Large Scale Geometry of Stratified Nilpotent Lie Groups

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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aus Karlsruhe

Tag der mündlichen Prüfung: 21. Dezember 2016

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Acknowledgements

The present thesis was developed during my time as PhD-student and research assistant at the Institute of Algebra and Geometry of the Karlsruhe Institute of Technology. First of all I want to thank my adviser Prof. Dr. Enrico Leuzinger for giving me the opportunity to pursue mathematics. I want to thank him for his guidance, his ability to ask always the right questions, his advise while putting this thesis down on paper and not at least for the freedom at work he granted and grants me.
I want to thank Prof. Dr. Wilderich Tuschmann, who agreed to act as second referee for this thesis. He and Enrico Leuzinger gave me the chance to present parts of my work at the "Chinese-German Workshop on Metric Riemannian Geometry" at the Shanghai Jiao Tong University.
Prof. Dr. Robert Young and Prof. Dr. Stefan Wenger I have to thank for the email-correspondences, which answered a lot of my questions.
Further I want to thank all members of the Institute of Algebra and Geometry for the nice atmosphere and for all the mathematical discussions and the non-job-related conversations.
Most of all I thank my wife Nina (just married!) for her support and encouragement during all the ups and downs during my time as PhD-student.

Moritz Gruber
Karlsruhe, Oktober 2016

Devenere locos, ubi nunc ingentia cernes
moenia surgentemque novae Karthaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo.

Vergil, *Aeneis*, Book 1, line 365 ff.

Introduction

Let us start with a question: How much area can a closed curve of a given length ℓ bound?
This problem, often called *Dido's problem* or *isoperimetric problem*, is a very classical question in geometry. Efforts to answer it are inequalities of the shape

$$\text{area} \leq f(\text{length})$$

for some function $f : \mathbb{R}^+ \to \mathbb{R}^+$. But a full answer would be an inequality of this kind, such that for every $\ell > 0$ there really exists a closed curve γ of length ℓ which bounds a region of area $f(\ell)$. This is very difficult, even if one looks at curves in the Euclidean plane. Of course, it is known that

$$\text{area} \leq \frac{1}{4\pi}\text{length}^2$$

holds with equality if and only if the curve is circle (see [10] for a discussion of it), but it needs a lot of work to prove this. Consequently there is little hope to get such sharp inequalities for many other metric spaces.
This leads to a more asymptotic way of looking at the question. Hence one only looks for the growth type of the function f, this means one asks whether it is for example exponential or polynomial and if so of which degree. Luckily, it turns out that this interpretation of the problem is much more manageable. So it is an easy exercise to prove the quadratic growth in the case of the Euclidean plane.

The above example is one way to enter the world of *large scale geometry*. Instead of isometry invariants as in classical geometry, the large scale geometry is interested in *quasi-isometry invariants*. These are geometric properties of a metric space, which are preserved under *quasi-isometries* which are maps as follows:

Definition.
Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is a quasi-isometry*, if there are constants $A, b, c, b', c' > 0$ and another map $g : Y \to X$, such that*

$$\frac{1}{c} \cdot d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2) + b$$

for all $x_1, x_2 \in X$, and

$$\frac{1}{c'} \cdot d_Y(y_1, y_2) - b' \leq d_X(g(y_1), g(y_2)) \leq c' \cdot d_Y(y_1, y_2) + b'$$

for all $y_1, y_2 \in Y$, and

$$d_X((g \circ f)(x), x) \le A \quad \textit{and} \quad d_Y((f \circ g)(y), y) \le A$$

for all $x \in X$ *and all* $y \in Y$.

Visually, two metric spaces are quasi-isometric if they look the same from far away.

The quasi-isometry invariants examined in this thesis are the *filling functions* and the *higher divergence functions*, which are a modification of the filling functions. Both are generalisations of the isoperimetric problem.
Our definition of the filling functions uses Lipschitz chains. These are finite formal sums

$$a = \sum_i z_i \alpha_i(\Delta^m) \quad \text{with } z_i \in \mathbb{Z}$$

of Lipschitz maps α_i from a m-simplex to the metric space we examine. We call a Lipschitz chain a a cycle if it is closed, i.e. if $\partial a = 0$. Then the $(m+1)$-dimensional filling function describes how difficult it is to fill Lipschitz m-cycles by Lipschitz $(m+1)$-chains, this means to find an $(m+1)$-chain with the m-cycle as boundary. The difficulty is measured by the maximally needed $(m+1)$-volume of the filling Lipschitz $(m+1)$-chain in terms of the m-volume of the filled Lipschitz m-cycle, up to an equivalence relation which only notices the growth rate.
The higher divergence functions do something similar. They, roughly speaking, measure the difficulty to fill an outside an r-ball lying Lipschitz m-cycle with an outside a ρr-ball, $0 < \rho \le 1$, lying Lipschitz $(m+1)$-chain. The difference to the filling functions, the no-go-area in the space, destroys a possibly existing homogeneity of the metric space and so makes it more difficult to construct fillings.

The filling functions of non-positively curved spaces are well understood. In the case of the Euclidean space, which has constant curvature 0, the $(m+1)$-dimensional filling function grows like $\ell^{\frac{m+1}{m}}$. Further it is known that the $(m+1)$-dimensional filling function of a Hadamard space does not grow faster than $\ell^{\frac{m+1}{m}}$ (see Stefan Wenger [37]). And in the case of symmetric spaces of non-compact type, there is the more explicit result of Enrico Leuzinger in [22], that the $(m+1)$-dimensional filling function grows exactly as $\ell^{\frac{m+1}{m}}$ as long as m is smaller than the rank of the symmetric space and it grows linearly in the dimensions above.
For the higher divergence functions the situation is much the same (see for example [36], [7]), albeit not elaborated in such a way.
So, leaving the world of non-positively curved spaces suggests itself for finding new interesting results. As strictly positively curved spaces are of bounded diameter and so quasi-isometric to points, one has to look at spaces with the whole

spectrum of curvature. A rich class of such spaces form the nilpotent Lie groups. These Lie groups have all three types of curvature in every point (see Joseph A. Wolf [39]). José Burillo in [9] and Robert Young in [42] and [43] computed for one of the most prominent examples of nilpotent Lie groups, namely the complex Heisenberg Group, that one can see all three types of the curvature.

Theorem.
Let $G = H^n_{\mathbb{C}}$ be the complex Heisenberg Group of dimension $2n+1$. Then holds:

$$\begin{aligned} F^{j+1}(\ell) &\sim \ell^{\frac{j+1}{j}} && \textit{for } j < n, \\ F^{n+1}(\ell) &\sim \ell^{\frac{n+2}{n}} && , \\ F^{j+1}(\ell) &\sim \ell^{\frac{j+2}{j+1}} && \textit{for } j > n. \end{aligned}$$

In the dimensions below $n+1$ the behaviour is Euclidean, which is related to the abundance of flat subspaces up to dimension n. In the dimensions above $n+1$ the behaviour is sub-Euclidean and hence related to the behaviour of the filling functions of spaces with negative curvature. Most interesting, in dimension $n+1$ one can observe the new, super-Euclidean growth rate, which highlights the occurrence of positive curvature.
One of the mainly used properties of the complex Heisenberg Group to prove the above result is that its Lie algebra allows a grading. This means it can be written as

$$\mathfrak{g} = V_1 \oplus V_2 \quad \text{with } [V_1, V_1] = V_2 \text{ and } [V_1, V_2] = [V_2, V_2] = 0 \ .$$

This property generalises to the class of *stratified nilpotent Lie groups*. With this property one gets additional structures on the Lie groups. The most helpful of them are the family of *scaling automorphism* $s_t : G \to G$, and the sub-Riemannian metric, for which the scaling automorphisms are homotheties.

We apply the techniques of Burillo and Young to prove a similar division of Euclidean, super- and sub-Euclidean growth of the filling functions of stratified nilpotent Lie groups under some algebraic condition on their Lie algebras. Further we use the results for the filling functions to establish lower and upper bounds on the higher divergence functions of these stratified nilpotent Lie groups.
We will see that the generalised Heisenberg Groups over the Hamilton quaternions and over the octonions satisfy the conditions for our theorems. So we get a direct generalisation of the behaviour of the filling functions of the complex Heisenberg Groups to the filling functions of its relatives defined over the quaternions and the octonions.

Structure of this thesis

In Chapter 1 we provide the basic definitions and properties of the examined invariants (filling function, Dehn functions and higher divergence functions). Further we introduce the class of stratified nilpotent Lie groups and their asymptotic cones, the Carnot Groups. We give the definitions of some special families of stratified nilpotent Lie groups. We also provide some background to the h-principle and to metric currents, which are needed as technical tools.
In Chapter 2 we state the main results of this thesis. Further the geometric meaning of the algebraic conditions is discussed.
Chapter 3 acts as a short overview of the main tools for the proofs. In this chapter we also state the theorems of Burillo and Young, which we will use in the proofs.
In Chapter 4 we develop our main auxiliary technique: the horizontal approximation.
In the following Chapters we prove the main results. The proofs are divided into the results for the filling functions in Chapter 5, the results for the higher divergence functions in Chapter 6 and the results for the Heisenberg Groups and the symmetric spaces in Chapter 7.
In Chapter 8, the last chapter, we formulate a list of open questions which arise from our results.

Contents

1 Background

In this chapter we give the basic definitions of the objects we want to examine. We also state some known properties and results.

1.1 Filling functions

Filling functions describe the difficulty to fill a given cycle with a chain. The base case coincides with the isoperimetric problem discussed in the Introduction: How much area is needed to fill a loop by a disk? There is a just as well natural generalisation to *higher-order filling functions*, which quantify higher connectivity by describing the difficulty of filling m-cycles by $(m+1)$-chains. They are harder to compute, but give a finer distinction of the large scale geometry of the metric space.

Definition.
Let X be a metric space and $m \in \mathbb{N}$. Further denote by $\mathcal{H}^m$ the m-dimensional Hausdorff-measure of X. The m-dimensional volume *of a subset $A \subset X$ is*

$$\mathrm{vol}_m(A) := \mathcal{H}^m(A) \ .$$

We denote by Δ^m the m-simplex equipped with an Euclidean metric.

Definition.
Let X be a metric space and $m \in \mathbb{N}$.
A Lipschitz m-chain *a in X is a (finite) formal sum $a = \sum_j z_j \alpha_j$ of Lipschitz maps $\alpha_j : \Delta^m \to X$ with coefficients $z_j \in \mathbb{Z}$.*
The boundary *of a Lipschitz m-chain $a = \sum_j z_j \alpha_j$ is defined as the Lipschitz $(m-1)$-chain*

$$\partial a = \sum_j \big(z_j \sum_{i=0}^{m} (-1)^i \alpha_{j|\Delta_i^m}\big)$$

where Δ_i^m denotes the i^{th} face of Δ^m.
A Lipschitz m-chain a with zero-boundary, i.e. $\partial a = 0$, is called a Lipschitz m-cycle.
A filling *of a Lipschitz m-cycle a is a Lipschitz $(m+1)$-chain b with boundary $\partial b = a$.*

We define the mass *of a Lipschitz m-chain a as the total volume of its summands:*

$$\operatorname{mass}(a) := \sum_j z_j \operatorname{vol}_m(\alpha_j(\Delta^m)) \ .$$

If X is a Riemannian manifold, the volume of such a summand is given by $\operatorname{vol}_m(\alpha_j(\Delta^m)) = \int_{\Delta^m} J_{\alpha_j} \,\mathrm{d}\,\lambda$, where $\mathrm{d}\,\lambda$ denotes the m-dimensional Lebesgue-measure and J_{α_j} is the jacobian of α_j. This is well defined, as Lipschitz maps are, by Rademacher's Theorem, almost everywhere differentiable.

Given a m-cycle, one is interested in the best possible filling of it, i.e. the filling with the smallest mass. To deduce a property of the space X, one varies the cycle and examines how large the ratio of the mass of the optimal filling and the mass of the cycle can get. This leads to the *filling functions*:

Definition.
Let $n \in \mathbb{N}$ and let X be a n-connected metric space. For $m \leq n$ the $(m+1)^{th}$- filling function *of X is given by*

$$F_X^{m+1}(l) = \sup_a \inf_b \operatorname{mass}(b) \qquad \forall l \in \mathbb{R}^+,$$

where the infimum is taken over all $(m+1)$-chains b with $\partial b = a$ and the supremum is taken over all m-cycles a with $\operatorname{mass}(a) \leq l$.

As we are mostly interested in the large scale geometry of the space X, the exact description of the filling functions is of less importance to us. Indeed we only look at the asymptotic growth rate of the functions. We do this by the following equivalence relation, which makes the growth rate of the filling functions an quasi-isometry invariant.

Definition.
Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be functions. Then we write $f \preccurlyeq g$ if there is a constant $C > 0$ with

$$f(l) \leq Cg(Cl) + Cl + C \quad \forall l \in \mathbb{R}^+.$$

If $f \preccurlyeq g$ and $g \preccurlyeq f$ we write $f \sim g$. This defines an equivalence relation.

We read this notation $f \preccurlyeq g$ as "*f is bounded from above by g* " respectively "*g is bounded from below by f* " according whether we are more interested in f or g.

Proposition 1.1 (see for example [40, Lemma 1]).
Let X and Y be n-connected metric spaces. Then holds:

$$X \text{ quasi-isometric to } Y \;\Rightarrow\; F_X^{j+1} \sim F_Y^{j+1} \quad \forall j \leq n.$$

Let's look at the example of the filling functions of the n-dimensional Euclidean space. They were first computed by Herbert Federer and Wendell H. Fleming in [13].

Example.
The filling functions of the Euclidean space $\mathbb{R}^n$ are

$$F_{\mathbb{R}}^{j+1}(l) \sim l^{\frac{j+1}{j}} \quad \text{for } j \leq n-1.$$

This enables us to use the terms *Euclidean, sub-Euclidean* respectively *super-Euclidean filling function* for filling functions with the same, strictly slower respectively strictly faster growth rate than $l^{\frac{j+1}{j}}$.

The following theorem generalises the Euclidean case to spaces with non-positive curvature. For a proof see [37].

Theorem 1.2.
The filling functions of an n-dimensional Hadamard space X are

$$F_X^{j+1}(l) \preccurlyeq l^{\frac{j+1}{j}} \quad \text{for } j \leq n-1.$$

The fact that a Riemannian manifold with non-positive curvature has Euclidean or sub-Euclidean filling functions in all dimensions yields a sufficient criterion for positive curvature: Let M be a Riemannian manifold with a super-Euclidean filling function in some dimension, then there is some positive curvature on M.

1.2 Dehn functions

The *Dehn functions* form another family of filling invariants. The most basic example defined for finitely presented groups is the *classical Dehn function*, which measures the number of relations needed to reduce a word representing the identity to the empty word. Even if there are open problems, as for example the growth rate of the classical Dehn function of $SL_4(\mathbb{Z})$, the behaviour of this invariant is well understood (see for example [6], [8], [38], [41]). They originally were defined in a combinatorial way for finitely presented groups. Using the lemma of Švarc-Milnor, their equivalence classes by the above defined equivalence relation for functions $\mathbb{R}^+ \to \mathbb{R}^+$ coincide with the ones of a homotopical version of the filling functions. As this definition is more geometric, we prefer it.

Definition.
Let M be a m-connected Riemannian manifold or a simplicial complex and let Γ be a (finitely generated) group which acts properly discontinuously and cocompactly on M by isometries. Further let $\mathbb{D}^{m+1}$ be the closed unit ball in $\mathbb{R}^{m+1}$ and S^m its boundary.
The m-dimensional (Lipschitz) Dehn function *(of M) is given by*

$$\delta_M^m(l) = \sup_f \inf_h \mathrm{vol}_{m+1}(h) \qquad \forall l \in \mathbb{R}^+,$$

where the supremum is taken over all Lipschitz maps $f : S^m \to M$ with the bound $\mathrm{vol}_m(f) \leq l$ and the infimum is taken over all Lipschitz maps $h : \mathbb{D}^{m+1} \to M$ with $h_{|\partial\mathbb{D}^{m+1}} = f$.

Compared with the definitions in the previous section, the (Lipschitz) Dehn functions are homotopical analogues of the (homologically defined) filling functions. So it is not surprising, that the equivalence classes of the Dehn functions are invariant under quasi-isometries:

Proposition 1.3 (see for example [2, Corollary 3])**.**
Let X and Y be n-connected Riemannian manifolds or simplicial complexes. Then holds:

$$X \text{ quasi-isometric to } Y \;\Rightarrow\; \delta_X^j \sim \delta_Y^j \quad \forall j \leq n.$$

Whenever a group Γ acts on a Riemannian manifold or a simplicial complex M as in the definition above, Γ and M are quasi-isometric by the lemma of Švarc-Milnor. This implies, that any two m-connected Riemannian manifolds or simplicial complexes M_1, M_2 with such a Γ-action are quasi-isometric and so have equivalent m-dimensional Dehn functions. This again implies that the equivalence class of the m-dimensional Dehn function of M in fact only depends on Γ. So it is convenient to speak of the m-dimensional Dehn function of Γ and denote it by δ_Γ^m.

A subject recently undergoing intense study are the Dehn functions of non-cocompact lattices in semi-simple Lie groups of rank k. They are expected to be of Euclidean growth up to dimension $k-1$ and of exponential growth in the dimension above. Leuzinger and Young proved this conjecture in [24] for lattices of $\mathbb{Q}$-rank 1, but the general case is still open.

Last but not least in this section we are interested in the relationship of these two filling invariants, the filling functions and the Dehn functions.

Proposition 1.4 (compare [14], [22]).
Let M be a n-connected Riemannian manifold or a simplicial complex and let Γ be a group which acts properly discontinuously and cocompactly on M by isometries. Then holds:

$$\delta_\Gamma^1 \preccurlyeq F_M^2 \ ,$$

$$\delta_\Gamma^2 \preccurlyeq F_M^3 \ ,$$

$$\delta_\Gamma^j \sim F_M^{j+1} \quad \textit{for } j \geq 3.$$

Here one sees that the notation of the filling functions and the Dehn functions, more precisely the notation of the exponent which indicates the dimension, are not chosen very luckily: The exponent in the notation of the Dehn function denotes the dimension of the boundary which is to fill, and the exponent in the notation of the filling function denotes the dimension of the filling chain. But as these are the established notations in literature, we use them.

1.3 Higher divergence functions

Another way to examine the asymptotic geometry of a space and to find quasi-isometry invariants is to study the topology at infinity. A quantitative version of this are the *higher divergence functions*, which, roughly speaking, measure the difficulty to fill a m-cycle lying outside an r-ball with an outside a ρr-ball lying $(m+1)$-chain (for some $0 < \rho \leq 1$).

Let X be a simply connected metric space with basepoint $\mathfrak{o} \in X$.

Definition.
Let $r > 0$. We call a Lipschitz chain a in X r-avoidant, *if* $\mathrm{image}(a) \cap B_r(\mathfrak{o}) = \emptyset$.

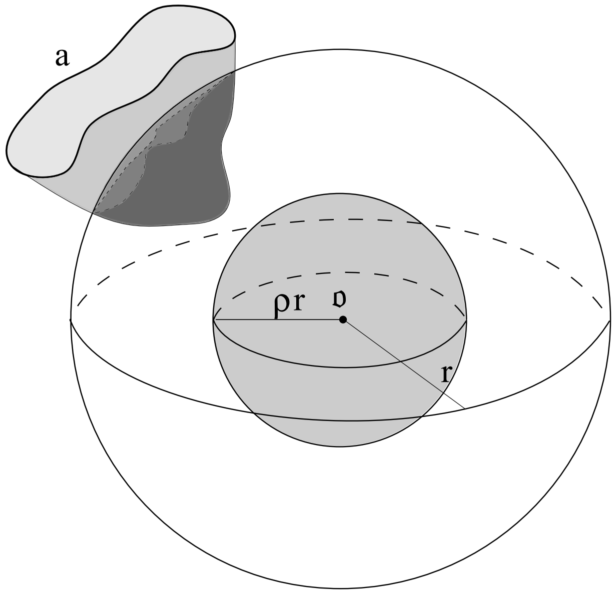

Figure 1: An r-avoidant cycle **a** with a ρr-avoidant filling (compare [7]).

One now wants to fill r-avoidant cycles by (nearly) r-avoidant chains. To do this, we need the cycle to be a boundary. In contrast to the case of the filling functions, here it doesn't suffice that X is highly connected as the avoidant-condition can be imagined as deleting the r-ball around the basepoint. This leads to the following definition:

Definition.
For $0 < \rho \leq 1$ we call X (ρ, n)-acyclic at infinity, *if every r-avoidant Lipschitz j-cycle a has a ρr-avoidant filling for all $0 \leq j \leq n$, i.e. there is a ρr-avoidant Lipschitz $(j+1)$-chain b with $\partial b = a$.*
The divergence dimension $\mathrm{divdim}(X)$ *of X is the largest integer n, such that X is (ρ, n)-acyclic at infinity for some ρ.*

It can be easily seen, that the divergence dimension is always smaller than $\dim X - 2$, as there are $(\dim X - 2)$-cycles homotopic to the boundary of the

r-ball $B_r(\mathfrak{o})$ around $\mathfrak{o}$. These cycles, of course, are not boundaries of chains in $X \setminus B_r(\mathfrak{o})$.

In the following let m be always less or equal to the divergence dimension of X.

Given an r-avoidant m-cycle one is interested, similar to the case of the filling functions, in the best possible avoidant filling of it. Now the virtual hole, arisen by making the ball around the basepoint a no-go area, complicates the geometry of the space. Therefore it is not so easy to produce from this a asymptotic property of the space. So there is need of some technical parameters, whose use will be explained later.

Definition.
For $0 < \rho \leq 1$ and $\alpha > 0$ we set

$$\operatorname{div}^m_{\rho,\alpha}(r) := \sup_a \operatorname{div}^m_\rho(a, \alpha r^m) := \sup_a \inf_b \operatorname{mass}(b) \qquad \forall r \in \mathbb{R}^+,$$

where the infimum is taken over all ρr-avoidant $(m+1)$-chains b with $\partial b = a$ and the supremum is taken over all r-avoidant m-cycles a with $\operatorname{mass}(a) \leq \alpha r^m$.

Then the m^{th}-divergence function *of X is the 2-parameter family*

$$\operatorname{Div}^m(X) := \{\operatorname{div}^m_{\rho,\alpha}\}_{\rho,\alpha} \ .$$

Above we asked m to be less or equal to the divergence dimension. Alternatively one could set the infimum over the empty set as ∞. In this case, the divergence dimension is the biggest number $n \in \mathbb{N}$, such that there is an $\rho \in (0,1]$ with $\operatorname{div}^j_{\rho,\alpha} < \infty$ for all $j \leq n$.

The functions $\operatorname{div}^m_{\rho,\alpha}$ are very explicit in terms of the metric. For example if one scales the metric by a constant $c > 0$ the functions will scale to $\operatorname{div}^m_{\rho,\alpha}(c\ \cdot)$. As we are mostly interested in the asymptotic behaviour, we look at the equivalence classes of the higher divergence functions $\operatorname{Div}^m(X)$ under the below defined equivalence relation for special 2-parameter families of functions. This makes $\operatorname{Div}^m(X)$ an quasi-isometry invariant.

Definition.

a) A positive 2-parameter m-family *is a 2-parameter family $F = \{f_{s,t}\}$ of functions $f_{s,t} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, indexed over $0 < s \leq 1, t > 0$, together with a fixed integer m.*

b) Let $m \in \mathbb{N}$ and let $F = \{f_{s,t}\}$ and $H = \{h_{s,t}\}$ be two positive 2-parameter m-families, indexed over $0 < s \leq 1, t > 0$.
Then we write $F \preceq H$, if there exists thresholds $0 < s_0 \leq 1$ and $t_0 > 0$, as well as constants $L, M \geq 1$, such that for all $s \leq s_0$ and all $t \geq t_0$ there is a constant $A \geq 1$ with:

$$f_{s,t}(x) \leq A h_{Ls,Mt}(Ax + A) + O(x^m) \ .$$

If both $F \preceq H$ and $H \preceq F$, so we write $F \sim H$. This defines an equivalence relation.

We read this notation $F \preccurlyeq H$ as "*F is bounded from above by H*" respectively "*H is bounded from below by F* " depending on whether we are more interested in F or H.

The nicest (and very special) case is that of a positive 2-parameter m-family $F = \{f_{s,t}\}_{s,t}$ bounded from above (or below) by a constant positive 2-parameter m-family H, i.e. $H = \{h\}_{s,t}$. This means that all functions $f_{s,t}$ are bounded from above (or below) by the same growth type. As this will often appear in the following, we just write h for the constant positive 2-parameter m-family $\{h\}_{s,t}$ (supposing, that it is clear which m we mean).

Remark.
We consider $\mathrm{Div}^m(X)$ *as positive 2-parameter m-family, indexed by ρ and α.*

The relation "$\preceq$" (and consequently "$\sim$") only captures the asymptotic behaviour of the functions for $r \to \infty$:
Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. If $B \geq 1$ and $\mathrm{div}^m_{\rho,\alpha}(B) \leq h(B)$ then $\mathrm{div}^m_{\rho,\alpha}(r) \leq B \cdot h(B + Br)$ for all $r \leq B$, because both sides are increasing. So we need to examine the relation "$\preceq$" (and consequently "$\sim$") only for r larger than an arbitrary constant $B = B(\rho, \alpha) \geq 1$.

The proof of the following proposition can be found in [1, Prop. 2.2]. It uses the fact that one can vary the parameter ρ by multiplying the constant L in the equivalence relation and that the constant A is chosen after the parameters ρ and α (at this point there is an error in the appropriate definition of equivalence in [1]).

Proposition 1.5.
Let $n \in \mathbb{N}$ and le X, Y be metric spaces with basepoints and let $\operatorname{divdim}(X) \geq n$ *and* $\operatorname{divdim}(Y) \geq n$. *Then holds:*

$$X \text{ quasi-isometric to } Y \;\Rightarrow\; \operatorname{Div}^j(X) \sim \operatorname{Div}^j(Y) \quad \forall j \leq n.$$

Again we look at the example of the n-dimensional Euclidean space and its higher divergence functions.

Example.
The higher divergence functions of the Euclidean space $\mathbb{R}^n$ are

$$\operatorname{Div}^j_{\mathbb{R}^n}(r) \sim r^{j+1} \quad \text{for } \; j \leq n-2 = \operatorname{divdim}(\mathbb{R}^n).$$

As in the case of the filling functions, this enables us to use the terms *Euclidean, sub-Euclidean* respectively *super-Euclidean j^{th}-divergence* for j^{th}-divergence functions with the same, strictly slower respectively strictly faster growth rate than r^{j+1}.

It remains to explain the importance of the upper bound αr^m on the mass of the r-avoidant m-cycles. In general there is the need of an upper bound in terms of r to prevent the possibility to choose cycles with exponential mass and at least exponential filling mass (e.g. a codimension ≥ 2 sphere of radius e^r in $\mathbb{R}^n$), which would make Div^m always exponential. The explicit choice of polynomial growth of degree m is mostly due to the fact, that the higher divergence were originally introduced for symmetric spaces. There the bound αr^m is exactly the right one to prove the following result of Leuzinger:

Theorem 1.6 (see [21])**.**
Let X be a symmetric space of non-compact type and rank k.
Then holds:

$$\operatorname{Div}^j_X(r) \preccurlyeq r^{j+1} \quad \text{for } \; j \leq k-2,$$

$$\operatorname{Div}^{k-1}_X(r) \sim e^r \; .$$

This shows, that higher divergence functions detect the rank of a symmetric space. A similar result for the more general case of Hadamard spaces is proved in [36] by Wenger.

Other recent results are due to Cornelia Druţu and Jason Behrstock, who computed in [4] higher divergence functions for mapping class groups. Their results show, that there is a change from super-Euclidean to Euclidean growth at the dimension equal to the rank of maximal abelian subgroups of the mapping class group.

1.4 Integral currents

For the proofs of some of our main results we need the notion of *integral currents*. These are functionals which can be seen as a completion of the space of Lipschitz chains. Here we will only give the basic definitions and will state the for our purpose most important properties. For a comprehensive study see [3] and [13].

We will state all the definitions in the most general setting of complete metric spaces, as we will need them in particular for Carnot groups (G, d_c). The original idea of *currents* is due to Federer and Fleming and the setting of the Euclidean space $\mathbb{R}^n$. As we will see later, many of their results can be transferred to Riemannian manifolds.

Let X be a complete metric space and $m \in \mathbb{N} \cup \{0\}$.
Denote by $\mathcal{D}^m(X)$ the $\mathbb{R}$-vector space of $(m+1)$-tuples $(f, \pi_1, ..., \pi_m)$ of real valued Lipschitz functions on X with the restriction that f has a bounded set of values.

Definition.
An m-dimensional metric functional *on X is a map*

$$T : \mathcal{D}^m(X) \to \mathbb{R}\ ,\ \ (f, \pi_1, ..., \pi_m) \mapsto T(f, \pi_1, ..., \pi_m)$$

such that for every $\omega, \eta \in \mathcal{D}^m(X)$ and every $t \geq 0$ holds:

(i) $|T(\omega + \eta)| \leq |T(\omega)| + |T(\eta)|$ *(subadditivity),*

(ii) $|T(t\omega)| = t \cdot |T(\omega)|$ *(positive 1-homogeneity).*

For metric functionals one can define the *boundary operator* and the *push-forward* along a Lipschitz map.

