Expectiles, Omega Ratios and Stochastic Ordering

Fabio Bellini[1] · Bernhard Klar[2] · Alfred Müller[3]

Abstract In this paper we introduce the *expectile order*, defined by $X \leq_e Y$ if $e_\alpha(X) \leq e_\alpha(Y)$ for each $\alpha \in (0, 1)$, where e_α denotes the α-expectile. We show that the expectile order is equivalent to the pointwise ordering of the Omega ratios, and we derive several necessary and sufficient conditions. In the case of equal means, the expectile order can be easily characterized by means of the stop-loss transform; in the more general case of different means we provide some sufficient conditions. In contrast with the more common stochastic orders such as $\leq_{st}$ and $\leq_{cx}$, the expectile order is not generated by a class of utility functions and is not closed with respect to convolutions. As an illustration, we compare the $\leq_{st}$, $\leq_{icx}$ and $\leq_e$ orders in the family of Lomax distributions and compare Lomax distributions fitted to real world data of natural disasters in the U.S. caused by different sources of weather risk like storms or floods.

Keywords Expectile order · Omega ratio · Stop-loss transform · Third-order stochastic dominance · Skew-normal distribution · Lomax distribution

Mathematics Subject Classification (2010) 60E15 · 60E05 · 91B82

In memory of Moshe Shaked

✉ Alfred Müller
mueller@mathematik.uni-siegen.de

Fabio Bellini
fabio.bellini@unimib.it

Bernhard Klar
bernhard.klar@kit.edu

[1] Dipartimento di Statistica e Metodi Quantitativi, Università di Milano Bicocca, Milano, Italy

[2] Department of Mathematics, Karlsruher Institut für Technologie (KIT), Karlsruhe, Germany

[3] Department of Mathematics, University of Siegen, Siegen, Germany

1 Introduction

A *stochastic order* quantifies the concept of one random variable being bigger than another one in some sense. A leading figure in the theory of stochastic order relations was Moshe Shaked, who passed away much too early. Among many other contributions he is coauthor of the most important reference on stochastic orders, the book Shaked and Shantikumar (2007). In this paper we introduce a new stochastic order based on the comparison of expectiles, and we want to dedicate this paper to the memory of Moshe Shaked.

In recent years there is an increasing interest in using expectiles as an alternative to quantiles to describe features of a distribution. They have been introduced by Newey and Powell (1987) as coefficients in linear regression models based on an asymmetric least square loss function. For a recent overview of the use of expectile curves in regression analysis we refer to Kneib (2013) and the extensive discussions to that paper, in particular by Eilers (2013) for an appraisal of expectiles and by Koenker (2013) for a critical viewpoint. For some interesting recent studies where expectile curves have been used to describe the relation between predictors and the response variable we refer to Schnabel and Eilers (2009) for a study of life expectancies and to Lopez-Cabrera and Schulz (2014) for an application to forecasting expectiles of electricity demands, see also Farooq and Steinwart (2015) and Schulze-Waltrup et al. (2015) for further examples.

If one uses expectile curves to describe the relation between the predictor and the response variable, then it is of course an important question what it means if the expectile curves are increasing for all levels α. Does this have the same meaning as all quantile curves being increasing? As a mathematical problem this boils down to the question of what it means that all expectiles of one distribution are smaller than the expectiles of another distribution. In this paper we will consider this problem by investigating the corresponding expectile order. It will turn out that this order is strictly weaker than the stochastic order which holds if all quantiles are ordered.

Another branch of research where expectiles became very popular recently is the theory of risk measures. Bellini et al. (2014) have shown that for $\alpha \geq 1/2$ an expectile is a coherent risk measure in the sense of Artzner et al. (1999). It has recently been shown that indeed expectiles with $\alpha \geq 1/2$ are the only risk measures that are coherent and elicitable, see e.g. Bellini and Bignozzi (2015) or Ziegel (2016). As coherent risk measures are increasing with respect to increasing convex order (see e.g. Bäuerle and Müller 2006 for details) there also should be a kind of relationship between expectile order and increasing convex order. It will turn out that there is only a simple relationship between these two concepts in the case of equal means, however. We will also show in the paper that ordering all expectiles is equivalent to ordering all *Omega ratios* for all possible benchmarks. These have been introduced in the financial literature by Keating and Shadwick (2002a) as an interesting concept for comparing the performance of investment decisions. As it is a difficult question how to choose an appropriate benchmark, our results are also interesting for that application, as expectile ordering thus implies unambiguous decisions for investors using the Omega ratio for decision making, independent of the chosen benchmark.

The rest of the paper is organized as follows. In Section 2 we will recall the basic definitions of expectiles, the related Omega ratios and stop-loss transforms and some important relationships between these concepts. Section 3 will be devoted to the main properties of expectile ordering, including subsections on the special case of distributions with equal means that will be used to derive some interesting sufficient conditions also for the general case. Section 4 will be devoted to the study of stochastic orderings of Lomax distributions, which is an interesting parametric family of distributions where the conditions for usual stochastic order, increasing convex order and expectile order are different. Finally, in Section 5 Lomax distributions are fitted to real world data of damage of natural disasters in the U.S., differentiating between different sources of weather risk. Comparison of these fitted distributions with respect to different stochastic orders is investigated.

2 Expectiles, Omega Ratios and Stop-loss Transforms

In this section we recall the definitions of expectiles, Omega ratios and stop-loss transforms. We show how they are related, and in particular we derive how they uniquely determine the distribution if considered as a function of the parameter.

Throughout the paper we assume that all mentioned random variables X have a finite mean (denoted as $X \in L^1$) and are defined on a common probability space $(\Omega, \mathcal{A}, P)$ unless stated otherwise. Recall that the expectiles $e_X(\alpha)$ of a random variable $X \in L^2$ have been defined by Newey and Powell (1987) as the minimizers of an asymmetric quadratic loss:

$$e_X(\alpha) = \arg\min_{t\in\mathbb{R}} \{E\ell_\alpha(X-t)\}, \tag{1}$$

where

$$\ell_\alpha(x) = \begin{cases} \alpha x^2, & \text{if } x \geq 0, \\ (1-\alpha)x^2, & \text{if } x < 0, \end{cases}$$

and $\alpha \in (0,1)$. For $X \in L^1$, Eq. 1 has to be modified (see Newey and Powell (1987)) to

$$e_X(\alpha) = \arg\min_{t\in\mathbb{R}} \{E\left[\ell_\alpha(X-t) - \ell_\alpha(X)\right]\}. \tag{2}$$

The minimizer in Eqs. 1 or 2 is always unique and is identified by the first order condition

$$\alpha E\left(X - e_X(\alpha)\right)_+ = (1-\alpha)E\left(X - e_X(\alpha)\right)_-, \tag{3}$$

where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$.

Expectiles are a smoothed version of quantiles; we collect their main properties from Newey and Powell (1987) and Bellini et al. (2014) in the following theorem.

Theorem 1 *Let $X \in L^1$ with distribution function F and $\alpha \in (0,1)$. Then:*

a) $e_{X+h}(\alpha) = e_X(\alpha) + h$, *for each* $h \in \mathbb{R}$,
b) $e_{\lambda X}(\alpha) = \lambda e_X(\alpha)$, *for each* $\lambda > 0$,
c) $X \leq Y$ *a.s.* $\Rightarrow e_X(\alpha) \leq e_Y(\alpha)$, *for each* $\alpha \in (0,1)$,
d) $X \leq Y$ *a.s. and* $P(X < Y) > 0 \Rightarrow e_X(\alpha) < e_Y(\alpha)$, *for each* $\alpha \in (0,1)$,
e) $e_{X_S}(\alpha)$ *is strictly increasing with respect to* α,
f) $e_X(\alpha)$ *is continuous with respect to* α,
g) $\lim_{\alpha\to 0^+} e_X(\alpha) = \operatorname{ess\,inf}(X)$, $\lim_{\alpha\to 1^-} e_X(\alpha) = \operatorname{ess\,sup}(X)$,
h) *if* $\alpha \leq 1/2$, *then* $e_{X+Y}(\alpha) \geq e_X(\alpha) + e_Y(\alpha)$; *if* $\alpha \geq 1/2$, *then*

$$e_{X+Y}(\alpha) \leq e_X(\alpha) + e_Y(\alpha),$$

i) $e_{-X}(\alpha) = -e_X(1-\alpha)$,
j) *for the right derivative of* e_X *it holds*

$$e_X'^{+}(\alpha) = \frac{E\left|X - e_X(\alpha)\right|}{(1-\alpha)F(e_X(\alpha)) + \alpha\overline{F}(e_X(\alpha))}$$

where $\overline{F} = 1 - F$.

