Additive control and observation systems

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Chapter 1

Introduction

Dynamical systems theory is a mathematical area unifying the treatment of an enormous set of applications ranging from automata theory to economics and population models as well as to physical processes of various kinds. Despite their variety, common ground of these models is that the system can mathematically be represented by a quantity called state. The dynamical system then describes the evolution of the state in time. In the present work we consider the subarea of time-invariant nonlinear control and observation problems with infinite dimensional state spaces. Here the focus lies on the interaction of the system with its environment via inputs and outputs. Our examples arise from partial differential equations modeling wave phenomena on bounded domains.

As for ordinary differential equations, in the finite dimensional situation one has a well established theory of systems with inputs and outputs. However, to treat e.g. partial differential equations, one has to pass to infinite dimensional state spaces. In the past decades a successful linear theory has been developed for such systems. On the one hand, there is the functional analytic approach based on operator semigroups which allows the unified treatment of large classes of problems in an efficient way. On the other hand, specific partial differential equations can very successfully be treated directly. It should be remarked that the applications of the general theory to concrete problems often require methods or results from the PDE approach. The theory for nonlinear infinite dimensional systems is much more restricted. There are almost no results on an abstract level, whereas the direct PDE approach mostly focuses on nonlinear state equations or feedbacks.

In engineering applications, problems are often cascades or even more complex networks of interacting subsystems. They can easily become quite complicated. Here abstraction is important to keep the overview, see e.g. Section 2.4 in [19] or Section 2.3 in [43]. In this thesis we present a general theory for a class of nonlinear control and observation systems.

Generally speaking, inputs influence the dynamical system. The ability to steer the state to certain points with the use of inputs is a desirable feature called controllability. There are several controllability concepts depending on what the reachable states are.

On the other hand, the system's state is not always visible from the outside world. Instead one measures an output, which might carry only reduced information. Mathematically, this is modeled as an output map, which receives the state and yields the output. The system is called observable if its state can be recovered from the output to some extent. Again, specifications of this informal description lead to various observability notions.

If the output is used as (part of) the input, we speak of a feedback control or a closed-loop system. Feedback controls are important especially for stabilization. To stabilize a system means to steer it to a state where it is at rest, possibly under uninfluenceable disturbances.

Before we come to the description of our aims and main results, we give an overview on the existing work. For the well understood area of linear and nonlinear systems on finite dimensional state spaces we refer to the monographs by E. Sontag [42] and J. Zabczyk [57]. Relevant for us are linearization principles that can be found e.g. in Sections 1.2, 2.8, 3.7 and 6.4 of [42]. In the linear finite dimensional setting, controllability and observability can be characterized in a strikingly simple way. One has to compute the rank of a certain (possibly large) matrix determined by the given operators. This characterization is called the Kalman-rank-condition named after R. Kalman [24], see also Sections 1.5 and 1.6 of [57]

The development of linear time-invariant systems on infinite dimensional spaces started with bounded control and observation operators. Such system can always be described by the state space representation

$$z'(t) = Az(t) + Bu(t) \quad \text{for all } t \geq 0, \tag{1.1}$$

$$z(0) = x_0, \tag{1.2}$$

$$y(t) = Cz(t) + Du(t) \quad \text{for all } t \geq 0, \tag{1.3}$$

where z is the state, u is the input, y is the output, A generates a strongly continuous semigroup, and B, C, D are bounded linear operators on appropriate spaces. Several books are available on the topic. We mention R. Curtain and H. Zwart [14] which contains all the relevant further references. For the well-developed semigroup theory, on which all results in the area of linear control problems are founded, we refer the reader to the books [33] by A. Pazy and [15] by K.-J. Engel and R. Nagel..

Inputs acting on (parts of) the boundary of the spatial domain can not be represented by bounded linear operators. The same is true for point controls as well as boundary or point observation. So there is a need for unbounded control and observation operators. When working in such a framework, it is not clear a priori if the mathematical model has a solution. Therefore it has to be determined which of these maps are 'admissible'. First general and abstract descriptions were given by A. J. Pritchard and D. Salamon in [34], [38] and [39]. With the articles [52] and [53], G. Weiss established the notions of admissible control and observation operator that are now widely accepted. For the time being we concentrate on control problems, since observation is a dual concept to control in the linear case. In short, Weiss' idea can be summarized as follows. Instead of the equations (1.1)–(1.3), he took its solution operators as the starting point. Guided by the finite dimensional case, he introduced abstract control systems encoding the fundamental properties of solutions to (1.1) and (1.2). Then he proved that these systems can be represented by a semigroup generator A and a control operator B. Further they also yield solutions of the corresponding state space representation (1.1)–(1.2). Conversely, such systems can be constructed by means of a semigroup generator and an admissible control operator. The approach can be compared to evolution equations, where the semigroup yields the solution of the Cauchy problem given by its generator. The research monograph [49] written by M. Tucsnak and G. Weiss comprehensively presents the theory of linear observation and control on infinite dimensional spaces and provides a large amount of examples. B. Jacob's and J. R. Partington's survey [22] is also very readable.

Control systems only deal with the state and take no account of the output. Similarly when working with observation systems one assumes that there is no input. The concept of well-posed linear time-invariant systems addresses the system as a whole, that is with inputs and outputs. In particular one can couple well-posed systems. Under different names, they first appeared in [39] as well as in [54] and [13]. An exhaustive discussion was given by G. Weiss and O. Staffans in the series of papers [55], [44] and [45]. Newcomers find a very readable and nearly complete overview of this field in the recent survey [50]. We also recommend the book [43] due to its detailed exposition. In Section 6 of [50] one finds a collection of the many problems arising in natural sciences that can be described as well-posed linear systems. Indeed, all kind of equations such as the wave equation, the heat and Schrödinger equation as well as Maxwell's equation fit into the framework. Hence, a successful and rich theory was build upon this definitions.

In general well-posed systems do not posses a unique state space representation as in (1.1)–(1.3). For the subclass of regular linear well-posed systems a description via (1.1)–(1.3) is possible. The precise definition goes back to [54]. Regularity has been characterized in [55] using the transfer function. The latter is an appreciated tool in applications, used also by engineers. However, already the finite dimensional case indicates that these methods in the frequency-domain can probably not be generalized to nonlinear systems.

Controllability and observability for infinite dimensional systems is a complicated matter. Other than in finite dimensions, there are several different controllability and observability notions. Apart from that, their verification mostly depends on the special structure of the problem. However, these properties have been checked for large classes of problem. We refer to Chapters 6 to 9 and 11 of [49].

By now there have only been a few papers which tackle nonlinear systems on infinite dimensional spaces on an abstract level. In [8] M. Baroun and B. Jacob together with L. Maniar and R. Schnaubelt introduce and represent a class of locally Lipschitz observation systems. Moreover, they prove a result on linearized observability for semilinear state equations and linear observation operators. See also [7] for earlier results. With the same methods feedback systems where studied in [23]. H. Bounit and A. Idrissi in [20] and [9] started the investigation of bilinear systems. They generalized the linear approach to problems where the scalar input is multiplied with the current state. We will treat problems like (1.1)–(1.3) with nonlinear B and C, as well as linear or semilinear A.