For $\omega = (f, \pi_1, ..., \pi_m) \in \mathcal{D}^m(X)$ denote by $d\omega$ the $(m+2)$-tuple

$$d\omega := (1, f, \pi_1, ..., \pi_m) \in \mathcal{D}^{m+1}(X) \ .$$

Definition.
Let T be an m-dimensional metric functional on X. Then the boundary of T *is given by*

$$\partial T(\omega) = T(d\omega)$$

for all $\omega \in \mathcal{D}^{m-1}(X)$.

Definition.
Let T be an m-dimensional metric functional on X and let $\varphi : X \to Y$ be a Lipschitz map from X into another complete metric space Y. The push-forward *of T along φ is the metric functional on Y given by*

$$\varphi_\# T(g, \tau_1, ..., \tau_m) := T(g \circ \varphi, \tau_1 \circ \varphi, ..., \tau_m \circ \varphi)$$

for all $(g, \tau_1, ..., \tau_m) \in \mathcal{D}^m(Y)$.

Let T be an m-dimensional metric functional on X, $\varphi : X \to Y$ be a Lipschitz map from X into another complete metric space Y and $\omega = (g, \tau_1, ..., \tau_{m-1}) \in \mathcal{D}^{m-1}(Y)$. Then one can execute the following computation:

$$\begin{aligned}\varphi_\#(\partial T)(\omega) =& \partial T(g \circ \varphi, \tau_1 \circ \varphi, ..., \tau_{m-1} \circ \varphi) = T(1, g \circ \varphi, \tau_1 \circ \varphi, ..., \tau_{m-1} \circ \varphi)\\ =& T(1 \circ \varphi, g \circ \varphi, \tau_1 \circ \varphi, ..., \tau_{m-1} \circ \varphi) = \varphi_\# T(1, g, \tau_1, ..., \tau_{m-1})\\ =& \partial(\varphi_\# T)(\omega)\end{aligned}$$

So we see, that boundary and push-forward commute.

Another important operation on metric functionals is the *restriction*:

Definition.
Let T be an m-dimensional metric functional on X and $\eta = (g, \tau_1, ..., \tau_k) \in \mathcal{D}^k(X)$, $k \le m$. The restriction *of T to η is the $(m-k)$-dimensional metric functional on X given by*

$$T \llcorner \eta(f, \pi_1, ..., \pi_{m-k}) := T(fg, \tau_1, ..., \tau_k, \pi_1, ..., \pi_{m-k})$$

for all $(f, \pi_1, ..., \pi_{m-k}) \in \mathcal{D}^{m-k}(X)$.

An important example of a restriction is the case $\eta = (\mathbb{1}_A) \in \mathcal{D}^0(X)$ for a Borel set $A \subset X$. The restriction of an m-dimensional metric functional T to η is given by

$$T \llcorner \eta(f, \pi_1, ..., \pi_m) = T(\mathbb{1}_A f, \pi_1, ..., \pi_m) = T(f_{|A}, \pi_1, ..., \pi_m) \ .$$

In the following we will denote it briefly by $T \llcorner A$.

Definition.
Let T be an m-dimensional metric functional on X. Then T is called of finite mass*, if there exists a finite Borel measure μ on X, such that*

$$|T(f, \pi_1, ..., \pi_m)| \leq \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \int_X |f| d\mu$$

for all $(f, \pi_1, ..., \pi_m) \in \mathcal{D}^m(X)$.
The minimal measure μ with this property is denoted by $\|T\|$. The number $\boldsymbol{M}(T) := \|T\|(X)$ is called the mass *of T.*

In literature often $\|T\|$ is called mass of T, too. But if one is interested in metric functionals (in particular in integral currents) as relatives of Lipschitz chains, then the measure $\boldsymbol{M}$ corresponds more closely with the measure "mass" of Lipschitz chains.

Definition.
The support *of an m-dimensional metric functional T on X is the set*

$$\mathrm{spt}(T) := \{x \in X \mid \|T\|(B_r(x)) > 0 \ \forall r > 0\} \ .$$

Now we are ready to do the first step of specialising:

Definition.
*An m-dimensional metric functional T on X is called an m-*dimensional current *if it satisfies the following properties:*

(i) T is multi-linear.

(ii) If for all $i \in \{1, ..., m\}$ the sequence $(\pi_i^j)_j$ converges pointwise to π_i for $j \to \infty$ and if $\sup_{i,j} \mathrm{Lip}(\pi_i^j) < \infty$, then holds:

$$T(f, \pi_1^j, ..., \pi_m^j) \xrightarrow{j \to \infty} T(f, \pi_1, ..., \pi_m) \qquad \textit{(continuity).}$$

(iii) *If there are Borel sets $B_i \subset X,\ i \in \{1, ..., m\}$, such that π_i is constant on B_i and the set $\{x \in X \mid f(x) \neq 0\}$ is contained in the union $\bigcup_{i=1}^m B_i$, then holds:*

$$T(f, \pi_1, ..., \pi_m) = 0 \qquad \textit{(locality)}.$$

(iv) *T has finite mass, i.e. $\boldsymbol{M}(T) < \infty$.*

The set of m-dimensional currents on X is denoted by $\boldsymbol{M}_m(X)$.

Example.
Let $X = \mathbb{R}^n$, $B \subset \mathbb{R}^n$ be a Borel set and $d\lambda^n$ the Lebesgue measure of $\mathbb{R}^n$. Then every function $\beta \in L^1(B, \mathbb{R})$ induces an n-dimensional current $[\![\beta]\!] \in \boldsymbol{M}_n(\mathbb{R}^n)$ by

$$[\![\beta]\!](f, \pi_1, ..., \pi_n) := \int_B \beta f \det(\left(\frac{\partial \pi_i}{\partial x_j}\right)_{i,j}) \; d\lambda^n \ .$$

It can be proved, that restriction and push-forward map currents onto currents (see [3]). But the boundary of a current is not always a current again. The properties $(i), (ii)$ and (iii) in the above definition are preserved, but the finite mass postulation can be violated.

Definition.
A current $T \in \boldsymbol{M}_m(X)$ is called normal, *if its boundary ∂T is again a current.*

As a convention, all currents in $\boldsymbol{M}_0(X)$ are called normal. The set of normal m-dimensional currents on X is denoted by $\boldsymbol{N}_m(X)$. On this set one defines the measure of *normal mass* by

$$\boldsymbol{N}(T) := \boldsymbol{M}(T) + \boldsymbol{M}(\partial T) \ .$$

As $\partial(\partial T) = 0$ for all metric functionals, one sees that normal currents are mapped under the boundary operator onto normal currents.

To define the *integral currents* we need the notion of *rectifiability*. For this we denote for $m \in \mathbb{N}$ the m-dimensional Hausdorff-measure of X by $\mathcal{H}^m$.

Definition.
Let U be a subset of X and $m \in \mathbb{N}$. Then U is called

a) m-rectifiable, *if there exists a bounded subset $V \subset \mathbb{R}^m$ and a surjective Lipschitz map $\phi : V \to U$.*

b) countably m-rectifiable, *if U is the union of countably many m-rectifiable subsets $U_i \subset X$.*

c) countably $\mathcal{H}^m$-rectifiable, *if there exists a countably m-rectifiable set $S \subset X$, such that $\mathcal{H}^m(U \setminus S) = 0$.*

This leads to the notion of rectifiability of currents:

Definition.
Let $m \geq 1$ and $T \in \boldsymbol{M}_m(X)$.

a) T is called integer rectifiable, *if*

i) the measure $\|T\|$ is concentrated on a countably $\mathcal{H}^m$-rectifiable set and vanishes on all Borel sets B with $\mathcal{H}^m(B) = 0$,

ii) and for every Lipschitz map $\varphi : X \to \mathbb{R}^m$ and every open subset $U \in X$ there is a function $\beta \in L^1(\mathbb{R}^m, \mathbb{Z})$, such that $\varphi_\#(T \llcorner U) = [\![\beta]\!]$.

b) T is called an integral current, *if T is normal and integer rectifiable. The set of m-dimensional integral currents on X is denoted by $\boldsymbol{I}_m(X)$.*

Integral currents are mapped under the boundary operator onto integral currents, as the following theorem states:

Theorem 1.7 (see [3, Theorem 8.6])**.**
Let $m \geq 1$ and $T \in \boldsymbol{I}_m(X)$ be an m-dimensional integral current. Then its boundary is again an integral current $\partial T \in \boldsymbol{I}_{m-1}(X)$.

Now we explain how every Lipschitz m-chain can be associated with a m-dimensional integral current.

Lets consider a Lipschitz m-chain

$$a = \sum_{j=1}^{n} z_j \alpha_j$$

with $z_j \in \mathbb{Z}$ and $\alpha_j : \Delta^m \to X$ Lipschitz.
Then a gives rise to a m-dimensional integral current

$$a_{\#} := \sum_{j=1}^{n} z_j \alpha_{j\#} [\![\mathbb{1}_{\Delta^m}]\!] \ \in \boldsymbol{I}_m(X) \ .$$

In the case that X is a Riemannian manifold, one gets coinciding mass for the Lipschitz chain and the associated integral current: $\mathrm{mass}(a) = \boldsymbol{M}(a_{\#})$ (see [3], [38]).

In the case of $X = \mathbb{R}^n$, there is a bijection between the classical integral currents defined by Federer and Fleming in [13] and the above defined (metric) integral currents. This means, that for every $m \leq n$ there is a map

$$\boldsymbol{I}_m(\mathbb{R}^n) \to \boldsymbol{I}_m^{FF}(\mathbb{R}^n) \ , \ T \mapsto \widetilde{T}$$

such that

$$\boldsymbol{M}(T) \leq \boldsymbol{M}(\widetilde{T}) \leq c(n,m) \cdot \boldsymbol{M}(T)$$

where $c(n,m) > 0$ is a constant only depending on the dimension m of the integral current T and the dimension of the space $\mathbb{R}^n$ (see [3]).

1.5 Filling invariants for integral currents

We will use integral currents to prove our results for the filling invariants for Lipschitz chains. In order to do this, we modify already existing results concerning filling invariants for integral currents. So we have to introduce isoperimetric inequalities and higher divergence functions for integral currents. This is mainly done by replacing the words 'Lipschitz chain' by 'integral current' in the respective definitions.

Definition.
Let X be a complete metric space and let $m \in \mathbb{N}$. Then X satisfies an isoperimetric inequality of rank δ for $\boldsymbol{I}_m(X)$*, if there is a constant $C > 0$, such that for*

every integral current $T \in \boldsymbol{I}_m(X)$ *with* $\partial T = 0$*, there exists an integral current* $S \in \boldsymbol{I}_{m+1}(X)$ *with* $\partial S = T$ *and*

$$\boldsymbol{M}(S) \leq C \cdot \boldsymbol{M}(T)^{\delta} \ .$$

To define higher divergence functions for integral currents we need, as in the case of Lipschitz chains, some auxiliary technical terms:

Definition.
Let $(X, \mathfrak{o})$ *be a complete metric space with basepoint and let* $r > 0$*. We call an integral current* T *on* X r-avoidant*, if* $\mathrm{spt}(T) \cap B_r(\mathfrak{o}) = \emptyset$*. Further* $(X, \mathfrak{o})$ *is* (ρ, n)-acyclic at infinity for integral currents*, if for every* r*-avoidant integral current* $T \in \boldsymbol{I}_j(X)$ *with* $\partial T = 0$ *there is an* ρr*-avoidant integral current* $S \in \boldsymbol{I}_{j+1}(X)$ *with* $\partial S = T$*, for all* $j \leq n$*.*
The divergence dimension of $(X, \mathfrak{o})$ for integral currents *is the maximal number* $n \in \mathbb{N}$*, such that there is an* $\rho \in (0, 1]$*, such that* $(X, \mathfrak{o})$ *is* (ρ, n)*-acyclic at infinity for integral currents.*

From now on let m be less or equal the divergence dimension of $(X, \mathfrak{o})$ for integral currents.

Definition.
For $0 < \rho \leq 1$ *and* $\alpha > 0$ *we set*

$$\widehat{\mathrm{div}}^m_{\rho,\alpha}(r) := \sup_a \inf_b \boldsymbol{M}(b)$$

where the infimum is taken over all ρr*-avoidant* $(m+1)$*-dimensional integral currents* $S \in \boldsymbol{I}_{m+1}(X)$ *with* $\partial S = T$ *and the supremum is taken over all* r*-avoidant* m*-dimensional integral currents* $T \in \boldsymbol{I}_m(X)$ *with* $\boldsymbol{M}(T) \leq \alpha r^m$ *and* $\partial T = 0$*.*

Then, for $m \in \mathbb{N}$*, the* m^{th}-divergence function of X for integral currents *is the 2-parameter family*

$$\widehat{\mathrm{Div}}^m(X) := \{\widehat{\mathrm{div}}^m_{\rho,\alpha}\}_{\rho,\alpha} \ .$$

As for the higher divergence functions $\mathrm{Div}^m(X)$ for Lipschitz chains, we consider $\widehat{\mathrm{Div}}^m(X)$ as a positive 2-parameter m-family and look at the equivalence classes with respect to the equivalence relation for positive 2-parameter m-families introduced in Section 1.3. Remember that we denote a constant positive 2-parameter

m-family $F = \{f\}_{s,t}$ briefly by f.

The main result we will use is the following theorem due to Wenger:

Theorem 1.8 (see [36, Prop. 1.8]).
Let X be a complete Riemannian manifold and $m \in \mathbb{N}$. If there is an $\delta < \frac{m+1}{m}$, such that X satisfies an isoperimetric inequality of rank δ for $\boldsymbol{I}_m(X)$, then holds:

$$\widehat{\mathrm{Div}}^m(X) \preccurlyeq r^{\delta m} \ .$$

This theorem implies sub-Euclidean upper bounds for the higher divergence functions for integral currents, whenever a sub-Euclidean isoperimetric inequality for integral currents is satisfied.

1.6 Asymptotic cones

For the previously defined invariants of the large scale geometry of a metric space X, one follows the idea to examine the asymptotic behaviour of functions. Another way to encode the large scale geometry of X is to examine how the appearance of X behaves if one "zooms out". Formally this leads to the concept of *asymptotic cones*. For the definition of these objects, we need some preparation.

Definition.
Let $\omega \subset \mathcal{P}(\mathbb{N})$ be a family of subsets of $\mathbb{N}$.

a) The family ω is a filter *on $\mathbb{N}$, if the following is satisfied:*

i) $\emptyset \notin \omega$,

ii) $A \in \omega,\ B \subset A \ \Rightarrow\ B \in \omega$,

iii) $A, B \in \omega \ \Rightarrow\ A \cap B \in \omega$.

b) A filter ω is called an ultrafilter, *if it is maximal, i.e. if there is another filter ω' with $\omega \subset \omega'$, then already holds $\omega = \omega'$.*

c) An ultrafilter is called non-principal, *if it doesn't contain any finite set.*

Another way to understand a non-principal ultrafilter ω is to consider it as a probability measure $\widehat{\omega}$ on $\mathbb{N}$. Then one has to demand, that for all subsets of $\mathbb{N}$, $\widehat{\omega}$ only takes values in $\{0,1\}$ and that for all finite subsets $B \subset \mathbb{N}$ holds $\widehat{\omega}(B) = 0$.

Definition.
Let ω be a non-principal ultrafilter and

$$x : \mathbb{N} \to \mathbb{R} \cup \{\infty\} \ , \ n \mapsto x_n$$

be a sequence.
The ω-limit *of x, denoted as $\lim_\omega x_n$, is the unique point $x_\omega \in \mathbb{R} \cup \{\infty\}$, such that for every open neighbourhood U of x_ω holds $x^{-1}(U) \in \omega$.*

In the alternative view of non-principal ultrafilters as probability measures, one can describe the ω-limit of a sequence $x = (x_n)_{n \in \mathbb{N}}$ as the unique point $x_\omega \in \mathbb{R} \cup \{\infty\}$, such that for every open neighbourhood U of x_ω holds

$$\widehat{\omega}(\{n \in \mathbb{N} \mid x_n \in U\}) = 1 \ .$$

In both views one sees immediately, that there have to be infinitely many elements of $x = (x_n)_{n \in \mathbb{N}}$ in every open neighbourhood of the ω-limit x_ω.

Definition.
Let $(X_n, d_n)_{n \in \mathbb{N}}$ be a sequence of metric spaces and let ω be a non-principal ultrafilter. The ultralimit *of $(X_n, d_n)_{n \in \mathbb{N}}$ is the metric space (X_ω, d_ω) defined as follows:*

Let $X^\infty := \bigtimes_{n \in \mathbb{N}} X_n$ and $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in X^\infty$. Then set

$$d_\omega(x, y) := \lim_\omega (d_n(x_n, y_n))$$

and

$$X_\omega := X^\infty / \sim_0$$

where $x \sim_0 y$ if and only if $d_\omega(x, y) = 0$.

If further $\mathfrak{o}_n \in X_n$, $n \in \mathbb{N}$, are basepoints, then the based ultralimit *is the (connected) metric space $((X_\omega^{\mathfrak{o}}, d_\omega), \mathfrak{o})$, where*

$$\mathfrak{o} = (\mathfrak{o}_n)_{n \in \mathbb{N}} \quad \textit{and} \quad X_\omega^{\mathfrak{o}} = \{x \in X_\omega \mid d_\omega(x, \mathfrak{o}) < \infty\} \ .$$

Now we are able to define the *asymptotic cone* of a pointed metric space with respect to a non-principal ultrafilter.

Definition.
Let (X, d) be a metric space and $\mathfrak{o} \in X$ be a basepoint. Further let ω be a non-principal ultrafilter and $r : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ be a sequence with $\lim_\omega r_n = \infty$.
The based ultralimit of $((X, \frac{1}{r_n}d), \mathfrak{o})$ is called asymptotic cone of X.

Ultralimits of metric spaces are a generalisation of limits with respect to (pointed) Gromov-Hausdorff convergence in the following sense:

Proposition 1.9 (compare [18])**.**
Let $((X_n, d_n), p_n)_{n\in\mathbb{N}}$ be a sequence of pointed proper metric spaces converging in the sense of pointed Gromov-Hausdorff convergence to a pointed proper metric space $((X, d), p)$.
Then $((X, d), p)$ is isometric to the based ultralimit $((X^p_\omega, d_\omega), p)$ for all non-principal ultrafilters ω.

The for us most interesting case in the above proposition is the one of an in the sense of pointed Gromov-Hausdorff convergence converging sequence $((X, \frac{1}{r_n}d), \mathfrak{o})$. In this case the limit space is isometric to *the*, now up to isometry unique, asymptotic cone of $(X, d, \mathfrak{o})$.

1.7 Stratified nilpotent Lie groups

The achievement about filling functions and higher divergence functions is mostly limited to spaces of non-positive sectional curvature. Further one requires by Theorem 1.2 positive curvature for the chance to produce examples of super-Euclidean filling functions. As spaces of strictly positive curvature are bounded, we have to examine spaces with curvature of different types. By results of Wolf [39] nilpotent Lie groups provide a rich class of such spaces.

A Lie group G with Lie algebra $\mathfrak{g}$ is called *nilpotent*, if its lower central series

$$G = G_1 \rhd G_2 \rhd G_3 \rhd ... \quad \text{with} \quad G_{j+1} = [G, G_j]$$

determines to the trivial group in finitely many steps. Here the bracket $[G, G_j]$ denotes the commutator group, i.e. the group generated by all commutators of elements of G and G_j. This condition is equivalent to the condition that the lower central series of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_1 \geq \mathfrak{g}_2 \geq \mathfrak{g}_3 \geq \dots \quad \text{with} \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j]$$

determines in finitely many steps to the null-space. Here the bracket $[\mathfrak{g}, \mathfrak{g}_j]$ denotes the linear subspace of $\mathfrak{g}$ generated by all brackets of element of $\mathfrak{g}$ and $\mathfrak{g}_j$. In both cases, group and algebra definition, the minimal number of steps in the lower central series needed to arrive at the trivial group or at the null-space, respectively, is the same, say d. It is called the *degree of nilpotency* of G and $\mathfrak{g}$. For brevity we call a nilpotent Lie group of nilpotency degree d in the following short *d-step nilpotent* Lie group.
In the examples for filling functions and higher divergence functions above, we examined the Euclidean space $\mathbb{R}^n$. This space with its additive group structure is the unique simply connected abelian Lie group of dimension n. As nilpotency is one natural way to generalise being abelian (this can be seen as 1-step nilpotent), nilpotent Lie groups lend themselves to be the next candidates.

Our main concern is for a special class of nilpotent Lie groups, the *stratified nilpotent Lie groups.* Their advantage is, that they additionally admit a family of self-similarities which are automorphisms. Further these self-similarities have nice properties concerning left-invariant (sub-)Riemannian metrics on the group.

Definition.
A stratified nilpotent Lie group *is a simply connected d-step nilpotent Lie group G with Lie algebra* $\mathfrak{g}$ *together with a grading*

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_d$$

with $[V_1, V_j] = V_{1+j}$ *where* $V_m = 0$ *if* $m > d$*.*

For example, every simply connected 2-step nilpotent Lie group G is such a stratified nilpotent Lie group with grading $\mathfrak{g} = V_1 \oplus [\mathfrak{g}, \mathfrak{g}]$, where V_1 is isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Recall that on a Lie group G any two left-invariant Riemannian metrics are equivalent. This means, if g and g' are left-invariant Riemannian metrics on G, then there is a constants $L > 0$, such that

$$\frac{1}{L} \cdot g \leq g' \leq L \cdot g \ .$$

From this it follows directly, that (G, g) and (G, g') are quasi-isometric. So for

our purpose to understand the asymptotic geometry of G, both metrics lead to the same results. Therefore it doesn't matter which left-invariant Riemannian metric we choose.
Most of time in which we will work explicitly with the Riemannian metric, we will choose, for technical reasons, a left-invariant Riemannian metric such that V_i is orthogonal to V_j whenever $i \neq j$. We call such a metric *fitting to the grading.*

Now we can introduce the above mentioned self-similarities on a stratified nilpotent Lie group.

Definition.
Let G be a stratified nilpotent Lie group with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$ of its Lie algebra. For every $t > 0$ we define the map

$$\hat{s}_t : \mathfrak{g} \to \mathfrak{g}\ ,\ \hat{s}_t(v_j) := t^j v_j \quad \textit{for}\ \ v_j \in V_j .$$

As $\hat{s}_t$ is an automorphism of the Lie algebra, there is an uniquely determined automorphism $s_t : G \to G$ of the Lie group G with $L(s_t) := d_e s_t = \hat{s}_t$. We call this automorphism s_t scaling automorphism*.*

As the elements of the first layer V_1 of the grading of the Lie algebra $\mathfrak{g}$ are scaled least, they play an outstanding role.

Definition.
The elements of the first layer V_1 are called horizontal*.*

Definition.

a) *Let M be a Riemannian manifold, G be a stratified nilpotent Lie group and $f : M \longrightarrow G$ be a Lipschitz map.*
Then f is horizontal*, if all the tangent vectors of its image lie in the subbundle*

$$\mathcal{H} := \bigcup_{g \in G} dL_g V_1$$

of the tangent bundle of G.

b) *Let X be a simplicial complex and $f : X \longrightarrow G$ be a Lipschitz map. Then f is* m-horizontal*, if f is horizontal on the interior of all j-simplices, $j \leq m$, of X.*

Lemma 1.10.
Let G be a stratified nilpotent Lie group.

a) *To be horizontal is a left-invariant property, i.e. if f is a horizontal map, then $L_g \circ f$ is a horizontal map for all $g \in G$ (see* [10, 2.2.1]*).*

b) *To be horizontal is invariant under scaling automorphisms, i.e. if f is a horizontal map, then $s_t \circ f$ is a horizontal map for all $t > 0$ (see* [10, 2.2.1]*).*

We now equip G with a left-invariant Riemannian metric g fitting to the grading of $\mathfrak{g}$ with associated length metric d_g. Then we get the following scaling estimates:

$$\mathrm{d}_g(s_t(p), s_t(q)) \begin{cases} \leq t \cdot \mathrm{d}_g(p,q) & \text{for } t \leq 1 \\ \geq t \cdot \mathrm{d}_g(p,q) & \text{for } t \geq 1 \end{cases} \qquad \forall p, q \in G.$$

In the above inequalities holds equality in both cases if and only if the distance of p and q is realised by a piecewise horizontal path. So we get in this special case:

$$\mathrm{d}_g(s_t(p), s_t(q)) = t \cdot \mathrm{d}_g(p,q) \quad \forall t > 0.$$

This leads to the following important property:

Lemma 1.11.
Let G be a stratified nilpotent Lie group with Riemannian metric g fitting to the grading of $\mathfrak{g}$ and let a be a horizontal Lipschitz m-chain in (G, d_g). Further let $t > 0$ and $s_t : G \to G$ be a scaling automorphism.
Then holds:

$$\mathrm{mass}(s_t(a)) = t^m \cdot \mathrm{mass}(a)\ .$$

Proof.
As the mass of a Lipschitz m-chain is defined as the sum of the m-dimensional volume of its summands, it suffices to prove

$$\mathrm{vol}_m(s_t \circ a) = t^m \cdot \mathrm{vol}_m(a)$$

for any horizontal Lipschitz m-chain $a = \alpha_1 : \Delta^m \to G$.
By the definition of the scaling automorphism, we have $\exp^{-1} \circ s_t = \hat{s}_t \circ \exp^{-1}$.

So we get:

$$\begin{aligned}
\mathrm{vol}_m(s_t \circ a) &= \int_{\Delta^m} J_{s_t \circ a}(x)\ d\lambda \\
&= \int_{\Delta^m} \det D\big((\exp^{-1} \circ L_{(s_t \circ a(x))^{-1}}) \circ (s_t \circ a)\big)(x)\ d\lambda \\
&\overset{(1)}{=} \int_{\Delta^m} \det D\big(\exp^{-1} \circ s_t \circ L_{a(x)^{-1}} \circ a\big)(x)\ d\lambda
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Delta^m} \det D\big(\hat{s}_t \circ \exp^{-1} \circ L_{a(x)^{-1}} \circ a\big)(x)\ d\lambda \\
&\overset{(2)}{=} \int_{\Delta^m} \det\Big(\begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} \cdot D(\exp^{-1} \circ L_{a(x)^{-1}} \circ a)(x)\Big)\ d\lambda \\
&= \int_{\Delta^m} t^m \cdot \det D((\exp^{-1} \circ L_{a(x)^{-1}}) \circ a)(x)\ d\lambda \\
&= t^m \cdot \int_{\Delta^m} J_a(x)\ d\lambda \\
&= t^m \cdot \mathrm{vol}_m(a)
\end{aligned}$$

The equality (1) holds as s_t is a group automorphism. As a is a horizontal m-chain, we have that $D(\exp^{-1} \circ L_{a(x)^{-1}} \circ a)(x)$ always lies in a m-dimensional subspace of V_1 and so is linearly scaled by $\hat{s}_t$. Therefore equality (2) holds. $\square$

On a stratified nilpotent Lie group there is another interesting metric. It is called the *Carnot-Carathéodory metric.* It is the left-invariant sub-Riemannian metric d_c induced by $\mathfrak{h} = V_1$. This means it is the length metric defined by the length with respect to the Riemannian metric g of horizontal curves:

$$d_c(p,q) := \inf\{\mathrm{Length}(c) \mid c \text{ piecewise horizontal curve with } c(0) = p,\ c(1) = q\}$$

A stratified nilpotent Lie group equipped with its Carnot-Carathéodory metric is called a *Carnot group.*
For the Carnot-Carathéodory metric, the nicest possible scaling behaviour holds:

$$\mathrm{d}_c(s_t(p), s_t(q)) = t \cdot \mathrm{d}_c(p,q) \quad \forall p, q \in G.$$

The Carnot-Carathéodory metric is a length metric using the same length functional as the Riemannian distance. Further is the class of admissible curves a subset of all the piecewise smooth curves which are admissible for the Riemannian distance. So we get the following relation between the two metric spaces (G, d_c) and (G, d_g), which later will become important:

Lemma 1.12.
Let G be a stratified nilpotent Lie group with left-invariant Riemannian metric g and associated length metric d_g and induced Carnot-Carathéodory metric d_c. Then the identity map

$$\iota : (G, \mathrm{d}_c) \to (G, \mathrm{d}_g), \;\; x \mapsto x$$

is 1-Lipschitz, i.e.

$$\mathrm{d}_g(x,y) \le \mathrm{d}_c(x,y) \quad \textit{for all } x, y \in G.$$

For the geometry of stratified nilpotent Lie groups the first layer V_1 of the grading of the Lie algebra plays an important role. In the following definition we look at it more analytically, such that we can define certain possible properties of subspaces of V_1. Later the presence of these properties will be very useful.

