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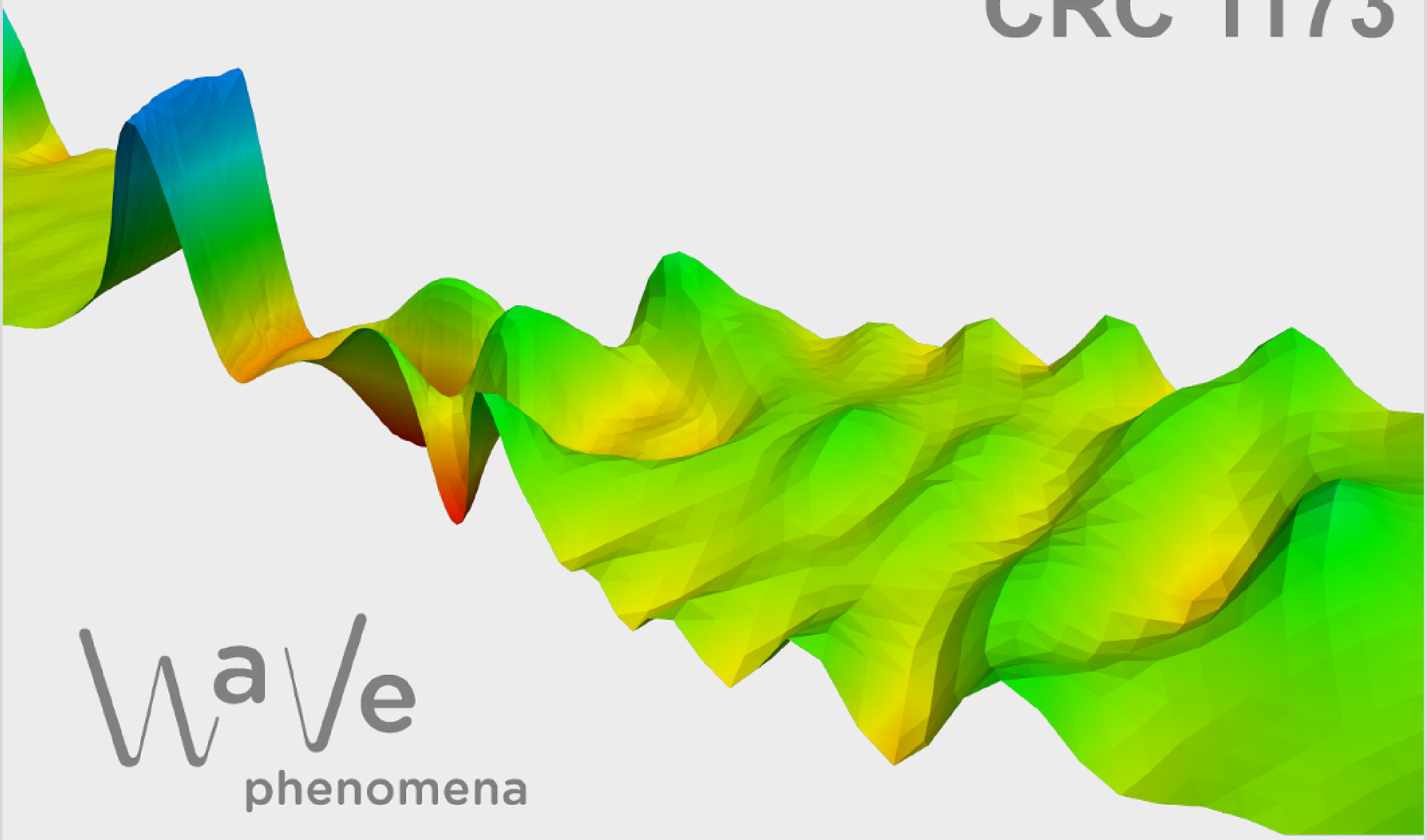

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Error analysis of implicit Runge–Kutta methods for quasilinear hyperbolic evolution equations

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Abstract We establish error bounds of implicit Runge–Kutta methods for a class of quasilinear hyperbolic evolution equations including certain Maxwell and wave equations. Our assumptions cover algebraically stable and coercive schemes such as Gauß and Radau collocation methods. We work in a refinement of the analytical setting of Kato's well-posedness theory.

Keywords hyperbolic evolution equations, quasilinear Maxwell equations, implicit Runge–Kutta methods, well-posedness, error analysis, time integration

Mathematics Subject Classification (2010) Primary: 65M12, 65J15 Secondary: 35Q61, 35L90

1 Introduction

Quasilinear hyperbolic evolution equations describe a wide range of phenomena in physics, including in particular the Maxwell system with nonlinear constitutive laws. There is a well established analytical theory for such problems. On the other hand, despite their importance, for quasilinear hyperbolic problems there are only very few rigorous convergence results concerning time integration methods. The implicit Euler method for nonlinear evolution equations has been studied in [9,13,18] for various cases. These papers establish convergence of order $1/2$ assuming that the numerical solution exists. In [3],

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Crandall and Souganidis showed that the approximations of the semi-implicit Euler method for (1) are well-posed and converge with order 1/2. To our knowledge even first order convergence of the implicit Euler scheme has not been proved until very recently.

Kato's approach from [11] provides a unified framework for a large class of quasilinear hyperbolic evolution equations. In this work a local well-posedness theory is established in spaces like H^3 in the case of first order systems. However, the setting of [11] cannot directly be applied to problems like quasilinear Maxwell equations, as noted on p. 53 of [11]. One has to invoke state dependent (energy-type) norms in addition, as in [10] for concrete hyperbolic PDEs and in [12] for a large class fo evolution equations. We work within a refinement of Kato's theory due to Müller [17] whose framework is closer to the applications we have in mind than that of [12]. Moreover, the setting of [17] takes into account the quasilinear nature of the problem by introducing an extra intermediate space on which the nonlinearity is defined. This approach allows to reduce the restrictions on the initial data, in particular if the nonlinearity is only defined on open sets of this intermediate space.

The framework of [17] was already used in the recent paper [8] by two of the present authors, who have proved well-posedness and first order convergence of the semi-implicit and implicit Euler approximations to quasilinear hyperbolic evolution equations. These results have been applied to certain quasilinear Maxwell and wave equations. In the very recent preprint [14] implicit Runge–Kutta schemes have been analyzed in Kato's original framework of [11]. For linear Maxwell equations such schemes were studied in [7] including the space discretization with discontinuous Galerkin methods.

In the present paper we study implicit Runge–Kutta methods for the quasilinear hyperbolic evolution equation

$$\Lambda(u(t))\partial_t u(t) = Au(t) + Q(u(t))u(t), \qquad u(0) = u_0, \tag{1}$$

on a Hilbert space X in the framework of [17]. Here A is a linear skew-adjoint operator, $\Lambda(v)$ is a symmetric positive definite operator for v in a neighborhood of zero and Q is a lower order term. In our applications to Maxwell equations, A is the Maxwell operator, Λ is given by the nonlinear constitutive relations and Q describes the conductivity, see Section 2. As noted above, the Maxwell system (and thus also (1) in general) is not covered by Kato's original setting of [11] and hence not by the results in [14].

In our main results we prove well-posedness and convergence of order s for an s-stage implicit Runge–Kutta method applied to (1). This is done first in the norm of the basic space (e.g., L^2) in Theorem 5.3 and then for a stronger norm in Theorem 6.3 under somehwat stronger assumptions on the data. It is mainly assumed that the scheme is algebraically stable and coercive and that the operators in (1) satisfy the assumptions of the analytical well-posedness result from [17]. Typical examples are Gauß and Radau collocation methods, see Section 3. To obtain full classical convergence order, one would need rather strong additional regularity assumptions, cf. Section 4.5 of [14], which we want to avoid here.

To treat (1), one first inverts $\Lambda(u(t))$. Our analysis (and also that of [12, 17]) then crucially depends on the dissipativity of $\Lambda(u(t))^{-1}A$ with respect to the scalar product on X with the (state-depending) weight $\Lambda(u(t))$. This fact is essential for the construction of one step of the scheme in Lemma 4.1. Even more importantly, the dissipativity provides the main energy-type bounds for the numerical solution given by Lemma 4.2 and Proposition 4.3 which involve the state dependent norms. On the other hand, these norms lead to substantial new diffculties throughout the paper since one is forced to switch between them within the estimates. For the reasoning it is also crucial to have a precise control of the constants and of the norms of the numerical solutions in the various spaces. Here we tried to be rather explicit in our formulations.

Energy techniques for implicit Runge–Kutta methods which are algebraically stable and coercive have been successfully applied to analyze stiff ordinary differential equations, cf. [6, Chapter IV] and references given there. Our analysis is motivated by [15], where the algebraic stability was an essential tool to prove rigorous error bounds for quasi-linear parabolic problems, and by [16], where linear wave equations on evolving surfaces have been considered using state dependent norms.

One should note however that Kato's setting in [11] and its variants from [12,17] do not work well for boundary value problems, in contrast to full space problems. In the case of Maxwell's equations one cannot treat perfectly conducting boundaries without very strong restrictions on the nonlinearities, [17]. On the other hand, one can handle Dirichlet boundary conditions for quasilinear Maxwell and wave equations, see [17] and [8]. This shortcoming is unfortunate since the more operator-theoretic approach in [11,12,17] fits very well to the tools from numerical analysis used in this paper and also in [8,14]. In future work we want to combine the present approach with more PDE type methods as in [1,5], for instance, to cover also Maxwell equations on domains with standard boundary conditions. In such a framework we will then also investigate the space discretization error which is not considered here.