We now depict the aims and main results of the thesis. Control problems are in the focus, since here up to now there are no results on a general level, whereas some progress was already made on observation systems as noted above. Nevertheless we also consider observation systems. In contrast to the linear case, duality arguments can not be used in the nonlinear setting. In fact, our proofs in both cases differ in many respects. Our first step is to extend Weiss' ansatz to nonlinear problems. To this end, we generalize the central notion of control system. We obtain a fairly general class of nonlinear systems with the property that the state can be split into two summands; one depending on the initial state and one on the input. We are able to show that many results such as the representation theorem remain valid in this setting but now the control operator B can be nonlinear. Our reasoning is based on the functional equations inherited from dynamical systems which consequently are the same for linear and our additive control systems. A difficulty we had to deal with is that statements not automatically extend from dense subsets to the whole space as they do for bounded linear operators. For example the representation theorem (Theorem 4.9) at first only holds for inputs from the class of step functions. To overcome the problem, we have to impose a polynomial growth condition. Moreover, equicontinuity on compact time intervals has to assumed, whereas in the linear case the operator norms of the input maps are increasing with time and so equicontinuity is immediately clear.

For nonlinear systems, in general we can not expect "global" controllability. On the other hand a well-known linearization principle from the finite dimensional theory says that controllability of the linearized system yields "local" controllability of the original system. We prove that this linearization principle is true for our class of systems which encompasses both admissibility and controllability.

At the beginning we consider the case that the evolution of the systems state in absence of inputs is governed by a linear strongly continuous semigroup. Then we turn our attention to semilinear state equations. Here we first have to establish a local existence and uniqueness theory for mild solutions. We now have to deal with much more technical difficulties due to the fact that solutions might only exist for finite times. Still the linearization principle holds also in this situation.

We remark that it is crucial for our reasoning that the part of the state depending on the initial state is governed by a linear strongly continuous semigroup (or a semilinear perturbation of it). In particular we have the interpolation and extrapolation spaces corresponding to its generator. The application of the Laplace transform is a central step in our proofs which also is only possible because of this linear component of the system. For the linearization theorem and the semilinear state equation we make use of the contraction mapping principle. These results also heavily depend on Duhamel's formula.

To treat the output, we define observation systems by a natural functional equation. Here the underlying state equation is linear. As in [8] we represent these systems by a semigroup generator and a nonlinear observation operator, but we further provide a more explicit representation on the domain of the generator. Again we need a polynomial growth condition on the observation operator in order to prove exponential boundedness of the outputs. Our linearization result is complementary to the one in [8], where a semilinear state equation and a linear observation operator was considered. Finally, we introduce and represent regular additive well-posed systems with inputs and outputs. However, it seems that here our approach is restricted to linear observation operators.

All our results are illustrated by the linear or semilinear wave equation with nonlinear control or observation, which act in the interior or via Dirichlet or Neumann boundary conditions.

Of course, results on particular systems were found before the abstract theory was developed. To our knowledge, except for the finite dimensional case, nonlinear control operators have not been considered before. As remarked above, the focus in the PDE literature lies on semilinear state equations. We refer to the classical treatise [10] by T. Cazenave and A. Haraux for semilinear evolution equations, i.e., systems without inputs and outputs. A standard book in the field is [11] by J.-M. Coron. It represents numerous known results and by that displays the state of the art. Another general reference is the (two-parted) research monograph [30], [31] by I. Lasiecka and R. Triggiani. J.-L. Lions' book [32] is an important early contribution.

We list here results on the main examples in the thesis, namely Dirichlet boundary control and mixed boundary control for the wave equation. For the well-posedness of the linear wave equation with Dirichlet boundary control, see [27] and also [26]. Controllability of the system was established in [28] where the control area is the whole boundary. The case that the control acts only on a part of the boundary was first considered in [32]. In [58], E. Zuazua studied the Dirichlet boundary controllability of the semilinear wave equation.

Our example on the wave equation with mixed boundary control is taken from [47]. In this article reference is made to the earlier work [35] as well as to [36] although the latter, from the "PDE-world", yields well-posedness of a related but different system. The results of [29] imply the exact controllability in this example. Here statements depend heavily on geometric properties of the controlled part of the boundary and we mention [6] for the most general such conditions. We have not found no results on the semilinear version of this system.

We give a short outline of the thesis. This introduction is the first chapter. In the preceding Chapter 2, we recall the general concept of dynamical systems. By specializing the situation we obtain the class of well-posed linear systems. As said before, our approach is based on this theory, so we also repeat several results on linear admissible control and observation operators. In the last section we explain how the systems treated in this work fit into the framework of dynamical systems.

Chapter 3 is mainly dedicated to Cauchy problems governed by a generator A. We discuss several solution concept for this type of equations. Strong solutions in the extrapolation space of A best fit our needs and we give a characterization of them. We also touch on semilinear Cauchy problems where matters are more involved, so that the detailed discussion is shifted to Chapter 5. We conclude with the introduction of the solution space of a linear control system. It enters into the concept of classical solutions.

The main results of this work are contained in the Chapters 4 to 6. In the first one of them we state the definition of additive control systems and prove the first representation theorem. Under mild continuity assumptions the control system yields the strong solution of the corresponding state space representation. This leads to the definition of L^p–admissible control operators. Next we verify the above indicated linearization principle.

Chapter 5 is devoted to perturbations of the state equation. In the framework of mild solutions we establish a well-posedness theory involving inputs. We also generalize the linearization theorem from the preceding chapter. Here we also have to prove that the derivative of the control operator is admissible for the perturbed semigroup.

Chapter 6 deals with the output. On the one hand we define additive observation systems and represent them via observation operators. Again a linearization result analog to the one of Chapter 4 is valid. In the final section we treat regular semilinear well-posed systems. Each of the Chapters 4 to 6 contains a section with applications mainly to wave equations.

We provide three appendices on extrapolation spaces, the Laplace transform and on boundary control systems, where we collect needed notions and results from the literature.

Chapter 2

Dynamical systems

In this chapter we discuss the fundamental concept of dynamical systems. The material is taken from Section 2 of [42], though we added feedthrough terms. See also [19]. We first introduce some notation which is used throughout the thesis.

Let J and U be nonempty sets. We will mostly have $J = [0, \infty)$. Denote by U^J the family of all maps $u : J \to U$. Let $\tau \geq 0$ and $u \in U^{[0,\infty)}$. The *left shift operator* $S_\tau^* : U^{[0,\infty)} \to U^{[0,\infty)}$ is given by

$$(S_\tau^* u)(t) := u(t + \tau).$$

If U is a vector space (i.e., including 0) we set

$$(S_\tau u)(t) = \begin{cases} 0, & t \in [0, \tau) \\ u(t - \tau), & t \in [\tau, \infty) \end{cases}, \qquad (P_\tau u)(t) = \begin{cases} u(t), & t \in [0, \tau) \\ 0, & t \in [\tau, \infty) \end{cases}.$$

The map $P_\tau u$ can be regarded as the *truncation* of u to the subinterval $[0, \tau)$, and S_τ is the *right shift operator*. Note that $S_0^* u = u = S_0 u$ and $P_0 u$ is the zero function.

Clearly S_τ^* is the left-inverse of S_τ, this means that $S_\tau^* S_\tau u = u$ for all $u \in U^{[0,\infty)}$ and $\tau \geq 0$. Moreover, for every $u \in U^{[0,\infty)}$ and all $\tau \geq 0$ we have

$$u = P_\tau u + S_\tau S_\tau^* u.$$

Note that P_τ as well as S_τ and S_τ^* are linear. Restricted to proper function spaces – such as $L^p([0, \infty), U)$ with a Banach space U – these operators are bounded with norm one, and S_τ^* is the dual of S_τ. In Definition 2.1 below for brevity we use the symbol $\mathbb{R}^2_\geq := \{(t, s) \in \mathbb{R}^2 \,|\, t \geq s\}$. It is needed only here and in Definition 2.6.

2.1 Time-invariant dynamical systems

We model a system which at every instant of time resides in some state and accepts inputs. Moreover, the current state can be observed via an output function. The current output may also depend on the current input. It is however independent of how the current state was reached, i.e., independent of states and inputs of the past.

