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APPROXIMATE INVERSE FOR THE COMMON OFFSET ACQUISITION GEOMETRY IN 2D SEISMIC IMAGING

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ABSTRACT. We explore how the concept of approximate inverse can be used and implemented to recover singularities in the sound speed from common offset measurements in two space dimensions. Numerical experiments demonstrate the performance of the method.

1. INTRODUCTION

In the inverse problem of seismology one seeks subsurface material parameters from measurements of reflected waves on a part of the propagation medium (typically an area on the earth's surface or in the ocean). To this end sources excite waves at certain positions and their reflections are recorded by receiver arrays. From a mathematical point of view we have to deal with a nonlinear parameter identification problem for a version of the elastic wave equation (with damping). This problem is solved by a multi-stage process which starts with determining the wave speed from a simpler model: the acoustic wave equation. By linearization (Born approximation) we are led to the generalized Radon transform (GRT) as a model for linear seismic imaging where the sound speed is averaged over reflection isochrones connecting sources and receivers (microphones) by points of equal travel time.

Let R be the GRT. Beylkin [1, 2] showed that there is a convolution type operator K and dual transform R^\sharp (generalized backprojection operator) such that $\Psi = R^\sharp K R$ is a perturbation of the identity operator. Thus, from the measurement $g = Rf$ we can compute Ψf by applying $R^\sharp K$, i.e., $\Psi f = R^\sharp K g$. Classical GRT migration entails the direct application of K and R^\sharp to the data g and is a state of the art reconstruction procedure in linearized seismology; see, for example, [25, Sec. 3] and [3, Chap. 5].

We advocate an alternative approach to this classical GRT migration scheme which we demonstrate in this paper for the common offset scanning geometry in two space dimensions. In this situation the GRT integrates over ellipses. Let F denote this elliptic Radon transform. Then, our imaging operator is $\Lambda = \Delta F^* \Phi F$ where Δ is the Laplacian, F^* is the formal adjoint in an L^2 -space with a smooth weight, see (9) below, and Φ is a smooth cutoff function such that $F^* \Phi F$ is well defined. We argue that Λ is a pseudodifferential operator of order 1 and hence emphasizes some singularities, see [16], that is, Λf images some of the discontinuities of f .

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The structure of Λ is suited to apply the method of approximate inverse [19, 22]: Let $e_{\mathbf{p}}$ be a smooth approximation to the Dirac distribution concentrated about \mathbf{p} and define $\psi_{\mathbf{p}} := F\Delta e_{\mathbf{p}}$. Then,

$$\langle g, \psi_{\mathbf{p}} \rangle = \langle \Phi Ff, \psi_{\mathbf{p}} \rangle = \langle \Lambda f, e_{\mathbf{p}} \rangle \approx \Lambda f(\mathbf{p}),$$

i.e., the L^2 -inner product of $\psi_{\mathbf{p}}$ with the data yields a stable approximation of the quantity we seek. Note that the adjoint F^* is not needed explicitly and that the evaluation of $\langle g, \psi_{\mathbf{p}} \rangle$ can be performed in parallel for different \mathbf{p} .

We should point out important work incorporating the GRT and microlocal analysis in seismic imaging. In [7], the GRT and a curvelet transform are used to reconstruct, and in [23], microlocal analysis is used in the presence of multipathing. The articles [6, 25] are good introductions with many further references.

In the following sections we motivate the selection of Λ , compute $\psi_{\mathbf{p}}$ semi-analytically for an explicitly given $e_{\mathbf{p}}$, and present numerical experiments to illustrate the performance of the scheme. For the sake of completeness and as a courtesy to the non-expert we derive the GRT from the acoustic wave equation in some detail. Our presentation in the next section is inspired by [4] and [24].

2. THE IMAGING OPERATOR

2.1. From the acoustic wave equation to the generalized Radon transform. Let $u(t; \mathbf{x}, \mathbf{x}_s)$ be the acoustic potential in $\mathbf{x} \in \mathbb{R}^2$ at time $t \geq 0$ satisfying the acoustic wave equation (with constant mass density)

$$(1) \quad \frac{1}{\nu^2} \partial_t^2 u - \Delta_{\mathbf{x}} u = \delta(\mathbf{x} - \mathbf{x}_s) \delta(t)$$

where $\nu = \nu(\mathbf{x})$ is the speed of sound and \mathbf{x}_s is the excitation (source) point. In seismic imaging one wants to recover ν from the backscattered (reflected) field $u(t; \mathbf{x}_r, \mathbf{x}_s)$, $(t; \mathbf{x}_r, \mathbf{x}_s) \in Y := [0, T_{\max}] \times \mathcal{R} \times \mathcal{S}$ where \mathcal{S} and \mathcal{R} are the sets of source and receiver (microphone) positions, respectively, and T_{\max} is the observation period.

Consider the ansatz

$$(2) \quad \frac{1}{\nu^2(\mathbf{x})} = \frac{1 + n(\mathbf{x})}{c^2(\mathbf{x})}$$

with a smooth and a priori known background velocity $c = c(\mathbf{x})$. Now, n is the quantity we seek. In the following we present a formal derivation of a linear integral equation for determining n , see (6) below, where we follow the lines of [4] and [24].

Let \tilde{u} denote the solution of the above wave equation with sound speed c , i.e.,

$$(3) \quad \frac{1}{c^2} \partial_t^2 \tilde{u} - \Delta_{\mathbf{x}} \tilde{u} = \delta(\mathbf{x} - \mathbf{x}_s) \delta(t).$$

We will use \tilde{u} to derive a linear equation for n .

Subtracting (1) from (3) and given (2) we find that

$$\frac{1}{c^2} \partial_t^2 (\tilde{u} - u) - \Delta_{\mathbf{x}} (\tilde{u} - u) = \frac{n}{c^2} \partial_t^2 u.$$

Replacing u by \tilde{u} on the right of the above equation we define the linear map

$$A: n \mapsto u_d|_Y$$

where u_d solves

$$\frac{1}{c^2} \partial_t^2 u_d - \Delta_{\mathbf{x}} u_d = \frac{n}{c^2} \partial_t^2 \tilde{u}$$

and Y is the set of receivers. We may view A as a kind of linearization of the map $\mathcal{F}: \nu^2 \mapsto u|_Y$, indeed, $\mathcal{F}(c^2) - \mathcal{F}(\nu^2) = \mathcal{F}(\nu^2 + n\nu^2) - \mathcal{F}(\nu^2) \approx An$. The linearized problem for constant density seismic imaging now reads: Determine n from

$$An = \mathcal{F}(c^2) - \mathcal{F}(\nu^2) = \tilde{u}|_Y - u|_Y$$

where $u|_Y$ has been recorded and $\tilde{u}|_Y$ has to be computed from (3).

Using Duhamel's principle,

$$(4) \quad An(t; \mathbf{x}_r, \mathbf{x}_s) = \int \frac{n(\mathbf{x})}{c^2(\mathbf{x})} \left(\int \partial_t^2 \tilde{u}(s; \mathbf{x}, \mathbf{x}_s) \tilde{u}(t-s; \mathbf{x}_r, \mathbf{x}) ds \right) d\mathbf{x}.$$

As c is assumed to be smooth we take the geometric optics approximation for granted, that is, $\mathbf{x} \in \text{supp } n$ can be connected to each \mathbf{x}_r and to each \mathbf{x}_s by one and only by one ray of geometric optics. Accordingly, \tilde{u} is a progressing wave in 2D:

$$(5) \quad \tilde{u}(t; \mathbf{x}, \mathbf{x}_s) \approx a(\mathbf{x}, \mathbf{x}_s) \Psi(t - \tau(\mathbf{x}, \mathbf{x}_s))$$

where

$$\Psi(t) = \begin{cases} t^{-1/2} & : t > 0, \\ 0 & : t \leq 0, \end{cases}$$

and where the travel time $\tau(\mathbf{x}, \mathbf{x}_s)$ solves the eikonal equation

$$|\nabla_{\mathbf{x}} \tau| = \frac{1}{c}$$

and the amplitude a satisfies

$$\text{div}(a^2 \nabla_{\mathbf{x}} \tau) = 0,$$

see, e.g., Symes [24, page 38]. See also Friedlander [11] and Courant and Hilbert [5].

Plugging (5) into (4) we find that

$$\begin{aligned} An(t; \mathbf{x}_r, \mathbf{x}_s) &\approx \partial_t^2 \int \frac{n(\mathbf{x})}{c^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_s) a(\mathbf{x}_r, \mathbf{x}) \Psi \star \Psi(t - \tau(\mathbf{x}, \mathbf{x}_s) - \tau(\mathbf{x}_r, \mathbf{x})) d\mathbf{x} \\ &= \partial_t^2 \int \frac{\pi n(\mathbf{x})}{c^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_s) a(\mathbf{x}_r, \mathbf{x}) H(t - \tau(\mathbf{x}, \mathbf{x}_s) - \tau(\mathbf{x}_r, \mathbf{x})) d\mathbf{x} \\ &= \partial_t \int \frac{\pi n(\mathbf{x})}{c^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_s) a(\mathbf{x}, \mathbf{x}_r) \delta(t - \tau(\mathbf{x}, \mathbf{x}_s) - \tau(\mathbf{x}, \mathbf{x}_r)) d\mathbf{x} =: \tilde{A}n(t; \mathbf{x}_r, \mathbf{x}_s) \end{aligned}$$

where the first equality holds since $\Psi \star \Psi = \pi H$ with H being the Heaviside function, see, e.g., [14, Chap. 1.5.5]. The second equality is true due to $H' = \delta$ where δ is the one-dimensional Dirac-distribution.

