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Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD

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Abstract: We compute the four-loop QCD corrections to the massless quark-anti-quark-photon form factor F_q in the large-N_c limit. From the pole part we extract analytic expressions for the corresponding cusp and collinear anomalous dimensions.

Keywords: Perturbative QCD, Scattering Amplitudes

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Contents

1 Introduction

Perturbation theory is a powerful tool to obtain reliable predictions for physical observables within the Standard Model of particle physics or its extensions. Due to the high precision of experimental results, e.g., at the CERN Large Hadron Collider (LHC) or at the B factories, it is on the one hand mandatory to advance the development of tools, which can be used for higher order calculations. On the other hand it is necessary to improve the understanding of the perturbative structure of quantum field theories. Form factors are ideal objects to obtain deeper insight into the latter. Especially in the context of QCD they are indispensable tools to investigate the infrared structure of scattering amplitudes to high orders in perturbation theory. Moreover, from the pole part it is possible to extract universal process-independent quantities [1–7] like the cusp anomalous dimension which can be extracted from the $1/\epsilon^2$ pole of the form factor. The finite parts of the form factors serve as building blocks for a variety of physical processes. For example, the quark-anti-quark-photon form factor enters the virtual corrections of the Drell-Yan process for the production of lepton pairs at hadron colliders.

In this paper we consider the quark-anti-quark-photon form factor which is conveniently obtained from the photon-quark vertex function Γ_q^μ by applying an appropriate projector. In $D = 4 - 2\epsilon$ space-time dimensions we have

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2} \,\mathrm{Tr}(\not{p}_2\, \Gamma_q^\mu \not{p}_1\, \gamma_\mu)\,, \tag{1.1}$$

with $q = p_1 + p_2$ where p_1 and p_2 are the incoming quark and anti-quark momenta and q is the momentum of the photon. We perform our calculation in the framework of QCD keeping the number of colours, N_c, generic. In the limit of large N_c the calculation of F_q is simplified since only planar Feynman diagrams contribute. This is the limit we consider in this paper. Besides N_c we also keep the number of active quark flavours, n_f as a parameter and thus have at four-loop order the colour structures N_c^4, $N_c^3 n_f$, $N_c^2 n_f^2$, $N_c n_f^3$ where each factor of n_f counts the number of closed fermion loops.

Two- and three-loop corrections to F_q have been computed in refs. [8–15]. To obtain the four-loop corrections two obstacles need to be overcome: (i) the reduction to a set of

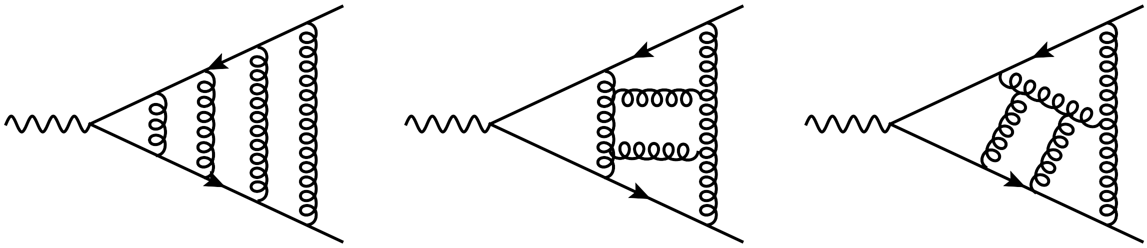

Figure 1. Sample four-loop Feynman diagrams contributing to F_q. Solid, curly and wavy lines represent quarks, gluons and photons, respectively. All particles are massless.

basis integrals and (ii) the analytic calculation of the latter (if possible). Recently, the first steps towards four loops have been initiated by computing the fermionic contributions to F_q in the planar limit [16]. The n_f^3 terms have been confirmed in ref. [17] using different methods both for the reduction and the computation of the master integrals. Let us mention that the four-loop corrections to the cusp anomalous dimension with two and three closed fermion loops have also been obtained in ref. [18]. In ref. [19] one finds a discussion of non-planar master intergrals relevant for the four-loop form factor in a $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

In the present paper, we evaluate the n_f^0 contribution and therefore complete the evaluation of the form factors and anomalous dimensions in the limit of large N_c.

