if u is bijective and

$$u \in C^m(\Omega, \mathbb{C}^d) \qquad \text{and} \qquad u^{-1} \in C^m(\Omega', \mathbb{C}^d).$$

We set

$$\operatorname{Diff}^m(\Omega, \Omega') := \Big\{ u : \Omega \to \Omega' \;\Big|\; u \text{ is a } C^m\text{-diffeomorphism} \Big\}.$$

Lebesgue Spaces. We define $L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^{d'})$ to be the set of Lebesgue measurable functions which are Lebesgue integrable on all compact subsets of Ω. With $L^2(\Omega, \mathbb{C}^{d'})$ and $L^\infty(\Omega, \mathbb{C}^{d'})$ we denote the Lebesgue spaces of *square integrable* and *essentially bounded* functions, respectively. Of course, we equip the space $L^2(\Omega, \mathbb{C}^{d'})$ with the inner product

$$(u \,|\, v)_{L^2(\Omega,\mathbb{C}^{d'})} := \int_\Omega u(x) \cdot \overline{v(x)} \,\mathrm{d}x$$

and the norm

$$\|u\|_{L^2(\Omega,\mathbb{C}^{d'})} := \Big(\int_\Omega |u(x)|^2 \,\mathrm{d}x \Big)^{1/2}$$

induced by this inner product. And in $L^\infty(\Omega,\mathbb{C}^{d'})$ we take the norm

$$\|u\|_{L^\infty(\Omega,\mathbb{C}^{d'})} := \inf\Big\{c>0 \;\Big|\; |u(x)| \le c \text{ for almost all } x\in\Omega\Big\},$$

where the function u on the right hand side is any representative of the equivalence class $u\in L^\infty(\Omega,\mathbb{C}^{d'})$. We will often use the abbreviation "a.a." for the term "almost all". For an element u in $L^1_{\mathrm{loc}}(\Omega,\mathbb{C}^{d'})$, $L^2(\Omega,\mathbb{C}^{d'})$ or $L^\infty(\Omega,\mathbb{C}^{d'})$, its *essential support* is also denoted by $\operatorname{supp}(u)$ and by definition the smallest closed set such that $u=0$ almost everywhere on $\Omega\setminus\operatorname{supp}(u)$. Sometimes we are encountering integrals of the form

$$\int_\Omega \chi(x)\,u(x)\,\mathrm{d}x,$$

with a scalar valued function $\chi:\Omega\to\mathbb{C}$ and a vector valued function $u:\Omega\to\mathbb{C}^{d'}$ such that $\chi\,u_j$ is integrable, and understand this integral taken componentwise. Here, u_j, $j=1,\dots,d'$, are the components of u.

Fourier Series. If $\Omega\subseteq\mathbb{R}^d$ has the special form of a *cuboid* Q_d, that is $\Omega=Q_d$, where

$$Q_d := \bigtimes_{j=1}^{d} (-L_j, L_j) \tag{1.13}$$

for some real numbers $L_j>0$, $j=1,\dots,d$, then an element u in $L^2(Q_d,\mathbb{C}^{d'})$ can be expanded into a *Fourier series*

$$u = \sum_{\mu\in\mathbb{Z}^d} u^{(\mu)}\,T^{(\mu)}_{Q_d},$$

where $T^{(\mu)}_{Q_d}$ are the *trigonometric monomials* given by

$$T^{(\mu)}_{Q_d}(x) := \frac{1}{\sqrt{|Q_d|}}\,\mathrm{e}^{\mathrm{i}q^{(\mu)}_{Q_d}\cdot x}, \qquad x\in Q_d, \quad \mu\in\mathbb{Z}^d,$$

with the *reciprocal lattice vector* $q^{(\mu)}_{Q_d}\in\mathbb{R}^d$ given by

$$q^{(\mu)}_{Q_d} := (\pi\mu_1/L_1,\dots,\pi\mu_d/L_d)^\top, \qquad \mu\in\mathbb{Z}^d, \tag{1.14}$$

and where $u^{(\mu)} \in \mathbb{C}^{d'}$ denote the *Fourier coefficients* of u given by

$$u^{(\mu)} := \int_{Q_d} u(x)\, T_{Q_d}^{(-\mu)}(x)\, \mathrm{d}x, \qquad \mu \in \mathbb{Z}^d. \tag{1.15}$$

It is easy to see, that for $u \in L^2(Q_d, \mathbb{C}^{d'})$ the Fourier coefficients $\overline{u}^{(\mu)}$ of the complex conjugate $\overline{u}$ of u and the Fourier coefficients $u^{(\mu)}$ of u are connected to each other by the relation

$$\overline{u}^{(\mu)} = \overline{u^{(-\mu)}}, \qquad \mu \in \mathbb{Z}^d. \tag{1.16}$$

It is well-known that $\{T_{Q_d}^{(\mu)} \mid \mu \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(Q_d)$. Furthermore, $L^2(Q_d, \mathbb{C}^{d'})$ is isometrically isomorphic to the set of *square summable sequences* $\ell^2(\mathbb{Z}^d, \mathbb{C}^{d'})$, i.e.,

$$L^2(Q_d, \mathbb{C}^{d'}) \;\cong\; \ell^2(\mathbb{Z}^d, \mathbb{C}^{d'}), \tag{1.17}$$

where the isometric isomorphism is given by relating $u \in L^2(Q_d, \mathbb{C}^{d'})$ to its Fourier coefficients $(u^{(\mu)})_{\mu \in \mathbb{Z}^d}$ and where for $p \in [1, \infty)$ the space $\ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})$ is given by

$$\ell^p(\mathbb{Z}^d, \mathbb{C}^{d'}) := \Big\{ (c^{(\mu)})_{\mu \in \mathbb{Z}^d} \in (\mathbb{C}^{d'})^{\mathbb{Z}^d} \;\Big|\; \sum_{\mu \in \mathbb{Z}^d} |c^{(\mu)}|^p < \infty \Big\},$$

equipped with the norm

$$\|(c^{(\mu)})_{\mu \in \mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})} := \Big(\sum_{\mu \in \mathbb{Z}^d} |c^{(\mu)}|^p \Big)^{1/p}.$$

The space of *trigonometric polynomials* is defined by

$$\mathcal{T}(Q_d, \mathbb{C}^{d'}) := \operatorname{span} \Big\{ e^{(j)}\, T_{Q_d}^{(\mu)} \mid j \in \{1, \dots, d'\},\ \mu \in \mathbb{Z}^d \Big\},$$

where $e^{(j)}$ denotes the j-th unit coordinate vector in $\mathbb{R}^{d'}$ and the linear combinations are taken with respect to complex numbers. For $\mu \in \mathbb{Z}^d$ we denote by $p_{Q_d}^{(\mu)} \in \mathbb{R}^3$ the *lattice vector* given by

$$p_{Q_d}^{(\mu)} := (\mu_1 2L_1, \dots, \mu_d 2L_d)^\top, \qquad \mu \in \mathbb{Z}^d. \tag{1.18}$$

Periodic Functions. A function $u : \mathbb{R}^d \to \mathbb{C}^{d'}$ is called *periodic*, if

$$u(x + p_{Q_d}^{(\mu)}) = u(x), \qquad x \in \mathbb{R}^d, \quad \mu \in \mathbb{Z}^d. \tag{1.19}$$

If u is in $L^1_{\mathrm{loc}}(Q_d, \mathbb{C}^{d'})$, then for periodicity we require that (1.19) holds almost everywhere in Q_d. For $m \in \mathbb{N}_0$, we define

$$C^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) := \Big\{ u \in C^m(Q_d, \mathbb{C}^{d'}) \;\Big|\; \exists\, v \in C^m(\mathbb{R}^d, \mathbb{C}^{d'}) : \\ v \text{ is periodic and } u = v|_{Q_d} \Big\}$$

and set

$$C^\infty_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) := \bigcap_{k=0}^{\infty} C^k_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}).$$

Remark 1.1 *$C^\infty_0(Q_d, \mathbb{C}^{d'})$ is a subspace of $C^\infty_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$.*

Convention 1.2 *In regard to the notation for the function spaces introduced in this section and in following ones, if $m = 0$, then we will drop the superscript in the symbol for the function spaces. Moreover, we will mostly suppress the symbol for the co-domain, if we consider only scalar valued functions, i.e., for example we will write $C^m(\Omega)$ instead of $C^m(\Omega, \mathbb{C})$.*

Modified Differential Operators. Besides the partial differential operator ∂^α, we now specify further basic differential operators. For $u : \mathbb{R}^d \to \mathbb{C}$ and $F : \mathbb{R}^3 \to \mathbb{C}^3$, both sufficiently smooth, we have for its *gradient* ∇u, its *rotation* $\operatorname{curl} F$ and its *divergence* $\operatorname{div} F$ (in a cartesian coordinate system)

$$\nabla u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_d\, u \end{pmatrix}, \qquad \operatorname{curl} F = \nabla \times F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix},$$

$$\operatorname{div} F = \nabla \cdot F = \sum_{j=1}^{3} \partial_j F_j,$$

respectively. Often we will use *modified versions* of the last differential operators in the following form. For $\beta \in \mathbb{R}^3$ we define

$$\nabla_\beta := \nabla + \mathrm{i}\beta \tag{1.20a}$$

where i denotes the imaginary unit, and, considering again $u : \mathbb{R}^3 \to \mathbb{C}$ and $F : \mathbb{R}^3 \to \mathbb{C}^3$, both sufficiently smooth,

$$\operatorname{curl}_\beta F := \nabla_\beta \times F, \quad \operatorname{div}_\beta F := \nabla_\beta \cdot F \quad \text{and} \quad \Delta_\beta u := \operatorname{div}_\beta \nabla_\beta u. \tag{1.20b}$$

The Generic Constant. And last but not least, to make estimates in the proofs more transparent, we denote by $C > 0$ a generic constant, meaning that C may change in each occurence.

2. Sobolev Spaces for Q-periodic Functions

In this comprehensive and exhausting chapter we provide the framework for a detailed analysis of electromagnetic scattering in a Q-periodic setting. We follow thereby closely the concept of [34]. Although therein the authors consider Sobolev spaces for bounded Lipschitz domains, their ideas seem to be best suited for our purposes as they consistently make use of periodic Sobolev spaces for functions on cuboids. The step from here to cell sets of Lipschitz layer type (the domains that we are mainly interested in) is then even easier than the corresponding step to bounded Lipschitz domains in the sense that the technical argumentation with a certain partition of unity can now be almost neglected.

In Section 2.1 we start with basic results for Sobolev spaces for functions on arbitrary open sets. Here, we intend to pick up the reader and to introduce into the notation. Then we look a little more in detail at functions on cuboids for reasons which were mentioned above and present results which are more extensive. Afterwards, we are ready to define the important trace spaces, with corresponding trace and extension operators, entirely in the spirit of [34]. This approach appears to be not so much represented in the literature and at the end of Section 2.1 we connect it with the approach used for instance in [22].

The introduction of the Q-periodic setting is topic of Section 2.2. We will present results which hold in a more general context, similarly as in the case for arbitrary open sets. Already here we are able to provide several forms of Helmholtz decompositions which are an important tool in the context of Maxwell's equations.

Finally, in Section 2.3 we adopt the concept of [34] to introduce cell sets of Lipschitz layer type, define trace spaces as well as corresponding trace and extension operators for each surface separately, prove Green's formula and derive compactness and many other results which are useful for further analyses.

Special attention should be paid to the fact that we will be working in Q-periodic Sobolev spaces instead of Q-quasi-periodic ones. As a consequence, instead of the usual differential operators its modified versions as in (1.20) will come into play and make the derivation of the results more involved.

2.1. Basic Results for Sobolev Spaces

2.1.1. Functions on Open Sets

Throughout this subsection let $d, d' \in \mathbb{N}$. We start with the basic results for scalar and vector valued functions defined on an arbitrary open set $\Omega \subseteq \mathbb{R}^d$. Later, in the formulation of Sobolev spaces on the boundary of a domain, of course we have to make some restrictions concerning the boundary $\partial\Omega$ of Ω.

Definition 2.1 *Let $\Omega \subseteq \mathbb{R}^d$ be open.*

(a) For $\alpha \in \mathbb{N}_0^d$, a function $u \in L^2(\Omega, \mathbb{C}^{d'})$ possesses a variational derivative (with respect to α), if there exists $v \in L^2(\Omega, \mathbb{C}^{d'})$ such that

$$\int_\Omega u(x)\, \partial^\alpha \chi(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x)\, \chi(x)\, \mathrm{d}x$$

for all $\chi \in C_0^\infty(\Omega)$. Then we set $\partial^\alpha u := v$.

(b) Supposed $d = 3$, a function $u \in L^2(\Omega, \mathbb{C}^3)$ possesses a variational curl, if there exists $v \in L^2(\Omega, \mathbb{C}^3)$ such that

$$\int_\Omega u(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x = \int_\Omega v(x) \cdot \chi(x)\, \mathrm{d}x$$

for all $\chi \in C_0^\infty(\Omega, \mathbb{C}^3)$. Then we set $\operatorname{curl} u := v$.

(c) Supposed $d = 3$, a function $u \in L^2(\Omega, \mathbb{C}^3)$ possesses a variational divergence, if there exists $v \in L^2(\Omega)$ such that

$$\int_\Omega u(x) \cdot \nabla \chi(x) \,\mathrm{d}x = - \int_\Omega v(x)\, \chi(x) \,\mathrm{d}x$$

for all $\chi \in C_0^\infty(\Omega)$. Then we set $\operatorname{div} u := v$.

It is well-known that the variational derivative, variational curl and variational divergence, respectively, is unique, if it exists. Note that for $\alpha = 0$ we have $\partial^\alpha u = u$.

Definition 2.2 *Let $\Omega \subseteq \mathbb{R}^d$ be open.*

(a) For $m \in \mathbb{N}_0$, we define

$$H^m(\Omega, \mathbb{C}^{d'}) := \Big\{ u \in L^2(\Omega, \mathbb{C}^{d'}) \;\Big|\; \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le m : \\ \partial^\alpha u \in L^2(\Omega, \mathbb{C}^{d'}) \Big\},$$

where $\partial^\alpha u$ has to be understood in the sense of Definition 2.1, and equip this space with the inner product

$$(u \,|\, v)_{H^m(\Omega, \mathbb{C}^{d'})} := \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha u(x) \cdot \overline{\partial^\alpha v(x)} \,\mathrm{d}x$$

and with the norm $\|\cdot\|_{H^m(\Omega,\mathbb{C}^{d'})} := \sqrt{(\cdot\,|\,\cdot)_{H^m(\Omega,\mathbb{C}^{d'})}}$, i.e., the norm induced by the inner product.

(b) Supposed $d = 3$, we define

$$H(\operatorname{curl}, \Omega) := \Big\{ u \in L^2(\Omega, \mathbb{C}^3) \;\Big|\; u \text{ has variational curl} \Big\}$$

and equip this space with the inner product

$$(u \,|\, v)_{H(\operatorname{curl},\Omega)} := (u \,|\, v)_{L^2(\Omega,\mathbb{C}^3)} + (\operatorname{curl} u \,|\, \operatorname{curl} v)_{L^2(\Omega,\mathbb{C}^3)}$$

and with the norm $\|\cdot\|_{H(\operatorname{curl},\Omega)}$, induced by the inner product.

(c) Supposed $d = 3$, we define

$$H(\operatorname{div}, \Omega) := \Big\{ u \in L^2(\Omega, \mathbb{C}^3) \;\Big|\; u \textit{ has variational divergence} \Big\}$$

and equip this space with the inner product

$$(u \,|\, v)_{H(\operatorname{div},\Omega)} := (u \,|\, v)_{L^2(\Omega,\mathbb{C}^3)} + (\operatorname{div} u \,|\, \operatorname{div} v)_{L^2(\Omega)}$$

and with the norm $\|\cdot\|_{H(\operatorname{div},\Omega)}$, induced by the inner product.

It is well-known that $H^m(\Omega, \mathbb{C}^{d'})$, $H(\operatorname{curl}, \Omega)$ and $H(\operatorname{div}, \Omega)$ are Hilbert spaces. Moreover, we have by definition that $H^0(\Omega, \mathbb{C}^{d'}) = L^2(\Omega, \mathbb{C}^{d'})$.

Recall that a function $u : \Omega \to \mathbb{C}^{d'}$ is called *Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$|u(x) - u(y)| \leq L\,|x - y|, \qquad x, y \in \Omega.$$

In this case, L is one *Lipschitz constant* of u. Clearly, if $u : \Omega \to \mathbb{C}^{d'}$ is Lipschitz continuous, then for $x \in \Omega$ the function

$$u(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d) : I \to \mathbb{C}^{d'},$$

$j = 1, \ldots, d$, is Lipschitz continuous as well, where $I \subseteq \mathbb{R}$ denotes some interval depending on Ω.

We will need that a Lipschitz continuous function $u : \Omega \to \mathbb{C}^{d'}$ belongs to $H^1(\Omega, \mathbb{C}^{d'})$, if Ω is additionally bounded, see the next proposition. For this let $I = [a, b] \subseteq \mathbb{R}$ be an interval. We recall that a Lipschitz continuous function $g : I \to \mathbb{C}$ is *absolutely continuous* and therefore in I almost everywhere differentiable with integrable derivative g' and with the equation

$$g(b) - g(a) = \int_a^b g'(t)\,\mathrm{d}t, \tag{2.1}$$

holding, see for instance the Paragraphs 9.22 and 9.23 in [51]. Furthermore, thanks to Rademacher's result, see [45], we have that $g' \in L^\infty(I)$.

Proposition 2.3 *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Furthermore, let $u : \Omega \to \mathbb{C}^{d'}$ be Lipschitz continuous. Then $u \in H^1(\Omega, \mathbb{C}^{d'})$.*
Moreover, $\partial_j u$ in the variational sense coincides almost everywhere with the almost everywhere given partial derivative $\partial_j u$ in the classical sense.

Proof: We only show the assertion for the scalar valued case, as then the generalization to the vector valued case is obvious.
At first, we consider the case $d \in \mathbb{N}$ with $d > 1$. Let $\chi \in C_0^\infty(\Omega)$. We extend u and χ by zero to $\mathbb{R}^d$. Then, by rewriting $x \in \mathbb{R}^d$ in the form $x = (x_1, x')^\top$, by decomposing $\operatorname{supp} \chi(\cdot, x')$ in non-intersecting intervals and by applying (2.1) together with the product rule in each interval, which yields the integration by parts formula due to vanishing boundary terms because of the compact support of χ contained in Ω, we obtain

$$\begin{aligned}
\int_\Omega u(x)\,\partial_1\chi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} u(x)\,\partial_1\chi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} u(t,x')\,\partial_1\chi(t,x')\,\mathrm{d}t\,\mathrm{d}x' \\
&= -\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \partial_1 u(t,x')\,\chi(t,x')\,\mathrm{d}t\,\mathrm{d}x' \\
&= -\int_{\mathbb{R}^d} \partial_1 u(x)\,\chi(x)\,\mathrm{d}x = -\int_\Omega \partial_1 u(x)\,\chi(x)\,\mathrm{d}x.
\end{aligned}$$

Note that due to Rademacher's result and due to the boundedness of Ω there holds $\partial_1 u \in L^\infty(\Omega) \subseteq L^2(\Omega)$.
Analogously, one shows $\int_\Omega u\,\partial_j\chi\,\mathrm{d}x = -\int_\Omega \partial_j u\,\chi\,\mathrm{d}x$ for $j = 2,\ldots,d$.
And finally, the assertion for $d = 1$ is shown in the same way. $\square$

Corollary 2.4 *Let Ω, $\Omega' \subseteq \mathbb{R}^d$ be open, Ω' additionally be bounded, and the function $\tilde{\Phi} : \Omega' \to \Omega$ be Lipschitz continuous. Then for $u \in C^\infty(\Omega, \mathbb{C}^{d'})$ we have $v := u \circ \tilde{\Phi} \in H^1(\Omega', \mathbb{C}^{d'})$. In particular, in the case $d' = 1$ its variational gradient $\nabla v := (\partial_1 v, \ldots, \partial_d v)^\top$ is given by*

$$\nabla v = (\tilde{\Phi}')^\top (\nabla u \circ \tilde{\Phi})$$

and exists almost everywhere in Ω' in the classical sense. Here, $\tilde{\Phi}'$ is the Jacobian of $\tilde{\Phi}$, see also Section 1.3.

Proof: Since $v : \Omega' \to \mathbb{C}^{d'}$ is Lipschitz continuous, the assertion follows directly from Proposition 2.3. $\square$

Definition 2.5 *Let $\Omega \subseteq \mathbb{R}^d$ be open.*

(a) *For* $m \in \mathbb{N}_0$, *we define the space* $H_0^m(\Omega, \mathbb{C}^{d'})$ *as the closure of* $C_0^\infty(\Omega, \mathbb{C}^{d'})$ *in* $H^m(\Omega, \mathbb{C}^{d'})$.

(b) *Supposed* $d = 3$, *we define the space* $H_0(\mathrm{curl}, \Omega)$ *as the closure of* $C_0^\infty(\Omega, \mathbb{C}^3)$ *in* $H(\mathrm{curl}, \Omega)$.

(c) *Supposed* $d = 3$, *we define the space* $H_0(\mathrm{div}, \Omega)$ *as the closure of* $C_0^\infty(\Omega, \mathbb{C}^3)$ *in* $H(\mathrm{div}, \Omega)$.

Theorem 2.6 *Let* $\Omega \subseteq \mathbb{R}^d$ *be a bounded open set and let* $m \in \mathbb{N}_0$. *Then the embedding* $H_0^m(\Omega, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} L^2(\Omega, \mathbb{C}^{d'})$ *is compact.*

For a proof regarding the scalar valued case we refer to [34, Theorem 4.14]. Again, the generalization to the vector valued case is obvious.

2.1.2. Functions on Cuboids

The authors in [34] use the following *periodic* Sobolev spaces with their properties to define Sobolev spaces for functions on bounded Lipschitz domains and to derive important results for the trace and extension operators. Furthermore, this method seems to be best suited for deriving analogous results for the Q-periodic setting, as we will see later.

Throughout this subsection let $Q_d \subseteq \mathbb{R}^d$ be a cuboid as given in (1.13). Here again, d and d' are assumed to be some natural numbers.

Definition 2.7 *(a)* *Let* $s \geq 0$. *The space* $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ *is defined by*

$$H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) := \left\{ u \in L^2(Q_d, \mathbb{C}^{d'}) \;\middle|\; \sum_{\mu \in \mathbb{Z}^d} \left(1 + |\mu|^2\right)^s |u^{(\mu)}|^2 < \infty \right\}$$

with inner product

$$(u \,|\, v)_{H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})} := \sum_{\mu \in \mathbb{Z}^d} \left(1 + |\mu|^2\right)^s u^{(\mu)} \cdot \overline{v^{(\mu)}}$$

and induced norm $\|\cdot\|_{H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})}$. *Here,* $u^{(\mu)}$ *denote the Fourier coefficients of* u, *see Section 1.3.*

(b) *The space* $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ *is defined by*

$$H_{\mathrm{per}}(\mathrm{curl}, Q_3) := \left\{ u \in L^2(Q_3, \mathbb{C}^3) \;\middle|\; \sum_{\mu \in \mathbb{Z}^3} \left(|u^{(\mu)}|^2 + \left| q_{Q_3}^{(\mu)} \times u^{(\mu)} \right|^2 \right) < \infty \right\}$$

with inner product

$$(u \,|\, v)_{H_{\mathrm{per}}(\mathrm{curl}, Q_3)} := \sum_{\mu \in \mathbb{Z}^3} \left(u^{(\mu)} \cdot \overline{v^{(\mu)}} + \left(q_{Q_3}^{(\mu)} \times u^{(\mu)} \right) \cdot \left(q_{Q_3}^{(\mu)} \times \overline{v^{(\mu)}} \right) \right),$$

induced norm $\| \cdot \|_{H_{\mathrm{per}}(\mathrm{curl}, Q_3)}$ *and where* $q_{Q_3}^{(\mu)}$ *is given by* (1.14).

Note that $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ are Hilbert spaces. Furthermore, there holds $H^0_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) = L^2(Q_d, \mathbb{C}^{d'})$. Again, we will write $H^s_{\mathrm{per}}(Q_d)$ instead of $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ if $d' = 1$.

Denseness and Compactness Results. The spaces of trigonometric polynomials $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ and $\mathcal{T}(Q_3, \mathbb{C}^3)$ are dense in the spaces $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$, respectively, see the next proposition.

Proposition 2.8 (a) *The space of trigonometric polynomials* $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ *is dense in the space* $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$.

(b) *The space of trigonometric polynomials* $\mathcal{T}(Q_3, \mathbb{C}^3)$ *is dense in the space* $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$.

Proof: *(a)*. A proof can be found in [36], or by following the arguments in part *(b)*, with corresponding, quite obvious, modifications.
(b). Let $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ with Fourier coefficients $u^{(\mu)} \in \mathbb{C}^3$, $\mu \in \mathbb{Z}^3$. For $n \in \mathbb{N}$, we set $u_n := \sum_{|\mu| \le n} u^{(\mu)} \, T_{Q_3}^{(\mu)} \in \mathcal{T}(Q_3, \mathbb{C}^3)$. Then

$$\|u - u_n\|^2_{H_{\mathrm{per}}(\mathrm{curl}, Q_3)} = \sum_{|\mu| > n} \left(|u^{(\mu)}|^2 + \left| q_{Q_3}^{(\mu)} \times u^{(\mu)} \right|^2 \right) \longrightarrow 0, \quad n \to \infty,$$

because of convergence of the series $\sum_{\mu \in \mathbb{Z}^3} \left(|u^{(\mu)}|^2 + \left| q_{Q_3}^{(\mu)} \times u^{(\mu)} \right|^2 \right)$. □

Often we will need the estimates from the following lemma.

Lemma 2.9 *For $q_{Q_d}^{(\mu)}$ from (1.14) there holds*

(i) $\exists\, c > 0\; \forall\, \mu \in \mathbb{Z}^d : |q_{Q_d}^{(\mu)}| \leq c\sqrt{1+|\mu|^2}$,

(ii) $\exists\, c > 0\; \forall\, \mu \in \mathbb{Z}^d \setminus \{0\} : |q_{Q_d}^{(\mu)}| \geq c\sqrt{1+|\mu|^2}$.

Proof: *(i)*. Let $\mu \in \mathbb{Z}^d$ and recall (1.3). Then $|q_{Q_d}^{(\mu)}|^2 \leq \frac{\pi^2}{L^2}(1+|\mu|^2)$. *(ii)*. Let $\mu \in \mathbb{Z}^d \setminus \{0\}$. Then $|q_{Q_d}^{(\mu)}| \geq \frac{\pi}{L}|\mu| \geq \frac{\pi}{L}\frac{1}{\sqrt{2}}\sqrt{1+|\mu|^2}$. □

The next lemma gives a useful decomposition for the dot product defined in (1.5).

Lemma 2.10 *Let $a, b \in \mathbb{C}^3$. Moreover, let $\rho \in \mathbb{C}^3$ such that $\rho \cdot \rho = 1$. Then*

$$a \cdot b = (\rho \cdot a)\,(\rho \cdot b) + (\rho \times a) \cdot (\rho \times b).$$

Proof: Using (A.1a) and (A.1b), we obtain

$$\begin{aligned}(\rho \times a) \cdot (\rho \times b) &= \rho \cdot \big(b \times (\rho \times a)\big) = \rho \cdot \big((a \cdot b)\rho - (\rho \cdot b)a\big) \\ &= (a \cdot b)(\rho \cdot \rho) - (\rho \cdot a)(\rho \cdot b),\end{aligned}$$

as asserted. □

Remark 2.11 *For $a, b \in \mathbb{C}^2$ and $\rho \in \mathbb{C}^2$ with $\rho \cdot \rho = 1$, we obtain from Lemma 2.10, and with formula (1.6), that*

$$a \cdot b = (\rho \cdot a)\,(\rho \cdot b) + (\rho \times a)\,(\rho \times b).$$

Indeed, here we only have to identify vectors from $\mathbb{C}^2$ with vectors from $\mathbb{C}^3$ whose third component is zero.

Clearly, for $s > 0$ the space $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is compactly embedded in $L^2(Q_d, \mathbb{C}^{d'})$, see the next proposition. Unfortunately, this is not the case for the space $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$. However, we are able to derive a similar compactness result for a certain – *divergence free* – subspace of $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ as given in the following definition.

Definition 2.12 *Let $\beta \in \mathbb{R}^3$. The space $H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3)$ is defined by*

$$H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3) := \Big\{ u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3) \;\Big|\; \forall \mu \in \mathbb{Z}^3 : (q_{Q_3}^{(\mu)} + \beta) \cdot u^{(\mu)} = 0 \Big\}.$$

If $\beta = 0$, then the subscript β in the symbol $H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3)$ will be dropped.

Proposition 2.13 *(a) Let $0 \le t < s$. Then we have that the embedding $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} H^t_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is compact. In particular, $H^1_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is compactly embedded in $L^2(Q_d, \mathbb{C}^{d'})$.*

(b) The embedding $H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3) \overset{\mathrm{id}}{\hookrightarrow} L^2(Q_3, \mathbb{C}^3)$ is compact.

Proof: *(a).* For a proof we refer to [36, Theorem 8.3] with obvious modifications for the cuboids considered here, and a straightforward generalization to the vector valued case.
(b). We denote the embedding from the proposition by J and define for $n \in \mathbb{N}$ the operator $J_n : H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3) \to L^2(Q_3, \mathbb{C}^3)$ by

$$J_n u := \sum_{|\mu| \le n} u^{(\mu)}\, T_{Q_3}^{(\mu)}, \qquad u \in H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3).$$

Note that J_n is compact, because of its finite dimensional range. Let $u \in H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3)$. Then, thanks to Lemma 2.10, we have for $\mu \in \mathbb{Z}^3$, with $|\mu|$ large enough,

$$|u^{(\mu)}|^2 = \frac{1}{|q_{Q_3}^{(\mu)} + \beta|^2} \Big| \underbrace{(q_{Q_3}^{(\mu)} + \beta) \cdot u^{(\mu)}}_{=0} \Big|^2 + \frac{1}{|q_{Q_3}^{(\mu)} + \beta|^2} \Big| (q_{Q_3}^{(\mu)} + \beta) \times u^{(\mu)} \Big|^2.$$

Moreover, we have $|q_{Q_3}^{(\mu)} + \beta| \ge \frac{1}{2} |q_{Q_3}^{(\mu)}|$ for such μ. Thus, together with Lemma 2.9,

$$\|(J_n - \mathrm{id})\, u\|^2_{L^2(Q_3, \mathbb{C}^3)} = \sum_{|\mu| > n} |u^{(\mu)}|^2 = \sum_{|\mu| > n} \frac{1}{|q_{Q_3}^{(\mu)} + \beta|^2} \Big| (q_{Q_3}^{(\mu)} + \beta) \times u^{(\mu)} \Big|^2$$

$$\leq C\,\frac{1}{n^2}\sum_{|\mu|>n}\big(|u^{(\mu)}|^2+|q_{Q_3}^{(\mu)}\times u^{(\mu)}|^2\big)\leq C\,\frac{1}{n^2}\|u\|^2_{H_{\text{per}}(\text{curl},Q_3)}.$$

Hence, $(J_n)_{n\in\mathbb{N}}$ converges in operator norm to J, and from this we conclude that J is compact. □

A Useful Characterization. Now, we continue with a characterization of the space $H^m_{\text{per}}(Q_d,\mathbb{C}^{d'})$ and $H_{\text{per}}(\text{curl},Q_3)$, respectively, which is more useful to work with in some cases, in particular for the derivation of a product rule in those spaces.

Definition 2.14 *(a) For $m\in\mathbb{N}_0$, the space $\mathcal{H}^m_{\text{per}}(Q_d,\mathbb{C}^{d'})$ is defined by*

$$\mathcal{H}^m_{\text{per}}(Q_d,\mathbb{C}^{d'}):=\Big\{u\in L^2(Q_d,\mathbb{C}^{d'})\;\Big|\;\forall\alpha\in\mathbb{N}_0^d,\text{ with }|\alpha|\leq m,$$
$$\exists\, v\in L^2(Q_d,\mathbb{C}^{d'})\,\forall\chi\in C^\infty_{\text{per}}(Q_d):$$
$$\int_{Q_d}u(x)\,\partial^\alpha\chi(x)\,\mathrm{d}x=(-1)^{|\alpha|}\int_{Q_d}v(x)\,\chi(x)\,\mathrm{d}x\Big\}.$$

For $u\in\mathcal{H}^m_{\text{per}}(Q_d,\mathbb{C}^{d'})$ we set for the moment $\partial^\alpha_{\text{per}}u:=v$, see also the next remark. Furthermore, we equip this space with the inner product $(\cdot\,|\,\cdot)_{\mathcal{H}^m_{\text{per}}(Q_d,\mathbb{C}^{d'})}$ and norm $\|\cdot\|_{\mathcal{H}^m_{\text{per}}(Q_d,\mathbb{C}^{d'})}$ according to Definition 2.2.

(b) We define the space $\mathcal{H}_{\text{per}}(\text{curl},Q_3)$ to be

$$\mathcal{H}_{\text{per}}(\text{curl},Q_3):=\Big\{u\in L^2(Q_3,\mathbb{C}^3)\;\Big|\;\exists\, v\in L^2(Q_3,\mathbb{C}^3)$$
$$\forall\chi\in C^\infty_{\text{per}}(Q_3,\mathbb{C}^3):\int_{Q_3}u(x)\cdot\text{curl}\,\chi(x)\,\mathrm{d}x=\int_{Q_3}v(x)\cdot\chi(x)\,\mathrm{d}x\Big\}.$$

For $u\in\mathcal{H}_{\text{per}}(\text{curl},Q_3)$ we set for the moment $\text{curl}_{\text{per}}\,u:=v$, see also the next remark. Furthermore, we equip this space with the inner product $(\cdot\,|\,\cdot)_{\mathcal{H}_{\text{per}}(\text{curl},Q_3)}$ and norm $\|\cdot\|_{\mathcal{H}_{\text{per}}(\text{curl},Q_3)}$ according to Definition 2.2.

For the definition of $C^\infty_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ see Section 1.3.

Remark 2.15 *Thanks to Remark 1.1, for u from the space $\mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $\mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$, the element v in the definition of those spaces is unique and coincides with $\partial^\alpha u$ and $\mathrm{curl}\, u$ from Definition 2.1, respectively, and therefore we will write again $\partial^\alpha u$ instead of $\partial^\alpha_{\mathrm{per}} u$ and $\mathrm{curl}\, u$ instead of $\mathrm{curl}_{\mathrm{per}}\, u$. In particular,*

$$\mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} H^m(Q_d, \mathbb{C}^{d'}) \quad \textit{and} \quad \mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3) \overset{\mathrm{id}}{\hookrightarrow} H(\mathrm{curl}, Q_3).$$

Of course, the spaces $C^n_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ are subspaces of $\mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ for all $n \in \mathbb{N}_0 \cup \{\infty\}$ with $n \geq m$. Lipschitz continuous functions are another example for elements in $\mathcal{H}^1_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ as shown in the next proposition, compare also with Proposition 2.3.

Proposition 2.16 *Let $u : \mathbb{R}^d \to \mathbb{C}^{d'}$ be periodic and Lipschitz continuous. Then $u|_{Q_d} \in \mathcal{H}^1_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$.*
Moreover, $\partial_j u$ in the variational sense coincides almost everywhere with the almost everywhere given partial derivative $\partial_j u$ in the classical sense.

Proof: Again, we only focus on the scalar valued case as the generalization to the vector valued case is obvious.
We consider at first the case $d > 1$ and choose some $\chi \in C^\infty_{\mathrm{per}}(Q_d)$. Supposing that u is differentiable in $x = (x_1, x')^\top \in Q_d$, we apply the product rule and obtain $u(x_1, x')\, \partial_1 \chi(x_1, x') = -\chi(x_1, x')\, \partial_1 u(x_1, x') + \partial_1(u(x_1, x')\chi(x_1, x'))$. Note that by (2.1) we have

$$\int_{-L_1}^{L_1} \partial_1(u(t, x')\chi(t, x'))\, \mathrm{d}t \\ = u(L_1, x')\chi(L_1, x') - u(-L_1, x')\chi(-L_1, x') = 0,$$

because of the periodicity of the integrands. Following now the arguments from the proof of Proposition 2.3, we obtain the assertion. □

Clearly, u from $\mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $\mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$, and also $\partial^\alpha u$ and $\mathrm{curl}\, u$, respectively, can be expanded into a Fourier series. The next lemma confirms the well-known and useful connection between the Fourier coefficients of u with $\partial^\alpha u$ and with $\operatorname{curl} u$, respectively, from the context of classical functions. For this recall also (1.8).

Lemma 2.17 *(a) Let $m \in \mathbb{N}_0$. For $u \in \mathcal{H}^m_{\text{per}}(Q_d, \mathbb{C}^{d'})$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ there holds the following relationship between the Fourier coefficients of u and $\partial^\alpha u$*

$$(\partial^\alpha u)^{(\mu)} = \left(\mathrm{i}\, q_{Q_d}^{(\mu)}\right)^\alpha u^{(\mu)}, \qquad \mu \in \mathbb{Z}^d.$$

(b) For $u \in \mathcal{H}_{\text{per}}(\operatorname{curl}, Q_3)$ there holds the following relationship between the Fourier coefficients of u and $\operatorname{curl} u$

$$(\operatorname{curl} u)^{(\mu)} = \mathrm{i}\, q_{Q_3}^{(\mu)} \times u^{(\mu)}, \qquad \mu \in \mathbb{Z}^3.$$

Proof: *(a).* Let $u \in \mathcal{H}^m_{\text{per}}(Q_d, \mathbb{C}^{d'})$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. Furthermore, let $\mu \in \mathbb{Z}^d$. Then $T_{Q_d}^{(-\mu)} \in C^\infty_{\text{per}}(Q_d)$ and by definition of the space $\mathcal{H}^m_{\text{per}}(Q_d, \mathbb{C}^{d'})$, together with (1.15), we obtain

$$\begin{aligned}
(\partial^\alpha u)^{(\mu)} &= \int_{Q_d} \partial^\alpha u(x)\, T_{Q_d}^{(-\mu)}(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_{Q_d} u(x)\, \partial^\alpha T_{Q_d}^{(-\mu)}(x)\, \mathrm{d}x \\
&= \int_{Q_d} u(x) \left(\mathrm{i}\, q_{Q_d}^{(\mu)}\right)^\alpha T_{Q_d}^{(-\mu)}(x)\, \mathrm{d}x = \left(\mathrm{i}\, q_{Q_d}^{(\mu)}\right)^\alpha u^{(\mu)}.
\end{aligned}$$

(b). Let $u \in \mathcal{H}_{\text{per}}(\operatorname{curl}, Q_3)$ and $\mu \in \mathbb{Z}^3$. Furthermore, let $j \in \{1, 2, 3\}$ and let $e^{(j)}$ denote the j-th unit coordinate vector in $\mathbb{R}^3$. Then $e^{(j)} T_{Q_3}^{(-\mu)} \in C^\infty_{\text{per}}(Q_3, \mathbb{C}^3)$ and again by the definition of the space $\mathcal{H}_{\text{per}}(\operatorname{curl}, Q_3)$, together with (1.15), we obtain

$$\begin{aligned}
(\operatorname{curl} u)_j^{(\mu)} &= \int_{Q_3} \operatorname{curl} u(x) \cdot e^{(j)} T_{Q_3}^{(-\mu)}(x)\, \mathrm{d}x \\
&= \int_{Q_3} u(x) \cdot \operatorname{curl}\left(e^{(j)} T_{Q_3}^{(-\mu)}\right)(x)\, \mathrm{d}x \\
&= \int_{Q_3} u(x) \cdot \Big(-\mathrm{i}\, q_{Q_3}^{(\mu)} \times e^{(j)}\Big) T_{Q_3}^{(-\mu)}(x)\, \mathrm{d}x \\
&= e^{(j)} \cdot \left(\mathrm{i}\, q_{Q_3}^{(\mu)} \times \int_{Q_3} u(x)\, T_{Q_3}^{(-\mu)}(x)\, \mathrm{d}x\right) = \left(\mathrm{i}\, q_{Q_3}^{(\mu)} \times u^{(\mu)}\right)_j,
\end{aligned}$$

and the proof is complete. □

Clearly, the trigonometric polynomials from $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ and $\mathcal{T}(Q_3, \mathbb{C}^3)$ are further examples for elements in $\mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $\mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$, respectively. Furthermore, in the next proposition we will see that in those subspaces the corresponding norms from Definition 2.7 and Definition 2.14 are equivalent or even equal. For this, the next lemma has preliminary character. A recall of (1.8) might be appropriate.

Lemma 2.18 *For all $m \in \mathbb{N}_0$ there exists a constant $c > 0$ such that for all $\mu \in \mathbb{Z}^d$*

$$\sum_{|\alpha|=m} \mu^{2\alpha} \le |\mu|^{2m} \le c \sum_{|\alpha|=m} \mu^{2\alpha},$$

where $\alpha \in \mathbb{N}_0^d$ denotes a multi-index. Moreover, we have

$$\frac{1}{2}\big(1 + |\mu|^{2m}\big) \le \big(1 + |\mu|^2\big)^m \le 2^m \big(1 + |\mu|^{2m}\big), \quad \mu \in \mathbb{Z}^d,\ m \in \mathbb{N}_0.$$

Proof: The assertion for the first inequalities follows by induction with respect to d and by an application of the binomial theorem. In fact, let $d = 1$. For arbitrary $m \in \mathbb{N}_0$ we have $|\alpha| = m$, if and only if $\alpha_1 = m$ and therefore $\sum_{|\alpha|=m} \mu^{2\alpha} = \mu_1^{2\alpha_1} = |\mu|^{2m}$, as asserted, with $c = 1$.
For the inductive step from d to $d+1$, we suppose that the assertion is true for some $d \in \mathbb{N}$. Without loss of generality we assume that $m \in \mathbb{N}$; otherwise if $m = 0$ the inequalities hold trivially. Then for $\mu \in \mathbb{Z}^{d+1}$, and with $\beta \in \mathbb{N}_0^2$, $\gamma \in \mathbb{N}_0^d$ and $\mu' := (\mu_1, \dots, \mu_d)^\top$,

$$\begin{aligned}
\sum_{|\alpha|=m} \mu^{2\alpha} &= \sum_{|\beta|=m} \sum_{|\gamma|=\beta_1} \mu_1^{2\gamma_1} \cdots \mu_d^{2\gamma_d} \mu_{d+1}^{2\beta_2} \\
&\le \sum_{|\beta|=m} \big(\mu_1^2 + \cdots + \mu_d^2\big)^{\beta_1} \mu_{d+1}^{2\beta_2} = \sum_{|\beta|=m} |\mu'|^{2\beta_1} \mu_{d+1}^{2\beta_2} \\
&\le \big(|\mu'|^2 + \mu_{d+1}^2\big)^m = |\mu|^{2m},
\end{aligned}$$

where in the second last step we have applied the binomial theorem. And for the second inequality we obtain

$$|\mu|^{2m} = \big(|\mu'|^2 + \mu_{d+1}^2\big)^m = \sum_{k=0}^{m} \tbinom{m}{k} |\mu'|^{2k} \mu_{d+1}^{2(m-k)} \le c \sum_{|\beta|=m} |\mu'|^{2\beta_1} \mu_{d+1}^{2\beta_2}$$

$$\leq C \sum_{|\beta|=m} \sum_{|\gamma|=\beta_1} \mu_1^{2\gamma_1} \cdots \mu_d^{2\gamma_d} \mu_{d+1}^{2\beta_2} = C \sum_{|\alpha|=m} \mu^{2\alpha}.$$

The last inequalities from the lemma are easy to show. □

Proposition 2.19 *(a) Let $m \in \mathbb{N}_0$. On the space of trigonometric polynomials $\mathcal{T}(Q_d, \mathbb{C}^{d'})$, the norms $\|\cdot\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$ and $\|\cdot\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$ are equivalent, i.e., there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \|u\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \leq \|u\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \leq c_2 \|u\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})},$$

for all $u \in \mathcal{T}(Q_d, \mathbb{C}^{d'})$.

(b) On the space of trigonometric polynomials $\mathcal{T}(Q_3, \mathbb{C}^3)$, the norms $\|\cdot\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)}$ and $\|\cdot\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)}$ coincide, i.e.,

$$\|u\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} = \|u\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)}, \qquad u \in \mathcal{T}(Q_3, \mathbb{C}^3).$$

Proof: *(a).* First of all we note that for arbitrary $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ there holds

$$\left(\tfrac{\pi}{\overline{L}}\right)^{2|\alpha|} \mu^{2\alpha} \leq (q_{Q_d}^{(\mu)})^{2\alpha} \leq |q_{Q_d}^{(\mu)}|^{2|\alpha|}, \qquad \mu \in \mathbb{Z}^d,$$

where for the number $\overline{L}$ we refer to (1.3). Now, let $u \in \mathcal{T}(Q_d, \mathbb{C}^{d'})$, that is, there exists some $n \in \mathbb{N}$ such that $u = \sum_{|\mu|\leq n} u^{(\mu)} T_{Q_d}^{(\mu)}$. Then for arbitrary $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ we have $\|\partial^\alpha u\|^2_{L^2(Q_d,\mathbb{C}^{d'})} = \sum_{|\mu|\leq n} \big(q_{Q_d}^{(\mu)}\big)^{2\alpha} |u^{(\mu)}|^2$, see also Lemma 2.17. Therefore, we obtain with the second inequality from above, together with Lemma 2.9, on the one hand

$$\begin{aligned}\|u\|^2_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} &= \sum_{|\alpha|\leq m} \|\partial^\alpha u\|^2_{L^2(Q_d,\mathbb{C}^{d'})} = \sum_{|\alpha|\leq m} \sum_{|\mu|\leq n} \big(q_{Q_d}^{(\mu)}\big)^{2\alpha} |u^{(\mu)}|^2 \\ &\leq c \Big(\sum_{|\alpha|\leq m} 1 \Big) \sum_{|\mu|\leq n} (1 + |\mu|^2)^m |u^{(\mu)}|^2 = C\, \|u\|^2_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})},\end{aligned}$$

and with the first inequality from above, together with Lemma 2.18, on the other hand

$$\|u\|^2_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} = \sum_{|\mu|\leq n} \big(1 + |\mu|^2\big)^m |u^{(\mu)}|^2 \leq C \sum_{|\mu|\leq n} \big(1 + |\mu|^{2m}\big) |u^{(\mu)}|^2$$

$$\leq C \sum_{|\mu|\leq n} \Big(1 + \sum_{|\alpha|=m} \mu^{2\alpha}\Big)|u^{(\mu)}|^2 \leq C \sum_{|\mu|\leq n} \Big(1 + \sum_{|\alpha|=m} (q_{Q_d}^{(\mu)})^{2\alpha}\Big)|u^{(\mu)}|^2$$
$$(*)$$
$$= C \Big(\|u\|^2_{L^2(Q_d,\mathbb{C}^{d'})} + \sum_{|\alpha|=m} \|\partial^\alpha u\|^2_{L^2(Q_d,\mathbb{C}^{d'})}\Big) \leq C\, \|u\|^2_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}.$$

(b). Let $u \in \mathcal{T}(Q_3, \mathbb{C}^3)$, that is, there exists some $n \in \mathbb{N}$ such that $u = \sum_{|\mu|\leq n} u^{(\mu)}\, T_{Q_3}^{(\mu)}$. Then $\|\operatorname{curl} u\|^2_{L^2(Q_3,\mathbb{C}^3)} = \sum_{|\mu|\leq n} |q_{Q_3}^{(\mu)} \times u^{(\mu)}|^2$, see Lemma 2.17, and we obtain

$$\begin{aligned}\|u\|^2_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)} &= \|u\|^2_{L^2(Q_3,\mathbb{C}^3)} + \|\operatorname{curl} u\|^2_{L^2(Q_3,\mathbb{C}^3)}\\ &= \sum_{|\mu|\leq n} |u^{(\mu)}|^2 + \sum_{|\mu|\leq n} |q_{Q_3}^{(\mu)} \times u^{(\mu)}|^2 = \|u\|^2_{H_{\mathrm{per}}(\mathrm{curl},Q_3)},\end{aligned}$$

and the proof is complete. □

Now, we come to the characterization for the spaces $H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$.

Theorem 2.20 *(a) For $m \in \mathbb{N}_0$ we have*

$$H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) = \mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$$

with equivalent norms $\|\cdot\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$ and $\|\cdot\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$ therein.

(b) We have

$$H_{\mathrm{per}}(\mathrm{curl}, Q_3) = \mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$$

with coinciding norms $\|\cdot\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)}$ and $\|\cdot\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)}$ therein.

Proof: *(a).* Let $u \in \mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Then u and $\partial^\alpha u$ belong to $L^2(Q_d, \mathbb{C}^{d'})$, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. Therefore, we proceed as in the proof of Proposition 2.19 to obtain the inequalities $(*)$ therein, where we sum now over $\mu \in \mathbb{Z}^d$, and interchange at the end the sum signs because of convergent series $\sum_{\mu\in\mathbb{Z}^d} (q_{Q_d}^{(\mu)})^{2\alpha} |u^{(\mu)}|^2 = \|\partial^\alpha u\|^2_{L^2(Q_d,\mathbb{C}^{d'})}$, the latter one thanks to Parseval's identity. This shows that $u \in H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $\|u\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \leq c\, \|u\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$, with $c > 0$ independent of u.

For the other direction, let $u \in H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Since by Proposition 2.8 the space of trigonometric polynomials $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ is dense in $H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ with $\|u_n - u\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \to 0$, as $n \to \infty$. In particular, we have $u_n \to u$ in $L^2(Q_d, \mathbb{C}^{d'})$ as $n \to \infty$. Let $\alpha \in \mathbb{N}_0^m$ with $|\alpha| \le m$. By Proposition 2.19, $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(Q_d, \mathbb{C}^{d'})$ and therefore convergent to some $v \in L^2(Q_d, \mathbb{C}^{d'})$. Now let $\chi \in C^\infty_{\mathrm{per}}(Q_d)$. Then

$$\begin{aligned}\int_{Q_d} u(x)\,\partial^\alpha \chi(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_{Q_d} u_n(x)\,\partial^\alpha \chi(x)\,\mathrm{d}x \\ &= (-1)^{|\alpha|} \lim_{n\to\infty} \int_{Q_d} \partial^\alpha u_n(x)\,\chi(x)\,\mathrm{d}x = \int_{Q_d} v(x)\cdot\chi(x)\,\mathrm{d}x,\end{aligned}$$

where the boundary terms on the right hand side of the second equation vanish due to the periodicity of all integrands. Hence, there exists $\partial^\alpha u = v$, and due to the choice of α we have $u \in \mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Moreover, we have implicitly shown that $\|u_n\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \to \|u\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$, as $n \to \infty$. Thus, again thanks to Proposition 2.19,

$$\begin{aligned}\|u\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} &= \lim_{n\to\infty} \|u_n\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \\ &\le C \lim_{n\to\infty} \|u_n\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} = C\,\|u\|_{H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}.\end{aligned}$$

(b). Let $u \in \mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$. Then u and $\mathrm{curl}\, u$ belong to $L^2(Q_3, \mathbb{C}^3)$, and by Parsevals's identity, together with Lemma 2.17, we have

$$\|u\|^2_{L^2(Q_3,\mathbb{C}^3)} = \sum_{\mu\in\mathbb{Z}^3} |u^{(\mu)}|^2 \quad \text{and} \quad \|\mathrm{curl}\, u\|^2_{L^2(Q_3,\mathbb{C}^3)} = \sum_{\mu\in\mathbb{Z}^3} |q^{(\mu)}_{Q_3} \times u^{(\mu)}|^2,$$

which shows that $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$.
Now let $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$. Analogous to part *(a)*, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{T}(Q_3, \mathbb{C}^3)$ with $\|u_n - u\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} \to 0$, as $n \to \infty$. In particular, we have $u_n \to u$ in $L^2(Q_3, \mathbb{C}^3)$ as $n \to \infty$. Moreover, by Proposition 2.19, $(\mathrm{curl}\, u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(Q_3, \mathbb{C}^3)$ and therefore convergent to some $v \in L^2(Q_3, \mathbb{C}^3)$. Now let $\chi \in C^\infty_{\mathrm{per}}(Q_3, \mathbb{C}^3)$. Then

$$\int_{Q_3} u \cdot \mathrm{curl}\,\chi\,\mathrm{d}x = \lim_{n\to\infty} \int_{Q_3} u_n \cdot \mathrm{curl}\,\chi\,\mathrm{d}x$$

$$= \lim_{n\to\infty} \int_{Q_3} \operatorname{curl} u_n \cdot \chi \, \mathrm{d}x = \int_{Q_3} v \cdot \chi \, \mathrm{d}x,$$

where the boundary terms on the right hand side of the second equation again vanish due to the periodicity of all integrands. Hence, $u \in \mathcal{H}_{\mathrm{per}}(\mathrm{curl}, Q_3)$ with $\operatorname{curl} u = v$. Moreover, we have implicitly shown that $\|u_n\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)} \to \|u\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)}$, as $n \to \infty$. Thus, again thanks to Proposition 2.19,

$$\begin{aligned} \|u\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)} &= \lim_{n\to\infty} \|u_n\|_{\mathcal{H}_{\mathrm{per}}(\mathrm{curl},Q_3)} \\ &= \lim_{n\to\infty} \|u_n\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} = \|u\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)}, \end{aligned}$$

and the proof is complete. □

As a first application of the previous characterization we will derive a product rule for the spaces $H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$. For this recall (1.9).

Proposition 2.21 *(a) Let $m \in \mathbb{N}_0$. If $u \in H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $\psi \in C^\infty_{\mathrm{per}}(Q_d)$, then the product $\psi\, u \in H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$ there holds Leibniz' product rule*

$$\partial^\alpha(\psi\, u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi\, \partial^\beta u.$$

(b) If $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ and $\psi \in C^\infty_{\mathrm{per}}(Q_3)$, then the product $\psi\, u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ and

$$\operatorname{curl}(\psi\, u) = \nabla\psi \times u + \psi \operatorname{curl} u.$$

In particular, for fixed ψ the multiplication by ψ establishes a linear and bounded operator in $H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$, respectively.

Proof: *(a).* Note that $H^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) = \mathcal{H}^m_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$, see Theorem 2.20, and that ψ and all its partial derivatives are bounded as smooth and periodic functions. Therefore, $\partial^{\alpha-\beta}\psi\, \partial^\beta u \in L^2(Q_d, \mathbb{C}^{d'})$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$ and all $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$. Since $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ is dense in

$H^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'}) = \mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})$ and the norms therein are equivalent, see Proposition 2.8 and Theorem 2.20, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{T}(Q_d,\mathbb{C}^{d'})$ converging to u with respect to $\|\cdot\|_{\mathcal{H}^m_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}$. Let $\alpha\in\mathbb{N}_0^d$ with $|\alpha|\le m$ and let $\beta\in\mathbb{N}_0^d$ with $\beta\le\alpha$. Then

$$u_n \to u, \quad \partial^\beta u_n \to \partial^\beta u \quad \text{in } L^2(Q_d,\mathbb{C}^{d'}),$$

and in particular

$$\psi\, u_n \to \psi\, u, \quad \partial^{\alpha-\beta}\psi\,\partial^\beta u_n \to \partial^{\alpha-\beta}\psi\,\partial^\beta u \quad \text{in } L^2(Q_d,\mathbb{C}^{d'}),$$

as $n\to\infty$. Now, let $\chi\in C^\infty_{\mathrm{per}}(Q_d)$. Then we obtain

$$\begin{aligned}\int_{Q_d}\psi\, u\,\partial^\alpha\chi\,\mathrm{d}x &= \lim_{n\to\infty}\int_{Q_d}\psi\, u_n\,\partial^\alpha\chi\,\mathrm{d}x = (-1)^{|\alpha|}\lim_{n\to\infty}\int_{Q_d}\partial^\alpha(\psi\, u_n)\,\chi\,\mathrm{d}x\\ &= (-1)^{|\alpha|}\lim_{n\to\infty}\int_{Q_d}\sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\psi\,\partial^\beta u_n\,\chi\,\mathrm{d}x\\ &= (-1)^{|\alpha|}\int_{Q_d}\sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\psi\,\partial^\beta u\,\chi\,\mathrm{d}x.\end{aligned}$$

(b). Let $\chi\in C^\infty_{\mathrm{per}}(Q_3,\mathbb{C}^3)$. Then $\psi\,\chi\in C^\infty_{\mathrm{per}}(Q_3,\mathbb{C}^3)$ and

$$\begin{aligned}\int_{Q_3}(\psi\, u)(x)\cdot\operatorname{curl}\chi(x)\,\mathrm{d}x &= \int_{Q_3}u(x)\cdot\psi(x)\,\operatorname{curl}\chi(x)\,\mathrm{d}x\\ &= \int_{Q_3}u(x)\cdot\operatorname{curl}\big(\psi(x)\,\chi(x)\big)\,\mathrm{d}x - \int_{Q_3}u(x)\cdot\big(\nabla\psi(x)\times\chi(x)\big)\,\mathrm{d}x\\ &= \int_{Q_3}\operatorname{curl}u(x)\cdot\big(\psi(x)\,\chi(x)\big)\,\mathrm{d}x + \int_{Q_3}\big(\nabla\psi(x)\times u(x)\big)\cdot\chi(x)\,\mathrm{d}x\\ &= \int_{Q_3}\big(\psi(x)\,\operatorname{curl}u(x)+\nabla\psi(x)\times u(x)\big)\cdot\chi(x)\,\mathrm{d}x,\end{aligned}$$

as asserted.
And finally, the linearity and boundedness of the multiplication operators are easy to see. □

Trace and Extension Operators. In the following considerations the cuboids Q_2 and Q_3 are related to one another by

$$Q_3 \cap \big(\mathbb{R}^2 \times \{0\}\big) = Q_2 \times \{0\}.$$

While for the scalar valued case the trace space is given by $H^{1/2}_{\mathrm{per}}(Q_2)$, the trace spaces for vector valued functions are more delicate. For a motivation of the following definition we refer to [34].

Definition 2.22 *Let $s \in \mathbb{R}$.*

(i) The space $H^s_{\mathrm{per}}(\mathrm{Div}, Q_2)$ is defined as the completion of $\mathcal{T}(Q_2, \mathbb{C}^2)$ with respect to the norm

$$\|\varphi\|_{H^s_{\mathrm{per}}(\mathrm{Div},Q_2)} := \Big(\sum_{\mu \in \mathbb{Z}^2} \big(1 + |\mu|^2\big)^s \big(|\varphi^{(\mu)}|^2 + |q^{(\mu)}_{Q_2} \cdot \varphi^{(\mu)}|^2\big)\Big)^{1/2}.$$

(ii) The space $H^s_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ is defined as the completion of $\mathcal{T}(Q_2, \mathbb{C}^2)$ with respect to the norm

$$\|\varphi\|_{H^s_{\mathrm{per}}(\mathrm{Curl},Q_2)} := \Big(\sum_{\mu \in \mathbb{Z}^2} \big(1 + |\mu|^2\big)^s \big(|\varphi^{(\mu)}|^2 + |q^{(\mu)}_{Q_2} \times \varphi^{(\mu)}|^2\big)\Big)^{1/2}.$$

Here, $q^{(\mu)}_{Q_2}$ is given by (1.14) *and $\varphi^{(\mu)}$ denote the Fourier coefficients of φ. For the cross product recall* (1.6)*.*

For $s < 0$, the elements in $H^s_{\mathrm{per}}(\mathrm{Div}, Q_2)$ and $H^s_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ do in general not belong to $L^2(Q_2, \mathbb{C}^2)$ and it is not clear in which sense for those elements there exists a Fourier series expansion. Later in Corollary 2.34 we will see that such an expansion exists and how convergence has to be understood.

Theorem 2.23 *(a) The trace operator*

$$\gamma_{0,\mathrm{per}} : H^1_{\mathrm{per}}(Q_3, \mathbb{C}^{d'}) \to H^{1/2}_{\mathrm{per}}(Q_2, \mathbb{C}^{d'}), \quad u \mapsto u|_{Q_2 \times \{0\}},$$

is well-defined, linear and bounded. Furthermore, there exists a bounded right inverse $\eta_{0,\mathrm{per}}$ *of* $\gamma_{0,\mathrm{per}}$*; that is, a linear and bounded operator* $\eta_{0,\mathrm{per}} : H^{1/2}_{\mathrm{per}}(Q_2, \mathbb{C}^{d'}) \to H^1_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$ *with* $\gamma_{0,\mathrm{per}} \circ \eta_{0,\mathrm{per}} = \mathrm{id}$*. In other words, the function* $u := \eta_{0,\mathrm{per}}\,\varphi \in H^1_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$ *coincides with* $\varphi \in H^{1/2}_{\mathrm{per}}(Q_2, \mathbb{C}^{d'})$ *on* $Q_2 \times \{0\}$*.*

(b) Let in addition $\hat{e} := (0,0,1)^\top$*. Then the following assertions are true.*

(i) The trace operator

$$\gamma_{t,\mathrm{per}} : H_{\mathrm{per}}(\mathrm{curl}, Q_3) \to H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2), \quad u \mapsto \hat{e} \times u(\cdot, 0),$$

is well-defined, linear and bounded. Furthermore, there exists a bounded right inverse $\eta_{t,\mathrm{per}} : H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2) \to H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ *of* $\gamma_{t,\mathrm{per}}$*.*

(ii) The trace operator

$$\gamma_{T,\mathrm{per}} : H_{\mathrm{per}}(\mathrm{curl}, Q_3) \to H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q_2), \quad u \mapsto \big(\hat{e} \times u(\cdot, 0)\big) \times \hat{e},$$

is well-defined, linear and bounded. Furthermore, there exists a bounded right inverse $\eta_{T,\mathrm{per}} : H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q_2) \to H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ *of* $\gamma_{T,\mathrm{per}}$*.*

For a proof we refer to [34, Therorem 5.7 and Theorem 5.21], with slight modifications for the cuboids considered here. Therein, the assertions for $\gamma_{0,\mathrm{per}}$ and $\eta_{0,\mathrm{per}}$ were shown for $d' = 1$. Of course, for the case $d' > 1$ the application of these operators has to be understood componentwise. Moreover, we only use one symbol $\gamma_{0,\mathrm{per}}$ and $\eta_{0,\mathrm{per}}$, although applications with different $d' \in \mathbb{N}$ simultaneously are possible. Then from the context it should always be clear in which concrete spaces these operators are currently working.

Lemma 2.24 *Let* $\psi \in C^\infty_{\mathrm{per}}(Q_3)$ *with* $\psi \equiv 1$ *in a neighborhood of* $Q_2 \times \{0\} \subseteq Q_3$*. Then the following assertions are true.*

(a) If $u \in H^1_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$*, then* $\gamma_{0,\mathrm{per}}(\psi\, u) = \gamma_{0,\mathrm{per}}\, u$*.*

(b) If $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$*, then*

$$\gamma_{t,\mathrm{per}}(\psi\, u) = \gamma_{t,\mathrm{per}}\, u \quad \textit{and} \quad \gamma_{T,\mathrm{per}}(\psi\, u) = \gamma_{T,\mathrm{per}}\, u.$$

Proof: We only show the assertion for part *(b)*, as the argumentation for part *(a)* is completely analogous.
Since $u \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ and since by Proposition 2.8 the space of trigonometric polynomials $\mathcal{T}(Q_3, \mathbb{C}^3)$ is dense in $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{T}(Q_3, \mathbb{C}^3)$ with $u_n \to u$ with respect to $\|\cdot\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)}$, as $n \to \infty$. Then, thanks to Theorem 2.20, $u_n \to u$ and $\mathrm{curl}\, u_n \to \mathrm{curl}\, u$ in $L^2(Q_3, \mathbb{C}^3)$, as $n \to \infty$. Hence, by exploiting Proposition 2.21 as well,

$$\begin{aligned}
\|\psi\, u_n - \psi\, u\|^2_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} &= \|\psi\, u_n - \psi\, u\|^2_{H(\mathrm{curl},Q_3)} \\
&= \int_{Q_3} |\psi(x)|^2 |u_n(x) - u(x)|^2 \,\mathrm{d}x + \int_{Q_3} \Big| \nabla\psi(x) \times \big(u_n(x) - u(x)\big) \\
&\qquad + \psi(x)\big(\,\mathrm{curl}\, u_n(x) - \mathrm{curl}\, u(x)\big)\Big|^2 \,\mathrm{d}x \;\to\; 0, \quad n \to \infty,
\end{aligned}$$

that is, $\psi\, u_n \to \psi\, u$ in the space $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$. Note that the traces $\gamma_{t,\mathrm{per}}(\psi\, u_n)$ and $\gamma_{t,\mathrm{per}}\, u_n$ can be evaluated pointwise on $Q_2 \times \{0\}$, and therefore we have $\gamma_{t,\mathrm{per}}(\psi\, u_n) = \gamma_{t,\mathrm{per}}\, u_n$, for all $n \in \mathbb{N}$, because $\psi \equiv 1$ in a neighborhood of $Q_2 \times \{0\}$ by assumption. Moreover, by continuity of $\gamma_{t,\mathrm{per}}$, we obtain $\gamma_{t,\mathrm{per}}(\psi\, u_n) \to \gamma_{t,\mathrm{per}}(\psi\, u)$ and $\gamma_{t,\mathrm{per}}\, u_n \to \gamma_{t,\mathrm{per}}\, u$, as $n \to \infty$. Thus, $\gamma_{t,\mathrm{per}}(\psi\, u) = \gamma_{t,\mathrm{per}}\, u$.
The assertion for $\gamma_{T,\mathrm{per}}$ is shown analogously. □

As we want to transfer trace and extension results from [34] to the Q-periodic framework (which will be introduced in Section 2.2), we need the following observation concerning the extension operator $\eta_{t,\mathrm{per}}$ from the proof of [34, Theorem 5.21]. Therein, the operator

$$\eta_{t,\mathrm{per}} : H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2) \to H_{\mathrm{per}}(\mathrm{curl}, Q_3), \quad \varphi \to \eta_{t,\mathrm{per}}\, \varphi =: u$$

was constructed by taking

$$u^{(\mu)} := \frac{\delta_{\tilde{\mu}}}{1 + |\mu|^2}\, (\varphi^{(\tilde{\mu})} \times a^{(\mu)}), \qquad \mu \in \mathbb{Z}^3, \tag{2.2}$$

as Fourier coefficients for u, where $\varphi^{(\nu)}$, $\nu \in \mathbb{Z}^2$, are the coefficients of φ with vanishing third component,

$$a^{(\mu)} := \begin{cases} \frac{1}{|\tilde{\mu}|^2}\left(|\mu|^2 \hat{e} - \mu_3 \mu\right), & \tilde{\mu} \neq 0, \\ \hat{e}, & \tilde{\mu} = 0, \end{cases} \qquad \mu \in \mathbb{Z}^3, \tag{2.3}$$

and

$$\delta_\nu := \left(\sum_{j=-\infty}^{\infty} \frac{1}{1 + |\nu|^2 + j^2} \right)^{-1}, \qquad \nu \in \mathbb{Z}^2.$$

Observation 2.25 *If $\varphi \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2)$, then $u := \eta_{t,\mathrm{per}}\,\varphi$ is symmetric, that is,*

$$u = u^*(\cdot^*) \qquad \text{on } Q_3,$$

where the symbol "$*$" denotes the reflection operator given by $\mathbb{C}^3 \ni a = (a_1, a_2, a_3)^\top \mapsto a^* = (a_1, a_2, -a_3)^\top \in \mathbb{C}^3$, see also Section 1.3.

Proof: Formally, for all $x \in Q_3$ there holds

$$u^*(x^*) = \sum_{\mu \in \mathbb{Z}^3} (u^{(\mu)})^* \, T^{(\mu)}_{Q_3}(x^*) = \frac{1}{|Q_3|} \sum_{\tilde{\mu} \in \mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\tilde{\mu}\cdot\tilde{x}} \Big(\sum_{\mu_3 \in \mathbb{Z}} (u^{(\tilde{\mu},\mu_3)})^* \, \mathrm{e}^{\mathrm{i}(-\mu_3)x_3} \Big).$$

Therefore, the proof is complete, if we have shown that $(u^{(\tilde{\mu},\mu_3)})^* = u^{(\tilde{\mu},-\mu_3)}$, for all $\mu \in \mathbb{Z}^3$. For this, because of

$$(u^{(\mu)})^* = -\frac{\delta_{\tilde{\mu}}}{1 + |\mu|^2} \left((\varphi^{(\tilde{\mu})})^* \times (a^{(\mu)})^* \right) = -\frac{\delta_{\tilde{\mu}}}{1 + |\mu|^2} \left(\varphi^{(\tilde{\mu})} \times (a^{(\mu)})^* \right)$$

by (2.2), it suffices to show that $-(a^{(\tilde{\mu},\mu_3)})^* = a^{(\tilde{\mu},-\mu_3)}$ for all $\mu \in \mathbb{Z}^3$. So, let $\mu \in \mathbb{Z}^3$. If $\tilde{\mu} \neq 0$, then by (2.3)

$$-(a^{(\tilde{\mu},\mu_3)})^* = -\frac{1}{|\tilde{\mu}|^2} \left(|\mu|^2 \, \hat{e}^* - \mu_3 \mu^* \right) = -\frac{1}{|\tilde{\mu}|^2} \begin{pmatrix} -\mu_3 \mu_1 \\ -\mu_3 \mu_2 \\ -|\mu|^2 + \mu_3^2 \end{pmatrix} = a^{(\tilde{\mu},-\mu_3)}.$$

And if $\tilde{\mu} = 0$, then $-(a^{(\tilde{\mu},\mu_3)})^* = -\hat{e}^* = \hat{e} = a^{(\tilde{\mu},-\mu_3)}$. $\square$

Duality Results. As mentioned before, the elements in $H_{\text{per}}^{-1/2}(\text{Div}, Q_2)$ and $H_{\text{per}}^{-1/2}(\text{Curl}, Q_2)$ do in general not belong to $L^2(Q_2, \mathbb{C}^2)$. It is the objective of the following presentation to derive a useful characterization for those trace spaces. For this, the dual space $H_{\text{per}}^{1/2}(Q_2, \mathbb{C}^2)^*$ of $H_{\text{per}}^{1/2}(Q_2, \mathbb{C}^2)$ plays an important role.

Definition 2.26 *Let $s \geq 0$. We define $H_{\text{per}}^{-s}(Q_d, \mathbb{C}^{d'})$ to be the dual space of $H_{\text{per}}^{s}(Q_d, \mathbb{C}^{d'})$ equipped with its canonical norm*

$$\|\ell\|_{H_{\text{per}}^{-s}(Q_d,\mathbb{C}^{d'})} := \sup_{\psi \in H_{\text{per}}^{s}(Q_d,\mathbb{C}^{d'}) \setminus \{0\}} \frac{|\langle \ell, \psi \rangle|}{\|\psi\|_{H_{\text{per}}^{s}(Q_d,\mathbb{C}^{d'})}}$$

for all $\ell \in H_{\text{per}}^{-s}(Q_d, \mathbb{C}^{d'})$. Here, $\langle \ell, \psi \rangle$ denotes the duality pairing as introduced in Section 1.3.

The following theorem says that the spaces $H_{\text{per}}^{s}(Q_d, \mathbb{C}^{d'})$ can be characterized by certain spaces of sequences. The case $s < 0$ will be the more important one.

Definition 2.27 *Let $s \in \mathbb{R}$. We define the space $\mathcal{C}^s_{\mathbb{C}^{d'}}$ by*

$$\mathcal{C}^s_{\mathbb{C}^{d'}} := \left\{ (c^{(\mu)})_{\mu \in \mathbb{Z}^d} \in \left(\mathbb{C}^{d'}\right)^{\mathbb{Z}^d} \;\Big|\; \sum_{\mu \in \mathbb{Z}^d} \left(1 + |\mu|^2\right)^s |c^{(\mu)}|^2 < \infty \right\}$$

and equip this space with the norm

$$\|(c^{(\mu)})_{\mu \in \mathbb{Z}^d}\|_{\mathcal{C}^s_{\mathbb{C}^{d'}}} := \left(\sum_{\mu \in \mathbb{Z}^d} \left(1 + |\mu|^2\right)^s |c^{(\mu)}|^2 \right)^{1/2}.$$

For ease of notation, if $d' = 1$, we will write $\mathcal{C}^s$ instead of $\mathcal{C}^s_{\mathbb{C}^{d'}}$.

Theorem 2.28 *(i) For $s \geq 0$ we have $H_{\text{per}}^{s}(Q_d, \mathbb{C}^{d'}) \cong \mathcal{C}^s_{\mathbb{C}^{d'}}$. An isometric isomorphism is given by* (1.17).

(ii) For $s > 0$ we have $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \cong \mathcal{C}^{-s}_{\mathbb{C}^{d'}}$. An isometric isomorphism is given by

$$H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \ni \ell \;\mapsto\; (c^{(\mu)})_{\mu\in\mathbb{Z}^d} \in \mathcal{C}^{-s}_{\mathbb{C}^{d'}},$$
$$c_j^{(\mu)} := \ell\big(e^{(j)} T^{(-\mu)}_{Q_d}\big), \quad \mu \in \mathbb{Z}^d,\ j = 1, \dots, d', \tag{2.4}$$

with inverse

$$\mathcal{C}^{-s}_{\mathbb{C}^{d'}} \ni (c^{(\mu)})_{\mu\in\mathbb{Z}^d} \;\mapsto\; \ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}),$$
$$\ell(u) := \sum_{\mu\in\mathbb{Z}^d} u^{(\mu)} \cdot c^{(-\mu)}, \quad u \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}).$$

Here, $T^{(\mu)}_{Q_d}$ denote the trigonometric monomials and $u^{(\mu)}$ the Fourier coefficients of u, see also Section 1.3, and $e^{(j)}$ is the j-th unit coordinate vector in $\mathbb{R}^{d'}$.

Proof: *(i).* This is easy to see.
(ii). We follow the lines in the proof of [36, Theorem 8.10]. Let $(c^{(\mu)})_{\mu\in\mathbb{Z}^d}$ be a sequence in $\mathcal{C}^{-s}_{\mathbb{C}^{d'}}$ and define $\ell : H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \to \mathbb{C}$ by

$$\ell(u) := \sum_{\mu\in\mathbb{Z}^d} u^{(\mu)} \cdot c^{(-\mu)}, \qquad u \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}),$$

where $u^{(\mu)}$ are the Fourier coefficients of u. Then, using the Cauchy-Schwarz inequality, we have

$$|\ell(u)|^2 \le \sum_{\mu\in\mathbb{Z}^d} (1+|\mu|^2)^{-s} |c^{(\mu)}|^2 \sum_{\mu\in\mathbb{Z}^d} (1+|\mu|^2)^{s} |u^{(\mu)}|^2.$$

Hence, ℓ is well-defined and bounded with

$$\|\ell\|_{H^{-s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \le \Big(\sum_{\mu\in\mathbb{Z}^d} (1+|\mu|^2)^{-s} |c^{(\mu)}|^2 \Big)^{1/2}.$$

In particular, for $n \in \mathbb{N}$ the function $u_n := \sum_{|\mu|\le n} (1+|\mu|^2)^{-s} \overline{c^{(-\mu)}} T^{(\mu)}_{Q_d}$ has norm

$$\|u_n\|^2_{H^{s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} = \sum_{j=1}^{d'} \|u_{n,j}\|^2_{H^{s}_{\mathrm{per}}(Q_d)} = \sum_{j=1}^{d'} \sum_{|\mu|\le n} (1+|\mu|^2)^{-s} |c_j^{(-\mu)}|^2$$

$$= \sum_{|\mu|\le n} (1+|\mu|^2)^{-s}|c^{(\mu)}|^2$$

and we obtain

$$\|\ell\|_{H^{-s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \ge \frac{|\ell(u_n)|}{\|u_n\|_{H^{s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}} = \frac{\sum_{|\mu|\le n}(1+|\mu|^2)^{-s}\overline{c^{(-\mu)}}\cdot c^{(-\mu)}}{\Big(\sum_{|\mu|\le n}(1+|\mu|^2)^{-s}|c^{(\mu)}|^2\Big)^{1/2}}$$
$$= \Big(\sum_{|\mu|\le n}(1+|\mu|^2)^{-s}|c^{(\mu)}|^2\Big)^{1/2}.$$

Passing to limits yields $\|\ell\|_{H^{-s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} = \Big(\sum_{\mu\in\mathbb{Z}^d}(1+|\mu|^2)^{-s}|c^{(\mu)}|^2\Big)^{1/2}$. Hence, the mapping $\mathcal{C}^{-s}_{\mathbb{C}^{d'}} \ni (c^{(\mu)})_{\mu\in\mathbb{Z}^d} \mapsto \ell \in H^{-s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})$, which we just constructed, is well-defined, linear and isometric, and thus bounded and injective.
To show its surjectivity, let $\ell \in H^{-s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})$ and define for $\mu\in\mathbb{Z}^d$ and $j\in\{1,\ldots,d'\}$ the numbers $c_j^{(\mu)} := \ell(e^{(j)}T^{(-\mu)}_{Q_d})$. For $n\in\mathbb{N}$ define the functions u_n as above and note that

$$\ell(u_n) = \sum_{|\mu|\le n}(1+|\mu|^2)^{-s}\ell\big(\overline{c^{(-\mu)}}T^{(\mu)}_{Q_d}\big)$$
$$= \sum_{|\mu|\le n}(1+|\mu|^2)^{-s}\Big(\sum_{j=1}^{d'}\overline{c_j^{(-\mu)}}\ell(e^{(j)}T^{(\mu)}_{Q_d})\Big) = \sum_{|\mu|\le n}(1+|\mu|^2)^{-s}|c^{(\mu)}|^2.$$

Hence, using the last estimate from above and passing to limits, we see that $(c^{(\mu)})_{\mu\in\mathbb{Z}^d}$ belongs to $\mathcal{C}^{-s}_{\mathbb{C}^{d'}}$. Now, let $u\in H^{s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})$ and define for $n\in\mathbb{N}$ the functions $u_n := \sum_{|\mu|\le n} u^{(\mu)}T^{(\mu)}_{Q_d}$. Consulting the proof of Proposition 2.8, we know that $u_n\to u$ in $H^{s}_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})$, as $n\to\infty$. Therefore,

$$\ell(u) = \lim_{n\to\infty}\ell\Big(\sum_{|\mu|\le n}u^{(\mu)}T^{(\mu)}_{Q_d}\Big) = \sum_{\mu\in\mathbb{Z}^d}\ell(u^{(\mu)}T^{(\mu)}_{Q_d}) = \sum_{\mu\in\mathbb{Z}^d}u^{(\mu)}\cdot c^{(-\mu)},$$

where we have again used that $\ell(u^{(\mu)}T^{(\mu)}_{Q_d}) = \sum_{j=1}^{d'}u_j^{(\mu)}\ell(e^{(j)}T^{(\mu)}_{Q_d})$. □

Compared to [36, Theorem 8.10], the reason for the slightly different definition of the sequence $(c^{(\mu)})_{\mu\in\mathbb{Z}^d}$ in the last theorem is the following theorem. Here, in difference to [36, Theorem 8.11], we would like to constuct an explicit embedding from $L^2(Q_d, \mathbb{C}^{d'})$ into $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$.

Theorem 2.29 *Let $s > 0$. The space $L^2(Q_d, \mathbb{C}^{d'})$ can be embedded into $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ via the linear mapping*

$$\jmath_{\mathbb{C}^{d'}} : L^2(Q_d, \mathbb{C}^{d'}) \hookrightarrow H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}), \quad w \to \jmath_{\mathbb{C}^{d'}} w := (\cdot \,|\, \overline{w})_{L^2(Q_d, \mathbb{C}^{d'})}$$

(for ease of notation, if $d' = 1$, then we will write $\jmath$ instead of $\jmath_{\mathbb{C}^{d'}}$). Furthermore, the space $\jmath_{\mathbb{C}^{d'}}\big(\mathcal{T}(Q_d, \mathbb{C}^{d'})\big)$ is dense in $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Here, $\mathcal{T}(Q_d, \mathbb{C}^{d'})$ denotes the space of trigonometric polynomials, see also Section 1.3.

Proof: We follow the lines in the proof of [36, Theorem 8.11]. Let $w \in L^2(Q_d, \mathbb{C}^{d'})$ and $w^{(\mu)}$ be the Fourier coefficients of w, for all $\mu \in \mathbb{Z}^d$. It is easy to check that $(w^{(\mu)})_{\mu \in \mathbb{Z}^d}$ belongs to $\mathcal{C}^{-s}_{\mathbb{C}^{d'}}$. Therefore, by expanding $u \in H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ into its Fourier series and substituting this series into the formula for $\jmath_{\mathbb{C}^{d'}} w$, we obtain that $\jmath_{\mathbb{C}^{d'}} w$ belongs to $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$, and thus, that $\jmath_{\mathbb{C}^{d'}}$ is well-defined. Its linearity is clear. And from $(\jmath_{\mathbb{C}^{d'}} w)(e^{(j)} T^{(-\mu)}_{Q_d}) = w^{(\mu)}_j$, for all $j = 1, \ldots, d'$ and all $\mu \in \mathbb{Z}^d$, we conclude from Theorem 2.28 that $\jmath_{\mathbb{C}^{d'}}$ is also injective.
Let $\ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $c^{(\mu)}_j := \ell(e^{(j)} T^{(-\mu)}_{Q_d})$, for all $\mu \in \mathbb{Z}^d$ and all $j = 1, \ldots, d'$. For $n \in \mathbb{N}$ we define $\ell_n := \jmath_{\mathbb{C}^{d'}} u_n$, where $u_n := \sum_{|\mu| \le n} c^{(\mu)} T^{(\mu)}_{Q_d}$. Note that $\ell_n(e^{(j)} T^{(-\mu)}_{Q_d}) = c^{(\mu)}_j$ for all $j = 1, \ldots, d'$ and all $|\mu| \le n$, and zero otherwise. Hence, by Theorem 2.28,

$$\|\ell - \ell_n\|^2_{H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})} = \sum_{|\mu| > n} (1 + |\mu|^2)^{-s} |c^{(\mu)}|^2 \longrightarrow 0, \quad \text{as } n \to \infty,$$

and the proof is complete. □

Now, we have all the ingredients to prove the following theorem.

Theorem 2.30 *Let $s > 0$. The spaces*

$$\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2) := \Bigg\{ \ell \in H^{-s}_{\mathrm{per}}(Q_2, \mathbb{C}^2) \;\Bigg|\; \text{for } (c^{(\mu)})_{\mu \in \mathbb{Z}^2} \text{ from (2.4) we have}$$

$$\sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^{-s}\Big(|c^{(\mu)}|^2+|q_{Q_2}^{(\mu)}\cdot c^{(\mu)}|^2\Big)<\infty\Bigg\},$$

$$\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2):=\Bigg\{\ell\in H^{-s}_{\mathrm{per}}(Q_2,\mathbb{C}^2)\;\Bigg|\;\textit{for } (c^{(\mu)})_{\mu\in\mathbb{Z}^2}\textit{ from } (2.4)\textit{ we have}$$

$$\sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^{-s}\Big(|c^{(\mu)}|^2+|q_{Q_2}^{(\mu)}\times c^{(\mu)}|^2\Big)<\infty\Bigg\},$$

endowed with the norms $\|\cdot\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)}$ *and* $\|\cdot\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2)}$ *given by the square root of the series above, are Banach spaces which contain* $\jmath_{\mathbb{C}^2}\big(\mathcal{T}(Q_2,\mathbb{C}^2)\big)$ *as a dense subspace, respectively. Here,* $\jmath_{\mathbb{C}^2}$ *denotes the embedding from Theorem 2.29 and* $a\times b:=a_1b_2-a_2b_1$*, for* $a,b\in\mathbb{C}^2$*, see also* (1.6)*. Furthermore,*

$$H^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)\cong\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)\quad\textit{and}\quad H^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2)\cong\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2).$$

Proof: We only show the assertions for $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)$ since the argumentation for $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2)$ is completely analogous. To simplify notation we will write Q instead of Q_2.
(i). To show that $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)$ is a Banach space, we follow the lines in the proof of [36, Theorem 8.2]. So, let $(\ell^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)$ with corresponding sequences $(c_n^{(\mu)})_{\mu\in\mathbb{Z}^2}\in\mathcal{C}^{-s}_{\mathbb{C}^2}$, $n\in\mathbb{N}$, see Theorem 2.28. Then to given $\varepsilon>0$ there exists $N(\varepsilon)\in\mathbb{N}$ such that for all $n,m\geq N(\varepsilon)$ we have $\|\ell^{(n)}-\ell^{(m)}\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)}\leq\varepsilon$, which means that

$$\sum_{\mu\in\mathbb{Z}^2}\frac{\big|c_n^{(\mu)}-c_m^{(\mu)}\big|^2+\big|q_Q^{(\mu)}\cdot\big(c_n^{(\mu)}-c_m^{(\mu)}\big)\big|^2}{\big(1+|\mu|^2\big)^s}\leq\varepsilon^2$$

for all $n,m\geq N(\varepsilon)$. From this we conclude that for $k\in\mathbb{N}$

$$\sum_{|\mu|\leq k}\frac{\big|c_n^{(\mu)}-c_m^{(\mu)}\big|^2+\big|q_Q^{(\mu)}\cdot\big(c_n^{(\mu)}-c_m^{(\mu)}\big)\big|^2}{\big(1+|\mu|^2\big)^s}\leq\varepsilon^2$$

for all $n, m \geq N(\varepsilon)$, which yields that for all $\mu \in \mathbb{Z}^2$ the sequence $\big(c_n^{(\mu)}\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}^2$ and therefore convergent to some $\alpha^{(\mu)} \in \mathbb{C}^2$. Passing to the limit $m \to \infty$ we obtain

$$\sum_{|\mu|\leq k} \frac{\big|c_n^{(\mu)} - \alpha^{(\mu)}\big|^2 + \big|q_Q^{(\mu)} \cdot \big(c_n^{(\mu)} - \alpha^{(\mu)}\big)\big|^2}{\big(1+|\mu|^2\big)^s} \leq \varepsilon^2$$

for all $k \in \mathbb{N}$ and all $n \geq N(\varepsilon)$. Hence, by applying the triangle inequality of $\|\cdot\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)}$,

$$\begin{aligned}
&\Bigg(\sum_{|\mu|\leq k} \frac{|\alpha^{(\mu)}|^2 + \big|q_Q^{(\mu)} \cdot \alpha^{(\mu)}\big|^2}{\big(1+|\mu|^2\big)^s}\Bigg)^{1/2} \\
&\quad\leq \Bigg(\sum_{|\mu|\leq k} \frac{\big|c_n^{(\mu)} - \alpha^{(\mu)}\big|^2 + \big|q_Q^{(\mu)} \cdot \big(c_n^{(\mu)} - \alpha^{(\mu)}\big)\big|^2}{\big(1+|\mu|^2\big)^s}\Bigg)^{1/2} \\
&\qquad\qquad + \Bigg(\sum_{|\mu|\leq k} \frac{\big|c_n^{(\mu)}\big|^2 + \big|q_Q^{(\mu)} \cdot c_n^{(\mu)}\big|^2}{\big(1+|\mu|^2\big)^s}\Bigg)^{1/2} \\
&\quad\leq \varepsilon + \|\ell^{(n)}\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)} \leq \varepsilon + c,
\end{aligned}$$

for all $k \in \mathbb{N}$ and some constant $c > 0$, the latter one because Cauchy sequences are bounded. Therefore, $\sum_{\mu\in\mathbb{Z}^2} \frac{|\alpha^{(\mu)}|^2 + |q_Q^{(\mu)} \cdot \alpha^{(\mu)}|^2}{(1+|\mu|^2)^s} < \infty$. In particular, $(\alpha^{(\mu)})_{\mu\in\mathbb{Z}^2}$ belongs to $\mathcal{C}^{-s}_{\mathbb{C}^2}$ and therefore there exists $\ell \in H^{-s}_{\mathrm{per}}(Q,\mathbb{C}^2)$ with $\ell(e^{(j)}T_Q^{(-\mu)}) = \alpha_j^{(\mu)}$ for all $\mu \in \mathbb{Z}^2$ and $j = 1, 2$, see Theorem 2.28. Using the estimates from above, we conclude that $\ell \in \mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q)$ and $\ell^{(n)} \to \ell$ in $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q)$.

(ii). We show that $\jmath_{\mathbb{C}^2}\big(\mathcal{T}(Q,\mathbb{C}^2)\big)$ is a dense subspace of $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q)$ and proceed as in the corresponding part in the proof of Theorem 2.29. So, let $\ell \in \mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q)$ be given with associated coefficients $c_j^{(\mu)} = \ell(e^{(j)}T_Q^{(-\mu)})$, for $\mu \in \mathbb{Z}^2$ and $j = 1, 2$. For $n \in \mathbb{N}$ we define $\ell_n := \jmath_{\mathbb{C}^2} u_n$, where $u_n := \sum_{|\mu|\leq n} c^{(\mu)} T_Q^{(\mu)}$. Note that $\ell_n(e^{(j)}T_Q^{(-\mu)}) = c_j^{(\mu)}$ for $j = 1, 2$ and

all $|\mu| \leq n$, and zero otherwise. Therefore, by definition of the norm $\|\cdot\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2)}$, we obtain

$$\|\ell - \ell^{(n)}\|^2_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)} = \sum_{|\mu|>n} \frac{\big|c^{(\mu)}\big|^2 + \big|q_Q^{(\mu)} \cdot c^{(\mu)}\big|^2}{\big(1+|\mu|^2\big)^s} \longrightarrow 0, \quad \text{as } n \to \infty.$$

(iii). For the last assertion of the theorem it sufficies to show that $\big(\mathcal{T}(Q,\mathbb{C}^2), \|\cdot\|_{H^{-s}_{\mathrm{per}}(\mathrm{Div},Q)}\big) \cong \big(\jmath_{\mathbb{C}^2}\big(\mathcal{T}(Q,\mathbb{C}^2)\big), \|\cdot\|_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)}\big)$, because then $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)$ is a completion of $\big(\mathcal{T}(Q,\mathbb{C}^2), \|\cdot\|_{H^{-s}_{\mathrm{per}}(\mathrm{Div},Q)}\big)$ and two completions of the same normed space are isometrically isomorphic.
To construct an isometric isomorphism, let $u = \sum_{|\mu|\leq n} u^{(\mu)} T_Q^{(\mu)}$ belong to $\mathcal{T}(Q,\mathbb{C}^2)$, we $n \in \mathbb{N}$ is some natural number. We set $\ell := \jmath_{\mathbb{C}^2} u$. Then we have that $\ell(e^{(j)} T_Q^{(-\mu)}) = u_j^{(\mu)}$ for $j = 1,2$ and all $|\mu| \leq n$, and zero otherwise. Hence,

$$\|\ell\|^2_{\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div},Q)} = \sum_{|\mu|\leq n} \frac{|u^{(\mu)}|^2 + |q_Q^{(\mu)} \cdot u^{(\mu)}|^2}{\big(1+|\mu|^2\big)^s} = \|u\|^2_{H^{-s}_{\mathrm{per}}(\mathrm{Div},Q)},$$

Hence, we have shown that the just constructed linear mapping $u \mapsto \ell$ is isometric. The surjectivity of this mapping is easy to see. □

Corollary 2.31 *Let $s > 0$. Then*

$$H^{-s}_{\mathrm{per}}(\mathrm{Div},Q_2) \cong \mathcal{C}^{-s}_{\mathrm{Div}} \qquad \text{and} \qquad H^{-s}_{\mathrm{per}}(\mathrm{Curl},Q_2) \cong \mathcal{C}^{-s}_{\mathrm{Curl}},$$

where the spaces $\mathcal{C}^{-s}_{\mathrm{Div}}$ and $\mathcal{C}^{-s}_{\mathrm{Curl}}$ are defined by

$$\mathcal{C}^{-s}_{\mathrm{Div}} := \Big\{ (c^{(\mu)}) \in \big(\mathbb{C}^2\big)^{\mathbb{Z}^2} \,\Big|\, \sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} \big(|c^{(\mu)}|^2 + |q_{Q_2}^{(\mu)} \cdot c^{(\mu)}|^2\big) < \infty \Big\},$$

$$\mathcal{C}^{-s}_{\mathrm{Curl}} := \Big\{ (c^{(\mu)}) \in \big(\mathbb{C}^2\big)^{\mathbb{Z}^2} \,\Big|\, \sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} \big(|c^{(\mu)}|^2 + |q_{Q_2}^{(\mu)} \times c^{(\mu)}|^2\big) < \infty \Big\}$$

with norms $\|(c^{(\mu)})\|_{\mathcal{C}^{-s}_{\mathrm{Div}}}$ and $\|(c^{(\mu)})\|_{\mathcal{C}^{-s}_{\mathrm{Curl}}}$ given by the square root of the series, respectively.

Proof: Using the isometric isomorphism from Theorem 2.28, it is easy to see that $\mathcal{C}^{-s}_{\mathrm{Div}} \cong \mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$. In fact, given $(c^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}^{-s}_{\mathrm{Div}}$ we have obviously that this sequence belongs to $\mathcal{C}^{-s}_{\mathbb{C}^2}$. By the isometric isomorphism from Theorem 2.28, the corresponding linear and bounded functional $\ell \in H^{-s}_{\mathrm{per}}(Q_2, \mathbb{C}^2)$ satisfies $\ell(e^{(j)} T^{(-\mu)}_{Q_2}) = c^{(\mu)}_j$, for all $\mu \in \mathbb{Z}^2$ and $j = 1, 2$. From this we obtain immediately that ℓ belongs to $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$. Moreover, it is easy to check that this just constructed mapping $\mathcal{C}^{-s}_{\mathrm{Div}} \ni (c^{(\mu)})_{\mu\in\mathbb{Z}^2} \mapsto \ell \in \mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ is isometrically isomorphic. The analogous result for the spaces $\mathcal{C}^{-s}_{\mathrm{Curl}}$ and $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ we obtain with the same arguments.
And finally, an application of Theorem 2.30 completes the proof. □

Corollary 2.32 *Let $s > 0$. Then*

$$L^2(Q_2, \mathbb{C}^2) \cap H^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2) = \Bigg\{ \varphi \in L^2(Q_2, \mathbb{C}^2) \;\Bigg|\; \sum_{\mu\in\mathbb{Z}^2} \Big(|\varphi^{(\mu)}|^2 + \big(1 + |\mu|^2\big)^{-s} |q^{(\mu)}_{Q_2} \cdot \varphi^{(\mu)}|^2 \Big) < \infty \Bigg\},$$

$$L^2(Q_2, \mathbb{C}^2) \cap H^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2) = \Bigg\{ \varphi \in L^2(Q_2, \mathbb{C}^2) \;\Bigg|\; \sum_{\mu\in\mathbb{Z}^2} \Big(|\varphi^{(\mu)}|^2 + \big(1 + |\mu|^2\big)^{-s} |q^{(\mu)}_{Q_2} \times \varphi^{(\mu)}|^2 \Big) < \infty \Bigg\},$$

where $\varphi^{(\mu)} \in \mathbb{C}^2$, with $\varphi^{(\mu)}_j = \big(\varphi_j \,|\, T^{(\mu)}_{Q_2}\big)_{L^2(Q_2)}$ for all $\mu \in \mathbb{Z}^2$ and $j = 1, 2$, denote the Fourier coefficients of φ. Furthermore,

$$\Big| (\varphi \,|\, \psi)_{L^2(Q_2, \mathbb{C}^2)} \Big| \le c \, \|\varphi\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2)} \, \|\psi\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q_2)}$$

for $\varphi \in L^2(Q_2, \mathbb{C}^2) \cap H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2)$, $\psi \in L^2(Q_2, \mathbb{C}^2) \cap H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$. Here, the constant $c > 0$ can be chosen as $c = \max\big\{1, \frac{\sqrt{2}\max\{L_1, L_2\}}{\pi}\big\}$.

Proof: From the equalities we only show the first one since the argumentation for the second one is completely analogous. The direction

"$\supseteq$" is easy to see. For the direction "$\subseteq$" we observe that the statement $\varphi \in L^2(Q_2, \mathbb{C}^2) \cap H^{-s}_{\text{per}}(\text{Div}, Q_2)$ means that $\ell := \jmath_{\mathbb{C}^2}\varphi \in \mathcal{H}^{-s}_{\text{per}}(\text{Div}, Q_2)$ and that its associated sequence $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^2}$ is just the sequence of the Fourier coefficients of φ, see Theorem 2.29 and Theorem 2.30. Therefore,

$$\sum_{\mu\in\mathbb{Z}^2} \left(1+|\mu|^2\right)^{-s} \Big(|\varphi^{(\mu)}|^2 + |q^{(\mu)}_{Q_2} \cdot \varphi^{(\mu)}|^2\Big) < \infty,$$

and together with $\sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2 < \infty$ the proof for the inclusion "$\subseteq$" is complete.
To show the inequality, let φ and ψ belong to the given intersections and let $\varphi^{(\mu)}$ and $\psi^{(\mu)}$, for $\mu \in \mathbb{Z}^2$, denote their Fourier coefficients, respectively. To simplify notation, set $L := \max\{L_1, L_2\}$, $q^{(\mu)} := q^{(\mu)}_{Q_2}$ and $Q := Q_2$. At first let $\mu \in \mathbb{Z}^2 \setminus \{0\}$. Then

$$|q^{(\mu)}|^2 = \pi^2\Big(\frac{\mu_1^2}{L_1^2} + \frac{\mu_2^2}{L_2^2}\Big) \geq \tfrac{1}{2}\tfrac{\pi^2}{L^2}\, 2\,|\mu|^2 \geq \tfrac{1}{2}\tfrac{\pi^2}{L^2}\big(1+|\mu|^2\big).$$

Moreover, with $\hat{q}^{(\mu)} := \frac{1}{|q^{(\mu)}|} q^{(\mu)}$, we have thanks to Remark 2.11

$$\varphi^{(\mu)} \cdot \overline{\psi^{(\mu)}} = (\hat{q}^{(\mu)} \cdot \varphi^{(\mu)})(\hat{q}^{(\mu)} \cdot \overline{\psi^{(\mu)}}) + (\hat{q}^{(\mu)} \times \varphi^{(\mu)})(\hat{q}^{(\mu)} \times \overline{\psi^{(\mu)}}).$$

Therefore, using $(ab+cd)^2 \leq (a^2+d^2)(b^2+c^2)$ for $a, b, c, d \in \mathbb{R}$,

$$\begin{aligned}
|\varphi^{(\mu)} \cdot \overline{\psi^{(\mu)}}|^2 &\leq \Big(|\hat{q}^{(\mu)} \cdot \varphi^{(\mu)}||\psi^{(\mu)}| + |\varphi^{(\mu)}||\hat{q}^{(\mu)} \times \psi^{(\mu)}|\Big)^2 \\
&= \Big(\frac{|q^{(\mu)} \cdot \varphi^{(\mu)}|}{|q^{(\mu)}|^{1/2}} \frac{|\psi^{(\mu)}|}{|q^{(\mu)}|^{1/2}} + \frac{|\varphi^{(\mu)}|}{|q^{(\mu)}|^{1/2}} \frac{|q^{(\mu)} \times \psi^{(\mu)}|}{|q^{(\mu)}|^{1/2}}\Big)^2 \\
&\leq \Big(\frac{|\varphi^{(\mu)}|^2}{|q^{(\mu)}|} + \frac{|q^{(\mu)} \cdot \varphi^{(\mu)}|^2}{|q^{(\mu)}|}\Big)\Big(\frac{|\psi^{(\mu)}|^2}{|q^{(\mu)}|} + \frac{|q^{(\mu)} \times \psi^{(\mu)}|^2}{|q^{(\mu)}|}\Big) \\
&\leq 2\frac{L^2}{\pi^2}\ \frac{|\varphi^{(\mu)}|^2 + |q^{(\mu)} \cdot \varphi^{(\mu)}|^2}{\sqrt{1+|\mu|^2}}\ \frac{|\psi^{(\mu)}|^2 + |q^{(\mu)} \times \psi^{(\mu)}|^2}{\sqrt{1+|\mu|^2}}.
\end{aligned}$$

Note that this estimate for $|\varphi^{(\mu)} \cdot \overline{\psi^{(\mu)}}|^2$ remains valid also for $\mu = 0$ if we replace $2\frac{L^2}{\pi^2}$ by c^2 with $c > 0$ from the corollary. Hence, by the Cauchy-Schwarz inequality,

$$\big|(\varphi \,|\, \psi)_{L^2(Q,\mathbb{C}^2)}\big| = \Big|\sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \cdot \overline{\psi^{(\mu)}}\Big| \leq c\,\|\varphi\|_{H^{-1/2}_{\text{per}}(\text{Div},Q)}\,\|\psi\|_{H^{-1/2}_{\text{per}}(\text{Curl},Q)},$$

and the proof is complete. □

Thanks to the results from above, elements from the spaces $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$, $H^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ and $H^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ possess a *series represention* as shown in the next corollary. In particular the series representation for $\varphi \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ turns out to be helpful in connection with the explicit formula for the Calderon operator, as we will see later. As a preparation, we need to specify the meaning of the product of a vector $\alpha \in \mathbb{C}^{d'}$ and a linear functional $\ell \in H^{-s}_{\mathrm{per}}(Q_d)$ which is done in the following definition. Furthermore, at this point we would like to take the opportunity to introduce also the complex conjugate $\overline{\ell}$ of $\ell \in H^{-s}_{\mathrm{per}}(Q_d)$.

Definition 2.33 *Let $d' \in \mathbb{N}$ and $s > 0$.*

(i) For $\alpha \in \mathbb{C}^{d'}$ and $\ell \in H^{-s}_{\mathrm{per}}(Q_d)$ we define $\alpha\ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ by

$$\langle \alpha\ell, \psi \rangle := \sum_{j=1}^{d'} \alpha_j \, \langle \ell, \psi_j \rangle, \qquad \psi \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}).$$

(ii) For $\ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ its complex conjugate $\overline{\ell} \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is defined by

$$\langle \overline{\ell}, \psi \rangle := \overline{\langle \ell, \overline{\psi} \rangle}, \qquad \psi \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}).$$

Corollary 2.34 *(a) For $\varphi \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ we have the unique series representation*

$$\varphi = \sum_{\mu \in \mathbb{Z}^d} \varphi^{(\mu)} \jmath(T^{(\mu)}_{Q_d}),$$

where $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^d} \in \mathcal{C}^{-s}_{\mathbb{C}^{d'}}$ denotes the sequence from Theorem 2.28 and convergence has to be understood in $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$.

(b) For $\varphi \in H^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ and $\psi \in H^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ we have the unique series representation

$$\varphi = \sum_{\mu \in \mathbb{Z}^2} \varphi^{(\mu)} \jmath(T^{(\mu)}_{Q_2}) \qquad \text{and} \qquad \psi = \sum_{\mu \in \mathbb{Z}^2} \psi^{(\mu)} \jmath(T^{(\mu)}_{Q_2}),$$

where $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}^{-s}_{\mathrm{Div}}$ *and* $(\psi^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}^{-s}_{\mathrm{Curl}}$ *denote the sequences from Corollary 2.31, convergence has to be understood in* $H^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ *and* $H^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$*, respectively, and where we have identified* $H^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ *with* $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Div}, Q_2)$ *and* $H^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$ *with* $\mathcal{H}^{-s}_{\mathrm{per}}(\mathrm{Curl}, Q_2)$.

Here, $\jmath$ *denotes the embedding from Theorem 2.29.*

Proof: *(a).* Let $\varphi \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ and $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^d} \in \mathcal{C}^{-s}_{\mathbb{C}^{d'}}$ be its associated sequence from Theorem 2.28. At first we show that the series on the right hand side, that is $\big(\sum \varphi^{(\mu)}\jmath(T^{(\mu)}_{Q_d})\big)$, is convergent in $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. For this let $\psi \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Then, by Definition 2.33, the definition of the embedding $\jmath$ from Theorem 2.29 and an application of the inequality of Cauchy-Schwarz, we obtain for $m, n \in \mathbb{N}$, with $m > n$,

$$\begin{aligned}
\Big|\Big\langle \sum_{n\le|\mu|<m} \varphi^{(\mu)}\jmath(T^{(\mu)}_{Q_d}), \psi\Big\rangle\Big| &= \Big|\sum_{n\le|\mu|<m} \big\langle \varphi^{(\mu)}\jmath(T^{(\mu)}_{Q_d}), \psi\big\rangle\Big| \\
&= \Big|\sum_{n\le|\mu|<m}\sum_{j=1}^{d'} \varphi^{(\mu)}_j \big\langle \jmath(T^{(\mu)}_{Q_d}), \psi_j\big\rangle\Big| \\
&= \Big|\sum_{n\le|\mu|<m}\sum_{j=1}^{d'} \varphi^{(\mu)}_j \Big(\psi_j \,\Big|\, T^{(-\mu)}_{Q_d}\Big)_{L^2(Q_d)}\Big| = \Big|\sum_{n\le|\mu|<m} \varphi^{(\mu)} \cdot \psi^{(-\mu)}\Big| \\
&\le \Big(\sum_{n\le|\mu|<m} (1+|\mu|^2)^{-s}|\varphi^{(\mu)}|^2\Big)^{1/2} \Big(\sum_{n\le|\mu|<m} (1+|\mu|^2)^{s}|\psi^{(\mu)}|^2\Big)^{1/2} \\
&\le \Big(\sum_{n\le|\mu|<m} (1+|\mu|^2)^{-s}|\varphi^{(\mu)}|^2\Big)^{1/2} \|\psi\|_{H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})},
\end{aligned}$$

where $\psi^{(\mu)}$ denote the Fourier coefficients of ψ, and thus

$$\Big\|\sum_{n\le|\mu|<m} \varphi^{(\mu)}\jmath(T^{(\mu)}_{Q_d})\Big\|_{H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})} \le \Big(\sum_{n\le|\mu|<m} (1+|\mu|^2)^{-s}|\varphi^{(\mu)}|^2\Big)^{1/2} \longrightarrow 0,$$

as $m, n \to \infty$, because of $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^d} \in \mathcal{C}^{-s}_{\mathbb{C}^{d'}}$. Therefore, Cauchy's convergence test for series in Banach spaces implies now that $\big(\sum \varphi^{(\mu)}\jmath(T^{(\mu)}_{Q_d})\big)$

converges in $H^{-s}_{\text{per}}(Q_d, \mathbb{C}^{d'})$, say to $\ell \in H^{-s}_{\text{per}}(Q_d, \mathbb{C}^{d'})$.
We show that $\ell = \varphi$. For this, thanks to the continuity of ℓ and φ, it suffices to restrict our considerations to the subspace $\mathcal{T}(Q_d, \mathbb{C}^{d'})$, because it is dense in $H^{s}_{\text{per}}(Q_d, \mathbb{C}^{d'})$. So, let $\psi = \sum_{|\mu| \le n} \psi^{(\mu)} T^{(\mu)}_{Q_d} \in \mathcal{T}(Q_d, \mathbb{C}^{d'})$, where $n \in \mathbb{N}$ is some natural number. Then, on the one hand, we obtain from Theorem 2.28 that

$$\langle \varphi, \psi \rangle = \sum_{|\mu| \le n} \psi^{(\mu)} \cdot \varphi^{(-\mu)},$$

and, on the other hand, we obtain from the definition of ℓ, the implication (1.11), Definition 2.33 and again by the definition of the embedding $\jmath$ from Theorem 2.29 that

$$\begin{aligned}
\langle \ell, \psi \rangle &= \lim_{m\to\infty} \Big\langle \sum_{|\mu| \le m} \varphi^{(\mu)} \jmath(T^{(\mu)}_{Q_d}), \psi \Big\rangle = \lim_{m\to\infty} \sum_{|\mu| \le m} \big\langle \varphi^{(\mu)} \jmath(T^{(\mu)}_{Q_d}), \psi \big\rangle \\
&= \lim_{m\to\infty} \sum_{|\mu| \le m} \sum_{j=1}^{d'} \varphi^{(\mu)}_j \big\langle \jmath(T^{(\mu)}_{Q_d}), \psi_j \big\rangle = \lim_{m\to\infty} \sum_{|\mu| \le m} \varphi^{(\mu)} \cdot \psi^{(-\mu)} \\
&= \sum_{|\mu| \le n} \psi^{(\mu)} \cdot \varphi^{(-\mu)}.
\end{aligned}$$

(b). Let $\varphi \in \mathcal{H}^{-s}_{\text{per}}(\text{Div}, Q_2)$ and $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2} \in \mathcal{C}^{-s}_{\text{Div}}$ be its associated sequence, see Theorem 2.30 and Corollary 2.31. It is easy to check that $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2}$ belongs to $\mathcal{C}^{-s}_{\mathbb{C}^2}$. But now, from part *(a)* we know that $\ell := \sum_{\mu \in \mathbb{Z}^2} \varphi^{(\mu)} \jmath(T^{(\mu)}_{Q_2})$ belongs to $H^{-s}_{\text{per}}(Q_2, \mathbb{C}^2)$ and has to coincide with φ, because also $\varphi \in H^{-s}_{\text{per}}(Q_2, \mathbb{C}^2)$, with $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2}$ as its associated sequence, and therefore ℓ is just the series representation for φ from part *(a)*. The series representation for $\psi \in H^{-s}_{\text{per}}(\text{Curl}, Q_2)$ is proven completely analogous. □

Remark 2.35 *Using Corollary 2.31 together with Corollary 2.32, in the series representation from Corollary 2.34 the coefficients $\varphi^{(\mu)}$ and $\psi^{(\mu)}$ are just the Fourier coefficients of φ and ψ if $\varphi \in L^2(Q_2, \mathbb{C}^2) \cap H^{-s}_{\text{per}}(\text{Div}, Q_2)$ and $\psi \in L^2(Q_2, \mathbb{C}^2) \cap H^{-s}_{\text{per}}(\text{Curl}, Q_2)$, respectively.*

Remark 2.36 *Using part (ii) from Definition 2.33 together with the definition of the sequence* $(c^{(\mu)})_{\mu\in\mathbb{Z}^d}$ *from Theorem 2.28, the series representation of the complex conjugate* $\overline{\ell}$ *of* $\ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ *is given by*

$$\overline{\ell} = \sum_{\mu\in\mathbb{Z}^d} \overline{c^{(-\mu)}}\,\jmath(T^{(\mu)}_{Q_d}),$$

where $(c^{(\mu)})_{\mu\in\mathbb{Z}^d}$ *is the associated sequence of* ℓ*. Compare also with* (1.16)*.*

Multiplication Operators for the Trace Spaces. Similarly as in Proposition 2.21, we want to derive multiplication operators for the trace spaces $H^s_{\mathrm{per}}(\mathrm{Div}, Q_2)$ and $H^s_{\mathrm{per}}(\mathrm{Curl}, Q_2)$. For this purpose we make explicitly use of the theorem of Young for series, see Theorem A.2. We will see that this method can also be applied to the space $H^s_{\mathrm{per}}(Q_d)$, where $s > 0$ is now not necessarily a natural number. The results are needed later when we will consider periodic and *smooth* surfaces and apply certain cut-off functions to exploit results which hold for surfaces of bounded and smooth domains.

Lemma 2.37 *Let* $d \in \mathbb{N}$ *and* $s \geq 0$*. Then the following assertions are true.*

(i) $\forall \mu, \nu \in \mathbb{Z}^d : \big(1 + |\mu|^2\big)^s \big(1 + |\mu - \nu|^2\big)^s \geq \frac{1}{4^s}\big(1 + |\nu|^2\big)^s.$

(ii) $s > \frac{d}{2} \;\Rightarrow\; \sum_{\mu\in\mathbb{Z}^d} \frac{1}{(1+|\mu|^2)^s} < \infty.$

(iii) Let $\tau \geq 0$*. If* $\chi \in H^\sigma_{\mathrm{per}}(Q_d)$ *with* $\sigma > \frac{d}{2} + 2\tau$*, then there holds*

$$\sum_{\mu\in\mathbb{Z}^d} \big(1 + |\mu|^2\big)^\tau |\chi^{(\mu)}| < \infty.$$

Proof: *(i).* Let $\mu, \nu \in \mathbb{Z}^d$. If $|\mu| \leq \frac{1}{2}|\nu|$, then we have

$$\begin{aligned}\big(1 + |\mu|^2\big)^s \big(1 + |\mu - \nu|^2\big)^s &\geq \Big(1 + \big(|\nu| - |\mu|\big)^2\Big)^s \geq \Big(1 + \big(\tfrac{1}{2}|\nu|\big)^2\Big)^s \\ &= \big(\tfrac{4}{4} + \tfrac{1}{4}|\nu|^2\big)^s \geq \tfrac{1}{4^s}\big(1 + |\nu|^2\big)^s,\end{aligned}$$

and otherwise if $|\mu| > \frac{1}{2}|\nu|$, then we obtain

$$\left(1+|\mu|^2\right)^s\left(1+|\mu-\nu|^2\right)^s \geq \left(1+\tfrac{1}{4}|\nu|^2\right)^s \geq \tfrac{1}{4^s}\left(1+|\nu|^2\right)^s.$$

(ii). At first we observe that for $\mu \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{|\mu|_\infty = n} 1 = (2n+1)^d - (2n-1)^d &= \sum_{k=0}^{d} \tbinom{d}{k}(2n)^{d-k} - \sum_{k=0}^{d} \tbinom{d}{k}(2n)^{d-k}(-1)^k \\ &\leq c\,n^{d-1},\end{aligned}$$

with a constant $c > 0$ not depending on n. Therefore,

$$\begin{aligned}\sum_{\mu\in\mathbb{Z}^d} \frac{1}{\left(1+|\mu|^2\right)^s} &= 1 + \sum_{n=1}^{\infty}\sum_{|\mu|_\infty=n} \frac{1}{\left(1+|\mu|^2\right)^s} \\ &\leq 1 + \sum_{n=1}^{\infty}\sum_{|\mu|_\infty=n} \frac{1}{\left(1+|\mu|_\infty^2\right)^s} \leq 1 + \sum_{n=1}^{\infty} \frac{c\,n^{d-1}}{\left(1+n^2\right)^s} \\ &\leq 1 + c\sum_{n=1}^{\infty} \frac{1}{n^{2s-d+1}}.\end{aligned}$$

From this the assertion follows, since $2s-d+1 > 1$ by assumption.
(iii). We note that the sequence $\left(\frac{1}{(1+|\mu|^2)^{\sigma/2-\tau}}\right)_{\mu\in\mathbb{Z}^d}$ belongs to $\ell^2(\mathbb{Z}^d)$, see part *(ii)*. Therefore, by means of the inequality of Cauchy-Schwarz, we obtain

$$\begin{aligned}\sum_{\mu\in\mathbb{Z}^d} \left(1+|\mu|^2\right)^\tau |\chi^{(\mu)}| &= \sum_{\mu\in\mathbb{Z}^d} \left(1+|\mu|^2\right)^{\sigma/2} |\chi^{(\mu)}| \frac{1}{\left(1+|\mu|^2\right)^{\sigma/2-\tau}} \\ &\leq C\,\|\chi\|_{H^\sigma_{\mathrm{per}}(Q_d)} < \infty,\end{aligned}$$

as desired. □

Theorem 2.38 *Let $s \in \mathbb{R}$. Furthermore, let $\sigma \in \mathbb{R}$, with $\sigma > \frac{d}{2} + |s|$, and let $\chi \in H^\sigma_{\mathrm{per}}(Q_d)$. Then the following assertions are true.*

(i) For $s \geq 0$ the mapping $H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \ni \varphi \mapsto \chi\varphi \in H^s_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is well-defined, linear and bounded.

(ii) For $s > 0$ the mapping $H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}) \ni \ell \mapsto \chi\ell \in H^{-s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$ is well-defined, linear and bounded. Here, $\chi\ell$ is defined by

$$\langle \chi\ell, \psi\rangle := \langle \ell, \chi\psi\rangle, \qquad \psi \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'}).$$

Proof: *(i).* Since $\sigma > 0$, we have $\chi \in L^2(Q_d)$. Therefore, also $T^{(-\mu)}_{Q_d}\chi$ belongs to $L^2(Q_d)$ for all $\mu \in \mathbb{Z}^d$. Now, let $\varphi \in H^{s}_{\mathrm{per}}(Q_d, \mathbb{C}^{d'})$. Furthermore, let $\mu \in \mathbb{Z}^d$. Then

$$\begin{aligned}\int_{Q_d} \chi\,\varphi\, T^{(-\mu)}_{Q_d}\,\mathrm{d}x &= \int_{Q_d} \Big(\sum_{\nu\in\mathbb{Z}^d} \varphi^{(\nu)} T^{(\nu)}_{Q_d}\Big)\,\chi\, T^{(-\mu)}_{Q_d}\,\mathrm{d}x \\ &= \sum_{\nu\in\mathbb{Z}^d} \varphi^{(\nu)} \int_{Q_d} \chi\, T^{(-(\mu-\nu))}_{Q_d}\,\mathrm{d}x = \sum_{\nu\in\mathbb{Z}^d} \varphi^{(\nu)}\,\chi^{(\mu-\nu)} =: \psi^{(\mu)},\end{aligned}$$

where we have exploited the continuity of the L^2-inner product in the second equation. Hence, $\psi^{(\mu)} \in \mathbb{C}^{d'}$ is well-defined for all $\mu \in \mathbb{Z}^d$. Using now part *(i)* from Lemma 2.37 and Theorem A.2 (which is also true for a convolution of a vector valued sequence with a scalar valued sequence), we obtain

$$\begin{aligned}\sum_{\mu\in\mathbb{Z}^d} \big(1+|\mu|^2\big)^s |\psi^{(\mu)}|^2 &= \sum_{\mu\in\mathbb{Z}^d} \Big| \sum_{\nu\in\mathbb{Z}^d} \big(1+|\mu|^2\big)^{s/2} \varphi^{(\nu)}\,\chi^{(\mu-\nu)}\Big|^2 \\ &\leq \sum_{\mu\in\mathbb{Z}^d} \Big(\sum_{\nu\in\mathbb{Z}^d} \big(1+|\mu|^2\big)^{s/2} |\varphi^{(\nu)}|\,|\chi^{(\mu-\nu)}|\Big)^2 \\ &\leq C \sum_{\mu\in\mathbb{Z}^d} \Big(\sum_{\nu\in\mathbb{Z}^d} \big(1+|\nu|^2\big)^{s/2} |\varphi^{(\nu)}|\,\big(1+|\mu-\nu|^2\big)^{s/2} |\chi^{(\mu-\nu)}|\Big)^2 \\ &\leq C\,\|\varphi\|^2_{H^s_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})} \Big(\sum_{\mu\in\mathbb{Z}^d} \big(1+|\mu|^2\big)^{s/2} |\chi^{(\mu)}|\Big)^2.\end{aligned}$$

From this we conclude, with $\tau := s/2$ and part *(iii)* of Lemma 2.37, that

$$\sum_{\mu\in\mathbb{Z}^d} \big(1+|\mu|^2\big)^s |\psi^{(\mu)}|^2 \leq C\,\|\varphi\|^2_{H^s_{\mathrm{per}}(Q_d,\mathbb{C}^{d'})}.$$

In particular $\big(\psi^{(\mu)}\big)_{\mu\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d, \mathbb{C}^{d'})$ and by the definition of $\psi^{(\mu)}$ we have that $\big(\psi^{(\mu)}\big)_{\mu\in\mathbb{Z}^d}$ are the Fourier coefficients of $\psi := \chi\varphi$. Thus, in

summary we have shown that $\chi\varphi \in H^s_{\text{per}}(Q_d, \mathbb{C}^{d'})$ with $\|\chi\varphi\|_{H^s_{\text{per}}(Q_d,\mathbb{C}^{d'})} \leq C\,\|\varphi\|_{H^s_{\text{per}}(Q_d,\mathbb{C}^{d'})}$ and the constant $C > 0$ independent of φ.
(ii). Let $\ell \in H^{-s}_{\text{per}}(Q_d, \mathbb{C}^{d'})$. Furthermore, let $\psi \in H^s_{\text{per}}(Q_d, \mathbb{C}^{d'})$. Then, by part *(i)*, also $\chi\psi \in H^s_{\text{per}}(Q_d, \mathbb{C}^{d'})$ and we obtain

$$|\langle \chi\ell, \psi\rangle| = |\langle \ell, \chi\psi\rangle| \leq \|\ell\|\,\|\chi\psi\|_{H^s_{\text{per}}(Q_d,\mathbb{C}^{d'})} \leq C\,\|\ell\|\,\|\psi\|_{H^s_{\text{per}}(Q_d,\mathbb{C}^{d'})}.$$

Hence, $\chi\ell \in H^{-s}_{\text{per}}(Q_d, \mathbb{C}^{d'})$ with $\|\chi\ell\| \leq C\,\|\ell\|$. □

Remark 2.39 *The requirement from Theorem 2.38 at least for part (i) is too restrictive, see Theorems A.36 and A.41.*

Theorem 2.40 *Let $s > 0$. Furthermore, let $\sigma > s + 2$ and $\chi \in H^\sigma_{\text{per}}(Q_2)$. Then the following assertions are true.*

(i) The mapping $\mathcal{H}^{-s}_{\text{per}}(\text{Div}, Q_2) \ni \ell \mapsto \chi\ell \in \mathcal{H}^{-s}_{\text{per}}(\text{Div}, Q_2)$ is well-defined, linear and bounded.

(ii) The mapping $\mathcal{H}^{-s}_{\text{per}}(\text{Curl}, Q_2) \ni \ell \mapsto \chi\ell \in \mathcal{H}^{-s}_{\text{per}}(\text{Curl}, Q_2)$ is well-defined, linear and bounded.