Set $u_{\text{data}} := \tilde{u} - u$. Our intermediate linear problem now reads

$$\tilde{A}n(t; \mathbf{x}_r, \mathbf{x}_s) = u_{\text{data}}(t; \mathbf{x}_r, \mathbf{x}_s)$$

and integrating both sides with respect to t over the observation period from 0 to T we finally obtain

$$(6) \quad Fn(T; \mathbf{x}_r, \mathbf{x}_s) = y(T; \mathbf{x}_r, \mathbf{x}_s)$$

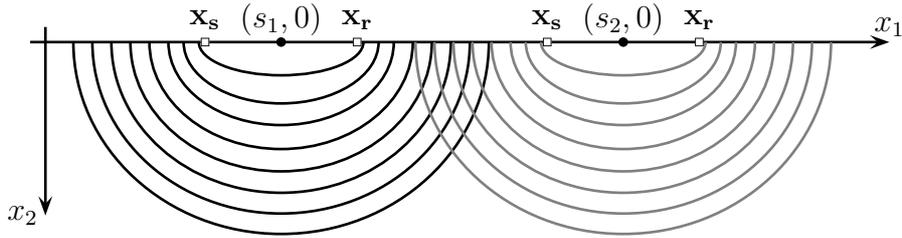


FIGURE 1. Common offset scanning geometry.

where

$$y(T; \mathbf{x}_r, \mathbf{x}_s) := \frac{1}{\pi} \int_0^T u_{\text{data}}(t; \mathbf{x}_r, \mathbf{x}_s) dt$$

and

$$(7) \quad Fn(T; \mathbf{x}_r, \mathbf{x}_s) = \int \frac{n(\mathbf{x})}{c^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_s) a(\mathbf{x}, \mathbf{x}_r) \delta(T - \tau(\mathbf{x}, \mathbf{x}_s) - \tau(\mathbf{x}, \mathbf{x}_r)) d\mathbf{x}$$

is a generalized Radon transform which integrates over reflection isochrones $\{\mathbf{x} : T = \tau(\mathbf{x}, \mathbf{x}_s) + \tau(\mathbf{x}, \mathbf{x}_r)\}$.

2.2. The elliptic Radon transform. In the following we assume that

- the background velocity c is constant, say, $c = 1$,
- n is compactly supported in the lower half space $x_2 > 0$ (the positive direction of the x_2 -axis points downwards to the interior of the earth),
- the common offset scanning geometry is used where source and receiver positions are located on the line $x_2 = 0$ and they are parameterized by $s \in \mathbb{R}$ via $(s - \alpha, 0)^\top$ and $(s + \alpha, 0)^\top$, respectively. Here, $\alpha > 0$ is the common offset.

Under these assumptions the reflection isochrones are ellipses with foci

$$\mathbf{x}_s(s) = (s - \alpha, 0)^\top \quad \text{and} \quad \mathbf{x}_r(s) = (s + \alpha, 0)^\top,$$

see Figure 1. Further,

$$\tau(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \quad \text{and} \quad a(\mathbf{x}, \mathbf{y}) = \frac{1}{|\mathbf{x} - \mathbf{y}|}.$$

In this situation the generalized Radon transform (7) integrates over ellipses and may be written as

$$(8) \quad Fn(s, t) = \int A(s, \mathbf{x}) n(\mathbf{x}) \delta(t - \varphi(s, \mathbf{x})) d\mathbf{x}, \quad t > 2\alpha,$$

with

$$\varphi(s, \mathbf{x}) := |\mathbf{x}_s(s) - \mathbf{x}| + |\mathbf{x}_r(s) - \mathbf{x}| \quad \text{and} \quad A(s, x) = \frac{1}{|\mathbf{x}_s(s) - \mathbf{x}| |\mathbf{x}_r(s) - \mathbf{x}|}.$$

The lower bound on t in (8) is needed because the major axis of the ellipse must be longer than half the distance between the foci.

2.3. Local reconstruction operator and approximate inverse. As we do not know an inversion formula for F we do not try to reconstruct n directly from its integrals $g = Fn$ over ellipses. Instead, we employ ideas from Lambda tomography [9] and define the reconstruction operator

$$\Lambda = \Delta F^* \Phi F$$

where $\Phi = \Phi(s, t)$ is a smooth compactly supported cutoff function, Δ is the Laplacian, and F^* is the *formal* (weighted) L^2 -adjoint of F satisfying

$$(9) \quad \langle F^* w, u \rangle_{L^2(\mathbb{R}^2)} = \langle w, Fu \rangle_{L^2(\mathbb{R} \times]2\alpha, \infty[, t^2 dt ds)}.$$

The use of the measure $t^2 dt ds$ in the image space of F compensates a damping factor, see (12) below.

Let us briefly explain our choice of Λ : under the Bolker assumption (see, e.g., [15, p. 371]), any hypersurface Radon transform R in an d -dimensional space and its (formal, smoothly weighted) L^2 -adjoint R^* are Fourier integral operators of order $(1 - d)/2$. Furthermore, if they can be composed, then R^*R is a pseudodifferential operator. Our transform F on \mathbb{R}^2 satisfies the Bolker assumption (this follows from [18, Theorem 4] which establishes the microlocal properties of this operator for functions supported on both sides of the axis $x_2 = 0$) so, $F^*\Phi F$ is of order -1 when Φ is a smooth cutoff function (which is needed so that F^* and F can be composed). Thus, Λ has order 1 and we expect

$$\Lambda n = \Delta F^* \Phi g$$

to emphasize singularities (e.g., jumps along curves) of n which are tangent to ellipses being integrated over (rigorously, covectors in the wavefront set of f that are normal to the ellipse which follows from [18, Theorem 3]). Further, Λ is local in the following sense: To reconstruct $\Lambda n(\mathbf{p})$ one only needs integrals over ellipses near to \mathbf{p} .

The structure of Λ is ideally suited to stabilize the evaluation of $\Lambda n(\mathbf{p})$ by the concept of approximate inverse, see [19]. Instead of $\Lambda n(\mathbf{p})$ we try to compute

$$\Lambda_\gamma n(\mathbf{p}) := \langle \Lambda n, e_{\mathbf{p}, \gamma, k} \rangle_{L^2(\mathbb{R}^2)}$$

where

$$e_{\mathbf{p}, \gamma, k}(\mathbf{x}) = C_{k, \gamma} \begin{cases} (\gamma^2 - \Theta^2)^k & : \Theta < \gamma, \\ 0 & : \Theta \geq \gamma, \end{cases} \quad \Theta = |\mathbf{x} - \mathbf{p}|,$$

with $\gamma, k > 0$ and

$$C_{k, \gamma} = \frac{k+1}{\pi \gamma^{2(k+1)}}.$$

We have $\text{supp } e_{\mathbf{p}, \gamma, k} = \overline{B_\gamma(\mathbf{p})}$, the closure of the ball about \mathbf{p} with radius γ . Moreover, $\int e_{\mathbf{p}, \gamma, k}(\mathbf{x}) d\mathbf{x} = 1$ such that $e_{\mathbf{p}, \gamma, k} \rightarrow \delta(\cdot - \mathbf{p})$ as $\gamma \rightarrow 0$. Thus, γ serves as a scaling or regularization parameter while k is a design parameter which determines the smoothness of $e_{\mathbf{p}, \gamma, k}$. By its properties $e_{\mathbf{p}, \gamma, k}$ is a mollifier.

Lemma 2.1. *For $k \geq 3$ we have that*

$$\Lambda_\gamma n(\mathbf{p}) = \langle \Phi F n, \psi_{\mathbf{p}, \gamma, k} \rangle_{L^2(\mathbb{R} \times]2\alpha, \infty[, t^2 dt ds)}$$

with

$$(10) \quad \psi_{\mathbf{p}, \gamma, k}(s, t) = 4k \left((k-1) F(|\cdot - \mathbf{p}|^2 e_{\mathbf{p}, \gamma, k-2})(s, t) - F e_{\mathbf{p}, \gamma, k-1}(s, t) \right).$$

Moreover, the following translation invariance holds

$$\psi_{\mathbf{p}, \gamma, k}(s, t) = \psi_{(0, p_2), \gamma, k}(s - p_1, t).$$

Proof. By duality we see that

$$\psi_{\mathbf{p},\gamma,k} = F\Delta e_{\mathbf{p},\gamma,k}$$

and evaluating $\Delta e_{\mathbf{p},\gamma,k}$ yields (10).