In the next section we provide some technical details, in particular to the calculation of the most complicated master integral, and we discuss our results in section 3. We provide explicit expressions for the four-loop cusp and collinear anomalous dimensions in the planar limit. Furthermore, we provide results for the finite part of $\log(F_q)$.

2 Technical details

For the calculation of F_q we use a well-tested automated chain of programs which work hand-in-hand. The Feynman amplitudes are generated with `qgraf` [20]. Since there is no possibility to select already at this point the planar diagrams also non-planar amplitudes are generated and we obtain in total 1, 15, 337 and 9784 diagrams at one, two, three and four loops. Sample Feynman diagrams at four loops can be found in figure 1. Next, we transform the output to `FORM` [21] notation using `q2e` and `exp` [22, 23]. The program `exp` furthermore maps each Feynman diagram to predefined integral families for massless four-loop vertices with two different non-vanishing external momenta; 68 of them are of planar type. At this point we perform the Dirac algebra and decompose the numerator into terms which appear in the denominator. This allows us to express each Feynman integral as a linear combination of scalar functions which belong to the corresponding family. After exploiting the symmetries connected to the exchange of the external momenta we can reduce the number of families, for which integral tables have to be generated, from 68 to 38. Note that for the fermionic contributions, which have been considered in ref. [16], only 24 families are needed.

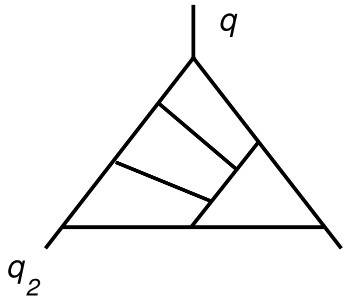

Figure 2. Massless four-loop vertex diagram with two external (incoming) momenta q and q_2 with $q^2 \neq 0 \neq q_2^2$ and $(q + q_2)^2 = 0$. The master integral I_{99} corresponds to $q_2^2 = 0$.

For the reduction to master integrals we use the program `FIRE` [24–26] which we apply in combination with `LiteRed` [27, 28]. We observe that the non-fermionic diagrams lead to more complex integrals for which the reduction time significantly increases. Let us remark that we have adopted Feynman gauge for the calculation of the non-fermionic parts.

Once reduction tables for each family are available we apply `tsort`, which is part of the latest `FIRE` version [26]. It is based on ideas presented in ref. [25] to establish relations between primary master integrals and thus minimize their number. In this way we arrive at 99 master integrals. In the remainder of this section we describe in detail the calculation of the most complicated integral corresponding to the graph of figure 2, I_{99}.

To have the possibility to apply differential equations for the evaluation of the form factor master integral corresponding to the graph of figure 2 we introduce a second mass scale by considering $q_2^2 = xq^2$ and derive differential equations with respect to the ratio of the two scales, x. This strategy was advocated in [29] and used for the four-loop form-factor integrals in [16, 30]. We derive differential equations for the corresponding 332 master integrals using `LiteRed` [27, 28].

To solve our differential equations we use an important observation made in ref. [31]. It has been suggested to turn from the basis of primary master integrals to a so-called canonical basis where the corresponding integrals satisfy a system differential equations which has a particular structure: the dependence on ϵ appears as a linear prefactor and the matrix in front of the vector of master integrals has only simple poles in x, i.e. has only so-called Fuchsian singularities. Such a system can then easily be solved in terms of iterated integrals.