Clearly, expectiles depend only on the distribution of the random variable X; they can be seen as statistical functionals defined on $\mathcal{M}_1(\mathbb{R})$, the set of distribution functions with finite mean on $\mathbb{R}$. Expectiles are the quantiles of a suitably transformed distribution $\widetilde{F}(t)$, as it has been noted by Jones (1994). Indeed, Eq. 3 can be written in the equivalent form

$$\alpha = \frac{E\,(X-t)_-}{E\,|X-t|} =: \widetilde{F}(t).$$

Expectiles are also related with the so called *Omega ratio*, which has been introduced in the financial literature by Keating and Shadwick (2002a) as

$$\Omega_X(t) = \frac{E\,(X-t)_+}{E\,(X-t)_-}. \tag{4}$$

As it was pointed out e.g. in Remillard (2013), Eq. 3 can be written as

$$\Omega_X(e_X(\alpha)) = \frac{1-\alpha}{\alpha}, \tag{5}$$

which gives the following one-to-one relation between expectiles and Omega ratios:

$$e_X(\alpha) = \Omega_X^{-1}\left(\frac{1-\alpha}{\alpha}\right), \quad \Omega_X(t) = \frac{1-e_X^{-1}(t)}{e_X^{-1}(t)}. \tag{6}$$

The following properties of the function Ω_X are immediate.

Theorem 2 *Let* $X \in L^1$, $m = \operatorname{ess\,inf}(X)$, $M = \operatorname{ess\,sup}(X)$. *The function* $\Omega_X : (m, M) \to (0, +\infty)$ *is strictly positive, continuous and strictly decreasing, with* $\lim_{t\to m^+} \Omega_X(t) = +\infty$, $\lim_{t\to M^-} \Omega_X(t) = 0$ *and* $\Omega_X(EX) = 1$.

To derive more properties of the Omega ratio we consider the strongly related *stop-loss transform*

$$\pi_X(t) := E(X-t)_+,$$

which is well known in the actuarial literature (see e.g. Müller 1996) as it describes the expected cost of a stop-loss insurance contract with deductible t for a risk X. From

$$E(X-t)_- = t - EX + E(X-t)_+ \tag{7}$$

we immediately get

$$\Omega_X(t) = \frac{\pi_X(t)}{t - EX + \pi_X(t)}$$

and therefore vice versa we can derive the stop-loss transform from the Omega ratio via the formula

$$\pi_X(t) = \frac{\Omega_X(t)\cdot(t-EX)}{1-\Omega_X(t)}, \quad t \neq EX,$$

which can be continuously extended in $t = EX$. Assuming differentiability we can derive the distribution function from the Omega ratio via

$$F_X(t) = 1 + \pi_X'(t) = \frac{1-\Omega_X(t)+\Omega_X'(t)\cdot(t-EX)}{(1-\Omega_X(t))^2}, \quad t \neq EX.$$

This formula holds in general, if we replace the derivative by the right derivative. Using Eq. 6 and taking into account that $EX = e_X(1/2)$ we derive the following explicit formula for the distribution function F_X in terms of e_X. It is basically equivalent to a corresponding formula already mentioned in Newey and Powell (1987) as Theorem 1 (iv), where a very similar formula is stated for continuously differentiable distribution functions.

Theorem 3 *Let $e_X(\alpha)$ be the expectile function of a random variable $X \in L^1$. Then the distribution function F_X of X is given by*

$$F_X(t) = \frac{1 - \Omega_X(t) + \Omega_X'^{+}(t) \cdot (t - e_X(1/2))}{(1 - \Omega_X(t))^2}, \quad t \neq e_X(1/2),$$

where

$$\Omega_X(t) = \frac{1 - e_X^{-1}(t)}{e_X^{-1}(t)}.$$

3 Expectile Order

Let us recall some basic definitions and results from the theory of stochastic orders. For a comprehensive review we refer to Müller and Stoyan (2002) or Shaked and Shantikumar (2007). In the following, in inequalities between expectations it is always tacitly assumed that the expectations exist.

Definition 4 For given random variables X, Y we define the order relations

$$\begin{aligned}
X \leq_{st} Y \quad &, \text{ if } Ef(X) \leq Ef(Y) \text{ for all increasing } f.\\
X \leq_{cx} Y \quad &, \text{ if } Ef(X) \leq Ef(Y) \text{ for all convex } f.\\
X \leq_{cv} Y \quad &, \text{ if } Ef(X) \leq Ef(Y) \text{ for all concave } f.\\
X \leq_{icx} Y \quad &, \text{ if } Ef(X) \leq Ef(Y) \text{ for all increasing convex } f.\\
X \leq_{icv} Y \quad &, \text{ if } Ef(X) \leq Ef(Y) \text{ for all increasing concave } f.
\end{aligned}$$

It is well known that the usual stochastic order $\leq_{st}$ is equivalent to the pointwise ordering of the quantiles. Bellini (2012) has shown the following results for expectiles:

Theorem 5 a) $X \leq_{st} Y \Rightarrow e_X(\alpha) \leq e_Y(\alpha)$, *for each* $\alpha \in (0, 1)$,
b) $X \leq_{cv} Y \Rightarrow e_X(\alpha) \leq e_Y(\alpha)$, *for each* $\alpha \in (0, 1/2]$,
c) $X \leq_{cx} Y \Rightarrow e_X(\alpha) \leq e_Y(\alpha)$, *for each* $\alpha \in [1/2, 1)$.

Theorem 5 shows that the usual stochastic order $\leq_{st}$ implies ordering of all expectiles. It is then very natural to introduce the main definition of the paper:

Definition 6 Two random variables $X, Y \in L^1$ are ordered in expectile order (written $X \leq_e Y$) if $e_X(\alpha) \leq e_Y(\alpha)$ for all $\alpha \in (0, 1)$.

Some immediate properties of the expectile order are the following:

Theorem 7 *Let $X, Y \in L^1$.*

a) $X \leq_e Y \Rightarrow X + h \leq_e Y + h$, *for each* $h \in \mathbb{R}$,
b) $X \leq_e Y \Rightarrow \lambda X \leq_e \lambda Y$, *for each* $\lambda > 0$,

c) $X \leq_{st} Y \Rightarrow X \leq_e Y$,
d) *If* $X_n \leq_e Y_n$, $X_n \to X$ *and* $Y_n \to Y$ *weakly, with* $E|X_n| \to E|X|$ *and* $E|Y_n| \to E|Y|$, *then* $X \leq_e Y$.

Proof a), b) and c) are immediate. d) follows from Theorem 10 in Bellini et al. (2014). □

The expectile order is equivalent to the pointwise ordering of Omega ratios, and can also be characterized by means of the stop-loss transform.

Theorem 8 *Let* $X, Y \in L^1$. *Let* $m = \max\{\text{ess}\inf(X), \text{ess}\inf(Y)\}$ *and* $M = \min\{\text{ess}\sup(X), \text{ess}\sup(Y)\}$. *The following are equivalent:*

a) $X \leq_e Y$,
b) $\Omega_X(x) \leq \Omega_Y(x)$, *for each* $x \in (m, M)$,
c) $\pi_X(x)(x - EY) \leq \pi_Y(x)(x - EX)$, *for each* $x \in (m, M)$.

Proof Let $\beta = (1-\alpha)/\alpha$. From Eq. 5, the condition $e_X(\alpha) \leq e_Y(\alpha)$ is equivalent to

$$\Omega_X(x) = \beta, \Omega_Y(y) = \beta \Rightarrow x \leq y. \tag{8}$$

Since Ω_X and Ω_Y are strictly decreasing and since Eq. 8 holds for each $\beta \in (0, +\infty)$, it follows that $\Omega_X(x) \leq \Omega_Y(x)$ for all $x \in \mathbb{R}$.