The translation invariance is a direct consequence of the translation invariances of the Laplacian and of F with respect to s : $Fn(s-r, t) = F(n(\cdot-r, \cdot))(s, t)$. \square

Lemma 2.1 tells us that we can compute $\Lambda_\gamma n(\mathbf{p})$ from the data g if we can evaluate the function $\psi_{\mathbf{p},\gamma,k}$ which is called a reconstruction kernel for F (belonging to the mollifier $e_{\mathbf{p},\gamma,k}$). The translation invariance of the reconstruction kernel might be used to speed up the computation of $\Lambda_\gamma n(\mathbf{p})$ as one only needs to evaluate $\psi_{(0,p_2),\gamma,k}$ numerically. Note that we do not need to know F^* explicitly to compute $\Lambda_\gamma n(\mathbf{p})$.

3. COMPUTING THE RECONSTRUCTION KERNEL

Calculating the reconstruction kernel $\psi_{\mathbf{p},\gamma,k}$ according to (10) requires the application of our imaging operator F , see (8), to functions which are supported in a ball. In a first step we therefore present an explicit calculation of F acting on the indicator function of a ball. This will be used for the general case, which is discussed at the end of this section.

Let χ be the indicator function of $B_r(\mathbf{p})$ with $p_2 > 0$, and radius r where $0 < r < p_2$, that is, $B_r(\mathbf{p})$ is completely contained in the lower half space (recall that the positive x_2 -direction points downwards). We are going to evaluate

$$F\chi(s, t) = \int A(s, \mathbf{x})\chi(\mathbf{x})\delta(t - \varphi(s, \mathbf{x}))d\mathbf{x}, \quad t > 2\alpha.$$

Using prolate spheroidal coordinates depending on $(s, 0)^\top$ given by

$$(11) \quad \mathbf{x}(s, t, \phi) = (x_1, x_2)^\top$$

where

$$x_1 = s + \frac{t}{2} \cos \phi, \quad \text{and} \quad x_2 = \sqrt{\frac{t^2}{4} - \alpha^2} \sin \phi,$$

we find from our calculations in Appendix A that

$$F\chi(s, t) = \int_0^\pi A(s, \mathbf{x}(s, t, \phi))\chi(\mathbf{x}(s, t, \phi)) \frac{t^2/4 - \alpha^2 \cos^2 \phi}{\sqrt{t^2 - 4\alpha^2}} d\phi.$$

As

$$A(s, \mathbf{x}(s, \varrho, \phi)) = \frac{1}{|\mathbf{x}_s(s) - \mathbf{x}| |\mathbf{x}_r(s) - \mathbf{x}|} = \frac{1}{t^2/4 - \alpha^2 \cos^2 \phi}$$

we obtain

$$(12) \quad F\chi(s, t) = \frac{1}{\sqrt{t^2 - 4\alpha^2}} \int_0^\pi \chi(\mathbf{x}(s, t, \phi)) d\phi.$$

Note that we can restrict the integration over ϕ to the interval $[0, \pi]$ because χ is supported in the lower half space where $x_2 = \sqrt{t^2/4 - \alpha^2} \sin \phi$ is non-negative.

To evaluate $F\chi(s, t)$ further we provide the following quantities

$$T_- = T_-(s, r, \mathbf{p}) = \min \{ \varphi(s, \mathbf{x}) : \mathbf{x} \in \partial B_r(\mathbf{p}) \},$$

$$T_+ = T_+(s, r, \mathbf{p}) = \max \{ \varphi(s, \mathbf{x}) : \mathbf{x} \in \partial B_r(\mathbf{p}) \}.$$

Let $E(s, t)$ be the ellipse with foci $\mathbf{x}_s(s)$, $\mathbf{x}_r(s)$ and travel time (major diameter) $t > 2\alpha$. Then,

$$E(s, t) \cap B_r(\mathbf{p}) \neq \emptyset \quad \text{if and only if} \quad T_- < t < T_+$$

which is a direct consequence of the definition of T_{\pm} . From that we get

$$(13) \quad F\chi(s, t) = \begin{cases} 0 & : t \notin]T_-, T_+[\\ \frac{\phi_2 - \phi_1}{\sqrt{t^2 - 4\alpha^2}} & : t \in]T_-, T_+[\end{cases}$$

where

$$\begin{aligned} \phi_1 &= \phi_1(s, t) = \min \{ \phi : \mathbf{x}(s, t, \phi) \in \overline{B_r(\mathbf{p})} \}, \\ \phi_2 &= \phi_2(s, t) = \max \{ \phi : \mathbf{x}(s, t, \phi) \in \overline{B_r(\mathbf{p})} \}. \end{aligned}$$

So it remains to find numerical approximations for T_- , T_+ , and ϕ_1 , ϕ_2 . For the first pair of values we parameterize $\partial B_r(\mathbf{p})$ by

$$\partial B_r(\mathbf{p}) = \{ \mathbf{p} + r(\cos \vartheta, \sin \vartheta)^\top : \vartheta \in [0, 2\pi[\},$$

so that

$$T_{-/ +} = \min / \max \{ \tilde{\varphi}(\vartheta) : \vartheta \in [0, 2\pi[\} \quad \text{where} \quad \tilde{\varphi}(\vartheta) := \varphi(s, \mathbf{p} + r(\cos \vartheta, \sin \vartheta)^\top).$$

From the geometric setting it becomes evident that $\tilde{\varphi}$ attains exactly one minimum and one maximum in $[0, 2\pi[$. As both extrema are clearly separated, we can apply Newton's method to get the two zeros of $\tilde{\varphi}'$.

Having T_{\mp} we solve

$$r^2 = |\mathbf{p} - \mathbf{x}(s, t, \phi)|^2$$

for ϕ . Given $t \in]T_-, T_+[$ and $s \in \mathbb{R}$ we have exactly the two solutions ϕ_1 and ϕ_2 . Plugging in prolate spheroidal coordinates we get the equation

$$r^2 = (p_1 - s)^2 + p_2^2 + \frac{t^2}{4} - \alpha^2 - \sqrt{t^2 - 4\alpha^2} p_2 \sin \phi - (p_1 - s) t \cos \phi + \alpha^2 \cos^2 \phi.$$

We substitute

$$\begin{aligned} z &= \cos \phi, \\ b &= -(p_1 - s) t, \\ c &= (p_1 - s)^2 + p_2^2 + \frac{t^2}{4} - \alpha^2 - r^2, \\ d &= -\sqrt{t^2 - 4\alpha^2} p_2, \end{aligned}$$

to obtain the equation

$$-d \sqrt{1 - z^2} = c + b z + \alpha^2 z^2$$

for z which has exactly two solutions in $[-1, 1]$. Using Newton's method again we compute the two solutions $z_2 < z_1$ in $[-1, 1]$ and get

$$\phi_i = \arccos z_i, \quad i = 1, 2.$$

Using this, we can calculate a numerical value for $F\chi(s, t)$, see (13).

The reconstruction kernel $\psi_{\mathbf{p}, \gamma, k}$ can be computed numerically in the same way as $F\chi$. For instance, let $k = 3$. Then,

$$\Delta e_{\mathbf{p}, \gamma, 3}(\mathbf{x}) = C_{3, \gamma} \left(-36 |\mathbf{x} - \mathbf{p}|^4 + 48\gamma^2 |\mathbf{x} - \mathbf{p}|^2 - 12\gamma^4 \right) \chi_{B_\gamma(\mathbf{p})}(\mathbf{x}).$$

Now F can be applied to each of the components of $\Delta e_{\mathbf{p}, \gamma, 3}$. Consider

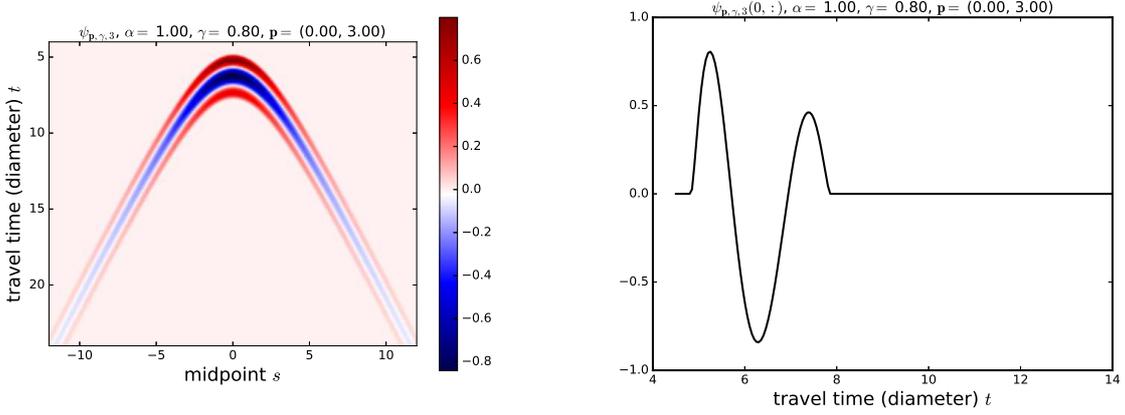


FIGURE 2. Reconstruction kernel $\psi_{\mathbf{p},\gamma,3}$ for $\gamma = 0.8$ and $\alpha = 1$ at $\mathbf{p} = (0, 3)$. The cross section on the right is taken for $s = 0$.

$$F(|\cdot - \mathbf{p}|^4 \chi_{B_\gamma(\mathbf{p})})(s, t) = \begin{cases} 0 & : t \notin]T_-, T_+[, \\ \frac{1}{\sqrt{t^2 - 4\alpha^2}} \int_{\phi_1}^{\phi_2} |\mathbf{x}(s, t, \phi) - \mathbf{p}|^4 d\phi & : t \in]T_-, T_+[. \end{cases}$$

Here,

$$|\mathbf{x}(s, t, \phi) - \mathbf{p}|^4 = \left(\left(s - p_1 + \frac{t}{2} \cos \phi \right)^2 + \left(\sqrt{\frac{t^2}{4} - \alpha^2} \sin \phi - p_2 \right)^2 \right)^2$$

is a trigonometric polynomial in ϕ whose antiderivative is known analytically. See Figure 2 for a graphical example.