In ref. [31], it was proposed that choosing integrals with constant leading singularities provides a canonical basis of the differential equations. In subsequent work, this was used in a variety of cases, see e.g. refs. [32–34]. As was motivated in ref. [16], this choice can be done in a systematic way at the level of the loop integrand. The first algorithm to convert a given differential system to a canonical form at the level of the differential equations has been provided in the case of one variable[1] in ref. [36] (therein called ϵ-form, see also ref. [32]). In the present paper we apply the private implementation of one of the authors

[1]The very recent paper [35] makes some interesting progress towards the algorithmic reduction in the multivariate case.

(R.N.L.) of the algorithm of ref. [36] to construct a canonical basis.[2] An independent evaluation of I_{99} can be found in ref. [30] where the canonical basis was chosen at the level of the loop integrand.

In our case the right-hand side of the differential system is determined by a 332×332 matrix. The matrix has a block-triangular form which means that non-zero elements appear only in and below the square blocks standing on the diagonal. Each diagonal block corresponds to the master integrals of a specific sector, i.e. to the master integrals having the same set of denominators. (The master integrals differ by powers of denominators from this set and/or by powers of irreducible numerators.) Each of these blocks is at most a 5×5 sub-matrix. Consequently, the reduction of individual blocks to ϵ-form within the approach of ref. [38] is simple. Note that already after this step one might claim that the solution is expressible in terms of harmonic polylogarithms. However, if we choose not to reduce the off-diagonal (rectangular) elements to an ϵ-form, the construction of the solution would require much more effort. First, we would have to construct the solution in a block-by-block fashion, using the results obtained on the previous steps, to write down the inhomogeneous terms in the subsequent differential systems. Second, we would have to perform integration by parts here and there to get rid of factors like $1/x^2$. Though these two steps are feasible and warrant the expressibility of the solution via harmonic logarithms, we choose to follow the prescription of section 7 of ref. [36] and reduce the whole system to an ϵ-form. Similar to ref. [38] we find that the differential-equation-based hierarchy of the set of the master integrals is too restrictive and the use of a sector-based hierarchy is necessary when factoring ϵ out of the whole matrix.

The family of one-scale Feynman integrals (with $(q+q_2)^2 = q_2^2 = 0$ and $q^2 \neq 0$) corresponding to figure 2 contains 76 master integrals. After introducing $q_2^2 \neq 0$ we define $x = q_2^2/q^2$ and obtain a family of Feynman integrals with 332 master integrals. Our strategy to compute the 76 one-scale master integrals is as follows

- Turn from a primary basis to a canonical basis.
- Solve differential equations for the canonical basis.
- Evaluate the *naive* values of the elements of the canonical basis at $x = 0$. By *naive* values of the integrals we understand the values obtained by setting $x = 0$ under the integral sign. It is exactly this contribution which is needed to obtain results for the primary master integrals.
- Obtain analytic result for primary master integrals of one-scale family, in particular I_{99}.

In the following we will elaborate on the individual steps.

Following ref. [36] we arrive at a canonical basis g which is obtained from the primary basis f by a linear transformation with a matrix T,

$$f = T \cdot g \,. \tag{2.1}$$

[2] Note that recently a public implementation of the algorithm of ref. [36] in a computer code `Fuchsia` [37] became available.

The vectors f and g have 332 entries and T is a 332×332 matrix. The dependence of f, g and T on x and ϵ has been suppressed.

It is convenient [31] to normalize the canonical master integrals such that they have uniformly transcendental ϵ-expansion which starts from ϵ^0. In our four-loop case, one has to compute expansion terms including ϵ^8 terms and we have

$$g(x,\epsilon) = \sum_{k=0}^{8} g_k(x)\epsilon^k. \tag{2.2}$$

The canonical basis g satisfies (by definition) a differential equation of the form

$$g'(x,\epsilon) = \epsilon\, A(x) \cdot g(x,\epsilon)\,, \tag{2.3}$$

where

$$A(x) = \frac{a}{x} + \frac{b}{x-1}\,, \tag{2.4}$$

with constant matrices a and b. It is straightforward to construct the generic solution of (2.3) order-by-order in ϵ in terms of harmonic polylogarithms (HPLs) [39] with letters 0 and 1, with 332×9 unknown constants.