It is further natural to assume that the transition of the state is evolutionary, a condition called *composition property*. This means, if $r < s < t$ are three instants of time, then the state at time t can either be calculated from the state at time r and inputs made between r

and t or from the state at time s using only inputs between s and t. Theoretically, we can stop the system, observe the current state and continue the evolution. *Consistency* ensures that the state of the system does not change in "zero time". Finally, *causality* tells us that the current state only depends on past input.

Definition 2.1. A *dynamical system* (*with outputs*) $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ consists of

- a *set of states* X, a *set of input-values* U and a *set of output-values* Y – each nonempty;
- a *state transition map* $\phi : D_\Sigma \to X$, defined on a set $D_\Sigma \subseteq \mathbb{R}^2_\geq \times X \times U^\mathbb{R}$, satisfying
  - 'consistency' $\forall\, x \in X, u \in U^\mathbb{R}, s \in \mathbb{R} :\ (s, s, x, u) \in D_\Sigma$ and $\phi(s, s, x, u) = x$.
  - 'composition property' $\forall\, x \in X, u \in U^\mathbb{R}, r, s, t \in \mathbb{R}$ with $t \geq s \geq r$:
  $$(t, r, x, u) \in D_\Sigma \implies (s, r, x, u) \in D_\Sigma \text{ and } (t, s, x_1, u) \in D_\Sigma \text{ and } \phi(t, s, x_1, u) = \phi(t, r, x, u),$$
  where $x_1 = \phi(s, r, x, u)$.
  - 'causality' $\forall\, x \in X, u_1, u_2 \in U^\mathbb{R}, s, t \in \mathbb{R}$ with $t > s$
  $$u_1|_{[s,t)} = u_2|_{[s,t)} \text{ and } (t, s, x, u_1) \in D_\Sigma \implies (t, s, x, u_2) \in D_\Sigma \text{ and } \phi(t, s, x, u_1) = \phi(t, s, x, u_2);$$
- an *output map* $\eta : \mathbb{R} \times X \times U \to Y$.

Remark 2.2. a) More generally one can replace $\mathbb{R}$ in the above definition by a so called *time set*, which is a subgroup $\mathcal{T}$ of $(\mathbb{R}, +)$, so that $0 \in \mathcal{T}$ as well as $-t \in \mathcal{T}$ and $t + \tau \in \mathcal{T}$ for all $t, \tau \in \mathcal{T}$. To avoid that the system is trivial one should require $\mathcal{T} \neq \{0\}$. Using time sets one can describe discrete-time and continuous-time uniformly. Since we do not consider discrete-time systems, we stick to $\mathbb{R}$.

b) It is natural to assume that for all $x \in X$ there is some input $u \in U^\mathbb{R}$ and times $s, t \in \mathbb{R}$ with $t > s$ such that $(t, s, x, u) \in D_\Sigma$. Else the set of states X might be chosen to large. ◊

Let the system modeled by the last definition reside in the state $x \in X$ at time s. Further assume that at instants $\tau \in [s, t)$ inputs $u(\tau)$ were made. Then the system's state at time t is $z(t) = \phi(t, s, x, u)$, provided that $(t, s, x, u) \in D_\Sigma$. In this situation the output is given by $y(t) = \eta(t, \phi(t, s, x, u), u(t))$.

We should mention that generally only the output y is known and that the input u and the initial state x can possibly be chosen or are also known. Writing $y(t) = y(t, s, x, u)$, we immediately derive the composition property

$$y(t, r, x, u) = y(t, s, \phi(s, r, x, u), u)$$

for all $(t, r, x, u) \in D_\Sigma$ and $s \in [r, t]$. Moreover, the causality of ϕ implies the following. Let $(t, s, x, u_1) \in D_\Sigma$ with $t > s$ and let $u_2 \in U^\mathbb{R}$ satisfy $u_1|_{[s,t)} = u_2|_{[s,t)}$. Then we infer

$$y(t, s, x, u_1) = \eta(t, z(t), u_1(t)) \quad \text{and} \quad y(t, s, x, u_2) = \eta(t, z(t), u_2(t)).$$

These values may differ, but only if $u_1(t) \neq u_2(t)$. In many cases the input u and the output y belong to some vector-valued L^p spaces and are thus defined only almost everywhere. Then all the above equations containing $u(t)$ or $y(t)$ should hold for almost every t. Examples for such system may be obtained from solving a PDE with functional analytic methods.

Now we are going to specialize the situation. Starting from Chapter 3 we will only consider 'time-invariant' systems, defined next. One uses the term *time-varying* for systems as in Definition 2.1. It is important to remark that even if one starts with a time-invariant system, the linearization along a non-constant trajectory in general will be a time-varying system.

Definition 2.3. A system $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ is called *time-invariant* if for all $x \in X$, $u \in U^{\mathbb{R}}$, $v \in U$ and $s, t \in \mathbb{R}$ with $t \geq s$ we have $\eta(t, x, v) = \eta(0, x, v)$ as well as

$$(t, s, x, u) \in D_\Sigma \implies (t - s, 0, x, S_s^* u) \in D_\Sigma \text{ and } \phi(t, s, x, u) = \phi(t - s, 0, x, S_s^* u).$$

For time-invariant systems, the evolution of the state thus solely depends on the time span $t - s$ and not on the actual times t and s. Consequently we only need inputs from $U^{[0,\infty)}$. To shorten the notation one drops the 0 in the argument of ϕ, i.e., one considers the state transition map $\phi_0 : D_\Sigma^0 \to X$ given by $\phi^0(\tau, x, u) = \phi(\tau, 0, x, u)$, where $D_\Sigma^0 = \{(t - s, x, S_s^* u) \,|\, (t, s, x, u) \in D_\Sigma\} \subseteq [0, \infty) \times X \times U^{[0,\infty)}$. For simplicity for ϕ^0 and D_Σ^0 we write ϕ and D_Σ respectively again. Similarly one drops the 0 in the argument of η. Definition 2.1 with this new state transition map and output map then reads as follows.

Definition 2.4. A *time-invariant dynamical system* $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ consists of

- a *set of states* X, a *set of input-values* U and a *set of output-values* Y;
- a *state transition map* $\phi : D_\Sigma \to X$ defined on $D_\Sigma \subseteq [0, \infty) \times X \times U^{[0,\infty)}$, satisfying
  - 'consistency' $\forall\, x \in X, u \in U^{[0,\infty)} : (0, x, u) \in D_\Sigma$ and $\phi(0, x, u) = x$.
  - 'composition property' $\forall\, x \in X, u \in U^{[0,\infty)}, t, \tau \in [0, \infty)$:

    $$(t + \tau, x, u) \in D_\Sigma \implies (\tau, x, u) \in D_\Sigma \text{ and } (t, x_1, S_\tau^* u) \in D_\Sigma \\ \text{and } \phi(t, x_1, S_\tau^* u) = \phi(t + \tau, x, u),$$

    where $x_1 = \phi(\tau, x, u)$.
  - 'causality' $\forall\, x \in X, u_1, u_2 \in U^{[0,\infty)}, t \in [0, \infty)$:

    $$u_1|_{[0,t)} = u_2|_{[0,t)} \text{ and } (t, x, u_1) \in D_\Sigma \implies (t, x, u_2) \in D_\Sigma \text{ and } \phi(t, x, u_1) = \phi(t, x, u_2);$$
- an *output map* $\eta : X \times U \to Y$.

Next we define linear systems. For this we need X, U and Y to be vector spaces (over the same field $\mathbb{K}$). The term 'linear time-invariant system' is often abbreviated LTI. We make no use of this acronym since we do not treat linear systems very much.