4. NUMERICAL EXPERIMENTS

We need to compute

$$(14) \quad \Lambda_\gamma n(\mathbf{p}) := \langle \Lambda n, e_{\mathbf{p},\gamma,3} \rangle_{L^2(\mathbb{R}^2)} = \langle \Phi F n, \psi_{\mathbf{p},\gamma,3} \rangle_{L^2(\mathbb{R} \times]2\alpha, \infty[; t^2 dt ds)}$$

from the discrete data

$$(15) \quad g(i, j) = \Phi(s_i, t_j) F n(s_i, t_j), \quad i = 1, \dots, N_s, \quad j = 1, \dots, N_t,$$

where

$$\{s_i\} \subset [-s_{\max}, s_{\max}] \quad \text{and} \quad \{t_j\} \subset [t_{\min}, t_{\max}], \quad t_{\min} > 2\alpha$$

are uniformly distributed with step sizes h_s and h_t , respectively. By a simple approximation of the double integral on the right of (14) we get

$$(16) \quad \begin{aligned} \Lambda_\gamma n(\mathbf{p}) &\approx \tilde{\Lambda}_\gamma n(\mathbf{p}) := h_s h_t \sum_{i=1}^{N_s} \sum_{j=1}^{N_t} g(i, j) \psi_{\mathbf{p},\gamma,3}(s_i, t_j) t_j^2 \\ &= h_s h_t \sum_{i=1}^{N_s} \sum_{t_j \in \mathcal{T}_i(\mathbf{p})} g(i, j) \psi_{\mathbf{p},\gamma,3}(s_i, t_j) t_j^2 \end{aligned}$$

with $\mathcal{T}_i(\mathbf{p}) =]T_-(s_i, \gamma, \mathbf{p}), T_+(s_i, \gamma, \mathbf{p})[$.

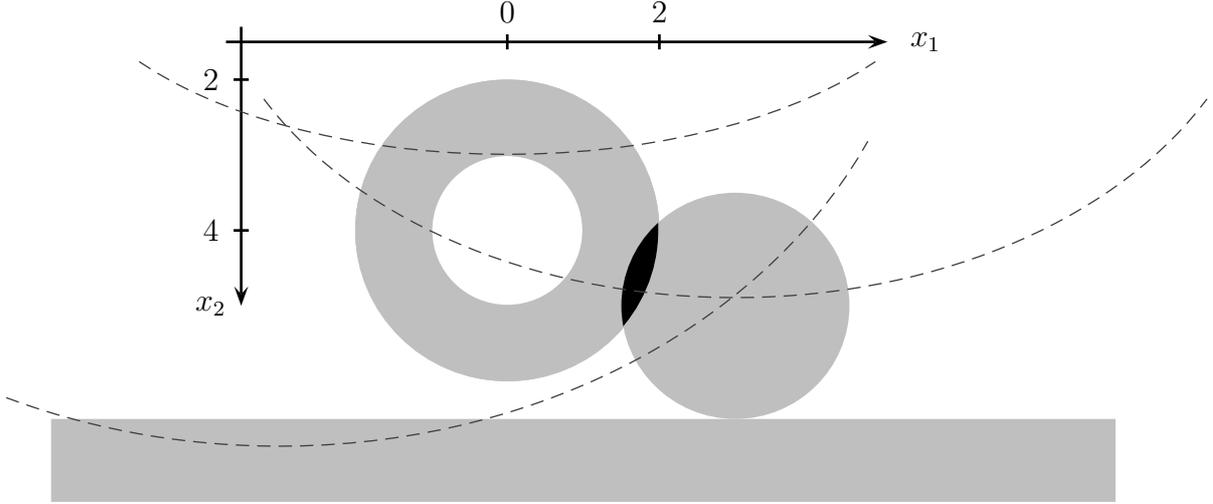


FIGURE 3. Visualization of the function n (17) to be reconstructed from elliptic means. Light gray area: $n = 1$, black: $n = 2$, white: $n = 0$. The light gray bar represents the half space $x_2 \geq 6.5$. The three dashed curves show elliptic arcs belonging to the common offset $\alpha = 5$ and $s = 0, t = 12$ (top), $s = 3, t = 14$ (middle), $s = -3, t = 17$ (bottom).

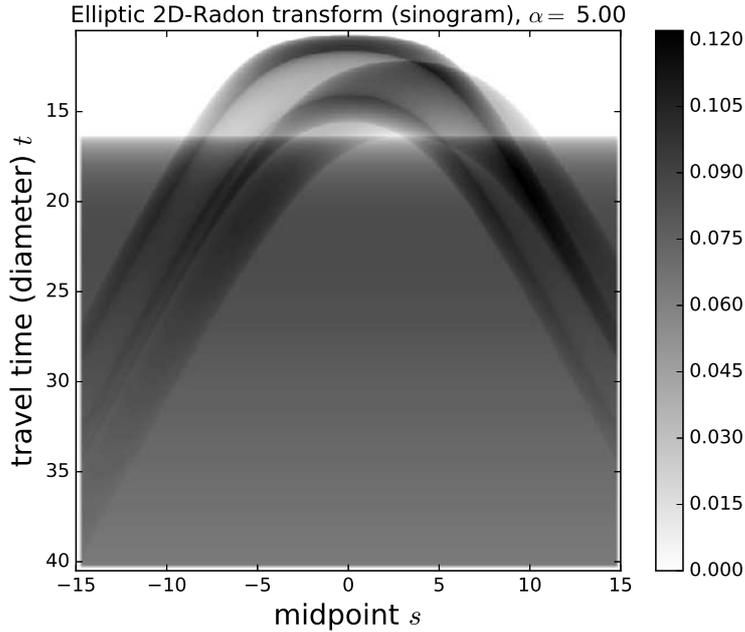


FIGURE 4. Elliptic Radon transform $\Phi F n$ for n from (17). The common offset is $\alpha = 5$.

For a first numerical experiment the function to be reconstructed is

$$(17) \quad n = \chi_{B((0,4),2)} - \chi_{B((0,4),1)} + \chi_{B((3,5),1.5)} + \chi_{x_2 \geq 6.5},$$

see Figure 3. The data $g = \Phi F n$ for the common offset $\alpha = 5$ are plotted in Figure 4. Observe the effect of the cutoff function Φ at the edges. We used the design of $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2)$

from [21, Sec. 5] modified such that

$$\text{supp } \Phi \subset [-s_{\max}, s_{\max}] \times [0.01, t_{\max}] \quad \text{and} \quad \Phi|_{[-s_{\max}+1, s_{\max}-1] \times [0.02, t_{\max}-1]} = 1.$$

For the computation of $\tilde{\Lambda}_\gamma n$ we used $s_{\max} = 15$, $t_{\max} = t_{\min} + 30$ with $t_{\min} = 4.9$ for $\alpha = 2$ and $t_{\min} = 10.5$ for $\alpha = 5$. All reconstructions have been computed from $N_s N_t = 600^2 = 360000$ elliptic means.

We approximated $\tilde{\Lambda}_\gamma n(\mathbf{p})$ for \mathbf{p} uniformly in $[-2.5, 5] \times [1.5, 7]$ using 150 equidistant points for each interval. The results for the scaling $\gamma = 0.2$ are shown in Figure 5. Restricting the data set reduces the visible singularities in the reconstruction: Figure 6 displays a reconstruction where $s_{\max} = 7.5$ and $t_{\max} = 25.50$. Compared to the bottom reconstruction of Figure 5 those singularities are missing whose tangents have a dominant vertical component: This data set does not include integrals over ellipses tangent to those singularities.

As F^* and F cannot be meaningfully composed we introduced the smooth cutoff function Φ in the definition of Λ . However, in our numerical experiments we can easily get rid of Φ by setting $\Phi(s_i, t_j) = 1$ in (15) for all i and j . Figure 7 displays the difference of such a reconstruction without a cutoff and the corresponding reconstruction with a smooth cutoff (which is the reconstruction from the bottom of Figure 5). We see that both reconstructions capture the singularities of n consistently: their difference does not contain a trace of these singularities. On the other hand, the reconstructions differ greatly in the added, unphysical singularities which are caused by the limited data and the numerical scheme. Introducing the cutoff function damps significantly the intensity of the artificial singularities. This is a general phenomenon that has been analyzed for tomography problems including limited angle X-ray CT [17, 13, 20], sonar, and photoacoustic tomography [12].

