Dynamic Compensation of Markov Jump Linear Systems without Mode Observation

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***Abstract*— In this paper, we address control of Markov Jump Linear Systems without mode observation via dynamic output feedback. Because the optimal nonlinear control law for this problem is intractable, we assume a linear controller. Under this assumption, the control law computation can be expressed in terms of an optimization problem that involves Bilinear Matrix Inequalities. Alternatively, it is possible to cast the problem as a Linear Matrix Inequality by introducing additional linearity constraints and requiring that some system parameters are constant. However, this latter approach is very restricting and it introduces additional conservatism that can yield poor performance. Thus, we propose an alternative iterative algorithm that does not pose any non-standard restrictions and demonstrate it in a numerical example.**

I. Introduction

In control engineering, systems with abrupt parameter changes constitute an important system class. These systems are often modeled as a set of continuous-valued dynamical systems, where the active system is selected according to a jumping parameter or *mode*. In literature, such systems are referred to as *Hybrid Systems* because they possess continuous-valued and discrete-valued dynamics [1]. In this paper, we address the control of a special class of Hybrid Systems, namely the Markov Jump Linear Systems (MJLS) that were first introduced by Krasovskii and Lidskii in 1961 [2]. In these systems, the continuous-valued dynamics are linear and the discrete-valued dynamics are modeled as a Markov chain that is independent of the continuous-valued dynamics. This modeling approach allows to consider economic processes [3], [4], systems with component failures [5], networked control systems [6], [7], multi-agent systems [8], and many more [9].

We focus specifically on optimal control of MJLS. While theory on optimal control of MJLS with observed mode is mature [9]–[11], there are still many unsolved challenges in control of MJLS with non-observable mode. Even though the continuous-valued dynamics are linear, the separation between control and estimation does not hold as it is the case in the Linear Quadratic Gaussian (LQG) control, i.e., there is a *dual effect*. Thus, the lack of mode observation leads to nonlinear optimization problems that suffer from the curse of dimensionality [12], [13]. For this reason, research concentrates on approximate control laws. We distinguish between (i) control laws that are based on approximation of the density of the hybrid system state, and (ii) control laws that make a structural assumption.

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Control laws that belong to the class (i) usually employ a multiple-model approach in order to estimate the continuous-valued state and the distribution of the discrete-valued state. This information is then used in order to compute the control input using one-step lookahead regulator gains determined by a set of coupled Riccati equations from the optimal LQG controller for MJLS with observable mode [14], [15].

The second class of approaches to optimal control of MJLS without mode observation makes an assumption on the structure of the control law. Usually, an affine control law is used. This approach was, e.g., used in [3], where the authors considered finite-horizon state-feedback control of MJLS without process noise. It was extended to systems with process noise in [5]. Furthermore, optimal control of MJLS with time-invariant linear control law assumption is treated in [16]–[18]. In [16], the authors considered state-feedback H_2 control of MJLS via Linear Matrix Inequalities (LMI) with clustered observations, i.e., the case where some of the mode values are observed and the others are not. The provided framework is able to model everything from full mode observation to a completely non-observable mode. For the case of a non-observed mode, we proposed an alternative iterative control law computation method in [17]. And in [18], we addressed static output-feedback control of MJLS without mode observation. For this scenario, we derived an iterative algorithm that computes the regulator gain and provided feasibility conditions in terms of an LMI. Important related work is also [19], where the authors consider H_∞ dynamic output feedback for MJLS with mode observation and discuss how the case clustered observations can be implemented by introducing additional constraints and restrictions. In case of no mode observation, these restrictions require that some of the system matrices are mode-independent.

In this paper, we extend our results on static output-feedback control of MJLS without mode observation from [18] to dynamic output-feedback control, i.e., the case where the controller receives noisy state measurements. As argued above, the optimal control law for this problem is nonlinear and intractable. For this reason, we make the assumption of a linear mode-independent control law. But even under this assumption, the computation of the controller is not trivial. Actually this problem can be addressed via Bilinear Matrix Inequalities (BMI) and in principle be solved using existing solvers [20], [21]. However, finding a solution may not always be tractable, because BMIs are NP-hard [22]. Alternatively, the BMI can be cast as a Linear Matrix Inequality (LMI) by introducing additional linearity constraints, which was proposed in [19]. However, not only these additional constraints introduce conservatism, which can yield poor performance and even render the LMI unsolvable although stabilizing controllers exist, but it is also necessary to require some of the system parameters to be constant within each cluster. Thus, we take a different approach similar to that in [23] and [17], [18], and propose an iterative algorithm for control law computation as alternative and extension of the method from [19]. Because the convergence of the presented algorithm does not imply stability of the closed-loop system, i.e., the proposed algorithm can converge even if the MJLS cannot be stabilized via linear mode-independent dynamic output feedback, it is necessary to check stability using the results from [24]. The convergence proof of the proposed algorithm is subject of our current research.

Outline. In the next section, we formulate the considered problem. The algorithm for control law computation is presented in Sec. III and demonstrated by means of a numerical example in Sec. IV. Finally, Sec. V concludes the paper.

II. Problem Formulation and Basic Concepts

Consider the stochastic system

$$\begin{aligned} \underline{\boldsymbol{x}}_{k+1} &= \mathbf{A}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k + \mathbf{B}_{\boldsymbol{\theta}_k}\underline{u}_k + \mathbf{H}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{w}}_k \ , \\ \underline{\boldsymbol{y}}_k &= \mathbf{C}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k + \mathbf{J}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{v}}_k \ , \end{aligned} \tag{1}$$

where $\underline{\boldsymbol{x}}_k \in \mathbb{R}^n$ is the state, $\underline{u}_k \in \mathbb{R}^m$ the control input, and $\underline{\boldsymbol{y}}_k \in \mathbb{R}^s$ the measurement. The matrices $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{H}_k$, $\mathbf{C}_k$, and $\mathbf{J}_k$ are selected from time-invariant sets $\{\mathbf{A}_i, \mathbf{B}_i, \mathbf{H}_i, \mathbf{C}_i, \mathbf{J}_i\}$, $i = 1, \ldots, M$, $M \in \mathbb{N}$ according to the value of the random variable $\boldsymbol{\theta}_k \in \{1, \ldots, M\}$, referred to as the mode, that forms an ergodic time-homogeneous Markov chain $\{\boldsymbol{\theta}_k\}$ with transition matrix $\mathbf{T} = (p_{ij})_{M\times M}$, $p_{ij} = \mathrm{P}(\boldsymbol{\theta}_{k+1} = j|\boldsymbol{\theta}_k = i)$. The dynamics (1) are subject to zero-mean independent and identically distributed Gaussian disturbances $\underline{\boldsymbol{w}}_k \in \mathbb{R}^p$ and $\underline{\boldsymbol{v}}_k \in \mathbb{R}^q$ with identity covariances[1].