Definition 2.5. A dynamical system $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ is called *linear* if for all $s, t \in \mathbb{R}$ with $t \geq s$ the set $D_{\Sigma,t,s} := \{(x, u) \in X \times U^{\mathbb{R}} \,|\, (t, s, x, u) \in D_\Sigma\}$ is a vector space, for all $t \in \mathbb{R}$ the map $\eta(t, \cdot, \cdot) : X \times U \to Y$ is linear and

$$\forall\, (x, u), (x_1, u_1), (x_2, u_2) \in D_{\Sigma,t,s},\ \alpha \in \mathbb{K} :\ \phi(t, s, \alpha x, \alpha u) = \alpha \phi(t, s, x, u) \\ \text{and } \phi(t, s, x_1 + x_2, u_1 + u_2) = \phi(t, s, x_1, u_1) + \phi(t, s, x_2, u_2).$$

In short, a system is linear if state transition map and output map are linear in the state and input arguments. For a linear time-invariant system Σ one demands that for all $t \in [0,\infty)$ the set $D_{\Sigma,t} := \{(x,u) \in X \times U^{[0,\infty)} \,|\, (t,x,u) \in D_\Sigma\}$ is a vector space,

$$\forall\,(x,u),(x_1,u_1),(x_2,u_2) \in D_{\Sigma,t}, \alpha \in \mathbb{K}: \ \phi(t,\alpha x,\alpha u) = \alpha\phi(t,x,u)$$
$$\text{and } \phi(t,x_1+x_2,u_1+u_2) = \phi(t,x_1,u_1) + \phi(t,x_2,u_2),$$

and that $\eta : X \times U \to Y$ is linear.

It is desirable to find a subset of $U^{\mathbb{R}}$ such that every input in it can be applied to any initial state for all times. This is the subject of our last definition.

Definition 2.6. Let $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ be a dynamical system and $\Omega \subseteq U^{\mathbb{R}}$. Then Σ is called Ω–*complete* if $\mathbb{R}^2_{\geq} \times X \times \Omega \subseteq D_\Sigma$.

Clearly, a time-invariant system Σ is Ω–complete if $[0,\infty) \times X \times \Omega \subseteq D_\Sigma$. If for a given system one finds an $\Omega \subseteq U^{\mathbb{R}}$ with $\mathbb{R}^2_{\geq} \times X \times \Omega \subseteq D_\Sigma$ that is rich enough for ones needs, then it might be a good idea to assume that $D_\Sigma = \mathbb{R}^2_{\geq} \times X \times \Omega$. If Ω is clear from the context, we simply say that Σ is complete.

2.2 Linear control and observation systems

Mainly all theory concerning control and observation problems fit into the framework introduced above. As an example, in this section we show that the functional equations demanded for a well-posed linear system can be deduced from our assumptions in Definitions 2.1 and 2.4. We introduce a metric structure on our sets X, U and Y in order to describe the regularity properties of such systems.

Let X, U, Y and $\Omega \subseteq U^{\mathbb{R}}$ be Banach spaces. Further let $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ be an Ω–complete *linear* system. For $t,s \in \mathbb{R}$ with $t \geq s$ we define the functions $\mathcal{U}_{t,s} : X \to X$ via $\mathcal{U}_{t,s}x = \phi(t,s,x,0)$. The resulting family $\mathcal{U} = (\mathcal{U}_{t,s})_{t\geq s}$ satisfies the functional equations of an evolution family, namely

$$\mathcal{U}_{s,s}x = x \qquad \text{and} \qquad \mathcal{U}_{t,r}x = \mathcal{U}_{t,s}\mathcal{U}_{s,r}x$$

for all $x \in X$ and $s,t,r \in \mathbb{R}$ with $t \geq s \geq r$. This follows immediately from consistency and the composition property. It is clear that $\mathcal{U}_{t,s}$ is linear for all $s,t \in \mathbb{R}$ with $t \geq s$. The family $\mathcal{U}$ actually is an evolution family in the sense of Definition 5.1.3 in [33] if in addition the following conditions are satisfied.

- $\mathcal{U}_{t,s} \in \mathcal{L}(X)$ for all $s,t \in \mathbb{R}$ with $t \geq s$.
- For every $x \in X$ the map $\{(t,s) \in \mathbb{R}^2 \,|\, t \geq s\} \to X;\ (t,s) \to \mathcal{U}_{t,s}x$ is continuous.

Given $s \in \mathbb{R}$, $x \in X$ and $u \in \Omega$, then $z(t) := z_{s,x,u}(t) := \phi(t,s,x,u)$ is the system's state at time $t \geq s$. Due to the linearity of Σ, we have $z(t) = \phi(t,s,x,0) + \phi(t,s,0,u)$. This decomposition suggests the definition of the operators $\Phi_{t,s} : \Omega \to X;\ \Phi_{t,s}u := \phi(t,s,0,u)$ for $s,t \in \mathbb{R}$ with $t \geq s$. Then we may write

$$z(t) = \phi(t,s,x,0) + \phi(t,s,0,u) = \mathcal{U}_{t,s}x + \Phi_{t,s}u \quad \text{for } t \geq s.$$

Hence the influence of initial state x and input u on the state z can be separated. The effect of the input is added to the unaffected evolution of the initial state.

Again the linearity of Σ yields that $\Phi_{t,s}$ is linear for every pair $s, t \in \mathbb{R}$ with $t \geq s$. Consistency and the composition property for the family $\Phi = (\Phi_{t,s})_{t \geq s}$ become

$$\Phi_{s,s}u = 0 \qquad \text{and} \qquad \Phi_{t,r}u = \phi(t, s, \phi(s, r, 0, u), 0) + \phi(t, s, 0, u) = \mathcal{U}_{t,s}\Phi_{s,r}u + \Phi_{t,s}u$$

for all $u \in \Omega$ and $r, s, t \in \mathbb{R}$ with $t \geq s \geq r$. The pair $(\mathcal{U}, \Phi)$ appears in Definition 3.1 of [40], where $\Omega = L^p_{\mathrm{loc}}([t_0, \infty), U)$ for some $t_0 \geq 0$ and $p \in [1, \infty)$. If natural continuity assumptions are added, it is called *nonautonomous control system*. Various special cases have been considered before e.g. in [12], [18], [21] and [1]. The setting of [40] was refined and applied in [41].

Let us specialize the situation further by assuming that the system Σ is time-invariant. In this setting a successful and encompassing theory was developed which we now want to describe. Here the evolution family $\mathcal{U}$ is time invariant, i.e., $\mathcal{U}_{t,s} = \mathcal{U}_{t-s,0}$ for all $s, t \in \mathbb{R}$ with $t \geq s$. It is easy to see that in this case $\mathbb{T}_t := \mathcal{U}_{t,0}$ yields a semigroup on X. Equivalently, we can also directly define the maps $\mathbb{T}_t : X \to X$ for $t \geq 0$ by

$$\mathbb{T}_t x := \phi(t, x, 0) \quad \text{for } x \in X.$$

The consistency and the composition property then imply that

$$\mathbb{T}_0 x = x \qquad \text{and} \qquad \mathbb{T}_{t+\tau}x = \mathbb{T}_t\mathbb{T}_\tau x \tag{2.1}$$

for $x \in X$ and $t \geq 0$. Again, the linearity of $\mathbb{T}$ is a consequence of the linearity of Σ. By adding the following assumptions, $\mathbb{T}$ becomes a strongly continuous linear semigroup on X.

