SECOND-ORDER PROPERTIES AND CENTRAL LIMIT THEOREMS FOR GEOMETRIC FUNCTIONALS OF BOOLEAN MODELS[1]

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Let Z be a Boolean model based on a stationary Poisson process η of compact, convex particles in Euclidean space $\mathbb{R}^d$. Let W denote a compact, convex observation window. For a large class of functionals ψ, formulas for mean values of $\psi(Z \cap W)$ are available in the literature. The first aim of the present work is to study the asymptotic covariances of general geometric (additive, translation invariant and locally bounded) functionals of $Z \cap W$ for increasing observation window W, including convergence rates. Our approach is based on the Fock space representation associated with η. For the important special case of intrinsic volumes, the asymptotic covariance matrix is shown to be positive definite and can be explicitly expressed in terms of suitable moments of (local) curvature measures in the isotropic case. The second aim of the paper is to prove multivariate central limit theorems including Berry–Esseen bounds. These are based on a general normal approximation result obtained by the Malliavin–Stein method.

1. Introduction. Let η be a *stationary (locally finite) Poisson process* on the space $\mathcal{K}^d$ of *convex bodies* in $\mathbb{R}^d$, that is, on the space of compact, convex subsets of $\mathbb{R}^d$. The *Boolean model* associated with η is the stationary random closed set Z defined by

$$Z := \bigcup_{K \in \eta} K, \tag{1.1}$$

where the Poisson process η is identified with its support. This is a fundamental model of stochastic geometry and continuum percolation with many applications in materials science and physics [3, 7, 19, 22, 31]. The intersection of Z with a compact and convex set $W \subset \mathbb{R}^d$ is a finite union of compact, convex sets, that is, an element of the *convex ring* $\mathcal{R}^d$. It is a common strategy in stochastic geometry to extract and explore local information about Z via functionals of the intersection

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$Z \cap W$. Perhaps the most prominent examples of such functionals on $\mathcal{R}^d$ are the intrinsic volumes $V_0, \dots, V_d$, which contain important geometric information about the sets to which they are applied. For instance, for a set $K \subset \mathbb{R}^d$ from the convex ring, $V_d(K)$ is the volume, $V_{d-1}(K)$ is half the surface area (if K is the closure of its interior), and $V_0(K)$ is the *Euler characteristic* of K; see [31], Section 14.2, for more details. The intrinsic volumes have several desirable properties. In particular, they are *additive*, in the sense that $V_i(K \cup L) = V_i(K) + V_i(L) - V_i(K \cap L)$ for all $K, L \in \mathcal{R}^d$ and $i \in \{0, \dots, d\}$. They are also translation invariant, and continuous if restricted to the space of convex bodies.

For a stationary and isotropic Boolean model, Miles [20] and Davy [5] obtained explicit formulas expressing the mean values $\mathbb{E} V_i(Z \cap W)$ in terms of the intensity measure of η. We refer to [31], Section 9.1, for a discussion and more recent developments related to this fundamental result.

In the following, we are especially interested in second-order properties and central limit theorems of the random vector $(V_0(Z \cap W), \dots, V_d(Z \cap W))$, for a compact and convex *observation window* W, but in fact we study more general additive functionals of $Z \cap W$, namely so called *geometric functionals*. A functional on the convex ring will be called geometric if it is additive, translation invariant, locally bounded, and measurable (see Section 3 for details).

While previous contributions focus on second-order properties and central limit theorems for volume and surface area, to the best of our knowledge we present here the first systematic mathematical investigation of second-order properties and central limit theorems of all intrinsic volumes and more general geometric functionals of a stationary Boolean model Z. The volume functional was first studied in [1, 17], while in [9] Berry–Esseen bounds and large deviation inequalities were established. The surface area was investigated in [21], and the results were extended in [10] to more general functionals and point processes. Integrals over Boolean models are considered in [2, 26], where the volume is included as a special case and also the surface area in the latter one. Volume and surface area of a more general Boolean model based on a Poisson process of cylinders have been investigated in [11, 12]. From a geometric point of view, volume and surface area are rather special functionals of Z. They arise as the restriction of deterministic measures to Z or the boundary of Z and do not involve the curvature of the (possibly intersecting) grains. A different though mathematically nonrigorous treatment of second moments of curvature measures of an isotropic Boolean model with an interesting application to morphological thermodynamics was presented in [18].

Our first main aim in this paper is to use the Fock space representation of Poisson functionals [15] to explore the covariance structure of geometric functionals of $Z \cap W$. Combined with some new integral geometric inequalities, which are derived by methods and results from convex and integral geometry, this approach appears to be perfectly tailored to our purposes. Under the minimal assumption that the second moments of the intrinsic volumes of the typical grain are finite, we show that for two geometric functionals ψ_1 and ψ_2 the ratio

$V_d(W)^{-1}\operatorname{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))$ tends to some limit $\sigma_{\psi_1,\psi_2} \in \mathbb{R}$ as the observation window is increased in a proper way. For the case that the third moments of the intrinsic volumes of the typical grain are finite, we establish a rate of this convergence in terms of the inradius of the observation window and show that it is optimal. Via the Fock space representation the asymptotic covariances can be expressed as series of second moments. In the important case of intrinsic volumes of an isotropic Boolean model they can be represented in terms of curvature based moment measures of the typical grain. In particular, the covariance structure of the two-dimensional isotropic Boolean model becomes surprisingly explicit. For a vector of geometric functionals of the Boolean model, it is shown that the asymptotic covariance matrix is positive definite under some additional conditions, which are for example satisfied for the intrinsic volumes. The second-order analysis is illustrated by explicit formulas for intrinsic volumes of a Boolean model with deterministic spherical grains, for which our formulas reduce to three-dimensional integration of explicitly known integrands.

Our second main aim is to prove univariate and multivariate central limit theorems for geometric functionals of $Z \cap W$. Under the same second moment assumptions as for the existence of the asymptotic covariances, we prove convergence in distribution. We also obtain rates of convergence under slightly stronger moment assumptions. For the multivariate central limit theorem, we argue that the rate is optimal. Following common belief, we guess that our convergence rate $V_d(W)^{-1/2}$ for the univariate case is optimal as well. In the univariate case, we do not need to assume that the functional on the convex ring is translation invariant. In the proofs, we use the Malliavin–Stein method for Poisson functionals that was recently developed in [23, 25]. In a sense, this method builds on the Fock space representation and the closely related Wiener–Itô chaos expansion of Poisson functionals. The main obstacle to the application of these results is the fact that, as a rule, geometric functionals of Z admit an infinite chaos expansion. We can resolve this by bounding the kernels of the chaos expansion by monotone functionals.

In the case of bounded grains, it is likely that the central limit theorem and the convergence of covariances can be derived with the theory of m-dependent random fields, perhaps even with rates of convergence. From there, one might proceed to the general case using a truncation argument as in [8, 12]. But such an approach would neither yield much information on the asymptotic covariance structure nor rates of convergence in the general case. Stabilization is another common approach to central limit theorems in stochastic geometry. We refer here to [2, 26, 27], where the first two references deal in particular with volume and surface area of the Boolean model without discussing rates of convergence. It is unclear whether the intrinsic volumes (other than volume or surface area) stabilize for Boolean models with unbounded grains. But even if they do, the quantitative bounds for the normal approximation derived by stabilization in [27] suggest that the rates would probably be suboptimal. Moreover, in our setting we would need to control boundary effects.

This paper is organized as follows. In the second section, we briefly summarize some notation and basic facts about the Boolean model and present a central limit theorem for the intrinsic volumes of the Boolean model to be generalized later. In the third section, we establish the existence of the asymptotic covariances of a vector of geometric functionals of $Z \cap W$ and determine the rate of convergence; see Theorem 3.1. Section 4 is devoted to the positive definiteness of the asymptotic covariance matrix; see Theorem 4.1. In Section 5, we focus on intrinsic volumes and introduce a family of curvature based moment measures of the typical grain to study infinite series of second moments arising in the Fock space representation. The main result of this section (Theorem 5.2) is of some independent interest and is applied in Section 6 to derive formulas for the asymptotic covariances of the intrinsic volumes of an isotropic Boolean model in terms of the moment measures mentioned above; see Theorem 6.1. Section 7 presents some explicit results for a Boolean model with deterministic spherical grains. In Section 8, we provide a general result on the normal approximation of Poisson functionals. We use this result in Section 9 to establish multivariate and univariate central limit theorems for geometric functionals of Z; see Theorems 9.1 and 9.3.

The extended arXiv-version [13] of this paper contains two additional appendices with a description of the curvature based moment measures from Section 5 in terms of mixed measures of translative integral geometry and with integral formulas for the exact (nonasymptotic) covariances of intrinsic volumes, which are rather explicit in the two-dimensional case.

2. Boolean models and intrinsic volumes. In this section, we collect a few basic facts about the stationary Poisson process η and the associated Boolean model Z in Euclidean space $\mathbb{R}^d$ before stating some of our main results for the special case of intrinsic volumes. For more details on Boolean models, we refer the reader to [3], Chapter 3, [22] or [31], Chapter 4, whereas background material on convex geometry can be found, for example, in [30] or [31], Chapter 14. All random objects occurring in this paper are defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A measure on $\mathcal{K}^d$ is *locally finite* if it assigns a finite number to $\{K \in \mathcal{K}^d : K \cap C \neq \varnothing\}$ for all C in the space $\mathcal{C}^d$ of compact subsets of $\mathbb{R}^d$. We consider the Poisson process η as a random element in the space $\mathbf{N}$ of all locally finite counting measures on $\mathcal{K}^d$, equipped with the smallest σ-field such that the mappings $\mu \mapsto \mu(A)$ are measurable for all A in the Borel σ-field (with respect to the Hausdorff metric) of $\mathcal{K}^d$. We assume that the *intensity measure* $\Lambda := \mathbb{E}\eta$ of η is invariant under the shifts $K \mapsto K + x := \{y + x : y \in K\}$, $x \in \mathbb{R}^d$. This is equivalent to the *stationarity* of η, that is to the distributional invariance of η under all shifts. We also assume that Λ is nontrivial and that $\Lambda(\{\varnothing\}) = 0$, which effectively excludes empty grains. Theorem 4.1.1 in [31] implies that

$$\Lambda(\cdot) = \gamma \iint \mathbf{1}\{K + x \in \cdot\}\, dx\, \mathbb{Q}(dK), \tag{2.1}$$

where $\gamma \in (0, \infty)$ is the *intensity* of η, "dx" denotes integration with respect to the d-dimensional Lebesgue measure λ_d, and $\mathbb{Q}$ is a probability measure on $\mathcal{K}^d$ satisfying $\mathbb{Q}(\{\varnothing\}) = 0$ as well as

$$\int V_d(K + C)\mathbb{Q}(dK) < \infty, \qquad C \in \mathcal{C}^d. \tag{2.2}$$

Here, as usual, $K + C := \{x + y : x \in K, y \in C\}$ is the *Minkowski sum* of K and C. Let Z_0 denote a *typical grain*, that is, a random convex set with distribution $\mathbb{Q}$. Then (2.2) can be written as

$$v_i := \mathbb{E}V_i(Z_0) < \infty, \qquad i = 0, \dots, d. \tag{2.3}$$

This is a direct consequence of the *Steiner formula* (see [31], equation (14.5))

$$V_d\big(K + B^d_r\big) = \sum_{i=0}^{d} \kappa_{d-i} r^{d-i} V_i(K), \qquad r \geq 0, K \in \mathcal{K}^d, \tag{2.4}$$

where B^d is the closed unit ball centered at the origin, $B^d_r := \{rx : x \in B^d\}$, and κ_n denotes the volume of the n-dimensional unit ball.

The Boolean model is given by $Z \equiv Z(\eta)$, where

$$Z(\mu) := \bigcup_{K \in \mu} K, \qquad \mu \in \mathbf{N},$$

and $K \in \mu$ means that $\mu(\{K\}) > 0$. Recall that the mapping $\mu \mapsto Z(\mu)$ from $\mathbf{N}$ to the space of all closed subsets of $\mathbb{R}^d$ (equipped with the Fell topology) is Borel measurable (see [31], Theorem 3.6.2). Without loss of generality, we can assume that $\mathbb{Q}$ is concentrated on $\mathcal{K}^d_o$, where $\mathcal{K}^d_o$ is the space of nonempty convex bodies such that the center of the circumscribed ball is at the origin. Since the center of the circumscribed ball of a convex body is always contained in the convex body, we have $0 \in K$ for all $K \in \mathcal{K}^d_o$.

Subsequently, we shall need integrability assumptions such as

$$\mathbb{E}V_i(Z_0)^2 < \infty, \qquad i = 0, \dots, d, \tag{2.5}$$

or

$$\mathbb{E}V_i(Z_0)^3 < \infty, \qquad i = 0, \dots, d. \tag{2.6}$$

We next introduce two basic characteristics of the Boolean model Z. The *volume fraction* $p := \mathbb{E}V_d(Z \cap [0, 1]^d)$ of Z can be expressed in the form $p = 1 - e^{-\gamma v_d}$. The mean *covariogram* of the typical grain is given by

$$C_d(x) := \mathbb{E}V_d\big(Z_0 \cap (Z_0 + x)\big), \qquad x \in \mathbb{R}^d. \tag{2.7}$$

It follows from (2.3) that $C_d(x) \leq v_d < \infty$ and that $C_d(x) \to 0$ as $\|x\| \to \infty$, where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. It is well known (see, e.g., [3], equation (3.18)) that the *covariance* of Z satisfies

$$\mathbb{P}(0 \in Z, x \in Z) = p^2 + (1 - p)^2\big(e^{\gamma C_d(x)} - 1\big). \tag{2.8}$$

For $W \in \mathcal{K}^d$, we define by $C_W(x) := V_d(W \cap (W + x))$, $x \in \mathbb{R}^d$, the set covariance function of W. Combining (2.8) with Fubini's theorem leads to the well-known formula

$$\text{(2.9)} \quad \operatorname{Var} V_d(Z \cap W) = (1 - p)^2 \int C_W(x)\big(e^{\gamma C_d(x)} - 1\big)\, dx, \qquad W \in \mathcal{K}^d.$$

Throughout this paper, we investigate the intersection $Z \cap W$ of the Boolean model Z with an expanding compact convex observation window W. More precisely, we consider sequences of convex bodies $(W_m)_{m \in \mathbb{N}}$ satisfying $\lim_{m \to \infty} r(W_m) = \infty$, where $r(W)$ denotes the *inradius* of $W \in \mathcal{K}^d$. We describe this situation by writing $r(W) \to \infty$ for short. Combining our Theorems 9.1 and 4.1 in the special case of intrinsic volumes of $Z \cap W$, we obtain the following multivariate central limit theorem.

THEOREM 2.1. *Assume that* (2.5) *is satisfied and let* $\mathcal{V} := (V_0, \dots, V_d)$. *Then there exists a* $(d + 1)$*-dimensional centered Gaussian random vector* N *with a covariance matrix* Σ *such that*

$$\frac{1}{\sqrt{V_d(W)}}\big(\mathcal{V}(Z \cap W) - \mathbb{E}\mathcal{V}(Z \cap W)\big) \xrightarrow{d} N \qquad \text{as } r(W) \to \infty.$$

If, additionally, the typical grain Z_0 *has nonempty interior with positive probability, the covariance matrix* Σ *is positive definite.*

To the best of our knowledge, this is the first central limit theorem for the intrinsic volumes of the Boolean model beyond volume and surface area. In fact, our Theorem 9.1 generalizes this result in several ways. It concerns a broader class of functionals and is also quantitative in the sense that it provides rates of convergence for a suitable distance under moment conditions slightly stronger than (2.6). Theorem 9.3 yields presumably optimal rates for the Wasserstein distance in the univariate case. As already mentioned above, our proofs rely on the Malliavin–Stein method for Poisson functionals. We are not aware of any other approach that might yield the same rates. In Section 6, we will derive formulas for the asymptotic covariances between the intrinsic volumes of an isotropic Boolean model.

3. Covariance structure of geometric functionals. In this paper, we study random variables of the form $\psi(Z \cap W)$, where ψ is a real-valued measurable function defined on the *convex ring* $\mathcal{R}^d$ whose elements are finite unions of compact, convex sets. Measurability again refers to the Borel σ-field generated by the Fell topology (or, equivalently, by the Hausdorff metric). We shall assume that ψ is *additive*, that is, $\psi(\varnothing) = 0$ and $\psi(K \cup L) = \psi(K) + \psi(L) - \psi(K \cap L)$ for all $K, L \in \mathcal{R}^d$. We shall also assume that ψ is *translation invariant*, that is, $\psi(K + x) = \psi(K)$ for all $(K, x) \in \mathcal{R}^d \times \mathbb{R}^d$, and *locally bounded* in the sense that its absolute value is (uniformly) bounded on compact, convex sets contained in a translate of the unit cube $Q_1 := [-1/2, 1/2]^d$ by a constant

$$M(\psi) := \sup\{|\psi(K)| : K \in \mathcal{K}^d, K \subset Q_1 + x, x \in \mathbb{R}^d\} < \infty. \tag{3.1}$$

Note that this definition simplifies in the translation-invariant case since one does not need the translations of Q_1.

In the following, we call a functional $\psi : \mathcal{R}^d \to \mathbb{R}$ *geometric* if it is additive, translation invariant, locally bounded and measurable. Examples of geometric functionals are (1) mixed volumes (see Section 5.1 in [30]) of the form $\psi(K) := V(K[k], K_1, \dots, K_{d-k})$, where $k \in \{0, \dots, d\}$, the notation $K[k]$ means that the body K is repeated k times, and $K_1, \dots, K_{d-k} \in \mathcal{R}^d$ are fixed. Up to normalization, intrinsic volumes are obtained for $K_i = B^d$, $i = 1, \dots, d-k$; (2) integrals of surface area measures (see Sections 4.1 and 4.2 in [30]) of the form $\psi_k(K) := \int_{\mathbb{S}^{d-1}} h(u) S_k(K, du)$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ (the boundary of B^d), $h : \mathbb{S}^{d-1} \to \mathbb{R}$ is measurable and bounded, and $k \in \{0, \dots, d-1\}$; (3) the centered support function $\psi(K) := h(K - s(K), u)$ in a fixed direction u, where $u \in \mathbb{R}^d$ and $s(K)$ is the Steiner point of K (see Section 1.7, Section 5.4, equation (5.100) in [30] and Lemma 6.1 in [31]); (4) total measures of translative integral geometry (see Section 6.4, especially page 234, and page 383 in [31]). These examples of geometric functionals are substantially more general than the intrinsic volumes. For instance, whereas intrinsic volumes are always rotation invariant, no such invariance is built into these four classes of examples in general. Moreover, it should be observed that linear combinations of mixed volumes are dense in the (normed) vector space of translation invariant, continuous valuations on convex bodies in $\mathbb{R}^n$ (see Section 6.5, page 406, in [30]).

Our main result of this section deals with the asymptotic behavior of the covariance between two geometric functionals of $Z \cap W$ for expanding convex observation window W. With a measurable functional $\psi : \mathcal{R}^d \to \mathbb{R}$, we associate another measurable function $\psi^* : \mathcal{K}^d \to \mathbb{R}$ by

$$\psi^*(K) := \mathbb{E}\psi(Z \cap K) - \psi(K), \qquad K \in \mathcal{K}^d, \tag{3.2}$$

if $\mathbb{E}|\psi(Z \cap K)| < \infty$ for all $K \in \mathcal{K}^d$. Under assumption (2.2), it follows from (3.10) below that ψ^* is well defined for a geometric functional ψ. For $C \in \mathcal{C}^d$ let N_C denote the number of all particles in η intersecting C.

THEOREM 3.1. *Let ψ_1 and ψ_2 be geometric functionals. If (2.5) is satisfied, then the limit*

$$\sigma_{\psi_1,\psi_2} = \lim_{r(W)\to\infty} \frac{\operatorname{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))}{V_d(W)} \tag{3.3}$$

exists, is finite, and is given by

$$\sigma_{\psi_1,\psi_2} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint \psi_1^*(K_1 \cap K_2 \cap \cdots \cap K_n)\psi_2^*(K_1 \cap K_2 \cap \cdots \cap K_n) \times \Lambda^{n-1}(d(K_2, \dots, K_n))\mathbb{Q}(dK_1). \tag{3.4}$$

Assume that (2.6) *holds and define*

$$c_\Lambda := 2^{d+2} \cdot 4^{2^d} \cdot 25^{2d} d! (\mathbb{E} 2^{N_{Q_1}} + 1)^2 \times \exp\left(2^{2^d} \cdot 25^d (d+1)! \gamma \sum_{i=0}^{d} \mathbb{E} V_i(Z_0)\right) \gamma \mathbb{E}\left(\sum_{i=0}^{d} V_i(Z_0)\right)^3.$$

Then, for $W \in \mathcal{K}^d$ *with* $r(W) \geq 1$,

$$\left|\frac{\operatorname{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))}{V_d(W)} - \sigma_{\psi_1,\psi_2}\right| \leq \frac{c_\Lambda M(\psi_1) M(\psi_2)}{r(W)}. \tag{3.5}$$

We start with some preparations. Our main probabilistic tool is the following Fock space representation of Poisson functionals, derived in [15]. For any measurable $f : \mathbf{N} \to \mathbb{R}$ and $K \in \mathcal{K}^d$, the function $D_K f : \mathbf{N} \to \mathbb{R}$ is defined by

$$D_K f(\mu) := f(\mu + \delta_K) - f(\mu), \qquad \mu \in \mathbf{N}, \tag{3.6}$$

where δ_K is the Dirac measure located at K. The *difference operator* D_K and its iterations play a central role in the analysis of Poisson processes. For $n \geq 2$ and $(K_1, \dots, K_n) \in (\mathcal{K}^d)^n$ we define a function $D^n_{K_1,\dots,K_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{K_1,\dots,K_n} f := D^1_{K_1} D^{n-1}_{K_2,\dots,K_n} f,$$

where $D^1 := D$. Note that

$$D^n_{K_1,\dots,K_n} f(\mu) = \sum_{J \subset \{1,2,\dots,n\}} (-1)^{n-|J|} f\left(\mu + \sum_{j \in J} \delta_{K_j}\right),$$

where $|J|$ denotes the number of elements of J. This shows that the operator $D^n_{K_1,\dots,K_n}$ is symmetric in $K_1, \dots, K_n$, and that $(\mu, K_1, \dots, K_n) \mapsto D^n_{K_1,\dots,K_n} f(\mu)$ is measurable. From [15], Theorem 1.1, we obtain for any measurable $f, g : \mathbf{N} \to \mathbb{R}$ satisfying $\mathbb{E} f(\eta)^2 < \infty$ and $\mathbb{E} g(\eta)^2 < \infty$ that

$$\begin{aligned} &\operatorname{Cov}(f(\eta), g(\eta)) \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{E} D^n_{K_1,\dots,K_n} f(\eta) \mathbb{E} D^n_{K_1,\dots,K_n} g(\eta) \Lambda^n(d(K_1, \dots, K_n)). \end{aligned} \tag{3.7}$$

For given $W \in \mathcal{K}^d$ and a functional $\psi : \mathcal{R}^d \to \mathbb{R}$ we shall apply (3.7) to functions $f_{\psi,W} : \mathbf{N} \to \mathbb{R}$ defined by $f_{\psi,W}(\mu) := \psi(Z(\mu) \cap W)$. Induction yields the following lemma.

