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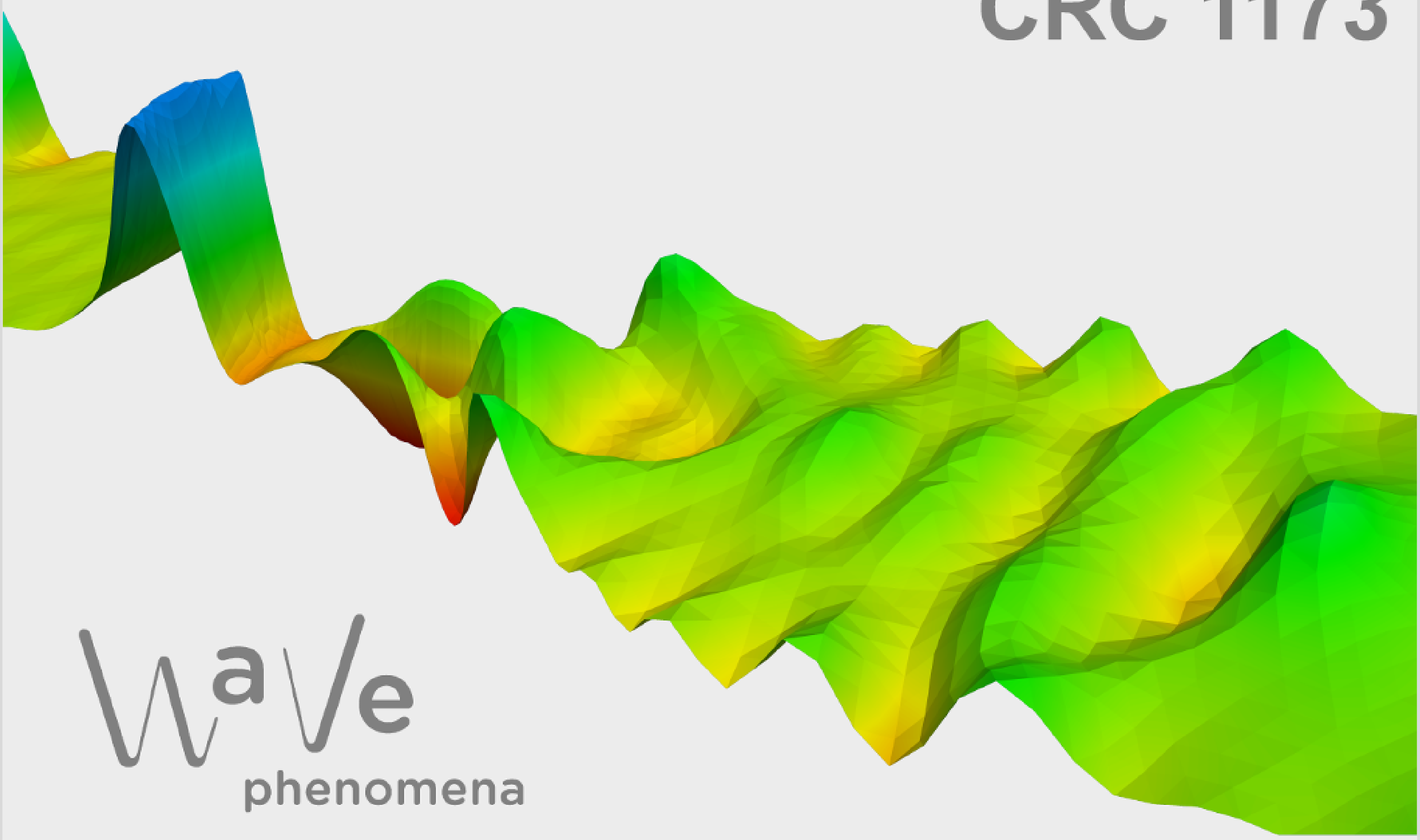

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Abstract

We show the local well-posedness of the Cauchy problem for the cubic nonlinear Schrödinger equation on modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ for $d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $s > d\left(1 - \frac{1}{q}\right)$ for $q > 1$ or $s \geq 0$ for $q = 1$. This improves [4, Theorem 1.1] by Bényi and Okoudjou where only the case $q = 1$ is considered. Our result is based on the algebra property of modulation spaces with indices as above for which we give an elementary proof via a new Hölder-like inequality for modulation spaces.

1. Introduction

We study the Cauchy problem for the cubic nonlinear Schrödinger equation (*NLS*)

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t}(x,t) + \Delta u(x,t) \pm |u|^2 u(x,t) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where the initial data u_0 is in a modulation space $M^s_{p,q}(\mathbb{R}^d)$. A definition of $M^s_{p,q}(\mathbb{R}^d)$ will be given in the next paragraph. As usual, we are interested in *mild solutions* u of (1), i.e. $u \in C\left([0,T), M^s_{p,q}(\mathbb{R}^d)\right)$ for a $T > 0$ which satisfy the corresponding integral equation

$$u(\cdot, t) = e^{\mathrm{i}t\Delta} u_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\Delta} \left(|u|^2 u(\cdot,\tau)\right) \mathrm{d}\tau \qquad (\forall t \in [0,T)). \tag{2}$$

Modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ were introduced by Feichtinger in [6]. Here, we give a short summary of their definition and properties. (We refer to Section 2 and the literature mentioned there for more information, the notation we use is explained at the end of the introduction.) Fix a so-called *window function* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The *short-time Fourier transform* $V_g f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is defined by

$$V_g f\,(x, \cdot) = \mathcal{F}\left(\overline{S_x g} f\right)(\cdot) \in \mathcal{S}'(\mathbb{R}^d) \qquad \forall x \in \mathbb{R}^d. \tag{3}$$

In fact, $V_g f : \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ can be represented by a continuous function $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$. Hence, taking a weighted, mixed L^P-norm is possible and we define

$$M^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, \|f\|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right\}, \text{ where } \|f\|_{M^s_{p,q}(\mathbb{R}^d)} = \left\| \xi \mapsto \langle \xi \rangle^s \left\| V_g f\,(\cdot, \xi) \right\|_p \right\|_q$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. It can be shown, that the $M^s_{p,q}(\mathbb{R}^d)$ are Banach spaces and that different choices of the window function g lead to equivalent norms.

