Chance-constrained Model Predictive Control based on Box Approximations

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Abstract— **In this paper, we consider finite-horizon predictive control of linear stochastic systems with chance constraints where the admissible region is a convex polytope. For this problem, we present a novel solution approach based on box approximations. The key notion of our approach consists of two steps. First, we apply a linear operation to the joint state probability density function such that its covariance is transformed into an identity matrix. This operation also defines the transformation of the state space and, therefore, of the admissible polytope. Second, we approximate the admissible region from the inside using axis-aligned boxes. By doing so, we obtain a conservative approximation of the constraint violation probability virtually in closed form (the expression contains Gaussian error functions). The presented control approach is demonstrated in a numerical example.**

I. Introduction

In Model Predictive Control (MPC), the objective is to compute (bounded) control inputs such that the closed-loop system is not only stable but also robust to disturbances and model uncertainties. To achieve this objective, constraints on the state are introduced into the MPC optimization problem. In conventional deterministic MPC, these constraints are hard because the uncertainties are assumed to be bounded, i.e., they are contained within some compact set. However, it is more suitable to model some uncertainty classes as unbounded stochastic processes. By doing so, it is possible to take their statistical properties into account and thus, to achieve a better performance. For most stochastic uncertainty models, hard state constraints become infeasible because a constraint violation can always occur, even if its probability may be negligible. In this case, state constraints are defined by means of chance constraints [1]. These constraints require that the system state is contained within the constraints with a predefined probability.

Many chance-constrained control problems are not convex, which makes them hard to solve. But even in the simplest convex form (linear time-invariant system, additive Gaussian noise, convex admissible region), chance-constrained control problems cannot be solved in a reasonable amount of time without approximations because the (iterative) solution requires the evaluation of a multivariate Gaussian integral in order to evaluate the constraint violation probability at each iteration step. Thus, the research on chance-constrained control concentrates on finding suitable approximations. Two different approximation classes can be identified in literature: conservative approximations and scenario approximations.

Conservative approximation methods are, for example, methods that use ellipsoidal bounds, methods based on Boole's inequality, and methods based on inequalities on concentration of measures. Methods based on ellipsoidal bounds are presented in [2], [3], [4]. Their main notion is to compute an ellipsoid that contains the probability of the state that is required to satisfy the constraints. This ellipsoid is then shifted in the state space by an appropriate choice of control inputs such that it remains completely within the admissible region. The main disadvantage of this control method is that it has a high level of conservatism as simulations imply, and that this level of conservatism cannot be influenced. Chance-constrained control and optimization methods based on Boole's inequality apply the inequality to transform integrals of multivariate Gaussian densities that have to be evaluated in order to check the constraints into univariate integrals [5]. However, this separation requires a method to weight the individual integrals, i.e., to perform risk allocation [6], [7]. As mentioned above, another conservative approximation method for chance-constrained control is a method based on concentration of measure. This method gives an upper and a lower bound for the norm of a random vector and, thus, allows to bound the probability of constraint violation. By doing so, it is possible to convert the control problem into a tractable quadratically constrained program with tighter domain for the decision variables than the initial control problem. This approach is considered in [8]. There are even more approximation methods that we do not discuss in this paper, e.g., [9], [10].

In contrast to general analytical conservative approximation methods that only work for linear systems with Gaussian disturbances and convex constraints, there are numerical approximation methods based on scenario approximation [11], [12], [13], [14]. These methods directly approximate the probability density of the joint system state using (usually equally-weighted) samples, which simplifies the evaluation of the integrals in order to evaluate constraint violation probability. If the samples have equal weights, it is sufficient to count the number of samples that violate the constraints and divide this number by the overall number of samples. The main advantage of control methods based on scenario approximation is that theoretically they can handle nonlinear systems, arbitrary disturbance densities, and non-convex constraints. However, these approximation methods can be very computationally demanding if the constraint

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violation probability is very small [15]. Besides that, scenario approximation methods based on stochastic sampling have the disadvantage that the approximation itself is stochastic. Thus, its validity can only be guaranteed with a certain confidence [14], [16].

In this paper, we address chance-constrained control problems with linear system dynamics and Gaussian disturbances, where the admissible region is defined as a full-dimensional polytope. The contribution can be summarized as follows. We propose a conservative approximation that is based on the approximation of the admissible region rather than the approximation of the joint state probability density. The proposed approximation scheme consists of two steps. First, we define a linear transformation for the joint state density such that its covariance becomes an identity matrix, and apply this transformation to the state space. In the second step, we approximate the admissible polytopes from the inside with axis-aligned boxes. This approximation allows us to express the probability mass of the joint state probability density contained within the boxes virtually in closed form (virtually, because the expression contains Gaussian error functions). This expression is then used during computation of the control inputs in order to evaluate the probability of constraint violation. Please note that the presented control approach is only recursively feasible if the control input space is unbounded.