LEMMA 3.2. *Let* $\psi : \mathcal{R}^d \to \mathbb{R}$ *be additive. Then, for* $n \in \mathbb{N}$, $K_1, \dots, K_n \in \mathcal{K}^d$, *and* $\mu \in \mathbf{N}$,

$$\begin{aligned} &D^n_{K_1,\dots,K_n} f_{\psi,W}(\mu) \\ &\quad = (-1)^n \big(\psi(Z(\mu) \cap K_1 \cap \cdots \cap K_n \cap W) - \psi(K_1 \cap \cdots \cap K_n \cap W)\big). \end{aligned}$$

LEMMA 3.3. *Let ψ be an additive, locally bounded and measurable functional and assume that* (2.2) *is satisfied. Then, for all $n \in \mathbb{N}$, $K_1, \dots, K_n \in \mathcal{K}^d$, and $W \in \mathcal{K}^d$,*

$$\mathbb{E}D^n_{K_1,\dots,K_n} f_{\psi,W}(\eta) = (-1)^n \psi^*(K_1 \cap \cdots \cap K_n \cap W) \tag{3.8}$$

and

$$\big|\mathbb{E}D^n_{K_1,\dots,K_n} f_{\psi,W}(\eta)\big| \le \beta(\psi) \sum_{i=0}^{d} V_i(K_1 \cap \cdots \cap K_n \cap W) \tag{3.9}$$

with $\beta(\psi) = 2^{2^d} \cdot 5^d M(\psi)(\mathbb{E}2^{N_{Q_1}} + 1)$. Moreover, for any $A \in \mathcal{K}^d$,

$$\mathbb{E}\psi(Z \cap A)^2 < \infty. \tag{3.10}$$

PROOF. We start by proving that there is a constant $c_1 > 0$ such that

$$\mathbb{E}\big|\psi(Z \cap A)\big| \le c_1 M(\psi) \sum_{i=0}^{d} V_i(A) \tag{3.11}$$

for $A \in \mathcal{K}^d$. Since (3.11) is obviously true for $A = \varnothing$, we assume $A \neq \varnothing$ in the following. We define $\mathcal{Q}(A) := \{Q_1 + z : z \in \mathbb{Z}^d, (Q_1 + z) \cap A \neq \varnothing\}$. By the inclusion–exclusion formula for additive functionals (see, e.g., [30], (6.2), page 330), we have

$$\big|\psi(Z \cap A)\big| = \left|\psi\left(Z \cap A \cap \bigcup_{Q \in \mathcal{Q}(A)} Q\right)\right| \le \sum_{\varnothing \neq \mathcal{I} \subset \mathcal{Q}(A)} \left|\psi\left(Z \cap A \cap \bigcap_{Q \in \mathcal{I}} Q\right)\right|.$$

For each nonempty subset $\mathcal{I} \subset \mathcal{Q}(A)$, we fix some cube $Q_{\mathcal{I}} \in \mathcal{I}$. Let $Z_1, \dots, Z_{N_{Q_{\mathcal{I}}}}$ denote the particles hitting $Q_{\mathcal{I}}$. Then, for $\varnothing \neq J \subset \{1, \dots, N_{Q_{\mathcal{I}}}\}$, assumption (3.1) yields that

$$\left|\psi\left(\bigcap_{j \in J} Z_j \cap A \cap \bigcap_{Q \in \mathcal{I}} Q\right)\right| \le M(\psi).$$

By the inclusion–exclusion formula and taking into account that $\psi(\varnothing) = 0$, we get

$$\begin{aligned}
\big|\psi(Z \cap A)\big| &\le \sum_{\varnothing \neq \mathcal{I} \subset \mathcal{Q}(A)} \left|\psi\left(\bigcup_{j=1}^{N_{Q_{\mathcal{I}}}} Z_j \cap A \cap \bigcap_{Q \in \mathcal{I}} Q\right)\right| \\
&\le \sum_{\varnothing \neq \mathcal{I} \subset \mathcal{Q}(A)} \sum_{\varnothing \neq J \subset \{1,\dots,N_{Q_{\mathcal{I}}}\}} \left|\psi\left(\bigcap_{j \in J} Z_j \cap A \cap \bigcap_{Q \in \mathcal{I}} Q\right)\right| \\
&\le \sum_{\varnothing \neq \mathcal{I} \subset \mathcal{Q}(A)} \mathbf{1}\left\{\bigcap_{Q \in \mathcal{I}} Q \neq \varnothing\right\} 2^{N_{Q_{\mathcal{I}}}} M(\psi). \qquad (3.12)
\end{aligned}$$

The cubes in $\mathcal{Q}(A)$ form a grid, hence

$$\left|\left\{\varnothing \neq \mathcal{I} \subset \mathcal{Q}(A) : \bigcap_{Q \in \mathcal{I}} Q \neq \varnothing\right\}\right| \leq c_2 |\mathcal{Q}(A)|$$

with $c_2 := 2^{2^d}$. By stationarity of η, we have $\mathbb{E}2^{N_{Q_\mathcal{I}}} = \mathbb{E}2^{N_{Q_1}}$, and thus

$$\mathbb{E}|\psi(Z \cap A)| \leq c_3 M(\psi) |\mathcal{Q}(A)| \tag{3.13}$$

with $c_3 := c_2 \mathbb{E}2^{N_{Q_1}}$. Here, we have used that $\mathbb{E}z^{N_C} = \exp((z-1)\gamma \mathbb{E}V_d(Z_0 + C^*)) < \infty$ holds for $C \in \mathcal{K}^d$ and $z \geq 0$, where $C^* := \{-x : x \in C\}$ is the reflection of C in the origin.

Since $|\mathcal{Q}(A)| \leq V_d(A + \sqrt{d}B^d)$, Steiner's formula (2.4) yields

$$|\mathcal{Q}(A)| \leq \sum_{i=0}^{d} \kappa_{d-i} d^{(d-i)/2} V_i(A) \leq c_4 \sum_{i=0}^{d} V_i(A) \tag{3.14}$$

with $c_4 := 5^d$. In the last step, we used that $\kappa_{d-i} d^{(d-i)/2} \leq 5^d$, $i \in \{0, \dots, d\}$, which can be deduced by elementary calculus from the representation of κ_{d-i} in terms of the Gamma function and from Stirling's formula. Now (3.13) together with (3.14) yields (3.11) with $c_1 := c_3 c_4$. Combining (3.11) with Lemma 3.2 and the definition of ψ^* in (3.2) shows (3.8). For $A \in \mathcal{K}^d$, we can argue as in the derivation of (3.12), and then use (3.14), to get

$$|\psi(A)| \leq c_2 |\mathcal{Q}(A)| M(\psi) \leq c_2 c_4 M(\psi) \sum_{i=0}^{d} V_i(A). \tag{3.15}$$

Combining (3.11) and (3.15) for $A = K_1 \cap \cdots \cap K_n \cap W$ with Lemma 3.2 yields (3.9).

In order to show that $\psi(Z \cap A)$ is square integrable, we first derive an upper bound for

$$M_A(\psi) := \sup\{|\psi(L)| : L \in \mathcal{K}^d, L \subset A\}. \tag{3.16}$$

Let $L \in \mathcal{K}^d$ with $L \subset A$. Then, using the inclusion–exclusion formula for additive functionals and (3.1), we get

$$|\psi(L)| = \left|\psi\left(L \cap \bigcup_{Q \in \mathcal{Q}(A)} Q\right)\right| \leq 2^{|\mathcal{Q}(A)|} M(\psi),$$

and hence $M_A(\psi) \leq 2^{|\mathcal{Q}(A)|} M(\psi)$. Again by the inclusion–exclusion formula, we have

$$|\psi(Z \cap A)| \leq (2^{N_A} - 1) M_A(\psi) \leq 2^{N_A} M_A(\psi), \tag{3.17}$$

and, therefore,

$$\mathbb{E}\psi(Z \cap A)^2 \leq \mathbb{E}[4^{N_A}] 4^{|\mathcal{Q}(A)|} M(\psi)^2 < \infty,$$

which completes the proof. $\square$

LEMMA 3.4. *Define* $\beta_1 := 2^{2^d} \cdot 25^d d!$. *Then, for all* $k \in \{0, \dots, d\}$ *and* $W, K \in \mathcal{K}^d$,

$$\int V_k\big(W \cap (K + x)\big)\, dx \le \beta_1 \sum_{i=0}^{d} V_i(W) \sum_{r=k}^{d} V_r(K). \tag{3.18}$$

PROOF. Using the same notation as in the proof of Lemma 3.3 and the fact that V_k is increasing and translation invariant, we obtain that

$$\begin{aligned}\int V_k\big(W \cap (K + x)\big)\, dx &\le \sum_{\varnothing \ne \mathcal{I} \subset \mathcal{Q}(W)} \int V_k\Big(W \cap \bigcap_{Q \in \mathcal{I}} Q \cap (K + x)\Big)\, dx \\ &\le \sum_{\varnothing \ne \mathcal{I} \subset \mathcal{Q}(W)} \mathbf{1}\Big\{\bigcap_{Q \in \mathcal{I}} Q \ne \varnothing\Big\} \int V_k\big(K \cap (Q_1 + x)\big)\, dx.\end{aligned}$$

Let B' denote a ball of radius $\sqrt{d}/2$. Then the kinematic formula (see [31], Theorem 5.1.3, and note that $c_{j,d}^{k,d-k+j} \le 1$) and the rotation invariance of B' yield that

$$\int V_k\big(K \cap (Q_1 + x)\big)\, dx \le \int V_k\big(K \cap (B' + x)\big)\, dx \le c_5 \sum_{r=k}^{d} V_r(K)$$

with $c_5 := 5^d d!$. On the other hand, it was shown in the proof of Lemma 3.3 that

$$\Big|\Big\{\varnothing \ne \mathcal{I} \subset \mathcal{Q}(W) : \bigcap_{Q \in \mathcal{I}} Q \ne \varnothing\Big\}\Big| \le c_2 c_4 \sum_{i=0}^{d} V_i(W).$$

Combining the preceding inequalities, we obtain the assertion of the lemma. □

LEMMA 3.5. *For* $A \in \mathcal{K}^d$ *and* $n \in \mathbb{N}$,

$$\int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \cdots \cap K_n) \Lambda^n\big(d(K_1, \dots, K_n)\big) \le \alpha^n \sum_{k=0}^{d} V_k(A),$$

where $\alpha = \gamma(d+1)\beta_1 \sum_{i=0}^{d} \mathbb{E} V_i(Z_0)$ *with* β_1 *as in Lemma* 3.4.

PROOF. In the following calculation and also later, we use the convention $\int c\, d\Lambda^0 := c$. We apply (2.1) and (3.18) to get

$$\begin{aligned}&\int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \cdots \cap K_n) \Lambda^n\big(d(K_1, \dots, K_n)\big) \\ &\quad = \sum_{k=0}^{d} \gamma \iiint V_k\big(A \cap K_1 \cap \cdots \cap K_{n-1} \cap (K_n + y)\big)\, dy \mathbb{Q}(dK_n)\end{aligned}$$

$$
\begin{aligned}
&\times \Lambda^{n-1}\big(d(K_1,\dots,K_{n-1})\big)\\
&\le \sum_{k=0}^{d} \gamma \iint \beta_1 \sum_{i=0}^{d} V_i(A\cap K_1\cap\cdots\cap K_{n-1})\\
&\qquad \times \sum_{r=k}^{d} V_r(K_n)\mathbb{Q}(dK_n)\Lambda^{n-1}\big(d(K_1,\dots,K_{n-1})\big)\\
&\le \gamma(d+1)\beta_1 \sum_{i=0}^{d} \mathbb{E}V_i(Z_0)\int \sum_{k=0}^{d} V_k(A\cap K_1\cap\cdots\cap K_{n-1})\\
&\qquad \times \Lambda^{n-1}\big(d(K_1,\dots,K_{n-1})\big).
\end{aligned}
$$

By iterating this step $(n-1)$ more times, we obtain the assertion. □

LEMMA 3.6. *Define* $\beta_2 := 2\cdot 25^d$. *Then, for* $K, W \in \mathcal{K}^d$,

$$
\lambda_d\big(\{x\in\mathbb{R}^d : (K+x)\cap\partial W \neq \varnothing\}\big) \le \beta_2 \sum_{i=0}^{d-1} V_i(W)\sum_{r=0}^{d} V_r(K).
$$

PROOF. Let $W \neq \varnothing$ and let $\mathcal{Q}(\partial W) := \{Q_1+z : z\in\mathbb{Z}^d, (Q_1+z)\cap\partial W\neq\varnothing\}$. Then we have

$$
\begin{aligned}
&\lambda_d\big(\{x\in\mathbb{R}^d : (K+x)\cap\partial W \neq \varnothing\}\big)\\
&\quad\le \sum_{Q\in\mathcal{Q}(\partial W)} \int \mathbf{1}\{(K+x)\cap Q\neq\varnothing\}\,dx\\
&\quad= \sum_{Q\in\mathcal{Q}(\partial W)} V_d(K+Q_1) \le \big|\mathcal{Q}(\partial W)\big| c_4 \sum_{r=0}^{d} V_r(K)
\end{aligned}
$$

with the same constant c_4 as in (3.14). Let $\operatorname{dist}(x, A) := \inf\{\|x-y\| : y\in A\}$ for $x\in\mathbb{R}^d$ and a closed set $A\subset\mathbb{R}^d$, and let $\partial_r^- W := \{x\in W : \operatorname{dist}(x,\partial W)\le r\}$ for $r\ge 0$. Then

$$
V_d\big(\partial_r^- W\big) \le V_d\big(W+B_r^d\big) - V_d(W). \tag{3.19}
$$

To see this, let $p_W : \mathbb{R}^d \to W$ denote the metric projection to W and consider the map $T : (W+B_r^d)\setminus W \to \mathbb{R}^d$, $x\mapsto 2p_W(x)-x$. Let $x\in\partial_r^- W$ and choose a point $y\in\partial W$ such that $\|x-y\| = \operatorname{dist}(x,\partial W)\le r$. Using that $y-x$ is an outer normal of W at y, it is easy to see that $T(2y-x)=x$. Hence, $\partial_r^- W\subset T((W+B_r^d)\setminus W)$. Since the metric projection is 1-Lipschitz, it is not hard to prove that T has the same property. Therefore, (3.19) follows. This yields that

$$
\begin{aligned}
\left|\mathcal{Q}(\partial W)\right| &\le \lambda_d\left(\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial W) \le \sqrt{d}\}\right) \le 2\left(V_d\left(W + B^d_{\sqrt{d}}\right) - V_d(W)\right) \\
&\le 2c_4 \sum_{i=0}^{d-1} V_i(W),
\end{aligned}
$$

where Steiner's formula was used. □

LEMMA 3.7. *Let $W \in \mathcal{K}^d$ be such that $r(W) > 0$ and let $k \in \{0, \dots, d-1\}$. Then*

$$
\frac{V_k(W)}{V_d(W)} \le \frac{2^d - 1}{\kappa_{d-k} r(W)^{d-k}} \le \frac{2^d d!}{r(W)^{d-k}}.
$$

PROOF. Steiner's formula and the fact that $V_i(W) \ge 0$, for $i = 0, \dots, d-1$, imply that

$$
\begin{aligned}
(2^d - 1)V_d(W) &= V_d(2W) - V_d(W) \\
&\ge V_d\left(W + r(W)B^d\right) - V_d(W) \\
&= \sum_{i=0}^{d-1} \kappa_{d-i} r(W)^{d-i} V_i(W) \ge \kappa_{d-k} r(W)^{d-k} V_k(W).
\end{aligned}
$$

Now the inequality $\kappa_n \ge 1/n!$, $n \in \mathbb{N}$, concludes the proof. □

PROOF OF THEOREM 3.1. Let $W \in \mathcal{K}^d$ with $r(W) \ge 1$. In order to compute the numerator in (3.3), we shall apply (3.7) with $f = f_{\psi_1, W}$ and $g = f_{\psi_2, W}$. From (3.10), we conclude that indeed $\mathbb{E} f(\eta)^2 < \infty$ and $\mathbb{E} g(\eta)^2 < \infty$. Since Z is stationary, the translation invariance of a functional $\psi : \mathcal{R}^d \to \mathbb{R}$ implies that $\psi^* : \mathcal{K}^d \to \mathbb{R}$ defined by (3.2) is translation invariant as well. From (3.8), we get

$$
\begin{aligned}
\frac{1}{n!} \int & \mathbb{E} D^n_{K_1,\dots,K_n} f_{\psi_1,W}(\eta) \mathbb{E} D^n_{K_1,\dots,K_n} f_{\psi_2,W}(\eta) \Lambda^n\left(d(K_1, \dots, K_n)\right) \\
&= \frac{\gamma}{n!} \iiint \psi_1^*\left((K + x) \cap K_2 \cap \dots \cap K_n \cap W\right) \\
&\qquad \times \psi_2^*\left((K + x) \cap K_2 \cap \dots \cap K_n \cap W\right) \\
&\qquad \times \Lambda^{n-1}\left(d(K_2, \dots, K_n)\right) \mathbb{Q}(dK)\, dx.
\end{aligned}
$$

For $n \in \mathbb{N}$, we define $f_{W,n} : \mathcal{K}^d \to \mathbb{R}$ by

$$
\begin{aligned}
f_{W,n}(K) := \frac{1}{V_d(W)} \iint & \psi_1^*\left((K + x) \cap K_2 \cap \dots \cap K_n \cap W\right) \\
& \times \psi_2^*\left((K + x) \cap K_2 \cap \dots \cap K_n \cap W\right) \\
& \times \Lambda^{n-1}\left(d(K_2, \dots, K_n)\right) dx,
\end{aligned}
$$

and $f_n : \mathcal{K}^d \to \mathbb{R}$ by

$$f_n(K) := \int \psi_1^*(K \cap K_2 \cap \cdots \cap K_n)\psi_2^*(K \cap K_2 \cap \cdots \cap K_n)\Lambda^{n-1}\big(d(K_2, \dots, K_n)\big).$$

Our aim is to prove that

$$\sum_{n=1}^{\infty} \frac{\gamma}{n!} \int f_{W,n}(K)\mathbb{Q}(dK) \to \sum_{n=1}^{\infty} \frac{\gamma}{n!} \int f_n(K)\mathbb{Q}(dK)$$

as $r(W) \to \infty$. Since we want to apply the dominated convergence theorem, we provide an upper bound for $\sum_{n=1}^{\infty} \frac{\gamma}{n!} |f_{W,n}|$, which is independent of W. It follows from (3.9) in Lemma 3.3, the translation invariance of V_i and Λ and the monotonicity of the intrinsic volumes that

$$\begin{aligned}
|f_{W,n}(K)| &\le \sum_{i,j=0}^{d} \frac{\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \iint V_i\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big) \\
&\qquad \times V_j\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big) \\
&\qquad \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\, dx \\
&\le \sum_{i,j=0}^{d} \frac{\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \int V_i(K \cap K_2 \cap \cdots \cap K_n)\Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
&\qquad \times \int V_j\big((K+x) \cap W\big)\, dx
\end{aligned}$$

for $K \in \mathcal{K}^d$ and $n \in \mathbb{N}$. Combining this estimate with Lemmas 3.4 and 3.5, we get

$$\frac{1}{n!}|f_{W,n}(K)| \le (d+1)\beta_1\beta(\psi_1)\beta(\psi_2)\left(\sum_{i=0}^{d} V_i(K)\right)^2 \sum_{r=0}^{d} \frac{V_r(W)}{V_d(W)} \frac{\alpha^{n-1}}{n!}. \tag{3.20}$$

By (2.5), the right-hand side of (3.20) is integrable. Moreover, Lemma 3.7 shows that it is uniformly bounded for $W \in \mathcal{K}^d$ with $r(W) \ge 1$, and the same holds if we sum over all $n \in \mathbb{N}$.

Next, we bound $|f_{W,n}(K) - f_n(K)|$ for $K \in \mathcal{K}^d$ and $n \in \mathbb{N}$. By using the translation invariance of ψ_1^*, ψ_2^* and Λ, we have

$$\begin{aligned}
&f_{W,n}(K) - f_n(K) \\
&\quad = \frac{1}{V_d(W)} \iint \big(\psi_1^*\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big) \\
&\qquad\qquad \times \psi_2^*\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big) \\
&\qquad\qquad - \mathbf{1}\{x \in W\}\psi_1^*\big((K+x) \cap K_2 \cap \cdots \cap K_n\big) \\
&\qquad\qquad \times \psi_2^*\big((K+x) \cap K_2 \cap \cdots \cap K_n\big)\big)\, dx\, \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big).
\end{aligned}$$

Note that the integrand is zero if $x \in W$ and $K + x \subset W$. The same holds for the case that $x \notin W$ and $(K+x) \cap W = \varnothing$. This means that the integrand can be only nonzero if $(K+x) \cap \partial W \neq \varnothing$. On the other hand, the integrand is always bounded by

$$\begin{aligned}
&\big|\psi_1^*\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big)\psi_2^*\big((K+x) \cap K_2 \cap \cdots \cap K_n \cap W\big)\big| \\
&\quad + \big|\psi_1^*\big((K+x) \cap K_2 \cap \cdots \cap K_n\big)\psi_2^*\big((K+x) \cap K_2 \cap \cdots \cap K_n\big)\big| \\
&\quad \leq 2\beta(\psi_1)\beta(\psi_2)\left(\sum_{i=0}^{d} V_i\big((K+x) \cap K_2 \cap \cdots \cap K_n\big)\right)^2,
\end{aligned}$$

where we have used Lemma 3.3 and the monotonicity of the intrinsic volumes. Hence, we obtain that

$$\begin{aligned}
&\big|f_{W,n}(K) - f_n(K)\big| \\
&\quad \leq \frac{2\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \iint \mathbf{1}\big\{(K+x) \cap \partial W \neq \varnothing\big\} \\
&\qquad\qquad \times \left(\sum_{i=0}^{d} V_i\big((K+x) \cap K_2 \cap \cdots \cap K_n\big)\right)^2 dx \\
&\qquad\qquad \times \Lambda^{n-1}\big(d(K_2, \ldots, K_n)\big) \\
&\quad \leq \frac{2\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \sum_{i=0}^{d} V_i(K) \int \mathbf{1}\big\{(K+x) \cap \partial W \neq \varnothing\big\}\, dx \\
&\qquad \times \int \sum_{r=0}^{d} V_r(K \cap K_2 \cap \cdots \cap K_n)\Lambda^{n-1}\big(d(K_2, \ldots, K_n)\big),
\end{aligned}$$

where we have used the fact that V_i is increasing and the translation invariance of V_i and Λ in the last step. Now Lemmas 3.5 and 3.6 yield that

$$\big|f_{W,n}(K) - f_n(K)\big| \leq \frac{2\beta_2\beta(\psi_1)\beta(\psi_2)\alpha^{n-1}}{V_d(W)} \left(\sum_{i=0}^{d} V_i(K)\right)^3 \sum_{r=0}^{d-1} V_r(W).$$

Together with Lemma 3.7 and $r(W) \geq 1$, this shows that, for $K \in \mathcal{K}^d$ and $n \in \mathbb{N}$,

$$\big|f_{W,n}(K) - f_n(K)\big| \leq \beta(\psi_1, \psi_2)\alpha^{n-1}\left(\sum_{i=0}^{d} V_i(K)\right)^3 \frac{1}{r(W)}$$

with $\beta(\psi_1, \psi_2) := 2^{d+2} \cdot 4^{2^d} \cdot 25^{2d} d! M(\psi_1)M(\psi_2)(\mathbb{E}2^{N_{Q_1}} + 1)^2$. Therefore, for $K \in \mathcal{K}^d$,

$$\left|\sum_{n=1}^{\infty} \frac{\gamma}{n!} f_{W,n}(K) - \sum_{n=1}^{\infty} \frac{\gamma}{n!} f_n(K)\right| \leq \gamma\beta(\psi_1, \psi_2)e^{\alpha}\left(\sum_{i=0}^{d} V_i(K)\right)^3 \frac{1}{r(W)}. \tag{3.21}$$

Now an application of the dominated convergence theorem yields the convergence result for $r(W) \to \infty$ stated in the theorem.

Under the stronger moment assumption (2.6), (3.5) follows from (3.21) by carrying out the integration with respect to K and collecting all the constants. $\square$

If the geometric functional is the volume, the asymptotic variance has a significantly easier representation than in (3.4), namely

$$\sigma_{d,d} := \lim_{r(W)\to\infty} \frac{\operatorname{Var} V_d(Z \cap W)}{V_d(W)} = (1-p)^2 \int \big(e^{\gamma C_d(x)} - 1\big)\, dx. \tag{3.22}$$

This follows from an application of the dominated convergence theorem to the exact variance formula (2.9). The inequalities $e^t - 1 \le te^t$, $t \ge 0$ and $C_d(x) \le v_d$ imply that

$$\int \big(e^{\gamma C_d(x)} - 1\big)\, dx \le \gamma e^{\gamma v_d} \int \mathbb{E} V_d\big(Z_0 \cap (Z_0 + x)\big)\, dx = \gamma e^{\gamma v_d} \mathbb{E} V_d(Z_0)^2 < \infty.$$

Together with $C_W(x)/V_d(W) \le 1$, this means that $e^{\gamma C_d(x)} - 1$ is integrable and is an upper bound for $(C_W(x)/V_d(W))(e^{\gamma C_d(x)} - 1)$. Now the observation that $C_W(x)/V_d(W) \to 1$ as $r(W) \to \infty$ for any $x \in \mathbb{R}^d$ [this follows from $V_d(W) - C_W(x) \le V_d(\partial^-_{\|x\|} W)$, (3.19), Steiner's formula, and Lemma 3.7] yields (3.22). In Section 6, formulas as (3.22) are derived for the other intrinsic volumes.