- $\mathbb{T}_t \in \mathcal{L}(X)$ for every $t \geq 0$.
- For all $x \in X$ the map $[0, \infty) \to X$; $t \mapsto \mathbb{T}_t x$ is continuous.[1]

Similar to the definition of Φ above, for $t \geq 0$ and $u \in \Omega$ we set $\Phi_t u := \phi(t, 0, u)$. The resulting maps $\Phi_t : \Omega \to X$ are called *input maps*. They are linear because Σ is linear. Again, for fixed $x \in X$ and $u \in \Omega$ the state $z(t) = z_{x,u}(t) = \phi(t, x, u)$ can be written as

$$z(t) = \mathbb{T}_t x + \Phi_t u \quad \text{for } t \geq 0. \tag{2.2}$$

Once more using consistency and the composition property, for the family $\Phi = (\Phi_t)_{t \geq 0}$ we obtain the rules

$$\Phi_0 u = 0 \qquad \text{and} \qquad \Phi_{t+\tau}u = \mathbb{T}_t\Phi_\tau u + \Phi_t S^*_\tau u \tag{2.3}$$

for all $t, \tau \geq 0$ and each $u \in \Omega$. We recognize the functional equations that Weiss postulates for an *abstract linear control system* $(\mathbb{T}, \Phi)$ in Definition 2.1 of [52]. Assuming further that

- $\Phi_t \in \mathcal{L}(\Omega, X)$ for all $t \geq 0$,

[1] It is well known that with the help of the semigroup properties, continuity of $t \mapsto \mathbb{T}_t x$ at 0 extends to continuity on all of $[0, \infty)$.

the pair $(\mathbb{T}, \Phi)$ indeed fulfills the definition in [52], except that there the space of input functions is restricted to $\Omega = L^p([0,\infty), U)$ for some $p \in [1,\infty]$. In Chapter 4 we define and discuss 'additive control systems' as a nonlinear generalization.

The output y can be dealt with in the same fashion. We discuss only the time-invariant case. Here the output at time $t \geq 0$ is given by $y(t) := y_{x,u}(t) := \eta(\phi(t,x,u), u(t))$. The linearity of Σ implies that

$$\eta(\phi(t,x,u),u(t)) = \eta(\phi(t,x,0)+\phi(t,0,u),u(t)) = \eta(\phi(t,x,0),0)+\eta(\phi(t,0,u),u(t))$$

for $x \in X$, $u \in \Omega$ and $t \geq 0$. We thus define the linear operators $\Psi_\infty : X \to Y^{[0,\infty)}$ and $F_\infty : \Omega \to Y^{[0,\infty)}$ through

$$\Psi_\infty x := \eta(\phi(\,\cdot\,,x,0),0) \qquad \text{and} \qquad F_\infty u := \eta(\phi(\,\cdot\,,0,u),u(\,\cdot\,)).$$

We then can write

$$y(t) = \Psi_\infty x(t) + F_\infty u(t) \quad \text{for } x \in X, u \in \Omega \text{ and } t \geq 0.$$

Again the dependence of y on x and u is separated, which could be expected for linear operators.

Let us first investigate Ψ_∞, the so called *(extended) output map*. Calculations using the composition property lead to

$$S_\tau^* \Psi_\infty x = \Psi_\infty \mathbb{T}_\tau x \quad \text{for all } t,\tau \geq 0 \text{ and every } x \in X. \tag{2.4}$$

To introduce a continuity condition for Ψ_∞ its range has to lie in a topological space. We think of $\Gamma = L^p([0,\infty), Y)$ for some $p \in [1,\infty)$. However, in principle the output could be e.g. constant for all times and consequently just locally integrable on $[0,\infty)$.

To avoid the use of the Fréchet space $L^p_{\text{loc}}([0,\infty), Y)$, we impose the continuity condition on the truncated maps $\Psi_t := P_t\Psi_\infty$. Assume that the families $\mathbb{T}$ and $\Psi = (\Psi_t)_{t\geq 0}$ satisfy the following conditions.

- $\mathbb{T}$ is a strongly continuous semigroup on X.
- Let Ψ_t belong to $\mathcal{L}(X,\Gamma)$ for all $t \geq 0$.[2]

Then $(\mathbb{T}, \Psi)$ is an *abstract linear observation system* in the sense of Definition 2.1 in [53]. We will use an analogous definition for our possibly nonlinear 'additive observation systems' in Chapter 6. The composition property translates to

$$S_\tau^* \Psi_{t+\tau} x = P_{t+\tau-\tau} S_\tau^* \Psi_\infty x = P_t \Psi_\infty(\mathbb{T}_\tau x) = \Psi_t(\mathbb{T}_\tau x)$$

for all $t,\tau \geq 0$ and every $x \in X$.

The operator F_∞ is called *(extended) input-output map*. A short calculation shows that we have the composition property

$$S_\tau^* F_\infty u = \Psi_\infty \Phi_\tau u + F_\infty S_\tau^* u \quad \text{for } u \in \Omega \text{ and } t,\tau \geq 0. \tag{2.5}$$

[2]This is actually the same as to assume that $\Psi_\infty : X \to L^p_{\text{loc}}([0,\infty), Y)$ is continuous.

This looks more complicated and, unlike in the previous discussion, the other operators Ψ_∞ and Φ_τ appear. As before we introduce the truncated maps $F_t := P_t F_\infty$ for $t \geq 0$. The composition property for the family $F = (F_t)_{t\geq 0}$ then becomes

$$S_\tau^* F_{t+\tau} u = \Psi_t \Phi_\tau u + F_t S_\tau^* u \quad \text{for } u \in \Omega \text{ and } t, \tau \geq 0.$$

As before we add continuity assumptions.

- Let $\mathbb{T}$ be a strongly continuous semigroup on X.
- Assume that $\Phi_t \in \mathcal{L}(\Omega, X)$, $\Psi_t \in \mathcal{L}(X, \Gamma)$ and $F_t \in \mathcal{L}(\Omega, \Gamma)$ for all $t \geq 0$.

Then the quadruple $(\mathbb{T}, \Phi, \Psi, F)$ is an *abstract linear system on Ω, X and Γ* in terms of Definition 1.1 of [54]. The names abstract control/observation system and abstract linear system are outdated, the latter was replaced by *well-posed linear systems*, see Definition 2.2 in [44] and the text before it. In the following, we will call the tuples $(\mathbb{T}, \Phi)$ and $(\mathbb{T}, \Psi)$ *linear control system* and *linear observation system* respectively.

Since our theory is based on the work of G. Weiss, let us repeat some linear results. We follow the papers [52] and [53]. It should be remarked that similar results were proved independently by D. Salamon in [39].

It can be observed that the state z of many linear dynamical systems obtained by modeling natural phenomena satisfies the differential equation

$$z'(t) = Az(t) + Bu(t); \quad z(0) = x_0, \tag{2.6}$$

with linear operators A and B. If state space and input space are finite dimensional, this equation can be solved. In this case the operators A and B can be seen as matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ for some $n, m \in \mathbb{N}$ and the solution is given by

$$z(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-s)} Bu(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

The same is certainly true if $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(U, X)$ are bounded on the spaces X and U which may have infinite dimensions. However, for partial differential equations with boundary control 'unbounded' A and B have to be considered. This means that A is merely a closed operator in X with a domain $\mathrm{D}(A)$ not being the whole space X, and B does not map into X. It is well known that (2.6) with $B = 0$ is well-posed if A is the generator of a strongly continuous semigroup on X. So one is seeking necessary (and sufficient) conditions for B under which (2.6) is well-posed.

Motivated by the question what properties a solution of (2.6) should have and guided by the finite dimensional case, Weiss introduced the concept of linear control systems we encountered above. He then proved the following representation theorem.