To fix these constants we use boundary conditions at the point $x = 1$. The primary master integrals are regular at this point where the integrals become propagator-type integrals which are well known [40]. In particular, all the corresponding 28 master integrals are known analytically [40, 41] in an ϵ-expansion up to weight 12 and have been cross checked numerically [42]. We obtain the boundary values of the elements of the canonical basis g at $x = 1$ by inverting eq. (2.1) and considering the limit $x \to 1$. The matrix T^{-1} involves elements which develop poles up to order $1/(1-x)^6$. This requires that the first seven expansion terms for $x \to 1$ of the primary master integrals f have to be computed. The corresponding reduction tables are again generated with `FIRE`. Note that the resulting two-point integrals with external momentum q have scalar products in the numerator involving the momentum q_2 which complicates the calculation. It is an important cross check of the calculation that all poles in $1/(1-x)$ cancel in the combination $T^{-1} \cdot f$ and we obtain the values for g at $x = 1$, in an ϵ-expansion up to ϵ^8. This fixes the solution of the system of differential equations in eq. (2.3).

In a next step we analyze the leading asymptotic behaviour of g near $x = 0$ to finally extract the naive value for $x = 0$. This can be achieved by picking the x^0 terms from the small-x asymptotics, where no expansion in ϵ is performed. The absence of x^n terms with negative integer n in such an asymptotics serves as an important check.

On the one hand, we obtain the $x \to 0$ behaviour of g from the differential equations (2.3) where the term $b/(x-1)$ on the right-hand side can be neglected and the solution has the form $g(x,\epsilon) = h(\epsilon)x^{\epsilon a}$. The quantity $x^{\epsilon a} = e^{\epsilon \log(x) a}$ can be evaluated using the `Mathematica` command `MatrixExp[]` which leads to a 332×332 matrix where each element is a linear combination of terms $x^{k\epsilon}$ with integer k. In general both non-positive k and positive k might appear. However, in the case of Feynman integrals only terms with non-positive k can be present. Such a restriction has been observed in many calculations and can be naturally explained within the expansion-by-regions approach. In a specific region each component of each loop momentum typically scales as integer non-negative power of $(q_2^2)^{1/2} = (xq^2)^{1/2}$. Therefore, the integration measure $d^d l = d^{4-2\epsilon} l$ scales as $x^{a+b\epsilon}$ with *integer non-positive* b. So, the absence of terms $x^{k\epsilon}$ with positive k serves as a nice check.

On the other hand, the leading asymptotic behaviour in the limit $x \to 0$ can also be obtained with the help of the `Mathematica` package `HPL` [43] from our analytic expression for the canonical basis. Matching the resulting expression to $g(x,\epsilon) = h(\epsilon)x^{\epsilon a}$ provides values for the vector $h(\epsilon)$ in an ϵ-expansion up to ϵ^8 and terms $x^{k\epsilon}$ with $k = 0, -1, -2, \ldots$. Note that this step involves powers of $\log(x)$ terms; their cancellation in the matching provides a welcome check for our calculation. Finally, the naive value for $g(x,\epsilon)$ at $x = 0$ is obtained by setting all terms $x^{k\epsilon}$ with $k \neq 0$ to zero in the expression for $x^{\epsilon a}$.