Outline: The remainder of this paper is organized as follows. In the next section, we formulate the considered problem and discuss its tractability issues. In Sec. III, we describe the proposed approach. The demonstration of the presented control method is given in Sec. IV. Finally, Sec. V concludes the paper.

II. Problem Formulation

We consider finite-horizon control of linear stochastic systems whose dynamics are given by

$$\underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_k \underline{\boldsymbol{x}}_k + \mathbf{B}_k \underline{u}_k + \underline{\boldsymbol{w}}_k \ , \tag{1}$$

where $\underline{\boldsymbol{x}}_k \in \mathbb{R}^n$ denotes the system state and $\underline{u}_k \in \mathbb{R}^m$ the control input. The system is perturbed by zero-mean i.i.d. Gaussian noise $\underline{\boldsymbol{w}}_k \in \mathbb{R}^n$ with positive definite covariance $\mathrm{E}\left\{\underline{\boldsymbol{w}}_k \underline{\boldsymbol{w}}_k^\top\right\} = \mathbf{\Sigma}_w$. The initial system state $\underline{\boldsymbol{x}}_0$ is also Gaussian with

$$\overline{\underline{x}}_0 = \mathrm{E}\left\{\underline{\boldsymbol{x}}_0\right\} \quad \text{and} \quad \mathbf{\Sigma}_x = \mathrm{E}\left\{(\underline{\boldsymbol{x}}_0 - \overline{\underline{x}}_0)(\underline{\boldsymbol{x}}_0 - \overline{\underline{x}}_0)^\top\right\} \ .$$

The performance of the controlled system is measured by an LQG cost function

$$J = \mathrm{E}\left\{ \underline{\boldsymbol{x}}_K^\top \mathbf{Q}_K \underline{\boldsymbol{x}}_K + \sum_{k=0}^{K-1} \left[\underline{\boldsymbol{x}}_k^\top \mathbf{Q}_k \underline{\boldsymbol{x}}_k + \underline{u}_k^\top \mathbf{R}_k \underline{u}_k \right] \right\} \ , \tag{2}$$

where $\mathbf{Q}_k$ is positive semidefinite and $\mathbf{R}_k$ is positive definite.

Robustness of the controlled system is defined by means of joint chance constraints defined as

$$\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P}) \geq 1 - \alpha \ , \tag{3}$$

where $\mathcal{P}$ is a full-dimensional convex polytope $\mathcal{P} = \{\xi \in \mathbb{R}^{n(K+1)} \mid \mathbf{F}\xi \leq \underline{c}\}$ and $0 < \alpha < 0.5$ is a predefined confidence level. The probability $\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P})$ can be evaluated according to

$$\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P}) = \int_{\underline{x} \in \mathcal{P}} f(\underline{x}) \, \mathrm{d}\underline{x} \ , \tag{4}$$

where $f(\underline{x})$ is the joint probability distribution of $\underline{\boldsymbol{x}}_{0:K}$.

The described control problem can be summarized as follows.

Problem 1: (Control problem with convex polytopic constraints)

$$\begin{aligned}
\min_{\underline{u}_{0:K-1}} \quad & J = \mathrm{E}\left\{ \underline{\boldsymbol{x}}_K^\top \mathbf{Q}_K \underline{\boldsymbol{x}}_K + \sum_{k=0}^{K-1} \left[\underline{\boldsymbol{x}}_k^\top \mathbf{Q}_k \underline{\boldsymbol{x}}_k + \underline{u}_k^\top \mathbf{R}_k \underline{u}_k \right] \right\} \\
\text{s.t.} \quad & \underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_k \underline{\boldsymbol{x}}_k + \mathbf{B}_k \underline{u}_k + \underline{\boldsymbol{w}}_k \ , \\
& \underline{\boldsymbol{x}}_0 \sim \mathcal{N}(\overline{\underline{x}}_0, \mathbf{\Sigma}_x) \ , \quad \underline{\boldsymbol{w}}_k \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma}_w) \ , \\
& \mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P}) \geq 1 - \alpha \ .
\end{aligned}$$

Although Problem 1 is convex [11], [12] and seems to be quite simple, its solution is not applicable in real-time control due to necessity to evaluate the integral (4) of a multivariate Gaussian at each iteration step of the solution algorithm in order to compute $\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P})$. Thus, in the following section, we propose a conservative approximation scheme to address this issue.