Item c) follows from Eqs. 4 and 7. □

Notice that in Remillard (2013), Proposition 4.4.3. p. 130, it is claimed without proof that $\Omega_X(x) \leq \Omega_Y(x)$, for each $x \in \mathbb{R}$ if and only if $X \leq_{st} Y$. The statement is wrong, as we will see in Theorem 12 in the equal mean case, and a counterexample is provided by Example 16. Indeed, it will turn out that $\leq_e$ is strictly weaker than $\leq_{st}$ and more similar to the third order stochastic dominance.

For $X \leq_e Y$, we clearly need $EX \leq EY$. Note that, in this case, the conditions b), c) in Theorem 8 have only to be checked for $x \in (m, EX)$ and $x \in (EY, M)$ since they are obviously satisfied for $EX \leq x \leq EY$.

From Theorem 2 we immediately get the following necessary conditions as a corollary.

Corollary 9 *If* $X \leq_e Y$ *then* $\text{ess}\inf(X) \leq \text{ess}\inf(Y)$ *and* $\text{ess}\sup(X) \leq \text{ess}\sup(Y)$.

In the case of unbounded X and Y we can derive further necessary conditions for $X \leq_e Y$ in terms of the tail behavior of X and Y.

Theorem 10 *Let* $X, Y \in L^1$ *and assume that* $\text{ess}\inf(X) = \text{ess}\inf(Y) = -\infty$ *and* $\text{ess}\sup(X) = \text{ess}\sup(Y) = \infty$. *If* $X \leq_e Y$ *then*

$$\limsup_{t\to\infty} \frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \geq 1 \quad \text{and} \quad \limsup_{t\to-\infty} \frac{F_X(t)}{F_Y(t)} \geq 1.$$

Proof We will show that

$$\limsup_{t\to\infty} \frac{\Omega_Y(t)}{\Omega_X(t)} \leq \limsup_{t\to\infty} \frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \quad \text{and} \quad \limsup_{t\to-\infty} \frac{\Omega_Y(t)}{\Omega_X(t)} \leq \limsup_{t\to-\infty} \frac{F_X(t)}{F_Y(t)}. \tag{9}$$

From this the assertion follows immediately, as $X \leq_e Y$ holds if and only if $\Omega_X(t) \leq \Omega_Y(t)$ for all $t \in \mathbb{R}$. Assume that $\beta := \limsup_{t\to\infty} \bar{F}_Y(t)/\bar{F}_X(t) < \infty$. Then for any $\varepsilon > 0$ there is some $t_0 < \infty$ such that

$$\bar{F}_Y(t) \leq (\beta + \varepsilon)\bar{F}_X(t) \quad \text{for all } t > t_0.$$

This implies

$$E(Y-t)_+ = \int_t^\infty \bar{F}_Y(z)dz \leq (\beta+\varepsilon)\int_t^\infty \bar{F}_X(z)dz = (\beta+\varepsilon)E(X-t)_+.$$

As

$$\lim_{t\to\infty} \frac{E(Y-t)_-}{E(X-t)_-} = \lim_{t\to\infty} \frac{EY-t}{EX-t} = 1$$

this implies

$$\limsup_{t\to\infty} \frac{\Omega_Y(t)}{\Omega_X(t)} = \limsup_{t\to\infty} \frac{E(Y-t)_+}{E(X-t)_+} \leq \beta + \varepsilon.$$

Thus the first assertion of Eq. 9 holds. The proof of the second assertion is similar. □

Example 11 Assume that X and Y are normally distributed with $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$. If $\sigma_1 \neq \sigma_2$, then it follows from Theorem 10 that X and Y cannot be compared with respect to $\leq_e$, as e.g. $\sigma_1 > \sigma_2$ implies

$$\lim_{t\to\infty} \frac{\bar{F}_Y(t)}{\bar{F}_X(t)} = 0 \quad \text{and} \quad \lim_{t\to-\infty} \frac{F_X(t)}{F_Y(t)} = \infty.$$

Hence it is necessary that $\sigma_1 = \sigma_2$. In this case $X \leq_e Y$ holds if $\mu_1 \leq \mu_2$, as in this case also $X \leq_{st} Y$ holds. Thus for normal distributions the orderings $X \leq_e Y$ and $X \leq_{st} Y$ are equivalent.

3.1 The Case of Equal Means

In the equal mean case the expectile order can be easily characterized. Indeed, Theorem 8 shows that $X \leq_e Y$ if and only if the pertaining stop-loss transforms cross only once, namely in μ.

Theorem 12 *Let $X, Y \in L^1$ with $EX = EY = \mu$. Then $X \leq_e Y$ if and only if $\pi_X(x) \geq \pi_Y(x)$, for each $x \in (m, \mu)$ and $\pi_X(x) \leq \pi_Y(x)$, for each $x \in (\mu, M)$. In particular, it is necessary that $E(X-EX)_- = E(Y-EY)_-$ and $E(X-EX)_+ = E(Y-EY)_+$.*

Thus, in the equal mean case, a necessary and sufficient condition for the expectile order is the concave ordering of the left deviation from the mean and the convex ordering of the right deviation from the mean.

Corollary 13 *Let $X, Y \in L^1$ with $EX = EY = \mu$. Then $X \leq_e Y$ if and only if $(X-\mu)_- \leq_{cv} (Y-\mu)_-$ and $(X-\mu)_+ \leq_{cx} (Y-\mu)_+$.*

These conditions are related to the notion of *third degree stochastic dominance (TSD)* introduced by Whitmore (1970) as follows: for random variables X and Y it holds $X \leq_{TSD} Y$ if $EX \leq EY$ and for all y,

$$\int_{-\infty}^{y}\int_{-\infty}^{z} F_X(t)\,dt\,dz \leq \int_{-\infty}^{y}\int_{-\infty}^{z} F_Y(t)\,dt\,dz.$$

Indeed, we get immediately the following result.

Theorem 14 *If $EX = EY$, $Var(X) \leq Var(Y)$ and $X \leq_e Y$, then $X \leq_{TSD} Y$.*

Proof It follows from Theorem 12 that

$$y \mapsto h(y) := \int_{-\infty}^{y} \int_{-\infty}^{z} F_Y(t)\, dt\, dz - \int_{-\infty}^{y} \int_{-\infty}^{z} F_x(t)\, dt\, dz$$

is increasing on $(-\infty, \mu]$ and decreasing on $[\mu, \infty)$ with

$$\lim_{y \to -\infty} h(y) = 0 \quad \text{and} \quad \lim_{y \to \infty} h(y) = Var(Y) - Var(X),$$

thus the result follows.

We now show by means of an example that the expectile order is not closed with respect to mixtures, in the sense that $F_1 \leq_e F_2$ but

$$\frac{1}{2} F_1 + \frac{1}{2} G \not\leq_e \frac{1}{2} F_2 + \frac{1}{2} G.$$

Example 15 **($\leq_e$ is not closed under mixtures)**
Let F_1 and F_2 be the distribution functions of X_1 and X_2 with

$$P(X_1 = 0) = P(X_1 = 4) = \frac{1}{2}, \quad P(X_2 = 1/2) = \frac{2}{3}, \; P(X_2 = 5) = \frac{1}{3}.$$

From Theorem 12 it follows that $X_1 \leq_e X_2$. Letting $G = \delta_1$, and denoting the mixtures with Y_1 and Y_2, we get

$$P(Y_1 = 0) = P(Y_1 = 4) = \tfrac{1}{4}, \; P(Y_1 = 1) = \tfrac{1}{2}$$
$$P(Y_2 = 1/2) = \tfrac{1}{3}, \; P(Y_2 = 5) = \tfrac{1}{6}, \; P(Y_2 = 1) = \tfrac{1}{2}.$$

Then $EY_1 = EY_2 = 3/2$ and

$$E(Y_1 - EY_1)_+ = 5/8 > E(Y_2 - EY_2)_+ = 7/12.$$

Therefore it follows from Theorem 12 that Y_1 and Y_2 can not be ordered with respect to expectile ordering.