For the definition of $\chi\ell$ see also Theorem 2.38.

Proof: *(i).* Let $\ell \in \mathcal{H}^{-s}_{\text{per}}(\text{Div}, Q_2)$ with coefficients $(c^{(\mu)})_{\mu\in\mathbb{Z}^2}$ according to (2.4). By Theorem 2.38 we have $\chi\ell \in H^{-s}_{\text{per}}(Q_2, \mathbb{C}^2)$ with coefficients

$$\begin{aligned} d_j^{(\mu)} &= \langle \chi\ell, e^{(j)} T_{Q_2}^{(-\mu)}\rangle = \langle \ell, \chi e^{(j)} T_{Q_2}^{(-\mu)}\rangle = \sum_{\nu\in\mathbb{Z}^2} \chi^{(\nu)} \langle \ell, e^{(j)} T_{Q_2}^{(-(\mu-\nu))}\rangle \\ &= \sum_{\nu\in\mathbb{Z}^2} \chi^{(\nu)} c_j^{(\mu-\nu)} = \sum_{\nu\in\mathbb{Z}^2} \chi^{(\mu-\nu)} c_j^{(\nu)}, \end{aligned}$$

where we have made the following considerations for the third equation: by part *(a)* of Proposition 2.13 we have $\chi \in H^s_{\text{per}}(Q_2)$, yielding that $e^{(j)}\chi = \sum_{\nu\in\mathbb{Z}^2} \chi^{(\nu)} e^{(j)} T_{Q_2}^{(\nu)}$ with convergence in $H^s_{\text{per}}(Q_2, \mathbb{C}^2)$ and where $\chi^{(\nu)}$ denote the Fourier coefficients of χ; since $T_{Q_2}^{(-\mu)} \in C^\infty_{\text{per}}(Q_2)$, we conclude by the continuity of the multiplication operator with $T_{Q_2}^{(-\mu)}$, see

Theorem 2.38, that $\chi e^{(j)} T_{Q_2}^{(-\mu)} = \sum_{\nu\in\mathbb{Z}^2} \chi^{(\nu)} e^{(j)} T_{Q_2}^{(\nu)} T_{Q_2}^{(-\mu)}$ with convergence in $H^s_{\mathrm{per}}(Q_2, \mathbb{C}^2)$; and finally we exploit the continuity of ℓ.
We have to show that $(d^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}^{-s}_{\mathrm{Div}}$. For this we apply part *(i)* of Lemma 2.37 and Theorem A.2 to obtain on the one hand, similarly as in the proof of Theorem 2.38

$$
\begin{aligned}
\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |d^{(\mu)}|^2 &= \sum_{\mu\in\mathbb{Z}^2} \Big| \sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} c^{(\nu)} \chi^{(\mu-\nu)} \Big|^2 \\
&\le \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} |c^{(\nu)}|\, |\chi^{(\mu-\nu)}| \Big)^2 \\
&\le C \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\nu|^2\big)^{-s/2} |c^{(\nu)}| \,\big(1+|\mu-\nu|^2\big)^{s/2} |\chi^{(\mu-\nu)}| \Big)^2 \\
&\le C \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |c^{(\mu)}|^2 \Big) \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{s/2} |\chi^{(\mu)}| \Big)^2
\end{aligned}
$$

From this we conclude, with $\tau := s/2$ and part *(iii)* of Lemma 2.37, that

$$
\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |d^{(\mu)}|^2 \le C\, \|(c^{(\mu)})_{\mu\in\mathbb{Z}^d}\|^2_{\mathcal{C}^{-s}_{\mathrm{Div}}} .
$$

Similarly, and in addition with the decomposition $q_{Q_2}^{(\mu)} = q_{Q_2}^{(\nu)} + q_{Q_2}^{(\mu-\nu)}$, we obtain on the other hand

$$
\begin{aligned}
\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |q_{Q_2}^{(\mu)} \cdot d^{(\mu)}|^2 &= \sum_{\mu\in\mathbb{Z}^2} \Big| \sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} q_{Q_2}^{(\nu)} \cdot c^{(\nu)} \chi^{(\mu-\nu)} \\
&\quad + \sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} c^{(\nu)} \cdot (q_{Q_2}^{(\mu-\nu)} \chi^{(\mu-\nu)}) \Big|^2 \\
&\le 2 \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} |q_{Q_2}^{(\nu)} \cdot c^{(\nu)}|\, |\chi^{(\mu-\nu)}| \Big)^2 \\
&\quad + 2 \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s/2} |c^{(\nu)}|\, |q_{Q_2}^{(\mu-\nu)}|\, |\chi^{(\mu-\nu)}| \Big)^2 \\
&\le C_1 \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\nu|^2\big)^{-s/2} |q_{Q_2}^{(\nu)} \cdot c^{(\nu)}| \,\big(1+|\mu-\nu|\big)^{s/2} |\chi^{(\mu-\nu)}| \Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C_2 \sum_{\mu\in\mathbb{Z}^2} \Big(\sum_{\nu\in\mathbb{Z}^2} \big(1+|\nu|^2\big)^{-s/2} |c^{(\nu)}| \big(1+|\mu-\nu|^2\big)^{(s+1)/2} |\chi^{(\mu-\nu)}| \Big)^2 \\
&\le C_1 \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |q_{Q_2}^{(\mu)} \cdot c^{(\mu)}|^2 \Big) \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{s/2} |\chi^{(\mu)}| \Big)^2 \\
&+ C_2 \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |c^{(\mu)}|^2 \Big) \Big(\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{(s+1)/2} |\chi^{(\mu)}| \Big)^2 .
\end{aligned}
$$

From this we conclude again, with $\tau := s/2$ as well as $\tau := (s+1)/2$ and part *(iii)* of Lemma 2.37, that

$$
\sum_{\mu\in\mathbb{Z}^2} \big(1+|\mu|^2\big)^{-s} |q_{Q_2}^{(\mu)} \cdot d^{(\mu)}|^2 \le C \, \|(c^{(\mu)})_{\mu\in\mathbb{Z}^d}\|^2_{\mathcal{C}^{-s}_{\mathrm{Div}}} .
$$

Summing up both results yields the assertion.
(ii). This is shown completely analogous. □

2.1.3. Functions on Bounded Lipschitz Domains

In this subsection we give a brief introduction into the concepts the authors in [34] used to work in Sobolev spaces for functions on bounded Lipschitz domains. Since those concepts are based on results for Sobolev spaces for functions on cuboids, it seems "natural" to pick up whose ideas for establishing later an analogous framework for Q-periodic functions.

We start by recalling the notion of a *Lipschitz domain*, see also the beginning of [34, Section 5.1].

Definition 2.41 *We call an open set $\Omega \subseteq \mathbb{R}^3$ with compact boundary $\partial\Omega$ a Lipschitz domain, if there exists a finite number of open cylinders U_j of the form $U_j = \{R_j x + z^{(j)} \mid x \in \mathbb{B}_2(0,\alpha_j) \times (-2\beta_j, 2\beta_j)\}$ with $z^{(j)} \in \mathbb{R}^3$, rotations $R_j \in \mathbb{R}^{3\times 3}$ and Lipschitz-continuous functions $f_j : \mathbb{B}_2[0,\alpha_j] \to \mathbb{R}$ with $|f_j(x_1,x_2)| \le \beta_j$ for all $(x_1,x_2)^\top \in \mathbb{B}_2[0,\alpha_j]$ such that $\partial\Omega \subseteq \bigcup_{j=1}^m U_j$ and*

$$
\begin{aligned}
\partial\Omega \cap U_j &= \big\{ R_j\, x + z^{(j)} \mid \tilde{x} \in \mathbb{B}_2(0,\alpha_j),\, x_3 = f_j(\tilde{x}) \big\}, \\
\Omega \cap U_j &= \big\{ R_j\, x + z^{(j)} \mid \tilde{x} \in \mathbb{B}_2(0,\alpha_j),\, x_3 < f_j(\tilde{x}) \big\},
\end{aligned}
$$

$$U_j \setminus \overline{\Omega} = \big\{ R_j\, x + z^{(j)} \mid \tilde{x} \in \mathbb{B}_2(0, \alpha_j),\, x_3 > f_j(\tilde{x}) \big\}.$$

Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain. We call $\{(U_j, f_j) \mid j = 1, \ldots, m\}$ a *local coordinate system* of $\partial\Omega$. Without loss of generality we assume that $\beta_j \geq \alpha_j$. Then we introduce the mappings

$$\tilde{\Psi}_j(x) := R_j \begin{pmatrix} x_1 \\ x_2 \\ f_j(\tilde{x}) + x_3 \end{pmatrix} + z^{(j)}, \qquad x \in \mathbb{B}_3(0, \alpha_j),$$

and their restrictions Ψ_j to $\mathbb{B}_2(0, \alpha_j)$, that is,

$$\Psi_j(x) := R_j \begin{pmatrix} x_1 \\ x_2 \\ f_j(x) \end{pmatrix} + z^{(j)}, \qquad x \in \mathbb{B}_2(0, \alpha_j).$$

Thanks to Rademacher's result, see [45], we have that f_j is differentiable almost everywhere on $\mathbb{B}_2[0, \alpha_j]$ and that its gradient is essentially bounded by the Lipschitz constant of f_j. Therefore, Ψ_j is differentiable almost everywhere on $\mathbb{B}_2(0, \alpha_j)$ and the surface patch $\partial\Omega \cap U_j$ can be parametrized by $y = \Psi_j(x)$ for $x \in \mathbb{B}_2(0, \alpha_j)$, with outward pointing normal unit vector $\mathrm{n}(y)$ at a.a. $y = \Psi_j(x)$ given by

$$\mathrm{n}(y) = \frac{1}{\rho_j(x)} \left(\frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right),$$

where

$$\rho_j(x) := \left| \frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right| = \sqrt{1 + |\nabla f_j(x)|^2}.$$

We set $U_j' = \tilde{\Psi}_j(\mathbb{B}_3(0, \alpha_j))$. Then $\partial\Omega \subseteq \bigcup_{j=1}^m U_j'$, $\mathbb{B}_3(0, \alpha_j) \cap (\mathbb{R}^2 \times \{0\}) = \mathbb{B}_2(0, \alpha_j) \times \{0\}$, and

$$\begin{aligned} \partial\Omega \cap U_j' &= \{\tilde{\Psi}_j(x) \mid x \in \mathbb{B}_3(0, \alpha_j),\, x_3 = 0\} = \{\Psi_j(x) \mid x \in \mathbb{B}_2(0, \alpha_j)\}, \\ \Omega \cap U_j' &= \{\tilde{\Psi}_j(x) \mid x \in \mathbb{B}_3(0, \alpha_j),\, x_3 < 0\}, \\ U_j' \setminus \overline{\Omega} &= \{\tilde{\Psi}_j(x) \mid x \in \mathbb{B}_3(0, \alpha_j),\, x_3 > 0\}. \end{aligned}$$

Note that the Jacobian $\tilde{\Psi}_j'(x) \in \mathbb{R}^{3 \times 3}$ is given by

$$\tilde{\Psi}_j'(x) = R_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f_j}{\partial x_1}(\tilde{x}) & \frac{\partial f_j}{\partial x_2}(\tilde{x}) & 1 \end{pmatrix}, \quad \text{for a.a. } x \in \mathbb{B}_3(0, \alpha_j).$$

Hence, these Jacobians are regular with constant determinant $\det \tilde{\Psi}_j'(x) = 1$ and $\tilde{\Psi}_j$ are isomorphisms from $\mathbb{B}_3(0,\alpha_j)$ to U_j' for every $j = 1,\dots,m$. For the case of vector valued functions, we will also need

$$F_j(x) := \left[\frac{\partial \Psi_j}{\partial x_1}(x) \middle| \frac{\partial \Psi_j}{\partial x_2}(x) \middle| \frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right], \text{ for a.a. } x \in \mathbb{B}_2(0,\alpha_j).$$

We recall the notion of a *partition of unity*, see also [34, Section A.3] and references therein. Here, the cut-off functions are chosen such that their square roots are smooth functions as well.

Theorem 2.42 *Let $d \in \mathbb{N}$ and $K \subseteq \mathbb{R}^d$ be compact. For every finite family $\{O_j \mid j = 1,\dots,m\}$ of open sets with $K \subseteq \bigcup_{j=1}^m O_j$ there exist $\chi_j \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\chi_j) \subseteq O_j$, $j = 1,\dots,m$, and $\sum_{j=1}^m \chi_j(x) = 1$ for all $x \in K$. We call the family $\{(O_j,\chi_j) \mid j = 1,\dots,m\}$ a partition of unity on K.*
Here, the cut-off functions χ_j can be chosen such that $\sqrt{\chi_j} \in C^\infty(\mathbb{R}^d)$.

Proof: We refer to the reference from above for the existence of a partition of unity $\{(O_j,\tilde{\chi}_j) \mid j = 1,\dots,m\}$ on K. We set

$$\eta = \sum_{j=1}^m \tilde{\chi}_j^2 + (1 - \sum_{j=1}^m \tilde{\chi}_j)^2 \quad \text{on } \mathbb{R}^d.$$

Then $\eta > 0$ on $\mathbb{R}^d$ and $\eta = \sum_{j=1}^m \tilde{\chi}_j^2$ on K, which can be easily verified. Hence, $\chi_j := \tilde{\chi}_j^2/\eta$ form also a partition of unity on K with $\sqrt{\chi_j} = \tilde{\chi}_j/\sqrt{\eta} \in C^\infty(\mathbb{R}^d)$. □

Assumption 2.43 *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with corresponding local coordinate system $\{(U_j,f_j) \mid j = 1,\dots,m\}$, corresponding mappings $\tilde{\Psi}_j$ from $\mathbb{B}_3(0,\alpha_j)$ to U_j' and their restrictions Ψ_j from $\mathbb{B}_2(0,\alpha_j)$ to $U_j' \cap \partial\Omega$. Furthermore, let $\{(U_j',\chi_j) \mid j = 1,\dots,m\}$ be a partition of unity on $\partial\Omega$ according to Theorem 2.42. And finally, let $Q_d := \bigtimes_{j=1}^d (-L_j, L_j)$, $d = 2,3$, where we assume that $L_j > 0$, $j = 1,\dots,3$, is chosen so big that all of the balls $\mathbb{B}_3(0,\alpha_j)$ are contained in Q_3.*

Together with a partition of unity, the parametrizations $\tilde{\Psi}_j$ allow us to transfer the concept of periodic Sobolev spaces on two-dimensional cuboids to Sobolev spaces for functions on the boundary $\partial\Omega$ of Ω. While for the scalar valued case this procedure is straightforward, for the vector valued case some modifications are necessary to make the transformation *curl-preserving*, see Section 2.3 for details. Nevertheless, we would like already here to state an important property of Lipschitz continuous functions, which will later turn out as a crucial part of this curl-preserving transformation.

Proposition 2.44 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and let $u, v : \Omega \to \mathbb{C}$ be Lipschitz continuous. Then $u\nabla v \in H(\mathrm{curl}, \Omega)$ with variational curl given by*

$$\mathrm{curl}(u\nabla v) = \nabla u \times \nabla v.$$

In particular, $\mathrm{curl}(u\nabla v)$ exists almost everywhere on Ω as a classical function.

Proof: First of all, thanks to Rademacher's result, we have that ∇u and ∇v belong to $L^\infty(\Omega, \mathbb{C}^3)$, which implies that $\nabla u \times \nabla v \in L^\infty(\Omega, \mathbb{C}^3)$. And, since Ω is bounded, we have $L^\infty(\Omega, \mathbb{C}^3) \subseteq L^2(\Omega, \mathbb{C}^3)$.
Moreover, due to Proposition 2.3, we know that u, $v \in H^1(\Omega)$. Therefore, by Theorem 2.46, we can choose a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1(\Omega)$, as $n \to \infty$. In particular, $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^2(\Omega)$ and $L^2(\Omega, \mathbb{C}^3)$, as $n \to \infty$, respectively.
Now let $\chi \in C_0^\infty(\Omega, \mathbb{C}^3)$. Note that for all $n \in \mathbb{N}$ the functions $[\mathrm{curl}(u_n\chi)]_j$, $j = 1, 2, 3$, belong to $C_0^\infty(\Omega)$ and therefore, by the definition of the variational derivative,

$$\int_\Omega \nabla v \cdot \mathrm{curl}(u_n\chi)\,\mathrm{d}x = -\int_\Omega v \underbrace{\mathrm{div}\,\mathrm{curl}(u_n\chi)}_{=0}\,\mathrm{d}x = 0.$$

Then, with the considerations above,

$$\begin{aligned}\int_\Omega (u\nabla v) \cdot \mathrm{curl}\,\chi\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega (u_n\nabla v) \cdot \mathrm{curl}\,\chi\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_\Omega \nabla v \cdot (u_n\,\mathrm{curl}\,\chi)\,\mathrm{d}x\end{aligned}$$

$$= \lim_{n\to\infty} \left(\int_\Omega \nabla v \cdot \operatorname{curl}(u_n\chi)\,\mathrm{d}x - \int_\Omega \nabla v \cdot (\nabla u_n \times \chi)\,\mathrm{d}x \right)$$
$$= \lim_{n\to\infty} \int_\Omega (\nabla u_n \times \nabla v)\cdot\chi\,\mathrm{d}x = \int_\Omega (\nabla u \times \nabla v)\cdot \chi\,\mathrm{d}x,$$

and the proof is complete. □

Let $\Omega \subseteq \mathbb{R}^3$ be given as in Assumption 2.43 and let $d' \in \mathbb{N}$. The space $L^2(\partial\Omega, \mathbb{C}^{d'})$ of square integrable functions on the surface $\partial\Omega$ of Ω can be characterized by $\varphi \in L^2(\partial\Omega, \mathbb{C}^{d'})$, if and only if $\tilde{\varphi}_j \in L^2(Q_2, \mathbb{C}^{d'})$ for all $j = 1, \dots, m$, where for $j = 1, \dots, m$ the functions $\tilde{\varphi}_j$ are given by

$$\tilde{\varphi}_j(x) := \begin{cases} \sqrt{\chi_j\big(\Psi_j(x)\big)}\,\varphi\big(\Psi_j(x)\big), & x \in \mathbb{B}_2(0,\alpha_j), \\ 0, & x \in Q_2 \setminus \mathbb{B}_2(0,\alpha_j). \end{cases} \tag{2.5}$$

Furthermore, we define the subspaces of $L^2(\partial\Omega, \mathbb{C}^3)$ and $L^2(Q_2, \mathbb{C}^3)$ of *tangential vector fields* by

$$L^2_{\mathrm{t}}(\partial\Omega) := \big\{\varphi \in L^2(\partial\Omega, \mathbb{C}^3) \mid \mathrm{n}(y)\cdot\varphi(y) = 0 \text{ for a.a. } y \in \partial\Omega\big\},$$
$$L^2_{\mathrm{t}}(Q_2) := \big\{u \in L^2(Q_2, \mathbb{C}^3) \mid u_3(x) = 0 \text{ for a.a. } x \in Q_2\big\},$$

respectively. Again, we will mostly suppress the symbol for the co-domain in $L^2(\partial\Omega, \mathbb{C}^{d'})$ if we consider only scalar valued functions.

Definition 2.45 *Let $\Omega \subseteq \mathbb{R}^3$ be given as in Assumption 2.43.*

(a) We define the space $H^{1/2}(\partial\Omega, \mathbb{C}^{d'})$ by

$$H^{1/2}(\partial\Omega, \mathbb{C}^{d'}) := \Big\{\varphi \in L^2(\partial\Omega, \mathbb{C}^{d'}) \;\Big|\; \forall j \in \{1,\dots,m\} : \tilde{\varphi}_j \in H^{1/2}_{\mathrm{per}}(Q_2, \mathbb{C}^{d'})\Big\}$$

with norm

$$\|\varphi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^{d'})} := \left(\sum_{j=1}^m \|\tilde{\varphi}_j\|^2_{H^{1/2}_{\mathrm{per}}(Q_2,\mathbb{C}^{d'})}\right)^{1/2},$$

where for $j = 1, \dots, m$ the functions $\tilde{\varphi}_j$ are given by (2.5).

(b) We define the spaces $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ *and* $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ *as the completion of*

$$\big\{\varphi \in L^2_{\mathrm{t}}(\partial\Omega) \mid \tilde{\varphi}^t_j \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q_2),\ j = 1, \dots, m\big\},$$
$$\big\{\varphi \in L^2_{\mathrm{t}}(\partial\Omega) \mid \tilde{\varphi}^T_j \in H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q_2),\ j = 1, \dots, m\big\},$$

with respect to the norms

$$\|\varphi\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} := \left(\sum_{j=1}^{m} \|\tilde{\varphi}^t_j\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q_2)} \right)^{1/2},$$
$$\|\varphi\|_{H^{-1/2}(\mathrm{Curl},\partial\Omega)} := \left(\sum_{j=1}^{m} \|\tilde{\varphi}^T_j\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Curl},Q_2)} \right)^{1/2},$$

where

$$\tilde{\varphi}^t_j(x) := \rho_j(x) \sqrt{\chi_j\big(\Psi_j(x)\big)}\, F_j^{-1}(x)\, \varphi\big(\Psi_j(x)\big),$$
$$\tilde{\varphi}^T_j(x) := \sqrt{\chi_j\big(\Psi_j(x)\big)}\, F_j^{\top}(x)\, \varphi\big(\Psi_j(x)\big),$$

for a.a. $x \in \mathbb{B}_2(0, \alpha_j)$ *and extended by zero into* Q_2*, respectively.*

We remark that the spaces from Definition 2.45 do not depend on the local coordinate system and on the partition of unity, see [34, Corollary 5.15].

Now, we come to the trace operators. They are defined by continuous extension which is possible thanks to the following denseness result.

Theorem 2.46 *Let* $\Omega \subseteq \mathbb{R}^3$ *be given as in Assumption 2.43 and additionally be bounded. Then the following assertions are true.*

(a) The space $C^\infty(\overline{\Omega}, \mathbb{C}^{d'})$ *is dense in* $H^m(\Omega, \mathbb{C}^{d'})$*.*

(b) The space $C^\infty(\overline{\Omega}, \mathbb{C}^3)$ *is dense in* $H(\mathrm{curl}, \Omega)$*.*

For a proof we refer to [34, Theorem 5.3 and Theorem 5.19]. Though therein is only shown that $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, the idea can be applied with slight adaptions to the spaces $H^m(\Omega)$ for $m \in \mathbb{N}$ and $m > 1$ as well, see also Theorem 2.93. And again, the generalization to the case $d' > 1$ is obvious.

Corollary 2.47 *Let $\Omega \subseteq \mathbb{R}^3$ be given as in Assumption 2.43 and additionally be bounded. Furthermore, let $m, n \in \mathbb{N}$ with $m > n$. Then*

$$H^m(\Omega, \mathbb{C}^{d'}) \overset{\text{id}}{\hookrightarrow} H^n(\Omega, \mathbb{C}^{d'}),$$

with $H^m(\Omega, \mathbb{C}^{d'})$ being dense in $H^n(\Omega, \mathbb{C}^{d'})$.

Proof: This follows immediately from the definition of $H^m(\Omega, \mathbb{C}^{d'})$ and Theorem 2.46. □

Theorem 2.48 *Let $\Omega \subseteq \mathbb{R}^3$ be given as in Assumption 2.43 and additionally be bounded.*

(a) The trace operator

$$\gamma_0 : C^\infty(\overline{\Omega}, \mathbb{C}^{d'}) \to H^{1/2}(\partial\Omega, \mathbb{C}^{d'}), \quad u \to u|_{\partial\Omega},$$

has a bounded extension from $H^1(\Omega, \mathbb{C}^{d'})$ to $H^{1/2}(\partial\Omega, \mathbb{C}^{d'})$, which we also denote by γ_0. Furthermore, there exists a bounded right inverse $\eta_0 : H^{1/2}(\partial\Omega, \mathbb{C}^{d'}) \to H^1(\Omega, \mathbb{C}^{d'})$ of γ_0.

(b) The following assertions are true.

(i) The trace operator

$$\gamma_t : C^\infty(\overline{\Omega}, \mathbb{C}^3) \to H^{-1/2}(\mathrm{Div}, \partial\Omega), \quad u \to \mathrm{n} \times u|_{\partial\Omega},$$

has a bounded extension from $H(\mathrm{curl}, \Omega)$ to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$, which we also denote by γ_t. Furthermore, there exists a bounded right inverse $\eta_t : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H(\mathrm{curl}, \Omega)$ of γ_t.

(ii) The trace operator

$$\gamma_T : C^\infty(\overline{\Omega}, \mathbb{C}^3) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega), \quad u \to (\mathrm{n} \times u|_{\partial\Omega}) \times \mathrm{n},$$

has a bounded extension from $H(\mathrm{curl}, \Omega)$ to $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$, which we also denote by γ_T. Furthermore, there exists a bounded right inverse $\eta_T : H^{-1/2}(\mathrm{Curl}, \partial\Omega) \to H(\mathrm{curl}, \Omega)$ of γ_T.

For a proof we refer to [34, Theorem 5.10 and Theorem 5.24], with similar remarks as after Theorem 2.23.

Corollary 2.49 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain, with characteristic quantities as in Assumption 2.43. Then the following assertions are true.*

(a) The space

$$\mathcal{D}_0(\partial\Omega, \mathbb{C}^{d'}) := \Big\{ u|_{\partial\Omega} \;\Big|\; u \in C^\infty(\overline{\Omega}, \mathbb{C}^{d'}) \Big\}$$

is dense in $H^{1/2}(\partial\Omega, \mathbb{C}^{d'})$.

(b) The spaces

$$\mathcal{D}_t(\partial\Omega, \mathbb{C}^3) := \Big\{ \mathrm{n} \times u|_{\partial\Omega} \;\Big|\; u \in C^\infty(\overline{\Omega}, \mathbb{C}^3) \Big\},$$
$$\mathcal{D}_T(\partial\Omega, \mathbb{C}^3) := \Big\{ (\mathrm{n} \times u|_{\partial\Omega}) \times \mathrm{n} \;\Big|\; u \in C^\infty(\overline{\Omega}, \mathbb{C}^3) \Big\}$$

are dense in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$, respectively.

For a proof we refer to the proof of Corollary 2.108 which is very similar.

Moreover, further important results, such as compact embeddings, characterizations for the kernels of the trace operators, Green's formula and its consequences, can be found in [34, Section 5.1].

We close this section by pointing out *mollifiers* as a key tool when working in Sobolev spaces. In particular they are needed to prove the denseness results from above. Since we intend to derive analogous results for Q-periodic Sobolev spaces (see Section 2.2), we would like to take a closer look at this tool. For a construction of such mollifiers, we follow [34] and consider $\chi : \mathbb{R} \to \mathbb{R}$ defined by $\chi(\xi) := \mathrm{e}^{-1/\xi}$ for $\xi > 0$ and by $\chi(\xi) := 0$ for $\xi \leq 0$. Then we set

$$\phi(\xi) := C \, \frac{\chi(1-\xi^2)}{\chi(1-\xi^2) + \chi(\xi^2 - 1/4)}, \qquad \xi \in \mathbb{R},$$

with $C > 0$ chosen such that $\int_0^1 \phi(\xi^2)\, \xi^2 \,\mathrm{d}\xi = 1/(4\pi)$, and define for $\delta > 0$

$$\phi_\delta(x) := \frac{1}{\delta^3}\, \phi\left(\frac{1}{\delta^2}|x|^2\right), \qquad x \in \mathbb{R}^3. \tag{2.6}$$

Then $\phi_\delta \in C_0^\infty(\mathbb{R}^3)$ with $\mathrm{supp}(\phi_\delta) \subseteq \mathbb{B}_3[0,\delta]$ and $\int_{\mathbb{R}^3} \phi_\delta(x)\,\mathrm{d}x = 1$.

Theorem 2.50 *Let $\delta > 0$ and $a \in \mathbb{R}^3$. For $u \in L^2(\mathbb{R}^3)$ set*

$$u_a^\delta(x) := \int_{\mathbb{R}^3} u(y)\,\phi_\delta(x + \delta a - y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Then the following statements are true:

(i) $u_a^\delta \in C^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. *Moreover, if $K \subseteq \mathbb{R}^3$ is compact and if u is zero outside of K, then* $\operatorname{supp}(u_a^\delta) \subseteq K - \{\delta a\} + \mathbb{B}_3[0, \delta]$.

(ii) $\|u_a^\delta - u\|_{L^2(\mathbb{R}^3)} \to 0$, *as* $\delta \to 0$.

For a proof, we refer to [34, Theorem 4.7].

2.1.4. Functions on Bounded and Smooth Domains

Our numerical scheme for solving electromagnetic scattering problems on biperiodic surfaces is based on integral equation methods where the solution is sought in form of a potential ansatz with an unknown density. To ensure solvability of those equations, we have to impose stronger requirements on the regularity of the surface, since a key ingredient will be [21, Lemma 11]. It is the objective of this subsection to establish the connection between the setting used in [21] and the setting we have introduced in Subsection 2.1.3.

Definition 2.51 *We call an open set $\Omega \subseteq \mathbb{R}^3$ with compact boundary $\partial\Omega$ a smooth domain, if the same statements from Definition 2.41 hold, with the difference that instead of Lipschitz continuity we require now the functions $f_j \in C^\infty(\mathbb{B}_2[0, \alpha_j])$.*

Of course, smooth domains are in particular Lipschitz domains and all results obtained so far for Lipschitz domains carry over.

Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain. We introduce the same quantities as in the subsequent considerations of Definition 2.41 and observe that now the mappings $\tilde{\Psi}_j$, Ψ_j, ρ_j and F_j are smooth. Recalling also Theorem 2.42, we fix those quantities, associated with Ω, by the following assumption, see also Assumption 2.43.

Assumption 2.52 *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain with corresponding local coordinate system $\{(U_j, f_j) \mid j = 1, \dots, m\}$, corresponding mappings $\tilde{\Psi}_j$ from $\mathbb{B}_3(0, \alpha_j)$ to U_j' and their restrictions Ψ_j from $\mathbb{B}_2(0, \alpha_j)$ to $U_j' \cap \partial\Omega$. Furthermore, let $\{(U_j', \chi_j) \mid j = 1, \dots, m\}$ be a partition of unity on $\partial\Omega$ according to Theorem 2.42. And finally, let $Q_d := \bigtimes_{j=1}^{d}(-L_j, L_j)$, $d = 2, 3$, where we assume that $L_j > 0$, $j = 1, \dots, 3$, is chosen so big that all of the balls $\mathbb{B}_3(0, \alpha_j)$ are contained in Q_3.*

Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain with characteristic quantities as in Assumption 2.52. For $m \in \mathbb{N}_0$ and $d' \in \mathbb{N}$ we define

$$C^m\big(\partial\Omega, \mathbb{C}^{d'}\big) := \Big\{\varphi \in C\big(\partial\Omega, \mathbb{C}^{d'}\big) \mid \forall j \in \{1, \dots, m\} : (\chi_j \varphi) \circ \Psi_j \in C^m\big(\mathbb{B}_2(0, \alpha_j), \mathbb{C}^{d'}\big)\Big\}$$

and the corresponding space of *smooth* functions

$$C^\infty\big(\partial\Omega, \mathbb{C}^{d'}\big) := \bigcap_{k=0}^{\infty} C^k\big(\partial\Omega, \mathbb{C}^{d'}\big).$$

Moreover, we define the spaces $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ as in Definition 2.45. Concerning the spaces $H^{1/2}(\partial\Omega, \mathbb{C}^{d'})$ we allow now more regularity, see the next definition.

Definition 2.53 *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain, with characteristic quantities as in Assumption 2.52, and let $d' \in \mathbb{N}$. For $s \geq 0$ we define the space $H^s(\partial\Omega, \mathbb{C}^{d'})$ by*

$$H^s(\partial\Omega, \mathbb{C}^{d'}) := \Big\{\varphi \in L^2(\partial\Omega, \mathbb{C}^{d'}) \mid \forall j \in \{1, \dots, m\} : \tilde{\varphi}_j \in H^s_{\mathrm{per}}(Q_2, \mathbb{C}^{d'})\Big\}$$

with norm

$$\|\varphi\|_{H^s(\partial\Omega, \mathbb{C}^{d'})} := \left(\sum_{j=1}^{m} \|\tilde{\varphi}_j\|^2_{H^s_{\mathrm{per}}(Q_2, \mathbb{C}^{d'})}\right)^{1/2},$$

where for $j = 1, \dots, m$ the functions $\tilde{\varphi}_j$ are given by (2.5).
For $s > 0$ we define $H^{-s}(\partial\Omega, \mathbb{C}^{d'})$ to be the dual space of $H^s(\partial\Omega, \mathbb{C}^{d'})$ equipped with its canonical norm

$$\|\ell\|_{H^{-s}(\partial\Omega, \mathbb{C}^{d'})} := \sup_{\psi \in H^s(\partial\Omega, \mathbb{C}^{d'}) \setminus \{0\}} \frac{|\langle \ell, \psi \rangle_{s, \partial\Omega}|}{\|\psi\|_{H^s(\partial\Omega, \mathbb{C}^{d'})}}$$

for all $\ell \in H^{-s}(\partial\Omega, \mathbb{C}^{d'})$. Here, $\langle \cdot, \cdot \rangle_{s,\partial\Omega}$ denotes the duality pairing as introduced in Section 1.3, and with index "$s, \partial\Omega$" to make them distinguishable.
For $s \in \mathbb{R}$ we define the spaces of tangential vector fields by

$$H_t^s(\partial\Omega) := \Big\{ \varphi \in H^s(\partial\Omega, \mathbb{C}^3) \mid \varphi \cdot \mathrm{n} = 0 \Big\},$$

where for $s > 0$ and $\ell \in H^{-s}(\partial\Omega, \mathbb{C}^3)$ the product $\ell \cdot \mathrm{n} \in H^{-s}(\partial\Omega)$ is defined by

$$\langle \ell \cdot \mathrm{n}, \psi \rangle_{s,\partial\Omega} := \langle \ell, \psi \mathrm{n} \rangle_{s,\partial\Omega}, \qquad \psi \in H^s(\partial\Omega).$$

Note that for $s > 0$ and $\psi \in H^s(\partial\Omega)$ the product $\psi\mathrm{n}$ is well-defined by Theorem 2.55, as the normal vector n is a smooth function.

Proposition 2.54 *Let $s \in \mathbb{R}$. Then the following assertions are true.*

(i) The space $\mathcal{D}_0(\partial\Omega, \mathbb{C}^{d'})$ is dense in $H^s(\partial\Omega, \mathbb{C}^{d'})$.

(ii) If $\sigma \in \mathbb{R}$, with $\sigma < s$, then the space $H^s(\partial\Omega, \mathbb{C}^{d'})$ is compactly embedded into $H^\sigma(\partial\Omega, \mathbb{C}^{d'})$.

(iii) If $\sigma \in \mathbb{R}$, with $\sigma < s$, then the space $H_t^s(\partial\Omega)$ is embedded into $H_t^\sigma(\partial\Omega)$.

Proof: *(i)* and *(ii)*. For the case $s \geq 0$, the first assertion follows from the definition of the space $H^s(\partial\Omega, \mathbb{C}^{d'})$, the smooth parametrization Ψ and the fact that the trigonometric polynomials are dense in $H^s_{\mathrm{per}}(Q_2, \mathbb{C}^{d'})$, see Proposition 2.8. And for the case $0 \leq t < s$, the second assertions is shown similarly as in the proof of [34, Corollary 5.9] and by means of Proposition 2.13. Note that in the sense of Gelfand triples we have

$$H^s \overset{\imath_2}{\hookrightarrow} H^t \overset{\imath_1}{\hookrightarrow} L^2 \overset{\imath_1^*}{\hookrightarrow} H^{-t} \overset{\imath_2^*}{\hookrightarrow} H^{-s},$$

where $\imath_j^*$ denotes the adjoint operator of the embedding $\imath_j$, which is, by the denseness property of $\imath_j$, thus itself injective and has, by applying the same argument to $\imath_j^*$ and $\imath_j^{**} = \imath_j$ (the latter equality holds thanks to the reflexibility of Hilbert spaces), dense range as well. With these observations we have shown the assertion also for the remaining cases, if

we take additionally into account that for compact $\imath_j$ also $\imath_j^*$ is compact. For details relating to properties of Gelfand triples we refer to [53].
(iii). The assertions follows from part *(ii)* and the definition of the dot product. □

Multiplication Operators for the Trace Spaces. For Lipschitz domains the spaces $C^m(\partial\Omega, \mathbb{C}^{d'})$ are only well-defined for the case $m = 0$. Thus, consulting Theorem 2.40, the regularity of $\chi \in C(\partial\Omega, \mathbb{C}^{d'})$ is too less to define a multiplication operator in $H^{1/2}(\partial\Omega, \mathbb{C}^{d'})$, $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$, because $(\chi_j \chi) \circ \Psi_j$ is only Lipschitz continuous and therefore only in $H^1(\mathbb{B}_2(0, \alpha_j), \mathbb{C}^{d'})$, see Proposition 2.3. The situation changes if we consider smooth domains as in this subsection.

In the following presentation we assume $\Omega \subseteq \mathbb{R}^3$ to be a bounded and smooth domain.

Theorem 2.55 *Let $\chi \in C^\infty(\partial\Omega)$. Then the following assertions are true.*

(a) For $s \in \mathbb{R}$ the mapping $H^s(\partial\Omega, \mathbb{C}^{d'}) \ni \varphi \mapsto \chi\varphi \in H^s(\partial\Omega, \mathbb{C}^{d'})$ is well-defined, linear and bounded.

(b) (i) The mapping $\mathcal{D}_t(\partial\Omega, \mathbb{C}^3) \ni \varphi \mapsto \chi\varphi \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ into itself.

(ii) The mapping $\mathcal{D}_T(\partial\Omega, \mathbb{C}^3) \ni \varphi \mapsto \chi\varphi \in H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ into itself.

For a proof we refer to the proof of Theorem 2.132, which is very similar.

An Alternative Approach. To introduce the setting of [21] for smooth surfaces, we need some preparation. Recall the trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ from Theorem 2.48, which is also well-defined for the domain Ω considered here, and define for $m, d' \in \mathbb{N}$

$$\tilde{H}^{m-\frac{1}{2}}\big(\partial\Omega, \mathbb{C}^{d'}\big) := \Big\{\varphi \in L^2\big(\partial\Omega, \mathbb{C}^{d'}\big) \;\big|\; \exists u \in H^m\big(\Omega, \mathbb{C}^{d'}\big) : \varphi = \gamma_0 u\Big\},$$

equipped with the norm

$$\|\varphi\|_{\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})} := \inf\Big\{\|u\|_{H^m(\Omega,\mathbb{C}^{d'})} \;\Big|\; u \in H^m(\Omega,\mathbb{C}^{d'}), \gamma_0 u = \varphi\Big\},$$

and let $\tilde{H}^{-m+\frac{1}{2}}(\partial\Omega)$ be the dual space of $\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})$ with respect to $L^2(\partial\Omega,\mathbb{C}^{d'})$ as pivot space, see also [42, page 44]. The corresponding duality pairings will now be denoted by $\langle\cdot,\cdot\rangle^{\sim}_{m-\frac{1}{2},\partial\Omega}$.

Proposition 2.56 *Let $d' \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then the following assertions are true.*

(i) $H^{1/2}(\partial\Omega,\mathbb{C}^{d'}) \simeq \tilde{H}^{1/2}(\partial\Omega,\mathbb{C}^{d'})$ *with both sets being equal. In particular,* $H^{-1/2}(\partial\Omega,\mathbb{C}^{d'}) \simeq \tilde{H}^{-1/2}(\partial\Omega,\mathbb{C}^{d'})$

(ii) $\tilde{H}^{m+\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'}) \overset{\text{id}}{\hookrightarrow} \tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})$ *with* $\tilde{H}^{m+\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})$ *being dense in* $\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})$.

Proof: *(i).* Let $\varphi \in \tilde{H}^{1/2}(\partial\Omega,\mathbb{C}^{d'})$. Then there exists $u \in H^1(\Omega,\mathbb{C}^{d'})$ such that $\varphi = \gamma_0 u \in H^{1/2}(\partial\Omega,\mathbb{C}^{d'})$. Moreover, let $\tilde{u} \in H^1(\Omega,\mathbb{C}^{d'})$ with $\gamma_0\tilde{u} = \varphi$. Then $\|\varphi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^{d'})} \le \|\gamma_0\|\,\|\tilde{u}\|_{H^1(\Omega,\mathbb{C}^{d'})}$, which implies that

$$\frac{\|\varphi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^{d'})}}{\|\gamma_0\|} \le \inf\Big\{\|\tilde{u}\|_{H^1(\Omega,\mathbb{C}^{d'})} \;\Big|\; \tilde{u} \in H^1(\Omega,\mathbb{C}^{d'}), \gamma_0\tilde{u} = \varphi\Big\},$$

meaning that $\|\varphi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^{d'})} \le \|\gamma_0\|\,\|\varphi\|_{\tilde{H}^{1/2}(\partial\Omega,\mathbb{C}^{d'})}$.
Conversely, let $\varphi \in H^{1/2}(\partial\Omega,\mathbb{C}^{d'})$. Then $\eta_0\varphi =: u \in H^1(\Omega,\mathbb{C}^{d'})$ with $\gamma_0 u = \varphi$; here, η_0 denotes the extension operator from Theorem 2.48. Hence, $\varphi \in \tilde{H}^{1/2}(\partial\Omega,\mathbb{C}^{d'})$. Moreover,

$$\begin{aligned}\|\varphi\|_{\tilde{H}^{1/2}(\partial\Omega,\mathbb{C}^{d'})} &= \inf\Big\{\|\tilde{u}\|_{H^1(\Omega,\mathbb{C}^{d'})} \;\Big|\; \tilde{u} \in H^1(\Omega,\mathbb{C}^{d'}), \gamma_0\tilde{u} = \varphi\Big\}\\ &\le \|\eta_0\varphi\|_{H^1(\Omega,\mathbb{C}^{d'})} \le \|\eta_0\|\,\|\varphi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^{d'})}.\end{aligned}$$

To show that also the corresponding dual spaces are isomorphic, we recall that in the Gelfand triple setting, as in the proof of Proposition 2.54, the space $H^{-1/2}$ can be considered as the completion of L^2 with respect to the norm

$$\|\varphi\|_{-\frac{1}{2}} := \sup_{0\neq\psi\in H^{1/2}} \frac{|\,(\varphi\,|\,\imath_1\psi)_{L_2}\,|}{\|\psi\|_{H^{1/2}}}, \qquad \varphi \in L^2,$$

see again [53] for details. And analogous, the space $\tilde{H}^{-1/2}$ can be considered as the completion of L^2 with respect to the norm $\|\cdot\|^{\sim}_{-\frac{1}{2}}$, which is correspondingly defined and equivalent to $\|\cdot\|_{-\frac{1}{2}}$. Since two completions of a normed space with respect to two equivalent norms are isomorphic, see Proposition A.5, the proof for part *(i)* is complete.

(ii). We start with the case $m \in \mathbb{N}$. Let $\varphi \in \tilde{H}^{m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'})$. Then there exists $u \in H^{m+1}(\Omega, \mathbb{C}^{d'})$ such that $\gamma_0 u = \varphi$. By Corollary 2.47, we have also $u \in H^m(\Omega, \mathbb{C}^{d'})$, which yields that $\varphi = \gamma_0 u \in \tilde{H}^{m-\frac{1}{2}}(\Omega, \mathbb{C}^{d'})$. Observing that the space $\{u \in H^{m+1}(\Omega, \mathbb{C}^{d'}) \mid \gamma_0 u = \varphi\}$ is a subspace of the space $\{u \in H^m(\Omega, \mathbb{C}^{d'}) \mid \gamma_0 u = \varphi\}$ and that the estimate $\|u\|_{H^m(\Omega,\mathbb{C}^{d'})} \leq \|u\|_{H^{m+1}(\Omega,\mathbb{C}^{d'})}$ holds for all $u \in H^{m+1}(\Omega, \mathbb{C}^{d'})$, we see that $\|\varphi\|_{\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})}$ is a lower bound of the set $\{\|u\|_{H^{m+1}(\Omega,\mathbb{C}^{d'})} \mid u \in H^{m+1}(\Omega, \mathbb{C}^{d'}), \gamma_0 u = \varphi\}$ and therefore

$$\|\varphi\|_{\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})} \leq \|\varphi\|_{\tilde{H}^{m+\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})},$$

which proves the statement for the embedding.

To show the denseness, let $\varphi \in \tilde{H}^{m-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'})$. Then there exists $u \in H^m(\Omega, \mathbb{C}^{d'})$ such that $\gamma_0 u = \varphi$. Moreover, by Corollary 2.47, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^{m+1}(\Omega, \mathbb{C}^{d'})$ converging to u in $H^m(\Omega, \mathbb{C}^{d'})$. Consider $\varphi_n := \gamma_0 u_n \in \tilde{H}^{m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'}) \subseteq \tilde{H}^{m-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'})$, $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\|\varphi_n - \varphi\|_{\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})} \\
&\qquad = \inf\left\{\|v\|_{H^m(\Omega,\mathbb{C}^{d'})} \;\middle|\; v \in H^m(\Omega, \mathbb{C}^{d'}), \gamma_0 v = \varphi_n - \varphi\right\} \\
&\qquad \leq \|u_n - u\|_{H^m(\Omega,\mathbb{C}^{d'})} \longrightarrow 0, \quad n \to \infty.
\end{aligned}$$

For the remaining case $m \in \mathbb{Z} \setminus \mathbb{N}$ we use the same arguments for Gelfand triples as in the proof of Proposition 2.54. □

As a consequence of Proposition 2.56, for $m \in \mathbb{N}_0$ and $\ell \in \tilde{H}^{-m-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'})$ the requirement $\ell \overset{!}{\in} \tilde{H}^{-m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'})$ makes sense as follows: the linear and bounded functional $\ell : \tilde{H}^{m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'}) \to \mathbb{C}$ can now be considered as a linear and bounded functional with respect to the norm

$$\|\ell\| = \sup_{\varphi\in\tilde{H}^{m+\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})\setminus\{0\}} \frac{|\langle\ell, \varphi\rangle^{\sim}_{m+\frac{1}{2},\partial\Omega}|}{\|\varphi\|_{\tilde{H}^{m-\frac{1}{2}}(\partial\Omega,\mathbb{C}^{d'})}}$$

and therefore continuously be extended to a linear and bounded functional $\ell : \tilde{H}^{m-\frac{1}{2}}(\partial\Omega, \mathbb{C}^{d'}) \to \mathbb{C}$.

And last but not least, for $m \in \mathbb{N}_0$ and $\ell \in \tilde{H}^{-m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^3)$ the product $\ell \cdot \mathrm{n}$, where n denotes the unit normal vector on $\partial\Omega$, is for the case $m = 0$ the usual dot product for functions, while for the case $m > 0$ it has to be understood in the following way

$$\langle \ell \cdot \mathrm{n}, \psi \rangle^{\sim}_{m-\frac{1}{2},\partial\Omega} := \langle \ell, \psi\, \mathrm{n} \rangle^{\sim}_{m-\frac{1}{2},\partial\Omega}, \qquad \psi \in \tilde{H}^{m-\frac{1}{2}}(\partial\Omega).$$

Note that for $\psi \in \tilde{H}^{m-\frac{1}{2}}(\partial\Omega)$ the product $\psi\, \mathrm{n} \in \tilde{H}^{m-\frac{1}{2}}(\partial\Omega, \mathbb{C}^3)$ is well-defined by Proposition 2.56 and Theorem 2.55, as n is a smooth function on $\partial\Omega$.

Now, we are in the position to introduce the announced setting. For the next two definitions we refer to (2.3), (2.4) and (2.5) from [3], slightly modified for a more general setup.

Definition 2.57 *For $m \in \mathbb{N}_0$ and $\varphi \in \tilde{H}^{-m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^3)$, with $\mathrm{n} \cdot \varphi = 0$, we define the tangential divergence $\operatorname{div}_{\partial\Omega} \varphi \in \tilde{H}^{-m-\frac{1}{2}}(\partial\Omega)$ by*

$$\langle \operatorname{div}_{\partial\Omega} \varphi, \psi \rangle^{\sim}_{m+\frac{1}{2},\partial\Omega} := \begin{cases} -\langle \varphi, \gamma_0(\nabla \tilde{\psi}_{m+1}) \rangle^{\sim}_{m-\frac{1}{2},\partial\Omega}, & m \geq 1, \\ -\int_{\partial\Omega} \varphi \cdot \nabla \tilde{\psi}_1 \, \mathrm{d}s, & m = 0, \end{cases}$$

for all $\psi \in \tilde{H}^{m+\frac{1}{2}}(\partial\Omega)$, where $\tilde{\psi}_{m+1} \in \tilde{H}^{m+1}(\Omega)$ is any extension of ψ such that $\gamma_0 \tilde{\psi}_{m+1} = \psi$, see also the preliminary considerations above.

Definition 2.58 *For $m \in \mathbb{N}_0$ the space $H^{-m+\frac{1}{2}}(\operatorname{div}_{\partial\Omega}, \partial\Omega)$ is defined by*

$$H^{-m+\frac{1}{2}}(\operatorname{div}_{\partial\Omega}, \partial\Omega) := \Big\{ \varphi \in \tilde{H}^{-m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^3) \;\Big|\; \varphi \cdot \mathrm{n} = 0,\ \operatorname{div}_{\partial\Omega} \varphi \in \tilde{H}^{-m+\frac{1}{2}}(\partial\Omega) \Big\}$$

and equipped with the norm

$$\|\varphi\|_{H^{-m+\frac{1}{2}}(\operatorname{div}_{\partial\Omega}, \partial\Omega)} := \|\varphi\| + \|\operatorname{div}_{\partial\Omega} \varphi\|, \qquad \varphi \in H^{-m+\frac{1}{2}}(\operatorname{div}_{\partial\Omega}, \partial\Omega),$$

where the norm on the right hand side is the operator norm in the space $\tilde{H}^{-m+\frac{1}{2}}(\partial\Omega, \mathbb{C}^3)$ and $\tilde{H}^{-m+\frac{1}{2}}(\partial\Omega)$, respectively.

The embedding from [34, Lemma 5.27] turns out to be the correct candidate for establishing the connection between the space $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ from Definition 2.45 and $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ from Definition 2.58.

Theorem 2.59 *We have that* $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega) \simeq H^{-1/2}(\mathrm{Div}, \partial\Omega)$ *in virtue of the mapping* $H^{-1/2}(\mathrm{Div}, \partial\Omega) \ni \varphi \mapsto \ell_\varphi \in H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ *given by*

$$\langle \ell_\varphi, \psi \rangle_{\frac{1}{2},\partial\Omega} = \langle \varphi, \gamma_T \tilde{\psi} \rangle_{\partial\Omega}, \qquad \psi \in H^{1/2}(\partial\Omega, \mathbb{C}^3),$$

where $\tilde{\psi} \in H^1(\Omega, \mathbb{C}^3)$ *is any extension of* ψ.

Proof: First of all, recall Proposition 2.56 for an identification of the spaces $\tilde{H}^{-1/2}(\partial\Omega, \mathbb{C}^{d'})$ and $H^{-1/2}(\partial\Omega, \mathbb{C}^{d'})$.
(i). Let $\varphi \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$. For well-definedness we have to show that $\ell_\varphi \cdot \mathrm{n} = 0$ and that $\mathrm{div}_{\partial\Omega}\, \ell_\varphi \in H^{-1/2}(\partial\Omega)$. For the first one let $\psi \in H^{1/2}(\partial\Omega)$. By Corollary 2.49, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ such that $\psi_n := u_n|_{\partial\Omega} \to \psi$ in $H^{1/2}(\partial\Omega)$, for $n \to \infty$. Note that $\mathrm{n} \times (u_n|_{\partial\Omega}\, \mathrm{n}) = 0$ for all $n \in \mathbb{N}$. Let $\eta_n \in H^1(\Omega, \mathbb{C}^3)$ be an extension of $\psi_n\, \mathrm{n} \in \mathcal{D}_0(\partial\Omega, \mathbb{C}^3)$, $n \in \mathbb{N}$. Then $\gamma_T \eta_n = \mathrm{n} \times (u_n|_{\partial\Omega}\, \mathrm{n}) \times \mathrm{n} = 0$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\langle \ell_\varphi \cdot \mathrm{n}, \psi \rangle_{\frac{1}{2},\partial\Omega} &= \lim_{n\to\infty} \langle \ell_\varphi \cdot \mathrm{n}, \psi_n \rangle_{\frac{1}{2},\partial\Omega} = \lim_{n\to\infty} \langle \ell_\varphi, \psi_n\, \mathrm{n} \rangle_{\frac{1}{2},\partial\Omega} \\ &= \lim_{n\to\infty} \langle \varphi, \gamma_T \eta_n \rangle_{\partial\Omega} = 0.\end{aligned}$$

For the second one, let $\psi \in \tilde{H}^{3/2}(\partial\Omega)$. Then

$$\begin{aligned}\langle \mathrm{div}_{\partial\Omega}\, \ell_\varphi, \psi \rangle_{\frac{3}{2},\partial\Omega} &= -\langle \ell_\varphi, \gamma_0(\nabla \tilde{\psi}_2) \rangle_{\frac{1}{2},\partial\Omega} = -\langle \varphi, \gamma_T(\nabla \tilde{\psi}_2) \rangle_{\partial\Omega} \\ &= \langle \mathrm{Div}\, \varphi, \gamma_0 \tilde{\psi}_2 \rangle_{\frac{1}{2},\partial\Omega} = \langle \mathrm{Div}\, \varphi, \psi \rangle_{\frac{1}{2},\partial\Omega},\end{aligned}$$

where we have applied [34, Lemma 5.27] in the second step and [34, Definition 5.29] in the third step, with $\mathrm{Div}\, \varphi \in H^{-1/2}(\partial\Omega)$. Hence, $\mathrm{div}_{\partial\Omega}\, \ell_\varphi : \tilde{H}^{3/2}(\partial\Omega) \to \mathbb{C}$ is linear and bounded with respect to the norm in $H^{-1/2}(\partial\Omega)$ as desired.
(ii). To show boundedness, we know already from step *(i)* that $\mathrm{div}_{\partial\Omega}\, \ell_\varphi = \mathrm{Div}\, \varphi$ (considered as continuous extension) and therefore $\| \mathrm{div}_{\partial\Omega}\, \ell_\varphi \| =$

$\|\operatorname{Div}\varphi\|$, for all $\varphi \in H^{-1/2}(\operatorname{Div}, \partial\Omega)$. Hence, for $\varphi \in H^{-1/2}(\operatorname{Div}, \partial\Omega)$ and $\psi \in H^{1/2}(\partial\Omega, \mathbb{C}^3)$ we obtain, with

$$\begin{aligned}\|\nabla(\eta_0\psi)\|_{H(\operatorname{curl},\Omega)} = \|\nabla(\eta_0\psi)\|_{L^2(\Omega,\mathbb{C}^3)} &\leq \|\eta_0\psi\|_{H^1(\Omega,\mathbb{C}^3)} \\ &\leq \|\eta_0\|\,\|\psi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^3)},\end{aligned}$$

where η_0 denotes the extension operator from Theorem 2.48, that

$$\begin{aligned}|\langle \operatorname{Div}\varphi, \psi\rangle| &= |\langle \varphi, \gamma_T \nabla(\eta_0\psi)\rangle_{\partial\Omega}| \\ &\leq \|\varphi\|_{H^{-1/2}(\operatorname{Div},\partial\Omega)}\,\|\gamma_T\|\,\|\eta_0\|\,\|\psi\|_{H^{1/2}(\partial\Omega,\mathbb{C}^3)}.\end{aligned}$$

(iii). To show surjectivity, let $\varphi \in H^{-1/2}(\operatorname{div}_{\partial\Omega}, \partial\Omega)$. Thanks to [3], there exists $\tilde{\varphi} \in H(\operatorname{curl}, \Omega)$ such that $\mathrm{n} \times \tilde{\varphi}|_{\partial\Omega} = \varphi$. We set $\hat{\varphi} := \gamma_t\tilde{\varphi} \in H^{-1/2}(\operatorname{Div}, \partial\Omega)$, see Theorem 2.48. To show that $\ell_{\hat{\varphi}} = \varphi$ let $\psi \in H^{1/2}(\partial\Omega, \mathbb{C}^3)$ and set $\tilde{\psi} := \eta_0\psi \in H^1(\Omega, \mathbb{C}^3)$, with η_0 the extension operator again from Theorem 2.48. Then on the one hand, we obtain from [3, (2.1)] that

$$\langle \varphi, \psi\rangle_{\frac{1}{2},\partial\Omega} = \int_\Omega \big(\operatorname{curl}\tilde{\varphi}\cdot\tilde{\psi} - \tilde{\varphi}\cdot\operatorname{curl}\tilde{\psi}\big)\,\mathrm{d}x$$

and on the other hand, by [34, Lemma 5.27 and (5.19)], that

$$\langle \ell_{\hat{\varphi}}, \psi\rangle_{\frac{1}{2},\partial\Omega} = \langle \hat{\varphi}, \gamma_T\tilde{\psi}\rangle_{\partial\Omega} = \int_\Omega \big(\operatorname{curl}\tilde{\varphi}\cdot\tilde{\psi} - \tilde{\varphi}\cdot\operatorname{curl}\tilde{\psi}\big)\,\mathrm{d}x,$$

as desired. □

2.2. Functions on Cell Sets

Later in the formulation of our scattering problem, the scatterer will be the graph of a Q-periodic and Lipschitz continuous function, and the corresponding cell set under consideration will have a special form, see Section 2.3. Nevertheless, in this section we would like to present results which hold more generally for arbitrary cell sets.

From now on let Q denote a rectangle in $\mathbb{R}^2$, given by

$$Q := (-L_1, L_1) \times (-L_2, L_2),$$

where $L_j > 0$ are some positive real numbers, $j = 1, 2$. This rectangle will be used to pick out a unit cell of the underlying periodic media as we will see later.

2.2.1. Q-periodic Domains and Functions

For the next definitions see also [7, Section 2.1].

We recall (1.18) and (1.14) for the definition of the lattice and reciprocal lattice vector $p_{Q_d}^{(\mu)}$ and $q_{Q_d}^{(\mu)}$, respectively. To define periodicity in $\mathbb{R}^3$ only in x_1, x_2-direction with respect to Q, we similarly introduce

$$p^{(\mu)} := \begin{pmatrix} \mu_1 2L_1 \\ \mu_2 2L_2 \\ 0 \end{pmatrix} \quad \text{and} \quad q^{(\mu)} := \begin{pmatrix} \mu_1 \pi/L_1 \\ \mu_2 \pi/L_2 \end{pmatrix}, \qquad \mu \in \mathbb{Z}^2. \tag{2.7}$$

A set $\Omega \subseteq \mathbb{R}^3$ is called *Q-periodic*, if

$$x \in \Omega \qquad \Rightarrow \qquad \forall \mu \in \mathbb{Z}^2 : x + p^{(\mu)} \in \Omega.$$

Let $\Omega \subseteq \mathbb{R}^3$ be Q-periodic and $d' \in \mathbb{N}$. A function $u : \Omega \to \mathbb{C}^{d'}$ is called *Q-periodic*, if

$$u(x + p^{(\mu)}) = u(x), \qquad x \in \Omega, \quad \mu \in \mathbb{Z}^2. \tag{2.8}$$

And u is said to be *Q-quasi-periodic with phase shift $\alpha \in \mathbb{R}^3$*, if

$$u(x + p^{(\mu)}) = \mathrm{e}^{\mathrm{i}\alpha \cdot p^{(\mu)}} u(x), \qquad x \in \Omega, \quad \mu \in \mathbb{Z}^2. \tag{2.9}$$

If Ω is additionally open and if u is in $L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^{d'})$, then for Q-periodicity and Q-quasi-periodicity we require that (2.8) and (2.9) holds almost everywhere in Ω, respectively.

We note that for $x \in \Omega$, a Q-periodic function u can be (formally) expanded into a Fourier series at the plane $\mathbb{R}^2 \times \{x_3\}$

$$u(\cdot, x_3) = \sum_{\mu \in \mathbb{Z}^2} u^{(\mu)}(x_3)\, T_Q^{(\mu)} \qquad \text{on } \mathbb{R}^2,$$

with Fourier coefficients $(u^{(\mu)}(x_3))_{\mu\in\mathbb{Z}^2}$ given by

$$u^{(\mu)}(x_3) := \int_Q u(z,x_3)\, T_Q^{(-\mu)}(z)\,\mathrm{d}z, \qquad \mu \in \mathbb{Z}^2,$$

where $T_Q^{(\mu)}$ are the trigonometric monomials from Section 1.3 and where we have extended u by zero in $(Q\times\{x_3\})\cap\Omega^c$.

It is easy to see, that for a Q-quasi-periodic function u the function v given by

$$v(x) := \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}}\, u(x), \qquad x\in\Omega, \tag{2.10}$$

is Q-periodic, and vice versa. Hence, for $x\in\Omega$, a Q-quasi-periodic function u can be similarly rewritten in terms of a Fourier series expansion

$$u(\cdot,x_3) = \sum_{\mu\in\mathbb{Z}^2} u^{(\mu)}(x_3)\,\mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot\cdot}\, T_Q^{(\mu)} \qquad \text{on } \mathbb{R}^2, \tag{2.11}$$

with

$$u^{(\mu)}(x_3) = \int_Q u(z,x_3)\,\mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot z}\, T_Q^{(-\mu)}\,\mathrm{d}z, \qquad \mu\in\mathbb{Z}^2.$$

In the Q-periodic context, the notion of a *cell set* is fundamental, see the following definition.

Definition 2.60 *An open set $D\subseteq\mathbb{R}^3$ is called a cell set, if there exists a closed and Q-periodic set $\Omega\subseteq\mathbb{R}^3$ such that $\overline{D} = (\overline{Q}\times\mathbb{R})\cap\Omega$. In this case, Ω is unique and we set $E_Q(D) := \mathring{\Omega}$ to be the Q-periodic extension of D.*

Let $D\subseteq\mathbb{R}^3$ be a cell set, $d'\in\mathbb{N}$ and $m\in\mathbb{N}_0$. Similarly as in Section 1.3, we define now for the Q-periodic framework the function spaces

$$C_Q^m(D,\mathbb{C}^{d'}) := \Big\{u\in C^m(D,\mathbb{C}^{d'}) \;\Big|\; \exists\, v\in C^m(E_Q(D),\mathbb{C}^{d'}) : \\ v \text{ is } Q\text{-periodic and } u = v|_D\Big\},$$

$$C_Q^m(\overline{D},\mathbb{C}^{d'}) := \Big\{u\in C_Q^m(D,\mathbb{C}^{d'}) \;\Big|\; \forall\, 0\le|\alpha|\le m : \\ \partial^\alpha u \text{ can be continuously extended to } \overline{D}\Big\},$$

$$C_{Q,0}^m(D,\mathbb{C}^{d'}) := \Big\{u\in C^m(D,\mathbb{C}^{d'}) \;\Big|\; \exists\, v\in C^m(E_Q(D),\mathbb{C}^{d'}) :$$

$$v \text{ is } Q\text{-periodic, } \operatorname{supp}(v) \subseteq E_Q(D),$$
$$\left.\operatorname{supp}(v) \cap (\overline{Q} \times \mathbb{R}) \text{ is compact and } u = v|_D \right\}.$$

And the corresponding spaces of smooth functions are

$$C_Q^\infty(D, \mathbb{C}^{d'}) := \bigcap_{k=0}^{\infty} C_Q^k(D, \mathbb{C}^{d'}),$$

$$C_Q^\infty(\overline{D}, \mathbb{C}^{d'}) := \bigcap_{k=0}^{\infty} C_Q^k(\overline{D}, \mathbb{C}^{d'}), \qquad C_{Q,0}^\infty(D, \mathbb{C}^{d'}) := \bigcap_{k=0}^{\infty} C_{Q,0}^k(D, \mathbb{C}^{d'}).$$

Note that for $u \in C^0(D, \mathbb{C}^{d'})$, by continuity, the v in the definition of this space is unique. Thus, this holds in particular for the other subspaces. Often we will call this v the *Q-periodic extension* of u and denote it by $\widetilde{u}$. Furthermore, as before, in the names for these function spaces we will often neglect the superscript "m" if $m = 0$. And again, we will mostly drop the symbol for the co-domain in the case of scalar valued functions, i.e., for instance we will mostly write $C_{Q,0}^\infty(D)$ instead of $C_{Q,0}^\infty(D, \mathbb{C})$.

Remark 2.61 *$C_0^\infty(D, \mathbb{C}^{d'})$ is a subspace of $C_{Q,0}^\infty(D, \mathbb{C}^{d'})$. Furthermore, on a cuboid $Q_3 \subseteq \mathbb{R}^3$ we have*

$$C_0^\infty(Q_3, \mathbb{C}^{d'}) \subseteq C_{Q,0}^\infty(Q_3, \mathbb{C}^{d'}) \subseteq C_{\text{per}}^\infty(Q_3, \mathbb{C}^{d'}).$$

2.2.2. Basic Results

In this subsection we will define Sobolev spaces for Q-periodic functions on cell sets and derive their most important properties.

Definition 2.62 *Let $D \subseteq \mathbb{R}^3$ be a cell set.*

(a) For $m \in \mathbb{N}_0$ we define the space $H_Q^m(D, \mathbb{C}^{d'})$ to be

$$H_Q^m(D, \mathbb{C}^{d'}) := \Bigg\{ u \in L^2(D, \mathbb{C}^{d'}) \;\Bigg|\; \forall \alpha \in \mathbb{N}_0^3, \textit{with } |\alpha| \leq m,$$
$$\exists\, v \in L^2(D, \mathbb{C}^{d'}) \,\forall \chi \in C_{Q,0}^\infty(D) :$$

$$\left.\int_D u(x)\,\partial^\alpha\chi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_D v(x)\,\chi(x)\,\mathrm{d}x\right\}.$$

For $u \in H_Q^m(D,\mathbb{C}^{d'})$ *we set for the moment* $\partial_Q^\alpha u := v$, *see also the next remark. Furthermore, we equip this space with the inner product* $(\cdot\,|\,\cdot)_{H_Q^m(D,\mathbb{C}^{d'})}$ *and the norm* $\|\cdot\|_{H_Q^m(D,\mathbb{C}^{d'})}$ *correspondingly to Definition 2.2.*

(b) *We define the space* $H_Q(\mathrm{curl}, D)$ *to be*

$$H_Q(\mathrm{curl}, D) := \Bigg\{u \in L^2(D,\mathbb{C}^3)\;\Bigg|\;\exists\, v \in L^2(D,\mathbb{C}^3)$$

$$\left.\forall \chi \in C^\infty_{Q,0}(D,\mathbb{C}^3): \int_D u(x)\cdot \mathrm{curl}\,\chi(x)\,\mathrm{d}x = \int_D v(x)\cdot\chi(x)\,\mathrm{d}x\right\}.$$

For $u \in H_Q(\mathrm{curl}, D)$ *we set for the moment* $\mathrm{curl}_Q\, u := v$, *see also the next remark. Furthermore, we equip this space with the inner product* $(\cdot\,|\,\cdot)_{H_Q(\mathrm{curl},D)}$ *and the norm* $\|\cdot\|_{H_Q(\mathrm{curl},D)}$ *correspondingly to Definition 2.2.*

(c) *We define the space* $H_Q(\mathrm{div}, D)$ *to be*

$$H_Q(\mathrm{div}, D) := \Bigg\{u \in L^2(D,\mathbb{C}^3)\;\Bigg|\;\exists\, v \in L^2(D)$$

$$\left.\forall \chi \in C^\infty_{Q,0}(D): \int_D u(x)\cdot \nabla\chi(x)\,\mathrm{d}x = -\int_D v(x)\;\chi(x)\,\mathrm{d}x\right\}.$$

For $u \in H_Q(\mathrm{div}, D)$ *we set for the moment* $\mathrm{div}_Q\, u := v$, *see also the next remark. Furthermore, we equip this space with the inner product* $(\cdot\,|\,\cdot)_{H_Q(\mathrm{div},D)}$ *and the norm* $\|\cdot\|_{H_Q(\mathrm{div},D)}$ *correspondingly to Definition 2.2.*

Remark 2.63 *Thanks to Remark 2.61, for* u *from the space* $H_Q^m(D,\mathbb{C}^{d'})$, $H_Q(\mathrm{curl}, D)$ *and* $H_Q(\mathrm{div}, D)$, *the element* v *in the definition of those spaces is unique and coincides with* $\partial^\alpha u$, $\mathrm{curl}\, u$ *and* $\mathrm{div}\, u$ *from Definition 2.1,* *respectively, and therefore we will write again* $\partial^\alpha u$ *instead of* $\partial^\alpha_Q u$*,* $\operatorname{curl} u$ *instead of* $\operatorname{curl}_Q u$ *and* $\operatorname{div} u$ *instead of* $\operatorname{div}_Q u$*. In particular,*

$$H^m_Q(D,\mathbb{C}^{d'}) \overset{\text{id}}{\hookrightarrow} H^m(D,\mathbb{C}^{d'}),$$
$$H_Q(\operatorname{curl},D) \overset{\text{id}}{\hookrightarrow} H(\operatorname{curl},D) \quad \textit{and} \quad H_Q(\operatorname{div},D) \overset{\text{id}}{\hookrightarrow} H(\operatorname{div},D).$$

Taking Theorem 2.20 into consideration as well, we have on a cuboid $Q_3 \subseteq \mathbb{R}^3$

$$H^m_{\text{per}}(Q_3,\mathbb{C}^{d'}) \overset{\text{id}}{\hookrightarrow} H^m_Q(Q_3,\mathbb{C}^{d'}) \overset{\text{id}}{\hookrightarrow} H^m(Q_3,\mathbb{C}^{d'}),$$

with equivalent norms in $H^m_{\text{per}}(Q_3,\mathbb{C}^{d'})$ *and*

$$H_{\text{per}}(\operatorname{curl},Q_3) \overset{\text{id}}{\hookrightarrow} H_Q(\operatorname{curl},Q_3) \overset{\text{id}}{\hookrightarrow} H(\operatorname{curl},Q_3),$$

with coinciding norms in $H_{\text{per}}(\operatorname{curl},Q_3)$*.*

With the same arguments as in [34], one shows that the spaces $H^m_Q(D,\mathbb{C}^{d'})$, $H_Q(\operatorname{curl},D)$ and $H_Q(\operatorname{div},D)$ are Hilbert spaces. Moreover, note that $C^\infty_{Q,0}(D,\mathbb{C}^{d'})$ is a subspace of $H^m_Q(D,\mathbb{C}^{d'})$ and that $C^\infty_{Q,0}(D,\mathbb{C}^3)$ is a subspace of $H_Q(\operatorname{curl},D)$ and of $H_Q(\operatorname{div},D)$.

Further examples for elements in $H_Q(\operatorname{curl},D)$ are given by the next two propositions.

Proposition 2.64 *Let* $D \subseteq \mathbb{R}^3$ *be a cell set. Furthermore, let* $u \in H^1_Q(D)$ *and* $\beta \in \mathbb{C}^3$*. Then* $\beta u \in H_Q(\operatorname{curl},D)$ *with* $\operatorname{curl}(\beta u) = \nabla u \times \beta$*.*

Proof: Let $\chi \in C^\infty_{Q,0}(D,\mathbb{C}^3)$. Then, by applying (A.3d) and exploiting the fact that $u \in H^1_Q(D)$, we obtain

$$\begin{aligned}\int_D \beta u \cdot \operatorname{curl}\chi \,\mathrm{d}x &= \int_D u\,\beta \cdot \operatorname{curl}\chi\,\mathrm{d}x = \int_D u \,\operatorname{div}(\chi\times\beta)\,\mathrm{d}x \\ &= \int_D u \sum_{j=1}^3 \partial_j(\chi\times\beta)_j\,\mathrm{d}x = -\int_D \Big(\sum_{j=1}^3 (\partial_j u)\,(\chi\times\beta)_j\Big)\,\mathrm{d}x\end{aligned}$$

$$= -\int_D \nabla u \cdot (\chi \times \beta)\,\mathrm{d}x = \int_D (\nabla u \times \beta) \cdot \chi\,\mathrm{d}x$$

and the proof is complete. □

Proposition 2.65 *Let $D \subseteq \mathbb{R}^3$ be a cell set. Then we have*

$$H^1_Q(D, \mathbb{C}^3) \xhookrightarrow{\mathrm{id}} H_Q(\mathrm{curl}, D),$$

with

$$\mathrm{curl}\, u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}, \qquad u \in H^1_Q(D, \mathbb{C}^3).$$

Proof: Let $u \in H^1_Q(D, \mathbb{C}^3)$, i.e., $u_j \in H^1_Q(D)$, $j = 1, 2, 3$. Furthermore, let $\chi \in C^\infty_{Q,0}(D, \mathbb{C}^3)$. Then, using the definition of the space $H^1_Q(D)$, we obtain

$$\int_D u \cdot \mathrm{curl}\, \chi\,\mathrm{d}x = \int_D \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 \chi_3 - \partial_3 \chi_2 \\ \partial_3 \chi_1 - \partial_1 \chi_3 \\ \partial_1 \chi_2 - \partial_2 \chi_1 \end{pmatrix} \mathrm{d}x$$
$$= \int_D \Big(\partial_3 u_1\, \chi_2 - \partial_2 u_1\, \chi_3 + \partial_1 u_2\, \chi_3 - \partial_3 u_2\, \chi_1 + \partial_2 u_3\, \chi_1 - \partial_1 u_3\, \chi_2 \Big)\,\mathrm{d}x$$
$$= \int_D \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \cdot \chi\,\mathrm{d}x.$$

Hence, $u \in H_Q(\mathrm{curl}, D)$ with $\mathrm{curl}\, u$ given as in the formula from the proposition. By means of this formula it is easy to see that the identity mapping yields indeed a bounded operator from $H^1_Q(D, \mathbb{C}^3)$ to $H_Q(\mathrm{curl}, D)$, which completes the proof. □

Definition 2.66 *Let $D \subseteq \mathbb{R}^3$ be a cell set.*

(a) For $m \in \mathbb{N}_0$ we define the space $H^m_{Q,0}(D, \mathbb{C}^{d'})$ as the closure of $C^\infty_{Q,0}(D, \mathbb{C}^{d'})$ in $H^m_Q(D, \mathbb{C}^{d'})$.

(b) *We define the space* $H_{Q,0}(\mathrm{curl}, D)$ *as the closure of* $C^\infty_{Q,0}(D, \mathbb{C}^3)$ *in* $H_Q(\mathrm{curl}, D)$.

(c) *We define the space* $H_{Q,0}(\mathrm{div}, D)$ *as the closure of* $C^\infty_{Q,0}(D, \mathbb{C}^3)$ *in* $H_Q(\mathrm{div}, D)$.

Clearly, as closed subspaces of Hilbert spaces, we have that the spaces $H^m_{Q,0}(D, \mathbb{C}^{d'})$, $H_{Q,0}(\mathrm{curl}, D)$ and $H_{Q,0}(\mathrm{div}, D)$ are itself Hilbert spaces.

As we want to exploit results from the periodic setting, see Subsection 2.1.2, for this the next proposition turns out to be very helpful.

Proposition 2.67 *Let* $Q_3 \subseteq \mathbb{R}^3$ *be a cuboid as given in (1.13).*

(a) *For* $m \in \mathbb{N}_0$ *we have*

$$H^m_{Q,0}(Q_3, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} H^m_{\mathrm{per}}(Q_3, \mathbb{C}^{d'}),$$

with equivalent norms in $H^m_{Q,0}(Q_3, \mathbb{C}^{d'})$.

(b) *We have*

$$H_{Q,0}(\mathrm{curl}, Q_3) \overset{\mathrm{id}}{\hookrightarrow} H_{\mathrm{per}}(\mathrm{curl}, Q_3),$$

with coinciding norms in $H_{Q,0}(\mathrm{curl}, Q_3)$.

Proof: *(a).* Let $u \in H^m_{Q,0}(Q_3, \mathbb{C}^{d'})$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(Q_3, \mathbb{C}^{d'})$ such that $u_n \to u$ with respect to $\|\cdot\|_{H^m_Q(Q_3,\mathbb{C}^{d'})}$, as $n \to \infty$. By Remark 2.61, we know that $u_n \in \mathcal{H}^m_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$ for all $n \in \mathbb{N}$. Hence, for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le m$, Lemma 2.17 yields for the Fourier coefficients of u_n and $\partial^\alpha u_n$ the connection

$$(\partial^\alpha u_n)^{(\mu)} = \big(\mathrm{i}\, q^{(\mu)}_{Q_3}\big)^\alpha (u_n)^{(\mu)}, \qquad \mu \in \mathbb{Z}^3, \quad n \in \mathbb{N}.$$

Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le m$ and let $\mu \in \mathbb{Z}^3$. By the convergence above, we have

$$(\partial^\alpha u)^{(\mu)} = \int_{Q_3} \partial^\alpha u(x)\, T^{(-\mu)}_{Q_3}(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{Q_3} \partial^\alpha u_n(x)\, T^{(-\mu)}_{Q_3}(x)\,\mathrm{d}x$$

$$= (\mathrm{i}\, q_{Q_3}^{(\mu)})^\alpha \lim_{n\to\infty} \int_{Q_3} u_n(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x$$
$$= (\mathrm{i}\, q_{Q_3}^{(\mu)})^\alpha \int_{Q_3} u(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x = (\mathrm{i}\, q_{Q_3}^{(\mu)})^\alpha\, u^{(\mu)}.$$

Since u and $\partial^\alpha u$ belong to $L^2(Q_3, \mathbb{C}^{d'})$, we obtain by Parseval's identity

$$\|u\|^2_{L^2(Q_3,\mathbb{C}^{d'})} = \sum_{\mu\in\mathbb{Z}^3} |u^{(\mu)}|^2 \quad \text{and} \quad \|\partial^\alpha u\|^2_{L^2(Q_3,\mathbb{C}^{d'})} = \sum_{\mu\in\mathbb{Z}^3} (q_{Q_3}^{(\mu)})^{2\alpha} |u^{(\mu)}|^2.$$

Now, we are in a position to regain the last two chains of inequalities from part *(a)* in the proof of Proposition 2.19, where now therein the interchange of the sum signs is allowed due to convergence of the series with respect to μ. Thus, we have shown that $u \in H^m_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$ and that the norms $\|\cdot\|_{H^m_{\mathrm{per}}(Q_3,\mathbb{C}^{d'})}$ and $\|\cdot\|_{H^m_Q(Q_3,\mathbb{C}^{d'})}$ in $H^m_{Q,0}(Q_3, \mathbb{C}^{d'})$ are equivalent.
(b). Let $u \in H_{Q,0}(\mathrm{curl}, Q_3)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(Q_3, \mathbb{C}^3)$ such that $u_n \to u$ with respect to $\|\cdot\|_{H_Q(\mathrm{curl},Q_3)}$, as $n \to \infty$, and analogous to part *(a)* we have for the Fourier coefficients of u_n and $\mathrm{curl}\, u_n$ the connection

$$(\mathrm{curl}\, u_n)^{(\mu)} = \mathrm{i}\, q_{Q_3}^{(\mu)} \times (u_n)^{(\mu)}, \qquad \mu \in \mathbb{Z}^3, \quad n \in \mathbb{N}.$$

Let $\mu \in \mathbb{Z}^3$. By the convergence above, we have

$$(\mathrm{curl}\, u)^{(\mu)} = \int_{Q_3} \mathrm{curl}\, u(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{Q_3} \mathrm{curl}\, u_n(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x$$
$$= \mathrm{i}\, q_{Q_3}^{(\mu)} \times \lim_{n\to\infty} \int_{Q_3} u_n(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x$$
$$= \mathrm{i}\, q_{Q_3}^{(\mu)} \times \int_{Q_3} u(x)\, T_{Q_3}^{(-\mu)}(x)\,\mathrm{d}x = \mathrm{i}\, q_{Q_3}^{(\mu)} \times u^{(\mu)}.$$

Since u and $\mathrm{curl}\, u$ belong to $L^2(Q_3, \mathbb{C}^3)$, we obtain by Parseval's identity

$$\|u\|^2_{L^2(Q_3,\mathbb{C}^3)} = \sum_{\mu\in\mathbb{Z}^3} |u^{(\mu)}|^2 \quad \text{and} \quad \|\mathrm{curl}\, u\|^2_{L^2(Q_3,\mathbb{C}^3)} = \sum_{\mu\in\mathbb{Z}^3} |q_{Q_3}^{(\mu)} \times u^{(\mu)}|^2.$$

From this we conclude finally $\|u\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} = \|u\|_{H_Q(\mathrm{curl},Q_3)} < \infty$, and the proof is complete. □

Restriction and Extension (by Zero) Operators. We continue with the introduction of certain restriction and extension (by zero) operators. Although they appear mostly in the background, their work should not be underestimated. For their notation recall Section 1.3, page 11.

Let $D, D' \subseteq \mathbb{R}^3$ be cell sets such that $D \subsetneq D'$. Since $u \in L^2(D, \mathbb{C}^{d'})$ implies that $\|u|_0^{D'}\|_{L^2(D',\mathbb{C}^{d'})} = \|u\|_{L^2(D,\mathbb{C}^{d'})}$ and, conversely, since $u \in L^2(D', \mathbb{C}^{d'})$ implies that $\|u|_D\|_{L^2(D,\mathbb{C}^{d'})} \leq \|u\|_{L^2(D',\mathbb{C}^{d'})}$, there holds that the mappings

$$L^2(D, \mathbb{C}^{d'}) \xrightarrow{\cdot|_0^{D'}} L^2(D', \mathbb{C}^{d'}) \quad \text{and} \quad L^2(D', \mathbb{C}^{d'}) \xrightarrow{\cdot|_D} L^2(D, \mathbb{C}^{d'})$$

are linear and bounded. The next proposition shows a similar result for the spaces from Definition 2.62 and Definition 2.66.

Proposition 2.68 *Let $D, D' \subseteq \mathbb{R}^3$ be cell sets such that $D \subsetneq D'$.*

(a) The following assertions are true.

(i) The mapping $H_Q^m(D', \mathbb{C}^{d'}) \ni u \mapsto u|_D \in H_Q^m(D, \mathbb{C}^{d'})$ is well-defined, linear and bounded with

$$\|u|_D\|_{H_Q^m(D,\mathbb{C}^{d'})} \leq \|u\|_{H_Q^m(D',\mathbb{C}^{d'})}.$$

Moreover, $\partial^\alpha(u|_D) = (\partial^\alpha u)|_D$ for all $u \in H_Q^m(D', \mathbb{C}^{d'})$ and all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$.

(ii) The mapping $H_{Q,0}^m(D, \mathbb{C}^{d'}) \ni u \mapsto u|_0^{D'} \in H_{Q,0}^m(D', \mathbb{C}^{d'})$ is well-defined, linear and bounded with

$$\|u|_0^{D'}\|_{H_Q^m(D',\mathbb{C}^{d'})} = \|u\|_{H_Q^m(D,\mathbb{C}^{d'})}.$$

Moreover, $\partial^\alpha(u|_0^{D'}) = (\partial^\alpha u)|_0^{D'}$ for all $u \in H_{Q,0}^m(D, \mathbb{C}^{d'})$ and all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$.

(b) The following assertions are true.

(i) The mapping $H_Q(\mathrm{curl}, D') \ni u \mapsto u|_D \in H_Q(\mathrm{curl}, D)$ is well-defined, linear and bounded with

$$\|u|_D\|_{H_Q(\mathrm{curl},D)} \leq \|u\|_{H_Q(\mathrm{curl},D')}.$$

Moreover, $\mathrm{curl}(u|_D) = (\mathrm{curl}\, u)|_D$ for $u \in H_Q(\mathrm{curl}, D')$.

(ii) The mapping $H_{Q,0}(\mathrm{curl}, D) \ni u \mapsto u|_0^{D'} \in H_{Q,0}(\mathrm{curl}, D')$ *is well-defined, linear and bounded with*

$$\|u|_0^{D'}\|_{H_Q(\mathrm{curl},D')} = \|u\|_{H_Q(\mathrm{curl},D)}.$$

Moreover, $\mathrm{curl}(u|_0^{D'}) = (\mathrm{curl}\, u)|_0^{D'}$ *for all* $u \in H_{Q,0}(\mathrm{curl}, D)$.

Proof: We only show the assertion for part *(b)* as the argumentation for part (a) is completely analogous.
(i). Let $u \in H_Q(\mathrm{curl}, D')$ and $\chi \in C^\infty_{Q,0}(D, \mathbb{C}^3)$. Then $\chi|_0^{D'} \in C^\infty_{Q,0}(D', \mathbb{C}^3)$ and

$$\begin{aligned}\int_D u|_D(x) \cdot \mathrm{curl}\, \chi(x)\, \mathrm{d}x &= \int_{D'} u(x) \cdot \mathrm{curl}\, \chi|_0^{D'}(x)\, \mathrm{d}x \\ &= \int_{D'} \mathrm{curl}\, u(x) \cdot \chi|_0^{D'}(x)\, \mathrm{d}x = \int_D (\mathrm{curl}\, u)|_D(x) \cdot \chi(x)\, \mathrm{d}x.\end{aligned}$$

Moreover,

$$\begin{aligned}\|u|_D\|^2_{H_Q(\mathrm{curl},D)} &= \int_D |u|_D(x)|^2\, \mathrm{d}x + \int_D |\mathrm{curl}(u|_D)(x)|^2\, \mathrm{d}x \\ &= \int_D |u(x)|^2\, \mathrm{d}x + \int_D |\mathrm{curl}\, u(x)|^2\, \mathrm{d}x \\ &\leq \int_{D'} |u(x)|^2\, \mathrm{d}x + \int_{D'} |\mathrm{curl}\, u(x)|^2\, \mathrm{d}x = \|u\|^2_{H_Q(\mathrm{curl},D')}.\end{aligned}$$

(ii). Let $u \in H_{Q,0}(\mathrm{curl}, D)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(D, \mathbb{C}^3)$ converging with respect to $\|\cdot\|_{H_Q(\mathrm{curl},D)}$ to u. Note that $u_n|_0^{D'} \in C^\infty_{Q,0}(D', \mathbb{C}^3)$ with $\mathrm{curl}(u_n|_0^{D'}) = (\mathrm{curl}\, u_n)|_0^{D'}$ for all $n \in \mathbb{N}$. Choose some $\chi \in C^\infty_{Q,0}(D', \mathbb{C}^3)$. Then, using the convergence from above and the definition of the variational curl, we obtain

$$\begin{aligned}\int_{D'} (u|_0^{D'}) \cdot \mathrm{curl}\, \chi\, \mathrm{d}x &= \lim_{n\to\infty} \int_{D'} u_n|_0^{D'} \cdot \mathrm{curl}\, \chi\, \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{D'} \mathrm{curl}(u_n|_0^{D'}) \cdot \chi\, \mathrm{d}x = \lim_{n\to\infty} \int_{D'} (\mathrm{curl}\, u_n)|_0^{D'} \cdot \chi\, \mathrm{d}x \\ &= \int_{D'} (\mathrm{curl}\, u)|_0^{D'} \cdot \chi\, \mathrm{d}x,\end{aligned}$$

i.e., $u|_0^{D'} \in H_Q(\operatorname{curl}, D')$ with $\operatorname{curl}(u|_0^{D'}) = (\operatorname{curl} u)|_0^{D'}$. Furthermore,

$$\begin{aligned}\|u|_0^{D'}\|^2_{H_Q(\operatorname{curl},D')} &= \|u|_0^{D'}\|^2_{L^2(D',\mathbb{C}^3)} + \|\operatorname{curl}(u|_0^{D'})\|^2_{L^2(D',\mathbb{C}^3)} \\ &= \|u\|^2_{L^2(D,\mathbb{C}^3)} + \|\operatorname{curl} u\|^2_{L^2(D,\mathbb{C}^3)} = \|u\|^2_{H_Q(\operatorname{curl},D)}.\end{aligned}$$

Applying the last chain of equations to the difference $u_n|_0^{D'} - u|_0^{D'}$, we see that $(u_n|_0^{D'})_{n\in\mathbb{N}}$ converges in $H_Q(\operatorname{curl}, D')$ to $u|_0^{D'}$ and the proof is complete. □

Compact Embeddings. In the context of compact embeddings, another extension operator for the vector valued case is needed, which extends functions from a certain subspace of $H_Q(\operatorname{curl}, D)$ by zero to cuboids. For the introduction of this subspace $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$ in the next definition and for the statements in the next proposition, it might be useful to recall (1.20b) and the definition of the space $H_{\mathrm{per}}(\operatorname{curl}, \operatorname{div}_\beta 0, Q_3)$ from Definition 2.12. Moreover, as a preparation for later purposes, see also the beginning of Subsection 2.2.3, we introduce this subspace already here in its most general form, that is, with respect to a matrix-weighted L^2-inner product similarly as in [34].

Definition 2.69 *Let $D \subseteq \mathbb{R}^3$ be a cell set. Suppose $A \in L^\infty(D, \mathbb{C}^{3\times 3})$ such that $A(x)$ is symmetric for a.a. $x \in D$ and $\operatorname{Re}(\overline{z}^\top A(x)z) \geq c|z|^2$ for all $z \in \mathbb{C}^3$, a.a. $x \in D$ and some constant $c > 0$. Furthermore, let $\beta \in \mathbb{R}^3$.*

(i) The space $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$ is defined by

$$H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D) := \Big\{ u \in H_{Q,0}(\operatorname{curl}, D) \;\Big|\; \forall \psi \in H^1_{Q,0}(D) : (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = 0 \Big\}.$$

(ii) The space $H_Q(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$ is defined by

$$H_Q(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D) := \Big\{ u \in H_Q(\operatorname{curl}, D) \;\Big|\; \forall \psi \in H^1_{Q,0}(D) : (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = 0 \Big\}.$$

If $\beta = 0$ and $A = I$, where I denotes the identity matrix, then we will drop the subscript β and A in the symbol $H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D)$ and $H_Q(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D)$, respectively.

Proposition 2.70 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set and let $L_3 > 0$ such that $D \subseteq Q \times (-L_3, L_3) =: Q_3$. Then the mapping*

$$H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \ni u \;\mapsto\; u|_0^{Q_3} \in H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3)$$

is well-defined, linear and bounded with

$$\|u|_0^{Q_3}\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} = \|u\|_{H_Q(\mathrm{curl},D)}.$$

Proof: Let $u \in H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D)$. From Proposition 2.68 and Proposition 2.67 we know that $u|_0^{Q_3}$ belongs to $H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ and that the equality for the norms hold. It remains to show that $(u|_0^{Q_3})^{(\mu)} \cdot (q_{Q_3}^{(\mu)} + \beta) = 0$ for all $\mu \in \mathbb{Z}^3$, where $(u|_0^{Q_3})^{(\mu)}$ denote the Fourier coefficients of $u|_0^{Q_3}$. For this let $\psi \in H^1_{Q,0}(D)$. By the propositions from above, together with Theorem 2.20 and Lemma 2.17, we have between the Fourier coefficients of $\psi|_0^{Q_3}$ and $\nabla(\psi|_0^{Q_3})$ the connection $[\nabla(\psi|_0^{Q_3})]^{(\nu)} = \mathrm{i} q_{Q_3}^{(\nu)} (\psi|_0^{Q_3})^{(\nu)}$ for all $\nu \in \mathbb{Z}^3$. Note that, again thanks to Proposition 2.68, there holds $(\nabla_\beta \psi)|_0^{Q_3} = \nabla_\beta(\psi|_0^{Q_3})$. Hence,

$$\begin{aligned} 0 = (u \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} &= \Big(u|_0^{Q_3} \,\Big|\, \nabla_\beta(\psi|_0^{Q_3})\Big)_{L^2(Q_3,\mathbb{C}^3)} \\ &= \sum_{\mu,\nu \in \mathbb{Z}^3} (u|_0^{Q_3})^{(\mu)} \cdot \overline{\mathrm{i}(q_{Q_3}^{(\nu)} + \beta)(\psi|_0^{Q_3})^{(\nu)}} \Big(T_{Q_3}^{(\mu)} \,\Big|\, T_{Q_3}^{(\nu)}\Big)_{L^2(Q_3,\mathbb{C}^3)}. \end{aligned}$$

Since $\psi \in H^1_{Q,0}(D)$ was arbitrary, we conclude from the last equality that $(u|_0^{Q_3})^{(\mu)} \cdot (q_{Q_3}^{(\mu)} + \beta) = 0$ for all $\mu \in \mathbb{Z}^3$, as desired. □

Now, we come to the first compactness result in the Q-periodic setting.

Theorem 2.71 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set and $\beta \in \mathbb{R}^3$. Then the following assertions are true.*

(a) The embedding $H^m_{Q,0}(D, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} L^2(D, \mathbb{C}^{d'})$ is compact.

(b) The embedding $H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \overset{\mathrm{id}}{\hookrightarrow} L^2(D, \mathbb{C}^3)$ *is compact.*

Proof: Choose some $L_3 > 0$ such that $D \subseteq Q \times (-L_3, L_3) =: Q_3$.
(a). We decompose the embedding from the theorem by

$$H^m_{Q,0}(D, \mathbb{C}^{d'}) \xrightarrow{\cdot|_0^{Q_3}} H^m_{Q,0}(Q_3, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} H^m_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$$
$$\overset{\mathrm{id}}{\hookrightarrow} L^2(Q_3, \mathbb{C}^{d'}) \xrightarrow{\cdot|_D} L^2(D, \mathbb{C}^{d'})$$

and observe that the first, second and last mapping are bounded, thanks to Propositon 2.68, Proposition 2.67 and the remarks in front of Proposition 2.68, respectively. Now the assertion follows, as the embedding from $H^m_{\mathrm{per}}(Q_3, \mathbb{C}^{d'})$ to $L^2(Q_3, \mathbb{C}^{d'})$ is compact, see Proposition 2.13.
(b). We decompose the embedding from the theorem by

$$H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \xrightarrow{\cdot|_0^{Q_3}} H_{\mathrm{per}}(\mathrm{curl}, \mathrm{div}_\beta\, 0, Q_3)$$
$$\overset{\mathrm{id}}{\hookrightarrow} L^2(Q_3, \mathbb{C}^3) \xrightarrow{\cdot|_D} L^2(D, \mathbb{C}^3),$$

which yields, with analogous arguments as in *(a)*, in particular with Propostition 2.70, the assertion. □

Friedrich's Inequality. It is well-known that in $H^1_0(\Omega)$ there holds an *inequality of Friedrich's type*, saying that $\|u\|_{L^2(\Omega)} \le c\, \|\nabla u\|_{L^2(\Omega, \mathbb{C}^3)}$ for all $u \in H^1_0(\Omega)$. Here Ω denotes any open and bounded subset of $\mathbb{R}^d$, and the constant $c > 0$ is independent of u. In the Q-periodic setting, i.e. in $H^1_{Q,0}(D)$, where $D \subseteq \mathbb{R}^3$ is a bounded cell set, such an inequality can be derived as well, see the next theorem. Therein a slightly more general version is proven which turns out to be more useful later when we transfer problems from the Q-quasi-periodic into the Q-periodic setting and therefore obtain modified operators for ∇, curl and div as in (1.20a) and (1.20b).

Theorem 2.72 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set and $\beta \in \mathbb{R}^3$. Then there exists $c > 0$ such that*

$$\|u\|_{L^2(D)} \le c\, \|\nabla_\beta u\|_{L^2(D, \mathbb{C}^3)}, \quad \textit{for all } u \in H^1_{Q,0}(D).$$

Here, the operator ∇_β is given by $\nabla_\beta = \nabla + \mathrm{i}\beta$, see also (1.20a).

Proof: We follow the lines in the proof of [34, Theorem 4.15]. Choose some $L_3 > 0$ such that $D \subseteq Q \times (-L_3, L_3) =: Q_3$. Let $u \in C^\infty_{Q,0}(D)$. Extend u by zero to Q_3 and choose some $x \in Q_3$. Then

$$\begin{aligned} e^{i\beta\cdot x} u(x) &= e^{i\beta\cdot(x_1,x_2,-L_3)^\top} \underbrace{u(x_1,x_2,-L_3)}_{=0} + \int_{-L_3}^{x_3} \partial_3(e^{i\beta\cdot\cdot} u)(x_1,x_2,\xi)\,d\xi \\ &= \int_{-L_3}^{x_3} e^{i(\beta_1 x_1+\beta_2 x_2+\beta_3\xi)} \big(\partial_3 u(x_1,x_2,\xi) + i\beta_3 u(x_1,x_2,\xi)\big)\,d\xi, \end{aligned}$$

and by the inequality of Cauchy-Schwarz

$$\begin{aligned} |u(x)|^2 &\le (x_3+L_3) \int_{-L_3}^{x_3} \big|\partial_3 u(x_1,x_2,\xi) + i\beta_3 u(x_1,x_2,\xi)\big|^2\,d\xi \\ &\le 2L_3 \int_{-L_3}^{L_3} \big|\partial_3 u(x_1,x_2,\xi) + i\beta_3 u(x_1,x_2,\xi)\big|^2\,d\xi. \end{aligned}$$

Hence,

$$\int_{-L_3}^{L_3} |u(x)|^2\,dx_3 \le (2L_3)^2 \int_{-L_3}^{L_3} \big|\partial_3 u(x_1,x_2,\xi) + i\beta_3 u(x_1,x_2,\xi)\big|^2\,d\xi,$$

and integration with respect to x_1 and x_2 yields

$$\begin{aligned} \|u\|^2_{L^2(D)} &= \|u\|^2_{L^2(Q_3)} \\ &\le (2L_3)^2 \int_{-L_3}^{L_3}\int_{-L_3}^{L_3}\int_{-L_3}^{L_3} \big|\partial_3 u(x_1,x_2,\xi) + i\beta_3 u(x_1,x_2,\xi)\big|^2\,d\xi\,dx_2\,dx_1 \\ &\le (2L_3)^2 \|\nabla u + i\beta u\|^2_{L^2(Q_3,\mathbb{C}^3)} = (2L_3)^2 \|\nabla_\beta u\|^2_{L^2(D,\mathbb{C}^3)}. \end{aligned}$$

Since $C^\infty_{Q,0}(D)$ is dense in $H^1_{Q,0}(D)$, the last inequality holds even for all $u \in H^1_{Q,0}(D)$. □

Corollary 2.73 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set and $\beta \in \mathbb{R}^3$. Then there exists $c > 0$ such that*

$$\|u\|_{H^1_Q(D)} \le c\,\|\nabla_\beta u\|_{L^2(D,\mathbb{C}^3)}, \quad \textit{for all } u \in H^1_{Q,0}(D).$$

Proof: Let $u \in H^1_{Q,0}(D)$. Then an application of the triangle inequality yields $\|\nabla u\|_{L^2(D,\mathbb{C}^3)} \le \|\nabla u + \mathrm{i}\beta u\|_{L^2(D,\mathbb{C}^3)} + \|\mathrm{i}\beta u\|_{L^2(D,\mathbb{C}^3)}$, and we conclude, together with Theorem 2.72,

$$\begin{aligned}\|u\|^2_{H^1_Q(D)} &= \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D,\mathbb{C}^3)}\\ &\le \|u\|^2_{L^2(D)} + 2\|\nabla u + \mathrm{i}\beta u\|^2_{L^2(D,\mathbb{C}^3)} + 2|\beta|^2\|u\|^2_{L^2(D)}\\ &\le c\,\|\nabla u + \mathrm{i}\beta u\|^2_{L^2(D,\mathbb{C}^3)},\end{aligned}$$

as asserted. □

In the proof of the next proposition we have a first application of Friedrich's inequality. The following lemma has preliminary character.

Lemma 2.74 *Let $D \subseteq \mathbb{R}^3$ be a cell set and $\beta \in \mathbb{R}^3$. Furthermore, let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $H^1_{Q,0}(D)$ such that $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D)$ and $(\nabla_\beta p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D,\mathbb{C}^3)$. Then for the L^2-limits $p := \lim_{n\to\infty} p_n$ and $u := \lim_{n\to\infty} \nabla_\beta p_n$ there holds $p \in H^1_{Q,0}(D)$ with $\nabla_\beta p = u$.*

Proof: We know that $\nabla p_n + \mathrm{i}\beta p_n \to u$ and $\mathrm{i}\beta p_n \to \mathrm{i}\beta p$, both in $L^2(D,\mathbb{C}^3)$, as $n \to \infty$. Hence, $\nabla p_n = (\nabla p_n + \mathrm{i}\beta p_n) - \mathrm{i}\beta p_n \to u - \mathrm{i}\beta p$ in $L^2(D,\mathbb{C}^3)$. Let $\chi \in C^\infty_{Q,0}(D)$. Then

$$\begin{aligned}\int_D p\,\nabla\chi\,\mathrm{d}x &= \lim_{n\to\infty}\int_D p_n\nabla\chi\,\mathrm{d}x = -\lim_{n\to\infty}\int_D \nabla p_n\chi\,\mathrm{d}x\\ &= -\int_D (u - \mathrm{i}\beta p)\,\chi\,\mathrm{d}x,\end{aligned}$$

showing that $p \in H^1_Q(D)$ with $\nabla p = u - \mathrm{i}\beta p$, where the latter one is equivalent to $\nabla_\beta p = u$. Moreover, $\|p_n - p\|^2_{L^2(D)} + \|\nabla p_n - \nabla p\|^2_{L^2(D,\mathbb{C}^3)} \to 0$, that is, $p_n \to p$ with respect to $\|\cdot\|_{H^1_Q(D)}$. Since $H^1_{Q,0}(D)$ is a closed subspace of $H^1_Q(D)$, p belongs indeed to $H^1_{Q,0}(D)$. □

Proposition 2.75 *Let $D \subseteq \mathbb{R}^3$ be a cell set and $\beta \in \mathbb{R}^3$. Then the following assertions are true.*

(i) There holds

$$\psi \in H^1_Q(D) \Rightarrow \nabla_\beta \psi \in H_Q(\mathrm{curl}, D), \ \textit{with} \ \mathrm{curl}(\nabla_\beta \psi) = \nabla\psi \times \mathrm{i}\beta.$$

(ii) There holds

$$\nabla_\beta H^1_{Q,0}(D) \subseteq H_{Q,0}(\mathrm{curl}, D).$$

If D is additionally bounded, then $\nabla_\beta H^1_{Q,0}(D)$ is a closed subspace of $H_{Q,0}(\mathrm{curl}, D)$.

Proof: *(i)*. Let $\psi \in H^1_Q(D)$. Take some $\chi \in C^\infty_{Q,0}(D, \mathbb{C}^3)$. Note that $(\mathrm{curl}\,\chi)_j \in C^\infty_{Q,0}(D)$ for all $j = 1, 2, 3$. Moreover, $\nabla\psi \times \mathrm{i}\beta \in L^2(D, \mathbb{C}^3)$. Then, by the definition of the variational derivative, on the one hand

$$\begin{aligned}\int_D \nabla\psi \cdot \mathrm{curl}\,\chi \,\mathrm{d}x &= \sum_{j=1}^3 \int_D \partial_j \psi \,(\mathrm{curl}\,\chi)_j \,\mathrm{d}x \\ &= -\sum_{j=1}^3 \int_D \psi\, \partial_j (\mathrm{curl}\,\chi)_j \,\mathrm{d}x = -\int_D \psi \,\mathrm{div}(\mathrm{curl}\,\chi)\,\mathrm{d}x = 0,\end{aligned}$$

and on the other hand

$$\begin{aligned}\int_D \mathrm{i}\beta\psi \cdot \mathrm{curl}\,\chi \,\mathrm{d}x &= \mathrm{i}\int_D \psi \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} \cdot \begin{pmatrix}\partial_2\chi_3 - \partial_3\chi_2\\ \partial_3\chi_1 - \partial_1\chi_3\\ \partial_1\chi_2 - \partial_2\chi_1\end{pmatrix} \mathrm{d}x \\ &= \mathrm{i}\int_D (\beta_1\chi_2\partial_3\psi - \beta_1\chi_3\partial_2\psi + \beta_2\chi_3\partial_1\psi - \beta_2\chi_1\partial_3\psi \\ &\qquad + \beta_3\chi_1\partial_2\psi - \beta_3\chi_2\partial_1\psi)\,\mathrm{d}x \\ &= \mathrm{i}\int_D \begin{pmatrix}\beta_3\partial_2\psi - \beta_2\partial_3\psi\\ \beta_1\partial_3\psi - \beta_3\partial_1\psi\\ \beta_2\partial_1\psi - \beta_1\partial_2\psi\end{pmatrix} \cdot \chi \,\mathrm{d}x = \int_D (\nabla\psi \times \mathrm{i}\beta)\cdot\chi\,\mathrm{d}x.\end{aligned}$$

Combining both results yields the assertion.
(ii). Let $\psi \in H^1_{Q,0}(D)$. From part *(i)* we know that $\nabla_\beta\psi \in H_Q(\mathrm{curl}, D)$ with $\mathrm{curl}(\nabla_\beta\psi) = \nabla\psi \times \mathrm{i}\beta$. Moreover, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(D)$, converging to ψ with respect to $\|\cdot\|_{H^1_Q(D)}$. Note that $\nabla_\beta\psi_n \in C^\infty_{Q,0}(D, \mathbb{C}^3)$ and $\mathrm{curl}(\nabla_\beta\psi_n) = \nabla\psi_n \times \mathrm{i}\beta$ for all $n \in \mathbb{N}$. Therefore, $\mathrm{curl}(\nabla_\beta\psi_n) \to \mathrm{curl}(\nabla_\beta\psi)$ in $L^2(D, \mathbb{C}^3)$, as $n \to \infty$, and we obtain

$$\|\nabla_\beta\psi_n - \nabla_\beta\psi\|_{L^2(D,\mathbb{C}^3)} + \|\,\mathrm{curl}(\nabla_\beta\psi_n) - \mathrm{curl}(\nabla_\beta\psi)\|_{L^2(D,\mathbb{C}^3)} \to 0.$$

Hence, $(\nabla_\beta \psi)_{n\in\mathbb{N}}$ converges with respect to $\|\cdot\|_{H_Q(\mathrm{curl},D)}$ to $\nabla_\beta \psi$, which shows that $\nabla_\beta \psi$ belongs to $H_{Q,0}(\mathrm{curl}, D)$.
To show the closedness property, let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $H^1_{Q,0}(D)$ such that $\nabla_\beta p_n \to u$ in $H_{Q,0}(\mathrm{curl}, D)$, as $n \to \infty$. In particular, $(\nabla_\beta p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D, \mathbb{C}^3)$. Therefore, by Friedrich's inequality from Theorem 2.72, $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D)$ and thus convergent to some $p \in L^2(D)$. We have to show that $p \in H^1_{Q,0}(D)$ and that $\nabla_\beta p = u$. But this follows immediately from Lemma 2.74. □

Lipschitz Continuous Transformations. In the next section we will introduce cell sets of Lipschitz layer type and derive in this setting trace and extension operators. As mentioned before, this is done by exploiting results from Subsection 2.1.2. For this a necessary tool is a transformation which maps Q-periodic functions defined on such cell sets to periodic functions defined on Q_3. In this context, Q-periodic and Lipschitz continuous functions are again important examples for elements in $H^1_Q(D)$, if D is additionally bounded.

Proposition 2.76 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set. Furthermore, let $v : E_Q(D) \to \mathbb{C}^{d'}$ be Q-periodic and Lipschitz continuous. Then $u := v|_D \in H^1_Q(D, \mathbb{C}^{d'})$.*
Moreover, $\partial_j u$ in the variational sense coincides almost everywhere with the almost everywhere given partial derivative $\partial_j u$ in the classical sense.

Proof: The proof follows exactly the lines from the proof of Proposition 2.3 if we replace therein Ω by D, C_0^∞ by $C_{Q,0}^\infty$ and $\mathbb{R}^d$ by $Q \times \mathbb{R}$, and if we take additionally into account that the boundary terms on $\overline{D} \cap (\partial Q \times \mathbb{R})$, for the case that this set is not empty, cancel out each other due to the Q-periodicity of the integrands. If this set is empty, then we are exactly in the situation of Proposition 2.3. □

Corollary 2.77 *Let D, $D' \subseteq \mathbb{R}^3$ be cell sets, D' additionally be bounded, $u \in C_Q^\infty(D, \mathbb{C}^{d'})$ and $\widetilde{u}$ its Q-periodic extension. Furthermore, let the mapping $\tilde{\Phi} : E_Q(D') \to E_Q(D)$ be Lipschitz continuous such that $\widetilde{v} := \widetilde{u} \circ \tilde{\Phi}$*

is Q-periodic. Then $v := \widetilde{v}|_{D'} \in H^1_Q(D', \mathbb{C}^{d'})$. In particular, in the case $d' = 1$ its variational gradient ∇v is given by

$$\nabla v = (\tilde{\Phi}')^\top [(\nabla u) \circ \tilde{\Phi}]$$

and exists almost everywhere in D' in the classical sense.

Proof: Since $\widetilde{v} : E_Q(D') \to \mathbb{C}^{d'}$ is Q-periodic and Lipschitz continuous, the assertion follows directly from Proposition 2.76. □

Product Rules. Later we will often multiply elements from $H^m_Q(D, \mathbb{C}^{d'})$, $H_Q(\mathrm{curl}, D)$ and $H_Q(\mathrm{div}, D)$ with certain cut-off functions to derive further results. For this purpose, the following presentation will be of special interest.

Proposition 2.78 *Let $D \subseteq \mathbb{R}^3$ be a cell set. Furthermore, let $\psi \in C^\infty_Q(D)$ be bounded. Then the following assertions are true.*

(a) Let $m \in \mathbb{N}_0$. If $u \in H^m_Q(D, \mathbb{C}^{d'})$, then the product $\psi\, u \in H^m_Q(D, \mathbb{C}^{d'})$ and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$ there holds Leibniz' product rule

$$\partial^\alpha(\psi\, u) = \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta} \partial^{\alpha-\beta}\psi\, \partial^\beta u.$$

(b) If $u \in H_Q(\mathrm{curl}, D)$, then the product $\psi\, u \in H_Q(\mathrm{curl}, D)$ and

$$\mathrm{curl}(\psi\, u) = \nabla\psi \times u + \psi\, \mathrm{curl}\, u.$$

(c) If $u \in H_Q(\mathrm{div}, D)$, then the product $\psi\, u \in H_Q(\mathrm{div}, D)$ and

$$\mathrm{div}(\psi\, u) = \psi\, \mathrm{div}\, u + u \cdot \nabla\psi.$$

In particular, for fixed and bounded $\psi \in C^\infty_Q(D)$, the multiplication by ψ is a linear and bounded operator in $H^m_Q(D, \mathbb{C}^{d'})$, $H_Q(\mathrm{curl}, D)$ and $H_Q(\mathrm{div}, D)$, respectively.

Proof: *(a).* We only show the assertion for the case $d' = 1$ since the generalization to the case $d' > 1$ is obvious.
In contrast to the corresponding proof of Proposition 2.21, here we cannot use an approximation argument so far. Therefore, we show the assertion by induction with respect to m. In fact, for $m = 1$ and $j \in \{1, 2, 3\}$ we obtain for $\chi \in C^\infty_{Q,0}(D)$ that $\partial_j(\psi\chi) = \chi\partial_j\psi + \psi\partial_j\chi$ and thus, since $\partial_j(\psi\chi) \in C^\infty_{Q,0}(D)$,

$$\begin{aligned}\int_D (\psi\, u)\, \partial_j\chi \,\mathrm{d}x &= \int_D u\, \partial_j(\psi\, \chi) \,\mathrm{d}x - \int_D \partial_j\psi\, u\, \chi \,\mathrm{d}x \\ &= -\int_D (\psi\, \partial_j u + u\, \partial_j\psi)\, \chi \,\mathrm{d}x.\end{aligned}$$

For the inductive step from m to $m+1$, we at first observe that for $\hat{\alpha} \in \mathbb{N}_0^3$ with $|\hat{\alpha}| = m$, for $\beta \in \mathbb{N}_0^3$ with $\beta \leq \hat{\alpha}$, for $j \in \{1, 2, 3\}$ and for

$$\begin{aligned}\beta' &:= (\beta_1, \ldots, \beta_{j-1}, 0, \beta_{j+1}, \ldots, \beta_3)^\top, \\ \beta'' &:= (\beta_1, \ldots, \beta_{j-1}, \hat{\alpha}_j + 1, \beta_{j+1}, \ldots, \beta_3)^\top,\end{aligned}$$

there holds, with $e^{(j)}$ denoting the j-th unit coordinate vector in $\mathbb{R}^3$,

$$\begin{aligned}&\sum_{\beta_j=0}^{\hat{\alpha}_j} \tbinom{\hat{\alpha}_j}{\beta_j} \left[\partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi + \partial^{\hat{\alpha}-\beta} u\, \partial^{\beta+e^{(j)}}\psi\right] \\ &= \sum_{\beta_j=0}^{\hat{\alpha}_j} \tbinom{\hat{\alpha}_j}{\beta_j} \partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi + \sum_{\beta_j=1}^{\hat{\alpha}_j+1} \tbinom{\hat{\alpha}_j}{\beta_j-1} \partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi \\ &= \partial^{\hat{\alpha}+e^{(j)}-\beta'} u\, \partial^{\beta'}\psi + \sum_{\beta_j=1}^{\hat{\alpha}_j} \left[\tbinom{\hat{\alpha}_j}{\beta_j} + \tbinom{\hat{\alpha}_j}{\beta_j-1}\right] \partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi \\ &\qquad + \partial^{\hat{\alpha}+e^{(j)}-\beta''} u\, \partial^{\beta''}\psi \\ &= \sum_{\beta_j=0}^{\hat{\alpha}_j+1} \tbinom{\hat{\alpha}_j+1}{\beta_j} \partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi,\end{aligned}$$

where we have used that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. Now, we assume that the assertion is true for some $m \in \mathbb{N}_0$. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = m + 1$

and write $\alpha = \hat{\alpha} + e^{(j)}$ with $j \in \{1,2,3\}$. Note that $|\hat{\alpha}| = m$. Let $\chi \in C^\infty_{Q,0}(D)$. Then, with $\beta \in \mathbb{N}_0^3$ such that $\beta \leq \hat{\alpha}$, with $j \in \{1,2,3\}$ and with $\partial_j(\chi\,\partial^\beta\psi) = \chi\,\partial^{\beta+e^{(j)}}\psi + \partial^\beta\psi\,\partial_j\chi$, we obtain

$$\begin{aligned}
\int_D \psi\, u\, \partial^\alpha \chi \,\mathrm{d}x &= \int_D \psi\, u\, \partial^{\hat{\alpha}}(\partial_j\chi)\,\mathrm{d}x \\
&= (-1)^{|\hat{\alpha}|} \int_D \Big(\sum_{\beta\leq\hat{\alpha}} \tbinom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha}-\beta} u\, \partial^\beta\psi \Big)\, \partial_j\chi\,\mathrm{d}x \\
&= (-1)^{|\hat{\alpha}|+1} \int_D \sum_{\beta\leq\hat{\alpha}} \tbinom{\hat{\alpha}}{\beta} \Big[\partial^{\hat{\alpha}+e^{(j)}-\beta} u\, \partial^\beta\psi + \partial^{\hat{\alpha}-\beta} u\, \partial^{\beta+e^{(j)}}\psi \Big]\, \chi\,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_D \sum_{\beta\leq\alpha} \Big(\tbinom{\alpha}{\beta} \partial^{\alpha-\beta} u\, \partial^\beta\psi \Big) \chi\,\mathrm{d}x,
\end{aligned}$$

where we have applied the observation from above in the last step. For this see also (1.9).

(b). The proof follows exactly the lines in the proof of Proposition 2.21 if we replace therein $C^\infty_{\mathrm{per}}(Q_3, \mathbb{C}^3)$ by $C^\infty_{Q,0}(D, \mathbb{C}^3)$.

(c). Let $\chi \in C^\infty_{Q,0}(D)$. Then $\psi\chi \in C^\infty_{Q,0}(D)$ and, with $\nabla(\psi\chi) = \psi\nabla\chi + \chi\nabla\psi$, we obtain

$$\begin{aligned}
\int_D (\psi u)\cdot\nabla\chi\,\mathrm{d}x &= \int_D u\cdot\psi\nabla\chi\,\mathrm{d}x \\
&= -\int_D (\operatorname{div} u)\,\psi\chi\,\mathrm{d}x - \int_D u\cdot\chi\nabla\psi\,\mathrm{d}x \\
&= -\int_D (\psi\,\operatorname{div} u + u\cdot\nabla\psi)\,\chi\,\mathrm{d}x,
\end{aligned}$$

as asserted.

And finally, the linearity and boundedness of the multiplication operators are easy to see. □

Functions with Compact Support. Later when we multiply elements from $H^m_Q(D, \mathbb{C}^{d'})$, $H_Q(\mathrm{curl}, D)$ and $H_Q(\mathrm{div}, D)$ with certain cut-off functions, the product will often again be multiplied with another cut-off function. As a consequence, we will then end up in situations where the latter product vanishes in a neighborhood of $\partial E_Q(D)$. For those situations the following results will be useful, in particular the integration by parts formulas.

In the following definition we specify the spaces for functions with compact support.

Definition 2.79 *Let $D \subseteq \mathbb{R}^3$ be a cell set and $m \in \mathbb{N}$.*

(a) The subspace $\mathcal{H}^m_{Q,0}(D, \mathbb{C}^{d'})$ of $H^m_Q(D, \mathbb{C}^{d'})$ is defined by

$$\mathcal{H}^m_{Q,0}(D, \mathbb{C}^{d'}) := \Big\{ u \in H^m_Q(D, \mathbb{C}^{d'}) \;\Big|\; \textit{there exists a bounded cell set } \Omega \subseteq D \textit{ such that } \overline{E_Q(\Omega)} \subseteq E_Q(D) \textit{ and } \operatorname{supp}(u) \subseteq \overline{\Omega} \Big\}.$$

(b) The subspace $\mathcal{H}_{Q,0}(\mathrm{curl}, D)$ of $H_Q(\mathrm{curl}, D)$ is defined by

$$\mathcal{H}_{Q,0}(\mathrm{curl}, D) := \Big\{ u \in H_Q(\mathrm{curl}, D) \;\Big|\; \textit{there exists a bounded cell set } \Omega \subseteq D \textit{ such that } \overline{E_Q(\Omega)} \subseteq E_Q(D) \textit{ and } \operatorname{supp}(u) \subseteq \overline{\Omega} \Big\}.$$

(c) The subspace $\mathcal{H}_{Q,0}(\mathrm{div}, D)$ of $H_Q(\mathrm{div}, D)$ is defined by

$$\mathcal{H}_{Q,0}(\mathrm{div}, D) := \Big\{ u \in H_Q(\mathrm{div}, D) \;\Big|\; \textit{there exists a bounded cell set } \Omega \subseteq D \textit{ such that } \overline{E_Q(\Omega)} \subseteq E_Q(D) \textit{ and } \operatorname{supp}(u) \subseteq \overline{\Omega} \Big\}.$$

Lemma 2.80 *Let Ω, $D \subseteq \mathbb{R}^3$ be cell sets such that Ω is bounded and $\overline{E_Q(\Omega)} \subseteq E_Q(D)$. Then there exists $\chi \in C^\infty_{Q,0}(D)$ with $\chi(x) = 1$ for all $x \in \Omega$.*

Proof: By compactness of $\overline{\Omega}$ and by Q-periodicity of $E_Q(\Omega)$ and $E_Q(D)$, there exists $\varepsilon > 0$ such that $\operatorname{dist}\big(E_Q(\Omega), \partial E_Q(D)\big) > \varepsilon$. Let $\phi_{\frac{\varepsilon}{4}}$ be the mollifier from (2.6) and set

$$\tilde{\chi}(x) := \int_{E_Q(\Omega) + \mathbb{B}_3(0, \frac{\varepsilon}{4})} \phi_{\frac{\varepsilon}{4}}(x - y)\, \mathrm{d}y, \qquad x \in E_Q(D).$$

Then $\operatorname{supp}(\tilde{\chi}) \subseteq \overline{E_Q(\Omega) + \mathbb{B}_3(0, \frac{\varepsilon}{2})}$ and $\tilde{\chi}(x) = 1$ for all $x \in E_Q(\Omega)$. Hence, $\chi := \tilde{\chi}|_D$ has the desired properties. □

Lemma 2.81 *Let $D \subseteq \mathbb{R}^3$ be a cell set. Then the following assertions are true.*

(i) $u \in \mathcal{H}_{Q,0}(\operatorname{curl}, D) \quad \Rightarrow \quad \int_D \operatorname{curl} u(x)\,\mathrm{d}x = 0.$

(ii) $u \in \mathcal{H}_{Q,0}(\operatorname{div}, D) \quad \Rightarrow \quad \int_D \operatorname{div} u(x)\,\mathrm{d}x = 0.$

Proof: We only show the assertion for part *(i)*, as for part *(ii)* the same arguments can be applied.
By assumption, there exists a bounded cell set $\Omega \subseteq D$ such that $\overline{E_Q(\Omega)} \subseteq E_Q(D)$ and $\operatorname{supp}(u) \subseteq \overline{\Omega}$. Furthermore, due to Lemma 2.80, there exists $\chi_0 \in C^\infty_{Q,0}(D)$ with $\chi_0(x) = 1$ for all $x \in \Omega$. Let $e^{(j)}$ denote the j-th unit coordinate vector in $\mathbb{R}^3$ and let $(\operatorname{curl} u)_j$ be the j-th component of u, $j = 1, 2, 3$. Let $j \in \{1, 2, 3\}$ and set $\chi(x) := e^{(j)}\chi_0(x)$, for $x \in D$. Then $\chi \in C^\infty_{Q,0}(D, \mathbb{C}^3)$ with $\operatorname{curl} \chi = 0$ on Ω, and therefore

$$\begin{aligned}\int_D (\operatorname{curl} u)_j(x)\,\mathrm{d}x &= \int_D \operatorname{curl} u(x) \cdot \chi(x)\,\mathrm{d}x = \int_D u(x) \cdot \operatorname{curl} \chi(x)\,\mathrm{d}x \\ &= \int_{\operatorname{supp}(u)} u(x) \cdot \operatorname{curl} \chi(x)\,\mathrm{d}x = 0.\end{aligned}$$

From this the assertion follows. □

Now, we are in a position to prove the following integration by parts formulas.