Next, we demonstrate how $\tilde{\Lambda}_\star n$ is affected by perturbed data which we generated according to

$$(18) \quad g^\delta(i, j) = g(i, j) + \delta \|g\|_\star \frac{\mathbf{n}(i, j)}{\|\mathbf{n}\|_\star}, \quad \delta > 0,$$

where g is the exact data (15) and \mathbf{n} is an $N_s \times N_t$ array of uniformly distributed random numbers with values in $[-1, 1]$. The discrete norm

$$\|g\|_\star^2 := h_s h_t \sum_{i=1}^{N_s} \sum_{j=1}^{N_t} |g(i, j)|^2 t_j^2$$

approximates the norm in $L^2([-s_{\max}, s_{\max}] \times [t_{\min}, t_{\max}], t^2 ds dt)$. We have that

$$\frac{\|g - g^\delta\|_\star}{\|g\|_\star} \leq \delta,$$

i.e., δ measures the *relative noise*. Figure 8 shows reconstructions for a data set with noise level $\delta = 8\%$ using three different scaling parameters. We see clearly the smoothing or regularizing effect of γ : increasing γ reduces the influence of noise while blurring the contours.

In another experiment we study how the reconstructions are affected by a wrong common offset in the inversion procedure, that is, a wrong α in computing the kernel $\psi_{\mathbf{p}, \gamma, 3}$ (10). If we choose the common offset α_{recon} for the reconstruction larger than

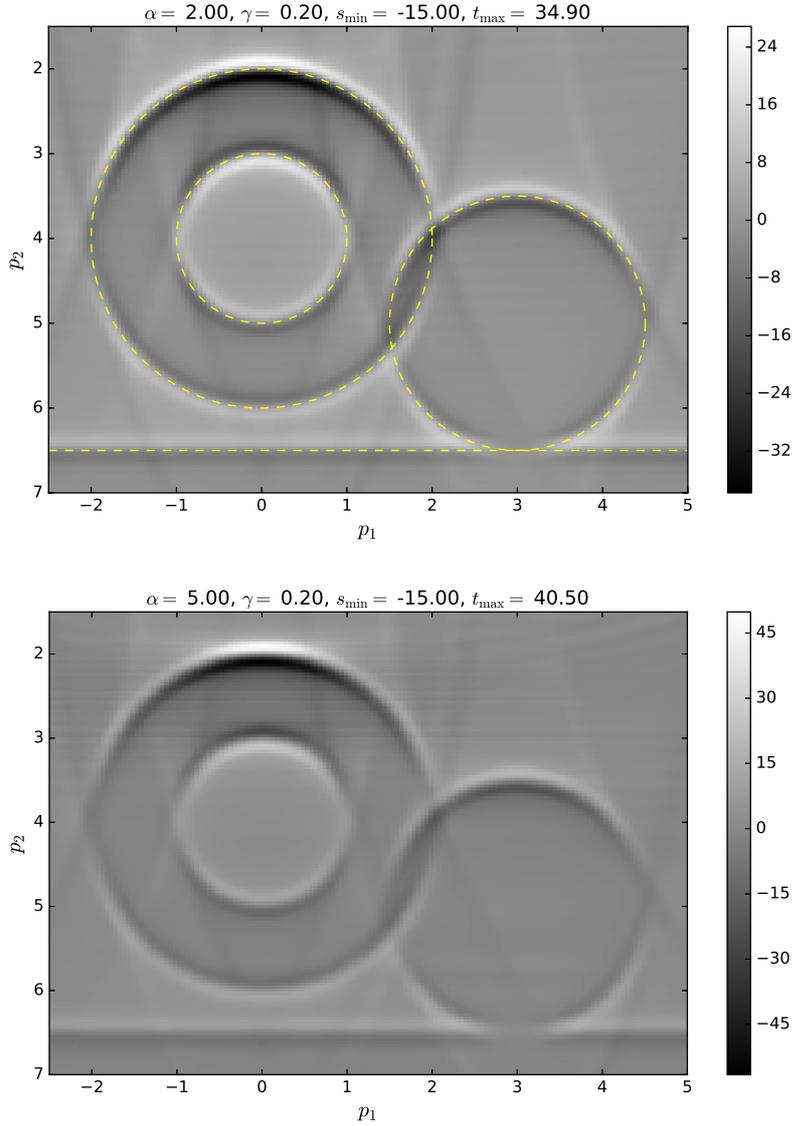


FIGURE 5. Reconstructions $\tilde{\Lambda}_\gamma n$, $\gamma = 0.2$, with n from (17). Top: common offset $\alpha = 2$, bottom: common offset $\alpha = 5$. Further, $N_s = N_t = 600$. The dashed lines in the top image indicate the singular support of n and are not part of the reconstruction.

α_{data} , the common offset for recording the data, then the reconstructed singularities appear closer to the surface than the real singularities (top of Figure 9). The reason is that the travel time t for a given point \mathbf{p} increases with the common offset. In other words: the diameter of an ellipse passing through \mathbf{p} increases with the distance of the foci. This effect is more pronounced at singularities located closer to the surface because for large t and fixed foci the corresponding ellipse approximates a circle with radius $t/2$, see Figure 1. Accordingly, if $\alpha_{\text{recon}} < \alpha_{\text{data}}$ then the reconstructed singularities appear farther down than the real singularities (bottom of Figure 9).

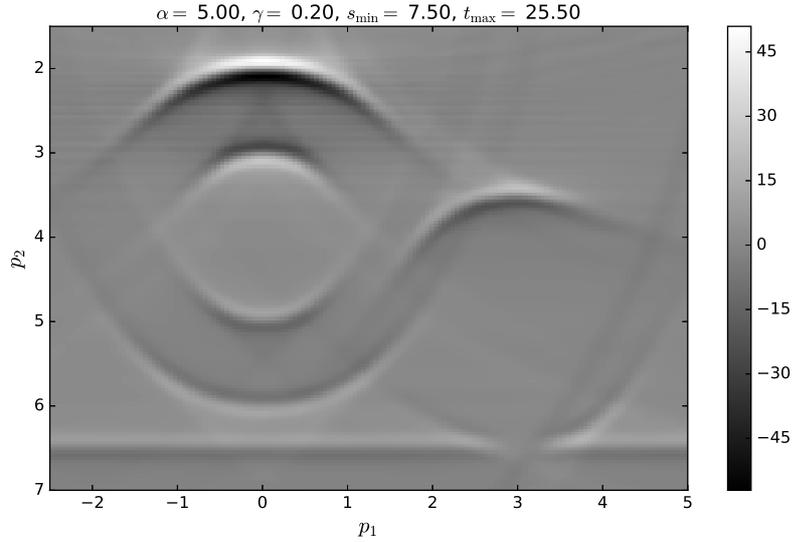


FIGURE 6. Reconstruction $\tilde{\Lambda}_\gamma n$, $\gamma = 0.2$, with n from (17). The common offset is $\alpha = 5$. The other used parameters are $s_{\max} = 7.5$, $t_{\max} = 25.50$, $N_s = N_t = 600$.

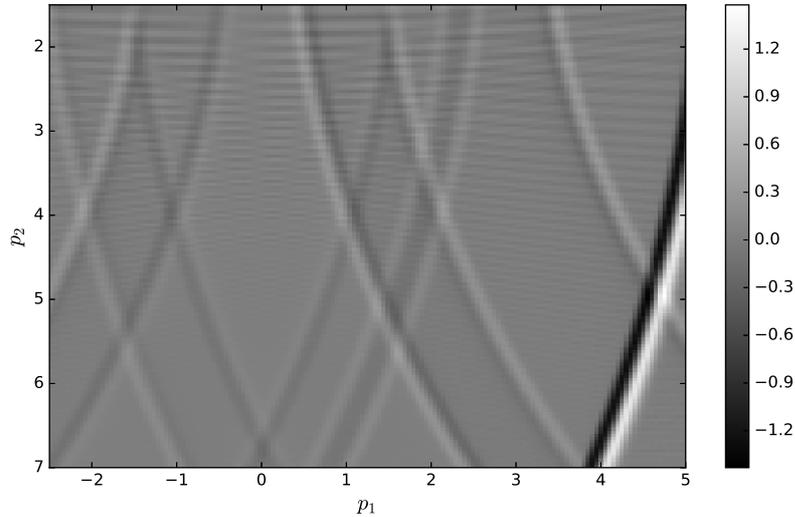


FIGURE 7. Difference of a reconstruction without the smooth cutoff function Φ and of a reconstruction with Φ .

We emphasize that singularities are displaced only in vertical direction. Their horizontal position is not affected which can be seen at the horizontal line singularity which is cleanly moved up and down, respectively.