III. Proposed Solution

In this section, we describe the proposed solution to Problem 1. For this purpose, we first introduce an augmented system state and reformulate the considered control problem. Finally, we describe the conservative approximation of the constraint violation probability.

A. Control Problem Reformulation

In the considered finite-horizon setup, we can introduce the concatenated state, input, and noise vectors

$$\underline{\boldsymbol{X}} = \begin{bmatrix} \underline{\boldsymbol{x}}_0 \\ \underline{\boldsymbol{x}}_1 \\ \vdots \\ \underline{\boldsymbol{x}}_K \end{bmatrix} \ , \quad \underline{U} = \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \\ \underline{u}_{K-1} \end{bmatrix} \ , \text{ and } \quad \underline{\boldsymbol{W}} = \begin{bmatrix} \underline{\boldsymbol{w}}_0 \\ \underline{\boldsymbol{w}}_1 \\ \vdots \\ \underline{\boldsymbol{w}}_{K-1} \end{bmatrix} \ .$$

Then, the dynamics of the augmented state can be written as

$$\underline{\boldsymbol{X}} = \mathbf{G}_{xx} \underline{\boldsymbol{x}}_0 + \mathbf{G}_{xu} \underline{U} + \mathbf{G}_{xw} \underline{\boldsymbol{W}} \ ,$$

where $\mathbf{G}_{xx} \in \mathbb{R}^{(K+1)n \times n}$, $\mathbf{G}_{xu} \in \mathbb{R}^{(K+1)n \times Km}$, and $\mathbf{G}_{xw} \in \mathbb{R}^{(K+1)n \times Kn}$ can be derived from the matrices $\mathbf{A}_k$ and $\mathbf{B}_k$.

The probability density of the augmented state $\underline{\boldsymbol{X}}$ is Gaussian [4] with

$$\begin{aligned}
\overline{\underline{X}} &= \mathbf{G}_{xx} \overline{\underline{x}}_0 + \mathbf{G}_{xu} \underline{U} \ , \\
\mathbf{\Sigma}_X &= \mathbf{G}_{xx} \mathbf{\Sigma}_x \mathbf{G}_{xx}^\top + \mathbf{G}_{xw} \mathbf{\Sigma}_W \mathbf{G}_{xw}^\top \ ,
\end{aligned}$$

where

$$\mathbf{\Sigma}_W = \mathrm{diag}\begin{bmatrix} \mathbf{\Sigma}_w & \mathbf{\Sigma}_w & \ldots & \mathbf{\Sigma}_w \end{bmatrix} \ .$$

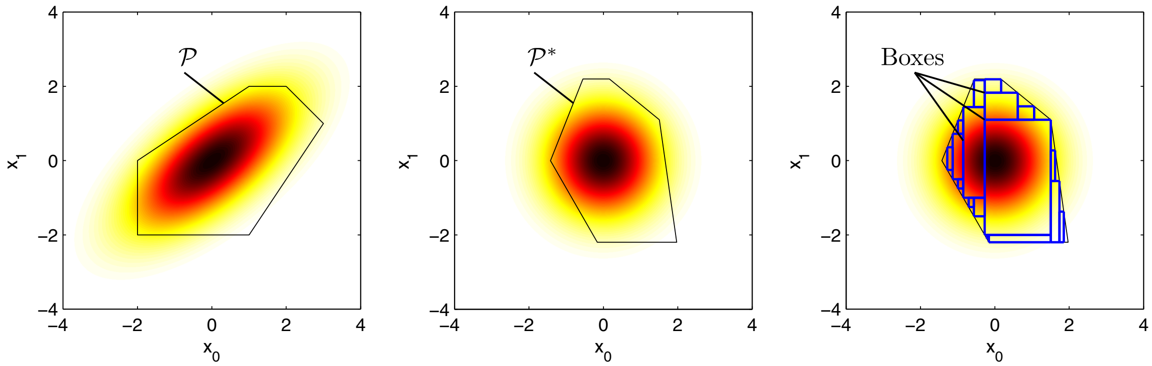

Fig. 1: Depicted is the transformation of the joint state covariance of the scalar system states $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ and the admissible polytope $\mathcal{P}$ into the identity matrix and the polytope $\mathcal{P}^\star$, respectively. Subsequently, $\mathcal{P}^\star$ is approximated of with boxes from the inside.