Proposition 2.82 *Let $D \subseteq \mathbb{R}^3$ be a cell set and $\psi \in C^\infty_Q(D)$ be bounded. Then the following assertions are true.*

(a) If $u \in H^m_Q(D, \mathbb{C}^{d'})$ and $\psi\, u \in \mathcal{H}^m_{Q,0}(D, \mathbb{C}^{d'})$, then for all $\alpha \in \mathbb{N}^3_0$ with $|\alpha| \leq m$ there holds

$$\int_D \partial^\alpha u(x)\,\psi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_D u(x)\,\partial^\alpha \psi(x)\,\mathrm{d}x.$$

(b) If $u \in H_Q(\mathrm{curl}, D)$ *and* $\psi\, u \in \mathcal{H}_{Q,0}(\mathrm{curl}, D)$*, then*

$$\int_D \psi(x)\; \mathrm{curl}\, u(x)\, \mathrm{d}x = -\int_D \nabla\psi(x) \times u(x)\, \mathrm{d}x.$$

(c) If $u \in H_Q(\mathrm{div}, D)$ *and* $\psi\, u \in \mathcal{H}_{Q,0}(\mathrm{div}, D)$*, then*

$$\int_D \psi(x)\; \mathrm{div}\, u(x)\, \mathrm{d}x = -\int_D \nabla\psi(x) \cdot u(x)\, \mathrm{d}x.$$

Proof: *(a).* By assumption, there exists a bounded cell set $\Omega \subseteq D$ such that $\overline{E_Q(\Omega)} \subseteq E_Q(D)$ and $\mathrm{supp}(\psi\, u) \subseteq \overline{\Omega}$. Furthermore, due to Lemma 2.80, there exists $\chi \in C^\infty_{Q,0}(D)$ with $\chi(x) = 1$ for all $x \in \Omega$. In particular, $\psi\,\chi \in C^\infty_{Q,0}(D)$. Now, let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le m$. Then, applying Leibniz' product rule,

$$\begin{aligned} \int_D \partial^\alpha u(x)\, \psi(x)\, \mathrm{d}x &= \int_D \partial^\alpha u(x)\, \psi(x)\, \chi(x)\, \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_D u(x)\, \partial^\alpha \psi(x)\, \chi(x)\, \mathrm{d}x \\ &\qquad + (-1)^{|\alpha|} \sum_{0 \ne \beta \le \alpha} \binom{\alpha}{\beta} \int_\Omega u(x)\, \partial^{\alpha-\beta}\psi(x)\, \underbrace{\partial^\beta \chi(x)}_{=0}\, \mathrm{d}x. \end{aligned}$$

Taking now into account that for the integral in the left summand only the set Ω is relevant and that $\chi \equiv 1$ on Ω, we have shown the assertion.
(b), (c). The assertion follows immediately by combining the statements from Proposition 2.78 and Lemma 2.81. □

The next proposition shows how the spaces $\mathcal{H}^m_{Q,0}(D)$ and $\mathcal{H}_{Q,0}(\mathrm{curl}, D)$ are related to the spaces $H^m_{Q,0}(D)$ and $H_{Q,0}(\mathrm{curl}, D)$, respectively.

Proposition 2.83 *Let* $D \subseteq \mathbb{R}^3$ *be a cell set. Then the following assertions are true.*

(a) $\mathcal{H}^m_{Q,0}(D, \mathbb{C}^{d'}) \subseteq H^m_{Q,0}(D, \mathbb{C}^{d'})$.

(b) $\mathcal{H}_{Q,0}(\mathrm{curl}, D) \subseteq H_{Q,0}(\mathrm{curl}, D)$.

Proof: We start with part *(b)* and postpone part *(a)* to the end of this proof.

(b). Let $u \in \mathcal{H}_{Q,0}(\mathrm{curl}, D)$. By assumption, there exists a bounded cell set $\Omega \subseteq D$ such that $\overline{E_Q(\Omega)} \subseteq E_Q(D)$ and $\mathrm{supp}(u) \subseteq \overline{\Omega}$. Due to the compactness of $\overline{\Omega}$ and the Q-periodicity of $E_Q(\Omega)$ and $E_Q(D)$, there exists $\delta > 0$ such that $\mathrm{dist}\big(E_Q(\Omega), \partial E_Q(D)\big) > \delta$. Let $0 < \varepsilon < \delta/3$ and set $u_\varepsilon := \widetilde{u}_\varepsilon|_D$, where

$$\widetilde{u}_\varepsilon(x) := \int_\Omega \widetilde{\phi}_\varepsilon(x-y)\, u(y)\,\mathrm{d}y, \qquad x \in E_Q(D),$$

and $\widetilde{\phi}_\varepsilon$ is the Q-periodic extension of the mollifier given in (2.6). Note that $\widetilde{u}_\varepsilon$ is a smooth Q-periodic function with $\widetilde{u}_\varepsilon(x) = 0$ for $x \in E_Q(D) \setminus \big(E_Q(\Omega) + \mathbb{B}_3(0, \frac{\delta}{3})\big)$. Hence, $u_\varepsilon \in C^\infty_{Q,0}(D, \mathbb{C}^3)$.

Let $x \in D$. Then $\widetilde{\phi}_\varepsilon(x - \bullet) \in C^\infty_Q(D)$ and by part *(b)* of Proposition 2.82

$$\begin{aligned}\mathrm{curl}\, u_\varepsilon(x) &= \int_\Omega \nabla_x \widetilde{\phi}_\varepsilon(x-y) \times u(y)\,\mathrm{d}y = -\int_\Omega \nabla_y \widetilde{\phi}_\varepsilon(x-y) \times u(y)\,\mathrm{d}y \\ &= \int_\Omega \widetilde{\phi}_\varepsilon(x-y)\ \mathrm{curl}\, u(y)\,\mathrm{d}y =: (\mathrm{curl}\, u)_\varepsilon(x).\end{aligned}$$

Now, let $\widetilde{D} := \bigcup_{|\mu|_\infty \le 1} \big(\{p^{(\mu)}\} + D\big)$ be the union of D and its eight neighbors from $E_Q(D)$, where $p^{(\mu)}$ is the lattice vector given in (2.7). Moreover, let $v \in \{u, \mathrm{curl}\, u\}$ and $\widetilde{v}$ be the Q-periodic extension of v from D to $\widetilde{D}$ and then be extended by zero to $\mathbb{R}^3 \setminus \widetilde{D}$. And finally set

$$\widetilde{v}_\varepsilon(x) := \int_{\mathbb{R}^3} \phi_\varepsilon(x-y)\, \widetilde{v}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Then $\widetilde{v}_\varepsilon(x) = v_\varepsilon(x)$ for all $x \in D$ and moreover $\widetilde{v} \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ with $\widetilde{v} = v$ almost everywhere in D. Then from Theorem 2.50 we conclude that $\widetilde{v}_\varepsilon \to \widetilde{v}$ in $L^2(\mathbb{R}^3, \mathbb{C}^3)$ and therefore in particular $v_\varepsilon = \widetilde{v}_\varepsilon|_D \to \widetilde{v}|_D = v$ in $L^2(D, \mathbb{C}^3)$, as $\varepsilon \to 0$. This means that $u_\varepsilon \to u$ with respect to $\|\cdot\|_{H(\mathrm{curl}, D)}$, as $\varepsilon \to 0$, and the proof for part *(b)* is complete.

(a). The proof follows very closely the lines from part *(b)*, where we now cite part *(a)* from Proposition 2.82. The details are omitted. □

2.2.3. Helmholtz Decompositions

In the next section, when we introduce cell sets $D \subseteq \mathbb{R}^3$ of Lipschitz layer type, we will need a compactness result as in Theorem 2.71, but now for a "divergence free" subset of functions from the space $H_Q(\mathrm{curl}, D)$ which do not have to vanish at the boundary. To derive such a result, we will apply the curl-preserving transformation from the next section to trace back to cuboids and to exploit already derived results therein. As we will see later, this transformation will force us to consider matrix-weighted L^2-inner products, similarly as in [34]. Therefore, we have to improve the result from Theorem 2.71 by allowing any matrix valued function $A \in L^\infty(D, \mathbb{C}^{3\times 3})$ in the sense of the setting from Definition 2.69. This is one reason for the introduction of the Helmholtz decompositions here. Another reason is its importance in the context of proofs for unique solvability of boundary value problems in variational form in the space $H_Q(\mathrm{curl}, D)$, see for instance Theorem 4.14.

Definition 2.84 *Let $D \subseteq \mathbb{R}^3$ be a cell set. Suppose $A \in L^\infty(D, \mathbb{C}^{3\times 3})$ such that $A(x)$ is symmetric for a.a. $x \in D$ and $\mathrm{Re}(\overline{z}^\top A(x) z) \geq c|z|^2$ for all $z \in \mathbb{C}^3$, a.a. $x \in D$ and some constant $c > 0$. Furthermore, let $\beta \in \mathbb{R}^3$. The space $L^2(\mathrm{div}_{\beta,A}\, 0, D)$ is defined by*

$$L^2(\mathrm{div}_{\beta,A}\, 0, D) := \Big\{ u \in L^2(D, \mathbb{C}^3) \;\Big|\; \forall \psi \in H^1_{Q,0}(D) : (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = 0 \Big\}.$$

If $\beta = 0$ and $A = I$, where I denotes the identity matrix, then we will drop the subscript β and A in the symbol $L^2(\mathrm{div}_{\beta,A}\, 0, D)$, respectively.

For the direct sum in the next theorem recall (1.10).

Theorem 2.85 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set and $\beta \in \mathbb{R}^3$. Then*

(i) $L^2(D, \mathbb{C}^3) = L^2(\mathrm{div}_{\beta,A}\, 0, D) \oplus \nabla_\beta H^1_{Q,0}(D)$,

(ii) $H_{Q,0}(\mathrm{curl}, D) = H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D) \oplus \nabla_\beta H^1_{Q,0}(D)$,

(iii) $H_Q(\mathrm{curl}, D) = H_Q(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D) \oplus \nabla_\beta H^1_{Q,0}(D)$.

Remark 2.86 *The Helmholtz decompositions from Theorem 2.85 have to be read in the following way: $u = \tilde{u} + v$, where v is of the form $v = \nabla_\beta p$ with $p \in H^1_{Q,0}(D)$ uniquely determined!*

Remark 2.87 *Provided X is a Banach space, it is a well-known result from functional analysis that to a direct sum $X = U \oplus V$ there corresponds a unique projection $P \in \mathcal{L}(X)$ such that $P(X) = U$ and $(I - P)(X) = V$.*

Proof: (Theorem 2.85) *(i).* We show that $L^2(\operatorname{div}_{\beta,A} 0, D)$ is a closed subspace of $L^2(D, \mathbb{C}^3)$. In fact, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in the space $L^2(\operatorname{div}_{\beta,A} 0, D)$, converging to some $u \in L^2(D, \mathbb{C}^3)$ with respect to the norm $\|\cdot\|_{L^2(D,\mathbb{C}^3)}$. Note that $Au_n \to Au$ in $L^2(D, \mathbb{C}^3)$, as $n \to \infty$. Let $\psi \in H^1_{Q,0}(D)$. Then

$$(Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = \lim_{n\to\infty} (Au_n \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3} = 0.$$

We show that $\nabla_\beta H^1_{Q,0}(D)$ is a closed subspace of $L^2(D, \mathbb{C}^3)$. In fact, let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $H^1_{Q,0}(D)$ such that $\nabla_\beta p_n \to u$ in $L^2(D, \mathbb{C}^3)$. In particular, $(\nabla_\beta p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D, \mathbb{C}^3)$. Therefore, by Friedrich's inequality from Theorem 2.72, $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D)$ and thus convergent to some $p \in L^2(D)$. We have to show that $p \in H^1_{Q,0}(D)$ and that $\nabla_\beta p = u$. But this follows immediately from Lemma 2.74.

We show that $L^2(\operatorname{div}_{\beta,A} 0, D) \cap \nabla_\beta H^1_{Q,0}(D) = \{0\}$. In fact, let u belong to the intersection of $L^2(\operatorname{div}_{\beta,A} 0, D)$ and $\nabla_\beta H^1_{Q,0}(D)$. Then $u = \nabla_\beta \psi$ with $\psi \in H^1_{Q,0}(D)$ and we obtain

$$\begin{aligned} 0 = \operatorname{Re} (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} &= \operatorname{Re} \int_D \overline{\nabla_\beta \psi(x)}^\top A(x) \nabla_\beta \psi(x) \, \mathrm{d}x \\ &\geq c \int_D |\nabla_\beta \psi(x)|^2 \, \mathrm{d}x = c \, \|\nabla_\beta \psi\|^2_{L^2(D,\mathbb{C}^3)}. \end{aligned}$$

Hence, $\nabla_\beta \psi = 0$, and with Friedrich's inequality we obtain $\psi = 0$, i.e., $u = 0$.

And finally, we show that $L^2(D, \mathbb{C}^3) \subseteq L^2(\operatorname{div}_{\beta,A} 0, D) + \nabla_\beta H^1_{Q,0}(D)$. In fact, let $u \in L^2(D, \mathbb{C}^3)$. Consider the sesquilinear form $a : H^1_{Q,0}(D) \times H^1_{Q,0}(D) \to \mathbb{C}$ and the linear functional $\ell : H^1_{Q,0}(D) \to \mathbb{C}$ given by

$$a(\psi, p) := (\nabla_\beta \psi \,|\, A \nabla_\beta p)_{L^2(D,\mathbb{C}^3)} \quad \text{and} \quad \ell(\psi) := (\nabla_\beta \psi \,|\, Au)_{L^2(D,\mathbb{C}^3)} \,.$$

Then $|\ell(\psi)| \leq \|\nabla_\beta \psi\|_{L^2(D,\mathbb{C}^3)} \|Au\|_{L^2(D,\mathbb{C}^3)} \leq c\,\|\psi\|_{H^1_Q(D)}$ and analogously $|a(\psi,p)| \leq c\,\|\psi\|_{H^1_Q(D)} \|p\|_{H^1_Q(D)}$ for all $\psi, p \in H^1_{Q,0}(D)$. Furthermore,

$$\begin{aligned}\operatorname{Re} a(\psi,\psi) &= \operatorname{Re} \int_D (\nabla_\beta \psi(x))^\top\, \overline{A(x)\nabla_\beta \psi(x)}\, \mathrm{d}x \\ &\geq C \int_D |\nabla_\beta \psi(x)|^2\, \mathrm{d}x \geq C\,\|\psi\|^2_{H^1_Q(D)},\end{aligned}$$

for all $\psi \in H^1_{Q,0}(D)$, where we have applied Corollary 2.73. Therefore, we are in the situation of Theorem A.8 and obtain a unique $p \in H^1_{Q,0}(D)$ such that $(A\nabla_\beta p \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)}$ for all $\psi \in H^1_{Q,0}(D)$. Set $\tilde{u} := u - \nabla_\beta p$. Then, for arbitrary $\psi \in H^1_{Q,0}(D)$, we arrive at

$$(A\tilde{u} \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} - (A\nabla_\beta p \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = 0.$$

(ii). We show that $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$ is a closed subspace of the space $H_{Q,0}(\operatorname{curl}, D)$. In fact, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$, converging to some $u \in H_{Q,0}(\operatorname{curl}, D)$ with respect to $\|\cdot\|_{H_Q(\operatorname{curl},D)}$. In particular, $u_n \to u$ and $Au_n \to Au$, both in $L^2(D,\mathbb{C}^3)$. Let $\psi \in H^1_{Q,0}(D)$. Then

$$(Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = \lim_{n\to\infty} (Au_n \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3} = 0.$$

We have to show that $\nabla_\beta H^1_{Q,0}(D)$ is a closed subspace of $H_{Q,0}(\operatorname{curl}, D)$. But this was already done in Proposition 2.75
We have to show that $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D) \cap \nabla_\beta H^1_{Q,0}(D) = \{0\}$. But this follows with exactly the same arguments as in the corresponding step from part *(i)*.
We have to show that $H_{Q,0}(\operatorname{curl}, D) \subseteq H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D) + \nabla_\beta H^1_{Q,0}(D)$. In fact, let $u \in H_{Q,0}(\operatorname{curl}, D)$. We consider again the sesquilinear form $a : H^1_{Q,0}(D) \times H^1_{Q,0}(D) \to \mathbb{C}$ and the linear functional $\ell : H^1_{Q,0}(D) \to \mathbb{C}$ given by

$$a(\psi,p) := (\nabla_\beta \psi \,|\, A\nabla_\beta p)_{L^2(D,\mathbb{C}^3)} \quad \text{and} \quad \ell(\psi) := (\nabla_\beta \psi \,|\, Au)_{L^2(D,\mathbb{C}^3)}$$

and repeat the arguments from the corresponding step in part *(i)* to obtain a unique $p \in H^1_{Q,0}(D)$ such that $(A\nabla_\beta p \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = (Au \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)}$ for all $\psi \in H^1_{Q,0}(D)$. We set again $\tilde{u} := u - \nabla_\beta p$ and easily check that $\tilde{u}$ belongs to $H_{Q,0}(\operatorname{curl}, \operatorname{div}_{\beta,A} 0, D)$.

(iii). The assertion is proven very similar to part *(ii).* We ommit the details. □

Now, we are in a position to improve the result from Theorem 2.71 by allowing any matrix valued function $A \in L^\infty(D, \mathbb{C}^{3\times 3})$ as in the setting of Definition 2.69.

Theorem 2.88 *Let $D \subseteq \mathbb{R}^3$ be a bounded cell set. Then the embedding $H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D) \overset{\mathrm{id}}{\hookrightarrow} L^2(D, \mathbb{C}^3)$ is compact.*

Proof: We follow the lines in the proof of [34, Lemma 5.31]. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\beta,A}\, 0, D)$. We decompose $u_n = \tilde{u}_n + \nabla_\beta p_n$ according to

$$H_{Q,0}(\mathrm{curl}, D) = H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \oplus \nabla_\beta H^1_{Q,0}(D),$$

for all $n \in \mathbb{N}$, and have, thanks to Remark 2.87, that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D)$ with respect to $\|\cdot\|_{H_Q(\mathrm{curl},D)}$. From Theorem 2.71 we conclude that there exists a subsequence (denoted by the same symbol) such that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is convergent in $L^2(D, \mathbb{C}^3)$. On the other hand, $\tilde{u}_n = u_n - \nabla_\beta p_n$ is just the decomposition with respect to

$$L^2(D, \mathbb{C}^3) = L^2(\mathrm{div}_{\beta,A}\, 0, D) \oplus \nabla_\beta H^1_{Q,0}(D).$$

Taking again Remark 2.87 into account, we conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D, \mathbb{C}^3)$ and thus convergent therein. □

2.3. Functions on Cell Sets of Lipschitz Layer Type

In this section we focus on cell sets of special type, which will naturally arise later when we pose scattering problems on Q-periodic surfaces. Moreover, for those cell sets we can introduce trace and extension operators by only slightly modifying the concepts in [34].

2.3.1. Geometrical Setting and First Consequences

Definition 2.89 *We call an open set $D \subseteq \mathbb{R}^3$ a cell set of Lipschitz layer type, if D is a cell set and if there exist Q-periodic and Lipschitz-continuous functions $f_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 0, 1$, such that $f_0(\xi) < f_1(\xi)$ for all $\xi \in \mathbb{R}^2$ and*

$$E_Q(D) = \big\{x \in \mathbb{R}^3 \mid f_0(\tilde{x}) < x_3 < f_1(\tilde{x})\big\}.$$

Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with corresponding functions f_j, $j = 0, 1$, as in Definition 2.89. We set

$$\widetilde{\Gamma}_j := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in \mathbb{R}^2 \text{ and } x_3 = f_j(\tilde{x})\big\}, \qquad j = 0, 1, \tag{2.12a}$$

$$\Gamma_j := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = f_j(\tilde{x})\big\}, \qquad j = 0, 1, \tag{2.12b}$$

$$\Gamma_j^\varepsilon := D \cap \big(\widetilde{\Gamma}_j + \mathbb{B}_3(0, \varepsilon)\big), \quad j = 0, 1, \quad 0 < \varepsilon < \operatorname{dist}(\Gamma_0, \Gamma_1). \tag{2.12c}$$

We continue with the following result, which transfers the notion of a partition of unity to the Q-periodic setting. This tool plays an important role for the derivation of denseness results and for the construction of trace and extension operators. We choose a partition of unity which consists of three members. This has the advantage that functions defined on D can be cut-off only near the boundary Γ_0 and Γ_1, which is absolutely sufficient and provides enough information for the definition of the trace operators.

Theorem 2.90 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type and recall* (2.12a) *for the definition of the set $\widetilde{\Gamma}_j$, $j = 0, 1$. Then there exist $\widetilde{\phi}_j \in C_Q^\infty(\mathbb{R}^3)$, $j = 0, 1, 2$, such that*

(i) $\widetilde{\phi}_j \geq 0$ *on* $\mathbb{R}^3$, $j = 0, 1, 2$,

(ii) $\sum_{j=0}^2 \widetilde{\phi}_j(x) = 1$ *for all* $x \in \overline{E_Q(D)}$,

(iii) $\widetilde{\phi}_0 \equiv 0$ *in a neighborhood of* $\widetilde{\Gamma}_1$ *and* $\widetilde{\phi}_1 \equiv 0$ *in a neighborhood of* $\widetilde{\Gamma}_0$, *as well as* $\widetilde{\phi}_2 \equiv 0$ *in a neighborhood of* $\widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1$.

Proof: Since D is of Lipschitz layer type, we have $\delta := \operatorname{dist}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1) > 0$. Let $0 < \varepsilon < \frac{\delta}{3}$. Set $\Omega_0 := D \cap \big(\widetilde{\Gamma}_0 + \mathbb{B}_3(0,\varepsilon)\big)$, $\Omega_1 := D \cap \big(\widetilde{\Gamma}_1 + \mathbb{B}_3(0,\varepsilon)\big)$ and $\Omega_2 := D \setminus (\overline{\Omega_0 \cup \Omega_1})$. Note that Ω_j are cell sets, $j = 0, 1, 2$. Let $\phi_{\varepsilon/3}$ be the mollifier given in (2.6) and define similarly as in the proof of Lemma 2.80

$$\widetilde{\psi}_j(x) := \int_{E_Q(\Omega_j)+\mathbb{B}_3(0,\varepsilon/3)} \phi_{\varepsilon/3}(x-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^3, \quad j = 0, 1, 2.$$

Then the functions $\widetilde{\phi}_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 0, 1, 2$, given by

$$\widetilde{\phi}_0 := \widetilde{\psi}_0, \quad \widetilde{\phi}_1 := \widetilde{\psi}_1(1 - \widetilde{\psi}_0) \quad \text{and} \quad \widetilde{\phi}_2 := \widetilde{\psi}_2(1 - \widetilde{\psi}_0)(1 - \widetilde{\psi}_1),$$

have the desired properties, if we take into account the fact as well that the equation $\sum_{j=0}^{2} \widetilde{\phi}_j = 1 - \prod_{j=0}^{2}(1 - \widetilde{\psi}_j)$ holds. □

Now, we fix characteristic quantities describing D, similarly as the authors of [34] have done for general Lipschitz domains, see Subsection 2.1.3.

Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with corresponding functions f_j and boundary patches Γ_j from (2.12b), $j = 0, 1$, let $L_3 \in \mathbb{R}$ such that

$$L_3 > \max\left\{ f_1(\xi) - f_0(\xi) \mid \xi \in \overline{Q} \right\} > 0$$

and set

$$Q_3 := Q \times (-L_3, L_3), \quad Q_3^- := Q \times (-L_3, 0), \quad Q_3^+ := Q \times (0, L_3).$$

We introduce the mappings

$$\begin{aligned} \tilde{\Psi}_0(x) &:= \big(x_1, x_2, f_0(\tilde{x}) - x_3\big)^\top, \qquad x \in Q_3^-, \\ \tilde{\Psi}_1(x) &:= \big(x_1, x_2, f_1(\tilde{x}) + x_3\big)^\top, \qquad x \in Q_3^-, \end{aligned}$$

and their extensions (and restrictions) to $Q \times \{0\}$, that is,

$$\Psi_0(x) := \begin{pmatrix} x_1 \\ x_2 \\ f_0(\tilde{x}) \end{pmatrix} \quad \text{and} \quad \Psi_1(x) := \begin{pmatrix} x_1 \\ x_2 \\ f_1(\tilde{x}) \end{pmatrix}, \qquad x \in Q.$$

Again thanks to Rademacher's result, we have that f_j is differentiable almost everywhere on Q and that its gradient is essentially bounded by the

Lipschitz constant of f_j. Therefore, Ψ_j is differentiable almost everywhere on Q and the surface Γ_j can be parametrized by $y = \Psi_j(x)$ for $x \in Q$, with outward pointing normal unit vector $\mathrm{n}_j(y)$ at a.a. $y = \Psi_j(x) \in \Gamma_j$ given by

$$\mathrm{n}_j(y) = \mp \frac{1}{\rho_j(x)} \left(\frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right),$$

where in "$\mp$" the minus sign holds for $j = 0$ and the plus sign for $j = 1$ and where

$$\rho_j(x) := \left| \frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right| = \sqrt{1 + |\nabla f_j(x)|^2}, \quad \text{for a.a. } x \in Q. \tag{2.13}$$

We set $U_j := \tilde{\Psi}_j(Q_3^-)$, $j = 0, 1$. Note that the Jacobian $\tilde{\Psi}_j'(x) \in \mathbb{R}^{3\times 3}$ is given by

$$\tilde{\Psi}_j'(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f_j}{\partial x_1}(\tilde{x}) & \frac{\partial f_j}{\partial x_2}(\tilde{x}) & \mp 1 \end{pmatrix}, \quad \text{for a.a. } x \in Q_3^-,$$

where again in "$\mp$" the minus sign holds for $j = 0$ and the plus sign for $j = 1$. Hence, these Jacobians are again regular with constant determinant $\det \tilde{\Psi}_j'(x) = \mp 1$ and $\tilde{\Psi}_j$ are isomorphisms from Q_3^- to U_j for $j = 0, 1$. Furthermore, we define

$$F_j(x) := \left[\frac{\partial \Psi_j}{\partial x_1}(x) \middle| \frac{\partial \Psi_j}{\partial x_2}(x) \middle| \frac{\partial \Psi_j}{\partial x_1}(x) \times \frac{\partial \Psi_j}{\partial x_2}(x) \right], \text{ for a.a. } x \in Q.$$

Finally, we choose $\widetilde{\phi}_j \in C_Q^\infty(\mathbb{R}^3)$ from Theorem 2.90 and set $\phi_j := \widetilde{\phi}_j|_D$, $j = 0, 1, 2$. Then

- $\phi_j \in C_Q^\infty(D)$ and $\phi_j \geq 0$ on D, $j = 0, 1, 2$,
- $\sum_{j=0}^2 \phi_j(x) = 1$ for all $x \in \overline{D}$,
- $\phi_0 \equiv 0$ in a neighborhood of Γ_1 and $\phi_1 \equiv 0$ in a neighborhood of Γ_0, as well as $\phi_2 \equiv 0$ in a neighborhood of $\Gamma_0 \cup \Gamma_1$.

Assumption 2.91 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with corresponding number L_3, boundary Γ_j, mapping $\tilde{\Psi}_j$ from Q_3^- to U_j, with their restriction Ψ_j from Q to Γ_j, $j = 0, 1$, and partition of unity ϕ_j, $j = 0, 1, 2$, from above.*

The next lemma puts two kinds of neighborhoods in relation to one another. Its statements are used several times in the sequel and their proofs make explicitly use of the Lipschitz continuity of the considered surface patches of D.

Lemma 2.92 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with corresponding functions f_0 and f_1 as in Definition 2.89. Recall* (2.12a) *for the set $\widetilde{\Gamma}_j$ and define for $t > 0$*

$$V_j(t) := \Big\{ x \in \mathbb{R}^3 \;\big|\; |x_3 - f_j(\tilde{x})| < t \Big\}, \qquad j = 0, 1.$$

Choose some $\delta \in \mathbb{R}$ such that $0 < \delta < \operatorname{dist}(\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)$. Furthermore, let $a := (0, 0, a_3)^\top \in \mathbb{R}^3$ be given with $a_3 \geq L + 2$, where $L > 0$ denotes the maximum of the Lipschitz constants of f_0 and f_1, and let $j \in \{0, 1\}$. Then the following statements are true.

(i) *For all $t > 0$ there exists $\varepsilon > 0$ such that $\widetilde{\Gamma}_j + \mathbb{B}_3(0, \varepsilon) \subseteq V_j(t)$.*

(ii) $V_j(t) + \mathbb{B}_3(0, \varepsilon) \subseteq \widetilde{\Gamma}_j + \mathbb{B}_3(0, \delta')$, *for all* $0 \leq \varepsilon < t + \varepsilon \leq \delta' \leq \delta$.

(iii) *For all $0 < \varepsilon \leq \frac{1}{L+1+a_3} \frac{\delta}{2}$ there holds*

- $x \in D \setminus V_1(\delta) \,\wedge\, y \in \mathbb{B}_3[x + \varepsilon a, \varepsilon] \;\Rightarrow\; y_3 \leq f_1(\tilde{y}) - \frac{\delta}{2}$,
- $x \in D \setminus V_0(\delta) \,\wedge\, y \in \mathbb{B}_3[x - \varepsilon a, \varepsilon] \;\Rightarrow\; y_3 \geq f_0(\tilde{y}) + \frac{\delta}{2}$.

(iv) *For all $\varepsilon > 0$ there holds*

- $x \in D \,\wedge\, y \in \mathbb{B}_3[x + \varepsilon a, \varepsilon] \;\Rightarrow\; y_3 \geq f_0(\tilde{y}) + \varepsilon$,
- $x \in D \,\wedge\, y \in \mathbb{B}_3[x - \varepsilon a, \varepsilon] \;\Rightarrow\; y_3 \leq f_1(\tilde{y}) - \varepsilon$.

(v) *In particular, for all $0 < \varepsilon \leq \frac{1}{L+1+a_3} \frac{\delta}{2}$ there holds*

- $x \in D \setminus V_1(\delta) \;\Rightarrow\; \mathbb{B}_3[x + \varepsilon a, \varepsilon] \subseteq E_Q(D) \setminus \big(V_1(\frac{\delta}{2}) \cup V_0(\varepsilon)\big)$,
- $x \in D \setminus V_0(\delta) \;\Rightarrow\; \mathbb{B}_3[x - \varepsilon a, \varepsilon] \subseteq E_Q(D) \setminus \big(V_0(\frac{\delta}{2}) \cup V_1(\varepsilon)\big)$.

(vi) *For all $\varepsilon > 0$ there holds*

- $x \in D \cap V_1(\varepsilon) \,\wedge\, y \in D \;\Rightarrow\; y \notin \mathbb{B}_3(x + \varepsilon a, \varepsilon)$,
- $x \in D \cap V_0(\varepsilon) \,\wedge\, y \in D \;\Rightarrow\; y \notin \mathbb{B}_3(x - \varepsilon a, \varepsilon)$.

Proof: For part *(iii)* and *(iv)* we were inspired by the proof from [34, Theorem 5.3].

(i). Let $t > 0$ and set $\varepsilon := \frac{t}{L+1}$. Let $x \in \widetilde{\Gamma}_j + \mathbb{B}_3(0, \varepsilon)$, i.e., $x = y + z$ with $y \in \widetilde{\Gamma}_j$ and $z \in \mathbb{B}_3(0, \varepsilon)$. Then

$$|x_3 - f_j(\tilde{x})| = |f_j(\tilde{y}) + z_3 - f_j(\tilde{y} + \tilde{z})| \leq L|\tilde{z}| + |z_3| < (L+1)\varepsilon = t.$$

(ii). At first we consider the case $\varepsilon = 0$, i.e., $\mathbb{B}_3(0, \varepsilon) = \emptyset$. Let $x \in V_j(t)$. Then by definition $|x_3 - f_j(\tilde{x})| < t$. Now define $y := (\tilde{x}, f_j(\tilde{x}))^\top$. Then there holds for $z := x - y$ that $|z| = |x - y| = |x_3 - f_j(\tilde{x})| < t \leq \delta'$. Now, let $\varepsilon > 0$ such that $t + \varepsilon \leq \delta'$. Then from the case $\varepsilon = 0$ we conclude that $V_j(t) \subseteq \widetilde{\Gamma}_j + \mathbb{B}_3(0, \delta' - \varepsilon)$. Therefore, $V_j(t) + \mathbb{B}_3(0, \varepsilon) \subseteq \widetilde{\Gamma}_j + \mathbb{B}_3(0, \delta' - \varepsilon) + \mathbb{B}_3(0, \varepsilon)$. Since $\mathbb{B}_3(0, \delta' - \varepsilon) + \mathbb{B}_3(0, \varepsilon) = \mathbb{B}_3(0, \delta')$, the assertion follows.

(iii). Let $x \in D \setminus V_1(\delta)$. Take some $y \in \mathbb{B}_3[x + \varepsilon a, \varepsilon]$. Hence, $|\tilde{x} - \tilde{y}| \leq \varepsilon$. Then

$$\begin{aligned}
y_3 &\leq x_3 + \varepsilon a_3 + \varepsilon \leq f_1(\tilde{x}) - \delta + \varepsilon(1 + a_3) \\
&= f_1(\tilde{x}) - f_1(\tilde{y}) + f_1(\tilde{y}) - \delta + \varepsilon(1 + a_3) \\
&\leq |f_1(\tilde{x}) - f_1(\tilde{y})| + f_1(\tilde{y}) - \delta + \varepsilon(1 + a_3) \\
&\leq L|\tilde{x} - \tilde{y}| + f_1(\tilde{y}) - \delta + \varepsilon(1 + a_3) \\
&\leq f_1(\tilde{y}) - \delta + \varepsilon(L + 1 + a_3) \leq f_1(\tilde{y}) - \tfrac{\delta}{2}.
\end{aligned}$$

The second assertion is shown analogously.

(iv). Take some $x \in D$ and some $y \in \mathbb{B}_3[x + \varepsilon a, \varepsilon]$. Hence, $|\tilde{x} - \tilde{y}| \leq \varepsilon$. Then

$$\begin{aligned}
y_3 &\geq x_3 + \varepsilon a_3 - \varepsilon > f_0(\tilde{x}) + \varepsilon(a_3 - 1) \\
&= f_0(\tilde{y}) - \big(f_0(\tilde{y}) - f_0(\tilde{x})\big) + \varepsilon(a_3 - 1) \\
&\geq f_0(\tilde{y}) - |f_0(\tilde{y}) - f_0(\tilde{x})| + \varepsilon(a_3 - 1) \\
&\geq f_0(\tilde{y}) - L|\tilde{y} - \tilde{x}| + \varepsilon(a_3 - 1) \\
&\geq f_0(\tilde{y}) + \varepsilon(a_3 - 1 - L) \geq f_0(\tilde{y}) + \varepsilon.
\end{aligned}$$

The second assertion is shown analogously.

(v). This is a combination of part *(iii)* and *(iv)*.

(vi). Let $x \in D \cap V_1(\varepsilon)$ and $y \in D$. Then $x_3 > f_1(\tilde{x}) - \varepsilon$. If $|\tilde{y} - \tilde{x}| > \varepsilon$,

then we obtain immediately $|x+\varepsilon a-y| \geq |\tilde{x}-\tilde{y}| > \varepsilon$. So let $|\tilde{y}-\tilde{x}| \leq \varepsilon$. Then we conclude

$$\begin{aligned}
|x+\varepsilon a-y| &\geq |x_3+\varepsilon a_3-y_3| \geq x_3+\varepsilon a_3-y_3 \\
&> f_1(\tilde{x})-\varepsilon+\varepsilon a_3-y_3 \\
&= f_1(\tilde{y})-y_3-\big(f_1(\tilde{y})-f_1(\tilde{x})\big)+\varepsilon(a_3-1) \\
&\geq -|f_1(\tilde{y})-f_1(\tilde{x})|+\varepsilon(a_3-1) \\
&\geq -L|\tilde{y}-\tilde{x}|+\varepsilon(a_3-1) \geq \varepsilon(a_3-L-1) \geq \varepsilon.
\end{aligned}$$

The second assertion is shown analogously. □

Denseness Results. Recall the integral identities (A.6) and consider $D \subseteq \mathbb{R}^3$ to be a cell set of Lipschitz layer type. Due to the Q-periodicity, it is easy to show that for $\varphi, \psi \in C^1_Q(D) \cap C_Q(\overline{D})$ and $u, v \in C^1_Q(D, \mathbb{C}^3) \cap C_Q(\overline{D}, \mathbb{C}^3)$ there holds

$$\int_D (\varphi \nabla \psi + \psi \nabla \varphi)\, \mathrm{d}x = \sum_{j=0}^{1} \int_{\Gamma_j} \varphi\, \psi\, \mathrm{n}\, \mathrm{d}s, \tag{2.14a}$$

$$\int_D (\operatorname{curl} u \cdot v - u \cdot \operatorname{curl} v)\, \mathrm{d}x = \sum_{j=0}^{1} \int_{\Gamma_j} (\mathrm{n} \times u) \cdot v\, \mathrm{d}s, \tag{2.14b}$$

$$\int_D (\psi \operatorname{div} u + u \cdot \nabla \psi)\, \mathrm{d}x = \sum_{j=0}^{1} \int_{\Gamma_j} \psi\, (\mathrm{n} \cdot u)\, \mathrm{d}s, \tag{2.14c}$$

where n denotes the outward pointing normal unit vector on D. Therefore, $C^\infty_Q(\overline{D})$ is a subspace of $H^m_Q(D)$ and $C^\infty_Q(\overline{D}, \mathbb{C}^3)$ is a subspace of $H_Q(\operatorname{curl}, D)$ and of $H_Q(\operatorname{div}, D)$. The next important theorem shows that these subspaces are even dense therein.

Theorem 2.93 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type. Then the following assertions are true.*

(a) The space $C^\infty_Q(\overline{D}, \mathbb{C}^{d'})$ is dense in $H^m_Q(D, \mathbb{C}^{d'})$.

(b) The space $C^\infty_Q(\overline{D}, \mathbb{C}^3)$ is dense in $H_Q(\operatorname{curl}, D)$.

(c) The space $C_Q^\infty(\overline{D}, \mathbb{C}^3)$ *is dense in* $H_Q(\operatorname{div}, D)$.

Proof: Recall $\widetilde{\Gamma}_j$, $j = 0, 1$, from (2.12a) and ϕ_j, $j = 0, 1, 2$, from Assumption 2.91. Note that $\phi_j \in C_Q^\infty(D)$. Moreover, there exists $\delta > 0$ such that $\operatorname{supp}(\phi_0) \subseteq \overline{D} \setminus \overline{\Gamma_1^\delta}$, $\operatorname{supp}(\phi_1) \subseteq \overline{D} \setminus \overline{\Gamma_0^\delta}$ and $\operatorname{supp}(\phi_2) \subseteq \overline{D} \setminus \big(\overline{\Gamma_0^\delta} \cup \overline{\Gamma_1^\delta}\big)$, with Γ_j^δ from (2.12c), $j = 0, 1$. Set $a := (0, 0, L + 2)^\top$, where $L > 0$ denotes the maximum of the Lipschitz constants from f_0 and f_1.
We start with part *(b)*, continue with part *(c)* and postpone part *(a)* to the end of this proof.
(b) Let $u \in H_Q(\operatorname{curl}, D)$. Set $u^{(j)} := \phi_j u$. Then $u^{(j)} \in H_Q(\operatorname{curl}, D)$ and $\operatorname{supp}(u^{(j)}) \subseteq \operatorname{supp}(\phi_j)$, $j = 0, 1, 2$. Furthermore, $\sum_{j=0}^{2} u^{(j)} = u$. We define for $0 < \varepsilon < \frac{1}{2}\delta$

$$\begin{aligned}
u_\varepsilon^{(0)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, u^{(0)}(y)\, \mathrm{d}y, && x \in D,\\
u_\varepsilon^{(1)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x - \varepsilon a - y)\, u^{(1)}(y)\, \mathrm{d}y, && x \in D,\\
u_\varepsilon^{(2)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x - y)\, u^{(2)}(y)\, \mathrm{d}y, && x \in D,
\end{aligned}$$

where $\widetilde{\phi}_\varepsilon$ denotes the Q-periodic extension of ϕ_ε from (2.6). Note that $u_\varepsilon^{(j)} \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$, $j = 0, 1, 2$. Furthermore, by part *(iv)* from Lemma 2.92, we have for all $x \in D$ that $\widetilde{\phi}_\varepsilon(x + \varepsilon a - \cdot)$ and $\widetilde{\phi}_\varepsilon(x - \varepsilon a - \cdot)$ vanish in a neighborhood of $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$, respectively. Therefore, thanks to Proposition 2.82, we obtain for $x \in D$

$$\begin{aligned}
\operatorname{curl} u_\varepsilon^{(0)}(x) &= \int_D \nabla_x \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u^{(0)}(y)\, \mathrm{d}y\\
&= -\int_D \nabla_y \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u^{(0)}(y)\, \mathrm{d}y\\
&= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \operatorname{curl} u^{(0)}(y)\, \mathrm{d}y =: (\operatorname{curl} u^{(0)})_\varepsilon(x).
\end{aligned}$$

And analogously, we have for $x \in D$

$$\operatorname{curl} u_\varepsilon^{(1)}(x) = \int_D \widetilde{\phi}_\varepsilon(x - \varepsilon a - y) \operatorname{curl} u^{(1)}(y)\, \mathrm{d}y =: (\operatorname{curl} u^{(1)})_\varepsilon(x),$$

$$\operatorname{curl} u_\varepsilon^{(2)}(x) = \int_D \widetilde{\phi}_\varepsilon(x-y) \operatorname{curl} u^{(2)}(y)\,\mathrm{d}y =: (\operatorname{curl} u^{(2)})_\varepsilon(x).$$

Now we similarly proceed as at the end of the proof of Proposition 2.83 and set $\widetilde{D} := \bigcup_{|\mu|_\infty \le 1} \big(\{p^{(\mu)}\} + D\big)$. Moreover, let $v^{(j)} \in \{u^{(j)}, \operatorname{curl} u^{(j)}\}$ and $\widetilde{v}^{(j)}$ be the Q-periodic extension of $v^{(j)}$ from D to $\widetilde{D}$ and then be extended by zero to $\mathbb{R}^3 \setminus \widetilde{D}$, $j = 0,1,2$. And finally set

$$\begin{aligned}
\widetilde{v}_\varepsilon^{(0)}(x) &:= \int_{\mathbb{R}^3} \phi_\varepsilon(x+\varepsilon a - y)\, \widetilde{v}^{(0)}(y)\,\mathrm{d}y, && x \in \mathbb{R}^3,\\
\widetilde{v}_\varepsilon^{(1)}(x) &:= \int_{\mathbb{R}^3} \phi_\varepsilon(x-\varepsilon a - y)\, \widetilde{v}^{(1)}(y)\,\mathrm{d}y, && x \in \mathbb{R}^3,\\
\widetilde{v}_\varepsilon^{(2)}(x) &:= \int_{\mathbb{R}^3} \phi_\varepsilon(x - y)\, \widetilde{v}^{(2)}(y)\,\mathrm{d}y, && x \in \mathbb{R}^3.
\end{aligned}$$

Let $j \in \{0,1,2\}$. Then $\widetilde{v}_\varepsilon^{(j)}(x) = v_\varepsilon^{(j)}(x)$ for all $x \in D$ and moreover $\widetilde{v}^{(j)} \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ with $\widetilde{v}^{(j)} = v^{(j)}$ almost everywhere in D. Then from Theorem 2.50 we conclude that $\widetilde{v}_\varepsilon^{(j)} \to \widetilde{v}^{(j)}$ in $L^2(\mathbb{R}^3, \mathbb{C}^3)$ and therefore in particular $v_\varepsilon^{(j)} = \widetilde{v}_\varepsilon^{(j)}|_D \to \widetilde{v}^{(j)}|_D = v^{(j)}$ in $L^2(D, \mathbb{C}^3)$, as $\varepsilon \to 0$. This means that $u_\varepsilon^{(j)} \to u^{(j)}$ with respect to $\|\cdot\|_{H(\operatorname{curl},D)}$, as $\varepsilon \to 0$.
From this we finally conclude that for $u_\varepsilon := \sum_{j=0}^2 u_\varepsilon^{(j)}$ there holds $u_\varepsilon \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$ and $u_\varepsilon = \sum_{j=0}^2 u_\varepsilon^{(j)} \to \sum_{j=0}^2 u^{(j)} = u$ in $H_Q(\operatorname{curl}, D)$, as $\varepsilon \to 0$, and the proof is complete.
(c) Let $u \in H_Q(\operatorname{div}, D)$. We define $u^{(j)}$ and $u_\varepsilon^{(j)}$, $j = 0,1,2$, as in part *(b)* and obtain with the same arguments, and again thanks to Proposition 2.82, for $x \in D$

$$\begin{aligned}
\operatorname{div} u_\varepsilon^{(0)}(x) &= \int_D \nabla_x \widetilde{\phi}_\varepsilon(x+\varepsilon a - y) \cdot u^{(0)}(y)\,\mathrm{d}y\\
&= -\int_D \nabla_y \widetilde{\phi}_\varepsilon(x+\varepsilon a - y) \cdot u^{(0)}(y)\,\mathrm{d}y\\
&= \int_D \widetilde{\phi}_\varepsilon(x+\varepsilon a - y) \operatorname{div} u^{(0)}(y)\,\mathrm{d}y =: (\operatorname{div} u^{(0)})_\varepsilon(x),
\end{aligned}$$

and analogously

$$\operatorname{div} u_\varepsilon^{(1)}(x) = \int_D \widetilde{\phi}_\varepsilon(x-\varepsilon a - y) \operatorname{div} u^{(1)}(y)\,\mathrm{d}y =: (\operatorname{div} u^{(1)})_\varepsilon(x),$$

$$\operatorname{div} u_\varepsilon^{(2)}(x) = \int_D \widetilde{\phi}_\varepsilon(x-y) \operatorname{div} u^{(2)}(y)\,\mathrm{d}y =: (\operatorname{div} u^{(2)})_\varepsilon(x).$$

Now we follow the lines as in part *(b)* to obtain the desired result.
(a) We only show the assertion for the case $d' = 1$ as the generalization for the case $d' > 1$ is obvious.
Let $u \in H_Q^m(D)$. We define $u^{(j)}$ and $u_\varepsilon^{(j)}$, $j = 0, 1, 2$, as in part *(b)* and obtain with the same arguments, and again thanks to Proposition 2.82, for $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le m$, and $x \in D$

$$\begin{aligned}
\partial^\alpha u_\varepsilon^{(0)}(x) &= \int_D \partial_x^\alpha\, \widetilde{\phi}_\varepsilon(x+\varepsilon a-y)\, u^{(0)}(y)\,\mathrm{d}y \\
&= (-1)^{|\alpha|} \int_D \partial_y^\alpha\, \widetilde{\phi}_\varepsilon(x+\varepsilon a-y)\, u^{(0)}(y)\,\mathrm{d}y \\
&= \int_D \widetilde{\phi}_\varepsilon(x+\varepsilon a-y)\, \partial^\alpha u^{(0)}(y)\,\mathrm{d}y =: (\partial^\alpha u^{(0)})_\varepsilon(x),
\end{aligned}$$

and analogously

$$\begin{aligned}
\partial^\alpha u_\varepsilon^{(1)}(x) &= \int_D \widetilde{\phi}_\varepsilon(x-\varepsilon a-y)\, \partial^\alpha u^{(1)}(y)\,\mathrm{d}y =: (\partial^\alpha u^{(1)})_\varepsilon(x), \\
\partial^\alpha u_\varepsilon^{(2)}(x) &= \int_D \widetilde{\phi}_\varepsilon(x-y)\, \partial^\alpha u^{(2)}(y)\,\mathrm{d}y =: (\partial^\alpha u^{(2)})_\varepsilon(x).
\end{aligned}$$

Now we follow again the lines as in part *(b)* to obtain the desired result and the proof is complete. □

Lower and Upper Boundary Patches. One of the most important properties of the geometrical structure of a cell set $D \subseteq \mathbb{R}^3$ of Lipschitz layer type is, that only the lower and upper boundary patch Γ_0 and Γ_1, described by the graph of the function f_0 and f_1, respectively, is interesting, since contributions of the side patches on intergrals over the whole boundary ∂D cancel out in the Q-periodic framework. Therefore it seems quite natural to focus on the lower and upper boundary patch separately and to introduce also separate trace and extension operators. However, to realize this concept, additional function spaces have to be defined.

Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $m \in \mathbb{N}_0$ and $j \in \{0,1\}$. We introduce

$$C^m_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^{d'}) := \Big\{ u \in C^m_Q(\overline{D}, \mathbb{C}^{d'}) \;\Big|\; \exists\, \varepsilon > 0 : \operatorname{supp}(u) \cap \overline{\Gamma^\varepsilon_j} = \emptyset \Big\},$$

$$C^\infty_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^{d'}) := \bigcap_{k=0}^{\infty} C^k_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^{d'}).$$

As before, in the names of these function spaces we will often neglect the superscript "m" if $m = 0$. And again, we will mostly drop the symbol for the co-domain in the case of scalar valued functions.

Definition 2.94 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type and $j \in \{0,1\}$.*

(a) We define $H^m_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'})$ as the closure of $C^\infty_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^{d'})$ in the space $H^m_Q(D, \mathbb{C}^{d'})$.

(b) We define $H_{Q,0,\Gamma_j}(\operatorname{curl}, D)$ as the closure of $C^\infty_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^3)$ in the space $H_Q(\operatorname{curl}, D)$.

Further Extension (by Zero) Operatos. Often we will need a variant of the extension (by zero) operator from Proposition 2.68. For cell sets of Lipschitz layer type, an upwards and downwards extension by zero is possible if the function under consideration vanishes in Γ^ε_1 and Γ^ε_0, respectively, see the next result.

Proposition 2.95 *Let $D, D' \subseteq \mathbb{R}^3$ be cell sets of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, such that $D \subsetneq D'$.*

(a) The following assertions are true.

(i) If $\Gamma_0 \subseteq \partial D'$, then the mapping

$$H^m_{Q,0,\Gamma_1}(D, \mathbb{C}^{d'}) \ni u \;\mapsto\; u|^{D'}_0 \in H^m_{Q,0,\Gamma'_1}(D', \mathbb{C}^{d'})$$

is well-defined, linear and bounded.

(ii) *If* $\Gamma_1 \subseteq \partial D'$*, then the mapping*

$$H^m_{Q,0,\Gamma_0}(D,\mathbb{C}^{d'}) \ni u \;\mapsto\; u|_0^{D'} \in H^m_{Q,0,\Gamma_0'}(D',\mathbb{C}^{d'})$$

is well-defined, linear and bounded.

In both cases we have $\|u|_0^{D'}\|_{H^m_Q(D',\mathbb{C}^{d'})} \le \|u\|_{H^m_Q(D,\mathbb{C}^{d'})}$ *and furthermore* $\partial^\alpha(u|_0^{D'}) = (\partial^\alpha u)|_0^{D'}$ *for all* $u \in H^m_{Q,0,\Gamma_j}(D,\mathbb{C}^{d'})$*,* $j = 0,1$*, and all* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le m$*.*

(b) *The following assertions are true.*

(i) *If* $\Gamma_0 \subseteq \partial D'$*, then the mapping*

$$H_{Q,0,\Gamma_1}(\mathrm{curl},D) \ni u \;\mapsto\; u|_0^{D'} \in H_{Q,0,\Gamma_1'}(\mathrm{curl},D')$$

is well-defined, linear and bounded.

(ii) *If* $\Gamma_1 \subseteq \partial D'$*, then the mapping*

$$H_{Q,0,\Gamma_0}(\mathrm{curl},D) \ni u \;\mapsto\; u|_0^{D'} \in H_{Q,0,\Gamma_0'}(\mathrm{curl},D')$$

is well-defined, linear and bounded.

In both cases we have $\|u|_0^{D'}\|_{H_Q(\mathrm{curl},D')} = \|u\|_{H_Q(\mathrm{curl},D)}$ *and furthermore* $\mathrm{curl}(u|_0^{D'}) = (\mathrm{curl}\,u)|_0^{D'}$ *for all* $u \in H_{Q,0,\Gamma_j}(\mathrm{curl},D)$*,* $j = 0,1$*.*

Proof: This is shown with exactly the same arguments as in part *(ii)* of the proof of Proposition 2.68. □

Friedrich's Inequality. Due to the special structure of a cell set $D \subseteq \mathbb{R}^3$ of Lipschitz layer type, we can generalize Friedrich's inequality from Theorem 2.72 and its Corollary 2.73 to situations where $u \in H^1_Q(D)$ vanishes only on one of both surfaces patches, see the next theorem.

Theorem 2.96 *Let* $D \subseteq \mathbb{R}^3$ *be a cell set of Lipschitz layer type and* $\beta \in \mathbb{R}^3$*. Then there exists* $c > 0$ *such that*

$$\|u\|_{L^2(D)} \le c\,\|\nabla_\beta u\|_{L^2(D,\mathbb{C}^3)}, \quad \textit{for all } u \in H^1_{Q,0,\Gamma_j}(D),\ j = 0,1.$$

Here, the operator ∇_β *is given by* $\nabla_\beta = \nabla + \mathrm{i}\beta$*, see also* (1.20a).

Proof: For $j = 0$ we can exactly follow the lines in the proof of Theorem 2.72 if we replace therein $C^\infty_{Q,0}(D)$ by $C^\infty_{Q,0,\Gamma_0}(\overline{D})$ and $H^1_{Q,0}(D)$ by $H^1_{Q,0,\Gamma_0}(D)$. The modification for $j = 1$ are then quite obvious. □

Corollary 2.97 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type and $\beta \in \mathbb{R}^3$. Then there exists $c > 0$ such that*

$$\|u\|_{H^1_Q(D)} \le c\,\|\nabla_\beta u\|_{L^2(D,\mathbb{C}^3)}, \quad \textit{for all } u \in H^1_{Q,0,\Gamma_j}(D),\ j = 0, 1.$$

Here, the operator ∇_β is given by $\nabla_\beta = \nabla + \mathrm{i}\beta$, see also (1.20a).

Proof: This is shown by the same arguments as in the proof of Corollary 2.73. □

We continue with the correspondents of Lemma 2.74 and Proposition 2.75. For example, one of these results will be needed later for the definition of the surface divergence.

Lemma 2.98 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $j \in \{0, 1\}$ and $\beta \in \mathbb{R}^3$.*
If $(p_n)_{n\in\mathbb{N}}$ is a sequence in $H^1_{Q,0,\Gamma_j}(D)$ such that $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D)$ and $(\nabla_\beta p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D,\mathbb{C}^3)$, then for the L^2-limits $p := \lim_{n\to\infty} p_n$ and $u := \lim_{n\to\infty} \nabla_\beta p_n$ there holds $p \in H^1_{Q,0,\Gamma_j}(D)$ with $\nabla_\beta p = u$.

Proof: The assertion is shown by copying the lines of the proof of Lemma 2.74 and by replacing therein $H^1_{Q,0}(D)$ with $H^1_{Q,0,\Gamma_j}(D)$. □

Proposition 2.99 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $j \in \{0, 1\}$ and $\beta \in \mathbb{R}^3$. Then $\nabla_\beta H^1_{Q,0,\Gamma_j}(D)$ is a closed subspace of $H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$.*

Proof: The assertion is shown by copying the lines of part *(ii)* in the proof of Proposition 2.75 and by replacing therein the space $H^1_{Q,0}(D)$ with $H^1_{Q,0,\Gamma_j}(D)$, the space $H_{Q,0}(\mathrm{curl}, D)$ with $H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$, the space $C^\infty_{Q,0}(D)$ with $C^\infty_{Q,0,\Gamma_j}(\overline{D})$, the space $C^\infty_{Q,0}(D, \mathbb{C}^3)$ with $C^\infty_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^3)$, Theorem 2.72 with Theorem 2.96 and finally if we replace Lemma 2.74 with Lemma 2.98. □

Functions with Compact Support. Again, we will often be in situations where the support of a function has some distance to the boundary of D. Here two more possibilities are conceivable. Recall Definition 2.79 for the space $\mathcal{H}_{Q,0}(\mathrm{curl}, D)$. In Proposition 2.83 we have seen that $\mathcal{H}_{Q,0}(\mathrm{curl}, D)$ is a subset of $H_{Q,0}(\mathrm{curl}, D)$, where D denoted therein an arbitrary cell set. Hence, we have this result in particular for cell sets of Lipschitz layer type. As mentioned above, we are now interested in spaces for functions which only vanish in the neighborhood of one, the lower or the upper, boundary patch, i.e., on Γ^ε_0 or on Γ^ε_1, respectively. In this situation we obtain a similar result, see the next proposition.

Definition 2.100 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, $m \in \mathbb{N}$ and $j \in \{0, 1\}$.*

(a) The subspace $\mathcal{H}^m_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'})$ of $H^m_Q(D, \mathbb{C}^{d'})$ is defined by

$$\mathcal{H}^m_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'}) := \Big\{ u \in H^m_Q(D, \mathbb{C}^{d'}) \;\Big|\; \exists\, \varepsilon > 0 : \mathrm{supp}(u) \cap \overline{\Gamma^\varepsilon_j} = \emptyset \Big\}.$$

(b) The subspace $\mathcal{H}_{Q,0,\Gamma_j}(\mathrm{curl}, D)$ of $H_Q(\mathrm{curl}, D)$ is defined by

$$\mathcal{H}_{Q,0,\Gamma_j}(\mathrm{curl}, D) := \Big\{ u \in H_Q(\mathrm{curl}, D) \;\Big|\; \exists\, \varepsilon > 0 : \mathrm{supp}(u) \cap \overline{\Gamma^\varepsilon_j} = \emptyset \Big\}.$$

Proposition 2.101 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type and let $j \in \{0, 1\}$. Then*

(a) $\mathcal{H}^m_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'}) \subseteq H^m_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'})$,

(b) $\mathcal{H}_{Q,0,\Gamma_j}(\mathrm{curl}, D) \subseteq H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$.

Proof: We start with part *(b)* and postpone part *(a)* to the end of this proof.

(b). We only prove the assertion for $j = 1$, because the argumentation for the case $j = 0$ is completely analogous.

Let $u \in \mathcal{H}_{Q,0,\Gamma_1}(\mathrm{curl}, D)$. By assumption, there exists $\delta > 0$ such that $\mathrm{supp}(u) \cap \overline{\Gamma_1^\delta} = \emptyset$. Set $a := (0, 0, L + 2)^\top$, where $L > 0$ denotes the maximum of the Lipschitz constants from f_0 and f_1. We define for $\varepsilon > 0$

$$u_\varepsilon(x) := \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, u(y)\, \mathrm{d}y, \qquad x \in D.$$

where $\widetilde{\phi}_\varepsilon$ denotes the Q-periodic extension of ϕ_ε from (2.6). By part *(vi)* from Lemma 2.92, u_ε vanishes in a neighborhood of Γ_1. Therefore, $u_\varepsilon \in C^\infty_{Q,0,\Gamma_1}(\overline{D}, \mathbb{C}^3)$ for all $\varepsilon > 0$. Furthermore, by part *(iv)* from Lemma 2.92, we have for all $x \in D$ that $\widetilde{\phi}_\varepsilon(x + \varepsilon a - \bullet)$ vanishes in a neighborhood of $\widetilde{\Gamma}_0$. Therefore, thanks to part *(b)* of Proposition 2.82, we obtain for $x \in D$

$$\begin{aligned}
\mathrm{curl}\, u_\varepsilon(x) &= \int_D \nabla_x \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u(y)\, \mathrm{d}y \\
&= -\int_D \nabla_y \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u(y)\, \mathrm{d}y \\
&= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, \mathrm{curl}\, u(y)\, \mathrm{d}y =: (\mathrm{curl}\, u)_\varepsilon(x).
\end{aligned}$$

Now we similarly proceed as at the end of the proof of Proposition 2.83 and set $\widetilde{D} := \bigcup_{|\mu|_\infty \le 1} \big(\{p^{(\mu)}\} + D\big)$ to be the union of D and its eight neighbors from $E_Q(D)$, where $p^{(\mu)}$ is the lattice vector given in (2.7). Moreover, let $v \in \{u, \mathrm{curl}\, u\}$ and $\widetilde{v}$ be the Q-periodic extension of v from D to $\widetilde{D}$ and then be extended by zero to $\mathbb{R}^3 \setminus \widetilde{D}$. And finally set

$$\widetilde{v}_\varepsilon(x) := \int_{\mathbb{R}^3} \phi_\varepsilon(x + \varepsilon a - y)\, \widetilde{v}(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Then $\widetilde{v}_\varepsilon(x) = v_\varepsilon(x)$ for all $x \in D$ and moreover $\widetilde{v} \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ with $\widetilde{v} = v$ almost everywhere in D. Then from Theorem 2.50 we conclude that $\widetilde{v}_\varepsilon \to \widetilde{v}$ in $L^2(\mathbb{R}^3, \mathbb{C}^3)$ and therefore in particular $v_\varepsilon = \widetilde{v}_\varepsilon|_D \to \widetilde{v}|_D = v$ in $L^2(D, \mathbb{C}^3)$, as $\varepsilon \to 0$. This means that $u_\varepsilon \to u$ with respect to $\|\cdot\|_{H(\mathrm{curl},D)}$, as $\varepsilon \to 0$, and the proof for part *(b)* is complete.

(a). The proof follows very closely the lines from part *(b)*, where we now cite part *(a)* from Proposition 2.82. The details are omitted. □

2.3.2. A curl-preserving Transformation

Since Q-periodic functions on a cell set D of Lipschitz layer type and periodic functions on a cuboid Q_3 are closely related to each other, we can and will often reuse results from Subsection 2.1.2 when we are deriving analogous results for Q-periodic functions defined on D. However, a suitable transformation between those function spaces is needed. While for the scalar valued case this procedure, realized by

$$u \mapsto u \circ \tilde{\Psi}_j,$$

is straightforward, for the vector valued case this transformation doesn't work because it does not map vector fields of $H_Q(\mathrm{curl}, D)$ into vector fields of $H_Q(\mathrm{curl}, Q_3^-)$, see also [34]. Hence, for the vector valued case a *curl-preserving* transformation, realized by

$$u \mapsto (\tilde{\Psi}_j')^\top (u \circ \tilde{\Psi}_j),$$

has to be introduced. This requires a little more effort, see the next lemma, with the following proposition as preparation. The lemma contains two more results which are later of importance when we will consider the trace operators.

Proposition 2.102 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type and let $\widetilde{u}, \widetilde{v} : E_Q(D) \to \mathbb{C}$ be Q-periodic and Lipschitz continuous. Then for $u := \widetilde{u}|_D$ and $v := \widetilde{v}|_D$ we have that $u\nabla v \in H_Q(\mathrm{curl}, D)$ with variational curl given by*

$$\mathrm{curl}(u\nabla v) = \nabla u \times \nabla v.$$

In particular, $\mathrm{curl}(u\nabla v)$ exists almost everywhere on D as a classical function.

Proof: Thanks to Proposition 2.76 and Theorem 2.93, we can, by replacing the space C^∞ by C_Q^∞ and C_0^∞ by $C_{Q,0}^\infty$, follow the proof from Proposition 2.44 line for line to obtain the assertion. □

Lemma 2.103 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. In the following arguments we will drop the index "j". Let $0 < \ell < L_3$ and $R := Q \times (-\ell, 0)$. For $u \in C_Q^\infty\big(\overline{\tilde{\Psi}(R)}, \mathbb{C}^3\big)$ set*

$$v(x) := \tilde{\Psi}'(x)^\top u\big(\tilde{\Psi}(x)\big), \qquad \text{for a.a. } x \in R \cup (Q \times \{0\}).$$

Furthermore, define $\hat{e} := (0,0,\mp 1)^\top \in \mathbb{R}^3$, where in "$\mp$" the minus sign holds for $j = 0$ and the plus sign for $j = 1$. Then $v \in H_Q(\mathrm{curl}, R)$ and, with $y = \Psi(x)$,

(i) $\mathrm{curl}\, v(x) = \big(\tilde{\Psi}'(x)\big)^{-1}\big(\mathrm{curl}\, u\big)\big(\tilde{\Psi}(x)\big)$, *for a.a. $x \in R$,*

(ii) $\rho(x)\big(\mathrm{n}(y) \times u(y)\big) = F(x)\big(\hat{e} \times v(x,0)\big)$, *for a.a. $x \in Q$,*

(iii) $\big(\mathrm{n}(y) \times u(y)\big) \times \mathrm{n}(y) = F(x)^{-\top}\big[\big(\hat{e} \times v(x,0)\big) \times \hat{e}\big]$, *for a.a. $x \in Q$.*

Proof: We proceed similarly as in the proof of [34, Lemma 5.22] and note that for any regular matrix $M = [a|b|c] \in \mathbb{R}^{3\times 3}$, with column vectors a, b and c, the inverse M^{-1} is given by

$$M^{-1} = \tfrac{1}{(a\times b)\cdot c}\,[b \times c|c \times a|a \times b]^\top = \tfrac{1}{\det M}\,[b \times c|c \times a|a \times b]^\top.$$

In the following arguments the subscript j denotes the j-th component of the vector under consideration.
(i). At first, with $y = \tilde{\Psi}(x)$, we show that

$$\sum_{j=1}^{3} \nabla(u_j \circ \tilde{\Psi})(x) \times \nabla\tilde{\Psi}_j(x) = \big(\tilde{\Psi}'(x)\big)^{-1}\,\mathrm{curl}\, u(y),$$

holds for a.a. $x \in R$. In fact, due to the smoothness of u and the Lipschitz continuity of $\tilde{\Psi}$, we have for a.a. $x \in R$ that the following equations hold in the classical sense

$$\begin{aligned}\textstyle\sum_{j=1}^{3} \nabla(u_j \circ \tilde{\Psi})(x) \times \nabla\tilde{\Psi}_j(x) &= \textstyle\sum_{j=1}^{3} \big[\big(\tilde{\Psi}'(x)\big)^\top \nabla u_j(y)\big] \times \nabla\tilde{\Psi}_j(x) \\ &= \textstyle\sum_{j,k=1}^{3} \partial_k u_j(y)\,\big[\nabla\tilde{\Psi}_k(x) \times \nabla\tilde{\Psi}_j(x)\big] \\ &= \big[(\nabla\tilde{\Psi}_2 \times \nabla\tilde{\Psi}_3)(x)\,|\,(\nabla\tilde{\Psi}_3 \times \nabla\tilde{\Psi}_1)(x)\,|\,(\nabla\tilde{\Psi}_1 \times \nabla\tilde{\Psi}_2)(x)\big]\,\mathrm{curl}\, u(y)\end{aligned}$$

$$= \big(\tilde{\Psi}'(x)\big)^{-1} \operatorname{curl} u(y),$$

where we applied the observation from the beginning to $\big(\tilde{\Psi}(x)\big)^{\top}$. Now, we note that $u_j \circ \tilde{\Psi}$ and $\tilde{\Psi}_j$ are Lipschitz continuous functions as in Proposition 2.102. Therefore, $(u_j \circ \tilde{\Psi})\, \nabla \tilde{\Psi}_j \in H_Q(\mathrm{curl}, R)$ with

$$\operatorname{curl}\big((u_j \circ \tilde{\Psi})\, \nabla \tilde{\Psi}_j\big) = \nabla(u_j \circ \tilde{\Psi}) \times \nabla \tilde{\Psi}_j, \qquad j = 1, 2, 3.$$

Hence, for arbitrary $\chi \in C^\infty_{Q,0}(R, \mathbb{C}^3)$ we obtain

$$\begin{aligned}\int_R v(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x &= \sum\nolimits_{j=1}^{3} \int_R u_j\big(\tilde{\Psi}(x)\big) \nabla \tilde{\Psi}_j(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x \\ &= \sum\nolimits_{j=1}^{3} \int_R \big[\nabla(u_j \circ \tilde{\Psi})(x) \times \nabla \tilde{\Psi}(x)\big] \cdot \chi(x)\, \mathrm{d}x \\ &= \int_R \big(\tilde{\Psi}'(x)\big)^{-1} \operatorname{curl} u\big(\tilde{\Psi}(x)\big) \cdot \chi(x)\, \mathrm{d}x,\end{aligned}$$

which shows that $v \in H_Q(\mathrm{curl}, R)$ with variational curl given as asserted.
(ii). The following equations have to be understood only almost everywhere in R. We have $v = (u \cdot \partial_1 \tilde{\Psi}, u \cdot \partial_2 \tilde{\Psi}, u \cdot \partial_3 \tilde{\Psi})^{\top}$ and therefore $\hat{e} \times v = (\pm u \cdot \partial_2 \Psi, \mp u \cdot \partial_1 \Psi, 0)^{\top}$ on the boundary $x_3 = 0$. Consequently,

$$F(\hat{e} \times v) = \pm(u \cdot \partial_2 \Psi)\, \partial_1 \Psi \mp (u \cdot \partial_1 \Psi)\, \partial_2 \Psi = \mp(\partial_1 \Psi \times \partial_2 \Psi) \times u = \rho(\mathrm{n} \times u).$$

(iii). Again, the following equations have to be understood only almost everywhere on the boundary of R for $x_3 = 0$. At first, we observe that

$$A := \begin{pmatrix} |\partial_2 \Psi|^2 & -\partial_1 \Psi \cdot \partial_2 \Psi & 0 \\ -\partial_1 \Psi \cdot \partial_2 \Psi & |\partial_1 \Psi|^2 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} = \rho^2 (F^{\top} F)^{-1} = \rho^2 F^{-1} F^{-\top}.$$

From above we have $\rho(\mathrm{n} \times u) = \pm v_2\, \partial_1 \Psi \mp v_1\, \partial_2 \Psi$. Hence,

$$\begin{aligned}\rho^2(\mathrm{n} \times u) \times \mathrm{n} &= -v_2\, \partial_1 \Psi \times (\partial_1 \Psi \times \partial_2 \Psi) + v_1\, \partial_2 \Psi \times (\partial_1 \Psi \times \partial_2 \Psi) \\ &= -v_2 \big[(\partial_1 \Psi \cdot \partial_2 \Psi) \partial_1 \Psi - |\partial_1 \Psi|^2 \partial_2 \Psi\big] + v_1 \big[|\partial_2 \Psi|^2 \partial_1 \Psi - (\partial_1 \Psi \cdot \partial_2 \Psi) \partial_2 \Psi\big] \\ &= FA\, (v_1, v_2, 0)^{\top} = \rho^2 F F^{-1} F^{-\top}\, (v_1, v_2, 0)^{\top} = \rho^2 F^{-\top}\big((\hat{e} \times v) \times \hat{e}\big).\end{aligned}$$

This completes the proof. □

Theorem 2.104 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $0 < \ell < L_3$ and $R := Q \times (-\ell, 0)$. In the following arguments we will drop the index "j" and consider $\tilde{\Psi}$ restricted to R. Then the following assertions are true.*

(a) The mapping

$$H^1_Q(\tilde{\Psi}(R)) \ni u \;\mapsto\; v := u \circ \tilde{\Psi} \in H^1_Q(R)$$

is well-defined and belongs to $\mathcal{L}_{\mathrm{is}}\big(H^1_Q(\tilde{\Psi}(R)), H^1_Q(R)\big)$. Its inverse is given by $H^1_Q(R) \ni v \;\mapsto\; u := v \circ \tilde{\Psi}^{-1} \in H^1_Q(\tilde{\Psi}(R))$. Furthermore,

$$\begin{aligned} \nabla v &= (\tilde{\Psi}')^{\top}[(\nabla u) \circ \tilde{\Psi}], && u \in H^1_Q(\tilde{\Psi}(R)),\\ \nabla u &= [(\tilde{\Psi}')^{-\top} \circ \tilde{\Psi}^{-1}][(\nabla v) \circ \tilde{\Psi}^{-1}], && v \in H^1_Q(R). \end{aligned}$$

(b) The mapping

$$H_Q(\mathrm{curl}, \tilde{\Psi}(R)) \ni u \;\mapsto\; v := (\tilde{\Psi}')^{\top}(u \circ \tilde{\Psi}) \in H_Q(\mathrm{curl}, R)$$

is well-defined and belongs to $\mathcal{L}_{\mathrm{is}}\big(H_Q(\mathrm{curl}, \tilde{\Psi}(R)), H_Q(\mathrm{curl}, R)\big)$. Its inverse is given by $H_Q(\mathrm{curl}, R) \ni v \;\mapsto\; u := (\tilde{\Psi}' \circ \tilde{\Psi}^{-1})^{-\top}(v \circ \tilde{\Psi}^{-1}) \in H_Q(\mathrm{curl}, \tilde{\Psi}(R))$. Furthermore,

$$\begin{aligned} \mathrm{curl}\, v &= (\tilde{\Psi}')^{-1}[(\mathrm{curl}\, u) \circ \tilde{\Psi}], && u \in H_Q(\mathrm{curl}, \tilde{\Psi}(R)),\\ \mathrm{curl}\, u &= (\tilde{\Psi}' \circ \tilde{\Psi}^{-1})[(\mathrm{curl}\, v) \circ \tilde{\Psi}^{-1}], && v \in H_Q(\mathrm{curl}, R). \end{aligned}$$

Proof: *(a)*. Let $u \in H^1_Q(\tilde{\Psi}(R))$. By Theorem 2.93, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_Q\big(\overline{\tilde{\Psi}(R)}\big)$, converging to u with respect to $\|\cdot\|_{H^1_Q(\tilde{\Psi}(R))}$. Set $v_n := u_n \circ \tilde{\Psi}$ for all $n \in \mathbb{N}$. Due to Corollary 2.77, $v_n \in H^1_Q(R)$ with $\nabla v_n = (\tilde{\Psi}')^{\top}[(\nabla u_n) \circ \tilde{\Psi}]$ for all $n \in \mathbb{N}$. Therefore, the convergence from above, together with the transformation formula, implies

$$\begin{aligned} u_n \circ \tilde{\Psi} &\to v && \text{in } L^2(R),\\ \nabla v_n &\to (\tilde{\Psi}')^{\top}[(\nabla u) \circ \tilde{\Psi}] && \text{in } L^2(R, \mathbb{C}^3), \end{aligned}$$

the latter because of essential boundedness of the entries in $(\tilde{\Psi}')^\top$, see Rademacher's result. Let $\chi \in C^\infty_{Q,0}(R)$. Then

$$\begin{aligned}\int_R v(x)\nabla\chi(x)\,\mathrm{d}x &= \lim_{n\to\infty}\int_R (u_n\circ\tilde{\Psi})(x)\nabla\chi(x)\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_R \nabla v_n(x)\chi(x)\,\mathrm{d}x = \int_R (\tilde{\Psi}')^\top(x)\,[(\nabla u)\circ\tilde{\Psi}](x)\,\chi(x)\,\mathrm{d}x,\end{aligned}$$

which shows that $v \in H^1_Q(R)$ with $\nabla v = (\tilde{\Psi}')^\top[(\nabla u)\circ\tilde{\Psi}]$. Hence, the mapping is well-defined. Its linearity is clear and its boundedness follows easily by means of the formulas for v and ∇v.
Analogously, one shows that the mapping $H^1_Q(R) \ni v \mapsto u := v\circ\tilde{\Psi}^{-1} \in H^1_Q(\tilde{\Psi}(R))$ is well-defined, linear and bounded. And finally, the bijectivity follows from $u = u\circ\tilde{\Psi}\circ\tilde{\Psi}^{-1}$ and $v = v\circ\tilde{\Psi}^{-1}\circ\tilde{\Psi}$.
(b). The assertions are shown with the same arguments as in part *(a)*, while we cite here Lemma 2.103 instead of Corollary 2.77. □

To use results from Subsection 2.1.2, after having transformed a Q-periodic function defined on a cell set D of Lipschitz layer type to a Q-periodic function defined on the half-cuboid Q_3^-, we often have to extend those functions in a further step to Q-periodic functions defined on the cuboid Q_3 by a reflection technique as described in the following proposition. For this recall (1.4) for the definition of the reflection operator "$*$".

Proposition 2.105 *Let $\ell > 0$ and set $R := Q\times(-\ell,\ell)$, $R^- := Q\times(-\ell,0)$ and $R^+ := Q\times(0,\ell)$. Then the following assertions are true.*

(a) For $v \in H^1_Q(R^-)$ define

$$\hat{v} := \begin{cases} v & \textit{on } R^-,\\ v(\cdot^*) & \textit{on } R^+,\end{cases} \quad \textit{and} \quad \hat{w} := \begin{cases} \nabla v & \textit{on } R^-,\\ (\nabla v)^*(\cdot^*) & \textit{on } R^+.\end{cases}$$

Then $\hat{v} \in H^1_Q(R)$ with $\nabla\hat{v} = \hat{w}$. Moreover, the mapping $H^1_Q(R^-) \ni v \mapsto \hat{v} \in H^1_Q(R)$ is linear and bounded.

(b) For $v \in H_Q(\mathrm{curl}, R^-)$ define

$$\hat{v} := \begin{cases} v & \textit{on } R^-,\\ v^*(\cdot^*) & \textit{on } R^+,\end{cases} \quad \textit{and} \quad \hat{w} := \begin{cases} \mathrm{curl}\, v & \textit{on } R^-,\\ -(\mathrm{curl}\, v)^*(\cdot^*) & \textit{on } R^+.\end{cases}$$

Then $\hat{v} \in H_Q(\mathrm{curl}, R)$ *with* $\mathrm{curl}\,\hat{v} = \hat{w}$. *Moreover, the mapping* $H_Q(\mathrm{curl}, R^-) \ni v \mapsto \hat{v} \in H_Q(\mathrm{curl}, R)$ *is linear and bounded.*

Proof: We only show the assertion for the vector valued case as the argumentation for the scalar valued case is completely analogous.
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, $x \to T(x) := x^* = (x_1, x_2, -x_3)^\top$. Then

$$T'(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad x \in \mathbb{R}^3,$$

with $|\det(T'(x))| = 1$ for all $x \in \mathbb{R}^3$. Moreover, there exists T^{-1} coinciding with T. We need this observation in the following arguments when we apply the transformation theorem.
Let $v \in H_Q(\mathrm{curl}, R^-)$. Take some $\chi \in C^\infty_{Q,0}(R, \mathbb{C}^3)$. Choose a sequence $(v_n)_{n\in\mathbb{N}}$ in $C^\infty_Q(\overline{R^-}, \mathbb{C}^3)$ with $v_n \to v$ in $H_Q(\mathrm{curl}, R^-)$, as $n \to \infty$. Due to $v_n \to v$ and $\mathrm{curl}\, v_n \to \mathrm{curl}\, v$ in $L^2(R^-, \mathbb{C}^3)$ and because of

$$\begin{aligned}\int_{R^+} \left|v_n^*(T(x)) - v^*(T(x))\right|^2 \mathrm{d}x &= \int_{R^-} \left|\big(v_n(y) - v(y)\big)^*\right|^2 \mathrm{d}y \\ &= \int_{R^-} |v_n(y) - v(y)|^2 \,\mathrm{d}y,\end{aligned}$$

for all $n \in \mathbb{N}$, we have $v_n^*(\cdot^*) \to v^*(\cdot^*)$ and analogously $(\mathrm{curl}\, v_n)^*(\cdot^*) \to (\mathrm{curl}\, v)^*(\cdot^*)$ in $L^2(R^+, \mathbb{C}^3)$, as $n \to \infty$. Let $n \in \mathbb{N}$. Then, using Equation (A.6b), we have

$$\begin{aligned}&\int_{R^-} v_n(x) \cdot \mathrm{curl}\,\chi(x)\,\mathrm{d}x \\ &\qquad = \int_{R^-} \mathrm{curl}\, v_n(x) \cdot \chi(x)\,\mathrm{d}x - \int_{Q\times\{0\}} \left(\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \times v_n(x)\right) \cdot \chi(x)\,\mathrm{d}s.\end{aligned}$$

Using in addition $\mathrm{curl}(\chi^* \circ T^{-1})(y) = -[(\mathrm{curl}\,\chi)^* \circ T^{-1}](y)$, for all $y \in T(R)$, we analogously obtain

$$\begin{aligned}\int_{R^+} v_n^*(T(x)) \cdot \mathrm{curl}\,\chi(x)\,\mathrm{d}x &= \int_{R^-} v_n^*(y) \cdot [(\mathrm{curl}\,\chi) \circ T^{-1}](y)\,\mathrm{d}y \\ &= \int_{R^-} v_n(y) \cdot [(\mathrm{curl}\,\chi)^* \circ T^{-1}](y)\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= -\int_{R^-} v_n(y) \cdot \operatorname{curl}(\chi^* \circ T^{-1})(y)\, \mathrm{d}y \\
&= -\int_{R^-} \operatorname{curl} v_n(y) \cdot (\chi^* \circ T^{-1})(y)\, \mathrm{d}y \\
&\qquad + \int_{Q\times\{0\}} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times v_n(y) \right) \cdot \underbrace{(\chi^* \circ T^{-1})(y)}_{=\chi^*(y)}\, \mathrm{d}s \\
&= -\int_{R^+} (\operatorname{curl} v_n)^*(T(x)) \cdot \chi(x)\, \mathrm{d}x \\
&\qquad + \int_{Q\times\{0\}} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times v_n(y) \right) \cdot \chi(y)\, \mathrm{d}s,
\end{aligned}
$$

where we have exploited the fact, that in the last cross product the third component is zero. Note that in the sum of both integral identities the boundary terms vanish. Hence,

$$
\begin{aligned}
&\int_R \hat{v}(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x \\
&\quad = \int_{R^-} v(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x + \int_{R^+} v^*(x^*) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x \\
&\quad = \lim_{n\to\infty} \left(\int_{R^-} v_n(x) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x + \int_{R^+} v_n^*(x^*) \cdot \operatorname{curl} \chi(x)\, \mathrm{d}x \right) \\
&\quad = \lim_{n\to\infty} \left(\int_{R^-} \operatorname{curl} v_n(x) \cdot \chi(x)\, \mathrm{d}x - \int_{R^+} (\operatorname{curl} v_n)^*(x^*) \cdot \chi(x)\, \mathrm{d}x \right) \\
&\quad = \int_{R^-} \operatorname{curl} v(x) \cdot \chi(x)\, \mathrm{d}x - \int_{R^+} (\operatorname{curl} v)^*(x^*) \cdot \chi(x)\, \mathrm{d}x,
\end{aligned}
$$

which shows that $\hat{v} \in H_Q(\operatorname{curl}, R)$ with variational curl as given in the formula from the assertion. Furthermore, it is easy to see that

$$
\begin{aligned}
\|v^*(\cdot^*)\|_{L^2(R^+,\mathbb{C}^3)} &= \|v\|_{L^2(R^-,\mathbb{C}^3)}, \\
\|(\operatorname{curl} v)^*(\cdot^*)\|_{L^2(R^+,\mathbb{C}^3)} &= \|\operatorname{curl} v\|_{L^2(R^-,\mathbb{C}^3)}.
\end{aligned}
$$

Therefore, $\|\hat{v}\|_{H(\operatorname{curl},R)} = \sqrt{2}\|v\|_{H(\operatorname{curl},R^-)}$ and the proof is complete. □

2.3.3. Trace and Extension Operators

Next, we want to derive trace and extension results for Q-periodic functions defined on a cell set D of Lipschitz layer type by following the ideas in [34] for bounded Lipschitz domains. This requires some technical efforts.

Let $d' \in \mathbb{N}$, $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0,1\}$. The space $L^2(\Gamma_j, \mathbb{C}^{d'})$ of square integrable functions on the surface patch Γ_j of D can be characterized by $\varphi \in L^2(\Gamma_j, \mathbb{C}^{d'})$, if and only if $\varphi \circ \Psi_j \in L^2(Q, \mathbb{C}^{d'})$, because of

$$\|\varphi\|^2_{L^2(\Gamma_j,\mathbb{C}^{d'})} = \int_{\Gamma_j} |\varphi(y)|^2 \,\mathrm{d}s = \int_Q |\varphi(\Psi_j(x))|^2 \,\rho_j(x)\,\mathrm{d}x, \tag{2.15a}$$

where ρ_j was defined in (2.13), and the estimate

$$1 \le \rho_j(x) \le \sqrt{1 + \|\nabla f_j\|^2_\infty}, \qquad \text{for a.a. } x \in Q. \tag{2.15b}$$

We define the subspace of $L^2(\Gamma_j, \mathbb{C}^3)$ of tangential vector fields by

$$L^2_{\mathrm{t}}(\Gamma_j) := \big\{\varphi \in L^2(\Gamma_j, \mathbb{C}^3) \mid \mathrm{n}(y)\cdot\varphi(y) = 0 \ \text{ for a.a. } y \in \Gamma_j\big\}.$$

Definition 2.106 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0,1\}$.*

(a) We define the space $H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'})$ by

$$H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}) := \Big\{\varphi \in L^2(\Gamma_j, \mathbb{C}^{d'}) \;\Big|\; \varphi \circ \Psi_j \in H^{1/2}_{\mathrm{per}}(Q, \mathbb{C}^{d'})\Big\}$$

with norm

$$\|\varphi\|_{H_Q^{1/2}(\Gamma_j,\mathbb{C}^{d'})} := \|\varphi \circ \Psi_j\|_{H^{1/2}_{\mathrm{per}}(Q,\mathbb{C}^{d'})}, \quad \varphi \in H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}).$$

(b) We define the spaces $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$ as the completion of

$$\big\{\varphi \in L^2_{\mathrm{t}}(\Gamma_j) \mid \tilde{\varphi}^t \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q)\big\},$$

$$\left\{\varphi \in L^2_{\mathrm{t}}(\Gamma_j) \mid \tilde{\varphi}^T \in H^{-1/2}_{\mathrm{per}}(\mathrm{Curl}, Q)\right\},$$

with respect to the norms

$$\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_j)} := \|\tilde{\varphi}^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)},$$
$$\|\varphi\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_j)} := \|\tilde{\varphi}^T\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Curl},Q)},$$

where

$$\tilde{\varphi}^t(x) := \rho_j(x)\, F_j^{-1}(x)\, \varphi\big(\Psi_j(x)\big), \qquad \textit{for a.a. } x \in Q, \tag{2.16a}$$
$$\tilde{\varphi}^T(x) := F_j^{\top}(x)\, \varphi\big(\Psi_j(x)\big), \qquad \textit{for a.a. } x \in Q, \tag{2.16b}$$

respectively.

Note that in contrast to Definition 2.45 no partition of unity is required here. Furthermore, due to (2.15) and Proposition 2.13, there exist constants $c_1, c_2 > 0$ such that for all $\varphi \in H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'})$ there holds

$$\begin{aligned}\|\varphi\|_{L^2(\Gamma_j,\mathbb{C}^{d'})} &\le c_1\, \|\varphi \circ \Psi_j\|_{L^2(Q,\mathbb{C}^{d'})}\\ &\le c_2\, \|\varphi \circ \Psi_j\|_{H^{1/2}_{\mathrm{per}}(Q,\mathbb{C}^{d'})} = c_2\, \|\varphi\|_{H_Q^{1/2}(\Gamma_j,\mathbb{C}^{d'})}.\end{aligned}$$

A closer look at Proposition 2.13 even shows that the embedding

$$H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}) \overset{\mathrm{id}}{\hookrightarrow} L^2(\Gamma_j, \mathbb{C}^{d'}) \tag{2.17}$$

is compact, see Theorem 2.122.

Theorem 2.107 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0, 1\}$.*

(a) The trace operator

$$\gamma_{0,\Gamma_j} : C_Q^\infty(\overline{D}, \mathbb{C}^{d'}) \to H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}), \quad u \mapsto u|_{\Gamma_j},$$

has a bounded extension from $H_Q^1(D, \mathbb{C}^{d'})$ to $H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'})$, which we also denote by γ_{0,Γ_j}. Furthermore, there exists a bounded right inverse $\eta_{0,\Gamma_j} : H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}) \to H_Q^1(D, \mathbb{C}^{d'})$ of γ_{0,Γ_j}. Moreover,

$$\eta_{0,\Gamma_0}\big(H_Q^{1/2}(\Gamma_0, \mathbb{C}^{d'})\big) \subseteq \mathcal{H}^1_{Q,0,\Gamma_1}(D, \mathbb{C}^{d'}),$$
$$\eta_{0,\Gamma_1}\big(H_Q^{1/2}(\Gamma_1, \mathbb{C}^{d'})\big) \subseteq \mathcal{H}^1_{Q,0,\Gamma_0}(D, \mathbb{C}^{d'}).$$

(b) The following assertions are true.

(i) The trace operator

$$\gamma_{t,\Gamma_j} : C_Q^\infty(\overline{D}, \mathbb{C}^3) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma_j), \quad u \mapsto \mathrm{n}_j \times u|_{\Gamma_j},$$

has a bounded extension from $H_Q(\mathrm{curl}, D)$ *to* $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$, *which we also denote by* γ_{t,Γ_j}. *Furthermore, there exists a bounded right inverse* $\eta_{t,\Gamma_j} : H_Q^{-1/2}(\mathrm{Div}, \Gamma_j) \to H_Q(\mathrm{curl}, D)$ *of* γ_{t,Γ_j}. *Moreover,*

$$\eta_{t,\Gamma_0}\big(H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)\big) \subseteq \mathcal{H}_{Q,0,\Gamma_1}(\mathrm{curl}, D),$$
$$\eta_{t,\Gamma_1}\big(H_Q^{-1/2}(\mathrm{Div}, \Gamma_1)\big) \subseteq \mathcal{H}_{Q,0,\Gamma_0}(\mathrm{curl}, D).$$

(ii) The trace operator

$$\gamma_{T,\Gamma_j} : C_Q^\infty(\overline{D}, \mathbb{C}^3) \to H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j), \quad u \mapsto (\mathrm{n}_j \times u|_{\Gamma_j}) \times \mathrm{n}_j,$$

has a bounded extension from $H_Q(\mathrm{curl}, D)$ *to* $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$, *which we also denote by* γ_{T,Γ_j}. *Furthermore, there exists a bounded right inverse* $\eta_{T,\Gamma_j} : H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j) \to H_Q(\mathrm{curl}, D)$ *of* γ_{T,Γ_j}. *Moreover,*

$$\eta_{T,\Gamma_0}\big(H_Q^{-1/2}(\mathrm{Curl}, \Gamma_0)\big) \subseteq \mathcal{H}_{Q,0,\Gamma_1}(\mathrm{curl}, D),$$
$$\eta_{T,\Gamma_1}\big(H_Q^{-1/2}(\mathrm{Curl}, \Gamma_1)\big) \subseteq \mathcal{H}_{Q,0,\Gamma_0}(\mathrm{curl}, D).$$

Proof: We start with part *(b)* and postpone part *(a)* to the end of this proof.
(b). Let $u \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$. We start with $j = 0$ and show that for

$$\tilde{\varphi}^t(x) := \rho_0(x)\, F_0^{-1}(x)\, (\mathrm{n}_0 \times u)\big(\Psi_0(x)\big), \qquad \text{for a.a. } x \in Q,$$

there holds $\tilde{\varphi}^t \in H_{\mathrm{per}}^{-1/2}(\mathrm{Div}, Q)$. For this we consider the product $\phi_0\, u$ and note that by construction of ϕ_0, we have $\phi_0\, u \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$ with $\phi_0\, u$ vanishing in a neighborhood of Γ_1 and with $\mathrm{n}_0 \times u|_{\Gamma_0} = \mathrm{n}_0 \times (\phi_0\, u)|_{\Gamma_0}$.

By Proposition 2.95 and Proposition 2.101, $(\phi_0\, u)\big|_0^{\tilde{\Psi}_0(Q_3^-)}$ belongs to $H_Q(\mathrm{curl}, \tilde{\Psi}_0(Q_3^-))$ and we have that this operation is bounded. Set

$$v(x) := \tilde{\Psi}_0'(x)^\top \Big[(\phi_0\, u)\,\Big|_0^{\tilde{\Psi}_0(Q_3^-)} \circ \tilde{\Psi}_0(x)\Big], \quad \text{for a.a. } x \in Q_3^- \cup (Q \times \{0\}).$$

Now, by Theorem 2.104, $v \in H_Q(\mathrm{curl}, Q_3^-)$ and, by Lemma 2.103, $\tilde{\varphi}^t = \hat{e} \times v(\cdot, 0)$ almost everywhere on Q, where $\hat{e} = (0, 0, -1)^\top$. Define

$$\hat{v}(x) := \begin{cases} v(x), & \text{for a.a. } x \in Q_3^- \cup (Q \times \{0\}), \\ v^*(x^*), & \text{for a.a. } x \in Q_3^+, \end{cases} \tag{$*$}$$

where the symbol "$*$" denotes the reflection operator given by $\mathbb{C}^3 \ni a = (a_1, a_2, a_3)^\top \mapsto a^* := (a_1, a_2, -a_3)^\top \in \mathbb{C}^3$, see also Section 1.3. Due to Proposition 2.105, we have $\hat{v} \in H_Q(\mathrm{curl}, Q_3)$. By Proposition 2.83, we even have $\hat{v} \in H_{Q,0}(\mathrm{curl}, Q_3)$. Moreover, $\hat{e} \times \hat{v}(\cdot, 0) = \hat{e} \times v(\cdot, 0)$ almost everywhere on Q. Therefore

$$\begin{aligned} \|\mathrm{n}_0 \times u|_{\Gamma_0}\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_0)} &= \|\tilde{\varphi}^t\|_{H_{\mathrm{per}}^{-1/2}(\mathrm{Div},Q)} = \|\hat{e} \times \hat{v}(\cdot, 0)\|_{H_{\mathrm{per}}^{-1/2}(\mathrm{Div},Q)} \\ &\le C_1 \|\hat{v}\|_{H_{\mathrm{per}}(\mathrm{curl},Q_3)} = C_1 \|\hat{v}\|_{H_Q(\mathrm{curl},Q_3)} \le C_2 \|v\|_{H_Q(\mathrm{curl},Q_3^-)} \\ &\le C_3 \Big\|(\phi_0\, u)\,\Big|_0^{\tilde{\Psi}_0(Q_3^-)}\Big\|_{H_Q(\mathrm{curl},\tilde{\Psi}_0(Q_3^-))} \le C_3 \|\phi_0\, u\|_{H_Q(\mathrm{curl},D)} \\ &\le C_4 \|u\|_{H_Q(\mathrm{curl},D)}, \end{aligned}$$

where we have applied Theorem 2.23, Proposition 2.67, Proposition 2.105, Theorem 2.104, Proposition 2.95 and finally Proposition 2.78.
Now to the construction of the extension operator η_{t,Γ_0}. Again, this is done by continuous extension. First of all we note that by Theorem 2.90 $\phi_0 \equiv 1$ in a neighborhood of Γ_0 in D. Therefore, there exists $0 < \delta < L_3$ such that $\tilde{\Psi}_0(Q \times (-\delta, 0))$ is a subset of this neighborhood. As part of the construction, this δ is assumed to be fix.
Let $\varphi \in \big\{\psi \in L_{\mathrm{t}}^2(\Gamma_0) \mid \tilde{\psi}^t \in H_{\mathrm{per}}^{-1/2}(\mathrm{Div}, Q)\big\}$. Then $\tilde{\varphi}^t \in H_{\mathrm{per}}^{-1/2}(\mathrm{Div}, Q)$ and by Theorem 2.23, there exists $\hat{v} \in H_{\mathrm{per}}(\mathrm{curl}, Q_3)$ such that $\gamma_{t,\mathrm{per}} \hat{v} = \tilde{\varphi}^t$. Note that by Observation 2.25 there holds $\hat{v}|_{Q_3^+} = \big(\hat{v}|_{Q_3^-}\big)^*(\cdot^*)$ on Q_3^+.
Next, consider

$$\chi_0(x) := \int_{Q \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)} \widetilde{\phi}_{\frac{\delta}{4}}(x - y)\, \mathrm{d}y, \qquad x \in Q_3,$$

where $\widetilde{\phi}_{\frac{\delta}{4}}$ denotes the Q-periodic extension of $\phi_{\frac{\delta}{4}}$ from (2.6). Then $\chi_0 \in C^\infty_{\text{per}}(Q_3)$ with $\chi_0 \equiv 1$ in a neighborhood of $Q \times \{0\}$ and with $\chi_0 \equiv 0$ in $Q \times \big((-L_3, -\frac{3}{4}\delta) \cup (\frac{3}{4}\delta, L_3)\big)$. We set $v := \chi_0 \hat{v}$. Then, by Proposition 2.21, $v \in H_{\text{per}}(\text{curl}, Q_3)$ and, by Lemma 2.24, $\gamma_{t,\text{per}} v = \gamma_{t,\text{per}} \hat{v} = \tilde{\varphi}^t$. Moreover, we still have $v|_{Q_3^+} = \big(v|_{Q_3^-}\big)^*(\cdot^*)$ on Q_3^+. Furthermore, due to Remark 2.63, $v \in H_Q(\text{curl}, Q_3)$. Finally, we define

$$w := T^{-1}\, v|_{Q_3^-},$$

with $T : H_Q(\text{curl}, \tilde{\Psi}_0(Q_3^-)) \to H_Q(\text{curl}, Q_3^-)$ denoting the curl-preserving transformation from Theorem 2.104, and we define u as restriction of w onto D. Then $u \in H_Q(\text{curl}, D)$ and all operators involved in the construction were bounded, see in particular Proposition 2.68.
Now, we show that $\gamma_{t,\Gamma_0} u = \varphi$. In fact, due to the choice of δ, we have $\phi_0 u = u$, meaning that $u \in \mathcal{H}_{Q,0,\Gamma_1}(\text{curl}, D)$, which yields that u can be extended by zero to an element of $H_Q(\text{curl}, \tilde{\Psi}_0(Q_3^-))$ coinciding with w. Therefore, $T(\phi_0 u) = v|_{Q_3^-}$. From above we know that $v|_{Q_3^+} = \big(v|_{Q_3^-}\big)^*(\cdot^*)$ on Q_3^+. Therefore, the image of $v|_{Q_3^-}$ under the reflection operator given by $(*)$ coincides with v and we are done, because from above we have $\gamma_{t,\text{per}} v = \tilde{\varphi}^t$.
For the case $j = 1$ we follow the lines from above completely analogous. And finally, the statements with respect to γ_{T,Γ_j} and η_{T,Γ_j}, $j = 0, 1$, are obtained analogously as well.
(a). We only show the assertion for the case $d' = 1$ as the generalization to the case $d' > 1$ is obvious.
Let $u \in C^\infty_Q(\overline{D})$. Similarly to part *(b)* we have that $\phi_0\, u$ vanishes in a neighborhood of Γ_1 and that therefore $(\phi_0\, u)\big|_0^{\tilde{\Psi}_0(Q_3^-)}$ belongs to $H^1_Q(\tilde{\Psi}_0(Q_3^-))$. We set

$$v(x) := (\phi_0\, u)\big|_0^{\tilde{\Psi}_0(Q_3^-)} \circ \tilde{\Psi}_0(x), \quad x \in Q_3^- \cup (Q \times \{0\}).$$

Note that $v(x, 0) = u(\Psi_0(x))$ for all $x \in Q$. By Theorem 2.104, v belongs to $H^1_Q(Q_3^-)$. We define

$$\hat{v}(x) := \begin{cases} v(x), & x \in Q_3^- \cup (Q \times \{0\}), \\ v(x^*), & x \in Q_3^+ \end{cases}$$

and have that $\hat{v} \in H^1_Q(Q_3)$, see Proposition 2.105. By Proposition 2.83, we even have $\hat{v} \in H^1_{Q,0}(Q_3)$. Moreover, $\hat{v}(\cdot, 0) = v(\cdot, 0)$. Using now the same arguments as in part *(b)*, we obtain

$$\|u|_{\Gamma_0}\|_{H_Q^{1/2}(\Gamma_j)} = \|\hat{v}(\cdot,0)\|_{H^{1/2}_{\mathrm{per}}(Q)} \leq C\,\|u\|_{H^1_Q(D)}.$$

Now to the construction of the extension operator η_{0,Γ_0}. Let $\varphi \in H_Q^{1/2}(\Gamma_0)$. Then $\varphi \circ \Psi_0 \in H^{1/2}_{\mathrm{per}}(Q)$ and by Theorem 2.23 there exists $\hat{v} \in H^1_{\mathrm{per}}(Q_3)$ such that $\gamma_{0,\mathrm{per}}\hat{v} = \varphi \circ \Psi_0$. Now we follow the arguments as in part *(b)* but for the scalar valued case and obtain $w \in H^1_Q(\tilde{\Psi}_0(Q_3^-))$. Again, we define u as restriction of w onto D. Then $u \in H^1_Q(D)$ and all involved operations were bounded. To show that $\gamma_{0,\Gamma_0} u = \varphi$, we apply now the separate operations in the construction of γ_{0,Γ_0} to u. Indeed, analogous to part *(b)*, these operations yield $v = \chi_0 \hat{v}$. Applying Lemma 2.24 and then $\gamma_{0,\mathrm{per}}$ from Theorem 2.23, we arrive at $\varphi \circ \Psi_0$ as desired.
Finally, the case $j = 1$ is shown completely analogous and the proof is complete. □

By means of the trace and extension operators we easily obtain the next denseness results.

Corollary 2.108 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0,1\}$. Then the following assertions are true.*

(a) The space

$$\mathcal{D}_{Q,0}(\Gamma_j, \mathbb{C}^{d'}) := \left\{ u|_{\Gamma_j} \,\middle|\, u \in C_Q^\infty(\overline{D}, \mathbb{C}^{d'}) \right\}$$

is dense in $H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'})$.

(b) The spaces

$$\mathcal{D}_{Q,t}(\Gamma_j, \mathbb{C}^3) := \left\{ \mathrm{n}_j \times u|_{\Gamma_j} \,\middle|\, u \in C_Q^\infty(\overline{D}, \mathbb{C}^3) \right\},$$
$$\mathcal{D}_{Q,T}(\Gamma_j, \mathbb{C}^3) := \left\{ (\mathrm{n}_j \times u|_{\Gamma_j}) \times \mathrm{n}_j \,\middle|\, u \in C_Q^\infty(\overline{D}, \mathbb{C}^3) \right\}$$

are dense in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$, respectively.

Proof: *(a).* The proof follows closely the argumentation as in part *(b)* from below and we leave the details to the reader.
(b). Let $j \in \{0,1\}$. We only show the assertion for $\mathcal{D}_{Q,t}(\Gamma_j,\mathbb{C}^3)$ since the argumentation for $\mathcal{D}_{Q,T}(\Gamma_j,\mathbb{C}^3)$ is completely analogous. By Theorem 2.107, $\mathcal{D}_{Q,t}(\Gamma_j,\mathbb{C}^3)$ is a subspace of $H_Q^{-1/2}(\mathrm{Div},\Gamma_j)$. Let $\varphi \in H_Q^{-1/2}(\mathrm{Div},\Gamma_j)$. Then again by Theorem 2.107, $u := \eta_{t,\Gamma_j}\varphi \in H_Q(\mathrm{curl},D)$. Since $C_Q^\infty(\overline{D},\mathbb{C}^3)$ is dense in $H_Q(\mathrm{curl},D)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_Q^\infty(\overline{D},\mathbb{C}^3)$ converging to u in $H_Q(\mathrm{curl},D)$. Therefore, again by Theorem 2.107, $(\gamma_{t,\Gamma_j}u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}_{Q,t}(\Gamma_j,\mathbb{C}^3)$ with $\gamma_{t,\Gamma_j}u_n \to \gamma_{t,\Gamma_j}u = \varphi$, as $n\to\infty$. □

Remark 2.109 *Consulting the proof of Theorem 2.107 and* (2.16)*, we have for $\varphi \in \mathcal{D}_{Q,t}(\Gamma_j,\mathbb{C}^3)$ and $\psi \in \mathcal{D}_{Q,T}(\Gamma_j,\mathbb{C}^3)$ that $\tilde{\varphi}^t$ and $\tilde{\psi}^T$ belong to $L^2(Q,\mathbb{C}^2)\cap H_{\mathrm{per}}^{-1/2}(\mathrm{Div},Q)$ and $L^2(Q,\mathbb{C}^2)\cap H_{\mathrm{per}}^{-1/2}(\mathrm{Curl},Q)$, respectively, $j=0,1$.*

Trace Operator in $H_Q(\mathrm{div},D)$. To show later the existence of a solution to the variational formulation of our scattering problem of interest, we will also need a trace theorem for elements in $H_Q(\mathrm{div},D)$. Thanks to the integral identity (A.6c), this can be done with almost no effort by interpreting the right hand side in (A.6c) as duality pairing, see below. We follow closely the lines at the end of [34, Subsection 5.1.1], see in particular [34, Definition 5.17], where the authors introduced the normal derivative for elements from a certain subspace of $H^1(\Omega)$, with $\Omega \subseteq \mathbb{R}^3$ therein a bounded Lipschitz domain.

Definition 2.110 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91. Furthermore, let $j \in \{0,1\}$. We define $H_Q^{-1/2}(\Gamma_j,\mathbb{C}^{d'})$ to be the dual space of $H_Q^{1/2}(\Gamma_j,\mathbb{C}^{d'})$ equipped with its canonical norm*

$$\|\ell\|_{H_Q^{-1/2}(\Gamma_j,\mathbb{C}^{d'})} := \sup_{\psi\in H_Q^{1/2}(\Gamma_j,\mathbb{C}^{d'})\setminus\{0\}} \frac{|\langle \ell,\psi\rangle|}{\|\psi\|_{H_Q^{1/2}(\Gamma_j,\mathbb{C}^{d'})}}$$

for all $\ell \in H_Q^{-1/2}(\Gamma_j,\mathbb{C}^{d'})$. Here, $\langle \ell,\psi\rangle$ denotes the duality pairing as introduced in Section 1.3.

In the following considerations let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0, 1\}$. Take some $u \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$ and $\psi \in H_Q^1(D)$. Since $C_Q^\infty(\overline{D})$ is dense in $H_Q^1(D)$, see Theorem 2.93, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C_Q^\infty(\overline{D})$ which converges to ψ with respect to $\|\cdot\|_{H_Q^1(D)}$. The continuity of the trace operators then implies that $\gamma_{0,\Gamma_j}\psi_n \to \gamma_{0,\Gamma_j}\psi$ in $H_Q^{1/2}(\Gamma_j)$, as $n \to \infty$. And by (2.17) we have that $(\gamma_{0,\Gamma_j}\psi_n)_{n\in\mathbb{N}}$ converges to $\gamma_{0,\Gamma_j}\psi$ also in $L^2(\Gamma_j)$. Therefore, taking also (2.14c) into account,

$$\begin{aligned}\int_D (\psi \operatorname{div} u + u \cdot \nabla\psi)\,\mathrm{d}x &= \lim_{n\to\infty} \int_D (\psi_n \operatorname{div} u + u \cdot \nabla\psi_n)\,\mathrm{d}x \\ = \lim_{n\to\infty} \sum_{j=0}^{1} \int_{\Gamma_j} (\mathrm{n} \cdot u)\,\gamma_{0,\Gamma_j}\psi_n\,\mathrm{d}s &= \sum_{j=0}^{1} \int_{\Gamma_j} (\mathrm{n} \cdot u)\,\gamma_{0,\Gamma_j}\psi\,\mathrm{d}s.\end{aligned}$$

Hence, the element $u \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$ can be assigned traces $\mathrm{n}_j \cdot u|_{\Gamma_j}$, which in turn can be considered as linear functionals from $H_Q^{1/2}(\Gamma_j)$ to $\mathbb{C}$ via the surface integral over Γ_j on the right hand side or, even better, via the integral over D on the left hand side. Since the latter integral is also well-defined for elements from $H_Q(\operatorname{div}, D)$ and since, by Theorem 2.93, $C_Q^\infty(\overline{D}, \mathbb{C}^3)$ is dense in $H_Q(\operatorname{div}, D)$, via this formula the just constructed trace operator can be continuously extended to the whole space $H_Q(\operatorname{div}, D)$, see the next theorem (definition).

Theorem 2.111 (and Definition) *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0, 1\}$. The mapping $\gamma_{n,\Gamma_j} : H_Q(\operatorname{div}, D) \to H_Q^{-1/2}(\Gamma_j)$ given by*

$$\langle \gamma_{n,\Gamma_j} u, \psi \rangle := \int_D \big(\tilde{\psi} \operatorname{div} u + u \cdot \nabla\tilde{\psi}\big)\,\mathrm{d}x, \qquad \psi \in H_Q^{1/2}(\Gamma_j),$$

is well-defined, linear and bounded. Here, for the case $j = 0$, $\tilde{\psi} \in H^1_{Q,0,\Gamma_1}(D)$ is any extension of ψ into D such that $\gamma_{0,\Gamma_0}\tilde{\psi} = \psi$, while for the case $j = 1$, $\tilde{\psi} \in H^1_{Q,0,\Gamma_0}(D)$ is any extension of ψ into D such that $\gamma_{0,\Gamma_1}\tilde{\psi} = \psi$. We call the mapping γ_{n,Γ_j} trace operator for elements in $H_Q(\operatorname{div}, D)$ with respect to Γ_j.

Proof: To see that the mapping γ_{n,Γ_0} is well-defined, first of all recall from Theorem 2.107 and Proposition 2.101 that by $\tilde{\psi} := \eta_{0,\Gamma_0}\psi$ such an extension exists. Let now $\tilde{\psi}_1$ and $\tilde{\psi}_2$ be such extensions. Then, by part *(a)* from Theorem 2.116 (inspecting the proof of this part we will note that we are using arguments that we have already on hand), $\tilde{\psi} := \tilde{\psi}_1 - \tilde{\psi}_2 \in H^1_{Q,0}(D)$. Hence, there exists a sequence $(\tilde{\psi}_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(D)$ which converges to $\tilde{\psi}$ with respect to $\|\cdot\|_{H^1_Q(D)}$, and the definition of the variational divergence yields

$$\int_D \left(\tilde{\psi}\operatorname{div} u + u\cdot\nabla\tilde{\psi}\right)\mathrm{d}x = \lim_{n\to\infty}\int_D \left(\tilde{\psi}_n\operatorname{div} u + u\cdot\nabla\tilde{\psi}_n\right)\mathrm{d}x = 0.$$

The linearity of γ_{n,Γ_0} is easy to see. To show its boundedness, let $u \in H_Q(\operatorname{div}, D)$ and $\psi \in H_Q^{1/2}(\Gamma_0)$. Then, by means of the inequality of Cauchy-Schwarz and the boundedness of η_{0,Γ_0}, we obtain

$$\begin{aligned}
|\langle\gamma_{n,\Gamma_0}u,\psi\rangle| &= \left|\int_D \left(\eta_{0,\Gamma_0}\psi\operatorname{div} u + u\cdot\nabla(\eta_{0,\Gamma_0}\psi)\right)\mathrm{d}x\right| \\
&\le \|\operatorname{div} u\|_{L^2(D)}\|\eta_{0,\Gamma_0}\psi\|_{L^2(D)} + \|u\|_{L^2(D,\mathbb{C}^3)}\|\nabla(\eta_{0,\Gamma_0}\psi)\|_{L^2(D,\mathbb{C}^3)} \\
&\le 2\|u\|_{H_Q(\operatorname{div},D)}\|\eta_{0,\Gamma_0}\psi\|_{H^1_Q(D)} \le 2\|u\|_{H_Q(\operatorname{div},D)}\|\eta_{0,\Gamma_0}\|\,\|\psi\|_{H_Q^{1/2}(\Gamma_0)}.
\end{aligned}$$

Hence, $\|\gamma_{n,\Gamma_0}u\|_{H_Q^{-1/2}(\Gamma_0)} \le 2\|\eta_{0,\Gamma_0}\|\,\|u\|_{H_Q(\operatorname{div},D)}$ which shows that the operator γ_{n,Γ_0} is indeed bounded.
The assertion for the case $j = 1$ is shown completely analogous. □

Remark 2.112 *Since the definition of the trace operator* γ_{n,Γ_j} *is motivated by* (A.6c)*, the sign changes in situation where the normal vector points into* D*.*

2.3.4. Greens Formula and Applications

In this subsection we will derive analogous formulas to (2.14), but now in the context for functions in $H^1_Q(D)$, $H_Q(\operatorname{curl}, D)$ and $H_Q(\operatorname{div}, D)$, respectively. The formula (2.18b) is often referred to as *Green's formula.*

Theorem 2.113 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Then the following assertions are true.*

(a) For $u, v \in H^1_Q(D)$ there holds the integration by parts formula in the following form

$$\int_D \big(u\nabla v + v\nabla u\big)\,\mathrm{d}x = \sum_{j=0}^{1} \int_{\Gamma_j} (\gamma_{0,\Gamma_j} u)\,(\gamma_{0,\Gamma_j} v)\,\mathrm{n}\,\mathrm{d}s. \tag{2.18a}$$

(b) For $j \in \{0,1\}$ the bilinear form

$$\langle\cdot,\cdot\rangle_{\Gamma_j} : \mathcal{D}_{Q,t}(\Gamma_j,\mathbb{C}^3) \times \mathcal{D}_{Q,T}(\Gamma_j,\mathbb{C}^3) \to \mathbb{C}$$

defined by

$$(\varphi,\psi) \to \langle\varphi,\psi\rangle_{\Gamma_j} := \big(\tilde{\varphi}^t \;\big|\; \overline{\tilde{\psi}^T}\,\big)_{L^2(Q,\mathbb{C}^2)},$$

where $\tilde{\varphi}^t$ and $\tilde{\psi}^T$ are given by (2.16), has a continuous extension from $H_Q^{-1/2}(\mathrm{Div},\Gamma_j) \times H_Q^{-1/2}(\mathrm{Curl},\Gamma_j)$ to $\mathbb{C}$, which we also denote by $\langle\cdot,\cdot\rangle_{\Gamma_j}$. There holds

$$|\langle\varphi,\psi\rangle_{\Gamma_j}| \le C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_j)}\,\|\psi\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_j)},$$

for all $\varphi \in H_Q^{-1/2}(\mathrm{Div},\Gamma_j)$, $\psi \in H_Q^{-1/2}(\mathrm{Curl},\Gamma_j)$ and $j = 0,1$, where $C > 0$ can be chosen as in Corollary 2.32. With these bilinear forms, there holds Green's formula in the following form

$$\begin{aligned}\int_D &\big(\operatorname{curl} u \cdot v - u \cdot \operatorname{curl} v\big)\,\mathrm{d}x \\ &= \sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} u, \gamma_{T,\Gamma_j} v\big\rangle_{\Gamma_j} = -\sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} v, \gamma_{T,\Gamma_j} u\big\rangle_{\Gamma_j}\end{aligned} \tag{2.18b}$$

for all $u, v \in H_Q(\mathrm{curl}, D)$.

(c) For $u \in H_Q(\mathrm{div}, D)$ and $\psi \in H^1_Q(D)$ there holds the formula

$$\int_D \big(\psi \operatorname{div} u + u \cdot \nabla\psi\big)\,\mathrm{d}x = \sum_{j=0}^{1} \langle \gamma_{n,\Gamma_j} u, \gamma_{0,\Gamma_j}\psi\rangle. \tag{2.18c}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing from Theorem 2.111.

Proof: *(a).* Let $u, v \in H^1_Q(D)$ and let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be sequences in $C^\infty_Q(\overline{D})$ such that $u_n \to u$, $v_n \to v$ in $H^1_Q(D)$, as $n \to \infty$. Then $\gamma_{0,\Gamma_j} u_n \to \gamma_{0,\Gamma_j} u$ in $H^{1/2}_Q(\Gamma_j)$, $j = 0, 1$. In particular, we have this convergence in $L^2(\Gamma_j)$, $j = 0, 1$. Therefore, by applying the integral identity (A.6a) and exploiting therein the Q-periodicity of the smooth functions,

$$\int_D (u\nabla v + v\nabla u)\,\mathrm{d}x = \lim_{n\to\infty} \int_D (u_n\nabla v_n + v_n\nabla u_n)\,\mathrm{d}x$$
$$= \lim_{n\to\infty} \sum_{j=0}^{1} \int_{\Gamma_j} (\gamma_{0,\Gamma_j} u_n)\,(\gamma_{0,\Gamma_j} v_n)\,\mathrm{n}\,\mathrm{d}s = \sum_{j=0}^{1} \int_{\Gamma_j} (\gamma_{0,\Gamma_j} u)\,(\gamma_{0,\Gamma_j} v)\,\mathrm{n}\,\mathrm{d}s.$$

(b). Due to Remark 2.109 and Corollary 2.32, the bilinear form is well-defined on $\mathcal{D}_{Q,t}(\Gamma_j, \mathbb{C}^3) \times \mathcal{D}_{Q,T}(\Gamma_j, \mathbb{C}^3)$ and bounded. Hence, Corollary 2.108 allows the continuous extension with the given estimate.
To verify the formula, take at first some $u, v \in C^\infty_Q(\overline{D}, \mathbb{C}^3)$ and let $j \in \{0, 1\}$. Set $\varphi := \mathrm{n}_j \times u|_{\Gamma_j}$ and $\psi := (\mathrm{n}_j \times v|_{\Gamma_j}) \times \mathrm{n}_j$. By the definition of γ_{t,Γ_j} and γ_{T,Γ_j} we can rewrite φ and ψ in the form $\varphi = \gamma_{t,\Gamma_j} u$ and $\psi = \gamma_{T,\Gamma_j} v$. Then

$$\int_{\Gamma_j} \gamma_{t,\Gamma_j} u \cdot \gamma_{T,\Gamma_j} v\,\mathrm{d}s = \int_Q \rho_j(x)\, F_j^{-1}(x)\, \varphi\big(\Psi_j(x)\big) \cdot F_j^\top(x)\, \psi\big(\Psi_j(x)\big)\,\mathrm{d}x$$
$$= \int_Q \tilde{\varphi}^t(x) \cdot \tilde{\psi}^T(x)\,\mathrm{d}x = \langle \varphi, \psi \rangle_{\Gamma_j} = \big\langle \gamma_{t,\Gamma_j} u, \gamma_{T,\Gamma_j} v \big\rangle_{\Gamma_j}.$$

Now, let $u, v \in H_Q(\mathrm{curl}, D)$ and let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be sequences in $C^\infty_Q(\overline{D}, \mathbb{C}^3)$ such that $u_n \to u$, $v_n \to v$ in $H_Q(\mathrm{curl}, D)$, as $n \to \infty$. Then $\gamma_{t,\Gamma_j} u_n \to \gamma_{t,\Gamma_j} u$ and $\gamma_{T,\Gamma_j} v_n \to \gamma_{T,\Gamma_j} v$ in $H^{-1/2}_Q(\mathrm{Div}, \Gamma_j)$ and $H^{-1/2}_Q(\mathrm{Curl}, \Gamma_j)$, respectively, $j = 0, 1$. Therefore, by applying the integral identity (A.6b), exploiting therein the Q-periodicity of the smooth functions and using the observation from above, we obtain

$$\int_D (\mathrm{curl}\, u \cdot v - u \cdot \mathrm{curl}\, v)\,\mathrm{d}x = \lim_{n\to\infty} \int_D (\mathrm{curl}\, u_n \cdot v_n - u_n \cdot \mathrm{curl}\, v_n)\,\mathrm{d}x$$
$$= \lim_{n\to\infty} \sum_{j=0}^{1} \int_{\Gamma_j} \gamma_{t,\Gamma_j} u_n \cdot \gamma_{T,\Gamma_j} v_n\,\mathrm{d}s$$

$$= \lim_{n\to\infty} \sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} u_n, \gamma_{T,\Gamma_j} v_n \big\rangle_{\Gamma_j} = \sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} u, \gamma_{T,\Gamma_j} v \big\rangle_{\Gamma_j}.$$

The second equation follows by interchanging the roles of u and v.
(c). Let $u \in H_Q(\operatorname{div}, D)$ and $\psi \in H^1_Q(D)$. Then there exist sequences $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_Q(\overline{D}, \mathbb{C}^3)$ and $(\psi_n)_{n\in\mathbb{N}}$ in $C^\infty_Q(\overline{D})$ converging to u and ψ with respect to $\|\cdot\|_{H_Q(\operatorname{div},D)}$ and $\|\cdot\|_{H^1_Q(D)}$, respectively. Using the partition of unity from Assumption 2.91, there holds $\psi = \sum_{j=0}^{2} \phi_j \psi = \sum_{j=0}^{2} \lim_{n\to\infty} \phi_j \psi_n$ and, together with Proposition 2.78,

$$\begin{aligned}\nabla\psi &= \sum_{j=0}^{2} \nabla(\phi_j\psi) = \sum_{j=0}^{2} (\phi_j \nabla\psi + \psi\nabla\phi_j) \\ &= \sum_{j=0}^{2} \lim_{n\to\infty} (\phi_j\nabla\psi_n + \psi_n\nabla\phi_j) = \sum_{j=0}^{2} \lim_{n\to\infty} \nabla(\phi_j\psi_n),\end{aligned}$$

where the limits are taken in $L^2(D)$ and $L^2(D,\mathbb{C}^3)$, respectively. For $n \in \mathbb{N}$ and $j \in \{0,1\}$ set $\tilde{\psi}^{(j)}_n := \eta_{0,\Gamma_j}\gamma_{0,\Gamma_j}\psi_n$. Note that $\tilde{\psi}^{(0)}_n \in H^1_{Q,0,\Gamma_1}(D)$ and $\tilde{\psi}^{(1)}_n \in H^1_{Q,0,\Gamma_0}(D)$, $n \in \mathbb{N}$. Furthermore, by the boundedness of the trace and extension operator the sequence $(\tilde{\psi}^{(j)}_n)_{n\in\mathbb{N}}$ converges to $\eta_{0,\Gamma_j}\gamma_{0,\Gamma_j}\psi =: \tilde{\psi}^{(j)}$ in $H^1_Q(D)$, $j = 0,1$. Let $n \in \mathbb{N}$. Then, by the definition of the cut-off functions ϕ_j, by the Q-periodicity of all involved functions and by (A.6c), we obtain

$$\begin{aligned}\sum_{j=0}^{2} \int_{\partial D} (\phi_j\psi_n)\, \mathrm{n}\cdot u_n \,\mathrm{d}s &= \int_{\Gamma_0} (\gamma_{0,\Gamma_0}\psi_n)\, \mathrm{n}\cdot u_n \,\mathrm{d}s + \int_{\Gamma_1} (\gamma_{0,\Gamma_1}\psi_n)\, \mathrm{n}\cdot u_n \,\mathrm{d}s \\ &= \int_D \big(\tilde{\psi}^{(0)}_n \operatorname{div} u_n + u_n\cdot\nabla\tilde{\psi}^{(0)}_n\big)\,\mathrm{d}x + \int_D \big(\tilde{\psi}^{(1)}_n \operatorname{div} u_n + u_n\cdot\nabla\tilde{\psi}^{(1)}_n\big)\,\mathrm{d}x.\end{aligned}$$

Hence, using the observations from above and again (A.6c),

$$\begin{aligned}\int_D \big(\psi \operatorname{div} u + u\cdot\nabla\psi\big)\,\mathrm{d}x &= \sum_{j=0}^{2} \int_D \big((\phi_j\psi)\operatorname{div} u + u\cdot\nabla(\phi_j\psi)\big)\,\mathrm{d}x \\ &= \sum_{j=0}^{2} \lim_{n\to\infty} \int_D \big((\phi_j\psi_n)\operatorname{div} u_n + u_n\cdot\nabla(\phi_j\psi_n)\big)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{2} \lim_{n\to\infty} \int_{\partial D} (\phi_j \psi_n)\, \mathrm{n}\cdot u_n \,\mathrm{d}s \\
&= \lim_{n\to\infty} \sum_{j=0}^{1} \int_D \big(\tilde{\psi}_n^{(j)} \operatorname{div} u_n + u_n \cdot \nabla \tilde{\psi}_n^{(j)}\big)\,\mathrm{d}x \\
&= \sum_{j=0}^{1} \int_D \big(\tilde{\psi}^{(j)} \operatorname{div} u + u \cdot \nabla \tilde{\psi}^{(j)}\big)\,\mathrm{d}x = \sum_{j=0}^{1} \langle \gamma_{n,\Gamma_j} u, \gamma_{0,\Gamma_j} \psi \rangle,
\end{aligned}
$$

where the last step holds by the definition of the trace operator γ_{n,Γ_j}, see Theorem 2.111. □

Duality Results. As a first application of Green's formula in $H_Q(\mathrm{curl}, D)$ we show that the spaces $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$ are dual to each other, up to isomorphism. The next corollary states more precisely what we mean by this formulation.