The following proposition shows that the rate of convergence stated in Theorem 3.1 is optimal.

PROPOSITION 3.8. *Assume that (2.5) is satisfied and that the typical grain is full-dimensional with positive probability. Then there is a constant $c_{d,d} > 0$ depending on Λ such that*

$$\left| \sigma_{d,d} - \frac{\operatorname{Var} V_d(Z \cap W)}{V_d(W)} \right| \ge \frac{c_{d,d}}{r(W)}$$

for $W \in \mathcal{K}^d$ with $r(W) \ge 1$.

PROOF. Recall from the proof of Lemma 3.6 that $\partial^-_r W = \{z \in W : \operatorname{dist}(z, \partial W) \le r\}$ for $r \ge 0$. For $s \ge 0$, we define $D_W(s) := \{z \in W : \operatorname{dist}(z, \partial W) = s\}$. Then

$$W_{-s} := \{z \in W : \operatorname{dist}(z, \partial W) \ge s\} = \{z \in \mathbb{R}^d : z + B^d_s \subset W\}$$

is convex, the boundary of W_{-s} is $D_W(s)$, and $s \mapsto W_{-s}$ is strictly decreasing with respect to set inclusion, for $s \in [0, r(W)]$.

It follows from (2.9) and (3.22) that

$$\sigma_{d,d} - \frac{\operatorname{Var}(V_d(Z \cap W))}{V_d(W)} = (1-p)^2 \int \frac{V_d(W) - V_d(W \cap (W + x))}{V_d(W)} \big(e^{\gamma C_d(x)} - 1\big)\, dx.$$

Since the typical grain is full-dimensional with positive probability, there are constants $\tau > 0$ and $r_0 \in (0, 1/2)$ such that $e^{\gamma C_d(x)} - 1 \geq \tau$ for all $x \in B^d_{r_0}$. This means that

$$
\begin{aligned}
\sigma_{d,d} - \frac{\operatorname{Var}(V_d(Z \cap W))}{V_d(W)} \\
\geq (1-p)^2 \frac{\tau}{V_d(W)} \int_{B^d_{r_0}} \bigl(V_d(W) - V_d\bigl(W \cap (W+x)\bigr)\bigr)\,dx.
\end{aligned}
\tag{3.23}
$$

Denoting by $B^d(x,r)$ the closed ball with center x and radius r, we have

$$
\begin{aligned}
&\int_{B^d_{r_0}} \bigl(V_d(W) - V_d\bigl(W \cap (W+x)\bigr)\bigr)\,dx \\
&\quad = \int_{B^d_{r_0}} \int_W \bigl(\mathbf{1}\{y \in W\} - \mathbf{1}\{y \in W, y \in W + x\}\bigr)\,dy\,dx \\
&\quad = \int_W \bigl(V_d\bigl(B^d(y,r_0)\bigr) - V_d\bigl(W \cap B^d(y,r_0)\bigr)\bigr)\,dy \\
&\quad \geq \int_{\partial^-_{r_0/2} W} \bigl(V_d\bigl(B^d(y,r_0)\bigr) - V_d\bigl(W \cap B^d(y,r_0)\bigr)\bigr)\,dy.
\end{aligned}
$$

Using that $V_d(B^d(y,r_0)) - V_d(W \cap B^d(y,r_0)) \geq \tilde{c} r_0^d$ for $y \in \partial^-_{r_0/2} W$ with $\tilde{c} > 0$, we obtain

$$
\int_{B^d_{r_0}} \bigl(V_d(W) - V_d\bigl(W \cap (W+x)\bigr)\bigr)\,dx \geq \tilde{c} r_0^d V_d\bigl(\partial^-_{r_0/2} W\bigr).
\tag{3.24}
$$

It follows from Lemma 3.2.34 in [6] that

$$
V_d\bigl(\partial^-_r W\bigr) = \int_0^r \mathcal{H}^{d-1}\bigl(D_W(s)\bigr)\,ds
$$

for $r \in [0, r(W)]$. The discussion at the beginning of this proof implies that $\mathcal{H}^{d-1}(D_W(\cdot))$ is strictly decreasing on $[0, r(W)]$. Together with $V_d(\partial^-_{r(W)} W) = V_d(W)$ we get for $r(W) \geq r_0/2$ that

$$
\begin{aligned}
V_d(W) &= \int_0^{r(W)} \mathcal{H}^{d-1}\bigl(D_W(s)\bigr)\,ds \leq \int_0^{r(W)} \mathcal{H}^{d-1}\Bigl(D_W\Bigl(\frac{r_0}{2r(W)} s\Bigr)\Bigr)\,ds \\
&= \int_0^{r_0/2} \mathcal{H}^{d-1}\bigl(D_W(t)\bigr) \frac{2r(W)}{r_0}\,dt = \frac{2r(W)}{r_0} V_d\bigl(\partial^-_{r_0/2} W\bigr).
\end{aligned}
$$

Combining this with (3.23) and (3.24) completes the proof. □

4. Positive definiteness. In this section, we consider the positive definiteness of the asymptotic covariance matrix for geometric functionals $\psi_0, \dots, \psi_d$ on $\mathcal{R}^d$. We assume that ψ_k, for $k \in \{0, \dots, d\}$, is positively homogeneous of degree k and

$$
\bigl|\psi_k(K)\bigr| \geq \tilde{\beta}(\psi_k) r(K)^k,
\tag{4.1}
$$

for $K \in \mathcal{K}^d$, with a constant $\tilde{\beta}(\psi_k) > 0$, which only depends on ψ_k. These conditions are motivated by the intrinsic volumes $V_0, \dots, V_d$, where they are obviously true. The additional assumptions on $\psi_0, \dots, \psi_d$ required in this section are used in an essential way in the proof of Theorem 4.1 [see (4.3) and (4.7) below], but are presumably not necessary conditions for the positive definiteness of the asymptotic covariance matrix. In particular, (4.1) is always satisfied if the absolute value of ψ_k on $\mathcal{K}^d$ is bounded from below by a functional $\tilde{\psi}_k : \mathcal{K}^d \to \mathbb{R}$ which is positive and monotone [i.e., $\tilde{\psi}_k(K) \geq \tilde{\psi}_k(L)$ for $K, L \in \mathcal{K}^d$ with $K \supset L$]. This applies to the second example given at the beginning of Section 3. If we assume that there is a constant $h_0 > 0$ with $h \geq h_0$, then

$$|\psi(K)| = \int_{\mathbb{S}^{d-1}} h(u) S_k(K, du) \geq h_0 \, d V\big(B^d[d-k], K[k]\big) \geq d h_0 \kappa_d r(K)^k$$

for $K \in \mathcal{K}^d$, which ensures that (4.1) is satisfied.

By Theorem 3.1, for $k, l \in \{0, \dots, d\}$, the asymptotic covariances σ_{ψ_k, ψ_l} exist under the assumption (2.5). The following theorem shows that the asymptotic covariance matrix is positive definite. In particular, the result applies to the intrinsic volumes $V_0, \dots, V_d$, which also means that their asymptotic variances are strictly positive.

THEOREM 4.1. *Let the preceding assumptions and* (2.5) *be satisfied. Moreover, assume that the typical grain Z_0 has nonempty interior with positive probability. Then the covariance matrix $\Sigma := (\sigma_{\psi_k, \psi_l})_{k,l=0,\dots,d}$ is positive definite.*

PROOF. For a vector $a = (a_0, \dots, a_d)^\top \in \mathbb{R}^{d+1}$, we have

$$a^\top \Sigma a = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint \left(\sum_{k=0}^{d} a_k \psi_k^*(K_1 \cap K_2 \cap \cdots \cap K_n) \right)^2 \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \mathbb{Q}(dK_1).$$

Since each summand is nonnegative, the matrix Σ is positive definite if we can prove that one summand is greater than zero for a given $a \in \mathbb{R}^{d+1} \setminus \{0\}$. Specifically, under the given assumptions we shall show that the summand obtained for $n = d+1$ is positive. In order to show this, we shall prove that for $K_1, \dots, K_{d+1}$ in the support of $\mathbb{Q}$ and having nonempty interiors, there is a set of translation vectors $x_2, \dots, x_{d+1} \in \mathbb{R}^d$ of positive λ_d^d measure (recall that λ_d denotes d-dimensional Lebesgue measure) for which

$$\sum_{k=0}^{d} a_k \psi_k^*\big(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_{d+1} + x_{d+1})\big) \neq 0.$$

For the rest of the proof, we argue with a nonempty convex body $L \in \mathcal{K}^d$. Properties which will be required of L will be provided by an application of Lemma 4.2

and L of the form $L = K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_{d+1} + x_{d+1}) \in \mathcal{K}^d$, for a set of translation vectors $x_2, \ldots, x_{d+1} \in \mathbb{R}^d$ of positive λ_d^d measure. This will finally prove the preceding assertion, and thus the theorem.

Let $N_1(L)$ be the number of grains of η that intersect L, but do not contain it, and let $N_2(L)$ be the number of grains of η that contain L. Then $N_1(L)$ and $N_2(L)$ are independent, Poisson distributed random variables with parameters

$$s_1(L) = \Lambda(\{K \in \mathcal{K}^d : K \cap L \neq \varnothing \text{ and } L \not\subset K\}) \quad \text{and}$$
$$s_2(L) = \Lambda(\{K \in \mathcal{K}^d : L \subset K\}).$$

If $N_2(L) \neq 0$, then $L \subset Z$ and, therefore, $\psi_k(Z \cap L) - \psi_k(L) = \psi_k(L) - \psi_k(L) = 0$. If $N_1(L) = N_2(L) = 0$, then $Z \cap L = \varnothing$, and hence $\psi_k(Z \cap L) - \psi_k(L) = 0 - \psi_k(L) = -\psi_k(L)$. This leads to

$$\begin{aligned} \psi_k^*(L) &= \mathbb{E}[\psi_k(Z \cap L) - \psi_k(L)] \\ &= -\exp(-s_1(L) - s_2(L))\psi_k(L) + R_k(L), \end{aligned} \tag{4.2}$$

where

$$R_k(L) = \mathbb{E}\mathbf{1}\{N_1(L) \geq 1, N_2(L) = 0\}(\psi_k(Z \cap L) - \psi_k(L)).$$

Next, we bound $R_k(L)$ from above. So assume that $N_1(L) \neq 0$. Let $K_1, \ldots, K_{N_1(L)}$ denote the grains of η which hit L, but do not contain L. With the definition of $M_L(\psi_k)$ from (3.16), we obtain from (3.17) that

$$|\psi_k(Z \cap L) - \psi_k(L)| \leq |\psi_k(Z \cap L)| + |\psi_k(L)| \leq 2^{N_1(L)} M_L(\psi_k).$$

In the following, let $R(K)$ stand for the radius of the circumscribed ball of $K \in \mathcal{K}^d$. For $A \in \mathcal{K}^d$ with $A \subset L$, let $\hat{a} \in \mathbb{R}^d$ be the center of the circumball of A, hence $A - \hat{a} \subset 2R(A)Q_1$. Since ψ_k is translation-invariant, homogeneous of degree k, and locally bounded, we get

$$|\psi_k(A)| = (2R(A))^k |\psi_k((2R(A))^{-1}(A - \hat{a}))| \leq (2R(A))^k M(\psi_k),$$

and hence

$$M_L(\psi_k) \leq (2R(L))^k M(\psi_k). \tag{4.3}$$

Thus, in the present case, we have

$$|\psi_k(Z \cap L) - \psi_k(L)| \leq 2^{N_1(L)} (2R(L))^k M(\psi_k).$$

Hence, the remainder term can be bounded from above by

$$\begin{aligned} |R_k(L)| &\leq \mathbb{E}[\mathbf{1}\{N_1(L) \geq 1, N_2(L) = 0\} 2^{N_1(L)} (2R(L))^k M(\psi_k)] \\ &= \exp(-s_2(L))(2R(L))^k M(\psi_k) \exp(s_1(L))(1 - \exp(-2s_1(L))) \\ &\leq \exp(-s_2(L))(2R(L))^k M(\psi_k) \exp(s_1(L)) 2s_1(L). \end{aligned}$$

Next, we derive an upper bound for $s_1(L)$. By definition and the reflection invariance of Lebesgue measure, we have

$$s_1(L) = \gamma \iint \mathbf{1}\{(L+x) \cap K \neq \varnothing, L+x \not\subset K\}\, dx\, \mathbb{Q}(dK).$$

To bound the inner integral from above, we can assume that $L \in \mathcal{K}_o^d$, by the translation invariance of Lebesgue measure. If the integrand is nonzero, then $x \in (K + R(L)B^d) \setminus K$ or $x \in \partial K^-_{R(L)}$. Then inequality (3.19) implies that the inner integral is bounded from above by $2V_d((K + R(L)B^d) \setminus K)$. Hence, if $R(L) \leq 1$, Steiner's formula and our moment assumption yield that

$$s_1(L) \leq c_6 R(L),$$

where c_6 denotes a constant depending on Λ. Hence, if $R(L)$ is sufficiently small, then $s_1(L) \leq 1$, and thus

$$\begin{aligned} |R_k(L)| &\leq 6 \cdot \big(2R(L)\big)^k M(\psi_k) s_1(L) \exp\big(-s_2(L)\big) \\ &\leq 6 \cdot 2^k \cdot c_6 M(\psi_k) R(L)^{k+1} \exp\big(-s_2(L)\big). \end{aligned} \tag{4.4}$$

We also have from (4.3) that

$$\begin{aligned} &\big|\exp\big(-s_1(L) - s_2(L)\big)\psi_k(L)\big| \\ &\quad \leq M_L(\psi_k) \exp\big(-s_2(L)\big) \leq \big(2R(L)\big)^k M(\psi_k) \exp\big(-s_2(L)\big). \end{aligned} \tag{4.5}$$

Hence, if $R(L)$ is sufficiently small, we deduce from (4.2), (4.4) and (4.5) that

$$\big|\psi_k^*(L)\big| \leq \bar{\beta}(\psi_k) R(L)^k \exp\big(-s_2(L)\big), \tag{4.6}$$

where $\bar{\beta}(\psi_k)$ is a constant depending on Λ and ψ_k. In addition,

$$\big|\exp\big(-s_1(L) - s_2(L)\big)\psi_k(L)\big| \geq \exp\big(-s_2(L)\big)\big(\tilde{\beta}(\psi_k)/3\big) r(L)^k, \tag{4.7}$$

if $s_1(L) \leq 1$, with $\tilde{\beta}(\psi_k)$ as in (4.1).

Let k_0 be the smallest $k \in \{0, \dots, d\}$ such that $a_k \neq 0$. Then, if $R(L)$ is sufficiently small, we get

$$\begin{aligned} &\left|\sum_{k=0}^{d} a_k \psi_k^*(L)\right| \\ &\quad = \left|\sum_{k=k_0}^{d} a_k \psi_k^*(L)\right| \\ &\quad = \left|-a_{k_0} \exp\big(-s_1(L) - s_2(L)\big)\psi_{k_0}(L) + a_{k_0} R_{k_0}(L) + \sum_{k=k_0+1}^{d} a_k \psi_k^*(L)\right| \end{aligned}$$

$$\geq |a_{k_0}| \left|\exp(-s_1(L) - s_2(L))\psi_{k_0}(L)\right| - \left|a_{k_0} R_{k_0}(L)\right| - \sum_{k=k_0+1}^{d} |a_k| \left|\psi_k^*(L)\right|$$

$$\geq \exp(-s_2(L))\bigl(|a_{k_0}|(\tilde{\beta}(\psi_{k_0})/3) r(L)^{k_0} - \beta^* R(L)^{k_0+1}\bigr),$$

where we used (4.4) and (4.7), for $k = k_0$, and (4.6) for $k \geq k_0 + 1$. Here, we denote by β^* a constant which depends on $a_{k_0}, \dots, a_d, \psi_{k_0}, \dots, \psi_d, \Lambda$. The lower bound thus obtained is positive if $R(L)$ is sufficiently small and $R(L)/r(L) \leq c_0$, for some constant c_0. The proof is completed by an application of Lemma 4.2 below. □

The following lemma on the ratio of circumradius and inradius of translates of convex bodies is a key argument in the proof of Theorem 4.1.

LEMMA 4.2. *For all $K_1, \dots, K_{d+1} \in \mathcal{K}^d$ with nonempty interior there is a constant $c_0 > 0$ such that*

$$\lambda_d^d\left(\left\{(x_2, \dots, x_{d+1}) \in (\mathbb{R}^d)^d : R(L) < c_0 r(L) \text{ and } R(L) \leq r \text{ for } L = K_1 \cap \bigcap_{i=2}^{d+1} (K_i + x_i)\right\}\right) > 0$$

for all $r > 0$.

PROOF. Let $u_1, \dots, u_{d+1} \in \mathbb{R}^d$ be unit vectors whose endpoints are the vertices of a regular simplex. For $i = 1, \dots, d+1$ let x_i be a point in the boundary of K_i which has u_i as an exterior normal vector. The support cone $S(K_i, x_i)$ of K_i at x_i (cf. [30], page 81) then satisfies

$$K_i - x_i \subset S(K_i, x_i) := \operatorname{cl}\left(\bigcup_{t>0} t(K_i - x_i)\right) \subset H^-(K_i, u_i) - x_i,$$

where $H^-(K_i, u_i)$ is the supporting half-space of K_i with exterior unit normal u_i and cl denotes the closure. By [31], Theorem 12.2.2, it follows that $t(K_i - x_i) \to S(K_i, x_i)$ in the topology of closed convergence as $t \to \infty$. Moreover, since $K_1, \dots, K_{d+1}$ have nonempty interiors, there are vectors $z_1, \dots, z_{d+1} \in \mathbb{R}^d$ such that the origin is an interior point of

$$S_0 := \bigcap_{i=1}^{d+1} \bigl(S(K_i, x_i) + z_i\bigr) \subset \bigcap_{i=1}^{d+1} \bigl(H^-(K_i, u_i) - x_i + z_i\bigr)$$

and the circumradius of the intersection on the right-hand side is less than 1 (say). Then [30], Theorem 1.8.10 and [31], Theorem 12.3.3, imply that

$$S_0 = \lim_{t \to \infty} \left(t \bigcap_{i=1}^{d+1} (K_i + x_i(t))\right),$$

where $x_i(t) := -x_i + t^{-1}z_i$ and the convergence is with respect to the Hausdorff distance. Since the inradius and the circumradius of the intersection of translates of convex bodies are continuous with respect to the translations as long as the intersection has nonempty interior, there is some $t_0 > 1$ such that the ratio between inradius and circumradius of

$$t \bigcap_{i=1}^{d+1} \big(K_i + x_i(t)\big)$$

is close to the corresponding ratio of S_0, for $t \geq t_0$ and, therefore, also

$$1 \leq \frac{R(\bigcap_{i=1}^{d+1}(K_i + x_i(t)))}{r(\bigcap_{i=1}^{d+1}(K_i + x_i(t)))} < \tilde{c}_0,$$

with a constant $\tilde{c}_0 > 1$ which depends only on $K_1, \dots, K_{d+1}$. Moreover, for $t \geq t_0 > 1$ we have

$$R\left(t \bigcap_{i=1}^{d+1} \big(K_i + x_i(t)\big)\right) \leq R\left(\bigcap_{i=1}^{d+1} \big(H^-(K_i, u_i) - x_i + z_i\big)\right) < 1$$

and thus

$$R\left(\bigcap_{i=1}^{d+1} \big(K_i + x_i(t)\big)\right) < \frac{1}{t}.$$

Therefore, if $r < 1/(2t_0)$ the proof of the lemma is completed by remarking that the intersections are continuous with respect to translations as long as the intersection has nonempty interior and by using the translation invariance of Lebesgue measure. Clearly, this proves the lemma for all $r > 0$. $\square$

5. Some integral formulas for intrinsic volumes. We shall see in the next section that in the particularly important case of intrinsic volumes and under the assumption of isotropy the asymptotic covariances of Theorem 3.1 can be expressed in terms of the numbers

$$\begin{aligned}\rho_{i,j} := \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint & V_i(K_1 \cap \cdots \cap K_n) V_j(K_1 \cap \cdots \cap K_n) \\ & \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\mathbb{Q}(dK_1).\end{aligned} \tag{5.1}$$

In this section, we study these numbers *without isotropy assumption* on Z. The results are of independent interest.

For $W \in \mathcal{K}^d$ and $i, j \in \{0, \dots, d\}$, we define

$$\begin{aligned}\rho_{i,j}(W) := \sum_{n=1}^{\infty} \frac{1}{n!} \int & V_i(K_1 \cap \cdots \cap K_n \cap W) V_j(K_1 \cap \cdots \cap K_n \cap W) \\ & \times \Lambda^{n}\big(d(K_1, \dots, K_n)\big),\end{aligned} \tag{5.2}$$

which is a finite window version of $\rho_{i,j}$. The numbers $\rho_{i,j}(W)$ are further studied in [13], Appendix B. The relationship between (5.1) and (5.2) is given in the next corollary.

COROLLARY 5.1. *Let $i, j \in \{0, \dots, d\}$. If (2.5) is satisfied, then $\rho_{i,j} < \infty$ and*

$$\lim_{r(W)\to\infty} \frac{\rho_{i,j}(W)}{V_d(W)} = \rho_{i,j}. \tag{5.3}$$

If (2.6) is satisfied, then there is a constant $c_{i,j}$ such that

$$\left|\rho_{i,j} - \frac{\rho_{i,j}(W)}{V_d(W)}\right| \le \frac{c_{i,j}}{r(W)}$$

for $W \in \mathcal{K}^d$ with $r(W) \ge 1$.

PROOF. This can be proved in a similar way as Theorem 3.1. □

The previous corollary describes $\rho_{i,j}$ as the limit of $V_d(W)^{-1}\rho_{i,j}(W)$ for observation windows with $r(W) \to \infty$. It is, however, more convenient to work with the series representation (5.1). We shall see that this series can be expressed in terms of a finite family of (curvature) measures $H_{i,j}$ to be introduced below.

For $j \in \{0, \dots, d\}$ and $K \in \mathcal{K}^d$, we let $\Phi_j(K; \cdot)$ denote the jth curvature measure of K (see [31], Section 14.2). In particular, $\Phi_d(K; \cdot)$ is the restriction of Lebesgue measure to K while $\Phi_{d-1}(K; \cdot)$ is half the $(d-1)$-dimensional Hausdorff measure restricted to the boundary of K (if the affine hull of K has full dimension). Furthermore, $\Phi_j(K; \mathbb{R}^d) = V_j(K)$ for all $j \in \{0, \dots, d\}$. For $j \in \{0, \dots, d-1\}$, $n \in \mathbb{N}$, and $K_1, \dots, K_n \in \mathcal{K}^d$ we define

$$\Phi_j(K_1, \dots, K_n; \cdot) := \Phi_j(K_1 \cap \cdots \cap K_n; \partial K_1 \cap \cdots \cap \partial K_n \cap \cdot). \tag{5.4}$$

Since $\Phi_j(K_1; \cdot)$, $j \in \{0, \dots, d-1\}$, is concentrated on the boundary ∂K_1 of K_1, this definition is consistent with the case $n = 1$. For $i \in \{1, \dots, d-1\}$ and $k \in \{1, \dots, d-i\}$, we define a measure $H_{i,d}^k$ on $\mathbb{R}^d$ by

$$\begin{aligned} H_{i,d}^k := \gamma \iiiint &\mathbf{1}\{y - z \in \cdot\}\mathbf{1}\{z \in K_1 \cap \cdots \cap K_k\}\Phi_i(K_1, \dots, K_k; dy)\, dz \\ &\times \Lambda^{k-1}\big(d(K_1, \dots, K_{k-1})\big)\mathbb{Q}(dK_k), \end{aligned} \tag{5.5}$$

with the appropriate interpretation of the case $k = 1$.