Notation. For Banach spaces X and Y we write $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y and $\|A\|_{Y\leftarrow X} := \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$ for the operator norm of $A \in \mathcal{L}(X,Y)$. We endow the domain $D(A)$ of a closed operator A in X with its graph norm $\|x\|_A := \|Ax\|_X + \|x\|_X$. The closed ball of radius r around 0 in X is denoted by $\overline{\mathcal{B}}_X(r)$. The number $c_{\mathrm{RK}} \geq 1$ stands for generic constants which only depend on the coefficients of the Runge-Kutta method, and C is a nonnegative number which depends on c_{RK}, the constants in Assumption 2.1, and the norm of the operators S, S^{-1}, S_Y and S_Y^{-1} in Assumption 2.2 below.

2 Analytical framework and Maxwell's equations

In this section we discuss our analytical framework, state the known well-posedness result for (1) from [17], and discuss the quasilinear Maxwell equations.

We use three Hilbert spaces $(X,(\cdot,\cdot)_X)$, $(Y,(\cdot,\cdot)_Y)$, and $(Z,(\cdot,\cdot)_Z)$ with continuous and dense embeddings $Z \hookrightarrow Y \hookrightarrow X$. In addition, Y is an exact interpolation space between Z and X. (For the Maxwell equations one employs spaces like $X = L^2$, $Y = H^2$, and $Z = H^3$.) We collect the main assumptions on the operators in (1). In particular, $\Lambda(y)$ and $Q(y)$ are bounded linear operators in X which are Lipschitz functions of y in a fixed ball $\overline{\mathcal{B}}_Y(R)$ of Y. The main linear operator A is skew-adjoint in X and also maps Z to Y. The initial value u_0 will be taken from Z and the evolution equation (1) is solved in Y, see Theorem 2.3. The assumptions below will allow us to derive energy-type estimates in terms of state dependent norms in X.

Assumption 2.1. *Let $R > 0$ be fixed.*

(a) *Let $A \in \mathcal{L}(Z,Y)$ be a skew-adjoint operator in X with $Y \hookrightarrow D(A) \hookrightarrow X$.*

(b) *There exist a family of invertible self-adjoint operators $\{\Lambda(y) : y \in \overline{\mathcal{B}}_Y(R)\}$ in $\mathcal{L}(X)$ such that the ranges $\operatorname{Ran}(I \mp \Lambda(y)^{-1}A)$ are dense in X and the inverses $\Lambda(y)^{-1}$ also belong to $\mathcal{L}(Y)$. Moreover, for all $x \in X$ and $y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R)$ we have the estimates*

$$(x, \Lambda(y)x)_X \ge \nu^{-1} \|x\|_X^2, \tag{2a}$$

$$\|\Lambda(y) - \Lambda(\widetilde{y})\|_{X \leftarrow X} \le \ell \, \|y - \widetilde{y}\|_Y, \tag{2b}$$

$$\|\Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1}\|_{Y \leftarrow Y} \le \ell_Y \|y - \widetilde{y}\|_Y, \tag{2c}$$

$$\|\Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1}\|_{X \leftarrow Y} \le \ell_X \|y - \widetilde{y}\|_X \tag{2d}$$

for constants $\nu, \ell, \ell_Y, \ell_X > 0$. Hence, $\Lambda(y) \ge \nu^{-1} I$ and $\|\Lambda(y)^{-1}\|_{X \leftarrow X} \le \nu$.

(c) *There are operators $\{Q(y) : y \in \overline{\mathcal{B}}_Y(R)\}$ in $\mathcal{L}(X)$ satisfying*

$$\mu_X := \sup_{y \in \overline{\mathcal{B}}_Y(R)} \|Q(y)\|_{X \leftarrow X} < \infty. \tag{3a}$$

Each $Q(y)$ also belongs to $\mathcal{L}(Z,Y)$ and there are constants $m_Y, m_X \ge 0$ with

$$\|Q(y) - Q(\widetilde{y})\|_{Y \leftarrow Z} \le m_Y \|y - \widetilde{y}\|_Y, \tag{3b}$$

$$\|Q(y) - Q(\widetilde{y})\|_{X \leftarrow Z} \le m_X \|y - \widetilde{y}\|_X \qquad \text{for all} \quad y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R). \tag{3c}$$

Below, R always refers to the radius from this assumption and the above constants may of course depend on R. Assumption 2.1 easily implies the bound

$$\|\Lambda(y)\|_{X \leftarrow X} \le \lambda_X := \|\Lambda(0)\|_{X \leftarrow X} + \ell R, \tag{4}$$

for all $y \in \overline{\mathcal{B}}_Y(R)$. We now write (1) in the equivalent short form

$$\partial_t u(t) = A_{u(t)} u(t), \qquad u(0) = u_0, \tag{5}$$

where for $v \in Y$ we introduce the operator

$$A_v := \Lambda(v)^{-1}(A + Q(v)) \tag{6}$$

with domain $D(A)$ in X. To control stronger norms, we also need the next assumption on A_v.

Assumption 2.2. *Let $r > 0$. We assume that there are continuous isomorphisms $S : Z \to X$ and $S_Y : Y \to X$ such that for all $v \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$ there are linear operators $B(v) \in \mathcal{L}(X)$ and $B_Y(v) \in \mathcal{L}(X)$ satisfying*

$$SA_vS^{-1} = A_v + B(v) \quad \text{and} \quad S_YA_vS_Y^{-1} = A_v + B_Y(v), \tag{7a}$$

$$\|B(v)\|_{X\leftarrow X} \le \beta_Z \quad \text{and} \quad \|B_Y(v)\|_{X\leftarrow X} \le \beta_Y \tag{7b}$$

for constants $\beta_Z = \beta_Z(r) > 0$ and $\beta_Y = \beta_Y(r) > 0$.

Throughout the paper we use the constant

$$c_0 = (\nu\lambda_X)^{1/2} \max\{\|S\|_{X\leftarrow Z}\|S^{-1}\|_{Z\leftarrow X}, \|S_Y\|_{X\leftarrow Y}\|S_Y^{-1}\|_{Y\leftarrow X}\} \ge (\nu\lambda_X)^{1/2} \ge 1. \tag{8}$$

The following well-posedness result is part of Theorem 3.41 of [17]. For this theorem one can omit the properties (2d) and (3c) in Assumption 2.1 and the isomorphism S_Y in Assumption 2.2. These conditions are only used later on to treat the numerical solutions. Here and below we take radii $r \ge 1$ to simplify some statements.

Theorem 2.3. *Let Assumptions 2.1 and 2.2 be fulfilled and let $r \ge 1$ be arbitrary. Then the following assertions hold.*

(a) *For each $u_0 \in \overline{\mathcal{B}}_Y((2c_0)^{-1}R) \cap \overline{\mathcal{B}}_Z((2c_0)^{-1}r)$ there exists a time $T_0 = T_0(r) \ge C/(r+\beta_Z(r)) > 0$ and a solution u in $C([0,T_0],Z)\cap C^1([0,T_0],Y)$ of (5) satisfying $\|u(t)\|_Y \le R$ and $\|u(t)\|_Z \le r$ for all $0 \le t \le T_0$.*
(b) *If $v \in C([0,T'],Z)\cap C^1([0,T'],Y)$ is another solution of (5) with $\|v(t)\|_Y \le R$ for all $t \in [0,T']$, then v coincides with u on $[0,\min\{T_0,T'\}]$.*

For the error analysis of the time integration methods we need a few additional properties of the operators collected below. The proofs can be found in Lemmas 3.1 and 3.6 of [8].

Lemma 2.4. *Let Assumption 2.1 be satisfied. For all $y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R)$ we have*

(a) $\|\Lambda(y)^{1/2}\|_{X\leftarrow X} \le \lambda_X^{1/2}$.
(b) $(x, \Lambda(y)^{1/2}x)_X \ge \nu^{-1/2}\|x\|_X^2$ *for all $x \in X$.*
(c) *There is a positive constant ℓ' such that*

$$\|\Lambda(y)^{1/2} - \Lambda(\widetilde{y})^{1/2}\|_{X\leftarrow X} \le \ell'\|y-\widetilde{y}\|_Y.$$

(d) *The operator A_y from (6) satisfies*

$$\|A_y\|_{Y\leftarrow Z} \le \alpha_Y, \tag{9a}$$

$$\|A_y - A_{\widetilde{y}}\|_{X\leftarrow Z} \le L_X\|y-\widetilde{y}\|_X, \tag{9b}$$

$$\|A_y - A_{\widetilde{y}}\|_{Y\leftarrow Z} \le L_Y\|y-\widetilde{y}\|_Y, \tag{9c}$$

where $\ell', \alpha_Y, L_X, L_Y > 0$ only depend on the constants in Assumption 2.1.