In the last step, using eq. (2.1), we compute the naive values of the elements of the primary basis f from the naive expansion of the canonical basis $g(x)$ near $x \to 0$. Note that some of the matrix elements of T involve singularities up to order $1/x^3$. Thus, the information about the naive values of $g(x)$ at $x = 0$ turns out to be insufficient so that we need to evaluate the naive expansion of $g(x)$ up to order x^3. It is obtained following the prescription outlined in ref. [44] where the expansion terms can be computed from the leading order asymptotics at $x \to 0$ after recursively solving matrix equations with 332×332 entries. After inserting the expansion of g in eq. (2.1) the poles cancel and the naive values of the primary master integrals at $x = 0$ are obtained. The naive value of one of the elements of our primary basis f is nothing but the one-scale master integral I_{99} (cf. figure 2) for which we obtain the following analytic result

$$\begin{aligned}
I_{99} = e^{4\epsilon\gamma_E}\left(\frac{\mu^2}{-q^2}\right)^{4\epsilon}\Bigg\{ &\frac{1}{\epsilon^7}\left[-\frac{1}{288}\right] + \frac{1}{\epsilon^6}\left[\frac{13}{576}\right] + \frac{1}{\epsilon^5}\left[-\frac{101}{576} - \frac{\pi^2}{48}\right] \\
&+ \frac{1}{\epsilon^4}\left[-\frac{17\zeta_3}{54} + \frac{5\pi^2}{36} + \frac{145}{96}\right] + \frac{1}{\epsilon^3}\left[\frac{1775\zeta_3}{432} - \frac{767\pi^4}{17280} - \frac{5\pi^2}{8} - \frac{1669}{144}\right] \\
&+ \frac{1}{\epsilon^2}\left[-\frac{83}{72}\pi^2\zeta_3 - \frac{21899\zeta_3}{864} - \frac{3659\zeta_5}{360} + \frac{31333\pi^4}{103680} + \frac{659\pi^2}{288} + \frac{11243}{144}\right] \\
&+ \frac{1}{\epsilon}\Bigg[-\frac{40231\zeta_3^2}{1296} + \frac{745\pi^2\zeta_3}{288} + \frac{18751\zeta_3}{144} + \frac{50191\zeta_5}{360} - \frac{277703\pi^6}{2177280} - \frac{14015\pi^4}{10368} \\
&\qquad - \frac{149\pi^2}{24} - \frac{22757}{48}\Bigg] \\
&+ \Bigg[\frac{39173\zeta_3^2}{324} - \frac{77399\pi^4\zeta_3}{25920} + \frac{4013\pi^2\zeta_3}{432} - \frac{259559\zeta_3}{432} - \frac{568\pi^2\zeta_5}{45} - \frac{1123223\zeta_5}{1440} \\
&\qquad - \frac{2778103\zeta_7}{4032} + \frac{3129533\pi^6}{4354560} + \frac{28201\pi^4}{5760} + \frac{173\pi^2}{36} + \frac{382375}{144}\Bigg] \\
&+ \epsilon\Bigg[\frac{4931 s_{8a}}{30} + \frac{2615}{144}\pi^2\zeta_3^2 - \frac{276671\zeta_3^2}{2592} - \frac{2702413\zeta_5\zeta_3}{1080} + \frac{154037\pi^4\zeta_3}{31104} \\
&\qquad - \frac{55327\pi^2\zeta_3}{432} + \frac{1100461\zeta_3}{432} + \frac{205\pi^2\zeta_5}{9} + \frac{155029\zeta_5}{48} + \frac{2732549\zeta_7}{1008} - \frac{665217829\pi^8}{1306368000} \\
&\qquad - \frac{131003\pi^6}{45360} - \frac{747929\pi^4}{51840} + \frac{2995\pi^2}{36} - \frac{2005247}{144}\Bigg]\Bigg\}, \qquad (2.5)
\end{aligned}$$

JHEP03(2017)139

where ζ_n is Riemann's zeta function evaluated at n and

$$s_{8a} = \zeta_8 + \zeta_{5,3} \approx 1.0417850291827918834\,. \tag{2.6}$$

$\zeta_{m_1,\ldots,m_k}$ are multiple zeta values given by

$$\zeta_{m_1,\ldots,m_k} = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_k=1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}\,. \tag{2.7}$$

As by-product we also obtain analytic results for the remaining 75 one-scale master integrals and we find agreement with the results obtained in ref. [30]. This constitutes a further cross check for our procedure. We want to stress that the calculation which is outlined in this section is largely independent from the one performed in ref. [30].