For $i, j \in \{1, \dots, d-1\}$, $k \in \{1, \dots, d-i\}$, $l \in \{1, \dots, d-j\}$, and $m \in \{0, \dots, k \wedge l\}$ we define a measure $H_{i,j}^{k,l,m}$ on $\mathbb{R}^d$ by

$$\begin{aligned} H_{i,j}^{k,l,m} := \gamma \iiiint &\mathbf{1}\{y - z \in \cdot\} \\ &\times \mathbf{1}\big\{y \in K_{k+1}^\circ \cap \cdots \cap K_{k+l-m}^\circ, z \in K_1^\circ \cap \cdots \cap K_{k-m}^\circ\big\} \\ &\times \Phi_i(K_1, \dots, K_k; dy)\Phi_j(K_{k+1-m}, \dots, K_{k+l-m}; dz) \\ &\times \Lambda^{k+l-m-1}\big(d(K_1, \dots, K_{k+l-m-1})\big)\mathbb{Q}(dK_{k+l-m}), \end{aligned} \tag{5.6}$$

where K° denotes the interior of $K \in \mathcal{K}^d$ and with the appropriate interpretation of the cases $m = k$ or $m = l$. Let

$$H_{i,d} := \sum_{k=1}^{d-i} \frac{1}{k!} H_{i,d}^k, \, i \in \{1, \dots, d-1\},$$

$$H_{i,j} := \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \wedge l} \frac{1}{m!(k-m)!(l-m)!} H_{i,j}^{k,l,m}, \qquad i, j \in \{1, \dots, d-1\},$$

and, for $j \in \{0, \dots, d-1\}$,

$$h_{0,j} := \sum_{l=1}^{d-j} \frac{\gamma}{l!} \iint \Phi_j\big(K_1, \dots, K_l; \mathbb{R}^d\big) \Lambda^{l-1}\big(d(K_1, \dots, K_{l-1})\big) \mathbb{Q}(dK_l). \tag{5.7}$$

Moreover, we define $H_{d,d}(dx) := (1 - e^{-\gamma C_d(x)})\, dx$, $H_{0,j} := H_{j,0} := h_{0,j}\delta_0$ for $j \in \{0, \dots, d-1\}$, and $H_{0,d} := H_{d,0} := (1 - e^{-\gamma v_d})\delta_0$, where δ_0 is the Dirac measure concentrated at the origin and $C_d(x)$ is the mean covariogram of the typcial grain as defined in (2.7).

Subsequently, we assume that

$$\mathbb{Q}\big(\{K \in \mathcal{K}^d : V_d(K) > 0\}\big) = 1, \tag{5.8}$$

that is, the typical grain almost surely has nonempty interior.

THEOREM 5.2. *Assume that* (2.5) *and* (5.8) *are satisfied. Then the measures* $H_{i,j}$ *are all finite. Moreover, the limits* (5.3) *are given by*

$$\rho_{i,j} = \int e^{\gamma C_d(x)} H_{i,j}(dx), \qquad i, j \in \{0, \dots, d\}. \tag{5.9}$$

For $i = d$ *or* $j = d$, *the result remains true without assumption* (5.8).

In particular, we thus have

$$\rho_{d,d} = \int \big(e^{\gamma C_d(x)} - 1\big)\, dx, \tag{5.10}$$

$\rho_{0,d} = e^{\gamma v_d} - 1$, and $\rho_{0,j} = e^{\gamma v_d} h_{0,j}$ for $j \in \{0, \dots, d-1\}$.

The proof of Theorem 5.2 is based on the following geometric result. Here, we use the abbreviation $[n] = \{1, \dots, n\}$.

LEMMA 5.3. *Let* $K_1, K_2', \dots, K_n' \in \mathcal{K}^d$, $n \in \mathbb{N}$, *have nonempty interiors, and let* $i \in \{0, \dots, d-1\}$. *Then*

$$\Phi_i(K_1 \cap \cdots \cap K_n; \cdot) = \sum_{\substack{\varnothing \neq I \subset [n] \\ |I| \leq d-i}} \Phi_i\Big(\bigcap_{r \in I} K_r; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K_s^\circ\Big),$$

for almost all translates K_i *of* K_i' *for* $i = 2, \dots, n$.

Hence, if (5.8) is satisfied and $K_1 \in \mathcal{K}^d$ has nonempty interior, then this lemma can be applied for Λ^{n-1}-a.e. $(K_2, \dots, K_n) \in (\mathcal{K}^d)^{n-1}$.

Before we prove Lemma 5.3, we provide two auxiliary results.

LEMMA 5.4. *Let $K_1, \dots, K_m \in \mathcal{K}^d$, $m \geq 2$, have nonempty interiors. Then, for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \dots, t_m) \in \mathbb{R}^{d(m-1)}$, if $K_1 \cap (K_2 + t_2) \cap \cdots \cap (K_m + t_m) \neq \varnothing$, then $(K_1)^\circ \cap (K_2 + t_2)^\circ \cap \cdots \cap (K_m + t_m)^\circ \neq \varnothing$.*

PROOF. The assertion is proved by induction over $m \geq 2$. For $m = 2$, the assertion holds, since any $t_2 \in \mathbb{R}^d$ such that $K_1 \cap (K_2 + t_2) \neq \varnothing$ and $K_1^\circ \cap (K_2^\circ + t_2) = \varnothing$ is contained in the boundary of $K_1 + (-K_2)$, which has d-dimensional Hausdorff measure zero. The induction step follows from the case $m = 2$ and Fubini's theorem. For further details, see [13]. $\square$

For the following lemma, we use basic notions from geometric measure theory (see, e.g., [6]).

LEMMA 5.5. *Let $K_1, \dots, K_m \in \mathcal{K}^d$, $m \in \mathbb{N}$. If $m \leq d$, then for $\mathcal{H}^{d(m-1)}$-almost all translates $(t_2, \dots, t_m) \in \mathbb{R}^{d(m-1)}$, the intersection $\partial K_1 \cap (\partial K_2 + t_2) \cap \cdots \cap (\partial K_m + t_m)$ has finite $(d - m)$-dimensional Hausdorff measure. For $m > d$, the intersection is the empty set for almost all translation vectors.*

PROOF. Since for $m = 1$ there is nothing to show, we assume that $m \in \{2, \dots, d\}$. Let $W := \partial K_1 \times \cdots \times \partial K_m \subset \mathbb{R}^{dm}$, let $Z \subset \mathbb{R}^{d(m-1)}$ be the compact image set of the Lipschitz map $T : W \to Z \subset \mathbb{R}^{d(m-1)}$, $(x_1, \dots, x_m) \mapsto (x_1 - x_2, \dots, x_1 - x_m)$. Then the assumptions of the coarea theorem ([6], Theorem 3.2.22 (2)) are satisfied. Thus, for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \dots, t_m) \in Z$, the set $T^{-1}\{(t_2, \dots, t_m)\}$ has finite $\mathcal{H}^{d-m}$ measure. Identify $\mathbb{R}^{dm}$ with $(\mathbb{R}^d)^m$ and denote by $\pi_1 : (\mathbb{R}^d)^m \to \mathbb{R}^d$ the projection to the first component, which is a Lipschitz map. Then $\partial K_1 \cap (\partial K_2 + t_2) \cap \cdots \cap (\partial K_m + t_m) = \pi_1(T^{-1}\{(t_2, \dots, t_m)\})$ has finite $(d - m)$-dimensional Hausdorff measure for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \dots, t_m) \in Z$. [If $(t_2, \dots, t_m) \notin Z$, the intersection is the empty set.]

The assertion for $m > d$ easily follows from the one for $m = d$. $\square$

PROOF OF LEMMA 5.3. There is nothing to show for $n = 1$ so that we assume that $n \geq 2$. By Lemmas 5.4 and 5.5, we can assume that $K_1, \dots, K_n$ have a common interior point and for $\varnothing \neq I \subset [n]$ each intersection $\bigcap_{r \in I} \partial K_r$ has finite $(d - |I|)$-dimensional Hausdorff measure if $|I| \leq d$, and otherwise is the empty set.

Since $\Phi_i(K_1 \cap \cdots \cap K_n, \cdot)$ is concentrated on the boundary of $K_1 \cap \cdots \cap K_n$, the measure property yields that

$$\Phi_i(K_1 \cap \cdots \cap K_n; \cdot) = \sum_{\varnothing \neq I \subset [n]} \Phi_i\left(K_1 \cap \cdots \cap K_n; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K_s^\circ\right).$$

The intersection $U := \bigcap_{s\notin I} K_s^\circ$ is open, $U' := \bigcap_{r\in I} \partial K_r \cap \bigcap_{s\notin I} K_s^\circ \subset U$, and $K_1 \cap \cdots \cap K_n \cap U = \bigcap_{r\in I} K_r \cap U$. Hence, since Φ_i is locally determined (see [30], page 215), it follows that

$$\Phi_i(K_1 \cap \cdots \cap K_n; \cdot) = \sum_{\varnothing \neq I \subset [n]} \Phi_i\biggl(\bigcap_{r\in I} K_r; \cdot \cap \bigcap_{r\in I} \partial K_r \cap \bigcap_{s\notin I} K_s^\circ\biggr).$$

Since $\bigcap_{r\in I} \partial K_r$ has finite $(d - |I|)$-dimensional Hausdorff measure for $|I| \in \{1, \dots, d\}$, and is the empty set for $|I| > d$, we conclude that if $d \geq |I| > d - i$, then $\bigcap_{r\in I} \partial K_r$ has i-dimensional Hausdorff measure zero. A special case of [4], Theorem 5.5, then yields that

$$\Phi_i\biggl(\bigcap_{r\in I} K_r; \cdot \cap \bigcap_{r\in I} \partial K_r \cap \bigcap_{s\notin I} K_s^\circ\biggr) = 0,$$

which completes the proof. $\square$

PROOF OF THEOREM 5.2. We start with showing that the measures $H_{i,j}$ are finite. Let $i, j \in \{1, \dots, d-1\}$, $k \in \{1, \dots, d-i\}$, $l \in \{1, \dots, d-j\}$, and $m \in \{0, \dots, k \wedge l\}$. Then

$$\begin{aligned} H_{i,j}^{k,l,m}(\mathbb{R}^d) &\leq \gamma \iiiint \mathbf{1}\{K_1 \cap \cdots \cap K_{k+l-m} \neq \varnothing\} \Phi_i(K_1, \dots, K_k; dy) \\ &\qquad \times \Phi_j(K_{k+1-m}, \dots, K_{k+l-m}; dz) \\ &\qquad \times \Lambda^{k+l-m-1}\bigl(d(K_1, \dots, K_{k+l-m-1})\bigr)\mathbb{Q}(dK_{k+l-m}) \\ &\leq \gamma \iint V_0(K_1 \cap \cdots \cap K_{k+l-m}) V_i(K_1) V_j(K_{k+l-m}) \\ &\qquad \times \Lambda^{k+l-m-1}\bigl(d(K_1, \dots, K_{k+l-m-1})\bigr)\mathbb{Q}(dK_{k+l-m}). \end{aligned}$$

For $k + l - m = 1$ the right-hand side is finite because of assumption (2.5). Otherwise, we obtain by Lemmas 3.5 and 3.4 that

$$\begin{aligned} &H_{i,j}^{k,l,m}(\mathbb{R}^d) \\ &\quad \leq \gamma^2 \alpha^{k+l-m-2} \iint \sum_{r=0}^{d} V_r\bigl((K_1 + x) \cap K_{k+l-m}\bigr) V_i(K_1) \\ &\qquad \times V_j(K_{k+l-m})\, dx\, \mathbb{Q}^2\bigl(d(K_1, K_{k+l-m})\bigr) \\ &\quad \leq (d+1)\gamma^2 \alpha^{k+l-m-2} \beta_1 \int \sum_{r=0}^{d} V_r(K_1) \sum_{r=0}^{d} V_r(K_{k+l-m}) V_i(K_1) V_j(K_{k+l-m}) \\ &\qquad \times \mathbb{Q}^2\bigl(d(K_1, K_{k+l-m})\bigr). \end{aligned}$$

Now it follows from (2.5) that the right-hand side is finite. Similar (but easier) arguments show that the other measures are also finite.

Note that $\rho_{i,j} = \rho_{j,i}$ for $i, j \in \{0, \dots, d\}$. To prove that the series (5.1) is given by (5.9), we distinguish different cases and start with $i = j = d$. Then we have

$$
\begin{aligned}
\rho_{d,d} &= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint V_d(K_1 \cap \dots \cap K_n)^2 \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\mathbb{Q}(dK_1) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \cdots \int \mathbf{1}\big\{y \in K_1 \cap (K_2 + x_2) \cap \dots \cap (K_n + x_n)\big\} \\
&\qquad \times \mathbf{1}\big\{z \in K_1 \cap (K_2 + x_2) \cap \cdots \\
&\qquad\qquad \cap (K_n + x_n)\big\}\, dy\, dz\, dx_2 \cdots dx_n \\
&\qquad \times \mathbb{Q}^n\big(d(K_1, \dots, K_n)\big) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint V_d\big((K_2 - y) \cap (K_2 - z)\big) \cdots V_d\big((K_n - y) \cap (K_n - z)\big) \\
&\qquad \times \mathbf{1}\{y \in K_1\}\mathbf{1}\{z \in K_1\}\, dy\, dz \mathbb{Q}^n\big(d(K_1, \dots, K_n)\big) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint \big(\mathbb{E} V_d\big(Z_0 \cap (Z_0 + y - z)\big)\big)^{n-1} \mathbf{1}\{y, z \in K_1\}\, dy\, dz \mathbb{Q}(dK_1) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \iiint \big(\mathbb{E} V_d\big(Z_0 \cap (Z_0 + y)\big)\big)^{n-1} \mathbf{1}\{y + z \in K_1\} \\
&\qquad \times \mathbf{1}\{z \in K_1\}\, dy\, dz \mathbb{Q}(dK_1) \\
&= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int C_d(y)^n\, dy = \int \big(e^{\gamma C_d(y)} - 1\big)\, dy.
\end{aligned}
$$

For $i = 0$ and $j = d$, we get by an even simpler calculation

$$
\rho_{0,d} = \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \big(\mathbb{E} V_d(Z_0)\big)^n = e^{\gamma v_d} - 1.
$$

This and the preceding calculation do not depend on assumption (5.8).

Next, we turn to $i = 0$ and $j \in \{0, \dots, d - 1\}$. Then, using $V_j(L) = \Phi_j(L; \mathbb{R}^d)$, for $L \in \mathcal{K}^d$ and Lemma 5.3, we get

$$
\begin{aligned}
\rho_{0,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{l=1}^{d-j} \sum_{\substack{J \subset [n] \\ |J| = l}} \iiint \mathbf{1}\Big\{z \in \bigcap_{s \notin J} K_s^{\circ}\Big\} \Phi_j(K_J; dz) \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
\times \mathbb{Q}(dK_1),
\end{aligned}
$$

where $\Phi_j(K_J;\cdot) = \Phi_j(K_{j_1},\dots,K_{j_l};\cdot)$ for $J = \{j_1,\dots,j_l\}$ [see (5.4)]. At this stage and also later, we use the covariance property

$$\int h(y)\Phi_i(K_1,\dots,K_l;dy) = \int h(y+x)\Phi_i(K_1-x,\dots,K_l-x;dy), \qquad x\in\mathbb{R}^d, \tag{5.11}$$

which holds for all measurable $h:\mathbb{R}^d\to[0,\infty]$. This follows from the definition (5.4) and [31], Theorem 14.2.2. Using (5.11) and then the invariance of Λ under translations, it is easy to check that, for instance,

$$\begin{aligned}&\iiint \mathbf{1}\{z\in K_{l+1}^\circ\cap\cdots\cap K_n^\circ\}\Phi_j(K_{\{1,\dots,l\}};dz)\Lambda^{n-1}\big(d(K_1,\dots,K_{n-1})\big)\mathbb{Q}(dK_n)\\ &\quad=\iiint \mathbf{1}\{z\in K_{l+1}^\circ\cap\cdots\cap K_n^\circ\}\Phi_j(K_{\{1,\dots,l\}};dz)\\ &\qquad\quad\times\Lambda^{n-1}\big(d(K_2,\dots,K_n)\big)\mathbb{Q}(dK_1).\end{aligned}$$

From such symmetry relations, we deduce that

$$\begin{aligned}\rho_{0,j}&=\gamma\sum_{l=1}^{d-j}\sum_{n=l}^{\infty}\frac{1}{n!}\binom{n}{l}\iiint \mathbf{1}\{z\in K_{l+1}^\circ\cap\cdots\cap K_n^\circ\}\\ &\qquad\times\Phi_j(K_1,\dots,K_l;dz)\\ &\qquad\times\Lambda^{n-1}\big(d(K_2,\dots,K_n)\big)\mathbb{Q}(dK_1)\\ &=\gamma\sum_{l=1}^{d-j}\sum_{n=l}^{\infty}\frac{1}{n!}\binom{n}{l}\gamma^{n-l}\iiint V_d(K_{l+1})\cdots V_d(K_n)\Phi_j\big(K_1,\dots,K_l;\mathbb{R}^d\big)\\ &\qquad\times\mathbb{Q}^{n-l}\big(d(K_{l+1},\dots,K_n)\big)\\ &\qquad\times\Lambda^{l-1}\big(d(K_2,\dots,K_l)\big)\mathbb{Q}(dK_1)\\ &=\gamma\sum_{l=1}^{d-j}\frac{1}{l!}\sum_{n=l}^{\infty}\frac{(\gamma v_d)^{n-l}}{(n-l)!}\iint \Phi_j\big(K_1,\dots,K_l;\mathbb{R}^d\big)\\ &\qquad\times\Lambda^{l-1}\big(d(K_2,\dots,K_l)\big)\mathbb{Q}(dK_1)=e^{\gamma v_d}h_{0,j}.\end{aligned}$$

Next, we address the case $i\in\{1,\dots,d-1\}$ and $j=d$. Using again Lemma 5.3 and a symmetry argument (as above), we obtain

$$\begin{aligned}\rho_{i,d}=\gamma\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{d-i}\binom{n}{k}\iiiint &\mathbf{1}\{y\in K_{k+1}^\circ\cap\cdots\cap K_n^\circ\}\\ &\times\mathbf{1}\{z\in K_1\cap\cdots\cap K_n\}\,dz\end{aligned}$$

$$\times \Phi_i(K_1, \dots, K_k; dy)$$
$$\times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\mathbb{Q}(dK_1).$$

Then we interchange the order of summation to get

$$\begin{aligned}
\rho_{i,d} = \gamma \sum_{k=1}^{d-i} \sum_{n=k}^{\infty} \frac{\gamma^{n-k}}{k!(n-k)!} \int \cdots \int & \mathbf{1}\big\{x_{k+1} \in \big(K_{k+1}^{\circ} - y\big) \cap (K_{k+1} - z)\big\} \cdots \\
& \times \mathbf{1}\big\{x_n \in \big(K_n^{\circ} - y\big) \cap (K_n - z)\big\} \\
& \times \mathbb{Q}(dK_{k+1}) \cdots \mathbb{Q}(dK_n)\, dx_{k+1} \cdots dx_n \\
& \times \mathbf{1}\{z \in K_1 \cap \cdots \cap K_k\} \Phi_i(K_1, \dots, K_k; dy)\, dz \\
& \times \Lambda^{k-1}\big(d(K_2, \dots, K_k)\big)\mathbb{Q}(dK_1) \\
= \gamma \sum_{k=1}^{d-i} \sum_{n=k}^{\infty} \frac{\gamma^{n-k}}{k!(n-k)!} \iiiint & \big(\mathbb{E} V_d\big(Z_0 \cap (Z_0 + y - z)\big)\big)^{n-k} \\
& \times \mathbf{1}\{z \in K_1 \cap \cdots \cap K_k\} \\
& \times \Phi_i(K_1, \dots, K_k; dy)\, dz \\
& \times \Lambda^{k-1}\big(d(K_2, \dots, K_k)\big)\mathbb{Q}(dK_1) \\
= \gamma \sum_{k=1}^{d-i} \frac{1}{k!} \iiiint & e^{\gamma C_d(y-z)} \mathbf{1}\{z \in K_1 \cap \cdots \cap K_k\} \\
& \times \Phi_i(K_1, \dots, K_k; dy)\, dz \Lambda^{k-1}\big(d(K_2, \dots, K_k)\big)\mathbb{Q}(dK_1),
\end{aligned}$$

which yields that

$$\rho_{i,d} = \sum_{k=1}^{d-i} \frac{1}{k!} \int e^{\gamma C_d(x)} H_{i,d}^k(dx) = \int e^{\gamma C_d(x)} H_{i,d}(dx).$$

Finally, we consider the case where $i, j \in \{1, \dots, d-1\}$. Again by Lemma 5.3, we get

$$\begin{aligned}
\rho_{i,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{\substack{I \subset [n] \\ |I| = k}} \sum_{\substack{J \subset [n] \\ |J| = l}} \iiiint & \mathbf{1}\Big\{y \in \bigcap_{r \notin I} K_r^{\circ}, z \in \bigcap_{s \notin J} K_s^{\circ}\Big\} \\
& \times \Phi_i(K_I; dy) \Phi_j(K_J; dz) \\
& \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\mathbb{Q}(dK_1) \\
= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \wedge l} \sum_{\substack{I, J \subset [n] \\ |I|=k, |J|=l, |I \cap J| = m}} \iiiint & \mathbf{1}\Big\{y \in \bigcap_{r \notin I} K_r^{\circ}, z \in \bigcap_{s \notin J} K_s^{\circ}\Big\}
\end{aligned}$$

$$\times \Phi_i(K_I; dy)\Phi_j(K_J; dz) \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \times \mathbb{Q}(dK_1).$$

A symmetry argument shows (as before) that for each choice of I, J such that $|I| = k$, $|J| = l$ and $|I \cap J| = m$, the preceding integral has the same value. There are $\binom{n}{k}\binom{k}{m}\binom{n-k}{l-m}$ possible choices of I, J with these properties. Thus, we obtain

$$\begin{aligned}
\rho_{i,j} &= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k\wedge l} \binom{n}{k}\binom{k}{m}\binom{n-k}{l-m} \\
&\quad \times \int \cdots \int \mathbf{1}\big\{y \in K_{k+1}^{\circ} \cap \cdots \cap K_n^{\circ}\big\} \\
&\qquad \times \mathbf{1}\big\{z \in K_1^{\circ} \cap \cdots \cap K_{k-m}^{\circ} \cap K_{k+l-m+1}^{\circ} \cap \cdots \cap K_n^{\circ}\big\} \\
&\qquad \times \Phi_i(K_1, \dots, K_k; dy) \\
&\qquad \times \Phi_j(K_{k+1-m}, \dots, K_{k+l-m}; dz) \\
&\qquad \times \Lambda^{k+l-m-1}\big(d(K_1, \dots, K_{k+l-m-1})\big) \\
&\qquad \times \mathbb{Q}(dK_{k+l-m})\Lambda^{n-(k+l-m)}(dK_{k+l-m+1}, \dots, K_n) \\
&= \gamma \sum_{n=1}^{\infty} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k\wedge l} \frac{\mathbf{1}\{n \geq k+l-m\}\gamma^{n-(k+l-m)}}{m!(k-m)!(l-m)!(n-(k+l-m))!} \\
&\quad \times \int \cdots \int \prod_{r=k+l-m+1}^{n} \mathbf{1}\big\{x_r \in \big(K_r^{\circ} - y\big) \cap \big(K_r^{\circ} - z\big)\big\}\, dx_{k+l-m+1} \cdots dx_n \\
&\qquad \times \mathbb{Q}(dK_{k+l-m+1}) \cdots \mathbb{Q}(dK_n) \\
&\qquad \times \mathbf{1}\big\{y \in K_{k+1}^{\circ} \cap \cdots \cap K_{k+l-m}^{\circ}\big\} \\
&\qquad \times \mathbf{1}\big\{z \in K_1^{\circ} \cap \cdots \cap K_{k-m}^{\circ}\big\}\Phi_i(K_1, \dots, K_k; dy) \\
&\qquad \times \Phi_j(K_{k+1-m}, \dots, K_{k+l-m}; dz) \\
&\qquad \times \Lambda^{k+l-m-1}\big(d(K_1, \dots, K_{k+l-m-1})\big)\mathbb{Q}(dK_{k+l-m}),
\end{aligned}$$

and hence

$$\begin{aligned}
\rho_{i,j} &= \gamma \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k\wedge l} \frac{1}{m!(k-m)!(l-m)!} \\
&\quad \times \iiiint \sum_{n=k+l-m}^{\infty} \frac{(\gamma C_d(y-z))^{n-(k+l-m)}}{(n-(k+l-m))!} \\
&\qquad \times \mathbf{1}\big\{y \in K_{k+1}^{\circ} \cap \cdots \cap K_{k+l-m}^{\circ}\big\}\mathbf{1}\big\{z \in K_1^{\circ} \cap \cdots \cap K_{k-m}^{\circ}\big\}
\end{aligned}$$

$$\times \Phi_i(K_1,\dots,K_k;dy)\Phi_j(K_{k+1-m},\dots,K_{k+l-m};dz)$$
$$\times \Lambda^{k+l-m-1}\big(d(K_1,\dots,K_{k+l-m-1})\big)\mathbb{Q}(dK_{k+l-m})$$
$$= \sum_{k=1}^{d-i}\sum_{l=1}^{d-j}\sum_{m=0}^{k\wedge l}\frac{1}{m!(k-m)!(l-m)!}\int e^{\gamma C_d(x)}H_{i,j}^{k,l,m}(dx).$$