Much of our analysis (and also that of [8,10,12,17]) relies on state dependent norms. Let $v, w \in \overline{\mathcal{B}}_Y(R)$. We define the inner product

$$(x, y)_v = (\Lambda(v)x, y)_X$$

and denote the space X endowed with this inner product by X_v. By (2a) and (4), the associated norm is uniformly equivalent to the X-norm, i.e.,

$$\lambda_X^{-1}\|x\|_v^2 \le \|x\|_X^2 \le \nu\|x\|_v^2, \qquad x \in X. \tag{10}$$

Formulas (2b) and (10) yield the Lipschitz property

$$\begin{aligned}\|x\|_v^2 &= (\Lambda(w)x, x)_X + ((\Lambda(v) - \Lambda(w)x, x)_X \\ &\le \|x\|_w^2 + \ell\,\|v - w\|_Y \|x\|_X^2 \\ &\le (1 + \ell\nu\,\|v - w\|_Y)\|x\|_w^2.\end{aligned} \tag{11}$$

Remark 2.5. *Assumption 2.1 and the Lumer-Phillips theorem (see Theorem II.3.15 in [4]) imply that $\Lambda(v)^{-1}A$ generates a contraction semigroup on X_v for each $v \in \overline{\mathcal{B}}_Y(R)$. From the bounded perturbation theorem, see Theorem III.1.3 in [4], we thus deduce that the operator A_v generates a strongly continuous semigroup on X.*

Example. We consider the Maxwell equations

$$\partial_t\mathbf{D}(t,x) = \nabla\times\mathbf{H}(t,x) - \sigma(\mathbf{E}(t,x))\mathbf{E}(t,x), \qquad t\in[0,T],\ x\in\mathbb{R}^3, \tag{12a}$$

$$\partial_t\mathbf{B}(t,x) = -\nabla\times\mathbf{E}(t,x), \qquad t\in[0,T],\ x\in\mathbb{R}^3, \tag{12b}$$

$$\nabla\cdot\mathbf{D}(t,x) = 0, \qquad t\in[0,T],\ x\in\mathbb{R}^3, \tag{12c}$$

$$\nabla\cdot\mathbf{B}(t,x) = 0, \qquad t\in[0,T],\ x\in\mathbb{R}^3, \tag{12d}$$

on $\mathbb{R}^3$ with a nonlinear conductivity $\sigma \in C^3(\mathbb{R}^3, \mathbb{R}^{3\times3})$ and constitutive relations of the form

$$\mathbf{D}(t,x) = \mathbf{E}(t,x) + P(\mathbf{E}(t,x)), \quad \mathbf{B}(t,x) = \mathbf{H}(t,x) + M(\mathbf{H}(t,x)).$$

Here, $P, M \in C^4(\mathbb{R}^3, \mathbb{R}^3)$ are vector fields such that $P'(\xi)$ and $M'(\xi)$ are symmetric and the matrices $I + P'(0)$ and $I + M'(0)$ are positive definite. An important special case is the Kerr nonlinearity with $P(\mathbf{E}) = \chi\,|\mathbf{E}|^2\,\mathbf{E}$ for the susceptility $\chi \in \mathbb{R}$ and $M = 0$, see [2]. We use the spaces

$$X := L^2(\mathbb{R}^3)^6, \qquad Y := H^2(\mathbb{R}^3)^6, \qquad Z := H^3(\mathbb{R}^3)^6.$$

For given initial data $u_0 = (E_0, H_0)$ in Z satisfying (12c) and (12d), we seek solutions in $C^1([0,T], Y) \cap C([0,T], Z)$ of (12a) and (12b). These solutions then automatically fulfill the divergence conditions (12c) and (12d). To tackle the problem in our framework, we set

$$\Lambda(v) = \begin{pmatrix} I + P'(\mathbf{E})) & 0 \\ 0 & I + M'(\mathbf{H})) \end{pmatrix}, \quad Q(v) = \begin{pmatrix} -\sigma(E)E \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}$$

for $v = (\mathbf{E}, \mathbf{H}) \in Y$. Observe that Y embeds into $L^\infty(\mathbb{R}^3)^6$. Hence, $\Lambda(v)$ is positive definite for v in a certain ball $\mathcal{B}_Y(R)$. In the Kerr case mentioned above, one has positive definiteness for all $v \in Y$ if $\chi > 0$.

In Theorem 4.9 of [17] and Remark 3.3 of [8] our Assumptions 2.1 and 2.2 without S_Y have been checked for the case $Q = 0$, i.e., for the system without conductivity. We note that nonzero $Q(v)$ can be treated as the operators $\Lambda(v)^{-1}$. For Assumption 2.2, let S_Y be the diagonal operator with entry $I+\Delta$. Then $B_Y(v)$ is the product $[S_Y, \Lambda(v)^{-1}(A + Q(v))]S_Y^{-1}$ for the commutator $[\cdot,\cdot]$. Using Sobolev's embedding and Hölder's inequality one checks that $B_Y(v)$ is an operator on X whose norm is uniformly bounded for $v \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$. Hence, Assumptions 2.1 and 2.2 are fulfilled in this example.

In [8] and [17] wave and Maxwell equations on a domain with Dirichlet boundary conditions were studied in a similar way. The Maxwell system with the usual boundary conditions of a perfect conductor can be treated in our framework only for special nonlinearities, see Proposition 4.8 in [17]. ⋄

3 Implicit Runge-Kutta methods

For the equation (5), the general s-stage Runge-Kutta method with s distinct nodes $0 \le c_i \le 1$ and weights $\mathcal{Q} = (a_{ij})_{i,j=1}^s$ and $b = (b_i)_{i=1}^s$ is given by

$$\dot U_{ni} = A_{U_{ni}} U_{ni}, \qquad i = 1,\dots,s, \tag{13a}$$

$$U_{ni} = u_n + \tau \sum_{j=1}^s a_{ij} \dot U_{nj}, \qquad i = 1,\dots,s, \tag{13b}$$

$$u_{n+1} = u_n + \tau \sum_{i=1}^s b_i \dot U_{ni} \tag{13c}$$

for $n = 0,\dots,N-1$ and a fixed stepsize $\tau > 0$. Here, $u_n \approx u(t_n)$ approximates the solution u to (5) at time $t_n = n\tau$ and $U_{ni} \approx u(t_n + c_i\tau)$ are the inner stages. In the next section we solve the above system for initial values u_0 in Z which belong to a certain ball of Y. To simplify notation, we set

$$U_n = \left(U_{n1},\dots,U_{ns}\right)^T, \qquad \mathcal{A}(U_n) = \operatorname{diag}\left(A_{U_{n1}},\dots,A_{U_{ns}}\right),$$

where $\mathcal{A}(U_n)$ has the domain $D(A)^s$ in X^s. We now reformulate (13) in a more compact form as

$$\begin{aligned} U_n &= \mathbb{1} \otimes u_n + \tau(\mathcal{Q} \otimes I)\mathcal{A}(U_n)U_n, \\ u_{n+1} &= u_n + \tau(b^T \otimes I)\mathcal{A}(U_n)U_n, \end{aligned} \tag{14}$$

where $\mathbb{1} = (1,\dots,1)^T$ and $\otimes$ denotes the Kronecker product. The first equation in (14) can be written as

$$(I_s \otimes I - \tau(\mathcal{Q} \otimes I)\mathcal{A}(U_n))U_n = \mathbb{1} \otimes u_n,$$

or equivalently

$$(\mathcal{Q}^{-1} \otimes I - \tau \mathcal{A}(U_n))U_n = (\mathcal{Q}^{-1}\mathbb{1}) \otimes u_n. \tag{15}$$

Here and throughout this paper we assume that the Runge-Kutta matrix has the inverse $\mathcal{Q}^{-1} = (\widetilde{a}_{ij})$.

Recall that a Runge–Kutta method is called *algebraically stable* if $b_i \geq 0$ for $i = 1, \dots, s$ and the matrix

$$\mathcal{M} = (m_{ij})_{i,j=1}^s \qquad \text{with} \qquad m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j \tag{16}$$

is positive semidefinite. It is well known that Gauß, Radau IA ($c_1 = 0$), and Radau IIA ($c_s = 1$) collocation methods are algebraically stable. See Definition IV.12.5 and Theorem IV.12.9 of [6].

For our analysis, we also need the *coercivity condition* that there exists a positive definite diagonal matrix $D \in \mathbb{R}^{s,s}$ and a positive scalar α such that

$$u^T D \mathcal{Q}^{-1} u \geq \alpha u^T D u \quad \text{for all} \qquad u \in \mathbb{R}^s. \tag{17}$$

This condition plays an important role in proving the existence of Runge–Kutta approximations, cf. Section IV.14 of [6]. For Gauß and Radau collocation methods, $d_1, \dots, d_n$ and the constant α are given explicitly in terms of the nodes c_i and the weights b_i, $i = 1, \dots, s$; see Theorem IV.14.5 of [6].

4 Well-posedness of the numerical scheme

In this section we construct solutions of the numerical scheme (13) and bound them in X, Y and Z. Let $W \in \{X, Y, Z\}$. In the following we denote the components of $U \in W^s$ by U_i, i.e., $U = (U_1, \dots, U_s)$, and W^s is equipped with the inner product

$$(U, V)_{W^s} = \sum_{i=1}^{s} (U_i, V_i)_W.$$

On W^s, for the operators from Section 2 and 3 we introduce the notation

$$\begin{aligned} \Lambda(U) &:= \operatorname{diag}(\Lambda(U_1), \dots, \Lambda(U_s)), & \mathcal{Q}(U) &:= \operatorname{diag}(Q(U_1), \dots, Q(U_s)), \\ \mathcal{B}(U) &:= \operatorname{diag}(B(U_1), \dots, B(U_s)), & \mathcal{D} &:= D \otimes I = \operatorname{diag}(d_1 I, \dots, d_s I), \\ \mathcal{S} &:= I_s \otimes S = \operatorname{diag}(S, \dots, S), & \mathcal{S}_Y &:= I_s \otimes S_Y = \operatorname{diag}(S_Y, \dots, S_Y). \end{aligned}$$

In the next lemma we consider one step of the scheme (13) for suitable u_n.