Corollary 2.114 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. To simplify notation we will drop the index "j".*

(i) The dual space $H_Q^{-1/2}(\mathrm{Div}, \Gamma)^*$ of $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ is isomorphic to $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$. An isomorphism is given by

$$J_1 : H_Q^{-1/2}(\mathrm{Curl}, \Gamma) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma)^*, \quad \psi \to J_1\psi := \langle \cdot, \psi \rangle_\Gamma.$$

(ii) The dual space $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)^*$ of $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ is isomorphic to $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. An isomorphism is given by

$$J_2 : H_Q^{-1/2}(\mathrm{Div}, \Gamma) \to H_Q^{-1/2}(\mathrm{Curl}, \Gamma)^*, \quad \varphi \to J_2\varphi := \langle \varphi, \cdot \rangle_\Gamma.$$

Proof: We follow the proof of [34, Theorem 5.26] and show at first that J_1 is surjective. For this let $\ell \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)^*$. Since $\ell \circ \gamma_{t,\Gamma}$ is an element of the dual space $H_Q(\mathrm{curl}, D)^*$ of the Hilbert space $H_Q(\mathrm{curl}, D)$, by the theorem of Riesz there exists a unique $v \in H_Q(\mathrm{curl}, D)$ such that for all $u \in H_Q(\mathrm{curl}, D)$ we have

$$\int_D (\mathrm{curl}\, u \cdot \mathrm{curl}\, v + u \cdot v)\, \mathrm{d}x = \ell\big(\gamma_{t,\Gamma} u\big).$$

From this we conclude

$$\int_D (\mathrm{curl}\, \chi \cdot \mathrm{curl}\, v + \chi \cdot v)\, \mathrm{d}x = 0, \qquad \text{for all } \chi \in C^\infty_{Q,0}(D, \mathbb{C}^3).$$

Hence, $\mathrm{curl}\, v \in H_Q(\mathrm{curl}, D)$ with $\mathrm{curl}^2 v = -v$. Set $\psi := \gamma_{T,\Gamma} \mathrm{curl}\, v$. To show that $J_1\psi = \ell$ let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ and set $u := \eta_{t,\Gamma}\, \varphi$. Then

$$\begin{aligned} \ell(\varphi) = \ell\big(\gamma_{t,\Gamma}\, u\big) &= \int_D (\mathrm{curl}\, u \cdot \mathrm{curl}\, v - u \cdot \mathrm{curl}^2 v)\, \mathrm{d}x \\ &= \big\langle \gamma_{t,\Gamma}\, u, \gamma_{T,\Gamma}\, \mathrm{curl}\, v \big\rangle_\Gamma = \langle \varphi, \psi \rangle_\Gamma = \big(J_1\psi\big)(\varphi). \end{aligned}$$

The surjectivity for J_2 is shown completely analogous.
To see that J_1 is injective, let $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ such that $J_1\psi = 0$. By a corollary of Hahn-Banach's theorem, there exists $\ell \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)^*$ with $\|\ell\| = 1$ and $\ell(\psi) = \|\psi\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma)}$. Due to the surjectivity of J_2, there exists $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ such that $J_2\varphi = \ell$. Therefore,

$$\|\psi\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma)} = \ell(\psi) = \langle \varphi, \psi \rangle_\Gamma = \big(J_1\psi\big)(\varphi) = 0.$$

Again, the injectivity of J_2 is obtained with the same arguments. □

Further Extension (by Zero) Operators. As a next application of Theorem 2.113 we show a generalization of the extension result from Proposition 2.95.

Proposition 2.115 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91 and let $D' \subseteq \mathbb{R}^3$ be a cell set such that $D \subsetneq D'$.*

(a) The following assertions are true.

(i) *If* $\Gamma_0 \subseteq \partial D'$, *then the mapping*

$$\ker(\gamma_{0,\Gamma_1}) \ni u \mapsto u|_0^{D'} \in H_Q^1(D', \mathbb{C}^{d'})$$

is well-defined, linear and bounded.

(ii) *If* $\Gamma_1 \subseteq \partial D'$, *then the mapping*

$$\ker(\gamma_{0,\Gamma_0}) \ni u \mapsto u|_0^{D'} \in H_Q^1(D', \mathbb{C}^{d'})$$

is well-defined, linear and bounded.

(iii) *The mapping*

$$\ker(\gamma_{0,\Gamma_0}) \cap \ker(\gamma_{0,\Gamma_1}) \ni u \mapsto u|_0^{D'} \in H_Q^1(D', \mathbb{C}^{d'})$$

is well-defined, linear and bounded.

In all cases we have $\|u|_0^{D'}\|_{H_Q^1(D',\mathbb{C}^{d'})} \leq \|u\|_{H_Q^1(D,\mathbb{C}^{d'})}$ *and furthermore* $\partial^\alpha(u|_0^{D'}) = (\partial^\alpha u)|_0^{D'}$ *for all* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \leq 1$.

(b) *The following assertions are true.*

(i) *If* $\gamma \in \{\gamma_{t,\Gamma_1}, \gamma_{T,\Gamma_1}\}$ *and* $\Gamma_0 \subseteq \partial D'$, *then the mapping*

$$\ker(\gamma) \ni u \mapsto u|_0^{D'} \in H_Q(\operatorname{curl}, D')$$

is well-defined, linear and bounded.

(ii) *If* $\gamma \in \{\gamma_{t,\Gamma_0}, \gamma_{T,\Gamma_0}\}$ *and* $\Gamma_1 \subseteq \partial D'$, *then the mapping*

$$\ker(\gamma) \ni u \mapsto u|_0^{D'} \in H_Q(\operatorname{curl}, D')$$

is well-defined, linear and bounded.

(iii) *If* $\gamma_1 \in \{\gamma_{t,\Gamma_1}, \gamma_{T,\Gamma_1}\}$ *and* $\gamma_0 \in \{\gamma_{t,\Gamma_0}, \gamma_{T,\Gamma_0}\}$, *then the mapping*

$$\ker(\gamma_1) \cap \ker(\gamma_0) \ni u \mapsto u|_0^{D'} \in H_Q(\operatorname{curl}, D')$$

is well-defined, linear and bounded.

In all cases we have $\|u|_0^{D'}\|_{H(\operatorname{curl},D')} \leq \|u\|_{H(\operatorname{curl},D)}$ *and furthermore* $\operatorname{curl}(u|_0^{D'}) = (\operatorname{curl} u)|_0^{D'}$.

Proof: We only show the assertion for part *(b)* as the argumentation for part *(a)* is completely analogous.
(i). Let $\gamma = \gamma_{t,\Gamma_1}$ and $u \in \ker(\gamma)$. Since $\ker(\gamma) \subseteq H_Q(\mathrm{curl}, D)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_Q^\infty(\overline{D}, \mathbb{C}^3)$ with $u_n \to u$ in $H_Q(\mathrm{curl}, D)$, as $n \to \infty$. In particular, $\gamma u_n \to \gamma u = 0$ in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_1)$. Let $\chi \in C_{Q,0}^\infty(D', \mathbb{C}^3)$. Note that by Theorem 2.113

$$\left|\left\langle \gamma u_n, \gamma_{T,\Gamma_1}\chi\right\rangle_{\Gamma_1}\right| \leq C \left\|\gamma u_n\right\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_1)} \left\|\gamma_{T,\Gamma_1}\chi\right\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_1)} \to 0,$$

as $n \to \infty$. Using now Green's formula (2.18b) from Theorem 2.113 and the fact that $\gamma_{T,\Gamma_0}\chi = 0$, we obtain therefore

$$\begin{aligned}
\int_{D'} u|_0^{D'}(x) \cdot \mathrm{curl}\,\chi(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_D u_n(x) \cdot \mathrm{curl}\,\chi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \left(\int_D \mathrm{curl}\,u_n(x) \cdot \chi(x)\,\mathrm{d}x + \left\langle \gamma u_n, \gamma_{T,\Gamma_1}\chi\right\rangle_{\Gamma_1} \right) \\
&= \int_D \mathrm{curl}\,u(x) \cdot \chi(x)\,\mathrm{d}x = \int_{D'} (\mathrm{curl}\,u)|_0^{D'}(x) \cdot \chi(x)\,\mathrm{d}x.
\end{aligned}$$

This proves that the mapping is well-defined. Its linearity is clear and its boundedness is easy to obtain, see the proof of Proposition 2.68.
Thanks to the second equality in (2.18b), we can proceed analogously for the case $\gamma = \gamma_{T,\Gamma_1}$.
The assertion in *(ii)* is shown with the same arguments. And the assertion in *(iii)* is proven by combining the arguments for *(i)* and *(ii)*. ☐

The Kernels of Trace Operators. Recall Definition 2.94 for the spaces $H^1_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'})$ and $H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$, $j = 0, 1$. It turns out that these spaces are the kernels of the corresponding trace operators on Γ_0 and Γ_1, respectively. This will be shown next.

Theorem 2.116 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91 and let $j \in \{0, 1\}$. Then the following assertions are true.*

(a) *(i)* $\ker(\gamma_{0,\Gamma_j}) = H^1_{Q,0,\Gamma_j}(D, \mathbb{C}^{d'})$,

(ii) $H^1_{Q,0}(D,\mathbb{C}^{d'}) = H^1_{Q,0,\Gamma_0}(D,\mathbb{C}^{d'}) \cap H^1_{Q,0,\Gamma_1}(D,\mathbb{C}^{d'})$.

(b) *(i)* $\ker(\gamma_{t,\Gamma_j}) = H_{Q,0,\Gamma_j}(\mathrm{curl}, D) = \ker(\gamma_{T,\Gamma_j})$,

(ii) $H_{Q,0}(\mathrm{curl}, D) = H_{Q,0,\Gamma_0}(\mathrm{curl}, D) \cap H_{Q,0,\Gamma_1}(\mathrm{curl}, D)$.

Proof: Set $a := (0,0,L+2)^\top$, where $L > 0$ denotes the maximum of the Lipschitz constants from f_0 and f_1.
We start with part *(b)* and postpone part *(a)* to the end of this proof.
(b). *(i).* We only show that $\ker(\gamma_{t,\Gamma_j}) = H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$, since the argumentation for the second equation is completely analogous.
Let $j \in \{0,1\}$ and $u \in H_{Q,0,\Gamma_j}(\mathrm{curl}, D)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0,\Gamma_j}(\overline{D},\mathbb{C}^3)$ with $u_n \to u$ in $H_Q(\mathrm{curl}, D)$, as $n \to \infty$. By the continuity of γ_{t,Γ_j} we obtain $\gamma_{t,\Gamma_j} u_n \to \gamma_{t,\Gamma_j} u$. And since $\gamma_{t,\Gamma_j} u_n = 0$ for all $n \in \mathbb{N}$, we conclude $\gamma_{t,\Gamma_j} u = 0$.
To show the other direction, let $u \in \ker(\gamma_{t,\Gamma_1})$, i.e., $u \in H_Q(\mathrm{curl}, D)$ with $\gamma_{t,\Gamma_1} u = 0$. Choose some cell set $D' \subseteq \mathbb{R}^3$ such that $D \subsetneq D'$ and $\Gamma_0 \subseteq \partial D'$. Then $u|_0^{D'} \in H_Q(\mathrm{curl}, D')$, see Proposition 2.115. We define for $\varepsilon > 0$

$$\begin{aligned} u_\varepsilon(x) &:= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, u(y)\,\mathrm{d}y, \qquad x \in D, \\ &= \int_{D'} \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, u|_0^{D'}(y)\,\mathrm{d}y, \end{aligned} \tag{$*$}$$

where $\widetilde{\phi}_\varepsilon$ denotes the Q-periodic extension of ϕ_ε from (2.6). From part *(vi)* of Lemma 2.92 we know that for $x \in D$ with $x_3 > f_1(\tilde{x}) - \varepsilon$ and for $y \in D$ there holds $y \notin \mathbb{B}_3(x + \varepsilon a, \varepsilon)$. Hence, $u_\varepsilon \in C^\infty_{Q,0,\Gamma_1}(\overline{D},\mathbb{C}^3)$. Combining part *(iv)* with part *(i)* of Lemma 2.92, we obtain that $\widetilde{\phi}_\varepsilon(x+\varepsilon a-\bullet)$ vanishes in a neighborhood of Γ_0 for all $x \in D$. Therefore, thanks to part *(b)* of Proposition 2.82, we obtain for $x \in D$

$$\begin{aligned} \mathrm{curl}\, u_\varepsilon(x) &= \int_{D'} \nabla_x \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u|_0^{D'}(y)\,\mathrm{d}y \\ &= -\int_{D'} \nabla_y \widetilde{\phi}_\varepsilon(x + \varepsilon a - y) \times u|_0^{D'}(y)\,\mathrm{d}y \\ &= \int_{D'} \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\,\mathrm{curl}\, u|_0^{D'}(y)\,\mathrm{d}y \\ &= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\,\mathrm{curl}\, u(y)\,\mathrm{d}y =: (\mathrm{curl}\, u)_\varepsilon(x). \end{aligned}$$

Now we proceed as at the end of the proof of Proposition 2.83 and set $\widetilde{D} := \bigcup_{|\mu|_\infty \le 1} \big(\{p^{(\mu)}\} + D\big)$. Moreover, let $v \in \{u, \operatorname{curl} u\}$ and $\widetilde{v}$ be the Q-periodic extension of v from D to $\widetilde{D}$ and then be extended by zero to $\mathbb{R}^3 \setminus \widetilde{D}$. And finally set

$$\widetilde{v}_\varepsilon(x) := \int_{\mathbb{R}^3} \phi_\varepsilon(x + \varepsilon a - y)\, \widetilde{v}(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Then $\widetilde{v}_\varepsilon(x) = v_\varepsilon(x)$ for all $x \in D$ and moreover $\widetilde{v} \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ with $\widetilde{v} = v$ almost everywhere in D. From Theorem 2.50 we conclude that $\widetilde{v}_\varepsilon \to \widetilde{v}$ in $L^2(\mathbb{R}^3, \mathbb{C}^3)$ and therefore in particular $v_\varepsilon = \widetilde{v}_\varepsilon|_D \to \widetilde{v}|_D = v$ in $L^2(D, \mathbb{C}^3)$, as $\varepsilon \to 0$. This means that $u_\varepsilon \to u$ with respect to $\|\cdot\|_{H(\mathrm{curl},D)}$, as $\varepsilon \to 0$, and this shows that $u \in H_{Q,0,\Gamma_1}(\mathrm{curl}, D)$.
To show the assertion for $u \in \ker(\gamma_{t,\Gamma_0})$, we use the same argumentation. We only have to interchange the indices "1" and "0" and to replace $\widetilde{\phi}_\varepsilon(x + \varepsilon a - y)$ by $\widetilde{\phi}_\varepsilon(x - \varepsilon a - y)$.
(ii). The direction "$\subseteq$" is easy to see. To show the other direction, let $u \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D) \cap H_{Q,0,\Gamma_1}(\mathrm{curl}, D)$. By part *(i)* we have $\gamma_{t,\Gamma_0} u = 0 = \gamma_{t,\Gamma_1} u$. Let $D' \subseteq \mathbb{R}^3$ be a cell set such that $D \subsetneq D'$ and $\Gamma_0 \cup \Gamma_1 \subseteq D'$. Again thanks to Proposition 2.115, $u|_0^{D'} \in H_Q(\mathrm{curl}, D')$. For $j \in \{0, 1, 2\}$ set $u^{(j)} := \widetilde{\phi}_j|_{D'}\, u|_0^{D'}$, where $\{\widetilde{\phi}_k \mid k = 0, 1, 2\}$ is the partition of unity from Theorem 2.90. One of its important properties implies that there exists $\delta > 0$ such that $\operatorname{supp}(u^{(1)}) \subseteq \overline{D} \setminus \overline{\Gamma_0^\delta}$, $\operatorname{supp}(u^{(0)}) \subseteq \overline{D} \setminus \overline{\Gamma_1^\delta}$ and $\operatorname{supp}(u^{(2)}) \subseteq \overline{D} \setminus (\overline{\Gamma_0^\delta} \cup \overline{\Gamma_1^\delta})$. Furthermore, $\sum_{j=0}^2 u^{(j)}|_D = u$. For $0 < \varepsilon < \frac{\delta}{2(L+2)}$ we set as in part *(i)*

$$\begin{aligned}
u_\varepsilon^{(0)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x - \varepsilon a - y)\, u^{(0)}|_D(y)\, \mathrm{d}y, && x \in D,\\
u_\varepsilon^{(1)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x + \varepsilon a - y)\, u^{(1)}|_D(y)\, \mathrm{d}y, && x \in D,\\
u_\varepsilon^{(2)}(x) &:= \int_D \widetilde{\phi}_\varepsilon(x - y)\, u^{(2)}|_D(y)\, \mathrm{d}y, && x \in D,
\end{aligned}$$

and note that analogous to $(*)$ we can rewrite these integrals as integrals over D'. Furthermore, we observe that we can use the same arguments from part *(i)* to see that $u_\varepsilon^{(1)}$ vanishes in a neighborhood of Γ_1 and that $\widetilde{\phi}_\varepsilon(x + \varepsilon a - \cdot)$ vanishes in a neighborhood of Γ_0 for all $x \in D$. To

see that $u_\varepsilon^{(1)}$ also vanishes in a neighborhood of Γ_0, let $x \in D$ with $x_3 < f_0(\tilde{x}) + (\frac{1}{4}\delta - \frac{1}{2}\varepsilon(L+2))$. This implies that $x + \varepsilon a \in V_0(\frac{\delta}{2})$, with V_0 from Lemma 2.92. Using part *(ii)* of this lemma, we have $x + \varepsilon a \in \Gamma_0^{\delta/2}$. Since $\varepsilon < \frac{\delta}{2}$, we obtain therefore $\mathbb{B}_3(x + \varepsilon a, \varepsilon) \subseteq \Gamma_0^\delta$, meaning that the integrand in the definition of $u_\varepsilon^{(1)}$ is indeed zero. Hence, $u_\varepsilon^{(1)} \in C_{Q,0}^\infty(D, \mathbb{C}^3)$ and furthermore for $\operatorname{curl} u_\varepsilon^{(1)}$ we obtain the same result as in part *(i)*.
For $u_\varepsilon^{(0)}$ and $u_\varepsilon^{(2)}$ we can argue completely analogous. Now we can follow the lines at the end of the proof of Theorem 2.93 and obtain $u_\varepsilon^{(j)} \to u^{(j)}|_D$ with respect to $\|\cdot\|_{H(\mathrm{curl},D)}$, as $\varepsilon \to 0$, $j = 0, 1, 2$. From this we finally conclude that for $u_\varepsilon := \sum_{j=0}^2 u_\varepsilon^{(j)}$ there holds $u_\varepsilon \in C_{Q,0}^\infty(D, \mathbb{C}^3)$ and $u_\varepsilon = \sum_{j=0}^2 u_\varepsilon^{(j)} \to \sum_{j=0}^2 u^{(j)}|_D = u$ in $H_Q(\mathrm{curl}, D)$, as $\varepsilon \to 0$, and the proof for part *(b)* is complete.
(a). The proof follows very closely the lines from part *(b)*, where we now cite part *(a)* from Proposition 2.82. The details are omitted. □

Corollary 2.117 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $0 < \ell < L_3$ and $R := Q \times (-\ell, 0)$. In the following arguments we consider $\tilde{\Psi}_j$ restricted to R. Moreover, let S_j and T_j denote the isomorphism from Theorem 2.104 for the scalar and vector valued case, respectively. For ease of notation we denote by Γ_0' and Γ_1' the lower and upper boundary patch of $\tilde{\Psi}_j(R)$, $j = 0, 1$, respectively. Then the following assertions are true.*

(a) For the scalar valued case we have

$$\begin{aligned} S_j\big(H^1_{Q,0}(\tilde{\Psi}_j(R))\big) &= H^1_{Q,0}(R), \quad j = 0, 1, \\ S_0\big(H^1_{Q,0,\Gamma_0'}(\tilde{\Psi}_0(R))\big) &= H^1_{Q,0,Q\times\{0\}}(R) = S_1\big(H^1_{Q,0,\Gamma_1'}(\tilde{\Psi}_1(R))\big), \\ S_0\big(H^1_{Q,0,\Gamma_1'}(\tilde{\Psi}_0(R))\big) &= H^1_{Q,0,Q\times\{-\ell\}}(R) = S_1\big(H^1_{Q,0,\Gamma_0'}(\tilde{\Psi}_1(R))\big). \end{aligned}$$

(b) For the vector valued case we have

$$T_j\big(H_{Q,0}(\mathrm{curl}, \tilde{\Psi}_j(R))\big) = H_{Q,0}(\mathrm{curl}, R), \quad j = 0, 1,$$

$$
\begin{aligned}
T_0\big(H_{Q,0,\Gamma_0'}(\mathrm{curl},\tilde\Psi_0(R))\big) &= H_{Q,0,Q\times\{0\}}(\mathrm{curl},R)\\
&= T_1\big(H_{Q,0,\Gamma_1'}(\mathrm{curl},\tilde\Psi_1(R))\big),\\
T_0\big(H_{Q,0,\Gamma_1'}(\mathrm{curl},\tilde\Psi_0(R))\big) &= H_{Q,0,Q\times\{-\ell\}}(\mathrm{curl},R)\\
&= T_1\big(H_{Q,0,\Gamma_0'}(\mathrm{curl},\tilde\Psi_1(R))\big).
\end{aligned}
$$

Proof: We only show the assertion for the vector valued case as the argumentation for the scalar valued case is completely analogous.
Let $v \in T_0\big(H_{Q,0}(\mathrm{curl},\tilde\Psi_0(R))\big)$, i.e., there exists $u \in H_{Q,0}(\mathrm{curl},\tilde\Psi_0(R))$ such that $v = T_0u$. Moreover, there exists $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(\tilde\Psi_0(R),\mathbb{C}^3)$ converging to u with respect to $\|\cdot\|_{H_Q(\mathrm{curl},\tilde\Psi_0(R))}$. Hence, $T_0u_n \to T_0u$ in $H_Q(\mathrm{curl},R)$ and therefore $0 = \gamma_{t,Q\times\{0\}}T_0u_n \to \gamma_{t,Q\times\{0\}}T_0u$ and $0 = \gamma_{t,Q\times\{-\ell\}}T_0u_n \to \gamma_{t,Q\times\{-\ell\}}T_0u$, as $n\to\infty$. Consequently, $v \in \ker(\gamma_{t,Q\times\{0\}}) \cap \ker(\gamma_{t,Q\times\{-\ell\}})$, which yields with Theorem 2.116 that $v \in H_{Q,0}(\mathrm{curl},R)$.
For the other direction let now $v \in H_{Q,0}(\mathrm{curl},R)$. Since T_0 is an isomorphism, we can use the same argumentation from above to obtain that $u := T_0^{-1}v$ belongs to $H_{Q,0}(\mathrm{curl},\tilde\Psi_0(R))$ and we are done.
The remaining equalities are shown analogously. □

Intermediate Layers. Thanks to Theorem 2.113, piecewise defined functions whose traces coincide can be clued together, as shown in the next proposition.

Proposition 2.118 *Let $D^-, D^+ \subseteq \mathbb{R}^3$ be cell sets of Lipschitz layer type with corresponding surfaces Γ_j^- and Γ_j^+, $j = 0,1$, respectively, such that $D^- \cap D^+ = \emptyset$, but $\Gamma_1^- = \Gamma_0^+ =: \Gamma$. For simplicity, set $D := D^- \cup \Gamma \cup D^+$.*

(a) *If $v \in H^1_Q(D^-)$ and $w \in H^1_Q(D^+)$ with $\gamma_{0,\Gamma}v = \gamma_{0,\Gamma}w$, then*

$$
u := \begin{cases} w, & \text{on } D^+,\\ v, & \text{on } D^- \end{cases}
$$

belongs to $H^1_Q(D)$ with $\nabla u = \nabla v$ on D^- and $\nabla u = \nabla w$ on D^+. Moreover, the mapping $H^1_Q(D^-) \times H^1_Q(D^+) \ni (v,w) \mapsto u \in H^1_Q(D)$ is linear and bounded.

(b) *If* $v \in H_Q(\mathrm{curl}, D^-)$ *and* $w \in H_Q(\mathrm{curl}, D^+)$ *with* $\gamma_{t,\Gamma} v = -\gamma_{t,\Gamma} w$ *or with* $\gamma_{T,\Gamma} v = \gamma_{T,\Gamma} w$*, then*

$$u := \begin{cases} w, & \text{on } D^+, \\ v, & \text{on } D^- \end{cases}$$

belongs to $H_Q(\mathrm{curl}, D)$ *with* $\mathrm{curl}\, u = \mathrm{curl}\, v$ *on* D^- *and* $\mathrm{curl}\, u = \mathrm{curl}\, w$ *on* D^+*. Moreover, the mapping* $H_Q(\mathrm{curl}, D^-) \times H_Q(\mathrm{curl}, D^+) \ni (v, w) \mapsto u \in H_Q(\mathrm{curl}, D)$ *is linear and bounded.*

Proof: We only show the assertion for part *(b)* as the argumentation for part *(a)* is completely analogous.
Moreover, we only show the assertion for the case $\gamma_{T,\Gamma} v = \gamma_{T,\Gamma} w$, because the assertion for the other case is shown by very similar arguments. So, let $\chi \in C^\infty_{Q,0}(D, \mathbb{C}^3)$. Note that $\gamma_{t,\Gamma}\chi|_{D^-} = -\gamma_{t,\Gamma}\chi|_{D^+}$ and $\gamma_{t,\Gamma_0^-}\chi|_{D^-} = \gamma_{t,\Gamma_1^+}\chi|_{D^+} = 0$. Using the second equality from Green's formula (2.18b), we obtain therefore

$$\begin{aligned}
\int_D u \cdot \mathrm{curl}\,\chi \,\mathrm{d}x &= \int_{D^-} v \cdot \mathrm{curl}\,\chi \,\mathrm{d}x + \int_{D^+} w \cdot \mathrm{curl}\,\chi \,\mathrm{d}x \\
&= \int_{D^-} \mathrm{curl}\, v \cdot \chi \,\mathrm{d}x + \langle \gamma_{t,\Gamma}\, \chi|_{D^-}, \gamma_{T,\Gamma}\, v \rangle_\Gamma \\
&\quad + \int_{D^+} \mathrm{curl}\, w \cdot \chi \,\mathrm{d}x + \langle \gamma_{t,\Gamma}\, \chi|_{D^+}, \gamma_{T,\Gamma}\, w \rangle_\Gamma \\
&= \int_{D^-} \mathrm{curl}\, v \cdot \chi \,\mathrm{d}x + \int_{D^+} \mathrm{curl}\, w \cdot \chi \,\mathrm{d}x,
\end{aligned}$$

which shows that $u \in H_Q(\mathrm{curl}, D)$ with variational curl as given in the proposition. The linearity and boundedness are easy to see. □

Otherwise, if a cell set D of Lipschitz layer type can be divided by an intermediate Lipschitz surface Γ into two cell sets of Lipschitz layer type which contact each other, then for the traces on both sides of Γ we have the following result.

Proposition 2.119 *Let* $D \subseteq \mathbb{R}^3$ *be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be* Q*-periodic and Lipschitz-continuous, and set*

$$\Gamma := \big\{ x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = f(\tilde{x}) \big\}.$$

Moreover, suppose that $\Gamma \subseteq D$ *and set*

$$D^+ := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } f(\tilde{x}) < x_3 < f_1(\tilde{x})\big\},$$
$$D^- := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } f_0(\tilde{x}) < x_3 < f(\tilde{x})\big\}.$$

Then the following assertions are true.

(a) If $u \in H^1_Q(D, \mathbb{C}^{d'})$*, then* $\gamma_{0,\Gamma} u|_{D^-} = \gamma_{0,\Gamma} u|_{D^+}$.

(b) If $u \in H_Q(\mathrm{curl}, D)$*, then*

$$\gamma_{t,\Gamma} u|_{D^-} = -\gamma_{t,\Gamma} u|_{D^+} \qquad \text{and} \qquad \gamma_{T,\Gamma} u|_{D^-} = \gamma_{T,\Gamma} u|_{D^+}.$$

Proof: We only show the assertion for part *(b)* as the argumentation for part *(a)* is completely analogous.
Let $u \in H_Q(\mathrm{curl}, D)$. Also here, we only show the first equation, since the proof for the second one uses analogous arguments.
Let $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$. Thanks to Theorem 2.107 we have $v^- := \eta_{T,\Gamma}\psi \in \mathcal{H}_{Q,0,\Gamma_0}(\mathrm{curl}, D^-)$ and $v^+ := \eta_{T,\Gamma}\psi \in \mathcal{H}_{Q,0,\Gamma_1}(\mathrm{curl}, D^+)$. Now, Proposition 2.118 implies that

$$v := \begin{cases} v^+, & \text{on } D^+, \\ v^-, & \text{on } D^- \end{cases}$$

belongs to $H_Q(\mathrm{curl}, D)$. In particular, $v \in \mathcal{H}_{Q,0}(\mathrm{curl}, D)$. Due to Proposition 2.83, there exist a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $C^\infty_{Q,0}(D, \mathbb{C}^3)$, converging to v in $H_Q(\mathrm{curl}, D)$. Now, by Proposition 2.68 we have $\chi_n|_{D^-} \to v^-$ in $H_Q(\mathrm{curl}, D^-)$ and $\chi_n|_{D^+} \to v^+$ in $H_Q(\mathrm{curl}, D^+)$, as $n \to \infty$. Therefore, by continuity of the trace operators, $\gamma_{T,\Gamma}\chi_n|_{D^-} \to \psi$ and $\gamma_{T,\Gamma}\chi_n|_{D^+} \to \psi$ in $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$, as $n \to \infty$. Hence, using Green's formula (2.18b),

$$\begin{aligned}
&\langle \gamma_{t,\Gamma} u|_{D^-} + \gamma_{t,\Gamma} u|_{D^+}, \psi\rangle_\Gamma \\
&\quad = \lim_{n\to\infty} \langle \gamma_{t,\Gamma} u|_{D^-}, \gamma_{T,\Gamma}\chi_n|_{D^-}\rangle_\Gamma + \lim_{n\to\infty} \langle \gamma_{t,\Gamma} u|_{D^+}, \gamma_{T,\Gamma}\chi_n|_{D^+}\rangle_\Gamma \\
&\quad = \lim_{n\to\infty} \Big[\int_{D^-} (\mathrm{curl}\, u \cdot \chi_n - u \cdot \mathrm{curl}\, \chi_n)\, \mathrm{d}x \\
&\qquad\qquad + \int_{D^+} (\mathrm{curl}\, u \cdot \chi_n - u \cdot \mathrm{curl}\, \chi_n)\, \mathrm{d}x \Big]
\end{aligned}$$

$$= \lim_{n\to\infty} \int_D (\operatorname{curl} u \cdot \chi_n - u \cdot \operatorname{curl} \chi_n) \,\mathrm{d}x = 0,$$

where the last equality holds by definition of the variational curl. Since $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ was arbitrarily chosen, Corollary 2.114 yields now the assertion. □

Surface Divergence. To show later existence of a solution to the variational formulation of our scattering problem of interest, we also need the notion of the *surface divergence*. In classical terms, its definition and some applications can be found in the appendix of [34]. Here we will focus on a definition in variational sense, and this in particular for the Q-periodic setting. However, for this purpose again [34] will be our basis.

Theorem 2.120 (and Definition) *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $j \in \{0, 1\}$ and $\beta \in \mathbb{R}^3$. For $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ the mapping*

$$H_Q^{1/2}(\Gamma_j) \ni \psi \;\mapsto\; -\langle \varphi, \gamma_{T,\Gamma_j} \nabla_{-\beta} \tilde{\psi} \rangle_{\Gamma_j} \in \mathbb{C}$$

is well-defined, linear and bounded. Here, $\tilde{\psi} \in H_Q^1(D)$ is any extension of ψ into D such that $\gamma_{0,\Gamma_j} \tilde{\psi} = \psi$. We call this mapping surface divergence of φ, in sign $\mathrm{Div}_\beta\, \varphi$ (if $\beta = 0$, then we will drop the index "β" in this symbol). Hence, for $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ we have that $\mathrm{Div}_\beta\, \varphi \in H_Q^{-1/2}(\Gamma_j)$ and

$$\langle \mathrm{Div}_\beta\, \varphi, \psi \rangle = -\langle \varphi, \gamma_{T,\Gamma_j} \nabla_{-\beta} \tilde{\psi} \rangle_{\Gamma_j}, \qquad \psi \in H_Q^{1/2}(\Gamma_j).$$

In particular, there holds the identity

$$\langle \mathrm{Div}_\beta\, \varphi, \gamma_{0,\Gamma_j} \psi \rangle = -\langle \varphi, \gamma_{T,\Gamma_j} \nabla_{-\beta} \psi \rangle_{\Gamma_j}, \tag{2.19}$$

for all $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ and all $\psi \in H_Q^1(D)$.

Proof: We only show the assertion for the case $j = 0$, as the assertion for the case $j = 1$ is shown completely analogous.
To see that the mapping is well-defined, first of all recall from Theorem 2.107 that by $\tilde{\psi} := \eta_{0,\Gamma_0} \psi$ such an extension exists. Let now $\tilde{\psi}_1$ and

$\tilde{\psi}_2$ be such extensions. Then, by part *(a)* from Theorem 2.116, $\tilde{\psi} := \tilde{\psi}_1 - \tilde{\psi}_2 \in H^1_{Q,0,\Gamma_0}(D)$. Furthermore, $\nabla_{-\beta} H^1_{Q,0,\Gamma_0}(D) \subseteq H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$, see Proposition 2.99. Hence, $\gamma_{T,\Gamma_0} \nabla_{-\beta} \tilde{\psi} = 0$.
To show the boundedness, let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ and $\psi \in H_Q^{1/2}(\Gamma_0)$. Then, by part *(b)* of Theorem 2.113, by the boundedness of γ_{T,Γ_0}, the estimate $\|\nabla_{-\beta}(\eta_{0,\Gamma_0}\psi)\|_{H_Q(\mathrm{curl},D)} \leq C \, \|\eta_{0,\Gamma_0}\psi\|_{H^1_Q(D)}$ (thanks to Proposition 2.75) and by the boundedness of η_{0,Γ_0}, we obtain

$$\begin{aligned} |\langle \mathrm{Div}_\beta \, \varphi, \psi \rangle| &= \left| \langle \varphi, \gamma_{T,\Gamma_0} \nabla_{-\beta}(\eta_{0,\Gamma_0}\psi) \rangle_{\Gamma_0} \right| \\ &\leq C \, \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_0)} \, \|\gamma_{T,\Gamma_0}\| \, \|\nabla_{-\beta}(\eta_{0,\Gamma_0}\psi)\|_{H_Q(\mathrm{curl},D)} \\ &\leq C \, \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_0)} \, \|\eta_{0,\Gamma_0}\| \, \|\psi\|_{H_Q^{1/2}(\Gamma_0)} . \end{aligned}$$

Hence, $\| \mathrm{Div}_\beta \, \varphi\|_{H_Q^{-1/2}(\Gamma_0)} \leq C$, which shows that $\mathrm{Div}_\beta \, \varphi$ is indeed bounded. For the symbol C, which appeared here several times, recall again the convention from the end of Section 1.3.
The identity (2.19) is clear, because $\psi \in H^1_Q(D)$ is an extension of $\gamma_{0,\Gamma_0}\psi$ with the required property. □

$H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ as a Subspace of $H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$. As in [34, Lemma 5.27], in the following presentation we will show that for a cell set $D \subseteq \mathbb{R}^3$ of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, the space $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ can be embedded into $H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$, $j = 0, 1$. We will need this result for the proof of existence of a solution to the variational formulation of our scattering problem of interest and, moreover, for the definition of vector surface potentials.

Recall Proposition 2.65. Hence, for $u \in H^1_Q(D, \mathbb{C}^3)$ its trace $\gamma_{T,\Gamma_j} u$ is well-defined and belongs to $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$, $j = 0, 1$.

Theorem 2.121 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0, 1\}$. Then we have*

$$H_Q^{-1/2}(\mathrm{Div}, \Gamma_j) \longrightarrow H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3),$$

where the embedding is given by $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j) \ni \varphi \mapsto \ell_\varphi \in H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$, *with* ℓ_φ *defined by*

$$\langle \ell_\varphi, \psi \rangle := \langle \varphi, \gamma_{T,\Gamma_j} \tilde{\psi} \rangle_{\Gamma_j}, \qquad \psi \in H_Q^{1/2}(\Gamma_j, \mathbb{C}^3).$$

Here, $\langle \cdot, \cdot \rangle$ *denotes the duality pairing with respect to* $H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$ *and* $\tilde{\psi} \in H_Q^1(D, \mathbb{C}^3)$ *is any extension of* ψ *such that* $\gamma_{0,\Gamma_j} \tilde{\psi} = \psi$.

Proof: Let $j \in \{0, 1\}$. Furthermore, let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$. We follow the lines in the proof of [34, Lemma 5.27] and show at first that $\ell_\varphi \in H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$ is well-defined. For this let $\tilde{\psi}_1, \tilde{\psi}_2 \in H_Q^1(D, \mathbb{C}^3)$ be two extensions of $\psi \in H_Q^{1/2}(\Gamma_j, \mathbb{C}^3)$. Then $\tilde{\psi} := \tilde{\psi}_1 - \tilde{\psi}_2$ belongs to $H^1_{Q,0,\Gamma_j}(D, \mathbb{C}^3)$, see Theorem 2.116 (applied to each component of $\tilde{\psi}$). Due to Definition 2.94, there exists a sequence $(\chi_n)_{n \in \mathbb{N}}$ in $C^\infty_{Q,0,\Gamma_j}(\overline{D}, \mathbb{C}^3)$ converging in $H_Q^1(D, \mathbb{C}^3)$ to $\tilde{\psi}$. By Proposition 2.65, this sequence converges even in $H_Q(\mathrm{curl}, D)$ to $\tilde{\psi}$. Therefore, $0 = \lim_{n \to \infty} \gamma_{T,\Gamma_j} \chi_n = \gamma_{T,\Gamma_j} \tilde{\psi}$. Moreover, for $\psi \in H_Q^{1/2}(\Gamma_j, \mathbb{C}^3)$ and $\tilde{\psi} := \eta_{0,\Gamma_j} \psi \in H_Q^1(D, \mathbb{C}^3)$ we obtain

$$\begin{aligned} |\langle \ell_\varphi, \psi \rangle| &= |\langle \varphi, \gamma_{T,\Gamma_j} \tilde{\psi} \rangle_{\Gamma_j}| \\ &\le c \, \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_j)} \, \|\gamma_{T,\Gamma_j}\| \, \|\eta_{0,\Gamma_j}\| \, \|\psi\|_{H_Q^{1/2}(\Gamma_j,\mathbb{C}^3)}. \end{aligned}$$

Both together shows indeed that ℓ_φ is well-defined.
Clearly, the mapping $H_Q^{-1/2}(\mathrm{Div}, \Gamma_j) \ni \varphi \mapsto \ell_\varphi \in H_Q^{-1/2}(\Gamma_j, \mathbb{C}^3)$ is linear, and its boundedness follows easily from the last estimate. It remains to show its injectivity. For this let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_j)$ such that $\ell_\varphi = 0$. Let $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma_j)$ and set $\tilde{\psi} := \eta_{T,\Gamma_j} \psi \in H_Q(\mathrm{curl}, D)$. Then there exists a sequence $(\chi_n)_{n \in \mathbb{N}}$ in $C_Q^\infty(\overline{D}, \mathbb{C}^3)$ converging to $\tilde{\psi}$ in $H_Q(\mathrm{curl}, D)$. Note that for $n \in \mathbb{N}$ the function $\chi_n|_{\Gamma_j}$ belongs to $\mathcal{D}_{Q,0}(\Gamma_j, \mathbb{C}^3)$, yielding that $\chi_n|_{\Gamma_j}$ itself belongs to $H_Q^{1/2}(\Gamma_j, \mathbb{C}^3)$, see Corollary 2.108. Hence, since $\ell_\varphi = 0$, we have

$$\langle \varphi, \psi \rangle_{\Gamma_j} = \lim_{n \to \infty} \langle \varphi, \gamma_{T,\Gamma_j} \chi_n \rangle_{\Gamma_j} = \lim_{n \to \infty} \langle \ell_\varphi, \chi_n|_{\Gamma_j} \rangle = 0.$$

Therefore, φ has to vanish because of the isomorphism J_2 from Corollary 2.114. $\square$

2.3.5. Compactness Results

The following theorem is an analog of Theorem 2.71. It might be useful to recall Definition 2.69.

Theorem 2.122 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, and let $j \in \{0,1\}$. Furthermore, let $\beta \in \mathbb{R}^3$. Then the following assertions are true.*

(a) (i) The embedding $H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'}) \xhookrightarrow{\text{id}} L^2(\Gamma_j, \mathbb{C}^{d'})$ is compact.

(ii) The embedding $H_Q^1(D, \mathbb{C}^{d'}) \xhookrightarrow{\text{id}} L^2(D, \mathbb{C}^{d'})$ is compact.

(b) The embedding $H_Q(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \xhookrightarrow{\text{id}} L^2(D, \mathbb{C}^3)$ is compact.

Proof: *(a) (i).* Let at first $d' = 1$. We decompose the embedding in the following way

$$H_Q^{1/2}(\Gamma_j) \xrightarrow{S_j} H_{\mathrm{per}}^{1/2}(Q) \xhookrightarrow{\text{id}} L^2(Q) \xrightarrow{T_j} L^2(\Gamma_j),$$

where the operators S_j and T_j are given by $S_j\varphi := \varphi \circ \Psi_j$ and $T_j\psi := \psi \circ \Psi_j^{-1}$, respectively. By the definition of the space $H_Q^{1/2}(\Gamma_j)$, the operator S_j is bounded, and the boundedness of T_j we obtain from (2.15). Now the assertion follows from Proposition 2.13.

For $d' > 1$ we conclude from the case $d' = 1$ that for a bounded sequence in $H_Q^{1/2}(\Gamma_j, \mathbb{C}^{d'})$ each sequence for the components has a convergent subsequence in $L^2(\Gamma_j)$; and from this it is easy to see that the sequence where we started from has a convergent subsequence in $L^2(\Gamma_j, \mathbb{C}^{d'})$.

(a) (ii). We only show the assertion for the case $d' = 1$ as the generalization to the case $d' > 1$ is obvious, see also part *(i)*.

Let at first $u \in H_Q^1(D)$ be arbitrary and define $u^{(j)} := (\phi_j u)|_0^{U_j}$ with ϕ_j and U_j from Assumption 2.91, $j = 0, 1$. Note that $u^{(0)} \in \mathcal{H}^1_{Q,0,\Gamma_1}(D)$ and $u^{(1)} \in \mathcal{H}^1_{Q,0,\Gamma_0}(D)$. Let $j \in \{0,1\}$. We define

$$v^{(j)} := u^{(j)} \circ \tilde{\Psi}_j \in H_Q^1(Q_3^-)$$

as in Theorem 2.104 and observe that $v^{(j)}$ vanishes in a neighborhood of $Q \times \{-L_3\}$. We set

$$\hat{v}^{(j)} := \begin{cases} v^{(j)}, & \text{on } Q_3^-, \\ v^{(j)}(\cdot^*), & \text{on } Q_3^+. \end{cases}$$

Due to Proposition 2.105 and Proposition 2.83, we have $\hat{v}^{(j)} \in H^1_{Q,0}(Q_3)$. Let now $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_Q(D)$. The quantities $u_n^{(j)}$, $v_n^{(j)}$ and $\hat{v}_n^{(j)}$ correspond to the quantities from above. Then $(\hat{v}_n^{(j)})_{n\in\mathbb{N}}$ is bounded in $H^1_{Q,0}(Q_3)$ and possesses therefore, thanks to Theorem 2.71, a converging subsequence in $L^2(Q_3)$ (which we denote by the same symbol). Hence $(v_n^{(j)})_{n\in\mathbb{N}}$ converges in $L^2(Q_3^-)$, yielding that $(u_n^{(j)})_{n\in\mathbb{N}}$ converges in $L^2(U_j)$. And this implies that $(\phi_j u_n)_{n\in\mathbb{N}}$ converges in $L^2(D)$.
Recall ϕ_2 from Assumption 2.91. Then $(\phi_2 u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1_{Q,0}(D)$ and possesses therefore, again due to Theorem 2.71, a subsequence (denoted by the same symbol) which converges in $L^2(D)$.
Since $\sum_{j=0}^2 \phi_j \equiv 1$, in summary we have shown that the bounded sequence $(u_n)_{n\in\mathbb{N}}$ from above contains a subsequence which converges in $L^2(D)$.
(b). We follow the lines in the proof of [34, Theorem 5.32], but in contrast we have to apply the argument of Lax-Milgram twice to get rid off the extra summand in $(*_4)$ caused by $\mathrm{i}\beta$ in ∇_β.
Let at first $u \in H_Q(\mathrm{curl}, \mathrm{div}_\beta\, 0, D)$ be arbitrary and define $u^{(j)} := (\phi_j u)|_0^{U_j}$ with ϕ_j and U_j from Assumption 2.91, $j = 0, 1$. Note that $u^{(0)} \in \mathcal{H}_{Q,0,\Gamma_1}(\mathrm{curl}, D)$ and $u^{(1)} \in \mathcal{H}_{Q,0,\Gamma_0}(\mathrm{curl}, D)$. Let $j \in \{0,1\}$. Note that for $\psi \in H^1_{Q,0}(U_j)$ we have $\phi_j\psi \in H^1_{Q,0}(D)$ and that we therefore obtain

$$\begin{aligned} 0 = (u \,|\, \nabla_\beta(\phi_j\psi))_{L^2(D,\mathbb{C}^3)} &= \left(u|_0^{U_j} \,\middle|\, \psi\nabla\phi_j + \phi_j\nabla\psi + \mathrm{i}\beta\phi_j\psi \right)_{L^2(U_j,\mathbb{C}^3)} \\ &= \left(u^{(j)} \,\middle|\, \nabla_\beta\psi \right)_{L^2(U_j,\mathbb{C}^3)} + \left(u|_0^{U_j} \,\middle|\, \psi\nabla\phi_j \right)_{L^2(U_j,\mathbb{C}^3)}. \end{aligned}$$

Consider the sesquilinear form $a : H^1_{Q,0}(U_j) \times H^1_{Q,0}(U_j) \to \mathbb{C}$ and the linear functional $\ell : H^1_{Q,0}(U_j) \to \mathbb{C}$ given by

$$a(\psi, p) := (\nabla_\beta\psi \,|\, \nabla_\beta p)_{L^2(U_j,\mathbb{C}^3)} \quad \text{and} \quad \ell(\psi) := -\left(u|_0^{U_j} \,\middle|\, \psi\nabla\phi_j \right)_{L^2(U_j,\mathbb{C}^3)}.$$

Then $|\ell(\psi)| \le C\,\|u\|_{L^2(U_j,\mathbb{C}^3)}\,\|\psi\|_{L^2(U_j)} \le C\,\|\psi\|_{H^1_Q(U_j)}$ and analogously $|a(\psi,p)| \le C\,\|\psi\|_{H^1_Q(U_j)}\,\|p\|_{H^1_Q(U_j)}$ for all $\psi, p \in H^1_{Q,0}(U_j)$. Furthermore,

$\operatorname{Re} a(\psi,\psi) = \|\nabla_\beta \psi\|^2_{L^2(U_j,\mathbb{C}^3)} \geq C\,\|\psi\|^2_{H^1_Q(U_j)}$ for all $\psi \in H^1_Q(U_j)$, where we have applied Corollary 2.73. Hence, by Theorem A.8, there exists a unique $p^{(j)} \in H^1_{Q,0}(U_j)$ such that

$$\left(\nabla_\beta p^{(j)} \,\middle|\, \nabla_\beta \psi\right)_{L^2(U_j,\mathbb{C}^3)} = -\left(u|_0^{U_j} \,\middle|\, \psi \nabla \phi_j\right)_{L^2(U_j,\mathbb{C}^3)} \tag{$*_1$}$$

for all $\psi \in H^1_{Q,0}(U_j)$. From this equation we conclude, if we choose $p^{(j)}$ for ψ,

$$\|\nabla_\beta p^{(j)}\|^2_{L^2(U_j,\mathbb{C}^3)} \leq C\,\|u\|_{L^2(D,\mathbb{C}^3)}\,\|p^{(j)}\|_{L^2(U_j)}, \tag{$*_2$}$$

$$\|\nabla_\beta p^{(j)}\|_{L^2(U_j,\mathbb{C}^3)} \leq C\,\|u\|_{L^2(D,\mathbb{C}^3)}, \tag{$*_3$}$$

where for the second inequality we have applied Theorem 2.96. We set $\tilde{u}^{(j)} := u^{(j)} - \nabla_\beta p^{(j)}$ and have that $\tilde{u}^{(j)}$ belongs to $H_Q(\operatorname{curl}, \operatorname{div}_\beta 0, U_j)$. Moreover, if $j = 0$, then $\tilde{u}^{(j)}$ is zero above from Γ_1, and if $j = 1$, then $\tilde{u}^{(j)}$ is zero below from Γ_0; otherwise by taking $\psi \in H^1_{Q,0}(U_j)$ such that its gradient does not vanish above from Γ_1 and below from Γ_0, respectively, we would get a contradiction to $(*_1)$. We define

$$\tilde{v}^{(j)} := (\tilde{\Psi}'_j)^\top (\tilde{u}^{(j)} \circ \tilde{\Psi}_j) \in H_Q(\operatorname{curl}, Q_3^-)$$

as in Theorem 2.104 and observe that $\tilde{v}^{(j)}$ vanishes in a neighborhood of $Q \times \{-L_3\}$. Take some $\psi \in H^1_{Q,0}(Q_3^-)$ and set $\tilde{\psi} := \psi \circ (\tilde{\Psi}_j)^{-1}$. Note that $\tilde{\psi} \in H^1_{Q,0}(U_j)$, thanks to Corollary 2.117. Therefore, together with the identity $(\nabla \tilde{\psi}) \circ \tilde{\Psi}_j = (\tilde{\Psi}_j)^{-\top} \nabla \psi$ and the transformation formula, we obtain

$$\begin{aligned} 0 = \left(\tilde{u}^{(j)} \,\middle|\, \nabla_\beta \tilde{\psi}\right)_{L^2(U_j,\mathbb{C}^3)} &= \left(\tilde{u}^{(j)} \circ \tilde{\Psi}_j \,\middle|\, (\tilde{\Psi}'_j)^{-\top} \nabla \psi + \mathrm{i}\beta\psi\right)_{L^2(Q_3^-,\mathbb{C}^3)} \\ &= \left(A\tilde{v}^{(j)} \,\middle|\, \nabla \psi\right)_{L^2(Q_3^-,\mathbb{C}^3)} + \left((\tilde{\Psi}'_j)^{-\top} \tilde{v}^{(j)} \,\middle|\, \mathrm{i}\beta\psi\right)_{L^2(Q_3^-,\mathbb{C}^3)}, \end{aligned} \tag{$*_4$}$$

where we have set $A(x) := (\tilde{\Psi}'_j(x))^{-1} (\tilde{\Psi}'_j(x))^{-\top}$ for almost all $x \in Q_3^-$. Now we consider the sesquilinear form $a : H^1_{Q,0}(Q_3^-) \times H^1_{Q,0}(Q_3^-) \to \mathbb{C}$ and the linear functional $\ell : H^1_{Q,0}(Q_3^-) \to \mathbb{C}$ given by

$$a(\psi, q) := (\nabla \psi \,|\, A \nabla q)_{L^2(Q_3^-,\mathbb{C}^3)}, \;\; \ell(\psi) := -\left(\mathrm{i}\beta\psi \,\middle|\, (\tilde{\Psi}'_j)^{-\top} \tilde{v}^{(j)}\right)_{L^2(Q_3^-,\mathbb{C}^3)}.$$

Note that $A \in L^\infty(Q_3^-, \mathbb{C}^{3\times 3})$ satisfies the assumptions from for instance Definition 2.84. Therefore, we can proceed analogous to above to obtain a unique $q^{(j)} \in H^1_{Q,0}(Q_3^-)$ such that

$$\Big(A\nabla q^{(j)} \,\Big|\, \nabla\psi \Big)_{L^2(Q_3^-,\mathbb{C}^3)} = -\Big((\tilde{\Psi}_j')^{-\top} \tilde{v}^{(j)} \,\Big|\, \mathrm{i}\beta\psi \Big)_{L^2(Q_3^-,\mathbb{C}^3)}$$

for all $\psi \in H^1_{Q,0}(Q_3^-)$. And from this equation we conclude again

$$\|\nabla q^{(j)}\|^2_{L^2(Q_3^-,\mathbb{C}^3)} \le C\,\|\tilde{v}^{(j)}\|_{L^2(Q_3^-,\mathbb{C}^3)}\,\|q^{(j)}\|_{L^2(Q_3^-)}, \tag{$*_5$}$$

$$\|\nabla q^{(j)}\|_{L^2(Q_3^-,\mathbb{C}^3)} \le C\,\|\tilde{v}^{(j)}\|_{L^2(Q_3^-,\mathbb{C}^3)}, \tag{$*_6$}$$

where we have used for the first estimate also the coercivity of A. We define $w^{(j)} := \tilde{v}^{(j)} - \nabla q^{(j)} \in H_Q(\mathrm{curl}, Q_3^-)$ and observe again that $w^{(j)}$ vanishes in a neighborhood of $Q \times \{-L_3\}$. We set

$$\hat{w}^{(j)} := \begin{cases} w^{(j)}, & \text{on } Q_3^-, \\ (w^{(j)})^*(\cdot^*), & \text{on } Q_3^+, \end{cases}$$

$$\hat{A}_{lk} := \begin{cases} A_{lk}, & \text{on } Q_3^-, \\ -A_{lk}(\cdot^*), & \text{on } Q_3^+,\ l,k \in \{1,2\} \text{ or } l = k = 3, \\ A_{lk}(\cdot^*), & \text{on } Q_3^+, \text{ else.} \end{cases}$$

Due to Proposition 2.105 and Proposition 2.83, we have that $\hat{w}^{(j)}$ belongs to $H_{Q,0}(\mathrm{curl}, Q_3)$. Moreover, for $\psi \in H^1_{Q,0}(Q_3)$ an elementary calculation yields

$$-A(x^*)\,w^{(j)}(x^*) \cdot \overline{(\nabla\psi)^*(x)} = \hat{A}(x)\,\hat{w}^{(j)}(x) \cdot \overline{\nabla\psi(x)}, \quad \text{for a.a. } x \in Q_3^+.$$

Let $\psi \in H^1_{Q,0}(Q_3)$ and set $\varphi := \psi|_{Q_3^-} - \psi(\cdot^*)|_{Q_3^-}$, which belongs to $H^1_{Q,0}(Q_3^-)$. Moreover, $\nabla\varphi(x) = \nabla\psi(x) - (\nabla\psi)^*(x^*)$ for almost all $x \in Q_3^-$. Therefore, using the definition of $w^{(j)}$,

$$\begin{aligned} 0 &= \Big(Aw^{(j)} \,\Big|\, \nabla\varphi \Big)_{L^2(Q_3^-,\mathbb{C}^3)} \\ &= \int_{Q_3^-} A(x)\,w^{(j)}(x) \cdot \overline{\nabla\psi(x)}\,\mathrm{d}x - \int_{Q_3^+} A(x^*)\,w^{(j)}(x^*) \cdot \overline{(\nabla\psi)^*(x)}\,\mathrm{d}x \end{aligned}$$

$$= \int_{Q_3^-} \hat{A}(x)\,\hat{w}^{(j)}(x)\cdot\overline{\nabla\psi(x)}\,\mathrm{d}x + \int_{Q_3^+} \hat{A}(x)\,\hat{w}^{(j)}(x)\cdot\overline{\nabla\psi(x)}\,\mathrm{d}x$$
$$= \int_{Q_3} \hat{A}(x)\,\hat{w}^{(j)}(x)\cdot\overline{\nabla\psi(x)}\,\mathrm{d}x,$$

meaning that $\hat{w}^{(j)} \in H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\hat{A}}\,0, Q_3)$.
Let now $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H_Q(\mathrm{curl}, \mathrm{div}_\beta\,0, D)$ and let $j \in \{0,1\}$. The quantities $u_n^{(j)}$, $p_n^{(j)}$, $\tilde{u}_n^{(j)}$, $\tilde{v}_n^{(j)}$, $q_n^{(j)}$, $w_n^{(j)}$ and $\hat{w}_n^{(j)}$ correspond to the quantities from above. Using $(*_3)$ and Corollary 2.73, we see that $(p_n^{(j)})_{n\in\mathbb{N}}$ is a bounded sequence in $H^1_{Q,0}(U_j)$. By Theorem 2.71, there exists a subsequence (which we denote by the same symbol) which converges in $L^2(U_j)$. Then the estimate $(*_2)$ applied to $p_l^{(j)} - p_m^{(j)}$ yields that $(\nabla_\beta p_n^{(j)})_{n\in\mathbb{N}}$ is convergent in $L^2(U_j, \mathbb{C}^3)$.
Now, $(\tilde{u}_n^{(j)})_{n\in\mathbb{N}}$ is bounded in $H_Q(\mathrm{curl}, \mathrm{div}_\beta\,0, U_j)$. Hence, $(\tilde{v}_n^{(j)})_{n\in\mathbb{N}}$ is bounded in $H_Q(\mathrm{curl}, Q_3^-)$. Using $(*_6)$ and again Corollary 2.73, we see again that $(q_n^{(j)})_{n\in\mathbb{N}}$ is a bounded sequence in $H^1_{Q,0}(Q_3^-)$. By Theorem 2.71, there exists a subsequence of $(q_n^{(j)})_{n\in\mathbb{N}}$ (which we denote by the same symbol) which converges in $L^2(Q_3^-)$. The estimate $(*_5)$ yields now that $(\nabla q_n^{(j)})_{n\in\mathbb{N}}$ is convergent in $L^2(Q_3^-, \mathbb{C}^3)$.
Furthermore, $(\hat{w}_n^{(j)})_{n\in\mathbb{N}}$ is now bounded in $H_{Q,0}(\mathrm{curl}, \mathrm{div}_{\hat{A}}\,0, Q_3)$. Therefore, by Theorem 2.88, there exists a subsequence of $(\hat{w}_n^{(j)})_{n\in\mathbb{N}}$ (which we denote by the same symbol) which converges in $L^2(Q_3, \mathbb{C}^3)$. Then $(w_n^{(j)})_{n\in\mathbb{N}}$ converges in $L^2(Q_3^-, \mathbb{C}^3)$ and therefore $(\tilde{v}_n^{(j)})_{n\in\mathbb{N}}$ converges in $L^2(Q_3^-, \mathbb{C}^3)$ which yields that $(\tilde{u}_n^{(j)})_{n\in\mathbb{N}}$ converges in $L^2(U_j, \mathbb{C}^3)$. From this we conclude that $(u_n^{(j)})_{n\in\mathbb{N}}$ is convergent in $L^2(U_j, \mathbb{C}^3)$ and therefore $(\phi_j u_n)_{n\in\mathbb{N}}$ is convergent in $L^2(D, \mathbb{C}^3)$.
Finally, take ϕ_2 from Assumption 2.91 and define $u_n^{(2)} := \phi_2 u_n$, $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Note that $u_n^{(2)} \in H_{Q,0}(\mathrm{curl}, D)$. Moreover, by repeating from the beginning of this part *(b)* of the proof the first chain of equalities and by considering the same sesquilinear form and linear functional, but now with U_j replaced by D, we obtain a decomposition $u_n^{(2)} = \tilde{u}_n^{(2)} + \nabla_\beta p_n^{(2)}$ with $p_n^{(2)} \in H^1_{Q,0}(D)$ and $\tilde{u}_n^{(2)} \in H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\,0, D)$, where for the latter one we also take Proposition 2.75 into account. Using again the corresponding estimates $(*_3)$ and $(*_2)$, we obtain analogous to above that $(\nabla_\beta p_n^{(2)})_{n\in\mathbb{N}}$

possesses a subsequence (denoted by the same symbol) which converges in $L^2(D, \mathbb{C}^3)$. Furthermore, we see again that $(\tilde{u}_n^{(2)})_{n\in\mathbb{N}}$ is a bounded sequence in $H_{Q,0}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D)$, which implies thanks to Theorem 2.88 that this sequence possesses a subsequence (denoted by the same symbol), which converges in $L^2(D, \mathbb{C}^3)$. Therefore, also $(\phi_2 u_n)_{n\in\mathbb{N}}$ is convergent in $L^2(D, \mathbb{C}^3)$.

Since again $\sum_{j=0}^{2} \phi_j \equiv 1$, adding it all up we have shown that the bounded sequence $(u_n)_{n\in\mathbb{N}}$ from above contains a subsequence which converges in $L^2(D, \mathbb{C}^3)$. □

Further Helmholtz Decompositions. Due to the special structure of a cell set of Lipschitz layer type, further Helmholtz decompositions are possible and also needed. Their main application, in connection with the compactness results from Theorem 2.122, will be in the next chapter when we introduce vector surface potentials and their corresponding boundary operators.

For the following results compare also with Definition 2.69 and Theorem 2.85.

Definition 2.123 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $\beta \in \mathbb{R}^3$ and $j \in \{0, 1\}$. The space $H_{Q,0,\Gamma_j}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D)$ is defined by*

$$H_{Q,0,\Gamma_j}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) := \Big\{ u \in H_{Q,0,\Gamma_j}(\mathrm{curl}, D) \;\Big|\; \forall \psi \in H^1_{Q,0,\Gamma_j}(D) : (u \,|\, \nabla_\beta \psi)_{L^2(D,\mathbb{C}^3)} = 0 \Big\}.$$

Theorem 2.124 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let $\beta \in \mathbb{R}^3$ and $j \in \{0, 1\}$. Then*

$$H_{Q,0,\Gamma_j}(\mathrm{curl}, D) = H_{Q,0,\Gamma_j}(\mathrm{curl}, \mathrm{div}_\beta\, 0, D) \oplus \nabla_\beta H^1_{Q,0,\Gamma_j}(D).$$

Proof: We can exactly follow the lines in the proof of part *(ii)* from Theorem 2.85, if we replace therein the cited results with their analogs for cell sets of Lipschitz layer type, i.e., in particular Corollary 2.97 instead of Corollary 2.73. We ommit the details. □

2.3.6. Conclusions for Flat Surfaces

Later in the variational formulation for our scattering problem of interest, the unit cell, i.e., the domain of integration, will be a cell set $D \subseteq \mathbb{R}^3$ of Lipschitz layer type where the upper surface patch Γ_1 is flat, that is, where the function f_1 which describes Γ_1 is given by $f_1(\xi) := h$, $\xi \in \mathbb{R}^2$, where h is some real number such that $h > \max_{\xi\in\mathbb{R}^2} f_0(\xi)$. Hence,

$$\Psi_1(x) = \begin{pmatrix} x_1 \\ x_2 \\ h \end{pmatrix}, \qquad x \in Q,$$

which implies that $\rho_j(x) = 1$ and $F_j(x) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ for all $x \in Q$. For the definition of those quantities see the presentation before Assumption 2.91. This gives rise to the following convention.

Convention 2.125 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. The function f_1 we suppose to be given by $f_1(\xi) := h$, for all $\xi \in \mathbb{R}^2$, where h is some real number such that $h > \max_{\xi\in\mathbb{R}^2} f_0(\xi)$. To make this situation more apparent, the surface patch Γ_1 is renamed as Γ_h. We identify*

- *$H_Q^{1/2}(\Gamma_h, \mathbb{C}^{d'})$ with $H_{\text{per}}^{1/2}(Q, \mathbb{C}^{d'})$, and thus the space $H_Q^{-1/2}(\Gamma_h, \mathbb{C}^{d'})$ with $H_{\text{per}}^{-1/2}(Q, \mathbb{C}^{d'})$,*
- *$H_Q^{-1/2}(\text{Div}, \Gamma_h)$ with $H_{\text{per}}^{-1/2}(\text{Div}, Q)$ and*
- *$H_Q^{-1/2}(\text{Curl}, \Gamma_h)$ with $H_{\text{per}}^{-1/2}(\text{Curl}, Q)$.*

Consequently, an element φ from the space $H_Q^{-1/2}(\Gamma_h, \mathbb{C}^{d'})$, $H_Q^{-1/2}(\text{Div}, \Gamma_h)$ and $H_Q^{-1/2}(\text{Curl}, \Gamma_h)$ has a series representation of the form

$$\varphi = \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \jmath(T_Q^{(\mu)}),$$

with $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^2}$ *from* $\mathcal{C}^{-1/2}_{\mathbb{C}^{d'}}$, $\mathcal{C}^{-1/2}_{\mathrm{Div}}$ *and* $\mathcal{C}^{-1/2}_{\mathrm{Curl}}$, *respectively; see also Corollary 2.34.*

As a consequence of those identifications, we can and will derive some convenient formulas for the trace operators, the surface divergence and the embedding from Theorem 2.121, what is the objective of this subsection. In particular, those formulas will be useful when we prove existence of a solution to the variational formulation.

We start with a result which has preliminary character.

Proposition 2.126 *Let* $D \subseteq \mathbb{R}^3$ *be a cell set of Lipschitz layer type as in Convention 2.125. Furthermore, let* $u \in C^\infty_Q(\overline{D}, \mathbb{C}^{d'})$ *and define for* $\alpha \in \mathbb{N}_0^3$

$$\varphi(x) := \partial^\alpha u(x_1, x_2, h), \qquad x \in Q,$$

i.e., φ *denotes the restriction from the continuous extension (from* D *to* $\overline{D}$*) of* $\partial^\alpha u$ *onto* Γ_h*. Then* $\varphi \in C^\infty_{\mathrm{per}}(Q, \mathbb{C}^{d'})$ *with*

$$\partial^\beta \varphi = \partial^{(\beta_1,\beta_2,0)}(\partial^\alpha u)(\cdot, h), \qquad \beta \in \mathbb{N}_0^2.$$

Proof: Let $\alpha \in \mathbb{N}_0^3$ and set for simplicity $v := \partial^\alpha u$. Furthermore, let $j \in \{1, \dots, d'\}$, take some $(a, x_2)^\top \in Q$ and let $n \in \mathbb{N}$. Then, by applying Taylor's formula for functions of several real variables, see for instance [5], we obtain for $x_1 \in (-L_1, L_1) \setminus \{a\}$

$$\begin{aligned}\frac{v_j(x_1, x_2, h - \frac{1}{n}) - v_j(a, x_2, h - \frac{1}{n})}{x_1 - a} &= \partial_1 v_j(a, x_2, h - \tfrac{1}{n}) \\ &+ (x_1 - a) \int_0^1 (1 - \theta)\, \partial_1^2 v_j\big(a + \theta(x_1 - a), x_2, h - \tfrac{1}{n}\big)\, \mathrm{d}\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\varphi_j(x_1, x_2) - \varphi_j(a, x_2)}{x_1 - a} &= \lim_{n\to\infty} \frac{v_j(x_1, x_2, h - \frac{1}{n}) - v_j(a, x_2, h - \frac{1}{n})}{x_1 - a} \\ &= \partial_1 v_j(a, x_2, h) + (x_1 - a) \int_0^1 (1 - \theta)\, \partial_1^2 v_j\big(a + \theta(x_1 - a), x_2, h\big)\, \mathrm{d}\theta,\end{aligned}$$

which converges to $\partial_1 v_j(a, x_2, h)$, as $x_1 \to a$. Hence, $\partial_1\varphi_j = \partial_1 v_j(\cdot, h)$, and $\partial_1\varphi_j$ is therefore also continuous. The remaining part of the assertion follows now by induction. $\square$

Now, we come to the announced formulas for the trace operators.

Lemma 2.127 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type as in Convention 2.125. Furthermore, recall $q^{(\mu)}$ from (2.7), and (1.7) for the definition of $a^\perp$, for $a \in \mathbb{C}^2$.*

(a) If $\beta \in \mathbb{R}^3$ and $u \in C_Q^\infty(\overline{D})$, then $\gamma_{t,\Gamma_h}\nabla_\beta u$ and $\gamma_{T,\Gamma_h}\nabla_\beta u$ belong to $C^\infty_{\mathrm{per}}(Q, \mathbb{C}^2)$ and possess the Fourier series expansion

$$\gamma_{t,\Gamma_h}\nabla_\beta u = \mathrm{i}\sum_{\mu\in\mathbb{Z}^2}(q^{(\mu)} + \tilde{\beta})^\perp\, u^{(\mu)}\, T_Q^{(\mu)},$$
$$\gamma_{T,\Gamma_h}\nabla_\beta u = \mathrm{i}\sum_{\mu\in\mathbb{Z}^2}(q^{(\mu)} + \tilde{\beta})\, u^{(\mu)}\, T_Q^{(\mu)}.$$

Here, $u^{(\mu)} \in \mathbb{C}$ denote the Fourier coefficients of $u(\cdot, h)$, $\tilde{\beta}$ is given by (1.4) and convergence is uniform. Moreover, by replacing $T_Q^{(\mu)}$ with $\jmath(T_Q^{(\mu)})$, where $\jmath$ denotes the embedding from Theorem 2.29, we have convergence in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_h)$, respectively.

(b) If $u \in C_Q^\infty(\overline{D}, \mathbb{C}^3)$, then $\gamma_{t,\Gamma_h}u$ and $\gamma_{T,\Gamma_h}u$ belong to $C^\infty_{\mathrm{per}}(Q, \mathbb{C}^2)$ and possess the Fourier series expansion

$$\gamma_{t,\Gamma_h}u = \sum_{\mu\in\mathbb{Z}^2}\begin{pmatrix}-u_2^{(\mu)}\\ u_1^{(\mu)}\end{pmatrix}T_Q^{(\mu)} \quad \text{and} \quad \gamma_{T,\Gamma_h}u = \sum_{\mu\in\mathbb{Z}^2}\begin{pmatrix}u_1^{(\mu)}\\ u_2^{(\mu)}\end{pmatrix}T_Q^{(\mu)}.$$

Here, $u^{(\mu)} \in \mathbb{C}^3$ denote the Fourier coefficients of $u(\cdot, h)$ and convergence is uniform. Moreover, by replacing $T_Q^{(\mu)}$ with $\jmath(T_Q^{(\mu)})$, we have convergence in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_h)$, respectively.

Proof: *(a).* Let $u \in C_Q^\infty(\overline{D})$. We observe that $\partial^\alpha u(\cdot, h) \in C^\infty_{\mathrm{per}}(Q)$ for any $\alpha \in \mathbb{N}_0^3$, see Proposition 2.126. Let $\big(\sum_{\mu\in\mathbb{Z}^2} u^{(\mu)}T_Q^{(\mu)}\big)$ be the Fourier series expansion with respect to $u(\cdot, h)$. Note that by the observation above the series converges uniformly to $u(\cdot,h)$. Furthermore, by the same observation, there holds

$$\partial_j u(\cdot,h) = \sum_{\mu\in\mathbb{Z}^2} \mathrm{i}\, q_j^{(\mu)} u^{(\mu)} T_Q^{(\mu)}, \qquad j=1,2,$$

where convergence is uniform too. Finally, let $\left(\sum_{\mu\in\mathbb{Z}^2} \theta^{(\mu)} T_Q^{(\mu)}\right)$ be the Fourier series expansion with respect to $\partial_3 u(\cdot,h)$. Here we have again uniform convergence by the observation above. Then

$$\nabla_\beta u(\cdot,h) = \sum_{\mu\in\mathbb{Z}^2} \begin{pmatrix} \mathrm{i}\,(q_1^{(\mu)} + \beta_1) u^{(\mu)} \\ \mathrm{i}\,(q_2^{(\mu)} + \beta_2) u^{(\mu)} \\ \theta^{(\mu)} + \mathrm{i}\beta_3 u^{(\mu)} \end{pmatrix} T_Q^{(\mu)}$$

and

$$\gamma_{t,\Gamma_h} \nabla_\beta u = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) \times \nabla_\beta u(\cdot,h) = \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} (q^{(\mu)} + \tilde{\beta})^\perp\, u^{(\mu)}\, T_Q^{(\mu)},$$

where uniform convergence has passed on and we have implicitly made use of Convention 2.125. Since $u^{(\mu)}$ are the Fourier coefficients of $u(\cdot,h) \in C^\infty_{\mathrm{per}}(Q)$, we conclude from the last equation that $\gamma_{t,\Gamma_h} \nabla_\beta u$ belongs to $C^\infty_{\mathrm{per}}(Q,\mathbb{C}^2)$. And now, we obtain from Remark 2.35 that its series representation converges even in $H_Q^{-1/2}(\mathrm{Div},\Gamma_h)$ to $\gamma_{t,\Gamma_h}\nabla_\beta u$, if we replace therein the trigonometric monomials $T_Q^{(\mu)}$ with $\jmath(T_Q^{(\mu)})$.
The assertion for $\gamma_{T,\Gamma_h}\nabla_\beta u$ is shown by the same arguments.
(b). Let $u \in C_Q^\infty(\overline{D},\mathbb{C}^3)$. Then, again thanks to Proposition 2.126, $u(\cdot,h) \in C^\infty_{\mathrm{per}}(Q,\mathbb{C}^3)$. Let $\left(\sum_{\mu\in\mathbb{Z}^2} u^{(\mu)} T_Q^{(\mu)}\right)$ be the Fourier series expansion with respect to $u(\cdot,h)$. Again, the series converges uniformly to $u(\cdot,h)$. Therefore,

$$\gamma_{t,\Gamma_h} u = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) \times u(\cdot,h) = \sum_{\mu\in\mathbb{Z}^2} \left(\begin{smallmatrix} -u_2^{(\mu)} \\ u_1^{(\mu)} \end{smallmatrix}\right) T_Q^{(\mu)},$$

where again uniform convergence has passed on and we have implicitly made use of Convention 2.125. Now we follow the arguments as in part *(a)* and obtain that $\gamma_{t,\Gamma_h} u \in C^\infty_{\mathrm{per}}(Q,\mathbb{C}^2)$ and that its series representation converges even in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$, if we replace therein $T_Q^{(\mu)}$ with $\jmath(T_Q^{(\mu)})$. The assertion for $\gamma_{T,\Gamma_h} u$ is shown by the same arguments. □

As announced above, in the following lemma we give a convenient formula for the surface divergence and the embedding from Theorem 2.121.

Remark 2.128 *The embedding from Theorem 2.121 reads now as*

$$H_Q^{-1/2}(\mathrm{Div}, \Gamma_h) \hookrightarrow H_Q^{-1/2}(\Gamma_h, \mathbb{C}^2),$$

where $\mathbb{C}^3$ was replaced by $\mathbb{C}^2$, because for the flat surface Γ_h elements from the trace space $H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$ do not have a component in x_3-direction.

Lemma 2.129 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type as in Convention 2.125.*

(i) Let $\beta \in \mathbb{R}^3$. Furthermore, recall $q^{(\mu)}$ from (2.7) and $\tilde{\beta}$ from (1.4). If $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$, then $\mathrm{Div}_\beta\, \varphi \in H_Q^{-1/2}(\Gamma_h)$ possesses the series representation

$$\mathrm{Div}_\beta\, \varphi = \mathrm{i} \sum_{\mu \in \mathbb{Z}^2} \varphi^{(\mu)} \cdot (q^{(\mu)} + \tilde{\beta})\, \jmath(T_Q^{(\mu)}),$$

where $(\varphi^{(\mu)}) \in \mathcal{C}_{\mathrm{Div}}^{-1/2}$ are the associated coefficients of φ, see also Convention 2.125.

(ii) The embedding from Theorem 2.121, that is (see also Remark 2.128)

$$H_Q^{-1/2}(\mathrm{Div}, \Gamma_h) \hookrightarrow H_Q^{-1/2}(\Gamma_h, \mathbb{C}^2),$$

is given by

$$\varphi \to \ell_\varphi = \varphi = \sum_{\mu \in \mathbb{Z}^2} \varphi^{(\mu)} \jmath(T_Q^{(\mu)})$$

where $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2} \in \mathcal{C}_{\mathrm{div}}^{-1/2}$ are the associated coefficients of $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$.

Proof: *(i).* Let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$. Then φ possesses the series representation $\varphi = \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \jmath(T_Q^{(\mu)})$ with coefficients $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}_{\mathrm{Div}}^{-1/2}$. It is easy to see that the sequence $\big(\mathrm{i}(q^{(\mu)} + \tilde{\beta}) \cdot \varphi^{(\mu)}\big)_{\mu\in\mathbb{Z}^2}$ belongs to $\mathcal{C}^{-1/2}$. Therefore, due to Corollary 2.34,

$$\ell := \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \cdot (q^{(\mu)} + \tilde{\beta})\, \jmath(T_Q^{(\mu)})$$

belongs to $H_{\mathrm{per}}^{-1/2}(Q)$. We have to show that $\mathrm{Div}_\beta\, \varphi$ and ℓ coincide. Since $\mathcal{D}_{Q,0}(\Gamma_h)$ is dense in $H_Q^{1/2}(\Gamma_h)$, see Corollary 2.108, and since $\mathrm{Div}_\beta\, \varphi$ and ℓ are continuous, it suffices to check coincidence for this dense subspace. So, let $\psi \in \mathcal{D}_{Q,0}(\Gamma_h)$, i.e., there exists $\tilde{\psi} \in C_Q^\infty(\overline{D})$ such that $\psi = \tilde{\psi}|_{\Gamma_h}$. We expand $\tilde{\psi}(\cdot, h)$ into its Fourier series $\tilde{\psi}(\cdot, h) = \sum_{\mu\in\mathbb{Z}^2} \tilde{\psi}^{(\mu)} T_Q^{(\mu)}$ and obtain from Lemma 2.127 that $\gamma_{T,\Gamma_h} \overline{\nabla_\beta \tilde{\psi}(\cdot, h)} = -\mathrm{i} \sum_{\nu\in\mathbb{Z}^2} (q^{(\nu)} + \tilde{\beta}) \overline{\tilde{\psi}^{(\nu)}}\, \jmath(T_Q^{(-\nu)})$. Therefore, by definition of $\mathrm{Div}\, \varphi$, see Theorem 2.120, by definition of $\langle \cdot, \cdot \rangle_{\Gamma_h}$, see part *(b)* from Theorem 2.113, and by an application of (1.16), we obtain

$$\begin{aligned}
\langle \mathrm{Div}_\beta\, \varphi, \overline{\psi} \rangle &= -\big\langle \varphi, \gamma_{T,\Gamma_h} \overline{\nabla_\beta \tilde{\psi}} \big\rangle_{\Gamma_h} \\
&= \mathrm{i} \sum_{\mu,\nu\in\mathbb{Z}^2} \overline{\tilde{\psi}^{(\nu)}} \Big\langle \varphi^{(\mu)} \jmath(T_Q^{(\mu)}), (q^{(\nu)} + \tilde{\beta})\, \jmath(T_Q^{(-\nu)}) \Big\rangle_{\Gamma_h} \\
&= \mathrm{i} \sum_{\mu,\nu\in\mathbb{Z}^2} \overline{\tilde{\psi}^{(\nu)}}\, \varphi^{(\mu)} \cdot (q^{(\nu)} + \tilde{\beta}) \Big(T_Q^{(\mu)} \,\Big|\, T_Q^{(\nu)} \Big)_{L^2(Q)} \\
&= \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \cdot (q^{(\mu)} + \tilde{\beta})\, \overline{\tilde{\psi}}^{(-\mu)} = \ell\big(\overline{\tilde{\psi}(\cdot, h)}\big) = \ell\big(\overline{\psi}\big),
\end{aligned}$$

as desired. For the second last step we have applied Theorem 2.28.
(ii). We proceed similarly as in part *(a)*. So, let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_h)$. Then φ possesses the series representation $\varphi = \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \jmath(T_Q^{(\mu)})$ with coefficients $(\varphi^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \mathcal{C}_{\mathrm{Div}}^{-1/2}$. It is easy to see that this sequence belongs to $\mathcal{C}_{\mathbb{C}^2}^{-1/2}$ as well. Therefore, $\varphi \in H_Q^{-1/2}(\Gamma_h, \mathbb{C}^2)$. To show that the continuous mappings φ and ℓ_φ, the latter one from Theorem 2.121, coincide, is suffices to restrict our considerations to the dense subspace $\mathcal{D}_{Q,0}(\Gamma_h, \mathbb{C}^2)$ of $H_Q^{-1/2}(\Gamma_h, \mathbb{C}^2)$. So, let $\psi \in \mathcal{D}_{Q,0}(\Gamma_h, \mathbb{C}^2)$, i.e., there exists $\tilde{\psi} \in C_Q^\infty(\overline{D}, \mathbb{C}^2)$ such that $\psi = \tilde{\psi}|_{\Gamma_h}$. We expand $\tilde{\psi}(\cdot, h)$ into its

Fourier series $\tilde{\psi}(\cdot, h) = \sum_{\nu\in\mathbb{Z}^2} \tilde{\psi}^{(\nu)} T_Q^{(\nu)}$ and obtain from Lemma 2.127 that $\gamma_{T,\Gamma_h}\tilde{\psi} = \sum_{\nu\in\mathbb{Z}^2} \tilde{\psi}^{(\nu)} \jmath(T_Q^{(\nu)})$. Therefore, by definition of ℓ_φ from Theorem 2.121 and the definition of $\langle\cdot,\cdot\rangle_{\Gamma_h}$ from part *(b)* of Theorem 2.113, we obtain

$$\begin{aligned}\langle \ell_\varphi, \psi\rangle = \langle \varphi, \gamma_{T,\Gamma_h}\tilde{\psi}\rangle_{\Gamma_h} &= \sum_{\mu,\nu\in\mathbb{Z}^2} \varphi^{(\mu)} \cdot \tilde{\psi}^{(\nu)} \left(T_Q^{(\mu)} \,\middle|\, T_Q^{(-\nu)} \right)_{L^2(Q)} \\ &= \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \cdot \tilde{\psi}^{(-\mu)} = \langle \varphi, \tilde{\psi}(\cdot, h)\rangle = \langle \varphi, \psi\rangle,\end{aligned}$$

as desired, where we have again applied Theorem 2.28 in the second last step. □

2.3.7. Some Results for Smooth Surfaces

In this subsection let $f \in C^\infty_{\mathrm{per}}(Q)$ be real valued and define

$$\Gamma := \left\{ x \in \mathbb{R}^3 \;\middle|\; \tilde{x} \in Q \text{ and } x_3 = f(\tilde{x}) \right\}.$$

Furthermore, we introduce the set $\widetilde{\Gamma}$, the parametrization $\Psi : Q \to \Gamma$ and the mapping $F : Q \to \mathbb{R}^{3\times 3}$ as above from Assumption 2.91 and observe that now the functions are smooth.

For $m \in \mathbb{N}_0$ and $d' \in \mathbb{N}$ we define

$$C^m_Q\big(\Gamma, \mathbb{C}^{d'}\big) := \Big\{ \varphi := \widetilde{\varphi}|_\Gamma \;\Big|\; \widetilde{\varphi} \in C\big(\widetilde{\Gamma}, \mathbb{C}^{d'}\big) \text{ is } Q\text{-periodic and } \varphi \circ \Psi \in C^m_{\mathrm{per}}(Q, \mathbb{C}^{d'}) \Big\}$$

and the corresponding space of *smooth* functions

$$C^\infty_Q\big(\Gamma, \mathbb{C}^{d'}\big) := \bigcap_{k=0}^{\infty} C^k_Q\big(\Gamma, \mathbb{C}^{d'}\big).$$

Moreover, we define the spaces $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ as in Definition 2.106. Concerning the spaces $H_Q^{1/2}(\Gamma, \mathbb{C}^{d'})$ we allow now again more regularity, see the next definition.

Definition 2.130 *Let the surface* Γ *be given as above. For* $s \geq 0$ *we define the space* $H^s_Q(\Gamma, \mathbb{C}^{d'})$ *by*

$$H^s_Q(\Gamma, \mathbb{C}^{d'}) := \left\{ \varphi \in L^2(\Gamma, \mathbb{C}^{d'}) \;\middle|\; \varphi \circ \Psi \in H^s_{\mathrm{per}}(Q, \mathbb{C}^{d'}) \right\}$$

with norm

$$\|\varphi\|_{H^s_Q(\Gamma,\mathbb{C}^{d'})} := \|\varphi \circ \Psi\|_{H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})}.$$

For $s > 0$ *we define* $H^{-s}_Q(\Gamma, \mathbb{C}^{d'})$ *to be the dual space of* $H^s_Q(\Gamma, \mathbb{C}^{d'})$ *equipped with its canonical norm*

$$\|\ell\|_{H^{-s}_Q(\Gamma,\mathbb{C}^{d'})} := \sup_{\psi \in H^s_Q(\Gamma,\mathbb{C}^{d'})\setminus\{0\}} \frac{|\langle \ell, \psi\rangle_{Q,s,\Gamma}|}{\|\psi\|_{H^s_Q(\Gamma,\mathbb{C}^{d'})}}$$

for all $\ell \in H^{-s}_Q(\Gamma, \mathbb{C}^{d'})$*. Here,* $\langle \cdot, \cdot \rangle_{Q,s,\Gamma}$ *denotes the duality pairing as introduced in Section 1.3, and with index "*Q, s, Γ*" to make them distinguishable.*
For $s \in \mathbb{R}$ *we define the spaces of tangential vector fields by*

$$H^s_{Q,t}(\Gamma) := \left\{ \varphi \in H^s_Q(\Gamma, \mathbb{C}^3) \;\middle|\; \varphi \cdot \mathrm{n} = 0 \right\},$$

where for $s > 0$ *and* $\ell \in H^{-s}_Q(\Gamma, \mathbb{C}^3)$ *the product* $\ell \cdot \mathrm{n} \in H^{-s}_Q(\Gamma)$ *is defined by*

$$\langle \ell \cdot \mathrm{n}, \psi \rangle_{Q,s,\Gamma} := \langle \ell, \psi \mathrm{n} \rangle_{Q,s,\Gamma}, \qquad \psi \in H^s_Q(\Gamma).$$

Note that for $s > 0$ and $\psi \in H^s_Q(\Gamma)$ the product $\psi\mathrm{n}$ is well-defined by Theorem 2.132, as the normal vector n is a smooth function.

Proposition 2.131 *Let* $s \in \mathbb{R}$*. Then the following assertions are true.*

(i) The space $\mathcal{D}_{Q,0}(\Gamma, \mathbb{C}^{d'})$ *is dense in* $H^s_Q(\Gamma, \mathbb{C}^{d'})$*.*

(ii) If $\sigma \in \mathbb{R}$*, with* $\sigma < s$*, then the space* $H^s_Q(\Gamma, \mathbb{C}^{d'})$ *is compactly embedded into* $H^\sigma_Q(\Gamma, \mathbb{C}^{d'})$*.*

(iii) If $\sigma \in \mathbb{R}$*, with* $\sigma < s$*, then the space* $H^s_{Q,t}(\Gamma)$ *is embedded into* $H^\sigma_{Q,t}(\Gamma)$*.*

Proof: This is shown with the same arguments as in the proof for Proposition 2.54. □

Clearly, all results we have achieved so far for Lipschitz continuous surfaces hold in particular for smooth surfaces.

As already mentioned at the beginning of Subsection 2.1.4, we intend to exploit results from [21]. Since those results are given for the trace space $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$, where $\Omega \subseteq \mathbb{R}^3$ is a bounded and smooth domain, and since $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega) \simeq H^{-1/2}(\mathrm{Div}, \partial\Omega)$ due to Theorem 2.59, we have somehow to relate the spaces $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ to each other. A key ingredient will be a certain partition of unity on Γ. Hence, we have to ensure that $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ multiplied by $\chi \in C_Q^\infty(\Gamma)$, with $\mathrm{supp}(\chi) \subseteq \Gamma$ as well as $\mathrm{supp}(\chi) \subseteq \partial\Omega$, and extended by zero to $\partial\Omega$ belongs to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$, and that the operator describing this mapping is linear and bounded.

Note that these investigations seem not to be trivial, since for the case of Lipschitz continuous surfaces the spaces $C_Q^m(\Gamma)$ are only well-defined for $m = 0$. Then for $\chi \in C_Q(\Gamma)$ the function $\chi \circ \Psi$ is only Lipschitz continuous and therefore, by Proposition 2.16, only in $H^1_{\mathrm{per}}(Q)$. Hence, according to Theorem 2.40, the regularity of this product is too less to give rise to a linear and bounded operator in $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. For smooth surfaces as considered in this subsection the situation is better, see the next theorem.

Theorem 2.132 *Let $\chi \in C_Q^\infty(\Gamma)$. Then the following assertions are true.*

(a) For $s \in \mathbb{R}$ the mapping $H_Q^s(\Gamma, \mathbb{C}^{d'}) \ni \varphi \mapsto \chi\varphi \in H_Q^s(\Gamma, \mathbb{C}^{d'})$ is well-defined, linear and bounded.

(b) (i) The mapping $\mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3) \ni \varphi \mapsto \chi\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ into itself.

(ii) The mapping $\mathcal{D}_{Q,T}(\Gamma, \mathbb{C}^3) \ni \varphi \mapsto \chi\varphi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$ into itself.

Proof: We consider only part *(b)*, as the argumentation for part *(a)* is very similar.
(i). Note that by assumption $\chi \circ \Psi$ belongs to $C^\infty_{\mathrm{per}}(Q)$. Let $\varphi \in \mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3)$. Then $\tilde{\varphi}^t \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q)$ (recall (2.16a) for the definition of $\tilde{\varphi}^t$). Moreover,

$$(\widetilde{\chi\varphi})^t(x) = \rho(x)F^{-1}(x)\chi(\Psi(x))\varphi(\Psi(x)) = \chi(\Psi(x))\tilde{\varphi}^t(x), \qquad x \in Q.$$

Therefore, thanks to Theorem 2.40, we have that $(\widetilde{\chi\varphi})^t \in H^{-1/2}_{\mathrm{per}}(\mathrm{Div}, Q)$ with $\|(\widetilde{\chi\varphi})^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} \leq C\,\|\tilde{\varphi}^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)}$ and the constant $C > 0$ independent of φ. Thus, $\chi\varphi \in H^{-1/2}_Q(\mathrm{Div}, \Gamma)$ with

$$\begin{aligned}\|\chi\varphi\|_{H^{-1/2}_Q(\mathrm{Div},\Gamma)} &= \|(\widetilde{\chi\varphi})^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} \\ &\leq C\,\|\tilde{\varphi}^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} = C\,\|\varphi\|_{H^{-1/2}_Q(\mathrm{Div},\Gamma)},\end{aligned}$$

which shows that the mapping is well-defined and bounded (its linearity is clear). Since $\mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3)$ is dense in $H^{-1/2}_Q(\mathrm{Div}, \Gamma)$, this multiplication operator can be continuously extended as desired.
(ii). The assertion is shown completely analogous. □

Now, we come to the main theorem of this subsection.

Theorem 2.133 *Let $\Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma$, such that Γ_0 is relatively closed and Γ_1 is relatively open in Γ, and let $\Omega \subseteq \mathbb{R}^3$ be a bounded and smooth domain such that $\Gamma_1 \subseteq \partial\Omega$. Furthermore, let $\chi \in C^\infty(\partial\Omega)$ with* $\mathrm{supp}(\chi) \subseteq \Gamma_0$. *Then the following assertions are true.*

(i) The mapping $\mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3) \ni \varphi \;\mapsto\; (\chi\varphi)|_0^{\partial\Omega} \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H^{-1/2}_Q(\mathrm{Div}, \Gamma)$ into $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

(ii) The mapping $\mathcal{D}_t(\partial\Omega, \mathbb{C}^3) \ni \varphi \mapsto (\chi\varphi)|_\Gamma \in H^{-1/2}_Q(\mathrm{Div}, \partial\Omega)$ is well-defined, linear and bounded and can be continuously extended to a linear and bounded operator from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ into $H^{-1/2}_Q(\mathrm{Div}, \Gamma)$.

Proof: As a preparation, recall Assumption 2.52. Without loss of generality, there exists $m' \le m$ such that $\Gamma_0 \subseteq \bigcup_{j=1}^{m'} U'_j \subseteq \Gamma_1$ and $\Gamma_0 \cap \bigcup_{j=m'+1}^{m} U'_j = \emptyset$. Moreover, without loss of generality we assume that $\mathbb{B}_2(0,\alpha_j) \subseteq Q$ for $j = 1,\dots,m'$.
(i). Let $\varphi \in \mathcal{D}_{Q,t}(\Gamma,\mathbb{C}^3)$. Then $\chi\varphi$ belongs to $\mathcal{D}_{Q,t}(\Gamma,\mathbb{C}^3)$ as well. We consider $\chi\varphi$ also extended by zero to $\partial\Omega$ and use the same symbols. Then $\chi\varphi$ belongs to $\mathcal{D}_t(\partial\Omega,\mathbb{C}^3)$ which is a dense subspace of $H^{-1/2}(\mathrm{Div},\partial\Omega)$. We consider the Fourier coefficients of $\tilde{(\chi\varphi)}^t_j$ determined with respect to Q, $j = 1,\dots,m'$. It sufficies to show that

$$\sum_{j=1}^{m'} \|\tilde{(\chi\varphi)}^t_j\|^2_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} \le C\,\|\tilde{(\chi\varphi)}^t\|^2_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)},$$

where by definition

$$\tilde{(\chi\varphi)}^t_j(x) = \begin{cases} \rho_j(x)\sqrt{\chi_j(\Psi_j(x))}\,F_j^{-1}(x)\,(\chi\varphi)(\Psi_j(x)), & x \in \mathbb{B}_2(0,\alpha_j),\\ 0, & x \in Q\setminus\mathbb{B}_2(0,\alpha_j),\end{cases}$$

$$\tilde{(\chi\varphi)}^t(x) = \rho(x)\,F^{-1}(x)\,(\chi\varphi)(\Psi(x)), \qquad x \in Q.$$

Note that $\tilde{(\chi\varphi)}^t_j = 0$ for $j = m'+1,\dots,m$. Let $j \in \{1,\dots,m'\}$. Due to our assumptions, the parametrization Ψ_j can be built up by means of the parametrization Ψ as follows

$$\Psi_j(u_1,u_2) = \begin{pmatrix} u_1\\ u_2\\ f(u_1+z_1^{(j)},u_2+z_2^{(j)})\end{pmatrix} + \begin{pmatrix} z_1^{(j)}\\ z_2^{(j)}\\ 0\end{pmatrix}, \qquad u \in \mathbb{B}_2(0,\alpha_j),$$

with $\rho_j(u) = \rho(u_1+z_1^{(j)},u_2+z_2^{(j)})$ and $F_j^{-1}(u) = F^{-1}(u_1+z_1^{(j)},u_2+z_2^{(j)})$ for $u \in \mathbb{B}_2(0,\alpha_j)$. Therefore,

$$\tilde{(\chi\varphi)}^t_j(u) = \sqrt{\chi_j(\Psi(u_1+z_1^{(j)},u_2+z_2^{(j)}))}\,\tilde{(\chi\varphi)}^t(u_1+z_1^{(j)},u_2+z_2^{(j)}) \quad (*)$$

for all $u \in \mathbb{B}_2(0,\alpha_j)$. Hence, by an application of the transformation formula we obtain with $(x_1,x_2)^\top = (u_1+z_1^{(j)},u_2+z_2^{(j)})^\top$ that

$$\frac{1}{\sqrt{|Q|}}\int_Q \tilde{(\chi\varphi)}^t_j(u)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot u}\,\mathrm{d}u$$

$$= \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot(z_1^{(j)},z_2^{(j)})^\top} \frac{1}{\sqrt{|Q|}} \int_Q \sqrt{\chi_j(\Psi(x))}\,\tilde{(\chi\varphi)}^t(x)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot x}\,\mathrm{d}x,$$

which shows that the Fourier coefficients differ only by a phase shift factor. Therefore,

$$\begin{aligned}\|\tilde{(\chi\varphi)}^t_j\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} &= \|(\sqrt{\chi_j}\circ\Psi)\,\tilde{(\chi\varphi)}^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)} \\ &\le C\,\|\tilde{(\chi\varphi)}^t\|_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)},\end{aligned}$$

where we have applied Theorem 2.40 after recalling that $\sqrt{\chi_j}\circ\Psi$ is a smooth function by the choice of our partition of unity on $\partial\Omega$. This shows that the mapping is well-defined and bounded (its linearity is clear) and thus can be continuously extended to a linear and bounded operator from $H_Q^{-1/2}(\mathrm{Div},\Gamma)$ into $H^{-1/2}(\mathrm{Div},\partial\Omega)$ as desired.

(ii). Let $\varphi\in\mathcal{D}_t(\partial\Omega,\mathbb{C}^3)$. We consider $\chi\varphi$ also restricted to Γ and use the same symbols. Then $\chi\varphi$ belongs to $\mathcal{D}_{Q,t}(\Gamma,\mathbb{C}^3)$. Note that for $y\in\Gamma_0$ we have $(\chi\varphi)(y)=\sum_{j=1}^{m'}\chi_j(y)\,(\chi\varphi)(y)$. Therefore, using in addition $(*)$ and again the transformation formula,

$$\begin{aligned}\frac{1}{\sqrt{|Q|}}\int_Q \tilde{(\chi\varphi)}^t(x)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot x}\,\mathrm{d}x &= \frac{1}{\sqrt{|Q|}}\int_{\Psi^{-1}(\Gamma_0)} \tilde{(\chi\varphi)}^t(x)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot x}\,\mathrm{d}x \\ &= \sum_{j=1}^{m'}\frac{1}{\sqrt{|Q|}}\int_{\Psi^{-1}(\Gamma_0\cap\mathrm{supp}(\chi_j))}\chi_j(\Psi(x))\,\tilde{(\chi\varphi)}^t(x)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot x}\,\mathrm{d}x \\ &= \sum_{j=1}^{m'}\mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot(z_1^{(j)},z_2^{(j)})^\top}\frac{1}{\sqrt{|Q|}}\int_Q\sqrt{\chi_j(\Psi(u))}\,\tilde{(\chi\varphi)}^t_j(u)\,\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot u}\,\mathrm{d}u.\end{aligned}$$

Hence, $\big|\big[\tilde{(\chi\varphi)}^t\big]^{(\mu)}\big|^2\le C\sum_{j=1}^{m'}\big|\big[(\sqrt{\chi_j}\circ\Psi)\,\tilde{(\chi\varphi)}^t_j\big]^{(\mu)}\big|^2$ and

$$\big|\big[q^{(\mu)}\cdot\tilde{(\chi\varphi)}^t\big]^{(\mu)}\big|^2\le C\sum_{j=1}^{m'}\big|\big[q^{(\mu)}\cdot(\sqrt{\chi_j}\circ\Psi)\,\tilde{(\chi\varphi)}^t_j\big]^{(\mu)}\big|^2,$$

which yields that

$$\|\tilde{(\chi\varphi)}^t\|^2_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)}\le C\sum_{j=1}^{m'}\|(\sqrt{\chi_j}\circ\Psi)\,\tilde{(\chi\varphi)}^t_j\|^2_{H^{-1/2}_{\mathrm{per}}(\mathrm{Div},Q)}$$

$$\leq C \sum_{j=1}^{m} \|(\tilde{\chi\varphi})_j^t\|^2_{H_{\mathrm{per}}^{-1/2}(\mathrm{Div},Q)},$$

where we have again applied Theorem 2.40 and the fact that $(\tilde{\chi\varphi})_j^t = 0$ for $j = m' + 1, \ldots, m$. Thus, the mapping is well-defined and bounded (its linearity is clear) and can again be continuously extended to a linear and bounded operator from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ into $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ and the proof is complete. $\square$

3. Electromagnetic Scattering – Variational Formulation

One of the two main approaches to treat questions about existence and uniqueness of solutions to boundary value problems are functional analytic methods based on variational formulations. To make this approach accessible to scattering problems we have to truncate the domain and to impose another boundary condition by means of the Calderon operator.

In this chapter we will take this route and start in Section 3.1, after a short derivation of the time-harmonic Maxwell's equations, with the geometrical setting as well as the introduction of upward (and downward) propagating waves as analogs of the Silver-Müller radiation condition. After these preparations, we are in a position to give a precise weak formulation of the scattering problem (1.1) and to show uniqueness of solutions.

In Section 3.2 we use a special extension operator, given by the unique solvability of a certain exterior boundary value problem, to define the Calderon operator. The latter operator allows us to rewrite our scattering problem from the previous section equivalently into its variational form – the starting point for investigations of existence of solutions.

This will be the topic of Section 3.3. For this, we follow the idea from [35] and [42], which is to split up the solution space into a direct sum, where one summand is "curl-free" and the other one is "divergence-free". By means of this decomposition, we are able to divide the scattering problem given in its variational form into two smaller ones, which are easier to analyse. Nevertheless, some technical efforts have to be overcome for the second auxiliary problem before we finally can state the existence result.

3.1. Problem Formulation and Uniqueness of Solution

3.1.1. Time-Harmonic Maxwell's Equations

The following presentation is an extract of [34, Chapter 1].

In general, electromagnetic wave phenomena are described by *Maxwell's equations*, which connect five vector fields, namely the *electric field* $\mathcal{E}$, the *electric displacement* $\mathcal{D}$, the *magnetic field* $\mathcal{H}$, the *magnetic flux density* $\mathcal{B}$ and the *current density* $\mathcal{J}$, and one scalar field, namely the *charge density* ϱ, to each other by

$$\begin{aligned} \frac{\partial \mathcal{B}}{\partial t} + \operatorname{curl}_x \mathcal{E} &= 0 && \text{(Faraday's Law of Induction)},\\ \frac{\partial \mathcal{D}}{\partial t} - \operatorname{curl}_x \mathcal{H} &= -\mathcal{J} && \text{(Ampere's Law)},\\ \operatorname{div}_x \mathcal{D} &= \varrho && \text{(Gauss' Electric Law)},\\ \operatorname{div}_x \mathcal{B} &= 0 && \text{(Gauss' Magnetic Law)}. \end{aligned}$$

We assume that all fields behave periodically with respect to *time* $t \geq 0$, with the same *frequency* $\omega > 0$. Then the complex valued functions

$$\mathcal{E}(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} E(x), \quad \mathcal{H}(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} H(x), \quad \text{etc.}, \tag{3.1}$$

as well as their real and imaginary parts, satisfy the *time-harmonic Maxwell's equations*

$$\begin{aligned} -\mathrm{i}\omega B + \operatorname{curl} E &= 0,\\ \mathrm{i}\omega D + \operatorname{curl} H &= J,\\ \operatorname{div} D &= \rho,\\ \operatorname{div} B &= 0. \end{aligned}$$

Incorporating now the *constitutive equations* for an *isotropic* and *homogeneous* medium

$$D = \varepsilon E \quad \text{and} \quad B = \mu H,$$

where $\varepsilon > 0$ denotes the *electric permittivity* and $\mu > 0$ the *magnetic permeability*, and *Ohm's law*

$$J = \sigma E + J_{\mathrm{e}},$$

where $\sigma \geq 0$ is the *conductivity* and J_{e} is the *external current density*, we arrive at

$$\begin{aligned}
\operatorname{curl} E - \mathrm{i}\omega\mu H &= 0, \\
\operatorname{curl} H + (\mathrm{i}\omega\varepsilon - \sigma)E &= J_{\mathrm{e}}, \\
\operatorname{div} E &= \rho/\varepsilon, \\
\operatorname{div} H &= 0.
\end{aligned}$$

3.1.2. Geometrical Setting

Material Parameters. In the sequel, $Q \subseteq \mathbb{R}^2$ will denote the rectangle given by

$$Q := (-L_1, L_1) \times (-L_2, L_2)$$

for some constants $L_j > 0$, $j = 1, 2$. Furthermore, $\alpha \in \mathbb{R}^3$ will be a vector of the form

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3.$$

Recall from the beginning of Section 2.2.1 the definition for Q-(quasi)-periodicity (with phase shift α). Throughout this thesis the term "biperiodic" will be considered as a synonym of the term "Q-periodic".

We are interested in time-harmonic electromagnetic wave scattering at impenetrable biperiodic surfaces. We suppose the *scatterer* $\widetilde{\Gamma}_0 \subseteq \mathbb{R}^3$ to be the graph of a Q-periodic Lipschitz continuous function $f_0 : \mathbb{R}^2 \to \mathbb{R}$, i.e.,

$$\widetilde{\Gamma}_0 := \left\{ x \in \mathbb{R}^3 \mid \tilde{x} \in \mathbb{R}^2 \text{ and } x_3 = f_0(\tilde{x}) \right\}, \tag{3.2}$$

which is illuminated from above. Since the scatterer is impenetrable, the domain of interest is above from $\widetilde{\Gamma}_0$. For the material parameters we make the following assumption.

Assumption 3.1 *We assume that* $\varepsilon, \mu > 0$ *and* $\sigma \geq 0$. *Furthermore, we suppose* J_{e} *and* ρ *to be zero.*

Definition 3.2 *We call the number* $k \in \mathbb{C}$, *satisfying* $\mathrm{Re}(k) > 0$ *and* $\mathrm{Im}(k) \geq 0$ *as well as*

$$k^2 = \omega^2 \varepsilon \mu + \mathrm{i}\omega\mu\sigma,$$

wave number.

Consequences for Maxwell's Equations. As *incident fields* we will consider Q-quasi-periodic vector fields, with phase shift α, which are smooth solutions to the time-harmonic Maxwell's equations and impact the scatterer from above. We denote them by E^{i}_{α} and H^{i}_{α}. As a consequence, by taking also the Q-periodicity of the scatterer into account, the *scattered fields* E^{s}_{α} and H^{s}_{α} can be assumed as Q-quasi-periodic, with phase shift α, as well, which satisfy the time-harmonic Maxwell's equations too. Due to Assumption 3.1, those equations read now as

$$\begin{cases} \operatorname{curl} E_\alpha - \mathrm{i}\omega\mu H_\alpha = 0, \\ \operatorname{curl} H_\alpha + (\mathrm{i}\omega\varepsilon - \sigma) E_\alpha = 0, \\ \operatorname{div} E_\alpha = 0, \\ \operatorname{div} H_\alpha = 0. \end{cases} \tag{3.3}$$

Using the first equation in (3.3) to substitute H_α in the second equation in (3.3) and using (A.2b), it is easy to check that E_α and H_α solve (3.3), if and only if E_α satisfies

$$\operatorname{curl}\operatorname{curl} E_\alpha - k^2 E_\alpha = 0$$

and $H_\alpha := \frac{1}{\mathrm{i}\omega\mu} E_\alpha$. Therefore, it suffices to concentrate on vector fields $u_\alpha : \mathbb{R}^3 \to \mathbb{C}^3$ which are Q-quasi-periodic, with phase shift α, and solve

$$\operatorname{curl}\operatorname{curl} u_\alpha - k^2 u_\alpha = 0. \tag{3.4}$$

The Unit Cell. A key ingredient to any scattering problem is a suitable radiation condition, see [7]. For this we follow in the next section the Rayleigh expansion ansatz in [7], which requires the introduction of a certain planar auxiliary surface. It turns out to be a great convenience to introduce the following sets, which will later keep the notation simple.

Let $\widetilde{\Gamma} \subseteq \mathbb{R}^3$ be the graph of an arbitrary Q-periodic Lipschitz continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and let Γ denote the patch of $\widetilde{\Gamma}$ whose orthogonal projection onto $\mathbb{R}^2$ gives Q, i.e.,

$$\Gamma := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = f(\tilde{x})\big\}.$$

For $a, b \in \mathbb{R}$, we define

$$D_\Gamma^a := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } f(\tilde{x}) < x_3 < a\big\}, \tag{3.5a}$$
$$D_\Gamma^\infty := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } f(\tilde{x}) < x_3 < \infty\big\}, \tag{3.5b}$$
$$D_a^\Gamma := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } a < x_3 < f(\tilde{x})\big\}, \tag{3.5c}$$
$$D_{-\infty}^\Gamma := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } -\infty < x_3 < f(\tilde{x})\big\}, \tag{3.5d}$$
$$D_a^b := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } a < x_3 < b\big\}, \tag{3.5e}$$
$$D_a^\infty := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } a < x_3 < \infty\big\}, \tag{3.5f}$$
$$D_{-\infty}^a := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } -\infty < x_3 < a\big\} \tag{3.5g}$$

and last but not least

$$\Gamma_a := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = a\big\}. \tag{3.6}$$

Note that there are choices of a and b such that some of those sets are empty.

Now, recall f_0 from above, whose graph describes our scatterer $\widetilde{\Gamma}_0$. Since the scattered wave u_α^{s} is assumed to be Q-quasi-periodic as well, it sufficies to restrict our considerations to the surface patch

$$\Gamma_0 := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = f_0(\tilde{x})\big\}.$$

We fix $h^\pm \in \mathbb{R}$ such that $h^+ > \max_{\xi \in \mathbb{R}^2} f_0(\xi)$ and $h^- < \min_{\xi \in \mathbb{R}^2} f_0(\xi)$. Furthermore, we define

$$D := D_{\Gamma_0}^{h^+} \tag{3.7}$$

and call D the *unit cell*. Note that D is a cell set of Lipschitz layer type according to Definition 2.89. This set will be later the domain of greatest interest for the variational formulation of our scattering problem.

The Scattering Problem in Classical Terms. With the preliminary considerations from above, we will now give a first (vague) formulation of the scattering problem we are interested in. We start with the incident field for which we will make the following assumption.

Assumption 3.3 *The incident field* u_α^{i} *is assumed to be a smooth function* $u_\alpha^{\mathrm{i}} \in C^\infty(\overline{D_{\Gamma_0}^\infty}, \mathbb{C}^3)$*, which is* Q*-quasi-periodic, with phase shift* α*, and solves* (3.4)*.*

Given an incident field u_α^{i} as in Assumption 3.3, we are looking for a function $u_\alpha : \overline{D_{\Gamma_0}^\infty} \to \mathbb{C}^3$, the *total field*, such that

$$\begin{cases} \operatorname{curl}\operatorname{curl} u_\alpha - k^2 u_\alpha = 0 & \text{in } D_{\Gamma_0}^\infty, \\ \mathrm{n} \times u_\alpha = 0 & \text{on } \Gamma_0, \\ u_\alpha^{\mathrm{s}} := u_\alpha - u_\alpha^{\mathrm{i}} & \text{is upward propagating in } D_{h+}^\infty, \end{cases} \tag{3.8}$$

where n denotes the unit normal vector on Γ_0, pointing in the downward direction. In the next subsection we will state more precisely what we mean by the term "upward propagating". And in the subsection after the next, we will be in the position to give a precise (weak) formulation of our scattering problem.

Connection to the Q-periodic Framework. To work out answers to the questions of existence and uniqueness of solutions to our scattering problem and to develop a high order solver for its numerical solution, the necessary tools were provided in Chapter 2 not for the Q-quasi-periodic but for the Q-periodic framework. As already mentioned in Subsection 2.2.1, both situations are closely related to each other by the transformation (2.10).

Convention 3.4 *Let* $\Omega \subseteq \mathbb{R}^3$ *be a cell set and* $d' \in \mathbb{N}$*. Recalling* (2.10)*, for given* $u_\alpha : \Omega \to \mathbb{C}^{d'}$*, which is* Q*-quasi-periodic with phase shift* α*, we denote by* u *its* Q*-periodic counterpart, i.e.,* u *is the* Q*-periodic function* $u : \Omega \to \mathbb{C}^{d'}$ *given by*

$$u(x) := \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} u_\alpha(x), \qquad x \in \Omega.$$

Conversely, for given Q*-periodic function* $u : \Omega \to \mathbb{C}^{d'}$*, we denote by* u_α *its* Q*-quasi-periodic counterpart, with phase shift* α*.*

Having Convention 3.4 in mind and using (1.20b), it is easy to check that u_α satisfies (3.4), if and only if u satisfies

$$\operatorname{curl}_\alpha \operatorname{curl}_\alpha u - k^2 u = 0. \tag{3.9}$$

3.1.3. Upward and Downward Propagating Waves

We follow the Rayleigh expansion ansatz in [7, Section 2.2] to define a suitable radiation condition. Since a bounded and Q-quasi-periodic solution u_α to the Helmholtz equation in D^∞_{h+} is an analytic and Q-quasi-periodic function on any plane $\{x_3 = h\}$, where $h > h^+$, u_α can be expanded on such planes into a Fourier series of the form (2.11), see [7]. Inserting this expansion into the Helmholtz equation, in [7] were derived conditions on the Fourier coefficients to ensure u_α to be an upward or downward propagating wave. Since solutions of the time harmonic Maxwell system are divergence free solutions to the vector Helmholtz equation, see [34, Lemma 1.3] and the remarks after it, we choose the next conditions for an upward and downward propagating wave in our electromagnetic context. For this, recall $q^{(\mu)}$ from (2.7) and define for $\mu \in \mathbb{Z}^2$

$$\rho^{(\mu)} := \left(k^2 - |d^{(\mu)}|^2\right)^{1/2} \in \mathbb{C}, \qquad \text{where } \; d^{(\mu)} := \tilde{\alpha} + q^{(\mu)} \in \mathbb{R}^2. \tag{3.10}$$

Assumption 3.5 *Throughout this thesis we assume that $\rho^{(\mu)} \neq 0$ for all $\mu \in \mathbb{Z}^2$.*

Definition 3.6 *(i) A function $u : D^\infty_a \to \mathbb{C}^3$ is said to satisfy the upward propagating Rayleigh expansion radiation condition (URC), if there exists a sequence $(u^{(\mu)})_{\mu\in\mathbb{Z}^2}$ in $\mathbb{C}^3$ such that*

$$u(x) = \sum_{\mu\in\mathbb{Z}^2} u^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\,\tilde{x} + \rho^{(\mu)}(x_3 - a))}, \qquad x \in D^\infty_a, \tag{3.11a}$$

with uniform convergence to u on D^∞_b for all $b > a$.

(ii) A function $u : D^a_{-\infty} \to \mathbb{C}^3$ is said to satisfy the downward propagating Rayleigh expansion radiation condition (DRC), if there exists a sequence $(u^{(\mu)})_{\mu\in\mathbb{Z}^2}$ in $\mathbb{C}^3$ such that

$$u(x) = \sum_{\mu\in\mathbb{Z}^2} u^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\,\tilde{x} + \rho^{(\mu)}(a - x_3))}, \qquad x \in D^a_{-\infty}, \tag{3.11b}$$

with uniform convergence to u on $D^b_{-\infty}$ for all $b < a$.

Multiplying (3.11a) by the factor $\mathrm{e}^{-\mathrm{i}\omega t}$ from (3.1) yields for the summands

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\omega t}\, u^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\,\tilde{x}+\rho^{(\mu)}(x_3-a))}\\ &\qquad = u^{(\mu)}\mathrm{e}^{-\mathrm{Im}(\rho^{(\mu)})(x_3-a)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\,\tilde{x}+\mathrm{Re}(\rho^{(\mu)})(x_3-a)-\omega t)},\end{aligned}$$

for all $x \in D_a^\infty$ and all $\mu \in \mathbb{Z}^2$. Note that in this equation for fixed $\mu \in \mathbb{Z}^2$ and for fixed $\tilde{x} \in Q$ the term $q^{(\mu)} \cdot \tilde{x} + \mathrm{Re}(\rho^{(\mu)})(x_3 - a) - \omega t$ is constant for growing $t > 0$, only if $x_3 > a$ is growing as well. Hence, each summand in (3.11a) represents indeed an upward propagating wave. Similarly, we see that the summands in (3.11b) are downward propagating waves.

Moreover, note that the function u from Definition 3.6 is Q-periodic. And, having still Convention 3.4 in mind, it is easy to see that a function $u : D_a^\infty \to \mathbb{C}^3$ satisfies the (URC) from Definition 3.6, if and only if its Q-quasi-periodic counterpart u_α satisfies

$$u_\alpha(x) = \sum_{\mu\in\mathbb{Z}^2} u^{(\mu)}\, \mathrm{e}^{\mathrm{i}(d^{(\mu)}\cdot\,\tilde{x}+\rho^{(\mu)}(x_3-a))}, \qquad x \in D_a^\infty, \tag{3.12}$$

with the same sequence of coefficients $(u^{(\mu)})_{\mu\in\mathbb{Z}^2}$. Condition (3.12) will be referred to as $(\mathrm{URC})_\alpha$. Of course, the analogous result we have for functions satisfying the (DRC).

Remark 3.7 *It is easy to check that $u : D^a_{-\infty} \to \mathbb{C}^3$ satisfies (DRC) if and only if $u^*(\cdot^*) : D^\infty_{-a} \to \mathbb{C}^3$ satisfies (URC). Here, recall* (1.4) *for the definition of z^* for some $z \in \mathbb{C}^3$. Therefore, in the following presentation it suffices to restrict our attention to functions satisfying (URC).*

Functions Satisfying (URC). Now, we will show that functions which satisfy the (URC) are smooth functions. Recall Lemma 2.9 for some convenient estimates for the quantity $q^{(\mu)}$. Similar estimates for the quantity $\rho^{(\mu)}$ will be of interest and are derived in the next lemma.

Lemma 3.8 *For the quantity $\rho^{(\mu)}$ from* (3.10) *there holds* *(i)* $\exists\, C > 0\ \forall\, \mu \in \mathbb{Z}^2 :\ |\rho^{(\mu)}| \leq C\sqrt{1+|\mu|^2}$,

(ii) $\exists\, C > 0\ \forall\, \mu \in \mathbb{Z}^2 :\ |\rho^{(\mu)}| \geq \mathrm{Im}\big(\rho^{(\mu)}\big) \geq C\sqrt{1+|\mu|^2}$.