Our main result is

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Theorem 1 (Local well-posedness). *Let $d \in \mathbb{N}$ and $1 \le p, q \le \infty$. For $q > 1$ let $s > d\left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \ge 0$. Assume that $u_0 \in M^s_{p,q}(\mathbb{R}^d)$. Then, there exists a unique maximal mild solution $u \in C\left([0, T^*), M^s_{p,q}(\mathbb{R}^d)\right)$ of (1) and the* blow-up alternative

$$T^* < \infty \qquad \Rightarrow \qquad \limsup_{t \to T^*} \|u(\cdot, t)\|_{M^s_{p,q}(\mathbb{R}^d)} = \infty$$

holds. Furthermore, for any $0 < T' < T^*$ there exists a neighborhood V of u_0 in $M^s_{p,q}$, such that the initial data to solution map

$$V \to C\left([0, T'], M^s_{p,q}(\mathbb{R}^d)\right), \qquad v_0 \mapsto v,$$

is Lipschitz continuous.

Let us remark that the only known local well-posedness results *in* modulation spaces until now are [13, Theorem 1.1] by Wang, Zhao and Guo for $M^0_{2,1}(\mathbb{R}^d)$ and its generalization [4, Theorem 1.1] due to Bényi and Okoudjou for $M^s_{p,1}(\mathbb{R}^d)$ with $1 \le p \le \infty$ and $s \ge 0$. Local well-posedness results without persistence (i.e. initial data in a modulation space, but the solution is not a curve on it) include [9, Theorem 1.4] for $u_0 \in M^0_{2,q}(\mathbb{R}^d)$ with $2 \le q < \infty$.

Theorem 1 generalizes [4, Theorem 1.1] to $q \ge 1$: Although our theorem is stated for the cubic nonlinearity, this is for simplicity of the presentation only. The proof allows for an easy generalization to *algebraic nonlinearities* considered in [4], which are of the form

$$f(u) = g(|u|^2)u = \sum_{k=0}^{\infty} c_k |u|^{2k} u, \qquad \text{where } g \text{ is an entire function.} \tag{4}$$

Also, Theorems 1.2 and 1.3 in [4], which concern the nonlinear wave and the nonlinear Klein-Gordon equation respectively, can be generalized in the same spirit.

This is due to Bényi's and Okoudjou's and our proofs being based on the well-known Banach's contraction principle, an estimate for the norm of the Schrödinger propagator and the fact that the considered modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ are *Banach *-algebras*[1] with respect to pointwise multiplication. Let us state the two latter ingredients formally and comment on them.

The first is given by

Proposition 2 (Algebra property). *Let $d \in \mathbb{N}$ and $1 \le p, q \le \infty$. For $q > 1$ let $s > d\left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \ge 0$. Then $M^s_{p,q}(\mathbb{R}^d)$ is a Banach *-algebra with respect to pointwise multiplication and complex conjugation. These operations are well-defined due to the following embedding*

$$M^s_{p,q}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \middle| f \text{ bounded} \right\}.$$

Proposition 2 had been observed already in 1983 by Feichtinger in his pioneering work on modulation spaces, cf. [6, Proposition 6.9] where he proves it using a rather abstract approach via Banach convolution triples. This might explain why the algebra property seems to be not well-known in the PDE community. In [4, Corollary 2.6] Proposition 2 for $q = 1$ is stated without referring to Feichtinger and a proof via the theory of pseudodifferential operators is said to be along the lines of [2, Theorem 3.1]. In contrast to these approaches, our proof of the algebra property is elementary. It follows from the new Hölder-like inequality stated in

Theorem 3 (Hölder-like inequality). *Let $d \in \mathbb{N}$ and $1 \le p, p_1, p_2, q \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q > 1$ let $s > d\left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \ge 0$. Then there exists a constant $C = C(d, s, q) > 0$ such that*

$$\|fg\|_{M^s_{p,q}(\mathbb{R}^d)} \le C \|f\|_{M^s_{p_1,q}(\mathbb{R}^d)} \|g\|_{M^s_{p_2,q}(\mathbb{R}^d)}. \tag{5}$$

[1] For us a Banach *-algebra X is a Banach algebra over $\mathbb{C}$ on which a continuous *involution* $*$ is defined, i.e. $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. We neither require X to have a unit nor $C = 1$ in the estimates $\|x \cdot y\| \le C \|x\| \|y\|$, $\|x^*\| \le C \|x\|$.

for all $f \in M^s_{p_1,q}(\mathbb{R}^d)$, $g \in M^s_{p_2,q}(\mathbb{R}^d)$. The pointwise multiplication is well-defined due to the embedding formulated in Proposition 2.

Crucial for the proof of Theorem 3 is the algebra property of the sequence spaces $l^q_s(\mathbb{Z}^d)$ stated in Lemma 9 (s, q and d are as in Theorem 3, $l^s_q(\mathbb{Z}^d)$ is defined at the end of the introduction).