Finally, we also express the cost function (2) in terms of the augmented system state and its probability density. It holds

$$J = \operatorname{trace}\left[\widetilde{\mathbf{Q}}\boldsymbol{\Sigma}_X\right] + \underline{U}^\top \widetilde{\mathbf{R}}\underline{U} \ , \tag{5}$$

with cost matrices $\widetilde{\mathbf{Q}} = \operatorname{diag}\begin{bmatrix}\mathbf{Q}_0 & \mathbf{Q}_1 & \dots & \mathbf{Q}_K\end{bmatrix}$ and $\widetilde{\mathbf{R}} = \operatorname{diag}\begin{bmatrix}\mathbf{R}_0 & \mathbf{R}_1 & \dots & \mathbf{R}_{K-1}\end{bmatrix}$. With these prerequisites, we can restate Problem 1 as follows.

Problem 2: (Reformulated chance-constrained control problem)

$$\begin{aligned}\min_{\underline{U}} \quad & J = \operatorname{trace}\left[\widetilde{\mathbf{Q}}\boldsymbol{\Sigma}_X\right] + \underline{U}^\top \widetilde{\mathbf{R}}\underline{U} \\ \text{s. t.} \quad & \boldsymbol{\Sigma}_X = \mathbf{G}_{xx}\boldsymbol{\Sigma}_x\mathbf{G}_{xx}^\top + \mathbf{G}_{xw}\boldsymbol{\Sigma}_W\mathbf{G}_{xw}^\top \ , \\ & \underline{\boldsymbol{X}} \sim \mathcal{N}(\overline{\underline{X}}, \boldsymbol{\Sigma}_X) \ , \\ & \mathrm{P}(\mathbf{F}\underline{\boldsymbol{X}} \leq \underline{c}) \geq 1 - \alpha\end{aligned}$$

In the next section, we describe the conservative approximation that allows us to solve Problem 2.

B. Conservative Constraint Violation Approximation

In order to evaluate the probability of constraint violation when solving Problem 2, it is necessary to compute a multivariate Gaussian integral, which is not practical in real-time control. Thus, we propose an approximation scheme that consists of two steps. In the first step, we define a linear transformation that transforms the augmented state covariance $\boldsymbol{\Sigma}_X$ into an identity matrix. This linear transformation also defines the transformation of the admissible polytope $\mathcal{P}$. In the second step, we conservatively approximate the transformed polytope $\mathcal{P}^\star$ from the inside using axis-aligned boxes [17]. This approximation allows us to give a virtually closed-form expression for the approximated constraint violation probability. We refer to this expression as virtually closed-form because it contains Gaussian error functions. However, these functions can be implemented as lookup tables such that their evaluation has only a small footprint in terms of computational time.

Before we describe the transformation of the polytope, we state the following lemma.

Lemma 1: If the initial state covariance $\boldsymbol{\Sigma}_x$ and the process noise covariance $\boldsymbol{\Sigma}_w$ are both positive definite, the joint state covariance $\boldsymbol{\Sigma}_X$ is also positive definite.

Proof: We can write $\boldsymbol{\Sigma}_X$ as

$$\boldsymbol{\Sigma}_X = \operatorname{diag}\begin{bmatrix}\boldsymbol{\Sigma}_x & \boldsymbol{\Sigma}_w & \dots & \boldsymbol{\Sigma}_w\end{bmatrix} + \mathbf{M} \ .$$

By definition, the first term is positive definite. The second term $\mathbf{M}$ is a sum of positive semidefinite matrices (these matrices can all be written as $\mathbf{M}_i = \mathbf{N}_i^\top \mathbf{N}_i, i \in \mathbb{N}$). Therefore, it is also positive semidefinite. Thus, the joint state covariance is positive definite. ■

According to Lemma 1, the joint state covariance is positive definite and thus can be written as $\boldsymbol{\Sigma}_X = \mathbf{T}^\top\mathbf{T}$, where $\mathbf{T} = \boldsymbol{\Sigma}_X^{\frac{1}{2}}$ is invertible. Such a representation of $\boldsymbol{\Sigma}_X$ can be obtained using Cholesky decomposition. The linear transformation

$$\underline{\boldsymbol{\xi}} = \mathbf{T}^{-1}\underline{\boldsymbol{X}} \tag{6}$$

transforms the probability density of the joint system state into an identity matrix, i.e., $\underline{\boldsymbol{\xi}}$ is distributed according to $\mathcal{N}(\overline{\underline{X}}, \mathbf{I})$. Solving the linear transformation (6) for $\underline{\boldsymbol{X}}$ and substituting it in the expression that defines the admissible polytope $\mathcal{P}$ yields the transformation of the polytope

$$\mathcal{P} : \mathbf{F}\underline{\boldsymbol{X}} \leq \underline{c} \quad \longrightarrow \quad \mathcal{P}^\star : \mathbf{F}\mathbf{T}\underline{\boldsymbol{\xi}} \leq \underline{c} \ .$$