Proof: We observe that $|q^{(\mu)}| \to \infty$, as $|\mu| \to \infty$, which implies that $|\tilde{\alpha} + q^{(\mu)}| \to \infty$ and $\arg(\rho^{(\mu)}) \to \frac{\pi}{2}$, as $|\mu| \to \infty$. The convergence for $\arg(\rho^{(\mu)})$ results from Definition 3.2.
(i). By the observation above, there exists $N \in \mathbb{N}$ such that for all $\mu \in \mathbb{Z}^2$ with $|\mu| \geq N$ we have $|\tilde{\alpha} + q^{(\mu)}|^2 \geq |\mathrm{Re}(k^2)|$, $|\tilde{\alpha} + q^{(\mu)}|^4 \geq (\mathrm{Im}(k^2))^2$ and $|q^{(\mu)}| \geq |\tilde{\alpha}|$. Let $\mu \in \mathbb{Z}^2$ with $|\mu| \geq N$. Then $0 \leq |\tilde{\alpha} + q^{(\mu)}|^2 - \mathrm{Re}(k^2) \leq 2|\tilde{\alpha} + q^{(\mu)}|^2$ and we obtain, together with *(i)* from Lemma 2.9,

$$\begin{aligned} |\rho^{(\mu)}|^2 &= \sqrt{\big(|\tilde{\alpha} + q^{(\mu)}|^2 - \mathrm{Re}(k^2)\big)^2 + \big(\mathrm{Im}(k^2)\big)^2} \\ &\leq \sqrt{5}\,|\tilde{\alpha} + q^{(\mu)}|^2 \leq 4\sqrt{5}\,|q^{(\mu)}|^2 \leq \tilde{C}^2\big(1+|\mu|^2\big). \end{aligned}$$

Set $M := \max\big\{|\rho^{(\mu)}| \,\big|\, |\mu| < N\big\}$ and $m := \min\big\{\sqrt{1+|\mu|^2} \,\big|\, |\mu| < N\big\}$. Note that $m > 0$. Now, let $\mu \in \mathbb{Z}^2$ with $|\mu| < N$. Then

$$|\rho^{(\mu)}| \leq M = \tfrac{M}{\sqrt{1+|\mu|^2}}\sqrt{1+|\mu|^2} \leq \tfrac{M}{m}\sqrt{1+|\mu|^2}.$$

Finally the assertion follows by choosing $C := \max\big\{\frac{M}{m}, \tilde{C}\big\}$.
(ii). By the observation above, there exists $N \in \mathbb{N}$ such that for all $\mu \in \mathbb{Z}^2$ with $|\mu| \geq N$ we have $|\tilde{\alpha} + q^{(\mu)}|^2 \geq 2|\mathrm{Re}(k^2)|$, $|q^{(\mu)}| \geq 2|\tilde{\alpha}|$ and $\sin\big(\arg(\rho^{(\mu)})\big) \geq \frac{1}{2}$. Let $\mu \in \mathbb{Z}^2$ with $|\mu| \geq N$. Then $|\tilde{\alpha}+q^{(\mu)}|^2 - \mathrm{Re}(k^2) \geq \frac{1}{2}|\tilde{\alpha} + q^{(\mu)}|^2 \geq 0$ and we obtain, together with *(ii)* from Lemma 2.9,

$$\begin{aligned} \mathrm{Im}(\rho^{(\mu)}) &= \sqrt[4]{\big(|\tilde{\alpha} + q^{(\mu)}|^2 - \mathrm{Re}(k^2)\big)^2 + \big(\mathrm{Im}(k^2)\big)^2}\ \sin\big(\arg(\rho^{(\mu)})\big) \\ &\geq \tfrac{1}{\sqrt{2}}\tfrac{1}{2}|\tilde{\alpha} + q^{(\mu)}| \geq \tfrac{1}{\sqrt{2}}\tfrac{1}{4}|q^{(\mu)}| \geq \tfrac{1}{\sqrt{2}}\tfrac{1}{4}\tilde{C}\sqrt{1+|\mu|^2}. \end{aligned}$$

Set $m := \min\big\{\mathrm{Im}(\rho^{(\mu)}) \,\big|\, |\mu| < N\big\}$ and $M := \max\big\{\sqrt{1+|\mu|^2} \,\big|\, |\mu| < N\big\}$. Note that $m > 0$. Now, let again $\mu \in \mathbb{Z}^2$ with $|\mu| < N$. Then

$$\mathrm{Im}(\rho^{(\mu)}) \geq m = m\tfrac{1}{\sqrt{1+|\mu|^2}}\sqrt{1+|\mu|^2} \geq \tfrac{m}{M}\sqrt{1+|\mu|^2}.$$

And finally the assertion follows by choosing $C := \min\big\{\frac{m}{M}, \frac{1}{\sqrt{2}}\frac{1}{4}\tilde{C}\big\}$. □

The next lemma gives a useful quantification for the convergence rate for the coefficients $u^{(\mu)}$ of a function u which satisfies the (URC).

Lemma 3.9 *Let $u : D_a^\infty \to \mathbb{C}^3$ satisfy (URC). Then for its coefficients $(u^{(\mu)})_{\mu\in\mathbb{Z}^2}$ there holds*

$$\big(u^{(\mu)}\,\mathrm{e}^{\mathrm{i}\rho^{(\mu)}(b-a)}\big)_{\mu\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2,\mathbb{C}^3), \quad \textit{for all } b > a.$$

Proof: By definition of the (URC), we have uniform convergence of the series representation for u on $D^\infty_{(a+b)/2}$ to u. Therefore, by the continuity of all summands, u is continuous on $D^\infty_{(a+b)/2}$. Hence, $u(\cdot, b)$ has a Fourier series expansion, i.e., $u(\cdot,b) = \sum_{\mu\in\mathbb{Z}^2} \hat{u}^{(\mu)}\,T_Q^{(\mu)}$. Since uniform convergence is stronger than L^2-convergence (on bounded measurable sets), we obtain, by uniqueness of the Fourier coefficients, $\big(u^{(\mu)}\,\mathrm{e}^{\mathrm{i}\rho^{(\mu)}(b-a)}\big)_{\mu\in\mathbb{Z}^2} = \big(\frac{1}{|Q|^{1/2}}\,\hat{u}^{(\mu)}\big)_{\mu\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2,\mathbb{C}^3)$. □

Using the statements of the last two lemmas, we obtain the following important result, saying that a function which satisfies the (URC) is a smooth function.

Proposition 3.10 *Let the function $u : D_a^\infty \to \mathbb{C}^3$ satisfy (URC). Then $u \in C_Q^\infty(D_a^\infty,\mathbb{C}^3)$ with*

$$\partial^\beta u_j(x) = \sum_{\mu\in\mathbb{Z}^2} \partial^\beta u_j^{(\mu)}\,\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}(x_3-a))}, \qquad x\in D_a^\infty,$$

for all $j = 1,2,3$ and all $\beta\in\mathbb{N}_0^3$, where convergence holds uniformly on D_b^∞ for all $b > a$.

Proof: Thanks to Lemma 3.8, there exists a constant $\tilde{C} > 0$ such that for all $\mu\in\mathbb{Z}^2$ we have $\frac{1}{2}\mathrm{Im}(\rho^{(\mu)}) \geq \tilde{C}\sqrt{1+|\mu|^2}$. Let $b > a$. Moreover, let $n\in\mathbb{N}$ and $x\in D_b^\infty$. For $\mu\in\mathbb{Z}^2$ we define $h(\mu) := \tilde{C}(b-a)\sqrt{1+|\mu|^2}$. Note that $b - a > 0$. Therefore, $\frac{(h(\mu))^{n+2}}{\mathrm{e}^{h(\mu)}}$ is bounded for all $\mu\in\mathbb{Z}^2$. Furthermore,

$$\begin{aligned}
&|u^{(\mu)}|\,\big(\sqrt{1+|\mu|^2}\big)^n\,\big|\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}(x_3-a))}\big| \\
&\qquad = \frac{\big(\sqrt{1+|\mu|^2}\big)^{n+2}}{\mathrm{e}^{\frac{1}{2}\mathrm{Im}(\rho^{(\mu)})(x_3-a)}}\,|u^{(\mu)}|\,\mathrm{e}^{-\frac{1}{2}\mathrm{Im}(\rho^{(\mu)})(x_3-a)}\,\frac{1}{1+|\mu|^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left(\sqrt{1+|\mu|^2}\right)^{n+2}}{\mathrm{e}^{\tilde{C}\sqrt{1+|\mu|^2}(b-a)}}\,|u^{(\mu)}|\,\mathrm{e}^{-\frac{1}{2}\operatorname{Im}(\rho^{(\mu)})(b-a)}\,\frac{1}{1+|\mu|^2}\\
&= \frac{1}{[\tilde{C}(b-a)]^{n+2}}\,\frac{(h(\mu))^{n+2}}{\mathrm{e}^{b(\mu)}}\,|u^{(\mu)}|\,\mathrm{e}^{-\operatorname{Im}(\rho^{(\mu)})(\hat{b}-a)}\,\frac{1}{1+|\mu|^2}\\
&\leq C\,|u^{(\mu)}|\,\mathrm{e}^{-\operatorname{Im}(\rho^{(\mu)})(\hat{b}-a)}\,\frac{1}{1+|\mu|^2},
\end{aligned}
$$

for all $\mu \in \mathbb{Z}^2$, where $\hat{b} := \frac{1}{2}(a+b) > a$. Using the parts *(i)* from Lemma 2.9 and Lemma 3.8, using Lemma 3.9 and the Cauchy-Schwarz inequality (note that $(\frac{1}{1+|\mu|^2})_{\mu\in\mathbb{Z}^2}$ belongs to $\ell(\mathbb{Z}^2)$ according to Lemma 2.37), we obtain from the last estimate, that the series of each partial derivative ∂_i from the j-th component of the continuously differentiable summands in (3.11a) converge uniformly on D_b^∞. In particular, they converge locally uniformly on D_a^∞. Therefore $u_j \in C^1(D_a^\infty)$, and hence $u \in C^1(D_a^\infty, \mathbb{C}^3)$. Now, by induction and again by the last estimate, we finally obtain the assertion. □

Remark 3.11 *Applying the argumentation in the proof of Lemma 3.9 to the series representation in Proposition 3.10, we see that for a function* $u : D_a^\infty \to \mathbb{C}^3$ *which satisfies (URC) there holds for its coefficients* $u^{(\mu)}$

$$
\Big((\sqrt{1+|\mu|^2})^n u^{(\mu)}\,\mathrm{e}^{\mathrm{i}\rho^{(\mu)}(b-a)}\Big)_{\mu\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2,\mathbb{C}^3), \quad \textit{for all } b > a,\ n \in \mathbb{N}_0.
$$

3.1.4. Weak Formulation and Uniqueness of Solution

We are now in a position to give a precise (weak) formulation of our scattering problem under consideration, see also (3.8) and Convention 3.4.

Problem 3.12 *Given an incident field* u^{i} *as in Assumption 3.3, find* $u : D_{\Gamma_0}^\infty \to \mathbb{C}^3$ *such that for all* $h > h^+$ *there holds* $u \in H_Q(\operatorname{curl}, D_{\Gamma_0}^h)$ *and*

$$
\begin{cases}
\forall v \in H_{Q,0}(\operatorname{curl}, D_{\Gamma_0}^h) : \displaystyle\int_{D_{\Gamma_0}^h} \left(\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u\cdot\overline{v}\right)\mathrm{d}x = 0,\\
\gamma_{t,\Gamma_0} u = 0,\\
u^{\mathrm{s}} := u - u^{\mathrm{i}} \textit{ satisfies (URC) in } D_{h^+}^\infty.
\end{cases}
$$

For a solution to Problem 3.12 we can derive the following properties, which turn out to be quite useful, in particular later when we show that Problem 3.12 is equivalent to the variational formulation based on the Calderon operator, see next section.

Proposition 3.13 *Let* $u : D^{\infty}_{\Gamma_0} \to \mathbb{C}^3$ *be a solution to Problem 3.12. Furthermore, let* $h > h^+$*. Then the following assertions are true.*

(i) $\operatorname{curl}_\alpha u$ *belongs to* $H_Q(\operatorname{curl}, D^h_{\Gamma_0})$ *with*

$$\operatorname{curl}(\operatorname{curl}_\alpha u) = k^2 u - (\mathrm{i}\alpha \times \operatorname{curl}_\alpha u), \quad \textit{i.e.,}\ \operatorname{curl}_\alpha \operatorname{curl}_\alpha u = k^2 u,$$

holding in $L^2(D^h_{\Gamma_0}, \mathbb{C}^3)$.

(ii) $u|_{D^{\infty}_{h+}}$ *belongs to* $C^{\infty}_Q(D^{\infty}_{h+}, \mathbb{C}^3)$ *and solves the equation in (i) in the classical sense. In particular, this is true for* $u^{\mathrm{s}}|_{D^{\infty}_{h+}}$.

(iii) For all $v \in H_{Q,0,\Gamma_0}(\operatorname{curl}, D)$ *there holds*

$$\int_D \big(\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u \cdot \overline{v}\big)\,\mathrm{d}x = -\langle \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha u|_D, \gamma_{T,\Gamma_{h+}} \overline{v}\rangle_{\Gamma_{h+}}.$$

Proof: *(i).* Let $\chi \in C^{\infty}_{Q,0}(D^h_{\Gamma_0}, \mathbb{C}^3)$ and set $v := \overline{\chi}$. Then v belongs to $H_{Q,0}(\operatorname{curl}, D^h_{\Gamma_0})$ and we obtain, using the first equation in Problem 3.12,

$$\begin{aligned}
&\int_{D^h_{\Gamma_0}} \operatorname{curl}_\alpha u \cdot \operatorname{curl}\chi\,\mathrm{d}x \\
&\qquad = \int_{D^h_{\Gamma_0}} \big[\operatorname{curl}_\alpha u \cdot (\overline{\operatorname{curl}\overline{\chi} + \mathrm{i}\alpha \times \overline{\chi}}) + \operatorname{curl}_\alpha u \cdot (\mathrm{i}\alpha \times \chi)\big]\,\mathrm{d}x \\
&\qquad = \int_{D^h_{\Gamma_0}} \big[\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - (\mathrm{i}\alpha \times \operatorname{curl}_\alpha u) \cdot \chi\big]\,\mathrm{d}x \\
&\qquad = \int_{D^h_{\Gamma_0}} \big[k^2 u - (\mathrm{i}\alpha \times \operatorname{curl}_\alpha u)\big] \cdot \chi\,\mathrm{d}x.
\end{aligned}$$

(ii). Since u^{s} satisfies (URC) in D^{∞}_{h+}, we have due to Proposition 3.10 that $u^{\mathrm{s}}|_{D^{\infty}_{h+}} \in C^{\infty}_Q(D^{\infty}_{h+}, \mathbb{C}^3)$. Moreover, u^{i} belongs to $C^{\infty}_Q(D^{\infty}_{\Gamma_0}, \mathbb{C}^3)$ by definition. Therefore, $u|_{D^{\infty}_{h+}} = (u^{\mathrm{i}} + u^{\mathrm{s}})|_{D^{\infty}_{h+}} \in C^{\infty}_Q(D^{\infty}_{h+}, \mathbb{C}^3)$.

(iii). Let $v \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$. Due to part *(i)* and according to part *(i)* of part *(b)* of Proposition 2.68, we know that $\mathrm{curl}_\alpha u \in H_Q(\mathrm{curl}, D)$ and that $\mathrm{curl}_\alpha \mathrm{curl}_\alpha u = k^2 u$ in $L^2(D, \mathbb{C}^3)$. Therefore, an application of Green's formula (2.18b) yields

$$\begin{aligned}
&\int_D \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x \\
&\quad = \int_D \big[\mathrm{curl}_\alpha u \cdot \mathrm{curl}\, \overline{v} + \mathrm{curl}_\alpha u \cdot (\overline{v} \times \mathrm{i}\alpha) - k^2 u \cdot \overline{v} \big]\, \mathrm{d}x \\
&\quad = \int_D \underbrace{(\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u)}_{=0} \cdot \overline{v}\, \mathrm{d}x - \langle \gamma_{t,\Gamma_0} \mathrm{curl}_\alpha u|_D, \underbrace{\gamma_{T,\Gamma_0} \overline{v}}_{=0} \rangle_{\Gamma_0} \\
&\qquad\quad - \langle \gamma_{t,\Gamma_{h+}} \mathrm{curl}_\alpha u|_D, \gamma_{T,\Gamma_{h+}} \overline{v} \rangle_{\Gamma_{h+}} \\
&\quad = -\langle \gamma_{t,\Gamma_{h+}} \mathrm{curl}_\alpha u|_D, \gamma_{T,\Gamma_{h+}} \overline{v} \rangle_{\Gamma_{h+}},
\end{aligned}$$

and the proof is complete. □

Theorem 3.14 *If* $\mathrm{Im}(k) > 0$*, then Problem 3.12 has at most one solution.*

Proof: Suppose $u, v : D^\infty_{\Gamma_0} \to \mathbb{C}^3$ are solutions to Problem 3.12 to a given incident field $u^{\mathrm{i}} : D^\infty_{\Gamma_0} \to \mathbb{C}^3$. Set $w := u - v$. Then, for all $h > h^+$, $w \in H_Q(\mathrm{curl}, D^h_{\Gamma_0})$ and satisfies the first equation in Problem 3.12. Furthermore, $\gamma_{t,\Gamma_0} w = 0$ and $w = (u - u^{\mathrm{i}}) - (v - u^{\mathrm{i}})$ satisfies (URC) in $D^\infty_{h^+}$, i.e.,

$$w(x) = \sum_{\mu \in \mathbb{Z}^2} w^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3 - h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)} \cdot \tilde{x}}, \qquad x \in D^\infty_{h^+},$$

with a certain sequence $(w^{(\mu)})_{\mu \in \mathbb{Z}^2}$ in $\mathbb{C}^3$. From Propostion 3.10 we know that $w \in C^\infty_Q(D^\infty_{h^+}, \mathbb{C}^3)$ and that the series (and its derivatives) converges uniformly on D^∞_h for all $h > h^+$. Moreover, by Proposition 3.13, there holds $\mathrm{curl}_\alpha \mathrm{curl}_\alpha w - k^2 w = 0$ even in the classical sense. From this we conclude that $\mathrm{div}_\alpha w = 0$. For any $h > h^+$, from the uniform convergence on Γ_h we obtain therefore

$$0 = \mathrm{div}_\alpha w(\cdot, h) = \frac{\mathrm{i}}{\sqrt{|Q|}} \sum_{\mu \in \mathbb{Z}^2} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \cdot w^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(h - h^+)}\, T_Q^{(\mu)}.$$

Since $\operatorname{div}_\alpha w(\cdot, h)$ is continuous on Γ_h, it can be expanded into a Fourier series and by uniqueness of Fourier coefficients we conclude from the last equation

$$\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \cdot w^{(\mu)} = 0, \quad \text{for all } \mu \in \mathbb{Z}^2. \tag{$*$}$$

Now, let $h > h^+$ and $\mu \in \mathbb{Z}^2$. Then

$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \left(\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times w^{(\mu)}\right)\right] \cdot \overline{w^{(\mu)}} = \left[w_3^{(\mu)} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ 0 \end{pmatrix} - \rho^{(\mu)} \begin{pmatrix} w_1^{(\mu)} \\ w_2^{(\mu)} \\ 0 \end{pmatrix}\right] \cdot \overline{w^{(\mu)}}$$
$$= -\overline{\rho^{(\mu)}}\, |w_3^{(\mu)}|^2 - \rho^{(\mu)} \big(|w_1^{(\mu)}|^2 + |w_2^{(\mu)}|^2\big),$$

where we have applied the complex conjugate of $(*)$. Again by uniform convergence of the series representation of w (and its derivatives), we obtain now

$$\int_{\Gamma_h} \left[\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \times \operatorname{curl}_\alpha w\right] \cdot \overline{w}\, \mathrm{d}s$$
$$= |Q| \int_Q \left(\sum_{\mu \in \mathbb{Z}^2} \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \times \operatorname{curl}_\alpha \left(w^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(h-h^+)}\, T_Q^{(\mu)}(\tilde{x})\right)\right)$$
$$\cdot \left(\sum_{\nu \in \mathbb{Z}^2} \overline{w^{(\nu)}}\, \overline{\mathrm{e}^{\mathrm{i}\rho^{(\nu)}(h-h^+)}}\, T_Q^{(-\nu)}(\tilde{x})\right) \mathrm{d}\tilde{x}$$
$$= -\mathrm{i}|Q| \sum_{\mu \in \mathbb{Z}^2} \left(\rho^{(\mu)} \big(|w_1^{(\mu)}|^2 + |w_2^{(\mu)}|^2\big) + \overline{\rho^{(\mu)}}\, |w_3^{(\mu)}|^2\right) \mathrm{e}^{-2\operatorname{Im}(\rho^{(\mu)})(h-h^+)}.$$

Let $a > h^+$ and consider for $\mu \in \mathbb{Z}^2$ the functions $g_\mu : (a, \infty) \to \mathbb{C}$ defined by

$$g_\mu(\xi) := \left(\rho^{(\mu)} \big(|w_1^{(\mu)}|^2 + |w_2^{(\mu)}|^2\big) + \overline{\rho^{(\mu)}}\, |w_3^{(\mu)}|^2\right) \mathrm{e}^{-2\operatorname{Im}(\rho^{(\mu)})(\xi-h^+)}$$

for $\xi \in (a, \infty)$. Then $|g_\mu(\xi)| \le C\sqrt{1 + |\mu|^2}\, |w^{(\mu)}|^2\, \mathrm{e}^{-2\operatorname{Im}(\rho(\mu))(a-h^+)}$ for all $\xi \in (a, \infty)$, because of Lemma 3.8. Thus, by Remark 3.11, the series

$(\sum g_\mu)$ of continuous functions g_μ converges uniformly to a continuous function g. Hence, the interchange of limits is allowed and we obtain

$$\lim_{h\to\infty}\int_{\Gamma_h}\Big[\Big(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\Big)\times \operatorname{curl}_\alpha w\Big]\cdot\overline{w}\,\mathrm{d}s = 0.$$

Using the previous observations, an application of Green's formula (2.18b) yields now, with $-\langle\gamma_{t,\Gamma_0}\operatorname{curl}_\alpha w, \gamma_{T,\Gamma_0}\overline{w}\rangle_{\Gamma_0} = \langle\gamma_{t,\Gamma_0}\overline{w}, \gamma_{T,\Gamma_0}\operatorname{curl}_\alpha w\rangle_{\Gamma_0} = 0$, that

$$\begin{aligned}\int_{D^h_{\Gamma_0}}&\big(\operatorname{curl}_\alpha w\cdot\overline{\operatorname{curl}_\alpha w} - k^2 w\cdot\overline{w}\big)\,\mathrm{d}x\\ &= \int_{D^h_{\Gamma_0}}\big(\underbrace{\operatorname{curl}_\alpha\operatorname{curl}_\alpha w - k^2 w}_{=0}\big)\cdot\overline{w}\,\mathrm{d}x - \int_{\Gamma_h}\Big[\Big(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\Big)\times\operatorname{curl}_\alpha w\Big]\cdot\overline{w}\,\mathrm{d}s.\end{aligned}$$

Dividing the last equation by k, taking then the imaginary part and passing finally to the limit for $h\to\infty$, we obtain $w=0$ almost everywhere on $D^\infty_{\Gamma_0}$ and the proof is complete. □

3.2. Calderon Operator and Variational Formulation

A Special Extension Operator. As a preparation for the definition of the Calderon operator, we now construct a certain extension operator, which turns out to be the unique solution of the following exterior boundary value problem.

Problem 3.15 *Given* $\varphi\in H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+})$*, find* $u: D^\infty_{h^+}\to\mathbb{C}^3$ *such that* u *satisfies (URC),* $u\in H_Q(\mathrm{curl}, D^h_{h^+})$*, for all* $h>h^+$*, and*

$$\begin{cases}\operatorname{curl}_\alpha\operatorname{curl}_\alpha u - k^2 u = 0 \quad \textit{in } D^\infty_{h^+},\\ \gamma_{t,\Gamma_{h^+}} u = \varphi.\end{cases}$$

Note that by Proposition 3.10 the function u in Problem 3.15 is smooth and that therefore the first equation therein holds in the classical sense.

Recall Convention 2.125 for the identification of the spaces $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$ and $H_{\mathrm{per}}^{-1/2}(\mathrm{Div}, Q)$. Hence, we have for $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$ its series representation

$$\varphi = \sum_{\mu \in \mathbb{Z}^2} \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \end{pmatrix} \jmath(T_Q^{(\mu)}), \tag{3.13}$$

with associated sequence $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2} \in \mathcal{C}_{\mathrm{Div}}^{-1/2}$ and convergence to be understood in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$. Moreover,

$$\|\varphi\|^2_{H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})} = \sum_{\mu \in \mathbb{Z}^2} \left(1 + |\mu|^2\right)^{-1/2} \left(|\varphi^{(\mu)}|^2 + |q^{(\mu)} \cdot \varphi^{(\mu)}|^2\right) < \infty.$$

Taking for $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$ this sequence $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2}$, we recall (3.10) and define the sequence $(E^{(\mu)})_{\mu \in \mathbb{Z}^2}$ in $\mathbb{C}^3$ by

$$E^{(\mu)} := \frac{1}{\sqrt{|Q|}} \frac{1}{\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \\ 0 \end{pmatrix}, \qquad \mu \in \mathbb{Z}^2 \tag{3.14}$$

and consider the extension

$$E(x) := \sum_{\mu \in \mathbb{Z}^2} E^{(\mu)} \, \mathrm{e}^{\mathrm{i}(q^{(\mu)} \cdot \tilde{x} + \rho^{(\mu)}(x_3 - h^+))}, \qquad x \in D_{h+}^{\infty}. \tag{3.15}$$

Note that the coefficients $E^{(\mu)}$ in (3.14) are well-defined thanks to Assumption 3.5 and that they are motivated by plugging the ansatz for E from (3.15) into Problem 3.15 and using in particular $\gamma_{t,\Gamma_{h+}} E \overset{!}{=} \varphi$ and $\mathrm{div}_\alpha E \overset{!}{=} 0$. Our next goal is to show that E from (3.15) is the unique solution to Problem 3.15.

Lemma 3.16 *Let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$, define for $\mu \in \mathbb{Z}^2$ the coefficients $E^{(\mu)}$ by* (3.14) *and furthermore the coefficients*

$$F^{(\mu)} := \mathrm{i} \begin{pmatrix} q_1^{(\mu)} \\ q_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times E^{(\mu)}, \; F_\alpha^{(\mu)} := \mathrm{i} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times E^{(\mu)}, \; G^{(\mu)} := \mathrm{i} \begin{pmatrix} q_1^{(\mu)} \\ q_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times F_\alpha^{(\mu)}.$$

Then the following assertions are true.

(i) $\exists\, C > 0 \;\forall\, \mu \in \mathbb{Z}^2 : \; |E^{(\mu)}| \le C\, |\varphi^{(\mu)}|$.

(ii) $\exists\, C > 0 \;\forall\, \mu \in \mathbb{Z}^2 : \; |F^{(\mu)}| \le C\, \big(|\varphi^{(\mu)}| + |q^{(\mu)} \cdot \varphi^{(\mu)}|\big)$.

(iii) $\exists\, C > 0 \;\forall\, \mu \in \mathbb{Z}^2 : \; |F_\alpha^{(\mu)}| \le C\, \big(|\varphi^{(\mu)}| + |q^{(\mu)} \cdot \varphi^{(\mu)}|\big)$.

(iv) $\exists\, C > 0 \;\forall\, \mu \in \mathbb{Z}^2 : \; |G^{(\mu)}| \le C\, \big(|\varphi^{(\mu)}| + |q^{(\mu)} \cdot \varphi^{(\mu)}|\big)$.

Proof: *(i)*. Let $\mu \in \mathbb{Z}^2$. Then, using the definition (3.14) for the coefficients $E^{(\mu)}$, we have

$$|E^{(\mu)}| \le \frac{1}{\sqrt{|Q|}}\, \frac{1}{|\rho^{(\mu)}|}\, \sqrt{|q^{(\mu)}|^2 + |\rho^{(\mu)}|^2}\; |\varphi^{(\mu)}|.$$

Now, the assertion follows immediately from Lemma 2.9 and Lemma 3.8. *(ii)*. Let $\mu \in \mathbb{Z}^2$. Using the definition (3.14) for the coefficients $E^{(\mu)}$ and (A.1b), we have

$$F^{(\mu)} = \mathrm{i}\, \frac{q^{(\mu)} \cdot \varphi^{(\mu)}}{\sqrt{|Q|}\, \rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} - \mathrm{i}\, \frac{|q^{(\mu)}|^2 + (\rho^{(\mu)})^2 + q^{(\mu)} \cdot \tilde{\alpha}}{\sqrt{|Q|}\, \rho^{(\mu)}} \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \\ 0 \end{pmatrix}.$$

Note that $|q^{(\mu)}|^2 + (\rho^{(\mu)})^2 = |q^{(\mu)}|^2 + k^2 - |\tilde{\alpha} + q^{(\mu)}|^2 = k^2 - |\tilde{\alpha}|^2 - 2\tilde{\alpha} \cdot q^{(\mu)}$. Therefore

$$|F^{(\mu)}| \le \frac{|k^2| + |\tilde{\alpha}|^2 + |\tilde{\alpha}|\, |q^{(\mu)}|}{\sqrt{|Q|}\, |\rho^{(\mu)}|}\, |\varphi^{(\mu)}| + \frac{|q^{(\mu)} \cdot \varphi^{(\mu)}|}{\sqrt{|Q|}} \sqrt{\frac{|q^{(\mu)}|^2}{|\rho^{(\mu)}|^2} + 1}.$$

Now, we obtain the assertion again immediately from Lemma 2.9 and Lemma 3.8.
(iii). Since $F_\alpha^{(\mu)} = F^{(\mu)} + \mathrm{i}\, (\alpha_1, \alpha_2, 0)^\top \times E^{(\mu)}$, the assertion follows easily

from *(i)* and *(ii)*.
(iv). Let $\mu \in \mathbb{Z}^2$. We obtain, similarly to step *(ii)*,

$$F_\alpha^{(\mu)} = \mathrm{i}\,\frac{d^{(\mu)} \cdot \varphi^{(\mu)}}{\sqrt{|Q|}\,\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} - \mathrm{i}\,\frac{k^2}{\sqrt{|Q|}\,\rho^{(\mu)}} \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \\ 0 \end{pmatrix},$$

where we have exploited that $|d^{(\mu)}|^2 + (\rho^{(\mu)})^2 = k^2$. Hence,

$$G^{(\mu)} = -\frac{q^{(\mu)} \cdot \varphi^{(\mu)} + \tilde{\alpha} \cdot \varphi^{(\mu)}}{\sqrt{|Q|}\,\rho^{(\mu)}} \begin{pmatrix} q_1^{(\mu)} \\ q_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} + \frac{k^2}{\sqrt{|Q|}\,\rho^{(\mu)}} \begin{pmatrix} q_1^{(\mu)} \\ q_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \\ 0 \end{pmatrix}$$

and therefore

$$|G^{(\mu)}| \le \frac{1}{\sqrt{|Q|}} \sqrt{\frac{|q^{(\mu)}|^2}{|\rho^{(\mu)}|^2} + 1}\, \Big[\big(|k^2| + |\tilde{\alpha}|^2\big) |\varphi^{(\mu)}| + |\tilde{\alpha}|\, \big| q^{(\mu)} \cdot \varphi^{(\mu)} \big| \Big].$$

And again, the assertion follows now immediately from Lemma 2.9 and Lemma 3.8. □

Remark 3.17 *The coefficients* $F^{(\mu)}$, $F_\alpha^{(\mu)}$ *and* $G^{(\mu)}$ *from Lemma 3.16 correspond to the application of* curl, curl_α *and* $\mathrm{curl}\,\mathrm{curl}_\alpha$ *to the* μ*-th summand of* E *from* (3.15), *respectively.*

Lemma 3.18 *Let* $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+})$ *and define for* $\mu \in \mathbb{Z}^2$ *the coefficients* $E^{(\mu)}$, $F^{(\mu)}$, $F_\alpha^{(\mu)}$ *and* $G^{(\mu)}$ *according to Lemma 3.16. Furthermore, let* $(H^{(\mu)})_{\mu\in\mathbb{Z}^2} \in \left\{ (E^{(\mu)})_{\mu\in\mathbb{Z}^2}, (F^{(\mu)})_{\mu\in\mathbb{Z}^2}, (F_\alpha^{(\mu)})_{\mu\in\mathbb{Z}^2}, (G^{(\mu)})_{\mu\in\mathbb{Z}^2} \right\}$ *and consider the series*

$$H(x) := \sum_{\mu\in\mathbb{Z}^2} H^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3 - h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}}, \qquad x \in D_{h^+}^\infty.$$

Then the following statements are true.

(i) $H \in C^\infty(D^\infty_{h^+}, \mathbb{C}^3)$ *and the series converges uniformly to* H *on* D^∞_h *for all* $h > h^+$*. In particular,* E *from* (3.15) *satisfies (URC) and furthermore*

$$\operatorname{curl}_\alpha \operatorname{curl}_\alpha E(x) = k^2 E(x), \qquad x \in D^\infty_{h^+},$$

in the classical sense.

(ii) *Let* $h > h^+$*. The series converges in* $L^2(D^h_{h^+}, \mathbb{C}^3)$ *to* $H|_{D^h_{h^+}}$*. In particular, the series representation for* E *and* $\operatorname{curl}_\alpha E$ *converges in* $H_Q(\operatorname{curl}, D^h_{h^+})$ *to* E *and* $\operatorname{curl}_\alpha E$*, respectively.*

(iii) *For* E *from* (3.15) *there exists* $C > 0$ *such that for all* $h > h^+$ *we have* $\operatorname{curl}_\alpha E \in H_Q(\operatorname{curl}, D^h_{h^+})$ *and*

$$\| \operatorname{curl}_\alpha E \|_{H(\operatorname{curl}, D^h_{h^+})} \le C \, \| E \|_{H(\operatorname{curl}, D^h_{h^+})}.$$

Proof: *(i).* Thanks to Proposition 3.10 and Remark 3.17, it sufficies to show that E from (3.15) satisfies (URC). For this we proceed similarly to the proof of Proposition 3.10 and observe that there exists a constant $\tilde{C} > 0$ such that for all $\mu \in \mathbb{Z}^2$ we have $\operatorname{Im}(\rho^{(\mu)}) \ge \tilde{C}\sqrt{1 + |\mu|^2}$. Let $h > h^+$ and $x \in D^\infty_h$. For $\mu \in \mathbb{Z}^2$ we define $b(\mu) := \tilde{C}(h - h^+)\sqrt{1 + |\mu|^2}$. Note that $h - h^+ > 0$. Therefore, $\frac{(b(\mu))^{5/2}}{\mathrm{e}^{b(\mu)}}$ is bounded for all $\mu \in \mathbb{Z}^2$. Now, let $\mu \in \mathbb{Z}^2$. Then, using the observations from above and Lemma 3.16, we obtain

$$\begin{aligned}
\left| E^{(\mu)} \, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3 - h^+)} \, \mathrm{e}^{\mathrm{i}q^{(\mu)} \cdot \tilde{x}} \right| &= \mathrm{e}^{-\operatorname{Im}(\rho(\mu))(x_3 - h^+)} \, |E^{(\mu)}| \\
&\le C \, \mathrm{e}^{-\tilde{C}\sqrt{1+|\mu|^2}(h - h^+)} \, |\varphi^{(\mu)}| \\
&= \frac{C}{[\tilde{C}\,(h-h^+)]^{5/2}} \, \frac{(b(\mu))^{5/2}}{\mathrm{e}^{b(\mu)}} \, \frac{|\varphi^{(\mu)}|}{\left(1 + |\mu|^2\right)^{1/4}} \, \frac{1}{1 + |\mu|^2}.
\end{aligned}$$

Hence, since $x \in D^\infty_h$ was arbitrarily chosen,

$$\sup_{x \in D^\infty_h} \left| E^{(\mu)} \, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3 - h^+)} \, \mathrm{e}^{\mathrm{i}q^{(\mu)} \cdot \tilde{x}} \right| \le C \, \frac{|\varphi^{(\mu)}| + |q^{(\mu)} \cdot \varphi^{(\mu)}|}{\left(1 + |\mu|^2\right)^{1/4}} \, \frac{1}{1 + |\mu|^2}.$$

Now, Cauchy-Schwarz's inequality yields

$$\sum_{\mu \in \mathbb{Z}^2} \sup_{x \in D^\infty_h} \left| E^{(\mu)} \, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3 - h^+)} \, \mathrm{e}^{\mathrm{i}q^{(\mu)} \cdot \tilde{x}} \right| \le C \, \|\varphi\|^2_{H_Q^{-1/2}(\operatorname{Div}, \Gamma_{h^+})},$$

which shows that E satisfies (URC) since normal convergence implies uniform convergence.
We show that E solves the differential equation in the classical sense. In fact, since E satisfies (URC), Proposition 3.10 implies now that all differential operations can be applied componentwise. In particular,

$$\operatorname{curl}_\alpha \operatorname{curl}_\alpha E(x) = \sum_{\mu\in\mathbb{Z}^2} H^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}}, \qquad x \in D_{h^+}^{\infty},$$

where, by recalling Remark 3.17, (3.14) and (3.10),

$$H^{(\mu)} := \mathrm{i}^2 \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \left[\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times E^{(\mu)} \right] = k^2\, E^{(\mu)}, \qquad \mu \in \mathbb{Z}^2.$$

From this the assertion follows immediately.
(ii). Let $n, m \in \mathbb{N}$ such that $n < m$. Then, using Fubini's theorem, Lemma 3.8 and Lemma 3.16, we obtain

$$\begin{aligned}
&\int_{D_{h^+}^h} \Big| \sum_{n<|\mu|\le m} H^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}} \Big|^2 \,\mathrm{d}x \\
&\quad = |Q| \int_{h^+}^{h} \Big\| \sum_{n<|\mu|\le m} H^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3-h^+)}\, T_Q^{(\mu)} \Big\|^2_{L^2(Q,\mathbb{C}^3)} \,\mathrm{d}x_3 \\
&\quad = |Q| \sum_{n<|\mu|\le m} |H^{(\mu)}|^2 \int_{h^+}^{h} \mathrm{e}^{-2\,\mathrm{Im}(\rho^{(\mu)})(x_3-h^+)} \,\mathrm{d}x_3 \\
&\quad \le |Q| \sum_{n<|\mu|\le m} |H^{(\mu)}|^2 \int_{h^+}^{h} \mathrm{e}^{-\tilde{C}\sqrt{1+|\mu|^2}\,(x_3-h^+)} \,\mathrm{d}x_3 \\
&\quad = |Q| \sum_{n<|\mu|\le m} |H^{(\mu)}|^2 \frac{1}{\tilde{C}\sqrt{1+|\mu|^2}} \Big(1 - \mathrm{e}^{-\tilde{C}\sqrt{1+|\mu|^2}\,(h-h^+)}\Big) \\
&\quad \le C \sum_{n<|\mu|\le m} \frac{1}{\sqrt{1+|\mu|^2}} \big(|\varphi^{(\mu)}|^2 + |q^{(\mu)}\cdot\varphi^{(\mu)}|^2\big).
\end{aligned}$$

Since $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+})$, Cauchy's convergence test for series in Banach spaces yields now that the series converges in $L^2(D_{h^+}^h, \mathbb{C}^3)$ to some

$\tilde{H} \in L^2(D^h_{h^+}, \mathbb{C}^3)$. By Riesz-Fischer's theorem, there exists a subsequence of the sequence of partial sums, converging pointwise almost everywhere on $D^h_{h^+}$ to $\tilde{H}$. From part *(i)* we know that this subsequence of partial sums also converges (in particular) pointwise to H on $D^h_{h^+}$. Hence, H and $\tilde{H}$ coincide almost everywhere on $D^h_{h^+}$.
(iii). Let $h > h^+$. From part *(ii)*, together with Remark 3.17, we obtain $E \in H_Q(\mathrm{curl}, D^h_{h^+})$. Note that $\mathrm{curl}_\alpha E(x) = \mathrm{curl}\, E(x) + \mathrm{i}\alpha \times E(x)$. Furthermore, by part *(i)*,

$$\begin{aligned}\mathrm{curl}\,\mathrm{curl}_\alpha E(x) &= \mathrm{curl}_\alpha \mathrm{curl}_\alpha E(x) - \mathrm{i}\alpha \times \mathrm{curl}_\alpha E(x) \\ &= k^2 E(x) - \mathrm{i}\alpha \times (\mathrm{i}\alpha \times E(x)) - \mathrm{i}\alpha \times \mathrm{curl}\, E(x).\end{aligned}$$

Moreover, from part *(ii)* we know that E and $\mathrm{curl}\, E$ belong to $L^2(D^h_{h^+}, \mathbb{C}^3)$. Together with the last observation, the asserted inequality follows immediately. □

Theorem 3.19 *Let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+})$, define for $\mu \in \mathbb{Z}^2$ the coefficients $E^{(\mu)}$ by* (3.14) *and consider E from* (3.15)*, i.e.,*

$$E(x) := \sum_{\mu \in \mathbb{Z}^2} E^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)} \cdot \tilde{x} + \rho^{(\mu)}(x_3 - h^+))}, \qquad x \in D^\infty_{h^+}.$$

Then E is the unique solution to Problem 3.15. Moreover, for all $h > h^+$ the mapping $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+}) \ni \varphi \mapsto E \in H_Q(\mathrm{curl}, D^h_{h^+})$ is bounded (with a constant not depending on h).

Proof: *(i)*. We have to show that E satisfies (URC) and that E solves the first equation in Problem 3.15 in the classical sense. But this follows from Lemma 3.18.
(ii). Let $h > h^+$. We have to show that E belongs to $H_Q(\mathrm{curl}, D^h_{h^+})$. In fact, from part *(i)* we know that E satisfies (URC). Therefore, Proposition 3.10 gives us that $\mathrm{curl}\, E$ exists in the classical sense and it allows us furthermore to apply the curl operation componentwise such that we end up in the situation of Lemma 3.18, if we take Remark 3.17 into account as well. Hence, E and $\mathrm{curl}\, E$ belong to $L^2(D^h_{h^+}, \mathbb{C}^3)$.
(iii). We show that $\gamma_{t,\Gamma_{h^+}} E = \varphi$. In fact, let $h > h^+$. In Lemma 3.18 we have shown that the series representation for E converges also in

$H_Q(\mathrm{curl}, D^h_{h+})$. Moreover, $\gamma_{t,\Gamma_{h+}}$ from $H_Q(\mathrm{curl}, D^h_{h+})$ to $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$ is bounded. Therefore, by recalling (3.14), (3.13) and Convention 2.125, we obtain

$$\gamma_{t,\Gamma_{h+}} E = \sum_{\mu\in\mathbb{Z}^2} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \times E^{(\mu)} \sqrt{|Q|}\, \jmath(T_Q^{(\mu)}) = \sum_{\mu\in\mathbb{Z}^2} \begin{pmatrix} \varphi_1^{(\mu)} \\ \varphi_2^{(\mu)} \end{pmatrix} \jmath(T_Q^{(\mu)}) = \varphi.$$

(iv). We show uniqueness of Problem 3.15. Suppose $u : D^\infty_{h+} \to \mathbb{C}^3$ is a solution to Problem 3.15 for $\varphi = 0$. Let $h > h^+$. Since u satisfies (URC) we have

$$\gamma_{t,\Gamma_h} u(\tilde{x}) = \sum_{\mu\in\mathbb{Z}^2} \begin{pmatrix} -u_2^{(\mu)} \\ u_1^{(\mu)} \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(h-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}}, \qquad \tilde{x} \in Q.$$

Let $\nu \in \mathbb{Z}^2$ and set

$$v(x) := \begin{pmatrix} -\overline{u_2^{(\nu)}} \\ \overline{u_1^{(\nu)}} \\ 0 \end{pmatrix} \overline{\mathrm{e}^{\mathrm{i}\rho^{(\nu)}(x_3-h^+)}}\, \mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tilde{x}}, \qquad x \in \overline{D^\infty_{h+}}.$$

Then v belongs to $H_Q(\mathrm{curl}, D^h_{h+})$, even to $C^\infty(D^\infty_{h+}, \mathbb{C}^3)$. Therefore, we can apply Green's formula from Theorem 2.113 and obtain

$$\int_{\Gamma_h} \gamma_{t,\Gamma_h} u \cdot \gamma_{T,\Gamma_h} v \,\mathrm{d}s + \langle \underbrace{\gamma_{t,\Gamma_{h+}} u}_{=0}, \gamma_{T,\Gamma_{h+}} v\rangle_{\Gamma_{h+}}$$
$$= \int_{D^h_{h+}} (\mathrm{curl}\, u \cdot v - u \cdot \mathrm{curl}\, v)\,\mathrm{d}x.$$

Since the right hand side converges to zero as h tends to h^+, we have

$$|Q| \left(|u_1^{(\nu)}|^2 + |u_2^{(\nu)}|^2 \right) \mathrm{e}^{-2\mathrm{Im}(\rho^{(\nu)})(h-h^+)} = \int_{\Gamma_h} \gamma_{t,\Gamma_h} u \cdot \gamma_{T,\Gamma_h} v \,\mathrm{d}s \to 0,$$

as $h \to h^+$. From this we conclude that $u_1^{(\mu)} = u_2^{(\mu)} = 0$ for all $\mu \in \mathbb{Z}^2$. Moreover, u satisfies $\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u = 0$ in the classical sense. Therefore, $\mathrm{div}_\alpha u = 0$. Furthermore, thanks to Proposition 3.10, all differential operations go through componentwise to the series representation for u. Since $\operatorname{div}_\alpha u(\cdot,h)$ is continuous, it can be expanded into a Fourier series. Hence,

$$0 = \operatorname{div}_\alpha u(\cdot,h) = \sqrt{|Q|}\sum_{\mu\in\mathbb{Z}^2} \mathrm{i}\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \cdot u^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(h-h^+)}\, T_Q^{(\mu)},$$

and by uniqueness of the Fourier coefficients we obtain $(d_1^{(\mu)}, d_2^{(\mu)}, \rho^{(\mu)})^\top \cdot u^{(\mu)} = 0$ for all $\mu \in \mathbb{Z}^2$. And from this we obtain finally $u_3^{(\mu)} = 0$ for all $\mu \in \mathbb{Z}^2$, since $\rho^{(\mu)} \neq 0$.

(v). We show boundedness of the solution operator. For this let $h > h^+$. Moreover, for $n \in \mathbb{N}$ we set

$$E_n(x) := \sum_{|\mu|\le n} E^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}}, \qquad x \in D_{h^+}^h,$$

and obtain very similar to part *(ii)* from the proof of Lemma 3.18 that for all $n \in \mathbb{N}$, and with $F^{(\mu)}$ from Lemma 3.16,

$$\begin{aligned} \|E_n\|^2_{H(\mathrm{curl},D_{h^+}^h)} &= |Q| \sum_{|\mu|\le n} \left(|E^{(\mu)}|^2 + |F^{(\mu)}|^2\right) \int_{h^+}^h \mathrm{e}^{-2\,\mathrm{Im}(\rho^{(\mu)})(x_3-h^+)}\, \mathrm{d}x_3 \\ &\le C \sum_{|\mu|\le n} \frac{1}{\sqrt{1+|\mu|^2}} \left(|\varphi^{(\mu)}|^2 + |q^{(\mu)}\cdot\varphi^{(\mu)}|^2\right) \\ &\le C\, \|\varphi\|^2_{H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+})}. \end{aligned}$$

Using that $E_n \to E$ in $H_Q(\mathrm{curl}, D_{h^+}^h)$, as $n \to \infty$, see Lemma 3.18, from the last estimate the assertion follows now immediately. □

The Calderon Operator. Theorem 3.19 yields a linear and bounded operator

$$H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+}) \ni \varphi \;\mapsto\; E \in H_Q(\mathrm{curl}, D_{h^+}^h),$$

where E is given by (3.15) and is the unique solution to Problem 3.15. Here, h is any real number such that $h > h^+$. Furthermore, taking this E, by part *(iii)* of Lemma 3.18 the operator

$$H_Q(\mathrm{curl}, D_{h^+}^h) \ni E \;\mapsto\; \mathrm{curl}_\alpha E \in H_Q(\mathrm{curl}, D_{h^+}^h)$$

is also linear and bounded. Finally, recall that $\gamma_{t,\Gamma_{h+}}$ is a linear and bounded operator from $H_Q(\mathrm{curl}, D^h_{h+})$ to $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$. These observations give rise to the following definition.

Definition 3.20 *The linear and bounded operator*

$$H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+}) \ni \varphi \;\mapsto\; \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha E \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$$

given by the composition of the three linear and bounded operators from above is called Calderon operator. We denote this operator by Λ_α.

Remark 3.21 *Note that in the definition of the Calderon operator there goes into the trace operator* $\gamma_{t,\Gamma_{h+}}$. *Since* $\operatorname{curl}_\alpha E$ *belongs for any* $h > h^+$ *to* $H_Q(\mathrm{curl}, D^h_{h+})$, $\gamma_{t,\Gamma_{h+}}$ *operates by definition with respect to the normal vector* $(0,0,-1)^\top$, *because it's showing outside of* D^h_{h+}.

Remark 3.22 *For the Calderon operator we have the explicit formula*

$$\begin{aligned}\Lambda_\alpha \varphi &= \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} d_2^{(\mu)} & (d_2^{(\mu)})^2 - k^2 \\ k^2 - (d_1^{(\mu)})^2 & -d_1^{(\mu)} d_2^{(\mu)} \end{pmatrix} \varphi^{(\mu)}\, \jmath(T_Q^{(\mu)}) \\ &= \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \left[k^2 (\varphi^{(\mu)})^\perp - (d^{(\mu)} \cdot \varphi^{(\mu)})(d^{(\mu)})^\perp \right] \jmath(T_Q^{(\mu)})\end{aligned}$$

for all $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$. *Here, convergence has to be understood in* $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$, *and* $a^\perp := (-a_2, a_1)^\top$ *for* $a \in \mathbb{C}^2$, *see* (1.7). *For the coefficients* $\varphi^{(\mu)}$ *of* φ *see* (3.13).

Proof: Let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$. Moreover, let $\mu \in \mathbb{Z}^2$ and recall (3.14) for the definition of the coefficients $E^{(\mu)}$. Then

$$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \times \left[\mathrm{i} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times E^{(\mu)} \right] = -\mathrm{i} \left[E_3^{(\mu)} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ 0 \end{pmatrix} - \rho^{(\mu)} \begin{pmatrix} E_1^{(\mu)} \\ E_2^{(\mu)} \\ 0 \end{pmatrix} \right]$$

$$= \frac{\mathrm{i}}{\sqrt{|Q|}\,\rho^{(\mu)}} \left[(d_2^{(\mu)}\varphi_1^{(\mu)} - d_1^{(\mu)}\varphi_2^{(\mu)}) \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ 0 \end{pmatrix} - (\rho^{(\mu)})^2 \begin{pmatrix} \varphi_2^{(\mu)} \\ -\varphi_1^{(\mu)} \\ 0 \end{pmatrix} \right]$$

$$= \frac{\mathrm{i}}{\sqrt{|Q|}\,\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} d_2^{(\mu)} & (d_2^{(\mu)})^2 - k^2 \\ k^2 - (d_1^{(\mu)})^2 & -d_1^{(\mu)} d_2^{(\mu)} \\ 0 & 0 \end{pmatrix} \varphi^{(\mu)},$$

where we have applied $(\rho^{(\mu)})^2 = k^2 - |d^{(\mu)}|^2$, see (3.10). Now, thanks to Lemma 3.18 and Remark 3.17, we have

$$\operatorname{curl}_\alpha E(x) = \sum_{\mu\in\mathbb{Z}^2} F_\alpha^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(x_3-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{x}}, \qquad x \in D_{h^+}^\infty,$$

with $F_\alpha^{(\mu)}$ from Lemma 3.16, and where we have convergence also in $H_Q(\operatorname{curl}, D_{h^+}^h)$ for all $h > h^+$. Therefore, we can apply the trace operator $\gamma_{t,\Gamma_{h^+}}$ componentwise and obtain

$$\gamma_{t,\Gamma_{h^+}} \operatorname{curl}_\alpha E = \sum_{\mu\in\mathbb{Z}^2} \gamma_{t,\Gamma_{h^+}} F_\alpha^{(\mu)}\, \mathrm{e}^{\mathrm{i}\rho^{(\mu)}(\cdot_3-h^+)}\, \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\tilde{\cdot}}.$$

Using the calculations from above and recalling Convention 2.125 together with (3.13), the explicit formula for the Calderon operator follows now immediately. $\square$

The Variational Formulation. By means of the Calderon operator, we are now in a position to give an equivalent formulation of Problem 3.12 in variational form. We will take this formulation to show later existence of solutions.

Problem 3.23 *Given an incident field u^{i} as in Assumption 3.3, find $u \in H_{Q,0,\Gamma_0}(\operatorname{curl}, D)$ such that for all $v \in H_{Q,0,\Gamma_0}(\operatorname{curl}, D)$*

$$\int_D \big(\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u\cdot\overline{v}\big)\,\mathrm{d}x + \Big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h^+}} u), \gamma_{T,\Gamma_{h^+}}\overline{v}\Big\rangle_{\Gamma_{h^+}}$$
$$= \Big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h^+}}(u^{\mathrm{i}}|_D)) - \gamma_{t,\Gamma_{h^+}} \operatorname{curl}_\alpha(u^{\mathrm{i}}|_D), \gamma_{T,\Gamma_{h^+}}\overline{v}\Big\rangle_{\Gamma_{h^+}}.$$

Remark 3.24 *If* $u \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$ *solves Problem 3.23, then we have*

$$\int_D \big(\mathrm{curl}_\alpha\, u \cdot \overline{\mathrm{curl}_\alpha\, v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x = 0$$

for all $v \in H_{Q,0}(\mathrm{curl}, D)$*. Hence, from Proposition 3.13 we conclude that* $\mathrm{curl}_\alpha\, u \in H_Q(\mathrm{curl}, D)$ *with* $\mathrm{curl}_\alpha \mathrm{curl}_\alpha\, u = k^2 u$ *holding in* $L^2(D, \mathbb{C}^3)$*.*

Theorem 3.25 *Problem 3.12 and Problem 3.23 are equivalent.*

Proof: *(i).* Let $u : D^\infty_{\Gamma_0} \to \mathbb{C}^3$ be a solution to Problem 3.12. Using Proposition 3.13, we see that $u^{\mathrm{s}}|_{D^\infty_{h^+}}$ is a solution to Problem 3.15 for $\varphi = \gamma_{t,\Gamma_{h^+}} u^{\mathrm{s}}|_{D^h_{h^+}}$ for any $h > h^+$. Since E defined by (3.15) solves this problem too, we have by uniqueness $E = u^{\mathrm{s}}|_{D^\infty_{h^+}}$. Therefore,

$$\Lambda_\alpha \varphi \stackrel{\mathrm{def}}{=} \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha\, E = \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u^{\mathrm{s}}|_{D^h_{h^+}}) = -\gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u^{\mathrm{s}}|_D).$$

Moreover, thanks to Proposition 2.119,

$$\Lambda_\alpha \varphi = \Lambda_\alpha \gamma_{t,\Gamma_{h^+}} (u^{\mathrm{s}}|_{D^h_{h^+}}) = -\Lambda_\alpha \gamma_{t,\Gamma_{h^+}} (u^{\mathrm{s}}|_D).$$

This yields

$$\begin{aligned} \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u|_D) &= \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u|_D - u^{\mathrm{i}}|_D) + \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u^{\mathrm{i}}|_D) \\ &= \Lambda_\alpha \gamma_{t,\Gamma_{h^+}} (u|_D - u^{\mathrm{i}}|_D) + \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u^{\mathrm{i}}|_D). \end{aligned}$$

Let $v \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$. Then, by part *(iii)* of Proposition 3.13, we have

$$\int_D \big(\mathrm{curl}_\alpha\, u \cdot \overline{\mathrm{curl}_\alpha\, v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x = -\langle \gamma_{t,\Gamma_{h^+}} \mathrm{curl}_\alpha (u|_D), \gamma_{T,\Gamma_{h^+}} \overline{v} \rangle_{\Gamma_{h^+}}.$$

Using the observations from above, from the last equation the assertion follows now immediately.

(ii). Let $u \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$ be a solution to Problem 3.23. Thanks to Remark 3.24, $\mathrm{curl}_\alpha\, u$ belongs to $H_Q(\mathrm{curl}, D)$. Therefore, we can apply Green's formula (2.18b) to $\mathrm{curl}_\alpha\, u$ and an arbitrary $v \in H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$, and obtain

$$\int_D \big(\mathrm{curl}_\alpha\, u \cdot \overline{\mathrm{curl}_\alpha\, v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x$$

$$= \int_D \big(\underbrace{\operatorname{curl}_\alpha \operatorname{curl}_\alpha u - k^2 u}_{=0} \big) \cdot \overline{v} \,\mathrm{d}x - \langle \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha u, \gamma_{T,\Gamma_{h+}} \overline{v} \rangle_{\Gamma_{h+}},$$

where we have again applied Remark 3.24. Let $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma_{h^+})$ and set $v := \eta_{T,\Gamma_{h+}} \overline{\psi} \in H_Q(\mathrm{curl}, D)$. Note that v is even in $H_{Q,0,\Gamma_0}(\mathrm{curl}, D)$, see Theorem 2.107. Then the variational equation in Problem 3.23 yields, together with the last equation from above,

$$\Big\langle \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha (u - u^{\mathrm{i}}|_D) - \Lambda_\alpha \gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D),\, \psi \Big\rangle_{\Gamma_{h+}} = 0.$$

Since $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma_{h^+})$ was arbitrary, Corollary 2.114 and a well-known conclusion from Hahn-Banach's theorem yield that

$$\gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha (u - u^{\mathrm{i}}|_D) = \Lambda_\alpha \gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D).$$

Let $u^{+,\mathrm{s}} : D_{h^+}^\infty \to \mathbb{C}^3$ denote the solution of Problem 3.15 to the boundary data $\varphi = -\gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D)$. Furthermore, set $u^+ := u^{\mathrm{i}}|_{D_{h^+}^\infty} + u^{+,\mathrm{s}}$ and define $\hat{u} : D_{\Gamma_0}^\infty \to \mathbb{C}^3$ by

$$\hat{u} := \begin{cases} u^+, & \text{on } D_{h^+}^\infty, \\ u, & \text{on } D. \end{cases}$$

Note that by construction $u^{+,\mathrm{s}}$ satisfies (URC) and that $u^{+,\mathrm{s}}$ is smooth with $\operatorname{curl}_\alpha \operatorname{curl}_\alpha u^{+,\mathrm{s}} = k^2 u^{+,\mathrm{s}}$. Furthermore, $u^{+,\mathrm{s}} \in H_Q(\mathrm{curl}, D_{h^+}^h)$, for all $h > h^+$, with $\gamma_{t,\Gamma_{h+}} u^{+,\mathrm{s}} = \varphi = -\gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D)$ and $\gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha u^{+,\mathrm{s}} = \Lambda_\alpha \varphi = -\Lambda_\alpha \gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}})|_D$. Therefore, $u^+ \in H_Q(\mathrm{curl}, D_{h^+}^h)$, for all $h > h^+$, and

$$\begin{aligned} \gamma_{t,\Gamma_{h+}} u^+ &= \gamma_{t,\Gamma_{h+}} (u^+ - u^{\mathrm{i}}|_{D_{h^+}^h}) + \gamma_{t,\Gamma_{h+}} (u^{\mathrm{i}}|_{D_{h^+}^h}) \\ &= -\gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D) - \gamma_{t,\Gamma_{h+}} (u^{\mathrm{i}}|_D) = -\gamma_{t,\Gamma_{h+}} u. \end{aligned}$$

Hence, by Proposition 2.118, we have $\hat{u} \in H_Q(\mathrm{curl}, D_{\Gamma_0}^h)$, for all $h > h^+$. Moreover, $\gamma_{t,\Gamma_0} \hat{u} = \gamma_{t,\Gamma_0} u = 0$. Finally, let $h > h^+$ and $v \in H_{Q,0}(\mathrm{curl}, D_{\Gamma_0}^h)$. Then, by applying Green's formula (2.18b) and using the observations from above,

$$\int_{D_{h^+}^h} \big(\operatorname{curl}_\alpha u^+ \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u^+ \cdot \overline{v} \big) \,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{D^h_{h+}} \big(\underbrace{\operatorname{curl}_\alpha \operatorname{curl}_\alpha u^+ - k^2 u^+}_{=0}\big)\cdot \overline{v}\,\mathrm{d}x - \langle \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha u^+, \gamma_{T,\Gamma_{h+}}\overline{v}\rangle_{\Gamma_{h+}} \\
&= -\langle \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha (u^+ - u^{\mathrm{i}}|_{D^h_{h+}}), \gamma_{T,\Gamma_{h+}}\overline{v}\rangle_{\Gamma_{h+}} - \langle \gamma_{t,\Gamma_{h+}} (u^{\mathrm{i}}|_{D^h_{h+}}), \gamma_{T,\Gamma_{h+}}\overline{v}\rangle_{\Gamma_{h+}} \\
&= \langle \Lambda_\alpha \gamma_{t,\Gamma_{h+}} (u - u^{\mathrm{i}}|_D), \gamma_{T,\Gamma_{h+}}\overline{v}\rangle_{\Gamma_{h+}} + \langle \gamma_{t,\Gamma_{h+}} (u^{\mathrm{i}}|_D), \gamma_{T,\Gamma_{h+}}\overline{v}\rangle_{\Gamma_{h+}}.
\end{aligned}
$$

Therefore, the first equation in Problem 3.12 follows now from

$$
\begin{aligned}
\int_{D^h_{\Gamma_0}} \big(\operatorname{curl}_\alpha \hat{u}\cdot \overline{\operatorname{curl}_\alpha v} - k^2 \hat{u}\cdot\overline{v}\big)\,\mathrm{d}x &= \int_D \big(\operatorname{curl}_\alpha u\cdot \overline{\operatorname{curl}_\alpha v} - k^2 u\cdot\overline{v}\big)\,\mathrm{d}x \\
&\quad + \int_{D^h_{h+}} \big(\operatorname{curl}_\alpha u^+\cdot \overline{\operatorname{curl}_\alpha v} - k^2 u^+\cdot\overline{v}\big)\,\mathrm{d}x
\end{aligned}
$$

together with the variational equation from Problem 3.23. □

3.3. Existence of Solution

From Theorem 3.14 we know, that for special values of k Problem 3.12 has at most one solution. It is the objective of this section to investigate when there also exists a solution. For this we will take its equivalent formulation in variational form, namely Problem 3.23, as a basis and apply standard tools from functional analysis.

Since the functionals, which will appear here, are *antilinear* (see Section 1.3 for a definition), we will need a variant of the theorem of Lax–Milgram as given by Corollary A.9.

3.3.1. The Idea of Proof

Recall Problem 3.23. For ease of notation, we set

$$V := H_{Q,0,\Gamma_0}(\operatorname{curl}, D)$$

and define for $u, v \in V$ the sesquilinear form $B(u,v) : V \times V \to \mathbb{C}$ and the antilinear functional $\ell : V \to \mathbb{C}$ by

$$B(u,v) := \int_D \big(\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u \cdot \overline{v} \big) \mathrm{d}x + \big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}} u), \gamma_{T,\Gamma_{h+}} \overline{v} \big\rangle_{\Gamma_{h+}}, \tag{3.16}$$

$$\ell(v) := \big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}}(u^{\mathrm{i}}|_D)) - \gamma_{t,\Gamma_{h+}} \operatorname{curl}_\alpha(u^{\mathrm{i}}|_D), \gamma_{T,\Gamma_{h+}} \overline{v} \big\rangle_{\Gamma_{h+}}. \tag{3.17}$$

Then Problem 3.23 reads as: find $u \in V$ such that

$$B(u,v) = \ell(v), \qquad \text{for all } v \in V.$$

Inspired by [35] and [42, Section 10.3], we define the subspaces W and $\tilde{X}$ of the space V by

$$W := H^1_{Q,0,\Gamma_0}(D) \text{ and } \tilde{X} := \big\{ \tilde{u} \in V \mid \forall \psi \in W : B(\tilde{u}, \nabla_\alpha \psi) = 0 \big\}. \tag{3.18}$$

Suppose for the moment that the space V can be decomposed into

$$V = \tilde{X} \oplus \nabla_\alpha W$$

and that $u = \tilde{u} + \nabla_\alpha p \in V$ is the solution to Problem 3.23. Then we obtain for arbitrary $v = \tilde{v} + \nabla_\alpha \psi \in V$ that

$$\begin{aligned} B(u,v) = \ell(v) \;&\Leftrightarrow\; B(\tilde{u}, v) = \ell(v) - B(\nabla_\alpha p, v) \\ &\Leftrightarrow\; B(\tilde{u}, \tilde{v}) + \underbrace{B(\tilde{u}, \nabla_\alpha \psi)}_{=0} = \ell(\tilde{v}) + \ell(\nabla_\alpha \psi) - B(\nabla_\alpha p, \tilde{v}) - B(\nabla_\alpha p, \nabla_\alpha \psi). \end{aligned}$$

This gives rise to introduce the following two auxiliary problems and to solve them separately.

Problem 3.26 *Find $p \in W$ such that*

$$B(\nabla_\alpha p, \nabla_\alpha \psi) = \ell(\nabla_\alpha \psi), \qquad \textit{for all } \psi \in W.$$

Problem 3.27 *For given $p \in W$ find $\tilde{u} \in \tilde{X}$ such that*

$$B(\tilde{u}, \tilde{v}) = \ell(\tilde{v}) - B(\nabla_\alpha p, \tilde{v}), \qquad \textit{for all } \tilde{v} \in \tilde{X}.$$

In the following presentation we will show that both problems are uniquely solvable and that indeed $V = \tilde{X} \oplus \nabla_\alpha W$. Then, by the chain of equivalences from above, $\tilde{u} + \nabla_\alpha p =: u$ solves Problem 3.23, where p is the solution to Problem 3.26 and $\tilde{u}$ is the solution to Problem 3.27 for this p.

3.3.2. The First Auxiliary Problem

Motivated by Problem 3.26, we introduce the sesquilinear forms $B_1, B_2 : W \times W \to \mathbb{C}$, defined by

$$B_1(p,\psi) := (\nabla p \,|\, \nabla\psi)_{L^2(D,\mathbb{C}^3)} + |\alpha|^2 \,(p \,|\, \psi)_{L^2(D)} - \frac{1}{k^2}\left\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}} \nabla_\alpha p), \gamma_{T,\Gamma_{h+}} \overline{\nabla_\alpha \psi}\right\rangle_{\Gamma_{h+}}, \tag{3.19a}$$

$$B_2(p,\psi) := (\nabla p \,|\, \mathrm{i}\alpha\,\psi)_{L^2(D,\mathbb{C}^3)} + (\mathrm{i}\alpha\, p \,|\, \nabla\psi)_{L^2(D,\mathbb{C}^3)}, \tag{3.19b}$$

and consider the following problem.

Problem 3.28 *To given antilinear and bounded $g : W \to \mathbb{C}$, find $p \in W$ such that*

$$B_1(p,\psi) + B_2(p,\psi) = g(\psi), \qquad \textit{for all } \psi \in W. \tag{3.20}$$

Remark 3.29 *It is easy to verify that there holds the equation*

$$B_1(p,\psi) + B_2(p,\psi) = -\frac{1}{k^2}\, B(\nabla_\alpha p, \nabla_\alpha \psi), \qquad \textit{for all } p,\psi \in W.$$

Hence, choosing the antilinear and bounded functional $g : W \to \mathbb{C}$ from the form

$$g(\psi) := -\frac{1}{k^2}\,\ell(\nabla_\alpha \psi), \qquad \textit{for all } \psi \in W,$$

Problem 3.26 is a special case of Problem 3.28.

In the next Proposition we will show that B_1 and B_2 are bounded and that B_1 is in addition coercive. This gives rise to rewrite the variational equation (3.20) to an operator equation. The operator therein will turn out to be a compact perturbation of the identity operator. Finally we show that this compact perturbation is injective. Then, an application of Riesz' third theorem will finish the proof for the solvability of Problem 3.28.

For the terms appearing in the next lemma consult for instance Theorem A.8.

Proposition 3.30 *The sesquilinear forms* B_1 *and* B_2*, given by* (3.19a) *and* (3.19b)*, respectively, are bounded. Furthermore,* B_1 *is coercive.*

Proof: *(i).* We show that B_2 is bounded. In fact, let $p, \psi \in W$. Then

$$\begin{aligned}|B_2(p,\psi)| &\le |\alpha|\, \|\nabla p\|_{L^2(D,\mathbb{C}^3)}\, \|\psi\|_{L^2(D)} + |\alpha|\, \|p\|_{L^2(D)}\, \|\nabla\psi\|_{L^2(D,\mathbb{C}^3)} \\ &\le |\alpha|\, \|p\|_{H^1(D)}\, \|\psi\|_{H^1(D)}.\end{aligned}$$

(ii). We show that B_1 is bounded. In fact, let $p, \psi \in W$. At first, we observe that by the boundedness of Λ_α and $\gamma_{t,\Gamma_{h+}}$, see Definition 3.20 and Theorem 2.107,

$$\begin{aligned}&\|\Lambda_\alpha(\gamma_{t,\Gamma_{h+}} \nabla_\alpha p)\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_{h+})} \le C\, \|\nabla_\alpha p\|_{H(\mathrm{curl},D)} \\ &\quad \le C\left(\|\nabla p\|_{H(\mathrm{curl},D)} + \|\mathrm{i}\alpha\, p\|_{H(\mathrm{curl},D)}\right) \\ &\quad = C\left(\|\nabla p\|_{L^2(D,\mathbb{C}^3)} + \sqrt{\|\alpha\, p\|^2_{L^2(D,\mathbb{C}^3)} + \|(\nabla p)\times\alpha\|^2_{L^2(D,\mathbb{C}^3)}}\right) \\ &\quad \le C\, \|p\|_{H^1(D)}.\end{aligned}$$

Therefore, by consulting also Theorem 2.113, we obtain now from (3.19a) easily

$$|B_1(p,\psi)| \le C\, \|p\|_{H^1(D)}\, \|\psi\|_{H^1(D)}.$$

(iii). We show that B_1 is coercive. In fact, let $p \in C^\infty_{Q,0,\Gamma_0}(\overline{D})$. By Lemma 2.127, the series representation therein for $\gamma_{t,\Gamma_{h+}} \nabla_\alpha p$ converges in $H_Q^{-1/2}(\mathrm{Div},\Gamma_{h+})$. Therefore, we can apply the Calderon operator componentwise, which yields

$$\Lambda_\alpha \gamma_{t,\Gamma_{h+}} \nabla_\alpha p = \mathrm{i} \sum_{\mu\in\mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \Big[-\mathrm{i}\, k^2 p^{(\mu)} d^{(\mu)} - \mathrm{i}\, p^{(\mu)} \big(\underbrace{d^{(\mu)}\cdot (d^{(\mu)})^\perp}_{=0}\big)(d^{(\mu)})^\perp\Big]\jmath(T_Q^{(\mu)}).$$

Here, $p^{(\mu)}$ denote the Fourier coefficients of $p(\cdot, h^+)$. Again by Lemma 2.127, we have

$$\gamma_{T,\Gamma_{h+}} \overline{\nabla_\alpha p} = -\mathrm{i} \sum_{\nu\in\mathbb{Z}^2} \overline{p^{(\nu)}} d^{(\nu)} \jmath(T_Q^{(-\nu)}),$$

with convergence in $H_Q^{-1/2}(\mathrm{Curl},\Gamma_{h+})$. Hence,

$$\left\langle \Lambda_\alpha \gamma_{t,\Gamma_{h+}} \nabla_\alpha p, \gamma_{T,\Gamma_{h+}} \overline{\nabla_\alpha p}\right\rangle_{\Gamma_{h+}}$$

$$
\begin{aligned}
&= \sum_{\mu,\nu\in\mathbb{Z}^2} \Big\langle \frac{k^2 p^{(\mu)}}{\rho^{(\mu)}} d^{(\mu)} \jmath(T_Q^{(\mu)}), -\mathrm{i}\, \overline{p^{(\nu)}} d^{(\nu)} \jmath(T_Q^{(-\nu)}) \Big\rangle_{\Gamma_{h+}} \\
&= -\mathrm{i}\, k^2 \sum_{\mu,\nu\in\mathbb{Z}^2} \frac{p^{(\mu)}\overline{p^{(\nu)}}}{\rho^{(\mu)}} d^{(\mu)} \cdot d^{(\nu)} \left(T_Q^{(\mu)} \,\middle|\, T_Q^{(\nu)} \right)_{L^2(Q)} \\
&= -\mathrm{i}\, k^2 \sum_{\mu\in\mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} |d^{(\mu)}|^2 |p^{(\mu)}|^2,
\end{aligned}
$$

where we have applied the definition of the bilinear form $\langle\cdot,\cdot\rangle_{\Gamma_{h+}}$ from Theorem 2.113. Note that $\mathrm{Re}\,(\frac{\mathrm{i}}{\rho^{(\mu)}}) \geq 0$, for all $\mu \in \mathbb{Z}^2$. Therefore,

$$
\begin{aligned}
\mathrm{Re}\,[B_1(p,p)] &= \|\nabla p\|^2_{L^2(D,\mathbb{C}^3)} + |\alpha|^2 \|p\|^2_{L^2(D)} + \sum_{\mu\in\mathbb{Z}^2} \mathrm{Re}\big(\tfrac{\mathrm{i}}{\rho^{(\mu)}}\big) |d^{(\mu)}|^2 |p^{(\mu)}|^2 \\
&\geq \min\{1, |\alpha|^2\}\, \|p\|^2_{H^1(D)}.
\end{aligned}
$$

If $\alpha = 0$, then one shows the last inequality by means of the inequality of Friedrich's type, see Theorem 2.96.
Finally, let $p \in W$. By definition, $C^\infty_{Q,0,\Gamma_0}(\overline{D})$ is dense in W. Therefore, the assertion follows from the last estimate by a standard approximation argument. □

As a consequence of Proposition 3.30, by a conclusion of the theorem of Lax–Milgram, see Corollary A.9, there exists a unique linear and bounded operator $K_1 : W \to W$ and a unique $b \in W$ such that

$$
\begin{aligned}
B_1(K_1 p, \psi) &= B_2(p, \psi), && \text{for all } p, \psi \in W, \\
B_1(b, \psi) &= g(\psi), && \text{for all } \psi \in W.
\end{aligned} \tag{3.21}
$$

Now, it is not difficult to see that Problem 3.28 is equivalent to: to given $b \in W$, find $p \in W$ such that

$$
(I + K_1)\, p = b, \tag{3.22}
$$

where $I : W \to W$ denotes the identity operator.

Proposition 3.31 *The operator $K_1 : W \to W$, given by* (3.21)*, is compact. Moreover, the operator $I + K_1 : W \to W$ is injective.*

Proof: *(i).* We show compactness of K_1. For this let $(p_n)_{n\in\mathbb{N}}$ be a bounded sequence in W. By Theorem 2.122, there exists a subsequence $(p_{n_1(l)})_{l\in\mathbb{N}}$ of $(p_n)_{n\in\mathbb{N}}$ which converges in $L^2(D)$. Since $K_1 : W \to W$ is bounded, also $(K_1 p_{n_1(l)})_{l\in\mathbb{N}}$ is bounded in W, and again by Theorem 2.122, there exists a subsequence $(K_1 p_{n_2(l)})_{l\in\mathbb{N}}$ of $(K_1 p_{n_1(l)})_{l\in\mathbb{N}}$ which converges in $L^2(D)$. Hence, using the coercivity of B_1, Equation (3.21) and the definition of B_2, we obtain

$$\begin{aligned}
&\|K_1(p_{n_2(l)} - p_{n_2(m)})\|^2_{H^1(D)} \\
&\quad\le C\left|B_1\big(K_1(p_{n_2(l)} - p_{n_2(m)}), K_1(p_{n_2(l)} - p_{n_2(m)})\big)\right| \\
&\quad= C\left|B_2\big(p_{n_2(l)} - p_{n_2(m)}, K_1(p_{n_2(l)} - p_{n_2(m)})\big)\right| \\
&\quad\le C\Big(\|\nabla(p_{n_2(l)} - p_{n_2(m)})\|_{L^2(D,\mathbb{C}^3)}\,\|\mathrm{i}\alpha K_1(p_{n_2(l)} - p_{n_2(m)})\|_{L^2(D,\mathbb{C}^3)} \\
&\qquad\quad + \|\mathrm{i}\alpha(p_{n_2(l)} - p_{n_2(m)})\|_{L^2(D,\mathbb{C}^3)}\,\|\nabla K_1(p_{n_2(l)} - p_{n_2(m)})\|_{L^2(D,\mathbb{C}^3)}\Big).
\end{aligned}$$

Since the terms where the symbol "∇" appears are bounded and the other terms are convergent, we conclude that $(K_1 p_{n_2(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence in W and therefore convergent in W.
(ii). We show injectivity of $I + K_1$. For this let $p \in W$ with $(I + K_1)\,p = 0$. This is equivalent to $B_1(p,\psi) + B_2(p,\psi) = 0$ for all $\psi \in W$. In particular

$$\begin{aligned}
0 &= B_1(p,p) + B_2(p,p) \\
&= \|\nabla_\alpha p\|^2_{L^2(D,\mathbb{C}^3)} - \frac{1}{k^2}\big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}}\nabla_\alpha p), \gamma_{T,\Gamma_{h+}}\overline{\nabla_\alpha p}\big\rangle_{\Gamma_{h+}}.
\end{aligned}$$

From part *(iii)* in the proof of Proposition 3.30 we know that

$$\mathrm{Re}\left(\frac{1}{k^2}\big\langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}}\nabla_\alpha p), \gamma_{T,\Gamma_{h+}}\overline{\nabla_\alpha p}\big\rangle_{\Gamma_{h+}}\right) \le 0.$$

This implies together with the equation before that $\|\nabla_\alpha p\|^2_{L^2(D,\mathbb{C}^3)} = 0$. Finally, Friedrich's inequality, see Theorem 2.96, yields $p = 0$ and the proof is complete. □

Theorem 3.32 *Problem 3.28 and Problem 3.26 are uniquely solvable.*

Proof: The assertion for Problem 3.28 follows from the equivalent problem represented by (3.22) and from Proposition 3.31 together with Riesz' third

theorem, see for instance [36, Theorem 3.3].
And the assertion for Problem 3.26 is now a consequence of the unique solvability of Problem 3.28 and Remark 3.29. □

Proposition 3.33 *The spaces* $\tilde{X}$ *and* $\nabla_\alpha W$ *are closed subspaces of* V*. Furthermore, we have* $V = \tilde{X} \oplus \nabla_\alpha W$*.*

Proof: *(i).* To see that $\nabla_\alpha W$ is a closed subspace of V, we repeat the argumentation in the proof of Proposition 2.75 with corresponding adaptions for the situation considered here, where the functions vanish only on Γ_0. For instance, we have to cite Theorem 2.96 instead of Theorem 2.72.
(ii). We show that $\tilde{X}$ is a closed subspace of V. For this let $(\tilde{u}_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{X}$ which converges to some $\tilde{u} \in V$ with respect to $\|\cdot\|_{H_Q(\mathrm{curl},D)}$. Let $\psi \in W$. Then $B(\tilde{u}_n, \nabla_\alpha\psi) = 0$ for all $n \in \mathbb{N}$ and we obtain

$$0 = \lim_{n\to\infty} B(\tilde{u}_n, \nabla_\alpha\psi) = B(\tilde{u}, \nabla_\alpha\psi),$$

where the last step holds thanks to the convergence of $(\tilde{u}_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{H_Q(\mathrm{curl},D)}$ and the definition of B, see (3.16).
(iii). We show that $V = \tilde{X} + \nabla_\alpha W$. The inclusion "$\supseteq$" is obviously true. For the inclusion "$\subseteq$", let $u \in V$. Define $p \in W$ as the unique solution of Problem 3.28 to given $g(\psi) := -\frac{1}{k^2}B(u, \nabla_\alpha\psi)$, for all $\psi \in W$, see Theorem 3.32. Set $\tilde{u} := u - \nabla_\alpha p$. Then, by the first equation in Remark 3.29,

$$\begin{aligned}\frac{1}{k^2}\,B(\tilde{u}, \nabla_\alpha\psi) &= \frac{1}{k^2}\,B(u, \nabla_\alpha\psi) - \frac{1}{k^2}\,B(\nabla_\alpha p, \nabla_\alpha\psi)\\ &= -g(\psi) + B_1(p,\psi) + B_2(p,\psi) = 0\end{aligned}$$

for all $\psi \in W$, and hence $\tilde{u} \in \tilde{X}$.
To see that $\tilde{X} \cap \nabla_\alpha W = \emptyset$, let $\tilde{u} \in \tilde{X} \cap \nabla_\alpha W$. Then $\tilde{u} = \nabla_\alpha p$ for some $p \in W$ and we obtain

$$0 = -\frac{1}{k^2}\,B(\tilde{u}, \nabla_\alpha\psi) = -\frac{1}{k^2}\,B(\nabla_\alpha p, \nabla_\alpha\psi) = B_1(p,\psi) + B_2(p,\psi)$$

for all $\psi \in W$. From this we conclude, since Problem 3.28 is uniquely solvable, that $p = 0$. Hence, $\tilde{u} = \nabla_\alpha p = 0$ and the proof is complete. □

3.3.3. The Second Auxiliary Problem

Unfortunately, the investigation of the solvability of Problem 3.27 is more involved, because the Calderon operator considered in the situation here does not give rise to a coercive sesquilinear form, see the following observation, and has therefore to be split up into a coercive and compact part. Especially the proof of the compactness result requires some technical efforts. For this, we need in particular to introduce several auxiliary operators and to study their mapping properties. Nevertheless, the main procedure is very similar to the investigation of Problem 3.26.

Observation 3.34 *The sesquilinear form*

$$\tilde{X} \times \tilde{X} \ni (\tilde{u}, \tilde{v}) \;\mapsto\; \langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}} \tilde{u}), \gamma_{T,\Gamma_{h+}} \overline{\tilde{v}} \rangle_{\Gamma_{h+}} \in \mathbb{C}$$

is in general not coercive.

Proof: Let $\tilde{u} \in C^\infty_{Q,0,\Gamma_0}(\overline{D}, \mathbb{C}^3)$. By Lemma 2.127, the series representation therein for $\gamma_{t,\Gamma_{h+}} \tilde{u}$ and $\gamma_{T,\Gamma_{h+}} \tilde{u}$ converges in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h+})$ and $H_Q^{-1/2}(\mathrm{Curl}, \Gamma_{h+})$, respectively. In particular, the Calderon operator can be applied componentwise yielding

$$\Lambda_\alpha \gamma_{t,\Gamma_{h+}} \tilde{u} = \mathrm{i} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \big[- k^2 \tilde{u}^{(\mu)} - \big(d^{(\mu)} \cdot (\tilde{u}^{(\mu)})^\perp\big)(d^{(\mu)})^\perp \big] \, \jmath(T_Q^{(\mu)}),$$

where here and for the rest of this proof $\tilde{u}^{(\mu)} \in \mathbb{C}^2$ denote the Fourier coefficients of $\tilde{u}(\cdot, h^+)$ orthogonally projected from $\mathbb{C}^3$ onto $\mathbb{C}^2$. Therefore,

$$\langle \Lambda_\alpha \gamma_{t,\Gamma_{h+}} \tilde{u}, \gamma_{T,\Gamma_{h+}} \overline{\tilde{u}} \rangle_{\Gamma_{h+}} = \mathrm{i} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \big(- k^2 |\tilde{u}^{(\mu)}|^2 + |d^{(\mu)} \cdot (\tilde{u}^{(\mu)})^\perp|^2 \big),$$

where we have applied the definition of the bilinear form $\langle \cdot, \cdot \rangle_{\Gamma_{h+}}$ from Theorem 2.113. Note that $\mathrm{Re}\big(\frac{\mathrm{i}}{\rho^{(\mu)}}\big) \geq 0$, for all $\mu \in \mathbb{Z}^2$. However, the summands containing the factor $-k^2$ destroy in general coercivity due to Definition 3.2. □

As mentioned before, because of Observation 3.34 we have to split up the Calderon operator Λ_α into a coercive and compact part. For this purpose we write

$$\Lambda_\alpha \varphi = \Lambda_\alpha^{(1)} \varphi + \Lambda_\alpha^{(2)} \varphi, \qquad \varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+}),$$

where the operators $\Lambda_\alpha^{(1)}, \Lambda_\alpha^{(2)} : H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+}) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+})$ are given by

$$\Lambda_\alpha^{(1)} \varphi := -\mathrm{i} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} (d^{(\mu)} \cdot \varphi^{(\mu)}) (d^{(\mu)})^\perp \jmath(T_Q^{(\mu)}),$$

$$\Lambda_\alpha^{(2)} \varphi := \mathrm{i} k^2 \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} (\varphi^{(\mu)})^\perp \jmath(T_Q^{(\mu)}),$$

see also Remark 3.22, in particular for the coefficients $(\varphi^{(\mu)})_{\mu \in \mathbb{Z}^2} \in \mathcal{C}_{\mathrm{Div}}^{-1/2}$. To see that the operators are well-defined, we have to show that the coefficients $\big(-\mathrm{i} \frac{d^{(\mu)} \cdot \varphi^{(\mu)}}{\rho^{(\mu)}} (d^{(\mu)})^\perp \big)_{\mu \in \mathbb{Z}^2}$ and $\big(\frac{\mathrm{i} k^2}{\rho^{(\mu)}} (\varphi^{(\mu)})^\perp \big)_{\mu \in \mathbb{Z}^2}$ belong to $\mathcal{C}_{\mathrm{Div}}^{-1/2}$. But, thanks to Lemma 2.9 and Lemma 3.8, this is easy to see. Moreover, the convergences have to be understood in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_{h^+})$ and the operators are linear and bounded.

Now motivated by Problem 3.27, we introduce the sesquilinear forms $B_3, B_4 : \tilde{X} \times \tilde{X} \to \mathbb{C}$, defined by

$$\begin{aligned} B_3(\tilde{u}, \tilde{v}) &:= (\mathrm{curl}\, \tilde{u} \,|\, \mathrm{curl}\, \tilde{v})_{L^2(D,\mathbb{C}^3)} + (\mathrm{i}\alpha \times \tilde{u} \,|\, \mathrm{i}\alpha \times \tilde{v})_{L^2(D,\mathbb{C}^3)} \\ &\quad + (\tilde{u} \,|\, \tilde{v})_{L^2(D,\mathbb{C}^3)} + \langle \Lambda_\alpha^{(1)} (\gamma_{t,\Gamma_{h^+}} \tilde{u}), \gamma_{T,\Gamma_{h^+}} \overline{\tilde{v}} \rangle_{\Gamma_{h^+}}, \qquad (3.23a) \\ B_4(\tilde{u}, \tilde{v}) &:= -(1+k^2)\, (\tilde{u} \,|\, \tilde{v})_{L^2(D,\mathbb{C}^3)} + (\mathrm{i}\alpha \times \tilde{u} \,|\, \mathrm{curl}\, \tilde{v})_{L^2(D,\mathbb{C}^3)} \\ &\quad + (\mathrm{curl}\, \tilde{u} \,|\, \mathrm{i}\alpha \times \tilde{v})_{L^2(D,\mathbb{C}^3)} + \langle \Lambda_\alpha^{(2)} (\gamma_{t,\Gamma_{h^+}} \tilde{u}), \gamma_{T,\Gamma_{h^+}} \overline{\tilde{v}} \rangle_{\Gamma_{h^+}}, \\ &\qquad (3.23b) \end{aligned}$$

and consider the following problem.

Problem 3.35 *To given antilinear and bounded $g : \tilde{X} \to \mathbb{C}$, find $\tilde{u} \in \tilde{X}$ such that*

$$B_3(\tilde{u}, \tilde{v}) + B_4(\tilde{u}, \tilde{v}) = g(\tilde{v}), \quad \textit{for all } \tilde{v} \in \tilde{X}. \qquad (3.24)$$

Remark 3.36 *It is easy to verify that there holds the equation*

$$B_3(\tilde{u}, \tilde{v}) + B_4(\tilde{u}, \tilde{v}) = B(\tilde{u}, \tilde{v}), \qquad \text{for all } \tilde{u}, \tilde{v} \in \tilde{X}.$$

Hence, choosing the antilinear and bounded functional $g : \tilde{X} \to \mathbb{C}$ from the form

$$g(\tilde{v}) := \ell(\tilde{v}) - B(\nabla_\alpha p, \tilde{v}), \qquad \text{for all } \tilde{v} \in \tilde{X},$$

for some $p \in W$, Problem 3.27 is a special case of Problem 3.35.

Similarly as for Problem 3.28, in the following presentation we will show that B_3 and B_4 are bounded and that B_3 is in addition coercive. Again, this gives rise to rewrite the variational equation (3.24) to an operator equation. The operator therein will turn out to be a compact perturbation of the identity operator. Unfortunately, to verify the compactness property, more work has to be done. Finally, we will show again that this compact perturbation is injective. Thus, as before, an application of Riesz' third theorem will yield the solvability of Problem 3.35.

We start with a useful characterization of the space $\tilde{X}$.

Proposition 3.37 *The space $\tilde{X}$ from* (3.18) *can be characterized by*

$$\tilde{X} = \big\{ \tilde{u} \in V \mid \operatorname{div}_\alpha \tilde{u} = 0 \text{ and } \operatorname{Div}_\alpha(\Lambda_\alpha \gamma_{t,\Gamma_{h+}} \tilde{u}) = -k^2 \gamma_{n,\Gamma_{h+}} \tilde{u} \big\}.$$

Moreover, the embedding $\tilde{X} \overset{\text{id}}{\hookrightarrow} L^2(D, \mathbb{C}^3)$ is compact.

Proof: *(i).* To show the characterization, first of all, for $\psi \in H^1_Q(D)$ we have $\mathrm{i}\alpha \times (\nabla\psi + \mathrm{i}\alpha\psi) = \mathrm{i}\alpha \times \nabla\psi$ and, thanks to Proposition 2.75, that $\operatorname{curl}(\nabla_\alpha \psi) = \nabla\psi \times \mathrm{i}\alpha$, which implies that $\operatorname{curl}_\alpha(\nabla_\alpha\psi) = 0$.
Let $\tilde{u} \in \tilde{X}$. Choosing some $\chi \in C^\infty_{Q,0}(D) \subseteq W$, from $B(\tilde{u}, \nabla_\alpha\chi) = 0$ we conclude, with the observation above, that

$$\int_D \tilde{u} \cdot \nabla\overline{\chi}\, \mathrm{d}x = -\int_D (-\mathrm{i}\alpha \cdot \tilde{u})\, \overline{\chi}\, \mathrm{d}x.$$

Hence, $\tilde{u} \in H_Q(\operatorname{div}, D)$ with $\operatorname{div} \tilde{u} = -\mathrm{i}\alpha \cdot \tilde{u}$, which is equivalent to $\operatorname{div}_\alpha \tilde{u} = 0$. Now, let $\psi \in W$. Then from $B(\tilde{u}, \nabla_\alpha\psi) = 0$ we conclude, with (2.18c) and (2.19), that

$$0 = -k^2 \int_D \tilde{u} \cdot \overline{\nabla_\alpha \psi}\, \mathrm{d}x + \langle \Lambda_\alpha(\gamma_{t,\Gamma_{h+}} \tilde{u}), \gamma_{T,\Gamma_{h+}} \overline{\nabla_\alpha \psi} \rangle_{\Gamma_{h+}}$$

$$= k^2\int_D \overline{\psi}\underbrace{\operatorname{div}_\alpha \tilde{u}}_{=0}\,\mathrm{d}x - k^2\langle\gamma_{n,\Gamma_{h+}}\tilde{u},\gamma_{0,\Gamma_{h+}}\overline{\psi}\rangle - \langle\operatorname{Div}_\alpha\Lambda_\alpha(\gamma_{t,\Gamma_{h+}}\tilde{u}),\gamma_{0,\Gamma_{h+}}\overline{\psi}\rangle$$
$$= -\big\langle k^2\gamma_{n,\Gamma_{h+}}\tilde{u} + \operatorname{Div}_\alpha\Lambda_\alpha(\gamma_{t,\Gamma_{h+}}\tilde{u}),\gamma_{0,\Gamma_{h+}}\overline{\psi}\big\rangle.$$

From this we obtain, together with the surjectivity of the trace operator $\gamma_{0,\Gamma_{h+}} \in \mathcal{L}(H^1_Q(D), H^{1/2}_Q(\Gamma_{h+}))$ and a well-known conclusion from Hahn-Banach's theorem, that indeed $\operatorname{Div}_\alpha(\Lambda_\alpha\gamma_{t,\Gamma_{h+}}\tilde{u}) = -k^2\gamma_{n,\Gamma_{h+}}\tilde{u}$.
To show the reverse inclusion, let $\tilde{u} \in V$ such that $\operatorname{div}_\alpha\tilde{u} = 0$ and $\operatorname{Div}_\alpha(\Lambda_\alpha\gamma_{t,\Gamma_{h+}}\tilde{u}) = -k^2\gamma_{n,\Gamma_{h+}}\tilde{u}$. Furthermore, let $\psi \in W$. Exploiting again (2.18c), we then have

$$0 = \int_D \overline{\psi}\operatorname{div}_\alpha\tilde{u}\,\mathrm{d}x = -\int_D \tilde{u}\cdot\overline{\nabla_\alpha\psi}\,\mathrm{d}x + \langle\gamma_{n,\Gamma_{h+}}\tilde{u},\gamma_{0,\Gamma_{h+}}\overline{\psi}\rangle$$

and again by (2.19) that

$$\langle\operatorname{Div}_\alpha(\Lambda_\alpha\gamma_{t,\Gamma_{h+}}\tilde{u}),\gamma_{0,\Gamma_{h+}}\overline{\psi}\rangle = -\langle\Lambda_\alpha(\gamma_{t,\Gamma_{h+}}\tilde{u}),\gamma_{T,\Gamma_{h+}}\overline{\nabla_\alpha\psi}\rangle_{\Gamma_{h+}}.$$

From this, together with $\operatorname{curl}_\alpha\nabla_\alpha\psi = 0$, we conclude finally $B(\tilde{u},\nabla_\alpha\psi) = 0$, as desired.
(ii). To show compactness of the embedding, by consulting the definition of the space $\tilde{X}$ from (3.18), it is easy to see that $\tilde{X}$ is a subspace of the space $H_Q(\operatorname{curl},\operatorname{div}_\alpha 0, D)$ from Definition 2.69. Therefore, the assertion follows now immediately from Theorem 2.122. □

Proposition 3.38 *The sesquilinear forms B_3 and B_4, given by* (3.23a) *and* (3.23b)*, respectively, are bounded. Furthermore, B_3 is coercive.*

Proof: *(i).* We show that B_3 is bounded. In fact, let $\tilde{u},\tilde{v} \in \tilde{X}$. Then, by the boundedness of the operators $\Lambda^{(1)}_\alpha$, $\gamma_{t,\Gamma_{h+}}$ and $\gamma_{T,\Gamma_{h+}}$, there holds the estimate $\|\Lambda^{(1)}_\alpha(\gamma_{t,\Gamma_{h+}}\tilde{u})\|_{H^{-1/2}_Q(\operatorname{Div},\Gamma_{h+})} \le C\,\|\tilde{u}\|_{H_Q(\operatorname{curl},D)}$ and furthermore $\|\gamma_{T,\Gamma_{h+}}\overline{\tilde{v}}\|_{H^{-1/2}_Q(\operatorname{Curl},\Gamma_{h+})} \le C\,\|\tilde{v}\|_{H_Q(\operatorname{curl},D)}$, and we obtain

$$|B_3(\tilde{u},\tilde{v})| \le \|\operatorname{curl}\tilde{u}\|_{L^2(D,\mathbb{C}^3)}\,\|\operatorname{curl}\tilde{v}\|_{L^2(D,\mathbb{C}^3)}$$
$$+ (1+|\alpha|^2)\|\tilde{u}\|_{L^2(D,\mathbb{C}^3)}\,\|\tilde{v}\|_{L^2(D,\mathbb{C}^3)}$$

$$+ C\,\|\Lambda_\alpha^{(1)}(\gamma_{t,\Gamma_{h^+}}\tilde{u})\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+})}\,\|\gamma_{T,\Gamma_{h^+}}\overline{\tilde{v}}\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_{h^+})}$$
$$\le C\,\|\tilde{u}\|_{H_Q(\mathrm{curl},D)}\,\|\tilde{v}\|_{H_Q(\mathrm{curl},D)}.$$

Again, for the constant C, which appeared here several times, see the convention at the end of Section 1.3.
(ii). We show that B_4 is bounded. In fact, let $\tilde{u},\tilde{v}\in\tilde{X}$. Then, with the same arguments as above, we obtain

$$\begin{aligned}|B_4(\tilde{u},\tilde{v}| &\le |1+k^2|\,\|\tilde{u}\|_{L^2(D,\mathbb{C}^3)}\,\|\tilde{v}\|_{L^2(D,\mathbb{C}^3)}\\ &\quad + |\alpha|\,\|\tilde{u}\|_{L^2(D,\mathbb{C}^3)}\,\|\operatorname{curl}\tilde{v}\|_{L^2(D,\mathbb{C}^3)}\\ &\quad + |\alpha|\,\|\operatorname{curl}\tilde{u}\|_{L^2(D,\mathbb{C}^3)}\,\|\tilde{v}\|_{L^2(D,\mathbb{C}^3)}\\ &\quad + C\,\|\Lambda_\alpha^{(2)}(\gamma_{t,\Gamma_{h^+}}\tilde{u})\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+})}\,\|\gamma_{T,\Gamma_{h^+}}\overline{\tilde{v}}\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_{h^+})}\\ &\le C\,\|\tilde{u}\|_{H_Q(\mathrm{curl},D)}\,\|\tilde{v}\|_{H_Q(\mathrm{curl},D)}.\end{aligned}$$

(iii). We show that B_3 is coercive. In fact, let $\tilde{u}\in C^\infty_{Q,0,\Gamma_0}(\overline{D},\mathbb{C}^3)$. By Lemma 2.127, the series representation therein for $\gamma_{t,\Gamma_{h^+}}\tilde{u}$ converges in $H_Q^{-1/2}(\mathrm{Div},\Gamma_{h^+})$. Therefore, we can apply the operator $\Lambda_\alpha^{(1)}$ componentwise, which yields

$$\Lambda_\alpha^{(1)}\gamma_{t,\Gamma_{h^+}}\tilde{u} = -\mathrm{i}\sum_{\mu\in\mathbb{Z}^2}\frac{1}{\rho^{(\mu)}}\big(d^{(\mu)}\cdot(\tilde{u}^{(\mu)})^\perp\big)(d^{(\mu)})^\perp\jmath(T_Q^{(\mu)}).$$

Here, $\tilde{u}^{(\mu)}\in\mathbb{C}^2$ denote the Fourier coefficients of $\tilde{u}(\cdot,h^+)$, orthogonally projected from $\mathbb{C}^3$ onto $\mathbb{C}^2$. Again by Lemma 2.127, we have $\gamma_{T,\Gamma_{h^+}}\overline{\tilde{u}} = \sum_{\nu\in\mathbb{Z}^2}\overline{\tilde{u}^{(\nu)}}\jmath(T_Q^{(-\nu)})$, with convergence in $H_Q^{-1/2}(\mathrm{Curl},\Gamma_{h^+})$. Hence,

$$\begin{aligned}&\big\langle\Lambda_\alpha^{(1)}\gamma_{t,\Gamma_{h^+}}\tilde{u},\gamma_{T,\Gamma_{h^+}}\overline{\tilde{u}}\big\rangle_{\Gamma_{h^+}}\\ &\quad=\sum_{\mu,\nu\in\mathbb{Z}^2}\Big\langle\frac{-\mathrm{i}\,d^{(\mu)}\cdot(\tilde{u}^{(\mu)})^\perp}{\rho^{(\mu)}}(d^{(\mu)})^\perp\jmath(T_Q^{(\mu)}),\overline{\tilde{u}^{(\nu)}}\jmath(T_Q^{(-\nu)})\Big\rangle_{\Gamma_{h^+}}\\ &\quad=-\mathrm{i}\sum_{\mu,\nu\in\mathbb{Z}^2}\frac{d^{(\mu)}\cdot(\tilde{u}^{(\mu)})^\perp}{\rho^{(\mu)}}\underbrace{(d^{(\mu)})^\perp\cdot\overline{\tilde{u}^{(\nu)}}}_{=-d^{(\mu)}\cdot\overline{(\tilde{u}^{(\nu)})^\perp}}\Big(T_Q^{(\mu)}\,\Big|\,T_Q^{(\nu)}\Big)_{L^2(Q)}\\ &\quad=\mathrm{i}\sum_{\mu\in\mathbb{Z}^2}\frac{1}{\rho^{(\mu)}}\,|d^{(\mu)}\cdot(\tilde{u}^{(\mu)})^\perp|^2,\end{aligned}$$

where we have applied the definition of the bilinear form $\langle\cdot,\cdot\rangle_{\Gamma_{h+}}$ from Theorem 2.113. Note that $\operatorname{Re}\big(\frac{\mathrm{i}}{\rho^{(\mu)}}\big) \geq 0$, for all $\mu \in \mathbb{Z}^2$. Therefore,

$$\begin{aligned}\operatorname{Re}[B_3(\tilde{u},\tilde{u})] &= \|\operatorname{curl}\tilde{u}\|^2_{L^2(D,\mathbb{C}^3)} + \|\mathrm{i}\alpha\times\tilde{u}\|^2_{L^2(D,\mathbb{C}^3)} + \|\tilde{u}\|^2_{L^2(D,\mathbb{C}^3)} \\ &\quad + \sum_{\mu\in\mathbb{Z}^2}\operatorname{Re}\big(\tfrac{\mathrm{i}}{\rho^{(\mu)}}\big)|d^{(\mu)}\cdot(\tilde{u}^{(\mu)})^{\perp}|^2 \\ &\geq \|\tilde{u}\|^2_{H_Q(\operatorname{curl},D)}.\end{aligned}$$

Finally, note that B_3 is even well-defined in V. Since, by definition, $C^{\infty}_{Q,0,\Gamma_0}(\overline{D},\mathbb{C}^3)$ is dense in V, by a standard approximation argument we conclude that B_3 is coercive in V, and therefore in particular in $\tilde{X}$. □

Note that $\tilde{X}$ is a Hilbert space, see Proposition 3.33. Hence, again as a consequence of Proposition 3.38, by a conclusion of the theorem of Lax–Milgram, see Corollary A.9, there exists a unique linear and bounded operator $K_3 : \tilde{X} \to \tilde{X}$ and a unique $\tilde{b} \in \tilde{X}$ such that

$$\begin{aligned} B_3(K_3\tilde{u},\tilde{v}) &= B_4(\tilde{u},\tilde{v}), &&\text{for all } \tilde{u},\tilde{v}\in\tilde{X}, \\ B_3(\tilde{b},\tilde{v}) &= g(\tilde{v}), &&\text{for all } \tilde{v}\in\tilde{X}.\end{aligned} \tag{3.25}$$

And again, it is not difficult to see that Problem 3.35 is equivalent to: to given $\tilde{b}\in\tilde{X}$, find $\tilde{u}\in\tilde{X}$ such that

$$(I + K_3)\,\tilde{u} = \tilde{b}, \tag{3.26}$$

where again $I : \tilde{X} \to \tilde{X}$ denotes the identity operator.

The next goal is to show that the operator K_3 is compact. For this, as we will see below, that the compactness of the mapping

$$\tilde{X} \ni \tilde{u} \;\mapsto\; \Lambda^{(2)}_{\alpha}\gamma_{t,\Gamma_{h+}}\tilde{u} \in H_Q^{-1/2}(\operatorname{Div},\Gamma_{h+}) \tag{3.27}$$

is a key ingredient. To prove this property, we recall Convention 2.125, together with Lemma 2.129, and observe that for $\varphi \in H_Q^{-1/2}(\operatorname{Div},\Gamma_{h+})$ we have

$$\|\varphi\|^2_{H_Q^{-1/2}(\operatorname{Div},\Gamma_{h+})} = \sum_{\mu\in\mathbb{Z}^2}(1+|\mu|^2)^{-1/2}\big(|\varphi^{(\mu)}|^2 + |(q^{(\mu)}+\tilde{\alpha}-\tilde{\alpha})\cdot\varphi^{(\mu)}|^2\big)$$

$$\leq (1+2|\alpha|^2) \sum_{\mu\in\mathbb{Z}^2} (1+|\mu|^2)^{-1/2} |\varphi^{(\mu)}|^2$$
$$+ 2 \sum_{\mu\in\mathbb{Z}^2} (1+|\mu|^2)^{-1/2} \big|(q^{(\mu)} + \tilde{\alpha}) \cdot \varphi^{(\mu)}\big|^2$$
$$= (1+2|\alpha|^2) \, \|\varphi\|^2_{H_Q^{-1/2}(\Gamma_{h^+},\mathbb{C}^2)} + 2 \, \| \operatorname{Div}_\alpha \varphi\|^2_{H_Q^{-1/2}(\Gamma_{h^+})}.$$

Therefore, the compactness of the mapping given by (3.27) follows easily, if we have shown that the mappings

$$\tilde{X} \ni \tilde{u} \;\mapsto\; \Lambda_\alpha^{(2)} \gamma_{t,\Gamma_{h^+}} \tilde{u} \in H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2), \tag{3.28a}$$
$$\tilde{X} \ni \tilde{u} \;\mapsto\; \operatorname{Div}_\alpha \big(\Lambda_\alpha^{(2)} \gamma_{t,\Gamma_{h^+}} \tilde{u}\big) \in H_Q^{-1/2}(\Gamma_{h^+}) \tag{3.28b}$$

are compact. This is the statement of the following lemma.

Lemma 3.39 *The mappings given by* (3.28a) *and* (3.28b) *are compact. In particular, the mapping given by* (3.27) *is compact.*

Proof: *(a).* To verify compactness of the mapping given by (3.28a), we note, thanks to the boundedness of the trace operator $\gamma_{t,\Gamma_{h^+}}$ and the embedding from Lemma 2.129, that it suffices to show compactness of the mapping $H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2) \ni \varphi \mapsto \Lambda_\alpha^{(2)} \varphi \in H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2)$. We call this mapping λ and consider for $n \in \mathbb{N}$ the compact mappings $\lambda_n : H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2) \to H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2)$ given by

$$\lambda_n \varphi := \mathrm{i}k^2 \sum_{|\mu|\leq n} \frac{1}{\rho^{(\mu)}} (\varphi^{(\mu)})^\perp \jmath(T_Q^{(\mu)}), \qquad \varphi \in H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2).$$

Then, for $\varphi \in H_Q^{-1/2}(\Gamma_{h^+}, \mathbb{C}^2)$ we obtain, by applying Lemma 3.8,

$$\|(\lambda_n - \lambda)\,\varphi\|^2_{H_Q^{-1/2}(\Gamma_{h^+},\mathbb{C}^2)} = |k|^4 \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \frac{|\varphi^{(\mu)}|^2}{|\rho^{(\mu)}|^2}$$
$$\leq C \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \frac{|\varphi^{(\mu)}|^2}{1+|\mu|^2} \leq \frac{C}{1+n^2} \, \|\varphi\|^2_{H_Q^{-1/2}(\Gamma_{h^+},\mathbb{C}^2)}.$$

Hence, $(\lambda_n)_{n\in\mathbb{N}}$ converges to λ in operator norm, as $n \to \infty$, which shows that λ is compact too.
(b). To verify compactness of the mapping given by (3.28b), let $\tilde{u} \in \tilde{X}$. Then, thanks to Proposition 3.37 and Theorem 2.111, we obtain

$$\begin{aligned}
&\| \operatorname{Div}_\alpha \big(\lambda_\alpha^{(2)} \gamma_{t,\Gamma_{h^+}} \tilde{u}\big) \|_{H_Q^{-1/2}(\Gamma_{h^+})} = |k|^2 \|\gamma_{n,\Gamma_{h^+}} \tilde{u}\|_{H_Q^{-1/2}(\Gamma_{h^+})} \\
&\leq C \,\|\tilde{u}\|_{H_Q(\operatorname{div},D)} = C \left(\|\tilde{u}\|^2_{L^2(D,\mathbb{C}^3)} + \| \underbrace{\operatorname{div}_\alpha \tilde{u}}_{=0} - \mathrm{i}\alpha \cdot \tilde{u}\|^2_{L^2(D)} \right)^{1/2} \\
&\leq C \,\|\tilde{u}\|_{L^2(D,\mathbb{C}^3)}.
\end{aligned}$$

Taking now the embedding from Proposition 3.37 into account, we have indeed shown that the mapping given by (3.28b) is compact.
And finally, the compactness of the mapping given by (3.27) follows now immediately from the compactness of the mappings given by (3.28a) and (3.28b), together with the estimate after (3.27). □

Proposition 3.40 *The operator $K_3 : \tilde{X} \to \tilde{X}$, given by* (3.25)*, is compact. Moreover, provided Problem 3.12 has at most one solution, the operator $I + K_3 : \tilde{X} \to \tilde{X}$ is injective.*

Proof: *(i).* We show compactness of K_3. For this let $(\tilde{u}_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\tilde{X}$. By Lemma 3.39, there exists a subsequence $(\tilde{u}_{n_1(l)})_{l\in\mathbb{N}}$ of $(\tilde{u}_n)_{n\in\mathbb{N}}$ such that $(\Lambda_\alpha^{(2)} \gamma_{t,\Gamma_{h^+}} \tilde{u}_{n_1(l)})_{l\in\mathbb{N}}$ is convergent in $H_Q^{-1/2}(\operatorname{Div}, \Gamma_{h^+})$. Moreover, thanks to the embedding from Proposition 3.37, there exists a subsequence $(\tilde{u}_{n_2(l)})_{l\in\mathbb{N}}$ of $(\tilde{u}_{n_1(l)})_{l\in\mathbb{N}}$ which converges in $L^2(D, \mathbb{C}^3)$. And last but not least, since the sequence $(K_3\tilde{u}_{n_2(l)})_{l\in\mathbb{N}}$ is bounded in $\tilde{X}$, there exists a subsequence $(\tilde{u}_{n_3(l)})_{l\in\mathbb{N}}$ of $(\tilde{u}_{n_2(l)})_{l\in\mathbb{N}}$ such that $(K_3\tilde{u}_{n_3(l)})_{l\in\mathbb{N}}$ converges in $L^2(D, \mathbb{C}^3)$, thanks again to Proposition 3.37. Hence, using the coercivity of B_3, Equation (3.25) and the definition of B_4, we obtain

$$\begin{aligned}
&\|K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\|^2_{H_Q(\operatorname{curl},D)} \\
&\quad\leq C \left| B_3\big(K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)}), K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\big)\right| \\
&\quad= C \left| B_4\big(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)}, K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\big)\right| \\
&\quad\leq C \Big(\|\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)}\|_{L^2(D,\mathbb{C}^3)} \,\|K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\|_{L^2(D,\mathbb{C}^3)}
\end{aligned}$$

$$+ \|\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)}\|_{L^2(D,\mathbb{C}^3)} \left\| \operatorname{curl}\left(K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\right)\right\|_{L^2(D,\mathbb{C}^3)}$$
$$+ \left\| \operatorname{curl}\left(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)}\right)\right\|_{L^2(D,\mathbb{C}^3)} \|K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\|_{L^2(D,\mathbb{C}^3)}$$
$$+ \|\Lambda_\alpha^{(2)}\gamma_{t,\Gamma_{h+}}(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_{h+})}$$
$$\|K_3(\tilde{u}_{n_3(l)} - \tilde{u}_{n_3(m)})\|_{H_Q(\mathrm{curl},D)}\Big) \longrightarrow 0, \quad \text{as } l, m \to \infty,$$

because both terms containing the curl-operator and the term measured in the $\|\cdot\|_{H_Q(\mathrm{curl},D)}$ norm are bounded and the remaining terms converge to zero by the considerations from above. Hence, $(K_3\tilde{u}_{n_3(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{X}$ and therefore convergent in $\tilde{X}$, because $\tilde{X}$ is a Hilbert space, see Proposition 3.33.
(ii). We show injectivity of $I + K_3$. For this let $\tilde{u} \in \tilde{X}$ with $(I + K_3)\tilde{u} = 0$. This is equivalent to $B_3(\tilde{u}, \tilde{v}) + B_4(\tilde{u}, \tilde{v}) = 0$ for all $\tilde{v} \in \tilde{X}$. Thanks to the first equation in Remark 3.36, the composition $V = \tilde{X} \oplus \nabla_\alpha W$ and $B(\tilde{u}, \nabla_\alpha\psi) = 0$, this is equivalent to $B(\tilde{u}, v) = 0$ for all $v = \tilde{v} + \nabla_\alpha\psi \in V$. That is, $\tilde{u}$ is a solution of Problem 3.23 to $u^{\mathrm{i}} = 0$. Since this problem has by assumption at most one solution, see also Theorem 3.25, it follows that $\tilde{u} = 0$. □

Theorem 3.41 *Provided Problem 3.12 has at most one solution, Problem 3.35 and Problem 3.27 possess exactly one solution.*

Proof: As in the proof of Theorem 3.32, the assertion for Problem 3.35 follows from the equivalent problem represented by (3.26) and from Proposition 3.40 together with Riesz' third theorem.
And again, the assertion for Problem 3.27 is now a consequence of the unique solvability of Problem 3.35 and Remark 3.36. □

3.3.4. Summa Summarum

Thanks to the preliminary considerations from above, we are now in a relaxed position to prove the main theorem of this chapter.

Theorem 3.42 *If Problem 3.12 has at most one solution, then it possesses exactly one solution.*

Proof: This follows as outlined at the end of Subsection 3.3.1, together with Theorem 3.25. □

4. Electromagnetic Scattering – Boundary Integral Equations

The integral equation method is the other main approach for investigations of existence and uniqueness of solutions to boundary value problems and well-suited for exterior (such as scattering) problems. In its indirect variant, we will use a fundamental solution (the Green's function) and look for the solution of the problem in form of vector potentials with an unknown density. In order to determine this density, we exploit certain jump relations of those potentials on the boundary and obtain a boundary integral equation, whose solvability has to be studied next. Furthermore, this equation can be used to derive high order numerical schemes.

In Section 4.1 we will recall the definition of the Q-(quasi-)periodic Green's function for the Helmholtz equation from [7] and collect its most important properties.

Then we continue in Section 4.2 to define vector potentials and investigate first properties. A key tool will be a special transmission problem as it provides the important jump relations and thus the boundary integral operators $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$. So far, we have followed very closely the ideas in [34] with corresponding adaptions to the Q-periodic framework. Those methods allow us also the write $\mathcal{L}_\alpha$ as a compact perturbation of an isomorphism. Unfortunately, such a result is not known for the operator $\mathcal{M}_\alpha$ in the case of Lipschitz surfaces and we have to impose more regularity on the surface. Then we are able to fall back on results of [21], which allow us by means of a special technique to show compactness of $\mathcal{M}_\alpha$.

Finally, in Section 4.3 we will derive the boundary integral equation and obtain its unique solvability in a straightforward manner. Technically more involved is the verification that the weak singularity of the kernels meets the requirements of the numerical scheme. These investigations will close this chapter.

4.1. The Q-periodic Green's Function

A key ingredient for the integral equation method is a suitable *Green's function*. In our framework a Q-(quasi-)periodic variant of Green's function for the Helmholtz equation is needed, as worked out in [7]. Based on this reference, it is the objective of this section to recall the definition of the Green's function and to collect its most important properties being relevant for the definition of vector potentials as well as for an application of the numerical method from Chapter 5.

Recall Convention 3.4 for the notation involving the phase shift α.

Let $\Omega := Q \times \mathbb{R}$ and set $\Omega_s := \{(x, x) \mid x \in \Omega\}$. We look for a function $G_{k,\alpha} : (\Omega \times \Omega) \setminus \Omega_s \to \mathbb{C}$, depending also on the wave number k, such that for fixed $y \in \Omega$ the function $G_{k,\alpha}(\cdot, y)$ has a Q-quasi-periodic extension to $\mathbb{R}^3$ with phase shift α and that

$$G_{k,\alpha}(x, y) = \Phi_k(x, y) + \Psi_{k,\alpha}(x - y), \qquad x, y \in (\Omega \times \Omega) \setminus \Omega_s, \tag{4.1}$$

where Φ_k denotes the *fundamental solution* to the Helmholtz equation in free fields conditions,

$$\Phi_k(x, y) := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x - y|}, \qquad x \neq y, \tag{4.2}$$

and $\Psi_{k,\alpha}$ is an analytic solution to the Helmholtz equation in Ω. Moreover, we require that $G_{k,\alpha}(\cdot, y)$ must be propagating away from $y \in \Omega$ and that this function is bounded on Ω except for neighborhoods of y.

In [7] there were tackled two approaches to solve this problem: the Green's function in form of a Fourier series expansion or in form of a superposition of point sources placed on a periodic array.

Recall the definition of $d^{(\mu)}$ and $\rho^{(\mu)}$ from (3.10) and of $p^{(\mu)}$ from (2.7).

Proposition 4.1 *The Q-quasi-periodic Green's function $G_{k,\alpha} : (\Omega \times \Omega) \setminus \Omega_s \to \mathbb{C}$, with phase shift α, has a representation*

(i) in form of a Fourier series expansion

$$G_{k,\alpha}(x,y) = \frac{\mathrm{i}}{2|Q|} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \mathrm{e}^{\mathrm{i}(d^{(\mu)} \cdot (\tilde{x} - \tilde{y}) + \rho^{(\mu)} |x_3 - y_3|)}, \quad x_3 \neq y_3,$$

where for fixed $y \in \Omega$ and $\varepsilon > 0$ convergence of the series (and all of its derivatives) is uniform with respect to x in $\{z \in \mathbb{R}^3 \mid z_3 \notin [y_3 - \varepsilon, y_3 + \varepsilon]\}$, and vice versa.

(ii) in form of a superposition of point sources

$$G_{k,\alpha}(x,y) = \frac{1}{4\pi} \sum_{\mu \in \mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\alpha \cdot p^{(\mu)}} \frac{\mathrm{e}^{\mathrm{i}k|x-y-p^{(\mu)}|}}{|x-y-p^{(\mu)}|}, \quad x - y \neq p^{(\mu)},$$

where for fixed $y \in \Omega$ convergence of the series (and all of its derivatives) is uniform with respect to x in every compact set $K \subseteq \mathbb{R}^3$ such that $y + p^{(\mu)} \notin K$, and vice versa.

In particular, the Green's function can be extended analytically to the line $x_3 = y_3$ for $x \neq y$.

For a proof we refer to the derivation in [7, Section 3.1]; for the statement regarding the convergence of the Fourier series expansion see also Lemma 4.3.

Remark 4.2 *The corresponding representations from Proposition 4.1 for the Q-periodic counterpart of $G_{k,\alpha}$ read as*

$$G_k(x,y) = \frac{\mathrm{i}}{2|Q|} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \mathrm{e}^{\mathrm{i}(q^{(\mu)} \cdot (\tilde{x} - \tilde{y}) + \rho^{(\mu)} |x_3 - y_3|)}, \qquad x_3 \neq y_3, \tag{4.3a}$$

$$G_k(x,y) = \frac{1}{4\pi} \mathrm{e}^{-\mathrm{i}\tilde{\alpha} \cdot (\tilde{x} - \tilde{y})} \sum_{\mu \in \mathbb{Z}^2} \mathrm{e}^{\mathrm{i}\alpha \cdot p^{(\mu)}} \frac{\mathrm{e}^{\mathrm{i}k|x-y-p^{(\mu)}|}}{|x-y-p^{(\mu)}|}, \quad x - y \neq p^{(\mu)}, \tag{4.3b}$$

see also Convention 3.4.

Fix for the moment $\mu \in \mathbb{Z}^2$ and $y \in \Omega$, and set $v(x) := \frac{\mathrm{e}^{\mathrm{i}k|x-y-p^{(\mu)}|}}{|x-y-p^{(\mu)}|}$ and $u(x) := \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot(\tilde{x}-\tilde{y})}v(x)$, for $x \neq y$. Note that $\Delta v = -k^2 v$. Then, on the one hand, because of (1.20b), we have

$$\Delta_\alpha u = \operatorname{div}_\alpha \nabla_\alpha u = \Delta u + 2\mathrm{i}\alpha \cdot \nabla u - |\alpha|^2 u,$$

and on the other hand from the definition of u we obtain

$$\nabla u(x) = \Big(-\mathrm{i}\begin{pmatrix}\alpha_1\\ \alpha_2\\ 0\end{pmatrix} v(x) + \nabla v(x)\Big)\,\mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot(\tilde{x}-\tilde{y})},$$
$$\Delta u(x) = \big(-|\alpha|^2 v(x) - 2\mathrm{i}\alpha\cdot\nabla v(x) + \Delta v(x)\big)\,\mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot(\tilde{x}-\tilde{y})}.$$

Both together yields $\Delta_\alpha u(x) = -k^2 u(x)$. An important consequence of Proposition 4.1 is that we can interchange differentiation and summation. This and the observation we just have made imply from (4.3b) that

$$\Delta_{\alpha,x} G_k(x,y) = -k^2 G_k(x,y) = \Delta_{\alpha,y} G_k(x,y), \quad x, y \in \mathbb{R}^3,\ x - y \neq p^{(\mu)}. \tag{4.4}$$

Furthermore, from this observation and the representation (4.3b) we obtain an explicit expression for the function $\Psi_{k,\alpha}$ in (4.1).

Lemma 4.3 *Let $y \in \mathbb{R}^3$ and $\varepsilon > 0$. Furthermore, let $\beta \in \mathbb{N}_0^3$. Then the series*

$$\Big(\sum_{\mu\in\mathbb{Z}^2} \frac{1}{\rho^{(\mu)}}\, \partial^\beta \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot(\tilde{\cdot}-\tilde{y})+\rho^{(\mu)}|\cdot_3-y_3|)} \Big)$$

converges uniformly in $\{z \in \mathbb{R}^3 \mid z_3 \notin [y_3-\varepsilon, y_3+\varepsilon]\}$. Here, the partial derivative ∂^β can be taken with respect to x or y.
The statement remains true if we interchange x and y.