The second crucial ingredient for the proof of Theorem 1 is the boundedness of the Schrödinger propagator $e^{\mathrm{i}t\Delta}$ on all modulation spaces $M^s_{p,q}(\mathbb{R}^d)$. Let us fix the window function $x \mapsto e^{-|x|^2}$ in the definition of the modulation space norm. Then we have (notation is explained at the end of the introduction)

Theorem 4 (Schrödinger propagator bound). *There is a constant $C > 0$ such that for any $d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ the inequality*

$$\left\| e^{\mathrm{i}t\Delta} \right\|_{\mathscr{L}(M^s_{p,q}(\mathbb{R}^d))} \leq C^d (1 + |t|)^{d\left|\frac{1}{2} - \frac{1}{p}\right|} \tag{6}$$

holds for all $t \in \mathbb{R}$. Furthermore, the exponent of the time dependence is sharp.

The boundedness has been obtained e.g. in [3, Theorem 1] whereas the sharpness was proven in [5, Proposition 4.1]. We sketch a simple proof of Theorem 4 in Section 2.

The remainder of our paper is structured as follows. We start with Section 2 providing an overview over modulation spaces, showing that Proposition 2 follows from Theorem 3 and sketching a simple proof of Theorem 4. In Section 3 we prove an algebra property of the weighted sequence spaces $l^q_s(\mathbb{Z}^d)$ for sufficiently large s. In the subsequent Section 4 we prove the Hölder-like inequality from Theorem 3. Finally, we prove Theorem 1 on the local well-posedness in Section 5.

Notation

We denote generic constants by C. To emphasize on which quantities a constant depends we write e.g. $C = C(d)$ or $C = C(d, s)$. Sometimes we omit a constant from an inequality by writing "$\lesssim$", e.g. $A \lesssim B$ instead of $A \leq C(d)B$. Special constants are $d \in \mathbb{N}$ for the *dimension*, $1 \leq p, q \leq \infty$ for the *Lebesgue* exponents and $s \in \mathbb{R}$ for the *regularity* exponent. By p' we mean the *dual* exponent of p, that is the number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. To simplify the subsequent claims we shall call a regularity exponent s *sufficiently large*, if

$$s \begin{cases} > \frac{d}{q'} & \text{for } q > 1, \\ \geq 0 & \text{for } q = 1. \end{cases} \tag{7}$$

We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of *Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^d)$ the space of *tempered distributions*. Furthermore, we denote the *Bessel potential spaces* or simply L^2-based *Sobolev spaces* by $H^s = H^s(\mathbb{R}^d)$ or by $H^s(\mathbb{T}^d)$, if we are on the d-dimensional Torus $\mathbb{T}^d$. For the space of bounded continuous functions we write C_b and for the space of smooth functions with compact support we write C^∞_c. The letters f, g, h denote either generic functions $\mathbb{R}^d \to \mathbb{C}$ or generic tempered distributions. Whereas $(a_k)_{k\in\mathbb{Z}^d}, (b_k)_{k\in\mathbb{Z}^d}, (c_k)_{k\in\mathbb{Z}^d}$ or $(a_k)_k, (b_k)_k, (c_k)_k$ or $(a_k), (b_k), (c_k)$ denote generic complex-valued sequences. By $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$ we denote the *Japanese bracket*.

For a Banach space X we write X^* for its dual and $\|\cdot\|_X$ for the norm it is canonically equipped with. By $\mathscr{L}(X)$ we denote the space of all bounded linear maps on X. By $[X, Y]_\theta$ we mean complex interpolation between X and another Banach space Y. For brevity we write $\|\cdot\|_p$ for the p-norm on the *Lebesgue space* $L^p = L^p(\mathbb{R}^d)$, the *sequence space* $l^p = l^p(\mathbb{Z}^d)$ or $l^p = l^p(\mathbb{N}_0)$ and $\|(a_k)\|_{q,s} := \|(\langle k\rangle^s a_k)\|_q$ for the norm on $\langle\cdot\rangle^s$-weighted sequence spaces $l^q_s = l^q_s(\mathbb{Z}^d)$. Also, we shorten the notation for modulation spaces: $M^s_{p,q}$ for $M^s_{p,q}(\mathbb{R}^d)$ and even $M_{p,q}$ for $M^0_{p,q}$. If the norm is clear from the context, we write $B_r(x)$ for a ball of radius r around $x \in X$ and set $B_r = B_r(0)$.

Furthermore, we denote the *Fourier transform* by $\mathcal{F}$ and the inverse Fourier transform by $\mathcal{F}^{(-1)}$, where we use the symmetric choice of constants and write also

$$\hat{f}(\xi) := (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x, \qquad \check{g}(x) := \left(\mathcal{F}^{(-1)}g\right)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} g(\xi)\mathrm{d}\xi.$$

Finally, we introduce the operations $S_x f(y) = f(y-x)$ of *translation* by $x \in \mathbb{R}^d$, $(M_k f)(y) = e^{ik\cdot y} f(y)$ of *modulation* by $k \in \mathbb{R}^d$ and $\overline{f}$ of *complex conjugation.*

2. Modulation spaces

As already mentioned in the introduction, modulation spaces were introduced by Feichtinger in [6] in the setting of locally compact Abelian groups. The textbook [8] by Gröchenig gives a thorough introduction, although it lacks the characterization of modulation spaces via *isometric decomposition operators* defined below. A presentation incorporating these operators is contained in the paper [12, Section 2, 3] by Wang and Hudzik. A survey on modulation spaces and nonlinear evolution equations is given in [10].