Having transformed the admissible polytope, we approximate it from the inside using M axis-aligned boxes $\mathcal{B}_i = [\underline{a}^{(i)}, \underline{b}^{(i)}], i = 1, ..., M$ where $\underline{a}^{(i)}$ and $\underline{b}^{(i)}$ are vectors that span box i. For this purpose, we employ the method proposed in [17]. In its simple form, the recursive algorithm presented in [17] proceeds as follows. First, it approximates the polytope using a maximum volume box $\mathcal{B}_1$ that fits into the polytope. The remaining volume of the polytope $\mathcal{P}\backslash\mathcal{B}_1$ is partitioned into several non-overlapping convex polytopes and the approximation is iteratively proceeded until the termination condition is reached. The discussion of different termination conditions can be found in [17]. The algorithm proposed in [17] can be viewed as a depth-first search because having fit a box in the current polytope, it takes one of the partitions and approximates it until the termination condition for this partition is satisfied. Then, it takes the next partition and repeats the procedure.

Fig. 1 demonstrates the transformation of the initial polytope $\mathcal{P}$ into the polytope $\mathcal{P}^\star$ and its subsequent box approximation. Please note that the joint state covariance is independent of control inputs $\underline{U}$. Thus, the transformation of the polytope and its approximation only have to be computed once per time step, and if the admissible polytope and the noise are time-invariant, then the approximation has to be computed only once.

Remark 1: The described box approximation method shall only demonstrate the key notion of the presented constraint violation probability approximation. More computationally efficient box approximation algorithms, can be used for practical implementations in order to reduce the computation time.

Remark 2: As mentioned above, the box approximation method proposed in [17] uses a depth-first search. In our numerical example, however, we use a breadth-first search approach as given in Algorithm 2. The algorithm proceeds as follows. After fitting a box into a polytope and dividing the remaining volume into non-overlapping partitions, we enqueue these partitions into a list. Then, the first partition of the list is taken, a box fitted into it, and its remaining volume divided into new partitions. These new partitions are then added to the end of the list and not to the front as in [17]. Using this approach allows us to approximate the largest partitions first. By doing so, we obtain a better approximation quality than the algorithm proposed in [17] if the number of boxes is fixed a priori.

Having obtained the box approximation of the transformed polytope, it is possible to evaluate the probability mass of the joint state density according to the following theorem.

Theorem 1: The probability mass of the joint state probability density within the box approximation of the transformed admissible polytope $\mathcal{P}^\star$ can be calculated according to

$$\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B}) = \frac{1}{(2\pi)^{\frac{n(K+1)}{2}}} \sum_{i=1}^{M} \prod_{j=1}^{n(K+1)} \left[\operatorname{erf}\left(\frac{\underline{b}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}} \right) - \operatorname{erf}\left(\frac{\underline{a}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}} \right) \right],$$

where $\underline{b}_j^{(i)}$ denotes the jth component of the vector $\underline{b}$ of the ith box, and $\underline{\mu} = \mathbf{T}^{-1}(\mathbf{G}_{xx}\overline{\underline{x}}_0 + \mathbf{G}_{xu}\underline{U})$.

Proof: The result can be obtained by evaluating

$$\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B}) = \int_{\mathcal{B}} \mathcal{N}(\underline{x}, \underline{\mu}, \mathbf{I})\, \mathrm{d}\underline{x} = \sum_{i=1}^{M} \int_{\mathcal{B}_i} \mathcal{N}(\underline{x}, \underline{\mu}, \mathbf{I})\, \mathrm{d}\underline{x} \,,$$

where $\mathcal{N}(\underline{t}, \underline{\mu}, \mathbf{I})$ denotes the normal distribution with the mean $\underline{\mu}$ and covariance $\mathbf{I}$ evaluated at $\underline{t}$. The probability mass contained within a single box is the solution of the following integral

$$\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B}_i) = \int_{\underline{a}^{(i)}}^{\underline{b}^{(i)}} \frac{1}{(2\pi)^{\frac{n(K+1)}{2}}} \exp\left(-\frac{1}{2} (\underline{x} - \underline{\mu})^\top (\underline{x} - \underline{\mu}) \right) \mathrm{d}\underline{x} \,.$$