Lemma 4.1. *Let Assumptions 2.1 and 2.2 be fulfilled and let the Runge-Kutta method satisfy the coercivity condition (17). Let $r \geq 1$ and set $R_1 = (2\gamma c_0)^{-1} R$ for the constants defined in (8) and below in (24). Then there exists a maximal step size $\tau_0 = \tau_0(r) \in (0, 1]$ such that for all $u_n \in \overline{\mathcal{B}}_Y(R_1) \cap \overline{\mathcal{B}}_Z(r)$ and $\tau \in (0, \tau_0]$ equation (15) has a solution U_n in Z^s satisfying*

$$\left\| U_{ni} \right\|_X \leq \gamma (\lambda_X \nu)^{1/2} \left\| u_n \right\|_X, \qquad \left\| U_{ni} \right\|_Z \leq 2\gamma c_0 \left\| u_n \right\|_Z, \tag{18}$$

$$\left\|U_{ni}\right\|_Y \le 2\gamma c_0 \left\|u_n\right\|_Y \le R, \qquad \left\|u_{n+1}\right\|_Y \le R, \tag{19}$$

for all $i = 1, \ldots, s$, where u_{n+1} is given by (13c). The number τ_0 depends only on r and the constants in the assumptions.

Proof. Step 1. The proof is based on Banach's fixed-point theorem. We start with some preparations. Let $r > 0$ and $\tau \in (0, 1]$. The set

$$M_{r,\tau} = \Big\{ V = (V_i)_{i=1}^s \in Z^s : \|V\|_{Z^s} \le \widehat{c}r,\ \left\|V_i - V_j\right\|_Y \le \widehat{c}r\tau, \\ \left\|V_i\right\|_Y \le R \quad \text{for all } i, j = 1, \ldots, s \Big\} \tag{20}$$

is endowed with the distance induced by the norm of X^s, where the constant $\widehat{c} > 0$ will be fixed below. This metric space is complete. Indeed, each Cauchy sequence (V^n) in $M_{r,\tau}$ has a limit V in X^s, and it is bounded in both Y^s and Z^s. Hence, a subsequence of (V^n) converges to V also weakly in Y^s and Z^s, so that V belongs to $M_{r,\tau}$.

Let $V \in M_{r,\tau}$. To define the fixed-point map, we introduce the operator

$$G(V) := \mathcal{O}\!\!\!\iota^{-1} \otimes I - \tau \mathcal{A}(V) \quad \text{with} \quad D(G(V)) = D(A)^s$$

in X^s. We next show that $G(V) - \frac{\alpha}{2} I_s \otimes I$ is m-accretive with respect to the equivalent inner product on X^s given by

$$(W, \widetilde{W})_{D\otimes\Lambda(V)} := (\mathcal{D}\Lambda(V)W, \widetilde{W})_{X^s} = \sum_{i=1}^s d_i(\Lambda(V_i)W_i, \widetilde{W}_i)_X.$$

We write $X^s_{D,V}$ for the Hilbert space X^s equipped with this inner product. Let $W = (W_i)_{i=1}^s \in X^s$. We first observe that

$$((G(V) - \tfrac{\alpha}{2} I_s \otimes I)W, W)_{D\otimes\Lambda(V)} = ((\mathcal{D}\Lambda(V)\mathcal{O}\!\!\!\iota^{-1} \otimes I - \tfrac{\alpha}{2}\mathcal{D}\Lambda(V))W, W)_{X^s} \\ -\tau((D \otimes A)W, W)_{X^s} - \tau(\mathcal{D}\mathcal{Q}(V)W, W)_{X^s}.$$

The second term on the right-hand side vanishes due to the skew-adjointness of A, see Assumption 2.1, and the third one is bounded by $\tau\mu_X \sum_{i=1}^s d_i \left\|W_i\right\|_X^2$ because of (3a). To treat the first one, we compute

$$\begin{aligned}
&((\mathcal{D}\Lambda(V)\mathcal{O}\!\!\!\iota^{-1} \otimes I - \tfrac{\alpha}{2}\mathcal{D}\Lambda(V))W, W)_{X^s} \\
&\quad = ((D\mathcal{O}\!\!\!\iota^{-1} \otimes I - \tfrac{\alpha}{2}\mathcal{D})\Lambda(V)^{1/2}W, \Lambda(V)^{1/2}W)_{X^s} \\
&\qquad + \sum_{i,j=1}^s d_i \widetilde{a}_{ij}((\Lambda(V_i)^{1/2} - \Lambda(V_j)^{1/2})W_j, \Lambda(V_i)^{1/2}W_i)_X \\
&\quad \ge \frac{\alpha}{2} \sum_{i=1}^s d_i \left\|\Lambda(V_i)^{1/2}W_i\right\|_X^2 \\
&\qquad + \sum_{i,j=1}^s d_i \widetilde{a}_{ij}((\Lambda(V_i)^{1/2} - \Lambda(V_j)^{1/2})W_j, \Lambda(V_i)^{1/2}W_i)_X,
\end{aligned}$$

where we employ the coercivity property (17) in the last step. Lemma 2.4, the definition (20) of $M_{r,\tau}$, condition (2a), and Hölder's inequality allow us to dominate the last term by

$$\sum_{i,j=1}^{s} d_i^{1/2} c_{\mathrm{RK}} \ell' \widehat{c} \tau r \, \|W_j\|_X \, \|\Lambda(V_i)^{1/2} W_i\|_X \le C r \tau \sum_{k=1}^{s} d_k \big\|\Lambda(V_k)^{1/2} W_k\big\|_X^2 .$$

There thus exists a number $\tau_0 = \tau_0(r) \in (0,1]$ such that the map $G(V) - \frac{\alpha}{2} I_s \otimes I$ is accretive in $X^s_{D,V}$ for all $\tau \in (0, \tau_0]$.

Remark 2.5 implies that the operator $\tau \mathcal{A}(V)$ generates a strongly continuous semigroup on X^s, so that also $-G(V)$ is a generator by bounded perturbation, see Theorem III.1.3 in [4]. In particular, the sum $\omega I_s \otimes I + G(V)$ is invertible in X^s for a sufficiently large number $\omega > 0$. The operator $G(V) - \frac{\alpha}{2} I_s \otimes I$ is thus m-accretive in $X^s_{D,V}$. Hence, $G(V)$ has an inverse in X^s which satisfies

$$\|G(V)^{-1}\|_{X^s_{D,V} \leftarrow X^s_{D,V}} \le \frac{2}{\alpha}, \tag{21}$$

cf. Proposition II.3.23 in [4].

For a given $u_n \in X$, we can now define the fixed-point map

$$\Phi : M_{r,\tau} \to X^s, \quad \Phi(V) = G(V)^{-1} (\mathcal{O}\!\!\!\iota^{-1} \mathbb{1}) \otimes u_n .$$

Observe that the equation $W = \Phi(V)$ is equivalent to

$$(\mathcal{O}\!\!\!\iota^{-1} \otimes I - \tau \mathcal{A}(V)) W = (\mathcal{O}\!\!\!\iota^{-1} \mathbb{1}) \otimes u_n, \tag{22}$$

which means that

$$W_i = u_n + \tau \sum_{j=1}^{s} a_{ij} A_{V_i} W_i, \qquad i = 1, \ldots, s.$$

Consequently, a fixed point of Φ solves (15).

Step 2. We establish that Φ maps $M_{r,\tau}$ into itself for all sufficiently small step sizes $\tau > 0$ and a suitable constant $\widehat{c}$, provided that u_n belongs to $\overline{\mathcal{B}}_Y(R_1) \cap \overline{\mathcal{B}}_Z(r)$. Let $V \in M_{r,\tau}$ and set $W = \Phi(V)$. The bound (21) yields

$$\big\|W\big\|_{D \otimes \Lambda(V)} = \|G(V)^{-1} (\mathcal{O}\!\!\!\iota^{-1} \mathbb{1}) \otimes u_n\|_{D \otimes \Lambda(V)} \le \frac{2}{\alpha} \big\|(\mathcal{O}\!\!\!\iota^{-1} \mathbb{1}) \otimes u_n\big\|_{D \otimes \Lambda(V)} .$$

By means of (2a) and (4), we derive the inequality

$$\|W\|_{X^s} \le \gamma (\nu \lambda_X)^{1/2} \, \|u_n\|_X \tag{23}$$

for the constant

$$\gamma = \max\Big\{1, \frac{2}{\alpha\sqrt{\delta}} \Big(\sum_{i=1}^{s} d_i \Big(\sum_{j=1}^{s} |\widetilde{a}_{ij}| \Big)^2 \Big)^{1/2} \Big\} \quad \text{with} \quad \delta := \min_{k=1,\ldots,s} d_k > 0. \tag{24}$$

We transfer these estimates to Z in order to check that W satisfies the first and third condition in (20). To this aim, we multiply (22) by the operator matrix $\mathcal{S} = I_s \otimes S$. Property (7a) of A now implies the equation

$$(\mathcal{O}\!\!\!\!\mathcal{I}^{-1} \otimes I - \tau\mathcal{A}(V) - \tau\mathcal{B}(V))\mathcal{S}W = (\mathcal{O}\!\!\!\!\mathcal{I}^{-1}\mathbb{1}) \otimes (Su_n). \tag{25}$$