This completes the proof of the theorem. □

Some of the measures in (5.5) and (5.6) can be expressed in terms of the mixed moment measures

$$M_{i,j} := \mathbb{E}\iint \mathbf{1}\{(y,z)\in\cdot\}\Phi_i(Z_0;dy)\Phi_j(Z_0;dz), \qquad i,j\in\{1,\dots,d\},$$

and the functions $C_j:\mathbb{R}^d\to[0,\infty)$, $j\in\{1,\dots,d-1\}$, defined by

$$C_j(x) := \mathbb{E}\Phi_j\big(Z_0;Z_0^\circ+x\big), \qquad x\in\mathbb{R}^d.$$

LEMMA 5.6. *Assume that (2.5) is satisfied. Then, for any* $i,j\in\{1,\dots,d-1\}$,

$$H_{i,d}^1 = \gamma\int \mathbf{1}\{y-z\in\cdot\}M_{i,d}\big(d(y,z)\big), \tag{5.12}$$

$$H_{i,j}^{1,1,0} = \gamma^2\int \mathbf{1}\{y-z\in\cdot\}C_i(y-z)M_{j,d}\big(d(z,y)\big), \tag{5.13}$$

$$H_{i,j}^{1,1,1} = \gamma\int \mathbf{1}\{y-z\in\cdot\}M_{i,j}\big(d(y,z)\big). \tag{5.14}$$

PROOF. Equations (5.12) and (5.14) follow directly from the definitions, while (5.13) follows from an easy calculation using the covariance property (5.11). □

In the next section, we will use the following consequences of Lemma 5.6:

$$H_{d-1,d} = \gamma\int \mathbf{1}\{y-z\in\cdot\}M_{d-1,d}\big(d(z,y)\big), \tag{5.15}$$

$$\begin{aligned} H_{d-1,d-1} &= \gamma^2\int \mathbf{1}\{y-z\in\cdot\}C_{d-1}(y-z)M_{d-1,d}\big(d(z,y)\big) \\ &\quad + \gamma\int \mathbf{1}\{y-z\in\cdot\}M_{d-1,d-1}\big(d(y,z)\big). \end{aligned} \tag{5.16}$$

6. Covariance structure in the isotropic case. In this section, we assume that the typical grain is *isotropic*, that is, its distribution $\mathbb{Q}$ is invariant under rotations and that the moment assumption (2.5) is satisfied. Our aim is to derive more explicit formulas for the asymptotic covariances

$$\sigma_{i,j} := \lim_{r(W)\to\infty}\frac{\operatorname{Cov}(V_i(Z\cap W),V_j(Z\cap W))}{V_d(W)}, \qquad i,j\in\{0,\dots,d\}; \tag{6.1}$$

confer the statement of Theorem 3.1.

Using the iterated version of the local kinematic formula ([31], Theorem 5.3.2), which is obtained by combining [31], Theorem 6.4.1, (b) and (6.15) and [31], Theorem 6.4.2, (6.20), we get for $j \in \{0, \dots, d-1\}$ and $l \in \{1, \dots, d-j\}$ that

$$\gamma \iint \Phi_j(K_1, \dots, K_l, \mathbb{R}^d)\Lambda^{l-1}(d(K_1, \dots, K_{l-1}))\mathbb{Q}(dK_l)$$

$$= \sum_{\substack{m_1,\dots,m_l=j \\ m_1+\dots+m_l=(l-1)d+j}}^{d-1} c_j^d \prod_{i=1}^{l} c_d^{m_i} \gamma v_{m_i},$$

where, as in [31], (5.4),

$$c_j^m := \frac{m!\kappa_m}{j!\kappa_j}, \qquad m, j \in \{0, \dots, d\}.$$

Combining this with (5.7) and Theorem 5.2, and under assumption (5.8), we deduce

$$\rho_{0,j} = e^{\gamma v_d} P_j(\gamma v_j, \dots, \gamma v_{d-1}), \qquad j \in \{0, \dots, d-1\}, \tag{6.2}$$

where P_j (a multivariate polynomial on $\mathbb{R}^{d-j}$ of degree d) is defined by

$$P_j(t_j, \dots, t_{d-1}) := c_j^d \sum_{l=1}^{d-j} \frac{1}{l!} \sum_{\substack{m_1,\dots,m_l=j \\ m_1+\dots+m_l=(l-1)d+j}}^{d-1} \prod_{i=1}^{l} c_d^{m_i} t_{m_i}.$$

The following main result of this section shows that the asymptotic covariances (6.1) are linear combinations of the numbers $\rho_{i,j}$ given by (5.1). To describe the coefficients, we define for any $j \in \{0, \dots, d-1\}$ and $l \in \{j, \dots, d\}$ a polynomial $P_{j,l}$ on $\mathbb{R}^{d-j}$ of degree $l-j$ by

$$P_{j,l}(t_j, \dots, t_{d-1}) := \mathbf{1}\{l = j\} + c_j^l \sum_{s=1}^{l-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s=sd+j-l}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} t_{m_i} \tag{6.3}$$

and complement this definition by $P_{d,d} := 1$.

THEOREM 6.1. *Assume that the typical grain is isotropic and suppose that* (2.5) *holds. Then*

$$\sigma_{i,j} = (1-p)^2 \sum_{k=i}^{d} \sum_{l=j}^{d} P_{i,k}(\gamma v_i, \dots, \gamma v_{d-1}) P_{j,l}(\gamma v_j, \dots, \gamma v_{d-1}) \rho_{k,l},$$

for all $i, j \in \{0, \dots, d\}$.

PROOF. The formula preceding Theorem 9.1.4 in [31] is the finite volume version of the fundamental result of [20] and [5] on the densities of intrinsic volumes. Using this result, we obtain for all $i \in \{0, \dots, d-1\}$ and $A \in \mathcal{K}^d$ that

$$\mathbb{E}V_i(Z \cap A) - V_i(A) = -(1-p) \sum_{k=i}^{d} V_k(A) P_{i,k}(\gamma v_i, \dots, \gamma v_{d-1}). \tag{6.4}$$

For $i = d$, equation (6.4) is a direct consequence of stationarity and the definition $P_{d,d} = 1$. Using this formula in (3.4), we obtain the assertion from (5.1). □

COROLLARY 6.2. *Assume that* (2.5) *is satisfied. Then, for* $i, j \in \{d-1, d\}$, *the assertions of Theorem* 6.1 *remain true in the general stationary case* (*without isotropy assumption*). *Moreover,*

$$\sigma_{d,d} = (1-p)^2 \int \big(e^{\gamma C_d(x)} - 1\big)\,dx,$$

$$\sigma_{d-1,d} = -(1-p)^2 \gamma v_{d-1} \int \big(e^{\gamma C_d(x)} - 1\big)\,dx + (1-p)^2 \gamma \int e^{\gamma C_d(x-y)} M_{d-1,d}\big(d(x,y)\big).$$

If, in addition, (5.8) *holds, then*

$$\begin{aligned}\sigma_{d-1,d-1} = {} & (1-p)^2 \gamma^2 v_{d-1}^2 \int \big(e^{\gamma C_d(x)} - 1\big)\,dx \\ & + (1-p)^2 \gamma^2 \int e^{\gamma C_d(x-y)} \big(C_{d-1}(x-y) - 2v_{d-1}\big) M_{d-1,d}\big(d(y,x)\big) \\ & + (1-p)^2 \gamma \int e^{\gamma C_d(x-y)} M_{d-1,d-1}\big(d(x,y)\big).\end{aligned}$$

PROOF. The formula preceding Theorem 9.1.4 in [31] does not require isotropy for $j = d-1$. Therefore, for $i, j \in \{d-1, d\}$, the proof of Theorem 6.1 applies without this assumption.

By definition (6.3), $P_{d-1,d-1} = P_{d,d} = 1$ and $P_{d-1,d}(\gamma v_{d-1}) = -\gamma v_{d-1}$. Therefore, we obtain from Theorem 6.1 that $\sigma_{d,d} = (1-p)^2 \rho_{d,d}$, $\sigma_{d-1,d} = (1-p)^2(\rho_{d-1,d} - \gamma v_{d-1} \rho_{d,d})$, and

$$\sigma_{d-1,d-1} = (1-p)^2\big(\rho_{d-1,d-1} - \gamma v_{d-1} \rho_{d-1,d} - \gamma v_{d-1} \rho_{d,d-1} + \gamma^2 v_{d-1}^2 \rho_{d,d}\big).$$

Inserting first (5.10), (5.9) and then (5.15) and (5.16), we obtain the result. □

Together with Corollary 6.2 the next corollary provides rather explicit formulas for the asymptotic covariance in the two-dimensional isotropic case.

COROLLARY 6.3. *Let $d = 2$, assume that the typical grain is isotropic, and suppose that* (2.5) *and* (5.8) *are satisfied. Then*

$$\begin{aligned}\sigma_{0,0} &= (1-2p)(1-p)\gamma + (1-p)(2p-3)\frac{\gamma^2 v_1^2}{\pi} \\ &\quad + (1-p)^2\Big(\gamma - \frac{\gamma^2 v_1^2}{\pi}\Big)^2 \int \big(e^{\gamma C_2(x)} - 1\big)\,dx \\ &\quad + (1-p)^2 \int \chi(x-y) M_{1,2}\big(d(y,x)\big) \\ &\quad + \frac{4}{\pi^2}(1-p)^2\gamma^3 v_1^2 \int e^{\gamma C_2(x-y)} M_{1,1}\big(d(x,y)\big), \\ \sigma_{0,1} &= (1-p)^2\gamma v_1 + (1-p)^2\Big(\gamma^2 v_1 - \frac{\gamma^3 v_1^3}{\pi}\Big)\int \big(e^{\gamma C_2(x)} - 1\big)\,dx \\ &\quad + (1-p)^2 \int \tilde{\chi}(x-y) M_{1,2}\big(d(y,x)\big) \\ &\quad - (1-p)^2 \frac{2\gamma^2 v_1}{\pi} \int e^{\gamma C_2(x-y)} M_{1,1}\big(d(x,y)\big), \\ \sigma_{0,2} &= p(1-p) - (1-p)^2\Big(\gamma - \frac{\gamma^2 v_1^2}{\pi}\Big)\int \big(e^{\gamma C_2(x)} - 1\big)\,dx \\ &\quad - (1-p)^2 \frac{2\gamma^2 v_1}{\pi} \int e^{\gamma C_2(x-y)} M_{1,2}\big(d(x,y)\big),\end{aligned}$$

where

$$\begin{aligned}\chi(z) &:= e^{\gamma C_2(z)}\Big(\frac{4\gamma^4 v_1^2}{\pi^2}\big(C_1(z) - v_1\big) + \frac{4\gamma^3 v_1}{\pi}\Big), \\ \tilde{\chi}(z) &:= e^{\gamma C_2(z)}\Big(\frac{3\gamma^3 v_1^2}{\pi} - \frac{2\gamma^3 v_1}{\pi} C_1(z) - \gamma^2\Big).\end{aligned}$$

The formula for $\sigma_{0,2}$ remains true without assumption (5.8).

PROOF. We have $P_{0,0}(t_0, t_1) = 1$, $P_{0,1}(t_0, t_1) = -\frac{2}{\pi}t_1$, $P_{0,2}(t_0, t_1) = -t_0 + \frac{1}{\pi}t_1^2$, $P_{1,1}(t_1) = 1$, $P_{1,2}(t_1) = -t_1$, and $P_{2,2}(t_1) = 1$. Moreover, we have $P_0(t_0, t_1) = t_0 + \frac{1}{\pi}t_1^2$ and $P_1(t_1) = t_1$. Using (6.2), Theorem 5.2 and Lemma 5.6, we obtain

$$\rho_{0,0} = e^{\gamma v_2}\Big(\gamma + \frac{\gamma^2 v_1^2}{\pi}\Big), \qquad \rho_{0,1} = e^{\gamma v_2}\gamma v_1, \qquad \rho_{0,2} = e^{\gamma v_2} - 1,$$

$$\rho_{1,1} = \gamma^2 \int e^{\gamma C_2(y-z)} C_1(y-z) M_{1,2}\big(d(z,y)\big) + \gamma \int e^{\gamma C_2(y-z)} M_{1,1}\big(d(y,z)\big),$$

$$\rho_{1,2} = \gamma \int e^{\gamma C_2(y-z)} M_{1,2}(d(y,z)), \qquad \rho_{2,2} = \int (e^{\gamma C_2(x)} - 1)\, dx.$$

The result follows by substituting these expressions into Theorem 6.1. □

The proof of Theorem 6.1 also yields the following nonasymptotic result for which definition (5.2) should be recalled. The case $d = 2$ is further discussed in Appendix B of [13].

THEOREM 6.4. *Assume that the typical grain is isotropic and that* (2.5) *holds. Let* $W \in \mathcal{K}^d$ *and* $i, j \in \{0, \dots, d\}$. *Then*

$$\begin{aligned}
&\operatorname{Cov}(V_i(Z \cap W), V_j(Z \cap W)) \\
&\qquad = (1-p)^2 \sum_{k=i}^{d} \sum_{l=j}^{d} P_{i,k}(\gamma v_i, \dots, \gamma v_{d-1}) P_{j,l}(\gamma v_j, \dots, \gamma v_{d-1}) \rho_{k,l}(W).
\end{aligned}$$

7. The spherical Boolean model. In this section, we show how some of the formulas of Section 6 can be used to determine explicitly the covariances of a stationary and isotropic Boolean model whose typical grain is the unit ball B^d. In this particular case, we get from Corollary 6.2 that

$$\begin{aligned}
\sigma_{d-1,d} = (1-p)^2 \gamma \Big[&-v_{d-1} \int (e^{\gamma C_d(x)} - 1)\, dx \\
&+ \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{B^d} e^{\gamma C_d(x-y)}\, dy \mathcal{H}^{d-1}(dx) \Big],
\end{aligned}$$

where $C_d(x) = V_d(B^d \cap (B^d + x))$ and $\mathcal{H}^j$ denotes the j-dimensional Hausdorff measure. Clearly, $\bar{C}_d(t) := V_d(B^d \cap (B^d + tv))$, for $t \ge 0$ and $v \in \mathbb{S}^{d-1}$, is independent of the choice of the unit vector v and

$$\begin{aligned}
\bar{C}_d(t) &= 2\kappa_{d-1} \int_{t/2}^{1} \sqrt{1-u^2}^{\,d-1}\, du \\
&= 2 \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \int_{t/2}^{1} \sqrt{1-u^2}^{\,d-1}\, du, \qquad t \in [0,2].
\end{aligned}$$

Introducing polar coordinates, we get

$$\begin{aligned}
F_d(\gamma) &:= v_{d-1} \int (e^{\gamma C_d(x)} - 1)\, dx = v_{d-1}\, d\kappa_d \int_0^2 (e^{\gamma \bar{C}_d(t)} - 1) t^{d-1}\, dt \\
&=: v_{d-1} f_d(\gamma),
\end{aligned}$$

where $v_{d-1} = d\kappa_d/2$. On the other hand, for an arbitrary unit vector $v \in \mathbb{S}^{d-1}$, by the rotation invariance of B^d we get

$$G_d(\gamma) := \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{B^d} e^{\gamma C_d(x-y)}\, dy \mathcal{H}^{d-1}(dx) = v_{d-1} \int_{B^d} e^{\gamma C_d(v-y)}\, dy.$$

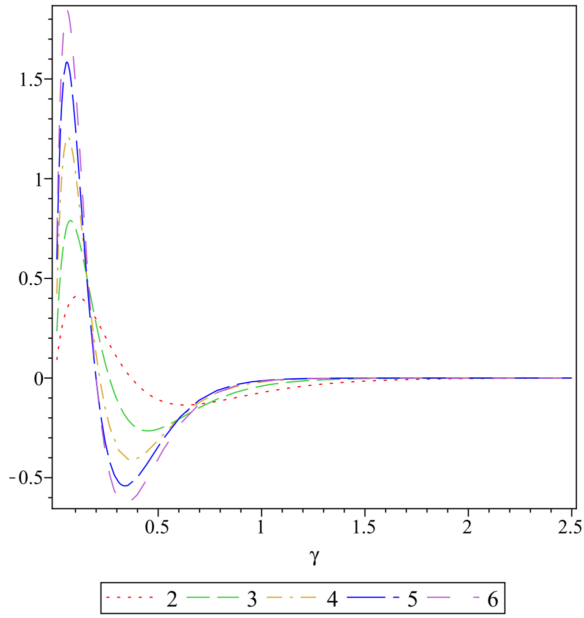

FIG. 1. $\sigma_{d-1,d}(\gamma)$ *for* $d = 2, \dots, d = 6$.

We parameterize y in the form

$$y = (1-t)v + \sqrt{1-(1-t)^2}\, s w, \qquad t \in [0,2], s \in [0,1], w \in v^{\perp} \cap \mathbb{S}^{d-1},$$

and hence we obtain

$$\begin{aligned} G_d(\gamma) &= v_{d-1}(d-1)\kappa_{d-1} \int_0^2 \int_0^1 \exp\bigl(\gamma \bar{C}_d\bigl(\sqrt{(2-t)^2 + t(2-t)s^2}\bigr)\bigr) \\ &\qquad\qquad \times s^{d-2} \sqrt{t(2-t)}^{\,d-1}\, ds\, dt \\ &=: v_{d-1} g_d(\gamma). \end{aligned}$$

Therefore, we have

$$\sigma_{d-1,d} = (1-p)^2 \gamma v_{d-1}\bigl(-f_d(\gamma) + g_d(\gamma)\bigr),$$

which shows that the sign of the covariance $\sigma_{d-1,d}$ is completely determined by the sign of the function $g_d - f_d$.

It is preferable to plot the covariances as functions of the intensity. Here, we get

$$\sigma_{d-1,d}(\gamma) = \gamma e^{-2\kappa_d \gamma} v_{d-1}\bigl(g_d(\gamma) - f_d(\gamma)\bigr).$$

Figure 1 shows the result for various dimensions.

Next, we determine the correlation coefficient $\mathrm{Cor}_{d-1,d}(\gamma)$, as a function of the intensity γ. For this, we also have to determine explicitly $\sigma_{d,d}$ and $\sigma_{d-1,d-1}$, which requires some further calculations. First, we have

$$\sigma_{d,d} = (1-p)^2 \int \big(e^{\gamma C_d(x)} - 1\big)\, dx = (1-p)^2 f_d(\gamma),$$

hence $\sqrt{\sigma_{d,d}} = (1-p)\sqrt{f_d(\gamma)}$; second,

$$\begin{aligned}\sigma_{d-1,d-1} = (1-p)^2\gamma^2\Big[&(v_{d-1})^2 f_d(\gamma)\\ &+ \int e^{\gamma C_d(x-y)} C_{d-1}(x-y) M_{d-1,d}\big(d(y,x)\big)\\ &- 2v_{d-1} \int e^{\gamma C_d(x-y)} M_{d-1,d}\big(d(y,x)\big)\\ &+ \frac{1}{\gamma} \int e^{\gamma C_d(x-y)} M_{d-1,d-1}\big(d(x,y)\big)\Big].\end{aligned}$$

Since $C_{d-1}(x)$ depends only on $\|x\|$, we denote it by $\bar{C}_{d-1}(\|x\|)$. For $0 < \|x\| \leq 2$, we then get

$$\bar{C}_{d-1}\big(\|x\|\big) = \frac{1}{2}\mathcal{H}^{d-1}\big(\mathbb{S}^{d-1} \cap \big(B^d + x\big)\big) = \frac{1}{2}(d-1)\kappa_{d-1} \int_{\|x\|/2}^{1} \sqrt{1-s^2}^{\,d-3}\, ds.$$

Let $v \in \mathbb{S}^{d-1}$ be fixed. Then, arguing as in the derivation of (7.1), we obtain

$$\begin{aligned}&\int e^{\gamma C_d(x-y)} C_{d-1}(x-y) M_{d-1,d}\big(d(y,x)\big)\\ &\quad= v_{d-1} \int_{B^d} e^{\gamma C_d(x-v)} C_{d-1}(x-v)\, dx\\ &\quad= v_{d-1}(d-1)\kappa_{d-1} \int_0^2 \int_0^1 s^{d-2} \sqrt{t(2-t)}^{\,d-1}\\ &\qquad\qquad \times \exp\big(\gamma \bar{C}_d\big(\sqrt{(2-t)^2 + t(2-t)s^2}\big)\big)\\ &\qquad\qquad \times \bar{C}_{d-1}\big(\sqrt{(2-t)^2 + t(2-t)s^2}\big)\, ds\, dt\\ &\quad=: v_{d-1}(d-1)\kappa_{d-1} h_d(\gamma).\end{aligned}$$

Furthermore, we have

$$2v_{d-1} \int e^{\gamma C_d(x-y)} M_{d-1,d}\big(d(x,y)\big) = 2(v_{d-1})^2 g_d(\gamma) = \frac{(d\kappa_d)^2}{2} g_d(\gamma).$$

Finally, since

$$M_{d-1,d-1} = \frac{1}{4} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \mathbf{1}\{(y,z) \in \cdot\}\mathcal{H}^{d-1}(dy)\mathcal{H}^{d-1}(dz),$$

we get (with an arbitrary unit vector v_0)

$$
\begin{aligned}
&\int e^{\gamma C_d(x-y)} M_{d-1,d-1}\big(d(x,y)\big) \\
&\quad = \frac{d\kappa_d}{4} \int_{\mathbb{S}^{d-2}} \int_0^{\pi} \exp\big(\gamma C_d\big(v_0 - [\cos\theta v_0 + \sin\theta v]\big)\big) \sin^{d-2}\theta \, d\theta \mathcal{H}^{d-2}(dv) \\
&\quad = \frac{d\kappa_d (d-1)\kappa_{d-1}}{4} \int_0^{\pi} \sin^{d-2}\theta \exp\big(\gamma \bar{C}_d\big(\sqrt{2(1-\cos\theta)}\big)\big) \, d\theta \\
&\quad = \frac{d\kappa_d (d-1)\kappa_{d-1}}{4} \int_0^{2} \sqrt{s(2-s)}^{d-3} \exp\big(\gamma \bar{C}_d\big(\sqrt{2(2-s)}\big)\big) \, ds \\
&\quad =: \frac{d\kappa_d (d-1)\kappa_{d-1}}{4} k_d(\gamma).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\frac{\sigma_{d-1,d-1}}{(1-p)^2\gamma^2} = {}& \left(\frac{d\kappa_d}{2}\right)^2 f_d(\gamma) - \frac{(d\kappa_d)^2}{2} g_d(\gamma) + \frac{d\kappa_d (d-1)\kappa_{d-1}}{2} h_d(\gamma) \\
& + \frac{d\kappa_d (d-1)\kappa_{d-1}}{4\gamma} k_d(\gamma).
\end{aligned}
$$

This finally implies that

$$
\begin{aligned}
&\mathrm{Cor}_{d-1,d}(\gamma) \\
&\quad = \left(\frac{d\kappa_d}{2}\big(g_d(\gamma) - f_d(\gamma)\big)\right) \\
&\qquad \Big/ \Bigg(\sqrt{f(\gamma)}\Bigg(\left(\frac{d\kappa_d}{2}\right)^2 f_d(\gamma) - \frac{(d\kappa_d)^2}{2} g_d(\gamma) \\
&\qquad\qquad + \frac{d\kappa_d (d-1)\kappa_{d-1}}{2} h_d(\gamma) + \frac{d\kappa_d (d-1)\kappa_{d-1}}{4\gamma} k_d(\gamma)\Bigg)^{1/2}\Bigg).
\end{aligned}
$$

From these considerations, we also deduce the plausible fact that

$$
\lim_{\gamma\downarrow 0} \mathrm{Cor}_{d-1,d}(\gamma) = \lim_{\gamma\downarrow 0} \frac{(1/2) d\kappa_d \kappa_d}{\sqrt{\gamma \int C_d(x)\, dx} \sqrt{(1/\gamma)(d\kappa_d/2)^2}} = 1,
$$

which is confirmed by our numerical calculations. Plots of $\mathrm{Cor}_{d-1,d}(\cdot)$ for $d = 2, \ldots, 6$ are shown in Figure 2.