Theorem 2.7. *Let X and U be Banach spaces and let $p \in [1, \infty)$. Further let $(\mathbb{T}, \Phi)$ be a linear control system on X and $L^p([0,\infty), U)$. Then there exists a unique linear operator $B \in \mathcal{L}(U, X_{-1})$ such that for all $u \in L^p([0,\infty), U)$ and every $t \geq 0$ we have*

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} Bu(s)\,\mathrm{d}s. \tag{2.7}$$

Moreover, for all $x_0 \in X$ and $u \in L^p([0,\infty),U)$ the function given by $z(t) = \mathbb{T}_t x_0 + \Phi_t u$ is the unique strong solution of (2.6), meaning that z belongs to $C([0,\infty),X)$ and we have

$$z(t) = x_0 + \int_0^t (Az(s) + Bu(s))\,\mathrm{d}s \quad \textit{for all } t \geq 0.$$

This is Theorem 3.9 of [52]. We refer to Appendix A for details on the space X_{-1} as well as on the spaces X_1 and $X_{1,d}$ needed below. Note that we have the continuous embeddings $X_1 \hookrightarrow X \hookrightarrow X_{-1}$, so that $\mathcal{L}(U,X_{-1})$ includes $\mathcal{L}(U,X)$. The notion of strong solution is discussed in the following chapter.

Next we turn our attention to the output. Here the state of a linear dynamical system is observed through a linear operator C. We add the equation

$$y(t) = Cz(t) \tag{2.8}$$

to (2.6). Again, there is no problem if C belongs to $\mathcal{L}(X,Y)$. But if C is defined on a subspace W of X, then even in the absence of an input (that is $u = 0$) it is not clear whether $z(t) = \mathbb{T}_t x_0$ lies in this space for all $x_0 \in X$ and $t \geq 0$. If W is $\mathbb{T}$–invariant, then for $x_0 \in W$ and $u = 0$ the solution of (2.6) & (2.8) is given by

$$y(t) = C\mathbb{T}_t x_0 \quad \text{for } t \geq 0. \tag{2.9}$$

We expect that this map has an extension to all of X which depends continuously on x_0. In Theorem 3.9 of [53], which we now repeat, a necessary condition for such 'admissible' operators C is given.

Theorem 2.8. *Let X and Y be Banach spaces and let $p \in [1,\infty]$. Further let $(\mathbb{T},\Psi)$ be a linear observation system on X and $L^p([0,\infty),Y)$. Then there is a unique linear operator $C \in \mathcal{L}(X_1,Y)$ such that for all $x_0 \in X_1$ and every $t \geq 0$ we have*

$$(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0.$$

Note that $\mathcal{L}(X,Y)$ is a subspace of $\mathcal{L}(X_1,Y)$. In the spirit of this approach we now define 'admissible' control and observation operators. In short, via (2.7) and (2.9) they must yield a linear control systems respectively a linear observation system. These abstract conditions have been checked for many operators in applications. Often this is difficult, see e.g. [49].

Definition 2.9. Let X, U and Y be Banach spaces and let $p \in [1,\infty)$. Moreover, assume that A generates a strongly continuous semigroup $\mathbb{T}$ on X.

A linear operator $B \in \mathcal{L}(U,X_{-1})$ is called *L^p–admissible control operator for* $\mathbb{T}$ if for all $t \geq 0$ the map $\Phi_t : L^p([0,\infty),U) \to X_{-1}$ defined by (2.7) actually lies in $\mathcal{L}(L^p([0,\infty),U),X)$.

A linear operator $C \in \mathcal{L}(X_1,Y)$ is called *L^p–admissible observation operator for* $\mathbb{T}$ if for all $\tau \geq 0$ the map $\Psi_\tau : X_1 \to L^p([0,\infty),Y)$ defined by

$$(\Psi_\tau x_0)(t) = \begin{cases} C\mathbb{T}_t x_0, & t \in [0,\tau) \\ 0, & t \in [\tau,\infty) \end{cases}$$

has a continuous extension to an operator $\Psi_\tau \in \mathcal{L}(X, L^p([0,\infty),U))$.

Let $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ be L^p–admissible for $\mathbb{T}$ as in the last definition. It is easy to see that the families $(\Phi_t)_{t\geq 0}$ and $(\Psi_\tau)_{\tau\geq 0}$ yield linear control and observation systems $(\mathbb{T}, \Phi)$ and $(\mathbb{T}, \Psi)$ on X and $L^p([0,\infty), U)$ as well as on X and $L^p([0,\infty), Y)$ respectively. Hence the following proposition directly follows from Theorem 2.7.

Proposition 2.10. *Let $\mathbb{T}$ be a strongly continuous semigroup on a Banach space X and let $B \in \mathcal{L}(U, X_{-1})$ be an L^p–admissible control operator for $\mathbb{T}$. Then for all $x_0 \in X$ and $u \in L^p_{\text{loc}}([0,\infty), U)$ the function $z : [0,\infty) \to X$ given by $z(t) = \mathbb{T}_t x_0 + \Phi_t u$ is the unique strong solution of* (2.6).

Under the conditions of the last definition it is not hard to verify that if for one $\tau > 0$ we have $\Psi_\tau \in \mathcal{L}(X, L^p([0,\infty), U))$, then C is L^p–admissible for $\mathbb{T}$. We refer to Proposition 4.3.2 of [49]. Clearly this is equivalent to the existence of a number $K_\tau \geq 0$ with

$$\|\Psi_\tau x_0\|_{L^p([0,\infty),Y)} = \left(\int_0^\tau \|C\mathbb{T}_t x_0\|_Y^p \, \mathrm{d}t\right)^{1/p} \leq K_\tau \|x_0\|_X \quad \text{for all } x_0 \in X_1.$$

Similarly, B is L^p–admissible for $\mathbb{T}$ if $\Phi_t \in \mathcal{L}(L^p([0,\infty), U), X)$ for one $t > 0$. It is remarkable that it suffices to check that $\Phi_t u \in X$ for one $t > 0$ and all $u \in L^p([0,\infty), U)$. This can be prove using the closed graph theorem, see Proposition 4.2 of [52].

Assume that $C \in \mathcal{L}(X_1, Y)$ is an L^p–admissible control operator for $\mathbb{T}$. Unfortunately, in general the representation $(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0$ does not extend to all $x_0 \in X$. To obtain such a formula on X, the Lebesgue extension of C is introduced. This is the linear operator $C_L : D(C_L) \to Y$ with domain

$$\mathrm{D}(C_L) = \left\{ x \in X \;\middle|\; \tfrac{1}{\tau} C\int_0^\tau \mathbb{T}_s x \, \mathrm{d}s \text{ converges in } Y \text{ as } t \to 0^+ \right\}.$$

The value $C_L x$ is defined as the limit above. Since for $x_0 \in X_1$ the function $[0,\infty) \to X_1$; $t \mapsto \mathbb{T}_t x_0$ is continuous at $t = 0$, it is clear that X_1 is a subset of $\mathrm{D}(C_L)$. It is shown in Theorem 4.5 of [53] that $x \in X$ belongs to $\mathrm{D}(C_L)$ if and only if $\Psi_\infty x$ has a Lebesgue point at $t = 0$. We will take this characterization as the definition of our Lebesgue extension in Definition 6.2.

More generally Theorem 4.5 of [53] says that $\mathbb{T}_t x$ is contained in $\mathrm{D}(C_L)$ if and only if $\Psi_\infty x$ has a Lebesgue point at t, and then

$$(\Psi_\infty x)(t) = C_L \mathbb{T}_t x.$$

As a consequence, for every $x \in X$ this equation holds for almost all $t \geq 0$.