For system (1), we seek to find a linear time-invariant control law

$$\begin{aligned} \underline{\widehat{\boldsymbol{x}}}_{k+1} &= \mathbf{F}\underline{\widehat{\boldsymbol{x}}}_k + \mathbf{K}\underline{\boldsymbol{y}}_k \ , \\ \underline{u}_k &= \mathbf{L}\underline{\widehat{\boldsymbol{x}}}_k \end{aligned} \tag{2}$$

that is independent of the initial condition $(\underline{\boldsymbol{x}}_0, \boldsymbol{\theta}_0)$ and where $\underline{\widehat{\boldsymbol{x}}}_k \in \mathbb{R}^n$ denotes the internal controller state. The matrices $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{L}$ are to be determined such that the infinite-horizon average cost function

$$\mathcal{J} = \lim_{K\to\infty} \frac{1}{K}\mathrm{E}\left\{\sum_{\mathrm{k}=0}^{\mathrm{K}} \underline{\boldsymbol{x}}_{\mathrm{k}}^\top \mathbf{Q}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_{\mathrm{k}} + \underline{\mathrm{u}}_{\mathrm{k}}^\top \mathbf{R}_{\boldsymbol{\theta}_k}\underline{\mathrm{u}}_{\mathrm{k}}\right\} \tag{3}$$

is minimized, where the positive semidefinite $\mathbf{Q}_{\boldsymbol{\theta}_k}$ and positive definite $\mathbf{R}_{\boldsymbol{\theta}_k}$ are selected according to $\boldsymbol{\theta}_k$ and the expectation is taken with respect to $\boldsymbol{\theta}_k$, $\underline{\boldsymbol{w}}_k$, and $\underline{\boldsymbol{v}}_k$.

[1] Please note that this assumption is not restrictive because by choosing the matrices $\mathbf{H}_{\boldsymbol{\theta}_k}$ and $\mathbf{J}_{\boldsymbol{\theta}_k}$ appropriately, we can obtain any other covariance function.

Under the control law assumption (2), we can construct the closed-loop system

$$\underline{\widetilde{\boldsymbol{x}}}_{k+1} = \widetilde{\mathbf{A}}_{\boldsymbol{\theta}_k}\underline{\widetilde{\boldsymbol{x}}}_k + \widetilde{\mathbf{H}}_{\boldsymbol{\theta}_k}\underline{\widetilde{\boldsymbol{w}}}_k \tag{4}$$

with

$$\underline{\widetilde{\boldsymbol{x}}}_k = \begin{bmatrix}\underline{\boldsymbol{x}}_k \\ \underline{\widehat{\boldsymbol{x}}}_k\end{bmatrix} \ , \ \widetilde{\mathbf{A}}_{\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{A}_{\boldsymbol{\theta}_k} & \mathbf{B}_{\boldsymbol{\theta}_k}\mathbf{L} \\ \mathbf{K}\mathbf{C}_{\boldsymbol{\theta}_k} & \mathbf{F}\end{bmatrix} \ ,$$

$$\underline{\widetilde{\boldsymbol{w}}}_k = \begin{bmatrix}\underline{\boldsymbol{w}}_k \\ \underline{\boldsymbol{v}}_k\end{bmatrix} \ , \ \widetilde{\mathbf{H}}_{\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{H}_{\boldsymbol{\theta}_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}\mathbf{J}_{\boldsymbol{\theta}_k}\end{bmatrix} \ .$$

Then, for the cost function (3) in terms of the closed-loop dynamics, we obtain

$$\mathcal{J} = \lim_{K\to\infty} \frac{1}{K}\mathrm{E}\left\{\sum_{\mathrm{k}=0}^{\mathrm{K}} \underline{\widetilde{\boldsymbol{x}}}_{\mathrm{k}}^\top \widetilde{\mathbf{Q}}_{\boldsymbol{\theta}_k}\underline{\widetilde{\boldsymbol{x}}}_{\mathrm{k}}\right\}$$

with

$$\widetilde{\mathbf{Q}}_{\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{Q}_{\boldsymbol{\theta}_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^\top\mathbf{R}_{\boldsymbol{\theta}_k}\mathbf{L}\end{bmatrix} \ .$$

Before we propose the algorithm for computation of $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{L}$ in the next section, we introduce the notion of mean square stability of MJLS.

Definition 1 *The MJLS*

$$\underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_{\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k$$

is mean square (MS) stable, if it holds

$$\lim_{k\to\infty} \mathrm{E}\left\{\underline{\boldsymbol{x}}_{\mathrm{k}}^\top \underline{\boldsymbol{x}}_{\mathrm{k}}\right\} \to 0 \ .$$

Analogously, we can define mean square stabilizability.

Definition 2 *The MJLS* (1) *is MS stabilizable via mode-independent dynamic output feedback* (2)*, if there exist matrices* $\mathbf{F}$*,* $\mathbf{K}$*, and* $\mathbf{L}$ *such that the closed-loop system* (4) *is mean square stable.*

III. Control Law Computation

In the remainder of this paper, we assume that system (1) is (MS) stabilizable via linear dynamic output feedback (2). Let us define the second moment

$$\widetilde{\mathbf{X}}_k^i = \mathrm{E}\left\{\underline{\widetilde{\boldsymbol{x}}}_{\mathrm{k}}\underline{\widetilde{\boldsymbol{x}}}_{\mathrm{k}}^\top \mathbb{1}_{\boldsymbol{\theta}_{\mathrm{k}}=\mathrm{i}}\right\} = \begin{bmatrix}\mathbf{X}_{1,k}^i & \mathbf{X}_{12,k}^i \\ (\mathbf{X}_{12,k}^i)^\top & \mathbf{X}_{2,k}^i\end{bmatrix} \ ,$$

where $\mathbb{1}_{\boldsymbol{\theta}_k=i} = 1$ if $\boldsymbol{\theta}_k = i$ and 0 otherwise. According to [9], the closed-loop dynamics of the second moment are given by

$$\widetilde{\mathbf{X}}_{k+1}^j = \sum\nolimits_{i=1}^M p_{ij}\left[\widetilde{\mathbf{A}}_i\widetilde{\mathbf{X}}_k^i\widetilde{\mathbf{A}}_i^\top + \underline{\mu}_k^i\widetilde{\mathbf{H}}_i\widetilde{\mathbf{H}}_i^\top\right] \ ,$$

where $\underline{\mu}_k^i$ denotes the probability of being in mode i at time step k, i.e., $\underline{\mu}_k^i = \mathrm{P}(\boldsymbol{\theta}_k = i)$.