Müller (1997) has shown that so called *integral stochastic orders* (also known as general stochastic dominance rules) generated by a class of functions $\mathcal{F}$ in the form

$$X \leq_e Y \text{ if and only if } Eu(X) \leq Eu(Y) \text{ for all } u \in \mathcal{F}$$

are always closed with respect to mixtures, thus the preceding example shows that there can not exist a class of utilities $\mathcal{F}$ generating expectile order.

Notice also that in Example 15 we have $X_1 \leq_e X_2$ but not even $X_1 \leq_{icx} X_2$ does hold. Thus $\leq_e$ is strictly weaker than $\leq_{st}$ and neither implies nor it is implied by $\leq_{icx}$. The following example shows that the expectile order is not closed with respect to independent sums.

Example 16 **($\leq_e$ is not closed under convolutions)**
Assume that X has a two-point distribution

$$P(X = 0) = 0.4, \quad P(X = 1) = 0.6,$$

and Y has a three-point distribution

$$P(Y = 0) = 0.4, \quad P(Y = 0.8) = 0.5, \quad P(Y = 2) = 0.1.$$

Then, $EX = EY = 0.6$, and Theorem 12 yields $X \leq_e Y$. Now, let Z be independent of X and Y, having a two-point distribution in 0 and 1 with equal weights. For $\tilde{X} = X + Z$ and $\tilde{Y} = Y + Z$, we get

$$P(\tilde{X} = 0) = 0.2, \quad P(\tilde{X} = 1) = 0.5, \quad P(\tilde{X} = 2) = 0.3,$$
$$P(\tilde{Y} = 0) = P(\tilde{Y} = 1) = 0.2, \quad P(\tilde{Y} = 0.8) = P(\tilde{Y} = 1.8) = 0.25,$$
$$P(\tilde{Y} = 2) = P(\tilde{Y} = 3) = 0.05.$$

Then $E\tilde{X} = E\tilde{Y} = 1.1$ and

$$E(\tilde{X} - E\tilde{X})_+ = 0.27 < E(\tilde{Y} - E\tilde{Y})_+ = 0.315.$$

Hence, by Theorem 12,

$$X + Z \not\leq_e Y + Z.$$

Finally, the following example shows that it is not sufficient to check the inequality for the Omega ratios or the stop-loss transforms in the points of support of discrete distributions in order to characterize expectile order.

Example 17 Assume that X and Y have three-point distributions

$$P(X = 0) = 0.4, \quad P(X = 1) = 0.5, \quad P(X = 2) = 0.1,$$
$$P(Y = 0) = 0.5, \quad P(Y = 1) = 0.3, \quad P(Y = 2) = 0.2.$$

Here, $EX = EY = \mu = 7/10$, and

$$\pi_X(t) = \begin{cases} (7 - 6t)/10, & 0 \leq t \leq 1, \\ (2 - t)/10, & 1 \leq t \leq 2, \end{cases} \qquad \pi_Y(t) = \begin{cases} (7 - 5t)/10, & 0 \leq t \leq 1, \\ (2 - t)/5, & 1 \leq t \leq 2. \end{cases}$$

Hence, $\pi_X(0) \geq \pi_Y(0)$, and $\pi_X(k) \leq \pi_Y(k)$ for $k = 1, 2$, but since $\pi_X(\mu) \neq \pi_Y(\mu)$, Theorem 12 shows that $X \not\leq_e Y$ (indeed, the expectile curves cross in $\alpha = 1/2$).

However, to prove or disprove expectile order between discrete distributions it nevertheless suffices to check a finite number of inequalities together with some elementary calculations: due to Theorem 8, one has to check the inequality

$$\pi_X(t)(t - EY) - \pi_Y(t)(t - EX) \leq 0.$$

Since the stop-loss transform is piecewise linear for discrete distributions, the left hand side of the inequality is a piecewise quadratic function in t. Hence, one simply has to check if a parabola is nonpositive between the pertaining support points.

3.2 Sufficient Conditions for the Expectile Order

In this subsection we derive sufficient conditions for expectile ordering in terms of crossing conditions for survival functions. From Theorem 12 we can derive the following Lemma for the case of equal means.

Lemma 18 *Assume that* $EX = EY = \mu$, $E(X - \mu)_+ = E(Y - \mu)_+$ *and that there exist* $z_1 < \mu < z_2$ *such that*

$$\bar{F}_X(z) \leq \bar{F}_Y(z) \quad \text{if } z < z_1 \text{ or } z > z_2$$

whereas

$$\bar{F}_X(z) \geq \bar{F}_Y(z) \quad \text{if } z_1 < z < z_2.$$

Then $X \leq_e Y$.

Proof First notice that $\lim_{t\to-\infty}(\pi_X(t) + t) = \lim_{t\to-\infty} E(\max\{X, t\}) = EX$. Under the stated assumption if follows

$$\lim_{t\to-\infty}(\pi_X(t) - \pi_Y(t)) = EX - EY = 0$$

and therefore the function

$$t \mapsto \pi_X(t) - \pi_Y(t) = \int_t^\infty (\bar{F}_X(z) - \bar{F}_Y(z))dz$$

is increasing on $(-\infty, z_1)$ and on (z_2, ∞) and decreasing on (z_1, z_2) with limit

$$\lim_{t\to-\infty}(\pi_X(t) - \pi_Y(t)) = \lim_{t\to\infty}(\pi_X(t) - \pi_Y(t)) = 0$$

and $\pi_X(\mu) - \pi_Y(\mu) = 0$. Thus the function in non-negative on $(-\infty, \mu)$ and non-positive on (μ, ∞). The assertion therefore follows from Theorem 12.

The preceding Lemma can be generalized also to the case of unequal means:

Theorem 19 *Assume that there exist* $z_1 < EX < z_2$ *such that*

$$\bar{F}_X(z) \leq \bar{F}_Y(z) \quad \text{if } z < z_1 \text{ or } z > z_2$$

whereas

$$\bar{F}_X(z) \geq \bar{F}_Y(z) \quad \text{if } z_1 < z < z_2.$$

We define the following areas between the survival functions:

$$A := \textstyle\int_{-\infty}^{z_1} |\bar{F}_Y(z) - \bar{F}_X(z)|dz, \;\; B := \displaystyle\int_{z_1}^{EX} |\bar{F}_Y(z) - \bar{F}_X(z)|dz,$$

$$C := \textstyle\int_{EX}^{z_2} |\bar{F}_Y(z) - \bar{F}_X(z)|dz, \;\; D := \displaystyle\int_{z_2}^{\infty} |\bar{F}_Y(z) - \bar{F}_X(z)|dz.$$

If $A \geq B$ *and* $C \leq D$ *then* $X \leq_e Y$.

Proof Notice that the conditions of Lemma 18 hold if and only if $A = B$ and $C = D$. If $A > B$ we can define for each $\alpha \in (\bar{F}_X(z_1), 1)$ and $z < z_1$ a function $\bar{F}_{Y_\alpha}$ with

$$\bar{F}_{Y_\alpha}(z) = \begin{cases} \bar{F}_X(z), & \text{if } \bar{F}_X(z) > \alpha, \\ \alpha, & \text{if } \bar{F}_X(z) \leq \alpha < \bar{F}_Y(z), \\ \bar{F}_Y(z), & \text{if } \bar{F}_Y(z) \leq \alpha. \end{cases}$$

Obviously, for all $z < z_1$ we have $\bar{F}_{Y_\alpha}(z) \leq \bar{F}_Y(z)$ and this function is increasing in α. The function

$$\alpha \mapsto g(\alpha) := \int_{-\infty}^{z_1} |\bar{F}_{Y_\alpha}(z) - \bar{F}_Y(z)|dz, \quad \alpha \in (\bar{F}_X(z_1), 1)$$

is continuous and increasing from 0 to A. Thus there is an α^* with $g(\alpha^*) = B$.