A convenient equivalent norm on modulation spaces which we are going to use is constructed as follows: Set $Q_0 := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ and $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$. Consider a smooth partition of unity $(\sigma_k)_{k\in\mathbb{Z}^d} \in \left(C_c^\infty(\mathbb{R}^d)\right)^{\mathbb{Z}^d}$ satisfying

(i) $\exists c > 0 : \forall k \in \mathbb{Z}^d : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,

(ii) $\forall k \in \mathbb{Z}^d : \operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$,

(iii) $\sum_{k\in\mathbb{Z}^d} \sigma_k = 1$,

(iv) $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z}^d : \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \Rightarrow \|D^\alpha \sigma_k\|_\infty \leq C_m$

and define the *isometric decomposition operators* $\Box_k := \mathcal{F}^{(-1)}\sigma_k\mathcal{F}$. Let us mention the fact that $\Box_k f \in C^\infty(\mathbb{R}^d)$ for $f \in \mathcal{S}'(\mathbb{R}^d)$ by [7, Theorem 2.3.1]. We cite from [12, Proposition 1.9] the following often used

Lemma 5 (Bernstein multiplier estimate). *Let $d \in \mathbb{N}$, $1 \leq p \leq \infty$, $s > \frac{d}{2}$ and $\sigma \in H^s(\mathbb{R}^d)$. Then the multiplier operator $T_\sigma = \mathcal{F}^{(-1)}\sigma\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ corresponding to the symbol σ is bounded on $L^p(\mathbb{R}^d)$. More precisely, there is a constant $C = C(s,d) > 0$ such that*

$$\|T_\sigma\|_{\mathscr{L}(L^p(\mathbb{R}^d))} \leq C \|\sigma\|_{H^s(\mathbb{R}^d)}.$$

By Lemma 5, the family $(\Box_k)_{k\in\mathbb{Z}^d}$ is bounded in $\mathscr{L}(L^p(\mathbb{R}^d))$ independently of p. The aforementioned equivalent norm for the modulation space $M^s_{p,q}$ is given by

$$\|f\|_{M^s_{p,q}} \cong \left\| \left(\|\Box_k f\|_p \right)_{k\in\mathbb{Z}^d} \right\|_{q,s}. \tag{8}$$

Choosing a different partition of unity (σ_k) yields yet another equivalent norm.

Lemma 6 (Continuous embeddings). *Let $s_1 \geq s_2$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then*

(a) $M^{s_1}_{p_1,q_1}(\mathbb{R}^d) \subseteq M^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ and the embedding is continuous,

(b) $M_{p,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$.

Lemma 6 is well-known (cf. [12, Proposition 2.5, 2.7]). For convenience we sketch a

Proof. (a) One can change indices one by one. The inclusion for "s" is by monotonicity and the inclusion for "q" is by the embeddings of the l^q spaces. For the "p"-embedding consider $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\tau|_{B_{\sqrt{d}}} \equiv 1$ and $\operatorname{supp}(\tau) \subseteq B_d$. Define the shifted $\tau_k = S_k\tau$ and the corresponding multiplier operators $\tilde{\Box}_k = \mathcal{F}^{(-1)}\tau_k\mathcal{F}$. Clearly, $\tilde{\Box}_k\Box_k = \Box_k$ and $\tilde{\Box}_k f = \frac{1}{(2\pi)^{\frac{d}{2}}}(M_k\check{\sigma}) * f$. Hence

$$\|\Box_k f\|_{p_2} = \left\|\tilde{\Box}_k\Box_k f\right\|_{p_2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \|(M_k\check{\sigma}) * (\Box_k f)\|_{p_2} \overset{\text{Young}}{\leq} \frac{1}{(2\pi)^{\frac{d}{2}}} \|\check{\sigma}\|_r \|\Box_k f\|_{p_1},$$

where $\frac{1}{r} = 1 - \frac{1}{p_1} + \frac{1}{p_2}$. Recalling (8) finishes the proof.

(b) By part (a) it is enough to show $M_{\infty,1} \hookrightarrow C_b$. For any $f \in M_{\infty,1}$ we have $\underbrace{\sum_{|k|\leq N} \Box_k f}_{\in C^\infty} \to f$ in $\mathcal{S}'$ as $N \to \infty$. But simultaneously

$$\left\| \sum_{N_1 \leq |k| \leq N_2} \Box_k f \right\|_\infty \leq \sum_{N_1 \leq |k| \leq N_2} \|\Box_k f\|_\infty \leq \sum_{k\in\mathbb{Z}^d} \|\Box_k f\|_\infty < \infty.$$

So $f \in C_b$ and $\sum_{|k|\leq N} \Box_k f \to f$ in C_b as $N \to \infty$.

We are now ready to give a

Proof of Proposition 2. We have $l^q_s \hookrightarrow l^1$ for sufficiently large s, since

$$\sum_{k\in\mathbb{Z}^d} |a_k| = \sum_{k\in\mathbb{Z}^d} \frac{1}{\langle k\rangle^s} \langle k\rangle^s |a_k| \overset{\text{Hölder}}{\leq} \underbrace{\left(\sum_{k\in\mathbb{Z}^d} \frac{1}{\langle k\rangle^{sq'}} \right)^{\frac{1}{q'}}}_{<\infty \text{ for } s>\frac{d}{q'}} \left(\sum_{l\in\mathbb{Z}^d} \langle l\rangle^{sq} |a_l|^q \right)^{\frac{1}{q}}.$$

Then (8) yields $M^s_{p,q} \hookrightarrow M_{p,1}$ and by Lemma 6 (b) we have $M_{p,1} \hookrightarrow C_b$. This proves the claimed embedding.

Choosing σ_k real-valued in (8) shows that complex conjugation does not change the modulation space norm.

Choosing $p_1 = p_2 = 2p$ in Theorem 3 and applying Lemma 6 (a) shows the estimate for the continuity of pointwise multiplication and finishes the proof.

Lemma 7 (Dual space). *For $s \in \mathbb{R}$, $1 \leq p, q < \infty$ we have*

$$\left(M^s_{p,q}\right)^* = M^{-s}_{p',q'}$$

(see [12, Theorem 3.1]).