With these prerequisites, the control problem described in Sec. II can be formulated as

$$\begin{aligned} \min_{\mathbf{F},\mathbf{K},\mathbf{L}} \quad & \sum\nolimits_{i=1}^M \mathrm{trace}\left[\widetilde{\mathbf{Q}}_i\widetilde{\mathbf{X}}_\infty^i\right] \\ \mathrm{s.\,t.} \quad & \widetilde{\mathbf{X}}_\infty^j = \sum\nolimits_{i=1}^M p_{ij}\left[\widetilde{\mathbf{A}}_i\widetilde{\mathbf{X}}_\infty^i\widetilde{\mathbf{A}}_i^\top + \underline{\mu}_\infty^i\widetilde{\mathbf{H}}_i\widetilde{\mathbf{H}}_i^\top\right] \ , \end{aligned} \tag{5}$$

where we already took the limit $K \to \infty$ and $\underline{\mu}_\infty$ denotes the limit distribution of the Markov chain $\{\boldsymbol{\theta}_k\}$ [25]. Next, introducing the positive definite Lagrange multiplier

$$\widetilde{\mathbf{P}}^i_\infty = \begin{bmatrix} \mathbf{P}^i_1 & \mathbf{P}^i_{12} \\ (\mathbf{P}^i_{12})^\top & \mathbf{P}^i_2 \end{bmatrix} ,$$

we can formulate the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{M} \operatorname{trace}\Big[\widetilde{\mathbf{Q}}_i \widetilde{\mathbf{X}}^i_\infty - \widetilde{\mathbf{P}}^i_\infty \widetilde{\mathbf{X}}^i_\infty + \widetilde{\boldsymbol{\Lambda}}^i_\infty \Big[\widetilde{\mathbf{A}}_i \widetilde{\mathbf{X}}^i_\infty \widetilde{\mathbf{A}}_i^\top + \underline{\mu}^i_\infty \widetilde{\mathbf{H}}_i \widetilde{\mathbf{H}}_i^\top\Big]\Big] , \quad (6)$$

where $\widetilde{\boldsymbol{\Lambda}}^i_\infty = \sum_{j=1}^{M} p_{ij} \widetilde{\mathbf{P}}^j_\infty$. Necessary optimality conditions are

$$\frac{\partial \mathcal{H}}{\partial \widetilde{\mathbf{X}}^i_\infty} = \mathbf{0} \;,\; \frac{\partial \mathcal{H}}{\partial \widetilde{\mathbf{P}}^i_\infty} = \mathbf{0} \;,\; \frac{\partial \mathcal{H}}{\partial \mathbf{F}} = \mathbf{0} \;,\; \frac{\partial \mathcal{H}}{\partial \mathbf{K}} = \mathbf{0} \;,\; \frac{\partial \mathcal{H}}{\partial \mathbf{L}} = \mathbf{0} .$$

From $\partial\mathcal{H}/\partial\widetilde{\mathbf{P}}^i_\infty$, we obtain

$$\widetilde{\mathbf{X}}^j_\infty = \sum_{i=1}^{M} p_{ij} \Big[\widetilde{\mathbf{A}}_i \widetilde{\mathbf{X}}^i_\infty \widetilde{\mathbf{A}}_i^\top + \underline{\mu}^i_\infty \widetilde{\mathbf{H}}_i \widetilde{\mathbf{H}}_i^\top\Big] , \quad (7)$$

and from $\partial\mathcal{H}/\partial\widetilde{\mathbf{X}}^i_\infty$

$$\widetilde{\mathbf{P}}^i_\infty = \widetilde{\mathbf{Q}}_i + \widetilde{\mathbf{A}}_i^\top \widetilde{\boldsymbol{\Lambda}}^i_\infty \widetilde{\mathbf{A}}_i . \quad (8)$$

At this point, we introduce the substitutions

$$\begin{aligned} \mathbf{X}^i_{2,\infty} &= \underline{\mathbf{X}}_i , & \mathbf{P}^i_{2,\infty} &= \underline{\mathbf{P}}_i , \\ \mathbf{X}^i_{12,\infty} &= \underline{\mathbf{X}}_i , & \mathbf{P}^i_{12,\infty} &= -\underline{\mathbf{P}}_i , \\ (\mathbf{X}^i_{12,\infty})^\top &= \underline{\mathbf{X}}_i , & (\mathbf{P}^i_{12,\infty})^\top &= -\underline{\mathbf{P}}_i , \\ \mathbf{X}^i_{1,\infty} &= \overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i , & \mathbf{P}^i_{1,\infty} &= \overline{\mathbf{P}}_i + \underline{\mathbf{P}}_i . \end{aligned} \quad (9)$$

This particular choice constrains $\widetilde{\mathbf{X}}^i_\infty$ and $\widetilde{\mathbf{P}}^i_\infty$ to be positive definite. It can be made without loss of generality [26]. Furthermore, note that $\underline{\mathbf{X}}_i = \lim_{k\to\infty} \mathrm{E}\{\underline{\boldsymbol{x}}_k \widehat{\underline{\boldsymbol{x}}}_k \mathbb{1}_{\boldsymbol{\theta}_k = i}\}$ and $\overline{\mathbf{X}}_i = \lim_{k\to\infty} \mathrm{E}\{(\underline{\boldsymbol{x}}_k - \widehat{\underline{\boldsymbol{x}}}_k)(\underline{\boldsymbol{x}}_k - \widehat{\underline{\boldsymbol{x}}}_k)^\top \mathbb{1}_{\boldsymbol{\theta}_k = i}\}$. With this substitution, we obtain from (7) and (8)

$$\begin{aligned} \overline{\mathbf{X}}_j = \sum_{i=1}^{M} p_{ij} \Big[&\underline{\mu}^i_\infty \mathbf{H}_i \mathbf{H}_i^\top + \underline{\mu}^i_\infty \mathbf{K} \mathbf{J}_i \mathbf{J}_i^\top \mathbf{K}^\top \\ &+ (\mathbf{A}_i - \mathbf{K}\mathbf{C}_i) \overline{\mathbf{X}}_i (\mathbf{A}_i - \mathbf{K}\mathbf{C}_i)^\top \\ &+ (\mathbf{A}_i - \mathbf{F} + \mathbf{B}_i\mathbf{L} - \mathbf{K}\mathbf{C}_i) \underline{\mathbf{X}}_i (\mathbf{A}_i - \mathbf{F} + \mathbf{B}_i\mathbf{L} - \mathbf{K}\mathbf{C}_i)\Big] , \end{aligned} \quad (10)$$

$$\underline{\mathbf{X}}_j = \sum_{i=1}^{M} p_{ij} \Big[\underline{\mu}^i_\infty \mathbf{K}\mathbf{J}_i \mathbf{J}_i^\top \mathbf{K}^\top + \mathbf{K}\mathbf{C}_i \overline{\mathbf{X}}_i \mathbf{C}_i^\top \mathbf{K}^\top + (\mathbf{F} + \mathbf{K}\mathbf{C}_i) \underline{\mathbf{X}}_i (\mathbf{F} + \mathbf{K}\mathbf{C}_i)^\top\Big] , \quad (11)$$

$$\begin{aligned} \overline{\mathbf{P}}_i = &\mathbf{Q}_i + \mathbf{L}^\top \mathbf{R}_i \mathbf{L} + (\mathbf{A}_i + \mathbf{B}_i\mathbf{L})^\top \overline{\boldsymbol{\Lambda}}_i (\mathbf{A}_i + \mathbf{B}_i\mathbf{L}) \\ &+ (\mathbf{A}_i - \mathbf{F} + \mathbf{B}_i\mathbf{L} - \mathbf{K}\mathbf{C}_i)^\top \underline{\boldsymbol{\Lambda}}_i (\mathbf{A}_i - \mathbf{F} + \mathbf{B}_i\mathbf{L} - \mathbf{K}\mathbf{C}_i) , \end{aligned} \quad (12)$$

$$\underline{\mathbf{P}}_i = \mathbf{L}^\top \mathbf{R}_i \mathbf{L} + \mathbf{L}^\top \mathbf{B}_i^\top \overline{\boldsymbol{\Lambda}}_i \mathbf{B}_i \mathbf{L} + (\mathbf{F} - \mathbf{B}_i\mathbf{L})^\top \underline{\boldsymbol{\Lambda}}_i (\mathbf{F} - \mathbf{B}_i\mathbf{L}) , \quad (13)$$

where $\overline{\boldsymbol{\Lambda}}_i = \sum_{i=1}^{M} p_{ij} \overline{\mathbf{P}}_j$ and $\underline{\boldsymbol{\Lambda}}_i = \sum_{i=1}^{M} p_{ij} \underline{\mathbf{P}}_j$. The results in (10)-(13) are derived in Appendix VI-A.