If $C < D$ we can define $z_3 > z_2$ such that

$$\int_{EX}^{z_2} |\bar{F}_Y(z) - \bar{F}_X(z)|dz = \int_{z_2}^{z_3} |\bar{F}_Y(z) - \bar{F}_X(z)|dz = C.$$

We can now define a random variable Y^* with survival function

$$\bar{F}_{Y^*}(z) = \begin{cases} \bar{F}_{Y_{\alpha^*}}(z), & \text{if } z < z_1, \\ \bar{F}_Y(z), & \text{if } z_1 \le z < z_3, \\ \bar{F}_X(z), & \text{if } z \ge z_3. \end{cases}$$

Then $\bar{F}_{Y^*}(z) \le \bar{F}_Y(z)$ for all $z \in \mathbb{R}$. Therefore $Y^* \le_{st} Y$ and thus according to Theorem 7 c) also $Y^* \le_e Y$. On the other hand, X and Y^* fulfill the conditions of Lemma 18 and therefore it holds $X \le_e Y^*$. By transitivity we get $X \le_e Y$. □

Example 20 As an illustrative example, we consider random variables with skew-normal distributions. To this end, let $Z \sim SN(\alpha)$ be a standard skew-normal random variable as defined in Azzalini (1985) with density $f_Z(z) = 2\varphi(z)\Phi(\alpha z), z \in \mathbb{R}$, for any $\alpha \in \mathbb{R}$, where $\varphi(\cdot)$ and $\Phi(\cdot)$ denote the density and the cumulative distribution function of a standard normal random variable, respectively. Adding scale and location parameters, $X = \xi + \omega Z$ is then said to have a skew-normal distribution with parameters ξ, ω, α (shortly, $X \sim SN(\xi, \omega, \alpha)$). Mean, variance and skewness are given by

$$EX = \xi + \omega\delta\sqrt{2/\pi}, \quad Var(X) = \omega^2\left(1 - 2\delta^2/\pi\right),$$

$$skew(X) = \gamma_1(X) = \frac{4-\pi}{2}\frac{\left(\delta\sqrt{2/\pi}\right)^3}{\left(1 - 2\delta^2/\pi\right)^{3/2}},$$

where $\delta = \alpha(1+\alpha^2)^{-1/2} \in (-1, 1)$. The stop-loss transform of Z is given by

$$\begin{aligned}\pi_Z(t) &= \int_t^\infty (x-t)2\varphi(x)\Phi(\alpha x)dx \\ &= -2\int_t^\infty \varphi'(x)\Phi(\alpha x)dx - t\bar{F}_Z(t) \\ &= 2\varphi(t)\Phi(\alpha t) + 2\alpha\int_t^\infty \varphi(x)\varphi(\alpha x)dx - t\bar{F}_Z(t) \\ &= f_Z(t) - t\bar{F}_Z(t) + \sqrt{2/\pi}\,\delta\left(1 - \Phi\left(t\sqrt{1+\alpha^2}\right)\right),\end{aligned}$$

where $\bar{F}_Z(\cdot)$ denotes the survival function of Z. The stop-loss transform of X is then given by $\pi_X(t) = \omega \cdot \pi_Z((t-\xi)/\omega)$.

Now, let $X_i \sim SN(\xi_i, \omega, \alpha_i)$, $i = 1, 2$. Suppose $\xi_1 \le \xi_2$ and $\alpha_1 \le \alpha_2$, then $X_1 \le_{st} X_2$ (see Corollary 4.2 in Blasi and Scarlatti (2012), where also sufficient conditions for $\le_{icv}$ are given).

As examples for skew-normal random variables ordered with respect to expectile order, but not with respect to the usual stochastic order, let $X \sim SN(\xi_1, \omega_1, \alpha_1)$ and $Y \sim SN(\xi_2, \omega_2, \alpha_2)$. Put $\delta_1 = 0.9$ and $\delta_2 = 0.99$ (i.e. $\alpha_1 = 2.065 < \alpha_2 = 7.018$). Then choose the other parameters in such a way that

$$EX = EY = 0, \quad Var(X) = 1, \quad \pi_X(0) = \pi_Y(0),$$

which yields $\xi_1 = -1.032$, $\xi_2 = -1.279$, $\omega_1 = 1.437$, $\omega_2 = 1.620$. Further, $V(Y) = 0.986$, $\gamma_1(X) = 0.472$, $\gamma_1(Y) = 0.917$. A plot of $\bar{F}_X(t) - \bar{F}_Y(t)$ is given in the left part of Fig. 1, which shows that the crossing conditions of Lemma 18 are satisfied. Hence, $X \le_e Y$.

Next, consider $\tilde{Y} \sim SN(\tilde{\xi}_2, \omega_2, \alpha_2)$ with $\tilde{\xi}_2 = \xi_2 + 0.05$. Hence, $E\tilde{Y} = 0.05 > EX$, whereas variances and skewness remain the same. The right part of Fig. 1 shows a plot of $\bar{F}_X(t) - \bar{F}_{\tilde{Y}}(t)$ together with the areas A, B, C, D defined in Theorem 19, indicating that

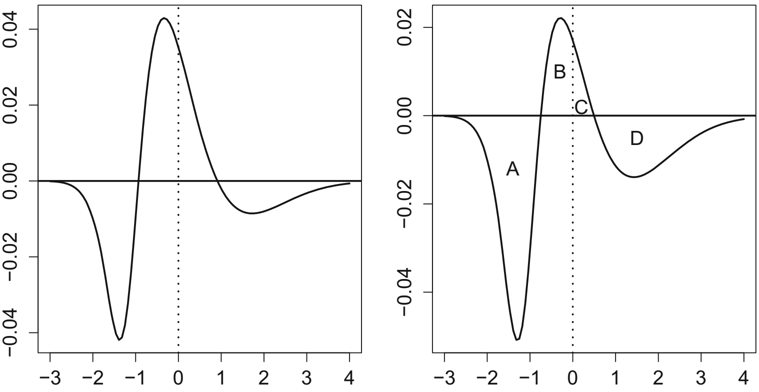

Fig. 1 Plot of $\bar{F}_X(t) - \bar{F}_Y(t)$ (*left*) and of $\bar{F}_X(t) - \bar{F}_{\tilde{Y}}(t)$ (*right*)

all conditions of the theorem are fulfilled. The requirement $A \geq B, C \leq D$ can easily be confirmed by numerical quadrature. Therefore, we obtain $X \leq_e \tilde{Y}$.

4 Stochastic Orderings for the Lomax Distribution

In this section, we consider in some detail the Lomax or Pareto type II distribution having density and distribution function

$$f(t) = f(t;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(1+\frac{t}{\lambda}\right)^{-(\alpha+1)}, \quad t \geq 0,$$
$$F(t) = F(t;\alpha,\lambda) = 1-\left(1+\frac{t}{\lambda}\right)^{-\alpha}, \quad t \geq 0,$$

where α and λ are positive parameters. Accordingly, hazard rate and stop-loss transform are given by

$$r(t) = f(t)/\left(1-F(t)\right) = \frac{\alpha}{\lambda+t}, \quad t \geq 0,$$
$$\pi(t) = \frac{\lambda}{\alpha-1}\left(1+\frac{t}{\lambda}\right)^{1-\alpha}, \quad t \geq 0.$$

In the following, assume $X \sim F(t;\alpha_1,\lambda_1)$ and $Y \sim F(t;\alpha_2,\lambda_2)$

4.1 Hazard Rate Order and Usual Stochastic Order

The random variable X is smaller than the random variable Y with respect to the hazard rate order (written $X \leq_{hr} Y$) if $r_X(t) \geq r_X(t)$ for all real t. For the Lomax distribution, this is the case if and only if

$$(\alpha_1-\alpha_2)t + \alpha_1\lambda_2 - \alpha_2\lambda_1 \geq 0, \quad t \geq 0,$$

hence, if and only if

$$\alpha_1 \geq \alpha_2 \tag{10}$$

and

$$\frac{\alpha_1}{\alpha_2} \geq \frac{\lambda_1}{\lambda_2}. \tag{11}$$

Since hazard rate order implies the usual stochastic order, Eqs. 10 and 11 are sufficient conditions for $X \leq_{st} Y$. However, they are also necessary: in order to have

$$F(t; \alpha_1, \lambda_1) \geq F(t; \alpha_2, \lambda_2) \tag{12}$$

for $t \to \infty$, condition (10) must hold. On the other hand, Eq. 12 is equivalent to

$$\alpha_1 \log\left(1 + \frac{t}{\lambda_1}\right) \geq \alpha_2 \log\left(1 + \frac{t}{\lambda_2}\right),$$

and a first order Taylor expansion around $t = 0$ shows that Eq. 11 is also necessary.