Remark about other acquisition geometries. Other important scanning arrangements are the common midpoint (CM) and the common source (CS) geometries.

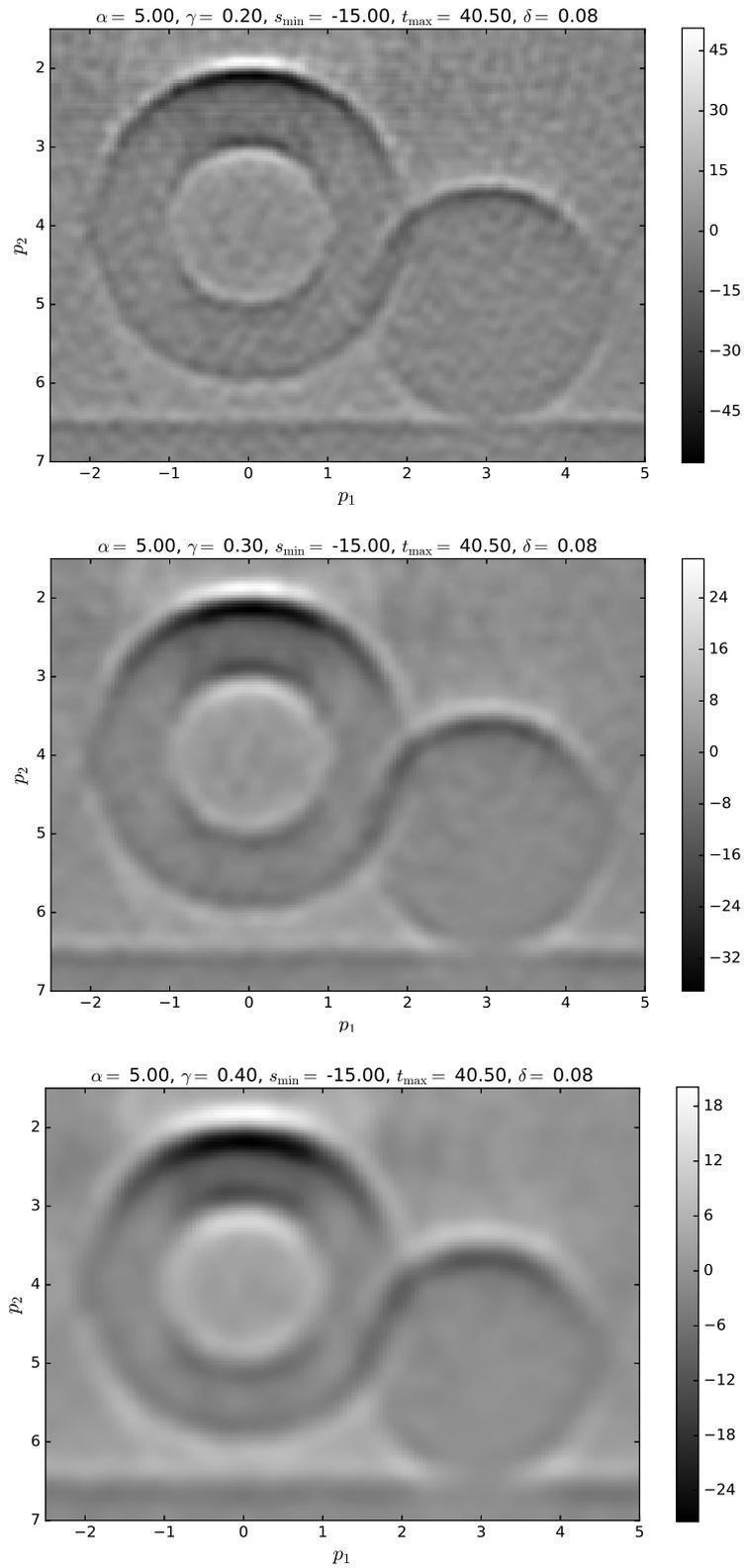


FIGURE 8. Reconstructions $\tilde{\Lambda}_\gamma n$ from the same perturbed data set with $\delta = 8\%$, see (18). Top: $\gamma = 0.2$, middle: $\gamma = 0.3$, bottom: $\gamma = 0.4$. All other parameters are as in Figure 5.

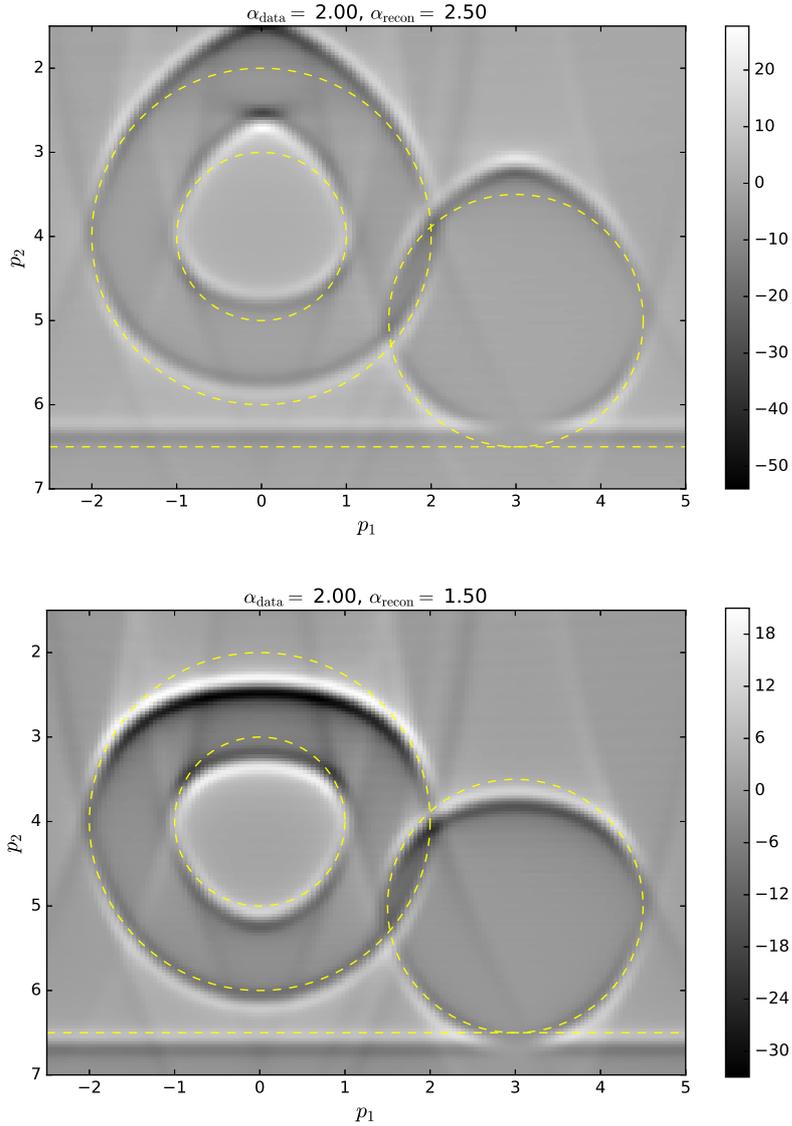


FIGURE 9. Reconstructions with a wrong common offset α_{recon} in the reconstruction kernel. The data g is generated with $\alpha_{\text{data}} = 2$ (see top of Figure 5 for the ‘correct’ reconstruction where $\alpha_{\text{data}} = \alpha_{\text{recon}}$). Top: $\alpha_{\text{recon}} = 2.5$, bottom: $\alpha_{\text{recon}} = 1.5$. The dashed lines indicate the singular support of n and are not part of the reconstruction.

- In CM, source and receiver positions are parameterized by $s \geq 0$ as $(x_{\text{mp}} - s, 0)$ and $(x_{\text{mp}} + s, 0)$, respectively, where $(x_{\text{mp}}, 0) \in \mathbb{R}^2$ is the (fixed) common midpoint, see Figure 10.
- In CS, the source is fixed, say at $(x_{\text{sc}}, 0) \in \mathbb{R}^2$, and the receivers are located at $(x_{\text{sc}} + s, 0)$ for $s \in \mathbb{R}$, see Figure 11.

Both geometries are known to allow only the reconstruction of a very restricted set of singularities, even if full data would be available. This fact can be rigorously understood by microlocal analysis which describes how Fourier integral operators and their adjoints

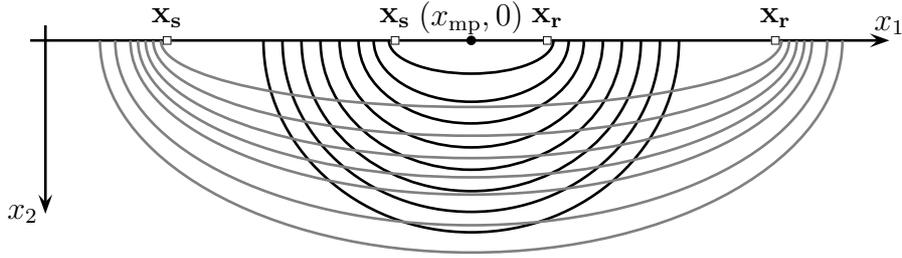


FIGURE 10. Common mid point scanning geometry.