Next, we use the necessary optimality condition $\partial\mathcal{H}/\partial\mathbf{F} = \mathbf{0}$ in order to eliminate $\mathbf{F}$ from $\mathcal{H}$, (7), and (8). It holds

$$\begin{aligned} \operatorname{vec}(\mathbf{F}) = &\Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i]\Big)^{-1} \Big[\operatorname{vec}\Big(\sum_{i=1}^{M} \underline{\boldsymbol{\Lambda}}_i \mathbf{A}_i \underline{\mathbf{X}}_i\Big) \\ &- \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \mathbf{C}_i^\top \otimes \underline{\boldsymbol{\Lambda}}_i]\Big) \operatorname{vec}(\mathbf{K}) \\ &+ \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i \mathbf{B}_i]\Big) \operatorname{vec}(\mathbf{L})\Big] , \end{aligned} \quad (14)$$

where $\operatorname{vec}(\cdot)$ is the vectorization operator and $\otimes$ denotes the Kronecker product [27]. The proof of this result is given in Appendix VI-B. It is important to emphasize that if the mode is available to the controller, (14) can be reduced to

$$\mathbf{F}_i = \mathbf{A}_i + \mathbf{B}_i \mathbf{L}_i - \mathbf{K}_i \mathbf{C}_i ,$$

which resembles the classical Kalman filter equations. However, due to the missing mode observation, we cannot apply this standard solution approach from [9].

Plugging (14) into (30) and using the necessary optimality conditions $\partial\mathcal{H}/\partial\mathbf{K} = \mathbf{0}$ and $\partial\mathcal{H}/\partial\mathbf{L} = \mathbf{0}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \mathbf{K}} &= \boldsymbol{\Phi}_{\mathbf{K}} \operatorname{vec}(\mathbf{K}) + \boldsymbol{\Psi}^\top \operatorname{vec}(\mathbf{L}) + \underline{\gamma}_{\mathbf{K}} = \mathbf{0} , \\ \frac{1}{2}\frac{\partial \mathcal{H}}{\partial \mathbf{L}} &= \boldsymbol{\Phi}_{\mathbf{L}} \operatorname{vec}(\mathbf{L}) + \boldsymbol{\Psi} \operatorname{vec}(\mathbf{K}) + \underline{\gamma}_{\mathbf{L}} = \mathbf{0} , \end{aligned} \quad (15)$$

with

$$\begin{aligned} \boldsymbol{\Phi}_{\mathbf{K}} &= \boldsymbol{\Phi}_{\mathbf{K}}(\overline{\mathbf{X}}, \underline{\mathbf{X}}, \overline{\boldsymbol{\Lambda}}, \underline{\boldsymbol{\Lambda}}) \\ &= \Big(\sum_{i=1}^{M} \Big[(\underline{\mu}^i_\infty \mathbf{J}_i \mathbf{J}_i^\top + \mathbf{C}_i (\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i) \mathbf{C}_i^\top) \otimes \underline{\boldsymbol{\Lambda}}_i\Big]\Big) \\ &- \Big(\sum_{i=1}^{M} [\mathbf{C}_i \underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i]\Big) \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i]\Big)^{-1} \\ &\times \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \mathbf{C}_i^\top \otimes \underline{\boldsymbol{\Lambda}}_i]\Big) , \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Phi}_{\mathbf{L}} &= \boldsymbol{\Phi}_{\mathbf{L}}(\overline{\mathbf{X}}, \underline{\mathbf{X}}, \overline{\boldsymbol{\Lambda}}, \underline{\boldsymbol{\Lambda}}) \\ &= \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes (\mathbf{R}_i + \mathbf{B}_i^\top (\overline{\boldsymbol{\Lambda}}_i + \underline{\boldsymbol{\Lambda}}_i) \mathbf{B}_i)]\Big) \\ &- \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \mathbf{B}_i^\top \underline{\boldsymbol{\Lambda}}_i]\Big) \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i]\Big)^{-1} \\ &\times \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i \mathbf{B}_i]\Big) , \end{aligned}$$

$$\begin{aligned} \boldsymbol{\Psi} &= \boldsymbol{\Psi}(\overline{\mathbf{X}}, \underline{\mathbf{X}}, \overline{\boldsymbol{\Lambda}}, \underline{\boldsymbol{\Lambda}}) \\ &= -\Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \mathbf{C}_i^\top \otimes \mathbf{B}_i^\top \underline{\boldsymbol{\Lambda}}_i]\Big) \\ &+ \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \mathbf{B}_i^\top \underline{\boldsymbol{\Lambda}}_i]\Big) \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\boldsymbol{\Lambda}}_i]\Big)^{-1} \\ &\times \Big(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \mathbf{C}_i^\top \otimes \underline{\boldsymbol{\Lambda}}_i]\Big) , \end{aligned}$$

$$\begin{aligned}\underline{\gamma}_{\mathbf{K}} &= \underline{\gamma}_{\mathbf{K}}(\overline{\mathbf{X}}, \underline{\mathbf{X}}, \overline{\mathbf{\Lambda}}, \underline{\mathbf{\Lambda}}) \\ &= -\mathrm{vec}\left(\sum_{i=1}^{M} \underline{\mathbf{\Lambda}}_i \mathbf{A}_i (\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i)\mathbf{C}_i^\top\right) \\ &+ \left(\sum_{i=1}^{M} [\mathbf{C}_i \underline{\mathbf{X}}_i \otimes \underline{\mathbf{\Lambda}}_i]\right)\left(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\mathbf{\Lambda}}_i]\right)^{-1} \\ &\times \mathrm{vec}\left(\sum_{i=1}^{M} \underline{\mathbf{\Lambda}}_i \mathbf{A}_i \underline{\mathbf{X}}_i\right) ,\end{aligned}$$

$$\begin{aligned}\underline{\gamma}_{\mathbf{L}} &= \underline{\gamma}_{\mathbf{L}}(\overline{\mathbf{X}}, \underline{\mathbf{X}}, \overline{\mathbf{\Lambda}}, \underline{\mathbf{\Lambda}}) \\ &= \mathrm{vec}\left(\sum_{i=1}^{M} \mathbf{B}_i^\top (\overline{\mathbf{\Lambda}}_i + \underline{\mathbf{\Lambda}}_i)\mathbf{A}_i \underline{\mathbf{X}}_i\right) \\ &- \left(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \mathbf{B}_i^\top \underline{\mathbf{\Lambda}}_i]\right)\left(\sum_{i=1}^{M} [\underline{\mathbf{X}}_i \otimes \underline{\mathbf{\Lambda}}_i]\right)^{-1} \\ &\times \mathrm{vec}\left(\sum_{i=1}^{M} \underline{\mathbf{\Lambda}}_i \mathbf{A}_i \underline{\mathbf{X}}_i\right) .\end{aligned}$$

The proof is given in Appendix VI-C.