Since V belongs to $M_{r,\tau}$, each norm $\|V_i\|_Z$ is bounded by $\widehat{c}r$ so that assumption (7b) provides the inequality $\|\mathcal{B}(V)\|_{X^s\leftarrow X^s} \le \beta_Z(\widehat{c}r)$. In view of (10), the norm of $\mathcal{B}(V)$ on $X^s_{D,V}$ is then dominated by $C\beta_Z(\widehat{c}r) =: \beta'(r)$. We replace $\tau_0(r)$ by $\min\{\tau_0(r), \alpha/(4\beta'(r))\}$ and take $\tau \in (0,\tau_0(r)]$. Because of (21), the sum $G(V) - \tau\mathcal{B}(V)$ thus has an inverse on $X^s_{D,V}$ with norm less or equal $4/\alpha$. Hence, formula (25) leads to

$$\big\|\mathcal{S}W\big\|_{D\otimes\Lambda(V)} \le \frac{4}{\alpha}\big\|(\mathcal{O}\!\!\!\!\mathcal{I}^{-1}\mathbb{1}) \otimes Su_n\big\|_{D\otimes\Lambda(V)}.$$

Using also Assumptions 2.1 and 2.2 and $\|u_n\|_Z \le r$, we arrive at the estimate

$$\|W\|_{Z^s} \le 2\gamma c_0\, \|u_n\|_Z \le \widehat{c}_1 r \tag{26}$$

for $c_0 \ge (\nu\lambda_X)^{1/2}$ from (8), all $\tau \in (0,\tau_0(r)]$, and $\widehat{c}_1 := 2\gamma c_0$. In (23) and (26) we bounded the norms of the linear map $u_n \mapsto W$ in $\mathcal{L}(X, X^s)$ and $\mathcal{L}(Z, Z^s)$, respectively. By interpolation, we now obtain the bound

$$\|W_i\|_Y \le 2\gamma c_0\|u_n\|_Y \le 2\gamma c_0 R_1 = R \tag{27}$$

for all $i = 1,\dots,s$. In the second inequality we use that u_n is contained in $\overline{\mathcal{B}}_Y(R_1)$. As a result, the vector W fulfills the last condition in (20).

For the second condition in (20), we employ (9a) and (26) to compute

$$\begin{aligned}\big\|W_i - W_j\big\|_Y &= \Big\|\tau\sum_{k=1}^{s}(a_{ik} - a_{jk})A_{V_k}W_k\Big\|_Y \le \tau c_{\mathrm{RK}}\alpha_Y\sum_{k=1}^{s}\|W_k\|_Z\\ &\le \tau\widehat{c}_2\|u_n\|_Z \le \tau\widehat{c}_2 r,\end{aligned}$$

where $\widehat{c}_2 := c_{\mathrm{RK}}\alpha_Y c_0$. With $\widehat{c} := \max\{\widehat{c}_1, \widehat{c}_2\}$, the above bounds imply that $W = \Phi(V)$ belongs to $M_{r,\tau}$.

Step 3. We show that $\Phi : M_{r,\tau} \to M_{r,\tau}$ is a strict contraction for the norm of X^s. We take $V, \widetilde{V} \in M_{r,\tau}$ and set $W = \Phi(V)$, $\widetilde{W} = \Phi(\widetilde{V})$ and $E = W - \widetilde{W}$. The definition (22) of Φ then yields the identity

$$G(V)E = (\mathcal{O}\!\!\!\!\mathcal{I}^{-1} \otimes I - \tau\mathcal{A}(V))E = \tau(\mathcal{A}(V) - \mathcal{A}(\widetilde{V}))\widetilde{W}.$$

From Assumption 2.1, (21) and (9b) we deduce

$$\begin{aligned}\|E\|_{X^s} &\le c_{\mathrm{RK}}\nu^{1/2}\,\big\|E\big\|_{D\otimes\Lambda(V)} \le \frac{2c_{\mathrm{RK}}\nu^{1/2}}{\alpha}\,\|\tau(\mathcal{A}(V) - \mathcal{A}(\widetilde{V}))\widetilde{W}\|_{D\otimes\Lambda(V)}\\ &\le \frac{2c_{\mathrm{RK}}(\nu\lambda_X)^{1/2}}{\alpha}\,\tau\|(\mathcal{A}(V) - \mathcal{A}(\widetilde{V}))\widetilde{W}\|_{X^s}\end{aligned}$$

$$\leq \frac{2c_{\mathrm{RK}}(\nu\lambda_X)^{1/2}L_X}{\alpha}\,\tau\|V-\widetilde{V}\|_{X^s}\|\widetilde{W}\|_{Z^s}$$
$$\leq \frac{2c_{\mathrm{RK}}\widehat{c}(\nu\lambda_X)^{1/2}L_X}{\alpha}\,r\tau\|V-\widetilde{V}\|_{X^s},$$

where we have also used that $\widetilde{W}$ belongs to $M_{r,\tau}$. Decreasing the maximal step size $\tau_0 = \tau_0(r) > 0$ from Step 2 if necessary, we infer the strict contractivity of Φ on $M_{r,\tau}$ for every $\tau \in (0, \tau_0]$. This map thus possesses a (unique) fixed point U_n in $M_{r,\tau}$ which then solves (15). The inequalities (18) and the first one in (19) now follow from estimates (23), (26), and (27).

Finally, the formulas (13) and the bounds (9a) and (26) yield

$$\|u_{n+1}\|_Y \leq R_1 + \tau c_{\mathrm{RK}}\alpha_Y \sum_{i=s}^{s}\|U_{ni}\|_Z \leq R_1 + C\tau r \leq R$$

for all $\tau \in (0, \tau_0]$, after decreasing $\tau_0 = \tau_0(r) > 0$ once more if necessary. □

Lemma 4.1 allows us to solve the numerical scheme (13) for step sizes $\tau \in (0, \tau_0(r)]$ as long as u_n stays in $\overline{\mathcal{B}}_Y(R_1) \cap \overline{\mathcal{B}}_Z(r)$. However, the estimates in this lemma are too coarse to show reasonable bounds for the numerical solution u_n by iteration. Instead, we next employ energy-type estimates for the inner products on X corresponding to u_n. The isomorphisms S_Y and S are used to transfer the bounds to Y and Z, respectively. The precise form of the constants in the next result is crucial for the iteration argument.

Lemma 4.2. *Let Assumptions 2.1 and 2.2 be fulfilled and let the Runge-Kutta method be algebraically stable and satisfy the coercivity condition (17). Let $r \geq 1$ and take the radius $R_1 = (2\gamma c_0)^{-1}R$ and the maximal step size $\tau_0 = \tau_0(r) \in (0, 1]$ from Lemma 4.1. For all $u_n \in \overline{\mathcal{B}}_Y(R_1) \cap \overline{\mathcal{B}}_Z(r)$ and $\tau \in (0, \tau_0]$ the equations (13) then have a solution $u_{n+1} \in Z$ satisfying*

$$\left\|u_{n+1}\right\|_{u_{n+1}} \leq e^{c_X r\tau}\left\|u_n\right\|_{u_n}, \tag{28a}$$
$$\left\|S_Y u_{n+1}\right\|_{u_{n+1}} \leq e^{c_Y(r+\beta_Y(r))\tau}\left\|S_Y u_n\right\|_{u_n}, \tag{28b}$$
$$\left\|S u_{n+1}\right\|_{u_{n+1}} \leq e^{c_Z(r+\beta_Z(r))\tau}\left\|S u_n\right\|_{u_n}. \tag{28c}$$

The numbers c_X, c_Y, and c_Z depend only on the constants in Assumption 2.1, on c_{RK}, and on c_0.

Proof. The existence of the solution u_{n+1} in Z follows from Lemma 4.1. By this result and the assumptions, the vectors u_n, u_{n+1} and U_{ni} belong to the ball $\overline{\mathcal{B}}_Y(R)$ which is needed for the following estimates. To show (28a), we take the inner product in $X_{u_{n+1}}$ of (13c) with itself and thus obtain

$$\left\|u_{n+1}\right\|_{u_{n+1}}^2 = \left\|u_n\right\|_{u_{n+1}}^2 + 2\tau\sum_{i=1}^{s} b_i(u_n, \dot{U}_{ni})_{u_{n+1}} + \tau^2\sum_{i,j=1}^{s} b_i b_j(\dot{U}_{ni}, \dot{U}_{nj})_{u_{n+1}}.$$

In the second term of the right-hand side we insert formula (13b) for u_n. Manipulating the resulting double sum, we deduce the identity

$$\begin{aligned}\|u_{n+1}\|^2_{u_{n+1}} = \|u_n\|^2_{u_{n+1}} + 2\tau \sum_{i=1}^{s} b_i (U_{ni}, \dot{U}_{ni})_{u_{n+1}} \\ + \tau^2 \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji})(\dot{U}_{ni}, \dot{U}_{nj})_{u_{n+1}}.\end{aligned} \tag{29}$$

The last term on the right-hand side is nonpositive due to the algebraic stability condition (16). For the first term, estimate (11) and equation (13c) yield that

$$\|u_n\|^2_{u_{n+1}} \le (1 + \ell\nu \|u_n - u_{n+1}\|_Y)\|u_n\|^2_{u_n} = \Big(1 + \ell\nu\tau \Big\| \sum_{i=1}^{s} b_i \dot{U}_{ni}\Big\|_Y\Big)\|u_n\|^2_{u_n}.$$