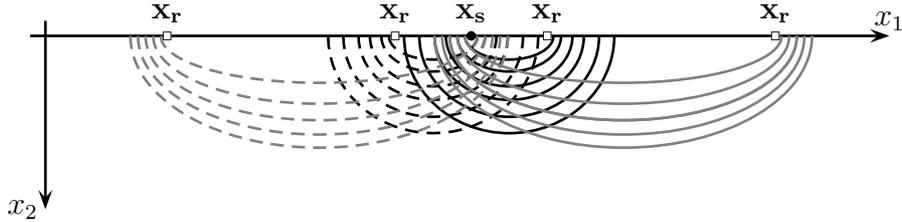


FIGURE 11. Common source scanning geometry.

map singularities [16]. Using this technique, Felea et al. [10] showed that the CM geometry in radar might even introduce singularities which are not present in the original image.

With the help of Figures 10 and 11 we can easily explain a fundamental difference of CM and CS when n is layered, i.e., only depends on x_2 . Recall that singularities are visible in the data only when there is an ellipse being integrated over which is tangent to the singularity. Therefore the distribution of ellipses allows CS to detect a horizontal line singularity completely while CM only detects one point of this line located at $x_1 = x_{mp}$.

Next we will highlight the shortcomings of CM and CS by numerical experiments. Our numerical scheme for the common offset geometry can easily be adapted to CM and CS based on the following relations

$$F^{CM}n(s, t) = F_s n(x_{mp}, t), \quad s \geq 0, \quad t > 2s,$$

$$F^{CS}n(s, t) = F_{|s/2|}n(x_{sc} + s/2, t), \quad s \in \mathbb{R}, \quad t > |s|,$$

where F_α denotes the common offset transform (8) with offset α and F^{CM} and F^{CS} are the corresponding transforms with the CM and CS parameterizations of ellipses, respectively.

Reconstructions from the CM and CS data displayed in Figure 12 can be examined in Figure 13. We emphasize that both reconstructions would not exhibit more singularities of n if the ranges of t and s would be increased. The number of used elliptic means in each case is $N_s N_t = 600^2$, cf. (15), and $\gamma = 0.2$. Thus, the reconstructions of Figure 13 are directly comparable to the reconstructions of Figure 5. The superiority of common offset over CM and CS is striking.

Data from the wave equation. We perform experiments where the discrete data g used in (16) are generated by solving the acoustic wave equation numerically, that is, g is an approximation to y in (6). Thus, we are closer to the real situation of seismic imaging, that is, the data are contaminated by a modeling and linearization error.

For the solution of the wave equations (1) and (3) as well as for recording the receiver signals we relied on PySIT [8] which is an open source toolbox for seismic inversion and seismic imaging. The computational domain is the rectangle $[0.1, 1] \times [0.1, 0.8]$ furnished

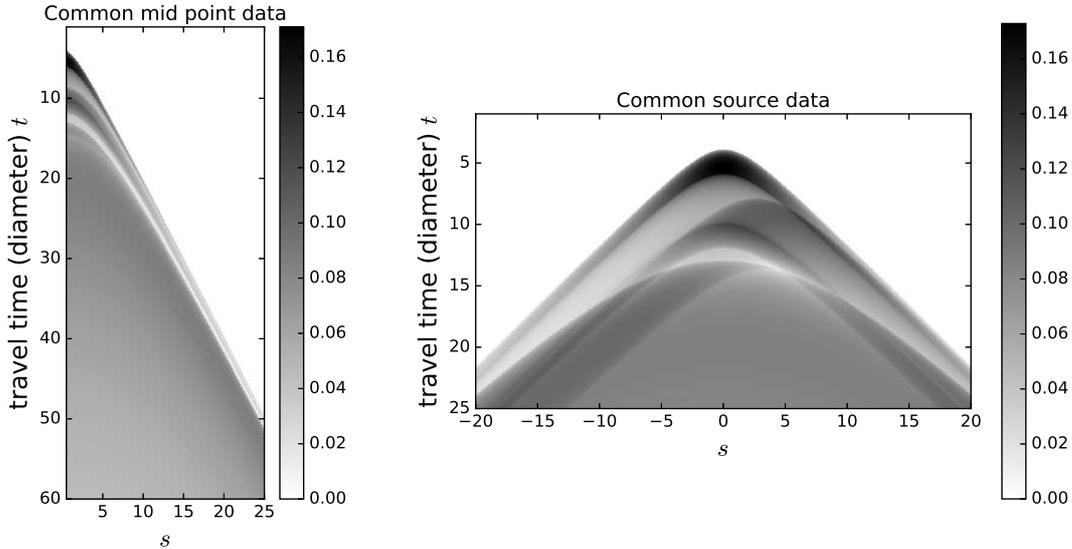


FIGURE 12. Transforms F^{CM}_n (left) and F^{CS}_n (right) for n from (17) where $x_{\text{mp}} = 0$ and $x_{\text{sc}} = 0$, respectively.

with absorbing boundary conditions using a perfectly matched layer (PML). It was discretized by a constant step size 0.01 in each coordinate direction resulting in a Cartesian grid \mathcal{G} with 90×70 points.

We considered two different sound speeds ν_1 and ν_2 each attaining exactly two different values with jumps along smooth curves:

$$\nu_1(x) = \begin{cases} 1 & : x_2 \leq 0.1 \sin(x_1) + 0.5, \\ 1.5 & : \text{otherwise,} \end{cases} \quad \nu_2(x) = \begin{cases} 1 & : x_2 \leq 0.1 \cos(x_1) + 0.5, \\ 1.5 & : \text{otherwise.} \end{cases}$$

We used 17 source/receiver pairs with common offset $\alpha = 0.05$ positioned at $(0.1, s \pm \alpha)$, $s \in \{0.15 + 0.05i : i = 0, \dots, 16\}$, to record the corresponding solutions u of (1) at the receivers. The (temporal) Dirac impulse modeling the source signal was approximated by a scaled (and truncated) Gaussian and the reference solution \tilde{u} of (3) was computed with constant sound speed $c = 1$. Thus, the right hand side y of (6) was available, see Figure 14.

In Figure 15 you see $\tilde{\Lambda}_\gamma n(\mathbf{p})$ of (16) for $\mathbf{p} \in \mathcal{G}$ and $\gamma = 0.06$ where g was replaced by the y 's of Figure 14. We clearly observe again that only those singularities can be detected which are tangent to ellipses being in the data set, i.e., being integrated over. This explains clearly why we recover more singularities at the example with the sine than with the cosine profile.

Software. The Python 2.7 code with which we have conducted the numerical experiments for the common offset geometry (Figure 5 to Figure 9) can be downloaded from http://www.waves.kit.edu/downloads/CRC1173_Preprint_2016-37_supplement.zip. We made use of the intrinsic parallel structure of the approximate inverse and organized the evaluation of $\tilde{\Lambda}_\gamma n$ at different reconstruction points in parallel using the `ProcessPoolExecutor` interface provided by the `concurrent.futures` module.

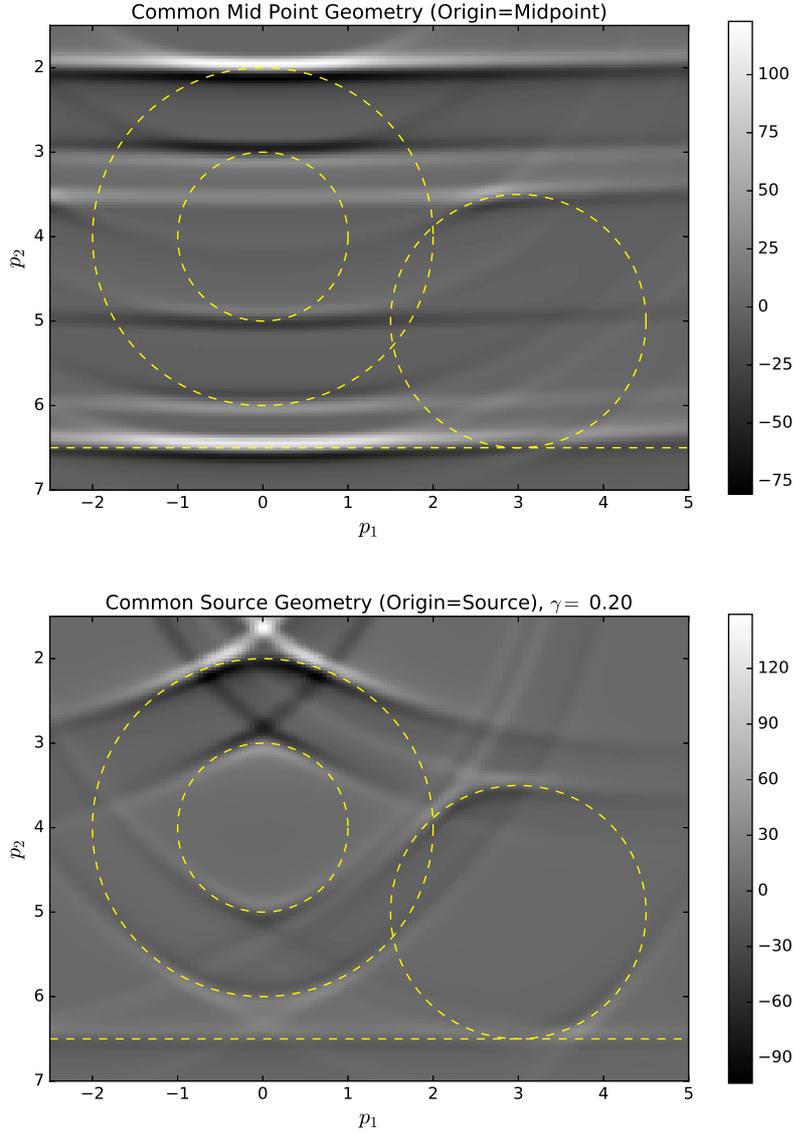


FIGURE 13. Reconstructions of n (17) from the CM (top) and CS (bottom) data which are displayed in Figure 12. The dashed lines indicate the singular support of n and are not part of the reconstruction.