Finally, we have that the optimal solution to (5) is determined by the set of coupled nonlinear equations (10), (11), (12), (13), (14), and (15). However, finding a solution to this set of coupled equations is non-trivial. Thus, we propose to apply an iterative scheme as argued in [23]. However, a direct application is not possible because [23] considers white switching in contrast to Markovian switching considered here and we have to make some important adaptations. The iterative scheme consists in iterating

$$\begin{aligned}\overline{\mathbf{X}}_j^{[\eta+1]} &= \sum_{i=1}^{M} p_{ij} \Big[\underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top + \underline{\mu}_\infty^i \mathbf{K}^{[\eta]} \mathbf{J}_i \mathbf{J}_i^\top (\mathbf{K}^{[\eta]})^\top \\ &+(\mathbf{A}_i - \mathbf{K}^{[\eta]}\mathbf{C}_i)\overline{\mathbf{X}}_i^{[\eta]}(\mathbf{A}_i - \mathbf{K}^{[\eta]}\mathbf{C}_i)^\top \\ &+(\mathbf{A}_i - \mathbf{F}^{[\eta]} + \mathbf{B}_i\mathbf{L}^{[\eta]} - \mathbf{K}^{[\eta]}\mathbf{C}_i)\underline{\mathbf{X}}_i^{[\eta]} \\ &\times(\mathbf{A}_i - \mathbf{F}^{[\eta]} + \mathbf{B}_i\mathbf{L}^{[\eta]} - \mathbf{K}^{[\eta]}\mathbf{C}_i)\Big] , \end{aligned} \tag{16}$$

$$\begin{aligned}\underline{\mathbf{X}}_j^{[\eta+1]} &= \sum_{i=1}^{M} p_{ij} \Big[\underline{\mu}_\infty^i \mathbf{K}^{[\eta]} \mathbf{J}_i \mathbf{J}_i^\top (\mathbf{K}^{[\eta]})^\top \\ &+\mathbf{K}^{[\eta]}\mathbf{C}_i\overline{\mathbf{X}}_i^{[\eta]}(\mathbf{C}_i^{[\eta]})^\top\mathbf{K}^\top \\ &+(\mathbf{F}^{[\eta]} + \mathbf{K}^{[\eta]}\mathbf{C}_i)\underline{\mathbf{X}}_i^{[\eta]}(\mathbf{F}^{[\eta]} + \mathbf{K}^{[\eta]}\mathbf{C}_i)^\top\Big] , \end{aligned} \tag{17}$$

$$\begin{aligned}\overline{\mathbf{P}}_i^{[\eta+1]} &= \mathbf{Q}_i + (\mathbf{L}^{[\eta]})^\top \mathbf{R}_i \mathbf{L}^{[\eta]} + (\mathbf{A}_i + \mathbf{B}_i\mathbf{L}^{[\eta]})^\top \overline{\mathbf{\Lambda}}_i^{[\eta]} \\ &\times (\mathbf{A}_i + \mathbf{B}_i\mathbf{L}^{[\eta]}) + (\mathbf{A}_i - \mathbf{F}^{[\eta]} + \mathbf{B}_i\mathbf{L}^{[\eta]} - \mathbf{K}^{[\eta]}\mathbf{C}_i)^\top \\ &\times \underline{\mathbf{\Lambda}}_i^{[\eta]}(\mathbf{A}_i - \mathbf{F}^{[\eta]} + \mathbf{B}_i\mathbf{L}^{[\eta]} - \mathbf{K}^{[\eta]}\mathbf{C}_i) , \end{aligned} \tag{18}$$

$$\begin{aligned}\underline{\mathbf{P}}_i^{[\eta+1]} &= (\mathbf{L}^{[\eta]})^\top \mathbf{R}_i \mathbf{L}^{[\eta]} + (\mathbf{L}^{[\eta]})^\top \mathbf{B}_i^\top \overline{\mathbf{\Lambda}}_i^{[\eta]} \mathbf{B}_i \mathbf{L}^{[\eta]} \\ &+ (\mathbf{F}^{[\eta]} - \mathbf{B}_i\mathbf{L}^{[\eta]})^\top \underline{\mathbf{\Lambda}}_i^{[\eta]}(\mathbf{F}^{[\eta]} - \mathbf{B}_i\mathbf{L}^{[\eta]}) , \end{aligned} \tag{19}$$

initialized with random positive definite $\overline{\mathbf{X}}_i^{[0]}$, $\underline{\mathbf{X}}_i^{[0]}$, $\overline{\mathbf{P}}_i^{[0]}$, and $\underline{\mathbf{P}}_i^{[0]}$. In our implementation, we obtained the fastest convergence when using $1\mathrm{e}{-4}\mathbf{I}$ instead of random values. At each iteration step, we update $\mathbf{K}^{[\eta-1]} \to \mathbf{K}^{[\eta]}$, and $\mathbf{L}^{[\eta-1]} \to \mathbf{L}^{[\eta]}$ using (15), where we set $\mathbf{\Phi_K} = \mathbf{\Phi_K}(\overline{\mathbf{X}}^{[\eta]}, \underline{\mathbf{X}}^{[\eta]}, \overline{\mathbf{\Lambda}}^{[\eta]}, \underline{\mathbf{\Lambda}}^{[\eta]})$, $\mathbf{\Psi} = \mathbf{\Psi}(\overline{\mathbf{X}}^{[\eta]}, \underline{\mathbf{X}}^{[\eta]}, \overline{\mathbf{\Lambda}}^{[\eta]}, \underline{\mathbf{\Lambda}}^{[\eta]})$, $\mathbf{\Phi_L} = \mathbf{\Phi_L}(\overline{\mathbf{X}}^{[\eta]}, \underline{\mathbf{X}}^{[\eta]}, \overline{\mathbf{\Lambda}}^{[\eta]}, \underline{\mathbf{\Lambda}}^{[\eta]})$, $\underline{\gamma}_{\mathbf{K}} = \underline{\gamma}_{\mathbf{K}}(\overline{\mathbf{X}}^{[\eta]}, \underline{\mathbf{X}}^{[\eta]}, \overline{\mathbf{\Lambda}}^{[\eta]}, \underline{\mathbf{\Lambda}}^{[\eta]})$, and $\underline{\gamma}_{\mathbf{L}} = \underline{\gamma}_{\mathbf{L}}(\overline{\mathbf{X}}^{[\eta]}, \underline{\mathbf{X}}^{[\eta]}, \overline{\mathbf{\Lambda}}^{[\eta]}, \underline{\mathbf{\Lambda}}^{[\eta]})$. To make the update $\mathbf{F}^{[\eta-1]} \to \mathbf{F}^{[\eta]}$, we use (14) with appropriate $\overline{\mathbf{X}}^{[\eta]}$, $\underline{\mathbf{X}}^{[\eta]}$, $\overline{\mathbf{\Lambda}}^{[\eta]}$, $\underline{\mathbf{\Lambda}}^{[\eta]}$, $\mathbf{K}^{[\eta]}$, and $\mathbf{L}^{[\eta]}$.