It is a very nice feature of linear control theory that results can be derived by duality. Assume that X and U are reflexive. In this case the family of duals $\mathbb{T}^* = (\mathbb{T}_t^*)_{t\geq 0}$ is a continuous semigroup on X^*, see Proposition A.4. Let $B \in \mathcal{L}(U, X_{-1})$ and $p \in (1,\infty)$. We identify X_{-1}^* with $X_{1,d}$ (see Proposition A.3 and the preceding text) so that we have $B^* \in \mathcal{L}(X_{1,d}, U^*)$. As usual we also identify $L^p([0,\infty), U)^*$ with $L^{p'}([0,\infty), U^*)$ where $p' \in (1,\infty)$ is the dual exponent for p. Then B is an L^p–admissible control operator for $\mathbb{T}$ if and only if B^* is an $L^{p'}$–admissible observation operator for $\mathbb{T}^*$. For the proof we refer to Theorem 6.9 in [53]. We mention that in this case

$$\Phi_\tau^* = Я_\tau \Psi_\tau^d \quad \text{for all } \tau \geq 0,$$

where $\Psi_\tau^d \in \mathcal{L}(X^*, L^{p'}([0,\infty), U^*))$ is the continuous extension of $\Psi_\tau^d x^* = P_\tau B^* \mathbb{T}^*_{(\cdot)} x^*$ for $x^* \in X_{1,d}$ and $Я_\tau$ is the *time-reflection operator* given by

$$(Я_\tau f)(t) = \begin{cases} f(\tau - t), & t \in [0,\tau) \\ 0, & t \in [\tau, \infty) \end{cases} \qquad \text{for suitable } f \text{ defined on } [0,\infty). \tag{2.10}$$

In the same way $C \in \mathcal{L}(X_1, Y)$ is an L^q–admissible observation operator for $\mathbb{T}$ if and only if $C^* \in \mathcal{L}(Y^*, X_{-1,d})$ is an $L^{q'}$–admissible control operator for $\mathbb{T}^*$ and then

$$\Psi_\tau^* = \Phi_\tau^d Я_\tau \quad \text{for all } \tau \geq 0,$$

where $\Phi_\tau^d \in \mathcal{L}(L^{q'}([0,\infty), Y^*), X^*)$ is defined by $\Phi_\tau^d u = \int_0^\tau \mathbb{T}^*_{\tau-s} C^* u(s)\,\mathrm{d}s$. Let us call $(\mathbb{T}^*, \Psi^d)$ and $(\mathbb{T}, \Phi^d)$ the *dual systems* corresponding to $(\mathbb{T}, \Phi)$ and $(\mathbb{T}, \Psi)$, respectively.

2.3 Additive dynamical systems

We saw that linearity of a dynamical system allows us to separate the effect of the input from the evolution of the system that is free of influences. This is not a property of linear systems alone, we can simply assume it. As we shall see in our examples, linear systems with modified input or output satisfy this assumption. First we introduce the term 'additive map'.

Let $\mathcal{M}$ and $\mathcal{N}$ be nonempty sets and let $(E, +)$ be a commutative group. We call a map $f \in E^{\mathcal{M}\times\mathcal{N}}$ *additive* if there are maps $f_1 \in E^{\mathcal{M}}$ and $f_2 \in E^{\mathcal{N}}$ such that $f(a,b) = f_1(a) + f_2(b)$ for all $(a,b) \in \mathcal{M} \times \mathcal{N}$. Additive maps can be characterized as follows.

Lemma 2.11. *For a function $f \in E^{\mathcal{M}\times\mathcal{N}}$ the following statements are equivalent.*

(a) f is additive.

(b) $\forall a, a^* \in \mathcal{M},\ b_1, b_2 \in \mathcal{N}:\ f(a,b_1) - f(a,b_2) = f(a^*,b_1) - f(a^*,b_2)$.

(c) $\forall a_1, a_2 \in \mathcal{M},\ b, b^* \in \mathcal{N}:\ f(a_1,b) - f(a_2,b) = f(a_1,b^*) - f(a_2,b^*)$.

Proof. It is obvious that (a) implies (b) and that (b) implies (c). If (c) is satisfied, choose any $(a_0, b_0) \in \mathcal{M} \times \mathcal{N}$ and define $f_1 \in E^{\mathcal{M}}$ and $f_2 \in E^{\mathcal{N}}$ via

$$f_1(a) = f(a, b_0) - f(a_0, b_0) \qquad \text{resp.} \qquad f_2(b) = f(a_0, b).$$

Then by condition (c) we have

$$f_1(a) + f_2(b) = f(a,b_0) - f(a_0,b_0) + f(a_0,b) = f(a,b) - f(a_0,b) + f(a_0,b) = f(a,b)$$

for all $(a,b) \in \mathcal{M} \times \mathcal{N}$. □

Informally speaking, (c) express that the difference $f(a_1,b) - f(a_2,b)$ is independent of b. Similarly (b) means that $f(a,b_1) - f(a,b_2)$ does not depend on a. It should be clear that (b) and (c) can be replaced by the conditions

(b') $\exists a^* \in \mathcal{M}\ \forall a \in \mathcal{M},\ b_1, b_2 \in \mathcal{N}:\ f(a,b_1) - f(a,b_2) = f(a^*,b_1) - f(a^*,b_2)$.

(c') $\exists b^* \in \mathcal{N}\ \forall a_1, a_2 \in \mathcal{M},\ b \in \mathcal{N}:\ f(a_1,b) - f(a_2,b) = f(a_1,b^*) - f(a_2,b^*)$.

In the proof above, f_1 was chosen such that $f_1(a_0) = 0$. Of course adding a constant to f_2 and subtracting the same from f_1 doesn't change the identity $f(a,b) = f_1(a) + f_2(b)$. Therefore one of the values $f_1(a_0)$ or $f_2(b_0)$ can be chosen. However $f_1(a_0) = f_2(b_0) = 0$ can only be achieved if $f(a_0, b_0) = 0$. This special case is characterized in the following lemma.

Lemma 2.12. *For a function $f \in E^{\mathcal{M}\times\mathcal{N}}$ the following statements are equivalent.*

(i) $\exists (a_0, b_0) \in \mathcal{M} \times \mathcal{N},\ f_1 \in E^{\mathcal{M}}, f_2 \in E^{\mathcal{N}}\ \forall (a,b) \in \mathcal{M}\times\mathcal{N}:\ f(a,b) = f_1(a) + f_2(b)$ *and* $f_1(a_0) = f_2(b_0) = 0$.

(ii) $\exists (a_0, b_0) \in \mathcal{M} \times \mathcal{N},\ f_1 \in E^{\mathcal{M}}, f_2 \in E^{\mathcal{N}}\ \forall (a,b) \in \mathcal{M}\times\mathcal{N}:\ f(a,b) = f_1(a) + f_2(b)$ *and* $f_2(b_0) = -f_1(a_0)$.

(iii) $\exists (a_0, b_0) \in \mathcal{M} \times \mathcal{N}\ \forall (a,b) \in \mathcal{M}\times\mathcal{N}:\ f(a,b) = f(a, b_0) + f(a_0, b)$.

In any of this equivalent cases we have $f(a_0, b_0) = 0$.