Definition.
Let G be a stratified nilpotent Lie group of dimension n with Lie algebra $\mathfrak{g}$. Let $\mathcal{H}$ be the horizontal distribution induced by the first layer V_1. Further let $n_1 = \dim V_1$. Then this distribution can be described as the set of common zeros of a set of 1-forms $\{\eta_1, ..., \eta_{n-n_1}\}$. These forms induce a (vector-valued) form

$$\Omega = (\omega_1, ..., \omega_{n-n_1}) : \Lambda^2 V_1 \to \mathfrak{g}/V_1 \cong \mathbb{R}^{n-n_1},$$

the curvature form, *where the ω_i denote the differentials $\omega_i := d\eta_i$ for $i = 1, ..., n - n_1$.*
Let $(\sigma_{ij}) \in \mathbb{R}^{(n-n_1)\times k}$. For a k-dimensional subspace $S \subset V_1$ consider the system of equations

$$\omega_i(\xi, X_j) = \sigma_{ij} \quad i = 1, ..., n - n_1 \textit{ and } j = 1, ..., k$$

where $\{X_j\}$ is a basis of S.
Then S is called Ω-regular, *if for any $(\sigma_{ij}) \in \mathbb{R}^{(n-n_1)\times k}$ the system of equations has a solution $\xi \in V_1$.*
Further a subspace $S \subset V_1$ is called Ω-isotropic, *if Ω restricted to $\Lambda^2 S$ is the zero form.*

Let $b_1, ..., b_n$ be a basis of the Lie algebra $\mathfrak{g}$, such that $b_1, ..., b_{n_1}$ span the first layer V_1. Then one can choose the 1-forms η_i as the dual forms of the remaining basis vectors $b_{n_1+1}, ..., b_n$:

$$\eta_i = b^*_{n_1+i} \qquad i \in \{1, ..., n - n_1\} \; .$$

Using the formula

$$(p+1)!(\mathrm{d}\,\gamma)(X_0,...,X_p) = \sum_{i<j}(-1)^{i+j+1}\gamma([X_i,X_j],X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_p)$$

for the differential of a left-invariant p-form γ, one gets

$$\omega_i(X_0,X_1) = \frac{1}{2}\cdot b^*_{n_1+i}([X_0,X_1])\ .$$

So we see, that an Ω-isotropic subspace S is nothing else than an abelian subalgebra, which is totally contained in the first layer V_1 of the Lie algebra.
Further we can interpret the property "Ω-regular", as something like *in general position.*
A more geometric interpretation of these subspaces will be given in Chapter 2.

We close this section mentioning two important properties of Carnot groups:

Proposition 1.13 (see [29]).
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric g with associated length metric d_g. Then the metric spaces $(G,\frac{1}{r}\,\mathrm{d}_g,e)$ converge in the pointed Gromov-Hausdorff sense for $r\to\infty$ to (G,d_c,e), where d_c denotes the Carnot-Carathéodory metric.

This means, that the group G equipped with its Carnot-Carathéodory metric, is the (up to isometry) unique asymptotic cone of (G,d_g).

It can be shown (see [27, Theorem 2]), that the Hausdorff-dimension D of (G,d_c) is given by

$$D = \sum_{j=1}^{d} j\cdot \dim V_j$$

where d is the degree nilpotency of G and the V_j are the summands of the grading of the Lie algebra $\mathfrak{g}$. We will see this number again, when we establish the equivalence classes of high-dimensional filling functions of stratified nilpotent Lie groups.

1.8 Heisenberg Groups and their generalisations

The most basic examples of non-abelian stratified nilpotent Lie groups are simply connected 2-step nilpotent Lie groups. And the most basic example of a simply connected 2-step nilpotent Lie group is the *classical Heisenberg Group.* One way to define it is as the subgroup

$$\boldsymbol{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

of $\mathrm{Gl}_3(\mathbb{R})$.
Then the natural way to generalise it is as the group of real upper triangular $(n \times n)$-matrices with ones on the diagonal.

Definition.
Let $n \in \mathbb{N}$ be greater or equal 3. The $\frac{n(n-1)}{2}$-dimensional Lie group

$$N_n = \{A = (a_{ij}) \in \mathrm{Gl}_n(\mathbb{R}) \mid a_{jj} = 1,\ a_{ij} = 0 \ \forall\ 1 \leq j < i \leq n\}$$

is called the group of unipotent upper triangular $(n \times n)$- matrices.

By construction one obtains the classical Heisenberg Group for $n = 3$:

$$N_3 = \boldsymbol{Heis}\ .$$

As a simply connected 2-step nilpotent Lie group, the Heisenberg group is automatically a stratified nilpotent Lie group. One would like to have this property for its generalisation, too. But as N_n is no longer 2-step nilpotent if $n \geq 4$, one has to construct the grading of the Lie algebra explicitly.

Lemma 1.14.
Let $n \in \mathbb{N}$, $n \geq 3$, and N_n be the group of unipotent upper triangular $(n \times n)$-matrices. Then N_n is simply connected, $(n-1)$-step nilpotent and for its Lie algebra $\mathfrak{n}_n$ holds:

(i) The Lie algebra is given by

$$\mathfrak{n}_n = \{B = (b_{ij}) \in \mathrm{Mat}(n, \mathbb{R}) \mid b_{jj} = 0,\ b_{ij} = 0 \ \forall\ 1 \leq j < i \leq n\}\ .$$

(ii) The decomposition

$$\mathfrak{n}_n = V_1 \oplus ... \oplus V_{n-1}$$

with

$$V_m = \{B = (b_{ij}) \in \mathrm{Mat}(n, \mathbb{R}) \mid b_{ij} = 0 \ \forall j \neq i + m\}\ , \quad m \in \{1, ..., n-1\}$$

is a grading in the sense of the definition of stratified nilpotent Lie groups, i.e. $[V_1, V_m] = V_{m+1}$.

Therefore N_n *is a stratified nilpotent Lie group.*

The proof of this lemma is a short computation.

Another way to generalise the classical Heisenberg Group $\boldsymbol{Heis}$ is inspired by its property to be 2-step nilpotent, to have an 1-dimensional centre and by the relations of the Lie algebra $\mathfrak{heis}$ of $\boldsymbol{Heis}$. This is given by

$$\mathfrak{heis} = \langle X, Y, Z \rangle_{\mathbb{R}}$$

with

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as vector space and with

$$[X, Y] = Z$$

as the only non-trivial bracket of the generators.

This generalises to the family of matrix Lie groups given as

$$\boldsymbol{H}^n = \left\{ \begin{pmatrix} 1 & x^T & z \\ 0 & I_n & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^n, \ z \in \mathbb{R} \right\} \subset \mathrm{Gl}_{n+2}(\mathbb{R})$$

where I_n denotes the $(n \times n)$ unit matrix.
One can check, that $\boldsymbol{H}^n$ is a $(2n+1)$-dimensional 2-step nilpotent Lie group and that its Lie algebra is given by

$$\mathfrak{h}^n = \langle X_1, ..., X_n, Y_1, ..., Y_n, Z \rangle_{\mathbb{R}}$$

with

$$X_j = \begin{pmatrix} 0 & e_j^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ Y_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad j, i \in \{1, ..., n\}$$

where e_k denotes k^{th} unit vector. The only non-trivial brackets of the generators are

$$[X_j, Y_j] = Z \ , \ j \in \{1, ..., n\} \ .$$

As it is 2-step nilpotent and simply connected it is a stratified nilpotent Lie group with the grading

$$\mathfrak{h}^n = \langle X_1, ..., X_n, Y_1, ..., Y_n \rangle_{\mathbb{R}} \oplus \langle Z \rangle_{\mathbb{R}} \ .$$

By using exponential coordinates, this leads to the following abstract way to construct these Lie groups:

Definition.
The complex Heisenberg Group $H^n_{\mathbb{C}}$ *of dimension* $2n+1$ *is as manifold*

$$H^n_{\mathbb{C}} := \mathbb{C}^n \times \operatorname{Im} \mathbb{C}$$

where $\mathbb{C}$ *denotes the complex numbers. The group law is given by*

$$(z, x)(w, y) := (z + w, x + y - \frac{1}{2} \sum_{i=1}^{n} \operatorname{Im}(z_i \overline{w_i})) \ .$$

This is a 2-step nilpotent Lie group with (real) Lie algebra $\mathfrak{h}^n_{\mathbb{C}} = V_1 \oplus V_2$ *where* $V_1 = \mathbb{C}^n, V_2 = \operatorname{Im} \mathbb{C} \cong \mathbb{R}$ *and with the bracket*

$$[(Z, X), (W, Y)] = (0, \sum_{i=1}^{n} \operatorname{Im}(Z_i \overline{W_i})) \ .$$

It can be seen as the unique simply connected Lie group with Lie algebra generated by
$B := \{j_1, ..., j_n, k_1, ..., k_n, K\}$
and with the only non-trivial brackets of the generators
$[k, j] = K$ *if both elements in the bracket have the same index.*

Remark.
The Lie algebras $\mathfrak{h}^n_{\mathbb{C}}$ *and* $\mathfrak{h}^n$ *are isomorphic.*

The isomorphism is given by sending $j_u \mapsto Y_u$ and $k_u \mapsto X_u$, $u \in \{1, ..., n\}$.

The advantage of this definition is, that we can get two new generalisations of this family of 2-step nilpotent Lie groups. We construct them by just replacing the division algebra $\mathbb{C}$ by its relatives: The quaternions and the octonions.
It is notable, that doing the same construction with $\mathbb{R}$, the real numbers, produces the abelian Lie group $(\mathbb{R}^n, +)$.

Definition.
The quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ *of dimension* $4n+3$ *is as manifold*

$$H^n_{\mathbb{H}} := \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$$

where $\mathbb{H}$ *denotes the Hamilton quaternions. The group law is given by*

$$(z,x)(w,y) := (z+w, x+y-\frac{1}{2}\sum_{i=1}^{n}\operatorname{Im}(z_i\overline{w_i}))\ .$$

This is a 2*-step nilpotent Lie group with (real) Lie algebra* $\mathfrak{h}^n_{\mathbb{H}} = V_1 \oplus V_2$ *where* $V_1 = \mathbb{H}^n, V_2 = \operatorname{Im}\mathbb{H}$ *and with the bracket*

$$[(Z,X),(W,Y)] = (0, \sum_{i=1}^{n}\operatorname{Im}(Z_i\overline{W_i}))\ .$$

It can be seen as the unique simply connected Lie group with Lie algebra generated by
$B := \{h_1,...,h_n,i_1,...,i_n,j_1,...,j_n,k_1,...,k_n,I,J,K\}$
and with the only non-trivial brackets of the generators

$[a,h] = A$ *for* $a \in \{i_1,...,i_n,j_1,...,j_n,k_1,...,k_n\}$, *(here* A *denotes the capital letter of the choice of* a*)*

and

$[k,j] = I, \quad [i,k] = J, \quad [j,i] = K$

if both elements in the bracket have the same index.

Definition.
The octonionic Heisenberg Group $H^n_{\mathbb{O}}$ *of dimension* $8n+7$ *is as manifold*

$$H^n_{\mathbb{O}} := \mathbb{O}^n \times \operatorname{Im}\mathbb{O}$$

where $\mathbb{O}$ *denotes the Cayley octonions. The group law is given by*

$$(z,x)(w,y) := (z+w, x+y-\frac{1}{2}\sum_{i=1}^{n}\operatorname{Im}(z_i\overline{w_i}))\ .$$

This is a 2*-step nilpotent Lie group with (real) Lie algebra* $\mathfrak{h}^n_{\mathbb{O}} = V_1 \oplus V_2$ *where*

$V_1 = \mathbb{O}^n, V_2 = \operatorname{Im}\mathbb{O}$ and with the bracket

$$[(Z,X),(W,Y)] = (0, \sum_{i=1}^{n} \operatorname{Im}(Z_i \overline{W_i})) \ .$$

It can be seen as the unique simply connected Lie group with Lie algebra generated by

$$B := \{d_1,...,d_n, e_1,...,e_n, f_1,...,f_n, g_1,...,g_n, h_1,...,h_n, i_1,...,i_n, j_1,...,j_n, k_1,...,k_n, \\ E, F, G, H, I, J, K\}$$

and with the only non-trivial brackets of the generators

$[a,d] = A$
for $a \in \{e_1,...,e_n, f_1,...,f_n, g_1,...,g_n, h_1,...,h_n, i_1,...,i_n, j_1,...,j_n, k_1,...,k_n\}$,
(here A denotes the capital letter of the choice of a)

and

$[i,f] = [k,h] = [j,g] = E,$
$[e,i] = [j,h] = [g,k] = F,$
$[k,f] = [e,j] = [h,i] = G,$
$[i,g] = [f,j] = [e,k] = H,$
$[g,h] = [f,e] = [k,j] = I,$
$[h,f] = [g,e] = [i,k] = J,$
$[f,g] = [e,h] = [j,i] = K$

if both elements in the bracket have the same index.

We constructed the complex Heisenberg Group $H^n_{\mathbb{C}}$ as 2-step nilpotent with 1-dimensional centre (and relations inspired by the classical Heisenberg Group ***Heis***). As for a 2-step nilpotent stratified nilpotent Lie group the centre of the Lie algebra is given by the second layer V_2 of the grading, we have $\operatorname{Im}\mathbb{H}$ respectively $\operatorname{Im}\mathbb{O}$ as centre of the quaternionic respectively octonionic Heisenberg Groups. These are not 1-dimensional (indeed they are 3- respectively 7-dimensional) and so we lost this property by our further generalisation. But we conserved the nilpotency of degree 2 and and as the groups are simply connected, we obtain two families of stratified nilpotent Lie groups.

To close this chapter, it should be mentioned, that the above defined generalised Heisenberg Groups are not only abstractly constructed Lie groups. They rather arise in geometry as natural generalisations of the complex Heisenberg

Groups. For seeing this one has to consider the complex hyperbolic spaces $\mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$, the quaternionic hyperbolic spaces $\mathrm{Sp}(n,1)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1))$ and the Cayley plane $\mathrm{F}_{4(-20)}\,/\,\mathrm{SO}(9)$ of real dimensions $2n$, $4n$ respectively 16. In these spaces the horospheres are biLipschitz equivalent to the Heisenberg Groups $H^{n-1}_{\mathbb{C}}$, $H^{n-1}_{\mathbb{H}}$ respectively $H^1_{\mathbb{O}}$. So they are of more general interest, for example for the geometry of non-cocompact lattices in the above mentioned hyperbolic spaces.
If one adds the real hyperbolic space $\mathrm{SO}(n,1)/\,\mathrm{SO}(n)$ of dimension n to the above list, the horospheres become biLipschitz equivalent to the *"real Heisenberg Group"* $(\mathbb{R}^{n-1},+)$. So, by the classification of rank 1 symmetric spaces (see for example [19]), the Heisenberg Groups can be seen as horospheres in rank 1 symmetric spaces of non-compact type.

1.9 The h-principle

For the proofs of our main theorems we will need *m-horizontal triangulations.* We will produce them by approximating ordinary triangulations by subdivided m-horizontal ones. To do this, we use a technique called *the h-principle.*

The h-principle (or homotopy principle) is a method to solve systems of partial differential equations (or inequalities) by homotopical techniques.
Such a system of partial differential equations is given by a set of equations imposed on a unknown smooth map f and its partial derivatives. For example consider a smooth map $f:\mathbb{R}^n\to\mathbb{R}^q$ and let the system of partial differential equations be of order r, this means that only partial derivatives up to order r are involved. Then a solution at a point $x\in\mathbb{R}^n$ can be viewed as a point

$$\big(x,f(x),f'(x),f^{(2)}(x),...,f^{(r)}(x)\big)\in\mathbb{R}^n\times\mathbb{R}^q\times\mathbb{R}^{qd(n,1)}\times...\times\mathbb{R}^{qd(n,r)}$$

where $d(n,s)$ denotes the number of all partial derivatives of order s. Denote the subset of such points by $\mathcal{R}$. Then every section with image in $\mathcal{R}$, i.e. a map

$$F:\mathbb{R}^n\to\mathcal{R}\subset\mathbb{R}^n\times\mathbb{R}^q\times\mathbb{R}^{qd(n,1)}\times...\times\mathbb{R}^{qd(n,r)}$$

with

$$pr_1\circ F=id_{\mathbb{R}^n}$$

where pr_1 denotes the projection to the first factor, consists of point-wise solutions of the system of partial differential equations. But in most cases its projection $pr(F):\mathbb{R}^n\to\mathbb{R}^q$ to the second factor will be no solution of the system of partial differential equations, not even locally, i.e. there is no open neighbourhood U of x such that F has the shape

$$F(y) = (y, pr(F)(y), pr(F)'(y), ..., pr(F)^{(r)}(y))$$

for all $y \in U$.
The approach of the h-principle is to examine whether such a *formal solution* F is homotopic in $\mathcal{R}$ to a *genuine* solution of the system of partial differential equations. If one knows, that for a special type of system of partial differential equations every formal solution is homotopic to a genuine solution, then the existence of a formal solution guarantees the existence of a genuine solution.

We want to give a short introduction to the vocabulary and the basic ideas of the h-principle. For a more detailed introduction see [12].

We start with some notations, which are used in the relevant literature (compare [12], [15]).

Definition.
Let $f : \mathbb{R}^n \to \mathbb{R}^q$ be a smooth map and $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. For $s \in \mathbb{N}$ denote by $f^{(s)}(x)$ the lexicographically ordered tuple of partial derivatives of order s, i.e. the entries of $f^{(s)}(x)$ are

$$D^\alpha f(x) \quad \textit{with} \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \quad \textit{and} \quad \sum_{i=1}^{n} \alpha_i = s \ ,$$

where $D^\alpha f(x)$ stands on the left hand side of $D^\beta f(x)$ iff there is a $t \in \{1, ..., n\}$ such that $\alpha_i = \beta_i$ for all $i < t$ and $\alpha_t < \beta_t$.

The r-jet of f at x *is the tuple*

$$J_f^r(x) := \left(f(x), f'(x), ..., f^{(r)}(x)\right) \ .$$

Let $d(n, s)$ denote the number of all partial derivatives of order s of a function $f : \mathbb{R}^n \to \mathbb{R}$. Then the r-jet $J_f^r(x)$ can be considered as a point in the space

$$\mathbb{R}^q \times \mathbb{R}^{qd(n,1)} \times \mathbb{R}^{qd(n,2)} \times ... \times \mathbb{R}^{qd(n,r)} = \mathbb{R}^{qN(r)}$$

with $N(r) = 1 + \sum_{j=1}^{r} d(n, j)$.

For $x \in \mathbb{R}^n$, the space $\{x\} \times \mathbb{R}^{qN(r)}$ contains all possible r-jets at the point x of maps $\mathbb{R}^n \to \mathbb{R}^q$.

Definition.
The space

$$J^r(\mathbb{R}^n, \mathbb{R}^q) := \mathbb{R}^n \times \mathbb{R}^{qN(r)}$$

is called the space of r-jets *of maps* $\mathbb{R}^n \to \mathbb{R}^q$.

Definition.
Let V *be a smooth manifold and* $p : X \to V$ *be a fibration. A* section *is a map* $F : V \to X$ *with* $p \circ F = id_V$.

Every map $f : \mathbb{R}^n \to \mathbb{R}^q$ can be viewed as a section of the fibration $p : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ by sending $x \mapsto (x, f(x))$. Analogously each section $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q,\ x \mapsto (x, y(x))$ of the fibration $p : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ can be seen as a function $\mathbb{R}^n \to \mathbb{R}^q$ by sending $x \mapsto y(x)$. From now on we use the term section, as we later treat a more general setting.

Definition.
Let $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ *be a section. The map*

$$J^r_f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)\ ,\ x \mapsto J^r_f(x)$$

is called the r-jet *of* f.

Let V be an n-dimensional smooth manifold. We want to define the space of r-jets of sections $V \to X$ for a $(n+q)$-dimensional smooth fibration $p : X \to V$. To do this we have to use the local trivialisations. So we have to consider r-jets up to change of coordinates.

For a subset $A \subset V$ we denote by $\mathcal{O}p(A)$ an arbitrary small (not specified) open neighbourhood of A. For a point $v \in V$ we denote by $\mathcal{O}p(v)$ the open neighbourhood $\mathcal{O}p(\{v\})$.

Definition.
Let $v \in V$ *and* $f : \mathcal{O}p(v) \to X$ *and* $g : \mathcal{O}p(v) \to X$ *be two local sections with* $f(v) = g(v) =: x$. *Then* f *and* g *are* r-tangent, *if there is neighbourhood* $U \subset X$ *of* x *and a local trivialisation* $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^q$ *such that*

$$J^r_{\varphi * f}(\varphi(v)) = J^r_{\varphi * g}(\varphi(v))$$

where $\varphi * f = \varphi \circ f \circ p \circ (\varphi_{|\mathcal{O}p(x)})^{-1}$ *and* $\varphi * g = \varphi \circ g \circ p \circ (\varphi_{|\mathcal{O}p(x)})^{-1}$.

The r-tangency class of a section $f : \mathcal{O}p(v) \to X$ at a point $v \in V$ is called the r-jet of f at v *and is denoted by $j_f^r(v)$.*

As we now have defined r-jets at points, we are able to define the space of r-jets:

Definition.
The space of r-jets $X^{(r)}$ *is the space of all r-tangency classes of local sections $f : \mathcal{O}p(v) \to X$, $v \in V$.*
Let $f : V \to X$ be a section. The map

$$j_f^r : V \to X^{(r)},\ v \to j_f^r(v)$$

is called the r-jet *of f.*

The above definition induces the smooth fibration $p^r : X^{(r)} \to V$.

One has to distinguish between two types of sections $F : V \to X^{(r)}$ of the space of r-jets: sections induced by sections $f : V \to X$ and the rest.

Definition.
Let $p_0^r : X^{(r)} \to X^{(0)} = X$ be the projection to 0-tangency classes and let $F : V \to X^{(r)}$ be a section.

a) The section $\operatorname{bs} F := p_0^r \circ F : V \to X$ *is called the* base section *of F.*

b) The section F is holonomic, *if $F = j_{\operatorname{bs} F}^r$. The space of all holonomic sections is denoted by* $\operatorname{Hol} X^{(r)}$.

If one denotes the space of sections $V \to X$ by $\operatorname{Sec} X$ and the space of sections $V \to X^{(r)}$ by $\operatorname{Sec} X^{(r)}$, one can consider the *jet map*

$$j^r : \operatorname{Sec} X \to \operatorname{Sec} X^{(r)},\ f \to j_f^r\ .$$

Then the space of holonomic sections is exactly the image of the jet map:

$$\operatorname{Hol} X^{(r)} = j^r(\operatorname{Sec} X)\ .$$

In the introduction to this section we motivated the h-principle from the view point of systems of partial differential equations. But there is no need to define

the set $\mathcal{R}$ by such a system. In practice it's often easier to describe the set $\mathcal{R}$ than the corresponding system of equations.
So one defines a *partial differential relation* $\mathcal{R}$ directly as a subset of the jet space:

Definition.

a) A partial differential relation *is a subset* $\mathcal{R} \subset X^{(r)}$ *of the jet space.*

b) A section $F : V \to \mathcal{R}$ *is called* formal solution of $\mathcal{R}$. *We denote the set of formal solutions of* $\mathcal{R}$ *by* $\operatorname{Sec}\mathcal{R}$. *If a solution* $F \in \operatorname{Sec}\mathcal{R}$ *is holonomic it is called* genuine solution of $\mathcal{R}$. *For the set of genuine solutions we write* $\operatorname{Hol}\mathcal{R}$.

Sometimes we will also call the base section $\operatorname{bs} F : V \to X$ of a genuine solution F a genuine solution of $\mathcal{R}$, too.

In the following all families of maps are at least continuous.

Definition.
Let $p : X \to V$ *be a fibration and let* $\mathcal{R} \subset X^{(r)}$ *be a partial differential relation.*

a) $\mathcal{R}$ *satisfies the* h-principle, *if every formal solution of* $\mathcal{R}$ *is homotopic in* $\operatorname{Sec}\mathcal{R}$ *to a genuine solution.*

b) $\mathcal{R}$ *satisfies the* parametric h-principle *if each homotopy* $F^t : V \to \mathcal{R}$ *in* $\operatorname{Sec}\mathcal{R}$ *with* $F^0, F^1 \in \operatorname{Hol}\mathcal{R}$ *can be deformed in* $\operatorname{Sec}\mathcal{R}$ *to a homotopy in* $\operatorname{Hol}\mathcal{R}$ *while* F^0 *and* F^1 *are fixed.*

c) Let $A \subset V$ *be a subset.* $\mathcal{R}$ *satisfies the* local h-principle near A *if every formal solution* $F : \mathcal{O}p(A) \to \mathcal{R}$ *is homotopic to a genuine solution by a homotopy* $F^t : \mathcal{O}p(A) \to \mathcal{R}$.

d) Let $A \subset V$ *be a subset.* $\mathcal{R}$ *satisfies the* C^0-dense local h-principle near A *if* $\mathcal{R}$ *satisfies the local h-principle near* A *and for every formal solution* $F_0 : \mathcal{O}p(A) \to \mathcal{R}$ *and for every arbitrary small neighbourhood* $U \subset X$ *of* $\operatorname{bs} F_0$, *there is a homotopy* $F^t : \mathcal{O}p(A) \to \mathcal{R}$ *from* F_0 *to a genuine solution such that* $\operatorname{bs} F^t$ *stays in* U, *i.e.* $\operatorname{bs} F^t(\mathcal{O}p(A)) \subset U$ *for all* $t \in [0,1]$.

Definition.
Let $\mathcal{R} \subset X^{(r)}$ be a partial differential relation and let I^n be the unit cube in $\mathbb{R}^n$.

a) $\mathcal{R}$ *is* locally integrable *if for every map $h : I^n \to V$ and every family of sections $F_p : h(p) \to \mathcal{R},\ p \in I^n$, and every family of holonomic extension $\widetilde{F}_p : \mathcal{O}p(h(p)) \to \mathcal{R},\ p \in \mathcal{O}p(\partial I^n)$ there is a family of holonomic extension*

$$\bar{F}_p : \mathcal{O}p(h(p)) \to \mathcal{R},\ p \in I^n$$

such that $\widetilde{F}_p = \bar{F}_p \quad \forall p \in \mathcal{O}p(\partial I^n)$.

b) Let $\mathrm{Diff}_V(X^{(r)})$ be the group of fiber-preserving diffeomorphisms of $X^{(r)}$. Then there is the projection map $\pi : \mathrm{Diff}_V(X^{(r)}) \to \mathrm{Diff}(V)$, restricting diffeomorphisms of $X^{(r)}$ to V.
Then $\mathcal{R}$ is Diff(V)-invariant *if there is a homomorphism $j : \mathrm{Diff}(V) \to \mathrm{Diff}_V(X^{(r)})$ with $\pi \circ j = id$ and if for every $h \in \mathrm{Diff}(V)$ the action $s \mapsto h_* s, s \in X^{(r)}$ leaves $\mathcal{R}$ invariant, where h_* denotes the image of the diffeomorphism h under the homomorphism $j : \mathrm{Diff}(V) \to \mathrm{Diff}_V(X^{(r)})$.*

c) $\mathcal{R}$ *is* microflexible *if the following holds: Let $K^m = [-1,1]^m$ and denote for $m < n = \dim V$ by Θ_m the pair $(K^n, K^m \cup \partial K^n)$. Then for every $m < n$ and every sufficiently small open ball $U \subset V$ and any pair $(A, B) \subset U \times U$ diffeomorphic to Θ_m and every holonomic section $F^0 : \mathcal{O}p(A) \to \mathcal{R}$ and every holonomic homotopy $F^t : \mathcal{O}p(B) \to \mathcal{R},\ t \in [0,1]$ which is constant over $\mathcal{O}p(\partial B)$, there is a number $\sigma > 0$ such that there is a holonomic homotopy $\widetilde{F}^t : \mathcal{O}p(A) \to \mathcal{R},\ t \in [0, \sigma]$ which extends F^t and is constant over $\mathcal{O}p(\partial A)$.*

A very important example of a $\mathrm{Diff}(V)$-invariant partial differential relation is the following:

Definition.
Let W, V be smooth manifolds, $n \geq \dim V$, and let $S \subset Gr_n W$ be a subset of the n-planes in the tangent bundle TW. An immersion $f : V \to W$ is called S-directed *if df maps the tangent bundle TV into S.*
We denote the corresponding differential relation by $\mathcal{R}_S$.

Lemma 1.15 (see [12, Chapter 7])**.**
The differential relation $\mathcal{R}_S$ of S-directed immersions is $\mathrm{Diff}(V)$-invariant for every $S \subset Gr_n W$.

1.10 Holonomic approximation

Holonomic approximation is one way to prove, that the (local) h-principle is satisfied. To do this, one first constructs holonomic approximations of formal solutions and then shows that these are homotopic. The proofs of the statements in this section can be found in [12].

In the following let V be a Riemannian manifold. Further we assume that $X^{(r)}$ is equipped with an Euclidean structure near F, this means with a family of Euclidean metrics on the fibres in a neighbourhood of the image of the section F. We denote the corresponding (local) distance-function by d.

Definition.
Let V be a smooth manifold. A polyhedron $P \subset V$ *is a closed subset of V, which can be obtained as an union of simplices of a smooth triangulation of V.*

Definition.
Let $\delta > 0$. A diffeotopy $h^t : V \to V$, $t \in [0,1]$, is called δ-small, *if*

$$d_V(v, h^t(v)) < \delta$$

for all $v \in V$ and all $t \in [0,1]$.