Please note that convergence of the proposed iterative algorithm does not imply stability of the closed-loop system, i.e., the algorithm may converge to a solution although the MJLS (1) is not MS stabilizable via (2). For this reason, it is necessary to check stability using, e.g., Corollary 2.6 from [24] that proposes to check whether the matrix

$$\mathbf{M} = \mathrm{diag}\left[\widetilde{\mathbf{A}}_1 \otimes \widetilde{\mathbf{A}}_1 \quad \dots \quad \widetilde{\mathbf{A}}_M \otimes \widetilde{\mathbf{A}}_M\right]\left(\mathbf{T}^\top \otimes \mathbf{I}\right)$$

is Schur stable, where $\widetilde{\mathbf{A}}_{\boldsymbol{\theta}_k}$ is the closed-loop system matrix. If $\mathbf{M}$ is Schur then the closed-loop system (4) is stable in the MS sense.

Also, note that we can solve (15) for $\mathrm{vec}\,(\mathbf{K})$ and $\mathrm{vec}\,(\mathbf{L})$ according to

$$\begin{bmatrix}\mathrm{vec}\,(\mathbf{K}) \\ \mathrm{vec}\,(\mathbf{L})\end{bmatrix} = -\begin{bmatrix}\mathbf{\Phi_K} & \mathbf{\Psi} \\ \mathbf{\Psi}^\top & \mathbf{\Phi_L}\end{bmatrix}^{-1}\begin{bmatrix}\underline{\gamma}_{\mathbf{K}} \\ \underline{\gamma}_{\mathbf{L}}\end{bmatrix} . \tag{20}$$

However, this equation involves a matrix inversion. Thus, it may be faster to solve

$$\begin{aligned}\min \quad & \left[\mathrm{vec}\,(\mathbf{K})^\top \quad \mathrm{vec}\,(\mathbf{L})\right]\left[\mathrm{vec}\,(\mathbf{K})^\top \quad \mathrm{vec}\,(\mathbf{L})\right]^\top \\ \mathrm{s.t.} \quad & (15)\end{aligned}$$

in order to obtain $\mathrm{vec}\,(\mathbf{K})$ and $\mathrm{vec}\,(\mathbf{L})$ instead. , using, e.g., the interior-point algorithm.

Remark 1 *Please note that if the jumping parameter $\boldsymbol{\theta}_k$ is white and not Markovian, we obtain the result from [23] as a special case.*

IV. Numerical Example

In order to demonstrate the presented algorithm, we performed a Monte Carlo simulation with $1\mathrm{e}4$ runs à 200 time steps each for different choices of transition, and process and measurement noise matrices. We compared the performance of the control law computed using the proposed algorithm with the time-invariant optimal controller from [9] that requires mode feedback[2]. The parameters of the MJLS were chosen to

$$\begin{aligned}&\mathbf{A}_1 = \begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix} , \ \mathbf{A}_2 = \begin{bmatrix}1.2 & 1.2 \\ 0 & 1\end{bmatrix} , \ \begin{matrix}\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I} , \\ \mathbf{R}_1 = \mathbf{R}_2 = 1 ,\end{matrix} \\ &\mathbf{B}_1 = \begin{bmatrix}0 \\ 1\end{bmatrix} , \ \mathbf{B}_2 = \begin{bmatrix}0.1 \\ 0.8\end{bmatrix} , \ \begin{matrix}\mathbf{C}_1 = \begin{bmatrix}1 & 0\end{bmatrix} , \\ \mathbf{C}_2 = \begin{bmatrix}0.8 & 0\end{bmatrix} .\end{matrix}\end{aligned} \tag{21}$$

Furthermore, we performed simulations with two different transition matrices

$$\mathbf{T}^{[1]} = \begin{bmatrix}0.7 & 0.3 \\ 0.2 & 0.8\end{bmatrix} \text{ and } \mathbf{T}^{[2]} = \begin{bmatrix}0.9 & 0.1 \\ 0.1 & 0.9\end{bmatrix} ,$$

[2]We chose not to compare our method with [19], because the mode-independent feedback from [19] requires $\mathbf{A}_{\boldsymbol{\theta}_k}$, $\mathbf{B}_{\boldsymbol{\theta}_k}$, and $\mathbf{C}_{\boldsymbol{\theta}_k}$ to be mode-independent, i.e., constant.

	$\mathbf{T}^{[1]}, \mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[1]}, \mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[2]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[1]}, \mathbf{H}^{[2]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[1]}, \mathbf{H}^{[2]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[2]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[2]}, \mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[2]}, \mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[2]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[2]}, \mathbf{H}^{[2]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$	$\mathbf{T}^{[2]}, \mathbf{H}^{[2]}_{\boldsymbol{\theta}_k}$, $\mathbf{J}^{[2]}_{\boldsymbol{\theta}_k}$
costs - optimal controller [9]	11.0848	11.3172	14.5255	14.7933	10.1433	10.2526	13.3096	13.5912
costs - proposed	11.2070	11.5388	15.4048	15.8356	10.2826	10.7506	14.2897	14.7641

TABLE I
RESULTS OF THE MONTE CARLO SIMULATION

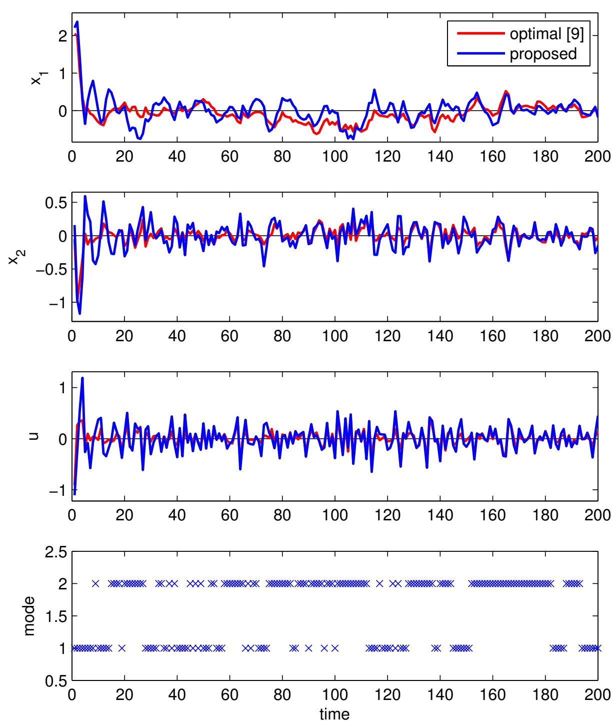

Fig. 1. State, control, and mode trajectories of an example run with $\mathbf{T}^{[1]}$, $\mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, and $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$.