Proof: In the following presentation let ∂^β be taken with respect to x or y. Furthermore, let x belong to the set from the lemma. Then $|x_3 - y_3| > \varepsilon$ and we obtain, thanks to Lemma 2.9 and Lemma 3.8,

$$\Big| \frac{1}{\rho^{(\mu)}}\, \partial^\beta \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot(\tilde{\cdot}-\tilde{y})+\rho^{(\mu)}|\cdot_3-y_3|)} \Big| \leq \frac{C}{\rho^{(\mu)}} \big(\sqrt{1+|\mu|^2}\big)^{|\beta|} \mathrm{e}^{-\mathrm{Im}(\rho^{(\mu)})|x_3-y_3|}$$
$$\leq C\, \frac{\big(\sqrt{1+|\mu|^2}\big)^{|\beta|-1}}{\mathrm{e}^{\tilde{C}\sqrt{1+|\mu|^2}\varepsilon}} = \frac{C}{(\tilde{C}\varepsilon)^{|\beta|+3}}\, \frac{\big(\tilde{C}\sqrt{1+|\mu|^2}\varepsilon\big)^{|\beta|+3}}{\mathrm{e}^{\tilde{C}\sqrt{1+|\mu|^2}\varepsilon}}\, \frac{1}{(1+|\mu|^2)^2}$$

$$\leq C \frac{1}{(1+|\mu|^2)^2},$$

because of the boundedness of the term $\frac{(\tilde{C}\sqrt{1+|\mu|^2}\varepsilon)^{|\beta|+1}}{\mathrm{e}^{\tilde{C}\sqrt{1+|\mu|^2}\varepsilon}}$. The assertion follows now from $\sum_{\mu \in \mathbb{Z}^2} \frac{1}{(1+|\mu|^2)^2} < \infty$, see also Lemma 2.37. □

Lemma 4.4 *Recalling the notation from Subsection 3.1.2, let* $h > h^+$ *and* $x \in D_h^\infty$. *Furthermore, let* $\beta \in \mathbb{N}_0^3$. *Then the Fourier series representation for* $\partial^\beta G_k(x,\cdot)e^{(j)}$ *converges also in* $H_Q(\mathrm{curl}, D)$, $j = 1,2,3$. *Here,* $e^{(j)}$ *denotes the* j*-th unit coordinate vector in* $\mathbb{R}^3$, $j = 1,2,3$, *and the partial derivative is taken either with respect to* x *or* y.

Proof: This is an immediate consequence of the uniform convergence of the Fourier series representation, see part *(i)* from Proposition 4.1. □

We close this section by citing the following theorem which will ensure the applicability of the numerical method from Chapter 5.

Theorem 4.5 *In our setting concerning the wave number* k *and the phase shift* α, *see also Assumption 3.5, there holds*

$$G_{k,\alpha}(x,y) = \frac{\cos(k|x-y|)}{4\pi|x-y|} + P(k^2|x-y|^2), \quad x \neq y, \quad |x-y| \leq \frac{\underline{L}}{2},$$

with an analytic function P. *For the number* $\underline{L}$ *recall* (1.3).

For a proof we refer to [7, Theorem 3.8].

4.2. Vector Potentials and Boundary Integral Operators

This section is devoted to solutions of Maxwell's equations which are of special form: *vector potentials* built up with an unknown *density*. Choosing those potentials as an ansatz for the solution to Problem 3.12 and making use of their properties established in this section, will enable us in the next section to pose an integral equation of Fredholm type of index zero for the unkown density. Those integral equations will be solved numerically with high order convergence in the next chapter.

4.2.1. Vector Potentials and First Properties

To define vector potentials and to derive their most important properties, we follow the lines in [34, Section 5.2] and adapt the presentation therein to the Q-periodic framework.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Q-periodic and Lipschitz-continuous and set

$$\begin{aligned} \Gamma &:= \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q \text{ and } x_3 = f(\tilde{x})\big\}, \\ \widetilde{\Gamma} &:= \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in \mathbb{R}^2 \text{ and } x_3 = f(\tilde{x})\big\}. \end{aligned} \tag{4.5}$$

Furthermore, let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. So far, the duality pairing $\langle \varphi, \psi \rangle_\Gamma$ has made only sense for $\psi \in H_Q^{-1/2}(\mathrm{Curl}, \Gamma)$. In the following presentation it will be convenient to give it a meaning also for the case that ψ is a scalar valued function, i.e., for $\psi \in H_Q^{1/2}(\Gamma)$.

Convention 4.6 *Let* Γ *be given by* (4.5). *For* $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ *and* $\psi \in H_Q^{1/2}(\Gamma)$ *we mean by* $\langle \varphi, \psi \rangle_\Gamma$ *the expression*

$$\langle \varphi, \psi \rangle_\Gamma := \sum_{j=1}^{3} \Big\langle \varphi, \gamma_{T,\Gamma}\big(e^{(j)} \tilde{\psi}\big) \Big\rangle_\Gamma e^{(j)},$$

where $\tilde{\psi} \in H_Q^1(D)$ *is any extension of* ψ *such that* $\gamma_{0,\Gamma}\tilde{\psi} = \psi$ *and where* $D \subseteq \mathbb{R}^3$ *is any cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, such that either* $\Gamma_0 = \Gamma$ *or* $\Gamma_1 = \Gamma$. *Again,* $e^{(j)}$ *denotes the* j*-th unit coordinate vector in* $\mathbb{R}^3$, $j = 1, 2, 3$. *Note that by Proposition 2.64 it doesn't matter whether we choose* D *such that* Γ_0 *coincides with* Γ *or whether we choose* D *such that* Γ_1 *coincides with* Γ. *Therefore, the right hand side of the expression above is well-defined.*
If ψ *belongs to* $H_Q^1(D)$, *then we set*

$$\langle \varphi, \psi \rangle_\Gamma := \langle \varphi, \gamma_{0,\Gamma}\psi \rangle_\Gamma.$$

Now, for a *density* $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ we introduce the function

$$w(x) := \big\langle \varphi, G_k(x, \cdot) \big\rangle_\Gamma, \qquad x \in \mathbb{R}^3 \setminus \widetilde{\Gamma}, \tag{4.6}$$

and call it *single layer vector potential.*

Proposition 4.7 *For the single layer vector potential w, see (4.6), the following assertions are true.*

(i) $w \in C_Q^\infty\big((Q \times \mathbb{R}) \setminus \Gamma, \mathbb{C}^3\big)$ *with* $\partial^\beta w(x) = \big\langle \varphi, \partial_x^\beta G_k(x, \cdot) \big\rangle_\Gamma$ *for all* $x \in (Q \times \mathbb{R}) \setminus \Gamma$ *and* $\beta \in \mathbb{N}_0^3$*.*

(ii) Let $a > \max_{\xi \in \mathbb{R}^2} f(\xi)$*,* $b < \min_{\xi \in \mathbb{R}^2} f(\xi)$ *and set* $D^+ := D_a^\infty$ *as well as* $D^- := D_{-\infty}^b$*, see also (3.5) for the notation here. Then*

$$\partial^\beta w(x) = \frac{\mathrm{i}}{2|Q|} \sum_{\mu \in \mathbb{Z}^2} \frac{1}{\rho^{(\mu)}} \Big\langle \varphi, \mathrm{e}^{-\mathrm{i}(q^{(\mu)} \cdot \tilde{\cdot} \pm \rho^{(\mu)} \cdot_3)} \Big\rangle_\Gamma \partial^\beta \mathrm{e}^{\mathrm{i}(q^{(\mu)} \cdot \tilde{x} \pm \rho^{(\mu)} x_3)}$$

for all $x \in D^\pm$ *and* $\beta \in \mathbb{N}_0^3$*. Furthermore, the series converges uniformly in* $D^\pm$*.*

Proof: *(i)* Obviously, w is Q-periodic. Let $j \in \{1, 2, 3\}$ and consider $\psi(x) := \big\langle \varphi, \gamma_{T,\Gamma}\big(e^{(j)} G_k(x, \cdot)\big) \big\rangle_\Gamma$, for $x \in (Q \times \mathbb{R}^3) \setminus \Gamma$. Due to Convention 4.6, it suffices to show that $\psi \in C_Q^\infty\big((Q \times \mathbb{R}) \setminus \Gamma\big)$ with $\partial^\beta \psi(x) = \big\langle \varphi, \gamma_{T,\Gamma}\big(e^{(j)} \partial_x^\beta G_k(x, \cdot)\big) \big\rangle_\Gamma$, for all $x \in (Q \times \mathbb{R}) \setminus \Gamma$ and $\beta \in \mathbb{N}_0^3$. For this purpose, let $x \in (Q \times \mathbb{R}) \setminus \Gamma$. Then there exists $\varepsilon > 0$ such that $\mathbb{B}_3(x, \varepsilon) \subseteq (Q \times \mathbb{R}) \setminus \Gamma$. Without loss of generality we assume that x is above from Γ, and hence also $\mathbb{B}_3(x, \varepsilon)$ is above from Γ. Let $D \subseteq \mathbb{R}^3$ be any cell set of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, such that $\Gamma_1 = \Gamma$.
Set $Z := \mathbb{B}_3(x, \varepsilon) \times D$ and consider for the moment the smooth function

$$g : Z \to \mathbb{C}, \quad z := (z', z'') \to g(z) := G_k(z', z''),$$

where Z has to be understood as a subset of $\mathbb{R}^6$. Then thanks to Taylor's formula for functions of several real variables, see for instance [5], we have

$$g(z + h) = g(z) + \partial g(z) h + \int_0^1 (1 - t) \partial^2 g(z + th)[h]^2 \, \mathrm{d}t,$$

and hence

$$\left|g(z+h)-g(z)-\partial g(z)h\right| \leq \int_0^1 (1-t)\left\|\partial^2 g(z+th)\right\||h|^2\,\mathrm{d}t,$$

for $z \in Z$ and $h \in \mathbb{R}^6$ such that $\{z+th \mid t \in [0,1]\} \subseteq Z$, and where ∂g and $\partial^2 g$ denote the first and the second derivative of g, which are in our case continuous mappings from Z to $\mathcal{L}(\mathbb{R}^6, \mathbb{C})$ and $\mathcal{L}(\mathbb{R}^6, \mathbb{R}^6; \mathbb{C})$, respectively, where the latter space denotes the set of all bilinear and bounded mappings from $\mathbb{R}^6 \times \mathbb{R}^6$ to $\mathbb{C}$. Choosing now $z \in Z$ such that $z' = x$ and $z'' \in D$, setting $y := z''$ and letting $h = h_1\hat{e}^{(1)}$, where $|h_1| < \varepsilon$ and $\hat{e}^{(1)}$ denotes the first unit coordinate vector in $\mathbb{R}^6$, we obtain from the last inequality

$$\left|G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G_k(x,y)\right| \leq C|h_1|^2,$$

where the constant $C > 0$ does not depend on $y \in D$. Analogously, the considerations we just have made yield, by replacing in the definition for g the expression $G_k(z', z'')$ by $\partial_{l,z''}G_k(z', z'')$, where l is some number in $\{1, 2, 3\}$,

$$\left|\partial_{l,y}G_k(x+h_1e^{(1)},y)-\partial_{l,y}G_k(x,y)-h_1\partial_{1,x}\partial_{l,y}G_k(x,y)\right| \leq C'|h_1|^2,$$

for $|h_1| < \varepsilon$ and where the constant $C' > 0$ does not depend on $y \in D$ and $l \in \{1, 2, 3\}$. Consequently,

$$\int_D \left|\left[G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G_k(x,y)\right]e^{(j)}\right|^2\mathrm{d}y \leq C|h_1|^4,$$

for $|h_1| < \varepsilon$. Analogously, by means of the observation

$$\begin{aligned}
&\left|\operatorname{curl}_y\left(\left[G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G_k(x,y)\right]e^{(j)}\right)\right| \\
&= \left|\nabla_y\left[G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G-k(x,y)\right]\times e^{(j)}\right| \\
&\leq \left|\nabla_y\left[G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G-k(x,y)\right]\right| \\
&\leq \sum_{l=1}^{3}\left|\partial_{l,y}G_k(x+h_1e^{(1)},y)-\partial_{l,y}G_k(x,y)-h_1\partial_{1,x}\partial_{l,y}G_k(x,y)\right| \\
&\leq C'|h_1|^2,
\end{aligned}$$

we obtain

$$\int_D \left|\operatorname{curl}_y\left(\left[G_k(x+h_1e^{(1)},y)-G_k(x,y)-h_1\partial_{1,x}G_k(x,y)\right]e^{(j)}\right)\right|^2\mathrm{d}y \\ \le C'|h_1|^4$$

for $|h_1|<\varepsilon$. After these preliminary considerations we obtain finally

$$\begin{aligned}\Big|\psi(x+h_1e^{(1)})-\psi(x)-h_1\Big\langle\varphi,\gamma_{T,\Gamma}\big(e^{(j)}\partial_{1,x}G_k(x,\cdot)\big)\Big\rangle\Big| \\ \le \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}\,\|\gamma_{T,\Gamma}\| \\ \left\|e^{(j)}\left[G_k(x+h_1e^{(1)},\cdot)-G_k(x,\cdot)-h_1\partial_{1,x}G_k(x,\cdot)\right]\right\|_{H_Q(\mathrm{curl},D)} \\ \le C''|h_1|^2,\end{aligned}$$

for $|h_1|<\varepsilon$. Hence, we have shown that $\partial_1\psi(x)$ exists and coincides with $\big\langle\varphi,\gamma_{T,\Gamma}\big(e^{(j)}\partial_{1,x}G_k(x,\cdot)\big)\big\rangle_\Gamma$, for $x\in(Q\times\mathbb{R})\setminus\Gamma$. Similarly, one shows that $\partial_1\psi:(Q\times\mathbb{R})\setminus\Gamma\to\mathbb{C}$ is continuous. Since $j\in\{1,2,3\}$ was arbitrarily chosen, we conclude that $\psi\in C^1_Q\big((Q\times\mathbb{R})\setminus\Gamma\big)$. The property $\psi\in C^\infty_Q\big((Q\times\mathbb{R})\setminus\Gamma\big)$ follows now by induction.

(ii). Fix some $h\in\mathbb{R}$ such that $\max_{\xi\in\mathbb{R}^2}f(\xi)<h<a$ and choose some cell set $D\subseteq\mathbb{R}^3$ of Lipschitz layer type, with characteristic quantities as in Assumption 2.91, such that $\Gamma_0=\Gamma$ and $\Gamma_1=\Gamma_h$, see also (3.6). Furthermore, let $\beta\in\mathbb{N}_0^3$ and $x\in D_a^\infty$. Then, thanks to the first part of this proposition and to Lemma 4.4,

$$\begin{aligned}\partial^\beta w(x) &= \sum_{j=1}^3\big\langle\varphi,\gamma_{T,\Gamma}\big(e^{(j)}\partial_x^\beta G_k(x,\cdot)\big)\big\rangle_\Gamma\,e^{(j)} \\ &= \frac{\mathrm{i}}{2|Q|}\sum_{j=1}^3\sum_{\mu\in\mathbb{Z}^2}\frac{1}{\rho^{(\mu)}}\Big\langle\varphi,\gamma_{T,\Gamma}\Big(e^{(j)}\partial_x^\beta\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot(\tilde{x}-\tilde{\cdot})+\rho^{(\mu)}(x_3-\cdot_3))}\Big)\Big\rangle_\Gamma e^{(j)} \\ &= \frac{\mathrm{i}}{2|Q|}\sum_{\mu\in\mathbb{Z}^2}\frac{1}{\rho^{(\mu)}}\Big\langle\varphi,\mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)}\Big\rangle_\Gamma\,\partial^\beta\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}x_3)},\end{aligned}$$

see also Convention 4.6. To verify uniform convergence, we observe that

$$\int_D\left|\mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{y}+\rho^{(\mu)}y_3)}e^{(j)}\right|^2\mathrm{d}y=\int_D\mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})y_3}\,\mathrm{d}y$$

$$
\begin{aligned}
&= \int_Q \int_{f(\xi)}^h \mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})y_3}\,\mathrm{d}y_3\,\mathrm{d}\xi \\
&= \frac{1}{2\mathrm{Im}(\rho^{(\mu)})} \int_Q \Big(\mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})h} \underbrace{- \mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})f(\xi)}}_{\leq 0} \Big)\,\mathrm{d}\xi \\
&\leq \frac{|Q|}{2\mathrm{Im}(\rho^{(\mu)})}\,\mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)}h}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\int_D \Big| \operatorname{curl} \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{y}+\rho^{(\mu)}y_3)} e^{(j)} \Big|^2\,\mathrm{d}y \\
&\quad = \int_D \Big| -\mathrm{i} \begin{pmatrix} q_1^{(\mu)} \\ q_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times e^{(j)} \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{y}+\rho^{(\mu)}y_3)} \Big|^2\,\mathrm{d}y \\
&\quad \leq \big(|q^{(\mu)}|^2 + |\rho^{(\mu)}|^2\big) \int_D \mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})y_3}\,\mathrm{d}y \\
&\quad \leq \frac{|Q|}{2\mathrm{Im}(\rho^{(\mu)})} \big(|q^{(\mu)}|^2 + |\rho^{(\mu)}|^2\big)\,\mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)}h},
\end{aligned}
$$

and therefore

$$
\Big\| \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3}\, e^{(j)} \Big\|^2_{H_Q(\mathrm{curl},D)} \leq |Q|\,\frac{1+|q^{(\mu)}|^2+|\rho^{(\mu)}|^2}{2\mathrm{Im}(\rho^{(\mu)})}\,\mathrm{e}^{2\mathrm{Im}(\rho^{(\mu)})h}.
$$

Hence, by Lemma 2.9 and Lemma 3.8,

$$
\begin{aligned}
&\Big| \frac{1}{\rho^{(\mu)}} \Big\langle \varphi, \gamma_{T,\Gamma}\big(\mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)} e^{(j)}\big) \Big\rangle_\Gamma \Big| \\
&\quad \leq \frac{1}{|\rho^{(\mu)}|} \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)} \|\gamma_{T,\Gamma}\| \Big\| \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)} e^{(j)} \Big\|_{H_Q(\mathrm{curl},D)} \\
&\quad \leq C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)} \frac{1}{\sqrt{1+|\mu|^2}} \sqrt{\frac{1+|\mu|^2}{\sqrt{1+|\mu|^2}}}\,\mathrm{e}^{\mathrm{Im}(\rho^{(\mu)})h} \\
&\quad = C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)} \frac{1}{\big(1+|\mu|^2\big)^{1/4}}\,\mathrm{e}^{\mathrm{Im}(\rho^{(\mu)})h}.
\end{aligned}
$$

Moreover, we have

$$
\Big| \partial^\beta \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}x_3)} \Big| = \Big| \mathrm{i}^{|\beta|} \big(q_1^{(\mu)}\big)^{\beta_1} \big(q_2^{(\mu)}\big)^{\beta_2} \big(\rho^{(\mu)}\big)^{\beta_3} \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}x_3)} \Big|
$$

$$\leq C\big(\sqrt{1+|\mu|^2}\big)^{|\beta|}\mathrm{e}^{-\mathrm{Im}(\rho(\mu))x_3}.$$

Using all these observations, we obtain

$$\begin{aligned}
&\Big|\frac{1}{\rho^{(\mu)}}\Big\langle\varphi,\gamma_{T,\Gamma}\big(\mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)}e^{(j)}\big)\Big\rangle_\Gamma\partial^\beta\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}x_3)}\,e^{(j)}\Big| \\
&\qquad\leq C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}\frac{1}{\big(1+|\mu|^2\big)^{1/4}}\big(1+|\mu|^2\big)^{|\beta|/2}\,\mathrm{e}^{-\mathrm{Im}(\rho^{(\mu)})(x_3-h)} \\
&\qquad\leq C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}\big(1+|\mu|^2\big)^{\frac{2|\beta|-1}{4}}\,\mathrm{e}^{-\tilde{C}\sqrt{1+|\mu|^2}(a-h)} \\
&\qquad= C\,\frac{\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}}{\big(\tilde{C}(a-h)\big)^{\frac{2|\beta|+7}{2}}}\underbrace{\frac{\big(\tilde{C}\sqrt{1+|\mu|^2}(a-h)\big)^{\frac{2|\beta|+7}{2}}}{\mathrm{e}^{\tilde{C}\sqrt{1+|\mu|^2}(a-h)}}}_{\text{bounded}}\frac{1}{(1+|\mu|^2)^2} \\
&\qquad\leq C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}\frac{1}{(1+|\mu|^2)^2}.
\end{aligned}$$

Since $x\in D_a^\infty$ was arbitrarily chosen and since $\sum_{\mu\in\mathbb{Z}^2}\frac{1}{(1+|\mu|^2)^2}<\infty$, we conclude from the last estimate that the series representation for $\partial^\beta w$ converges uniformly in D_a^∞.
And finally, the assertion for $D_{-\infty}^b$ is shown analogoulsy. □

Taking again some density $\varphi\in H_Q^{-1/2}(\mathrm{Div},\Gamma)$, we define now, by means of the single layer vector potential w, the functions

$$u(x):=\mathrm{curl}_\alpha\,\mathrm{curl}_\alpha\big\langle\varphi,G_k(x,\cdot)\big\rangle_\Gamma=\mathrm{curl}_\alpha^2\,w(x),\quad x\in\mathbb{R}^3\setminus\widetilde{\Gamma},\tag{4.7a}$$

$$v(x):=\mathrm{curl}_\alpha\big\langle\varphi,G_k(x,\cdot)\big\rangle_\Gamma=\mathrm{curl}_\alpha\,w(x),\qquad x\in\mathbb{R}^3\setminus\widetilde{\Gamma},\tag{4.7b}$$

and call them *electric* and *magnetic vector potential*, respectively.

Proposition 4.8 *The electric and magnetic vector potential u and v, see* (4.7)*, belong to $C_Q^\infty\big((Q\times\mathbb{R})\setminus\Gamma,\mathbb{C}^3\big)$ and satisfy the equation*

$$\mathrm{curl}_\alpha\,\mathrm{curl}_\alpha\,w-k^2w=0\qquad\text{in }(Q\times\mathbb{R})\setminus\Gamma$$

as well as the (URC) in D_a^∞ for any $a>\max_{\xi\in\mathbb{R}^2}f(\xi)$ and the (DRC) on $D_{-\infty}^a$ for any $a<\min_{\xi\in\mathbb{R}^2}f(\xi)$.

Proof: The first assertion is an immediate consequence of Proposition 4.7. For the second assertion, we observe that $v(x) = \operatorname{curl}_\alpha w(x)$, for $x \in \mathbb{R}^3 \setminus \Gamma$, where w denotes here the single layer vector potential from (4.6). Using now (A.4c), (A.4a) and (4.4), we obtain for $x \in (Q \times \mathbb{R}) \setminus \Gamma$, that

$$\begin{aligned}\operatorname{curl}_\alpha^2 v(x) &= \operatorname{curl}_\alpha \operatorname{curl}_\alpha^2 w(x) = \underbrace{\operatorname{curl}_\alpha \nabla_\alpha}_{=0} \operatorname{div}_\alpha w(x) - \operatorname{curl}_\alpha \Delta_\alpha w(x)\\ &= -\operatorname{curl}_\alpha(-k^2 w(x)) = k^2 v(x).\end{aligned}$$

The equation for u is shown analogously.
For the last assertions regarding the radiation condition, take at first some $a > \max_{\xi \in \mathbb{R}^2} f(\xi)$. Note that

$$\operatorname{curl}_\alpha(\psi\chi) = \nabla\psi \times \chi + \psi \operatorname{curl}\chi + \mathrm{i}\alpha \times (\psi\chi) = (\nabla_\alpha \psi) \times \chi + \psi \operatorname{curl}\chi$$

for smooth enough $\chi : D_a^\infty \to \mathbb{C}^3$ and $\psi : D_a^\infty \to \mathbb{C}$. Then, by Proposition 4.7, we obtain for $x \in D_a^\infty$

$$\begin{aligned}v(x) &= \operatorname{curl}_\alpha \left\langle \varphi, G_k(x,\cdot)\right\rangle_\Gamma\\ &= \sum_{\mu \in \mathbb{Z}^2} \frac{\mathrm{i}^2}{2|Q|} \frac{1}{\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \left\langle \varphi, \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)}\right\rangle_\Gamma \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}x_3)}\\ &= \sum_{\mu \in \mathbb{Z}^2} v^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{x}+\rho^{(\mu)}(x_3-a))},\end{aligned}$$

where $d^{(\mu)}$ was defined in (3.10),

$$v^{(\mu)} := -\frac{\mathrm{e}^{\mathrm{i}\rho^{(\mu)}a}}{2|Q|\rho^{(\mu)}} \begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \left\langle \varphi, \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)}\right\rangle_\Gamma, \qquad \mu \in \mathbb{Z}^2,$$

and where convergence is uniform in D_a^∞, and therefore in particular in D_b^∞ for all $b > a$. Applying the operator $\operatorname{curl}_\alpha$ now to this representation of v on D_a^∞, we obtain, again thanks to Proposition 4.7, $u = \sum_{\mu \in \mathbb{Z}^2} u^{(\mu)}\, \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}(\cdot_3-a))}$, where

$$u^{(\mu)} := -\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\rho^{(\mu)}a}}{2|Q|\rho^{(\mu)}} \left[\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \left(\begin{pmatrix} d_1^{(\mu)} \\ d_2^{(\mu)} \\ \rho^{(\mu)} \end{pmatrix} \times \left\langle \varphi, \mathrm{e}^{-\mathrm{i}(q^{(\mu)}\cdot\tilde{\cdot}+\rho^{(\mu)}\cdot_3)}\right\rangle_\Gamma\right)\right],$$

for $\mu \in \mathbb{Z}^2$, and where convergence is uniform as above.
And finally, the assertion regarding the (DRC) is shown in a very similar way, again thanks to Proposition 4.7. □

4.2.2. A Special Transmission Problem

Further properties of the vector potentials, like boundedness and the important jump relations, we will derive as in [34] by means of a special transmission problem. As we will see in a moment, its unique solution is connected to the vector potentials by the following version of the well-known Stratton-Chu formula. To prove this version, we need the result given by the next lemma.

Lemma 4.9 *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. For $u \in H_Q(\operatorname{curl}, D)$ and $\psi \in H^1_Q(D)$ we have*

$$\sum_{j=0}^{1} \langle \gamma_{t,\Gamma_j} u, \gamma_{0,\Gamma_j} \psi \rangle_{\Gamma_j} = \int_D \big(\psi \operatorname{curl} u + \nabla\psi \times u\big)\,\mathrm{d}x.$$

Proof: Due to Convention 4.6, we obtain, by means of Green's formula (2.18b),

$$\begin{aligned}
\sum_{j=0}^{1} \langle \gamma_{t,\Gamma_j} u, \gamma_{0,\Gamma_j} \psi \rangle_{\Gamma_j} &= \sum_{j=0}^{1} \sum_{n=1}^{3} \Big\langle \gamma_{t,\Gamma_j} u, \gamma_{T,\Gamma_j} \big(e^{(n)} \psi\big) \Big\rangle_{\Gamma_j} e^{(n)} \\
&= \sum_{n=1}^{3} \Big(\int_D \big(\operatorname{curl} u \cdot e^{(n)} \psi - u \cdot \underbrace{\operatorname{curl}(e^{(n)} \psi)}_{= \nabla\psi \times e^{(n)}} \big)\,\mathrm{d}x \Big) e^{(n)} \\
&= \sum_{n=1}^{3} \Big(\int_D \big(\psi \operatorname{curl} u + \nabla\psi \times u\big) \cdot e^{(n)}\,\mathrm{d}x \Big) e^{(n)} \\
&= \int_D \big(\psi \operatorname{curl} u + \nabla\psi \times u\big)\,\mathrm{d}x,
\end{aligned}$$

which is the desired result. □

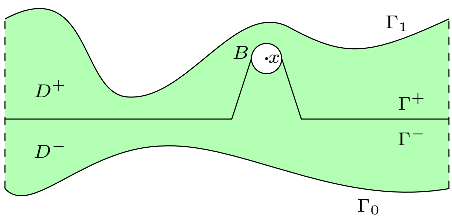
Theorem 4.10 (Stratton-Chu Formula) *Let $D \subseteq \mathbb{R}^3$ be a cell set of Lipschitz layer type with characteristic quantities as in Assumption 2.91. Furthermore, let G_k denote the Q-periodic Green's function. For any weak solution $u \in H_Q(\mathrm{curl}, D)$ of $\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u = 0$, i.e., for $u \in H_Q(\mathrm{curl}, D)$ satisfying*

$$\forall v \in H_{Q,0}(\mathrm{curl}, D) : \int_D \left(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \right) \mathrm{d}x = 0,$$

we have the representation

$$- \sum_{j=0}^{1} \mathrm{curl}_\alpha \left\langle \gamma_{t,\Gamma_j} u, G_k(x,\cdot) \right\rangle_{\Gamma_j} - \frac{1}{k^2} \sum_{j=0}^{1} \mathrm{curl}_\alpha^2 \left\langle \gamma_{t,\Gamma_j} \mathrm{curl}_\alpha u, G_k(x,\cdot) \right\rangle_{\Gamma_j}$$
$$= \begin{cases} u(x), & x \in D, \\ 0, & x \notin \overline{D}. \end{cases}$$

Proof: We follow the lines in the proof of [34, Theorem 5.49] and note at first that u is a smooth solution of $\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u = 0$, see also the first paragraph in the proof of [42, Theorem 9.2]. Second, we choose $x \in D$ and $r > 0$ such that for $B := \mathbb{B}_3(x, r)$ there holds $\overline{B} \subseteq D$. Next, we divide D into two cell sets of Lipschitz layer type D^+ and D^-. The cell set D^+ is bounded by the surfaces Γ^+ and Γ_1, while the borders for D^- are Γ_0 and Γ^-. Γ^- and Γ^+ share the same surface, except for ∂B. Here, Γ^+ consists of the upper part of ∂B and Γ^- of the lower part, respectively. This situation is visualized by the picture below, which can similarly be drawn also for cell sets of Lipschitz layer type with $\max_{\xi \in \mathbb{R}^2} f_0(\xi) > \min_{\xi \in \mathbb{R}^2} f_1(\xi)$.



Due to Lemma 4.9 we obtain now

$$\left\langle \gamma_{t,\Gamma_0} u, G_k(x,\cdot) \right\rangle_{\Gamma_0} + \int_{\Gamma^-} (\mathrm{n}^- \times u)\, G_k(x,\cdot)\, \mathrm{d}s$$

$$= \int_{D^-} \big(G_k(x,\cdot)\operatorname{curl} u + \nabla_y G_k(x,\cdot) \times u\big)\,\mathrm{d}y,$$

$$\big\langle \gamma_{t,\Gamma_1} u, G_k(x,\cdot)\big\rangle_{\Gamma_1} + \int_{\Gamma^+} (\mathrm{n}^+ \times u)\, G_k(x,\cdot)\,\mathrm{d}s = \int_{D^+} \big(G_k(x,\cdot)\operatorname{curl} u + \nabla_y G_k(x,\cdot) \times u\big)\,\mathrm{d}y,$$

where n^- and n^+ denote the normal unit vectors on Γ^- and Γ^+, pointing outward of D^- and D^+, respectively. Adding up both equations yields

$$\sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} u, G_k(x,\cdot)\big\rangle_{\Gamma_j} - \int_{\partial B} (\mathrm{n} \times u)\, G_k(x,\cdot)\,\mathrm{d}s = \int_{D\setminus B} \big(G_k(x,\cdot)\operatorname{curl} u + \nabla_y G_k(x,\cdot) \times u\big)\,\mathrm{d}y, \qquad (*_1)$$

where n denotes the normal unit vector on ∂B pointing outward of B. Analogously we obtain

$$\sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} \operatorname{curl}_\alpha u, G_k(x,\cdot)\big\rangle_{\Gamma_j} - \int_{\partial B} (\mathrm{n} \times \operatorname{curl}_\alpha u)\, G_k(x,\cdot)\,\mathrm{d}s = \int_{D\setminus B} \big(G_k(x,\cdot)\operatorname{curl}\operatorname{curl}_\alpha u + \nabla_y G_k(x,\cdot) \times \operatorname{curl}_\alpha u\big)\,\mathrm{d}y. \qquad (*_2)$$

For the next calculations we need the observation that for $v \in H_Q(\operatorname{curl}, D)$ and $\psi \in H^1_Q(D)$ there holds

$$\psi \operatorname{curl} v + \nabla\psi \times v = \psi \operatorname{curl}_\alpha v + \nabla_{-\alpha}\psi \times v,$$

what can easily be verified. Moreover, we note that $\big(\nabla_{-\alpha,y} G_k(x,\cdot)\big) \times u = -\operatorname{curl}_{\alpha,x}\big(G_k(x,\cdot)\,u\big)$. Now, we take a closer look at the right hand side of the equations $(*_1)$ and $(*_2)$ and calculate

$$\begin{aligned}
&\operatorname{curl}_\alpha \int_{D\setminus B} \Big(G_k(x,\cdot)\operatorname{curl} u + \nabla_y G_k(x,\cdot) \times u\Big)\,\mathrm{d}y \\
&= \operatorname{curl}_\alpha \int_{D\setminus B} \Big(G_k(x,\cdot)\operatorname{curl}_\alpha u + \nabla_{-\alpha,y} G_k(x,\cdot) \times u\Big)\,\mathrm{d}y \\
&= \int_{D\setminus B} \Big(\operatorname{curl}_{\alpha,x}\big(G_k(x,\cdot)\operatorname{curl}_\alpha u\big) - \operatorname{curl}^2_{\alpha,x}\big(G_k(x,\cdot)u\big)\Big)\,\mathrm{d}y
\end{aligned}$$

as well as, by means of $\operatorname{curl}_\alpha \operatorname{curl}_\alpha = \nabla_\alpha \operatorname{div}_\alpha - \Delta_\alpha$ and $\operatorname{curl}_\alpha \nabla_\alpha = 0$, see (A.4), and $\Delta_{\alpha,x} G_k(x,\cdot) = -k^2 G_k(x,\cdot)$, see (4.4),

$$\begin{aligned}
&\frac{1}{k^2}\operatorname{curl}_\alpha^2 \int_{D\setminus B} \Big(G_k(x,\cdot)\operatorname{curl}\operatorname{curl}_\alpha u + \nabla_y G_k(x,\cdot) \times \operatorname{curl}_\alpha u \Big)\,\mathrm{d}y \\
&\quad = \frac{1}{k^2}\operatorname{curl}_\alpha^2 \int_{D\setminus B} \Big(G_k(x,\cdot)\operatorname{curl}_\alpha\operatorname{curl}_\alpha u + \nabla_{-\alpha,y} G_k(x,\cdot) \times \operatorname{curl}_\alpha u \Big)\,\mathrm{d}y \\
&\quad = \frac{1}{k^2}\operatorname{curl}_\alpha^2 \int_{D\setminus B} \Big(k^2 G_k(x,\cdot)\, u - \operatorname{curl}_{\alpha,x}\big(G_k(x,\cdot)\operatorname{curl}_\alpha u \big) \Big)\,\mathrm{d}y \\
&\quad = \int_{D\setminus B} \Big(\operatorname{curl}_{\alpha,x}^2 \big(G_k(x,\cdot) u \big) - \operatorname{curl}_{\alpha,x}\big(G_k(x,\cdot)\operatorname{curl}_\alpha u \big) \Big)\,\mathrm{d}y.
\end{aligned}$$

Hence, adding up both equations yields zero. Using this result, we obtain from $(*_1)$ and $(*_2)$ that

$$\begin{aligned}
&\operatorname{curl}_\alpha \sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j} u, G_k(x,\cdot) \big\rangle_{\Gamma_j} + \frac{1}{k^2}\operatorname{curl}_\alpha^2 \sum_{j=0}^{1} \big\langle \gamma_{t,\Gamma_j}\operatorname{curl}_\alpha u, G_k(x,\cdot) \big\rangle_{\Gamma_j} \\
&\quad = \operatorname{curl}_\alpha \int_{\partial B} (\mathrm{n} \times u)\, G_k(x,\cdot)\,\mathrm{d}s + \frac{1}{k^2}\operatorname{curl}_\alpha^2 \int_{\partial B} (\mathrm{n} \times \operatorname{curl}_\alpha u)\, G_k(x,\cdot)\,\mathrm{d}s \\
&\quad = -u(x)
\end{aligned}$$

by the classical Stratton-Chu formula, see for instance [34, Theorem 3.27], whose proof justifies also its application for the slightly modified equation here.
Finally, the case $x \notin D$ is handled in the same way by applying Lemma 4.9 in all of D. □

Now we specify the transmission problem which was mentioned above. First of all, we fix some $L_3 > 0$ such that $f(\xi) \in (-L_3, L_3)$ for all $\xi \in Q$, and set $Q_3 := Q \times (-L_3, L_3)$. Here, f denotes the function describing Γ, see also (4.5). Furthermore, we fix the direction of the unit normal vector on a Lipschitz surface of the form (4.5) to point upwards and introduce the cell sets of Lipschitz layer type Q_3^- and Q_3^+ with corresponding boundaries Γ^- and Γ as well as Γ and Γ^+, respectively, such that $Q_3^- \cup \Gamma \cup Q_3^+ = Q_3$, see the picture below. Recalling Proposition 2.119, we have for $u \in H_Q(\operatorname{curl}, Q_3)$ that $\gamma_{t,\Gamma} u|_{Q_3^-} = -\gamma_{t,\Gamma} u|_{Q_3^+}$. To simplify

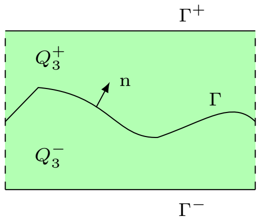


Figure 4.1.: The geometrical setting for the special transmission problem.

notation, in the following presentation we will sometimes use the symbols for the classical traces $\mathrm{n} \times u|_\pm$ instead of $\gamma_{t,\Gamma} u|_{Q_3^\pm}$, i.e., we define

$$\mathrm{n} \times u|_- := \gamma_{t,\Gamma} u|_{Q_3^-} \quad \text{and} \quad \mathrm{n} \times u|_+ := -\gamma_{t,\Gamma} u|_{Q_3^+} \qquad \text{on } \Gamma$$

and analogously

$$\mathrm{n} \times u|_{\Gamma^-} := -\gamma_{t,\Gamma^-} u \quad \text{on } \Gamma^- \quad \text{and} \quad \mathrm{n} \times u|_{\Gamma^+} := \gamma_{t,\Gamma^+} u \quad \text{on } \Gamma^+,$$

where the minus sign appears by definition of the trace operator γ_{t,Γ_j}, which is applied to functions from $H_Q(\mathrm{curl}, D)$ with surfaces Γ_j, $j = 0, 1$, and normal unit vectors pointing *outside* of the cell set $D \subseteq \mathbb{R}^3$ of Lipschitz layer type, recall also Theorem 2.107. For the next results, see also [34, Theorem 5.51].

To write in the following presentation the formulas more compact, the symbol $\sum_\pm a^\pm$, where a^- and a^+ are some (summable) mathematical objects, has to be understood as $\sum_\pm a^\pm := a^- + a^-$.

Problem 4.11 *Let $\eta \in \mathbb{C}$ with $\mathrm{Im}(\eta\overline{k}) > 0$ and $K^\pm : H_Q^{-1/2}(\mathrm{Div}, \Gamma^\pm) \to H_Q^{-1/2}(\mathrm{Curl}, \Gamma^\pm)$ be a linear and compact operator such that $\langle \overline{\psi}, K^\pm \psi \rangle_{\Gamma^\pm}$ are real valued and $\langle \overline{\psi}, K^\pm \psi \rangle_{\Gamma^\pm} > 0$ for all $\psi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma) \setminus \{0\}$. For given $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$, find $u \in H_Q(\mathrm{curl}, Q_3)$ such that*

$$\forall v \in H_{Q,0}(\mathrm{curl}, Q_3^\pm) : \int_{Q_3^\pm} \left(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \right) \mathrm{d}x = 0,$$

$$\mathrm{n} \times u|_- = \mathrm{n} \times u|_+, \quad \mathrm{n} \times \mathrm{curl}_\alpha u|_+ - \mathrm{n} \times \mathrm{curl}_\alpha u|_- = \varphi \qquad \text{on } \Gamma$$
$$\mathrm{n} \times \mathrm{curl}_\alpha u + \eta\, \mathrm{n} \times K^-(\mathrm{n} \times u) = 0 \qquad \text{on } \Gamma^-,$$
$$\mathrm{n} \times \mathrm{curl}_\alpha u - \eta\, \mathrm{n} \times K^+(\mathrm{n} \times u) = 0 \qquad \text{on } \Gamma^+.$$

In variational form this reads as: for given $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$*, find* $u \in H_Q(\mathrm{curl}, Q_3)$ *such that*

$$\int_{Q_3} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x - \eta \sum\nolimits_{\pm} \big\langle \gamma_{t,\Gamma^\pm} \overline{v}, K^\pm(\gamma_{t,\Gamma^\pm} u) \big\rangle_{\Gamma^\pm} = \langle \varphi, \gamma_{T,\Gamma} \overline{v} \rangle_\Gamma \tag{4.8}$$

for all $v \in H_Q(\mathrm{curl}, Q_3)$*.*

To verify (4.8), we observe, similarly to Proposition 3.13, that the first equation in Problem 4.11 implies that $\mathrm{curl}_\alpha u$ belongs to $H_Q(\mathrm{curl}, Q_3^\pm)$ with $\mathrm{curl}_\alpha \mathrm{curl}_\alpha u = k^2 u$ holding in $L^2(Q_3^\pm, \mathbb{C}^3)$. Therefore, an application of Green's formula (2.18b) yields for $v \in H_Q(\mathrm{curl}, Q_3^-)$

$$\begin{aligned} &\int_{Q_3^-} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x \\ &= \int_{Q_3^-} \big(\mathrm{curl}\,\mathrm{curl}_\alpha u \cdot \overline{v} - \mathrm{curl}_\alpha u \cdot (\mathrm{i}\alpha \times \overline{v}) - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x \\ &\qquad - \langle \gamma_{t,\Gamma^-} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma^-} \overline{v} \rangle_{\Gamma^-} - \langle \gamma_{t,\Gamma} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma} \overline{v} \rangle_\Gamma \\ &= \int_{Q_3^-} \big(\underbrace{\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u}_{=0} \big) \cdot \overline{v}\, \mathrm{d}x \\ &\qquad - \langle \gamma_{t,\Gamma^-} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma^-} \overline{v} \rangle_{\Gamma^-} - \langle \gamma_{t,\Gamma} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma} \overline{v} \rangle_\Gamma \end{aligned}$$

and similarly for $v \in H_Q(\mathrm{curl}, Q_3^+)$

$$\begin{aligned} &\int_{Q_3^+} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x \\ &\qquad = -\langle \gamma_{t,\Gamma} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma} \overline{v} \rangle_\Gamma - \langle \gamma_{t,\Gamma^+} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma^+} \overline{v} \rangle_{\Gamma^+}. \end{aligned}$$

Adding the last two equations and incorporating the boundary conditions yields finally for $v \in H_Q(\mathrm{curl}, Q_3)$

$$\int_{Q_3} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v} \big)\, \mathrm{d}x$$

$$
\begin{aligned}
&= \langle -\gamma_{t,\Gamma} \operatorname{curl}_\alpha u|_{Q_3^-} - \gamma_{t,\Gamma} \operatorname{curl}_\alpha u|_{Q_3^+}, \gamma_{T,\Gamma}\overline{v}\rangle_\Gamma \\
&\qquad - \langle \gamma_{t,\Gamma^-} \operatorname{curl}_\alpha u, \gamma_{T,\Gamma^-}\overline{v}\rangle_{\Gamma^-} - \langle \gamma_{t,\Gamma^+} \operatorname{curl}_\alpha u, \gamma_{T,\Gamma^+}\overline{v}\rangle_{\Gamma^+} \\
&= \langle -\mathrm{n} \times \operatorname{curl}_\alpha u|_- + \mathrm{n} \times \operatorname{curl}_\alpha u|_+, \gamma_{T,\Gamma}\overline{v}\rangle_\Gamma \\
&\qquad + \langle \mathrm{n} \times \operatorname{curl}_\alpha u, \gamma_{T,\Gamma^-}\overline{v}\rangle_{\Gamma^-} - \langle \mathrm{n} \times \operatorname{curl}_\alpha u, \gamma_{T,\Gamma^+}\overline{v}\rangle_{\Gamma^+} \\
&= \langle \varphi, \gamma_{T,\Gamma}\overline{v}\rangle_\Gamma - \eta \sum\nolimits_\pm \langle \mathrm{n} \times K^\pm(\mathrm{n} \times u), \gamma_{T,\Gamma^\pm}\overline{v}\rangle_{\Gamma^\pm} \\
&= \langle \varphi, \gamma_{T,\Gamma}\overline{v}\rangle_\Gamma - \eta \sum\nolimits_\pm \langle \gamma_{t,\Gamma^\pm}(K^\pm(\gamma_{t,\Gamma^\pm} u)), \gamma_{T,\Gamma^\pm}\overline{v}\rangle_{\Gamma^\pm} \\
&= \langle \varphi, \gamma_{T,\Gamma}\overline{v}\rangle_\Gamma + \eta \sum\nolimits_\pm \langle \gamma_{t,\Gamma^\pm}\overline{v}, K^\pm(\gamma_{t,\Gamma^\pm} u)\rangle_{\Gamma^\pm},
\end{aligned}
$$

as asserted.

Our next goal is to show that Problem 4.11 is uniquely solvable. For this purpose the *modified* Helmholtz decompositions

$$H_Q(\operatorname{curl}, Q_3) = H_Q(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3) \oplus \frac{1}{k}\nabla_\alpha H^1_{Q,0}(Q_3) \tag{4.9a}$$

$$= H_Q(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3) \oplus \frac{1}{\overline{k}}\nabla_\alpha H^1_{Q,0}(Q_3), \tag{4.9b}$$

see also Theorem 2.85 and Definition 2.69, will be applied. The following two lemmas have preliminary character.

Lemma 4.12 *The variational formulation of Problem 4.11 is equivalent to: for given* $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ *find* $(u_0, p) \in H_Q(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3) \times H^1_{Q,0}(Q_3)$ *such that*

$$
\begin{aligned}
&\int_{Q_3} \big(\operatorname{curl}_\alpha u_0 \cdot \overline{\operatorname{curl}_\alpha v_0} - k^2 u_0 \cdot \overline{v_0} + \nabla_\alpha p \cdot \overline{\nabla_\alpha q}\big)\, \mathrm{d}x \\
&\qquad - \eta \sum\nolimits_\pm \Big\langle \gamma_{t,\Gamma^\pm}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q}), K^\pm\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}p)\big)\Big\rangle_{\Gamma^\pm} \\
&\quad = \Big\langle \varphi, \gamma_{t,\Gamma}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q})\Big\rangle_\Gamma
\end{aligned} \tag{4.10}
$$

for all $(v_0, q) \in H_Q(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3) \times H^1_{Q,0}(Q_3)$.

Proof: Let $u \in H_Q(\operatorname{curl}, Q_3)$ be a solution to Problem 4.11. We write $u = u_0 + \frac{1}{k}\nabla_\alpha p$ according to (4.9a). Note that $(u_0, p) \in H_Q(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3) \times$

$H^1_{Q,0}(Q_3) =: X$. Let $(v_0, q) \in X$ and set $v := v_0 - \frac{1}{k}\nabla_\alpha q$. Substituting u and v into (4.8), exploiting (A.4a) and Definition 2.69 yields then (4.10). Conversely, let $(u_0, p) \in X$ be a solution to (4.10). Set $u := u_0 + \frac{1}{k}\nabla_\alpha p$. Furthermore, let $v \in H_Q(\mathrm{curl}, Q_3)$ and write $v = v_0 - \frac{1}{k}\nabla_\alpha q$ according to (4.9b). Then $(v_0, q) \in X$ and substituting (u_0, p) and (v_0, q) into (4.10) yields then (4.8). □

Lemma 4.13 *Recalling* (3.6) *and Convention 2.125, let $a \in \mathbb{R}$ and consider $K : H_Q^{-1/2}(\mathrm{Div}, \Gamma_a) \to H_Q^{-1/2}(\mathrm{Curl}, \Gamma_a)$, given by*

$$\varphi = \sum_{\mu\in\mathbb{Z}^2} \varphi^{(\mu)} \jmath\big(T_Q^{(\mu)}\big) \;\to\; K\varphi := \sum_{\mu\in\mathbb{Z}^2} \frac{1}{1+|\mu|^2}\, \varphi^{(\mu)} \jmath\big(T_Q^{(\mu)}\big).$$

Then K is well-defined, linear and compact. Moreover

$$\langle\overline{\varphi}, K\varphi\rangle_{\Gamma_a} \in \mathbb{R}, \qquad \varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_a),$$
$$\langle\overline{\varphi}, K\varphi\rangle_{\Gamma_a} > 0, \qquad \varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_a) \setminus \{0\}.$$

Proof: For $n \in \mathbb{N}$ we consider the operators $K_n : H_Q^{-1/2}(\mathrm{Div}, \Gamma_a) \to H_Q^{-1/2}(\mathrm{Curl}, \Gamma_a)$, given by $K_n\varphi := \sum_{|\mu|\le n} \frac{1}{1+|\mu|^2}\varphi^{(\mu)}\jmath\big(T_Q^{(\mu)}\big)$, and note that they are linear and compact. Recalling Remark 2.11, we have for $\mu \neq 0$ that $|\varphi^{(\mu)}|^2 = |\hat{q}^{(\mu)} \cdot \varphi^{(\mu)}|^2 + |\hat{q}^{(\mu)} \times \varphi^{(\mu)}|^2$, where $\hat{q}^{(\mu)} := \frac{1}{|q^{(\mu)}|} q^{(\mu)}$. Hence,

$$|q^{(\mu)} \times \varphi^{(\mu)}|^2 = |q^{(\mu)}|^2 |\hat{q}^{(\mu)} \times \varphi^{(\mu)}|^2 = |q^{(\mu)}|^2|\varphi^{(\mu)}|^2 - |q^{(\mu)} \cdot \varphi^{(\mu)}|^2, \quad \mu \neq 0.$$

Using this observation and Lemma 2.9, we obtain for $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_a)$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\big\|(K_n - K)\,\varphi\big\|^2_{H_Q^{-1/2}(\mathrm{Curl},\Gamma_a)} \\
&\quad= \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \left(\frac{|\varphi^{(\mu)}|^2}{(1+|\mu|^2)^2} + \frac{|q^{(\mu)} \times \varphi^{(\mu)}|^2}{(1+|\mu|^2)^2}\right) \\
&\quad= \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \left(\frac{1+|q^{(\mu)}|^2}{(1+|\mu|^2)^2}\,|\varphi^{(\mu)}|^2 - \frac{|q^{(\mu)} \cdot \varphi^{(\mu)}|^2}{(1+|\mu|^2)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \frac{1+|\mu|^2}{(1+|\mu|^2)^2} |\varphi^{(\mu)}|^2 \\
&\leq C \frac{1}{n^2} \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} |\varphi^{(\mu)}|^2 \\
&\leq C \frac{1}{n^2} \sum_{|\mu|>n} (1+|\mu|^2)^{-1/2} \Big(|\varphi^{(\mu)}|^2 + |q^{(\mu)} \cdot \varphi^{(\mu)}|^2 \Big) \\
&\leq C \frac{1}{n^2} \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma_a)}.
\end{aligned}
$$

From this we conclude that the operator K is well-defined and compact. Finally, let $\varphi \in H_Q^{-1/2}(\mathrm{Div},\Gamma_a)$. Then, by means of Remark 2.36,

$$
\begin{aligned}
\langle \overline{\varphi}, K\varphi \rangle_{\Gamma_a} &= \sum_{\mu,\nu\in\mathbb{Z}^2} \Big\langle \overline{\varphi^{(-\mu)}} \jmath\big(T_Q^{(\mu)}\big), \frac{1}{1+|\nu|^2} \varphi^{(\nu)} \jmath\big(T_Q^{(\nu)}\big) \Big\rangle_{\Gamma_a} \\
&= \sum_{\mu,\nu\in\mathbb{Z}^2} \frac{1}{1+|\nu|^2} \overline{\varphi^{(-\mu)}} \cdot \varphi^{(\nu)} \big(T_Q^{(\mu)} \mid T_Q^{(-\nu)} \big)_{L^2(Q)} \\
&= \sum_{\nu\in\mathbb{Z}^2} \frac{1}{1+|\nu|^2} |\varphi^{(\nu)}|^2 \geq 0,
\end{aligned}
$$

and the proof is complete. □

Theorem 4.14 *Problem 4.11 is uniquely solvable. The solution operator*

$$
H_Q^{-1/2}(\mathrm{Div},\Gamma) \ni \varphi \;\mapsto\; u \in H_Q(\mathrm{curl}, Q_3)
$$

is bounded; here, $u \in H_Q(\mathrm{curl}, Q_3)$ *denotes the unique solution to Problem 4.11. Furthermore, the operator*

$$
H_Q^{-1/2}(\mathrm{Div},\Gamma) \ni \varphi \;\mapsto\; \gamma_{t,\Gamma} u|_{Q_3^-} \in H_Q^{-1/2}(\mathrm{Div},\Gamma)
$$

is an isomorphism.

Proof: First of all we note that thanks to Lemma 4.13 the operators $K^{\pm}$ from Problem 4.11 indeed exists. Moreover, due to Lemma 4.12,

Problem 4.11 is equivalent to (4.10). Now, we divide the proof into several steps.
(i). We define the space $X := H_Q(\mathrm{curl}, \mathrm{div}_\alpha 0, Q_3) \times H^1_{Q,0}(Q_3)$ and equip it with the inner product

$$((u_0, p) \,|\, (v_0, q))_X := (u_0 \,|\, v_0)_{L^2(Q_3,\mathbb{C}^3)} + (\mathrm{curl}_\alpha u_0 \,|\, \mathrm{curl}_\alpha v_0)_{L^2(Q_3,\mathbb{C}^3)} \\ + (p \,|\, q)_{L^2(Q_3)} + (\nabla_\alpha p \,|\, \nabla_\alpha q)_{L^2(Q_3,\mathbb{C}^3)} \,,$$

for $(u_0, p), (v_0, q) \in X$. Then, (4.10) can be rewritten to: for given $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ find $(u_0, p) \in X$ such that

$$((u_0, p) \,|\, (v_0, q))_X - (1 + k^2)\,(u_0 \,|\, v_0)_{L^2(Q_3,\mathbb{C}^3)} - (p \,|\, q)_{L^2(Q_3)} \\ - \eta \sum\nolimits_\pm \Big\langle \gamma_{t,\Gamma^\pm}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q}), K^\pm\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}p)\big) \Big\rangle_{\Gamma^\pm} \\ = \Big\langle \varphi, \gamma_{T,\Gamma}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q}) \Big\rangle_\Gamma \tag{4.11}$$

for all $(v_0, q) \in X$. Thus, by the well-known representation theorem of Riesz, there exist $(w_0, r) \in X$ and $A \in \mathcal{L}(X)$ such that for $(u_0, p), (v_0, q) \in X$ we have

$$(A(u_0, p) \,|\, (v_0, q))_X = -(1 + k^2)\,(u_0 \,|\, v_0)_{L^2(Q_3,\mathbb{C}^3)} - (p \,|\, q)_{L^2(Q_3)} \\ - \eta \sum\nolimits_\pm \Big\langle \gamma_{t,\Gamma^\pm}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q}), K^\pm\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}\nabla_\alpha p)\big) \Big\rangle_{\Gamma^\pm} \\ ((w_0, r) \,|\, (v_0, q))_X = \Big\langle \varphi, \gamma_{T,\Gamma}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q}) \Big\rangle_\Gamma,$$

which yields that (4.11) can be equivalently rewritten to: find $(u_0, p) \in X$ such that

$$(u_0, p) + A\,(u_0, p) = (w_0, r).$$

(ii). We show that the operator A is compact. For this let $(u_0, p), (v_0, q) \in X$. Then, by using $a_1 b_1 + a_2 b_2 \le (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}$ for $a_j, b_j \ge 0$, $j = 1, 2$, we obtain

$$\big|\,(A(u_0, p) \,|\, (v_0, q))_X\,\big| \\ \le C_1 \|(u_0, p)\|_{L^2(Q_3,\mathbb{C}^3)\times L^2(Q_3)}\, \|(v_0, q)\|_{L^2(Q_3,\mathbb{C}^3)\times L^2(Q_3)} \\ + C_2 \sum\nolimits_\pm \big\|\gamma_{t,\Gamma^\pm}(\overline{v_0 - \tfrac{1}{k}\nabla_\alpha q})\big\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)}$$

$$\cdot \left\| K^{\pm}\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}\nabla_\alpha p)\big) \right\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma^\pm)}$$
$$\le C \Big(\|(u_0,p)\|_{L^2(Q_3,\mathbb{C}^3)\times L^2(Q_3)} + \sum\nolimits_{\pm} \left\| K^{\pm}\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}\nabla_\alpha p)\big) \right\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma^\pm)} \Big) \|(v_0,q)\|_X .$$

Here, we have chosen the ℓ^2-norm in $L^2(Q_3,\mathbb{C}^3) \times L^2(Q_3)$. From the last estimate, with the special choice $(v_0,q) := A(u_0,p)$ we obtain

$$\|A(u_0,p)\|_X \le C \Big(\|(u_0,p)\|_{L^2(Q_3,\mathbb{C}^3)\times L^2(Q_3)} + \sum\nolimits_{\pm} \left\| K^{\pm}\big(\gamma_{t,\Gamma^\pm}(u_0 + \tfrac{1}{k}\nabla_\alpha p)\big) \right\|_{H_Q^{-1/2}(\mathrm{Curl},\Gamma^\pm)} \Big).$$

Now, let $((u_0^{(n)}, p^{(n)}))_{n\in\mathbb{N}}$ be a bounded sequence in X. Then $(u_0^{(n)})_{n\in\mathbb{N}}$, $(p^{(n)})_{n\in\mathbb{N}}$ and $(u_0^{(n)} + \frac{1}{k}\nabla_\alpha p^{(n)})_{n\in\mathbb{N}}$ are bounded in $H_Q(\mathrm{curl}, \mathrm{div}_\alpha 0, Q_3)$, $H_Q^1(Q_3)$ and $H_Q(\mathrm{curl}, Q_3)$, respectively. With these observations, together with Theorem 2.122, we conclude now from the last estimate, that there exists a subsequence $((u_0^{(n_l)}, p^{(n_l)}))_{l\in\mathbb{N}}$ such that $(A(u_0^{(n_l)}, p^{(n_l)}))_{l\in\mathbb{N}}$ converges in X.

(iii). We show that the operator $I + A$ is injective. For this let $u \in H_Q(\mathrm{curl}, Q_3)$ be a solution of (4.8) to given data $\varphi = 0$. Then we obtain from (4.8), with $v := ku$,

$$\overline{k}\|\mathrm{curl}_\alpha u\|^2_{L^2(Q_3,\mathbb{C}^3)} - k|k|^2\|u\|^2_{L^2(Q_3,\mathbb{C}^3)} - \eta\overline{k}\sum\nolimits_{\pm} \underbrace{\big\langle \overline{\gamma_{t,\Gamma^\pm} u}, K^{\pm}(\gamma_{t,\Gamma^\pm} u)\big\rangle_{\Gamma^\pm}}_{>0 \text{ for } u\neq 0} = 0.$$

Taking from this equation the imaginary part yields then $\|u\|_{L^2(Q_3,\mathbb{C}^3)} = 0$, and thus $u = 0$.

As a consequence from step *(ii)* and *(iii)*, we obtain now from Riesz' third theorem, see for instance [36, Theorem 3.3], that Problem 4.11 is uniquely solvable.

(iv). We show that $\varphi \mapsto \gamma_{t,\Gamma} u|_{Q_3^-}$ is an isomporphism. But this follows from the unique solvability of the following two problems: for given $\psi \in H_Q^{-1/2}(\mathrm{Div},\Gamma)$ find $u^- \in H_Q(\mathrm{curl}, Q_3^-)$ such that

$$\forall v \in H_{Q,0}(\mathrm{curl}, Q_3^-) : \int_{Q_3^-} \big(\mathrm{curl}_\alpha u^- \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u^- \cdot \overline{v} \big)\,\mathrm{d}x = 0,$$

$$\mathrm{n} \times u^- = \psi \quad \text{on } \Gamma,$$
$$\mathrm{n} \times \operatorname{curl}_\alpha u^- + \eta\, \mathrm{n} \times K^-(\mathrm{n} \times u^-) = 0 \quad \text{on } \Gamma^-$$

and $u^+ \in H_Q(\operatorname{curl}, Q_3^+)$ such that

$$\forall v \in H_{Q,0}(\operatorname{curl}, Q_3^+) : \int_{Q_3^+} \left(\operatorname{curl}_\alpha u^+ \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u^+ \cdot \overline{v}\right) \mathrm{d}x = 0,$$
$$\mathrm{n} \times u^+ = \psi \quad \text{on } \Gamma,$$
$$\mathrm{n} \times \operatorname{curl}_\alpha u^+ - \eta\, \mathrm{n} \times K^+(\mathrm{n} \times u^+) = 0 \quad \text{on } \Gamma^+,$$

see step *(v)*. In fact, let u^- and u^+ be the solutions to $\psi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. Then $u := \begin{cases} u^+ & \text{in } Q_3^+, \\ u^- & \text{in } Q_3^- \end{cases}$ belongs to $H_Q(\operatorname{curl}, Q_3)$, see Proposition 2.118, and solves (4.8) for given data $\varphi := -\gamma_{t,\Gamma} \operatorname{curl}_\alpha u^+ - \gamma_{t,\Gamma} \operatorname{curl}_\alpha u^-$. Hence, $\varphi \mapsto u \mapsto \gamma_{t,\Gamma} u^- = \mathrm{n} \times u^- = \psi$, which shows surjectivity.
To check injectivity, let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ with $0 = \gamma_{t,\Gamma} u|_{Q_3^-} = -\gamma_{t,\Gamma} u|_{Q_3^+}$, where $u \in H_Q(\operatorname{curl}, Q_3)$ is the unique solution to Problem 4.11. Then $u^- := u|_{Q_3^-}$ and $u^+ := u|_{Q_3^+}$ solve the corresponding problems from above to $\psi = 0$. From their unique solvability we conclude that $u^- = u^+ = 0$, and thus $u = 0$. Therefore, $0 = -\gamma_{t,\Gamma} \operatorname{curl}_\alpha u|_{Q_3^+} - \gamma_{t,\Gamma} \operatorname{curl}_\alpha u|_{Q_3^-} = \varphi$.
It remains to show that $\varphi \mapsto \gamma_{t,\Gamma} u|_{Q_3^-}$ is continuous (because of the open mapping theorem). For this let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. We know that $(I + A)^{-1} \in \mathcal{L}(X)$, see for instance [36, Theorem 3.4]. Hence, in the setting from step *(i)*, the solution to Problem 4.11 is given by $u := u_0 + \frac{1}{k} \nabla_\alpha p$, where $(u_0, p) := (I + A)^{-1}(w_0, r)$. Furthermore, we know from Riesz' representation theorem that

$$\|(w_0, r)\|_X = \sup_{(v_0,q) \in X \setminus \{0\}} \frac{\left|\left\langle \varphi, \gamma_{T,\Gamma}\left(\overline{v_0 - \frac{1}{k}\nabla_\alpha q}\right)\right\rangle_\Gamma\right|}{\|(v_0, q)\|_X}.$$

Using this relation, we obtain

$$\begin{aligned} \|(w_0, r)\|_X &\le \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)} \|\gamma_{T,\Gamma}\| \sup_{(v_0,q) \in X \setminus \{0\}} \frac{\|v_0 - \frac{1}{k}\nabla_\alpha q\|_{H_Q(\operatorname{curl}, Q_3)}}{\|(v_0, q)\|_X} \\ &\le C\, \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}. \end{aligned}$$

From these observations, we finally conclude that

$$\begin{aligned}\|u\|_{H_Q(\mathrm{curl},Q_3)} &= \|u_0 + \tfrac{1}{k}\nabla_\alpha p\|_{H_Q(\mathrm{curl},Q_3)} \le C\,\|(u_0,p)\|_X \\ &\le C\,\|(I+A)^{-1}\|\|(w_0,r)\|_X \\ &\le C\,\|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}.\end{aligned}$$

Hence, on the one hand we have shown that the solution operator is bounded. On the other hand, by applying further the bounded operator $\gamma_{t,\Gamma}$ to $u|_{Q_3^-}$, we have also shown that the mapping $\varphi \mapsto \gamma_{t,\Gamma} u|_{Q_3^-}$ is bounded.

(v). We show that both problems from step *(iv)* are uniquely solvable. For this we consider only the first one, because the second one can be treated in the same way. We follow the hint in the corresponding proof of [34, Theorem 5.51] and make the ansatz $u^- = \hat{u} + u$, where $\hat{u} := \eta_{t,\Gamma}\psi$. Note that $\hat{u} \in H_Q(\mathrm{curl}, Q_3^-)$ vanishes in a neighborhood of Γ^-, see Theorem 2.107. Hence, the first problem in step *(iv)* is equivalent to: for given $\psi \in H_Q^{-1/2}(\mathrm{Div},\Gamma)$ find $u \in H_Q(\mathrm{curl}, Q_3^-)$ such that

$$\begin{aligned}\forall v \in H_{Q,0}(\mathrm{curl}, Q_3^-) : \int_{Q_3^-} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v}\big)\,\mathrm{d}x \\ = -\int_{Q_3^-} \big(\mathrm{curl}_\alpha \hat{u} \cdot \overline{\mathrm{curl}_\alpha v} - k^2 \hat{u} \cdot \overline{v}\big)\,\mathrm{d}x,\end{aligned}$$

$$\mathrm{n} \times u = 0 \quad \text{on } \Gamma,$$

$$\mathrm{n} \times \mathrm{curl}_\alpha u + \eta\, \mathrm{n} \times K^-(\mathrm{n} \times u) = 0 \quad \text{on } \Gamma^-.$$

The condition $\mathrm{n} \times u = 0$ on Γ suggests the test space $H_{Q,0,\Gamma}(\mathrm{curl}, Q_3^-)$. Now, we obtain, similarly to the verification of the variational form (4.8) after Problem 4.11, that for $v \in H_{Q,0,\Gamma}(\mathrm{curl}, Q_3^-)$

$$\begin{aligned}\int_{Q_3^-} \big(\mathrm{curl}_\alpha u \cdot \overline{\mathrm{curl}_\alpha v} - k^2 u \cdot \overline{v}\big)\,\mathrm{d}x &= \int_{Q_3^-} \big(\mathrm{curl}_\alpha \mathrm{curl}_\alpha u - k^2 u\big) \cdot \overline{v}\,\mathrm{d}x \\ &\quad - \langle \gamma_{t,\Gamma} \mathrm{curl}_\alpha u, \underbrace{\gamma_{T,\Gamma}\overline{v}}_{=0}\rangle_\Gamma - \langle \gamma_{t,\Gamma^-} \mathrm{curl}_\alpha u, \gamma_{T,\Gamma^-}\overline{v}\rangle_{\Gamma^-}, \\ \int_{Q_3^-} \big(\mathrm{curl}_\alpha \hat{u} \cdot \overline{\mathrm{curl}_\alpha v} - k^2 \hat{u} \cdot \overline{v}\big)\,\mathrm{d}x &= \int_{Q_3^-} \big(\mathrm{curl}_\alpha \mathrm{curl}_\alpha \hat{u} - k^2 \hat{u}\big) \cdot \overline{v}\,\mathrm{d}x\end{aligned}$$

$$- \langle \gamma_{t,\Gamma} \operatorname{curl}_\alpha \hat{u}, \underbrace{\gamma_{T,\Gamma} \overline{v}}_{=0} \rangle_\Gamma - \langle \underbrace{\gamma_{t,\Gamma^-} \operatorname{curl}_\alpha \hat{u}}_{=0}, \gamma_{T,\Gamma^-} \overline{v} \rangle_{\Gamma^-}.$$

Summing up both equations and exploiting that

$$\int_{Q_3^-} \big(\operatorname{curl}_\alpha \operatorname{curl}_\alpha u - k^2 u \big) \cdot \overline{v} \, \mathrm{d}x + \int_{Q_3^-} \big(\operatorname{curl}_\alpha \operatorname{curl}_\alpha \hat{u} - k^2 \hat{u} \big) \cdot \overline{v} \, \mathrm{d}x = 0,$$

yields the variational formulation

$$\begin{aligned} \int_{Q_3^-} \big(\operatorname{curl}_\alpha u \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u \cdot \overline{v} \big) \, \mathrm{d}x - \eta \, \langle \gamma_{t,\Gamma^-} \overline{v}, K^-(\gamma_{t,\Gamma^-} u) \rangle_{\Gamma^-} \\ = \int_{Q_3^-} \big(k^2 \hat{u} \cdot \overline{v} - \operatorname{curl}_\alpha \hat{u} \cdot \overline{\operatorname{curl}_\alpha v} \big) \, \mathrm{d}x. \end{aligned}$$

Now, we compare this formulation with (4.8) and recognize, by taking the decomposition

$$H_{Q,0,\Gamma}(\operatorname{curl}, Q_3^-) = H_{Q,0,\Gamma}(\operatorname{curl}, \operatorname{div}_\alpha 0, Q_3^-) \oplus \nabla_\alpha H^1_{Q,0,\Gamma}(Q_3^-),$$

from Theorem 2.124 into account as well, that we can apply the same method as in step *(i)*, *(ii)* and *(iii)* of this proof to show that the last variational formulation is uniquely solvable. □

4.2.3. Jump Relations and Boundary Integral Operators

We are now in a position to derive further properties of the vector potentials. These properties will enable us in the next section to derive the already mentioned boundary integral equations.

Theorem 4.15 *Let the surface* Γ *be given by* (4.5), *choose any cell set* $D \subseteq \mathbb{R}^3$ *of Lipschitz layer type, such that* $\Gamma \subseteq D$, *and denote the upper part of* D *with respect to (and without)* Γ *by* D^+ *and the lower part by* D^-, *see also Figure 4.2.*

(i) *The operator* $\tilde{\mathcal{L}}_\alpha : H_Q^{-1/2}(\mathrm{Div}, \Gamma) \to H_Q(\mathrm{curl}, D)$ *and the operators* $\tilde{\mathcal{M}}_\alpha^\pm : H_Q^{-1/2}(\mathrm{Div}, \Gamma) \to H_Q(\mathrm{curl}, D^\pm)$ *given by*

$$(\tilde{\mathcal{L}}_\alpha \varphi)(x) := \mathrm{curl}_\alpha^2 \langle \varphi, G_k(x, \cdot) \rangle_\Gamma, \qquad x \in D,$$
$$(\tilde{\mathcal{M}}_\alpha^\pm \varphi)(x) := \mathrm{curl}_\alpha \langle \varphi, G_k(x, \cdot) \rangle_\Gamma, \qquad x \in D^\pm,$$

are well-defined, linear and bounded.

(ii) *For* $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ *the electric and magnetic vector potential* u *and* v*, see* (4.7)*, satisfy* $u|_{D^\pm}, v|_{D^\pm} \in H_Q(\mathrm{curl}, D^\pm)$ *with the jump conditions*

$$\gamma_{t,\Gamma} u|_{D^-} + \gamma_{t,\Gamma} u|_{D^+} = 0 \qquad \text{and} \qquad \gamma_{t,\Gamma} v|_{D^-} + \gamma_{t,\Gamma} v|_{D^+} = -\varphi,$$

respectively.

(iii) *The boundary operators* $\mathcal{L}_\alpha, \mathcal{M}_\alpha : H_Q^{-1/2}(\mathrm{Div}, \Gamma) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma)$*, given by*

$$\mathcal{L}_\alpha \varphi := \gamma_{t,\Gamma} \big(\tilde{\mathcal{L}}_\alpha \varphi \big)|_{D^-},$$
$$\mathcal{M}_\alpha \varphi := \frac{1}{2} \Big(\gamma_{t,\Gamma} \big(\tilde{\mathcal{M}}_\alpha^- \varphi \big) - \gamma_{t,\Gamma} \big(\tilde{\mathcal{M}}_\alpha^+ \varphi \big) \Big),$$

are well-defined, linear and bounded. With these notations the jump conditions for the electric and magnetic vector potential u *and* v *from part (ii) read as*

$$\gamma_{t,\Gamma} u|_{D^\pm} = \mathcal{L}_\alpha \varphi \qquad \text{and} \qquad \mp \gamma_{t,\Gamma} v|_{D^\pm} = \pm \frac{1}{2} \varphi + \mathcal{M}_\alpha \varphi,$$

respectively.

(iv) *The boundary operator* $\mathcal{L}_\alpha$ *from part (iii) can be splitted into the sum* $\mathcal{L}_\alpha = \hat{\mathcal{L}} + \hat{\mathcal{K}}$ *with* $\hat{\mathcal{L}} \in \mathcal{L}_{\mathrm{is}}(H_Q^{-1/2}(\mathrm{Div}, \Gamma))$ *and a compact operator* $\hat{\mathcal{K}} : H_Q^{-1/2}(\mathrm{Div}, \Gamma) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma)$.

Proof: We follow the lines in the proof of [34, Theorem 5.52] and fix some $L_3 > 0$ such that $L_3 > \max_{\xi \in \mathbb{R}^2} f_1(\xi)$ and $-L_3 < \min_{\xi \in \mathbb{R}^2} f_0(\xi)$, and set $Q_3 := Q \times (-L_3, L_3)$; here f_j denotes the function which describes the surface Γ_j of D, $j = 0, 1$, see Assumption 2.91. Hence, Q_3 is a cuboid which contains D, including its surfaces Γ_j, $j = 0, 1$, and which is divided

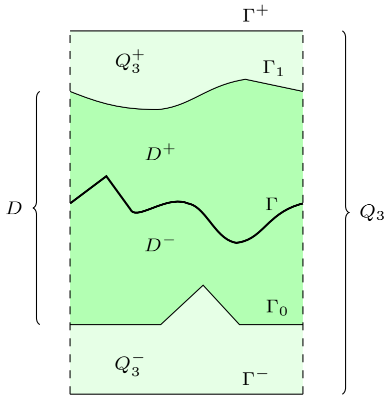


Figure 4.2.: The cell set D of Lipschitz layer type is contained in the cuboid Q_3. Both sets are divided by the surface Γ into D^- and D^+ as well as Q_3^- and Q_3^+.

by Γ into the upper part Q_3^+ and the lower part Q_3^- too. The situation is illustrated in Figure 4.2.
(i). Let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$ and let $w \in H_Q(\mathrm{curl}, Q_3)$ denote the solution to (4.8), see Theorem 4.14. We show that

$$\tilde{\mathcal{L}}_\alpha \varphi = k^2(w|_D + \tilde{w}) \qquad \text{in } H_Q(\mathrm{curl}, D), \qquad (*_1)$$

$$\tilde{\mathcal{M}}_\alpha^\pm \varphi = \mathrm{curl}_\alpha\, w|_{D^\pm} + \mathrm{curl}_\alpha\, \tilde{w}|_{D^\pm} \qquad \text{in } H_Q(\mathrm{curl}, D^\pm), \qquad (*_2)$$

where

$$\tilde{w}(x) := \sum\nolimits_\pm \Big(\mathrm{curl}_\alpha \langle \gamma_{t,\Gamma^\pm} w, G_k(x,\cdot)\rangle_{\Gamma^\pm} + \frac{1}{k^2} \mathrm{curl}_\alpha^2 \langle \gamma_{t,\Gamma^\pm} \mathrm{curl}_\alpha\, w, G_k(x,\cdot)\rangle_{\Gamma^\pm} \Big) \qquad (*_3)$$

for $x \in D$. To show these decompositions, we fix some $x \in D^+$ and apply the Stratton-Chu formula from Theorem 4.10 to w with respect to Q_3^+ as well as to Q_3^-, i.e., to $w^+ := w|_{Q_3}^+$ and $w^- := w|_{Q_3}^-$, respectively, and obtain

$$w(x) = -\,\mathrm{curl}_\alpha \left\langle \gamma_{t,\Gamma} w^+, G_k(x,\cdot) \right\rangle_\Gamma - \frac{1}{k^2} \mathrm{curl}_\alpha^2 \left\langle \gamma_{t,\Gamma} \mathrm{curl}_\alpha\, w^+, G_k(x,\cdot) \right\rangle_\Gamma$$

$$- \operatorname{curl}_\alpha \left\langle \gamma_{t,\Gamma^+} w, G_k(x,\cdot) \right\rangle_{\Gamma^+} - \frac{1}{k^2} \operatorname{curl}_\alpha^2 \left\langle \gamma_{t,\Gamma^+} \operatorname{curl}_\alpha w, G_k(x,\cdot) \right\rangle_{\Gamma^+},$$

$$0 = - \operatorname{curl}_\alpha \left\langle \gamma_{t,\Gamma} w^-, G_k(x,\cdot) \right\rangle_{\Gamma} - \frac{1}{k^2} \operatorname{curl}_\alpha^2 \left\langle \gamma_{t,\Gamma} \operatorname{curl}_\alpha w^-, G_k(x,\cdot) \right\rangle_{\Gamma}$$
$$- \operatorname{curl}_\alpha \left\langle \gamma_{t,\Gamma^-} w, G_k(x,\cdot) \right\rangle_{\Gamma^-} - \frac{1}{k^2} \operatorname{curl}_\alpha^2 \left\langle \gamma_{t,\Gamma^-} \operatorname{curl}_\alpha w, G_k(x,\cdot) \right\rangle_{\Gamma^-}.$$

Now we recall the boundary conditions $-\gamma_{t,\Gamma} \operatorname{curl}_\alpha w^+ - \gamma_{t,\Gamma} \operatorname{curl}_\alpha w^- = \varphi$ and $\gamma_{t,\Gamma} w^- = -\gamma_{t,\Gamma} w^+$ to obtain, by summing up both equations,

$$w(x) = \frac{1}{k^2} \operatorname{curl}_\alpha^2 \left\langle \varphi, G_k(x,\cdot) \right\rangle_{\Gamma} - \tilde{w}(x),$$

from which the asserted decomposition $(*_1)$ follows immediately for this case. For $x \in D \setminus D^+$ we argue similarly. Note that $\tilde{w}$ belongs to $C_Q^\infty(D, \mathbb{C}^3)$, see for instance Proposition 4.7, with all derivatives being integrable, since the domain of definition D of $\tilde{w}$ is away from some neighborhoods of the surfaces $\Gamma^\pm$ and therefore far away from any singularities of the Green's function G_k. Hence, the operator $\tilde{\mathcal{L}}_\alpha$ is well-defined. Its linearity is clear and its boundedness follows from the decomposition $(*_1)$ together with the boundedness of the solution operator from Theorem 4.14. To obtain the corresponding properties for the operators $\tilde{\mathcal{M}}_\alpha^\pm$, we observe that away from the boundary Γ we can apply Proposition 4.8 to $\tilde{\mathcal{L}}_\alpha \varphi$, which yields for $x \in D^\pm$

$$\begin{aligned} \operatorname{curl}_\alpha (\tilde{\mathcal{L}}_\alpha \varphi)(x) &= \operatorname{curl}_\alpha^2 \operatorname{curl}_\alpha \left\langle \varphi, G_k(x,\cdot) \right\rangle_{\Gamma} = k^2 \operatorname{curl}_\alpha \left\langle \varphi, G_k(x,\cdot) \right\rangle_{\Gamma} \\ &= k^2 (\tilde{\mathcal{M}}_\alpha^\pm \varphi)(x) \end{aligned} \tag{4.12}$$

and shows the decomposition $(*_2)$. Hence, we have shown that the operators $\tilde{\mathcal{M}}_\alpha^\pm$ are well-defined. Again, their linearity is clear and their boundednes follows similarly to above from properties of w.
(ii). The jump conditions follow easily from the decomposition $(*_1)$ as well as $(*_2)$ and the boundary conditions for the solution w to Problem 4.11. In fact, by using $\gamma_{t,\Gamma} \tilde{w}|_{D^-} = -\gamma_{t,\Gamma} \tilde{w}|_{D^+}$, there holds

$$\begin{aligned} \gamma_{t,\Gamma} u|_{D^-} &= \gamma_{t,\Gamma} (\tilde{\mathcal{L}}_\alpha \varphi)|_{D^-} = k^2 \big(\gamma_{t,\Gamma} w|_{Q_3^-} + \gamma_{t,\Gamma} \tilde{w}|_{D^-} \big) \\ &= -k^2 \big(\gamma_{t,\Gamma} w|_{Q_3^+} + \gamma_{t,\Gamma} \tilde{w}|_{D^+} \big) = -\gamma_{t,\Gamma} (\tilde{\mathcal{L}}_\alpha \varphi)|_{D^+} = -\gamma_{t,\Gamma} u|_{D^+} \end{aligned}$$

and, since $\gamma_{t,\Gamma} \operatorname{curl}_\alpha \tilde{w}|_{D^-} = -\gamma_{t,\Gamma} \operatorname{curl}_\alpha \tilde{w}|_{D^+}$, we have

$$\gamma_{t,\Gamma} v|_{D^-} + \gamma_{t,\Gamma} v|_{D^+} = \gamma_{t,\Gamma} (\tilde{\mathcal{M}}_\alpha^- \varphi) + \gamma_{t,\Gamma} (\tilde{\mathcal{M}}_\alpha^+ \varphi)$$

$$= \gamma_{t,\Gamma}\big(\operatorname{curl}_\alpha w|_{Q_3^-} + \operatorname{curl}_\alpha \tilde{w}|_{D^-}\big) + \gamma_{t,\Gamma}\big(\operatorname{curl}_\alpha w|_{Q_3^+} + \operatorname{curl}_\alpha \tilde{w}|_{D^+}\big)$$
$$= \gamma_{t,\Gamma} \operatorname{curl}_\alpha w|_{Q_3^-} + \gamma_{t,\Gamma} \operatorname{curl}_\alpha w|_{Q_3^+} = -\varphi.$$

(iii). The statements follow easily from step *(i)* and *(ii)*.
(iv). The decomposition of the operator $\mathcal{L}_\alpha$ follows by an application of the trace operator $\gamma_{t,\Gamma}$ to the decomposition $(*_1)$ and by Theorem 4.14. For the compactness of the operator $\hat{\mathcal{K}}$ it suffices to show that the mapping $T : H_Q^{-1/2}(\mathrm{Div},\Gamma) \to (C_Q^2(\overline{D},\mathbb{C}^3), \|\cdot\|_{C^2(\overline{D},\mathbb{C}^3)})$, given by $\varphi \to T\varphi := \tilde{w}$, where $\tilde{w}$ is defined by $(*_3)$ (with domain of definition extended to $\overline{D}$), is bounded, because then the mapping $\hat{\mathcal{K}} : H_Q^{-1/2}(\mathrm{Div},\Gamma) \to H_Q^{-1/2}(\mathrm{Div},\Gamma)$ can be split up into

$$H_Q^{-1/2}(\mathrm{Div},\Gamma) \xrightarrow{T} C_Q^2(\overline{D},\mathbb{C}^3) \hookrightarrow C_Q^{1,1}(\overline{D},\mathbb{C}^3) \overset{\text{cp.}}{\hookrightarrow} C_Q^{1,\lambda}(\overline{D},\mathbb{C}^3)$$
$$\hookrightarrow H_Q^1(D,\mathbb{C}^3) \hookrightarrow H_Q(\operatorname{curl},D) \xrightarrow{k^2\gamma_{t,\Gamma}} H_Q^{-1/2}(\mathrm{Div},\Gamma)$$

with the compact embedding $C_Q^{1,1}(\overline{D},\mathbb{C}^3) \overset{\text{cp.}}{\hookrightarrow} C_Q^{1,\lambda}(\overline{D},\mathbb{C}^3)$, see [2, Theorem 1.34] and the other mappings being bounded. For the Hölder spaces $C^{m,\lambda}(\overline{\Omega})$, for $0 < \lambda \le 1$, we refer to [2] as well, with slight modifications for $C_Q^{m,\lambda}(\overline{D},\mathbb{C}^3)$. For the norm $\|\cdot\|_{C^2(\overline{D},\mathbb{C}^3)}$ in the definition of T see also (1.12).
To show that the linear operator T is bounded, we introduce for $\psi^\pm \in H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)$ the functions

$$g^\pm(x) := \operatorname{curl}_\alpha \langle \psi^\pm, G_k(x,\cdot)\rangle_{\Gamma^\pm}, \qquad x \in \overline{D},$$
$$h^\pm(x) := \operatorname{curl}_\alpha^2 \langle \psi^\pm, G_k(x,\cdot)\rangle_{\Gamma^\pm}, \qquad x \in \overline{D}.$$

Note that by Convention 4.6 we have for $x \in \overline{D}$ and $\beta \in \mathbb{N}_0^3$

$$\partial^\beta g^\pm(x) = \partial^\beta \Big(\operatorname{curl}_\alpha \sum_{j=1}^3 \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} e^{(j)} \Big)$$
$$= \partial^\beta \sum_{j=1}^3 \Big(\nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \times e^{(j)}$$
$$+ \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \mathrm{i}\alpha \times e^{(j)} \Big)$$

$$= \sum_{j=1}^{3} \Big(\nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \times e^{(j)} \\ + \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \mathrm{i}\alpha \times e^{(j)} \Big),$$

where we also applied Proposition 4.7, and analogously

$$\partial^\beta h^\pm(x) = \sum_{j=1}^{3} \Bigg(\nabla\big(\nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \cdot e^{(j)}\big) \\ + \nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \mathrm{i}\alpha \cdot e^{(j)} \\ + \mathrm{i}\alpha \big(\nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \cdot e^{(j)}\big) \\ + \mathrm{i}\alpha \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \big(\mathrm{i}\alpha \cdot e^{(j)}\big) \\ - \Big[\big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \Delta_x \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \\ + 2\mathrm{i}\alpha \cdot \nabla \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm} \\ - |\alpha|^2 \big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm}\Big] e^{(j)} \Bigg).$$

Now we choose some cell sets $\Omega^\pm \subseteq \mathbb{R}^3$ of Lipschitz layer type, both with flat upper and lower surfaces, such that $\Omega^\pm \subseteq Q_3$, $\overline{\Omega^\pm} \cap \overline{D} = \emptyset$ and the upper surface of Ω^+ coincides with Γ^+ while the lower surface of Ω^- coincides with Γ^-. Then we obtain for $x \in \overline{D}$ and $\beta \in \mathbb{N}_0^3$ that

$$\big|\partial^\beta g^\pm(x)\big| \le \sum_{j=1}^{3} \Bigg(\sum_{l=1}^{3} \Big|\big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_{l,x} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm}\Big| \\ + |\alpha| \Big|\big\langle \psi^\pm, \gamma_{T,\Gamma^\pm}\big(e^{(j)} \partial_x^\beta G_k(x,\cdot)\big)\big\rangle_{\Gamma^\pm}\Big| \Bigg) \\ \le C \, \|\gamma_{T,\Gamma^\pm}\| \sum_{j=1}^{3} \Bigg(\sum_{l=1}^{3} \underbrace{\Big\| e^{(j)} \partial_{l,x} \partial_x^\beta G_k(x,\cdot) \Big\|_{H_Q(\mathrm{curl},\Omega^\pm)}}_{\le C_1} \\ + \underbrace{\Big\| e^{(j)} \partial_x^\beta G_k(x,\cdot) \Big\|_{H_Q(\mathrm{curl},\Omega^\pm)}}_{\le C_2} \Bigg) \|\psi^\pm\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)},$$

where the constants $C_j > 0$, $j = 1, 2$, do not depend on x. To see this, we note that for the calculation of the $H_Q(\operatorname{curl}, \Omega^\pm)$-norms some partial derivatives of G_k are used, while the function G_k is only considered on $\overline{D} \times \overline{Q^\pm}$, which is a set far away of any occurrence of singularities. Therefore, the supremum of the absolute value of any partial derivative of G_k with respect to $\overline{D} \times \overline{\Omega^\pm}$ is finite and can be drawn out the integrals during the calculation of the norms. Hence, for $g^\pm$, and similarly for $h^\pm$, we obtain for $\beta \in \mathbb{N}_0^3$

$$\|\partial^\beta g^\pm\|_\infty \leq C \|\psi^\pm\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)} \quad \text{and} \quad \|\partial^\beta h^\pm\|_\infty \leq C \|\psi^\pm\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)}.$$

Now, we choose $\psi^\pm := \gamma_{t,\Gamma^\pm} w$, with $w \in H_Q(\operatorname{curl}, Q_3)$ the unique solution to Problem 4.11 for given $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma)$. Then

$$\|\psi^\pm\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)} \leq \|\gamma_{t,\Gamma^\pm}\| \, \|w\|_{H_Q(\operatorname{curl},Q_3)} \leq C \|\gamma_{t,\Gamma^\pm}\| \, \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)},$$

where the last inequality holds by the boundedness of the solution operator, see Theorem 4.14. Finally, we choose $\psi^\pm := \gamma_{t,\Gamma^\pm} \operatorname{curl}_\alpha w$, with w from above. Since w is the solution to Problem 4.11, we have $\operatorname{curl}_\alpha w^\pm \in H_Q(\operatorname{curl}, Q_3^\pm)$ with $\operatorname{curl}_\alpha \operatorname{curl}_\alpha w^\pm = k^2 w^\pm$, where we have set $w^\pm := w|_{Q_3^\pm}$ for simplicity. Therefore,

$$\begin{aligned}\|\operatorname{curl}_\alpha w^\pm\|^2_{H_Q(\operatorname{curl},Q_3^\pm)} &\leq C\Big(\|\operatorname{curl}_\alpha w^\pm\|^2_{L^2(Q_3^\pm,\mathbb{C}^3)} + \|\operatorname{curl}^2_\alpha w^\pm\|^2_{L^2(Q_3^\pm,\mathbb{C}^3)}\Big) \\ &\leq C \|w^\pm\|^2_{H_Q(\operatorname{curl},Q_3^\pm)} \leq C \|w\|^2_{H_Q(\operatorname{curl},Q_3)},\end{aligned}$$

where the last inequality holds by part *(i)* in part *(b)* from Proposition 2.68. Hence, also for this choice of $\psi^\pm$ we obtain

$$\|\psi^\pm\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma^\pm)} \leq \|\gamma_{t,\Gamma^\pm}\| \, \|\operatorname{curl}_\alpha w^\pm\|_{H_Q(\operatorname{curl},Q_3^\pm)} \leq C \|\varphi\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)}.$$

In summary, we have now for $x \in \overline{D}$ and $\beta \in \mathbb{N}_0^3$, with $|\beta| \leq 2$, that

$$\begin{aligned}\left|\partial^\beta \tilde{w}(x)\right| \leq \sum\nolimits_\pm \Bigg(&\left|\partial^\beta \operatorname{curl}_\alpha \left\langle \gamma_{t,\Gamma^\pm} w, G_k(x,\cdot)\right\rangle_{\Gamma^\pm}\right| \\ &+ \frac{1}{|k|^2}\left|\partial^\beta \operatorname{curl}^2_\alpha \left\langle \gamma_{t,\Gamma^\pm} \operatorname{curl}_\alpha w, G_k(x,\cdot)\right\rangle_{\Gamma^\pm}\right|\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum\nolimits_{\pm} \left(\left\| \partial^{\beta} g^{\pm} \right\|_{\infty} + \frac{1}{|k|^2} \left\| \partial^{\beta} h^{\pm} \right\|_{\infty} \right) \\
&\leq C \left\| \varphi \right\|_{H_Q^{-1/2}(\mathrm{Div},\Gamma)},
\end{aligned}
$$

where $g^{\pm}$ and $h^{\pm}$ in the second inequality were defined with the corresponding choices of $\psi^{\pm}$ from above. Observing that the constant $C > 0$ can be chosen not depending on β, the proof is complete. □

4.2.4. On the Compactness of the Operator $\mathcal{M}_\alpha$

Unfortunately, for Lipschitz surfaces the boundary operator $\mathcal{M}_\alpha$ fails to be a compact operator in $H_Q^{-1/2}(\mathrm{Div}, \Gamma)$, since this operator can be considered as the counterpart of the double layer potential for the scalar valued case. For the latter operator, examples can be constructed involving domains with corners which violate the condition $\frac{|\mathrm{n}(x)\cdot(y-x)|}{|x-y|^{1+\varepsilon}} \leq C$, a crucial property for establishing compactness. But for smooth surfaces, by means of the following result from [21] with regard to an operator C_k, which is the analog of $\mathcal{M}_\alpha$ for smooth and bounded domains, we are able to show compactness of $\mathcal{M}_\alpha$. It might be useful to recall the alternative approach from Subsection 2.1.4 for the notation used therein.

Theorem 4.16 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded and smooth domain. Furthermore, let $s \in \mathbb{R}$. Then the operator C_k is linear and bounded as an operator $C_k : H_t^s(\partial\Omega) \to H_t^{s+1}(\partial\Omega)$ and as an operator $C_k : H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega) \to H^{1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$.*

For a proof we refer to [21, Lemma 11].

Corollary 4.17 *$C_k : H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega) \to H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ is compact.*

Proof: According to Theorem 4.16, we only have to show that the space $H^{1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$ is compactly embedded into $H^{-1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$. But, thanks to Proposition 2.54, this follows from the compact embedding $H^{1/2}(\partial\Omega, \mathbb{C}^3) \hookrightarrow H^{-1/2}(\partial\Omega, \mathbb{C}^3)$. In fact, let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^{1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$. By definition of the norm in $H^{1/2}(\mathrm{div}_{\partial\Omega}, \partial\Omega)$,

$(\varphi_n)_{n\in\mathbb{N}}$ and $(\operatorname{div}_{\partial\Omega}\varphi_n)_{n\in\mathbb{N}}$ are bounded sequences in $H^{1/2}(\partial\Omega,\mathbb{C}^3)$ and there exists a subsequence $(\varphi_{n_j})_{j\in\mathbb{N}}$ such that $(\varphi_{n_j})_{j\in\mathbb{N}}$ and $(\operatorname{div}_{\partial\Omega}\varphi_{n_j})_{j\in\mathbb{N}}$ are convergent in $H^{-1/2}(\partial\Omega,\mathbb{C}^3)$. Now, by definition of the norm in $H^{-1/2}(\partial\Omega,\mathbb{C}^3)$, $(\varphi_{n_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1/2}(\partial\Omega,\mathbb{C}^3)$ and therefore convergent therein. □

For the rest of this section we assume the surface Γ to be smooth, i.e.,

$$\Gamma := \Big\{x\in\mathbb{R}^3 \;\Big|\; \tilde{x}\in Q \text{ and } x_3 = f(\tilde{x})\Big\},$$

where $f\in C^\infty_{\mathrm{per}}(Q)$ is real valued, see also Subsection 2.3.7. Now we come to the important compactness result for the boundary operator $\mathcal{M}_\alpha$ for the case of smooth surfaces.

Theorem 4.18 *Let Γ be smooth. Then the operator $\mathcal{M}_\alpha$ is linear and bounded as an operator $\mathcal{M}_\alpha : H_Q^{-1/2}(\mathrm{Div},\Gamma)\to H^{1/2}_{Q,t}(\Gamma)$, and for $s\geq 0$ also as an operator $\mathcal{M}_\alpha : H^s_{Q,t}(\Gamma)\to H^{s+1}_{Q,t}(\Gamma)$. Furthermore, $\mathcal{M}_\alpha$ is compact as an operator $\mathcal{M}_\alpha : H_Q^{-1/2}(\mathrm{Div},\Gamma)\to H_Q^{-1/2}(\mathrm{Div},\Gamma)$.*

Proof: For the following arguments we were inspired by the proof of [7, Theorem 4.22] or the proof of [22, Lemma 4.15]. We only show the compactness property, as the other mapping properties are shown in a very similar way. We divide the proof into several steps.
(i). Let $\{(O_j,\hat{\chi}^{(j)}) \mid j=1,\ldots,N\}$ be a partition of unity on $\overline{Q}$ as in [7, Proof of Theorem 4.22] or in [22, Definition 2.29], i.e., it is a partition of unity as in Theorem 2.42, with the additional property: if $\operatorname{supp}\hat{\chi}^{(j)}\cap\operatorname{supp}\hat{\chi}^{(m)}\neq\emptyset$, then there exists a translation Q' of Q such that $\operatorname{supp}\hat{\chi}^{(j)}\cup\operatorname{supp}\hat{\chi}^{(m)}\subseteq Q'$. We set $\chi^{(j)}(y) := \hat{\chi}^{(j)}(\tilde{y})$ for $y\in\Gamma$ and $j\in\{1,\ldots,N\}$. Note that $\sum_{j=1}^n \chi^{(j)}(y) = \sum_{j=1}^N \hat{\chi}^{(j)}(\tilde{y}) = 1$ for $y\in\Gamma$ and that $\chi^{(j)}\circ\Psi = \hat{\chi}^{(j)}$ on Q for $j=1,\ldots,N$. Let $\mathcal{N} := \{1,\ldots,N\}\times\{1,\ldots,N\} = \mathcal{N}_1\cup\mathcal{N}_2$ with

$$\mathcal{N}_1 := \big\{(m,j)\in\mathcal{N} \mid \operatorname{supp}\chi^{(m)}\cap\operatorname{supp}\chi^{(j)}=\emptyset\big\} \quad\text{and}\quad \mathcal{N}_2 := \mathcal{N}\setminus\mathcal{N}_1.$$

For $(m,j)\in\mathcal{N}_2$ we choose smooth and bounded domains $\Omega_{m,j}\subseteq\mathbb{R}^3$ such that

$$\Gamma_{m,j} := \big\{x\in\mathbb{R}^3 \mid \tilde{x}\in\operatorname{supp}\hat{\chi}^{(m)}\cup\operatorname{supp}\hat{\chi}^{(j)} \text{ and } x_3 = f(\tilde{x})\big\}\subseteq\partial\Omega_{m,j}.$$

Without loss of generality we consider $\chi^{(j)}$ as Q-periodic functions and define its Q-quasi-periodic counterparts by

$$\chi_\alpha^{(j)}(y) := \mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot\tilde{y}}\chi^{(j)}(y), \qquad y \in \widetilde{\Gamma}, \quad j = 1, \dots, N,$$

see also (2.10). Moreover, without loss of generality we assume for $(m, j) \in \mathcal{N}_2$ that $\operatorname{supp}\hat{\chi}^{(m)} \cup \operatorname{supp}\hat{\chi}^{(j)} \subseteq Q$ (otherwise we have by assumption that $\operatorname{supp}\hat{\chi}^{(m)} \cup \operatorname{supp}\hat{\chi}^{(j)} \subseteq Q'$ and we integrate later over Q' instead of Q which makes no difference due to the Q-periodicity of the integrands).
(ii). We show that for $(m, j) \in \mathcal{N}_2$, $\varphi \in \mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3)$ and $x \notin \Gamma$

$$\operatorname{curl}_\alpha \left\langle \chi^{(m)}\varphi, G_k(x,\cdot)\right\rangle_\Gamma = \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} \operatorname{curl} \int_{\partial\Omega_{m,j}} \chi_\alpha^{(m)}\varphi\, G_{k,\alpha}(x,\cdot)\,\mathrm{d}s.$$

In fact, first of all we observe that for $x \neq y$, $\psi \in \mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3)$ and ψ_α according to Convention 3.4 we have

$$\begin{aligned}
\nabla_x G_k(x,y) &= \nabla_x\Big(\mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot(\tilde{x}-\tilde{y})} G_{k,\alpha}(x,y)\Big)\\
&= \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot(\tilde{x}-\tilde{y})}\Big(\nabla_x G_{k,\alpha}(x,y) - \mathrm{i}\alpha G_{k,\alpha}(x,y)\Big),\\
\nabla_x G_k(x,y) \times \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{y}}\psi_\alpha(y) &= \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}}\nabla_x G_{k,\alpha}(x,y) \times \psi_\alpha(y)\\
&\quad - \mathrm{i}\alpha \times \psi(y) G_k(x,y)
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{curl}_\alpha \int_\Gamma \psi\, G_k(x,\cdot)\,\mathrm{d}s &= \int_\Gamma \Big[\operatorname{curl}_x\big(\psi\, G_k(x,\cdot)\big) + \mathrm{i}\alpha \times \big(\psi\, G_k(x,\cdot)\big)\Big]\,\mathrm{d}s\\
&= \int_\Gamma \Big[\nabla_x G_k(x,\cdot) \times \psi + \mathrm{i}\alpha \times \big(\psi\, G_k(x,\cdot)\big)\Big]\,\mathrm{d}s\\
&= \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} \int_\Gamma \nabla_x G_{k,\alpha}(x,\cdot) \times \psi_\alpha\,\mathrm{d}s = \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} \operatorname{curl}\int_\Gamma \psi_\alpha\, G_{k,\alpha}(x,\cdot)\,\mathrm{d}s.
\end{aligned}$$

Hence, by the definition of the bilinear form $\langle\cdot,\cdot\rangle_\Gamma$ from Theorem 2.113 we obtain for $x \notin \Gamma$

$$\begin{aligned}
\operatorname{curl}_\alpha \left\langle \chi^{(m)}\varphi, G_k(x,\cdot)\right\rangle_\Gamma &= \operatorname{curl}_\alpha \int_\Gamma \chi^{(m)}\,\varphi\, G_k(x,\cdot)\,\mathrm{d}s\\
&= \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} \operatorname{curl}\int_\Gamma \chi_\alpha^{(m)}\varphi\, G_{k,\alpha}(x,\cdot)\,\mathrm{d}s
\end{aligned}$$

$$= \mathrm{e}^{-\mathrm{i}\tilde{\alpha}\cdot\tilde{x}} \operatorname{curl} \int_{\partial\Omega_{m,j}} \chi_\alpha^{(m)} \varphi \, G_{k,\alpha}(x,\cdot) \, \mathrm{d}s,$$

as desired.
(iii). Let $\varphi \in \mathcal{D}_{Q,t}(\Gamma, \mathbb{C}^3)$. Then with $\mathcal{M}_\alpha \varphi = \sum_{m,j=1}^{N} \chi^{(j)} \mathcal{M}_\alpha(\chi^{(m)}\varphi)$ we obtain

$$\begin{aligned} \mathcal{M}_\alpha \varphi = & \sum_{(m,j)\in\mathcal{N}_1} \chi^{(j)} \mathcal{M}_\alpha(\chi^{(m)}\varphi) \\ & + \sum_{(m,j)\in\mathcal{N}_2} \Big(\chi^{(j)} \mathcal{M}_\alpha(\chi^{(m)}\varphi) - \chi_{-\alpha}^{(j)} C_k^{(m,j)}(\chi_\alpha^{(m)}\varphi) \Big) \\ & + \sum_{(m,j)\in\mathcal{N}_2} \chi_{-\alpha}^{(j)} C_k^{(m,j)}(\chi_\alpha^{(m)}\varphi), \end{aligned}$$

where $C_k^{(m,j)}$ denotes the operator from [21] with respect to the surface $\partial\Omega_{m,j}$, which is the analog of $\mathcal{M}_\alpha$ but for the non-periodic setting considered there and built up with the kernel (4.2). Note that by Theorem 2.132 the operator $\chi^{(j)}\mathcal{M}_\alpha(\chi^{(m)}\bullet) : H_Q^{-1/2}(\mathrm{Div},\Gamma) \to H_Q^{-1/2}(\mathrm{Div},\Gamma)$ in the first summand is linear and bounded. Moreover, its kernel is smooth, yielding that this operator is compact. For the operator $\chi_{-\alpha}^{(j)} C_k^{(m,j)}(\chi_\alpha^{(m)}\bullet)$ from $H_Q^{-1/2}(\mathrm{Div},\Gamma)$ into itself in the third summand we recall Theorem 2.133, Theorem 2.59 and Corollary 4.17 and see that this operator can be decomposed into two linear and bounded outer operators and a compact inner operator, yielding that this operator is compact too. And last but not least, by means of the first equation from the step *(ii)*, Equation (4.1) and the definitions of the operators $\mathcal{M}_\alpha$ and $C_k^{(m,j)}$, we see that the kernel of the operator difference in the second summand is also smooth and gives rise to a compact operator as well.
In summary we have shown that the restriction of $\mathcal{M}_\alpha$ onto $\mathcal{D}_{Q,t}(\Gamma,\mathbb{C}^3)$ is compact. And the compactness of $\mathcal{M}_\alpha : H_Q^{-1/2}(\mathrm{Div},\Gamma) \to H_Q^{-1/2}(\mathrm{Div},\Gamma)$ follows now from Proposition A.6. □

4.3. The Boundary Integral Equation of Interest

In this section we will derive a boundary integral equation, which enables us to determine the solution to Problem 3.12 explicitly by an ansatz in form of vector potentials with an unknown density. Beside their derivation and the investigation of their solvability, we will prove that the kernels of the corresponding integral operator have a singularity of a certain kind – a property which is fundamental for an application of the numerical scheme introduced in the next chapter.

For the notation used in the following presentation concerning the geometrical setting, we recall the explanations from Subsection 3.1.2, in particular for the unit cell D and its variants $D^h_{\Gamma_0}$, and require now in the definition of Γ_0,

$$\Gamma_0 = \Big\{ x \in \mathbb{R}^3 \ \big| \ \tilde{x} \in Q \text{ and } x_3 = f_0(\tilde{x}) \Big\}, \tag{4.13}$$

instead of Lipschitz continuity the function f_0 to be in $C^\infty_{\text{per}}(Q)$. Sometimes we will also need the whole surface which was denoted by $\widetilde{\Gamma}_0$, see also (3.2).

4.3.1. Derivation and Solvability

In the next chapter we will introduce a high order solver for the electromagnetic scattering problem as given in Problem 3.12, see also (3.8). In Theorem 3.42 we have seen that this problem is uniquely solvable, at least for special values of the wave number k (see also Theorem 3.14). To determine in those cases the solution explicitly we make an ansatz for the scattered field in the form

$$u^{\text{s}} = \tilde{\mathcal{M}}^+_\alpha \varphi$$

for some $\varphi \in H_Q^{-1/2}(\text{Div}, \Gamma_0)$. Let $h > h^+$. From Theorem 4.15 we know that u^{s} and $\operatorname{curl}_\alpha u^{\text{s}}$ belong to $H_Q(\text{curl}, D^h_{\Gamma_0})$. Hence, from Proposition 4.8 we conclude, similarly as in the proof of part *(iii)* from Proposition 3.13, that for arbitrary $v \in H_{Q,0}(\text{curl}, D^h_{\Gamma_0})$

$$\int_{D^h_{\Gamma_0}} \big(\operatorname{curl}_\alpha u^{\text{s}} \cdot \overline{\operatorname{curl}_\alpha v} - k^2 u^{\text{s}} \cdot \overline{v} \big) \, \mathrm{d}x = 0.$$

Furthermore, again by Proposition 4.8, u^{s} satisfies the (URC) in D^{∞}_{h+}. Therefore, so far, $u := u^{\mathrm{s}} + u^{\mathrm{i}}$ seems to be a good candidate for the solution to Problem 3.12. From the boundary condition $\gamma_{t,\Gamma_0} u = 0$ therein, together with the jump relations from Theorem 4.15, we obtain

$$\tfrac{1}{2}\varphi + \mathcal{M}_\alpha \varphi = \gamma_{t,\Gamma_0} u^{\mathrm{i}} \qquad \text{in } H_Q^{-1/2}(\mathrm{Div}, \Gamma_0), \tag{4.14}$$

the boundary integral equation of greatest interest for our numerical method. The following theorem shows that this equation is uniquely solvable, and thus u is indeed the solution to Problem 3.12. For this, the next lemma has preliminary character.

Lemma 4.19 *Let* $\Gamma^\pm := \big\{x \in \mathbb{R}^3 \mid \tilde{x} \in Q$ *and* $x_3 = \pm f(\tilde{x})\big\}$, *where* $f \in C^\infty_{\mathrm{per}}(Q)$ *is Lipschitz continuous and real valued. Furthermore, let* $h > \max_{\xi\in\mathbb{R}^2} f(\xi)$. *Then the following assertions are true.*

(i) $u \in H_Q(\mathrm{curl}, D^h_{\Gamma^+}) \Leftrightarrow \hat{u} \in H_Q(\mathrm{curl}, D^{\Gamma^-}_{-h})$, *and in this case we have* $\mathrm{curl}\,\hat{u} = -(\mathrm{curl}\,u)^*(\cdot^*)$ *as well as* $\gamma_{t,\Gamma^-}\hat{u} = -(\gamma_{t,\Gamma^+} u)^*$.

(ii) $u \in H_Q(\mathrm{curl}, D^h_{\Gamma^+})$ *solves*

$$\forall v \in H_{Q,0}(\mathrm{curl}, D^h_{\Gamma^+}) : \int_{D^h_{\Gamma^+}} \big(\mathrm{curl}_\alpha\, u \cdot \overline{\mathrm{curl}_\alpha\, v} - k^2 u \cdot \overline{v}\big)\,\mathrm{d}x = 0$$

if and only if $\hat{u} \in H_Q(\mathrm{curl}, D^{\Gamma^-}_{-h})$ *solves*

$$\forall v \in H_{Q,0}(\mathrm{curl}, D^{\Gamma^-}_{-h}) : \int_{D^{\Gamma^-}_{-h}} \big(\mathrm{curl}_\alpha\, \hat{u} \cdot \overline{\mathrm{curl}_\alpha\, v} - k^2 \hat{u} \cdot \overline{v}\big)\,\mathrm{d}x = 0.$$

Here, we have defined $\hat{u} := u^*(\cdot^*)$, *where for* $z \in \mathbb{C}^3$ *the vector* z^* *is given by* (1.4).

Proof: We will similarly proceed as in the proof of Proposition 2.105 and take from there the observation $(\mathrm{curl}\,w)^*(\cdot^*) = -\mathrm{curl}(w^*(\cdot^*))$. Moreover, we note that for $a, b \in \mathbb{C}^3$ we have $a \times b^* = -(a^* \times b)^*$ and $(-a)^* = -a^*$, which can easily be verified. Furthermore, we recall the definition of α which yields $\alpha^* = \alpha$. And last but not least, $x \in D^h_{\Gamma^+}$ if and only if $x^* \in D^{\Gamma^-}_{-h}$, where the absolute value of the Jacobian of this transformation

is constant and equal to one.
(i). Let $u \in H_Q(\operatorname{curl}, D^h_{\Gamma+})$. Furthermore, let $\chi \in C^\infty_{Q,0}(D^{\Gamma^-}_{-h}, \mathbb{C}^3)$. Then, with the observations from above, we obtain

$$\begin{aligned}\int_{D^{\Gamma^-}_{-h}} \hat{u}(x) \cdot \operatorname{curl} \chi(x) \, \mathrm{d}x &= \int_{D^{\Gamma^-}_{-h}} u^*(x^*) \cdot \operatorname{curl} \chi(x) \, \mathrm{d}x \\ &= \int_{D^h_{\Gamma+}} u^*(x) \cdot \operatorname{curl} \chi(x^*) \, \mathrm{d}x = \int_{D^h_{\Gamma+}} u(x) \cdot (\operatorname{curl} \chi)^*(x^*) \, \mathrm{d}x \\ &= -\int_{D^h_{\Gamma+}} \operatorname{curl} u(x) \cdot \chi^*(x^*) \, \mathrm{d}x = -\int_{D^{\Gamma^-}_{-h}} (\operatorname{curl} u)^*(x^*) \cdot \chi(x) \, \mathrm{d}x,\end{aligned}$$

where we have applied that $\chi^*(\cdot^*) \in C^\infty_{Q,0}(D^h_{\Gamma+}, \mathbb{C}^3)$. The other direction is shown in the same way. The formula for the traces can be easily verified for smooth functions and then we use an approximation argument.
(ii). Let $u \in H_Q(\operatorname{curl}, D^h_{\Gamma+})$ solve the first variational equation in the lemma. From part *(i)* we know already that $\hat{u} \in H_Q(\operatorname{curl}, D^{\Gamma^-}_{-h})$. Let $v \in H_{Q,0}(\operatorname{curl}, D^{\Gamma^-}_{-h})$. Note that $v^*(\cdot^*) \in H_{Q,0}(\operatorname{curl}, D^h_{\Gamma+})$, which is shown by approximation and a similar calculation as in part *(i)*. Then, again with the observations from above, we have

$$\begin{aligned}\int_{D^{\Gamma^-}_{-h}} \big(\operatorname{curl}_\alpha \hat{u} \cdot \overline{\operatorname{curl}_\alpha v} - k^2 \hat{u} \cdot \overline{v}\big) \, \mathrm{d}x &= \int_{D^{\Gamma^-}_{-h}} \Big[\big(\operatorname{curl} u^*(x^*) + \mathrm{i}\alpha \times u^*(x^*)\big) \\ &\qquad \cdot \overline{\big(\operatorname{curl} v(x) + \mathrm{i}\alpha \times v(x)\big)} - k^2 u^*(x^*) \cdot \overline{v}(x)\Big] \, \mathrm{d}x \\ = \int_{D^{\Gamma^-}_{-h}} \Big[&- \operatorname{curl}_\alpha u(x^*) \cdot \overline{\big((\operatorname{curl} v)^*(x) + (\mathrm{i}\alpha \times v(x))^*\big)} \\ &- k^2 u(x^*) \cdot \overline{v^*(x)}\Big] \, \mathrm{d}x \\ = \int_{D^h_{\Gamma+}} \Big[&- \operatorname{curl}_\alpha u(x) \cdot \overline{\big((\operatorname{curl} v)^*(x^*) + (\mathrm{i}\alpha \times v(x^*))^*\big)} \\ &- k^2 u(x) \cdot \overline{v^*(x^*)}\Big] \, \mathrm{d}x \\ = \int_{D^h_{\Gamma+}} \big(&\operatorname{curl}_\alpha u(x) \cdot \overline{(\operatorname{curl}_\alpha v^*(\cdot^*))(x)} - k^2 u(x) \cdot \overline{v^*(x^*)}\big) \, \mathrm{d}x = 0,\end{aligned}$$

as desired. And the other direction is again shown by very similar arguments. □

Theorem 4.20 *Let u^{i} be an incident field as in Assumption 3.3. If Γ_0 is a smooth surface (as introduced above) and if Problem 3.12 has at most one solution, then the operator equation in* (4.14) *possesses exactly one solution.*

Proof: Thanks to the compactness of $\mathcal{M}_\alpha$ as an operator from the space $H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ onto itself and thanks to the third theorem of Riesz, it suffices to show that $\frac{1}{2}I + \mathcal{M}_\alpha : H_Q^{-1/2}(\mathrm{Div}, \Gamma_0) \to H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ is injective; here I denotes the identity operator in $H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$.
For this let $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ such that $\frac{1}{2}\varphi + \mathcal{M}_\alpha\varphi = 0$. We choose cell sets $D^\pm$ of Lipschitz layer type as in Theorem 4.15, with the Γ therein now being Γ_0. Furthermore, we set

$$\begin{aligned} v^\pm &:= \tilde{\mathcal{M}}_\alpha^\pm \varphi \qquad \text{in } D^\pm, \\ u &:= \tilde{\mathcal{L}}_\alpha \varphi \qquad \text{in } D. \end{aligned}$$

Note that by the jump relations from Theorem 4.15 we have $\gamma_{t,\Gamma_0} v^+ = -(\frac{1}{2}\varphi + \mathcal{M}_\alpha\varphi) = 0$. Arguing similarly as at the beginning of this section, we see that v^+ solves Problem 3.12 for an incident field which is zero. By assumption, this problem has at most one solution and we conclude that $v^+ = 0$. Thus, also $\mathrm{curl}_\alpha v^+ = 0$. Since by definition $u|_{D^+} = \mathrm{curl}_\alpha v^+$, we obtain therefore $u|_{D^+} = 0$. Again by the jump relations from Theorem 4.15, we have $\gamma_{t,\Gamma_0} u|_{D^-} = \gamma_{t,\Gamma_0} u|_{D^+} = 0$. Let $h < h^-$. Arguing again as at the beginning of this section, and taking again Proposition 4.8 into account, we see that $u^- := u|_{D^-}$ solves $u^- \in H_Q(\mathrm{curl}, D_h^{\Gamma_0})$ and

$$\begin{cases} \forall w \in H_{Q,0}(\mathrm{curl}, D_h^{\Gamma_0}) : \displaystyle\int_{D_h^{\Gamma_0}} \left(\mathrm{curl}_\alpha u^- \cdot \overline{\mathrm{curl}_\alpha w} - k^2 u^- \cdot \overline{w} \right) \mathrm{d}x = 0, \\ \qquad \gamma_{t,\Gamma_0} u^- = 0, \\ \qquad u^- \text{ satisfies (DRC) on } D_{-\infty}^{h^-}. \end{cases}$$

Now, Remark 3.7 and Lemma 4.19 come into play, which yield that $(u^-)^*(\cdot^*)$ solves Problem 3.12 (with another Γ_0) for an incident field again being zero. Therefore, one more time thanks to our assumption, $(u^-)^*(\cdot^*) = 0$, and hence $u^- = 0$. Thus, $u = 0$ and in particular $\mathrm{curl}_\alpha u|_{D^\pm} = 0$. Using (4.12), we obtain

$$0 = \mathrm{curl}_\alpha u|_{D^\pm} = \mathrm{curl}_\alpha(\tilde{\mathcal{L}}_\alpha \varphi)|_{D^\pm} = k^2 \tilde{\mathcal{M}}_\alpha^\pm \varphi = k^2 v^\pm.$$

Finally, we exploit the jump relations from Theorem 4.15 a last time and arrive at

$$-\varphi = \gamma_{t,\Gamma_0} v^+ + \gamma_{t,\Gamma_0} v^- = 0,$$

and the proof is complete. □

4.3.2. On the Weak Singularity of the Kernels

As mentioned above, the numerical method from the next chapter requires the singularity of the kernels of $\mathcal{M}_\alpha$ to be of a special kind, see Assumption 5.6. It is the objective of the following presentation to show that those kernels indeed satisfy this assumption.

We start with Equation (4.14) and rewrite it equivalently to: for given $\psi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ find $\varphi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_0)$ such that

$$\varphi + 2\mathcal{M}_\alpha \varphi = \psi,$$

where the operator $\mathcal{M}_\alpha$ was given by

$$\mathcal{M}_\alpha \varphi = \frac{1}{2}\Big(\gamma_{t,\Gamma_0}\big(\tilde{\mathcal{M}}_\alpha^- \varphi\big) - \gamma_{t,\Gamma_0}\big(\tilde{\mathcal{M}}_\alpha^+ \varphi\big)\Big),$$

see Theorem 4.15. Now, we make two observations: For $s \geq 0$ and $\psi \in H_Q^{-1/2}(\mathrm{Div}, \Gamma_0) \cap H_{Q,t}^s(\Gamma_0)$, we obtain from Theorem 4.18 and Proposition 2.131 that the solution $\varphi = \psi - 2\mathcal{M}_\alpha \varphi$ belongs to $H_{Q,t}^s(\Gamma_0)$ as well, i.e., the solution has the same regularity as the right hand side. Taking Sobolev's embedding theorem as well as the smoothness of Γ_0, recall (4.13), into account, for smooth enough right hand side ψ the action of $\mathcal{M}_\alpha$ to $\varphi \in H_{Q,t}^s(\Gamma_0)$ can therefore be described by an ordinary boundary integral of the form

$$(\mathcal{M}_\alpha \varphi)(x) = \mathrm{n}(x) \times \mathrm{curl}_\alpha \int_{\Gamma_0} G_k(x,y)\, \varphi(y)\, \mathrm{d}s(y), \qquad x \in \Gamma_0,$$

see for instance [34, Theorem 3.34]. Here, the unit normal vector $\mathrm{n}(x)$ points into the upward direction of Γ_0. And the second observation is, that for $\varphi \in H_{Q,t}^s(\Gamma_0)$ there holds $(\mathrm{n}(y) \times \varphi(y)) \times \mathrm{n}(y) = \varphi(y)$, for all $y \in \Gamma_0$. Therefore, we can and will build in this projection onto the tangential plane into the operator, which has at the moment no effect, and consider from now on the operator $\mathcal{M}_\alpha$ given by

$$(\mathcal{M}_\alpha\varphi)(x) = \mathrm{n}(x) \times \operatorname{curl}_\alpha \int_{\Gamma_0} G_k(x,\cdot)\,(\mathrm{n}\times\varphi)\times\mathrm{n}\,\mathrm{d}s, \quad x\in\Gamma_0. \quad (4.15)$$

The introduction of this artificial projection has numerical advantages, since now we can widen our solution space based on $H^s_Q(\Gamma_0,\mathbb{C}^3)$ instead of $H^s_{Q,t}(\Gamma_0)$ and need not care about tangential fields, because $\varphi = \psi - 2\mathcal{M}_\alpha\varphi$, with $\mathcal{M}_\alpha$ according to (4.15), is automatically a tangential field, if the right hand side is.

Lemma 4.21 *The boundary integral operator $\mathcal{M}_\alpha$ can be rewritten to*

$$\begin{aligned}(\mathcal{M}_\alpha\varphi)(x) = \int_{\Gamma_0} \mathrm{e}^{\mathrm{i}\tilde\alpha\cdot(\tilde y-\tilde x)}\Big[\nabla_x G_{k,\alpha}(x,y)\,\big(\mathrm{n}(x)-\mathrm{n}(y)\big)^\top \\ - \mathrm{n}(x)^\top\nabla_x G_{k,\alpha}(x,y)\,I_3\Big]\zeta(y)\,\mathrm{d}s(y), \qquad x\in\Gamma_0,\end{aligned}$$

for $\varphi\in H_Q^{-1/2}(\mathrm{Div},\Gamma_0)\cap H^s_{Q,t}(\Gamma_0)$ and $s\geq 0$ big enough. Here, $G_{k,\alpha}$ is the Q-quasi-periodic Green's function, see (4.1)*, $I_3 := \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$, $\zeta := (\mathrm{n}\times\varphi)\times\mathrm{n}$ and the unit normal vector* n *is pointing in the upward direction of* Γ_0.