Theorem 8 (Complex interpolation). *For $1 \leq p_1, q_1 < \infty$, $1 \leq p_2, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$ and $\theta \in (0,1)$ one has*

$$\left[M^{s_1}_{p_1,q_1}(\mathbb{R}^d), M^{s_2}_{p_2,q_2}(\mathbb{R}^d)\right]_\theta = M^s_{p,q}(\mathbb{R}^d),$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2$$

(see [6, Theorem 6.1 (D)]).

Using these results we sketch a

Proof of Theorem 4. We have $V_g(e^{\mathrm{i}t\Delta} f) = V_{e^{-\mathrm{i}t\Delta}g} f$ by duality, i.e. the Schrödinger time evolution of the initial data can be interpreted as the backwards time evolution of the window function. The price for changing from window g_0 to window g_1 is $\|V_{g_0} g_1\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$ by [8, Proposition 11.3.2 (c)]. For $g(x) = e^{-|x|^2}$ one explicitly calculates

$$\left\|V_{e^{-\mathrm{i}t\Delta}g} g\right\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = C^d \left(1+|t|\right)^{\frac{d}{2}},$$

which proves the claimed bound for $p \in \{1, \infty\}$. Conservation for $p = 2$ is easily seen from (8). Complex interpolation between the cases $p = 2$ and $p = \infty$ yields (6) for $2 \leq p \leq \infty$. The remaining case $1 < p < 2$ is covered by duality.

Optimality in the case $1 \leq p \leq 2$ is proven by choosing the window g and the argument f to be a Gaussian and explicitly calculating $\left\|e^{\mathrm{i}t\Delta} f\right\|_{M^s_{p,q}} \approx (1+|t|)^{d\left(\frac{1}{p}-\frac{1}{2}\right)}$. This implies the optimality for $2 < p \leq \infty$ by duality.

3. Algebra property of some weighted sequence spaces

Let us recall the definition of the $\langle\cdot\rangle^s$-weighted sequence spaces

$$l_q^s(\mathbb{Z}^d) = \left\{ (a_k) \in \mathbb{C}^{(\mathbb{Z}^d)} \middle| \, \|(a_k)\|_{q,s} < \infty \right\}, \qquad \text{where} \qquad \|(a_k)\|_{q,s} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{qs} |a_k|^q\right)^{\frac{1}{q}} & \text{for } 1 \le q < \infty, \\ \sup_{k\in\mathbb{Z}^d} \langle k\rangle^s |a_k| & \text{for } q = \infty, \end{cases}$$

and $s \in \mathbb{R}$, $d \in \mathbb{N}$. We have

Lemma 9 (Algebra property). *Let $1 \le q \le \infty$. For $q > 1$ let $s > d\left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \ge 0$. Then $l_s^q(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution*

$$(a_l) * (b_m) = \left(\sum_{m\in\mathbb{Z}^d} a_{k-m} b_m \right)_{k\in\mathbb{Z}^d}, \tag{9}$$

which is well-defined, as the series above always converge absolutely.

This result is most likely not new. For the sake of self-containedness of the presentation, and because we could not come up with any suitable reference, we will give a proof. The inspiration for Lemma 9 comes from the fact that $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$ is a Banach algebra with respect to pointwise multiplication and $l_s^2(\mathbb{Z}^d) = \mathcal{F}(H^s(\mathbb{T}^d))$. A proof for the algebra property of $H^s(\mathbb{R}^d)$ can be given using the Littlewood-Paley decomposition, see e.g. [1, Proposition II.A.2.1.1 (ii)]. We were able to adapt that proof to the $l_s^q(\mathbb{Z}^d)$ case, even for $q \neq 2$, by noting that we are already on the Fourier side.

Let us recall that the Littlewood-Paley decomposition of a tempered distribution is a series essentially such that the Fourier transform of l-th summand has its support in the annulus with radii comparable to 2^l. In the same spirit we formulate

Lemma 10 (Discrete Littlewood-Paley characterization). *Let $1 \le q \le \infty$ and $s \in \mathbb{R}$. Define $C(s) = 2^{|s|}$,*

$$A_0 := \{0\} \subseteq \mathbb{Z}^d, \qquad \text{and} \qquad A_l := \left\{ k \in \mathbb{Z}^d \middle| 2^{(l-1)} \le |k| < 2^l \right\} \quad \forall l \in \mathbb{N}.$$

(a) (Necessary condition) *For any $(a_k) \in l_s^q(\mathbb{Z}^d)$ there is a sequence $(C_l) \in l^q(\mathbb{N}_0)$ such that $\|C_l\|_q = 1$ and*

$$\left\| (\mathbb{1}_{A_l}(k) a_k)_k \right\|_q \le C(s) 2^{-ls} C_l \|(a_k)\|_{q,s} \qquad \forall l \in \mathbb{N}_0.$$

(b) (Sufficient condition) *Conversely, if for some $N \ge 0$ and $(C_l) \in l^q(\mathbb{N}_0)$ with $\|(C_l)\|_q \le 1$ the estimate*

$$\left\| (\mathbb{1}_{A_l}(k) a_k)_k \right\|_q \le \frac{1}{C(s)} 2^{-ls} C_l N \qquad \forall l \in \mathbb{N}_0$$

holds, then $(a_k) \in l_s^q(\mathbb{Z}^d)$ and $\|(a_k)\|_{q,s} \le N$.

Proof. Observe that $2^{l-1} \le \langle k\rangle < 2^{l+1}$ so $\langle k\rangle^t \le 2^{|t|} 2^{lt} = C(t) 2^{lt}$ for each $l \in \mathbb{N}_0$, $k \in A_l$ and $t \in \mathbb{R}$.