two different sets of matrices $\mathbf{H}_1^{[1]} = \mathbf{H}_2^{[1]} = 0.01 \cdot \mathbf{I}$ and $\mathbf{H}_1^{[2]} = \mathbf{H}_2^{[2]} = 0.1 \cdot \mathbf{I}$, and two different sets $\mathbf{J}_1^{[1]} = \mathbf{J}_2^{[1]} = 0.05$ and $\mathbf{J}_1^{[2]} = \mathbf{J}_2^{[2]} = 0.1$. Also, we drew the initial state $\underline{\boldsymbol{x}}_0$ from a Gaussian with mean $\bar{x}_0 = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ and covariance $\boldsymbol{\Xi} = 0.2^2 \cdot \mathbf{I}$. An example run with $\mathbf{T}^{[1]}$, $\mathbf{H}^{[1]}_{\boldsymbol{\theta}_k}$, and $\mathbf{J}^{[1]}_{\boldsymbol{\theta}_k}$ is depicted in Fig. 1. The simulation results are depicted in Table I. They indicate that the proposed control law performs quite well and that the performance decrease due to the non-observability of the mode is low. This can also be seen in the spectral radii of the controlled MJLS. The spectral radius of the uncontrolled MJLS with $\mathbf{T}^{[1]}$ is $\rho(\mathbf{M}(\mathbf{T}^{[1]})) = 1.2968$ and $\rho(\mathbf{M}(\mathbf{T}^{[2]})) = 1.3295$ with $\mathbf{T}^{[2]}$, respectively, i.e., the MJLS (1) is unstable for both transition matrices. For the optimal controller from [9], the spectral radii are $\rho(\mathbf{M}(\mathbf{T}^{[1]})) = 0.7654$ and $\rho(\mathbf{M}(\mathbf{T}^{[2]})) = 0.8089$. For the proposed algorithm, we obtain $\rho(\mathbf{M}(\mathbf{T}^{[1]})) = 0.9591$ and $\rho(\mathbf{M}(\mathbf{T}^{[2]})) = 0.9530$, which yields that the controller computed using the proposed algorithm can stabilize the MJLS. Of course, the stabilization property of the optimal controller with mode observation is better. The proposed control law required on average 20 iterations to converge. A reference implementation of the presented algorithm is available on GitHub [28].

V. CONCLUSION

In this paper, we addressed dynamic output-feedback control of Markov Jump Linear Systems without mode observation. We assumed a linear control law and, to our knowledge, proposed the first solution to this problem that does not make any non-standard restrictions. The computation of the control law is given in terms of an iterative algorithm. Simulations indicate that this algorithm always converges, if the MJLS is stabilizable via dynamic output feedback. However, a proof of this conjecture is not yet available and is subject of current research. Nevertheless, the presented algorithm can be used in order to compute the control law. Then, the stability of the closed-loop system can be checked using results available in literature.

VI. APPENDIX

A. *Proof of* (10)-(13)

From (7) and (8), we have

$$\mathbf{X}^j_{1,\infty} = \sum_{i=1}^{M} p_{ij} \Big[\mathbf{A}_i \mathbf{X}^i_{1,\infty} \mathbf{A}_i^\top + \mathbf{A}_i \mathbf{X}^i_{12,\infty} \mathbf{L}^\top \mathbf{B}_i^\top + \mathbf{B}_i \mathbf{L} (\mathbf{X}^i_{12,\infty})^\top \mathbf{A}_i^\top + \mathbf{B}_i \mathbf{L} \mathbf{X}^i_{2,\infty} \mathbf{L}^\top \mathbf{B}_i^\top + \underline{\mu}^i_\infty \mathbf{H}_i \mathbf{H}_i^\top \Big] , \quad (22)$$

$$\mathbf{X}^j_{12,\infty} = \sum_{i=1}^{M} p_{ij} \Big[\mathbf{A}_i \mathbf{X}^i_{1,\infty} \mathbf{C}_i^\top \mathbf{K}^\top + \mathbf{A}_i \mathbf{X}^i_{12,\infty} \mathbf{F}^\top + \mathbf{B}_i \mathbf{L} (\mathbf{X}^i_{12,\infty})^\top \mathbf{C}_i^\top \mathbf{K}^\top + \mathbf{B}_i \mathbf{L} \mathbf{X}^i_{2,\infty} \mathbf{F}^\top \Big] , \quad (23)$$

$$\mathbf{X}^j_{2,\infty} = \sum_{i=1}^{M} p_{ij} \Big[\mathbf{K} \mathbf{C}_i \mathbf{X}^i_{1,\infty} \mathbf{C}_i^\top \mathbf{K}^\top + \mathbf{K} \mathbf{C}_i \mathbf{X}^i_{12,\infty} \mathbf{F}^\top + \mathbf{F} (\mathbf{X}^i_{12,\infty})^\top \mathbf{C}_i^\top \mathbf{K}^\top + \mathbf{F} \mathbf{X}^i_{2,\infty} \mathbf{F}^\top + \underline{\mu}^i_\infty \mathbf{K} \mathbf{J}_i \mathbf{J}_i^\top \mathbf{K}^\top \Big] , \quad (24)$$

$$\mathbf{P}^i_{1,\infty} = \mathbf{A}_i^\top \boldsymbol{\Lambda}^i_{1,\infty} \mathbf{A}_i + \mathbf{A}_i^\top \boldsymbol{\Lambda}^i_{12,\infty} \mathbf{K} \mathbf{C}_i + \mathbf{C}_i^\top \mathbf{K}^\top (\boldsymbol{\Lambda}^i_{12,\infty})^\top \mathbf{A}_i + \mathbf{C}_i^\top \mathbf{K}^\top \boldsymbol{\Lambda}^i_{2,\infty} \mathbf{K} \mathbf{C}_i + \mathbf{Q}_i , \quad (25)$$

$$\mathbf{P}^i_{12,\infty} = \mathbf{A}_i^\top \boldsymbol{\Lambda}^i_{1,\infty} \mathbf{B}_i \mathbf{L} + \mathbf{A}_i^\top \boldsymbol{\Lambda}^i_{12,\infty} \mathbf{F} + \mathbf{C}_i^\top \mathbf{K}^\top (\boldsymbol{\Lambda}^i_{12,\infty})^\top \mathbf{B}_i \mathbf{L} + \mathbf{C}_i^\top \mathbf{K}^\top \boldsymbol{\Lambda}^i_{2,\infty} \mathbf{F} , \quad (26)$$

$$\mathbf{P}^i_{2,\infty} = \mathbf{L}^\top \mathbf{B}_i^\top \boldsymbol{\Lambda}^i_{1,\infty} \mathbf{B}_i \mathbf{L} + \mathbf{L}^\top \mathbf{B}_i^\top \boldsymbol{\Lambda}^i_{12,\infty} \mathbf{F} + \mathbf{F}^\top (\boldsymbol{\Lambda}^i_{12,\infty})^\top \mathbf{B}_i \mathbf{L} + \mathbf{F}^\top \boldsymbol{\Lambda}^i_{2,\infty} \mathbf{F} + \mathbf{L}^\top \mathbf{R}_i \mathbf{L} . \quad (27)$$