3 Results

This section is devoted to the analytic results of the cusp and collinear anomalous dimensions and the finite part of F_q. Generic formulae where the pole part of F_q is parametrized in terms of the cusp and collinear anomalous dimensions and the QCD beta function can, e.g., be found in refs. [13, 45]. In what follows we use eq. (2.3) of ref. [16] which displays the pole parts of $\log(F_q)$ up to four-loop order. In this formula it is assumed that the one-loop coefficient of the beta function is given by

$$\beta_0 = \frac{11N_c}{3} - \frac{2n_f}{3}\,, \tag{3.1}$$

with n_f being the number of active quarks and the coefficients of the anomalous dimensions are defined through

$$\gamma_x = \sum_{n\geq 0} \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n \gamma_x^n\,, \tag{3.2}$$

with $x \in \{\mathrm{cusp}, q\}$ and α_s is the renormalized coupling constant with n_f active flavours. From eq. (2.3) of ref. [16] one observes that the four-loop corrections of γ_{cusp} follows from the $1/\epsilon^2$ term of $\log(F_q)$ and γ_q from the linear pole terms.

In the following we start with explicit results for the cusp and collinear anomalous dimensions. The four-loop corrections to γ_{cusp} reads

$$\begin{aligned}\gamma_{\mathrm{cusp}}^3 = \Bigg(&-\frac{32\pi^4}{135} + \frac{1280\zeta_3}{27} - \frac{304\pi^2}{243} + \frac{2119}{81}\Bigg)N_c n_f^2 + \Bigg(\frac{128\pi^2\zeta_3}{9} + 224\zeta_5 - \frac{44\pi^4}{27} \\ &- \frac{16252\zeta_3}{27} + \frac{13346\pi^2}{243} - \frac{39883}{81}\Bigg)N_c^2 n_f + \Bigg(\frac{64\zeta_3}{27} - \frac{32}{81}\Bigg)n_f^3 + \Bigg(-32\zeta_3^2 \\ &- \frac{176\pi^2\zeta_3}{9} + \frac{20992\zeta_3}{27} - 352\zeta_5 - \frac{292\pi^6}{315} + \frac{902\pi^4}{45} - \frac{44416\pi^2}{243} + \frac{84278}{81}\Bigg)N_c^3\,.\end{aligned} \tag{3.3}$$

We note that, after taking into account the difference between fundamental and adjoint representation of the external fields, the leading transcendental piece of this expression

JHEP03(2017)139

agrees with the result for planar $\mathcal{N}=4$ super Yang-Mills [46, 47], in agreement with expectations from [48]. Note that $\gamma^3_{\rm cusp}$ entering the $1/\epsilon^2$ pole of $\log(F_q)$ is multiplied by C_F. For this reason there is only a N_c^3 factor in front of the n_f-independent term in eq. (3.3). The one-, two- and three-loop corrections in the large-N_c limit can be found in eq. (2.6) of ref. [16] where also the fermionic part of $\gamma^3_{\rm cusp}$ is shown. The four-loop coefficient of the collinear anomalous dimension is given by