Proof: First of all, by recalling also Remark 4.2, we obtain for $x,y\in\Gamma_0$, such that $x-y\neq p^{(\mu)}$,

$$\begin{aligned}\operatorname{curl}_{\alpha,x}\big(G_k(x,y)\,\zeta(y)\big) &= \operatorname{curl}_x\Big(\mathrm{e}^{\mathrm{i}\tilde\alpha\cdot(\tilde y-\tilde x)}G_{k,\alpha}(x,y)\,\zeta(y)\Big) \\ &\qquad + \mathrm{i}\alpha\times\big(G_k(x,y)\,\zeta(y)\big) \\ &= \mathrm{e}^{\mathrm{i}\tilde\alpha\cdot(\tilde y-\tilde x)}\Big(\nabla_x G_{k,\alpha}(x,y) - \mathrm{i}\alpha\,G_{k,\alpha}(x,y)\Big)\times\zeta(y) \\ &\qquad + \big(\mathrm{i}\alpha\,G_k(x,y)\big)\times\zeta(y) \\ &= \mathrm{e}^{\mathrm{i}\tilde\alpha\cdot(\tilde y-\tilde x)}\nabla_x G_{k,\alpha}(x,y)\times\zeta(y)\end{aligned}$$

and furthermore, by (A.1b) and because of $\mathrm{n}(y)\cdot\zeta(y)=0$,

$$\mathrm{n}(x)\times\Big(\nabla_x G_{k,\alpha}(x,y)\times\zeta(y)\Big) = \Big[\big(\mathrm{n}(x)-\mathrm{n}(y)\big)\cdot\zeta(y)\Big]\nabla_x G_{k,\alpha}(x,y)$$

$$-\Big[\mathrm{n}(x) \cdot \nabla_x G_{k,\alpha}(x,y)\Big]\zeta(y).$$

Note that $(a \cdot b)\, c = c\, a^\top b$ for all $a, b, c \in \mathbb{C}^3$. Using now the last results together with (4.15) and considerations as in the proof of [34, Theorem 3.34], the assertion follows immediately. □

Now, choose some cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $0 \le \chi \le 1$, $\chi \equiv 1$ in a neighborhood of the origin and with $\operatorname{supp}(\chi) \subseteq \mathbb{B}_3(0, \frac{\underline{L}}{2})$; see (1.3) for the definition of $\underline{L}$. We define $\widetilde{\chi}$ to be the Q-periodic extension of χ. Furthermore, let K denote the kernel of $\mathcal{M}_\alpha$, i.e.,

$$\begin{aligned} K(x,y) := \mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot(\tilde{y}-\tilde{x})} \Big[&\nabla_x G_{k,\alpha}(x,y) \big(\mathrm{n}(x) - \mathrm{n}(y)\big)^\top \\ &- \mathrm{n}(x)^\top \nabla_x G_{k,\alpha}(x,y)\, I_3\Big] \end{aligned}$$

for $x, y \in \widetilde{\Gamma}_0$ with $x \ne y + p^{(\mu)}$. By means of the Q-periodic cut-off function $\widetilde{\chi}$ we can split up K into

$$\begin{aligned} K(x,y) &= \widetilde{\chi}(x-y)\, K(x,y) + (1 - \widetilde{\chi}(x-y))\, K(x,y) \\ &=: K_1(x,y) + K_2(x,y), \qquad x, y \in \widetilde{\Gamma}_0, \quad x \ne y + p^{(\mu)}, \end{aligned}$$

with a Q-periodic (in both variables) and smooth part K_2 and a Q-periodic (in both variables) part K_1, the latter one containing the singularities.

Remark 4.22 *Note that the singularities of K_1 are isolated, meaning that K_1 has support only in a neighborhood of the singularties. Moreover, all singularities of K_1 are of the same nature, which can be described, thanks to the choice of $\widetilde{\chi}$, by the representation of $G_{k,\alpha}$ from Theorem 4.5.*

Recalling the representation of $G_{k,\alpha}$ from Theorem 4.5, we have for $x \ne y$ and $|x - y| \le \frac{\underline{L}}{2}$

$$\nabla_x G_{k,\alpha}(x,y) = -\frac{\cos(k|x-y|)}{4\pi|x-y|^3}(x-y) - k\frac{\sin(k|x-y|)}{4\pi|x-y|^2}(x-y) + H(x,y),$$

with some smooth function H. We set for $x, y \in \widetilde{\Gamma}_0$ and $x \ne y + p^{(\mu)}$

$$K_{1,1}(x,y) := -\frac{1}{4\pi}\widetilde{\chi}(x-y)\, \mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot(\tilde{y}-\tilde{x})} \Bigg[\frac{\cos(k|x-y|)}{|x-y|^3}$$

$$
\begin{aligned}
&\qquad + k\frac{\sin(k|x-y|)}{|x-y|^2}\Bigg](x-y)\big(\mathrm{n}(x)-\mathrm{n}(y)\big)^\top \\
&+ \frac{1}{4\pi}\widetilde{\chi}(x-y)\,\mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot(\tilde{y}-\tilde{x})}\left[\frac{\cos(k|x-y|)}{|x-y|^3}\right. \\
&\qquad \left. + k\frac{\sin(k|x-y|)}{|x-y|^2}\right]\mathrm{n}(x)^\top(x-y)\,I_3,
\end{aligned}
$$

where we consider the terms on the right hand side, where necessary, Q-periodically extended, and furthermore for $x, y \in \widetilde{\Gamma}_0$

$$
K_{1,2}(x,y) := \widetilde{\chi}(x-y)\,\mathrm{e}^{\mathrm{i}\tilde{\alpha}\cdot(\tilde{y}-\tilde{x})}\Big[H(x,y)\,\big(\mathrm{n}(x)-\mathrm{n}(y)\big)^\top - \mathrm{n}(x)^\top H(x,y)\,I_3\Big].
$$

Note that by Remark 4.22 we have $K_1 = K_{1,1} + K_{1,2}$ and that the singularities are now only contained in $K_{1,1}$. Taking into account that for $x, y \in \Gamma_0$ there holds, since Γ_0 is smooth,

$$
|\mathrm{n}(x)\cdot(x-y)| \le C\,|x-y|^2 \quad \text{and} \quad |\mathrm{n}(x)-\mathrm{n}(y)| \le C|x-y|,
$$

we see already from the definition of $K_{1,1}$ that their singularities are weak. Finally, we recall the parametrization Ψ_0 of Γ_0 and consider it Q-periodically extended, i.e.,

$$
\Psi_0(t) = \begin{pmatrix} t_1 \\ t_2 \\ f_0(t_1,t_2) \end{pmatrix}, \qquad t \in \mathbb{R}^2,
$$

and define for $t, \tau \in \mathbb{R}^2$, and additionally $t \neq \tau + \tilde{p}^{(\mu)}$ for the first definition,

$$
\begin{aligned}
k_1(t,\tau) &:= K_{1,1}\big(\Psi_0(t),\Psi_0(\tau)\big)\,\sqrt{1+|\nabla f_0(\tau)|^2}, \\
k_2(t,\tau) &:= \Big[K_{1,2}\big(\Psi_0(t),\Psi_0(\tau)\big) + K_2\big(\Psi_0(t),\Psi_0(\tau)\big)\Big]\,\sqrt{1+|\nabla f_0(\tau)|^2}.
\end{aligned}
$$

Proposition 4.23 *The functions k_1 and k_2 from above satisfy Assumption 5.6.*

Proof: By the considerations from above we have already shown that the assumptions for k_2 are satisfied. Thus, it remains to take a closer

look at k_1. Again by the considerations from above we have already that k_1 is Q-periodic with respect to both arguments and that their entries $k_1^{(i,j)}$ belong to $C^\infty((\overline{Q} \times \overline{Q}) \setminus \{(t,t) \mid t \in \overline{Q}\})$. Furthermore, the assumption for the bounds regarding the partial derivatives of $k_1^{(i,j)}$ can be easily verified by means of the product rule and by the kind of a certain reproducing structure while differentiating the terms $\frac{\cos(k|x-y|)}{|x-y|^n}(x_l - y_l)$ and $\frac{\sin(k|x-y|}{|x-y|^m}(x_l - y_l)$.
Let $0 < \varrho_0 < \pi$ be small enough and let $i, j \in \{1,2,3\}$. To show that

$$\ell : Q \times [-\varrho_0, \varrho_0] \times \mathbb{S}^1 \to \mathbb{C}, \quad (t,r,v) \to \ell(t,r,v) := |r|\, k_1^{(i,j)}(t, t+rv)$$

belongs to $C^\infty(Q \times [-\varrho_0, \varrho_0] \times \mathbb{S}^1)$, we only have to concentrate on those terms in k_1 which are relevant for the singularyties, namely

$$\begin{aligned} h_1(t,\tau) &:= \frac{\Psi_0(t) - \Psi_0(\tau)}{|\Psi_0(t) - \Psi_0(\tau)|^3} \left(\frac{\partial_1 \Psi_0(t) \times \partial_2 \Psi_0(t)}{\sqrt{1 + |\nabla f_0(t)|^2}} \right. \\ &\qquad \left. - \frac{\partial_1 \Psi_0(\tau) \times \partial_2 \Psi_0(\tau)}{\sqrt{1 + |\nabla f_0(\tau)|^2}} \right)^\top \in \mathbb{C}^{3\times 3}, \\ h_2(t,\tau) &:= \frac{\big(\partial_1 \Psi_0(t) \times \partial_2 \Psi_0(t)\big) \cdot \big(\Psi_0(t) - \Psi_0(\tau)\big)}{|\Psi_0(t) - \Psi_0(\tau)|^3}, \end{aligned}$$

for $t, \tau \in Q$ with $t \neq \tau$, because the other terms are smooth factors. Thanks to Taylor's theorem, see for instance [5], we have for a smooth function $g : Q \to \mathbb{C}^{d'}$ a representation for its differences in the following forms

$$\begin{aligned} g(\tau) - g(t) &= \int_0^1 \partial g\big(t + \xi(\tau - t)\big)(\tau - t)\, \mathrm{d}\xi, && (*_1) \\ &= \partial g(t)(\tau - t) + \int_0^1 (1-\xi)\, \partial^2 g\big(t + \xi(\tau - t)\big)[\tau - t, \tau - t]\, \mathrm{d}\xi && (*_2) \end{aligned}$$

for all $t, \tau \in Q$, where $\partial^2 g(s)$ is a bounded bilinear form from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{C}^{d'}$, see [5] for details. Moreover, we observe that

$$\Big| \int_0^1 \partial \Psi_0(t + \xi r v) v\, \mathrm{d}\xi \Big|^2 = \int_0^1 \int_0^1 v^\top \partial \Psi_0(t + \xi r v)^\top \partial \Psi_0(t + \theta r v) v\, \mathrm{d}\xi \mathrm{d}\theta.$$

Since Ψ_0 is regular and since $(\Psi_0(t))^\top \partial\Psi_0(t)$ is the first fundamental form of Γ_0, the latter expression is uniformly positive definite. Hence, due to continuity, there exist $r_0, \delta > 0$ such that

$$\Big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \Big|^2 \geq \delta, \qquad (t,r,v) \in Q \times [-r_0, r_0] \times \mathbb{S}^1.$$

In particular, there holds

$$\Big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \Big|^3 = \Big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \Big|^2 \sqrt{\Big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \Big|^2},$$

which yields that $Q \times [-r_0, r_0] \times \mathbb{S}^1 \ni (t,r,v) \mapsto \big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \big|^3 \in \mathbb{R}$ is a smooth function.

Applying now $(*_1)$ to the smooth functions $\Psi_0 : Q \to \mathbb{R}^3$ and $g : Q \to \mathbb{C}$, $t \to g(t) := \frac{(\partial_1\Psi_0(t) \times \partial_2\Psi_0(t))_j}{\sqrt{1+|\nabla f_0(t)|^2}}$ we obtain

$$h_1^{(i,j)}(t, t+rv) = \frac{\int_0^1 \partial\Psi_{0,i}(t+\xi r v) v \,\mathrm{d}\xi \int_0^1 \partial g(t+\xi r v) v \,\mathrm{d}\xi}{|r| \, \big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \big|^3}$$

with smooth enumerator and denominator and the latter one always away from zero, such that the quotient rule is applicable and yields a smooth fraction as well. For the enumerator in h_2 we apply $(*_2)$ and note that $\big(\partial_1\Psi_0(t) \times \partial_2\Psi_0(t)\big) \times \partial\Psi_0(t) v = 0$. Hence,

$$h_2(t, t+rv) = \frac{\big(\partial_1\Psi_0(t) \times \partial_2\Psi_0(t)\big) \cdot \int_0^1 (1-\xi)\, \partial^2\Psi_0(t+\xi r v)[v,v] \,\mathrm{d}\xi}{|r| \, \big| \int_0^1 \partial\Psi_0(t+\xi r v) v \,\mathrm{d}\xi \big|^3},$$

where with the same arguments as before this fraction is smooth. With those results it is now easy to see that ℓ satisfies the requirement from above and the proof is complete. □

5. The High-Order Numerical Scheme

It is the objective of this chapter to introduce a high order numerical method for a system of integral equations

$$\varphi^{(i)}(t) - \sum_{j=1}^{n} \int_Q k^{(i,j)}(t,\tau)\,\varphi^{(j)}(\tau)\,\mathrm{d}\tau = \psi^{(i)}(t), \quad t \in Q, \quad i = 1,\dots,n, \tag{5.1}$$

where the kernel functions $k^{(i,j)}$ of the underlying integral operators are weakly singular and Q-periodic with respect to both arguments. Precise assumptions will be given in Section 5.2. Such systems appear quite often in applications, e.g. for boundary value problems which can be solved by the integral equation method – and after having rewritten the boundary integrals by means of the parametrization to integrals (or system of integrals) over the parameter domain Q. An example is of course the electromagnetic scattering problem introduced in this thesis. Provided the surface is smooth (what we tacitly do because of Theorem 4.20), the numerical scheme achieves super-algebraic convergence rate. Most of the results were already prepublished in [9]. Therein, for the case of a single integral equation, other examples can be found including numerical experiments and complexity estimates. The scheme is a variant of the approach from [19, 20] and improves the scheme of [7] by reducing the overall complexity.

The numerical method can be interpreted as a collocation method based on trigonometric interpolation, and Section 5.1 collects corresponding results which will be needed later.

In Section 5.2, the main idea of the method is demonstrated and analysed on a single biperiodic integral equation. As already indicated in (1.2), a key ingredient will be a transformation into polar coordinates for the first integral to remove the singularity. However, now the corresponding integral operator takes on a non-standard form which makes its analysis much more involved, in particular for its approximation with respect to the fully discrete system. By means of certain auxiliary spaces and operators, we finally obtain all desired results in order to state the two main theorems of this section about stability and convergence.

The generalization to systems of biperiodic integral equations is then straighforward and shown in Section 5.3.

To simplify notation, especially for the convergence analysis, we assume in this chapter, without loss of generality, our rectangle Q to be of the form

$$Q = (-\pi, \pi) \times (-\pi, \pi).$$

5.1. Trigonometric Interpolation

As mentioned above, an essential component of the numerical method is trigonometric interpolation. In this section we will collect all relevant results.

First of all, we recall the space $H^s_{\mathrm{per}}(Q, \mathbb{C}^{d'})$ from Definition 2.7 and the well-known Sobolev's embedding theorem, which says that $H^s_{\mathrm{per}}(Q, \mathbb{C}^{d'})$ is compactly embedded in $(C_{\mathrm{per}}(Q, \mathbb{C}^{d'}), \|\cdot\|_\infty)$, provided $s > 1$. As an important consequence, it makes then sense to introduce an interpolation operator based on

$$\mathcal{T}_N(Q, \mathbb{C}^{d'}) := \operatorname{span}\left\{ e^{(j)}\, T_Q^{(\mu)} \mid j \in \{1, \ldots, d'\},\, \mu \in \mathbb{Z}_N^2 \right\},$$

which is a finite dimensional subspace of $\mathcal{T}(Q, \mathbb{C}^{d'})$ from Section 1.3. Here, for $N = (N_1, N_2)^\top \in \mathbb{N}^2$ we have set

$$\mathbb{Z}_N^2 := \left\{ \mu \in \mathbb{Z}^2 \mid -N_j < \mu_j \leq N_j,\, j = 1, 2 \right\}.$$

Furthermore, for the interpolation operator we also need an appropriate *grid of interpolation points* $(t_\mu^N)_{\mu\in\mathbb{Z}_N^2}$ and choose it as

$$t_\mu^N = (t_{\mu,1}^N, t_{\mu,2}^N)^\top := \left(\frac{\mu_1\,\pi}{N_1}, \frac{\mu_2\,\pi}{N_2}\right)^\top, \quad \mu \in \mathbb{Z}_N^2.$$

Lemma 5.1 *Suppose that* $s > 1$ *and* $0 \le \sigma \le s$. *Given* $u \in H^s_{\mathrm{per}}(Q, \mathbb{C}^{d'})$, *for every* $N \in \mathbb{N}^2$ *there exists a unique interpolation polynomial* $P_N u \in \mathcal{T}_N(Q, \mathbb{C}^{d'})$ *such that*

$$u(t_\mu^N) = P_N u(t_\mu^N)\,, \qquad \mu \in \mathbb{Z}_N^2\,.$$

The linear operator $P_N : H^s_{\mathrm{per}}(Q, \mathbb{C}^{d'}) \to H^\sigma_{\mathrm{per}}(Q, \mathbb{C}^{d'})$ *is bounded with*

$$\|P_N u - u\|_{H^\sigma_{\mathrm{per}}(Q,\mathbb{C}^{d'})} \le C\, \frac{\overline{N}^\sigma}{\underline{N}^s}\, \|u\|_{H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})},$$

where $C > 0$ *is a constant depending on* σ *and* s. *For* $\overline{N}$ *and* $\underline{N}$ *recall* (1.3).

For a proof we refer to [7, Lemma 5.1], which holds also for vector valued functions. Here, the symbol P_N relates to both the scalar and vector valued case. From the context it should always be clear on which spaces P_N is currently working.

An alternative way to express the interpolation operator is using the Lagrange basis representation,

$$P_N u = \sum_{\mu\in\mathbb{Z}_N^2} u(t_\mu^N)\, L_\mu^N\,, \tag{5.2}$$

with the Lagrange basis functions given by

$$L_\mu^N(t) = \frac{\pi}{2\,N_1\,N_2} \sum_{\nu\in\mathbb{Z}_N^2} T_Q^{(\nu)}(t - t_\mu^N)\,, \qquad t \in \mathbb{R}^2\,.$$

For $t \in Q \setminus \{t_\mu^N\}$, there also holds the expression

$$L_\mu^N(t) = \frac{1}{4\,N_1\,N_2} \prod_{j=1}^{2} \sin\left(N_j\,(t_j - t_{\mu,j}^N)\right) \left[\mathrm{i} + \cot\frac{t_j - t_{\mu,j}^N}{2}\right].$$

This follows from the corresponding one-dimensional result in [36, Section 11.3], with some obvious modifications due to a slightly different choice of the space $\mathcal{T}_N(Q)$.

Lemma 5.2 *The set* $\left\{e^{(j)}L_\mu^N \mid j \in \{1,\dots,d'\},\, \mu \in \mathbb{Z}_N^2\right\}$ *is an orthogonal basis of* $\left(\mathcal{T}_N(Q,\mathbb{C}^{d'}), \|\cdot\|_{L^2(Q,\mathbb{C}^{d'})}\right)$ *with*

$$\left(L_\mu^N \mid L_\nu^N\right)_{L^2(Q)} = \frac{\pi^2}{N_1\,N_2}\,\delta_{\mu,\nu}\,, \qquad \mu,\nu \in \mathbb{Z}_N^2\,.$$

Proof: From $P_N(\mathcal{T}_N(Q,\mathbb{C}^{d'})) = \mathcal{T}_N(Q,\mathbb{C}^{d'})$ and (5.2) it follows, that $\mathcal{T}_N(Q,\mathbb{C}^{d'}) = \operatorname{span}\left\{e^{(j)}L_\mu^N \mid j \in \{1,\dots,d'\},\, \mu \in \mathbb{Z}_N^2\right\}$. Moreover,

$$\begin{aligned}
(L_\mu^N, L_\nu^N)_{L^2(Q)} &= \frac{\pi^2}{4\,N_1^2\,N_2^2} \sum_{\lambda,\iota\in\mathbb{Z}_N^2} \left(T_Q^{(\lambda)}(\cdot - t_\mu^N) \mid T_Q^{(\iota)}(\cdot - t_\nu^N)\right)_{L^2(Q)} \\
&= \frac{(\pi^2)^2}{N_1^2\,N_2^2} \sum_{\lambda,\iota\in\mathbb{Z}_N^2} \left(T_Q^{(\lambda)} \mid T_Q^{(\iota)}\right)_{L^2(Q)} T_Q^{(\iota)}(t_\nu^N)\,T_Q^{(\lambda)}(-t_\mu^N) \\
&= \frac{\pi^3}{2\,N_1^2\,N_2^2} \sum_{\lambda\in\mathbb{Z}_N^2} T_Q^{(\lambda)}(t_\nu^N - t_\mu^N) = \frac{\pi^2}{N_1\,N_2}\,L_\mu^N(t_\nu^N) = \frac{\pi^2}{N_1\,N_2}\,\delta_{\mu,\nu}\,,
\end{aligned}$$

which completes the proof. □

In some instances, products of functions from $H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})$ with smooth and scalar valued functions occur. For $m \in \mathbb{N}_0$ and $\chi \in C^m_{\mathrm{per}}(Q,\mathbb{C}^{d'})$, we set

$$\|\chi\|_{\infty;m} := \sup_{t\in Q}|\chi(t)| + \max_{|\beta|=m}\,\sup_{t\in Q}|\partial^\beta\chi(t)|.$$

Lemma 5.3 *Let* $s \geq 0$ *and* $\sigma \in \mathbb{N}_{\geq s}$. *Suppose* $\varphi \in H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})$ *and let* $\chi \in C^\sigma_{\mathrm{per}}(Q)$. *Then* $\chi\varphi \in H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})$ *and*

$$\|\chi\varphi\|_{H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})} \leq C\|\chi\|_{\infty;\sigma}\|\varphi\|_{H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})},$$

where the constant $C > 0$ *is independent of* φ *and* χ.

Proof: The assertion follows from the equivalence of $\|\cdot\|_{H^s_{\mathrm{per}}(Q,\mathbb{C}^{d'})}$ with the Sobolev-Slobodeckiĭ norm, see Theorems A.36 and A.41. □

In particular, we are interested in an estimate of this kind when the smooth factor is a trigonometric monomial.

Lemma 5.4 *Let $\sigma \in \mathbb{N}$. Then $\|T_Q^{(\mu)}\|_{\infty;\sigma} \leq \frac{1}{2\pi}\,(1+|\mu|^\sigma)$ for all $\mu \in \mathbb{Z}^2$.*

Proof: Let $\beta \in \mathbb{N}_0^2$ with $|\beta| = \sigma$. Then, for $\mu \in \mathbb{Z}^2$ and $t \in \overline{Q}$, we have $\partial^\beta T_Q^{(\mu)}(t) = \mathrm{i}^{|\beta|}\,\mu_1^{\beta_1}\,\mu_2^{\beta_2}\,T_Q^{(\mu)}(t)$ and hence

$$\left|\partial^\beta T_Q^{(\mu)}(t)\right| \leq \frac{|\mu_1|^{\beta_1}\,|\mu_2|^{\beta_2}}{2\pi} \leq \frac{|\mu|_\infty^{|\beta|}}{2\pi}\,.$$

Since $|\mu|_\infty \leq |\mu|$, we obtain $\|T_Q^{(\mu)}\|_{\infty;\sigma} \leq \frac{1}{2\pi}\,(1+|\mu|^\sigma)$. □

In the later analysis, functions which are Q-periodic with respect to several independent variables will occur. Such functions can be expanded into a Fourier series with respect to one of these variables. The behaviour of the Fourier coefficients in such expansions will be of importance.

Lemma 5.5 *Let $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C}^{d'})$ be Q-periodic with respect to both arguments and define*

$$F^{(\lambda)}(t) := \int_Q F(t,\tau)\,T_Q^{(-\lambda)}(\tau)\,\mathrm{d}\tau, \qquad \lambda \in \mathbb{Z}^2, \quad t \in \mathbb{R}^2.$$

Then

$$F(t,\tau) = \sum_{\lambda\in\mathbb{Z}^2} F^{(\lambda)}(t)\,T_Q^{(\lambda)}(\tau), \qquad t,\tau \in \mathbb{R}^2,$$

holds pointwise, where $(F^{(\lambda)})_{\lambda\in\mathbb{Z}^2}$ is a sequence in $C^\infty_{\mathrm{per}}(Q,\mathbb{C}^{d'})$. Moreover, for any $m \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^2$ there exists a constant $C > 0$ such that

$$\sup_{\lambda\in\mathbb{Z}^2}\sup_{t\in\mathbb{R}^2} \left(1+|\lambda|^2\right)^m |\partial^\beta F^{(\lambda)}(t)| \leq C\,\|F\|_{\infty;|\beta|+2m}\,.$$

Proof: We expand F into a Fourier series with respect to the second argument. Pointwise convergence holds due to the smoothness of $F(t,\cdot)$ for all $t \in \mathbb{R}^2$. In particular, the definition of $F^{(\lambda)}$ and well-known facts about parameter-dependent integrals yield $F^{(\lambda)} \in C^\infty_{\mathrm{per}}(Q, \mathbb{C}^{d'})$. Furthermore,

$$|\partial^\beta F^{(\lambda)}(t)| \le 2\pi \, \|F\|_{\infty;|\beta|} \,, \qquad t \in \mathbb{R}^2, \quad \beta \in \mathbb{N}_0^2 \,.$$

For $m \in \mathbb{N}_0$, we have $|\lambda|^{2m} = \sum_{\alpha \in \mathbb{N}_0^2 : |\alpha| = m} \frac{m!}{\alpha_1! \alpha_2!} \lambda_1^{2\alpha_1} \lambda_2^{2\alpha_2}$. Let $\lambda \in \mathbb{Z}^2$, $t \in \mathbb{R}^2$, $\beta \in \mathbb{N}_0^2$ and $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = m$. Then

$$\begin{aligned}
&\tfrac{m!}{\alpha_1!\alpha_2!} \lambda_1^{2\alpha_1} \lambda_2^{2\alpha_2} \big|\partial^\beta F^{(\lambda)}(t)\big| \\
&\quad = \tfrac{m!}{\alpha_1!\alpha_2!} \Big| \int_Q \partial_t^\beta F(t,\tau) \underbrace{\big(-\mathrm{i}\big)^{2|\alpha|} \lambda_1^{2\alpha_1} \lambda_2^{2\alpha_2} \, T_Q^{(-\lambda)}(\tau)}_{= \partial^{(2\alpha_1, 2\alpha_2)} T_Q^{(-\lambda)}(\tau)} \, \mathrm{d}\tau \Big| \\
&\quad = \tfrac{m!}{\alpha_1!\alpha_2!} \Big| \int_Q \partial_t^\beta \partial_\tau^{(2\alpha_1, 2\alpha_2)} F(t,\tau) \, T_Q^{(-\lambda)}(\tau) \, \mathrm{d}\tau \Big| \le 2\pi \, \tfrac{m!}{\alpha_1!\alpha_2!} \|F\|_{\infty;|\beta|+2m} \,,
\end{aligned}$$

where we have used integration by parts in the third line. Hence,

$$|\lambda|^{2m} \big|\partial^\beta F^{(\lambda)}(t)\big| \le 2\pi \, \|F\|_{\infty;|\beta|+2m} \sum_{|\alpha|=m} \tfrac{m!}{\alpha_1!\alpha_2!} = 2^{m+1} \, \pi \, \|F\|_{\infty;|\beta|+2m} \,.$$

From $\big(1 + |\lambda|^2\big)^m \le 2^m \big(1 + |\lambda|^{2m}\big)$, $\lambda \in \mathbb{Z}^2$, we obtain

$$\big(1 + |\lambda|^2\big)^m \big|\partial^\beta F^{(\lambda)}(t)\big| \le 2^{m+1} \pi \left(\|F\|_{\infty;|\beta|} + 2^m \, \|F\|_{\infty;|\beta|+2m} \right) .$$

Since $\lambda \in \mathbb{Z}^2$ and $t \in \mathbb{R}^2$ were chosen arbitrarily, the proof is completed by observing the boundedness of the embedding from $C^{|\beta|+2m}_{\mathrm{per}}(Q, \mathbb{C}^{d'})$ into $C^{|\beta|}_{\mathrm{per}}(Q, \mathbb{C}^{d'})$. □

5.2. The Approach for a Single Biperiodic Integral Equation

In this section we reduce the system (5.1) to a single integral equation and introduce the numerical scheme for it. The generalization to systems is then straightforward and topic of the next section.

We consider an integral equation

$$\varphi(t) - \int_Q k(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau = \psi(t), \qquad t \in Q, \tag{5.3}$$

and impose the following assumptions on the kernel function and the right hand side:

Assumption 5.6 *Let $k = k_1 + k_2$, where $k_1 \in C^\infty(\overline{Q}\times\overline{Q}\setminus\{(t,t) \mid t \in \overline{Q}\})$ and $k_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ are Q-periodic with respect to both variables. For every multi-index $\alpha \in \mathbb{N}_0^2$, there exists $C > 0$ such that the estimate*

$$|\partial^\alpha k_1(t,\tau)| \le \frac{C}{\min_{\nu\in\mathbb{Z}^2} |t-\tau-2\pi\,\nu|^{1+|\alpha|}}\,, \quad t,\tau \in \mathbb{R}^2\,,\ t \neq \tau + 2\pi\nu,$$

is satisfied. For some $0 < \varrho_0 < \pi$, setting $\ell(t,r,v) = |r|\,k_1(t,t+rv)$, $t \in Q$, $r \in [-\varrho_0,\varrho_0]$, $v \in \mathbb{S}^1$, we assume that $\ell \in C^\infty(Q \times [-\varrho_0,\varrho_0] \times \mathbb{S}^1)$.

We also assume $\psi \in H^s_{\mathrm{per}}(Q)$ for some $s > 1$.

Hence, k_1 is assumed to have a particular type of weak singularity that can be removed by a transformation to polar coordinates around the singularity.

Examples of such kernels are the entries of the kernel function of $\mathcal{M}_\alpha$, see Proposition 4.23. Further examples are discussed in [9].

Isolating and Removing the Singularity. To make use of Assumption 5.6 in the numerical method, we require appropriate cut-off functions. For $0 < \delta < \varepsilon < \pi$ we define

$$\chi_{\delta,\varepsilon}(\tau) := \begin{cases} 1\,, & |\tau| \le \delta\,, \\ \tilde{\chi}\left(\frac{\varepsilon-|\tau|}{\varepsilon-\delta}\right), & \delta < |\tau| < \varepsilon\,, \\ 0\,, & |\tau| \ge \varepsilon\,, \end{cases} \qquad \tau \in Q,$$

with

$$\tilde{\chi}(s) := \frac{\mathrm{e}^{-1/s}}{\mathrm{e}^{-1/s} + \mathrm{e}^{-1/(1-s)}}\,, \quad s \in (0,1)\,.$$

Note that $\chi_{\delta,\varepsilon}$ is infinitely often differentiable. On all of $\mathbb{R}^2$, $\chi_{\delta,\varepsilon}$ is assumed to be Q-periodic. Furthermore, an argument by induction shows that

$$\partial^\alpha \chi_{\delta,\varepsilon}(\tau) = \sum_{\ell=1}^{m} \frac{p_\ell^\alpha(\tau)}{(\delta-\varepsilon)^\ell \, |\tau|^{2m-\ell}} \, \tilde{\chi}^{(\ell)}\left(\frac{\varepsilon - |\tau|}{\varepsilon-\delta}\right), \qquad \alpha \in \mathbb{N}_0^2\,,\ |\alpha| = m \in \mathbb{N}\,,$$

where p_ℓ^α are either homogeneous polynomials of degree m or the zero function. From this representation, we obtain the estimate

$$|\partial^\alpha \chi_{\delta,\varepsilon}(t)| \le C_\alpha \sum_{\ell=1}^{m} \frac{\varepsilon^m}{\delta^{2m-\ell}\,(\varepsilon-\delta)^\ell}\,, \qquad t \in \mathbb{R}^2\,. \tag{5.4}$$

Fixing numbers $0 < \delta_1 < \delta_2$, (5.4) implies

$$|\partial^\alpha \chi_{\delta_1 \varrho, \delta_2 \varrho}(t)| \le C_{\alpha,\delta_1,\delta_2}\, \varrho^{-m}\,, \qquad 0 < \varrho < \pi/\delta_2\,, \tag{5.5}$$

with a constant $C_{\alpha,\delta_1,\delta_2}$ independent of ϱ.

With the help of these cut-off functions, we localize the singularity in the integral operator from (5.3). Fixing numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $0 < \varrho < \varrho_0$, define

$$k_{\text{smooth}}(t,\tau) := k_1(t,\tau)\,(1 - \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau - t)) + k_2(t,\tau)\,, \tag{5.6}$$

and introduce the operators

$$\left.\begin{aligned} J_1\varphi(t) &:= \int_Q k_1(t,\tau)\,\chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\tau-t)\,\chi_{\varepsilon_2\varrho,\varrho}(\tau-t)\,\varphi(\tau)\,\mathrm{d}\tau\,, \\ J_2\varphi(t) &:= \int_Q k_{\text{smooth}}(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau\,, \end{aligned}\right\} \quad t \in Q\,.$$

Then (5.3) can be rewritten as

$$\varphi - J_1\varphi - J_2\varphi = \psi \qquad \text{on } Q\,. \tag{5.7}$$

The reason for introducing $\chi_{\varepsilon_2\varrho,\varrho}$ will be explained below.

Next, we rewrite $J_1\varphi(t)$ using polar coordinates around t. We set

$$\Pi(p) := r\,\frac{\varrho}{\pi}\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix}, \qquad p = (r,\vartheta)^\top \in Q\,,$$

and

$$k_{\text{polar}}(t,p) := \frac{|r|\,\varrho^2}{2\,\pi^2}\, k_1(t, t+\Pi(p))\, \chi_{\varepsilon_1\varrho,\varepsilon_2\varrho}(\Pi(p))\,, \quad t,\, p = (r,\vartheta)^\top \in Q\,.$$

Substituting $\tau = t + \Pi(p)$ in the expression for the operator J_1 gives

$$J_1\varphi(t) = \int_Q k_{\text{polar}}(t,p)\, \chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\, \varphi(t+\Pi(p))\,\mathrm{d}p\,, \qquad t \in Q\,. \quad (5.8)$$

By Assumption 5.6 and $k_{\text{polar}}(t,p) = 0$ for $|\Pi(p)| \geq \varepsilon_2\varrho$, we have that $k_{\text{polar}} \in C^\infty(\overline{Q}\times\overline{Q})$ can be extended Q-periodically with respect to both arguments to $C^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. The reason for introducing $\chi_{\varepsilon_2\varrho,\varrho}$ becomes clear now: $\chi_{\varepsilon_2\varrho,\varrho}(\Pi(\cdot))\ \varphi(t+\Pi(\cdot))$ can also be Q-periodically extended to $C^\infty(\mathbb{R}^2)$.

The Semidiscrete Problem and Modifications. We want to solve (5.7) numerically using a collocation method on the space $\mathcal{T}_N(Q)$. Thus, the semidiscrete problem is to find $\varphi_N \in \mathcal{T}_N(Q)$ such that

$$\varphi_N - P_N J_1 \varphi_N - P_N J_2 \varphi_N = P_N \psi \qquad \text{on } Q. \quad (5.9)$$

A fully discrete method is obtained in several steps. Firstly, both integrals are replaced by composite trapezoidal rules which are highly efficient for periodic functions. For M, $N \in \mathbb{N}^2$, we set for $\varphi \in H^s_{\text{per}}(Q)$

$$\begin{aligned} J_{1,M}\varphi(t) &:= \int_Q P_M\left[k_{\text{polar}}(t,\cdot)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(\cdot))\,\varphi(t+\Pi(\cdot))\right](p)\,\mathrm{d}p \\ &= \frac{\pi^2}{M_1M_2}\sum_{\nu\in\mathbb{Z}^2_M} k_{\text{polar}}(t,t^M_\nu)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(t^M_\nu))\,\varphi(t+\Pi(t^M_\nu)) \end{aligned} \quad (5.10)$$

$$\begin{aligned} J_{2,N}\varphi(t) &:= \int_Q P_N\left[k_{\text{smooth}}(t,\cdot)\,\varphi\right](\tau)\,\mathrm{d}\tau \\ &= \frac{\pi^2}{N_1N_2}\sum_{\nu\in\mathbb{Z}^2_N} k_{\text{smooth}}(t,t^N_\nu)\,\varphi(t^N_\nu)\,. \end{aligned} \quad (5.11)$$

While both operators are discrete in principle, only $J_{2,N}$ can be used directly. The expression for $J_{1,M}$ involves the evaluation of $\varphi(t+\Pi(t^M_\nu))$.

An exact evaluation requires the knowledge of $L_\mu^N(\Pi(t_\nu^M))$ for all $\mu \in \mathbb{Z}_N^2$ and $\nu \in \mathbb{Z}_M^2$, which amounts to $\mathrm{O}(\mathrm{N}^2\mathrm{M}^2)$ operations. In [19, 20, 18] the quadrature rule in radial direction is slightly perturbed and the values of $\varphi(t + \Pi(\cdot))$ in the quadrature points are obtained to high accuracy by fixed degree polynomial interpolation. However, this approach limits the asymptotic convergence rate.

The approach of [7] is a collocation method and uses the exact values of $\varphi(t + \Pi(\cdot))$ in the quadrature points. Here, we modify the scheme by reducing the cost in the approximation of J_1. We require the orthogonal projection O_M from $L^2(Q)$ onto $\mathcal{T}_M(Q)$,

$$O_M v := \sum_{\mu \in \mathbb{Z}_M^2} \left(v \mid T_Q^{(\mu)}\right)_{L^2(Q)} T_Q^{(\mu)} = \frac{M_1 M_2}{\pi^2} \sum_{\mu \in \mathbb{Z}_M^2} \left(v \mid L_\mu^M\right)_{L^2(Q)} L_\mu^M \tag{5.12}$$

for $v \in L^2(Q)$, where the second representation is due to Lemma 5.2. Let $1 < \varepsilon_3$ denote a number such that $\varepsilon_3 \varrho \leq \varrho_0$. A scaled projection for functions on $Q_\varrho := (-\varepsilon_3\varrho, \varepsilon_3\varrho)^2$ is given by

$$\tilde{O}_M v := O_M \left[v\left(\frac{\varepsilon_3 \varrho}{\pi} \cdot\right)\right] \left(\frac{\pi}{\varepsilon_3 \varrho} \cdot\right), \qquad v \in L^2(Q_\varrho)\,.$$

We define for $M,\, \tilde{M} \in \mathbb{N}^2$,

$$J_{1,M,\tilde{M}}\varphi(t) := \int_Q P_M \Big[k_{\mathrm{polar}}(t,\cdot) \; O_M \left[\left\{ \chi_{\varepsilon_2\varrho,\varrho}\, \tilde{O}_{\tilde{M}} \left[\chi_{\varrho,\varepsilon_3\varrho}\, \varphi(t+\cdot) \right] \right\} \circ \Pi \right] \Big](p)\, \mathrm{d}p\,. \tag{5.13}$$

The operator O_M was already used in [7]. It makes the derivation of (5.20) below possible which is central to the proof of Theorem 5.15. The projection $\tilde{O}_{\tilde{M}}$ reduces the complexity of the scheme when compared to the approach in [7], see also [9, Section 5].

Mapping Properties. For the remaining part of this section, we focus on the convergence analysis of the approach introduced above. We will start with properties of the operators J_2 and $J_{2,N}$ which are simpler to analyse.

Theorem 5.7 *Let $s \geq 0$. Then $J_2 : H^s_{\mathrm{per}}(Q) \to H^{s+1}_{\mathrm{per}}(Q)$ is well-defined, linear and bounded with $\|J_2\| \leq C\,\varrho^{-\max\{5,s+3\}}$ for all $\varrho \leq \varrho_0$ with the constant C dependent on the kernel k and the numbers ε_1, ε_2.*

Proof: We write k_{smooth} using its Fourier series representation from Lemma 5.5,

$$k_{\mathrm{smooth}}(t,\tau) = \sum_{\lambda\in\mathbb{Z}^2} k^{(\lambda)}(t)\, T_Q^{(\lambda)}(\tau), \qquad t,\tau \in \mathbb{R}^2.$$

Let $\varphi \in H^s_{\mathrm{per}}(Q)$ and set $\sigma := \lfloor s \rfloor$, see also the beginning of Section 1.3 for the meaning of this symbol. By Lemma 5.3, there holds

$$\left\|k^{(\lambda)}\right\|_{H^{s+1}_{\mathrm{per}}(Q)} = 2\pi \left\|k^{(\lambda)} T_Q^{(0)}\right\|_{H^{s+1}_{\mathrm{per}}(Q)} \leq C \left\|k^{(\lambda)}\right\|_{\infty;\sigma+2}$$

for all $\lambda \in \mathbb{Z}^2$. Therefore, by applying the triangle inequality with respect to the norm $\|\cdot\|_{H^{s+1}_{\mathrm{per}}(Q)}$,

$$\begin{aligned}
\|J_2\,\varphi\|_{H^{s+1}_{\mathrm{per}}(Q)} &\leq \sum_{\lambda\in\mathbb{Z}^2} \left| \int_Q T_Q^{(\lambda)}(\tau)\,\varphi(\tau)\,\mathrm{d}\tau \right| \left\|k^{(\lambda)}\right\|_{H^{s+1}_{\mathrm{per}}(Q)} \\
&= \sum_{\lambda\in\mathbb{Z}^2} |\varphi_{-\lambda}| \left\|k^{(\lambda)}\right\|_{H^{s+1}_{\mathrm{per}}(Q)} \\
&\leq C\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)} \left(\sum_{\lambda\in\mathbb{Z}^2} \left(1+|\lambda|^2\right)^{-\sigma} \left\|k^{(\lambda)}\right\|^2_{\infty;\sigma+2} \right)^{1/2} \\
&\leq C\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)} \sup_{\lambda\in\mathbb{Z}^2} \left[\left(1+|\lambda|^2\right)^{\max\{1-\frac{\sigma}{2},0\}} \left\|k^{(\lambda)}\right\|_{\infty;\sigma+2} \right] \\
&\qquad\qquad \times \left(\sum_{\lambda\in\mathbb{Z}^2} \left(1+|\lambda|^2\right)^{-2} \right)^{1/2} \\
&\leq C\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)} \left\|k_{\mathrm{smooth}}\right\|_{\infty;\sigma+2+\max\{2-\sigma,0\}} \\
&= C\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)} \left\|k_{\mathrm{smooth}}\right\|_{\infty;\max\{4,\sigma+2\}},
\end{aligned}$$

where the last estimate is due to Lemma 5.5. Recall that C a generic constant that may be different in each occurence.

Define the set $\Omega_\varrho = \{(t,\tau) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid |t-\tau+2\pi\nu| \geq \varepsilon_2\varrho \text{ for all } \nu \in \mathbb{Z}^2\}$. We proceed to bound for $m \in \mathbb{N}_0$ using Assumption 5.6 and (5.5)

$$\begin{aligned}\big\|k_{\text{smooth}}\big\|_{\infty;m} &\leq \|k_2\|_{\infty;m} + \|k_1\|_\infty \\ &+ C \sum_{|\alpha|+|\beta|=m} \|\partial^\alpha k_1(\cdot,\cdot)\|_{\infty;\Omega_\varrho} \, \|\partial^\beta (1-\chi_{\varepsilon_1\varrho,\varepsilon_2\varrho})\|_{\infty;\mathbb{R}^2} \leq C\varrho^{-m-1}\end{aligned}$$

for $\varrho \leq \varrho_0$, which completes the proof. □

Theorem 5.8 *Let $s > 1$ and $t \in [0,s]$. Then $J_{2,N} : H^s_{\text{per}}(Q) \to H^{t+1}_{\text{per}}(Q)$ is well-defined, linear and bounded. Moreover,*

$$\|(J_2 - J_{2,N})\,\varphi\|_{H^{t+1}_{\text{per}}(Q)} \leq C\,\varrho^{-2s-6}\,\frac{(\max\{N_1,N_2\})^t}{(\min\{N_1,N_2\})^s}\,\|\varphi\|_{H^s_{\text{per}}(Q)}$$

for all $\varphi \in H^s_{\text{per}}(Q)$, $\varrho \leq \varrho_0$ and all $N \in \mathbb{N}^2$, where C depends on k, ε_1 and ε_2.

Proof: Let $\sigma \in \mathbb{N}_{\geq s}$. From Cauchy-Schwarz's inequality and Lemma 5.1, we conclude

$$\begin{aligned}\Big|\int_Q \big(T_Q^{(\lambda)}\varphi - P_N[T_Q^{(\lambda)}\varphi]\big)(\tau)\,\mathrm{d}\tau\Big| &\leq 2\pi\,\big\|T_Q^{(\lambda)}\varphi - P_N[T_Q^{(\lambda)}\varphi]\big\|_{H^t_{\text{per}}(Q)} \\ &\leq C\,\frac{(\max\{N_1,N_2\})^t}{(\min\{N_1,N_2\})^s}\,\big\|T_Q^{(\lambda)}\varphi\big\|_{H^s_{\text{per}}(Q)}\,.\end{aligned}$$

From Lemmas 5.3 and 5.4, we obtain

$$\begin{aligned}\big\|T_Q^{(\lambda)}\varphi\big\|_{H^s_{\text{per}}(Q)} &\leq C\,\big\|T_Q^{(\lambda)}\big\|_{\infty;\sigma}\|\varphi\|_{H^s_{\text{per}}(Q)} \leq C\,\big(1+|\lambda|^\sigma\big)\|\varphi\|_{H^s_{\text{per}}(Q)} \\ &\leq C\,\big(1+|\lambda|^2\big)^{\frac{\sigma}{2}}\|\varphi\|_{H^s_{\text{per}}(Q)}\end{aligned}$$

so that

$$\begin{aligned}&\Big|\int_Q \big(T_Q^{(\lambda)}\varphi - P_N[T_Q^{(\lambda)}\varphi]\big)(\tau)\,\mathrm{d}\tau\Big| \\ &\qquad\leq C\,\frac{(\max\{N_1,N_2\})^t}{(\min\{N_1,N_2\})^s}\,\big(1+|\lambda|^2\big)^{\sigma/2}\,\|\varphi\|_{H^s_{\text{per}}(Q)}\,.\end{aligned}$$

Thus, by applying the triangle inequality with respect to $\|\cdot\|_{H^{t+1}_{\mathrm{per}}(Q)}$,

$$
\begin{aligned}
&\|(J_2 - J_{2,N})\,\varphi\|_{H^{t+1}_{\mathrm{per}}(Q)} \\
&\quad\le \sum_{\lambda\in\mathbb{Z}^2} \left| \int_Q \big(T_Q^{(\lambda)}\varphi - P_N[T_Q^{(\lambda)}\varphi]\big)(\tau)\,\mathrm{d}\tau \right| \left\|k^{(\lambda)}_{\mathrm{smooth}}\right\|_{H^{t+1}_{\mathrm{per}}(Q)} \\
&\quad\le C\,\frac{(\max\{N_1,N_2\})^t}{(\min\{N_1,N_2\})^s}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)} \sum_{\lambda\in\mathbb{Z}^2} \big(1+|\lambda|^2\big)^{\sigma/2} \left\|k^{(\lambda)}_{\mathrm{smooth}}\right\|_{\infty;\sigma+1}
\end{aligned}
$$

and again Lemma 5.5 completes the proof as the remaining argument is very similar to that at the end of the proof for Theorem 5.7. □

The derivation of a similar result for the approximation $J_{1,M,\tilde{M}}$ of J_1 as introduced in (5.13) is more complicated. Although the singularity has been removed, the integral operator now takes on a non-standard form which makes the analysis of its mapping properties much more involved.

To simplify the considerations, let us rewrite J_1 in terms of expressions that are easier to analyse. Writing k_{polar} as a Fourier series with respect to p,

$$
k_{\mathrm{polar}}(t,p) = \sum_{\lambda\in\mathbb{Z}^2} k^{(\lambda)}_{\mathrm{polar}}(t)\,T_Q^{(\lambda)}(p), \qquad t,p\in Q\,, \tag{5.14}
$$

we formally have

$$
J_1\varphi(t) = \sum_{\lambda\in\mathbb{Z}^2} k^{(\lambda)}_{\mathrm{polar}}(t) \int_Q T_Q^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,\varphi(t+\Pi(p))\,\mathrm{d}p\,.
$$

The later analysis will show that interchanging integration and summation is indeed justified.

Recalling $Q_\varrho = (-\varepsilon_3\varrho, \varepsilon_3\varrho)^2$, with corresponding trigonometric monomials

$$
T^{(\nu)}_{Q_\varrho}(\tau) = \frac{1}{2\varepsilon_3\varrho}\exp\left(\mathrm{i}\,\frac{\pi}{\varepsilon_3\varrho}\,\tau\cdot\nu\right), \qquad \tau\in Q_\varrho\,,
$$

consider now functions u of $t\in Q$ and $\tau\in Q_\varrho$. These can be expanded into Fourier series with respect to both variables,

$$
u(t,\tau) = \sum_{\mu,\nu\in\mathbb{Z}^2} u_{\mu,\nu}\,T_Q^{(\mu)}(t)\,T^{(\nu)}_{Q_\varrho}(\tau)\,.
$$

For $s \geq 0$, we introduce the vector space

$$\mathcal{H}^s_{Q,Q_\varrho} := \Big\{ u \in L^2(Q \times Q_\varrho) \mid p_{s,\sigma}(u) < \infty \text{ for all } \sigma \geq 0 \Big\},$$

where

$$p_{s,\sigma}(u) := \sum_{\mu,\nu \in \mathbb{Z}^2} (1 + |\mu|^2)^s (1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2)^\sigma |u_{\mu,\nu}|^2 , \qquad \sigma \geq 0 .$$

Remark 5.9 *The space $\mathcal{H}^s_{Q,Q_\varrho}$ is a subspace of $\mathcal{H}^t_{Q,Q_\varrho}$, for all $0 \leq t \leq s$.*

For $u \in \mathcal{H}^s_{Q,Q_\varrho}$ and $\sigma \geq 0$, we also set

$$q_{s,\sigma}(u) := \sum_{\mu,\nu \in \mathbb{Z}^2} (1 + |\tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2)^s (1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2)^\sigma |u_{\mu,\nu}|^2.$$

Between $p_{s,\sigma}$ and $q_{s,\sigma}$, there holds a certain equivalence relation. For $u \in \mathcal{H}^s_{Q,Q_\varrho}$ and $\sigma \geq 0$, we estimate

$$\begin{aligned} p_{s,\sigma}(u) &= \sum_{\mu,\nu \in \mathbb{Z}^2} \big(1 + |\mu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\ &\leq 2^s \sum_{\mu,\nu \in \mathbb{Z}^2} \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2 + |\tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\ &\leq 2^s \sum_{\mu,\nu \in \mathbb{Z}^2} \big(1 + |\tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2\big)^{\sigma+s} |u_{\mu,\nu}|^2 = 2^s \, q_{s,\sigma+s}(u) , \end{aligned} \tag{5.15}$$

and by similar arguments also $q_{s,\sigma}(u) \leq 2^s \, p_{s,\sigma+s}(u)$.

Two technical lemmas are required to establish the mapping properties of the operator J_1.

Lemma 5.10 *Denote by $\widehat{\chi_{\varrho,\varepsilon_3\varrho}}$ the Fourier transform of the extension of $\chi_{\varrho,\varepsilon_3\varrho}|_Q$ to $\mathbb{R}^2$ by 0. Then for any $\sigma \in \mathbb{N}_0$ and $\varepsilon_3 \varrho \leq \varrho_0$,*

$$\sup_{x \in \mathbb{R}^2} \Big[(1 + |x|^2)^\sigma \, \big| \widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) \big| \Big] \leq C \, \varrho^{-2\sigma+2} ,$$

where the constant C depends only on σ and ε_3.

Proof: We note that $\chi_{\varrho,\varepsilon_3\varrho}(\varrho t) = \chi_{1,\varepsilon_3}(t)$, for all $t \in Q$, and obtain

$$\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) = \int_{\mathbb{B}_2(0,\varepsilon_3\varrho)} \chi_{\varrho,\varepsilon_3\varrho}(t)\,\mathrm{e}^{-\mathrm{i}t\cdot x}\,\mathrm{d}t = \varrho^2 \int_{\mathbb{B}_2(0,\varepsilon_3)} \chi_{1,\varepsilon_3}(t)\,\mathrm{e}^{-\mathrm{i}\varrho\,t\cdot x}\,\mathrm{d}t\,.$$

Let $R > 0$ and consider $x = |x|\,\hat{x}$ with $|x| \geq R$. We rewrite the integral using the divergence theorem as

$$\begin{aligned}
&\int_{\mathbb{B}_2(0,\varepsilon_3)} \chi_{1,\varepsilon_3}(t)\,\mathrm{e}^{-\mathrm{i}\varrho\,t\cdot x}\,\mathrm{d}t \\
&\quad= \int_{\mathbb{B}_2(0,\varepsilon_3)} \left\{ \frac{\hat{x}\cdot\nabla\chi_{1,\varepsilon_3}(t)\,\mathrm{e}^{-\mathrm{i}\varrho|x|\,t\cdot\hat{x}}}{\mathrm{i}\,|x|\,\varrho} - \nabla_t \cdot \left[\frac{\hat{x}\,\chi_{1,\varepsilon_3}(t)\,\mathrm{e}^{-\mathrm{i}\varrho|x|\,t\cdot\hat{x}}}{\mathrm{i}\,|x|\,\varrho} \right] \right\} \mathrm{d}t \\
&\quad= \frac{1}{\mathrm{i}\varrho|x|} \int_{\mathbb{B}_2(0,\varepsilon_3)} \hat{x}\cdot\nabla\chi_{1,\varepsilon_3}(t)\,\mathrm{e}^{-\mathrm{i}\varrho|x|\,t\cdot\hat{x}}\,\mathrm{d}t\,.
\end{aligned}$$

We repeat this argument $2\sigma - 1$ times to obtain

$$\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x) = \frac{\varrho^2}{(\mathrm{i}\varrho|x|)^{2\sigma}} \int_{\mathbb{B}_2(0,\varepsilon_3)} h(t)\,\mathrm{e}^{-\mathrm{i}\varrho|x|\,t\cdot\hat{x}}\,\mathrm{d}t$$

with some function h depending on ε_3 and continuously on $\hat{x}$. The assertion follows by applying the triangular inequality for integrals and taking the maximum with respect to $\hat{x}$.
For $|x| \leq R$, we use $|\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(x)| \leq C\,(\varepsilon_3\varrho)^2$ and $\varepsilon_3\varrho \leq \varrho_0$. $\square$

Lemma 5.11 *Let $s \geq 0$, $\varepsilon_3\varrho \leq \varrho_0$.*

(i) For $\varphi \in H^s_{\mathrm{per}}(Q)$ define

$$\mathcal{M}\varphi(t,\tau) := \chi_{\varrho,\varepsilon_3\varrho}(\tau)\,\varphi(t+\tau)\,, \quad (t,\tau) \in Q \times Q_\varrho.$$

Then $\mathcal{M}\varphi \in \mathcal{H}^s_{Q,Q_\varrho}$, and for all $\sigma \geq 0$,

$$p_{s,\sigma}(\mathcal{M}\varphi) \leq C\,\varrho^{-2\sigma-2}\,\|\varphi\|^2_{H^s_{\mathrm{per}}(Q)}\,,$$

where the constant C depends only on σ and ε_3.

(ii) *For $\lambda \in \mathbb{Z}^2$ and $u \in \mathcal{H}^s_{Q,Q_\varrho}$, set*

$$\mathcal{J}^{(\lambda)} u(t) := \int_Q T^{(\lambda)}(p)\, \chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\, u(t, \Pi(p))\, \mathrm{d}p, \quad t \in Q.$$

$(\mathcal{J}^{(\lambda)})_{\lambda\in\mathbb{Z}^2}$ is a family of linear operators which map the space $\mathcal{H}^s_{Q,Q_\varrho}$ to $H^{s+1}_{\mathrm{per}}(Q)$ and with

$$\|\mathcal{J}^{(\lambda)} u\|_{H^{s+1}_{\mathrm{per}}(Q)} \le C\, \varrho^{-1}\,(1+|\lambda|^2)\,\sqrt{p_{s,3}(u)}, \quad u \in \mathcal{H}^s_{Q,Q_\varrho},\ \lambda \in \mathbb{Z}^2,$$

where $C > 0$ is a constant only depending on ε_2, ε_3 and s.

(iii) *$(\mathcal{J}^{(\lambda)} \circ \mathcal{M})_{\lambda\in\mathbb{Z}^2}$ is a family of linear and bounded operators mapping $H^s_{\mathrm{per}}(Q)$ to $H^{s+1}_{\mathrm{per}}(Q)$. In particular,*

$$\|\mathcal{J}^{(\lambda)}\mathcal{M}\,\varphi\|_{H^{s+1}_{\mathrm{per}}(Q)} \le C\, \varrho^{-5}\,(1+|\lambda|^2)\|\varphi\|_{H^s_{\mathrm{per}}(Q)}, \quad \varphi \in H^s_{\mathrm{per}}(Q),$$

and for all $\lambda \in \mathbb{Z}^2$, where the constant $C > 0$ only depends on ε_2, ε_3 and s.

Proof: *(i)*. Let $s \ge 0$ and $\varphi \in H^s_{\mathrm{per}}(Q)$. In a first step, we calculate the Fourier-coefficients $u_{\mu,\nu}$ of $u = \mathcal{M}\varphi$. Therefore, let $\mu, \nu \in \mathbb{Z}^2$. Then

$$\begin{aligned}
u_{\mu,\nu} &= \int_Q \int_{Q_\varrho} u(t,\tau)\, T_Q^{(-\mu)}(t)\, T_{Q_\varrho}^{(-\nu)}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t \\
&= \frac{1}{2\varepsilon_3\varrho} \int_{Q_\varrho} \chi_{\varrho,\varepsilon_3\varrho}(\tau)\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{\varepsilon_3\varrho}\nu\cdot\tau} \left(\frac{1}{2\pi} \int_Q \varphi(t+\tau)\, \mathrm{e}^{-\mathrm{i}\mu\cdot(t+\tau-\tau)}\, \mathrm{d}t \right) \mathrm{d}\tau \\
&= \frac{1}{2\varepsilon_3\varrho} \int_{\mathbb{R}^2} \chi_{\varrho,\varepsilon_3\varrho}(\tau)\, \mathrm{e}^{-\mathrm{i}(\frac{\pi}{\varepsilon_3\varrho}\nu-\mu)\cdot\tau} \left(\frac{1}{2\pi} \int_{\tau+Q} \varphi(t')\, \mathrm{e}^{-\mathrm{i}\mu\cdot t'}\, \mathrm{d}t' \right) \mathrm{d}\tau \\
&= \frac{1}{2\varepsilon_3\varrho}\, \widehat{\chi_{\varrho,\varepsilon_3\varrho}}(\tfrac{\pi}{\varepsilon_3\varrho}\nu - \mu)\varphi_\mu,
\end{aligned}$$

where the last step holds due to the Q-periodicity of φ. Now, in a second step, for $\sigma \ge 0$, there holds

$$p_{s,\sigma}(u) = \frac{1}{(2\varepsilon_3\varrho)^2} \sum_{\mu,\nu\in\mathbb{Z}^2} \left(1+|\mu|^2\right)^s \frac{\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^{\sigma+2}}{\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^2} |\varphi_\mu|^2 \left|\widehat{\chi_{\varrho,\varepsilon_3\varrho}}(\tfrac{\pi}{\varepsilon_3\varrho}\nu-\mu)\right|^2.$$

From

$$\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^2}\,\mathrm{d}x \ge h^2 \sum_{\substack{\nu\in\mathbb{Z}^2\\ \nu_1,\nu_2\neq 0}} \frac{1}{(1+|h\nu|^2)^2}\,, \qquad h>0\,,$$

and similar estimates for the remaining terms in the sum, we see that the value of the series $\sum_{\nu\in\mathbb{Z}^2}\left(1+|\frac{\pi}{\varepsilon_3\varrho}\nu-\mu|^2\right)^{-2}$ is uniformly bounded in μ and ϱ for $\varepsilon_3\varrho\le\varrho_0$. Thus from Lemma 5.10, the assertion follows.
(ii). Using the Fourier series expansion of u, there holds

$$\mathcal{J}^{(\lambda)}u = \sum_{\mu,\nu\in\mathbb{Z}^2} u_{\mu,\nu} \int_Q T_Q^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,T_{Q_\varrho}^{(\nu)}(\Pi(p))\,\mathrm{d}p\,T_Q^{(\mu)}.$$

Suppose $\nu\neq 0$. We write $(\nu_1,\nu_2)^\top = q_\nu(\cos\vartheta_\nu,\sin\vartheta_\nu)^\top$ for some $q_\nu>0$ and some $\vartheta_\nu\in(-\pi,\pi]$, and obtain

$$T_{Q_\varrho}^{(\nu)}(\Pi(p)) = \frac{1}{2\varepsilon_3\varrho}\exp\left(\mathrm{i}\,q_\nu\,(r/\varepsilon_3)\,\cos(\vartheta-\vartheta_\nu)\right), \qquad p=(r,\vartheta)\in Q.$$

Hence, the substitution $\vartheta'=\vartheta-\vartheta_\nu$ and the 2π-periodicity with respect to ϑ yield

$$\begin{aligned}\int_Q T_Q^{(\lambda)}(p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,T_{Q_\varrho}^{(\nu)}(\Pi(p))\,\mathrm{d}p &= \frac{1}{2\varepsilon_3\varrho}\mathrm{e}^{\mathrm{i}\,\lambda_2\,\vartheta_\nu}\int_Q T_Q^{(\lambda)}(r,\vartheta')\\ &\quad\times\left[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi\right](r,\vartheta'+\vartheta_\nu)\mathrm{e}^{\mathrm{i}\,q_\nu\,(r/\varepsilon_3)\,\cos\vartheta'}\,\mathrm{d}(r,\vartheta').\end{aligned}$$

The behaviour of the integral in this expression with respect to λ and ν can be estimated by the method of stationary phase. A detailed proof is given in [7, Lemma 6.2]. We obtain

$$\Big|\int_Q T_Q^{(\lambda)}(p)\,\left[\chi_{\varepsilon_2\varrho,\varrho}\,T_{Q_\varrho}^{(\nu)}\right]\circ\Pi(p)\,\mathrm{d}p\Big| \le C\,\frac{\left\|T_Q^{(\lambda)}\left[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi\right](\cdot,\cdot+\vartheta_\nu)\right\|_{\infty;2}}{q_\nu}\,.$$

Similarly as in the proof of Lemma 5.10 we observe that

$$\left[\chi_{\varepsilon_2\varrho,\varrho}\circ\Pi\right](r,\vartheta+\vartheta_\nu) = \chi_{\varepsilon_2,1}\left(\frac{r}{\pi}\begin{pmatrix}\cos(\vartheta+\vartheta_\nu)\\ \sin(\vartheta+\vartheta_\nu)\end{pmatrix}\right)$$

is independent of ϱ. Hence

$$\Big| \int_Q T_Q^{(\lambda)}(p) \, \Big[\chi_{\varepsilon_2 \varrho, \varrho} \, T_{Q_\varrho}^{(\nu)} \Big] \circ \Pi(p) \, \mathrm{d}p \Big| \\ \leq C \, \frac{\big\| T_Q^{(\lambda)} \big\|_{\infty;2}}{q_\nu} \leq \sqrt{2} \, C \, \frac{\big\| T_Q^{(\lambda)} \big\|_{\infty;2}}{(1+|\nu|^2)^{1/2}} \,. \tag{5.16}$$

Note that the final estimate is also true for $\nu = 0$. Using Lemma 5.4, gives

$$\left| \int_Q T_Q^{(\lambda)}(p) \chi_{\varepsilon_2 \varrho, \varrho}(\Pi(p)) T_{Q_\varrho}^{(\nu)}(\Pi(p)) \, \mathrm{d}p \right| \leq C \frac{1+|\lambda|^2}{(1+|\nu|^2)^{1/2}} \,.$$

We proceed with

$$\begin{aligned} \left\| \mathcal{J}^{(\lambda)} u \right\|^2_{H^{s+1}_{\mathrm{per}}(Q)} &\leq C^2 \sum_{\mu \in \mathbb{Z}^2} (1+|\mu|^2)^{s+1} \left(\sum_{\nu \in \mathbb{Z}^2} \tfrac{1+|\lambda|^2}{(1+|\nu|^2)^{1/2}} |u_{\mu,\nu}| \right)^2 \\ &= C^2 (1+|\lambda|^2)^2 \sum_{\mu \in \mathbb{Z}^2} (1+|\mu|^2)^s \left(\sum_{\nu \in \mathbb{Z}^2} \left(\tfrac{1+|\mu|^2}{1+|\nu|^2} \right)^{1/2} |u_{\mu,\nu}| \right)^2 \\ &\leq C^2 (1+|\lambda|^2)^2 \sum_{\mu \in \mathbb{Z}^2} (1+|\mu|^2)^s \\ &\leq C^2 \, \varrho^{-2} \, (1+|\lambda|^2)^2 \sum_{\mu \in \mathbb{Z}^2} (1+|\mu|^2)^s \left(\sum_{\nu \in \mathbb{Z}^2} \frac{(1+|\mu - \frac{\pi}{\varepsilon_3 \varrho} \nu|^2)^{3/2}}{1+|\mu - \frac{\pi}{\varepsilon_3 \varrho} \nu|^2} \, |u_{\mu,\nu}| \right)^2 . \end{aligned} \tag{5.17}$$

As in the proof of part *(i)*, the series $\sum_{\nu \in \mathbb{Z}^2} \left(1 + |\mu - \frac{\pi}{\varepsilon_3 \varrho} \nu|^2 \right)^{-2}$ is bounded independently of μ and $\varrho < 1$, so that we can apply the Hölder inequality for ℓ^2-series to obtain

$$\begin{aligned} \left\| \mathcal{J}^{(\lambda)} u \right\|^2_{H^{s+1}_{\mathrm{per}}(Q)} &\leq C^2 \, \varrho^{-2} \, (1+|\lambda|^2)^2 \\ &\quad \times \sum_{\mu,\nu \in \mathbb{Z}^2} (1+|\mu|^2)^s (1+|\mu - \tfrac{\pi}{\varepsilon_3 \varrho} \nu|^2)^3 |u_{\mu,\nu}|^2 \end{aligned}$$

$$= C^2\, \varrho^{-2}\, (1+|\lambda|^2)^2\, p_{s,3}(u)\,.$$

(iii). The assertion follows directly by combining *(i)* and *(ii).* □

With these preliminary considerations, we are now able to investigate the mapping properties of J_1.

Theorem 5.12 *Let $s \geq 0$. Then $J_1 : H^s_{\text{per}}(Q) \to H^{s+1}_{\text{per}}(Q)$ defined in (5.8) is a linear and bounded operator with*

$$\|J_1\varphi\|_{H^{s+1}_{\text{per}}(Q)} \leq C\, \varrho^{-5}\, \|\varphi\|_{H^s_{\text{per}}(Q)}$$

for $\varepsilon_3\varrho \leq \varrho_0$ with C depending only on k, s, ε_1, ε_2 and ε_3.

Proof: By definition, J_1 is a linear integral operator. To show boundedness from $H^s_{\text{per}}(Q)$ to $H^{s+1}_{\text{per}}(Q)$, we insert $\chi_{\varrho,\varepsilon_3\varrho}(\Pi(\cdot))$ in the integrand and expand k_{polar} into its Fourier series (5.14). With $\mathcal{J}^{(\lambda)}$ from Lemma 5.11 we obtain

$$J_1\varphi(t) = \sum_{\lambda\in\mathbb{Z}^2} k^{(\lambda)}_{\text{polar}}(t)\, \mathcal{J}^{(\lambda)}\, \mathcal{M}\varphi(t)\,.$$

This is justified by the estimates for any $\sigma \in \mathbb{N}_{\geq s+1}$ using Lemma 5.11

$$\begin{aligned}
\|J_1\varphi\|_{H^{s+1}_{\text{per}}(Q)} &\leq C \sum_{\lambda\in\mathbb{Z}^2} \|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma}\, \left\|\mathcal{J}^{(\lambda)}\mathcal{M}\,\varphi\right\|_{H^{s+1}_{\text{per}}(Q)} \\
&\leq C\, \varrho^{-5}\, \|\varphi\|_{H^s_{\text{per}}(Q)} \sum_{\lambda\in\mathbb{Z}^2} \big(1+|\lambda|^2\big) \|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma} \\
&\leq C\, \varrho^{-5}\, \|\varphi\|_{H^s_{\text{per}}(Q)} \sum_{\lambda\in\mathbb{Z}^2} \frac{1}{(1+|\lambda|^2)^2} \big(1+|\lambda|^2\big)^3 \|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma}\,.
\end{aligned}$$

The series converges as the two last factors are bounded by Lemma 5.5 with

$$\sup_{\lambda\in\mathbb{Z}^2} \big(1+|\lambda|^2\big)^3 \|k^{(\lambda)}_{\text{polar}}\|_{\infty;\sigma} \leq C\, \|k_{\text{polar}}\|_{\infty,\sigma+6}\,.$$

With t, $p = (r,\vartheta)^\top \in Q$ and setting $\hat{p} = (\cos\vartheta, \sin\vartheta)^\top$, we write $k_{\text{polar}}(t,p)$ as

$$k_{\text{polar}}(t,p) = \frac{\varrho}{2\pi}\, \ell\left(t, \frac{\varrho\, r}{\pi}, \hat{p}\right)\, \chi_{\varepsilon_1,\varepsilon_2}\left(\frac{r}{\pi}\, \hat{p}\right)$$

with the function ℓ from Assumption 5.6. It follows that $\|k_{\mathrm{polar}}\|_{\infty,\sigma+6} \le C$ uniformly for $\varepsilon_3\varrho \le \varrho_0$ with C depending only on σ, ε_1 and ε_2. This completes the proof. $\square$

We next derive an analogue of Theorem 5.8 for J_1, i.e. an estimate for the difference $J_1 - J_{1,M,\tilde{M}}$. To this end, we write

$$J_1 - J_{1,M,\tilde{M}} = \left[J_1 - \tilde{J}_{1,\tilde{M}}\right] + \left[\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}}\right],$$

where

$$\tilde{J}_{1,\tilde{M}}\varphi(t) := \int_Q k_{\mathrm{polar}}(t,p)\left[\left\{\chi_{\varepsilon_2\varrho,\varrho}\,\tilde{O}_{\tilde{M}}\left[\chi_{\varrho,\varepsilon_3\varrho}\,\varphi(t+\cdot)\right]\right\}\circ\Pi\right](p)\,\mathrm{d}p \tag{5.18}$$

for $t \in Q$. Note that, using the projection

$$\mathcal{O}_{\tilde{M}}u(\cdot,\cdot\cdot) := \sum_{\mu\in\mathbb{Z}^2}\sum_{\nu\in\mathbb{Z}^2_{\tilde{M}}} u_{\mu,\nu}\,T_Q^{(\mu)}(\cdot)\,T_{Q_\varrho}^{(\nu)}(\cdot\cdot)\,, \qquad u \in \mathcal{H}^s_{Q,Q_\varrho}\,, \tag{5.19}$$

we can write $\tilde{J}_{1,\tilde{M}}$ as $\tilde{J}_{1,\tilde{M}}\varphi = \sum_{\lambda\in\mathbb{Z}^2} k^{(\lambda)}_{\mathrm{polar}}\,\mathcal{J}^{(\lambda)}\mathcal{O}_{\tilde{M}}\mathcal{M}\,\varphi$.

Lemma 5.13 *Let $s \ge 0$, $\varepsilon_3\varrho \le \varrho_0$, $M \in \mathbb{N}^2$ and recall the definitions of $\mathcal{J}^{(\lambda)}$ and $\mathcal{M}$ from Lemma 5.11.*

(i) For all $u \in \mathcal{H}^s_{Q,Q_\varrho}$ and $\lambda \in \mathbb{Z}^2$,

$$\|\mathcal{J}^{(\lambda)}u\|_{H^{s+1}_{\mathrm{per}}(Q)} \le C\,\varrho^{-1}\,(1+|\lambda|^2)\,\sqrt{q_{s,s+3}(u)}\,,$$

where C depends only on s, ε_2 and ε_3.

(ii) For $0 \le t \le s$, $\sigma \ge 0$ and all $u \in \mathcal{H}^s_{Q,Q_\varrho}$,

$$q_{t,\sigma}\big((\mathcal{I}-\mathcal{O}_M)u\big) \le \big(\sqrt{2}\,\underline{M}\big)^{2(t-s)} q_{s,\sigma}(u)\,.$$

Here $\mathcal{I}$ denotes the identity operator and for $\underline{M}$ recall (1.3)*.*

(iii) Let $0 \le t \le s$, $\lambda \in \mathbb{Z}^2$. Then $\mathcal{J}^{(\lambda)}(\mathcal{I}-\mathcal{O}_M)\mathcal{M} : H^s_{\mathrm{per}}(Q) \to H^{t+1}_{\mathrm{per}}(Q)$ is bounded with

$$\|\mathcal{J}^{(\lambda)}(\mathcal{I}-\mathcal{O}_M)\mathcal{M}\,\varphi\|_{H^{t+1}_{\mathrm{per}}(Q)} \le C\,\varrho^{-2s-5}\,(1+|\lambda|^2)\,\underline{M}^{t-s}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)}$$

for all $\varphi \in H^s_{\mathrm{per}}(Q)$, where the constant $C > 0$ only depends on s, t, ε_2 and ε_3.

Proof: *(i).* This follows from Lemma 5.11 *(ii)* together with (5.15).
(ii). Let $0 \le t \le s$ and $\sigma \ge 0$. Then

$$
\begin{aligned}
q_{t,\sigma}\big((\mathcal{I} - \mathcal{O}_M)u\big) &= \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_M} \big(1 + |\tfrac{\pi}{\varepsilon_3 \varrho}\nu|^2\big)^t \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho}\nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\
&\le \sum_{\mu \in \mathbb{Z}^2} \sum_{\nu \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_M} \big(1 + |\nu|^2\big)^{t-s} \big(1 + |\tfrac{\pi}{\varepsilon_3 \varrho}\nu|^2\big)^s \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3 \varrho}\nu|^2\big)^\sigma |u_{\mu,\nu}|^2 \\
&\le \big(\sqrt{2}\,\underline{M}\big)^{2(t-s)} q_{s,\sigma}(u)
\end{aligned}
$$

holds for all $u \in \mathcal{H}^s_{Q,Q_\varrho}$.
(iii). Let $\varphi \in H^s_{\mathrm{per}}(Q)$ and set $u = \mathcal{M}\varphi$. Then, by Lemma 5.11 *(i)* and Remark 5.9, $u \in \mathcal{H}^t_{Q,Q_\varrho}$, and hence also $(\mathcal{I} - \mathcal{O}_M)u \in \mathcal{H}^t_{Q,Q_\varrho}$. From part *(i)* and *(ii)* together with (5.15) and Lemma 5.11 *(i)*, we obtain the estimate

$$
\begin{aligned}
\|\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_M)\,u\|_{H^{t+1}_{\mathrm{per}}(Q)} &\le C\,\varrho^{-1}\,(1 + |\lambda|^2)\,\sqrt{q_{t,t+3}\big((I - \mathcal{O}_M)u\big)} \\
&\le C\,\varrho^{-1}\,(1 + |\lambda|^2)\,\underline{M}^{t-s}\,\sqrt{q_{s,s+3}(u)} \\
&\le C\,\varrho^{-1}\,(1 + |\lambda|^2)\,\underline{M}^{t-s}\,\sqrt{p_{s,2s+3}(u)} \\
&\le C\,\varrho^{-2s-5}\,(1 + |\lambda|^2)\,\underline{M}^{t-s}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)},
\end{aligned}
$$

which is the desired result. □

Theorem 5.14 *Let $\tilde{M} \in \mathbb{N}^2$, $s \ge 0$ and $t \in [0, s]$. Then the operator $\tilde{J}_{1,\tilde{M}} : H^s_{\mathrm{per}}(Q) \to H^{t+1}_{\mathrm{per}}(Q)$ defined in* (5.18) *is well-defined, linear and bounded with*

$$
\|(J_1 - \tilde{J}_{1,\tilde{M}})\varphi\|_{H^{t+1}_{\mathrm{per}}(Q)} \le C\,\varrho^{-2s-5}\,\underline{\tilde{M}}^{t-s}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)}
$$

for all $\varphi \in H^s_{\mathrm{per}}(Q)$ and $\varepsilon_3 \varrho \le \varrho_0$, where $C > 0$ only depends on k, s, t, ε_2 and ε_3.

Proof: Let $\varphi \in H^s_{\mathrm{per}}(Q)$ and $\sigma \in \mathbb{N}_{\ge s+1}$. Proceeding analogously as in the proof of Theorem 5.12, from Lemma 5.13 *(iii)* we obtain

$$
\|(J_1 - \tilde{J}_{1,\tilde{M}})\varphi\|_{H^{t+1}_{\mathrm{per}}(Q)} \le C \sum_{\lambda \in \mathbb{Z}^2} \|k^{(\lambda)}_{\mathrm{polar}}\|_{\infty;\sigma} \big\|\mathcal{J}^{(\lambda)}(\mathcal{I} - \mathcal{O}_{\tilde{M}})\mathcal{M}\,\varphi\big\|_{H^{t+1}_{\mathrm{per}}(Q)}
$$

$$\leq C\,\varrho^{-2s-5}\,\underline{\tilde{M}}^{t-s}\|\varphi\|_{H^s_{\mathrm{per}}(Q)}\sum_{\lambda\in\mathbb{Z}^2}\big(1+|\lambda|^2\big)\|k^{(\lambda)}_{\mathrm{polar}}\|_{\infty;\sigma}\,.$$

For the remainder of this proof we proceed as at the end of the proof of Theorem 5.12. □

Theorem 5.15 *Let $M,\tilde{M}\in\mathbb{N}^2$, $s\geq 0$ and $t\in[0,s]$. Then the operator $J_{1,M,\tilde{M}}:H^s_{\mathrm{per}}(Q)\to H^{t+1}_{\mathrm{per}}(Q)$ defined in (5.13) is well-defined, linear and bounded. Moreover, there exists some $\tau>0$ such that*

$$\|(\tilde{J}_{1,\tilde{M}}-J_{1,M,\tilde{M}})\,\varphi\|_{H^{t+1}_{\mathrm{per}}(Q)}\leq C\,\varrho^{-4}\,\frac{\overline{M}^{\tau}}{\underline{M}^{s-t+\tau}}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)}$$

for all $\varphi\in H^s_{\mathrm{per}}(Q)$ and $\varepsilon_3\varrho\leq\varrho_0$, where $C>0$ only depends on k, s, t, τ, ε_2 and ε_3.

Proof: We follow the proof of [7, Theorem 6.5]. Let $\varphi\in H^s_Q$. We set

$$v(p):=\big\{\chi_{\varepsilon_2\varrho,\varrho}\,\tilde{O}_{\tilde{M}}[\chi_{\varrho,\varepsilon_3\varrho}\,\varphi(t+\cdot)]\big\}\circ\Pi(p),\qquad p\in Q,$$

and write the operators as

$$\tilde{J}_{1,\tilde{M}}\varphi=\sum_{\lambda\in\mathbb{Z}^2}k^{(\lambda)}_{\mathrm{polar}}\int_Q T^{(\lambda)}_Q(p)\,v(p)\,\mathrm{d}p\,,$$

$$J_{1,M,\tilde{M}}\varphi=\sum_{\lambda\in\mathbb{Z}^2}k^{(\lambda)}_{\mathrm{polar}}\int_Q P_M\left[T^{(\lambda)}_Q O_M v\right](p)\,\mathrm{d}p\,.$$

A central observation regarding this representation of $J_{1,M,\tilde{M}}$ is

$$\int_Q P_M\big[T^{(\lambda)}_Q O_M v\big](p)\,\mathrm{d}p=\sum_{\iota\in\mathbb{Z}^2_M}T^{(\lambda)}_Q(t^M_\iota)\,O_M v(t^M_\iota)\int_Q L^M_\iota(p)\,\mathrm{d}p$$

$$=\sum_{\iota\in\mathbb{Z}^2_M}T^{(\lambda)}_Q(t^M_\iota)\frac{\pi^2}{M_1M_2}O_M v(t^M_\iota)$$

$$=\sum_{\iota\in\mathbb{Z}^2_M}T^{(\lambda)}_Q(t^M_\iota)\frac{\pi^2}{M_1M_2}\Big(\sum_{\nu\in\mathbb{Z}^2_M}\frac{M_1M_2}{\pi^2}\int_Q L^M_\nu(p)\,v(p)\,\mathrm{d}p\,L^M_\nu(t^M_\iota)\Big)$$

$$= \sum_{\iota\in\mathbb{Z}^2_M} T_Q^{(\lambda)}(t_\iota^M) \int_Q L_\iota^M(p)\, v(p)\, \mathrm{d}p = \int_Q v(p)\, P_M T_Q^{(\lambda)}(p)\, \mathrm{d}p\,,$$

so that we obtain

$$(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}})\varphi = \sum_{\lambda\in\mathbb{Z}^2} k_{\text{polar}}^{(\lambda)} \int_Q v(p) \left[T_Q^{(\lambda)}(p) - P_M T_Q^{(\lambda)}(p)\right] \mathrm{d}p\,. \tag{5.20}$$

Moreover, let $\tau > 3$ and $\omega \geq s - t + \tau$. By Sobolev's Imbedding Theorem, the space $H^\tau_{\text{per}}(Q)$ is continuously imbedded in the space of twice continuously differentiable Q-periodic functions. Hence, by Lemma 5.1

$$\begin{aligned}\big\|T_Q^{(\lambda)} - P_M T_Q^{(\lambda)}\big\|_{\infty;2} &\leq C\, \big\|T_Q^{(\lambda)} - P_M T_Q^{(\lambda)}\big\|_{H^\tau_{\text{per}}(Q)} \\ &\leq C\, \frac{\overline{M}^\tau}{\underline{M}^\omega}\, \big\|T_Q^{(\lambda)}\big\|_{H^\omega_{\text{per}}(Q)} \leq C\, \frac{\overline{M}^\tau}{\underline{M}^{s-t+\tau}} \big(1 + |\lambda|^2\big)^{\omega/2}.\end{aligned}$$

Setting $u := \mathcal{M}\varphi$ with $\mathcal{M}$ from Lemma 5.11 and recalling $\mathcal{O}_{\tilde{M}}$ from (5.19), we obtain

$$\begin{aligned}&\int_Q \big(T_Q^{(\lambda)}(p) - P_M T_Q^{(\lambda)}(p)\big)\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\, \mathcal{O}_{\tilde{M}} u(\cdot, \Pi(p))\, \mathrm{d}p \\ &\quad = \sum_{\mu\in\mathbb{Z}^2} \sum_{\nu\in\mathbb{Z}^2_{\tilde{M}}} u_{\mu,\nu} \int_Q \big(T_Q^{(\lambda)} - P_M T_Q^{(\lambda)}\big)(p) \left[\chi_{\varepsilon_2\varrho,\varrho}\, T_{Q_\varrho}^{(\nu)}\right] \circ \Pi(p)\, \mathrm{d}p\, T_Q^{(\mu)}.\end{aligned}$$

Hence, by a slight modification of the estimate in (5.16), we can proceed as in (5.17) to obtain

$$\begin{aligned}&\Big\| \int_Q \big(T_Q^{(\lambda)}(p) - P_M T_Q^{(\lambda)}(p)\big)\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\, \mathcal{O}_{\tilde{M}} u(\cdot, \Pi(p))\, \mathrm{d}p \Big\|_{H^{t+1}_{\text{per}}(Q)} \\ &\leq C\, \frac{\overline{M}^\tau\, (1 + |\lambda|^2)^{\omega/2}}{\underline{M}^{s-t+\tau}} \left(\sum_{\mu\in\mathbb{Z}^2} \big(1 + |\mu|^2\big)^t \Big(\sum_{\nu\in\mathbb{Z}^2_{\tilde{M}}} \tfrac{(1+|\mu|^2)^{1/2}}{(1+|\nu|^2)^{1/2}} |u_{\mu,\nu}| \Big)^2 \right)^{1/2} \\ &\leq C\, \frac{\overline{M}^\tau\, (1 + |\lambda|^2)^{\omega/2}}{\underline{M}^{s-t+\tau}} \Big(\sum_{\substack{\mu\in\mathbb{Z}^2\\ \nu\in\mathbb{Z}^2_{\tilde{M}}}} \big(1 + |\mu|^2\big)^t \big(1 + |\mu - \tfrac{\pi}{\varepsilon_3\varrho}\nu|^2\big)^3 |u_{\mu,\nu}|^2 \Big)^{1/2} \\ &\leq C\, \frac{\overline{M}^\tau}{\underline{M}^{s-t+\tau}} \big(1 + |\lambda|^2\big)^{\omega/2} \sqrt{p_{t,3}(u)}.\end{aligned}$$

Now, by setting $\sigma := \lfloor t \rfloor$, from (5.20) we conclude that

$$\begin{aligned}\big\|(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}})\varphi\big\|_{H^{t+1}_{\mathrm{per}}(Q)} &\le C \sum_{\lambda\in\mathbb{Z}^2} \|k^{(\lambda)}_{\mathrm{polar}}\|_{\infty;\sigma+2} \\ &\times \left\| \int_Q \big(T_Q^{(\lambda)}(p) - P_M T_Q^{(\lambda)}(p)\big)\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\mathcal{O}_{\tilde{M}}u(\cdot,\Pi(p))\,\mathrm{d}p\right\|_{H^{t+1}_{\mathrm{per}}(Q)} \\ &\le C\,\frac{\overline{M}^\tau}{\underline{M}^{s-t+\tau}}\sqrt{p_{t,3}(u)}\sum_{\lambda\in\mathbb{Z}^2}\|k^{(\lambda)}_{\mathrm{polar}}\|_{\infty;\sigma+2}\big(1+|\lambda|^2\big)^{\omega/2}.\end{aligned}$$

Using Lemma 5.11 *(i)*, Lemma 5.5 and arguing as in the proof of Theorem 5.12, we establish the bound

$$\big\|(\tilde{J}_{1,\tilde{M}} - J_{1,M,\tilde{M}})\varphi\big\|_{H^{t+1}_{\mathrm{per}}(Q)} \le C\,\frac{\overline{M}^\tau}{\underline{M}^{s-t+\tau}}\,\varrho^{-4}\,\|\varphi\|_{H^t_{\mathrm{per}}(Q)}\,.$$

From this, the assertion follows due to the continuous imbedding of $H^s_{\mathrm{per}}(Q)$ in $H^t_{\mathrm{per}}(Q)$. □

The Fully Discrete System. We now consider the approximation of the solution of the integral equation (5.7) by the fully discrete version of (5.9) which is to find $\varphi_N \in \mathcal{T}_N(Q)$ such that

$$\varphi_N - P_N\,(J_{1,M,\tilde{M}} + J_{2,N})\varphi_N = P_N\psi\,. \tag{5.21}$$

Based on our results so far, we now prove stability and convergence for Equation (5.21). To simplify expressions in the following analysis, let us assume $N_1 = N_2$ and introduce the meshsize $h := \pi/N_1$. We next set $\tilde{M}_1 := \tilde{M}_2 := \lceil \varrho/h \rceil$, $M := \tilde{M}$ and furthermore

$$A := J_1 + J_2\,, \qquad A_h := P_N\,(J_{1,M,\tilde{M}} + J_{2,N})\,.$$

We will assume that $I - A$ is boundedly invertible on any $H^s_{\mathrm{per}}(Q)$, $s \ge 0$. This is no restriction with respect to our application of electromagnetic scattering in mind, see Theorems 4.20 and 4.18, and many other applications, see [9].

Theorem 5.16 *Let $t > 1$ and assume that $\varrho = h^\alpha$ for some $\alpha \in (0, \frac{1}{2t+6})$. Then there exists $h_0 > 0$ such that $I - A_h \in \mathcal{L}_{\mathrm{is}}(H^t_{\mathrm{per}}(Q))$ for $0 < h \le h_0$, with uniformly bounded inverse.*

Proof: We write

$$A - A_h = (J_1 - J_{1,M,\tilde{M}}) + (J_2 - J_{2,N}) + (I - P_N)\,(J_{1,M,\tilde{M}} + J_{2,N})\,.$$

From Theorems 5.8, 5.15 and 5.14, we have the estimates

$$\begin{aligned}
\|(J_1 - J_{1,M,\tilde{M}})\varphi\|_{H^t_{\text{per}}(Q)} &\le C\,h\,\varrho^{-1}\left(\varrho^{-2t-5} + \varrho^{-4}\right)\|\varphi\|_{H^t_{\text{per}}(Q)}\,,\\
\|(J_1 - J_{1,M,\tilde{M}})\varphi\|_{H^{t+1}_{\text{per}}(Q)} &\le C\left(\varrho^{-2t-5} + \varrho^{-4}\right)\|\varphi\|_{H^t_{\text{per}}(Q)}\,,\\
\|(J_2 - J_{2,N})\varphi\|_{H^t_{\text{per}}(Q)} &\le C\,h\,\varrho^{-2t-6}\,\|\varphi\|_{H^t_{\text{per}}(Q)}\,,\\
\|(J_2 - J_{2,N})\varphi\|_{H^{t+1}_{\text{per}}(Q)} &\le C\,\varrho^{-2t-6}\,\|\varphi\|_{H^t_{\text{per}}(Q)}\,.
\end{aligned}$$

By Lemma 5.1, the norm of $I - P_N : H^{t+1}_{\text{per}}(Q) \to H^t_{\text{per}}(Q)$ is bounded by the number Ch. Thus

$$\|(A - A_h)\,\varphi\|_{H^t_{\text{per}}(Q)} \le C\,h\,\varrho^{-2t-6}\,\|\varphi\|_{H^t_{\text{per}}(Q)} \longrightarrow 0 \quad (h \to 0)\,.$$

The assertion follows now from standard results for operator approximation, see for instance [36]. □

Theorem 5.17 *Let $\alpha \in (0, 1/3)$ and $\varrho = h^\alpha$. Assume that $t \ge 0$ and $s > \max\{1, t, \frac{10\alpha+3\alpha t+t}{1-3\alpha}\}$. Furthermore, let (5.21) be a stable approximation of (5.7) in $H^s_{\text{per}}(Q)$, i.e. there exists $c > 0$ such that $\|\varphi_h\|_{H^s_{\text{per}}(Q)} \le c\|\varphi\|_{H^s_{\text{per}}(Q)}$ for sufficiently small h. Then there exists $h_0 > 0$ such that*

$$\|\varphi - \varphi_h\|_{H^t_{\text{per}}(Q)} \le C\,h^{(s-t)(1-3\alpha)/2}\|\varphi\|_{H^s_{\text{per}}(Q)}$$

for all $0 < h \le h_0$.

Proof: From $\varphi_h = P_N\psi + A_h\varphi_h = P_N(\varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h)$ we obtain

$$\begin{aligned}
&(I - A)(\varphi - \varphi_h)\\
&= \varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h - \varphi_h - (J_{1,M,\tilde{M}} + J_{2,N} - A)\varphi_h\\
&= (I - P_N)(\varphi - A\varphi + J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h) - (J_{1,M,\tilde{M}} + J_{2,N} - A)\varphi_h\,.
\end{aligned}$$

From Theorems 5.8, 5.14 and 5.15, we have

$$\|J_{1,M,\tilde{M}}\varphi_h + J_{2,N}\varphi_h\|_{H^s_{\text{per}}(Q)} \le \left(\|A\| + C\,h\,\varrho^{-2s-6}\right)\|\varphi_h\|_{H^s_{\text{per}}(Q)}\,.$$

Similarly, $\|(J_{1,M,\tilde{M}} + J_{2,N} - A)\,\varphi_h\|_{H^1_{\mathrm{per}}(Q)} \leq C\,h^s\,\varrho^{-3s-5}\,\|\varphi_h\|_{H^s_{\mathrm{per}}(Q)}$. Thus from the boundedness of $(I-A)^{-1}$ in $L^2(Q)$, Lemma 5.1 and the stability estimate, we conclude

$$\|\varphi - \varphi_h\|_{L^2(Q)} \leq C\,\|(I-A)(\varphi - \varphi_h)\|_{L^2(Q)} \leq C\,h^s\,\varrho^{-3s-5}\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)}$$

for all $h \leq h_0$ such that also $\varrho \leq \varrho_0$.
For the general result, we observe that for $T \in \mathcal{T}_N(Q)$, the estimate $\|T\|_{H^t_{\mathrm{per}}(Q)} \leq C\,h^{-t}\,\|T\|_{L^2(Q)}$ follows directly from the definition of the norm in $H^t_{\mathrm{per}}(Q)$. Using the orthogonal projection O_N, we have

$$\begin{aligned}\|\varphi - \varphi_h\|_{H^t_{\mathrm{per}}(Q)} &\leq \|\varphi - O_N\varphi\|_{H^t_{\mathrm{per}}(Q)} + \|O_N\varphi - \varphi_h\|_{H^t_{\mathrm{per}}(Q)}\\ &\leq \|\varphi - O_N\varphi\|_{H^t_{\mathrm{per}}(Q)} + C\,h^{-t}\,\|O_N\varphi - \varphi_h\|_{L^2(Q)}\\ &\leq \|\varphi - O_N\varphi\|_{H^t_{\mathrm{per}}(Q)} + C\,h^{-t}\,\|\varphi - \varphi_h\|_{L^2(Q)}\,,\end{aligned}$$

where the last estimate follows from the Pythagorean theorem. For $\varphi - O_N\varphi$, bounds as for $\varphi - P_N\varphi$ have been shown in the proof of Lemma 5.1. Thus

$$\|\varphi - \varphi_h\|_{H^t_{\mathrm{per}}(Q)} \leq C\,h^{s-t}\left(1 + \varrho^{-3s-5}\right)\|\varphi\|_{H^s_{\mathrm{per}}(Q)}\,.$$

From $s \geq \frac{10\alpha+3\alpha t+t}{1-3\alpha}$ follows $(s-t)(1-3\alpha)/2 \geq \alpha(5+3t)$. Thus

$$h^{s-t}\,\varrho^{-3s-5} = h^{s-t-\alpha(3s+5)} = h^{(s-t)(1-3\alpha)}\,h^{-\alpha(5+3t)} \leq h^{(s-t)(1-3\alpha)/2}\,.$$

This concludes the proof. □

By assumption, the Equation (5.7) is uniquely solvable and the solution $\varphi := (I-A)^{-1}\,\psi$ belongs to $H^s_{\mathrm{per}}(Q)$ for any $s \geq 0$, provided the right hand side ψ does. In this case, Theorem 5.17 establishes a *super-algebraic* convergence rate, i.e., for fixed $t \geq 0$ and any $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\|\varphi - \varphi_h\|_{H^t_{\mathrm{per}}(Q)} \leq C_n\,h^n, \qquad 0 < h \leq h_0.$$

5.3. Extension to Systems of Biperiodic Integral Equations

As we want to apply the numerical scheme from the last section to electromagnetic scattering problems, we have it to generalize to systems of biperiodic integral equations as given in (5.1). This is topic of the present section. The procedure is straightforward and involves matrix operators, which will be indicated by bold letters.

Inspecting (5.1) again, now of course we require each kernel function $k^{(i,j)}$ and each right hand side $\psi^{(i)}$ to satisfy Assumption 5.6 (see also Proposition 4.23 for our application in mind). Then, by exploiting the results from Section 5.2, the operators

$$\begin{aligned} (J_1^{(i,j)}\varphi)(t) &:= \int_Q k_{\text{polar}}^{(i,j)}(t,p)\,\chi_{\varepsilon_2\varrho,\varrho}(\Pi(p))\,\varphi(t+\Pi(p))\,\mathrm{d}p, \\ (J_2^{(i,j)}\varphi)(t) &:= \int_Q k_{\text{smooth}}^{(i,j)}(t,\tau)\,\varphi(\tau)\,\mathrm{d}\tau, \end{aligned} \qquad t \in Q,$$

with $k_{\text{polar}}^{(i,j)}$ and $k_{\text{smooth}}^{(i,j)}$ as in Section 5.2, but now for $k_1^{(i,j)}$ and $k_2^{(i,j)}$ instead of k_1 and k_2, respectively, satisfy Theorem 5.12 and Theorem 5.7, respectively. And for their discrete analogs $\tilde{J}_{1,\tilde{M}}^{(i,j)}$, $J_{1,M,\tilde{M}}^{(i,j)}$ and $J_{2,N}^{(i,j)}$, see (5.18), (5.13) and (5.11), there hold the statements from Theorem 5.14, 5.15 and 5.8, respectively.

The following lemma gives a useful result concerning the mapping properties of certain matrix operators.

Lemma 5.18 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and for fixed $n \in \mathbb{N}$ let $\boldsymbol{X} := X^n$ and $\boldsymbol{Y} := Y^n$ be their product spaces, endowed with the norm*

$$\begin{aligned} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}} &:= \Big(\sum_{i=1}^n \|\boldsymbol{\varphi}_i\|_X^2\Big)^{1/2}, \qquad \boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n)^\top \in \boldsymbol{X}, \\ \|\boldsymbol{\psi}\|_{\boldsymbol{Y}} &:= \Big(\sum_{i=1}^n \|\boldsymbol{\psi}_i\|_Y^2\Big)^{1/2}, \qquad \boldsymbol{\psi} = (\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n)^\top \in \boldsymbol{Y}. \end{aligned}$$

Moreover, consider the matrix operator $\boldsymbol{T} := \big(T^{(i,j)}\big)_{i,j=1}^n$, with entries $T^{(i,j)} \in \mathcal{L}(X,Y)$ such that $\|T^{(i,j)}\varphi\|_Y \le C_{i,j}\,\|\varphi\|_X$ for all $\varphi \in X$ and all $i,j \in \{1,\dots,n\}$, and acting in terms of the usual matrix-vector product. Then $\boldsymbol{T} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{Y})$ with

$$\|\boldsymbol{T}\boldsymbol{\varphi}\|_{\boldsymbol{Y}} \le C\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}}, \qquad \boldsymbol{\varphi} \in \boldsymbol{X},$$

where $C^2 = 2^{n-1} \sum_{i,j=1}^{n} C_{i,j}^2$.

Proof: Let $\boldsymbol{\varphi} \in \boldsymbol{X}$. We estimate

$$\|\boldsymbol{T}\,\boldsymbol{\varphi}\|_{\boldsymbol{Y}}^2 = \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} T^{(i,j)} \boldsymbol{\varphi}_j \Big\|_Y^2 \leq 2^{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \big\| T^{(i,j)} \boldsymbol{\varphi}_j \big\|_Y^2$$
$$\leq \sum_{j=1}^{n} \|\boldsymbol{\varphi}_j\|_X^2 \, 2^{n-1} \sum_{i=1}^{n} C_{i,j}^2 \leq C\,\|\boldsymbol{\varphi}\|_{\boldsymbol{X}},$$

as asserted. □

For $s \geq 0$ we set

$$\boldsymbol{H}_{\mathrm{per}}^s(Q) := (H_{\mathrm{per}}^s(Q))^n$$

and endow this space with the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_{\mathrm{per}}^s(Q)} := \Big(\sum_{j=1}^{n} \|\boldsymbol{\varphi}_j\|_{H_{\mathrm{per}}^s(Q)}^2 \Big)^{1/2}, \quad \boldsymbol{\varphi} = (\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n)^\top \in \boldsymbol{H}_{\mathrm{per}}^s(Q).$$

Analogously, we set $\boldsymbol{L}^2(Q) := (L^2(Q))^n$ and $\boldsymbol{\mathcal{T}}_N(Q) := (\mathcal{T}_N(Q))^n$, for $N \in \mathbb{N}^2$, and consider therein the norm $\|\cdot\|_{\boldsymbol{L}^2(Q)} := \|\cdot\|_{\boldsymbol{H}_{\mathrm{per}}^0(Q)}$, because of $\boldsymbol{L}^2(Q) = \boldsymbol{H}_{\mathrm{per}}^0(Q)$ and the definition of the norm at the right hand side. It is easy to check that

$$H_{\mathrm{per}}^s(Q, \mathbb{C}^n) \cong \boldsymbol{H}_{\mathrm{per}}^s(Q),\; L^2(Q, \mathbb{C}^n) \cong \boldsymbol{L}^2(Q) \;\text{ and }\; \mathcal{T}_N(Q, \mathbb{C}^n) \cong \boldsymbol{\mathcal{T}}_N(Q).$$

This justifies the use of the bold variants in the following analysis, where we are especially interested in the case $n = 3$.

Now, in the spirit of Lemma 5.18, we define for $N \in \mathbb{N}^2$ the operators $\boldsymbol{O}_N$ and $\boldsymbol{P}_N$ given by

$$\boldsymbol{O}_N := \begin{pmatrix} O_N & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_N \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{P}_N := \begin{pmatrix} P_N & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_N \end{pmatrix},$$

and furthermore,

$$\boldsymbol{J}_1 := \big(J_1^{(i,j)}\big)_{i,j=1}^n, \quad \tilde{\boldsymbol{J}}_{1,\tilde{M}} := \big(\tilde{J}_{1,\tilde{M}}^{(i,j)}\big)_{i,j=1}^n, \quad \boldsymbol{J}_{1,M,\tilde{M}} := \big(J_{1,M,\tilde{M}}^{(i,j)}\big)_{i,j=1}^n,$$

$$\boldsymbol{J}_2 := \left(J_2^{(i,j)}\right)_{i,j=1}^n, \quad \boldsymbol{J}_{2,N} := \left(J_{2,N}^{(i,j)}\right)_{i,j=1}^n.$$

And last but not least, we connect again all indices to the meshsize h by assuming $N_1 = N_2$ and setting $h := \pi/N_1$, $\tilde{M}_1 := \tilde{M}_2 := \lceil \varrho/h \rceil$ and $M := \tilde{M}$, introduce

$$\boldsymbol{A} := \boldsymbol{J}_1 + \boldsymbol{J}_2, \qquad \boldsymbol{A}_h := \boldsymbol{P}_N \left(\boldsymbol{J}_{1,M,\tilde{M}} + \boldsymbol{J}_{2,N}\right),$$

and assume that $\boldsymbol{I} - \boldsymbol{A}$ is boundedly invertible on $\boldsymbol{H}^s_{\mathrm{per}}(Q)$ for any $s \geq 0$, where $\boldsymbol{I}$ denotes the identity on $\boldsymbol{H}^s_{\mathrm{per}}(Q)$. Then the system (5.1) reads as: for given $\boldsymbol{\psi} \in \boldsymbol{H}^s_{\mathrm{per}}(Q)$ find $\boldsymbol{\varphi} \in \boldsymbol{H}^s_{\mathrm{per}}(Q)$ such that

$$(\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{\varphi} = \boldsymbol{\psi}, \tag{5.22}$$

and we consider approximations of its solution by the fully discrete system: find $\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{T}}_N(Q)$ such that

$$(\boldsymbol{I} - \boldsymbol{A}_h)\,\boldsymbol{\varphi}_h = \boldsymbol{P}_N\,\boldsymbol{\psi}. \tag{5.23}$$

Theorem 5.19 *Let $t > 1$ and assume that $\varrho = h^\alpha$ for some $\alpha \in (0, \frac{1}{2t+6})$. Then there exists $h_0 > 0$ such that $\boldsymbol{I} - \boldsymbol{A}_h \in \mathcal{L}_{\mathrm{is}}(\boldsymbol{H}^t_{\mathrm{per}}(Q))$ for $0 < h \leq h_0$, with uniformly bounded inverse.*

Proof: We copy the proof for Theorem 5.16 line for line and replace the lean symbols by their bold analogs. Then, thanks to Lemma 5.18, in particular to the special form of the constant C therein, we factor out all common constants from the estimates for the lean operators. This allows us to continue our argumentation as in the remaining part of the proof of Theorem 5.16, which finally yields the assertion. □

Theorem 5.20 *Let $\alpha \in (0, 1/3)$ and $\varrho = h^\alpha$. Assume that $t \geq 0$ and $s > \max\{1, t, \frac{10\alpha+3\alpha t+t}{1-3\alpha}\}$. Furthermore, let (5.23) be a stable approximation of (5.22) in $\boldsymbol{H}^s_{\mathrm{per}}(Q)$, i.e. there exists $c > 0$ such that $\|\boldsymbol{\varphi}_h\|_{\boldsymbol{H}^s_{\mathrm{per}}(Q)} \leq c\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^s_{\mathrm{per}}(Q)}$ for sufficiently small h. Then there exists $h_0 > 0$ such that*

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\boldsymbol{H}^t_{\mathrm{per}}(Q)} \leq C\, h^{(s-t)(1-3\alpha)/2} \|\boldsymbol{\varphi}\|_{\boldsymbol{H}^s_{\mathrm{per}}(Q)}$$

for all $0 < h \leq h_0$.

Proof: We copy the proof of Theorem 5.17 line for line and proceed as in the proof of Theorem 5.19. □

A. Elementary Results from Calculus and Functional Analysis

Throughout this thesis we use elementary results from calculus and functional analysis. For convenience, those results are collected in this appendix. Some of them in Section A.2 and A.3 are extracted from the appendix of [34].

An exception makes Section A.5. Therein the results for the biperiodic case were not found in the literature and it seemed appropriate to give them the opportunity for an appearance at least here.

A.1. The Theorems of Fubini and Young for Series

Theorem A.1 (Fubini for series) *Let $d, d' \in \mathbb{N}$ and let $a^{(\mu\nu)} \in \mathbb{C}^{d'}$ for $\mu, \nu \in \mathbb{Z}^d$. Assume that $\left(\sum_{\nu\in\mathbb{Z}^d} |a^{(\mu\nu)}|\right)$ converges for all $\mu \in \mathbb{Z}^d$ and*

$$c := \sum_{\mu\in\mathbb{Z}^d} \Big(\sum_{\nu\in\mathbb{Z}^d} |a^{(\mu\nu)}| \Big) < \infty.$$

Then $\left(\sum_{\mu\in\mathbb{Z}^d} |a^{(\mu\nu)}|\right)$ converges for all $\nu \in \mathbb{Z}^d$ and

$$\sum_{\nu\in\mathbb{Z}^d} \Big(\sum_{\mu\in\mathbb{Z}^d} |a^{(\mu\nu)}| \Big) = c.$$

Proof: This is a special case of the theorem for double series, see for instance [4]. □

Theorem A.2 (Young for series) *Let* $d, d' \in \mathbb{N}$ *and let* $p \in [1, \infty)$. *Furthermore, let* $(a^{(\nu)})_{\nu \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d, \mathbb{C}^{d'})$, $(b^{(\nu)})_{\nu \in \mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})$ *and define*

$$c^{(\mu)} := \sum_{\nu \in \mathbb{Z}^d} a^{(\mu-\nu)} \cdot b^{(\nu)}, \qquad \mu \in \mathbb{Z}^d.$$

Then $(c^{(\mu)})_{\mu \in \mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})$ *and*

$$\left\|(c^{(\mu)})_{\mu \in \mathbb{Z}^d}\right\|_{\ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})} \leq \left\|(a^{(\mu)})_{\mu \in \mathbb{Z}^d}\right\|_{\ell^1(\mathbb{Z}^d, \mathbb{C}^{d'})} \left\|(b^{(\mu)})_{\mu \in \mathbb{Z}^d}\right\|_{\ell^p(\mathbb{Z}^d, \mathbb{C}^{d'})}.$$

Proof: *(i).* We start with the case $p = 1$ and show that

$$\sum_{\mu \in \mathbb{Z}^d} \Big(\sum_{\nu \in \mathbb{Z}^d} |a^{(\mu-\nu)}||b^{(\nu)}| \Big) = \left\|(a^{(\nu)})_{\nu \in \mathbb{Z}^d}\right\|_{\ell^1} \left\|(b^{(\nu)})_{\nu \in \mathbb{Z}^d}\right\|_{\ell^1}.$$

In fact, $\sum_{\mu \in \mathbb{Z}^d} |a^{(\mu-\nu)}||b^{(\nu)}| = |b^{(\nu)}| \left\|(a^{(\mu)})_{\mu \in \mathbb{Z}^d}\right\|_{\ell^1}$, for all $\nu \in \mathbb{Z}^d$, and $\sum_{\nu \in \mathbb{Z}^d} \big(\sum_{\mu \in \mathbb{Z}^d} |a^{(\mu-\nu)}||b^{(\nu)}| \big) = \left\|(a^{(\mu)})_{\mu \in \mathbb{Z}^d}\right\|_{\ell^1} \left\|(b^{(\nu)})_{\nu \in \mathbb{Z}^d}\right\|_{\ell^1}$, and thus Theorem A.1 yields the desired equation. With this result, the assertion follows immediately.

(ii). Now, let $p \in (1, \infty)$ and set $p' := p/(p-1)$. Since $(|a^{(\nu)}|)_{\nu \in \mathbb{Z}^d}$ and $(|b^{(\nu)}|^p)_{\nu \in \mathbb{Z}^d}$ belong to $\ell^1(\mathbb{Z}^d)$, we can apply the theorem for the case $p = 1$ and obtain

$$\sum_{\mu \in \mathbb{Z}^d} \Big(\sum_{\nu \in \mathbb{Z}^d} |a^{(\mu-\nu)}||b^{(\nu)}|^p \Big) \leq \left\|(a^{(\nu)})_{\nu \in \mathbb{Z}^d}\right\|_{\ell^1} \left\|(b^{(\nu)})_{\nu \in \mathbb{Z}^d}\right\|_{\ell^p}^p.$$

In particular, the sequence $\big(|a^{(\mu-\nu)}|^{1/p} |b^{(\nu)}|\big)_{\nu \in \mathbb{Z}^d}$ belongs to $\ell^p(\mathbb{Z}^d)$ and an application of Hölder's inequality yields

$$\begin{aligned}
\sum_{\nu \in \mathbb{Z}^d} |a^{(\mu-\nu)}||b^{(\nu)}| &= \sum_{\nu \in \mathbb{Z}^d} |a^{(\mu-\nu)}|^{1/p'} |a^{(\mu-\nu)}|^{1/p} |b^{(\nu)}| \\
&\leq \Big(\sum_{\nu \in \mathbb{Z}^2} |a^{(\mu-\nu)}| \Big)^{1/p'} \Big(\sum_{\nu \in \mathbb{Z}^2} |a^{(\mu-\nu)}||b^{(\nu)}|^p \Big)^{1/p},
\end{aligned}$$

for all $\mu \in \mathbb{Z}^d$, and thus

$$\sum_{\mu\in\mathbb{Z}^d}\Big(\sum_{\nu\in\mathbb{Z}^d}|a^{(\mu-\nu)}||b^{(\nu)}|\Big)^p \le \big\|(a^{(\nu)})_{\nu\in\mathbb{Z}^d}\big\|_{\ell^1}^{p/p'}\sum_{\mu\in\mathbb{Z}^d}\Big(\sum_{\nu\in\mathbb{Z}^d}|a^{(\mu-\nu)}||b^{(\nu)}|^p\Big)$$
$$\le \big\|(a^{(\nu)})_{\nu\in\mathbb{Z}^d}\big\|_{\ell^1}^{1+p/p'}\,\big\|(b^{(\nu)})_{\nu\in\mathbb{Z}^d}\big\|_{\ell^p}^{p},$$

which completes the proof. □

A.2. Differential Operators

For $x, y, z \in \mathbb{C}^3$ there holds

$$x\cdot(y\times z) = y\cdot(z\times x) = z\cdot(x\times y) \tag{A.1a}$$
$$x\times(y\times z) = (x\cdot z)\,y - (x\cdot y)\,z \tag{A.1b}$$

For sufficiently smooth scalar valued function $u : \mathbb{R}^3 \to \mathbb{C}$ and vector valued functions $F, G : \mathbb{R}^3 \to \mathbb{C}^3$ we have

$$\operatorname{curl}\nabla u = 0, \tag{A.2a}$$
$$\operatorname{div}\operatorname{curl} F = 0, \tag{A.2b}$$
$$\operatorname{curl}\operatorname{curl} F = \nabla\operatorname{div} F - \Delta F, \tag{A.2c}$$

where in the last equation the Laplacian operator is taken componentwise. Moreover, there holds

$$\operatorname{div}(uF) = F\cdot\nabla u + u\operatorname{div} F, \tag{A.3a}$$
$$\operatorname{curl}(uF) = \nabla u\times F + u\operatorname{curl} F, \tag{A.3b}$$
$$\nabla(F\cdot G) = (F')^\top G + (G')^\top F, \tag{A.3c}$$
$$\operatorname{div}(F\times G) = G\cdot\operatorname{curl} F - F\cdot\operatorname{curl} G, \tag{A.3d}$$
$$\operatorname{curl}(F\times G) = F\operatorname{div} G - G\operatorname{div} F + F'G - G'F, \tag{A.3e}$$

where $F'(x), G'(x) \in \mathbb{C}^{3\times 3}$ are the Jacobian matrices of F and G, respectively. Recalling the modified versions of the differential operators from the end of Section 1.3, the analogs of (A.2) read then as

$$\operatorname{curl}_\beta \nabla_\beta u = 0, \tag{A.4a}$$

$$\operatorname{div}_\beta \operatorname{curl}_\beta F = 0, \tag{A.4b}$$
$$\operatorname{curl}_\beta \operatorname{curl}_\beta F = \nabla_\beta \operatorname{div}_\beta F - \Delta_\beta F. \tag{A.4c}$$

In fact, to verify for instance Equation (A.4c), we obtain for smooth enough $F : \mathbb{R}^3 \to \mathbb{C}^3$ on the one hand

$$\begin{aligned}\operatorname{curl}_\beta \operatorname{curl}_\beta F &= \operatorname{curl}\operatorname{curl} F + \operatorname{curl}(\mathrm{i}\beta \times F) + \mathrm{i}\beta \times \operatorname{curl} F + \mathrm{i}\beta \times (\mathrm{i}\beta \times F)\\ &= \operatorname{curl}\operatorname{curl} F + \mathrm{i}\beta \operatorname{div} F - \mathrm{i}F'\beta - \mathrm{i}\beta \times \operatorname{curl} F + (\mathrm{i}\beta \cdot F)\mathrm{i}\beta - (\mathrm{i}\beta \cdot \mathrm{i}\beta)F,\end{aligned}$$

where we have applied (A.3e) in the second step. Note that

$$\mathrm{i}\beta \times \operatorname{curl} F = \mathrm{i}\begin{pmatrix}\beta_2(\partial_1 F_2 - \partial_2 F_1) - \beta_3(\partial_3 F_1 - \partial_1 F_3)\\ \beta_3(\partial_2 F_3 - \partial_3 F_2) - \beta_1(\partial_1 F_2 - \partial_2 F_1)\\ \beta_1(\partial_3 F_1 - \partial_1 F_3) - \beta_2(\partial_2 F_3 - \partial_3 F_2)\end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned}\Delta_\beta F &= \begin{pmatrix}\operatorname{div}\nabla F_1 + \operatorname{div}(\mathrm{i}\beta F_1) + \mathrm{i}\beta \cdot \nabla F_1 + \mathrm{i}\beta \cdot (\mathrm{i}\beta F_1)\\ \operatorname{div}\nabla F_2 + \operatorname{div}(\mathrm{i}\beta F_2) + \mathrm{i}\beta \cdot \nabla F_2 + \mathrm{i}\beta \cdot (\mathrm{i}\beta F_2)\\ \operatorname{div}\nabla F_3 + \operatorname{div}(\mathrm{i}\beta F_3) + \mathrm{i}\beta \cdot \nabla F_3 + \mathrm{i}\beta \cdot (\mathrm{i}\beta F_3)\end{pmatrix}\\ &= \Delta F + \mathrm{i}F'\beta + \mathrm{i}F'\beta + (\mathrm{i}\beta \cdot \mathrm{i}\beta)F\end{aligned}$$

and therefore

$$\begin{aligned}\nabla_\beta \operatorname{div}_\beta F - \Delta_\beta F &= \nabla \operatorname{div} F + \nabla(\mathrm{i}\beta \cdot F) + \mathrm{i}\beta \operatorname{div} F + \mathrm{i}\beta(\mathrm{i}\beta \cdot F) - \Delta_\beta F\\ &= \operatorname{curl}^2 F + \mathrm{i}\beta \operatorname{div} F - \mathrm{i}F'\beta + (\mathrm{i}\beta \cdot F)\mathrm{i}\beta - (\mathrm{i}\beta \cdot \mathrm{i}\beta)F + \nabla(\mathrm{i}\beta \cdot F) - \mathrm{i}F'\beta.\end{aligned}$$

Using finally the fact that

$$\begin{aligned}\nabla(\mathrm{i}\beta \cdot F) - \mathrm{i}F'\beta &= \mathrm{i}\left((F')^\top - F'\right)\beta\\ &= \mathrm{i}\begin{pmatrix}0 & \partial_1 F_2 - \partial_2 F_1 & -(\partial_3 F_1 - \partial_1 F_3)\\ -(\partial_1 F_2 - \partial_2 F_1) & 0 & \partial_2 F_3 - \partial_3 F_2\\ \partial_3 F_1 - \partial_1 F_3 & -(\partial_2 F_3 - \partial_3 F_2) & 0\end{pmatrix}\beta\\ &= \mathrm{i}\beta \times \operatorname{curl} F,\end{aligned}$$

we have indeed shown the Equation (A.4c). The remaining equations in (A.4) are shown analogously, but even easier.

A.3. Integral Identities

Theorem A.3 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. For $F \in C^1(\Omega, \mathbb{C}^3) \cap C(\overline{\Omega}, \mathbb{C}^3)$ there holds*

$$\int_\Omega \operatorname{div} F(x)\,\mathrm{d}x = \int_{\partial D} F(x) \cdot \mathrm{n}(x)\,\mathrm{d}s, \tag{A.5}$$

where n *denotes the outward pointing normal unit vector on $\partial\Omega$.*

For a proof for Lipschitz domains we refer to [40]. For smooth domains a proof can be found in [29].

The last theorem is often referred to as *Theorem of Gauss* or as *Divergence Theorem.* As a simple application we obtain the following result.

Theorem A.4 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Furthermore, let $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$ and A, $B \in C^1(\Omega, \mathbb{C}^3) \cap C(\overline{\Omega}, \mathbb{C}^3)$. Then*

$$\int_\Omega u\,\nabla v\,\mathrm{d}x + \int_\Omega v\,\nabla u\,\mathrm{d}x = \int_{\partial\Omega} u\,v\,\mathrm{n}\,\mathrm{d}s, \tag{A.6a}$$

$$\int_\Omega (B \cdot \operatorname{curl} A - A \cdot \operatorname{curl} B)\,\mathrm{d}x = \int_{\partial\Omega} (\mathrm{n} \times A) \cdot B\,\mathrm{d}s, \tag{A.6b}$$

$$\int_\Omega (u \operatorname{div} A + A \cdot \nabla u)\,\mathrm{d}x = \int_{\partial\Omega} u\,(\mathrm{n} \cdot A)\,\mathrm{d}s, \tag{A.6c}$$

where n *denotes the outward pointing normal unit vector on $\partial\Omega$.*

A.4. Results from Functional Analysis

Proposition A.5 *Let X be a vector space endowed with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Furthermore, let $\widetilde{X}_1$ and $\widetilde{X}_2$ be the completion of X with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then*

$$\widetilde{X}_1 \simeq \widetilde{X}_2.$$

Proof: By a well-known construction, we have $\widetilde{X}_i \cong \mathcal{C}_i(X)/\mathcal{N}_i(X)$, where $\mathcal{C}_i(X)$ and $\mathcal{N}_i(X)$ denote the set of all Cauchy sequences and all null sequences in $(X, \|\cdot\|_i)$, respectively, $i = 1, 2$. Since $\mathcal{C}_1(X) = \mathcal{C}_2(X)$ and $\mathcal{N}_1(X) = \mathcal{N}_2(X)$, clearly $\mathcal{C}_1(X)/\mathcal{N}_1(X) = \mathcal{C}_2(X)/\mathcal{N}_2(X)$. Moreover, by construction, the norms in $\mathcal{C}_1(X)/\mathcal{N}_1(X)$ and $\mathcal{C}_2(X)/\mathcal{N}_2(X)$ are equivalent. Therefore, we obtain

$$\widetilde{X}_1 \cong \mathcal{C}_1(X)/\mathcal{N}_1(X) \simeq \mathcal{C}_2(X)/\mathcal{N}_2(X) \cong \widetilde{X}_2,$$

as asserted. □

Proposition A.6 *Let X be a normed vector space, Y a Banach space and U be a dense subspace of X. Furthermore, let $T_0 : (U, \|\cdot\|_X) \to Y$ be compact. Then the continuous extension $T : X \to Y$ of T_0 is compact too.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Choose a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ in U such that $\|\hat{x}_n - x_n\| \le \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(\hat{x}_n)_{n\in\mathbb{N}}$ is a bounded sequence in U and there exists a subsequence $(\hat{x}_{n_j})_{j\in\mathbb{N}}$ with $(T_0\hat{x}_{n_j})_{j\in\mathbb{N}}$ converging in Y to some $y \in Y$. Therefore,

$$\|Tx_{n_j} - y\| \le \|T(x_{n_j} - \hat{x}_{n_j})\| + \|T_0\hat{x}_{n_j} - y\| \le \|T\|\frac{1}{n_j} + \|T_0\hat{x}_{n_j} - y\|,$$

for all $j \in \mathbb{N}$, which shows that $(Tx_{n_j})_{j\in\mathbb{N}}$ is convegent in Y. □

The following result can be regarded as a corollary of the well-known extension theorem for linear and bounded operators.