From the expression $\dot{U}_{ni} = A_{U_{ni}} U_{ni}$, inequality (9a) and Lemma 4.1, we infer the bound

$$\|\dot{U}_{ni}\|_Y \le \alpha_Y 2\gamma c_0 \, \|u_n\|_Z \le Cr \tag{30}$$

using the assumption $\|u_n\|_Z \le r$. Consequently, we have

$$\|u_n\|^2_{u_{n+1}} \le (1 + Cr\tau)\|u_n\|^2_{u_n}. \tag{31}$$

To control the second term in (29), we write

$$\begin{aligned}(U_{ni}, \dot{U}_{ni})_{u_{n+1}} &= (U_{ni}, A_{U_{ni}} U_{ni})_{U_{ni}} + \big((\Lambda(u_{n+1}) - \Lambda(U_{ni}))U_{ni}, \dot{U}_{ni}\big)_X \\ &= (U_{ni}, (A + Q(U_{ni}))U_{ni})_X + \big((\Lambda(u_{n+1}) - \Lambda(U_{ni}))U_{ni}, \dot{U}_{ni}\big)_X.\end{aligned} \tag{32}$$

The first term in the last line is bounded by $\mu_X \|U_{ni}\|^2_X$ due to Assumption 2.1 (a) and (c). Condition (2b) further implies

$$((\Lambda(u_{n+1}) - \Lambda(U_{ni}))U_{ni}, \dot{U}_{ni})_X \le \ell \, \|u_{n+1} - U_{ni}\|_Y \|U_{ni}\|_X \|\dot{U}_{ni}\|_X.$$

We next subtract (13b) from (13c) and then use (30) to deduce the estimate

$$\|u_{n+1} - U_{ni}\|_Y = \tau \Big\| \sum_{j=1}^{s} (b_j - a_{ij})\dot{U}_{nj}\Big\|_Y \le Cr\tau. \tag{33}$$

On the other hand, formula (13b) yields the identity

$$\dot{U}_{ni} = \frac{1}{\tau} \sum_{j=1}^{s} \widetilde{a}_{ij}(U_{nj} - u_n),$$

so that the inequality

$$\|\dot{U}_{ni}\|_X \le \frac{c_{\mathrm{RK}}}{\tau} \sum_{j=1}^{s} \|U_{nj} - u_n\|_X \le \frac{C}{\tau} \, \|u_n\|_X \tag{34}$$

follows from Lemma 4.1. Together with the norm equivalence (10) and the above bounds, equation (32) lead to

$$(U_{ni}, \dot{U}_{ni})_{u_{n+1}} \leq \mu_X \|U_{ni}\|_X^2 + Cr\,\|u_n\|_X\,\|U_{ni}\|_X \leq Cr\,\|u_n\|_{u_n}^2, \tag{35}$$

where we also employed Lemma 4.1. The relations (29), (31), and (35) now show the estimate $\|u_{n+1}\|_{u_{n+1}}^2 \leq (1 + c_X r\tau)\,\|u_n\|_{u_n}^2$. The number c_X only depends on c_0 and the constants given by Assumption 2.1 and the Runge-Kutta scheme. So we have established (28a).

For (28c), we multiply (13c) by S and obtain as above

$$\|Su_{n+1}\|_{u_{n+1}}^2 \leq \|Su_n\|_{u_{n+1}}^2 + 2\tau \sum_{i=1}^{s} b_i (SU_{ni}, S\dot{U}_{ni})_{u_{n+1}}.$$

The first term on the right-hand side can be controlled as in (31). We rewrite the second term as

$$\begin{aligned}(SU_{ni}, S\dot{U}_{ni})_{u_{n+1}} &= (SU_{ni}, SA_{U_{ni}}U_{ni})_{U_{ni}} + ((\Lambda(u_{n+1}) - \Lambda(U_{ni}))SU_{ni}, S\dot{U}_{ni})_X \\ &= \big(SU_{ni}, (A + Q(U_{ni}) + \Lambda(U_{ni})B(U_{ni}))SU_{ni}\big)_X \\ &\quad + \big((\Lambda(u_{n+1}) - \Lambda(U_{ni}))SU_{ni}, S\dot{U}_{ni}\big)_X\end{aligned}$$

by means of condition (7a). By Assumption 2.2 and Lemma 4.1 the norm of $B(U_{ni})$ on X is less or equal the number $\beta_Z(r)$. Moreover, Lemma 4.1 and Assumptions 2.1 and 2.2 provide the estimates

$$\|SU_{ni}\|_X \leq \|S\|_{X\leftarrow Z}\|U_{ni}\|_Z \leq C\,\|S\|_{X\leftarrow Z}\|S^{-1}Su_n\|_Z \leq C\,\|Su_n\|_{u_n}.$$

As in (34) we then derive the inequality $\|S\dot{U}_{ni}\|_X \leq C\tau^{-1}\,\|Su_n\|_{u_n}$. Arguing as above, one now establishes assertion (28c). Inequality (28b) can be shown in the same way using S_Y and B_Y instead of S and B. □

Given an arbitrary radius $r \geq 1$ we can now solve the system (13) within $\overline{\mathcal{B}}_Y(R_1) \cap \overline{\mathcal{B}}_Z(r)$ and up to a time $T_1(r) > 0$ provided that u_0 belongs to somewhat smaller balls. The constants in the next results behave similar as in Theorem 2.3.

Proposition 4.3. *Let Assumptions 2.1 and 2.2 be fulfilled and let the Runge-Kutta method be algebraically stable and satisfy the coercivity condition (17). Let $r \geq 1$ and take the radii $R_0 = (2c_0)^{-1}R_1 = (4\gamma c_0^2)^{-1}R$ and $r_0 = (2c_0)^{-1}r$ and the maximal step size $\tau_0 = \tau_0(r) \in (0,1]$ from Lemma 4.1. Fix the maximal time*

$$T_1 = \min\Big\{\frac{\ln 2}{c_Y(r + \beta_Y(r))}, \frac{\ln 2}{c_Z(r + \beta_Z(r))}\Big\}$$

for the constants from Lemma 4.2. Let $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z(r_0)$. We can then solve scheme (13) for $n \leq T_1/\tau$ and the solution u_n satisfies

$$\begin{aligned}\|u_n\|_X &\leq (\nu\lambda_X)^{1/2} e^{rc_X n\tau}\|u_0\|_X, \\ \|u_n\|_Y &\leq c_0 e^{c_Y(r+\beta_Y(r))n\tau}\|u_0\|_Y, \\ \|u_n\|_Z &\leq c_0 e^{c_Z(r+\beta_Z(r))n\tau}\|u_0\|_Z.\end{aligned}$$

Proof. Note that $c_0 e^{c_Y(r+\beta_Y(r))n\tau}\|u_0\|_Y \le R_1$ and $c_0 e^{c_Z(r+\beta_Z(r))n\tau}\|u_0\|_Z \le r$ for $n \le T_1/\tau$. Using also (10) and (8), one can now iterate the bounds in (28) to deduce the result. □

5 The convergence result in X

Let Assumptions 2.1 and 2.2 hold. For the Runge-Kutta method we assume that it is algebraically stable, satisfies the coercivity condition (17), and has stage order s and order at least $s+1$. These assumptions are satisfied for Gauss collocation methods with $s \ge 1$ and Radau collocation methods with $s \ge 2$.

Moreover, let $r \ge 1$, $\tau \in (0, \tau_0(r)]$, $R_0 = (4\gamma c_0^2)^{-1}R$, $R_1 = (2\gamma c_0)^{-1}R$ and $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z((2c_0)^{-1}r)$, cf. Lemma 4.1. Let $T = T(r) > 0$ be the minimum of the existence times $T_0 = T_0(r)$ from Theorem 2.3 and $T_1 = T_1(r)$ from Proposition 4.3. These results provide a solution u in $C([0,T], Z) \cap C([0,T], Y)$ of the evolution equation (5) and a solution u_n of the scheme (13) for $n \in \mathbb{N}_0$ with $n \le T/\tau$, both staying in the balls $\overline{\mathcal{B}}_Y(R_1)$ and $\overline{\mathcal{B}}_Z(r)$. We assume in addition that $u^{(s+1)} \in L^2([0,T], D(A))$ and $u^{(s+2)} \in L^2([0,T], X)$. We now set

$$\widetilde{u}_n := u(t_n), \qquad \widetilde{U}_{ni} := u(t_n + c_i\tau), \qquad \dot{\widetilde{U}}_{ni} := u'(t_n + c_i\tau)$$

for $n \in \mathbb{N}$ and $i = 1, \dots, s$ with the nodes $c_i \in [0,1]$ of the Runge-Kutta scheme. We then introduce the defects Δ_{ni} and δ_{n+1} by the equations

$$\dot{\widetilde{U}}_{ni} = A_{\widetilde{U}_{ni}} \widetilde{U}_{ni}, \qquad i = 1, \dots, s, \tag{36a}$$

$$\widetilde{U}_{ni} = \widetilde{u}_n + \tau \sum_{j=1}^{s} a_{ij} \dot{\widetilde{U}}_{ni} + \Delta_{ni}, \qquad i = 1, \dots, s, \tag{36b}$$

$$\widetilde{u}_{n+1} = \widetilde{u}_n + \tau \sum_{i=1}^{s} b_i \dot{\widetilde{U}}_{ni} + \delta_{n+1}. \tag{36c}$$

By assumption, the defects of the Runge–Kutta method are given by

$$\Delta_{ni} = \tau^s \int_{t_n}^{t_{n+1}} u^{(s+1)}(t)\kappa_i\Big(\frac{t - t_n}{\tau}\Big)\,dt, \tag{37a}$$

$$\delta_{n+1} = \tau^{s+1} \int_{t_n}^{t_{n+1}} u^{(s+2)}(t)\kappa\Big(\frac{t - t_n}{\tau}\Big)\,dt. \tag{37b}$$

Here κ_i and κ denote the Peano kernels corresponding to the quadrature rules defining the Runge–Kutta method. They are uniformly bounded with constants depending on the Runge–Kutta coefficients only.