Applying substitution $\underline{z} = \underline{x} - \underline{\mu}$ and exploiting the axis alignment property of the boxes, we obtain

$$\begin{aligned}
\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B}_i) &= \frac{1}{(2\pi)^{\frac{n(K+1)}{2}}} \int_{\underline{a}_1^{(i)} - \mu_1}^{\underline{b}_1^{(i)} - \mu_1} \exp\left(-\frac{z_1^2}{2} \right) \mathrm{d}z_1 \\
&\quad \int_{\underline{a}_2^{(i)} - \mu_2}^{\underline{b}_2^{(i)} - \mu_2} \exp\left(-\frac{z_2^2}{2} \right) \mathrm{d}z_1 \cdots \int_{\underline{a}_{n(K+1)}^{(i)} - \mu_{n(K+1)}}^{\underline{b}_{n(K+1)}^{(i)} - \mu_{n(K+1)}} \exp\left(-\frac{z_{n(K+1)}^2}{2} \right) \mathrm{d}z_{n(K+1)} \\
&= \prod_{j=1}^{n(K+1)} \left[\operatorname{erf}\left(\frac{\underline{b}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}} \right) - \operatorname{erf}\left(\frac{\underline{a}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}} \right) \right].
\end{aligned}$$

Summation over all M boxes concludes the proof. ∎

Finally, we can summarize our results as the following control problem that is a conservative approximation of Problem 2.

Problem 3: (Conservative approximation of chance-constrained control problem)

$$\begin{aligned}
\min_{\underline{U}} \quad & J = \operatorname{trace}\left[\widetilde{\mathbf{Q}} \mathbf{\Sigma}_X \right] + \underline{U}^\top \widetilde{\mathbf{R}} \underline{U} \\
\text{s.t.} \quad & \overline{\underline{X}} = \mathbf{G}_{xx} \mathbf{\Sigma}_x + \mathbf{G}_{xu} \underline{U} \,, \\
& \mathrm{P}(\mathbf{T}^{-1} \overline{\underline{X}} \in \mathcal{B}) \geq 1 - \alpha \,.
\end{aligned}$$

Problem 3 can become non-convex near the boundary of the admissible region because the box approximation is in general not convex near boundaries.

In the following theorem, we state and prove that Problem 2 is a conservative approximation of Problem 3.

Theorem 2: A feasible solution to Problem 3 is also a feasible solution to Problem 2.

Proof: Observe that a probability density is a non-negative function and that the domain of integration to evaluate $\mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{B})$ is contained within the domain of integration to evaluate $\mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{P}^\star)$, i.e., it holds

$$\mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{B}) \leq \mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{P}^\star) \,.$$

Thus, a feasible solution to Problem 3 satisfies

$$1 - \alpha \leq \mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{B}) \leq \mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{P}^\star) = \mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{P}) \,.$$

This means that a feasible solution to Problem 3 is also feasible for Problem 2. ∎

Remark 3: The conservatism level $\beta = \mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{P}^\star) - \mathrm{P}(\underline{\boldsymbol{\xi}} \in \mathcal{B})$ can be tuned by variation of the number of boxes used to approximate $\mathcal{P}^\star$, i.e., by selection of an appropriate termination condition for the box approximation algorithm.

Algorithm 1 summarizes the actions performed by the controller in order to compute the control inputs.

Finally, we give the gradient of $\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B})$ that can be used when solving Problem 3 in the next theorem.

Theorem 3: The gradient of $\mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B})$ with respect to $\underline{U}$ is given by

Algorithm 1: Control inputs computation.

Input: State estimate $\underline{\overline{x}}_0$, covariance $\mathbf{\Sigma}_x$, constraints $\mathbf{F}, \underline{c}$, termination condition ϵ
Output: Control inputs $\underline{u}_{0:K-1}$

$\mathbf{\Sigma}_X = \mathbf{G}_{xx}\mathbf{\Sigma}_x\mathbf{G}_{xx}^\top + \mathbf{G}_{xw}\mathbf{\Sigma}_W\mathbf{G}_{xw}^\top$;
$\mathbf{T} = \mathrm{chol}(\mathbf{\Sigma}_X)$;
$\mathcal{B} = \mathrm{InnerApproximation}(\mathbf{FT}, \underline{c}, \epsilon)$; `// Alg. 5 from [17]`
$\underline{u}_{0:K-1} \leftarrow$ Solve Problem 3;
return $\underline{u}_{0:K-1}$;

$$\frac{\partial\, \mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B})}{\partial \underline{U}} = \frac{\sqrt{2}}{(2\pi)^{\frac{(K+1)n}{2}}\sqrt{\pi}} \sum_{i=1}^{M} \sum_{p=1}^{(K+1)n}$$
$$\times \left[\exp\left(-\frac{1}{2}(\underline{a}_p^{(i)} - \underline{\mu}_j)^2\right) - \exp\left(-\frac{1}{2}(\underline{b}_p^{(i)} - \underline{\mu}_j)^2\right)\right] \mathbf{G}_{xu}^\top \left(\mathbf{T}^{-1}\right)^\top \underline{e}_j$$
$$\times \prod_{\substack{j=1 \\ j\neq p}}^{(K+1)n} \left[\mathrm{erf}\left(\frac{\underline{b}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{\underline{a}_j^{(i)} - \underline{\mu}_j}{\sqrt{2}}\right)\right] ,$$

where $\underline{e}_j$ denotes a column vector of all elements being zero and the jth element being 1.