Theorem 1.16 (Holonomic Approximation Theorem (compare [12, 13.4.1]))**.**
Let $\mathcal{R}$ be a locally integrable microflexible differential relation, $A \subset V$ be a polyhedron of positive codimension and let $F : \mathcal{O}p(A) \to \mathcal{R}$ be a section.
Then there exists for arbitrary $\delta, \varepsilon > 0$ a δ-small diffeotopy $h^t : V \to V$, $t \in [0,1]$, and a holonomic section $\widetilde{F} : \mathcal{O}p(h^1(A)) \to \mathcal{R}$ such that

$$\mathrm{d}(\widetilde{F}(v), F(v)) < \varepsilon \quad \forall v \in \mathcal{O}p(h^1(A)) \ .$$

Similar theorems holds for the relative and the parametric h-principle:

Theorem 1.17 (Relative Version of Theorem 1.16)**.**
Let $\mathcal{R}$ be a locally integrable microflexible differential relation, $A \subset V$ be a polyhedron of positive codimension and let $F : \mathcal{O}p(A) \to \mathcal{R}$ be a section. Further let $B \subset A$ be a subpolyhedron of A and $F : \mathcal{O}p(B) \to \mathcal{R}$ be already holonomic.
Then there exists for arbitrary $\delta, \varepsilon > 0$ a δ-small diffeotopy $h^t : V \to V$, $t \in [0,1]$, fixed on $\mathcal{O}p(B)$, and a holonomic section $\widetilde{F} : \mathcal{O}p(h^1(A)) \to \mathcal{R}$ coinciding on

$\mathcal{O}p(B)$ *with* F *such that*

$$\mathrm{d}(\widetilde{F}(v), F(v)) < \varepsilon \quad \forall v \in \mathcal{O}p(h^1(A)) \ .$$

Theorem 1.18 (Parametric Version of Theorem 1.16)**.**
Let $\mathcal{R}$ *be a locally integrable microflexible differential relation,* $A \subset V$ *be a polyhedron of positive codimension and let* $F_z : \mathcal{O}p(A) \to \mathcal{R}$ *be a (continuous) family of sections parametrized over* $[0,1]^m$ *with* $m \in \mathbb{N}$*, such that* F_z *is holonomic for* $z \in \mathcal{O}p(\partial[0,1]^m)$*.*
Then there exists for arbitrary $\delta, \varepsilon > 0$ *a family of* δ*-small diffeotopies* $h_z^t : V \to V$*,* $t \in [0,1]$*,* $z \in [0,1]^m$*, such that* $h_z^t = id_V$ *for* $z \in \mathcal{O}p(\partial[0,1]^m)$*. Further there is a (continuous) family of holonomic sections* $\widetilde{F}_z : \mathcal{O}p(h^1(A)) \to \mathcal{R}$ *coinciding for* $z \in \mathcal{O}p(\partial[0,1]^m)$ *with* F_z *such that*

$$\mathrm{d}(\widetilde{F}(v), F(v)) < \varepsilon \quad \forall v \in \mathcal{O}p(h^1(A)) \ .$$

Using the above theorems one can prove that (locally integrable) microflexible $\mathrm{Diff}(V)$-invariant differential relations satisfy several versions of the h-principle (see [12]):

Theorem 1.19 (Local h-principle (compare [12, 13.5.1]))**.**
Every locally integrable microflexible $\mathrm{Diff}(V)$*-invariant differential relation satisfies the* C^0*-dense local h-principle near any polyhedron* $A \subset V$ *of positive codimension.*

Theorem 1.20 (h-principle for open manifolds (compare [16, p.79], [12, 13.5.2]))**.**
Every microflexible $\mathrm{Diff}(V)$*-invariant differential relation* $\mathcal{R}$ *satisfies the parametric h-principle.*

1.11 h-principle for sheaves

As mentioned at the beginning of Section 1.9, we will use the h-principle to construct m-horizontal triangulations. To do this, we need some results of Mikhael Gromov formulated in an alternative language of the h-principle, the *h-principle for sheaves.* This is closely related to h-principle treated above. For proofs and other details compare [15].

Definition (compare [15] or [33]).

a) A quasi-topology *on a set A is a subset $\mathcal{T}_{quasi}$ of the set*

$$\{f : T \to A \mid T \text{ topological space}\}$$

of all maps from topological spaces into A, such that the following holds:

(i) If $f : T_1 \to A$ is in $\mathcal{T}_{quasi}$ and if $\varphi : T_2 \to T_1$ is continuous, then the composition $f \circ \varphi$ is in $\mathcal{T}_{quasi}$.

(ii) Let $f : T \to A$ be a map. If there is for every point $p \in T$ a neighbourhood U_p of p such that $f_{|U_p}$ is in $\mathcal{T}_{quasi}$, then f is in $\mathcal{T}_{quasi}$.

(iii) Let P_1, P_2 be two closed subsets of T with $T = P_1 \cup P_2$ and let $f : T \to A$ be a map. If $f_{|P_1}, f_{|P_2}$ are in $\mathcal{T}_{quasi}$, then f is in $\mathcal{T}_{quasi}$.

b) Let $(A, \mathcal{T}^A_{quasi})$ and $(B, \mathcal{T}^B_{quasi})$ be two quasi-topological spaces, i.e. sets equipped with quasi-topologies. A map $F : A \to B$ is called continuous in the sense of quasi-topologies *if for all topological spaces T and all maps $\varphi : T \to A$ in $\mathcal{T}^A_{quasi}$ the composition $F \circ \varphi$ is in $\mathcal{T}^B_{quasi}$.*

Definition.
Let V be a smooth manifold and $U \subset V$.

a) A sheaf Φ over V is called continuous *if every set $\Phi(U)$ is equipped with a quasi-topology, such that for all inclusions $\iota : U \hookrightarrow U' \subset V$ the maps $\Phi(\iota)$ are continuous in the sense of quasi-topologies.*

b) Let Φ be a continuous sheaf over V and denote by Φ^V the continuous sheaf over $V \times V$ defined by

(i) The sections are by V parametrized families of sections of Φ.

(ii) For $U, W \subset V$ open subsets $\Phi^V(U \times W)$ is defined as $(\Phi(U))^W$.

(iii) A map $Q \to \Phi^V(U \times W)$ is continuous if and only if the corresponding map $Q \times W \to \Phi(U)$ is continuous.

Then we define Φ^* as the restriction of Φ^V to the diagonal $\Delta \subset V \times V$.

The sections of Φ^* are continuous families of germs $\varphi_v \in \Phi(v)$. On the other hand, every section of Φ can be seen as a unique constant family of sections of Φ with parameter space V, i.e. as a section of Φ^*. This makes Φ into a subsheaf of Φ^*. Denote this inclusion homomorphism of sheaves by D.

Definition.
Let A, A' be quasi-topological spaces and $\alpha : A \to A'$ be a continuous map. Then α is a microfibration *if for all compact polyhedra P and all continuous maps $\varphi : P \to A$ and all homotopies $\Phi' : P \times [0,1] \to A'$ with $\Phi'_{|P\times\{0\}} = \varphi'$ defined as $\varphi' = \alpha \circ \varphi$, there is a positive $\sigma \in (0,1]$ and a map $\Phi : P \times [0,\sigma] \to A$ such that $\Phi_{|P\times\{0\}} = \varphi$ and $\alpha \circ \Phi = \Phi'_{|P\times[0,\sigma]}$.*

$$\begin{tikzcd}
P \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[r, "\alpha"] & A' \\
P \times [0,\sigma] \arrow[d, hook] \arrow[ur, dashed, "\exists\Phi"'] & & \\
P \times [0,1] \arrow[uurr, "\Phi'"'] & &
\end{tikzcd}$$

Definition.
Let Φ be a continuous sheaf over V.

*a) Φ satisfies the h-*principle *(for sheaves), if for any open subset $U \subset V$ every section $\varphi \in \Phi^*(U)$ can be homotoped to $\Phi(U) \subset \Phi^*(U)$.*

b) Φ satisfies the local h-principle *(for sheaves) near $A \subset V$, if $\Phi(\mathcal{O}p(A))$ satisfies the h-*principle *(for sheaves).*

c) Φ is microflexible *if for any pair $C' \subset C \subset V$ of compact subsets the restriction map $\Phi(C) \to \Phi(C')$ is a mircofibration.*

Remark.

*a) There is a close connection between differential relations and sheaves:
Let $\mathcal{R} \subset X^{(r)}$ be a differential relation for a fibration $p : X \to V$. Then for $k \in \mathbb{N}_{\geq r} \cup \{\infty\}$ the space of C^k-solutions of $\mathcal{R}$ forms a sheaf Φ over V. The sheaves $\Phi(U)$, $U \subset V$, are the C^k-solutions of $\mathcal{R}$ over U.*

b) Let Φ be the sheaf of (generic) solutions of a differential relation $\mathcal{R} \in X^{(r)}$ and Ψ be the sheaf of continuous sections $V \to \mathcal{R}$ (i.e. of formal solutions). Then there is a homomorphism $J : \Phi^* \to \Psi$ such that $(J \circ D)(f) = j_f^r : U \to \mathcal{R}$ for all open subsets $U \subset V$ and all $f \in \Phi(U)$. If this J is a weak homotopy equivalence, the h-principle for the sheaf Φ implies the h-principle for the relation $\mathcal{R}$.
Looking at the definition of the local h-principles, one can see immediately that this relationship stays true for the local case.

c) Let again Φ be the sheaf of solutions of a differential relation $\mathcal{R} \in X^{(r)}$. Then one can prove that the microflexibility of Φ implies the microflexibility of $\mathcal{R}$ (just take $C = A$, $C' = B \cup \partial A$ and $P = \{\bullet\}$ in the respective definitions).

2 Main results

In this chapter we state our main results. We do this separately for statements about filling function, higher divergence functions and more concrete results for generalised Heisenberg Groups and applications to rank 1 symmetric spaces.
Summarized, our results show that stratified nilpotent Lie groups behave Euclideanly in low dimensions, this means similar to flat spaces, and sub-Euclideanly in high dimensions, this means similar to negatively curved spaces. For some examples, including the generalised Heisenberg Groups, we prove the appearance of the third possible kind, i.e. dimensions of super-Euclidean behaviour. This corresponds to the three types of sectional curvature, which appear in every point of a nilpotent Lie group (compare [39]).

Remark.
In stratified nilpotent Lie groups equipped with the sub-Riemannian metric there are no Lipschitz chains (of positive mass) in high dimensions (i.e. dimensions greater than the maximal dimension of abelian subalgebras of the first layer of the grading of the Lie algebra, compare [25]*). So it's more reasonable to examine filling functions and higher divergence functions for the Riemannian metric. Therefore every statement about this two invariants for stratified nilpotent Lie groups will be with respect to a left-invariant Riemannian metric.*
In the sub-Riemannian case the problem is rather how to define suitable analogues to filling functions and higher divergence functions without using Lipschitz chains (or integral currents, for which the same problem turns up).
Nevertheless, the Carnot-Carathéodory metric is an useful tool to examine the geometry of stratified nilpotent Lie groups. Whenever we use the Carnot-Carathéodory metric, we will highlight it by the notation (G, d_c).

2.1 Filling functions of stratified nilpotent Lie groups

First we describe our results for the filling functions. We will prove, that the existence of a $(k+1)$-dimensional, Ω-regular abelian subalgebra in the first layer of the Lie algebra leads to Euclidean filling functions up to dimension $k+1$. For technical reasons we additionally assume the existence of a scalable lattice in G. Remember the notation $s_t : G \to G\ ,\ t > 0$, for the scaling automorphisms of a stratified nilpotent Lie group (see Section 1.7).

Theorem 1.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G and V_1 be the first layer of the grading and let $k \in \mathbb{N}$. If there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$, then holds:

$$F_G^{j+1}(l) \sim l^{\frac{j+1}{j}} \quad \textit{for all } j \leq k.$$

Let d denote the degree of nilpotency of G. Then further holds $F_G^{k+2}(l) \preccurlyeq l^{\frac{k+1+d}{k+1}}$.

The upper bound on F_G^{k+2} is the (higher dimensional) analogue to Gromov's bound $\delta_\Gamma(n) \preccurlyeq n^{1+d}$ on the (1-dimensional) Dehn function for nilpotent groups in [28, 5.A$'$.5] (see also [30]) as there is the relation $\delta^{k+1} \preccurlyeq F^{k+2}$ (see Proposition 1.4).

Remark.
Gromov proved in [28] *the following dimension-formula for Ω-isotropic, Ω-regular subspaces $S \subset V_1$:*

$$\dim V_1 - \dim S \geq \dim S(\dim \mathfrak{g} - \dim V_1) \quad (*)$$

For the existence of Ω-isotropic, Ω-regular subspaces this means, that the horizontal distribution has to be large, i.e. $\dim V_1 >> \operatorname{codim}_{\mathfrak{g}} V_1$. *This formula comes from the fact, that the Ω-isotropy and Ω-regularity of S implies that the linear map*

$$\Omega_\bullet : V_1 \to \operatorname{Hom}(S, \mathfrak{g}/V_1), X \mapsto \Omega(X, \_)$$

is surjective and vanishes on S. The left hand side in the above inequality equals the dimension of V_1/S and the right hand side the dimension of $\operatorname{Hom}(S, \mathfrak{g}/V_1)$. *As S is in the kernel of $\Omega_\bullet$ we get by the surjectivity of $\Omega_\bullet$ the inequality as necessary condition.*
And on the other hand Gromov proved, that $(*)$ is sufficient for generic Ω, *i.e. for a class of forms, which form an open and everywhere dense subset.*

Our second result refers to "high dimensions". We assume the existence of a scalable lattice and a $(k+1)$-dimensional abelian Ω-regular subalgebra in the first layer of the Lie algebra.. Then we can prove sub-Euclidean filling functions in the k dimensions below the dimension of the group (if the group is not abelian). So the geometry of stratified nilpotent Lie groups is not Euclidean in high dimensions.

Theorem 2.
Let G be an n-dimensional stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Denote by $D = \sum_{i=1}^{d} i \cdot \dim V_i$ the Hausdorff-dimension of the asymptotic cone of G and let $k \in \mathbb{N}$. If there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$, then holds:

$$F_G^{n-j}(l) \sim l^{\frac{D-j}{D-j-1}} \quad \textit{for all } j \leq k-1.$$

Note that every lattice Γ in a nilpotent Lie group G is cocompact. So G and Γ are quasi-isometric. Therefore their asymptotic cones are isometric and have coinciding Hausdorff-dimensions.
Therefore the above result extends a classical result of Nicolas Varopoulos (see [17], [34]).

Theorem (compare [34])**.**
Let Γ be a lattice in an n-dimensional nilpotent Lie group G. Further denote by D the Hausdorff dimension of the asymptotic cone of Γ. Then holds:

$$\delta_\Gamma^{n-1}(l) \sim l^{\frac{D}{D-1}} .$$

Varopoulos' result corresponds to the case $j = 0$ in Theorem 2 (remember the different meaning of the exponents in the notation of the filling functions and the Dehn functions, discussed in Section 1.2).

Now we turn to the special case of simply connected 2-step nilpotent Lie groups. As seen in Section 1.7, all simply connected 2-step nilpotent Lie groups are stratified nilpotent Lie groups and so fit to our situation. We will see that in every such group there is a lattice Γ which satisfies the condition $s_2(\Gamma) \subset \Gamma$. So this doesn't remain a restriction to the Lie group and we can drop this requirement. This leads to the following version:

Theorem 3.
Let G be an n-dimensional simply connected 2-step nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus V_2$. Let $n_2 = \dim V_2$ and let $k \in \mathbb{N}$. If there exists a $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$, then holds:

(i) $F_G^{j+1}(l) \sim l^{\frac{j+1}{j}}$ *for all* $j \leq k$,

(ii) $F_G^{k+2}(l) \preccurlyeq l^{\frac{k+3}{k+1}}$,

(iii) $F_G^{n-j}(l) \sim l^{\frac{n+n_2-j}{n+n_2-j-1}}$ *for all* $j \leq k-1$.

Now it would be interesting to know, what happens in the dimension above the maximal dimension of an Ω-regular, Ω-isotropic subspace of V_1. The above theorems only give us a (super-Euclidean) upper bound on the filling function in this dimension. The following theorem, which generalises a theorem of Wenger [38, Theorem 5.2], states a super-Euclidean lower bound in the special case, that the maximal dimension of Ω-regular, Ω-isotropic subspaces of V_1 coincides with the maximal dimension of Ω-isotropic subspaces of V_1.

Theorem 4.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Let $k_0, k_1 \in \mathbb{N}$, such that (k_0+1) is the maximal dimension of an Ω-regular, Ω-isotropic subspace of V_1 and (k_1+1) is the maximal dimension of an Ω-isotropic subspace of V_1. Further let one of the following two conditions be satisfied:

a) There is an $k_0 \leq k \leq k_1$ such that there is an integral current $T \in \boldsymbol{I}_{k+1}^{cpt}(G, \mathrm{d}_c)$ with $\partial T = 0$ and $T \neq 0$ but no integral current $S \in \boldsymbol{I}_{k+2}^{cpt}(G, \mathrm{d}_c)$ with $\partial S = T$.

b) The two numbers k_0 and k_1 coincide: $k_0 = k_1 =: k$.

Then holds:

$$F_G^{k+2}(l) \succsim l^{\frac{k+2}{k+1}} .$$

We will see in the proof, that condition *b)* implies condition *a)*. Nevertheless we allow condition *b)* to stand in the theorem, as it is easier to check directly for some specific groups.

A stratified nilpotent Lie group which fulfils the conditions of Theorem 4 has at least one super-Euclidean filling function. This wouldn't be possible, if the group would be a space of non-positive curvature. Of course a stratified nilpotent Lie group can't be a space of strictly positive curvature, as it is diffeomorphic to some $\mathbb{R}^N$ and therefore not bounded. So the above theorem recovers for (some) stratified nilpotent Lie groups the result of Wolf [39] about the appearance of all different types of sectional curvature.
We will see, that the Heisenberg Groups $H_{\mathbb{C}}^n$, $H_{\mathbb{H}}^n$ and $H_{\mathbb{O}}^n$ are such groups.

2.2 Higher divergence functions of stratified nilpotent Lie groups

Our results for the filling functions of stratified nilpotent Lie groups lead directly to lower bounds for the higher divergence functions. This is, as we will see in the proof, mainly due to the left-invariance of the Riemannian metric. In the low dimensions we obtain the following theorem:

Theorem 5.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G and V_1 be the first layer of the grading and let $k \in \mathbb{N}$. If there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$, then holds:

$$\mathrm{Div}_G^j(r) \succcurlyeq r^{j+1} \quad \textit{for all } j \leq k.$$

In the high dimensions we obtain additional upper bounds for the higher divergence functions, which coincide with the lower bounds. The lower bounds come in the same way as in the low dimensions, while the upper bounds are possible to establish because the filling functions are sub-Euclidean (in contrast to the Euclidean filling functions in the low dimensions).
To establish the upper bounds on the higher divergence functions it is important to know the divergence dimension of the stratified nilpotent Lie group. Every simply connected nilpotent Lie group of dimension n is polynomial Lipschitz equivalent to $\mathbb{R}^n$ via the exponential map $\exp : \mathbb{R}^n \cong \mathfrak{g} \to G$. This means that the Lipschitz constants of $\exp$ and $\exp^{-1}$ on balls of radius R grow at most polynomial in R. Using this, one can construct in G (ρr-avoidant) fillings with polynomial bounded mass of (r-avoidant) cycles from Euclidean ones (compare [28, Chapter 5]) and vice versa. Therefore G and $\mathbb{R}^n$ have the same divergence dimension: $\mathrm{divdim}(G) = n - 2$.

Theorem 6.
Let G be an n-dimensional stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Denote by $D = \sum_{i=1}^{d} i \cdot \dim V_i$ the Hausdorff-dimension of the asymptotic cone of G and let $k \in \mathbb{N}$. If there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$, then holds:

$$\mathrm{Div}_G^{n-j}(r) \sim r^{\frac{(D-j)(n-j-1)}{D-j-1}} \quad \textit{for all } 2 \leq j \leq k.$$

As we can reduce the conditions of Theorem 1 and Theorem 2 in the case of simply connected 2-step nilpotent Lie groups (see Theorem 3), we can do the same for Theorem 5 and Theorem 6 and obtain:

Theorem 7.
Let G be an n-dimensional simply connected 2-step nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with the grading $\mathfrak{g} = V_1 \oplus V_2$, let $n_2 = \dim V_2$ and let $k \in \mathbb{N}$. If there exists a $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$, then holds:

(i) $\mathrm{Div}^j_G(r) \succcurlyeq r^{j+1}$ *for all* $j \leq k$,

(ii) $\mathrm{Div}^{n-j-1}_G(r) \sim r^{\frac{(n+n_2-j)(n-j-1)}{n+n_2-j-1}}$ *for all* $2 \leq j \leq k$.

2.3 Results for generalised Heisenberg Groups

In [42] and [43] Young computes upper bounds for filling functions of the complex Heisenberg Groups $H^n_{\mathbb{C}}$. His technique for the dimensions $\leq n+1$ is the foundation of our upper bounds in Theorem 1. The proof of the upper bounds in Theorem 2 is based on the technique he uses for the dimensions $2n+1$ down to $n+2$. Both techniques of Young need a lot of conditions, which are a lot of work to check explicitly for each specific group. Our new conditions are much easier to check for a single group and we will do this for the quaternionic and octonionic Heisenberg Groups. Young verified his conditions for the complex Heisenberg Groups and together with the coinciding lower bounds computed by Burillo [9] they get:

$$F^{j+1}_{H^n_{\mathbb{C}}}(l) \sim \begin{cases} l^{\frac{j+1}{j}} & \text{for } 1 \leq j < n, \\ l^{\frac{n+2}{n}} & , \\ l^{\frac{j+2}{j+1}} & \text{for } n < j \leq 2n. \end{cases}$$

So the filling functions of the complex Heisenberg Groups are known in all dimensions. This comes as Young's first technique works for the first n dimensions, providing Euclidean bounds, plus the super-Euclidean bound in the dimension above and Young's second technique works for the n dimensions below the dimension of the group. And the conditions for the first technique are contained in the conditions for the second one, so the symmetry (n from below and n from above) is no coincidence.
As the dimensions of the quaternionic and octonionic Heisenberg Groups $H^n_{\mathbb{H}}$

and $H^n_{\mathbb{O}}$ are bigger than $2n+1 = \dim H^n_{\mathbb{C}}$, there remains a gap in the dimensions between $(n+1)$ and $(3n+3)$ respectively $(7n+7)$. For these dimensions we can't compute the filling functions with our techniques. But we obtain the following bound on the filling functions of the quaternionic and octonionic Heisenberg Groups:

Corollary 2.1.
Let $H^n_{\mathbb{H}}$ be the $4n+3$ dimensional quaternionic Heisenberg Group.
Then holds:

i) $F^{j+1}_{H^n_{\mathbb{H}}}(l) \sim l^{\frac{j+1}{j}}$ *for* $j < n$,

ii) $F^{n+1}_{H^n_{\mathbb{H}}}(l) \preccurlyeq l^{\frac{n+2}{n}}$,

iii) $F^{m+1}_{H^n_{\mathbb{H}}}(l) \sim l^{\frac{m+4}{m+3}}$ *for* $3n+3 < m < 4n+3$.

Corollary 2.2.
Let $H^n_{\mathbb{O}}$ be the $8n+7$ dimensional octonionic Heisenberg Group.
Then holds:

i) $F^{j+1}_{H^n_{\mathbb{O}}}(l) \sim l^{\frac{j+1}{j}}$ *for* $j < n$,

ii) $F^{n+1}_{H^n_{\mathbb{O}}}(l) \preccurlyeq l^{\frac{n+2}{n}}$,

iii) $F^{m+1}_{H^n_{\mathbb{O}}}(l) \sim l^{\frac{m+8}{m+7}}$ *for* $7n+7 < m < 8n+7$.

The proof of Theorem 5 and Theorem 6 (respectively Theorem 7) on the higher divergence functions only uses the bounds on the filling functions. So these theorems remain true if one replaces the conditions of them by the bounds on the filling functions established in Theorem 1 and Theorem 2 (respectively Theorem 3). Using the bounds for the complex Heisenberg Groups (computed in [42] and [43]) we get the following behaviour of the higher divergence functions:

Corollary 2.3.
Let $H^n_{\mathbb{C}}$ be the $2n+1$ dimensional complex Heisenberg Group.
Then holds:

i) $\mathrm{Div}^j_{H^n_{\mathbb{C}}}(r) \succcurlyeq r^{j+1}$ *for* $j < n$, *ii)* $\mathrm{Div}^{n}_{H^n_{\mathbb{C}}}(r) \succcurlyeq r^{n+2}$,

iii) $\mathrm{Div}^{m}_{H^n_{\mathbb{C}}}(r) \sim r^{\frac{(m+2)m}{m+1}}$ *for* $n+1 \leq m < 2n$.

And for the quaternionic and octonionic Heisenberg Groups we obtain:

Corollary 2.4.
Let $H^n_{\mathbb{H}}$ be the $4n+3$ dimensional quaternionic Heisenberg Group.
Then holds:

i) $\mathrm{Div}^{j}_{H^n_{\mathbb{H}}}(r) \succcurlyeq r^{j+1}$ *for* $j < n$,

ii) $\mathrm{Div}^{m}_{H^n_{\mathbb{H}}}(r) \sim r^{\frac{(m+4)m}{m+3}}$ *for* $3n+3 < m < 4n+2$.

Corollary 2.5.
Let $H^n_{\mathbb{O}}$ be the $8n+7$ dimensional octonionic Heisenberg Group.
Then holds:

i) $\mathrm{Div}^{j}_{H^n_{\mathbb{O}}}(r) \succcurlyeq r^{j+1}$ *for* $j < n$,

ii) $\mathrm{Div}^{m}_{H^n_{\mathbb{O}}}(r) \sim r^{\frac{(m+8)m}{m+7}}$ *for* $7n+7 < m < 8n+6$.

In the above Corollaries 2.1 and 2.2 concerning the filling functions of the quaternionic and the octonionic Heisenberg Groups we only stated upper bounds in dimension n. This is due to the fact, that the technique for the lower bounds, used by Burillo in [9] to compute the lower bounds in dimension n for the complex Heisenberg Groups, is very special. Indeed it can't be generalised to stratified nilpotent Lie groups, not even to the octonionic Heisenberg Groups as the example of $H^1_{\mathbb{O}}$ (discussed below) shows. But in the case of the quaternionic Heisenberg Groups we managed to compute the desired lower bound in dimension n:

Theorem 8.
Let $H^n_{\mathbb{H}}$ be the quaternionic Heisenberg Group of dimension $4n+3$.
Then holds:

$$F^{n+1}_{H^n_{\mathbb{H}}}(l) \sim l^{\frac{n+2}{n}} .$$

So the quaternionic Heisenberg Groups have at least one strictly super-Euclidean filling function.

By the discussion following Theorem 1, one can see, that n is the maximal possible dimension of an Ω-isotropic, Ω-regular subspace S of V_1 in the grading of the Lie algebra of the quaternionic respectively octonionic Heisenberg Group $H^n_{\mathbb{H}}$ respectively $H^n_{\mathbb{O}}$. This follows from the equalities

$$\dim V_1 - n = 4n - n = 3n = n(4n + 3 - 4n) = n(\dim \mathfrak{g} - \dim V_1)$$

for the quaternionic case, and

$$\dim V_1 - n = 8n - n = 7n = n(8n + 7 - 8n) = n(\dim \mathfrak{g} - \dim V_1)$$

for the octonionic case.
And as the left hand side is strictly decreasing and the right hand side is strictly increasing in the dimension of the Ω-regular, Ω-isotropic horizontal subspace, the necessary condition for the existence of such a subspace is not satisfied for any dimension greater than n.
One could think, that this is only a technical appearance, i.e. there could be the same behaviour of the filling functions in the dimensions above n. But Theorem 8 shows, that for the quaternionic Heisenberg Groups, this is not the case and we observe a change from Euclidean to super-Euclidean behaviour of the filling functions. The same holds in the case of the complex Heisenberg Groups (see [9] and [42]) and for the octonionic Heisenberg Groups (see Corollary 2.6). Unfortunately, in contrast to the complex and the quaternionic case, the technique of Burillo doesn't work in the octonionic case (compare the discussion in Section 2.4). But we will see, that for $H^n_{\mathbb{O}}$ the number n is the maximal dimension of Ω-isotropic subspaces of V_1, too. So Theorem 4 applies and we get at least a super-Euclidean lower bound:

Corollary 2.6.
Let $H^n_{\mathbb{O}}$ be the octonionic Heisenberg Group of dimension $8n + 7$.
Then holds:

$$F^{n+1}_{H^n_{\mathbb{O}}}(l) \succsim l^{\frac{n+1}{n}} \ .$$

The additional lower bound for the $(n + 1)$-dimensional filling function of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ stated in Theorem 8 and lower bound for the $(n + 1)$-dimensional filling function of the octonionic Heisenberg Group $H^n_{\mathbb{O}}$ stated in Corollary 2.6 induce lower bounds for the higher divergence function in dimension n. As in the case of the complex Heisenberg Group $H^n_{\mathbb{C}}$ we observe a change from Euclidean to super-Euclidean behaviour.