(a) For $(a_k) = 0$ there is nothing to show, so assume $\|(a_k)\|_{q,s} > 0$. Then for any $l \in \mathbb{N}_0$

$$\|(\mathbb{1}_{A_l}(k) a_k)\|_q = \left\| \left(\mathbb{1}_{A_l}(k) \frac{\langle k\rangle^s}{\langle k\rangle^s} a_k \right) \right\|_q \le \frac{C(s)}{2^{ls}} \|(\mathbb{1}_{A_l}(k) a_k)\|_{q,s} = C(s) 2^{-ls} C_l \|(a_k)\|_{q,s},$$

where $C_l := \frac{\left\|\left(\mathbb{1}_{A_l}(k) a_k\right)\right\|_{q,s}}{\|(a_k)\|_{q,s}}$.

(b) We have $(a_k) = (\sum_{l=0}^{\infty} \mathbb{1}_{A_l}(k) a_k)$. Thus, for $q < \infty$,

$$\|(a_k)\|_{q,s}^q = \sum_{l=0}^{\infty} \|(\langle k\rangle^s \mathbb{1}_{A_l}(k) a_k)\|_q^q \leq C(s)^q \sum_{l=0}^{\infty} 2^{lsq} \|(\mathbb{1}_{A_l}(k) a_k)\|_q^q \leq N^q \sum_{l=0}^{\infty} C_l^q \leq N^q.$$

Similarly, for $q = \infty$, we have

$$\|(a_k)\|_{\infty,s} = \sup_{l\in\mathbb{N}_0} \max_{k\in A_l} \langle k\rangle^s |a_k| \leq \sup_{l\in\mathbb{N}_0} C(s) 2^{ls} \|(\mathbb{1}_{A_l}(k) a_k)\|_\infty \leq N \sup_{l\in\mathbb{N}_0} C_l \leq N.$$

For the proof of Lemma 9 we will require yet another sufficient condition. The discrete Littlewood-Paley decomposition in Lemma 10 consisted of sequences having their supports in disjoint dyadic annuli. We now consider non-disjoint dyadic balls B_m.

Lemma 11 (Sufficient condition for balls). *Let $1 \leq q \leq \infty$ and $s > 0$. Define $C(s) = \frac{2^s}{1-2^{-s}}$ and*

$$B_m := \left\{k \in \mathbb{Z}^d \middle| \, |k| < 2^m\right\} \qquad \forall m \in \mathbb{N}_0.$$

For each $m \in \mathbb{N}_0$ let $(a_{k,m})_{k\in\mathbb{Z}^d}$ be such that $\operatorname{supp}((a_{k,m})_{k\in\mathbb{Z}^d}) \subseteq B_m$. *If for some $N \geq 0$ and $(C_m) \in l^q(\mathbb{N}_0)$ with $\|(C_m)\|_q \leq 1$ the estimate*

$$\left\|(a_{k,m})_{k\in\mathbb{Z}^d}\right\|_q \leq \frac{1}{C(s)} 2^{-ms} C_m N \qquad \forall m \in \mathbb{N}_0$$

holds, then

$$(a_k) := \left(\sum_{m=0}^{\infty} a_{k,m}\right)_k \in l_s^q(\mathbb{Z}^d) \qquad \textit{and} \qquad \|(a_k)\|_{q,s} \leq N.$$

Proof. We want to apply the sufficient condition for annuli. Observe, that $A_l \cap B_m = \emptyset$ if $l > m$. Hence

$$\|(\mathbb{1}_{A_l}(k) a_k)\|_q = \left\| \left(\sum_{m=0}^{\infty} \mathbb{1}_{A_l \cap B_m}(k) a_{k,m}\right)_k \right\|_q \leq \sum_{m=l}^{\infty} \|(a_{k,m})\|_q \leq \frac{1}{C(s)} N 2^{-ls} \underbrace{\sum_{m=l}^{\infty} 2^{-(m-l)s} C_m}_{=:\tilde{C}_l}$$

for all $l \in \mathbb{N}_0$. It remains to show that $(\tilde{C}_l) \in l^q(\mathbb{N}_0)$ and $\left\|(\tilde{C}_l)\right\|_q \leq \frac{1}{1-2^{-s}}$. We can assume $1 < q < \infty$, as the proof for the other cases is easier and follows the same lines. We have

$$\tilde{C}_l = \sum_{m=l}^{\infty} \left[2^{-(m-l)\frac{s}{q'}}\right] \times \left[2^{-(m-l)\frac{s}{q}} C_m\right] \overset{\text{Hölder}}{\leq} \left(\sum_{m=0}^{\infty} 2^{-ms}\right)^{\frac{1}{q'}} \times \left(\sum_{m=l}^{\infty} 2^{-(m-l)s} C_m^q\right)^{\frac{1}{q}}$$

for all $l \in \mathbb{N}_0$. Using the geometric series formula we recognize $\sum_{m=0}^{\infty} 2^{-ms} = \frac{1}{1-2^{-s}}$ and

$$\sum_{l=0}^{\infty} \sum_{m=l}^{\infty} 2^{-(m-l)s} C_m^q = \sum_{m=0}^{\infty} C_m^q 2^{-ms} \sum_{l=0}^{m} 2^{ls} = \sum_{m=0}^{\infty} C_m^q 2^{-ms} \left(\frac{2^{(m+1)s} - 1}{2^s - 1}\right) \leq \frac{1}{1-2^{-s}} \sum_{m=0}^{\infty} C_m^q.$$

Recalling $\|(C)_m\|_q \leq 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ finishes the proof.