Corollary A.7 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $U \subseteq X$ be a dense subspace of X and $V \subseteq Y$ be a dense subspace of Y. Moreover, suppose $T_0 \in \mathcal{L}(U, V)$ and $S_0 \in \mathcal{L}(V, U)$ such that*

$$T_0S_0 = \mathrm{id}_V \qquad \text{and} \qquad S_0T_0 = \mathrm{id}_U\,.$$

Then for the continuous extensions $T \in \mathcal{L}(X, Y)$ of T_0 and $S \in \mathcal{L}(Y, X)$ of S_0 we have

$$TS = \mathrm{id}_Y \qquad \text{and} \qquad ST = \mathrm{id}_X\,.$$

Proof: Let $y \in Y$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq V$ such that $y_n \to y$ in $(Y, \|\cdot\|_Y)$, as $n \to \infty$. Hence,

$$TSy = T\big(\lim_{n\to\infty} S_0 y_n\big) = \lim_{n\to\infty} T_0 S_0 y_n = \lim_{n\to\infty} y_n = y.$$

The other equality is shown completely analogous. □

Theorem A.8 (Lax–Milgram) *Let X be a Hilbert space over the field $\mathbb{C}$ and $\ell \in X^*$. Furthermore, let $a : X \times X \to \mathbb{C}$ be sesquilinear, bounded and coercive, that is, there exist $c_1, c_2 > 0$ such that*

$$|a(u,v)| \le c_1 \|u\|_X \|v\|_X \qquad \text{for all } u, v \in X,$$
$$\operatorname{Re} a(u,u) \ge c_2 \|u\|_X^2 \qquad \text{for all } u \in X.$$

Then there exists a unique $u \in X$ such that

$$a(\psi, u) = \ell(\psi) \qquad \text{for all } \psi \in X.$$

Furthermore, there exists a constant $c > 0$, independent of u and ℓ, such that $\|u\|_X \le c\|\ell\|_{X^*}$.

For a proof we refer to, e.g., [29, Section 6.2].

If the functional $\ell : X \to \mathbb{C}$ is antilinear (see Section 1.3 for a definition) and bounded, then the statement of Theorem A.8 has to be slightly modified, see the following corollary.

Corollary A.9 *Let X be a Hilbert space over the field $\mathbb{C}$ and $\ell : X \to \mathbb{C}$ be antilinear and bounded. Furthermore, let $a : X \times X \to \mathbb{C}$ be sesquilinear, bounded and coercive. Then there exists a unique $u \in X$ such that*

$$a(u, \psi) = \ell(\psi) \qquad \text{for all } \psi \in X.$$

Furthermore, there exists a constant $c > 0$, independent of u and ℓ, such that $\|u\|_X \le c\|\ell\|_{X^*}$.

Proof: Consider $\tilde{a}(u,v) := \overline{a(v,u)}$ and $\tilde{\ell}(u) := \overline{\ell(u)}$, for all $u, v \in X$. Then

$$\forall \psi \in X : a(u,\psi) = \ell(\psi) \quad \Leftrightarrow \quad \forall \psi \in X : \tilde{a}(\psi, u) = \tilde{\ell}(\psi),$$

and the assertion follows now immediately from Theorem A.8, because $\tilde{a}$ and $\tilde{\ell}$ satisfy its assumptions. □

A.5. Sobolev-Slobodeckiĭ spaces

In this thesis we could manage the analysis without Sobolev-Slobodeckiĭ spaces, even in the case of Lemma 5.3, albeit with a less strong statement (see Theorem 2.38). For stronger results as in Lemma 5.3, the use of Sobolev-Slobodeckiĭ spaces seems inevitable. For periodic functions in one dimension corresponding results can be found in [36]. However, for their counterparts in two dimensions it is hard to find analogs in the literature. It is the objective of this section to provide such results. Although that will be given for periodic functions in $\mathbb{R}^2$, the generalization to $\mathbb{R}^d$ then can be easily realized.

But before, we give a short introduction into general Sobolev-Slobodeckiĭ spaces and provide some key tools which will facilitate the handling therein.

A.5.1. Fractional Sobolev Spaces

In this subsection we give a short introduction into Sobolev-Slobodeckiĭ spaces. For this, a good reference is [53], which was also the basis here. For our applications, only Sobolev-Slobodeckiĭ spaces based on the Hilbert space $L^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$, are of interest.

Throughout this subsection let Ω be an open subset of $\mathbb{R}^d$.

We recall the spaces $C_0^m(\Omega)$ and $C_0^\infty(\Omega)$ from Section 1.3. Additionally, we define the spaces $C_b^m(\Omega)$ and $C_b^\infty(\Omega)$ to consist of all m-times and infinitely often continuously differentiable functions $\chi : \Omega \to \mathbb{C}$, where all partial derivatives are bounded, respectively. Clearly, $C_0^m(\Omega)$ is a subspace of $C_b^m(\Omega)$. In $C_b^m(\Omega)$ we choose the norm

$$\|\chi\|_{C_b^m(\Omega)} := \max_{|\alpha|\leq m} \sup_{x\in\Omega} |\partial^\alpha \chi(x)|,$$

similarly as in (1.12).

Recalling furthermore the notion of the variational derivative from Definition 2.1, the following lemma gives a useful criterion to decide whether $\varphi \in L^2(\Omega)$ possesses such a derivative.

Lemma A.10 *Let $\varphi, \psi \in L^2(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. If there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C^{|\alpha|}(\Omega)$ such that*

$$\|\varphi_n - \varphi\|_{L^2(\Omega)} \to 0 \qquad \text{and} \qquad \|\partial^\alpha \varphi_n - \psi\|_{L^2(\Omega)} \to 0,$$

as $n \to \infty$, then φ possesses the derivative $\partial^\alpha \varphi = \psi$ in the variational sense.

Proof: Let $\chi \in C_0^\infty(\Omega)$ and set $B := \operatorname{supp} \chi$. Then

$$\begin{aligned}\int_\Omega \varphi(x)\,\partial^\alpha \chi(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_B \varphi_n(x)\,\partial^\alpha \chi(x)\,\mathrm{d}x \\ &= \lim_{n\to\infty} (-1)^{|\alpha|} \int_B \partial^\alpha \varphi_n(x)\,\chi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \psi(x)\,\chi(x)\,\mathrm{d}x\end{aligned}$$

and the proof is complete. □

To make in the following presentation the notation easier, we introduce for $q \in (0,1)$ and for suitable $\varphi, \psi \in L^2(\Omega)$ the quantities

$$\begin{aligned}\langle \varphi, \psi \rangle_{q,\Omega} &:= \int_\Omega \int_\Omega \frac{\big(\varphi(x) - \varphi(y)\big)\overline{\big(\psi(x) - \psi(y)\big)}}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x, \\ |\varphi|_{q,\Omega} &:= \sqrt{\langle \varphi, \varphi \rangle_{q,\Omega}}\,.\end{aligned}$$

Moreover, for $\varphi \in L^1_{\text{loc}}(\Omega)$ we recall that its *essential support* $\operatorname{supp} \varphi$ is the smallest closed set such that $\varphi = 0$ almost everywhere on $\Omega \setminus \operatorname{supp} \varphi$.

Definition A.11 *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s = m + q$, where $m \in \mathbb{N}_0$ and $q \in (0,1)$. We define*

$$H^s(\Omega) := \left\{ \varphi \in H^m(\Omega) \,\middle|\, \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le m : |\partial^\alpha \varphi|_{q,\Omega} < \infty \right\}$$

and endow this space with the inner product

$$(\varphi \mid \psi)_{H^s(\Omega)} := (\varphi \mid \psi)_{H^m(\Omega)} + \sum_{|\alpha| \le m} \langle \partial^\alpha \varphi, \partial^\alpha \psi \rangle_{q,\Omega}\,.$$

Here, the space $H^m(\Omega)$ was defined in Definition 2.2. For $s \in \mathbb{R}$ with $s \geq 0$, we choose in $H^s(\Omega)$ the norm $\|\cdot\|_{H^s(\Omega)} := \sqrt{(\cdot \mid \cdot)_{H^s(\Omega)}}$. Furthermore, we set

$$H_c^s(\Omega) := \big\{\varphi \in H^s(\Omega) \mid \operatorname{supp}\varphi \subseteq \Omega \textit{ is compact}\big\}$$

and, for compact $K \subseteq \Omega$,

$$H_K^s(\Omega) := \big\{\varphi \in H^s(\Omega) \mid \operatorname{supp}\varphi \subseteq K\big\}.$$

It is well-known, that for $s \geq 0$ the space $H^s(\Omega)$ is a separable Hilbert space, see for instance [53, Theorem 3.1].

Regarding the space $H_K^s(\Omega)$ we make the following observation.

Proposition A.12 *Let $\varphi \in H_K^s(\Omega)$. Then for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \lfloor s \rfloor$ there holds*

$$\operatorname{supp}\partial^\alpha\varphi \subseteq K.$$

Proof: Let $\chi \in C_0^\infty(\Omega\backslash K)$. Then $\chi|_0^\Omega$ belongs to $C_0^\infty(\Omega)$ and, by definition of the derivatives in the variational sense, we conclude

$$\begin{aligned}\int_{\Omega\backslash K}(\partial^\alpha\varphi)\,\chi\,\mathrm{d}x &= \int_\Omega(\partial^\alpha\varphi)\,\chi|_0^\Omega\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega\varphi\,\partial^\alpha\chi|_0^\Omega\,\mathrm{d}x\\ &= (-1)^{|\alpha|}\Big(\int_K\varphi\,\underbrace{\partial^\alpha\chi|_0^\Omega}_{=0}\,\mathrm{d}x + \int_{\Omega\backslash K}\underbrace{\varphi}_{=0}\,\partial^\alpha\chi\,\mathrm{d}x\Big) = 0.\end{aligned}$$

Since $\chi \in C_0^\infty(\Omega \setminus K)$ was arbitrarily chosen, by a well-known theorem it follows that $\partial^\alpha\varphi = 0$ almost everywhere on $\Omega \setminus K$. □

In the next lemma we will see that the notation $|\cdot|_{q,\Omega}$ from above is advisable since it shows that $|\cdot|_{q,\Omega}$ is a seminorm.

Lemma A.13 *Let $q \in (0,1)$. Then $|\cdot|_{q,\Omega}$ is a seminorm in $H^q(\Omega)$.*

Proof: Let $\varphi, \psi \in H^q(\Omega)$ and $\lambda \in \mathbb{C}$ be arbitrary. Then it is easy to see that $|\varphi|_{q,\Omega} \geq 0$ and that $|\lambda\varphi|_{q,\Omega} = |\lambda|\,|\varphi|_{q,\Omega}$. To show the triangle inequality, we set

$$\Phi(x,y) := \frac{\varphi(x)-\varphi(y)}{|x-y|^{d/2+q}}, \qquad (x,y) \in \Omega\times\Omega,$$

and define $\Psi : \Omega \times \Omega \to \mathbb{C}$ analogously. Then $\Phi, \Psi \in L^2(\Omega \times \Omega)$ and we obtain

$$|\varphi + \psi|_{q,\Omega} = \left(\int_\Omega \int_\Omega \frac{\big|\big(\varphi(x) + \psi(x)\big) - \big(\varphi(y) + \psi(y)\big)\big|^2}{|x-y|^{d+2q}} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2}$$
$$= \|\Phi + \Psi\|_{L^2(\Omega\times\Omega)} \le \|\Phi\|_{L^2(\Omega\times\Omega)} + \|\Psi\|_{L^2(\Omega\times\Omega)} = |\varphi|_{q,\Omega} + |\psi|_{q,\Omega}\,,$$

as desired. □

Mollifiers. A main tool when working in Sobolev spaces are *mollifiers*. One possibility to construct them is to take $\tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ defined by

$$\tilde{\chi}(x) = \begin{cases} \exp\big(-1/(1-|x|^2)\big), & x \in \mathbb{B}_d(0,1), \\ 0, & x \in \mathbb{R}^d \setminus \mathbb{B}_d(0,1), \end{cases}$$

and to consider modifications χ and χ_ε of $\tilde{\chi}$ as follows

$$\chi(x) := \frac{1}{\|\tilde{\chi}\|_{L^1(\mathbb{R}^d)}}\,\tilde{\chi}(x) \quad \text{and} \quad \chi_\varepsilon(x) := \varepsilon^{-d}\,\chi\big(\tfrac{1}{\varepsilon}x\big),$$

for $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Then $0 \le \chi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$, $\chi_\varepsilon(x) > 0$ if and only if $|x| < \varepsilon$, $\chi_\varepsilon = 0$ on $\mathbb{R}^d \setminus \mathbb{B}_d(0,\varepsilon)$ and $\|\chi_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$. For $\varphi \in L^2(\Omega)$ we extend φ by zero to $\mathbb{R}^d$ and set

$$T_\varepsilon\varphi(x) := \int_{\mathbb{B}_d(x,\varepsilon)} \chi_\varepsilon(x-y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{B}_d(0,\varepsilon)} \chi_\varepsilon(z)\varphi(x-z)\,\mathrm{d}z$$
$$= \int_{\mathbb{B}_d(0,\varepsilon)} \chi_\varepsilon(z)\varphi(x+z)\,\mathrm{d}z, \qquad \text{for } x \in \mathbb{R}^d.$$

Then $T_\varepsilon\varphi \in C^\infty(\mathbb{R}^d)$,

$$\operatorname{supp} T_\varepsilon\varphi \subseteq \operatorname{supp}\varphi + \mathbb{B}_d[0,\varepsilon], \tag{A.7}$$

$$\|T_\varepsilon\varphi\|_{L^2(\Omega)} \le \|T_\varepsilon\varphi\|_{L^2(\mathbb{R}^d)} \le \|\chi_\varepsilon\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\Omega)} \tag{A.8}$$

and

$$T_\varepsilon\varphi \to \varphi \quad \text{in } L^2(\Omega), \quad \text{as } \varepsilon \to 0.$$

For $\varphi \in L^2(\Omega)$ with variational derivative $\partial^\alpha\varphi \in L^2(\Omega)$, for some $\alpha \in \mathbb{N}_0^d$, there also holds

$$\partial^\alpha T_\varepsilon\varphi = T_\varepsilon\,\partial^\alpha\varphi \to \partial^\alpha\varphi \quad \text{in } L^2(\Omega), \quad \text{as } \varepsilon \to 0. \tag{A.9}$$

Multiplication Operators. As a first application of the mollifiers, we now introduce multiplication operators with respect to differentiable and bounded functions. We start with an important consequence from the mean value theorem given in form of the following lemma.

Lemma A.14 *Let $\chi \in C_b^1(\mathbb{R}^d)$. Then there exists a constant $C \geq 0$ such that*

$$|\chi(x) - \chi(y)| \leq C\,\frac{|x-y|}{1+|x-y|}, \qquad x, y \in \mathbb{R}^d.$$

The constant C can be chosen as $C = 2\,\|\chi\|_{C_b^1(\mathbb{R}^d)}\,\max\left\{2, \sqrt{d}\right\}$.

Proof: Let $\chi \in C_b^1(\mathbb{R}^d)$. If $|x-y| > 1$ then

$$\begin{aligned} |\chi(x) - \chi(y)| \leq 2\,\|\chi\|_{C_b^1(\mathbb{R}^d)} &= 2\,\|\chi\|_{C_b^1(\mathbb{R}^d)}\frac{|x-y|}{1+|x-y|}\left(1 + \frac{1}{|x-y|}\right) \\ &\leq 4\,\|\chi\|_{C_b^1(\mathbb{R}^d)}\frac{|x-y|}{1+|x-y|}. \end{aligned}$$

And if $|x-y| \leq 1$ then $1 \leq 2/(1+|x-y|)$ and the mean value theorem yields

$$|\chi(x) - \chi(y)| \leq \sqrt{d}\,\|\chi\|_{C_b^1(\mathbb{R}^d)}\,|x-y| \leq 2\,\sqrt{d}\,\|\chi\|_{C_b^1(\mathbb{R}^d)}\,\frac{|x-y|}{1+|x-y|},$$

where we used the fact that $\|\partial\chi(x)\|^2 = \sum_{k=1}^d \left|\frac{\partial}{\partial x_k}\chi(x)\right|^2 \leq d\,\|\chi\|^2_{C_b^1(\mathbb{R}^d)}$. Now the assertion follows immediately. □

Theorem A.15 *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s \in \mathbb{R}$ with $s \geq 0$. Furthermore, let $\chi \in C_b^{\lceil s \rceil}(\mathbb{R}^d)$ and $\varphi \in H^s(\Omega)$. Then $\chi\,\varphi \in H^s(\Omega)$ and there holds Leibniz' product rule*

$$\partial^\alpha(\chi\,\varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta}\,\partial^{\alpha-\beta}\chi\,\partial^\beta\varphi, \qquad \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq \lfloor s \rfloor.$$

Moreover, there exists a constant $C \geq 0$, not depending on χ and φ, such that

$$\|\chi\,\varphi\|_{H^s(\Omega)} \leq C\,\|\chi\|_{C_b^{\lceil s \rceil}(\mathbb{R}^d)}\,\|\varphi\|_{H^s(\Omega)}.$$

Proof: Since χ is bounded, there holds $\chi\,\varphi \in L^2(\Omega)$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le \lfloor s \rfloor$. Furthermore, let $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$. By (A.9) we have

$$T_\varepsilon\varphi \to \varphi, \quad \partial^\beta T_\varepsilon\varphi = T_\varepsilon\partial^\beta\varphi \to \partial^\beta\varphi \qquad \text{in } L^2(\Omega), \text{ as } \varepsilon \to 0,$$

and, since χ and its derivatives are bounded, in particular

$$\chi\, T_\varepsilon\varphi \to \chi\,\varphi, \quad \partial^{\alpha-\beta}\chi\,\partial^\beta T_\varepsilon\varphi \to \partial^{\alpha-\beta}\chi\,\partial^\beta\varphi \qquad \text{in } L^2(\Omega), \text{ as } \varepsilon \to 0.$$

Let $\theta \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}\int_\Omega \chi(x)\,\varphi(x)\,\partial^\alpha\theta(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega \chi(x)\,T_\varepsilon\varphi(x)\,\partial^\alpha\theta(x)\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\lim_{\varepsilon\to 0}\int_\Omega \partial^\alpha(\chi\,T_\varepsilon\varphi)(x)\,\theta(x)\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\lim_{\varepsilon\to 0}\int_\Omega \sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\chi(x)\,\partial^\beta T_\varepsilon\varphi(x)\,\theta(x)\,\mathrm{d}x \\ &= (-1)^{|\alpha|}\int_\Omega \left(\sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\chi(x)\,\partial^\beta\varphi(x)\right)\theta(x)\,\mathrm{d}x.\end{aligned}$$

Hence, there exists $\partial^\alpha(\chi\,\varphi) \in L^2(\Omega)$ and is given by the Leibniz product rule. Moreover, we have shown that $\chi\,\varphi \in H^{\lfloor s \rfloor}(\Omega)$.
Now, let $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0,1)$. And again, let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$ and $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$. Then

$$\|\partial^{\alpha-\beta}\chi\,\partial^\beta\varphi\|^2_{L^2(\Omega)} \le \|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\|\partial^\beta\varphi\|^2_{L^2(\Omega)} \le \|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\,\|\varphi\|^2_{H^s(\Omega)}.$$

Note that by Fubini's theorem and by Lemma A.14 there holds

$$\begin{aligned}&\int_\Omega\int_\Omega |\partial^\beta\varphi(y)|^2\frac{|\partial^{\alpha-\beta}\chi(x)-\partial^{\alpha-\beta}\chi(y)|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x \\ &\le 4\,\|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\max\{4,d\}\int_\Omega|\partial^\beta\varphi(y)|^2\underbrace{\left(\int_{\mathbb{R}^d}\frac{1}{(1+|z|)^2|z|^{d+2q-2}}\,\mathrm{d}z\right)}_{=:C_1<\infty}\mathrm{d}y \\ &\le 4\max\{4,d\}\,C_1\,\|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\,\|\varphi\|^2_{H^s(\Omega)},\end{aligned}$$

and thus we continue with

$$\left|\partial^{\alpha-\beta}\chi\,\partial^\beta\varphi\right|^2_{q,\Omega} = \int_\Omega\int_\Omega\left|\frac{\partial^{\alpha-\beta}\chi(x)[\partial^\beta\varphi(x)-\partial^\beta\varphi(y)]}{|x-y|^{d+2q}}\right.$$

$$
\begin{aligned}
&\qquad + \left. \frac{\partial^\beta \varphi(y)[\partial^{\alpha-\beta}\chi(x) - \partial^{\alpha-\beta}\chi(y)]}{|x-y|^{d+2q}} \right|^2 \mathrm{d}y\,\mathrm{d}x \\
&\le 2\,\|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)} \left|\partial^\beta \varphi\right|^2_{q,\Omega} + 8\max\{4,d\}\,C_1\,\|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\,\|\varphi\|^2_{H^s(\Omega)} \\
&\le C_2\,\|\chi\|^2_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\,\|\varphi\|^2_{H^s(\Omega)},
\end{aligned}
$$

where $C_2 := 2 + 8\max\{4,d\}\,C_1$. Consequently, by applying Leibniz' product rule, the triangle inequality and the results above, we obtain

$$
\begin{aligned}
\|\chi\,\varphi\|^2_{H^s(\Omega)} &= \sum_{|\alpha|\le\lfloor s\rfloor} \|\partial^\alpha(\chi\,\varphi)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le\lfloor s\rfloor} |\partial^\alpha(\chi\,\varphi)|^2_{q,\Omega} \\
&\le \big(1 + C_2\big) \sum_{|\alpha|\le\lfloor s\rfloor} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|\chi\|_{C_b^{\lceil s\rceil}(\mathbb{R}^d)}\|\varphi\|_{H^s(\Omega)} \right)^2,
\end{aligned}
$$

and from this we see immediately that also the last assertion from the theorem holds. □

Denseness Results. Now, we will derive an important denseness result for the space $H^s(\Omega)$. The following lemmas have preliminary character.

Lemma A.16 *Let $\Omega \subseteq \mathbb{R}^d$ be open.*

(i) If $O \subseteq \Omega$ is open and bounded, then there exists a constant $C > 0$ such that

$$
\int_O \int_O |\varphi(y)|^2 \frac{1}{|x-y|^{d+2q-2}}\,\mathrm{d}y\,\mathrm{d}x \le C\,\|\varphi\|^2_{L^2(O)},
$$

for all $\varphi \in C_b(\Omega) \cup L^2(\Omega)$.

(ii) Let $q \in (0,1)$. If $\emptyset \ne K \subseteq \Omega$ is compact and $O \subseteq \Omega$ is open such that $K \subseteq O$, then there exists a constant $C > 0$ such that

$$
\int_{-x+\Omega\setminus O} \frac{1}{|z|^{d+2q}}\,\mathrm{d}z \le C, \qquad x \in K.
$$

Proof: *(i).* Since $\overline{O}$ is compact, there exists $r_0 := \max\left\{|x| \mid x \in \overline{O}\right\} < \infty$. Hence,

$$\int_O \int_O |\varphi(y)|^2 \frac{1}{|x-y|^{d+2q-2}} \,\mathrm{d}y\,\mathrm{d}x = \int_O |\varphi(y)|^2 \left(\int_{-y+O} \frac{1}{|z|^{d+2q-2}} \,\mathrm{d}z\right) \mathrm{d}y$$
$$\leq \int_O |\varphi(y)|^2 \underbrace{\left(\int_{\mathbb{B}_d(0,2r_0)} \frac{1}{|z|^{d+2q-2}} \,\mathrm{d}z\right)}_{=:C} \mathrm{d}y.$$

(ii). Set $r_0 := \operatorname{dist}(K, \mathbb{R}^d \setminus O)$. Then $r_0 > 0$. Note that $-K + \Omega \setminus O \subseteq \mathbb{R}^d \setminus \mathbb{B}_d(0, r_0)$, because: if $z \in -K + \Omega \setminus O$ then $z = -x + y$ with $x \in K$ and $y \in \Omega \setminus O$, meaning that $|z| = |-x+y| \geq r_0$. Let $x \in K$. Then,

$$0 \leq \int_{-x+\Omega\setminus O} \frac{1}{|z|^{d+2q}} \,\mathrm{d}z \leq \int_{-K+\Omega\setminus O} \frac{1}{|z|^{d+2q}} \,\mathrm{d}z$$
$$\leq \int_{\mathbb{R}^d\setminus\mathbb{B}_d(0,r_0)} \frac{1}{|z|^{d+2q}} \,\mathrm{d}z =: C < \infty,$$

as asserted. □

Lemma A.17 *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s \in \mathbb{R}$ with $s \geq 0$. Then*

$$C_0^{\lceil s \rceil}(\Omega) \subseteq H_c^s(\Omega).$$

Proof: For $s = m \in \mathbb{N}_0$, the assertion is clear. So, let $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0,1)$. Let $\chi \in C_0^{m+1}(\Omega)$. We know already that $\chi \in H_c^m(\Omega)$. Extend χ to $\mathbb{R}^d \setminus \Omega$ by zero and set $K := \operatorname{supp}\chi$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. Then $\partial^\alpha \chi \in C_0^1(\mathbb{R}^d)$ and therefore there exist $C_{1,\alpha} := \max_{z\in\mathbb{R}^d} |\partial^\alpha\chi(z)| < \infty$ and $C_{2,\alpha} := \max_{z\in\mathbb{R}^d} \|\partial(\partial^\alpha\chi)(z)\| < \infty$. In particular, by the mean value theorem,

$$|\partial^\alpha\chi(x) - \partial^\alpha\chi(y)| \leq C_{2,\alpha}\,|x-y|, \qquad x, y \in \mathbb{R}^d.$$

Moreover, there exists an open and bounded set $O \subseteq \mathbb{R}^d$ such that $K \subseteq O \subseteq \Omega$. Consequently,

$$|\partial^\alpha\chi|_{q,\Omega}^2 = \int_\Omega \int_\Omega \frac{|\partial^\alpha\chi(x) - \partial^\alpha\chi(y)|^2}{|x-y|^{d+2q}} \,\mathrm{d}y\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_O \int_\Omega \frac{|\partial^\alpha \chi(x) - \partial^\alpha \chi(y)|^2}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x + \int_{\Omega\setminus O} \int_K \frac{|\partial^\alpha \chi(y)|^2}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_O \int_O \frac{|\partial^\alpha \chi(x) - \partial^\alpha \chi(y)|^2}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x + 2 \int_K \int_{\Omega\setminus O} \frac{|\partial^\alpha \chi(x)|^2}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x \\
&\leq C_{2,\alpha}^2 \int_O \int_O \frac{1}{|x-y|^{d+2q-2}} \,\mathrm{d}y \,\mathrm{d}x + 2\, C_{1,\alpha}^2 \int_K \int_{\Omega\setminus O} \frac{1}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}
$$

Using Fubini's theorem and the substitution $z = y - x$ we obtain

$$
\int_K \int_{\Omega\setminus O} \frac{1}{|x-y|^{d+2q}} \,\mathrm{d}y \,\mathrm{d}x = \int_K \left(\int_{-x+\Omega\setminus O} \frac{1}{|z|^{d+2q}} \,\mathrm{d}z \right) \mathrm{d}x.
$$

Using this identity in the last estimate and applying Lemma A.16, the proof is complete. □

Lemma A.18 *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $s \in \mathbb{R}$ with $s \geq 0$. Furthermore, let $\varphi \in H_c^s(\Omega)$. Then there exists $\varepsilon_0 > 0$ such that $\left\{T_\varepsilon \varphi \mid \varepsilon \in (0, \varepsilon_0)\right\}$ is dense in $H_c^s(\Omega)$. In particular, $C_0^\infty(\Omega)$ is dense in $H_c^s(\Omega)$.*

Proof: Let $\varphi \in H_c^s(\Omega)$. Set $\varepsilon_0 := \operatorname{dist}(\operatorname{supp}\varphi, \mathbb{R}^d \setminus \Omega)$. Then $\varepsilon_0 > 0$, and for $\varepsilon \in (0, \varepsilon_0)$ by (A.7) we have $T_\varepsilon \varphi \in C_0^\infty(\Omega)$, and thus by Lemma A.17 also $T_\varepsilon \varphi \in H_c^s(\Omega)$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \lfloor s \rfloor$. Then by (A.9),

$$
\partial^\alpha T_\varepsilon \varphi \to \partial^\alpha \varphi \quad \text{in } L^2(\Omega), \quad \text{as } \varepsilon \to 0,
$$

yielding that $\|T_\varepsilon \varphi - \varphi\|_{H^{\lfloor s \rfloor}(\Omega)}$ converges to zero, as $\varepsilon \to 0$. Thus, it remains to consider the case $s = m + q$, with $m \in \mathbb{N}_0$ and $q \in (0,1)$, and to show that $|\partial^\alpha T_\varepsilon \varphi - \partial^\alpha \varphi|_{q,\Omega}$ converges to zero, as $\varepsilon \to 0$, with α from above. To this end, we observe, with $B := \mathbb{B}_d(0, \varepsilon)$,

$$
\begin{aligned}
&\left| \big(\partial^\alpha T_\varepsilon \varphi(x) - \partial^\alpha \varphi(x)\big) - \big(\partial^\alpha T_\varepsilon \varphi(y) - \partial^\alpha \varphi(y)\big) \right|^2 \\
&= \left| \int_B \chi_\varepsilon(z) \big\{ \big(\partial^\alpha \varphi(x+z) - \partial^\alpha \varphi(y+z)\big) - \big(\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)\big) \big\} \,\mathrm{d}z \right|^2 \\
&\leq \left(\int_B \chi_\varepsilon(z)^{\frac{1}{2}+\frac{1}{2}} \left| \big(\partial^\alpha \varphi(x+z) - \partial^\alpha \varphi(y+z)\big) - \big(\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)\big) \right| \mathrm{d}z \right)^2
\end{aligned}
$$

$$\leq \int_B \chi_\varepsilon(z) \big| \underbrace{\big(\partial^\alpha \varphi(x+z) - \partial^\alpha \varphi(y+z)\big) - \big(\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)\big)}_{=:f(x,y,z)} \big|^2 \,\mathrm{d}z \cdot 1.$$

Hence,

$$\begin{aligned}
|\partial^\alpha T_\varepsilon \varphi - \partial^\alpha \varphi|^2_{q,\Omega} &= \int_\Omega \int_\Omega \Bigg| \frac{\partial^\alpha T_\varepsilon \varphi(x) - \partial^\alpha \varphi(x)}{|x-y|^{d+2q}} \\
&\qquad - \frac{\partial^\alpha T_\varepsilon \varphi(y) - \partial^\alpha \varphi(y)}{|x-y|^{d+2q}} \Bigg|^2 \,\mathrm{d}y \,\mathrm{d}x \\
&\leq \int_{\mathbb{B}_d(0,\varepsilon)} \chi_\varepsilon(z) \int_{\Omega\times\Omega} \frac{\big|f(x,y,z)\big|^2}{|x-y|^{d+2q}} \,\mathrm{d}(x,y) \,\mathrm{d}z \\
&\leq 1 \cdot \sup_{|z|\leq\varepsilon} \int_{\Omega\times\Omega} \frac{\big|f(x,y,z)\big|^2}{|x-y|^{d+2q}} \,\mathrm{d}(x,y) \\
&= \sup_{|z|\leq\varepsilon} \int_{\Omega\times\Omega} \frac{\big|\big(\partial^\alpha \varphi(x+z) - \partial^\alpha \varphi(y+z)\big) - \big(\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)\big)\big|^2}{|x-y|^{d+2q}} \,\mathrm{d}(x,y).
\end{aligned}$$

By assumption, $\Omega \times \Omega \ni (x,y) \mapsto |\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|/|x-y|^{p+d/2}$ belongs to $L^2(\Omega \times \Omega)$ and is therefore mean continuous (see the second condition in Remark A.19). This means, that we can make the last expression in the last estimate as small as we like. Thus, we have shown the first part of the lemma.
Since $\big\{T_\varepsilon \varphi \mid \varepsilon \in (0, \varepsilon_0)\big\} \subseteq C_0^\infty(\Omega)$, also the second part of the lemma is true. □

The next statement was used in the proof of the last lemma and is also known as the *Kolmogorov-Riesz compactness principle*, see for instance [53, page 4].

Remark A.19 *Let M be a subset of $L^p(\Omega)$, $1 \leq p < \infty$. M is relatively compact if and only if the following three conditions are satisfied:*

- *(i) M is bounded in $L^p(\Omega)$, i.e., $\sup_{\varphi\in M} \|\varphi\|_{L^p(\Omega)} < \infty$.*
- *(ii) $\lim_{h\to 0} \int_\Omega |\varphi(x+h) - \varphi(x)|^p \,\mathrm{d}x = 0$ holds uniformly for $\varphi \in M$.*

(iii) $\lim_{r\nearrow\infty}\int_{\{|x|>r\}\cap\Omega}|\varphi(x)|^p\,\mathrm{d}x = 0$ *holds uniformly for* $\varphi\in M$.

For a proof we refer to [53] and references therein.

Theorem A.20 *Let* $\Omega\subseteq\mathbb{R}^d$ *be open and let* $s\in\mathbb{R}$ *with* $s\geq 0$. *Then* $C^\infty(\Omega)\cap H^s(\Omega)$ *is dense in* $H^s(\Omega)$.

Proof: Set $\Omega_n := \big\{x\in\Omega \;\big|\; |x|<n,\ \operatorname{dist}(x,\partial\Omega)>1/n\big\}$ for $n\in\mathbb{N}$. Then Ω_n is open and bounded, and $\Omega_n\subseteq\overline{\Omega_n}\subseteq\Omega_{n+1}\subseteq\Omega$ for all $n\in\mathbb{N}$. Note that $\bigcup_{n=1}^\infty\Omega_n=\Omega$ and $\Omega=\bigcup_{n=0}^\infty\Omega_{n+1}\setminus\overline{\Omega_{n-1}}$, where $\Omega_0:=\Omega_{-1}:=\emptyset$. There exist functions $0\leq\chi_n\in C_0^\infty(\Omega)$ such that $\operatorname{supp}\chi_n\subseteq\Omega_{n+1}\setminus\overline{\Omega_{n-1}}$ and $\sum_{n=1}^\infty\chi_n(x)=1$ for all $x\in\Omega$.
Let $\varphi\in H^s(\Omega)$ and $\varepsilon>0$. Note that $\chi_n\,\varphi\in H^s_c(\Omega)$, see Theorem A.15. Then, thanks to Lemma A.18, for all $n\in\mathbb{N}$ there exists $\delta_n>0$ such that $T_{\delta_n}(\chi_n\,\varphi)=:\theta_n\in C_0^\infty(\Omega)$, $\operatorname{supp}\theta_n\subseteq\operatorname{supp}(\chi_n\,\varphi)+\mathbb{B}_d[0,\delta_n]\subseteq\Omega_{n+1}\setminus\overline{\Omega_{n-1}}$ and $\|\theta_n-\chi_n\,\varphi\|_{H^s(\Omega)}\leq 2^{-n}\varepsilon$. Define $\theta(x):=\sum_{n=1}^\infty\theta_n(x)$ for all $x\in\Omega$. Since any $\mathbb{B}_d[0,r]\subseteq\Omega$ intersects only finitely many of the sets $\Omega_{n+1}\setminus\overline{\Omega_{n-1}}$, we obtain $\theta\in C^\infty(\Omega)$, because the sum is finite on each $\mathbb{B}_d[0,r]$.
Since $\|\theta_n-\chi_n\,\varphi\|_{H^s(\Omega)}\leq 2^{-n}\varepsilon$ and $H^s(\Omega)$ is a Banach space, there exists $\psi\in H^s(\Omega)$ such that $\psi=\sum_{n=1}^\infty(\theta_n-\chi_n\,\varphi)$. Let $\chi\in C_0^\infty(\Omega)$. Set $K:=\operatorname{supp}\chi$ and $N:=\max\big\{n\in\mathbb{N}\mid K\cap\big(\Omega_{n+1}\setminus\overline{\Omega_{n-1}}\big)\neq\emptyset\big\}$. Then for $n\geq N$ there holds $\theta-\varphi=\sum_{k=1}^n(\theta_k-\chi_k\,\varphi)$ almost everywhere on K. Consequently,

$$\begin{aligned}\int_\Omega[\psi-(\theta-\varphi)]\chi\,\mathrm{d}x &= \lim_{n\to\infty}\int_K\left[\sum_{k=1}^n(\theta_k-\chi_k\,\varphi)-(\theta-\varphi)\right]\chi\,\mathrm{d}x\\ &= \lim_{\substack{n\to\infty\\ N\leq n}}\int_K\left[\sum_{k=1}^n(\theta_k-\chi_k\,\varphi)-\sum_{k=1}^n(\theta_k-\chi_k\,\varphi)\right]\chi\,\mathrm{d}x=0,\end{aligned}$$

i.e., $\psi=\theta-\varphi$ almost everywhere on Ω. Hence, $\theta=\theta-\varphi+\varphi\in H^s(\Omega)$ and $\|\theta-\varphi\|_{H^s(\Omega)}\leq\sum_{n=1}^\infty\|\theta_n-\chi_n\,\varphi\|_{H^s(\Omega)}\leq\varepsilon$. $\square$

Transformation Theorem. Our next goal is to give a simplified version of the transformation theorem which is tailor-made for many situations in applications. We start with a lemma which ensures the application of the mean value theorem also for the case where the subset under consideration is not convex.

Lemma A.21 *Let $\Omega \subseteq \mathbb{R}^d$ be open and let $O \subseteq \mathbb{R}^d$ be open and bounded such that $\overline{O} \subseteq \Omega$. Then the following statements are true.*

(i) For all $\chi \in C^1(\Omega)$ there exists a constant $C \geq 0$ such that

$$|\chi(x) - \chi(y)| \leq C\,|x-y|, \qquad x, y \in \overline{O}.$$

(ii) For any open set $\Omega' \subseteq \mathbb{R}^d$ and any $\zeta \in \mathrm{Diff}^1(\Omega, \Omega')$ there exist constants $C, C' \geq 0$ such that, with $O' := \zeta(O)$ and $x' := \zeta(x)$, $x \in O$,

$$|x'-y'| \leq C\,|x-y| \quad \textit{and} \quad |x-y| \leq C'\,|x'-y'|,$$

for all $x, y \in O$ and all $x', y' \in O'$.

Proof: *(i).* For $x, y \in \mathbb{R}^d$ we set $[\![x,y]\!] := \{x + \xi(y-x) \mid \xi \in [0,1]\}$. Since $\overline{O}$ is compact, we find some open and bounded set $O_1 \subseteq \mathbb{R}^d$ such that $\overline{O} \subseteq O_1$ and $\overline{O_1} \subseteq \Omega$. Set $\delta := \operatorname{dist}\big(\overline{O}, \mathbb{R}^d \setminus O_1\big)$. Then $\delta > 0$. Choose some $\varepsilon \in (0, \delta)$. Thus, for $x, y \in \overline{O}$ with $|x-y| < \varepsilon$, we have $[\![x,y]\!] \subseteq O_1$ and therefore

$$|\chi(x) - \chi(y)| \leq \max_{z \in \overline{O_1}} \big\|\partial\chi(z)\big\| |x-y|.$$

And if $x, y \in \overline{O}$ such that $|x-y| \geq \varepsilon$, then

$$|\chi(x) - \chi(y)| \leq 2 \max_{z \in \overline{O}} |\chi(z)| \leq \frac{2}{\varepsilon} \max_{z \in \overline{O}} |\chi(z)| |x-y|.$$

Choosing $C := \max\big\{ \max_{z \in \overline{O_1}} \|\partial\chi(z)\|, 2/\varepsilon \max_{z \in \overline{O}} |\chi(z)|\big\}$, the first part is finished.

(ii). The proof follows exactly the proof from the first part. For the second estimate we note that $|x-y| = |\zeta^{-1}(x') - \zeta^{-1}(y')|$, for $x', y' \in \overline{O'}$, and that $\overline{O'} = \zeta(\overline{O}) \subseteq \Omega'$ is compact. □

The next formula for the chain rule in higher dimensions is convenient and meets our requirements.

Proposition A.22 *Let $O, O' \subseteq \mathbb{R}^d$ be open and let $m \in \mathbb{N}$. Suppose $\zeta \in C^m(O, O')$ and $\varphi \in C^m(O')$. Then for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$*

$$\partial^\alpha(\varphi \circ \zeta)(x) = \sum_{|\beta| \leq |\alpha|} p_{\alpha\beta}(x) \left((\partial^\beta \varphi) \circ \zeta\right)(x), \qquad x \in O,$$

where $p_{\alpha\beta} : O \to \mathbb{R}$ is a polynomial of degree less or equal to $|\beta|$ in the derivatives up to order $|\alpha|$ of the components of ζ. To be more precisely, the summands of $p_{\alpha\beta}(x)$ are of the form

$$c\left(\partial^{\gamma^{(1,1)}}\zeta_1(x) \cdots \partial^{\gamma^{(1,\beta_1)}}\zeta_1(x)\right) \cdots \left(\partial^{\gamma^{(d,1)}}\zeta_d(x) \cdots \partial^{\gamma^{(d,\beta_d)}}\zeta_d(x)\right),$$

where $c \in \mathbb{N}_0$, $\gamma^{(i,j)} \in \mathbb{N}_0^d$ with $|\gamma^{(i,j)}| \leq |\alpha|$ and the k-th factor in the expression above is equal to one if $\beta_k = 0$.

Proof: We show the statement by induction. For $|\alpha| = 0$, the equation is trivially satisfied with $p_{00} \equiv 1$. Now, suppose the formula holds for some $\alpha \in \mathbb{N}_0^d$ with $0 \leq |\alpha| < m$. Choose $i \in \{1, \ldots, d\}$ and denote by $e^{(i)}$ the i-th unit coordinate vector in $\mathbb{R}^d$. Then

$$\begin{aligned}\partial^{\alpha+e^{(i)}}[\varphi \circ \zeta] &= \partial^{e^{(i)}}\left\{\partial^\alpha[\varphi \circ \zeta]\right\} = \partial^{e^{(i)}}\left\{\sum_{|\beta| \leq |\alpha|} p_{\alpha\beta} \cdot \left([\partial^\beta \varphi] \circ \zeta\right)\right\} \\ &= \sum_{|\beta| \leq |\alpha|}\left\{\left[\partial^{e^{(i)}} p_{\alpha\beta}\right] \cdot \left([\partial^\beta \varphi] \circ \zeta\right) + p_{\alpha\beta} \sum_{j=1}^d \left(\left[\partial^{\beta+e^{(j)}}\varphi\right] \circ \zeta\right) \partial^{e^{(i)}}\zeta_j\right\}.\end{aligned}$$

From the last line, we already see that in the expression on the right hand side of the plus sign there appear only $p_{\alpha\beta} \cdot \partial^{e^{(i)}}\zeta_j$, meaning that the summands of the old polynomial $p_{\alpha\beta}$ are only multiplied by $\partial^{e^{(i)}}\zeta_j$, and hence, these products contribute already summands of the new polynomial in the form as asserted. For the expression on the left hand side of the plus sign, we proceed with an extra step to recognize the special form also for these contributions. In fact, for the summands in $\partial^{e^{(i)}} p_{\alpha\beta}$ we have

$$\partial^{e^{(i)}}\left[c \prod_{j=1}^d \prod_{k=1}^{\beta_j} \partial^{\gamma^{(j,k)}}\zeta_j\right] = c \sum_{j=1}^d \left\{\partial^{e^{(i)}}\left[\prod_{k=1}^{\beta_j} \partial^{\gamma^{(j,k)}}\zeta_j\right] \cdot \prod_{\substack{l=1 \\ l \neq j}}^d \left(\prod_{p=1}^{\beta_l} \partial^{\gamma^{(l,p)}}\zeta_l\right)\right\}$$

$$= c \sum_{j=1}^{d} \left\{ \left(\sum_{k=1}^{\beta_j} \left\{ \partial^{\gamma^{(j,k)}+e^{(i)}} \zeta_j \prod_{\substack{n=1 \\ n \neq k}}^{\beta_j} \partial^{\gamma^{(j,n)}} \zeta_j \right\} \right) \cdot \prod_{\substack{l=1 \\ l \neq j}}^{d} \left(\prod_{p=1}^{\beta_l} \partial^{\gamma^{(l,p)}} \zeta_l \right) \right\}$$

$$= c \sum_{j=1}^{d} \sum_{k=1}^{\beta_j} \left(\partial^{\gamma^{(j,k)}+e^{(i)}} \zeta_j \prod_{\substack{n=1 \\ n \neq k}}^{\beta_j} \partial^{\gamma^{(j,n)}} \zeta_j \right) \cdot \left(\prod_{\substack{l=1 \\ l \neq j}}^{d} \prod_{p=1}^{\beta_l} \partial^{\gamma^{(l,p)}} \zeta_l \right),$$

from which we see that also the summands of the new polynomial, generated by the derivatives of the summands of the old polynomial, are of the asserted form. And finally, $c \in \mathbb{N}_0$ results from the consideration, that during the process of generating the summands of the new polynomial, as described above, it may happen that some of those are equal. □

We define the pullback operators ${}^*\zeta$ and $({}^*\zeta)^{-1}$ in the next *transformation theorem* by continuous extension, thanks to Corollary A.7.

Theorem A.23 *Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be open, $s \in \mathbb{R}$ with $s \geq 0$ and let $\zeta \in \mathrm{Diff}^{\lfloor s \rfloor + 1}(\Omega, \Omega')$. Furthermore, let $O \subseteq \mathbb{R}^d$ be open and bounded such that $\overline{O} \subseteq \Omega$, and set $O' := \zeta(O)$. Then the mapping*

$$\big(C^\infty(O') \cap H^s(O'), \|\cdot\|_{H^s(O')}\big) \ni \varphi \mapsto \varphi \circ \zeta \in H^s(O)$$

is well-defined, linear and bounded. Its continuous extension to $H^s(O')$, denoted by ${}^*\zeta$, belongs to the space $\mathcal{L}_{\mathrm{is}}\big(H^s(O'), H^s(O)\big)$. The inverse $({}^*\zeta)^{-1} : H^s(O) \to H^s(O')$ is the continuous extension of

$$\big(C^\infty(O) \cap H^s(O), \|\cdot\|_{H^s(O)}\big) \ni \psi \mapsto \psi \circ \zeta^{-1} \in H^s(O')$$

to $H^s(O)$.

Proof: Note, by Theorem A.20 the subspace $C^\infty(O') \cap H^s(O')$ and $C^\infty(O) \cap H^s(O)$ is dense in $H^s(O')$ and $H^s(O)$, respectively. Furthermore,

$$\varphi \mapsto \varphi \circ \zeta \mapsto \varphi \circ \zeta \circ \zeta^{-1} = \varphi \quad \text{and} \quad \psi \mapsto \psi \circ \zeta^{-1} \mapsto \psi \circ \zeta^{-1} \circ \zeta = \psi,$$

for $\varphi \in C^\infty(O') \cap H^s(O')$ and $\psi \in C^\infty(O) \cap H^s(O)$. Therefore, thanks to Corollary A.7, we only have to show, that the mappings from the theorem, defined on the dense subset, are indeed well-defined, linear and bounded. And since both of them play the same role, it sufficies to consider only one of them, say the first one.

Without loss of generality, we assume $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0,1)$, as the case $s = m \in \mathbb{N}_0$ is a special case from the following explanation. We start by choosing some $\chi \in C_0^\infty(\Omega)$ with $\chi|_{\overline{O}} \equiv 1$. Now, let $\varphi \in C^\infty(O') \cap H^s(O')$. Furthermore, let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. Then, by Proposition A.22,

$$\partial^\alpha(\varphi \circ \zeta) = \sum_{|\beta| \leq |\alpha|} p_{\alpha\beta} \cdot \left([\partial^\beta \varphi] \circ \zeta \right).$$

Let $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq |\alpha|$. Note that $p_{\alpha\beta}\chi \in C_0^1(\Omega)$ and by extension by zero to $\mathbb{R}^d$ we have $p_{\alpha\beta}\chi \in C_b^1(\mathbb{R}^d)$. Set $C_\zeta := \| \det \partial(\zeta^{-1}) \|_{C_b(O)}$ and $C_p := \max \left\{ \|p_{\gamma\kappa}\|_{C_b^1(O)} \;\middle|\; |\kappa| \leq |\gamma|, |\gamma| \leq m \right\}$. By the transformation theorem for Lebesgue integrable functions we have

$$\begin{aligned} \left\| [\partial^\beta \varphi] \circ \zeta \right\|_{L^2(O)}^2 &\leq C_\zeta \left\| \partial^\beta \varphi \right\|_{L^2(O')}^2, \\ \left| [\partial^\beta \varphi] \circ \zeta \right|_{q,O}^2 &\leq C_1^{d+2q}\, C_\zeta^2 \left| \partial^\beta \varphi \right|_{q,O'}^2, \end{aligned}$$

where we also applied part *(ii)* from Lemma A.21 in the second estimate and where C_1 denotes the corresponding constant therein. Hence, $[\partial^\beta \varphi] \circ \zeta \in H^q(O)$ and by Theorem A.15 we obtain $p_{\alpha\beta}\chi \left([\partial^\beta \varphi] \circ \zeta \right) \in H^q(O)$ as well and moreover

$$\begin{aligned} \left\| p_{\alpha\beta} \left([\partial^\beta \varphi] \circ \zeta \right) \right\|_{L^2(O)}^2 + \left| p_{\alpha\beta} \left([\partial^\beta \varphi] \circ \zeta \right) \right|_{q,O}^2 &= \left\| p_{\alpha\beta}\chi \left([\partial^\beta \varphi] \circ \zeta \right) \right\|_{H^q(O)}^2 \\ &\leq C_2^2\, C_p^2 \left\| [\partial^\beta \varphi] \circ \zeta \right\|_{H^q(O)}^2 \\ &\leq C_2^2\, C_p^2\, C_\zeta \left(\left\| \partial^\beta \varphi \right\|_{L^2(O')}^2 + C_1^{d+2q}\, C_\zeta \left| \partial^\beta \varphi \right|_{q,O'}^2 \right) \\ &\leq \underbrace{C_2^2\, C_p^2\, C_\zeta \left(1 + C_1^{d+2q}\, C_\zeta\right)}_{=:C_3} \|\varphi\|_{H^s(O')}^2, \end{aligned}$$

where C_2 denotes the corresponding constant from Theorem A.15. With the triangle inequality for the L^2-norm and the seminorm we conclude that

$$\left\| \partial^\alpha(\varphi \circ \zeta) \right\|_{L^2(O)}^2 + \left| \partial^\alpha(\varphi \circ \zeta) \right|_{q,O}^2$$

$$\leq \Big(\sum_{|\beta| \leq |\alpha|} 1 \Big) \sum_{|\beta| \leq |\alpha|} \Big(\big\| p_{\alpha\beta}\big([\partial^\beta \varphi] \circ \zeta\big) \big\|^2_{L^2(O)} + \big| p_{\alpha\beta}\big([\partial^\beta \varphi] \circ \zeta\big) \big|^2_{q,O} \Big)$$
$$\leq C_3\, C_4^2\, \|\varphi\|^2_{H^s(O')},$$

where we have set $C_4 := \sum_{|\beta| \leq m} 1$. From this we get finally the estimate $\|\varphi \circ \zeta\|^2_{H^s(O)} \leq C_3\, C_4^3\, \|\varphi\|^2_{H^s(O')}$, as desired. □

Corollary A.24 *Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be open, $s \in \mathbb{R}$ with $s \geq 0$ and let $\zeta \in \mathrm{Diff}^{\lfloor s \rfloor + 1}(\Omega, \Omega')$. Furthermore, let $K \subseteq \Omega$ be compact and let $O \subseteq \mathbb{R}^d$ be open and bounded such that $K \subseteq O$ and $\overline{O} \subseteq \Omega$. Set $K' := \zeta(K)$ and $O' := \zeta(O)$. Finally, let ${}^*\zeta \in \mathcal{L}_{\mathrm{is}}\big(H^s(O'), H^s(O)\big)$ be the pullback operator from Theorem A.23. Then*

$${}^*\zeta|_{H^s_{K'}(O')} \in \mathcal{L}_{\mathrm{is}}\big(H^s_{K'}(O'), H^s_K(O)\big).$$

Proof: It sufficies to show that $\varphi \in H^s_{K'}(O')$ implies $\mathrm{supp}({}^*\zeta\, \varphi) \subseteq K$. For this let $\varphi \in H^s_{K'}(O')$. By definition, ${}^*\zeta\, \varphi = \lim_{n\to\infty} \varphi_n \circ \zeta$, where the limit is taken with respect to $\|\cdot\|_{H^s(O)}$, for some sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C^\infty(O') \cap H^s(O')$ converging to φ with respect to $\|\cdot\|_{H^s(O')}$. Take any $\chi \in C_0^\infty(O \setminus K)$ and set $x' := \zeta(x)$ for $x \in O$. Then

$$\int_{O\setminus K} ({}^*\zeta\, \varphi)(x)\, \chi(x)\, \mathrm{d}x = \lim_{n\to\infty} \int_{O\setminus K} \varphi_n\big(\zeta(x)\big)\, \chi(x)\, \mathrm{d}x$$
$$= \lim_{n\to\infty} \int_{O'\setminus K'} \varphi_n(x')\, \chi\big(\zeta^{-1}(x')\big)\, \big|\det \partial(\zeta^{-1})(x')\big|\, \mathrm{d}x'$$
$$= \int_{O'\setminus K'} \varphi(x')\, \chi\big(\zeta^{-1}(x')\big)\, \big|\det \partial(\zeta^{-1})(x')\big|\, \mathrm{d}x' = 0,$$

as $\mathrm{supp}\, \varphi \subseteq K'$. This shows that ${}^*\zeta\, \varphi$ is zero almost everywhere on $O \setminus K$, as desired. □

Auxiliary mappings. Let $\Omega' \subseteq \Omega \subseteq \mathbb{R}^d$ and both be open. For $\varphi \in L^2(\Omega)$ recall its restriction $\varphi|_{\Omega'}$ and for $\varphi \in L^2(\Omega')$ its extension (by zero) $\varphi|_0^\Omega$ from Section 1.3. Now, we would like to introduce restriction and extension (by zero) operators for the space $H^s(\Omega)$ and $H^s_K(\Omega')$, respectively.

Lemma A.25 *Let $\Omega' \subseteq \Omega \subseteq \mathbb{R}^d$ and both be open. Moreover, let $s \in \mathbb{R}$ with $s \geq 0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \lfloor s \rfloor$. Then*

(i) $\varphi \in H^s(\Omega) \;\Rightarrow\; \exists\, \partial^\alpha(\varphi|_{\Omega'}) = \big(\partial^\alpha \varphi\big)\big|_{\Omega'} \in L^2(\Omega')$,

(ii) $K \subseteq \Omega'$ *compact,* $\varphi \in H^s_K(\Omega') \;\Rightarrow\; \exists\, \partial^\alpha(\varphi|_0^\Omega) = (\partial^\alpha\varphi)|_0^\Omega \in L^2(\Omega)$.

Proof: *(i).* Let $\varphi \in H^s(\Omega)$. Clearly, $\big(\partial^\alpha\varphi\big)|_{\Omega'} \in L^2(\Omega')$. Take any $\chi \in C_0^\infty(\Omega')$. Of course, $\chi|_0^\Omega \in C_0^\infty(\Omega)$ and $\operatorname{supp}\chi|_0^\Omega \subseteq \Omega'$. Hence,

$$\int_{\Omega'} \varphi|_{\Omega'}\, \partial^\alpha \chi \, \mathrm{d}x = \int_\Omega \varphi\, \partial^\alpha \chi|_0^\Omega \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega (\partial^\alpha\varphi)\, \chi|_0^\Omega \, \mathrm{d}x$$
$$= (-1)^{|\alpha|} \int_{\Omega'} \big(\partial^\alpha\varphi\big)|_{\Omega'}\, \chi \, \mathrm{d}x.$$

(ii). Let $\varphi \in H^s_K(\Omega')$. By Proposition A.12, the support of $\partial^\alpha\varphi$ is contained in K. Since $\partial^\alpha\varphi \in L^2(\Omega')$, of course $(\partial^\alpha\varphi)|_0^\Omega \in L^2(\Omega)$. Take some $\chi \in C_0^\infty(\Omega)$. Choose $\eta \in C_0^\infty(\Omega')$ such that $\eta \equiv 1$ in a neighborhood of K. By extension by zero, we can consider $\eta \in C_0^\infty(\Omega)$ as well. Then $\chi = \eta\chi + (1-\eta)\chi$ with $\operatorname{supp}\eta\chi \subseteq \Omega'$ and $\operatorname{supp}(1-\eta)\chi \subseteq K^c$. Therefore, by considering also $\eta\chi \in C_0^\infty(\Omega')$,

$$\begin{aligned}
\int_\Omega \varphi|_0^\Omega\, \partial^\alpha\chi \, \mathrm{d}x &= \int_\Omega \varphi|_0^\Omega\, \partial^\alpha(\eta\chi)\, \mathrm{d}x + \int_\Omega \varphi|_0^\Omega\, \partial^\alpha\big((1-\eta)\chi\big)\, \mathrm{d}x \\
&= \int_{\Omega'} \varphi\, \partial^\alpha(\eta\chi)\, \mathrm{d}x + \int_K \varphi|_0^\Omega\, \underbrace{\partial^\alpha\big((1-\eta)\chi\big)}_{=0}\, \mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\Omega'} \partial^\alpha\varphi(\eta\chi)\, \mathrm{d}x + (-1)^{|\alpha|} \int_{\Omega\setminus K} 0 \cdot \big((1-\eta)\chi\big)\, \mathrm{d}x \\
&= (-1)^{|\alpha|} \Big(\int_{\Omega'} (\partial^\alpha\varphi)|_0^\Omega(\eta\chi)\, \mathrm{d}x + \int_{\Omega\setminus K} (\partial^\alpha\varphi)|_0^\Omega\big((1-\eta)\chi\big)\, \mathrm{d}x \Big) \\
&= (-1)^{|\alpha|} \Big(\int_\Omega (\partial^\alpha\varphi)|_0^\Omega(\eta\chi)\, \mathrm{d}x + \int_\Omega (\partial^\alpha\varphi)|_0^\Omega\big((1-\eta)\chi\big)\, \mathrm{d}x \Big) \\
&= (-1)^{|\alpha|} \int_\Omega (\partial^\alpha\varphi)|_0^\Omega\, \chi\, \mathrm{d}x.
\end{aligned}$$

This completes the proof. □

Theorem A.26 *Let $\Omega' \subseteq \Omega \subseteq \mathbb{R}^d$ and both be open. Moreover, let $s \in \mathbb{R}$ with $s \geq 0$. Then the following assertions are true.*

(i) The restriction operator $H^s(\Omega) \ni \varphi \mapsto \varphi|_{\Omega'} \in H^s(\Omega')$ is well-defined, linear and bounded, i.e.,

$$\|\varphi|_{\Omega'}\|_{H^s(\Omega')} \leq \|\varphi\|_{H^s(\Omega)}, \qquad \varphi \in H^s(\Omega).$$

(ii) Let in addition Ω be bounded and $K \subseteq \Omega'$ be compact. Then the extension operator $H^s_K(\Omega') \ni \varphi \mapsto \varphi|_0^\Omega \in H^s_K(\Omega)$ is well-defined, linear and bounded, i.e.,

$$\|\varphi|_0^\Omega\|_{H^s(\Omega)} \leq c\,\|\varphi\|_{H^s(\Omega')}, \qquad \varphi \in H^s_K(\Omega').$$

The constant c only depends on K, Ω', Ω and s.

Proof: At first we assume that $s = m + q$ with $m \in \mathbb{N}$ and $q \in (0, 1)$.
(i). The assertion for the restriction operator is obviously true.
(ii). In the following considerations, $c_2 > 0$ denotes the constant from the second part of Lemma A.16. Let $\varphi \in H^s_K(\Omega')$. Take some $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m$. Note that also $\operatorname{supp} \partial^\alpha \varphi \subseteq K$, see Proposition A.12. Then

$$\begin{aligned}\int_{\Omega'}\int_{\Omega\setminus\Omega'} \frac{\left|\partial^\alpha\varphi(x)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x &= \int_K\int_{\Omega\setminus\Omega'} \frac{\left|\partial^\alpha\varphi(x)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq c_2\,|\Omega\setminus\Omega'|\,\left\|\partial^\alpha\varphi\right\|^2_{L^2(\Omega')}.\end{aligned}$$

Hence, thanks to Lemma A.25,

$$\begin{aligned}&\left|\partial^\alpha(\varphi|_0^\Omega)\right|^2_{q,\Omega} = \int_\Omega\int_\Omega \frac{\left|\partial^\alpha(\varphi|_0^\Omega)(x) - \partial^\alpha(\varphi|_0^\Omega)(y)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega\int_\Omega \frac{\left|(\partial^\alpha\varphi)|_0^\Omega(x) - (\partial^\alpha\varphi)|_0^\Omega(y)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x \\ &= 2\int_{\Omega'}\int_{\Omega\setminus\Omega'} \frac{\left|\partial^\alpha\varphi(x)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega'}\int_{\Omega'} \frac{\left|\partial^\alpha\varphi(x) - \partial^\alpha\varphi(y)\right|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq 2\,c_2\,|\Omega\setminus\Omega'|\,\left\|\partial^\alpha\varphi\right\|^2_{L^2(\Omega')} + \left|\partial^\alpha\varphi\right|^2_{q,\Omega'}.\end{aligned}$$

Therefore, again with the help of the two lemmas,

$$\begin{aligned}\|\varphi|_0^\Omega\|_{H^s(\Omega)}^2 &= \|\varphi|_0^\Omega\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=m} \|\partial^\alpha(\varphi|_0^\Omega)\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=m} \left|\partial^\alpha(\varphi|_0^\Omega)\right|_{q,\Omega}^2 \\ &\le \|\varphi\|_{L^2(\Omega')}^2 + \Big(1 + 2\,c_2\,|\Omega\setminus\Omega'|\Big) \sum_{|\alpha|=m} \|\partial^\alpha\varphi\|_{L^2(\Omega')}^2 + \sum_{|\alpha|=m} |\partial^\alpha\varphi|_{q,\Omega'}^2 .\end{aligned}$$

Thus, the extension operator is well-defined. Its linearity is clear and its boundedness follows immediately from the last estimate.
Finally, the remaining cases for s are similarly treated, but easier. □

A Useful "Continuity" Result. We conclude this subsection with a "continuity" result concerning the seminorm, as stated in the following lemma. This result is needed later, when we show norm equivalences in periodic Sobolev spaces. Nevertheless, we outsource the corresponding step from there already here in form of this lemma, since the result may also be seen in a general context and its proof contains some useful ideas. The ideas were taken over from [53, page 63].

Lemma A.27 *Let $\Omega \subseteq \mathbb{R}^d$ be open, $q \in (0,1)$ and let $\varphi, \varphi_n \in L^2(\Omega)$ for all $n \in \mathbb{N}$, such that $\varphi_n \to \varphi$ in $L^2(\Omega)$, as $n \to \infty$. Moreover, suppose that $\big(|\varphi_n|_{q,\Omega}^2\big)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then*

$$|\varphi|_{q,\Omega}^2 < \infty \qquad \text{and} \qquad |\varphi_n|_{q,\Omega}^2 \longrightarrow |\varphi|_{q,\Omega}^2\,, \quad \text{as } n \to \infty.$$

Proof: By the Theorem of Riesz-Fischer, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$, converging pointwise to φ almost everywhere in Ω. Hence,

$$\frac{|\varphi_{n_k}(x) - \varphi_{n_k}(y)|^2}{|x-y|^{d+2q}} \longrightarrow \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2q}}, \qquad \text{as } k \to \infty,$$

for almost all $(x,y) \in \Omega\times\Omega$. Since $\big(|\varphi_n|_{q,\Omega}^2\big)_{n\in\mathbb{N}}$ is a Cauchy sequence, it is in particular bounded, say by $0 < M < \infty$, and together with Fatou's lemma

$$|\varphi|_{q,\Omega}^2 = \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2q}}\,\mathrm{d}y\,\mathrm{d}x$$

$$\leq \liminf_{k\to\infty} \int_\Omega \int_\Omega \frac{|\varphi_{n_k}(x) - \varphi_{n_k}(y)|^2}{|x-y|^{d+2q}} \, \mathrm{d}y \, \mathrm{d}x \leq M.$$

Moreover,

$$\left| \frac{|\varphi_{n_k}(x) - \varphi_{n_k}(y)|^2}{|x-y|^{d+2q}} - \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2q}} \right| \longrightarrow 0, \qquad \text{as } k \to \infty,$$

for almost all $(x, y) \in \Omega \times \Omega$. Applying now a corollary of Fatou's lemma, see for instance [6, Korollar 3.8], yields

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \left| \int_\Omega \int_\Omega \left(\frac{|\varphi_{n_k}(x) - \varphi_{n_k}(y)|^2}{|x-y|^{d+2q}} - \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2q}} \right) \mathrm{d}y \, \mathrm{d}x \right| \quad (*) \\ &\leq \limsup_{k\to\infty} \int_\Omega \int_\Omega \left| \frac{|\varphi_{n_k}(x) - \varphi_{n_k}(y)|^2}{|x-y|^{d+2q}} - \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2q}} \right| \mathrm{d}y \, \mathrm{d}x \leq 0. \end{aligned}$$

Because $\left(|\varphi_n|^2_{q,\Omega} \right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, it is convergent. Therefore, also $\left(|\varphi_{n_k}|^2_{q,\Omega} \right)_{k\in\mathbb{N}}$ is convergent and converges to the same limit. Hence, there exists

$$\lim_{k\to\infty} \left| |\varphi_{n_k}|^2_{q,\Omega} - |\varphi|^2_{q,\Omega} \right| = \left| \lim_{k\to\infty} |\varphi_{n_k}|^2_{q,\Omega} - |\varphi|^2_{q,\Omega} \right|.$$

Since in the case of convergence the limit and the limit superior coincide, we obtain from the last identity, together with $(*)$ and the remark from above about the limit of subsequences, indeed the convergence result from the lemma. □

A.5.2. Results for Biperiodic Sobolev Spaces

In this subsection we will transfer the results for Sobolev-Slobodeckiĭ spaces from above to the biperiodic setting. For this purpose, let throughout this subsection the rectangle $Q \subseteq \mathbb{R}^2$ be given by

$$Q := (-L_1, L_1) \times (-L_2, L_2),$$

where $L_j > 0$ are some real numbers, $j = 1, 2$, and recall the notion of periodic functions (with respect to Q) from the corresponding paragraph in Section 1.3.

Clearly, with $C_b^m(Q)$ from the last subsection there holds $C_{\text{per}}^m(Q) \subseteq C_b^m(Q)$, i.e., we can also equip $C_{\text{per}}^m(Q)$ with the norm $\|\cdot\|_{C_b^m(Q)}$.

As in (2.7) (see also (1.14)) we define

$$q^{(\mu)} := q_Q^{(\mu)} = \begin{pmatrix} \mu_1\pi/L_1 \\ \mu_2\pi/L_2 \end{pmatrix}, \qquad \mu \in \mathbb{Z}^2.$$

Now, we want to introduce Sobolev-Slobodeckiĭ spaces based on periodic functions and investigate their relation to $H_{\text{per}}^s(Q)$ from Definition 2.7. To this end, the space $\mathcal{H}_{\text{per}}^m(Q)$ from Definition 2.14 turns out to be the correct starting point, similarly as its analog $H^m(\Omega)$ in the last subsection. For ease of notation, we introduce for $q \in (0,1)$ and for suitable $\varphi, \psi \in L^2(Q)$ the quantities

$$\langle \varphi, \psi \rangle_{q,\text{per}} := \int_Q \int_Q \frac{\big(\varphi(t) - \varphi(\tau)\big)\overline{\big(\psi(t) - \psi(\tau)\big)}}{\Big(\Big|\sin \frac{\pi(t_1-\tau_1)}{2L_1}\Big| + \Big|\sin \frac{\pi(t_2-\tau_2)}{2L_2}\Big|\Big)^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t,$$

$$|\varphi|_{q,\text{per}} := \sqrt{\langle \varphi, \varphi \rangle_{q,\text{per}}}.$$

As we will see in Lemma A.33, such suitable $\varphi, \psi \in L^2(Q)$ are for instance the trigonometric monomials.

Definition A.28 *For $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0,1)$ we define*

$$\mathcal{H}_{\text{per}}^s(Q) := \Big\{ \varphi \in \mathcal{H}_{\text{per}}^m(Q) \;\Big|\; \forall \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \leq m : \big|\partial_{\text{per}}^\alpha \varphi\big|_{q,\text{per}} < \infty \Big\}$$

and endow this space with the inner product

$$(\varphi \mid \psi)_{\mathcal{H}_{\text{per}}^s(Q)} := (\varphi \mid \psi)_{\mathcal{H}_{\text{per}}^m(Q)} + \sum_{|\alpha| \leq m} \langle \partial^\alpha \varphi, \partial^\alpha \psi \rangle_{q,\text{per}}.$$

Here, the space $\mathcal{H}_{\text{per}}^m(Q)$ was defined in Definition 2.14. For $s \in \mathbb{R}$ with $s \geq 0$ we choose in $\mathcal{H}_{\text{per}}^s(Q)$ the norm $\|\cdot\|_{\mathcal{H}_{\text{per}}^s(Q)} := \sqrt{(\cdot \mid \cdot)_{\mathcal{H}_{\text{per}}^s(Q)}}$.

Again, the choice of the symbol $|\cdot|_{q,\text{per}}$ is advisable since the following lemma shows that $|\cdot|_{q,\text{per}}$ is a seminorm.

Lemma A.29 *Let $q \in (0,1)$. Then $|\cdot|_{q,\mathrm{per}}$ is a seminorm in $\mathcal{H}^q_{\mathrm{per}}(Q)$.*

Proof: The proof is a repetition of the proof of Lemma A.13. So, let $\varphi, \psi \in \mathcal{H}^q_{\mathrm{per}}(Q)$ and $\lambda \in \mathbb{C}$ be arbitrary. Then it is easy to see that $|\varphi|_{q,\mathrm{per}} \geq 0$ and that $|\lambda\varphi|_{q,\mathrm{per}} = |\lambda|\,|\varphi|_{q,\mathrm{per}}$. To show the triangle inequality, we set

$$\Phi(t,\tau) := \frac{\varphi(t)-\varphi(\tau)}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{q+1}}, \qquad (t,\tau) \in Q \times Q,$$

and define $\Psi : Q \times Q \to \mathbb{C}$ analogously. Then $\Phi, \Psi \in L^2(Q \times Q)$ and we obtain

$$\begin{aligned} |\varphi+\psi|_{q,\mathrm{per}} &= \left(\int_Q\int_Q \frac{\left|\big(\varphi(t)+\psi(t)\big) - \big(\varphi(\tau)+\psi(\tau)\big)\right|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t\right)^{1/2} \\ &= \|\Phi+\Psi\|_{L^2(Q\times Q)} \leq \|\Phi\|_{L^2(Q\times Q)} + \|\Psi\|_{L^2(Q\times Q)} \\ &= |\varphi|_{q,\mathrm{per}} + |\psi|_{q,\mathrm{per}}\,, \end{aligned}$$

as desired. □

Correlation between $\mathcal{H}^s_{\mathrm{per}}(Q)$ and $H^s_{\mathrm{per}}(Q)$. From Theorem 2.20 we know already that for the special case $s = m \in \mathbb{N}_0$ we have $\mathcal{H}^s_{\mathrm{per}}(Q) = H^s_{\mathrm{per}}(Q)$. In the following presentation we will derive that this equality even holds for all $s \in \mathbb{R}$ with $s \geq 0$. For this, we follow [36]. However, we have to overcome some effort when transfering the results therein to the two dimensional case. We proceed in several steps. The next lemmas are of particular importance to get a grip on the fractional part of the Sobolev-Slobodeckiĭ norm for the periodic case.

Lemma A.30 *(i)* $\forall t \in Q:\ \frac{1}{L}\,|t| \leq \left|\sin\frac{\pi t_1}{2L_1}\right| + \left|\sin\frac{\pi t_2}{2L_2}\right| \leq \frac{\pi}{L}\,|t|.$

(ii) Let $K \subseteq Q$ be compact. Then there exist constants $c_1, c_2 > 0$ such that

$$c_1\,|t-\tau| \leq \left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right| \leq c_2\,|t-\tau|, \qquad t,\tau \in K.$$

Proof: *(i).* We use $\frac{2}{\pi}|\xi| \le |\sin\xi| \le |\xi|$ for $\xi \in \mathbb{R}$ and $|\xi| \le \pi/2$ to estimate for $t \in Q$

$$\tfrac{1}{\overline{L}}|t| \le \tfrac{1}{\overline{L}}\big(|t_1| + |t_2|\big) \le \big|\sin\tfrac{\pi t_1}{2L_1}\big| + \big|\sin\tfrac{\pi t_2}{2L_2}\big| \le \tfrac{\pi}{2\underline{L}}\big(|t_1| + |t_2|\big) \le \tfrac{\pi}{\underline{L}}|t|.$$

(ii). For $\varrho \in (0,1)$ we observe that $\frac{\sin(\varrho\pi)}{\varrho\pi}|\xi| \le |\sin\xi| \le |\xi|$ for $\xi \in \mathbb{R}$ and $|\xi| \le \varrho\,\pi$. Since $K \subseteq Q$ is compact, there exists $\varrho_i \in (0,1)$ such that for all $t, \tau \in K$ there holds $|\frac{\pi(t_i-\tau_i)}{2L_i}| \le \varrho_i\pi$, $i = 1, 2$. Now, with the observation from the beginning, the assertion follows similarly as in part *(i)*. The constants can be chosen as $c_1 = \frac{1}{2\overline{L}}\min_{i=1,2}\frac{\sin(\varrho_i\pi)}{\varrho_i}$ and $c_2 = \frac{\pi}{\underline{L}}$. □

Lemma A.31 *Let $q \in (0,1)$. Then there exist constants $c_0 > 0$ and $c_1 > 0$ such that*

$$c_0\,|\mu|^{2q} \le \int_{Q'} \frac{\sin^2\left(\frac{q^{(\mu)}\cdot\tau}{2}\right)}{|\tau|^{2q+2}}\,\mathrm{d}\tau \le c_1\,|\mu|^{2q}, \qquad \mu \in \mathbb{Z}^2,$$

where $Q' := [-L_1, L_1] \times [0, L_2]$.

Proof: We note that the existence of the integral follows in particular from the second inequality. Furthermore, the inequalities

$$\tfrac{2}{\pi}|\xi| \le |\sin\xi| \le |\xi|, \qquad \xi \in \mathbb{R}, \quad |\xi| \le \pi/2,$$

will be helpful. Clearly, the assertion of the lemma holds if $\mu = 0$. So, let's assume $\mu \in \mathbb{Z}^2 \setminus \{0\}$. We start with the observation

$$\begin{aligned}\int_{Q'} \frac{\sin^2\left(\frac{q^{(\mu)}\cdot\tau}{2}\right)}{|\tau|^{2q+2}}\,\mathrm{d}\tau &= \int_{Q'} \frac{\sin^2\left(\frac{1}{2}\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot|q^{(\mu)}|\tau\right)}{|\tau|^{2q+2}}\,\mathrm{d}\tau \\ &= |q^{(\mu)}|^{2q}\int_{|q^{(\mu)}|Q'} \frac{\sin^2\left(\frac{1}{2}\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot\sigma\right)}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma,\end{aligned}$$

and proceed to derive bounds for the last integral. For this, on the one hand we estimate

$$\int_{|q^{(\mu)}|Q'} \frac{\sin^2\left(\frac{1}{2}\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot\sigma\right)}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma \le \int_{\mathbb{B}_2(0,\pi)} \frac{\frac{1}{4}|\sigma|^2}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma + \int_{\mathbb{R}^2\setminus\mathbb{B}_2(0,\pi)} \frac{1}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma$$

$$\begin{aligned}
&\leq \frac{\pi}{2}\int_0^{\pi} \frac{1}{r^{2q-1}}\,\mathrm{d}r + 2\pi\int_{\pi}^{\infty}\frac{1}{r^{2q+1}}\,\mathrm{d}r \\
&= \frac{\pi^{3-2q}}{2}\frac{1}{2-2q} + \frac{\pi}{q}\Big(\frac{1}{\pi}\Big)^{2q} =: \tilde{c}_1 > 0,
\end{aligned}$$

which yields with $c_1 := \tilde{c}_1(\pi/\underline{L})^{2q}$ the second inequality. On the other hand

$$\begin{aligned}
\int_{|q^{(\mu)}|Q'} \frac{\sin^2\big(\frac{1}{2}\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot\sigma\big)}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma &\geq \int_{\frac{\pi}{4\overline{L}}Q'} \frac{\sin^2\big(\frac{1}{2}\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot\sigma\big)}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma \\
&\geq \frac{4}{\pi^2}\int_{\frac{\pi}{4\overline{L}}Q'} \frac{\frac{1}{4}\big|\frac{q^{(\mu)}}{|q^{(\mu)}|}\cdot\sigma\big|^2}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma.
\end{aligned}$$

Now, we set $F(\hat{t}) := \int_{\frac{\pi}{4\overline{L}}Q'} \frac{|\hat{t}\cdot\sigma|^2}{|\sigma|^{2q+2}}\,\mathrm{d}\sigma$, for $\hat{t}\in\mathbb{S}^1$. We observe that F is continuous and that $F(\hat{t}) > 0$ for all $\hat{t}\in\mathbb{S}^1$. Since $\mathbb{S}^1$ is compact, there exists $\tilde{c}_0 := \min_{\hat{t}\in\mathbb{S}^1} F(\hat{t}) > 0$. Finally, setting $c_0 := (\tilde{c}_0/\pi^2)(\pi/\overline{L})^{2q}$ yields the first inequality and completes the proof. □

Lemma A.32 *Let $q\in(0,1)$. Define*

$$\gamma_\mu := 8\int_{Q'} \frac{\sin^2\big(\frac{q^{(\mu)}\cdot t}{2}\big)}{\Big(\big|\sin\frac{\pi t_1}{2L_1}\big| + \big|\sin\frac{\pi t_2}{2L_2}\big|\Big)^{2q+2}}\,\mathrm{d}t, \qquad \mu\in\mathbb{Z}^2,$$

where $Q' := [-L_1, L_1]\times[0, L_2]$. Then γ_μ is well-defined and there exist constants $C_0, C_1 > 0$ such that

$$C_0|\mu|^{2q} \leq \gamma_\mu \leq C_1|\mu|^{2q}, \qquad \mu\in\mathbb{Z}^2.$$

Proof: The assertion follows from Lemma A.31 together with part *(i)* from Lemma A.30. The constants can be chosen as $C_0 = 8\,c_0\big(\frac{\underline{L}}{\pi}\big)^{2q+2}$ and $C_1 = 8\,c_1\,\overline{L}^{2q+2}$, both greater than zero. □

Lemma A.33 *Let $q \in (0,1)$ and γ_μ be the number from Lemma A.32. Then*

$$\gamma_\mu = 8 \int_{Q'} \frac{\sin^2\left(\frac{q^{(\mu)}\cdot t}{2}\right)}{\left(\left|\sin\frac{\pi t_1}{2L_1}\right| + \left|\sin\frac{\pi t_2}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}t$$
$$= 2\int_{Q} \frac{1-\mathrm{e}^{\pm \mathrm{i} q^{(\mu)}\cdot t}}{\left(\left|\sin\frac{\pi t_1}{2L_1}\right| + \left|\sin\frac{\pi t_2}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}t.$$

Moreover

$$\left\langle T_Q^{(\mu)}, T_Q^{(\nu)}\right\rangle_{q,\mathrm{per}} = \gamma_\mu \delta_{\mu,\nu}, \qquad \mu,\nu \in \mathbb{Z}^2.$$

Here $Q' := [-L_1, L_1]\times[0, L_2]$ and $\delta_{\mu,\nu}$ denotes the Kronecker delta.

Proof: For convenience, we introduce $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\sigma) := \left|\sin\tfrac{\pi\sigma_1}{2L_1}\right| + \left|\sin\tfrac{\pi\sigma_2}{2L_2}\right|, \qquad \sigma = (\sigma_1,\sigma_2)^\top \in \mathbb{R}^2,$$

and observe that f is an even and periodic function. Let $\mu,\nu\in\mathbb{Z}^2$. First of all there holds, with $Q_1 := [0,L_1]\times[0,L_2]$, $Q_2 := [-L_1,0]\times[0,L_2]$ and the substitution $\sigma = -\tau$,

$$\int_Q \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau = \int_{Q_1} \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau + \int_{Q_1} \frac{1-\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot\sigma}}{\left[f(\sigma)\right]^{2q+2}}\,\mathrm{d}\sigma$$
$$+ \int_{Q_2} \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau + \int_{Q_2} \frac{1-\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot\sigma}}{\left[f(\sigma)\right]^{2q+2}}\,\mathrm{d}\sigma$$
$$= \int_{Q_1} \frac{2-2\cos(q^{(\nu)}\cdot\tau)}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau + \int_{Q_2} \frac{2-2\cos(q^{(\nu)}\cdot\tau)}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau$$
$$= \int_{Q'} \frac{4\sin^2\left(\frac{q^{(\nu)}\cdot\tau}{2}\right)}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau.$$

Thus, together with the definition of γ_μ and the calculation

$$\int_Q \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\left[f(\tau)\right]^{2q+2}}\,\mathrm{d}\tau = \int_Q \frac{1-\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot(-\tau)}}{\left[f(-\tau)\right]^{2q+2}}\,\mathrm{d}\tau = \int_Q \frac{1-\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot t}}{\left[f(t)\right]^{2q+2}}\,\mathrm{d}t,$$

we have shown the first two identities from the lemma. Now, we observe that

$$\big[T_Q^{(\mu)}(t) - T_Q^{(\mu)}(\tau)\big]\overline{\big[T_Q^{(\nu)}(t) - T_Q^{(\nu)}(\tau)\big]} \\ = \frac{1}{|Q|}\Big(\mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t} + \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot \tau} - \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot t - q^{(\nu)}\cdot\tau)} - \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot \tau - q^{(\nu)}\cdot t)}\Big).$$

Moreover, by interchanging the integration order, using that f is even and interchanging finally the role of t and τ, we easily check the equation

$$\int_Q\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t = \int_Q\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot \tau}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t$$

and

$$\int_Q\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot t-q^{(\nu)}\cdot \tau)}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t = \int_Q\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot \tau-q^{(\nu)}\cdot t)}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t.$$

Therefore, we conclude that

$$\Big\langle T_Q^{(\mu)}, T_Q^{(\nu)}\Big\rangle_{q,\mathrm{per}} = \frac{2}{|Q|}\int_Q\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t} - \mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot t-q^{(\nu)}\cdot \tau)}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t.$$

Now, we calculate

$$\begin{aligned}\int_Q \frac{\mathrm{e}^{\mathrm{i}(q^{(\mu)}\cdot t-q^{(\nu)}\cdot \tau)}}{\big[f(t-\tau\big]^{2q+2}}\,\mathrm{d}\tau &= \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot t}\int_Q \frac{\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau \\ &= \mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot t}\int_{t-Q} \frac{\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot(\sigma-t)}}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma = \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t}\int_{t-Q} \frac{\mathrm{e}^{\mathrm{i}q^{(\nu)}\cdot\sigma}}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma \\ &= \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t}\int_{t-Q} \frac{\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot(-\sigma)}}{\big[f(-\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma \\ &= \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t}\int_{-t+Q} \frac{\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\big[f(\tau)\big]^{2q+2}}\,\mathrm{d}\tau = \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t}\int_{Q} \frac{\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\big[f(\tau)\big]^{2q+2}}\,\mathrm{d}\tau,\end{aligned}$$

where we have applied in the second equality the substitution $\sigma = t - \tau$, in the fifth equality the substitution $\tau = -\sigma$ and in the last equality the periodicity of the integrand. Hence,

$$\begin{aligned}\left\langle T_Q^{(\mu)}, T_Q^{(\nu)} \right\rangle_{q,\mathrm{per}} &= \frac{2}{|Q|} \int_Q \mathrm{e}^{\mathrm{i}(q^{(\mu)}-q^{(\nu)})\cdot t} \left(\int_Q \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\big[f(\tau)\big]^{2q+2}}\,\mathrm{d}\tau \right) \mathrm{d}t \\ &= 2\left(T_Q^{(\mu)}, T_Q^{(\nu)}\right)_{L^2(Q)} \int_Q \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\nu)}\cdot\tau}}{\big[f(\tau)\big]^{2q+2}}\,\mathrm{d}\tau = \delta_{\mu,\nu}\gamma_\mu,\end{aligned}$$

which is the last identity from the lemma. □

Lemma A.34 *Let $q \in (0,1)$ and $\varphi \in \mathcal{H}^q_{\mathrm{per}}(Q)$. Then*

$$|\varphi|^2_{q,\mathrm{per}} = \sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2 \gamma_\mu,$$

with γ_μ from Lemma A.32 and where $\varphi^{(\mu)}$ denote the Fourier coefficients of φ.

Proof: Again, for convenience, we introduce $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\sigma) := \left|\sin\tfrac{\pi\sigma_1}{2L_1}\right| + \left|\sin\tfrac{\pi\sigma_2}{2L_2}\right|, \qquad \sigma = (\sigma_1,\sigma_2)^\top \in \mathbb{R}^2,$$

and observe that f is an even and periodic function. Moreover, we need

$$|1-\mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\sigma}|^2 = 1-\mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\sigma} + 1 - \mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot\sigma}, \qquad \mu\in\mathbb{Z}^2,\ \sigma\in\mathbb{R}^2.$$

Thus

$$\begin{aligned}|\varphi|^2_{q,\mathrm{per}} &= \int_Q\int_Q \frac{|\varphi(t)-\varphi(\tau)|^2}{\big[f(t-\tau)\big]^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_Q \int_{-\tau+Q} \frac{|\varphi(\tau+\sigma)-\varphi(\tau)|^2}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \int_Q \Big(\int_Q \frac{|\varphi(\tau+\sigma)-\varphi(\tau)|^2}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\tau \Big) \mathrm{d}\sigma\end{aligned}$$

$$= \int_Q \frac{1}{\big[f(\sigma)\big]^{2q+2}} \underbrace{\big\|\varphi(\cdot+\sigma)-\varphi(\cdot)\big\|^2_{L^2(Q)}}_{=\sum_{\mu\in\mathbb{Z}^2}|1-\mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\sigma}|^2|\varphi^{(\mu)}|^2} \,\mathrm{d}\sigma$$

$$= \sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2 \int_Q \frac{|1-\mathrm{e}^{\mathrm{i}q^{(\mu)}\cdot\sigma}|^2}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma$$

$$= \sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2 \, 2\int_Q \frac{1-\mathrm{e}^{-\mathrm{i}q^{(\mu)}\cdot\sigma}}{\big[f(\sigma)\big]^{2q+2}}\,\mathrm{d}\sigma = \sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2\gamma_\mu$$

where we have used in the third equality the periodicity of the integrand, in the fifth equality a corollary of the monotone convergence theorem and in the second last and last equality Lemma A.33. □

Lemma A.35 *Let $q \in (0,1)$ and $\varphi \in H^q_{\mathrm{per}}(Q)$. Then*

$$|\varphi|^2_{q,\mathrm{per}} = \sum_{\mu\in\mathbb{Z}^2} |\varphi^{(\mu)}|^2\gamma_\mu,$$

with γ_μ from Lemma A.32 and where again $\varphi^{(\mu)}$ denote the Fourier coefficients of φ.

Proof: We set $\varphi_N := \sum_{|\mu|_\infty\le N} \varphi^{(\mu)} T_Q^{(\mu)}$ for $N\in\mathbb{N}$. Clearly, $\varphi_N \to \varphi$ in $L^2(Q)$, as $N\to\infty$. Thanks to Lemma A.29, the reverse triangle inequality (which also holds for seminorms) is applicable, and using the orthogonality relation for the trigonometric monomials in Lemma A.33, we obtain for $M,N\in\mathbb{N}$ with $M<N$

$$\big|\, |\varphi_N|_{q,\mathrm{per}} - |\varphi_M|_{q,\mathrm{per}} \,\big|^2 \le |\varphi_N-\varphi_M|^2_{q,\mathrm{per}}$$

$$= \Big\langle \sum_{M<|\mu|_\infty\le N} \varphi^{(\mu)}T_Q^{(\mu)}, \sum_{M<|\nu|_\infty\le N} \varphi^{(\nu)}T_Q^{(\nu)} \Big\rangle_{q,\mathrm{per}}$$

$$= \sum_{\substack{M<|\mu|_\infty\le N\\ M<|\nu|_\infty\le N}} \varphi^{(\mu)}\overline{\varphi^{(\nu)}} \left\langle T_Q^{(\mu)}, T_Q^{(\nu)}\right\rangle_{q,\mathrm{per}} = \sum_{M<|\mu|_\infty\le N} |\varphi^{(\mu)}|^2\,\gamma_\mu$$

$$\le C_1 \sum_{M<|\mu|_\infty\le N} |\mu|^{2q}\,|\varphi^{(\mu)}|^2$$

$$\leq C_1 \sum_{M<|\mu|_\infty\leq N} \big(1+|\mu|^2\big)^q |\varphi^{(\mu)}|^2 \longrightarrow 0, \qquad \text{as } M,N\to\infty,$$

where we have used in the second inequality Lemma A.32 and where the convergence follows as φ belongs to $H^q_{\mathrm{per}}(Q)$. From this we conclude that $\big(|\varphi_N|^2_{q,\mathrm{per}}\big)_{N\in\mathbb{N}}$ is a Cauchy sequence, and due to Lemma A.27 (after convincing ourselves that its proof also works for the denominator in $|\cdot|^2_{q,\mathrm{per}}$) we obtain

$$|\varphi_N|^2_{q,\mathrm{per}} \to |\varphi|^2_{q,\mathrm{per}} < \infty, \qquad \text{as } N\to\infty.$$

Moreover, with the considerations above it is easy to see, that $|\varphi_N|^2_{q,\mathrm{per}} = \sum_{0\leq|\mu|_\infty\leq N} |\varphi^{(\mu)}|^2\gamma_\mu$ and that the series on the right hand side possesses a convergent majorant. Therefore the limit exists and has to coincide with $|\varphi|^2_{q,\mathrm{per}}$, as we wanted to show. □

Now, we come to the main theorem of this subsection.

Theorem A.36 *Let $s\in\mathbb{R}$ with $s\geq 0$. Then*

$$H^s_{\mathrm{per}}(Q) = \mathcal{H}^s_{\mathrm{per}}(Q).$$

Furthermore, on $H^s_{\mathrm{per}}(Q)$ (and thus on $\mathcal{H}^s_{\mathrm{per}}(Q)$) the norms $\|\cdot\|_{H^s_{\mathrm{per}}(Q)}$ and $\|\cdot\|_{\mathcal{H}^s_{\mathrm{per}}(Q)}$ are equivalent.

Proof: The assertion for $s=m\in\mathbb{N}_0$ follows from Theorem 2.20. Therefore, we assume that $s=m+q$ with $m\in\mathbb{N}_0$ and $q\in(0,1)$. Moreover, without loss of generality, we assume $m\in\mathbb{N}$, as the case $s=q$ is a special case of the following considerations.
Let $\varphi\in\mathcal{H}^s_{\mathrm{per}}(Q)$. Thanks to Lemma 2.18 there holds (recall that $0^0=1$)

$$\begin{aligned}\big(1+|\mu|^2\big)^s &\leq 2^s|\mu|^{2q}|\mu|^{2m}\\ &\leq 2^s\binom{m}{\lfloor m/2\rfloor}\left(\frac{L}{\pi}\right)^{2m}|\mu|^{2q}\sum_{|\alpha|=m}\pi^{2m}\left|\frac{\mu_1}{L_1}\right|^{2\alpha_1}\left|\frac{\mu_2}{L_2}\right|^{2\alpha_2},\end{aligned}$$

for all $\mu\in\mathbb{Z}^2\backslash\{0\}$. Then, by Lemma 2.17, Lemma A.32 and Lemma A.34,

$$\sum_{\mu\in\mathbb{Z}^2}\big(1+|\mu|^2\big)^s|\varphi^{(\mu)}|^2 \leq \|\varphi\|^2_{L^2(Q)} + \sum_{\mu\in\mathbb{Z}^2\backslash\{0\}}\big(1+|\mu|^2\big)^s|\varphi^{(\mu)}|^2$$

$$\leq \|\varphi\|^2_{L^2(Q)} + 2^s \binom{m}{\lfloor m/2 \rfloor} \left(\frac{L}{\pi}\right)^{2m} \sum_{|\alpha|=m} \sum_{\mu \in \mathbb{Z}^2} |\mu|^{2q} \left|(\partial^\alpha \varphi)^{(\mu)}\right|^2$$

$$\leq \|\varphi\|^2_{L^2(Q)} + \tfrac{2^s}{C_0} \binom{m}{\lfloor m/2 \rfloor} \left(\frac{L}{\pi}\right)^{2m} \sum_{|\alpha|=m} \sum_{\mu \in \mathbb{Z}^2} \gamma^{(\mu)} \left|(\partial^\alpha \varphi)^{(\mu)}\right|^2$$

$$= \|\varphi\|^2_{L^2(Q)} + \tfrac{2^s}{C_0} \binom{m}{\lfloor m/2 \rfloor} \left(\frac{L}{\pi}\right)^{2m} \sum_{|\alpha|=m} |\partial^\alpha \varphi|^2_{q,\mathrm{per}},$$

and thus $\|\varphi\|^2_{H^s_{\mathrm{per}}(Q)} \leq \max\left\{1, \tfrac{2^s}{C_0} \binom{m}{\lfloor m/2 \rfloor} \left(\frac{L}{\pi}\right)^{2m}\right\} \|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)}$.

Conversely, let $\varphi \in H^s_{\mathrm{per}}(Q)$. Thanks to the embedding $H^s_{\mathrm{per}}(Q) \hookrightarrow H^m_{\mathrm{per}}(Q)$ and Theorem 2.20 we have that $\varphi \in H^m_{\mathrm{per}}(Q) = \mathcal{H}^m_{\mathrm{per}}(Q)$ and that there exists a constant $c > 0$, independent of φ, such that

$$\|\varphi\|_{\mathcal{H}^m_{\mathrm{per}}(Q)} \leq c\,\|\varphi\|_{H^m_{\mathrm{per}}(Q)} \leq c\,\|\varphi\|_{H^s_{\mathrm{per}}(Q)}.$$

Let $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq m$. Then, by Lemma 2.17,

$$\left|(\partial^\alpha \varphi)^{(\mu)}\right|^2 \leq \left(\tfrac{\pi}{L}\right)^{2|\alpha|} |\mu|^{2|\alpha|} |\varphi^{(\mu)}|^2 \leq \left(1 + \tfrac{\pi}{L}\right)^{2m} \left(1 + |\mu|^2\right)^m |\varphi^{(\mu)}|^2,$$

for all $\mu \in \mathbb{Z}^2$. From this we conclude $\partial^\alpha \varphi \in H^q_{\mathrm{per}}(Q)$. Thus, Lemma A.35 is applicable and using again Lemma A.32 together with the last estimate from above we obtain

$$\sum_{|\alpha|\leq m} |\partial^\alpha \varphi|^2_{q,\mathrm{per}} = \sum_{\mu \in \mathbb{Z}^2} \sum_{|\alpha|\leq m} \gamma_\mu \left|(\partial^\alpha \varphi)^{(\mu)}\right|^2 \leq C_1 \sum_{\mu \in \mathbb{Z}^2} \sum_{|\alpha|\leq m} |\mu|^{2q} \left|(\partial^\alpha \varphi)^{(\mu)}\right|^2$$

$$\leq C_1 \sum_{\mu \in \mathbb{Z}^2} \sum_{|\alpha|\leq m} \left(1 + \tfrac{\pi}{L}\right)^{2m} \left(1 + |\mu|^2\right)^s |\varphi^{(\mu)}|^2$$

$$\leq C_1 \left(1 + \tfrac{\pi}{L}\right)^{2m} \Big(\sum_{|\alpha|\leq m} 1\Big) \|\varphi\|^2_{H^s_{\mathrm{per}}(Q)}.$$

Hence $\|\varphi\|_{\mathcal{H}^s_{\mathrm{per}}(Q)} \leq \left(c^2 + C_1\left(1 + \tfrac{\pi}{L}\right)^{2m} \left(\sum_{|\alpha|\leq m} 1\right)\right)^{1/2} \|\varphi\|_{H^s_{\mathrm{per}}(Q)}$, and the proof is complete. □

Definition A.37 *Let $s \in \mathbb{R}$ with $s \geq 0$ and $K \subseteq Q$ be compact. We define*

$$\mathcal{H}^s_{\mathrm{per},K}(Q) := \left\{\varphi \in \mathcal{H}^s_{\mathrm{per}}(Q) \mid \operatorname{supp} \varphi \subseteq K\right\}.$$

If $K \subseteq Q$ is compact, then $H_K^s(Q)$ from the last subsection coincides with $\mathcal{H}^s_{\mathrm{per},K}(Q)$ for all $s \in \mathbb{R}$ with $s \geq 0$, as shown in the next theorem. The following lemma has preliminary character.

Lemma A.38 *Let $K \subseteq Q$ be compact and $q \in (0,1)$. Then $H_K^q(Q) = \mathcal{H}^q_{\mathrm{per},K}(Q)$ and on both spaces the seminorms $|\cdot|_{q,Q}$ and $|\cdot|_{q,\mathrm{per}}$ are equivalent.*

Proof: Choose some open subset $O \subseteq Q$ such that $K \subseteq O$ and $\overline{O} \subseteq Q$. In the following considerations we proceed similarly as in the proof of Theorem A.26. We define $f, g : (\overline{Q} \setminus O) \times K \to \mathbb{R}$ by

$$(t,\tau) \to f(t,\tau) := \frac{1}{|t-\tau|^{2q+2}}$$
$$(t,\tau) \to g(t,\tau) := \frac{1}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}.$$

Then $f, g > 0$ and, by continuity of f, g as well as by compactness of their domain of definition, there exist constants $c_{f,1}, c_{f,2}, c_{g,1}, c_{g,2} > 0$ such that

$$0 < c_{f,1} \leq f(t,\tau) \leq c_{f,2} \quad \text{and} \quad 0 < c_{g,1} \leq g(t,\tau) \leq c_{g,2},$$

for all $(t,\tau) \in (\overline{Q} \setminus O) \times K$. Therefore, if $\varphi \in H_K^q(Q)$ then

$$\begin{aligned}
\underbrace{2\,c_{f,1}\left|\overline{Q}\setminus O\right|}_{=:C_{f,1}} \|\varphi\|^2_{L^2(Q)} &= 2\,c_{f,1}\left|\overline{Q}\setminus O\right| \int_K |\varphi(\tau)|^2 \,\mathrm{d}\tau \\
&= 2\,c_{f,1} \int_{\overline{Q}\setminus O}\int_K |\varphi(\tau)|^2 \,\mathrm{d}\tau\,\mathrm{d}t \\
&\leq 2 \int_{\overline{Q}\setminus O}\int_K \frac{|\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \,\mathrm{d}\tau\,\mathrm{d}t \leq \underbrace{2\,c_{f,2}\left|\overline{Q}\setminus O\right|}_{=:C_{f,2}} \|\varphi\|^2_{L^2(Q)},
\end{aligned}$$

and analogously, if $\varphi \in \mathcal{H}^q_{\mathrm{per},K}(Q)$ then

$$C_{g,1}\,\|\varphi\|^2_{L^2(Q)} \leq 2 \int_{\overline{Q}\setminus O}\int_K \frac{|\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}} \,\mathrm{d}\tau\,\mathrm{d}t$$

$$\leq C_{g,2} \, \|\varphi\|^2_{L^2(Q)}.$$

Moreover, there holds for $\varphi \in H^q_K(Q)$

$$\begin{aligned}
|\varphi|^2_{q,Q} &= \int_{\overline{Q}\setminus O} \int_Q \frac{|\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t + \int_O \int_Q \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&= \int_{\overline{Q}\setminus O} \int_K \frac{|\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t + \int_O \int_{\overline{Q}\setminus O} \frac{|\varphi(t)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&\qquad + \int_O \int_O \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&= 2 \int_{\overline{Q}\setminus O} \int_K \frac{|\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t + \int_{\overline{O}} \int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t
\end{aligned}$$

and analogously for $\varphi \in \mathcal{H}^q_{\mathrm{per},K}(Q)$

$$\begin{aligned}
|\varphi|^2_{q,\mathrm{per}} &= 2 \int_{\overline{Q}\setminus O} \int_K \frac{|\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&\quad + \int_{\overline{O}} \int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t.
\end{aligned}$$

Furthermore, by Lemma A.30 part *(ii)*, there exist constants $C_1, C_2 > 0$ such that for all $\varphi \in H^q_K(Q) \cup \mathcal{H}^q_{\mathrm{per},K}(Q)$

$$\begin{aligned}
C_1 & \int_{\overline{O}} \int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&\leq \int_{\overline{O}} \int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \\
&\leq C_2 \int_{\overline{O}} \int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t < \infty.
\end{aligned}$$

Altogether, we obtain for $\varphi \in H^q_K(Q)$ and with $C := \max\left\{\frac{C_{f,2}}{C_{g,1}}, \frac{1}{C_1}\right\}$

$$|\varphi|^2_{q,Q} = \int_Q \int_Q \frac{|\varphi(t)-\varphi(\tau)|^2}{|t-\tau|^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t$$

$$\leq C_{f,2}\,\|\varphi\|^2_{L^2(Q)} + \frac{1}{C_1}\int_{\overline{O}}\int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t$$

$$\leq C\left(C_{g,1}\,\|\varphi\|^2_{L^2(Q)} + \int_{\overline{O}}\int_{\overline{O}} \frac{|\varphi(t)-\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t\right)$$

$$\leq C\int_Q\int_Q \frac{|\varphi(t)-\varphi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t = C\,|\varphi|^2_{q,\mathrm{per}}$$

and analogously for $\varphi \in \mathcal{H}^q_{\mathrm{per},K}(Q)$

$$|\varphi|^2_{q,\mathrm{per}} \leq \max\left\{\frac{C_{g,2}}{C_{f,1}}, C_2\right\}|\varphi|^2_{q,Q}.$$

Hence, the last two estimates show that $H^q_K(Q) = \mathcal{H}^q_{\mathrm{per},K}(Q)$ and in particular that also the seminorms therein are equivalent. □

Theorem A.39 *Let $K \subseteq Q$ be compact and $s \in \mathbb{R}$ with $s \geq 0$. Then*

$$H^s_K(Q) = \mathcal{H}^s_{\mathrm{per},K}(Q)$$

and on both spaces the norms $\|\cdot\|_{H^s(Q)}$ and $\|\cdot\|_{\mathcal{H}^s_{\mathrm{per}}(Q)}$ are equivalent.

Proof: Without loss of generality we assume that $s > 0$. Moreover, since $K \subseteq Q$ is compact, there exists $\chi_K \in C^\infty_0(Q)$ such that $\chi_K(t) = 1$, for all $t \in K$. In particular, $\partial^\beta\chi_K(t) = 0$, for all t in the interior of K and for all $\beta \in \mathbb{N}^2_0 \setminus \{0\}$.
We start with the case $s = m \in \mathbb{N}$. Let $\varphi \in \mathcal{H}^m_{\mathrm{per},K}(Q)$ and $\alpha \in \mathbb{N}^2_0$ with $|\alpha| \leq m$. By Remark 2.15, there exists $\partial^\alpha\varphi = \partial^\alpha_{\mathrm{per}}\varphi \in L^2(Q)$. Therefore, $\varphi \in H^m_K(Q)$. Now, let $\varphi \in H^m_K(Q)$ and note that $\operatorname{supp}\varphi \subseteq K$. Let $\alpha \in \mathbb{N}^2_0$ with $|\alpha| \leq m$. Then, by Proposition A.12, $\operatorname{supp}\partial^\alpha\varphi \subseteq K$. Take some $\chi \in C^\infty_{\mathrm{per}}(Q)$ and observe that $\chi_K \cdot \chi \in C^\infty_0(Q)$. Then

$$\begin{aligned}\int_Q (\partial^\alpha\varphi)(t)\,\chi(t)\,\mathrm{d}t &= \int_Q (\partial^\alpha\varphi)(t)\,\chi_K(t)\chi(t)\,\mathrm{d}t\\ &= (-1)^{|\alpha|}\int_Q \varphi(t)\,\partial^\alpha(\chi_K\cdot\chi)(t)\,\mathrm{d}t\end{aligned}$$

$$= (-1)^{|\alpha|} \int_Q \varphi(t) \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta \chi_K(t) \, \partial^{\alpha-\beta} \chi(t) \, \mathrm{d}t$$
$$= (-1)^{|\alpha|} \int_Q \varphi(t) \, \chi_K(t) \, \partial^\alpha \chi(t) \, \mathrm{d}t = (-1)^{|\alpha|} \int_Q \varphi(t) \, \partial^\alpha \chi(t) \, \mathrm{d}t.$$

Hence, there exists $\partial^\alpha_{\mathrm{per}} \varphi \in L^2(Q)$, coinciding with $\partial^\alpha \varphi$. Consequently, $\varphi \in \mathcal{H}^m_{\mathrm{per},K}(Q)$. Thus, we have shown that $H^m_K(Q) = \mathcal{H}^m_{\mathrm{per},K}(Q)$ and by the definition of the norms there holds also $\|\cdot\|_{H^m(Q)} = \|\cdot\|_{\mathcal{H}^m_{\mathrm{per}}(Q)}$ therein. Now we turn to the remaining case $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0,1)$. Since we already know that $H^m_K(Q) = \mathcal{H}^m_{\mathrm{per},K}(Q)$, it sufficies to take a closer look at the seminorms. Let $\varphi \in H^s_K(Q)$. Again, take some $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le m$ and note that $\operatorname{supp} \partial^\alpha \varphi \subseteq K$. Then, by Lemma A.38, there exists a constant $c_2 > 0$, not depending on $\partial^\alpha \varphi$, such that $|\partial^\alpha \varphi|_{q,\mathrm{per}} \le c_2 |\partial^\alpha \varphi|_{q,Q}$. And by the same argument there exists a constant $c_1 > 0$ such that for all $\varphi \in \mathcal{H}^s_{\mathrm{per},K}(Q)$ and for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le m$ there holds $c_1 |\partial^\alpha \varphi|_{q,Q} \le |\partial^\alpha \varphi|_{q,\mathrm{per}}$. From this we conclude that $H^s_K(Q) = \mathcal{H}^s_{\mathrm{per},K}(Q)$ and that the norms $\|\cdot\|_{H^s(Q)}$ and $\|\cdot\|_{\mathcal{H}^s_{\mathrm{per}}(Q)}$ therein are equivalent, as desired. □

Multiplication and Transformation. We proceed similarly as in the non-periodic case and show that the multiplication with differentiable and periodic functions gives rise to a linear and bounded operator in $\mathcal{H}^s_{\mathrm{per}}(Q)$. Afterwards, we are able to carry over the transformation theorem from the last subsection to the periodic framework.

Lemma A.40 *Let $\chi \in C^1_{\mathrm{per}}(Q)$. Then there exists a constant $C > 0$ such that*

$$\int_Q \int_Q |\varphi(\tau)|^2 \frac{|\chi(t) - \chi(\tau)|^2}{\left(\left|\sin \frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin \frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}} \, \mathrm{d}\tau \, \mathrm{d}t \le C \, \|\varphi\|^2_{L^2(Q)},$$

for all $\varphi \in C_b(Q) \cup L^2(Q)$. Here, $C = \frac{\pi}{1-q} \, 2^{2-q} \, \overline{L}^4 \, \|\chi\|^2_{C^1_b(Q)}$.

Proof: We denote by $\widetilde{\chi}$ the periodic extension of χ to $\mathbb{R}^2$ as in the definition of $C^1_{\mathrm{per}}(Q)$. In particular $\widetilde{\chi} \in C^1_b(Q)$ and the mean value theorem is applicable. Hence,

$$\begin{aligned}
&\int_Q \int_Q |\varphi(\tau)|^2 \frac{|\chi(t)-\chi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right| + \left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad = \int_Q |\varphi(\tau)|^2 \left(\int_{-\tau+Q} \frac{|\chi(\sigma+\tau)-\chi(\tau)|^2}{\left(\left|\sin\frac{\pi\sigma_1}{2L_1}\right| + \left|\sin\frac{\pi\sigma_2}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\sigma\right)\mathrm{d}\tau \\
&\quad = \int_Q |\varphi(\tau)|^2 \left(\int_{Q} \frac{|\widetilde{\chi}(\sigma+\tau)-\widetilde{\chi}(\tau)|^2}{\left(\left|\sin\frac{\pi\sigma_1}{2L_1}\right| + \left|\sin\frac{\pi\sigma_2}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\sigma\right)\mathrm{d}\tau \\
&\quad \le 2\,\overline{L}^{2q+2}\,\|\chi\|^2_{C^1_b(Q)} \int_Q |\varphi(\tau)|^2 \left(\int_Q \frac{1}{|\sigma|^{2q}}\,\mathrm{d}\sigma\right)\mathrm{d}\tau,
\end{aligned}$$

where we have applied in the last step the mean value theorem and part *(i)* from Lemma A.30. From this the assertion follows immediately. □

Theorem A.41 *Let $s \in \mathbb{R}$ with $s \ge 0$. Furthermore, let $\chi \in C^{\lceil s\rceil}_{\mathrm{per}}(Q)$ and $\varphi \in \mathcal{H}^s_{\mathrm{per}}(Q)$. Then $\chi\,\varphi \in \mathcal{H}^s_{\mathrm{per}}(Q)$ and there holds Leibniz' product rule*

$$\partial^\alpha(\chi\,\varphi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta}\,\partial^{\alpha-\beta}\chi\,\partial^\beta\varphi, \qquad \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le \lfloor s\rfloor.$$

Moreover, there exists a constant $C \ge 0$, not depending on χ and φ, such that

$$\|\chi\,\varphi\|_{\mathcal{H}^s_{\mathrm{per}}(Q)} \le C\,\|\chi\|_{C_b^{\lceil s\rceil}(Q)}\,\|\varphi\|_{\mathcal{H}^s_{\mathrm{per}}(Q)}.$$

Proof: Since χ is bounded, there holds $\chi\,\varphi \in L^2(Q)$. Moreover, since $\mathcal{T}(Q)$ is dense in $H^s_{\mathrm{per}}(Q) = \mathcal{H}^s_{\mathrm{per}}(Q)$ and the norms therein are equivalent, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{T}(Q)$ converging to φ with respect to $\|\cdot\|_{\mathcal{H}^s_{\mathrm{per}}(Q)}$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le \lfloor s\rfloor$. Furthermore, let $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$. Then

$$\varphi_n \to \varphi, \quad \partial^\beta\varphi_n \to \partial^\beta\varphi \qquad \text{in } L^2(Q), \text{ as } n \to \infty,$$

and, since χ and its derivatives are bounded, in particular

$$\chi\,\varphi_n \to \chi\,\varphi, \quad \partial^{\alpha-\beta}\chi\,\partial^{\beta}\varphi_n \to \partial^{\alpha-\beta}\chi\,\partial^{\beta}\varphi \qquad \text{in } L^2(Q), \text{ as } n\to\infty.$$

Let $\theta \in C^{\infty}_{\mathrm{per}}(Q)$. Then

$$\begin{aligned}\int_Q \chi(t)\varphi(t)\partial^{\alpha}\theta(t)\,\mathrm{d}t &= \lim_{n\to\infty}\int_Q \chi(t)\varphi_n(t)\partial^{\alpha}\theta(t)\,\mathrm{d}t\\ &= (-1)^{|\alpha|}\lim_{n\to\infty}\int_Q \partial^{\alpha}(\chi\,\varphi_n)(t)\theta(t)\,\mathrm{d}t\\ &= (-1)^{|\alpha|}\lim_{n\to\infty}\int_Q \sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\chi(t)\,\partial^{\beta}\varphi_n(t)\,\theta(t)\,\mathrm{d}t\\ &= (-1)^{|\alpha|}\int_Q\left(\sum_{\beta\le\alpha}\tbinom{\alpha}{\beta}\partial^{\alpha-\beta}\chi(t)\,\partial^{\beta}\varphi(t)\right)\theta(t)\,\mathrm{d}t.\end{aligned}$$

Hence, there exists $\partial^{\alpha}(\chi\,\varphi)\in L^2(Q)$ and is given by the Leibniz product rule. Moreover, we have shown that $\chi\,\varphi\in\mathcal{H}^{\lfloor s\rfloor}_{\mathrm{per}}(Q)$.
Now, let $s = m+q$ with $m\in\mathbb{N}_0$ and $q\in(0,1)$. And again, let $\alpha\in\mathbb{N}_0^d$ with $|\alpha|\le m$ and $\beta\in\mathbb{N}_0^d$ with $\beta\le\alpha$. Then

$$\|\partial^{\alpha-\beta}\chi\,\partial^{\beta}\varphi\|^2_{L^2(Q)} \le \|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\|\partial^{\beta}\varphi\|^2_{L^2(Q)} \le \|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)}.$$

Note that by Lemma A.40 there holds

$$\begin{aligned}\int_Q\int_Q |\partial^{\beta}\varphi(\tau)|^2\frac{|\partial^{\alpha-\beta}\chi(t)-\partial^{\alpha-\beta}\chi(\tau)|^2}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right|+\left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\,\mathrm{d}\tau\,\mathrm{d}t&\\ \le C_1\,\|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)},&\end{aligned}$$

where C_1 does not depend on φ and χ, and thus we continue with

$$\begin{aligned}\left|\partial^{\alpha-\beta}\chi\,\partial^{\beta}\varphi\right|^2_{q,\mathrm{per}} = \int_Q\int_Q&\left|\frac{\partial^{\alpha-\beta}\chi(t)[\partial^{\beta}\varphi(t)-\partial^{\beta}\varphi(\tau)]}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right|+\left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\right.\\ &\left.+\frac{\partial^{\beta}\varphi(\tau)[\partial^{\alpha-\beta}\chi(t)-\partial^{\alpha-\beta}\chi(\tau)]}{\left(\left|\sin\frac{\pi(t_1-\tau_1)}{2L_1}\right|+\left|\sin\frac{\pi(t_2-\tau_2)}{2L_2}\right|\right)^{2q+2}}\right|^2\mathrm{d}\tau\,\mathrm{d}t\end{aligned}$$

$$\leq 2\,\|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\,\left|\partial^\beta\varphi\right|^2_{q,\mathrm{per}} + 2\,C_1\,\|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)}$$
$$\leq C_2\,\|\chi\|^2_{C_b^{\lceil s\rceil}(Q)}\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)},$$

where $C_2 := 2\,(1 + C_1)$. Consequently, by applying Leibniz' product rule, the triangle inequality and the results above, we obtain

$$\|\chi\,\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)} = \sum_{|\alpha|\leq\lfloor s\rfloor} \|\partial^\alpha(\chi\,\varphi)\|^2_{L^2(Q)} + \sum_{|\alpha|\leq\lfloor s\rfloor} \left|\partial^\alpha(\chi\,\varphi)\right|^2_{q,\mathrm{per}}$$
$$\leq \left(1 + C_2\right) \sum_{|\alpha|\leq\lfloor s\rfloor} \left(\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \|\chi\|_{C_b^{\lceil s\rceil}(Q)} \|\varphi\|_{\mathcal{H}^s_{\mathrm{per}}(Q)} \right)^2 ,$$

and from this we see immediately that also the last assertion from the theorem holds. □

Theorem A.42 *Let the sets $\Omega, \Omega' \subseteq Q$ be open, $s \in \mathbb{R}$ with $s \geq 0$ and $\zeta \in \mathrm{Diff}^{\lfloor s\rfloor+1}(\Omega, \Omega')$. Furthermore, let $O \subseteq \Omega$ be open and bounded such that $\overline{O} \subseteq \Omega$, and set $O' := \zeta(O)$. Then the mapping*

$$H^s_{\mathrm{per}}(Q) \ni \varphi \mapsto \varphi \circ \zeta \in H^s(O)$$

is well-defined, linear and bounded.

Proof: Similar to Theorem A.23 we use a density argument, i.e., we show that the mapping $\mathcal{T}(Q) \ni \varphi \mapsto \varphi \circ \zeta \in H^s(O)$ is well-defined, linear and bounded; then the assertion from the theorem follows by continuous extension, because $\mathcal{T}(Q)$ is dense in $H^s_{\mathrm{per}}(Q) = \mathcal{H}^s_{\mathrm{per}}(Q)$ and the norms therein are equivalent.
Without loss of generality, we assume $s = m + q$ with $m \in \mathbb{N}_0$ and $q \in (0, 1)$, as the case $s = m \in \mathbb{N}_0$ is a special case from the following explanation. We start by choosing some $\chi \in C_0^\infty(\Omega)$ with $\chi|_{\overline{O}} \equiv 1$. Now, let $\varphi \in \mathcal{T}(Q)$. Furthermore, let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. Then, by Proposition A.22,

$$\partial^\alpha(\varphi \circ \zeta) = \sum_{|\beta|\leq|\alpha|} p_{\alpha\beta} \cdot \left([\partial^\beta\varphi] \circ \zeta\right).$$

Let $\beta \in \mathbb{N}_0^d$ with $|\beta| \le |\alpha|$. Note that $p_{\alpha\beta}\chi \in C_0^1(\Omega)$ and by extension by zero to $\mathbb{R}^2$ we have $p_{\alpha\beta}\chi \in C_b^1(\mathbb{R}^2)$. Set $C_\zeta := \|\det \partial(\zeta^{-1})\|_{C_b(O)}$ and $C_p := \max\big\{\|p_{\gamma\kappa}\|_{C_b^1(O)} \;\big|\; |\kappa| \le |\gamma|, |\gamma| \le m\big\}$. By the transformation theorem for Lebesgue integrable functions there holds

$$\begin{aligned}\big\|[\partial^\beta\varphi]\circ\zeta\big\|^2_{L^2(O)} &\le C_\zeta\,\big\|\partial^\beta\varphi\big\|^2_{L^2(Q)},\\ \big|[\partial^\beta\varphi]\circ\zeta\big|^2_{q,O} &\le C_1^{2q+2}\,C_\zeta^2\;\big|\partial^\beta\varphi\big|^2_{q,\mathrm{per}}\,,\end{aligned}$$

where we also have applied part *(ii)* from Lemma A.21 and part *(ii)* from Lemma A.30 in the second estimate and where C_1 summarizes the corresponding constants therein. Hence, $[\partial^\beta\varphi]\circ\zeta \in H^q(O)$ and by Theorem A.15 we obtain $p_{\alpha\beta}\chi\,\big([\partial^\beta\varphi]\circ\zeta\big) \in H^q(O)$ as well and moreover

$$\begin{aligned}\big\|p_{\alpha\beta}\big([\partial^\beta\varphi]\circ\zeta\big)\big\|^2_{L^2(O)} + \big|p_{\alpha\beta}\big([\partial^\beta\varphi]\circ\zeta\big)\big|^2_{q,O} &= \big\|p_{\alpha\beta}\chi\,\big([\partial^\beta\varphi]\circ\zeta\big)\big\|^2_{H^q(O)}\\ &\le C_2^2\,C_p^2\;\big\|[\partial^\beta\varphi]\circ\zeta\big\|^2_{H^q(O)}\\ &\le C_2^2\,C_p^2\,C_\zeta\,\Big(\big\|\partial^\beta\varphi\big\|^2_{L^2(Q)} + C_1^{2q+2}\,C_\zeta\;\big|\partial^\beta\varphi\big|^2_{q,\mathrm{per}}\Big)\\ &\le \underbrace{C_2^2\,C_p^2\,C_\zeta\big(1 + C_1^{2q+2}\,C_\zeta\big)}_{=:C_3}\;\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)},\end{aligned}$$

where C_2 denotes the corresponding constant from Theorem A.15. With the triangle inequality for the L^2-norm and the seminorm we conclude that

$$\begin{aligned}&\big\|\partial^\alpha(\varphi\circ\zeta)\big\|^2_{L^2(O)} + \big|\partial^\alpha(\varphi\circ\zeta)\big|^2_{q,O}\\ &\quad\le \Big(\sum_{|\beta|\le|\alpha|} 1\Big)\sum_{|\beta|\le|\alpha|}\Big(\big\|p_{\alpha\beta}\big([\partial^\beta\varphi]\circ\zeta\big)\big\|^2_{L^2(O)} + \big|p_{\alpha\beta}\big([\partial^\beta\varphi]\circ\zeta\big)\big|^2_{q,O}\Big)\\ &\quad\le C_3\,C_4^2\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)},\end{aligned}$$

where we have set $C_4 := \sum_{|\beta|\le m} 1$. From this we get finally $\|\varphi\circ\zeta\|^2_{H^s(O)} \le C_3\,C_4^3\,\|\varphi\|^2_{\mathcal{H}^s_{\mathrm{per}}(Q)}$, as desired. □

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