Corollary 2.7.
Let $H^n_{\mathbb{H}}$ be the $4n+3$ dimensional quaternionic Heisenberg Group and $H^n_{\mathbb{O}}$ be the $8n+7$ dimensional octonionic Heisenberg Group.
Then holds:

$$\mathrm{Div}^n_{H^n_{\mathbb{H}}}(r) \succcurlyeq r^{n+2}$$

and

$$\mathrm{Div}^n_{H^n_{\mathbb{O}}}(r) \succnsim r^{n+1} \ .$$

2.4 Application to lattices in rank 1 symmetric spaces

For $n \geq 2$ the Corollaries 2.1 and 2.2 show that the 2-dimensional filling function F^2 of $H^n_{\mathbb{H}}$ and $H^n_{\mathbb{O}}$ is of quadratic type in both cases (see also [42]). Furthermore, one can use the fact that there is a gap between linear and quadratic Dehn functions (see [5]). This means that a sub-quadratic Dehn function has to be linear. By the relation $\delta^1 \preccurlyeq F^2$ and the fact that the above groups are not hyperbolic (and therefore can't have linear Dehn funcitons), their Dehn functions δ^1 are quadratic, too. Christophe Pittet computed the Dehn function of $H^1_{\mathbb{H}}$ (which is cubic) in [31] and so the Dehn function of $H^n_{\mathbb{H}}$ is known for all $n \in \mathbb{N}$. Unfortunately there is an error in Pittet's computation of the Dehn function of $H^1_{\mathbb{O}}$ (mentioned in [23]). And annoyingly this error can't be repaired, because the used proposition in [31] needs a 2-form ω of the shape

$$\omega = \sum_{x_i \in B_1, Y_i \in B_2} \alpha_i (Y_i^* \wedge x_i^*) \neq 0$$

with differential

$$\mathrm{d}\,\omega = \sum_{x_i \in B_1, Y_i \in B_2} \alpha_i \ \mathrm{d}(Y_i^* \wedge x_i^*) = 0$$

where $B_1 = \{d, e, f, g, h, i, j, k\}$ and $B_2 = \{E, F, G, H, I, J, K\}$. The condition on the differential leads to a system of linear equations with no non-trivial solution. So there is no such 2-form and Pittet's technique doesn't work for $H^1_{\mathbb{O}}$. One could think, that Burillo's technique [9] could solve this problem, but this leads to the same requirement of a 2-form with the properties described above. This means the 2-dimensional filling function of $H^1_{\mathbb{O}}$ is still unknown, while the 2-dimensional filling function of $H^n_{\mathbb{O}}$ is known for all $n \geq 2$.

Nevertheless our results have an application to the (higher dimensional) Dehn functions of non-uniform lattices is the complex and quaternionic hyperbolic spaces:

Corollary 2.8.
Let $n \in \mathbb{N}_{\geq 3}$ and X be the complex hyperbolic space $\mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ *of dimension* $2n$ *or the quaternionic hyperbolic space* $\mathrm{Sp}(n,1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1))$ *of dimension* $4n$*. Further let Γ be a group acting properly discontinuously by isometries, such that the quotient space X/Γ is of finite volume, but not compact. Then holds:*

$$\delta_\Gamma^j(l) \sim l^{\frac{j+1}{j}} \qquad \textit{for } 1 \leq j < n-1.$$

Remark.
Let X be the complex hyperbolic plane $\mathrm{SU}(2,1)/\,\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ *of dimension* 4 *or the quaternionic hyperbolic plane* $\mathrm{Sp}(2,1)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ *of dimension* 8 *and Γ a group acting properly discontinuously on X by isometries, such that the quotient space has finite volume but is not compact. It is proved in* [31] *that:*

$$\delta_\Gamma(l) \sim \delta_\Gamma^1(l) \sim l^3 \ .$$

2.5 Geometrical Interpretation

All of our theorems have conditions concerning the existence of Ω-regular, Ω-isotropic subspaces in the first layer of the grading of Lie algebra. The proofs following in the next chapters will use this algebraic conditions in a more or less technical manner. But it is interesting what the geometrical meaning of these subspaces is. Furthermore, one can explain geometrically the change of the behaviour of the filling invariants at the maximal dimension of such subspaces.

A good grasp of the meaning of the maximal dimension of an Ω-regular, Ω-isotropic subspace $S \subset V_1$ one can get from the view point of differential geometry. More explicitly, one has to look at the sectional curvature. For the sectional curvature of a Lie group equipped with a left-invariant Riemannian metric, there is the following formula (see [26]):

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\{e_1, ..., e_n\}$ be an orthonormal basis of $\mathfrak{g}$. Define the numbers α_{uvw} by

$$[e_u, e_v] = \sum_{w=1}^{n} \alpha_{uvw} e_w \ .$$

Then holds for the sectional curvature K:

$$K(e_i,e_j)=\sum_{k=1}^{n}\Big(\frac{1}{2}\alpha_{ijk}(-\alpha_{ijk}+\alpha_{jki}+\alpha_{kij}) \\ -\frac{1}{4}(\alpha_{ijk}-\alpha_{jki}+\alpha_{kij})(\alpha_{ijk}+\alpha_{jki}-\alpha_{kij})-\alpha_{kii}\alpha_{kjj}\Big)$$

Now consider G as 2-step nilpotent with the grading

$$\mathfrak{g}=V_1\oplus[V_1,V_1]$$

of its Lie algebra. Further let $\{e_1,...,e_{n_1}\}$ be an orthonormal basis of V_1 and $\{e_{n_1+1},...,e_n\}$ an orthonormal basis of $V_2:=[V_1,V_1]$. As G is 2-step nilpotent, one gets

$$[e_u,e_v]=0 \quad \text{if} \quad u\geq n_1+1 \quad \text{or} \quad v\geq n_1+1$$

and $[e_u,e_v]\in V_2$ for all $v,u\in\{1,...,n\}$. Therefore $\alpha_{uvw}=0$, whenever $u\notin\{1,...,n_1\}$ or $v\notin\{1,...,n_1\}$ or $w\leq n_1$.

For the sectional curvature in the case $i,j\leq n_1$ follows:

$$\begin{aligned}K(e_i,e_j)&=\sum_{k=n_1+1}^{n}\Big(\frac{1}{2}\alpha_{ijk}(-\alpha_{ijk}+0+0)-\frac{1}{4}(\alpha_{ijk}-0+0)(\alpha_{ijk}+0-0)-0\cdot 0\Big)\\ &=\sum_{k=n_1+1}^{n}\Big(-\frac{1}{2}(\alpha_{ijk})^2-\frac{1}{4}(\alpha_{ijk})^2\Big)\\ &=-\frac{3}{4}\sum_{k=n_1+1}^{n}(\alpha_{ijk})^2\\ &=:K_{1,1}\end{aligned}$$

And in the case $i\leq n_1$ and $j\geq n_1+1$:

$$\begin{aligned}K(e_i,e_j)&=\sum_{k=1}^{n_1}\Big(0(-0+0+\alpha_{kij})-\frac{1}{4}(0-0+\alpha_{kij})(0+0-\alpha_{kij})-0\cdot 0\Big)\\ &=\sum_{k=1}^{n_1}\frac{1}{4}(\alpha_{kij})^2\\ &=\frac{1}{4}\sum_{k=1}^{n_1}(\alpha_{kij})^2\\ &=:K_{1,2}\end{aligned}$$

In the case, that both vectors are from the basis of V_2, the sectional curvature equals 0.

One can see, that $K_{1,1} \leq 0$ with equality if and only if $[e_i, e_j] = 0$; and $K_{1,2} \geq 0$ with equality if and only if for all $k \in \{1, ..., n_1\}$ holds: $\pi_{e_j}([e_k, e_i]) = 0$, where π_{e_j} denotes the projection on the subspace $\langle e_j \rangle$.

Let $S \subset V_1$ be an Ω-isotropic, Ω-regular subspace of maximal dimension, say of dimension m, and let the basis $\{e_1, ..., e_n\}$ be chosen in way, such that $S = \langle e_1, ..., e_m \rangle$. Then one gets:

1) $K(e_i, e_j) = 0$ if $i, j \leq m$.

2) For all e_j with $m + 1 \leq j \leq n_1$ there exists an $i \in \{1, ..., m\}$, such that $K(e_i, e_j) < 0$.

3) For all e_j with $j \geq n_1+1$ there exists an $i \in \{1, ..., m\}$, such that $K(e_i, e_j) > 0$.

The first property comes by the Ω-isotropy, the second by the maximality of the dimension and the third by the Ω-regularity.

This shows, that every plane in S has sectional curvature $= 0$. But whenever one extends S by another direction, one gets a plane with sectional curvature $\neq 0$.

So one can explain the Euclidean behaviour of the filling invariants up to the maximal dimension of Ω-regular, Ω-isotropic subspaces by the flatness of these subspaces. The super-Euclidean behaviour in the dimension above is related to the positive curvature, which occurs whenever one adds a direction not contained in the first layer of the grading.

Another way to see the necessity of the Ω-regularity is the following example:

The paper [9] of Burillo contains the interesting result:
For $n \geq 4$ the group N_n of unipotent upper triangular $(n \times n)$-matrices the 2-dimensional filling function fulfils

$$F^2_{N_n}(l) \succcurlyeq l^3 \not\sim l^{\frac{1+1}{1}}$$

which is a strictly super-Euclidean behaviour.
This is no contradiction to Theorem 1, as the first layer of the grading $\mathfrak{n}_n = V_1 \oplus ... \oplus V_{n-1}$ has dimension n and the dimension of $\mathfrak{n}_n$ is $\frac{n(n-1)}{2}$.

Therefore

$$\dim V_1 - m = n - m \geq m(\frac{n^2 - 3n}{2}) = m(\frac{n(n-1)}{2} - n) = m(\dim \mathfrak{n}_n - \dim V_1)$$

holds never true for $m \geq 2$. By the discussion after Theorem 1, there can't exist a 2-dimensional Ω-isotropic, Ω-regular subspace S of V_1 and therefore Theorem 1 doesn't apply.

On the other hand, there is a $\lfloor \frac{n}{2} \rfloor$-dimensional Ω-isotropic subspace of V_1, generated by the matrices $E_{2k-1,2k} = (e_{i,j})$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, with only non-zero entry $e_{2k-1,2k} = 1$.

This shows, as $\lfloor \frac{n}{2} \rfloor \geq 2$ for $n \geq 4$, that the condition of the Ω-regularity is of crucial importance for the Euclidean behaviour of the filling functions.

3 Strategy of the Proofs

In this chapter we give a short sketch of the plan of action to prove our results and state the theorems of Burillo and Young which we are going to use.

3.1 Filling functions

For the proofs of the bounds on the filling functions we will use the following theorems. The first of them, due to Burillo, will be crucial to establish lower bounds on the filling functions.

Theorem 3.1 (see [9, Prop. 1.2]).
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric and let $m \in \mathbb{N}$. If there exists a Lipschitz m-chain b and a closed G-invariant m-form γ in G and constants $C, r, s > 0$ such that

1) $\operatorname{mass}(s_t(\partial b)) \leq Ct^r$,

2) $\int_b \gamma > 0$,

3) $s_t^* \gamma = t^s \gamma$,

then holds $F_G^m(l) \succcurlyeq l^{\frac{s}{r}}$.

The following two theorems, both due to Young, are essential to establish upper bounds on the filling functions. For these theorems we need the notion of horizontal maps introduced in Section 1.7.

Theorem 3.2 (see [42, Thm. 3]).
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric, let (τ, f) be a triangulation of G and let $\phi : \tau \to G$ be a m-horizontal map in bounded distance to f. Further let (η, h) be a triangulation of $G \times [1, 2]$ which restricts on $G \times \{1\}$ to (τ, f) and on $G \times \{2\}$ to $(\tau, s_2 \circ f)$ and let $\psi : \eta \to G$ be an m-horizontal map which extends ϕ and $s_2 \circ \phi$ (i.e. $\psi^{(m)}_{|h^{-1}(G\times\{1\})} = \phi^{(m)}$ and $\psi^{(m)}_{|h^{-1}(G\times\{2\})} = s_2 \circ \phi^{(m)}$). Then holds:

$$F_G^{j+1}(l) \preccurlyeq l^{\frac{j+1}{j}} \quad \textit{for all } j \leq m-1.$$

Theorem 3.3 (see [43, Prop. 8]).
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric and let $\Gamma \subset G$ be a lattice with $s_2(\Gamma) \subset \Gamma$ and let (τ, f) be a Γ-adapted triangulation, i.e. f is Γ-equivariant. Further let $(\widetilde{\tau}, \widetilde{f})$ be a $s_2(\Gamma)$-adapted triangulation of $G \times [1,2]$, such that for $i \in \{1,2\}$ the restriction to $G \times \{i\}$ coincides with the triangulation $(\tau, s_i \circ f)$. Denote by D the Hausdorff-dimension of the asymptotic cone (G, d_c) of G. If there is a $s_2(\Gamma)$-equivariant, m-horizontal, piecewise smooth map $\psi : G \times [1,2] \cong \widetilde{\tau} \to G$ with $\psi(g,2) = s_2(\psi(s_{\frac{1}{2}}(g), 1))$, then holds:

$$F_G^{n-j}(l) \preccurlyeq l^{\frac{D-j}{D-j-1}} \quad \textit{for all } j \leq m-1.$$

To check that our conditions imply the conditions of these theorems, we will use the h-principle and microflexibility. Roughly speaking, the existence of the Ω-regular, Ω-isotropic subspace will give us small horizontal submanifolds, which we are able to agglutinate to the desired triangulation.

We will give a proof by contradiction to establish the super-Euclidean lower bound on the filling function in the dimension above the maximal dimension of Ω-regular, Ω-isotropic subspaces. More precisely, we will show that an Euclidean upper bound on the filling function of the Riemannian manifold (G, d) implies an Euclidean isoperimetric inequality for integral currents in the metric space (G, d_c). Then we will show, that our conditions exclude this possibility.

3.2 Higher Divergence Functions

The lower bounds on the higher divergence functions follow mainly by the homogeneity of Lie groups. The idea is to move the hard-to-fill cycle out of the r-ball around the base point. For the upper bounds in the high dimensions we will use that sub-Euclidean fillings stay near the filled boundary. For technical reasons we will have to use the terminology of integral currents, as these form the completion of the space of Lipschitz chains.
In particular, our proofs don't use the conditions like the existence of an Ω-regular, Ω-isotropic subspace directly, rather they need the deduced bounds on the filling functions.

3.3 Generalised Heisenberg Groups and the application

Most of our results for the generalised Heisenberg Groups $H^n_{\mathbb{K}}$ for $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, follow as direct corollaries of our general theorems for stratified nilpotent Lie groups. In fact we will show, that $H^n_{\mathbb{K}}$ has in all three cases an n-dimensional Ω-regular, Ω-isotropic subspace of the first layer V_1.

More work is to do to establish the bounds on the $(n+1)$-dimensional filling functions:

For the lower bound on the $(n+1)$-dimensional filling function of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ we will use the above stated theorem of Burillo. For this we have to construct the needed Lipschitz chain and the differential form explicitly.

The lower bound on the $(n+1)$-dimensional filling function of the octonionic Heisenberg Group $H^n_{\mathbb{O}}$ is again a corollary of our general Theorem 4 for stratified nilpotent Lie groups. To see this, we will show that the maximal dimension of Ω-regular, Ω-isotropic subspaces and the maximal dimension of Ω-isotropic subspaces coincide for $H^n_{\mathbb{O}}$.

The application to rank 1 symmetric spaces follows directly from the results for the generalised Heisenberg Groups by using the 'horosphere-trick' (see [22]).

4 Horizontal approximation

To apply Theorem 3.2 and Theorem 3.3 of Young, we need to approximate a triangulation (τ, f) of a stratified nilpotent Lie group G by a $(k+1)$-horizontal map $\psi : \tau \to G$. In this chapter we collect the techniques to do this.
We consider G equipped with the sub-Riemannian metric d_c. We further will use a lot the properties Ω-isotropic and Ω-regular, so remember the definition of the curvature form Ω (see Section 1.7).

4.1 Some definitions

Let W be a simplicial complex and let V be a smooth manifold. We call a map $f : W \to V$ smooth, immersion or horizontal if the respective property holds for f restricted to each single simplex of W.

Definition.
Let W' be a simplicial complex and let V be a smooth manifold. A map $f : W' \to V$ is a folded immersion *if W' has a locally finite covering by compact subcomplexes, $W' = \bigcup W'_i$, such that f is smooth on each simplex of W' and sends each W'_i homeomorphically to a smooth compact submanifold (with boundary) of V.*
If $V = G$ is a stratified nilpotent Lie group, we call a folded immersion f horizontal *(and/or* Ω-regular*) if $f(W'_i)$ is horizontal (and/or Ω-regular) for all i.*

Definition.
Let G be a stratified nilpotent Lie group and let T and T' be m-dimensional simplicial complexes. We say $f' : T' \to G$ approximates *$f : T \to G$, if for every neighbourhood $U \subset G \times T$ of the graph $\{(f(x), x) \mid x \in T\}$ of f and every neighbourhood $V \subset T \times T$ of the diagonal $\{(x, x) \mid x \in T\}$ there are proper homotopy equivalences $\varphi : T \to T'$ and $\varphi' : T' \to T$ such that:*

i) The graph of $\varphi \circ \varphi'$ is contained in V.

ii) The graph of $f' \circ \varphi$ is contained in U.

4.2 The folded approximation theorem

To construct approximations of maps into stratified nilpotent Lie groups by horizontal ones, we need the following Lemma:

Lemma 4.1 (local h-principle, [16, 4.2.A′], compare [11] and [15, 2.3.2])**.**
Let G be a stratified nilpotent Lie group with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$ of the Lie algebra. Further let T be an m-dimensional simplicial complex.
Then the sheaf of horizontal, Ω-regular, smooth immersions $f : T \to G$ is microflexible and satisfies the local h-principle. In particular, for every Ω-regular, Ω-isotropic m-dimensional subspace $S \subset V_1$ and every $g \in G$ there exists a germ of smooth integral submanifolds $W_i \subset G$ at g with $T_g(W_i) = dL_g S$.

This is a corollary of the Main Theorem of [15, 2.3.2] as the differential operator which sends smooth maps $f : T \to G$ to the induced forms $\{f^*(\eta_i)\}_i$ is infinitesimal invertible on Ω-regular horizontal immersions (compare [15, 2.3.1]).

Proposition 4.2 (Folded Approximation Theorem, [16, 4.4])**.**
Let G be a stratified nilpotent Lie group and let T be a m-dimensional simplicial complex.
Then a continuous map $f_0 : T \to G$ admits an approximation by folded horizontal Ω-regular immersions $f' : T' \to G$ if and only if there is a continuous map $T \ni x \mapsto S_x$, where S_x is the translate of an Ω-regular, Ω-isotropic m-dimensional subspace $S \subset V_1$.

Proof.
Let $f_0 : T \to G$ be continuous and let $f' : T' \to G$ be a folded horizontal Ω-regular immersion approximating f_0. Then, by definition, there is a homotopy equivalence $\varphi : T \to T'$. The pullback under φ of the folded tangent bundle over T' provides a continuous map $T \ni x \mapsto S_x \subset dL_{f_0(x)}V_1$. This proves the "only if" part.
Let's turn towards the "if" direction:
As the sheaf of horizontal Ω-regular immersions $f : T \to G$ are microflexible, Diff(V)-invariant and by Lemma 4.1 locally integrable, we can use Theorem 1.16 to prove the above Proposition for the $(m-1)$-skeleton $T^{(m-1)}$ of T:
$T^{(m-1)}$ is a subpolyhedron of codimension 1 and

$$F_0 : \mathcal{O}p(T^{(m-1)}) \to \mathcal{R},\ x \mapsto (f_0(x), S_x)$$

is a formal solution near $T^{(m-1)}$. With Theorem 1.19 there is a genuine solution $F : \mathcal{O}p(T^{(m-1)}) \to \mathcal{R}$ homotopic to F_0 and such that $f = \operatorname{bs} F$ is arbitrary close to f_0.
It remains to prove the existence of the approximation in the top-dimensional case under the assumption that f_0 is an horizontal Ω-regular immersion near the $(m-1)$-skeleton of T.
We can treat each simplex $\Delta^m \subset T$ separately. So let $f_0 : \Delta^m \to G$ be an horizontal Ω-regular immersion near $\partial\Delta^m \subset T^{(m-1)}$.
We consider Δ^m as $\Delta^m = (\partial\Delta^m \times [0,1])/\sim$, with $\sim$ the equivalence relation defined by $(x,1) \sim (y,1)\ \forall x, y \in \partial\Delta^m$. Denote by $\Delta(t)$ the layer $\partial\Delta^m \times \{t\}$.

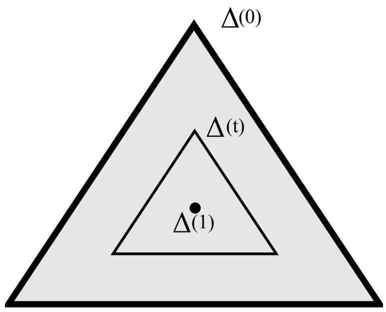

Figure 2: The layers of Δ^m for $m = 2$.

Each of this layers $\Delta(t)$ has dimension $\leq m-1$. We look at the maps

$$F_0^t : \mathcal{O}p(\Delta(t)) \to \mathcal{R}, x \to (f_0(x), S_x)$$

and by Theorem 1.19 we get horizontal Ω-regular immersions $f^t : \mathcal{O}p(\Delta(t)) \to G$ close to f_0 (for $t = 0$ we take $f^0 = f_0$). With Theorem 1.18, the parametric version of Theorem 1.16, we can choose the family $\{f^t\}$ to be continuous in t. Now let $\varepsilon > 0$ be sufficiently small, such that the 2ε-neighbourhood of the t-layer

$$N_{2\varepsilon}(t) := (\partial\Delta^m \times ((t-2\varepsilon, t+2\varepsilon) \cap [0,1]))/\sim$$

is contained in $\mathcal{O}p(\Delta(t))$ for all t. We define the (holonomic) homotopy

$$H^0 : \left(N_{\frac{\varepsilon}{3}}(0) \cup N_{\frac{\varepsilon}{3}}(\varepsilon)\right) \times [0,1] \to G\ ,$$

$$(x,t) \mapsto \begin{cases} f^0(x), & \text{if } x \in N_{\frac{\varepsilon}{3}}(0) \\ f^t(x), & \text{if } x \in N_{\frac{\varepsilon}{3}}(\varepsilon) \end{cases} .$$

The microflexibility gives a positive $t_1 \in (0,1]$ such that we can extend H^0 on $\mathcal{O}p(\Delta(0)) \times [0, t_1]$. Then we replace f^0 by $H^0_{t_1}$.

We define the (holonomic) homotopy

$$H^1 : \big(N_{\frac{\varepsilon}{3}}(t_1-\varepsilon) \cup N_{\frac{\varepsilon}{3}}(t_1+\varepsilon)\big) \times [0,1] \to G \ ,$$

$$(x,t) \mapsto \begin{cases} f^{t_1}(x), \ \text{if } x \in N_{\frac{\varepsilon}{3}}(t_1+\varepsilon) \\ f^{t_1+t}(x), \ \text{if } x \in N_{\frac{\varepsilon}{3}}(t_1-\varepsilon) \end{cases}$$

where $f^s = f^1$ for all $s \geq 1$.
The microflexibility gives a positive $t_2 \in (0,1]$ such that we can extend H^1 on $\mathcal{O}p(\Delta(t_1)) \times [0,t_2]$. We replace f^{t_1} by $H^1_{t_2}$.
We define the (holonomic) homotopy

$$H^2 : \big(N_{\frac{\varepsilon}{3}}(t_1+t_2-\varepsilon) \cup N_{\frac{\varepsilon}{3}}(t_1+t_2+\varepsilon)\big) \times [0,1] \to G \ ,$$

$$(x,t) \mapsto \begin{cases} f^{t_1+t_2}(x), \ \text{if } x \in N_{\frac{\varepsilon}{3}}(t_1+t_2-\varepsilon) \\ f^{t_1+t_2+t}(x), \ \text{if } x \in N_{\frac{\varepsilon}{3}}(t_1+t_2+\varepsilon) \end{cases} \quad .$$

The microflexibility gives a positive $t_3 \in (0,1]$ such that we can extend H^2 on $\mathcal{O}p(\Delta(t_1+t_2)) \times [0,t_3]$. We replace $f^{t_1+t_2}$ by $H^2_{t_3}$.
We continue this procedure until we reach f^1. As the number t_1 depends continuously on the layers, i.e. on $t \in [0,1]$, there is the minimum $\min\{t_i\} > 0$ and so we only need finitely many steps (less than $\lceil \frac{1}{\min\{t_i\}} \rceil$ many).
We define $T' := \bigcup \overline{N_{\frac{2}{3}\varepsilon}(t_i)}$ as the disjoint union of the closed $\frac{2}{3}\varepsilon$-neighbourhoods of the $\Delta(t_i)$. Further we make identifications corresponding to the intersections of their images under the $H^i_{t_{i+1}}$ and give it a simplicial structure such that the $\Delta(t_i)$ are contained in the $(m-1)$-skeleton. T' is obviously homotopy equivalent to T. Then we get the approximating map as

$$f' := \bigcup H^i_{t_{i+1}} : T' \to G$$

where $f'_{|\overline{N_\varepsilon(t_i)}} = H^i_{t_{i+1}}$. □

The above proof shows, that we need *folded* immersions only in the top-dimension (i.e. in dimension m, the dimension of the Ω-regular, Ω-isotropic subspace S). In lower dimensions we can consider T as a subcomplex of an m-dimensional simplicial complex $\widetilde{T}$ and the local h-principle yields an Ω-regular horizontal immersion

$$\widetilde{f} : \mathcal{O}p(T) \to G$$

approximating f_0 on T and the desired approximation is given by $f := \widetilde{f}_{|T}$.

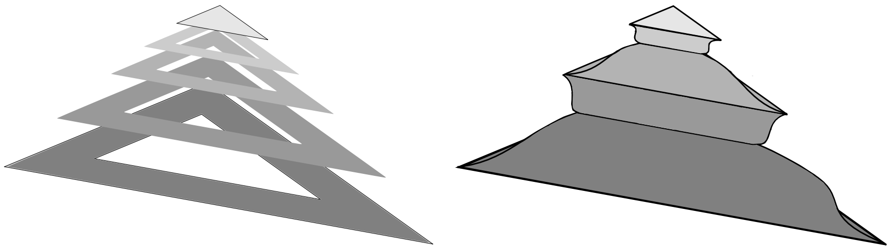

Figure 3: Schematical illustration of he images of the f^{t_i} (left) and the images of the $H^i_{t_{i+1}}$ (right) for $m = 2$.

Corollary 4.3 ([16, 4.4 Corollary]).
Let G be a stratified nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $m \in \mathbb{N}$. Further let S be a m-dimensional Ω-isotropic, Ω-regular horizontal subspace of $\mathfrak{g}$. Then every continuous map $f_0 : T \to G$ from an m-dimensional simplicial complex T into the stratified nilpotent Lie group G can be approximated by continuous, piecewise smooth, piecewise horizontal maps $f : T \to G$.

Proof.
We define $S_x = dL_{f_0(x)}S$ and get a horizontal approximation $f' : T' \to G$ by Proposition 4.2. As f' approximates f_0, there is a homotopy equivalence $\varphi : T \to T'$ and we can use $f = f' \circ \varphi$ as the desired approximation. $\square$

Remark.

a) *Later we want to use horizontal approximations in the Riemannian manifold (G, d_g). For the change from the Carnot-Carathéodory metric to the Riemannian metric one can use the fact, that both metrics d_c and d_g induce the same topology on G (see* [20, Proposition 2.26]*). What one really needs to transport the above results to G equipped with a left-invariant Riemannian metric is, that every continuous map $f : T \to (G, \mathrm{d}_c)$ is also continuous as map $f : T \to (G, \mathrm{d}_g)$. This holds in any case if the by d_c induced topology is finer than the topology of the Riemannian manifold. This is true, as the identity map $\iota : (G, \mathrm{d}_c) \to (G, \mathrm{d}_g)$ is 1-Lipschitz (compare Lemma 1.12) and therefore continuous. So every open set in (G, d_g) is also open as subset of (G, d_c).*
Further is the notion of being horizontal in both cases the same. So the above lemma yields a piecewise horizontal approximation of f_0 with respect to the Riemannian metric.

b) The above Proposition 4.2 and its Corollary 4.3 hold true for each microflexible differential relation $\mathcal{R}_S$ of S-directed immersions on a smooth manifold M. The points one has to change are to demand the map $x \mapsto S_x$ to be a continuous map into S and the resulting immersion will be S-directed instead of horizontal. Then the proof goes exactly the same way as above, one has just to replace the G by M, the translates of the Ω-regular, Ω-isotropic subspaces by S and horizontal by S-directed.

5 Proofs for Filling Functions

In this chapter we give the proofs for the bounds on the filling functions of stratified nilpotent Lie groups. We start with the proof of Theorem 1.

5.1 The proof of Theorem 1

Theorem 1 states the following behaviour of the filling functions of a stratified d-step nilpotent Lie group G equipped with a left-invariant Riemannian metric: If there is a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace of the first layer of the grading of the Lie algebra and a lattice $\Gamma \leq G$ with $s_2(\Gamma) \subset \Gamma$, then holds: The filling functions are Euclidean up to dimension $k+1$ and bounded from above by a rational function of degree $\frac{k+1+d}{k+1}$ in dimension $k+2$.

To prove Theorem 1 we split its statement into three parts: The upper bounds in the dimensions from dimension 2 up to dimension $k+1$, the lower bounds in these dimensions and the upper bound in dimension $k+2$. We prove the first part in Proposition 5.1, the second part in Proposition 5.2 and finally the third part in Proposition 5.3.