The errors are next given by

$$e_n := u_n - \widetilde{u}_n, \qquad E_{ni} := U_{ni} - \widetilde{U}_{ni}, \qquad \dot{E}_{ni} := \dot{U}_{ni} - \dot{\widetilde{U}}_{ni}$$

for $n \in \mathbb{N}$ and $i = 1, \dots, s$. Formulas (36) and (13) lead to the expressions

$$\dot{E}_{ni} = A_{U_{ni}} E_{ni} + (A_{U_{ni}} - A_{\widetilde{U}_{ni}}) \widetilde{U}_{ni}, \qquad i = 1, \dots, s, \tag{38a}$$

$$E_{ni} = e_n + \tau \sum_{j=1}^{s} a_{ij} \dot{E}_{ni} - \Delta_{ni}, \qquad i = 1, \dots, s, \tag{38b}$$

$$e_{n+1} = e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni} - \delta_{n+1}. \tag{38c}$$

As in (14) we rewrite the system (38) in the compact form

$$\dot{E}_n = \mathcal{A}(U_n) E_n + (\mathcal{A}(U_n) - \mathcal{A}(\widetilde{U}_n)) \widetilde{U}_n, \tag{39a}$$

$$E_n = \mathbb{1} \otimes e_n + \tau(\mathcal{O} \otimes I) \dot{E}_n - \Delta_n, \tag{39b}$$

$$e_{n+1} = e_n + \tau(b^T \otimes I) \dot{E}_n - \delta_{n+1}. \tag{39c}$$

For the convergence proof we use the energy technique from [15,16] in combination with solution-dependent norms, starting with the basic error recursion.

Lemma 5.1. *Under the assumptions stated at the beginning of this section, the error* $e_n = u_n - u(t_n)$ *satisfies*

$$\begin{aligned} \|e_{n+1}\|_{u_{n+1}}^2 &\le (1 + Cr\tau) \|e_n\|_{u_n}^2 \\ &\quad + Cr\tau \left(\|E_n\|_{X^s}^2 + \|\Delta_n\|_{D(A)^s}^2 \right) + C\tau \left\| \tfrac{1}{\tau} \delta_{n+1} \right\|_X^2 \end{aligned} \tag{40}$$

for all $\tau \in (0, \tau_0]$ *and* $n \in \mathbb{N}$ *with* $n \le T/\tau$.

Proof. Taking the u_{n+1}-inner product of (38c) with itself, we compute

$$\begin{aligned} \|e_{n+1}\|_{u_{n+1}}^2 &= \Big\| e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni} \Big\|_{u_{n+1}}^2 \\ &\quad - 2 \Big(e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni}, \delta_{n+1} \Big)_{u_{n+1}} + \|\delta_{n+1}\|_{u_{n+1}}^2. \end{aligned} \tag{41}$$

The last term is dominated by the right-hand side of (40). Equation (38b) and the algebraic stability of the Runge-Kutta scheme imply the inequality

$$\Big\| e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni} \Big\|_{u_{n+1}}^2 \le \|e_n\|_{u_{n+1}}^2 + 2\tau \sum_{i=1}^{s} b_i (E_{ni} + \Delta_{ni}, \dot{E}_{ni})_{u_{n+1}}, \tag{42}$$

cf. (29). As in inequality (31) one sees that

$$\|e_n\|_{u_{n+1}}^2 \le (1 + Cr\tau) \|e_n\|_{u_n}^2.$$

To bound the second term on the right-hand side of (42), we write

$$(E_{ni} + \varDelta_{ni}, \dot{E}_{ni})_{u_{n+1}} = (E_{ni} + \varDelta_{ni}, \dot{E}_{ni})_{U_{ni}} + ((\Lambda(u_{n+1}) - \Lambda(U_{ni})(E_{ni} + \varDelta_{ni}), \dot{E}_{ni})_X. \tag{43}$$

In the first term on the right-hand side we replace $\dot{E}_{ni}$ by (38a). Estimate (3a) and the skew-adjointness of A then yield

$$\begin{aligned}(E_{ni} + \varDelta_{ni}, A_{U_{ni}} E_{ni})_{U_{ni}} &= (E_{ni} + \varDelta_{ni}, \big(A + Q(U_{ni})\big) E_{ni})_X \\ &\le \mu_X \|E_{ni}\|_X^2 - (A\varDelta_{ni}, E_{ni})_X + \mu_X \|\varDelta_{ni}\|_X \|E_{ni}\|_X \\ &\le C\big(\|\varDelta_{ni}\|_{D(A)}^2 + \|E_{ni}\|_X^2\big).\end{aligned} \tag{44}$$

Using (9b) and $\|\widetilde{U}_{ni}\|_Z \le r$, we also infer

$$\begin{aligned}(E_{ni} + \varDelta_{ni}, (A_{U_{ni}} - A_{\widetilde{U}_{ni}})\widetilde{U}_{ni})_{U_{ni}} &\le C \|E_{ni} + \varDelta_{ni}\|_X \|E_{ni}\|_X \|\widetilde{U}_{ni}\|_Z \\ &\le Cr \left(\|\varDelta_{ni}\|_X^2 + \|E_{ni}\|_X^2\right).\end{aligned} \tag{45}$$

To control the second term in (43), we replace $\dot{E}_{ni}$ by

$$\dot{E}_{ni} = \frac{1}{\tau} \sum_{j=1}^{s} \widetilde{a}_{ij} (E_{nj} + \varDelta_{nj} - e_n),$$

see (38b). The Lipschitz condition (2b) and the inequality (33) now lead to

$$\begin{aligned}&((\Lambda(u_{n+1}) - \Lambda(U_{ni}))(E_{ni} + \varDelta_{ni}), \dot{E}_{ni})_X \\ &\quad\le \frac{1}{\tau} c_{\mathrm{RK}} \ell \|u_{n+1} - U_{ni}\|_Y \|E_{ni} + \varDelta_{ni}\|_X \sum_{j=1}^{s} (\|E_{nj}\|_X + \|\varDelta_{nj}\|_X + \|e_n\|_X) \\ &\quad\le Cr\big(\|E_n\|_{X^s}^2 + \|\varDelta_n\|_{X^s}^2 + \|e_n\|_X^2\big).\end{aligned}$$

Combining the above formulas, we arrive at the estimate

$$\Big\|e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni}\Big\|_{u_{n+1}}^2 \le (1 + Cr\tau)\|e_n\|_{u_n}^2 + Cr\tau\Big(\|E_n\|_{X^s}^2 + \|\varDelta_n\|_{D(A)^s}^2\Big).$$

The second term in (41) is bounded as in the linear case, see (3.18) in [7], by

$$\Big(e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni}, \delta_{n+1}\Big)_{u_{n+1}} \le C\tau\Big(\|e_n\|_X^2 + \|E_n\|_{X^s}^2 + \|\varDelta_n\|_{X^s}^2 + \|\tfrac{1}{\tau}\delta_{n+1}\|_X^2\Big).$$

The norm equivalence (10) then yields the assertion (40). □

In the next lemma we control the errors E_n of the inner stages by $\|e_n\|_X$ and the defects.

Lemma 5.2. *Let the assumptions stated at the beginning of this section be fulfilled. Possibly after decreasing the maximal step size $\tau_0 = \tau_0(r) \in (0,1]$, we obtain*

$$\sum_{i=1}^{s} \|E_{ni}\|_X^2 \le C\big(\|\Delta_n\|_{X^s}^2 + \|e_n\|_X^2\big) \tag{46}$$

for all $\tau \in (0,\tau_0]$ and $n \in \mathbb{N}$ with $n \le T/\tau$.

Proof. The formulas (39) implies the identity

$$E_n = \mathbb{1} \otimes e_n + \tau(\mathcal{O} \otimes I)\mathcal{A}(U_n)E_n + \tau(\mathcal{O} \otimes I)(\mathcal{A}(U_n) - \mathcal{A}(\widetilde{U}_n))\widetilde{U}_n - \Delta_n.$$

Multiplying by $D\mathcal{O}^{-1} \otimes I$, we obtain

$$\begin{aligned}(D\mathcal{O}^{-1} \otimes I)E_n &= (D\mathcal{O}^{-1} \otimes I)(\mathbb{1} \otimes e_n - \Delta_n) \\ &\quad + \tau\mathcal{D}\mathcal{A}(U_n)E_n + \tau\mathcal{D}(\mathcal{A}(U_n) - \mathcal{A}(\widetilde{U}_n))\widetilde{U}_n.\end{aligned} \tag{47}$$

We now take the inner product of this equation with the vector $\Lambda(U_n)E_n$. To treat the left-hand side, we write

$$\Lambda(U_n) = I_s \otimes \Lambda(u_{n+1}) + \big(\Lambda(U_n) - I_s \otimes \Lambda(u_{n+1})\big).$$