We are now ready to give a

Proof of Lemma 9. As already mentioned in the proof of Proposition 2 (see Section 2), $l_s^q \hookrightarrow l^1$ for sufficiently large s (recall (7)). Hence, by Young's inequality, the series in (9) is absolutely convergent and the case $s = 0$ is obvious. Consider now the case $s > 0$.

To that end, let us study what happens to the parts of the Littlewood-Paley decompositions of (a_l) and (b_m) under convolution. Let the annuli A_i and the balls B_j $(i, j \in \mathbb{N}_0)$ be defined as in the Lemmas 10 and 11. By the preceding remark, all of the occurring series are absolutely convergent and hence the following manipulations are justified:

$$\begin{aligned}
(a_l) * (b_m) &= \left(\sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l) a_l\right)_l * \left(\sum_{j=0}^{\infty} \mathbb{1}_{A_j}(m) b_m\right)_m \\
&= \sum_{i=0}^{\infty} (\mathbb{1}_{A_i}(l) a_l)_l * \left(\sum_{j=0}^{i} \mathbb{1}_{A_j}(m) b_m\right)_m + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} (\mathbb{1}_{A_i}(l) a_l)_l * \big(\mathbb{1}_{A_j}(m) b_m\big)_m \\
&= \sum_{i=0}^{\infty} (\mathbb{1}_{A_i}(l) a_l)_l * (\mathbb{1}_{B_i}(m) b_m)_m + \sum_{j=1}^{\infty} \left(\sum_{i=0}^{j-1} \mathbb{1}_{A_i}(l) a_l\right)_l * \big(\mathbb{1}_{A_j}(m) b_m\big)_m \\
&= \sum_{i=0}^{\infty} \underbrace{(\mathbb{1}_{A_i}(l) a_l)_l * (\mathbb{1}_{B_i}(m) b_m)_m}_{=:(a_{k,i})_k} + \sum_{j=0}^{\infty} \underbrace{\big(\mathbb{1}_{B_j}(l) a_l\big)_l * \big(\mathbb{1}_{A_{j+1}}(m) b_m\big)_m}_{=:(b_{k,j})_k}
\end{aligned}$$

Observe that $\operatorname{supp}((a_{k,i})_k) \subseteq B_{i+1}$ and $\operatorname{supp}((b_{k,j})_k) \subseteq B_{j+2}$ by the properties of convolution and so the sufficient condition for balls could be applied. Indeed we have

$$\left\|(a_{k,i})_k\right\|_q \leq \|(\mathbb{1}_{B_i}(m) b_m)_m\|_1 \, \|(\mathbb{1}_{A_i}(l) a_l)_l\|_q \lesssim 2^{-is} C_i \, \|(b_m)\|_{q,s} \, \|(a_l)\|_{q,s} \, ,$$

where we used Young's inequality, the embedding $l_s^q \hookrightarrow l^1$ and the necessary condition for $(a_l) \in l_s^q$ from Lemma 10 (C_i was called C_l there). Hence, $\sum_{i=0}^{\infty} (a_{k,i})_k \in l_s^q$ with $\|\sum_{i=0}^{\infty} (a_{k,i})_k\|_{q,s} \lesssim \|(a_l)\|_{q,s} \, \|(b_m)\|_{q,s}$ by Lemma 11. The same argument applies to $\sum_{j=0}^{\infty} (b_{k,j})_k$ and finishes the proof.

4. Proof of the Hölder-like inequality, Theorem 3.

We have already shown $M_{p,q}^s \hookrightarrow C_b$ in the proof of Proposition 2 in Section 2, so it remains to prove (5). To that end, we shall use (8). Fix a $k \in \mathbb{Z}^d$. By the definition of the operator $\square_k$ we have

$$\square_k(fg) = \frac{1}{(2\pi)^{\frac{d}{2}}} \mathcal{F}^{(-1)} \left(\sigma_k (\hat{f} * \hat{g})\right) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{l,m \in \mathbb{Z}^d} \mathcal{F}^{(-1)} \left(\sigma_k ((\sigma_l \hat{f}) * (\sigma_m \hat{g}))\right).$$

As the supports of the partition of unity are compact, many summands vanish. Indeed, for any $k, l, m \in \mathbb{Z}^d$

$$\operatorname{supp}\left(\sigma_k \left((\sigma_l \hat{f}) * (\sigma_m \hat{g})\right)\right) \subseteq \operatorname{supp}(\sigma_k) \cap (\operatorname{supp}(\sigma_l) + \operatorname{supp}(\sigma_m)) \subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l+m)$$

and so $\sigma_k \left((\sigma_l \hat{f}) * (\sigma_m \hat{g})\right) \equiv 0$ if $|(k-l) - m| > 3\sqrt{d}$. Hence, the double series over $l, m \in \mathbb{Z}^d$ boils down to a finite sum of discrete convolutions

$$\square_k(fg) = \frac{1}{(2\pi)^{\frac{d}{2}}} \mathcal{F}^{(-1)} \left(\sigma_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\sigma_l \hat{f}) * (\sigma_{k-l+m} \hat{g})\right) = \square_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\square_l f) \cdot (\square_{k+m-l} g),$$

where $M = \left\{ m \in \mathbb{Z}^d \,\middle|\, |m| \leq 3\sqrt{d} \right\}$ and $\#M \leq \left(6\sqrt{d}+1\right)^d < \infty$. That was the job of $\square_k$ and we now get rid of it,

$$\|\square_k(fg)\|_p \lesssim \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \|(\square_l f) \cdot (\square_{k+m-l} g)\|_p \, ,$$

using the Bernstein multiplier estimate from Lemma 5.