Proposition 5.1.
Let G be a stratified nilpotent Lie group with Lie algebra $\mathfrak{g}$ and V_1 as first layer of the grading of the Lie algebra. Further let $k \in \mathbb{N}$ and let d be the degree of nilpotency of G. If there exists a lattice Γ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$, then holds:

$$F_G^{j+1}(l) \preccurlyeq l^{\frac{j+1}{j}} \quad \textit{for all } j \leq k.$$

Proof.
First we look at the quotient $M = G/\Gamma$. As G is a simply connected nilpotent Lie group and as Γ is a lattice in G, we know by [32, Theorem 2.18] that Γ is torsion free. So M is a smooth manifold.
Let (τ_M, f_M) be a triangulation of M. This triangulation (τ_M, f_M) lifts to a Γ-invariant triangulation (τ, f) of G.
By [42, Lemma 4.5] we get an $s_2(\Gamma)$-invariant triangulation $(\eta, \tilde{f})$ of $G \times [1,2]$ which restricts on $G \times \{1\}$ to (τ, f) and on $G \times \{2\}$ to $(\tau, s_2 \circ f)$. Here, the $s_2(\Gamma)$-action on $G \times \{1,2\}$ is defined by $\varphi_\gamma(g,i) = (s_{\frac{1}{3-i}}(\gamma)g, i)$ for $(g,i) \in G \times \{1,2\}$ and $\gamma \in s_2(\Gamma)$.

Define the map $\psi_0 : \eta \to G$ by $\psi_0 = \mathrm{pr}_G \circ \tilde{f}$ where $\mathrm{pr}_G : G \times [1,2] \to G$ denotes the projection to the first factor. Let further $f^{(k+1)} : \tau^{(k+1)} \to G$ and $\psi_0^{(k+1)} : \eta^{(k+1)} \to G$ be the restrictions of f and ψ_0 to the $(k+1)$-skeletons of τ and η.
As there is a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$, the group G fulfils the conditions of Corollary 4.3 for $m = k+1$. Therefore (mentioning the remark following Corollary 4.3) we can approximate $f^{(k+1)}$ by a horizontal map

$$\phi^{(k+1)} : \tau^{(k+1)} \to G \ .$$

Further, as ψ_0 extends f and $s_2 \circ f$, i.e.

$$\psi_{0|\tilde{f}^{-1}(G\times\{1\})} = f \quad \text{and} \quad \psi_{0|\tilde{f}^{-1}(G\times\{2\})} = s_2 \circ f$$

we can, using again Corollary 4.3, approximate $\psi_0^{(k+1)}$ by a horizontal map

$$\psi^{(k+1)} : \eta \to G$$

which extends $\phi^{(k+1)}$ and $s_2 \circ \phi^{(k+1)}$, i.e.

$$\psi^{(k+1)}_{|\tilde{f}^{-1}(G\times\{1\})} = \phi^{(k+1)} \quad \text{and} \quad \psi^{(k+1)}_{|\tilde{f}^{-1}(G\times\{2\})} = s_2 \circ \phi^{(k+1)} \ .$$

Now we extend $\phi^{(k+1)}$ and $\psi^{(k+1)}$ to the whole simplicial complexes such that $\psi : \eta \to G$ extends $\phi : \tau \to G$ and $s_2 \circ \phi : \tau \to G$. We do this by filling successively the boundary of each r-simplex $\widetilde{\Delta^r}$ of η, $r \geq k+2$, by a Lipschitz map $\psi^{(r)} : \widetilde{\Delta^r} \to G$ with $\psi^{(r)}(\partial\widetilde{\Delta^r}) = \psi^{(r-1)}(\partial\widetilde{\Delta^r})$. This can be done as G is contractible.
So we have the triangulations (τ, f) and $(\eta, \tilde{f})$ and the $(k+1)$-horizontal maps ϕ and ψ in bounded distance to f and $\tilde{f}$ as required in Theorem 3.2.
So G fulfils the conditions for Young's filling theorem and we get the bound $F_G^{j+1}(l) \preccurlyeq l^{\frac{j+1}{j}}$ for all $j \leq k$. □

To prove the remaining lower bounds in Theorem 1 we use Theorem 3.1 of Burillo:

We have to construct a $(j+1)$-form γ and a closed $(j+1)$-chain b for the constants $r = j$ and $s = j+1$ for $1 \leq j \leq k$. To do this we use again the results of the previous chapter, in particular the local integrability, i.e. the existence of germs of horizontal submanifolds.

Proposition 5.2.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Let $\mathfrak{g}$ be the Lie algebra of G and V_1 be the first layer of the grading of the Lie algebra and let $k \in \mathbb{N}$. If there exists a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$ then holds:

$$F_G^{j+1}(l) \succcurlyeq l^{\frac{j+1}{j}} \quad \textit{for all } j \leq k.$$

Proof.
Let $1 \leq j \leq k$. We will show that there exist γ and b which fulfil the conditions of Burillo's filling theorem for $r = j$ and $s = j+1$ (Theorem 3.1).
Let $X_1, ..., X_{k+1}$ be a basis of S and define $S_j = \langle X_1, ..., X_{j+1} \rangle$. By Lemma 4.1 there is an integral submanifold M to $S_j \subset S$, i.e. $T_pM = dL_pS_j \quad \forall p \in M$, where dL_g denotes the differential of the left-multiplication by $g \in G$. Let $\varepsilon > 0$ and $b = B_\varepsilon^M(id)$ be the ε-ball in M. Further let $\gamma = X_1^* \wedge X_2^* \wedge ... \wedge X_{j+1}^*$, where X_i^* denotes the G-invariant dual-form to X_i defined by $X_u^*(X_v) = \delta_{uv}$. Then γ is a closed G-invariant $(j+1)$-form as all of the X_i lie in $S_j \subset S \subset V_1$ which has trivial intersection with $[\mathfrak{g}, \mathfrak{g}]$.
It remains to check the conditions of Theorem 3.1:

1) $\mathrm{mass}(s_t(\partial b)) = \mathrm{mass}(\partial b) \cdot t^j$ as $\partial b \subset M$ is a horizontal j-cycle.
As constant we can choose $C = \mathrm{mass}(\partial b)$.

2) $\int_b \gamma > 0$ as γ is a multiple of the volume form of M.

3) $s_t^* \gamma = t^{j+1} \gamma$ as all X_i lie in V_1 and γ is a $(j+1)$-form.

So the conditions of Theorem 3.1 are fulfilled for $r = j$ and $s = j+1$. Therefore holds for all $j \leq k$: $F_G^{j+1}(l) \succcurlyeq l^{\frac{j+1}{j}}$. □

It now remains to prove the upper bound in the dimension above the dimension of the Ω-regular Ω-isotropic subspace S:

Proposition 5.3.
Let G be a stratified d-step nilpotent Lie group equipped with a left-invariant Riemannian metric. Let $\mathfrak{g}$ be the Lie algebra of G and V_1 be the first layer of the grading of the Lie algebra and let $k \in \mathbb{N}$. If there exists a lattice Γ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace $S \subset V_1$, then holds:

$$F_G^{k+2}(l) \preccurlyeq l^{\frac{k+1+d}{k+1}} .$$

Proof.
With the proof of Proposition 5.1 we have the triangulations (τ, f), $(\eta, \tilde{f})$ and the $(k+1)$-horizontal maps ϕ, ψ in bounded distance to f anf $\tilde{f}$. Therefore the conditions of [42, Theorem 7] are fulfilled for a function $\sim t^{k+1+d}$ (compare also [42, Discussion before Theorem 8]).
Let $\Delta = \Delta^{k+2}$ be a $(k+2)$-simplex of τ. Then ϕ is already horizontal on $\partial\Delta$. Identify Δ with the cone $\mathrm{Cone}(\partial\Delta) = (\partial\Delta \times [0,1])/(\partial\Delta \times \{0\})$ over the boundary. Define the map

$$h_\Delta : \Delta \to G, \overline{(x,a)} \mapsto s_a(\phi(x)) \ .$$

Then h_Δ coincides with ϕ on $\partial\Delta$. Further holds

$$\mathrm{mass}((s_t \circ h_\Delta)(\Delta)) \leq t^{k+1+d} \cdot \mathrm{mass}(h_\Delta(\Delta))$$

as in every point the tangent space to h_Δ is the span of a $(k+1)$-dimensional horizontal subspace and another vector v. As G is d-step nilpotent, for v holds $\|s_t(v)\| \leq t^d\|v\|$. Replace ϕ by $\phi'_{|\Delta} = h_\Delta$ for every $(k+2)$-simplex Δ of τ and extend ϕ' to all of τ (compare with the proof of Proposition 5.1).
With the same techniques one can construct a sufficient map ψ'.
By [42, Theorem 7] follows: $F_G^{k+2}(l) \preccurlyeq l^{\frac{k+1+d}{k+1}}$. □

So we obtain the lower bounds on the filling functions by Proposition 5.2 and Proposition 5.3. Together with the upper bounds from Proposition 5.1 this proves Theorem 1.

5.2 The proof of Theorem 2

Theorem 2 states sub-Euclidean filling functions in the k highest dimensions, if there is a $(k+1)$-dimensional Ω-isotropic, Ω-regular subspace of the first layer of the grading of the Lie algebra and a lattice $\Gamma \leq G$ with $s_2(\Gamma) \subset \Gamma$.

Similarly as for the proof of Theorem 1, we split the statement of Theorem 2 into parts. In Proposition 5.4 we prove the upper bounds and in Proposition 5.5 we prove the lower bounds on the filling functions.

Proposition 5.4.
Let G be an n-dimensional stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Let $\mathfrak{g}$ be the Lie algebra of G with grading

$\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Further let $D = \sum_{i=1}^{d} i \cdot \dim V_i$ be the Hausdorff-dimension of the asymptotic cone of G and let $k \in \mathbb{N}$. If there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$ and a $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$, then holds:

$$F_G^{n-j}(l) \preccurlyeq l^{\frac{D-j}{D-j-1}} \quad \textit{for all } j \leq k-1.$$

Proof.
We want to use Young's filling theorem for high dimensions (Theorem 3.3). So we have to check that our conditions imply the conditions of this theorem. This means we have to construct an $s_2(\Gamma)$-adapted triangulation $(\tilde{\tau}, \tilde{f})$ of $G \times [1,2]$, such that the restrictions to $G \times \{i\}$, $i = 1,2$ are transformed into each other by the scaling s_2. Further we need a piecewise smooth, $s_2(\Gamma)$-equivariant, k-horizontal map $\psi : \tilde{\tau} \to G$ with $\psi(x,2) = s_2(\psi(s_{\frac{1}{2}}(x),1))$. Here we used the notation (x,t) for points in $\tilde{\tau}$ as this simplicial complex is homeomorhic to $G \times [1,2]$. Further does $s_{\frac{1}{2}}$ denote the change from the triangulation of $G \times \{2\}$ to the triangulation of $G \times \{1\}$.

We look at the quotient $M = G/\Gamma$. As G is a simply connected nilpotent Lie group and as Γ is a lattice in G, we know by [32, Theorem 2.18] that Γ is torsion free. So M is a smooth manifold.
Let (τ_M, f_M) be a triangulation of M. Then (τ_M, f_M) lifts to a Γ-adapted triangulation (τ, f) of G.
Let $M_2 = G/s_2(\Gamma)$. This is again a smooth manifold. The Γ-adapted triangulation (τ, f) projects down to a triangulation (τ_{M_2}, f_{M_2}) of M_2.
Denote by $s_i(\tau)$ the triangulation $(\tau, s_i \circ f)$. Let $\varphi_\gamma : \tau \to \tau$, $\gamma \in \Gamma$, be the Γ-action on τ. Then we define the $s_2(\Gamma)$-action on $s_1(\tau)$ by $\varphi_\gamma^1 = \varphi_{s_{\frac{1}{2}}(\gamma)}$ and on $s_2(\tau)$ by $\varphi_\gamma^2 = \varphi_\gamma$ for $\gamma \in s_2(\Gamma)$. In respect to this actions, both triangulations are $s_2(\Gamma)$-adapted.
As in [42, Proposition 4.5] we can extend the projected triangulation to a triangulation $(\tilde{\tau}_{M_2}, \tilde{f}_{M_2})$ of $M_2 \times [1,2]$, such that this restricts to $s_i(\tau)/s_2(\Gamma)$ on $M_2 \times \{i\}$, $i = 1,2$. Then $(\tilde{\tau}_{M_2}, \tilde{f}_{M_2})$ lifts to a $s_2(\Gamma)$-adapted triangulation $(\tilde{\tau}, \tilde{f})$ of $G \times [1,2]$ with the required restrictions.

To construct the map ψ we use the h-principle on M_2. The translates of the $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$ descend to a continuous subbundle of the tangent bundle of M_2 (consisting of $(k+1)$-planes). As the property to be microflexible is local, it descends to M_2, too. So we can use Proposition 4.2 (remember the Remark at the end of Chapter 4).
As $\tilde{\tau}_{M_2}$ is homeomorphic to $M_2 \times [1,2]$, we can write each point of $\tilde{\tau}_{M_2}$ as (x,t), where the second entry is the image of the point in the second factor of $M_2 \times [1,2]$.

Denote by pr_{M_2} the projection to the first factor of $M_2 \times [1,2]$. Define the map

$$\psi_0 : \{(x,1) \in \tilde{\tau}_{M_2}\} \to M_2 \ , \ (x,1) \mapsto \mathrm{pr}_{M_2}\left(\tilde{f}_{M_2}(x,1)\right) = \mathrm{pr}_{M_2}\left(f(x)s_2(\Gamma)\right) \ .$$

We approximate ψ_0 on $\{(x,1) \in \tilde{\tau}_{M_2}\}$ by a $(k+1)$-horizontal immersion $\psi^1_{M_2}$ using Proposition 4.2. Then we lift $\psi^1_{M_2}$ to a $s_2(\Gamma)$-equivariant map $\psi' : \tau \to G$ and define

$$\psi^2_{M_2} : \{(x,2) \in \tilde{\tau}_{M_2}\} \to M_2 \ , \ (x,2) \mapsto \mathrm{pr}_{M_2}\left((s_2 \circ \psi')(s_{\frac{1}{2}}(x))s_2(\Gamma)\right)$$

where $s_{\frac{1}{2}}(x)$ denotes the change from the triangulation $s_2(\tau)$ to the triangulation $s_1(\tau)$.
Then we extend $\psi^1_{M_2}$ and $\psi^2_{M_2}$ to a Lipschitz map ψ'_{M_2} from $\tilde{\tau}_{M_2}$ to M_2 by defining $\psi'_{M_2|\tilde{\tau}^{(0)}_{M_2}} = \psi_{0|\tilde{\tau}^{(0)}_{M_2}}$ on the vertices and then successively filling the boundaries of the simplices.
To complete the construction we use (the relative version of) the Holonomic Approximation Theorem (Theorem 1.16) and approximate ψ'_{M_2} on all of $\tilde{\tau}_{M_2}$, fixed on $\{(x,i) \in \tilde{\tau}_{M_2} \mid i \in \{1,2\}\}$, by a k-horizontal, piecewise smooth map

$$\psi_{M_2} : \tilde{\tau}_{M_2} \to M_2$$

with $\psi_{M_2}(x,i) = \psi'_{M_2}(x,i)$ for all $(x,i) \in \tilde{\tau}_{M_2}$ with $i \in \{1,2\}$.
This map rises to a $s_2(\Gamma)$-equivariant, piecewise smooth, k-horizontal map $\psi : \tilde{\tau} \to G$ with $\psi(x,2) = s_2(\psi(s_{\frac{1}{2}}(x),1))$.

So the conditions of Theorem 3.3 are fulfilled and we get $F^{n-j}_G(l) \preccurlyeq l^{\frac{D-j}{D-j-1}}$ for all $j \le k-1$. □

It remains to prove the lower bounds. As in the proof of the low dimensions (Theorem 1), we will use Burillo's filling theorem (see Theorem 3.1).

Proposition 5.5.
Let G be an n-dimensional stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Further let $D = \sum_{i=1}^{d} i \cdot \dim V_i$ be the Hausdorff-dimension of the asymptotic cone of G and let $k \in \mathbb{N}$. If there exists a $(k+1)$-dimensional Ω-isotropic subspace $S \subset V_1$, then holds:

$$F^{n-j}_G(l) \succcurlyeq l^{\frac{D-j}{D-j-1}} \quad \text{for all } j \le k.$$

Proof.
As all left-invariant Riemannian metrics on G are biLipschitz equivalent, we can choose one of our liking. So let the left-invariant Riemannian metric on G fitting to the grading of $\mathfrak{g}$, i.e. $V_s \perp V_t$ for $s \neq t$.
Let $j \leq k$. We will use the filling theorem of Burillo (see Theorem 3.1) with $m = n - j$.
To do this, we consider the grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. We choose an orthonormal basis $B_1 = \{v_1^{(1)}, ..., v_{\dim V_1}^{(1)}\}$ of V_1, such that the vectors $v_1^{(1)}, ..., v_{k+1}^{(1)}$ span the $(k+1)$-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$. Then we choose on each summand V_i an orthonormal basis

$$B_i = \{v_1^{(i)}, ..., v_{\dim V_i}^{(i)}\}$$

which gives us an orthonormal basis $B = \bigcup_{i=1}^{d} B_i$ of the Lie algebra $\mathfrak{g}$.
As m-chain b we choose now the image under the exponential map of the unit cube in all coordinates of B except $v_1^{(1)}, ..., v_j^{(1)}$, i.e.

$$b = \exp(\{\sum_{i=1}^{d} \sum_{q=1}^{\dim V_i} \alpha_{i,q} \cdot v_q^{(i)} \mid 0 \leq \alpha_{i,q} \leq 1 \ \forall i \ \forall q, \text{ and } \alpha_{1,q} = 0 \text{ for } 1 \leq q \leq j\}) \ .$$

The vectors of the i^{th} layer V_i of the Lie algebra are scaled under the scaling automorphism $L(s_t) : \mathfrak{g} \to \mathfrak{g}$ in the way

$$L(s_t)(v^{(i)}) = t^i v^{(i)} \ .$$

So we have for the cube b

$$\text{mass}(s_t(b)) = t^{D-j} \, \text{mass}(b) \ .$$

The proof of this scaling behaviour follows the same lines as the proof of Lemma 1.11. The only difference is, that the tangent space of the image of the chain b always is a translate of $\mathfrak{g}/W$ for $W = \langle v_1^{(1)}, v_2^{(1)}, ..., v_j^{(1)} \rangle$. Therefore the differential of the sclaing automorphism $\hat{s}_t$ is the matrix

$$D\hat{s}_t = \begin{pmatrix} t & & & & & & & & \\ & \ddots & & & & & & & \\ & & t & & & & & & \\ & & & t^2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & t^2 & & & \\ & & & & & & t^3 & & \\ & & & & & & & \ddots & \\ & & & & & & & & t^d \end{pmatrix}$$

with $(\dim V_1 - j)$ many t's on the diagonal and $\dim V_i$ many t^i's on the diagonal for $2 \leq i \leq d$. So the determinant of this matrix is t^{D-j} and this implies the scaling behaviour.
Then the boundary ∂b of b consists of all unit cubes contained in b of dimension one smaller than b and with one additional coordinate set 1 or 0. This further by now constant coordinate has scaled under the scaling automorphism at least linearly, and so we get

$$\mathrm{mass}(s_t(\partial b)) \leq t^{D-j-1} \cdot \mathrm{mass}(\partial b) \ .$$

Next we need to construct the G-invariant, closed m-form γ. We do this by choosing γ as the volume form of b:

$$\gamma = (v_j^{(1)})^* \wedge ... \wedge (v_{\dim V_d}^{(d)})^* \ .$$

Here v^* denotes the dual form of $v \in \mathfrak{g}$. This form γ is by definition G-invariant and

$$\int_b \gamma = \mathrm{mass}(b) > 0$$

as γ is the volume form of b.
For the scaling behaviour of γ we get

$$s_t^* \gamma = t^{D-j} \gamma$$

with the same argument as above for the scaling behaviour of the cube b.

So it remains to show, that γ is closed. For that, we recall the formula for the differential of a G-invariant p-form ω:

$$(p+1)!(\mathrm{d}\,\omega)(X_0, ..., X_p) = \sum_{s<t} (-1)^{s+t+1} \omega([X_s, X_t], X_0, ..., \widehat{X}_s, ..., \widehat{X}_t, ..., X_p) \ .$$

The key essence of this formula is that it suffices to examine "pre-images" of basis vectors under the Lie bracket to compute the differential of γ. This follows as a dual form v^* only sees the projection to the subspace spanned by v. To be precise, the differential of v^* is

$$2 \cdot (\mathrm{d}v^*)(X,Y) \overset{(1)}{=} v^*([X,Y]) \overset{(2)}{=} \sum_{\substack{u,w \in B \\ [u,w]=v}} u^*(X) \cdot w^*(Y) = \sum_{\substack{u,w \in B \\ [u,w]=v}} (u^* \wedge w^*)(X,Y)$$

where (1) holds by the above formula and (2) is the evaluation by computing the "length" of the projection to the subspace $\langle v \rangle \subset \mathfrak{g}$.
For the left-invariant m-form γ this leads to

$$\mathrm{d}\gamma = \frac{1}{(m+1)!} \sum_{i=1}^{d} \sum_{q=1}^{\dim V_i} \sum_{\substack{v,w \in B \\ [v,w]=v_q^{(i)}}} (\text{-}1)^{i+q+1} v^* \wedge w^* \wedge (v_{j+1}^{(1)})^* \wedge \ldots \wedge \widehat{(v_q^{(i)})^*} \wedge \ldots \wedge (v_{\dim V_d}^{(d)})^*.$$

Now let $i \geq 2$ (the case $i = 1$ is trivial, as V_1 has zero intersection with the image of the Lie bracket). Let $v_q^{(i)} \in V_j$ be one of the above chosen basis vectors. For each pair

$$x = \sum_{s=1}^{d} \sum_{t=1}^{\dim(V_s)} \alpha_{st} v_t^{(s)} \ , \quad y = \sum_{s=1}^{d} \sum_{t=1}^{\dim(V_s)} \beta_{st} v_t^{(s)} \in \mathfrak{g}$$

with $[x,y] = v_q^{(i)}$ we get:

$$[x,y] = \sum_{s_1,s_2=1}^{d} \sum_{t_1,t_2=1}^{\dim(V_s)} \alpha_{s_1 t_1} \beta_{s_2 t_2} [v_{t_1}^{(s_1)}, v_{t_2}^{(s_2)}] \ .$$

By this and by the linearity of differential forms we can assume without loss of generality, that x and y are basis vectors in B.
We first look at the case $i = 2$. In this case we have $x, y \in B_1$ as $[V_1, V_u] = V_{u+1}$. The first $k+1$ vectors $v_1^{(1)}, \ldots, v_{k+1}^{(1)}$ span the Ω-isotropic subspace $S \subset V_1$ and therefore $[v_s^{(1)}, v_t^{(1)}] = 0$ for $1 \leq s, t \leq k+1$. So at least one of the vectors x and y has to be in $B_1 \setminus \{v_1^{(1)}, \ldots, v_{k+1}^{(1)}\}$. But all dual forms of basis vectors $v_\ell^{(1)}$ with $\ell > k+1 > j-1$ are part of γ (and are not the deleted $v_i^{(2)}$). Therefore all these summands in $\mathrm{d}\gamma$ are zero.
Now let $i \geq 3$. Then at least one of the vectors x and y has to be a basis vector in a layer V_ℓ with $2 \leq \ell \leq i-1$. But again all the dual forms of such basis vectors are part of γ and unequal to the deleted $v_q^{(i)}$. Therefore all these summands in $\mathrm{d}\gamma$ are zero, too.
This implies

$$\mathrm{d}\gamma = 0$$

and we can apply Burillo's theorem and gain

$$F_G^{n-j}(l) \succcurlyeq l^{\frac{D-j}{D-j-1}}$$

which is the desired bound. $\square$

5.3 The proof of Theorem 3

Theorem 3 reduces the conditions of Theorem 1 and Theorem 2 in the case of 2-step nilpotent Lie groups. It omits the condition of the existence of the lattice Γ, but doesn't weaken the statements about the filling functions. Therefore Theorem 3 follows from Theorem 1 and Theorem 2 if in every 2-step nilpotent Lie group there exists a lattice with the requested scaling property. So we only have to prove the following lemma:

Lemma 5.6.
Let G be a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then there exists a lattice $\Gamma \subset G$ with $s_2(\Gamma) \subset \Gamma$.

Proof.
Let $\mathfrak{g} = V_1 \oplus V_2$ be the grading of the Lie algebra. Then take a basis

$$B_1 = \{v_1^{(1)}, v_1^{(2)}, ..., v_1^{(\dim V_1)}\}$$

of the first layer. For V_2 we complete $\{\frac{1}{2}[a,b] \mid a, b \in B_1\}$ to a basis

$$B_2 = \{v_2^{(1)}, v_2^{(2)}, ..., v_2^{(\dim V_2)}\}$$

of V_2. This leads to a Basis $B = B_1 \cup B_2$ of the Lie algebra $\mathfrak{g}$.
Now let

$$\mathcal{Z} := \langle \{b \mid b \in B\} \rangle_{\mathbb{Z}} \subset \mathfrak{g}$$

be the $\mathbb{Z}$-span of this basis. Then $\mathcal{Z}$ is, by construction, closed under the Lie bracket $[\cdot,\cdot]$ and fulfils $L(s_2)(\mathcal{Z}) \subset \mathcal{Z}$ as $2t \in \mathbb{Z}$ for all $t \in \mathbb{Z}$.
Then define

$$\Gamma := \langle \exp(\mathcal{Z}) \rangle \ .$$

Then $\Gamma \leq G$ is a lattice :
The structural constants with respect to the basis B of $\mathfrak{g}$ are rational. The set $\mathcal{Z}$ is lattice of maximal rank in the $\mathbb{Q}$-span $\mathfrak{g}_{\mathbb{Q}}$ of B. Therefore the group generated

by $\exp(\mathcal{Z})$ is a lattice in G (see [32, Theorem 2.12]).

The Baker-Campbell-Hausdorff formula in the case of 2-step nilpotent Lie groups reduces to

$$\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x, y])$$

and so for $h = \exp(x), g = \exp(y) \in \Gamma$ with $x, y \in \mathcal{Z}$ holds:

$$hg = \exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x, y])$$

The product hg is in $\exp(\mathcal{Z})$ as $\frac{1}{2}[x, y] \in \mathcal{Z}$. (One can see this by writing $x = \sum_{b\in B} \alpha_b b$ and $y = \sum_{b\in B} \beta_b b$). Therefore holds $\Gamma = \exp(\mathcal{Z})$.

Every $g \in \Gamma$ can be written as

$$g = \exp(\sum_{b\in B_1} n_b b + \sum_{b\in B_2} m_b b)$$

and so

$$s_2(g) = \exp\big(L(s_2)(\sum_{b\in B_1} n_b b + \sum_{b\in B_2} m_b b)\big) = \exp(\sum_{b\in B_1} 2n_b b + \sum_{b\in B_2} 4m_b b) \in \Gamma\ .$$

Therefore Γ fulfils $s_2(\Gamma) \subset \Gamma$. $\square$

5.4 The proof of Theorem 4

Theorem 4 states the following for a stratified nilpotent Lie group G equipped with a left-invariant Riemannian metric: The filling function in dimension $k+2$ is super-Euclidean, if there is either a unfillable $(k+1)$-dimensional integral current in the Carnot group (G, d_c) or the maximal dimension of Ω-regular, Ω-isotropic subspaces is $k + 1$ and coincides with the maximal dimension of Ω-isotropic subspaces.

Recall that we denote the set of integral m-currents by $\boldsymbol{I}_m(G)$ and set of integral m-currents with compact support by $\boldsymbol{I}_m^{cpt}(G)$.

For the proof of Theorem 4 we need two propositions. The first is due to Wenger:

Proposition 5.7 ([38, Proposition 3.6]).
Let $G = (G, \mathrm{d})$ be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric and denote by $G_\infty = (G, \mathrm{d}_c)$ the same group equipped with its Carnot-Carathéodory metric. If G satisfies an Euclidean isoperimetric inequality for $\boldsymbol{I}_m^{cpt}(G)$, then G_∞ satisfies an Euclidean isoperimetric inequality for $\boldsymbol{I}_m^{cpt}(G_\infty)$.

The original proposition in [38] presumes an Euclidean isoperimetric inequality for $\boldsymbol{I}_m(G)$. But an examination of the proof yields, that only an Euclidean isoperimetric inequality for $\boldsymbol{I}_m^{cpt}(G)$ is needed.

Further we need the following proposition:

Proposition 5.8.
Let G be a Lie group equipped with a left-invariant Riemannian metric. Let $m \in \mathbb{N}$ and $\delta \geq 1$. If $F_G^{m+1} \preccurlyeq l^\delta$, then G satisfies an isoperimetric inequality of rank δ for $\boldsymbol{I}_m^{cpt}(G)$.

Proof.
Let $T \in \boldsymbol{I}_m^{cpt}(G)$ with $T \neq 0$ and $\partial T = 0$. Note, that in particular ∂T is associated to a Lipschitz chain, i.e. $\partial T = a_\#$ for the Lipschitz chain $a = 0$. Now embed G isometrically in some $\mathbb{R}^N$ and look at T and ∂T from now on as integral currents of $\mathbb{R}^N$.
Let $\eta > 0$ be arbitrary small. With [13, Lemma 5.7] we get:
There is an integral current $S \in \boldsymbol{I}_{m+1}(\mathbb{R}^N)$, such that

i) $T - \partial S$ is a Lipschitz chain,

ii) $\boldsymbol{N}(S) = \boldsymbol{M}(S) + \boldsymbol{M}(\partial S) \leq \eta$ and

iii) $\mathrm{spt}(S) \subset U_\eta(\mathrm{spt}(T))$.