Proof: First, we apply the chain rule and obtain

$$\frac{\partial\, \mathrm{P}(\underline{\boldsymbol{x}}_{0:K} \in \mathcal{B})}{\partial \underline{U}} = \frac{1}{(2\pi)^{\frac{(K+1)n}{2}}} \sum_{i=1}^{M} \sum_{p=1}^{(K+1)n} \frac{\partial g(\underline{b}_p^{(i)}, \underline{a}_p^{(i)}, \underline{\mu}_p)}{\partial \underline{U}}$$
$$\times \prod_{\substack{j=1 \\ j\neq p}}^{(K+1)n} g(\underline{b}_j^{(i)}, \underline{a}_j^{(i)}, \underline{\mu}_j) ,$$

with $g(x, y, z) = \mathrm{erf}\left(\frac{x-z}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{y-z}{\sqrt{2}}\right)$. Differentiation of $g(\underline{b}_p^{(i)}, \underline{a}_p^{(i)}, \underline{\mu}_p)$ can be calculated according to

$$\frac{\partial g(\underline{b}_p^{(i)}, \underline{a}_p^{(i)}, \underline{\mu}_p)}{\partial \underline{U}} = \frac{\partial g(\underline{b}_p^{(i)}, \underline{a}_p^{(i)}, \underline{\mu}_p)}{\partial \underline{\mu}_p} \frac{\partial \underline{\mu}_p}{\partial \underline{U}} ,$$

with

$$\frac{\partial g(\underline{b}_p^{(i)}, \underline{a}_p^{(i)}, \underline{\mu}_p)}{\partial \underline{\mu}_p} =$$
$$\sqrt{\frac{2}{\pi}} \left[\exp\left(-\frac{1}{2}(\underline{a}_p^{(i)} - \underline{\mu}_j)^2\right) - \exp\left(-\frac{1}{2}(\underline{b}_p^{(i)} - \underline{\mu}_j)^2\right)\right] ,$$

and

$$\frac{\partial \underline{\mu}_p}{\partial \underline{U}} = \mathbf{G}_{xu}^\top \left(\mathbf{T}^{-1}\right)^\top \underline{e}_j ,$$

where we used the identity $\underline{\mu}_j = \underline{e}_j^\top \underline{\mu}$. Combination of these intermediate steps yields the result of Theorem 3. ∎

Remark 4: The presented approximation method can be used not only for chance-constrained problems with polytopic constraints but also for problems, where the admissible region is the entire state space except for a finite number of convex polytopes (imagine polytopic obstacles, for example). For these problems, it is possible to approximate the obstacles with boxes from the outside as presented in [17], and then use this approximation in order to efficiently evaluate constraint violation probability.

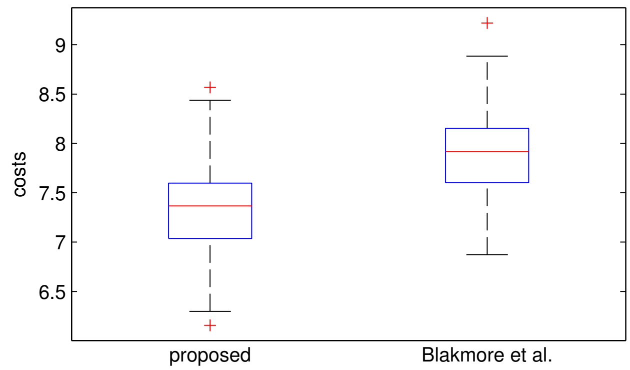

Fig. 2: Monte Carlo simulation results. Depicted are the costs of the proposed controller and the controller from [7]. The box boundaries indicate the 25% and the 75% quantiles, the red line is the median.

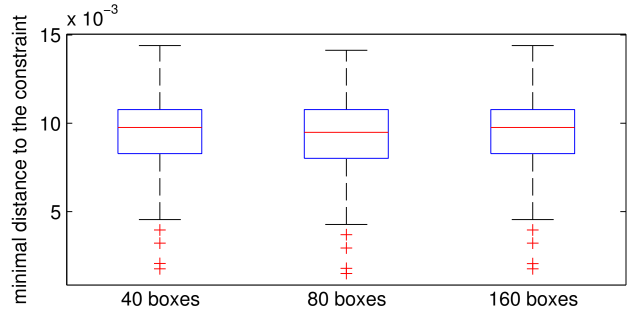

Fig. 3: Minimal distance to the constraints for different numbers of boxes.