4.2 Increasing Convex Order

Here, we have to assume $\alpha_1, \alpha_2 > 1$. For $X \leq_{icx} Y$, we have to show that $\pi_X(t) \leq \pi_Y(t)$ for $t \geq 0$. Looking at the behaviour of the stop-loss transform for $t \to \infty$, we see that Eq. 10 is necessary for $X \leq_{icx} Y$ as well. A second necessary condition is $EX = \pi_X(0) \leq \pi_Y(0) = EY$, or

$$\frac{\alpha_1 - 1}{\alpha_2 - 1} \geq \frac{\lambda_1}{\lambda_2}. \tag{13}$$

Now, assume that Eqs. 10 and 13 hold. Then, we get from the above results about the usual stochastic order that

$$\bar{F}(t; \alpha_1 - 1, \lambda_1) \leq \bar{F}(t; \alpha_2 - 1, \lambda_2), \quad t \geq 0.$$

Noting that

$$\pi(t) = \frac{\lambda}{\alpha - 1} \cdot \bar{F}(t; \alpha - 1, \lambda),$$

where $\bar{F}(t) = 1 - F(t)$ denotes the survival function of F, we obtain

$$\pi_X(t) \leq \pi_Y(t), \quad t \geq 0. \tag{14}$$

Hence, conditions (10) and (13) are also sufficient for $X \leq_{icx} Y$.

4.3 Expectile Order

Again, let $\alpha_1, \alpha_2 > 1$. Using Theorem 10, we see that condition (10) is necessary for $X \leq_e Y$. Further, $EX \leq EY$, i.e. condition (13), is necessary as well. Under these conditions, $X \leq_{icx} Y$, and Eq. 14 yields $\pi_X(t) \leq \pi_Y(t)$ for $t > EY$. Therefore, the condition in Theorem 8 c) is fulfilled for $t > EY$. Hence, for $X \leq_e Y$, we additionally need

$$G(t) = \pi_X(t)(EY - t) \geq \pi_Y(t)(EX - t) = H(t), \quad 0 \leq t < EX. \tag{15}$$

Since $\pi'(t) = -\bar{F}(t)$ and $\bar{F}'(t; \alpha, \lambda) = -\frac{\alpha}{\lambda} \bar{F}(t; \alpha + 1, \lambda)$, we obtain

$$G'(t) = -\bar{F}_X(t)(EY - t) - \pi_X(t)$$

and

$$G''(t) = \frac{\alpha_1}{\lambda_1} \bar{F}(t; \alpha_1 + 1, \lambda_1)(EY - t) + 2\bar{F}_X(t), \tag{16}$$

and corresponding expressions for H' and H''. Obviously, $G(0) = H(0)$ and $G'(0) = H'(0)$. Therefore, Eq. 15 can only be satisfied if

$$G''(0) = \frac{\alpha_1}{\lambda_1} \frac{\lambda_2}{\alpha_2 - 1} + 2 \geq \frac{\alpha_2}{\lambda_2} \frac{\lambda_1}{\alpha_1 - 1} + 2 = H''(0),$$

or, equivalently, if

$$\sqrt{\frac{\alpha_1(\alpha_1 - 1)}{\alpha_2(\alpha_2 - 1)}} \geq \frac{\lambda_1}{\lambda_2}. \tag{17}$$

Note that, since the function $h(x) = \frac{\alpha_1 - x}{\alpha_2 - x}$ with $x < \alpha_2$ is increasing in x for $\alpha_1 \geq \alpha_2 > 1$, condition (17) is weaker than Eq. 11, but stronger than Eq. 13. Summing up, Eqs. 10 and 17 are necessary conditions for $X \leq_e Y$.

Remark 21 Since $h(x)$ is strictly increasing in x for $\alpha_1 > \alpha_2$, Eq. 17 can never be satisfied if X and Y have equal means, i.e. if $(\alpha_1 - 1)/(\alpha_2 - 1) = \lambda_1/\lambda_2$ (except in the trivial case $\alpha_1 = \alpha_2, \lambda_1 = \lambda_2$). Hence, if X and Y are random variables from different Lomax distributions, but with equal means, they can never be ordered in expectile order.

In the following, our aim is to derive sufficient conditions for $X \leq_e Y$ which are weaker than the conditions for $X \leq_{st} Y$. For this, we can assume that

$$\frac{\alpha_1}{\alpha_2} < \frac{\lambda_1}{\lambda_2} \tag{18}$$

(otherwise, $X \leq_{st} Y$, which implies $X \leq_e Y$). Assuming Eq. 18, we get

$$\lambda_1 > \lambda_2 \tag{19}$$

(since $\alpha_1 \geq \alpha_2$). Now, Eqs. 17 and 18 imply

$$\frac{\alpha_1}{\lambda_1}(EY - t) = \frac{\alpha_1}{\lambda_1} \frac{\lambda_2}{(\alpha_2 - 1)} - \frac{\alpha_1}{\lambda_1} t > \frac{\alpha_2}{\lambda_2} \frac{\lambda_1}{(\alpha_1 - 1)} - \frac{\alpha_2}{\lambda_2} t = \frac{\alpha_2}{\lambda_2}(EX - t) \tag{20}$$

for $t > 0$. Further, we have the following theorem:

Theorem 22 *Under* Eqs. 10 *and* 18*,*

$$\bar{F}_X(t) = \left(1 + \frac{t}{\lambda_1}\right)^{-\alpha_1} \geq \left(1 + \frac{t}{\lambda_2}\right)^{-\alpha_2} = \bar{F}_Y(t) \tag{21}$$

for $0 \leq t \leq t_1$*, where* $t_1 = 2(\alpha_2\lambda_1 - \alpha_1\lambda_2)/(\alpha_1 - \alpha_2)$ *if* $\alpha_1 > \alpha_2$*, and* $t_1 = \infty$ *if* $\alpha_1 = \alpha_2$.

A proof is given in the Appendix. Now, Eq. 21 together with Eq. 19 imply

$$\begin{aligned} \bar{F}(t; \alpha_1 + 1, \lambda_1) \qquad &= \left(1 + \tfrac{t}{\lambda_1}\right)^{-1} \bar{F}_X(t) \\ &\geq \left(1 + \tfrac{t}{\lambda_2}\right)^{-1} \bar{F}_Y(t) = \bar{F}(t; \alpha_2 + 1, \lambda_2) \end{aligned} \tag{22}$$

for $0 \leq t \leq t_1$. With an eye to Eq. 16, we see that Eqs. 20, 21 and 22 imply $G''(t) \geq H''(t)$ for $0 \leq t \leq t_1$, which, in turn, implies $G(t) \geq H(t)$ for $0 \leq t \leq t_1$. Hence, we have the following result:

Theorem 23 *If* Eqs. 10, 17 *and* 18 *hold, and if*

$$EX = \frac{\lambda_1}{\alpha_1 - 1} \leq t_1 = \frac{2(\alpha_2\lambda_1 - \alpha_1\lambda_2)}{\alpha_1 - \alpha_2},$$

then $X \leq_e Y$, but X and Y are not ordered with respect to the usual stochastic order.

As a typical example, take $\alpha_1 = 3, \lambda_1 = \sqrt{3}, \alpha_2 = 2, \lambda_2 = 1$. Then, $EX = \sqrt{3}/2 < EY = 1$ and

$$\frac{\alpha_1}{\alpha_2} = \frac{3}{2} < \sqrt{\frac{\alpha_1(\alpha_1 - 1)}{\alpha_2(\alpha_2 - 1)}} = \sqrt{3} < \frac{\alpha_1 - 1}{\alpha_2 - 1} = 2, \quad \frac{\lambda_1}{\lambda_2} = \sqrt{3}.$$

Further, $t_1 = 2\sqrt{3}(2 - \sqrt{3}) \approx 0.928 > EX \approx 0.866$. Hence, all conditions in Theorem 23 are fulfilled, and X is smaller than Y with respect to expectile order, but not with respect to stochastic order. Figure 2 shows the functions $G(t)$ and $H(t)$ defined in Eq. 15 and a plot of $\log \bar{F}_X(t) - \log \bar{F}_Y(t)$.