As G is an isometrically embedded Riemannian manifold, it is a local Lipschitz neighbourhood retract. This means, there is a neighbourhood U of G in $\mathbb{R}^N$ and a locally Lipschitz map

$$\varphi : U \to G \;\text{ with }\; \varphi(g) = g \quad \forall g \in G.$$

Now let $\eta > 0$ be sufficiently small, such that $\mathrm{spt}(S) \subset U$. Then the map $\bar{\varphi} := \varphi_{|\,\mathrm{spt}(S)} : \mathrm{spt}(S) \to G$ is locally Lipschitz.

For $x \in \operatorname{spt}(S)$ we denote by $U(x)$ the maximal neighbourhood of x such that $\bar{\varphi}$ is Lipschitz on $U(x)$.
As $T - \partial S$ is associated a Lipschitz chain, it has compact support. Therefore there is a finite cover

$$\bigcup_{i=1}^{d} U(x_i) \quad \text{with} \quad x_1, ..., x_d \in \operatorname{spt}(T - \partial S) \; .$$

So one can subdivide $T - \partial S$ in finitely many smaller simplices (i.e. simplicial chains $\alpha_j : \Delta \to \operatorname{spt}(T - \partial S)$), such that each $\alpha_j(\Delta)$ is completely contained in one of the $U(X_i)$.
As now $\bar{\varphi}$ is Lipschitz on each $\alpha_j(\Delta)$, we obtain the Lipschitz chain

$$\varphi^*(T - \partial S) = \sum_{j=1}^{l} \bar{\varphi} \circ \alpha_j$$

on G.

Further the support of T is compact. Therefore the closure of the 2η-neighbourhood of $\operatorname{spt}(T)$ is compact. So we can divide $\operatorname{spt}(S)$ in finitely many Borel sets A_v, such that each of these sets is completely contained in one of the neighbourhoods $U(x)$, $x \in \operatorname{spt}(S)$. One can do this as $\mathbb{R}^N$ is second-countable. Then, as $\bar{\varphi}$ is Lipschitz on each $A_v \subset U(x_v)$, we obtain the integral current

$$\varphi^*(S) = \sum_{v=1}^{n} \bar{\varphi}_{\#}(S \llcorner A_v)$$

on G.

Further holds

$$\operatorname{mass}(\varphi^*(T - \partial S)) \sim \boldsymbol{M}(T)$$

as $\bar{\varphi}$ is Lipschitz on each simplex and $\boldsymbol{M}(\partial S) \leq \eta$.

Now let $l := \boldsymbol{M}(T)$ and let b be a Lipschitz chain in G with $\partial b = \varphi^*(T - \partial S)$ and $\operatorname{mass}(b) \preccurlyeq l^{\delta}$. Such a chain exists due to the condition on the filling function. So we get

$$\begin{aligned} \partial(b_{\#} + \varphi^*(S)) &= \partial b_{\#} + \partial\varphi^*(S) = \partial b_{\#} + \varphi^*(\partial S) \\ &= \varphi^*(T - \partial S) + \varphi^*(\partial S) = T - \varphi^*(\partial S) + \varphi^*(\partial S) \\ &= T \end{aligned}$$

and

$$\boldsymbol{M}(b_{\#} + \varphi^*(S)) \preccurlyeq l^\delta$$

as $\bar{\varphi}$ is Lipschitz on each restriction and $\boldsymbol{M}(S) \leq \eta$. □

Now we are prepared for the proof of Theorem 4. We start with the easy case, that condition *a)* is fulfilled and prove that this implies the desired bound on the filling function.

Proposition 5.9.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$. Let $k_0, k_1 \in \mathbb{N}$, such that $(k_0 + 1)$ is the maximal dimension of an Ω-regular, Ω-isotropic subspace of V_1 and $(k_1 + 1)$ is the maximal dimension of an Ω-isotropic subspace of V_1. If there is an $k_0 \leq k \leq k_1$, such that there is an integral current $T \in \boldsymbol{I}_{k+1}^{cpt}(G, \mathrm{d}_c)$ with $\partial T = 0$ and $T \neq 0$, but no integral current $S \in \boldsymbol{I}_{k+2}^{cpt}(G, \mathrm{d}_c)$ with $\partial S = T$, then holds

$$F_G^{k+2}(l) \succsim l^{\frac{k+2}{k+1}} \ .$$

Proof.
So there is an $k_0 \leq k \leq k_1$ such that there is an integral current $T \in \boldsymbol{I}_{k+1}^{cpt}(G, \mathrm{d}_c)$ with $\partial T = 0$ and $T \neq 0$ but no integral current $S \in \boldsymbol{I}_{k+2}^{cpt}(G, \mathrm{d}_c)$ with $\partial S = T$. This means, that (G, d_c) doesn't satisfy an isoperimetric inequality of rank δ for $\boldsymbol{I}_{k+1}^{cpt}(G, \mathrm{d}_c)$ for any $\delta < \infty$, and in particular no Euclidean isoperimetric inequality for $\boldsymbol{I}_{k+1}^{cpt}(G, \mathrm{d}_c)$. By Proposition 5.7, G doesn't satisfy an Euclidean isoperimetric inequality for $\boldsymbol{I}_{k+1}(G)$ and by Proposition 5.8 we have

$$F_G^{k+2}(l) \succsim l^{\frac{k+2}{k+1}}$$

as desired. □

We finish the proof of Theorem 4 by showing, that if condition *b)* is fulfilled this implies that condition *a)* is satisfied.

Lemma 5.10.
Let G be a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric. Further let $\mathfrak{g}$ be the Lie algebra of G with grading $\mathfrak{g} = V_1 \oplus ... \oplus V_d$.

Let $k_0, k_1 \in \mathbb{N}$*, such that* $(k_0 + 1)$ *is the maximal dimension of an* Ω*-regular,* Ω*-isotropic subspace of* V_1 *and* $(k_1 + 1)$ *is the maximal dimension of an* Ω*-isotropic subspace of* V_1*.*
If the two numbers k_0 *and* k_1 *coincide, then there is for* $k := k_0 = k_1$ *an integral current* $T \in \boldsymbol{I}^{cpt}_{k+1}(G, \mathrm{d}_c)$ *with* $\partial T = 0$ *and* $T \neq 0$ *but no integral current* $S \in \boldsymbol{I}^{cpt}_{k+2}(G, \mathrm{d}_c)$ *with* $\partial S = T$*.*

Proof. Suppose $k + 1$ is the maximal dimension of Ω-regular, Ω-isotropic subspaces of V_1. Using the h-principle (as in the proof of Theorem 1), we can construct an $(k+1)$-horizontal triangulation of G. The boundary of any $(k+2)$-simplex $\Delta^{(k+2)}$ forms an horizontal Lipschitz $(k + 1)$-chain $a = \partial\Delta^{(k+2)} \neq 0$ with $\partial a = 0$. Viewed as current, this gives us the integral current $T := a_{\#} \in \boldsymbol{I}^{cpt}_{k+1}(G, \mathrm{d}_c)$. As $k + 1$ is the maximal dimension of Ω-isotropic subspaces of V_1 too, we have by [25, Theorem 1.1], that (G, d_c) is purely $\mathcal{H}^{k+2}$-unrectifiable and so there are no non-trivial integral $(k + 2)$-dimensional currents on (G, d_c). So condition *a)* holds. □

The combination of Proposition 5.9 and Lemma 5.10 proves Theorem 4.

By this last proof, we finished the verification of the results for the filling functions in the general setting of stratified nilpotent Lie groups. We will use these results to prove the results for the higher divergence functions and the applications to the Heisenberg Groups.

6 Proofs for Higher Divergence Functions

We proceed with the statements about the higher divergence functions.

The statements about the higher divergence functions of a stratified nilpotent Lie group equipped with a left-invariant Riemannian metric can be naturally divided into two parts: The lower bounds and the upper bounds.
Due to this subdivision we split the proof in the separated treatment of lower and upper bounds. To establish the lower bounds we will prove the more general Proposition 6.1, which deduces lower bounds on the higher divergence functions from lower bounds on the filling functions in the setting of arbitrary Lie groups. Our technique for the upper bounds on the higher divergence functions (Proposition 6.2) needs sub-Euclidean lower bounds on the filling functions and so only works in the top dimensions.

6.1 Lower bounds

We consider a Lie group G equipped with a left-invariant Riemannian metric. So if we transport a cycle by left-multiplication, there is no change of the mass of the cycle or of the mass bounded by the cycle. We use this fact to prove the following proposition, which provides the lower bounds for the higher divergence functions stated in the Theorems 5, 6 and 7.

Proposition 6.1.
Let G be a Lie group equipped with a left-invariant Riemannian metric and let $m \in \mathbb{N}$, such that $\operatorname{divdim}(G) \geq m$. *If the filling function F_G^{m+1} is bounded from below by a function $h : \mathbb{R}^+ \to \mathbb{R}^+$, then holds:*

$$\operatorname{Div}_G^m(r) \succcurlyeq h(r^m) \ .$$

Proof.
By assumption holds for the $(m+1)$-dimensional filling function

$$F_G^{m+1}(l) \succcurlyeq h(l) \ .$$

So there is a constant $C \geq 1$ such that for every $l > 0$ there exists a m-cycle a_l with mass $\operatorname{mass}(a_l) = l$ such that every $(m+1)$-chain b with boundary $\partial b = a$ has to fulfil

$$C \cdot \operatorname{mass}(b) + Cl + C \geq h(l) \ .$$

Let $\alpha_0 = 1$ and $\rho_0 < 1$, such that G is (ρ_0, m)-acyclic at infinity. We have to show, that there are constants $L, M \geq 1$, such that for all $\rho \leq \rho_0$ and all $\alpha \geq \alpha_0$ there is a constant $A \geq 1$ with:

$$A \cdot \operatorname{div}^m_{L\rho, M\alpha}(Ar + A) + O(r^m) \geq h(r^m) \ .$$

To do this, we use the above 'hard-to-fill'-cycle a_l with $l = r^m$.
As a_l is a priori not r-avoidant, we transport it out of the r-ball around $1 \in G$ by left-multiplication with some suitable group element $g \in G$ and obtain the r-avoidant m-cycle $g \bullet a_l$. For the mass we obtain

$$\operatorname{mass}(g \bullet a_l) = \operatorname{mass}(a_l) = l$$

as the metric is left-invariant.
The left-invariance of the metric also guarantees that the property 'hard-to-fill' is preserved under the left-multiplication, i.e. every $(m+1)$-chain b with boundary $\partial b = g \bullet a_l$ has

$$\operatorname{mass}(b) \succcurlyeq h(l) \ .$$

As the ρr-avoidance of the filling is an additional restriction to the $(m+1)$-chain, the above inequality for the mass of b holds true for ρr-avoidant $(m+1)$-chains b with boundary $\partial b = g \bullet a_l$. Here we need the assumption $\operatorname{divdim} G \geq m$ for the existence of such ρr-avoidant fillings.
Now we choose $L = M = 1$ and for $\rho \leq \rho_0$ and $\alpha \geq \alpha_0$ we set $A = \alpha \cdot C$.
Then we get

$$\begin{aligned} A \cdot \operatorname{div}^m_{L\rho, M\alpha}(Ar + A) + O(r^m) &\geq A \cdot \operatorname{div}^m_{\rho,\alpha}(Ar + A) + A \cdot r^m + A \\ &\overset{(1)}{\geq} C \cdot \operatorname{div}^m_{\rho,1}(r) + C \cdot r^m + C \\ &\overset{(2)}{\geq} C \cdot \operatorname{mass}(b_0) + C \cdot r^m + C \\ &\overset{(3)}{\geq} h(r^m) \end{aligned}$$

where (1) holds true, as $\alpha \geq 1$, $A \geq C \geq 1$ and as $\operatorname{div}^m_{\rho,\alpha}(r)$ is increasing in α and r. Further (2) holds true, as we have the r-avoidant 'hard-to-fill' m-cycle a_l with $\operatorname{mass}(a_l) = l = r^m \leq \alpha \cdot r^m$ and as $\operatorname{divdim}(G) \geq m$ there is some optimal ρr-avoidant filling b_0 of a_l. And (3) holds true, as b_0 is a filling of a_l and therefore has to fulfil this inequality by the lower bound on the filling function. $\square$

With this Proposition 6.1 and our results for the filling functions (Theorem 1, Theorem 2 and Theorem 3) we get the lower bounds in Theorem 5, Theorem 6 and Theorem 7.

6.2 Upper bounds

Theorem 6 states the following for an n-dimensional stratified nilpotent Lie group G: If $k+1$ is the maximal dimension of Ω-regular, Ω-isotropic subspaces in the first layer of the grading of the Lie algebra $\mathfrak{g}$ of G, then there are sub-Euclidean upper bounds of the higher divergence functions in the k highest dimensions. In particular, this upper bounds coincide with the lower bounds established in the previous Section 6.1.
To obtain the upper bounds for the higher divergence functions, we prove a more general theorem: We deduce sub-Euclidean upper bounds for higher divergence functions from sub-Euclidean upper bounds for filling functions.

Proposition 6.2.
Let M be a complete Riemannian manifold and $m \in \mathbb{N}$, $m \leq \operatorname{divdim}(M)$. If there is a $\delta < \frac{m+1}{m}$, such that the $(m+1)$-dimensional filling function $F_M^{m+1}(l)$ is bounded from above by l^δ, then holds:

$$\operatorname{Div}_M^m(r) \preccurlyeq r^{\delta m} \ .$$

Proof.
First choose a basepoint $x_0 \in M$ and let $\alpha \geq 1$ and $\rho_0 = \frac{1}{4}$. Let $r_0 > 0$ be sufficiently large. Further let $C > 0$, such that

$$F_M^{m+1}(\alpha r^m) \leq C \cdot (\alpha r^m)^\delta \qquad \forall r \geq r_0.$$

Let a be an r-avoidant Lipschitz m-cycle of $\operatorname{mass}(a) \leq \alpha r^m =: l$.
The condition on the $(m+1)$-dimensional filling function of M implies the existence of a Lipschitz $(m+1)$-chain b (not necessarily avoidant) with

i) $\partial b = a$ and

ii) $\operatorname{mass}(b) \leq C \cdot l^\delta$.

Now let $T = a_\#$ be the Lipschitz cycle a considered as integral current. Then there is an integral current $S \in \boldsymbol{I}_{m+1}(M)$ with

i) $\partial S = T$ and

ii) $\boldsymbol{M}(S) \leq C \cdot l^\delta$.

For example, $S = b_\#$ would be an appropriate choice.

So the proof of [36, Lemma 3.1] (see also [35, Lemma 3.4]) together with the computation in the proof of [36, Proposition 1.8] yields the following:

For $x \in M$ and $t > 0$ denote by $B(x,t)$ the closed ball of radius t around x.
For sufficiently large r and every $\varepsilon > 0$ there is an integral current $S_\varepsilon \in \boldsymbol{I}_{m+1}(M)$ with:

i) $\partial S_\varepsilon = T$,

ii) $\boldsymbol{M}(S_\varepsilon) \leq C \cdot l^\delta + \varepsilon$,

iii) $\operatorname{spt} S_\varepsilon \subset M \setminus B(x_0, \frac{1}{2}r)$.

Now we have an $\frac{1}{2}r$-avoidant integral current S_ε, that "fills" the Lipschitz cycle a. From this integral current we construct a $\rho_0 r$-avoidant Lipschitz $(m+1)$-chain of the desired mass that fills a as follows:

For $A \subset \mathbb{R}^N$ and $t > 0$ denote by $B(A,t) := \{y \in \mathbb{R}^N \mid \exists x \in A : \| x - y\|_2 \leq t\}$.
At first we use the Nash embedding theorem to embed M isometrically in some $\mathbb{R}^N$. Consider S_ε as an integral current in $\mathbb{R}^N$. Then [13, Lemma 5.7] provides for every $\eta > 0$ the existence of a Lipschitz $(m+1)$-chain $b'_{\varepsilon,\eta}$ such that:

i) $\partial b'_{\varepsilon,\eta} = a$,

ii) $\operatorname{mass}(b'_{\varepsilon,\eta}) \leq C \cdot l^\delta + \varepsilon + \eta$,

iii) $b'_{\varepsilon,\eta} \subset B(\operatorname{spt}(S_\varepsilon), \eta)$.

As $M \subset \mathbb{R}^N$ is a local Lipschitz neighbourhood retract, we can retract $b'_{\varepsilon,\eta}$ to a Lipschitz $(m+1)$-chain $b_{\varepsilon,\eta}$ on M (compare proof of Proposition 5.8) with

i) $\partial b_{\varepsilon,\eta} = a$,

ii) $\operatorname{mass}(b_{\varepsilon,\eta}) \leq L^{m+1} C \cdot l^\delta + L^{m+1}\varepsilon + L^{m+1}\eta$,

iii) $b_{\varepsilon,\eta} \subset B(\operatorname{spt}(S_\varepsilon), L\eta)$,

where L denotes the Lipschitz constant of the retraction.

So for η sufficiently small, i.e. such small that $\rho_0 r < r - (L\eta + \frac{1}{2}r) = \frac{1}{2}r - L\eta$, the Lipschitz chain $b_{\varepsilon,\eta}$ is an $\rho_0 r$-avoidant filling of the cycle a. Further holds for the mass of $b_{\varepsilon,\eta}$:

$$\begin{aligned}\operatorname{mass}(b_{\varepsilon,\eta}) &\leq L^{m+1}C\cdot l^{\delta} + L^{m+1}\varepsilon + L^{m+1}\eta = L^{m+1}C\cdot(\alpha r^m)^{\delta} + L^{m+1}\varepsilon + L^{m+1}\eta\\ &\preccurlyeq r^{m\delta}\end{aligned}$$

This proves the claim. □

Theorem 2 provides for an n-dimensional stratified nilpotent Lie group G with a lattice Γ, such that $s_2(\Gamma)\subset\Gamma$, and with an Ω-isotropic, Ω-regular subspace $S\in V_1$ of dimension $k+1$, the following upper bound on the filling functions:

$$F_G^{n-j}(l)\preccurlyeq l^{\frac{D-j}{D-j-1}} \quad \text{for } j\leq k.$$

Whenever G is not abelian, i.e. not isomorphic to $\mathbb{R}^n$, the Hausdorff-dimension D is strictly larger than n. Therefore holds

$$\frac{D-j+1}{D-j}<\frac{n-j+1}{n-j}$$

and the above proposition applies for $m=n-j$ with $2\leq j\leq k$. Here j has to be less or equal 2, as the divergence dimension of an n-dimensional stratified nilpotent Lie group is $n-2$ (mentioned earlier in Chapter 2).

Therefore holds

$$\operatorname{Div}_G^{n-j}(r)\sim r^{\frac{(D-j)(n-j-1)}{D-j-1}} \quad \text{for all } 2\leq j\leq k,$$

where the lower bounds are obtained Proposition 6.1 and the upper bounds are obtained by Proposition 6.2. (For the case $G\cong\mathbb{R}^n$ see [1].)

Together with the lower bounds obtained in Section 6.1 this proves Theorem 5, Theorem 6 and Theorem 7.

7 Proofs for Heisenberg Groups and the Application

Most of the results for the generalised Heisenberg Groups are corollaries of the results for stratified nilpotent Lie groups. These ones we will treat in Section 7.1. The super-Euclidean lower bounds on the filling function in dimension n of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ follows not as such a corollary, so it needs a more extensive proof, which we treat in Section 7.3. The super-Euclidean lower bounds on the filling function in dimension n of the octonionic Heisenberg Group $H^n_{\mathbb{O}}$ follow as a corollary of Theorem 4. We treat its proof in Section 7.2.
Finally, it remains to prove Corollary 2.8 about the Dehn functions of non-cocompact lattices in the complex and quaternionic hyperbolic spaces. We do this in Section 7.4.

7.1 Proofs of the Corollaries 2.1, 2.2, 2.3, 2.4 and 2.5

The complex, quaternionic and otonionic Heisenberg groups are two step nilpotent. Therefore there exists, by Lemma 5.6, in each case a lattice with the needed scaling properties. In other words, we can use Theorem 3 and Theorem 7 to prove the corollaries concerning the filling functions and the higher divergence functions of the complex, quaternionic and otonionic Heisenberg Groups (except the lower bounds on F^{n+1}).
For the following remember the bases of the Lie algebras $\mathfrak{h}^n_{\mathbb{H}}$ and $\mathfrak{h}^n_{\mathbb{O}}$ introduced in Section 1.8.

Proposition 7.1.
Let G be the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ or the octonionic Heisenberg Group $H^n_{\mathbb{O}}$ and $\mathfrak{g}$ the respective Lie algebra. Then there is an n-dimensional Ω-regular, Ω-isotropic subspace $S \subset V_1$ of the first layer of the grading $\mathfrak{g} = V_1 \oplus [\mathfrak{g}, \mathfrak{g}]$.

Proof.

i) In the quaternionic case we define $Z_1 = I$, $Z_2 = J$, $Z_3 = K$ and $\eta_i = Z_p^*$, where Z_p^* denotes the dual form to Z_p. Then we get by the general formula for left-invariant differential m-forms on Lie groups

$$(m+1)!(\mathrm{d}\,\gamma)(Y_0, ..., Y_m) = \sum_{i<j} (-1)^{i+j+1} \gamma([Y_i, Y_j], X_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_m)$$

the following components of the curvature form Ω:

$$\omega_i = \mathrm{d}\,\eta_i = \frac{1}{2} \cdot \sum_{[X_l, X_m] = Z_i} X_l^* \wedge X_m^* \quad \text{where } X_q \in \{h_q, i_q, j_q, k_q\}.$$

We choose $S = \langle h_1, ..., h_n \rangle \subset V_1$, which is Ω-isotropic as $[h_u, h_v] = 0$ and therefore $\omega_i(h_u, h_v) = 0$ for all $u, v \in \{1, ..., n\}$.
It remains to give for every choice of σ_{pq} a solution ξ for $\omega_p(\xi, h_q) = \sigma_{pq}$ for $p = 1, 2, 3$ and $q = 1, ..., n$. Let X_{pq} denote the unique element in $\{i_q, j_q, k_q\}$ with $[X_{pq}, h_q] = Z_p$ (compare Section 1.8). Then one can check by a short computation, that such a solution is given by the following element:

$$\xi = \sum_{p,q} \sigma_{pq} X_{pq} \ .$$

ii) In the octonionic case we define $Z_1 = E$, $Z_2 = F$, $Z_3 = G$, $Z_4 = H$, $Z_5 = I$, $Z_6 = J$, $Z_7 = K$ and $\eta_p = Z_p^*$, where Z_p^* denotes the dual form to Z_p. Then we get by the general formula for left-invariant differential m-forms on Lie groups the following components of the curvature form Ω:

$$\omega_i = \mathrm{d}\,\eta_i = \frac{1}{2} \cdot \sum_{[X_l, X_m] = Z_i} X_l^* \wedge X_m^* \quad \text{where } X_p \in \{e_q, f_q, g_q, h_q, i_q, j_q, k_q\}$$

We choose $S = \langle d_1, ..., d_n \rangle \subset V_1$, which is Ω-isotropic as $[d_u, d_v] = 0$ and therefore $\omega_i(d_u, d_v) = 0$ for all $u, v \in \{1, ..., n\}$.
It remains to give for every choice of σ_{pq} a solution ξ for $\omega_p(\xi, d_q) = \sigma_{pq}$ for $p = 1, 2, 3, 4, 5, 6, 7$ and $q = 1, ..., n$. Let X_{pq} denote the unique element in $\{e_q, f_q, g_q, h_q, i_q, j_q, k_q\}$ with $[X_{pq}, d_q] = Z_p$ (compare Section 1.8). Then one can check by a short computation, that such a solution is given by the following element:

$$\xi = \sum_{p,q} \sigma_{pq} X_{pq} \ .$$

□

By the above Proposition 7.1, the quaternionic and the octonionic Heisenberg Groups fulfil the conditions of Theorem 3 and Theorem 7 as they are 2-step nilpotent. So Corollary 2.1 and Corollary 2.2 follow by Theorem 3 and Corollary 2.4 and Corollary 2.5 follow by Theorem 7 as the Hausdorff-dimension of $H^n_{\mathbb{H}}$ is $4n + 3$ and the Hausdorff-dimension of $H^n_{\mathbb{O}}$ is $8n + 7$.

For the proof of Corollary 2.3 concerning the higher divergence functions of the complex Heisenberg Group $H^n_{\mathbb{C}}$, we use [42, Corollary 1.1] and [43, Theorem 1], which together state the following behaviour of the filling functions:

$$F^{j+1}_{H^n_{\mathbb{C}}}(\ell) \sim \ell^{\frac{j+1}{j}} \quad \text{for } j < n,$$

$$F^{n+1}_{H^n_{\mathbb{C}}}(\ell) \sim \ell^{\frac{n+2}{n}} \quad ,$$

$$F^{j+1}_{H^n_{\mathbb{C}}}(\ell) \sim \ell^{\frac{j+2}{j+1}} \quad \text{for } j > n.$$

With Proposition 6.1 and Proposition 6.2 this proves Corollary 2.3.

7.2 The proof of Corollary 2.6

To prove Corollary 2.6 we use Theorem 4. We have seen above, that there is an n-dimensional Ω-regular, Ω-isotropic subspace in the first layer V_1 of $H^n_{\mathbb{O}}$. So the lemma below implies, that condition *b)* in Theorem 4 is satisfied.

Lemma 7.2.
There is no $(n+1)$-dimensional Ω-isotropic subspace $S \subset V_1$ in the octonionic Heisenberg Group $H^n_{\mathbb{O}}$.

Proof.
We prove this in two steps:

Step 1): Every n-dimensional Ω-isotropic subspace $W \subset V_1$ is Ω-regular.

Let $W = \langle w_1, ..., w_n \rangle_{\mathbb{R}}$ be an n-dimensional Ω-isotropic subspace spanned by

$$w_p = (w_p^{d_1}, w_p^{d_2}, ..., w_p^{k_n}) \quad \text{for } 1 \leq p \leq n,$$

with coordinates with respect to the basis $\{d_1, ..., k_n\}$ of the first layer V_1 of the grading of Lie algebra.
Let $Z_1 = E,\ Z_2 = F,\ Z_3 = G,\ Z_4 = H,\ Z_5 = I,\ Z_6 = J$ and $Z_7 = K$. Then holds for $l \in \{1, 2, 3, 4, 5, 6, 7\}$:

$$\begin{aligned}\mathrm{d}\,\eta_l(\_,w_p) = &\sum_{\substack{q\\ [X_m,d_q]=Z_l}} w_p^{d_q} X_m^* + \sum_{\substack{q\\ [X_m,e_q]=Z_l}} w_p^{e_q} X_m^* + \sum_{\substack{q\\ [X_m,f_q]=Z_l}} w_p^{f_q} X_m^* \\ &+ \sum_{\substack{q\\ [X_m,g_q]=Z_l}} w_p^{g_q} X_m^* + \sum_{\substack{q\\ [X_m,h_q]=Z_l}} w_p^{h_q} X_m^* + \sum_{\substack{q\\ [X_m,i_q,]=Z_l}} w_p^{i_q} X_m^* \\ &+ \sum_{\substack{q\\ [X_m,j_q]=Z_l}} w_p^{j_q} X_m^* + \sum_{\substack{q\\ [X_m,k_q]=Z_l}} w_p^{k_q} X_m^*\end{aligned}$$

So we get:

$$\mathrm{d}\,\eta_1(\_,w_p) = \sum_q \left(w_p^{d_q} e_q^* + w_p^{f_q} i_q^* + w_p^{h_q} k_q^* + w_p^{g_q} j_q^* - w_p^{e_q} d_q^* - w_p^{i_q} f_q^* - w_p^{k_q} h_q^* - w_p^{j_q} g_q^*\right)$$

$$\mathrm{d}\,\eta_2(\_,w_p) = \sum_q \left(w_p^{d_q} f_q^* + w_p^{i_q} e_q^* + w_p^{h_q} j_q^* + w_p^{k_q} g_q^* - w_p^{f_q} d_q^* - w_p^{e_q} i_q^* - w_p^{j_q} h_q^* - w_p^{g_q} k_q^*\right)$$

$$\mathrm{d}\,\eta_3(\_,w_p) = \sum_q \left(w_p^{d_q} g_q^* + w_p^{f_q} k_q^* + w_p^{j_q} e_q^* + w_p^{i_q} h_q^* - w_p^{g_q} d_q^* - w_p^{k_q} f_q^* - w_p^{e_q} j_q^* - w_p^{h_q} i_q^*\right)$$

$$\mathrm{d}\,\eta_4(\_,w_p) = \sum_q \left(w_p^{d_q} h_q^* + w_p^{g_q} i_q^* + w_p^{j_q} f_q^* + w_p^{k_q} e_q^* - w_p^{h_q} d_q^* - w_p^{i_q} g_q^* - w_p^{f_q} j_q^* - w_p^{e_q} k_q^*\right)$$

$$\mathrm{d}\,\eta_5(\_,w_p) = \sum_q \left(w_p^{d_q} i_q^* + w_p^{h_q} g_q^* + w_p^{e_q} f_q^* + w_p^{j_q} k_q^* - w_p^{i_q} d_q^* - w_p^{g_q} h_q^* - w_p^{f_q} e_q^* - w_p^{k_q} j_q^*\right)$$

$$\mathrm{d}\,\eta_6(\_,w_p) = \sum_q \left(w_p^{d_q} j_q^* + w_p^{f_q} h_q^* + w_p^{e_q} g_q^* + w_p^{k_q} i_q^* - w_p^{j_q} d_q^* - w_p^{h_q} f_q^* - w_p^{g_q} e_q^* - w_p^{i_q} k_q^*\right)$$

$$\mathrm{d}\,\eta_7(\_,w_p) = \sum_q \left(w_p^{d_q} k_q^* + w_p^{g_q} f_q^* + w_p^{h_q} e_q^* + w_p^{i_q} j_q^* - w_p^{k_q} d_q^* - w_p^{f_q} g_q^* - w_p^{e_q} h_q^* - w_p^{j_q} i_q^*\right)$$

As W is Ω-isotropic, the curvature form

$$\Big(\Omega(w_p, \_)\Big)_p = \Big(\big(\,\mathrm{d}\,\eta_1(w_p, \_), \mathrm{d}\,\eta_2(w_p, \_), ..., \mathrm{d}\,\eta_7(w_p, \_)\big)\Big)_p$$

can be interpreted as a linear isomorphism $V_1/W \cong \mathbb{R}^{7n} \to \mathbb{R}^{7n}$.
For $\sigma = (\sigma_{pq}) \in \mathbb{R}^{7n}$ this leads, as the w_p are linear independent, to a system $A\xi = \sigma$ of linear equations with an invertible matrix $A \in \mathbb{R}^{7n\times 7n}$. So there always exists a solution and W is Ω-regular.