APPENDIX A. WEIGHT CALCULATION

In this section we show how the integral

$$\tilde{F}w(s, t) := \int w(\mathbf{x})\delta(t - \varphi(s, \mathbf{x}))d\mathbf{x}$$

has to be evaluated where δ is the one-dimensional Dirac-distribution. Indeed, we will see that $\tilde{F}w(s, t)$ has to be understood as

$$(19) \quad \tilde{F}w(s, t) = \int_0^\pi w(\mathbf{x}(s, t, \phi)) \frac{t^2/4 - \alpha^2 \cos^2 \phi}{\sqrt{t^2 - 4\alpha^2}} d\phi$$

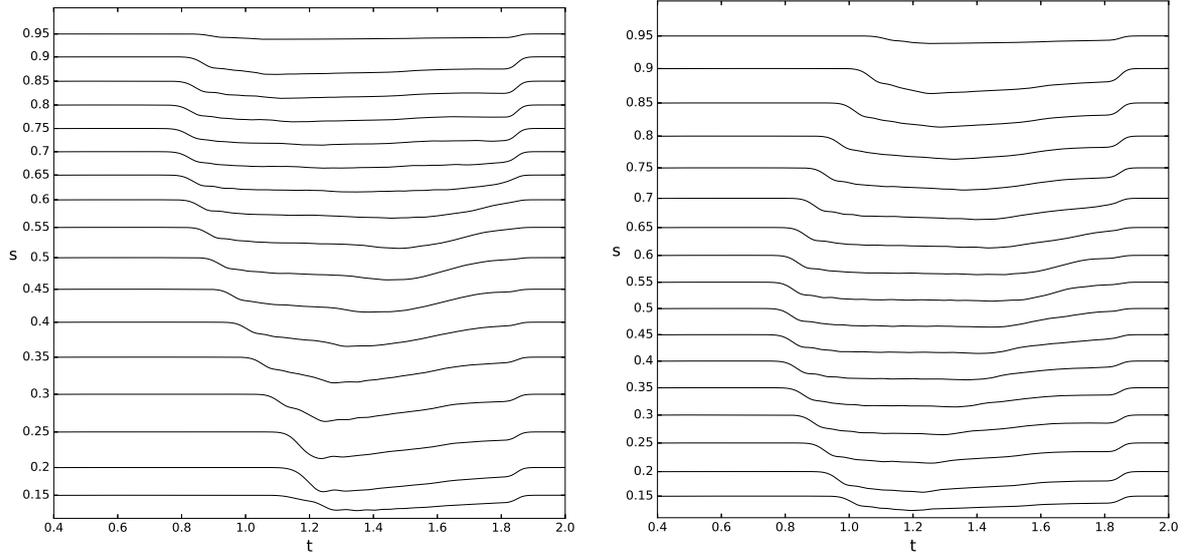


FIGURE 14. Data y of (6) as a function of time at the 17 receiver positions. The numbers at the ordinate give the midpoints s of the source/receiver pair at $(0.1, s \pm \alpha)$. Left: sine velocity profile ν_1 , right: cosine velocity profile ν_2 .

if w is integrable and supported in the lower half space. Here, $\mathbf{x}(s, t, \phi)$ are the prolate spheroidal coordinates from (11). Note that $\tilde{F}(An) = Fn$, see (8).

Our starting point is the limit representation of the one-dimensional Dirac distribution, that is,

$$(20) \quad \tilde{F}w(s, t) = \lim_{h \searrow 0} \frac{1}{h} \int_{\{\mathbf{x} \in \mathbb{R}^2 : t \leq \varphi(s, \mathbf{x}) \leq t+h\}} w(\mathbf{x}) \, d\mathbf{x}.$$

Next, we express the integral using the prolate spheroidal coordinates (with t replaced by τ). The Jacobian determinant of the corresponding coordinate transform is easily calculated to be

$$\det \begin{pmatrix} \frac{1}{2} \cos \phi & -\frac{\tau}{2} \sin \phi \\ \frac{\tau/2}{\sqrt{\tau^2 - 4\alpha^2}} \sin \phi & \frac{\sqrt{\tau^2 - 4\alpha^2}}{2} \cos \phi \end{pmatrix} = \frac{\tau^2/4 - \alpha^2 \cos^2 \phi}{\sqrt{\tau^2 - 4\alpha^2}}$$

which is positive because $\tau > 2\alpha$.

Applying this transformation and relying on $\varphi(s, \mathbf{x}(s, \tau, \phi)) = \tau$ we arrive at

$$\begin{aligned} \tilde{F}w(s, t) &= \lim_{h \searrow 0} \frac{1}{h} \int_{\{\mathbf{x} \in \mathbb{R}^d : t \leq \varphi(s, \mathbf{x}) \leq t+h\}} w(\mathbf{x}) \, d\mathbf{x} \\ &= \lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} \int_0^\pi w(\mathbf{x}(s, \tau, \phi)) \frac{\tau^2/4 - \alpha^2 \cos^2 \phi}{\sqrt{\tau^2 - 4\alpha^2}} \, d\phi \, d\tau \end{aligned}$$

which immediately implies (19) by the Lebesgue differentiation theorem.

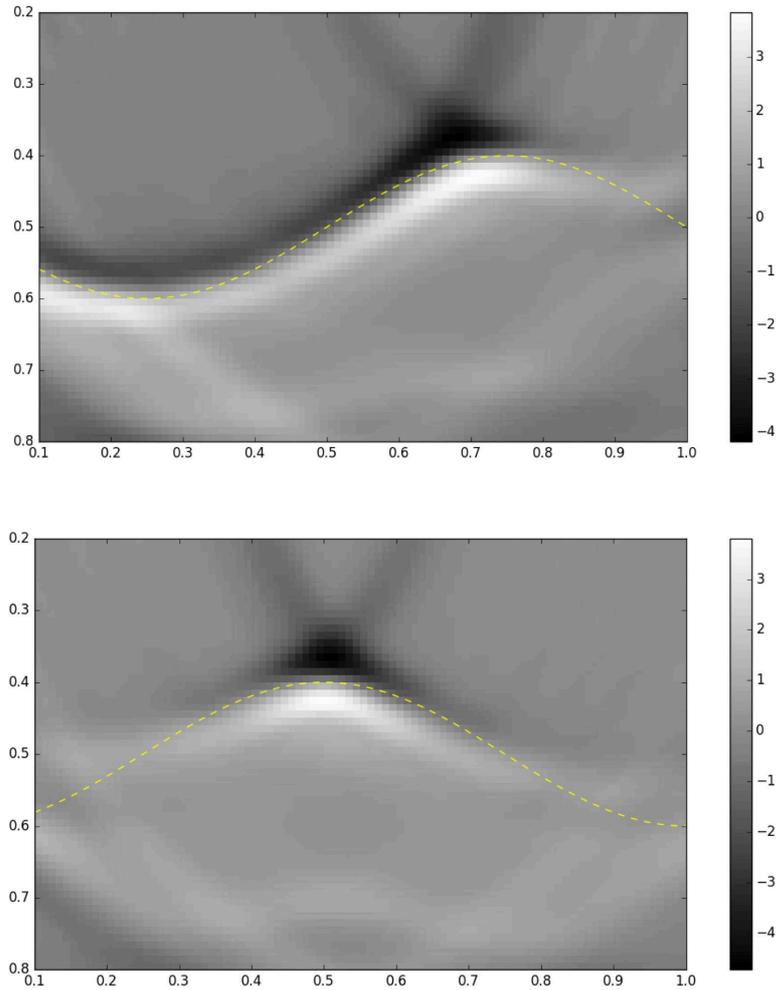


FIGURE 15. Reconstructions $\tilde{\Lambda}_\gamma n$, $\gamma = 0.06$, from the data shown in Figure 14. Top: reconstruction from data of ν_1 , bottom: reconstruction from data of ν_2 . The dashed lines indicate the true singular support of ν_1 and ν_2 , respectively, and are not part of the reconstructions.

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