5 Real Data Example

As an illustrative application, we consider data of billion-dollar weather and climate disasters taken from NOAA (2016). Data set 1 are the costs of severe storm disasters to affect the U.S. from 2000 to 9/2016 with CPI-adjusted losses exceeding $1 billion each across the United States. Data sets 2 and 3 are tropical cyclone and flooding disasters exceeding $1 billion for the same period. For reproducibility, we list the values:

$$\begin{aligned} x_1 = {} & (1.1, 1.1, 1.6, 1.8, 3.5, 2.1, 1.2, 1.0, 2.0, 1.2, 1.3, 1.3, 1.6, 1.4, 1.8, 3.8, 1.6, \\ & 1.4, 1.1, 1.0, 1.8, 2.5, 1.4, 2.1, 3.0, 2.7, 2.4, 3.5, 1.2, 1.1, 3.3, 1.2, 1.1, 1.4, \\ & 9.7, 10.9, 1.1, 2.2, 2.4, 3.0, 4.2, 1.0, 3.6, 1.1, 1.5, 1.6, 1.8, 1.9, 1.6, 3.4, 1.1, \\ & 1.2, 1.3, 2.9, 2.9, 1.9, 1.5, 1.3, 1.3, 1.1, 5.4, 2.6, 2.8, 4.2), \\ x_2 = {} & (68.3, 2.9, 2.7, 14.4, 33.6, 6.7, 1.5, 23.4, 22.8, 153.8, 3.1, 9.6, 26.2, 12.5, \\ & 21.1, 7.2, 1.5, 11.6), \\ x_3 = {} & (10.0, 1.0, 1.2, 1.3, 2.0, 2.6, 1.0, 1.5, 1.1, 2.1, 3.2, 2.5, 1.7, 11.2, 1.8). \end{aligned}$$

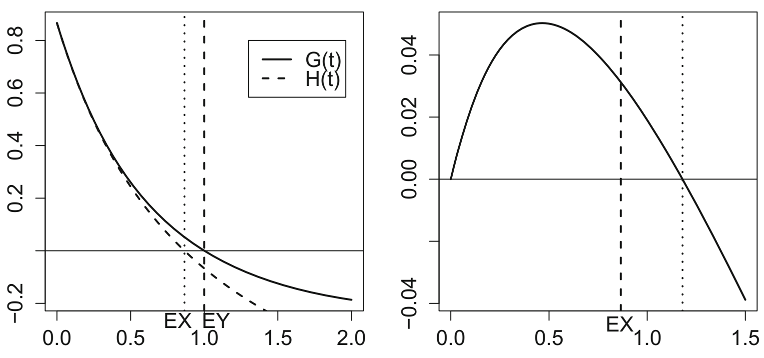

Fig. 2 Plot of the functions $G(t)$ and $H(t)$ defined in Eq. 15 (*left*), and plot of $\log \bar{F}_X(t) - \log \bar{F}_Y(t)$ (*right*)

Table 1 Sample sizes, arithmetic means, standard deviations and estimated parameter values for the three data sets in Section 5

	n	$\bar{x}$	s	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}/\hat{\lambda}$	$(\hat{\alpha}-1)/\hat{\lambda}$
x_1	64	2.25	1.76	3.083	2.649	1.164	0.786
x_2	18	23.49	36.32	1.972	23.660	0.083	0.041
x_3	15	2.95	3.18	1.305	0.964	1.354	0.317

The use of the Lomax distribution is widespread for this type of loss data. For example, Giles et al. (2013) fitted this distribution to data relating to insurance losses in excess of 5 million dollars due to major hurricanes between 1949 and 1980. This is also theoretically justified as the Lomax distribution is a generalized Pareto distribution and therefore a natural model for peak over threshold data, as shown by Pickands (1975).

We fitted the Lomax distribution to each data set using MLE after subtracting 1 (billion) from each of their sample values. Key statistical values and estimated parameter values of the fitted Lomax distributions are given in Table 1. The plots of the empirical cumulative distribution functions (after subtracting 1 from each sample value) and the cdf's of the fitted Lomax distributions in Fig. 3 indicate that the Lomax distribution is a suitable model for all three data sets.

In the following, denote the parameters of the Lomax distribution fitted to the i-th data set by $(\hat{\alpha}_i, \hat{\lambda}_i)$, and let X_i be a random variable with the corresponding distribution, $i = 1, 2, 3$. From Table 1, $\hat{\alpha}_1 > \hat{\alpha}_2 > \hat{\alpha}_3 > 1$.

First, we compare the first with the second fitted Lomax distribution. Since $\hat{\alpha}_1 > \hat{\alpha}_2$ and $\hat{\alpha}_1/\hat{\alpha}_2 > \hat{\lambda}_1/\hat{\lambda}_2$, i.e. Eqs. 10 and 11 hold, and using the results of Section 4.1, we obtain $X_1 \leq_{st} X_2$, which is not surprising in view of Fig. 3.

Next, we compare the first with the third fitted Lomax distribution. Note that $X_1 \not\leq_{st} X_3$ since condition (11) is violated. In contrast, condition (17), which is sufficient for $X_1 \leq_{icx} X_3$ and necessary for $X_1 \leq_e X_3$, is satisfied. Indeed, the plot of the corresponding expectile curves on the left-hand side of Fig. 4 clearly shows that $X_1 \leq_e X_3$. Note, however, that the additional sufficient condition in Theorem 23 ensuring $X_1 \leq_e X_3$ is not fulfilled.

Finally, we compare the second with the third fitted Lomax distribution. The plot on the right-hand side of Fig. 4 which shows the difference of the cdf's of X_2 and X_3 clearly suggests $X_2 \geq_{st} X_3$. In fact, it is also easy to see that the empirical distributions clearly satisfy

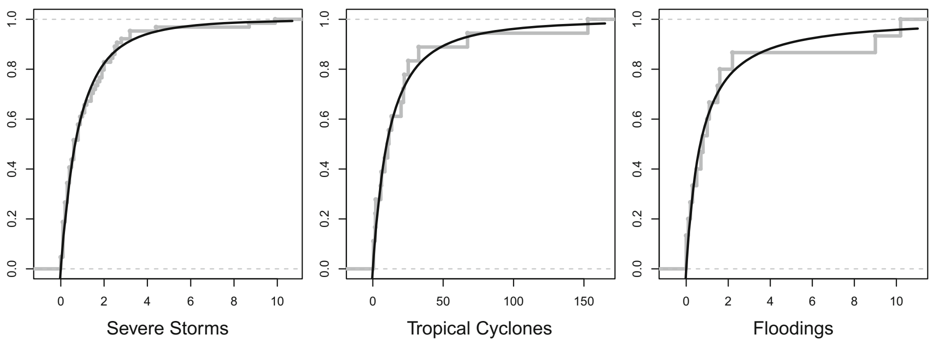

Fig. 3 Empirical cumulative distribution functions (*in grey*) and cdf's of the fitted Lomax distributions (*in black*) for the datasets in Section 5

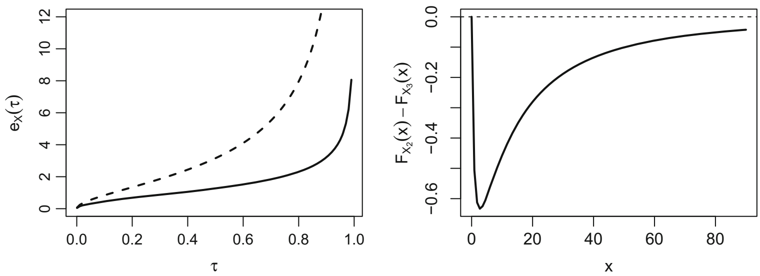

Fig. 4 Left: Expectile curve for the distribution of X_1 (*solid line*) and X_3 (*dotted line*). Right: Difference of the cdf's of X_2 and X_3

the conditions of $\geq_{st}$. However, theoretically, for the estimated Lomax distribution this is impossible since $\alpha_2 > \alpha_3$. On the other hand, condition (13) is not fulfilled, therefore even $X_2 \leq_{icx} X_3$ does not hold. Hence, both distributions are not comparable using the stochastic orders considered in Section 4, as we get a heavier tail for the estimated distribution of the third data set. Note, however, that the two distribution function cross at $x = 12\,437$ billion \$. Hence, for all practical purposes and the considered example, one may well assume X_2 stochastically dominates X_3.