In a similar way, the formulas from Corollary 6.3 can be specified in the case of a planar Boolean model with the unit circle as deterministic typical grain. Then we have

$$
\begin{aligned}
\chi(r,\gamma) &= 4\gamma^3 e^{\gamma \bar{C}_2(r)} \big(\gamma \bar{C}_1(r) - \pi\gamma + 1\big) \quad \text{and} \\
\tilde{\chi}(r,\gamma) &= \gamma^2 e^{\gamma \bar{C}_2(r)} \big(3\pi\gamma - 2\bar{C}_1(r)\gamma - 1\big),
\end{aligned}
$$

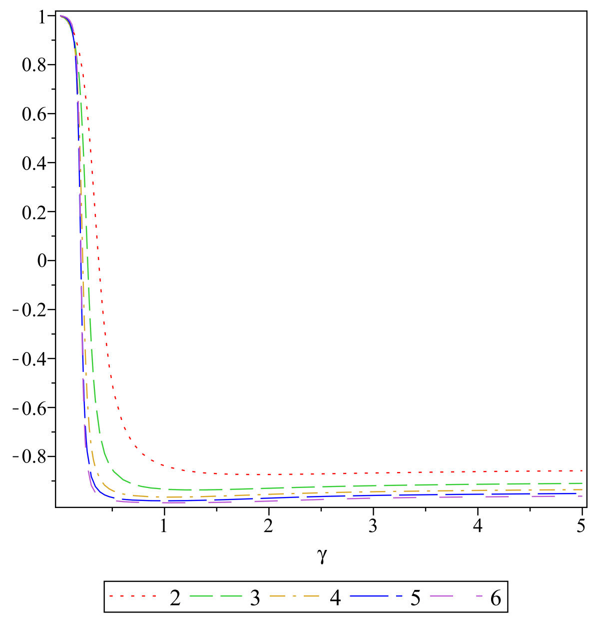

FIG. 2. $\mathrm{Cor}_{d-1,d}(\gamma)$ *for* $d = 2, \ldots, 6$.

and, for instance,

$$\begin{aligned}\sigma_{0,0}(\gamma) &= (1-2p)(1-p)\gamma + (1-p)(2p-3)\gamma^2\pi \\ &\quad + (1-p)^2\gamma^2(1-\pi\gamma)^2 f_2(\gamma) \\ &\quad + (1-p)^2 2\pi \int_0^2 \int_0^1 \chi\big(\sqrt{(2-t)^2 + t(2-t)s^2}, \gamma\big)\sqrt{t(2-t)}\, ds\, dt \\ &\quad + (1-p)^2\gamma^3 4\pi \int_0^\pi \exp\big(\gamma \bar{C}_2\big(\sqrt{2(1-\cos(t))}\big)\big)\, dt,\end{aligned}$$

where $p = p(\gamma) = 1 - e^{-\pi\gamma}$. Moreover,

$$\begin{aligned}\sigma_{0,1}(\gamma) &= (1-p)^2\gamma\pi + (1-p)^2\gamma^2\pi(1-\pi\gamma) f_2(\gamma) \\ &\quad + (1-p)^2 2\pi \int_0^2 \int_0^1 \tilde{\chi}\big(\sqrt{(2-t)^2 + t(2-t)s^2}, \gamma\big)\sqrt{t(2-t)}\, ds\, dt \\ &\quad - (1-p)^2\gamma^2 2\pi \int_0^\pi \exp\big(\gamma \bar{C}_2\big(\sqrt{2(1-\cos(t))}\big)\big)\, dt,\end{aligned}$$

$$\sigma_{0,2}(\gamma) = p(1-p) - (1-p)^2\gamma(1-\pi\gamma) f_2(\gamma) - (1-p)^2 2\gamma^2\pi g_2(\gamma),$$

$$\sigma_{2,2}(\gamma) = (1-p)^2 f_2(\gamma).$$

The variances and covariances as well as the correlation functions for the planar case are plotted in Figures 3 and 4.

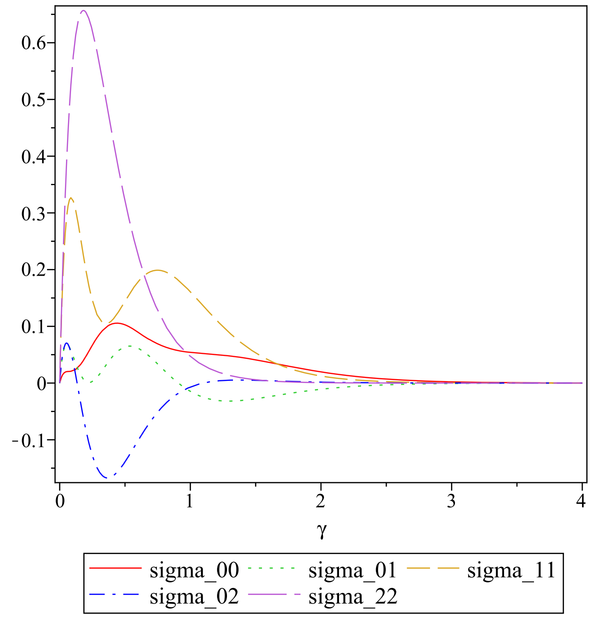

FIG. 3. *Variances/covariances for $d = 2$.*

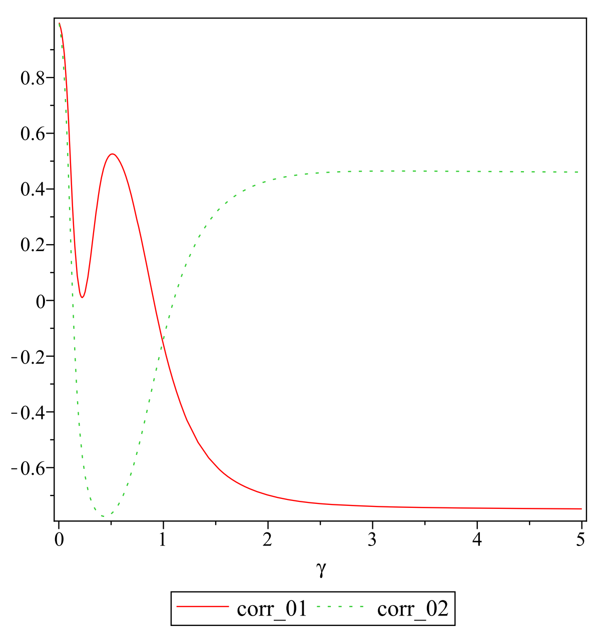

FIG. 4. *Correlation functions for $d = 2$.*

8. Normal approximation via the Malliavin–Stein method. In this section, we prepare the central limit theorems for geometric functionals of a Boolean model by proving a general result on the normal approximation of Poisson functionals. Our approach is based on recent findings in [23, 25] and uses similar arguments as in [28].

Throughout this section, let η be a Poisson process on a measurable space $(\mathbf{X}, \mathcal{X})$ with a σ-finite intensity measure λ; see [14], Chapter 12. Consider a $[-\infty, \infty]$-valued random variable F such that $\mathbb{P}(|F| < \infty) = 1$ and $F = f(\eta)$ $\mathbb{P}$-a.s. for some measurable $f : \mathbf{N} \to \mathbb{R}$. Any such f is called a *representative* of the *Poisson functional* F. If f is a (fixed) representative of F, we define

$$D^n_{x_1,\dots,x_n} F := D^n_{x_1,\dots,x_n} f(\eta), \qquad n \in \mathbb{N}, x_1, \dots, x_n \in \mathbf{X},$$

where D^n is the nth iterated difference operator used in Section 3. If $\tilde{f}$ is another representative of F, then the multivariate Mecke equation (see, e.g., [15], (2.10)) implies that $D^n_{x_1,\dots,x_n} f(\eta) = D^n_{x_1,\dots,x_n} \tilde{f}(\eta)$ $\mathbb{P}$-a.s. and for λ^n-a.e. $(x_1, \dots, x_n) \in \mathbf{X}^n$. Let L^2_η denote the space of all Poisson functionals F such that $\mathbb{E}F^2 < \infty$. For $F \in L^2_\eta$ we define $f_n : \mathbf{X}^n \to \mathbb{R}$ by

$$f_n(x_1, \dots, x_n) = \frac{1}{n!} \mathbb{E} D^n_{x_1,\dots,x_n} F.$$

It was shown in [15], Theorem 1.1, that f_n belongs to the space $L^2_s(\lambda^n)$ of λ^n-almost everywhere symmetric functions on $\mathbf{X}^n$ that are square-integrable with respect to λ^n. Now the Fock space representation (see [15], Theorem 1.1) tells us that

$$\operatorname{Var} F = \sum_{n=1}^{\infty} n! \| f_n \|_n^2, \tag{8.1}$$

where $\| \cdot \|_n$ denotes the norm in $L^2(\lambda^n)$. Moreover, it is known from [15], Theorem 1.3, that F has the representation

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n), \tag{8.2}$$

where $I_n(\cdot)$ stands for the nth multiple Wiener–Itô integral, and the right-hand side converges in $L^2(\mathbb{P})$. The identity (8.2) is called Wiener–Itô chaos expansion of F. The multiple Wiener–Itô integrals are defined for square integrable symmetric functions and are orthogonal in the sense that

$$\mathbb{E} I_n(f) I_m(g) = \begin{cases} n! \langle f, g \rangle_n, & n = m, \\ 0, & n \neq m, \end{cases}$$

for $f \in L^2_s(\lambda^n)$, $g \in L^2_s(\lambda^m)$, and $n, m \in \mathbb{N}$, where $\langle \cdot, \cdot \rangle_n$ denotes the scalar product in $L^2(\lambda^n)$.

If the condition

$$\sum_{n=1}^{\infty} n n! \| f_n \|_n^2 < \infty \tag{8.3}$$

is satisfied, the difference operator (3.6) has the representation

$$D_x F = \sum_{n=1}^{\infty} n I_{n-1}\big(f_n(x, \cdot)\big) \tag{8.4}$$

$\mathbb{P}$-a.s. for λ-a.e. $x \in \mathbf{X}$ (see, e.g., [15], Theorem 3.3). From now on, we write $F \in \operatorname{dom} D$ if $F \in L_\eta^2$ satisfies (8.3). The Ornstein–Uhlenbeck generator associates with any Poisson functional $F \in L_\eta^2$ such that $\sum_{n=1}^{\infty} n^2 n! \| f_n \|_n^2 < \infty$ the random variable

$$LF = -\sum_{n=1}^{\infty} n I_n(f_n),$$

and its pseudo-inverse is given by

$$L^{-1} F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n) \tag{8.5}$$

for $F \in L_\eta^2$. These operators together with the difference operator and the Skorohod integral, which is not used in this paper, are called Malliavin operators. Combining (8.4) and (8.5), we see that

$$D_x L^{-1} F = -\sum_{n=1}^{\infty} I_{n-1}\big(f_n(x, \cdot)\big) \tag{8.6}$$

$\mathbb{P}$-a.s. for λ-a.e. $x \in \mathbf{X}$. More details on the Wiener–Itô chaos expansion and the Malliavin operators can be found in [15] and the references therein. In [23, 25], the Malliavin operators and Stein's method are combined to derive bounds for the normal approximation of Poisson functionals. In the following, we evaluate bounds obtained by this technique, which is called the Malliavin–Stein method.

To measure the distance between two real-valued random variables Y_1, Y_2, we use the Wasserstein distance that is given by

$$\mathbf{d}_W(Y_1, Y_2) = \sup_{h \in \operatorname{Lip}(1)} \big| \mathbb{E} h(Y_1) - \mathbb{E} h(Y_2) \big|.$$

Here, $\operatorname{Lip}(1)$ stands for the set of all functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to one. For two m-dimensional random vectors Y_1, Y_2, we define

$$\mathbf{d}_3(Y_1, Y_2) = \sup_{h \in \mathcal{H}} \big| \mathbb{E} h(Y_1) - \mathbb{E} h(Y_2) \big|,$$

where $\mathcal{H}$ is the set of all three times continuously differentiable functions $h:\mathbb{R}^m \to \mathbb{R}$ such that

$$\max_{i,j=1,\dots,m} \sup_{x\in\mathbb{R}^m} \left| \frac{\partial^2 h}{\partial x_i\, \partial x_j}(x) \right| \le 1 \quad \text{and} \quad \max_{i,j,k=1,\dots,m} \sup_{x\in\mathbb{R}^m} \left| \frac{\partial^3 h}{\partial x_i\, \partial x_j\, \partial x_k}(x) \right| \le 1.$$

Convergence in the Wasserstein distance or in the $\mathbf{d}_3$-distance implies convergence in distribution.

In the following, we establish an upper bound for the $\mathbf{d}_3$-distance between a Gaussian random vector and a random vector $F = (F^{(1)}, \dots, F^{(m)})$ of Poisson functionals $F^{(1)}, \dots, F^{(m)} \in L^2_\eta$. Each of these components has a Wiener–Itô chaos expansion

$$F^{(k)} = \mathbb{E}F^{(k)} + \sum_{n=1}^{\infty} I_n\big(f_n^{(k)}\big)$$

with $f_n^{(k)} \in L^2_s(\lambda^n)$, $n \in \mathbb{N}$. We also state a bound for the Wasserstein distance between the normalization of a Poisson functional F and a standard Gaussian random variable.

We need to introduce some notation. Consider functions $g_1:\mathbf{X}^{n_1} \to \mathbb{R}$ and $g_2:\mathbf{X}^{n_2} \to \mathbb{R}$, where $n_1, n_2 \in \mathbb{N}$. The tensor product $g_1 \otimes g_2$ is the function on $\mathbf{X}^{n_1+n_2}$ which maps each $(x_1, \dots, x_{n_1+n_2})$ to $g_1(x_1, \dots, x_{n_1}) g_2(x_{n_1+1}, \dots, x_{n_1+n_2})$. This definition can be iterated in the obvious way. Fix two integers $i, j \ge 1$ and consider functions $f:\mathbf{X}^i \to \mathbb{R}$ and $g:\mathbf{X}^j \to \mathbb{R}$. Let σ be a partition of $I_{ij} := \{1, \dots, 2i + 2j\}$ and let $|\sigma|$ be the number of blocks (i.e., the disjoint sets constituting the partition) of σ. The function $(f \otimes f \otimes g \otimes g)_\sigma : \mathbf{X}^{|\sigma|} \to \mathbb{R}$ is defined by replacing all variables whose indices belong to the same block of σ by a new common variable. Let $\pi = \{J_1, \dots, J_4\}$ be the partition of I_{ij} into the sets $J_1 := \{1, \dots, i\}$, $J_2 := \{i + 1, \dots, 2i\}$, $J_3 := \{2i + 1, \dots, 2i + j\}$, and $J_4 := \{2i + j + 1, \dots, 2i + 2j\}$. Let Π_{ij} be the set of all partitions σ of I_{ij} such that $|J \cap J'| \le 1$ for all $J \in \pi$ and all $J' \in \sigma$. By $\widetilde{\Pi}_{ij}$ we denote the set of all partitions $\sigma \in \Pi_{ij}$ such that:

(i) $\{1, 2i + 1\}, \{i + 1, 2i + j + 1\} \in \sigma$ or $\{1, i + 1, 2i + 1, 2i + j + 1\} \in \sigma$;
(ii) each block of σ has at least two elements;
(iii) for every partition of $\{1, 2, 3, 4\}$ in two disjoint nonempty sets M_1, M_2 there are $u \in M_1$, $v \in M_2$ such that J_u and J_v are both intersected by one block of σ.

Let $\widetilde{\Pi}_{ij}^{(1)}$ (resp., $\widetilde{\Pi}_{ij}^{(2)}$) be the set of all partitions $\sigma \in \widetilde{\Pi}_{ij}$ such that $\{1, 2i + 1\}, \{i + 1, 2i + j + 1\} \in \sigma$ (resp. $\{1, i + 1, 2i + 1, 2i + j + 1\} \in \sigma$). In the terminology of diagram formulae as it is used in [24], Chapter 4, condition (iii) means that π and σ generate a "connected diagram."

Now we are able to state the main result of this section.

THEOREM 8.1. *Assume that* $F^{(k)} \in L^2_\eta$ *and*

$$\int \big|\big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma\big| \, d\lambda^{|\sigma|} < \infty \tag{8.7}$$

for all $\sigma \in \Pi_{ij}$, $i, j \in \mathbb{N}$, *and* $k, l \in \{1, \dots, m\}$. *Further, assume that there are* $a > 0$ *and* $b \geq 1$ *such that*

$$\int \big|\big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma\big| \, d\lambda^{|\sigma|} \leq \frac{ab^{i+j}}{(i!)^2(j!)^2} \tag{8.8}$$

for all $\sigma \in \widetilde{\Pi}_{ij}$, $i, j \in \mathbb{N}$, *and* $k, l \in \{1, \dots, m\}$. *Let* $F := (F^{(1)}, \dots, F^{(m)})$ *and let* N *be a centered Gaussian random vector with a given positive semidefinite covariance matrix* $(\sigma_{k,l})_{k,l=1,\dots,m}$. *Then*

$$\begin{aligned}\mathbf{d}_3(F - \mathbb{E}F, N) &\leq \frac{m}{2} \sum_{k,l=1}^{m} \big|\sigma_{k,l} - \operatorname{Cov}\big(F^{(k)}, F^{(l)}\big)\big| \\ &\quad + \left(\frac{m}{2} + \frac{m}{4} \sum_{n=1}^{m} \sqrt{\operatorname{Var} F^{(n)}}\right) 2^{13/2} m^2 \sum_{i=1}^{\infty} i^{17/2} \frac{b^i}{\lfloor i/14 \rfloor!} \sqrt{a}.\end{aligned}$$

In the univariate case, we have the following result for the Wasserstein distance.

COROLLARY 8.2. *Let* $F \in L^2_\eta$ *be such that* $\operatorname{Var} F > 0$ *and the assumptions* (8.7) *and* (8.8) *are satisfied and let* N *be a standard Gaussian random variable. Then*

$$\mathbf{d}_W\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq 2^{15/2} \sum_{i=1}^{\infty} i^{17/2} \frac{b^i}{\lfloor i/14 \rfloor!} \frac{\sqrt{a}}{\operatorname{Var} F}.$$

We prepare the proof of Theorem 8.1 by two lemmas and a proposition.

LEMMA 8.3. *Let* $i, j \in \mathbb{N}$, $f \in L^2_s(\lambda^i)$, $g \in L^2_s(\lambda^j)$, *and assume that*

$$\int \big|(f \otimes f \otimes g \otimes g)_\sigma\big| \, d\lambda^{|\sigma|} < \infty, \qquad \sigma \in \Pi_{ij}.$$

Then

$$\begin{aligned}&\operatorname{Var}\left(\int I_{i-1}\big(f(z,\cdot)\big) I_{j-1}\big(g(z,\cdot)\big) \lambda(dz)\right) \\ &\quad = \sum_{\sigma \in \widetilde{\Pi}_{ij}^{(1)}} \int (f \otimes f \otimes g \otimes g)_\sigma \, d\lambda^{|\sigma|},\end{aligned} \tag{8.9}$$

$$\begin{aligned}&\mathbb{E} \int I_{i-1}\big(f(z,\cdot)\big)^2 I_{j-1}\big(g(z,\cdot)\big)^2 \lambda(dz) \\ &\quad = \sum_{\sigma \in \widetilde{\Pi}_{ij}^{(2)}} \int (f \otimes f \otimes g \otimes g)_\sigma \, d\lambda^{|\sigma|}.\end{aligned} \tag{8.10}$$

PROOF. Combining the formulas

$$\mathbb{E}\Big(\int I_{i-1}\big(f(z,\cdot)\big)I_{j-1}\big(g(z,\cdot)\big)\lambda(dz)\Big)^2$$
$$= \iint \mathbb{E}I_{i-1}\big(f(y,\cdot)\big)I_{i-1}\big(f(z,\cdot)\big)I_{j-1}\big(g(y,\cdot)\big)I_{j-1}\big(g(z,\cdot)\big)\lambda(dy)\lambda(dz)$$

and

$$\mathbb{E}\int I_{i-1}\big(f(z,\cdot)\big)I_{j-1}\big(g(z,\cdot)\big)\lambda(dz) = \begin{cases} (i-1)!\langle f, g\rangle_i, & i=j, \\ 0, & i \neq j, \end{cases} \tag{8.11}$$

with Theorem 3.1 in [16] (see also [24], Corollary 7.2 and [32], Proposition 3.1) proves the first equation. The second identity is a consequence of

$$\mathbb{E}\int I_{i-1}\big(f(z,\cdot)\big)^2 I_{j-1}\big(g(z,\cdot)\big)^2\lambda(dz) = \int \mathbb{E}I_{i-1}\big(f(z,\cdot)\big)^2 I_{j-1}\big(g(z,\cdot)\big)^2\lambda(dz)$$

and, again, Theorem 3.1 in [16]. □

PROPOSITION 8.4. *Let $F^{(1)},\dots,F^{(m)} \in L^2_\eta$ be such that (8.7) holds. Let $F := (F^{(1)},\dots,F^{(m)})$ and let N be a centered Gaussian random vector with a given positive semidefinite covariance matrix $(\sigma_{k,l})_{k,l=1,\dots,m}$. Then*

$$\begin{aligned} &\mathbf{d}_3(F-\mathbb{E}F, N) \\ &\quad \le \frac{m}{2}\sum_{k,l=1}^m \big|\sigma_{k,l} - \operatorname{Cov}\big(F^{(k)},F^{(l)}\big)\big| \\ &\qquad + \Big(\frac{m}{2} + \frac{m}{4}\sum_{n=1}^m \sqrt{\operatorname{Var} F^{(n)}}\Big) \\ &\qquad \times \sum_{k,l=1}^m \sum_{i,j=1}^\infty ij \sqrt{\sum_{\sigma\in\widetilde{\Pi}_{i,j}} \int \big|\big(f_i^{(k)}\otimes f_i^{(k)}\otimes f_j^{(l)}\otimes f_j^{(l)}\big)_\sigma\big|\, d\lambda^{|\sigma|}}. \end{aligned} \tag{8.12}$$

PROOF. To avoid convergence issues, we start by proving (8.12) for $F_s := (F_s^{(1)},\dots,F_s^{(m)})$ with the truncated Poisson functionals $F_s^{(l)} := \mathbb{E}F^{(l)} + \sum_{n=1}^s I_n(f_n^{(l)})$, $l \in \{1,\dots,m\}$, for a fixed $s \in \mathbb{N}$. By construction, we have $F_s^{(1)},\dots,F_s^{(m)} \in \operatorname{dom} D$. From [25], Theorem 4.2, it is known that

$$\begin{aligned} \mathbf{d}_3(F_s-\mathbb{E}F_s, N) &\le \frac{m}{2}\sqrt{\sum_{k,l=1}^m \mathbb{E}\Big(\sigma_{k,l} - \int D_z F_s^{(k)}\big(-D_z L^{-1}F_s^{(l)}\big)\lambda(dz)\Big)^2} \\ &\quad + \frac{1}{4}\int \mathbb{E}\Big(\sum_{k=1}^m \big|D_z F_s^{(k)}\big|\Big)^2 \sum_{l=1}^m \big|D_z L^{-1}F_s^{(l)}\big|\lambda(dz). \end{aligned} \tag{8.13}$$

We bound the two summands on the above right-hand side separately. For the first one, we have

$$\sqrt{\sum_{k,l=1}^{m} \mathbb{E}\Big(\sigma_{k,l} - \int D_z F_s^{(k)}(-D_z L^{-1} F_s^{(l)})\lambda(dz)\Big)^2}$$

$$\leq \sum_{k,l=1}^{m} \Big(\mathbb{E}\Big(\sigma_{k,l} - \operatorname{Cov}(F_s^{(k)}, F_s^{(l)}) + \operatorname{Cov}(F_s^{(k)}, F_s^{(l)}) - \int D_z F_s^{(k)}(-D_z L^{-1} F_s^{(l)})\lambda(dz)\Big)^2\Big)^{1/2}$$

$$\leq \sum_{k,l=1}^{m} \Big(|\sigma_{k,l} - \operatorname{Cov}(F_s^{(k)}, F_s^{(l)})| + \sqrt{\mathbb{E}\Big(\operatorname{Cov}(F_s^{(k)}, F_s^{(l)}) - \int D_z F_s^{(k)}(-D_z L^{-1} F_s^{(l)})\lambda(dz)\Big)^2}\Big).$$

Put $g_n^{(l)}(z) := I_{n-1}(f_n^{(l)}(z, \cdot))$. From (8.4), (8.6), the covariance version of (8.1) [see (3.7)] and (8.11), we obtain that

$$a_s^{k,l} := \mathbb{E}\Big(\int D_z F_s^{(k)}(-D_z L^{-1} F_s^{(l)})\lambda(dz) - \operatorname{Cov}(F_s^{(k)}, F_s^{(l)})\Big)^2$$

$$= \mathbb{E}\Big(\int \sum_{i=1}^{s} i g_i^{(k)}(z) \sum_{j=1}^{s} g_j^{(l)}(z)\lambda(dz) - \sum_{n=1}^{s} n!\langle f_n^{(k)}, f_n^{(l)}\rangle_n\Big)^2$$

$$= \operatorname{Var}\Big(\sum_{i,j=1}^{s} i \int g_i^{(k)}(z) g_j^{(l)}(z)\lambda(dz)\Big).$$

Note that the right-hand side is well defined since (8.7) and Lemma 8.3 ensure that each of the summands is square integrable.