Invoking Hölder's inequality we further estimate

$$\left(\|\Box_k(fg)\|_p\right)_k \lesssim \sum_{m\in M}\left(\|\Box_l(f)\|_{p_1}\right)_l * \left(\|\Box_{n+m}(g)\|_{p_2}\right)_n$$

pointwise in k and hence

$$\|fg\|_{M^s_{p,q}} \lesssim \left\|\left(\|\Box_l f\|_{p_1}\right)_l\right\|_{q,s}\left(\sum_{m\in M}\left\|\left(\|\Box_{n+m} g\|_{p_2}\right)_n\right\|_{q,s}\right)$$

by the algebra property of l^q_s from Lemma 9. Finally, we remove the sum over m

$$\sum_{m\in M}\left\|\left(\|\Box_{n+m} g\|_{p_2}\right)_n\right\|_{q,s} \lesssim \|g\|_{M^s_{p_2,q}}$$

applying Peetre's inequality $\langle k+l\rangle^s \le 2^{|s|}\langle k\rangle^s\langle l\rangle^{|s|}$. See e.g. [11, Proposition 3.3.31].

Let us finish the proof remarking that the only estimate involving "p"s we used was Hölder's inequality and thus indeed $C = C(d,s,q)$. □

5. Proof of the local well-posedness, Theorem 1.

For $T > 0$ let $X(T) = C([0,T], M^s_{p,q}(\mathbb{R}^d))$. Proposition 2 immediately implies that X is a Banach *-algebra, i.e.,

$$\|uv\|_X = \sup_{0\le t\le T}\|uv(\cdot,t)\|_{M^s_{p,q}} \lesssim \left(\sup_{0\le s\le T}\|u(\cdot,s)\|_{M^s_{p,q}}\right)\left(\sup_{0\le t\le T}\|v(\cdot,t)\|_{M^s_{p,q}}\right) = \|u\|_X\|v\|_X.$$

For $R > 0$ we denote by $M(R,T) = \left\{u \in X \,\middle|\, \|u\|_{X(T)} \le R\right\}$ the closed ball of radius R in $X(T)$ centered at the origin. We show that for some $T, R > 0$ the right-hand side of (2),

$$(\mathcal{T}u)(\cdot,t) := e^{\mathrm{i}t\Delta}u_0 \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(|u|^2 u(\cdot,\tau)\right)\mathrm{d}\tau \qquad (\forall t\in[0,T]), \tag{10}$$

defines a contractive self-mapping $\mathcal{T} = \mathcal{T}(u_0) : M_{R,T} \to M_{R,T}$.

To that end let us observe that Theorem 4 implies the *homogeneous estimate*

$$\left\|t \mapsto e^{\mathrm{i}t\Delta}v\right\|_X \lesssim (1+T)^{\frac{d}{2}}\|v\|_{M^s_{p,q}} \qquad (\forall v\in M^s_{p,q}),$$

which, together with the algebra property of $X(T)$, proves the *inhomogeneous estimate*

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(|u|^2 u(\cdot,\tau)\right)\mathrm{d}\tau\right\|_{M^s_{p,q}} \lesssim (1+T)^{\frac{d}{2}}\int_0^t\left\||u|^2 u(\cdot,\tau)\right\|_{M^s_{p,q}}\mathrm{d}\tau \lesssim T(1+T)^{\frac{d}{2}}\|u\|_X^3,$$

holding for $0 \le t \le T$ and $u \in X$.

Applying the triangle inequality in (10) yields $\|\mathcal{T}u\|_X \le C(1+T)^{\frac{d}{2}}(\|u_0\|_{M^s_{p,q}} + TR^3)$ for any $u \in M(R,T)$. Thus, $\mathcal{T}$ maps $M(R,T)$ onto itself for $R = 2C\|u_0\|_{M^s_{p,q}}$ and T small enough. Furthermore,

$$|u|^2 u - |v|^2 v = (u-v)|u|^2 + (\overline{u}u - \overline{v}v)v = (u-v)(|u|^2 + \overline{u}v) + (\overline{u}-\overline{v})v^2$$

and hence

$$\|\mathcal{T}u - \mathcal{T}v\|_X \lesssim T(1+T)^{\frac{d}{2}}R^2\|u-v\|_X$$

for $u, v \in M(R,T)$, where we additionally used the algebra property of X and the homogeneous estimate. Taking T sufficiently small makes $\mathcal{T}$ a contraction.

Banach's fixed-point theorem implies the existence and uniqueness of a mild solution up to the minimal time of existence $T_* = T_*\left(\|u_0\|_{M^s_{p,q}}\right) \approx \|u_0\|^{-2}_{M^s_{p,q}} > 0$. Uniqueness of the maximal solution and the blow-up alternative now follow easily by the usual contradiction argument.

For the proof of the Lipschitz continuity let us notice that for any $r > \|u_0\|_{M^s_{p,q}}$, $v_0 \in B_r$ and $0 < T \le T_*(r)$ we have

$$\|u - v\|_{X(T)} = \|\mathcal{T}(u_0)u - \mathcal{T}(v_0)v\|_{X(T)} \lesssim (1+T)^{\frac{d}{2}} \|u_0 - v_0\|_{M^s_{p,q}} + T(1+T)^{\frac{d}{2}} R^2 \|u - v\|_{X(T)},$$

where v is the mild solution corresponding to the initial data v_0 and $R = 2Cr$ as above. Collecting terms containing $\|u-v\|_{X(T)}$ shows Lipschitz continuity with constant $L = L(r)$ for sufficiently small T, say $T_l = T_l(r)$. For arbitrary $0 < T' < T^*$ put $r = 2\|u\|_{X(T')}$ and divide $[0, T']$ into n subintervals of length $\le T_l$. The claim follows for $V = B_\delta(u_0)$ where $\delta = \frac{\|u_0\|_{M^s_{p,q}}}{L^n}$ by iteration. This concludes the proof. □

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