6 Conclusion

In this paper we have investigated properties of the expectile ordering. Necessary as well as sufficient conditions have been derived and it was related to other stochastic orderings. As an interesting example of a family of distributions we considered the Lomax distribution, for which we could not find any results on stochastic ordering properties in the literature. We also fitted Lomax distributions to real world data on the damage of natural weather disaster in the U.S. and compared the different sources of weather risk with respect to various stochastic orderings for that concrete application.

Several natural questions arise from this first study on expectile ordering for future investigations. The most natural one is to ask for the meaning of more spread out expectiles. One gets a natural definition of an expectile dispersion ordering, if one replaces quantiles by expectiles in the definition of dispersive ordering $\leq_{disp}$ (see e.g. Definition 1.7.1 in Müller and Stoyan (2002)). Several different related concepts could be defined and their usefulness and their relation to other orderings could be studied.

Appendix: Proof of Theorem 22

Since the assertion is true for $\alpha_1 = \alpha_2$, we assume $\alpha_1 > \alpha_2$ in the following. Consider the functions

$$\begin{aligned} g(t) &= \log \bar{F}_X(t) = -\alpha_1 \log\left(1 + \frac{t}{\lambda_1}\right), \\ h(t) &= \log \bar{F}_Y(t) = -\alpha_2 \log\left(1 + \frac{t}{\lambda_2}\right), \\ d(t) &= g(t) - h(t), \end{aligned}$$

with $d(0) = 0$. We have to show that

$$d(t) = g(t) - h(t) \geq 0 \quad \text{for} \quad 0 \leq t \leq t_1.$$

Since

$$d'(t) = \frac{\alpha_2}{t + \lambda_2} - \frac{\alpha_1}{t + \lambda_1},$$

we obtain $d'(0) = \alpha_2/\lambda_2 - \alpha_1/\lambda_1 > 0$, and

$$d'(t) = 0 \;\Leftrightarrow\; t = \frac{\alpha_2\lambda_1 - \alpha_1\lambda_2}{\alpha_1 - \alpha_2} =: t_0,$$

where $t_0 > 0$. Further, an application of l'Hôpital's rule yields $\lim_{t\to\infty} g(t)/h(t) = \alpha_1/\alpha_2 > 1$ or

$$\lim_{t\to\infty} \frac{d(t)}{h(t)} = \frac{\alpha_1}{\alpha_2} - 1 \;=\; c > 0,$$

which implies $d(t) < c/2 \cdot h(t)$ for t large enough. Hence, d is positive and increasing until t_0, then decreasing to minus infinity for $t \to \infty$.

Next, we show

$$d'(t_0 - t) - |d'(t_0 + t)| \;\geq\; 0, \quad 0 \leq t \leq t_0, \tag{23}$$

or, equivalently,

$$\frac{\alpha_2}{(\alpha_1 - \alpha_2)\lambda_2 + \gamma - u} + \frac{\alpha_2}{(\alpha_1 - \alpha_2)\lambda_2 + \gamma + u} - \frac{\alpha_1}{(\alpha_1 - \alpha_2)\lambda_1 + \gamma - u} - \frac{\alpha_1}{(\alpha_1 - \alpha_2)\lambda_1 + \gamma + u} \geq 0, \quad 0 \leq u \leq \gamma,$$

where $\gamma = \alpha_2\lambda_1 - \alpha_1\lambda_2 > 0$, and $u = (\alpha_1 - \alpha_2)t$. With $c_i = (\alpha_1 - \alpha_2)\lambda_i + \gamma, i = 1, 2$ this is equivalent to

$$\frac{\alpha_2 c_2}{c_2^2 - u^2} - \frac{\alpha_1 c_1}{c_1^2 - u^2} \geq 0, \quad 0 \leq u \leq \gamma,$$

or

$$c_1 c_2(\alpha_2 c_1 - \alpha_1 c_2) + (\alpha_1 c_1 - \alpha_2 c_2)u^2 \geq 0, \quad 0 \leq u \leq \gamma. \tag{24}$$

Since

$$\begin{aligned} c_1 c_2(\alpha_2 c_1 - \alpha_1 c_2) \;&=\; 0, \\ \alpha_1 c_1 - \alpha_2 c_2 \;&=\; (\alpha_1 - \alpha_2)\,(\alpha_1\lambda_1 - \alpha_2\lambda_2 + \gamma) \;>\; 0, \end{aligned}$$

inequality (24), and hence (23), are true. This means that the increase up to t_0 is steeper than the subsequent decrease, which implies $d(t_0 + t) \geq d(t_0 - t)$ for $0 \leq t \leq t_0$. It follows that $d \geq 0$ for $0 \leq t \leq 2t_0 = t_1$.

References

Artzner P, Delbaen F, Eber J-M, Heath D (1999) Coherent measures of risk. Math Financ 9:203–228

Azzalini A (1985) A class of distribution which includes the normal ones. Scand J Stat 12:171–178

Bäuerle N, Müller A (2006) Stochastic Orders and Risk Measures: Consistency and Bounds. Insurance: Mathematics and Economics 38:132–148

Bellini F (2012) Isotonicity results for generalized quantiles. Statistics and Probability Letters 82:2017–2024

Bellini F, Bignozzi V (2015) On elicitable risk measures. Quant Finan 15:725–733

Bellini F, Klar B, Müller A, Rosazza Gianin E (2014) Generalized quantiles as risk measures. Insurance: Mathematics and Economics 54:41–48

Blasi F, Scarlatti S (2012) From Normal vs Skew-Normal Portfolios: FSD and SSD Rules. Journal of Mathematical Finance 2:90–95

Eilers PHC (2013) Discussion: the beauty of expectiles. Stat Model 13:317–322

Farooq M, Steinwart I (2015) An SVM-like Approach for Expectile Regression. arXiv:1507.03887

Giles DE, Feng H, Godwin RT (2013) On the Bias of the Maximum Likelihood Estimator for the Two-Parameter Lomax Distribution. Communications in Statistics - Theory and Methods 42:1934–1950

Jones MC (1994) Expectiles and M-quantiles are quantiles. Statistics and Probability Letters 20:149–153

Keating C, Shadwick WF (2002) A Universal Performance Measure. The Finance Development Centre, London

Keating C, Shadwick WF (2002) An Introduction to Omega. The Finance Development Centre, London

Kneib T (2013) Beyond mean regression. Stat Model 13:275–303

Koenker R (2013) Discussion: Living beyond our means. Stat Model 13:323–333

Lopez-Cabrera B, Schulz F (2014) Forecasting generalized quantiles of electricity demand: A functional data approach. SFB 649 Discussion Paper, No. 2014-030. Humboldt University, Berlin

Müller A (1996) Ordering of risks: A comparative study via stop-loss transforms. Insurance: Mathematics and Economics 17:215–222

Müller A (1997) Stochastic Orders generated by Integrals: A Unified Study. Adv Appl Probab 29:414–428

Müller A, Stoyan D (2002) Comparison Methods for Stochastic Models and Risks. John Wiley & Sons Ltd., Chichester

Newey K, Powell J (1987) Asymmetric least squares estimation and testing. Econometrica 55:819–847

NOAA (2016) National Centers for Environmental Information (NCEI) U.S. Billion-Dollar Weather and Climate Disasters. https://www.ncdc.noaa.gov/billions/

Pickands J (1975) Statistical inference using extreme order statistics. Ann Stat 3:119–131

Remillard B (2013) Statistical Methods for Financial Engineering, Chapman and Hall/CRC

Schnabel SK, Eilers PHC (2009) An analysis of life expectancy and economic production using expectile frontier zones. Demogr Res 21:109–134

Schulze-Waltrup L, Sobotka F, Kneib T, Kauermann G (2015) Expectile and quantile regression: David and Goliath? Stat Model 15:433–456

Shaked M, Shantikumar JG (2007) Stochastic Orders, Springer Series in Statistics

Whitmore GA (1970) Third-Degree Stochastic Dominance. American Economic Review 60:457–459

Ziegel JF (2016) Coherence and elicitability. Math Financ 26:901–918