Since $\sqrt{\operatorname{Var}(Y_1 + Y_2)} \leq \sqrt{\operatorname{Var} Y_1} + \sqrt{\operatorname{Var} Y_2}$ for random variables Y_1, Y_2, we obtain

$$\sqrt{a_s^{k,l}} \leq \sum_{i,j=1}^{s} i \sqrt{\operatorname{Var}\Big(\int g_i^{(k)}(z) g_j^{(l)}(z)\lambda(dz)\Big)}$$

$$\leq \sum_{i,j=1}^{s} i \sqrt{\sum_{\sigma \in \widetilde{\Pi}_{ij}} \int |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\lambda^{|\sigma|}},$$

where we have applied (8.9) in Lemma 8.3 to get the final inequality.

By Jensen's inequality and the definitions of the Malliavin operators, we obtain for the second summand in (8.13) that

$$\begin{aligned}
&\int \mathbb{E}\Bigg(\sum_{k=1}^{m}|D_z F_s^{(k)}|\Bigg)^2 \sum_{l=1}^{m}|D_z L^{-1}F_s^{(l)}|\lambda(dz)\\
&\qquad\le m\sum_{k,l=1}^{m}\int \mathbb{E}\big(D_z F_s^{(k)}\big)^2|D_z L^{-1}F_s^{(l)}|\lambda(dz)\\
&\qquad\le m\sum_{k,l=1}^{m}\sum_{i,j=1}^{s} ij\int \mathbb{E}|g_i^{(k)}(z)||g_j^{(k)}(z)||D_z L^{-1}F_s^{(l)}|\lambda(dz)\\
&\qquad\le m\sum_{k,l=1}^{m}\sum_{i,j=1}^{s} ij\sqrt{\int \mathbb{E}g_i^{(k)}(z)^2 g_j^{(k)}(z)^2\lambda(dz)}\sqrt{\int \mathbb{E}\big(D_z L^{-1}F_s^{(l)}\big)^2\lambda(dz)}.
\end{aligned}$$

Combining (8.6) and (8.11) with (8.1), we get

$$\int \mathbb{E}\big(D_z L^{-1}F_s^{(l)}\big)^2\lambda(dz)=\sum_{n=1}^{s}(n-1)!\|f_n^{(l)}\|_n^2\le \operatorname{Var} F_s^{(l)}.$$

Now (8.10) in Lemma 8.3 completes the proof of (8.12) for F_s. By the triangle inequality for the $\mathbf{d}_3$-distance and [16], Lemma 5.5, we have that

$$\begin{aligned}
\mathbf{d}_3(F-\mathbb{E}F,N)&\le \mathbf{d}_3(F-\mathbb{E}F,F_s-\mathbb{E}F_s)+\mathbf{d}_3(F_s-\mathbb{E}F_s,N)\\
&\le m\sqrt{\mathbb{E}\|F-\mathbb{E}F\|^2+\mathbb{E}\|F_s-\mathbb{E}F_s\|^2}\sqrt{\mathbb{E}\|F-F_s\|^2}\\
&\quad+\mathbf{d}_3(F_s-\mathbb{E}F_s,N),
\end{aligned}$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^m$. Since $F_s^{(l)}\to F^{(l)}$ in L_η^2 as $s\to\infty$, the first summand vanishes as $s\to\infty$. Applying (8.12) to the second summand and letting $s\to\infty$ completes the proof. □

LEMMA 8.5. *For any integers $i,j\ge 1$,*

$$|\widetilde{\Pi}_{i,j}|\le \frac{(i!)^2(j!)^2\max\{i+1,j+1\}^{11}}{\lceil \max\{i,j\}/7\rceil!}.$$

PROOF. For a fixed partition $\sigma\in\widetilde{\Pi}_{ij}$, let k_{uv} with $u,v\in\{1,2,3,4\}$ and $u<v$ be the number of blocks $A\in\sigma$ such that $|A\cap J_u|=|A\cap J_v|=1$ and $A\cap(J_u\cup J_v)=A$. We define k_{uvw} for $u,v,w\in\{1,2,3,4\}$ with $u<v<w$ and k_{1234} in the same way. For a possible combination of fixed numbers $k_{12},\dots,k_{1234}$ the number of partitions $\sigma\in\widetilde{\Pi}_{ij}$ having this form is less than

$$\frac{(i!)^2(j!)^2}{k_{12}!k_{13}!k_{14}!k_{23}!k_{24}!k_{34}!k_{123}!k_{124}!k_{134}!k_{234}!k_{1234}!}\le\frac{(i!)^2(j!)^2}{\lceil \max\{i,j\}/7\rceil!}.$$

To get this inequality, we have used the fact that

$$k_{12}+k_{13}+k_{14}+k_{123}+k_{124}+k_{134}+k_{1234}=i,$$

whence one of the factors in the denominator is at least $\lceil i/7\rceil$. For a similar reason, there must be a factor in the denominator that is at least $\lceil j/7\rceil$.

Moreover, there are less than $\max\{i+1, j+1\}^{11}$ possible choices for $k_{12},\dots,$ k_{1234}, which completes the proof. □

Note that we have not used the first and the third condition of the definition of $\widetilde{\Pi}_{ij}$ in the proof of Lemma 8.5, whence the inequality even holds for a larger class of partitions. Now we are prepared for the proofs of Theorem 8.1 and Corollary 8.2.

PROOF OF THEOREM 8.1. We aim at applying Proposition 8.4. Combining Lemma 8.5 and assumption (8.8), we get

$$\begin{aligned}\sum_{i,j=1}^{\infty} ij\sqrt{\sum_{\sigma\in\widetilde{\Pi}_{ij}}\int\big|(f_i^{(k)}\otimes f_i^{(k)}\otimes f_j^{(l)}\otimes f_j^{(l)})_\sigma\big|\,d\lambda^{|\sigma|}}\\ \le\sum_{i,j=1}^{\infty} ij\sqrt{\frac{\max\{i+1,j+1\}^{11}b^{i+j}a}{\lceil\max\{i,j\}/7\rceil!}}.\end{aligned}$$

A straightforward computation and the inequality $\sqrt{m!}\ge\lfloor m/2\rfloor!$ for $m\in\mathbb{N}$ show that

$$\begin{aligned}\sum_{i,j=1}^{\infty} ij\sqrt{\frac{\max\{i+1,j+1\}^{11}b^{i+j}}{\lceil\max\{i,j\}/7\rceil!}}&\le 2^{13/2}\sum_{1\le j\le i} i^2\sqrt{\frac{\max\{i,j\}^{11}b^{i+j}}{\lceil\max\{i,j\}/7\rceil!}}\\ &\le 2^{13/2}\sum_{i=1}^{\infty} i^{17/2}\frac{b^i}{\lfloor i/14\rfloor!},\end{aligned}$$

where the right-hand side converges. Thus, Theorem 8.1 is a consequence of Proposition 8.4. □

PROOF OF COROLLARY 8.2. We define the truncated Poisson functional $F_s:=\mathbb{E}F+\sum_{n=1}^{s}I_n(f_n)$ for $s\in\mathbb{N}$. By the triangle inequality for the Wasserstein distance and combining the definition of the Wasserstein distance with the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\mathbf{d}_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var}F}},N\right)&\le\mathbf{d}_W\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var}F}},\frac{F_s-\mathbb{E}F_s}{\sqrt{\operatorname{Var}F}}\right)+\mathbf{d}_W\left(\frac{F_s-\mathbb{E}F_s}{\sqrt{\operatorname{Var}F}},N\right)\\ &\le\frac{\sqrt{\mathbb{E}(F-F_s)^2}}{\sqrt{\operatorname{Var}F}}+\mathbf{d}_W\left(\frac{F_s-\mathbb{E}F_s}{\sqrt{\operatorname{Var}F}},N\right).\end{aligned}$$

Here, the first summand vanishes as $s \to \infty$ since $F_s \to F$ in L^2_η as $s \to \infty$. For the second term, we know from [23], Theorem 3.1, that

$$\begin{aligned}
&\mathbf{d}_W\Big(\frac{F_s - \mathbb{E}F_s}{\sqrt{\operatorname{Var} F}}, N\Big) \\
&\quad \le \frac{\operatorname{Var} F - \operatorname{Var} F_s}{\operatorname{Var} F} + \frac{1}{\operatorname{Var} F}\sqrt{\mathbb{E}\Big(\operatorname{Var} F_s - \int D_z F_s(-D_z L^{-1} F_s)\lambda(dz)\Big)^2} \\
&\qquad + \frac{1}{(\operatorname{Var} F)^{3/2}} \int \mathbb{E}(D_z F_s)^2 |D_z L^{-1} F_s| \lambda(dz).
\end{aligned}$$

Now we can use the same arguments as in the proofs of Proposition 8.4 and Theorem 8.1. □

9. Central limit theorems for geometric functionals. In the following, we use the general normal approximation results of the previous section to derive central limit theorems for geometric functionals of the Boolean model (1.1). We establish central limit theorems under the minimal moment assumption (2.5), but we need a stronger moment assumption in order to derive rates of convergence. For the Berry–Esseen bounds, we assume that the typical grain Z_0 of the Boolean model satisfies the moment assumption

$$\mathbb{E}V_i(Z_0)^{3+\varepsilon} < \infty, \qquad i \in \{0, \dots, d\}, \tag{9.1}$$

for a fixed $\varepsilon \in (0, 1]$. This allows us to state central limit theorems with rates of convergence depending on ε.

THEOREM 9.1. *Let $\psi_1, \dots, \psi_m$ be geometric functionals on $\mathcal{R}^d$ and let $\Psi := (\psi_1, \dots, \psi_m)$. Assume that (2.5) is satisfied and let N be an m-dimensional centered Gaussian random vector with covariance matrix $(\sigma_{\psi_k,\psi_l})_{k,l=1,\dots,m}$ given by (3.3). Then*

$$\frac{1}{\sqrt{V_d(W)}}\big(\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)\big) \xrightarrow{d} N \qquad \text{as } r(W) \to \infty.$$

If (9.1) is satisfied, there is a constant $c_{\psi_1,\dots,\psi_m}$ depending on $\psi_1, \dots, \psi_m$, Λ, and ε such that

$$\mathbf{d}_3\Big(\frac{1}{\sqrt{V_d(W)}}\big(\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)\big), N\Big) \le \frac{c_{\psi_1,\dots,\psi_m}}{r(W)^{\min\{\varepsilon d/2, 1\}}} \tag{9.2}$$

for $W \in \mathcal{K}^d$ with $r(W) \ge 1$.

REMARK 9.2. We will see in the proof of Theorem 9.1 that the translation invariance of $\psi_1, \dots, \psi_m$ is only used to ensure the existence of an asymptotic covariance matrix. Hence, such a multivariate central limit theorem still holds for functionals $\psi_1, \dots, \psi_m$ which are not translation invariant if we can establish the existence of an asymptotic covariance matrix. In this case, the rate of convergence depends on the rate of convergence for the covariances.

In the univariate case, we can rescale by the square root of the variance, whence the existence of the asymptotic variance is not necessary. Thus, translation invariance of the functional is not required. We only need to assume that the variance does not degenerate as $r(W) \to \infty$, which, for instance, holds under the conditions of Section 4.

THEOREM 9.3. *Assume that* (2.5) *is satisfied and let* ψ *be an additive, locally bounded and measurable functional on* $\mathcal{R}^d$ *with constants* $r_0 \geq 1$ *and* $\sigma_0 > 0$ *such that*

$$\frac{\operatorname{Var} \psi(Z \cap W)}{V_d(W)} \geq \sigma_0 \tag{9.3}$$

for $W \in \mathcal{K}^d$ *with* $r(W) \geq r_0$. *Denote by* N *a standard Gaussian random variable. Then*

$$\frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\operatorname{Var} \psi(Z \cap W)}} \xrightarrow{d} N \qquad \text{as } r(W) \to \infty.$$

If (9.1) *is satisfied, there is a constant* c_ψ *depending on* $\psi, \Lambda, \sigma_0, r_0$, *and* ε *such that*

$$\mathbf{d}_W\left(\frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\operatorname{Var} \psi(Z \cap W)}}, N\right) \leq \frac{c_\psi}{V_d(W)^{\min\{\varepsilon/2, 1/2\}}} \tag{9.4}$$

for $W \in \mathcal{K}^d$ *with* $r(W) \geq r_0$.

REMARK 9.4. Together with the well-known fact that the Kolmogorov distance to a standard Gaussian random variable is always bounded by the square root of the Wasserstein distance to a standard Gaussian random variable (see Proposition 1.2 in [29], e.g.), it follows from (9.4) that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\operatorname{Var} \psi(Z \cap W)}} \leq x\right) - \mathbb{P}(N \leq x) \right| \leq \frac{\sqrt{c_\psi}}{V_d(W)^{\min\{\varepsilon/4, 1/4\}}}$$

for $W \in \mathcal{K}^d$ with $r(W) \geq r_0$. However, this approach leads to a weaker rate of convergence than for the Wasserstein distance, which might be suboptimal since for many central limit theorems one has the same rate for both distances.

REMARK 9.5. By replacing in (9.4) the volume of W by the volume of its inball, we obtain a rate of order $r(W)^{-\min\{\varepsilon d/2, d/2\}}$. Comparing (9.2) and (9.4), we see that for $\varepsilon = 1$ and $d \geq 3$ the rate of convergence in the multivariate case is weaker than in the univariate case. This is caused by the slow rate of convergence in Theorem 3.1 since we need to bound

$$\sum_{k,l=1}^{m} \left| \sigma_{\psi_k, \psi_l} - \frac{\operatorname{Cov}(\psi_k(Z \cap W), \psi_l(Z \cap W))}{V_d(W)} \right|$$

in order to apply Theorem 8.1. In the univariate analogue, which is Corollary 8.2, we normalize with the exact variance and do not have such a term. If we replace the Gaussian random vector N by a centered Gaussian random vector $N(W)$, having the covariance matrix of $V_d(W)^{-1/2}\Psi(Z\cap W)$, the sum above vanishes and we obtain

$$\mathbf{d}_3\Big(\frac{1}{\sqrt{V_d(W)}}\big(\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\big),N(W)\Big)\leq\frac{c_{\psi_1,\dots,\psi_m}}{V_d(W)^{\min\{\varepsilon/2,1/2\}}},$$

which is the same rate as in the univariate case.

For $k,l\in\{1,\dots,m\}$, we obtain by choosing $g(x)=x_kx_l/2$ as a test function in the definition of the $\mathbf{d}_3$-distance that

$$\begin{aligned}&\mathbf{d}_3\Big(\frac{1}{\sqrt{V_d(W)}}\big(\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\big),N\Big)\\&\quad\geq\frac{1}{2}\Big|\sigma_{\psi_k,\psi_l}-\frac{\operatorname{Cov}(\psi_k(Z\cap W),\psi_l(Z\cap W))}{V_d(W)}\Big|.\end{aligned}$$

Hence, Proposition 3.8 shows that the rate in (9.2) is optimal for $\varepsilon=1$ and $d\geq 2$.

We organize the proofs of Theorems 9.1 and 9.3 such that we first impose the moment assumption (9.1) and establish (9.2) and (9.4). In a second step, we prove that convergence in distribution is still obtained (without convergence rates) under the weaker moment assumption (2.5).

PROOF OF (9.2) IN THEOREM 9.1 UNDER ASSUMPTION (9.1). From now on, we write

$$f_i^{(k)}(K_1,\dots,K_i):=\frac{(-1)^i}{i!}\psi_k^*(K_1\cap\dots\cap K_i\cap W)$$

for $K_1,\dots,K_i\in\mathcal{K}^d$, $1\leq k\leq m$, and $i\geq 1$. It is a direct consequence of (3.8) that $f_i^{(k)}$ is the ith kernel of the Wiener–Itô chaos expansion of the Poisson functional $\psi_k(Z\cap W)$.

The integrability condition (8.7) is satisfied since the kernels are bounded by (3.9) for every $W\in\mathcal{K}^d$ and the measure of the grains hitting W is also finite.

In the sequel, we check assumption (8.8) for the cases $\sigma\in\widetilde{\Pi}_{ij}^{(1)}$ and $\sigma\in\widetilde{\Pi}_{ij}^{(2)}$ separately. We start with the first case. Let $k,l\in\{1,\dots,m\}$ and $\sigma\in\widetilde{\Pi}_{ij}^{(1)}$. From (3.9) in Lemma 3.3, it follows that

$$\begin{aligned}&\int\big|\big(f_i^{(k)}\otimes f_i^{(k)}\otimes f_j^{(l)}\otimes f_j^{(l)}\big)_\sigma\big|\,d\Lambda^{|\sigma|}\\&\quad\leq\frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2}\int\prod_{p=1}^4\sum_{r=0}^d V_r\Big(\bigcap_{n\in N_p(\sigma)}K_n\cap W\Big)\Lambda^{|\sigma|}\big(d(K_1,\dots,K_{|\sigma|})\big)\end{aligned}$$

with nonempty sets $N_p(\sigma) \subset \{1, \dots, |\sigma|\}$, $p = 1, \dots, 4$, depending on σ. Every $n \in \{1, \dots, |\sigma|\}$ is contained in at least two of these sets. By removing the index n from the sets until it occurs only in one set, we increase the integral and can use Lemma 3.5 to integrate over K_n. Due to the special structure of $\sigma \in \widetilde{\Pi}_{ij}^{(1)}$, we obtain by iterating this step and using the abbreviation

$$h_W(A) = \sum_{r=0}^{d} V_r(A \cap W)$$

that

$$\begin{aligned}
&\int \big| \big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma \big| \, d\Lambda^{|\sigma|} \\
&\quad \le \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2} \alpha^{|\sigma|-3} \\
&\qquad \times \int h_W(K_1) h_W(K_1 \cap K_2) h_W(K_2 \cap K_3) h_W(K_3) \Lambda^3\big(d(K_1, K_2, K_3)\big) \\
&\quad = \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2} \alpha^{|\sigma|-3} \int \left(\int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \right)^2 \Lambda(dK_2).
\end{aligned}$$

For a fixed $K_2 \in \mathcal{K}^d$, Lemma 3.4 implies the second inequality in

$$\begin{aligned}
&\int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \\
&\quad \le \gamma \mathbb{E}\left[\sum_{r=0}^{d} V_r(Z_0) \int \sum_{s=0}^{d} V_s\big((Z_0 + x) \cap K_2 \cap W\big) \, dx \right] \\
&\quad \le (d+1)\gamma\beta_1 \mathbb{E}\left[\left(\sum_{r=0}^{d} V_r(Z_0) \right)^2 \right] \sum_{s=0}^{d} V_s(K_2 \cap W).
\end{aligned}$$

Putting $c_7 := (d+1)\gamma\beta_1 \mathbb{E}[(\sum_{r=0}^{d} V_r(Z_0))^2]$ and applying Lemma 3.4 again, we get

$$\begin{aligned}
&\int \left(\int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \right)^2 \Lambda(dK_2) \\
&\quad \le c_7^2 \int \left(\sum_{r=0}^{d} V_r(K_2 \cap W) \right)^2 \Lambda(dK_2) \\
&\quad \le \gamma c_7^2 \mathbb{E}\left[\sum_{r=0}^{d} V_r(Z_0) \int \sum_{s=0}^{d} V_s\big((Z_0 + x) \cap W\big) \, dx \right] \\
&\quad \le (d+1)\gamma\beta_1 c_7^2 \mathbb{E}\left[\left(\sum_{r=0}^{d} V_r(Z_0) \right)^2 \right] \sum_{s=0}^{d} V_s(W) = c_8 \sum_{r=0}^{d} V_r(W)
\end{aligned}$$

with $c_8 := c_7^3$. Finally, since $|\sigma| \le i + j$ for $\sigma \in \widetilde{\Pi}_{ij}$ we have

$$\begin{aligned}\int \big|\big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma\big| \, d\Lambda^{|\sigma|} &\le \frac{c_8(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2}\alpha^{|\sigma|-3}\sum_{r=0}^{d} V_r(W) \\ &\le \frac{a_1 b_1^{i+j}}{(i!)^2(j!)^2}\end{aligned} \tag{9.5}$$

with $a_1 := \max_{1\le k,l\le m} \alpha^{-3} c_8(\beta(\psi_k)\beta(\psi_l))^2 \sum_{r=0}^{d} V_r(W)$ and $b_1 := \max\{\alpha, 1\}$. It follows from Lemma 3.7 that there is a constant c_9 depending on $\psi_1, \dots, \psi_m$ and Λ such that

$$\frac{a_1}{V_d(W)^2} \le \frac{c_9}{V_d(W)} \tag{9.6}$$

for $W \in \mathcal{K}^d$ with $r(W) \ge 1$.