$$\begin{aligned}
\gamma_q^3 = \bigg(&-\frac{680\zeta_3^2}{9}-\frac{1567\pi^6}{20412}+\frac{83\pi^2\zeta_3}{9}+\frac{557\zeta_5}{9}+\frac{3557\pi^4}{19440}-\frac{94807\zeta_3}{972}+\frac{354343\pi^2}{17496}\\
&+\frac{145651}{1728}\bigg)N_c^3 n_f+\bigg(-\frac{8\pi^4}{1215}-\frac{356\zeta_3}{243}-\frac{2\pi^2}{81}+\frac{18691}{13122}\bigg)N_c n_f^3+\bigg(-\frac{2}{3}\pi^2\zeta_3\\
&+\frac{166\zeta_5}{9}+\frac{331\pi^4}{2430}-\frac{2131\zeta_3}{243}-\frac{68201\pi^2}{17496}-\frac{82181}{69984}\bigg)N_c^2 n_f^2\\
&+\bigg(\frac{1175\zeta_3^2}{9}+\frac{82\pi^4\zeta_3}{45}-\frac{377\pi^2\zeta_3}{6}+\frac{867397\zeta_3}{972}+24\pi^2\zeta_5-1489\zeta_5+705\zeta_7\\
&+\frac{114967\pi^6}{204120}-\frac{59509\pi^4}{9720}-\frac{120659\pi^2}{17496}-\frac{187905439}{839808}\bigg)N_c^4\,. \qquad (3.4)
\end{aligned}$$

The one-, two- and three-loop corrections to γ_q and the fermionic four-loop terms in γ_q^3 are listed in eq. (2.7) of ref. [16]. The N_c^4 term is new.

Finally, we also present the finite part of the form factor. We parametrize the perturbative expansion in terms of the renormalized coupling constant and set $\mu^2=-q^2$. Furthermore, it is convenient to consider $\log(F_q)$ which leads to more compact expressions. Thus, we have the following parametrization

$$\log(F_q)=\sum_{n\geq 1}\left(\frac{\alpha_s}{4\pi}\right)^n \log(F_q)|^{(n)}. \qquad (3.5)$$

In the large-N_c limit the four-loop term reads

$$\begin{aligned}
&\log(F_q)|^{(4)}_{\text{large-}N_c,\text{ finite part}} =\\
&N_c^4\bigg(-14s_{8a}+10\pi^2\zeta_3^2-\frac{86647\zeta_3^2}{54}+766\zeta_5\zeta_3-\frac{251\pi^4\zeta_3}{6480}-\frac{57271\pi^2\zeta_3}{1296}+\frac{173732459\zeta_3}{23328}\\
&\quad+\frac{1517\pi^2\zeta_5}{216}-\frac{881867\zeta_5}{1080}-\frac{36605\zeta_7}{288}+\frac{674057\pi^8}{5443200}-\frac{135851\pi^6}{77760}+\frac{386729\pi^4}{31104}\\
&\quad-\frac{429317557\pi^2}{839808}-\frac{54900768805}{6718464}\bigg)+\ldots \qquad (3.6)
\end{aligned}$$

where the ellipses refer to the fermionic contributions which are given in eq. (2.8) of ref. [16]. For convenience of the reader we provide the results for the form factor F_q expanded in the bare strong coupling constant in an ancillary file which can be downloaded from `https://www.ttp.kit.edu/preprints/2016/ttp16-055/`. This file also contains the lower-loop results expanded to higher order in ϵ. Furthermore, it contains the dependence of the renormalization scale μ.

4 Conclusions and outlook

We have computed the photon-quark form factor to four loops up to the finite term in ϵ in the large-N_c limit which is obtained from the planar Feynman diagrams. From the pole parts we extract the cusp and collinear anomalous dimensions. We discuss in detail the calculation of the most complicated master integral (see figure 2) and present analytic results expanded in ϵ up to transcendental weight eight. An independent calculation of this integral, together with a discussion of the remaining master integrals can be found in refs. [16, 30] to the required order in ϵ. We want to remark that the same master integrals enter the Higgs gluon form factor in the planar limit. However, the corresponding reduction is significantly more complicated. The logical next step is the calculation of the non-planar contributions. Note that also here both the reduction and the computation of the master integrals turn out to be much more complicated.

Acknowledgments

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Note added. After submission of our mansucript to `arxiv.org` we were contacted by colleagues drawing our attention to a talk by Ben Ruijl at the University of Zurich where the result for $\gamma^3_{\rm cusp}$ has been shown. The result agrees with our eq. (3.3).

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