The assumptions (2a) and (2b) on Λ and the coercivity (17) of the Runge–Kutta method allow us to estimate

$$\begin{aligned}&(\Lambda(U_n)E_n, (D\mathcal{O}^{-1} \otimes I)E_n) \\ &\quad= ((I_s \otimes \Lambda(u_{n+1})^{1/2})E_n, (D\mathcal{O}^{-1} \otimes I)(I_s \otimes \Lambda(u_{n+1})^{1/2})E_n)_{X^s} \\ &\qquad+ (\big(\Lambda(U_n) - I_s \otimes \Lambda(u_{n+1})\big)E_n, (D\mathcal{O}^{-1} \otimes I)E_n)_{X^s} \\ &\quad\ge \frac{\alpha\delta}{\nu}\|E_n\|_{X^s}^2 - c_{\mathrm{RK}}\ell \sum_{i,j=1}^{s} \|U_{ni} - u_{n+1}\|_Y \|E_{ni}\|_X \|E_{nj}\|_X\end{aligned}$$

with $\delta = \min_i d_i > 0$. Inequality (33) yields $\|U_{ni} - u_{n+1}\|_Y \le Cr\tau$. After possibly decreasing $\tau_0(r) > 0$, we thus deduce the lower bound

$$(\Lambda(U_n)E_n, (D\mathcal{O}^{-1} \otimes I)E_n)_{X^s} \ge \frac{\alpha\delta}{2\nu}\|E_n\|_{X^s}^2.$$

For the first term on the right-hand side of (47), it follows

$$\begin{aligned}&(\Lambda(U_n)E_n, (D\mathcal{O}^{-1} \otimes I)(\mathbb{1} \otimes e_n - \Delta_n))_{X^s} \\ &\qquad\le \lambda_X c_{\mathrm{RK}} \|E_n\|_{X^s}\big(\|e_n\|_X + \|\Delta_n\|_{X^s}\big) \\ &\qquad\le C\varepsilon\|E_n\|_{X^s}^2 + C\varepsilon^{-1}\big(\|\Delta_n\|_{X^s}^2 + \|e_n\|_X^2\big)\end{aligned}$$

from (4), where $\varepsilon > 0$ is arbitrary. For the second term, the skew-adjointness of A and (3a) imply

$$\tau(\Lambda(U_n)E_n, \mathcal{D}\mathcal{A}(U_n))E_n)_{X^s} = \tau(E_n, \mathcal{D}(I \otimes A + \mathcal{Q}(U_n))E_n)_{X^s}$$

$$\leq \tau \mu_X c_{\mathrm{RK}} \big\| E_n \big\|_{X^s}^2 .$$

Using (4), (9b), and $\|\widetilde{U}_n\|_{Z^s} \leq r$, the third term can be bounded by

$$\tau(\Lambda(U_n)E_n, \mathcal{D}(\mathcal{A}(U_n) - \mathcal{A}(\widetilde{U}_n))\widetilde{U}_n)_{X^s} \leq C\tau \, \big\|E_n\big\|_{X^s}^2 \big\|\widetilde{U}_n\big\|_{Z^s} \leq Cr\tau \, \big\|E_n\big\|_{X^s}^2 .$$

After choosing a sufficiently small $\varepsilon > 0$ and possibly decreasing $\tau_0(r) > 0$ once more, we deduce the assertion from the above expressions. □

Our first main result now easily follows.

Theorem 5.3. *Let Assumptions 2.1 and 2.2 be fulfilled. Let the Runge-Kutta method be algebraically stable, satisfy the coercivity condition (17), and have stage order s and order at least $s+1$. Let $r \geq 1$, $R_0 = (4\gamma c_0^2)^{-1}R$, $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z((2c_0)^{-1}r)$, and choose $T = T(r) > 0$ as at the beginning of this section. Take the maximal step size $\tau_0 = \tau_0(r) > 0$ obtained in Lemma 5.2, $\tau \in (0, \tau_0]$, and $N \in \mathbb{N}$ with $N \leq T/\tau$. Let $u \in C([0,T], Z) \cap C([0,T], Y)$ be the solution of (5) and assume that $u^{(s+1)} \in L^2([0,T], D(A))$ and $u^{(s+2)} \in L^2([0,T], X)$. Then the error of the Runge-Kutta scheme (13) is bounded by*

$$\big\|e_N\big\|_X \leq C r^{1/2} e^{CrT} \tau^{s+1} \Big(\int_0^T \big(\big\|u^{(s+1)}(t)\big\|_{D(A)}^2 + \|u^{(s+2)}(t)\|_X^2 \big)\, dt \Big)^{1/2} .$$

The constants C only depend on the coefficients of the Runge-Kutta scheme, the isomorphisms S and S_Y, and the constants in Assumptions 2.1.

Proof. Inserting (46) into (40), we obtain

$$\begin{aligned} \big\|e_{n+1}\big\|_{u_{n+1}}^2 &\leq (1 + Cr\tau)\big\|e_n\big\|_{u_n}^2 + Cr\tau\big(\big\|\varDelta_n\big\|_{D(A)^s}^2 + \big\|\tfrac{1}{\tau}\delta_{n+1}\big\|_X^2\big) \\ &\leq e^{Cr\tau}\big\|e_n\big\|_{u_n}^2 + Cr\tau^{2s+2}\int_{t_n}^{t_{n+1}} (\|u^{(s+1)}(t)\|_{D(A)}^2 + \|u^{(s+2)}(t)\|_X^2)\, dt, \end{aligned}$$

where we used the formulas (37) to bound the defects. The asserted error bound now follows from a straightforward iteration, the identity $e_0 = 0$ and the norm equivalence (10). □

6 The convergence result in Y

Using the isomorphism S_Y from Assumption 2.2, we next extend our convergence result to the norm of Y under slightly stronger conditions. Since the proofs are analogous, we only sketch them.

Lemma 6.1. *Let the assunmptions stated at the beginning of Section 5 be fulfilled. Assume that $S_Y u^{(s+1)} \in L^2([0,T], D(A))$ and $S_Y u^{(s+2)} \in L^2([0,T], X)$. The error $e_n = u_n - u(t_n)$ then satisfies*

$$\begin{aligned} \big\|S_Y e_{n+1}\big\|_{u_{n+1}}^2 \leq (1 + Cr\tau)\big\|S_Y e_n\big\|_{u_n}^2 &+ C\tau \, \big\|\tfrac{1}{\tau} S_Y \delta_{n+1}\big\|_X^2 \\ &+ C(r + \beta_Y(r))\tau \, \big(\big\|S_Y E_n\big\|_{X^s}^2 + \big\|S_Y \varDelta_n\big\|_{D(A)^s}^2\big) \end{aligned}$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}$ with $n \leq T/\tau$.

Proof. We proceed as in the proof of Lemma 5.1, but first multiply the error recursion (38) by S_Y. There are two main changes. First, to estimate the expression corresponding to (45) we apply (9c) and derive

$$\begin{aligned}\big\|S_Y(A_{U_{ni}} - A_{\widetilde{U}_{ni}})\widetilde{U}_{ni}\big\|_X &\le \big\|S_Y\big\|_{X\leftarrow Y}\big\|(A_{U_{ni}} - A_{\widetilde{U}_{ni}})\widetilde{U}_{ni}\big\|_Y \\ &\le \big\|S_Y\big\|_{X\leftarrow Y} L_Y \,\big\|U_{ni} - \widetilde{U}_{ni}\big\|_Y \,\|\widetilde{U}_{ni}\|_Z \\ &\le Cr\,\big\|S_Y E_{ni}\big\|_X .\end{aligned}$$

To prove the bound corresponding to (44), we now have to use (7a) and (7b) for S_Y, which means that the constant now also depends on $\beta_Y(r)$. □

Lemma 6.2. *Let the assumptions stated at the beginning of Section 5 be fulfilled and let $S_Y u^{(s+1)} \in L^2([0,T], D(A))$. Possibly after decreasing the maximal step size $\tau_0 = \tau_0(r) \in (0,1]$, we obtain*

$$\sum_{i=1}^{s} \big\|S_Y E_{ni}\big\|_X^2 \le C\,(\big\|S_Y \varDelta_n\big\|_X^2 + \big\|S_Y e_n\big\|_X^2)$$

for all $\tau \in (0,\tau_0]$ and $n \in \mathbb{N}$ with $n \le T/\tau$.

Proof. We show this result as Lemma 5.2, again starting from the error recursion (39). One now multiplies it with $\mathcal{S}_Y$ and takes the inner product with $\Lambda(U_n)\mathcal{S}_Y E_n$. With modifications as indicated in the proof of Lemma 6.1 one then derives the assertion. □

Arguing as in the proof of Theorem 5.3, we finally deduce from the above two lemmas our convergence result in Y.

Theorem 6.3. *Let Assumptions 2.1 and 2.2 be fulfilled. Let the Runge-Kutta method be algebraically stable, satisfy the coercivity condition (17), and have stage order s and order at least $s+1$. Let $r \ge 1$, $R_0 = (4\gamma c_0^2)^{-1}R$, $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z((2c_0)^{-1}r)$, and choose $T = T(r) > 0$ as at the beginning of Section 5. Take the maximal step size $\tau_0 = \tau_0(r) > 0$ obtained in Lemma 6.2, $\tau \in (0,\tau_0]$, and $N \in \mathbb{N}$ with $N \le T/\tau$. Let $u \in C([0,T],Z) \cap C([0,T],Y)$ be the solution of (5) and assume that $S_Y u^{(s+1)} \in L^2([0,T], D(A))$ and $S_Y u^{(s+2)} \in L^2([0,T],X)$. Then the error of the Runge-Kutta scheme (13) is bounded by*

$$\begin{aligned}\big\|e_N\big\|_Y \le C(r+\beta_Y(r))^{\frac12} e^{C(r+\beta_Y(r))T}\tau^{s+1} \\ \cdot \Big(\int_0^T \big(\big\|S_Y u^{(s+1)}(t)\big\|_{D(A)}^2 + \|S_Y u^{(s+2)}(t)\|_X^2\big)dt\Big)^{\frac12}.\end{aligned}$$

The constants C only depend on the coefficients of the Runge-Kutta scheme, the isomorphisms S and S_Y, and the constants in Assumptions 2.1.

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