Step 2): There is no $(n+1)$-dimensional Ω-isotropic subspace $S \subset V_1$.

Assume there is an $(n+1)$-dimensional Ω-isotropic subspace $S = \langle s_1, ..., s_{n+1}\rangle_{\mathbb{R}}$ of V_1. Then $W = \langle s_1, ..., s_n\rangle_{\mathbb{R}}$ is Ω-isotropic and n-dimensional. Claim 1 implies that W is Ω-regular. So the linear map

$$\Omega^W_\bullet : V_1 \to \mathrm{Hom}(W, \mathfrak{g}/V_1),\ \ X \mapsto \Omega(X, \_)$$

is surjective and vanishes on S. Therefore:

$$\begin{aligned} 7n - 1 = 8n - (n+1) &= \dim(V_1/S) \\ &\geq \dim(V_1/\operatorname{kern}(\Omega^W_\bullet) = \dim(\mathrm{image}(\Omega^W_\bullet)) \\ &= \dim(\mathrm{Hom}(W, \mathfrak{g}/V_1)) = n(8n + 7 - 8n) \\ &= 7n \end{aligned}$$

But this is a contradiction and such an S can't exist. □

7.3 The proof of Theorem 8

Here we compute the $(n+1)$-dimensional filling function of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$. As $H^n_{\mathbb{H}}$ fulfils the conditions of Theorem 3, we already have the super-Euclidean upper bound:

$$F^{n+1}_{H^n_{\mathbb{H}}}(l) \preccurlyeq l^{\frac{n+2}{n}}\ .$$

Thus we only have to prove the corresponding lower bound.

Proposition 7.3.
Let $H^n_{\mathbb{H}}$ be the quaternionic Heisenberg Group of dimension $4n+3$.
Then holds:
$$F^{n+1}_{H^n_{\mathbb{H}}}(l) \succcurlyeq l^{\frac{n+2}{n}} \ .$$

Proof.
We will use Burillo's filling theorem (see Theorem 3.1). To this end, we have to construct a Lipschitz $(n+1)$-chain b in $H^n_{\mathbb{H}}$ and a closed $H^n_{\mathbb{H}}$-invariant $(n+1)$-form η on $H^n_{\mathbb{H}}$ with the correct scaling behaviour.
For this we avail ourselves of the constructions Burillo did in the proof of [9, Theorem 2.1] to obtain the lower bound for the filling function $F^{n+1}_{H^n_{\mathbb{C}}}$ of the complex Heisenberg Group. We denote the there constructed $(n+1)$-chain by b' and the corresponding $(n+1)$-form by γ.
Now let $x_1, ..., x_n, y_1, ..., y_n, Z$ be the usual basis of the Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$ of the complex Heisenberg Group $H^n_{\mathbb{C}}$ and $h_1, ..., k_n, I, J, K$ the basis of the Lie algebra $\mathfrak{h}^n_{\mathbb{H}}$ of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ (compare Section 1.8).
Then the complex Heisenberg Group $H^n_{\mathbb{C}}$ embeds as Lie subgroup into the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$. We do this on the Lie algebra level via the map
$$x_i \mapsto k_i \ , \ y_i \mapsto h_i \ , \ Z \mapsto K \ .$$
Hence we can consider the in [9] constructed $(n+1)$-chain b' as an $(n+1)$-chain in the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$. We set $b := b'$. The above embedding respects the grading of the Lie algebras, i.e. vectors of the first layer are mapped to vectors of the first layer and vectors of the second layer are mapped to vectors of the second layer. Therefore the boundary ∂b of the chain b has the same scaling behaviour in the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$ as before in the the complex Heisenberg Group $H^n_{\mathbb{C}}$, i.e. $\mathrm{mass}(s_t(\partial b)) \leq \mathrm{mass}(\partial b) \cdot t^n$.
Consequently it only remains to construct a closed, $H^n_{\mathbb{H}}$-invariant $(n+1)$-form η on $H^n_{\mathbb{H}}$, such that η restricts on the embedded $H^n_{\mathbb{C}}$ to the in [9] constructed closed, $H^n_{\mathbb{C}}$-invariant $(n+1)$-form γ on $H^n_{\mathbb{C}}$ (this would imply condition *2)* in Theorem 3.1) and satisfies
$$s_t^*\eta = t^{n+2}\eta \ .$$
The form γ is given (with respect to the above notation for $H^n_{\mathbb{C}} \subset H^n_{\mathbb{H}}$) by
$$\gamma = (-1)^n \cdot K^* \wedge h_1^* \wedge ... \wedge h_n^*$$
where for $v \in \mathfrak{h}^n_{\mathbb{H}}$ the symbol v^* denotes the dual form of v.

We start the construction of η by defining some special n-forms. Let $\mathfrak{S}_n$ be the set of all ($H^n_{\mathbb{H}}$-invariant) n-forms S of the shape

$$S = h_l^* \wedge ... \wedge h_m^* \wedge i_p^* \wedge ... \wedge i_q^*$$

with increasing index within the h^*-part, increasing index within the i^*-part and such that every number in $\{1, ..., n\}$ appears as index of an h^* or an i^*, as well with an even number of i^*'s. In particular, each integer between 1 and n appears exactly once as index.
For example in the case $n = 3$ we get:

$$\mathfrak{S}_3 = \{h_1^* \wedge h_2^* \wedge h_3^*,\ h_1^* \wedge i_2^* \wedge i_3^*,\ h_2^* \wedge i_1^* \wedge i_3^*,\ h_3^* \wedge i_1^* \wedge i_2^*\}\ .$$

Further let $\mathfrak{T}_n$ be the set of all ($H^n_{\mathbb{H}}$-invariant) n-forms T of the shape

$$T = h_l^* \wedge ... \wedge h_m^* \wedge i_p^* \wedge ... \wedge i_q^*$$

with increasing index within the h^*-part, increasing index within the i^*-part and such that every number in $\{1, ..., n\}$ appears as index of an h^* or an i^*, as well with an odd number of i^*'s. In particular, each integer between 1 and n appears exactly once as index.
For example in the case $n = 3$ we get:

$$\mathfrak{T}_3 = \{i_1^* \wedge i_2^* \wedge i_3^*,\ h_1^* \wedge h_2^* \wedge i_3^*,\ h_1^* \wedge h_3^* \wedge i_2^*,\ h_2^* \wedge h_3^* \wedge i_1^*\}\ .$$

Each of this n-forms in $\mathfrak{S}_n$ and $\mathfrak{T}_n$ can be obtained either from h_1^* or from i_1^* by adding successively h_{r+1}^* or i_{r+1}^* in the r^{th} step at the last position of the h^*-part respectively at the last position of the i^*-part. If one now gives h_1^* the sign "+" and i_1^* the sign "−" this induces a sign to each of the n-forms $S \in \mathfrak{S}_n$ and $T \in \mathfrak{T}_n$ by the following rule:

Let A_r be the signed r-form before adding h_{r+1}^* or i_{r+1}^*. Then:

- If A_r has an even number of i^*'s, we change the sign if we add i_{r+1}^*, but not if we add h_{r+1}^*.

- If A_r has an odd number of i^*'s, we change the sign if we add h_{r+1}^*, but not if we add i_{r+1}^*.

In the following we denote for $S \in \mathfrak{S}_n$ respectively $T \in \mathfrak{T}_n$ the signed n-form by $\widetilde{S}$ respectively $\widetilde{T}$.

We observe, that if S and T only differ at position r, for the signs holds:

$$\operatorname{sign}(\widetilde{S}) = (-1)^{n-r+1} \cdot \operatorname{sign}(\widetilde{T}) \ .$$

This is true, as after the step r of adding h^* or i^*, the two forms have different signs. In each of the following steps one of the forms will change its sign and the other one will not. So the signs of $\widetilde{S}$ and $\widetilde{T}$ differ if and only if the number $n-r$ of remaining steps is even.
Now we arrived at the point where we are able to define our candidate for the $H^n_{\mathbb{H}}$-invariant $(n+1)$-form η:

$$\eta := (-1)^n \cdot \left(\sum_{S \in \mathfrak{S}_n} K^* \wedge \widetilde{S} - \sum_{T \in \mathfrak{T}_n} J^* \wedge \widetilde{T} \right)$$

The only form in $\mathfrak{S}_n \cup \mathfrak{T}_n$, which is not zero when restricted to $H^n_{\mathbb{C}}$, is

$$h_1^* \wedge h_2^* \wedge ... \wedge h_n^* \ \in \mathfrak{S}_n \ .$$

As the sign of this form is "+", the form η coincides with γ on $H^n_{\mathbb{C}}$.
Further holds

$$s_t^* \eta = t^{n+2} \eta$$

as each summand in η consists of n dual forms of vectors of the first layer of the grading of the Lie algebra $\mathfrak{h}^n_{\mathbb{H}}$ which scale linearly and one dual form of a vector of the second layer of the grading which scales quadratically.

It remains to show that η is closed. As for this purpose the sign $(-1)^n$ has no effect, we will neglect it in the following.

For fixed $r \in \{1, ..., n\}$ let $(S_p)_{p \in P}$ be a numbering of the forms $S \in \mathfrak{S}_n$ containing h_r^*. Then for each $p \in P$ there is an unique $T_p \in \mathfrak{T}_n$ containing i_r^*, such that S_p and T_p only differ at position r. This means T_p arises from S_p by just replacing h_r^* by i_r^*. This gives a numbering $(T_p)_{p \in P}$ of the forms $T \in \mathfrak{T}_n$ containing i_r^* associated to the numbering $(S_p)_{p \in P}$. Analogously let $(S_q)_{q \in Q}$ be a numbering of the $S \in \mathfrak{S}_n$ containing i_r^*. Then for each $q \in Q$ there is an unique $T_q \in \mathfrak{T}_n$ containing h_r^*, such that S_q and T_q only differs at position r. This means that T_q arises from S_q by just replacing i_r^* by h_r^*. This gives a numbering $(T_q)_{q \in Q}$ of the forms $T \in \mathfrak{T}_n$ containing h_r^* associated to the numbering $(S_q)_{q \in Q}$.

Further let $\mathfrak{p}$ and $\mathfrak{q}$ be the $(n-1)$-forms obtained by deleting the position in which S_p and T_p differ and the position in which S_q and T_q differ, respectively. (Here it makes no difference whether one does this in the respective form $T \in \mathfrak{T}_n$ or $S \in \mathfrak{S}_n$.)

Then holds:

$$
\begin{aligned}
(n+2)!\,\mathrm{d}\eta &= \sum_{S\in\mathfrak{S}_n}\sum_{t=1}^{n}(k_t^*\wedge h_t^*+j_t^*\wedge i_t^*)\wedge\widetilde{S}-\sum_{T\in\mathfrak{T}_n}\sum_{t=1}^{n}(j_t^*\wedge h_t^*+i_t^*\wedge k_t^*)\wedge\widetilde{T}\\
&=\sum_{t=1}^{n}\left(\sum_{S\in\mathfrak{S}_n}(k_t^*\wedge h_t^*+j_t^*\wedge i_t^*)\wedge\widetilde{S}-\sum_{T\in\mathfrak{T}_n}(j_t^*\wedge h_t^*+i_t^*\wedge k_t^*)\wedge\widetilde{T}\right)\\
&=\sum_{t=1}^{n}\Bigg(\sum_{\substack{S\in\mathfrak{S}_n\\ \text{with } h_t^*}} j_t^*\wedge i_t^*\wedge\widetilde{S}+\sum_{\substack{S\in\mathfrak{S}_n\\ \text{with } i_t^*}} k_t^*\wedge h_t^*\wedge\widetilde{S}\\
&\qquad\qquad-\sum_{\substack{T\in\mathfrak{T}_n\\ \text{with } h_t^*}} i_t^*\wedge k_t^*\wedge\widetilde{T}-\sum_{\substack{T\in\mathfrak{T}_n\\ \text{with } i_t^*}} j_t^*\wedge h_t^*\wedge\widetilde{T}\Bigg)\\
&=\sum_{t=1}^{n}\Bigg(\sum_{p\in P}\Big[j_t^*\wedge i_t^*\wedge\widetilde{S}_p-j_t^*\wedge h_t^*\wedge\widetilde{T}_p\Big]\\
&\qquad\qquad+\sum_{q\in Q}\Big[k_t^*\wedge h_t^*\wedge\widetilde{S}_q-i_t^*\wedge k_t^*\wedge\widetilde{T}_q\Big]\Bigg)\\
&=\sum_{t=1}^{n}\Bigg(\sum_{p\in P}\Big[(-1)^{x_p}\cdot j_t^*\wedge i_t^*\wedge h_t^*\wedge\mathfrak{p}-(-1)^{y_p}\cdot j_t^*\wedge h_t^*\wedge i_t^*\wedge\mathfrak{p}\Big]\\
&\qquad+\sum_{q\in Q}\Big[(-1)^{x_q}\cdot k_t^*\wedge h_t^*\wedge i_t^*\wedge\mathfrak{q}-(-1)^{y_q}\cdot i_t^*\wedge k_t^*\wedge h_t^*\wedge\mathfrak{q}\Big]\Bigg)
\end{aligned}
$$

where for each $t\in\{1,...,n\}$ the family $(S_p)_{p\in P}$ is a numbering of the $S\in\mathfrak{S}_n$ containing h_t^* and the family $(S_q)_{q\in Q}$ is a numbering of the $S\in\mathfrak{S}_n$ containing i_t^*.

Before we continue the computation, we have to identify the signs of the summands. We do this by computing the congruence classes modulo 2 of exponents of the -1's. These exponents arise by pulling the h_t^*'s and i_t^*'s from inside of S_p, S_q, T_p and T_q to the third positions.

Denote by $\#_h^m A$ the number of h^*'s in $A\in\mathfrak{S}_n\cup\mathfrak{T}_n$ with index smaller than m and by $\#_i^m A$ the number of i^*'s in $A\in\mathfrak{S}_n\cup\mathfrak{T}_n$ with index smaller than m. Then for fixed t we have for the exponents x_p, x_q, y_p and y_q the following congruences modulo 2:

$x_p=\#_h^t S_p$

$$
\begin{aligned}
y_p &= \operatorname{sign}(\widetilde{S}_p) - \operatorname{sign}(\widetilde{T}_p) + \#_h^n T_p + \#_i^t T_p \\
&\equiv (n - t + 1) + (n + 1) + (t - 1 - \#_h^t S_p) \\
&\equiv 2n + 1 - \#_h^t S_p \\
&\equiv 1 + \#_h^t S_p
\end{aligned}
$$

$$
\begin{aligned}
x_q &= \operatorname{sign}(\widetilde{S}_q) - \operatorname{sign}(\widetilde{T}_q) + \#_h^n S_q + \#_i^t S_q \\
&\equiv (n - t + 1) + (n) + (t - 1 - \#_h^t T_q) \\
&\equiv 2n - \#_h^t T_q \\
&\equiv \#_h^t T_q
\end{aligned}
$$

$$
y_q = \#_h^t T_q
$$

We used for this, that S_p and T_p respectively S_q and T_q only differ at position t and so one has $\#_h^t S_p = \#_h^t T_p$ and $\#_h^t S_q = \#_h^t T_q$.

Further holds for the needed permutations the following:

$$
j_t^* \wedge i_t^* \wedge h_t^* = (-1)^1 \cdot j_t^* \wedge h_t^* \wedge i_t^*
$$

$$
k_t^* \wedge h_t^* \wedge i_t^* = (-1)^2 \cdot i_t^* \wedge k_t^* \wedge h_t^*
$$

Therefore we can continue with the computation:

$$
\begin{aligned}
(n+2)!\,\mathrm{d}\eta &= \sum_{t=1}^{n} \Bigg(\sum_{p \in P} \Big[(-1)^{x_p} \cdot j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^{y_p} \cdot j_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{p} \Big] \\
&\qquad + \sum_{q \in Q} \Big[(-1)^{x_q} \cdot k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - (-1)^{y_q} \cdot i_t^* \wedge k_t^* \wedge h_t^* \wedge \mathfrak{q} \Big] \Bigg) \\
&= \sum_{t=1}^{n} \Bigg(\sum_{p \in P} (-1)^{\#_h^t S_p} \cdot \Big[j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^1 \cdot j_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{p} \Big] \\
&\qquad + \sum_{q \in Q} (-1)^{\#_h^t T_q} \cdot \Big[k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - i_t^* \wedge k_t^* \wedge h_t^* \wedge \mathfrak{q} \Big] \Bigg) \\
&= \sum_{t=1}^{n} \Bigg(\sum_{p \in P} (-1)^{\#_h^t S_p} \cdot \Big[j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} - (-1)^{1+1} \cdot j_t^* \wedge i_t^* \wedge h_t^* \wedge \mathfrak{p} \Big] \\
&\qquad + \sum_{q \in Q} (-1)^{\#_h^t T_q} \cdot \Big[k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} - (-1)^2 \cdot k_t^* \wedge h_t^* \wedge i_t^* \wedge \mathfrak{q} \Big] \Bigg) \\
&= 0
\end{aligned}
$$

So η is closed and therefore the conditions of Theorem 3.1 are fulfilled and we get $F_{H_{\mathbb{H}}^n}^{n+1}(l) \succcurlyeq l^{\frac{n+2}{n}}$. □

Together with upper bound $F^{n+1}_{H^n_{\mathbb{H}}}(l) \preccurlyeq l^{\frac{n+2}{n}}$ from Corollary 2.1 this proves Theorem 8.

7.4 The proof of Corollary 2.8

We have to prove the remaining corollary concerning the non-cocompact lattices in the complex and quaternionic hyperbolic spaces. The main part of Corollary 2.8 comes by the filling functions of the complex and quaternionic Heisenberg Groups. So we only have to connect the Dehn functions of the lattices with the filling functions of the Heisenberg Groups.

Lemma 7.4 (Corollary 2.8)**.**
Let $n \in \mathbb{N}_{\geq 3}$ and X be the complex hyperbolic space $\mathrm{SU}(n,1)/\,\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ of dimension $2n$ or the quaternionic hyperbolic space $\mathrm{Sp}(n,1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1))$ of dimension $4n$. Further let Γ be a group acting properly discontinuously by isometries, such that the quotient space X/Γ is of finite volume, but not compact. Then holds:

$$\delta^j_\Gamma(l) \sim l^{\frac{j+1}{j}} \qquad \textit{for } 1 \leq j < n-1.$$

Proof.
For the complex hyperbolic case this is proved in [22, Theorem 5].
For the quaternionic case we use, as in the proof of [22, Theorem 5], that Γ acts geometrically on a space X_0 obtained from the quaternionic hyperbolic space $\mathrm{Sp}(n,1)/\,\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ by removing a Γ-invariant family of horoballs. By the Lemma of Švarc-Milnor, Γ is quasi-isometric to X_0 and so $\delta^j_\Gamma \sim \delta^j_{X_0} \sim F^{j+1}_{X_0}$ for all $1 \leq j \leq n-1$. The filling functions of X_0 are equivalent to the filling functions of any of its boundary components (see [22, Theorem 5], [28, 5.D.(5)(c)]), which are horospheres in $\mathrm{Sp}(n,1)/\,\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.
The horospheres in the quaternionic hyperbolic space $\mathrm{Sp}(n,1)/\,\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ are biLipschitz equivalent to the quaternionic Heisenberg Group $H^{n-1}_{\mathbb{H}}$.
By Corollary 2.1 we get $\delta^j_\Gamma(l) \sim F^{j+1}_{X_0}(l) \sim l^{\frac{j+1}{j}}$ for all $1 \leq j \leq n-1$. □

8 Some open questions

We computed the filling functions for stratified nilpotent Lie groups under the assumption of the existence of Ω-regular, Ω-isotropic subspaces of the first layer of the Lie algebra. Our results suggest a division of the behaviour of the filling functions in a part of Euclidean growth in the low dimensions, strictly sub-Euclidean growth in the highest dimensions and at least one dimension of strictly super-Euclidean growth in between. Whether this is still true without the algebraic condition, is an open question.
Burillo proved a cubic lower bound for the filling area function of the group N_n of unipotent upper triangular $n \times n$-matrices for $n \geq 4$. So Gromov's conjecture (see [28, 5.D.]) about the first super-Euclidean filling function of a nilpotent Lie group is wrong. The group N_n is $(n-1)$-step nilpotent and Gromov's heuristic argument is mainly based on observations for the complex Heisenberg Groups which are 2-step nilpotent. So we ask:

Question 1.
Does every stratified nilpotent Lie group of nilpotency degree 2 have Euclidean filling functions up to the maximal dimension of horizontal submanifolds and a super-Euclidean filling function in the dimension above?

In Section 2.3 we gave a positive answer to this question for the quaternionic and octonionic Heisenberg Groups. Further we reduced this problem by Theorem 4 to a question on the existence of boundaries without fillings in the corresponding Carnot Group (without consideration of mass).

An important reason why we are interested in (non-abelian) nilpotent Lie groups is the fact, that they have sectional curvature of both signs, negative and positive, at each point. All spaces of non-positive curvature have no super-Euclidean filling functions, so it would be interesting, if the positive curvature at every point could be seen by the filling functions:

Question 2.
Does every (non-abelian) nilpotent Lie group have a super-Euclidean filling function in some dimension?

This question may be easier to answer, if one restricts it to stratified nilpotent Lie groups:

Question 2b.
Does every stratified nilpotent Lie group have a super-Euclidean filling function in some dimension?

A more explicit problem concerns the filling function in dimension n for the octonionic Heisenberg Group $H^n_{\mathbb{O}}$. We proved lower bounds proportional to $\ell^{\frac{n+2}{n}}$ for the filling function of the quaternionic Heisenberg Group $H^n_{\mathbb{H}}$. But this technique fails for the octonionic Heisenberg Group $H^n_{\mathbb{O}}$ as we discussed above. Nonetheless we proved, that the filling function of $H^n_{\mathbb{O}}$ is super-Euclidean, but the exact growth type is still unknown. So:

Question 3.
What is the exact growth type of the filling function in dimension n of the octonionic Heisenberg Group $H^n_{\mathbb{O}}$?

Another question arises for the higher divergence functions of stratified nilpotent Lie groups. We computed the exact growth rate for higher divergence functions in the high dimensions, but in the lower dimensions we only have been able to establish Euclidean lower bounds.

Question 4.
Are there Euclidean upper bounds for the higher divergence functions of stratified nilpotent Lie groups in low dimensions?

References

[1] Aaron Abrams, Noel Brady, Pallavi Dani, Moon Duchin, and Robert Young. Pushing fillings in right-angled Artin groups. *J. Lond. Math. Soc. (2)*, 87(3):663–688, 2013.

[2] J. M. Alonso, X. Wang, and S. J. Pride. Higher-dimensional isoperimetric (or Dehn) functions of groups. *J. Group Theory*, 2(1):81–112, 1999.

[3] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[4] Jason Behrstock and Cornelia Drutu. Combinatorial higher dimensional isoperimetry and divergence for mapping class groups. *arXiv:1507.01518.*

[5] B. H. Bowditch. A short proof that a subquadratic isoperimetric inequality implies a linear one. *Michigan Math. J.*, 42(1):103–107, 1995.

[6] Noel Brady and Martin R. Bridson. There is only one gap in the isoperimetric spectrum. *Geom. Funct. Anal.*, 10(5):1053–1070, 2000.

[7] Noel Brady and Benson Farb. Filling-invariants at infinity for manifolds of nonpositive curvature. *Trans. Amer. Math. Soc.*, 350(8):3393–3405, 1998.

[8] Martin R. Bridson. The geometry of the word problem. In *Invitations to geometry and topology*, volume 7 of *Oxf. Grad. Texts Math.*, pages 29–91. Oxford Univ. Press, Oxford, 2002.

[9] José Burillo. Lower bounds of isoperimetric functions for nilpotent groups. In *Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994)*, volume 25 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 1–8. Amer. Math. Soc., Providence, RI, 1996.

[10] Luca Capogna, Donatella Danielli, Scott D. Pauls, and Jeremy T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2007.

[11] Giuseppina D'Ambra. Induced subbundles and Nash's implicit function theorem. *Differential Geom. Appl.*, 4(1):91–105, 1994.

[12] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[13] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[14] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[15] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[16] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[17] Mikhael Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[18] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds. *Geom. Funct. Anal.*, 5(3):582–603, 1995.

[19] Adam Korányi and Fulvio Ricci. A classification-free construction of rank-one symmetric spaces. *Bull. Kerala Math. Assoc.*, (Special Issue):73–88 (2007), 2005.

[20] Enrico Le Donne and Séverine Rigot. Besicovitch covering property on graded groups and applications to measure differentiation. *arXiv:1512.04936.*

[21] Enrico Leuzinger. Corank and asymptotic filling-invariants for symmetric spaces. *Geom. Funct. Anal.*, 10(4):863–873, 2000.

[22] Enrico Leuzinger. Optimal higher-dimensional Dehn functions for some CAT(0) lattices. *Groups Geom. Dyn.*, 8(2):441–466, 2014.

[23] Enrico Leuzinger and Christophe Pittet. On quadratic Dehn functions. *Math. Z.*, 248(4):725–755, 2004.

[24] Enrico Leuzinger and Robert Young. The distorsion dimension of $\mathbb{Q}$-rank 1 lattices. *arXiv:1509.09224v1.*

[25] Valentino Magnani. Unrectifiability and rigidity in stratified groups. *Arch. Math. (Basel)*, 83(6):568–576, 2004.

[26] John Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.*, 21(3):293–329, 1976.

[27] John Mitchell. On Carnot-Carathéodory metrics. *J. Differential Geom.*, 21(1):35–45, 1985.

[28] Graham A. Niblo and Martin A. Roller, editors. *Geometric group theory. Vol. 2*, volume 182 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1993.

[29] Pierre Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.

[30] Christophe Pittet. Isoperimetric inequalities for homogeneous nilpotent groups. In *Geometric group theory (Columbus, OH, 1992)*, volume 3 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 159–164. de Gruyter, Berlin, 1995.

[31] Christophe Pittet. Isoperimetric inequalities in nilpotent groups. *J. London Math. Soc. (2)*, 55(3):588–600, 1997.

[32] M. S. Raghunathan. *Discrete subgroups of Lie groups.* Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[33] E. H. Spanier and J. H. C. Whitehead. The theory of carriers and S-theory. Algebraic geometry and topology. In *A symposium in honor of S. Lefschetz*, pages 330–360. Princeton University Press, Princeton, N.J., 1957.

[34] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[35] Stefan Wenger. Isoperimetric inequalities of Euclidean type in metric spaces. *Geom. Funct. Anal.*, 15(2):534–554, 2005.

[36] Stefan Wenger. Filling invariants at infinity and the Euclidean rank of Hadamard spaces. *Int. Math. Res. Not.*, pages Art. ID 83090, 33, 2006.

[37] Stefan Wenger. A short proof of Gromov's filling inequality. *Proc. Amer. Math. Soc.*, 136(8):2937–2941, 2008.

[38] Stefan Wenger. Nilpotent groups without exactly polynomial Dehn function. *J. Topol.*, 4(1):141–160, 2011.

[39] Joseph A. Wolf. Curvature in nilpotent Lie groups. *Proc. Amer. Math. Soc.*, 15:271–274, 1964.

[40] Robert Young. Homological and homotopical higher-order filling functions. *Groups Geom. Dyn.*, 5(3):683–690, 2011.

[41] Robert Young. The Dehn function of $\mathrm{SL}(n;\mathbb{Z})$. *Ann. of Math. (2)*, 177(3):969–1027, 2013.

[42] Robert Young. Filling inequalities for nilpotent groups through approximations. *Groups Geom. Dyn.*, 7(4):977–1011, 2013.

[43] Robert Young. High-dimensional fillings in Heisenberg groups. *J. Geom. Anal.*, 26(2):1596–1616, 2016.

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