For $\sigma \in \widetilde{\Pi}_{i,j}^{(2)}$, we obtain from (3.9) in Lemmas 3.3 and 3.5 as above that

$$\begin{aligned}&\int \big|\big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma\big| \, d\Lambda^{|\sigma|} \\ &\quad\le \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2}\alpha^{|\sigma|-1}\int\left(\sum_{r=0}^{d} V_r(K_1\cap W)\right)^4 \Lambda(dK_1).\end{aligned}$$

A further application of Lemma 3.4 yields the second inequality in

$$\begin{aligned}&\int\left(\sum_{r=0}^{d} V_r(K_1\cap W)\right)^4 \Lambda(dK_1) \\ &\quad\le \gamma\mathbb{E}\left[\left(\sum_{r=0}^{d}\min\{V_r(Z_0), V_r(W)\}\right)^3 \int\sum_{s=0}^{d} V_s\big((Z_0+x)\cap W\big)\,dx\right] \\ &\quad\le (d+1)\gamma\beta_1\mathbb{E}\left[\left(\sum_{r=0}^{d}\min\{V_r(Z_0), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(Z_0)\right]\sum_{u=0}^{d} V_u(W).\end{aligned}$$

Consequently, we have

$$\int \big|\big(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)}\big)_\sigma\big| \, d\Lambda^{|\sigma|} \le \frac{a_2 b_2^{i+j}}{(i!)^2(j!)^2} \tag{9.7}$$

with

$$\begin{aligned}a_2 &= (d+1)\gamma\beta_1 \max_{1\le k,l\le m}\frac{(\beta(\psi_k)\beta(\psi_l))^2}{\alpha} \\ &\quad\times\mathbb{E}\left[\left(\sum_{r=0}^{d}\min\{V_r(Z_0), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(Z_0)\right]\sum_{u=0}^{d} V_u(W)\end{aligned}$$

and $b_2 = \max\{\alpha, 1\}$. Since

$$\frac{1}{V_d(W)^2}\mathbb{E}\left[\left(\sum_{r=0}^{d}\min\{V_r(Z_0), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(Z_0)\right]\sum_{u=0}^{d} V_u(W)$$
$$\leq \mathbb{E}\left[\left(\sum_{r=0}^{d} V_r(Z_0)\right)^3 \sum_{s=0}^{d} \frac{V_s(Z_0)^{\varepsilon} V_s(W)^{1-\varepsilon}\}}{V_d(W)}\right]\sum_{u=0}^{d} \frac{V_u(W)}{V_d(W)},$$

Lemma 3.7 and the moment assumption (9.1) imply that there is a constant c_{10} depending on $\psi_1, \dots, \psi_m$, Λ, and ε such that, for $W \in \mathcal{K}^d$ with $r(W) \geq 1$,

$$\frac{a_2}{V_d(W)^2} \leq \frac{c_{10}}{V_d(W)^{\varepsilon}}. \tag{9.8}$$

If we rescale the Poisson functionals by $V_d(W)^{-1/2}$, (9.5) and (9.7) imply that (8.8) holds with $a = \max\{a_1, a_2\} V_d(W)^{-2}$ and $b = \max\{b_1, b_2\}$. By (9.6) and (9.8), a is of the order $V_d(W)^{-\min\{1,\varepsilon\}}$. Now (9.2) is a consequence of Theorems 3.1 and 8.1. □

PROOF OF (9.4) IN THEOREM 9.3 UNDER ASSUMPTION (9.1). By the same arguments as in the previous proof and analogous choices for a_1, a_2, the conditions of Corollary 8.2 are satisfied with $a = \max\{a_1, a_2\}$. It follows from assumption (9.3) that

$$\frac{a}{(\operatorname{Var}\psi(Z \cap W))^2} = \frac{\max\{a_1, a_2\}}{(\operatorname{Var}\psi(Z \cap W))^2} \leq \frac{1}{\sigma_0^2}\frac{\max\{a_1, a_2\}}{V_d(W)^2}$$

for $W \in \mathcal{K}^d$ with $r(W) \geq r_0$. Combining this with (9.6) and (9.8) completes the proof. □

For $W \in \mathcal{K}^d$ with $r(W) > 0$, we define the set

$$M_W = \{K \in \mathcal{K}^d : V_j(K) \leq \sqrt{V_d(W)}, j \in \{0, \dots, d\}\}.$$

The restriction of η to M_W is a stationary Poisson process, which generates the stationary Boolean model

$$Z_W = \bigcup_{K \in \eta \cap M_W} K.$$

The idea of the proofs of Theorems 9.1 and 9.3 under the weaker moment assumption (2.5) is to approximate the Boolean model Z by the Boolean model Z_W. A similar truncation has been used in [12] to prove the central limit theorem for the volume of a more general Boolean model based on a Poisson process of cylinders.

The restriction of η to M_W has the intensity $\gamma_W := \gamma\mathbb{P}(Z_0 \in M_W)$ [note that $\gamma_W > 0$ for $r(W)$ sufficiently large] and its typical grain $Z_{0,W}$ has the distribution $\mathbb{P}(Z_{0,W} \in \cdot) := \mathbb{P}(Z_0 \in \cdot \cap M_W)/\mathbb{P}(Z_0 \in M_W)$. For the Boolean model Z_W, obviously all previous results hold if we replace Z_0 and γ by $Z_{0,W}$ and γ_W. But Lemmas 3.3 and 3.5 as well as the upper bounds in the proof of Theorem 9.1 under the stronger assumption (9.1) remain true for Z_W if we take the same constants as for Z, which we do in the sequel. The reason for this is that the constants do only depend on the product of intensity and grain distribution and that the intrinsic volumes are monotone.

The Boolean models Z_W and Z satisfy the following relation.

LEMMA 9.6. *Let ψ be an additive, locally bounded and measurable functional on $\mathcal{R}^d$ and assume that* (2.5) *is satisfied. Then*

$$\lim_{r(W)\to\infty} \frac{\mathbb{E}((\psi(Z\cap W)-\mathbb{E}\psi(Z\cap W))-(\psi(Z_W\cap W)-\mathbb{E}\psi(Z_W\cap W)))^2}{V_d(W)}=0$$

and

$$\limsup_{r(W)\to\infty} \frac{\mathbb{E}(\psi(Z\cap W)-\mathbb{E}\psi(Z\cap W))^2+\mathbb{E}(\psi(Z_W\cap W)-\mathbb{E}\psi(Z_W\cap W))^2}{V_d(W)} < \infty.$$

PROOF. Define, for $K_1,\dots,K_n\in\mathcal{K}^d$,

$$g_{n,W}(K_1,\dots,K_n) := \frac{(-1)^n}{n!\sqrt{V_d(W)}}\big(\mathbb{E}\psi(Z\cap K_1\cap\cdots\cap K_n\cap W)-\psi(K_1\cap\cdots\cap K_n\cap W)\big).$$

Further, we define

$$h_{n,W}(K_1,\dots,K_n) := \frac{(-1)^n}{n!\sqrt{V_d(W)}}\big(\mathbb{E}\psi(Z_W\cap K_1\cap\cdots\cap K_n\cap W)-\psi(K_1\cap\cdots\cap K_n\cap W)\big)$$

for $K_1,\dots,K_n\in M_W$ and $h_{n,W}(K_1,\dots,K_n):=0$ if there is a $j\in\{1,\dots,n\}$ with $K_j\notin M_W$.

In view of Lemma 3.2 and the Fock space representation (3.7), the assertions of this lemma are equivalent to

$$\lim_{r(W)\to\infty}\sum_{n=1}^{\infty} n!\|g_{n,W}-h_{n,W}\|_n^2=0 \quad\text{and}$$
$$\limsup_{r(W)\to\infty}\sum_{n=1}^{\infty} n!\big(\|g_{n,W}\|_n^2+\|h_{n,W}\|_n^2\big)<\infty, \tag{9.9}$$

which we shall prove in the following. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|g_{n,W} - h_{n,W}\|_n^2 \\
&\quad = \gamma \iiint \big(g_{n,W}(K_1 + x, K_2, \dots, K_n) \\
&\qquad\qquad - h_{n,W}(K_1 + x, K_2, \dots, K_n)\big)^2 dx\, \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big)\mathbb{Q}(dK_1).
\end{aligned}$$

Our aim is to apply the dominated convergence theorem to the outer integral. For any $K_1 \in \mathcal{K}_o^d$, it follows from Lemmas 3.3, 3.4 and 3.5 similarly as in (3.20) that

$$\begin{aligned}
&\iint \big(g_{n,W}(K_1 + x, K_2, \dots, K_n) \\
&\quad - h_{n,W}(K_1 + x, K_2, \dots, K_n)\big)^2\, dx\, \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
&\qquad \le 2 \iint \big(g_{n,W}(K_1 + x, K_2, \dots, K_n)^2 + h_{n,W}(K_1 + x, K_2, \dots, K_n)^2\big)\, dx \\
&\qquad\qquad \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
&\qquad \le 2(d+1)\beta_1 \beta(\psi)^2 \left(\sum_{i=0}^{d} V_i(K_1)\right)^2 \sum_{i=0}^{d} \frac{V_i(W)}{V_d(W)} \frac{\alpha^{n-1}}{(n!)^2}.
\end{aligned}$$

The right-hand side of the previous inequality is uniformly bounded for $r(W) \ge 1$ because of Lemma 3.7. Moreover, the sum over n is integrable with respect to K_1 due to (2.5). Thus, limit and summation in the first sum in (9.9) can be interchanged. By the same arguments, the second inequality above yields the second formula in (9.9).

Next, we show that, for any $K_1 \in \mathcal{K}_o^d$,

$$\begin{aligned}
&\lim_{r(W)\to\infty} \iint \big(g_{n,W}(K_1 + x, K_2, \dots, K_n) - h_{n,W}(K_1 + x, K_2, \dots, K_n)\big)^2\, dx \\
&\qquad \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) = 0. \qquad (9.10)
\end{aligned}$$

For $K_1, \dots, K_n \in M_W$, we have

$$\begin{aligned}
&g_{n,W}(K_1, \dots, K_n) - h_{n,W}(K_1, \dots, K_n) \\
&\quad = \frac{1}{n!} \frac{(-1)^n}{\sqrt{V_d(W)}} \mathbb{E}\big[\psi(Z \cap K_1 \cap \dots \cap K_n \cap W) \\
&\qquad\qquad - \psi(Z_W \cap K_1 \cap \dots \cap K_n \cap W)\big].
\end{aligned}$$

Let us denote by $Z_1, \dots, Z_{N_{K_1 \cap W}}$ the grains of η that intersect $K_1 \cap W$ and are not in M_W. Then $N_{K_1 \cap W}$ follows a Poisson distribution with mean $\Lambda(\{K \notin M_W : K \cap K_1 \cap W \neq \varnothing\})$. Since $Z \cap K_1 \cap W = (Z_W \cup Z_1 \cup \dots \cup Z_{N_{K_1 \cap W}}) \cap K_1 \cap W$, it follows from the inclusion–exclusion formula that

$$\begin{aligned}
&\big|\psi(Z\cap K_1\cap\cdots\cap K_n\cap W)-\psi(Z_W\cap K_1\cap\cdots\cap K_n\cap W)\big|\\
&\quad\le\sum_{\varnothing\neq J\subset\{1,\dots,N_{K_1\cap W}\}}\Big|\psi\Big(Z_W\cap\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\Big|\\
&\qquad+\sum_{\varnothing\neq J\subset\{1,\dots,N_{K_1\cap W}\}}\Big|\psi\Big(\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\Big|.
\end{aligned}$$

Recall the definitions of the constants c_1, c_2 and c_4 from Section 3. Denoting by $\mathbb{P}_W$ the distribution of the restriction of η to M_W, we obtain by (3.11) and the monotonicity of the intrinsic volumes that

$$\begin{aligned}
&\int\Big|\psi\Big(Z(\mu)\cap\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\Big|\mathbb{P}_W(d\mu)\\
&\quad\le c_1M(\psi)\sum_{i=0}^{d}V_i\Big(\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\\
&\quad\le c_1M(\psi)\sum_{i=0}^{d}V_i(K_1\cap\cdots\cap K_n\cap W).
\end{aligned}$$

Applying (3.15) and the monotonicity of the intrinsic volumes yields

$$\begin{aligned}
&\Big|\psi\Big(\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\Big|\\
&\quad\le c_2c_4M(\psi)\sum_{i=0}^{d}V_i\Big(\bigcap_{j\in J}Z_j\cap K_1\cap\cdots\cap K_n\cap W\Big)\\
&\quad\le c_2c_4M(\psi)\sum_{i=0}^{d}V_i(K_1\cap\cdots\cap K_n\cap W).
\end{aligned}$$

Since the restrictions of η to M_W and to its complement are stochastically independent, altogether we have that, for $K_1,\dots,K_n\in M_W$,

$$\begin{aligned}
&\big|g_{n,W}(K_1,\dots,K_n)-h_{n,W}(K_1,\dots,K_n)\big|\\
&\quad\le\frac{(c_1+c_2c_4)M(\psi)}{n!\sqrt{V_d(W)}}\mathbb{E}\big[2^{N_{K_1\cap W}}-1\big]\sum_{i=0}^{d}V_i(K_1\cap\cdots\cap K_n\cap W)\\
&\quad\le\frac{\hat\beta(\psi)}{n!\sqrt{V_d(W)}}\big(\exp\big(p_W(K_1)\big)-1\big)\sum_{i=0}^{d}V_i(K_1\cap\cdots\cap K_n\cap W)
\end{aligned}$$

with $p_W(K_1) = \Lambda(\{K \notin M_W : K \cap K_1 \neq \varnothing\}$ and $\hat{\beta}(\psi) = (c_1 + c_2 c_4) M(\psi)$.

If there is a $j \in \{1, \dots, n\}$ such that $K_j \notin M_W$, we have $g_{n,W} - h_{n,W} = g_{n,W}$, and it follows from Lemma 3.3 that

$$\left|g_{n,W}(K_1, \dots, K_n)\right| \leq \frac{\beta(\psi)}{n!\sqrt{V_d(W)}} \sum_{k=0}^{d} V_k(K_1 \cap \dots \cap K_n \cap W).$$

For a fixed $K_1 \in \mathcal{K}_o^d$ and $r(W)$ sufficiently large such that $K_1 \in M_W$, we have that

$$\begin{aligned}
&\iint \big(g_{n,W}(K_1 + x, K_2, \dots, K_n) - h_{n,W}(K_1 + x, K_2, \dots, K_n)\big)^2 \, dx \\
&\quad \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
&\quad \leq \iint \Bigg(\frac{\hat{\beta}(\psi)^2}{(n!)^2 V_d(W)} \left(\sum_{k=0}^{d} V_k\big((K_1 + x) \cap K_2 \cap \dots \cap K_n \cap W\big) \right)^2 \\
&\qquad \times \big(\exp\big(p_W(K_1)\big) - 1\big)^2 \\
&\qquad + \frac{\beta(\psi)^2}{(n!)^2 V_d(W)} \left(\sum_{k=0}^{d} V_k\big((K_1 + x) \cap K_2 \cap \dots \cap K_n \cap W\big) \right)^2 \\
&\qquad\qquad \times \sum_{i=2}^{n} \mathbf{1}\{K_i \notin M_W\} \Bigg) \, dx \\
&\qquad \times \Lambda^{n-1}\big(d(K_2, \dots, K_n)\big) \\
&\quad \leq \frac{\hat{\beta}(\psi)^2 (d+1) \beta_1 \alpha^{n-1}}{(n!)^2} \frac{1}{V_d(W)} \\
&\qquad \times \sum_{i=0}^{d} V_i(W) \left(\sum_{r=0}^{d} V_r(K_1) \right)^2 \big(\exp\big(p_W(K_1)\big) - 1\big)^2 \\
&\qquad + \frac{\beta(\psi)^2 (d+1) \beta_1 \alpha^{n-2}}{(n!)^2} \frac{1}{V_d(W)} \\
&\qquad \times \sum_{i=0}^{d} V_i(W) \left(\sum_{r=0}^{d} V_r(K_1) \right)^2 (n-1) p_W(K_1),
\end{aligned}$$

where we have used Lemmas 3.4 and 3.5.

Then Lemma 3.7 and $p_W(K_1) \to 0$ as $r(W) \to \infty$ show that the right-hand side vanishes for $r(W) \to \infty$. This proves (9.10) so that the dominated convergence theorem yields the first formula in (9.9), which completes the proof. □

PROOF OF THEOREM 9.1 UNDER ASSUMPTION (2.5). The triangle inequality for the $\mathbf{d}_3$-distance yields

$$\begin{aligned}
&\mathbf{d}_3\biggl(\frac{1}{\sqrt{V_d(W)}}\bigl(\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\bigr), N\biggr)\\
&\quad\le \mathbf{d}_3\biggl(\frac{1}{\sqrt{V_d(W)}}\bigl(\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\bigr),\\
&\qquad\qquad \frac{1}{\sqrt{V_d(W)}}\bigl(\Psi(Z_W\cap W)-\mathbb{E}\Psi(Z_W\cap W)\bigr)\biggr)\\
&\qquad + \mathbf{d}_3\biggl(\frac{1}{\sqrt{V_d(W)}}\bigl(\Psi(Z_W\cap W)-\mathbb{E}\Psi(Z_W\cap W)\bigr), N\biggr).
\end{aligned} \tag{9.11}$$

In the sequel, we show that both terms on the right-hand side of (9.11) vanish as $r(W)\to\infty$. By [16], Lemma 5.5, the first expression is bounded by

$$\begin{aligned}
&m\bigl(\mathbb{E}\|\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\|^2/V_d(W)\\
&\quad + \|\Psi(Z_W\cap W)-\mathbb{E}\Psi(Z_W\cap W)\|^2/V_d(W)\bigr)^{1/2}\\
&\quad \times \bigl(\mathbb{E}\|\Psi(Z\cap W)-\mathbb{E}\Psi(Z\cap W)\\
&\qquad - \Psi(Z_W\cap W)+\mathbb{E}\Psi(Z_W\cap W)\|^2/V_d(W)\bigr)^{1/2},
\end{aligned}$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^m$. Since, by Lemma 9.6, the first factor is bounded and the second factor vanishes as $r(W)\to\infty$, the first expression on the right-hand side of (9.11) vanishes as $r(W)\to\infty$.

By applying Theorem 8.1 to the vector $\Psi(Z_W\cap W)$ of Poisson functionals depending on the restriction of η to M_W, we shall prove that

$$\lim_{r(W)\to\infty}\mathbf{d}_3\biggl(\frac{1}{\sqrt{V_d(W)}}\bigl(\Psi(Z_W\cap W)-\mathbb{E}\Psi(Z_W\cap W)\bigr), N\biggr)=0. \tag{9.12}$$

Theorem 8.1 yields this without a rate of convergence if

$$\lim_{r(W)\to\infty}\frac{\operatorname{Cov}(\psi_k(Z_W\cap W),\psi_l(Z_W\cap W))}{V_d(W)}=\sigma_{\psi_k,\psi_l} \tag{9.13}$$

for $k,l\in\{1,\dots,m\}$ and if (8.8) holds with a fixed $b\ge 1$ and $a\ge 0$ depending on W such that a tends to zero as $r(W)\to\infty$.

Condition (9.13) is satisfied because of Lemma 9.6 and Theorem 3.1. Inequalities (9.5) and (9.7) also hold for the Boolean model Z_W with the same a_1, b_1, b_2 as in the proof of (9.2) under assumption (9.1) and

$$\begin{aligned}
a_2 &= c_{11}\max_{1\le k,l\le m}\frac{(\beta(\psi_k)\beta(\psi_l))^2\gamma_W}{\alpha}\\
&\quad\times\mathbb{E}\Biggl[\Biggl(\sum_{r=0}^d\min\{V_r(Z_{0,W}), V_r(W)\}\Biggr)^3\sum_{s=0}^d V_s(Z_{0,W})\Biggr]\sum_{u=0}^d V_u(W)
\end{aligned}$$

with $c_{11} := (d+1)\beta_1$. This is the case since the derivations of (9.5) and (9.7) require only finite second moments and we can use the constants related to Z as discussed before Lemma 9.6. Consequently, (8.8) is satisfied with $a = \max\{a_1, a_2\}/V_d(W)^2$ and $b = \max\{b_1, b_2\}$. Since (9.6) only requires that the second moments, which are contained in c_8, are finite, we obtain that $a_1/V_d(W)^2$ tends to zero as $r(W) \to \infty$. On the other hand, $\lim_{r(W)\to\infty} a_2/V_d(W)^2 = 0$ is equivalent to

$$\lim_{r(W)\to\infty} \gamma_W \mathbb{E}\left[\frac{1}{V_d(W)}\left(\sum_{r=0}^{d} \min\{V_r(Z_{0,W}), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(Z_{0,W})\right] = 0. \tag{9.14}$$

The expression in the limit can be rewritten as

$$\gamma \int \frac{1}{V_d(W)}\left(\sum_{r=0}^{d} \min\{V_r(K), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(K)\mathbf{1}\{K \in M_W\}\mathbb{Q}(dK).$$

For $K \in \mathcal{K}_o^d \cap M_W$, we have $V_r(K) \le \sqrt{V_d(W)}$ for $r \in \{0, \dots, d\}$ and, therefore,

$$\begin{aligned}
&\frac{1}{V_d(W)}\left(\sum_{r=0}^{d} \min\{V_r(K), V_r(W)\}\right)^3 \sum_{s=0}^{d} V_s(K)\mathbf{1}\{K \in M_W\} \\
&\quad \le (d+1)^2 \left(\sum_{r=0}^{d} V_r(K)\right)^2,
\end{aligned}$$

which is independent of W and integrable with respect to $\mathbb{Q}$. For any fixed $K \in \mathcal{K}_o^d$ the left-hand side vanishes as $r(W) \to \infty$ so that the dominated convergence theorem implies (9.14), and hence a tends to zero as $r(W) \to \infty$. Finally, Theorem 8.1 yields (9.12), which completes the proof of Theorem 9.1. □

REMARK 9.7. As discussed in Remark 9.5, (9.2) still holds if we replace the centered Gaussian random vector N with the asymptotic covariance matrix by a centered Gaussian random vector $N(W)$ with the exact covariance matrix. This can be done even if the functionals are not translation invariant since in this case we do not need Theorem 3.1. The second part of the proof of Theorem 9.1 still holds because (9.13) is not required in this situation. This means that under condition (2.5) for additive, locally bounded and measurable functionals $\psi_1, \dots, \psi_m$,

$$\lim_{r(W)\to\infty} \mathbf{d}_3\left(\frac{1}{\sqrt{V_d(W)}}(\Psi(Z\cap W) - \mathbb{E}\Psi(Z \cap W)), N(W)\right) = 0.$$

PROOF OF THEOREM 9.3 UNDER ASSUMPTION (2.5). For $m = 1$ and a centered Gaussian random variable $N(W)$ with variance $\operatorname{Var} \psi(Z \cap W)/V_d(W)$, the previous remark implies that

$$\lim_{r(W)\to\infty} \mathbf{d}_3\left(\frac{\psi(Z\cap W) - \mathbb{E}\psi(Z\cap W)}{\sqrt{V_d(W)}}, N(W)\right) = 0. \tag{9.15}$$

It follows from the definition of the $\mathbf{d}_3$-distance that for random vectors Y_1, Y_2 and any $c > 0$,

$$\mathbf{d}_3(cY_1, cY_2) \le \max\{1, c\}^3 \mathbf{d}_3(Y_1, Y_2).$$

With $c_W := \sqrt{V_d(W)}/\sqrt{\operatorname{Var} \psi(Z \cap W)}$ and a standard Gaussian random variable N, this yields

$$\mathbf{d}_3\left(\frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\operatorname{Var} \psi(Z \cap W)}}, N\right) \\ \le \max\{1, c_W\}^3 \mathbf{d}_3\left(\frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{V_d(W)}}, N(W)\right).$$

Since c_W is bounded by assumption (9.3), (9.15) completes the proof. □

REMARK 9.8. In Theorem 9.3, it is possible to weaken the assumption that the Poisson process is stationary. In the proof, we only need to find upper bounds for the kernels and some integrals. This is, for instance, still possible if the intensity measure is of the form

$$\Lambda(\cdot) = \iint \mathbf{1}\{K + x \in \cdot\} f(x)\, dx \mathbb{Q}(dK)$$

with a nonnegative bounded function $f : \mathbb{R}^d \to \mathbb{R}$. Now we always get upper bounds if we replace this intensity measure by the measure in (2.1) with $\gamma = \sup_{x \in \mathbb{R}^d} |f(x)| < \infty$.

For the multivariate central limit, this argument does not work in general since its proof makes use of Theorem 3.1, which depends on the translation invariance of the intensity measure. But if one can prove by other methods the existence of an asymptotic covariance matrix, it is still possible to weaken the stationarity assumption as described above.

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