Using the substitutions from (9), we obtain (11). Next, the correspondence

$$\overline{\mathbf{X}}_j = \lim_{k \to \infty} \mathrm{E}\left\{ (\underline{\boldsymbol{x}}_\mathrm{k} - \widehat{\underline{\boldsymbol{x}}}_\mathrm{k})(\underline{\boldsymbol{x}}_\mathrm{k} - \widehat{\underline{\boldsymbol{x}}}_\mathrm{k})^\top \mathbb{1}_{\boldsymbol{\theta}_\mathrm{k} = \mathrm{j}} \right\} \quad (28)$$

implies $\overline{\mathbf{X}}_j = \mathbf{X}^j_{1,\infty} - \mathbf{X}^j_{12,\infty} - (\mathbf{X}^j_{12,\infty})^\top + \mathbf{X}^j_{2,\infty}$, where we can use (22)-(24) and apply (9) in order to obtain (10). Accordingly, (27) yields (13). Finally, from the classical optimal control theory, we can infer that $\widetilde{\mathbf{P}}^i_\infty$ is the second moment of the costate $\widetilde{\underline{\boldsymbol{\xi}}}_k$ of the closed-loop state $\widetilde{\underline{\boldsymbol{x}}}_k$. Thus,

$$\overline{\mathbf{P}}_i = \lim_{k\to\infty} \mathrm{E}\left\{(\underline{\boldsymbol{\xi}}_\mathrm{k} - \widehat{\underline{\xi}}_\mathrm{k})(\underline{\boldsymbol{\xi}}_\mathrm{k} - \widehat{\underline{\xi}}_\mathrm{k})^\top \mathbb{1}_{\boldsymbol{\theta}_\mathrm{k}=\mathrm{i}}\right\} \quad (29)$$

is the analogy to (28), where $\underline{\boldsymbol{\xi}}_k$ is the costate of $\underline{\boldsymbol{x}}_k$ and $\widehat{\underline{\xi}}_k$ the costate of $\widehat{\underline{\boldsymbol{x}}}_k$, respectively. Taking the limit in (29), and using (25)-(27) and (9) yields (12), which concludes the proof.

B. *Proof of* (14)

The Hamiltonian (6) can be evaluated to

$$\begin{aligned}\mathcal{H} = \sum\nolimits_{i=1}^{M} &\mathrm{trace}\big[\mathbf{Q}_i(\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i) + \mathbf{L}^\top\mathbf{R}_i\mathbf{L}\underline{\mathbf{X}}_i - \overline{\mathbf{P}}_i\overline{\mathbf{X}}_i \\ &-\underline{\mathbf{P}}_i\overline{\mathbf{X}}_i - \overline{\mathbf{P}}_i\underline{\mathbf{X}}_i + \underline{\mu}^i_\infty(\overline{\mathbf{\Lambda}}_i + \underline{\mathbf{\Lambda}}_i)\mathbf{H}_i\mathbf{H}_i^\top \\ &+\underline{\mu}^i_\infty\underline{\mathbf{\Lambda}}_i\mathbf{K}\mathbf{J}_i\mathbf{J}_i^\top\mathbf{K}^\top + (\overline{\mathbf{\Lambda}}_i + \underline{\mathbf{\Lambda}}_i)\mathbf{A}_i(\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i)\mathbf{A}_i^\top \\ &+2(\overline{\mathbf{\Lambda}}_i + \underline{\mathbf{\Lambda}}_i)\mathbf{A}_i\underline{\mathbf{X}}_i\mathbf{L}^\top\mathbf{B}_i^\top + (\overline{\mathbf{\Lambda}}_i + \underline{\mathbf{\Lambda}}_i)\mathbf{B}_i\mathbf{L}\underline{\mathbf{X}}_i\mathbf{L}^\top\mathbf{B}_i^\top \\ &-2\underline{\mathbf{\Lambda}}_i\mathbf{A}_i(\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i)\mathbf{C}_i^\top\mathbf{K}^\top - 2\underline{\mathbf{\Lambda}}_i\mathbf{B}_i\mathbf{L}\underline{\mathbf{X}}_i\mathbf{C}_i^\top\mathbf{K}^\top \\ &-2\underline{\mathbf{\Lambda}}_i\mathbf{A}_i\underline{\mathbf{X}}_i\mathbf{F}^\top - 2\underline{\mathbf{\Lambda}}_i\mathbf{B}_i\mathbf{L}\underline{\mathbf{X}}_i\mathbf{F}^\top + 2\underline{\mathbf{\Lambda}}_i\mathbf{K}\mathbf{C}_i\underline{\mathbf{X}}_i\mathbf{F}^\top \\ &+\underline{\mathbf{\Lambda}}_i\mathbf{K}\mathbf{C}_i(\overline{\mathbf{X}}_i + \underline{\mathbf{X}}_i)\mathbf{C}_i^\top\mathbf{K}^\top + \underline{\mathbf{\Lambda}}_i\mathbf{F}\underline{\mathbf{X}}_i\mathbf{F}^\top\big] \ . \end{aligned} \quad (30)$$

Differentiation with respect to $\mathbf{F}$ and setting the result to $\mathbf{0}$ yields

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\mathbf{F}} = 2\sum\nolimits_{i=1}^{M} -\underline{\mathbf{\Lambda}}_i\mathbf{F}\underline{\mathbf{X}}_i + \underline{\mathbf{\Lambda}}_i\mathbf{K}\mathbf{C}_i\underline{\mathbf{X}}_i - \underline{\mathbf{\Lambda}}_i\mathbf{B}_i\mathbf{L}\underline{\mathbf{X}}_i \\ - \underline{\mathbf{\Lambda}}_i\mathbf{A}_i\underline{\mathbf{X}}_i = \mathbf{0} \ .\end{aligned}$$

Using $\mathrm{vec}(\mathbf{AXB}) = (\mathbf{B}^\top \otimes \mathbf{A})\mathrm{vec}(\mathbf{X})$ and resolving for $\mathrm{vec}(\mathbf{F})$, we obtain (14). Please note that if there was only one mode, i.e., (1) was an ordinary stochastic system, or if the parameters of (1) were white, we could premultiply with $\underline{\mathbf{X}}_i^{-1}$ and postmultiply with $\underline{\mathbf{\Lambda}}_i^{-1}$ to obtain one of the classical Kalman filter equations.

C. *Proof of* (15)

Using $\mathrm{trace}\left[\mathbf{A}^\top\mathbf{B}\right] = \mathrm{vec}(\mathbf{A})^\top \mathrm{vec}(\mathbf{B})$, we can write the Hamiltonian (30) as

$$\begin{aligned}\mathcal{H} = \sum\nolimits_{i=1}^{M} &\mathrm{trace}\big[\mathbf{Q}_i\mathbf{X}_1^i - \mathbf{P}_1^i\mathbf{X}_1^i - 2(\mathbf{P}_{12}^i)^\top\mathbf{X}_{12}^i - \mathbf{P}_2^i\mathbf{X}_2^i \\ &+\underline{\mu}^i_\infty\mathbf{\Lambda}_1^i\mathbf{H}_i\mathbf{H}_i^\top + \mathbf{\Lambda}_1^i\mathbf{A}_i\mathbf{X}_1^i\mathbf{A}_i^\top\big] + \mathrm{vec}(\mathbf{L})^\top\mathbf{\Phi}_\mathbf{L}\mathrm{vec}(\mathbf{L}) \\ &+ \mathrm{vec}(\mathbf{K})^\top\mathbf{\Phi}_\mathbf{K}\mathrm{vec}(\mathbf{K}) + 2\mathrm{vec}(\mathbf{L})^\top\mathbf{\Psi}\mathrm{vec}(\mathbf{K}) \ .\end{aligned}$$

Differentiation with respect to $\mathrm{vec}(\mathbf{K})$ and $\mathrm{vec}(\mathbf{L})$ yields (15).

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