IV. Numerical Example

To demonstrate the presented approach, we considered the following system dynamics

$$\underline{\boldsymbol{x}}_{k+1} = \underline{\boldsymbol{x}}_k + \underline{u}_k + \underline{\boldsymbol{w}}_k$$
$$\underline{\boldsymbol{y}}_k = \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_k ,$$

with $\mathbf{\Sigma}_w = 0.3^2$, $\mathbf{\Sigma}_v = 0.2^2$, $\underline{\overline{x}}_0 = 2$, and $\mathbf{\Sigma}_x = 0.5^2$. The cost function parameters were set to $\mathbf{Q}_k = 1$ and $\mathbf{R}_k = 1$. The constraints were chosen to $1 \leq \underline{\boldsymbol{x}}_k \leq 8$ for the entire optimization horizon. This constraint formulation can be reformulated in terms of convex polytopic constraints according to

$$\mathbf{F} = \mathbf{I} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix}^\top ,$$
$$\underline{c}_k = \begin{cases} 8, & k = 2t-1,\ t = 1, \ldots, K+1 , \\ -1, & k = 2t,\ t = 1, \ldots, K+1 , \end{cases}$$

where $\mathbf{I} \in \mathbb{R}^{K+1 \times K+1}$ and $\otimes$ denotes the Kronecker product. Please note that although the constraints are a box, they are sheared after application of the transformation $\mathbf{T}$.

We implemented the presented control law in a receding horizon framework with prediction horizon length $K = 10$

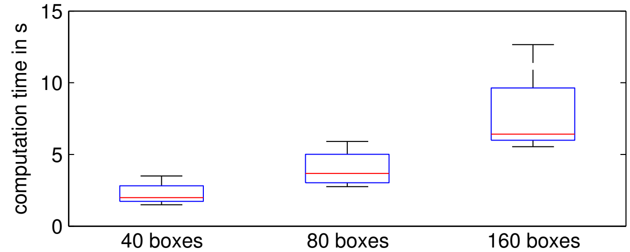

Fig. 4: Computation time for different numbers of boxes.

and fixed the number of boxes used to evaluate the chance constraints to $M = 40$. To improve the approximation quality when using a fixed number of boxes, we applied the breadth-first search algorithm given in Algorithm 2. For comparison, we used the control approach proposed by Blackmore et. al. in [7]. We performed a Monte Carlo simulation with 100 runs. The simulation time was chosen to $H = 50$ and control violation probability to $\alpha = 0.1$.

Algorithm 2: Breadth-first box approximation.

Input: Polytope $\mathcal{P}^*$, maximum number of boxes M
Output: Boxes $\mathcal{B}$

```
Υ ← [P*]; // create list of polytopes
for i = 1 to M do
    // approximate first list element
    B_i = single-inner(Υ[1]) (Algorithm 4 in [17]);
    // add not covered partitions
    for j = 1 to n(K + 1) do
        define H_{2j−1} according to (18) in [17];
        Υ ← [Υ, P* ∩ H_{2j−1}];
        define H_{2j} according to (19) in [17];
        Υ ← [Υ, P* ∩ H_{2j}];
    end
    Υ{1} ← {}; // delete first element
end
```

Fig. 2 shows the costs of both compared control laws. It can be seen that in the considered scenario, the proposed controller performs better than the controller from [7]. The empirical constraint violation of the proposed controller was $\alpha_s = 0.0052$ and $\alpha_s = 0.0022$ of the controller from [7]. The mean control input computation time of the proposed controller is 1.1903s with $M = 20$ boxes and 4.9249s with $M = 40$, while the controller from [7] requires $0.2022s$ to compute a control input (Intel Core i5-3320M@2.6 Ghz, 8 GB RAM, MATLAB 2013b). Furthermore, control input computation times are depicted in Fig. 4 for a Monte Carlo simulation with 30 runs. Finally, Fig. 3 shows the minimal distance to the constraints for different numbers of boxes for a Monte Carlo simulation with 30 runs à 20 time steps.

V. Conclusion

In this paper, we presented a new conservative approximation method for chance-constrained control of linear systems with Gaussian disturbances and convex polytopic admissible regions. The main idea of the presented approach is to approximate the admissible polytope from the inside using axis-aligned boxes after the polytope has been transformed using a linear transformation that transforms the joint state probability density into a standard normal distribution. By doing so, we can give a conservative approximation of the probability mass contained within the admissible polytope virtually in closed form. The presented approach allows for a tradeoff between approximation accuracy and computational time by choosing the number of boxes.

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