Proof. The implication (i) ⇒ (ii) is trivial. If (ii) holds then for all $(a,b) \in \mathcal{M}\times\mathcal{N}$ we have

$$f(a, b_0) = f_1(a) + f_2(b_0) = f_1(a) - f_1(a_0) \qquad \text{and} \qquad f(a_0, b) = f_1(a_0) + f_2(b).$$

Adding up these equations, we obtain $f(a,b_0) + f(a_0,b) = f_1(a) + f_2(b) = f(a,b)$. Thus (iii) is satisfied. In case (iii) is valid, we define $f_1 \in E^{\mathcal{M}}$ and $f_2 \in E^{\mathcal{N}}$ by $f_1(a) = f(a, b_0)$ and $f_2(b) = f(a_0, b)$ respectively. Then clearly $f(a_0,b_0) = f(a_0,b_0) + f(a_0,b_0)$ which means $0 = f(a_0, b_0)$. We conclude that $f_1(a_0) = f_2(b_0) = 0$ and (i) is fulfilled. □

Let $\Sigma = (\mathbb{R}, X, U, \phi, Y, \eta)$ be an Ω–complete time-invariant dynamical system. We may assume that $D_\Sigma = [0,\infty) \times X \times \Omega$. Let $\phi(t, \cdot, \cdot)$ be additive for every $t \ge 0$. Then there are functions $\mathbb{T}_t : X \to X$ and $\Phi_t : \Omega \to X$ such that

$$\phi(t,x,u) = \mathbb{T}_t(x) + \Phi_t(u) \quad \text{for all } x \in X, u \in \Omega.$$

Let $t \ge 0$, $x \in X$ and $u \in \Omega$. As argued above, we may assume that $\mathbb{T}_t(x) = \phi(t,x,0) - \phi(t,0,0)$ and $\Phi_t(u) = \phi(t,0,u)$. Note that then $\mathbb{T}_t(0) = 0$. From Definition 2.4 we further infer the identities

$$\begin{aligned}
\Phi_0(u) &= \phi(0,0,u) = 0,\\
\Phi_{t+\tau}(u) &= \phi(t+\tau,0,u) = \phi(t,\phi(\tau,0,u),S^*_\tau u) \pm \phi(t,0,S^*_\tau u)\\
&= \phi(t,\phi(\tau,0,u),0) - \phi(t,0,0) + \phi(t,0,S^*_\tau u) = \mathbb{T}_t(\Phi_\tau(u)) + \Phi_t(S^*_\tau u).
\end{aligned}$$

So the conditions (2.3) are satisfied as in the linear case, and we also have $\mathbb{T}_0(x) = \phi(0,x,0) - \phi(0,0,0) = x$. On order to fulfill (2.1) we assume that $\mathbb{T}_t$ is linear for every $t \ge 0$. Then we can derive

$$\begin{aligned}
\mathbb{T}_{t+\tau}x &= \phi(t,\phi(\tau,x,0),S^*_\tau 0) - \phi(t,\phi(\tau,0,0),S^*_\tau 0)\\
&= \mathbb{T}_t(\mathbb{T}_\tau x + \phi(\tau,0,0)) + \Phi_t(0) - \mathbb{T}_t(\phi(\tau,0,0)) - \Phi_t(0)\\
&= \mathbb{T}_t(\mathbb{T}_\tau x \pm \phi(\tau,0,0)) = \mathbb{T}_t\mathbb{T}_\tau x.
\end{aligned}$$

These are the type of control systems we are going to study. As for the linear systems we will add some regularity assumptions. We will further restrict the choice of Ω. See the Definitions 4.1 and 4.3.

We now describe the observation systems treated in this work. Let $t, \tau \geq 0$, $x \in X$ and $u \in \Omega$. Consider the output $y : [0, \infty) \times X \times \Omega \to Y$ given by

$$y(t, x, u) = \eta(\phi(t, x, u), u(t)).$$

The composition property from Definition 2.4 yields the identity

$$y(t + \tau, x, u) = \eta(\phi(t, \phi(\tau, x, u), S_\tau^* u), u(t + \tau)) = y(t, \phi(\tau, x, u), S_\tau^* u). \tag{2.11}$$

We could assume that $y(t, \cdot, \cdot)$ is additive, i.e., $y(t, x, u) = y(t, x, 0) + y(t, 0, u) - y(t, 0, 0)$. But would lead to the identity

$$\begin{aligned}\eta(\phi(t, x, 0) + \phi(t, 0, u), u(t)) = y(t, x, u) &= y(t, x, 0) + y(t, 0, u) - y(t, 0, 0)\\ &= \eta(\phi(t, x, 0), 0) + \eta(\phi(t, 0, u), u(t)) - \eta(\phi(t, 0, 0), 0),\end{aligned}$$

which is reasonable only if η is linear. Instead, we consider the operator $\Psi_\infty : X \to Y^{[0,\infty)}$ defined by

$$\Psi_\infty(x)(t) = y(t, x, 0) = \eta(\phi(t, x, 0), 0).$$

Assume that $\phi(t, 0, 0) = 0$ for all $t \geq 0$, so that with the above notation

$$\mathbb{T}_t x = \phi(t, x, 0) \qquad \text{and} \qquad \Phi_t(u) = \phi(t, 0, u).$$

Note that in this case we have $\mathbb{T}_t 0 = 0 = \Phi_t(0)$ for all $t \geq 0$. Plugging in $u = 0$ into the composition property (2.11) we obtain

$$\Psi_\infty(x)(t + \tau) = y(t, \phi(\tau, x, 0), S_\tau^* 0) = y(t, \mathbb{T}_\tau x, 0) = \Psi_\infty(\mathbb{T}_\tau x)(t).$$

It follows that $S_\tau^* \Psi_\infty(x) = \Psi_\infty(\mathbb{T}_\tau x)$ for all $\tau \geq 0$ and $x \in X$ which equals (2.4). The systems introduced here are discussed in Sections 6.1 – 6.3.

On the other hand, the composition property (2.11) with $x = 0$ yields

$$y(t + \tau, 0, u) = y(t, \phi(\tau, 0, u), S_\tau^* u).$$

For the method used in Section 6.4 it is crucial, that we have the additive structure as in (2.5). The reasoning above indicates that we have to assume that η is linear. In this case, we set

$$F_\infty(u)(t) = y(t, 0, u) = \eta(\phi(t, 0, u), u(t)) \quad \text{for } u \in \Omega.$$

Then with (2.11) we infer

$$\begin{aligned}F_\infty(u)(t + \tau) &= y(t, \phi(\tau, 0, u), S_\tau^* u) = y(t, \phi(\tau, 0, u), 0) + y(t, 0, S_\tau^* u)\\ &= y(t, \Phi_\tau(u), 0) + F_\infty(S_\tau^* u)(t) = \Psi_\infty(\Phi_\tau(u))(t) + F_\infty(S_\tau^* u)(t).\end{aligned}$$

for all $u \in \Omega$ and $t, \tau \geq 0$. This means that (2.5) is satisfied. From the causality property in Definition 2.4 one easily derives that F_∞ also fulfills $P_t F_\infty(u) = P_t F_\infty(P_t u)$ for every $u \in \Omega$ and all $t \geq 0$.

Chapter 3

Solvability of Cauchy problems

Throughout let X be a Banach space. In the first two sections we define and compare several solution concepts for evolution equations. A related topic is the discussion of the solution space Z in Section 3.3.

3.1 Inhomogeneous Cauchy problems

We define what we mean by a 'solution' of the inhomogeneous Cauchy problem

$$z'(t) = Az(t) + f(t); \quad z(0) = x_0, \tag{3.1}$$

and study the properties of such functions. The following results are essentially known but since they are crucial for our thesis we give a detailed exposition for convenience. Good references are Section 4.2 in [33] and Section 3.8 in [43].

We have to specify the objects in 3.1. Let $x_0 \in X$. It is known that the homogeneous problem (that is (3.1) with $f = 0$) is well-posed if and only if A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. So let A be such a generator.

The generator A at hand, we can construct a space X_{-1} such that there is an extension $A \in \mathcal{L}(X, X_{-1})$. Seen as an unbounded operator in X_{-1} this extension is the generator of the extension of $\mathbb{T}$ to X_{-1}. For more details see Appendix A. In linear control theory, we have $f = B \circ u$ for some $B \in \mathcal{L}(U, X_{-1})$ and $u \in L^p([0,\infty), U)$, where $p \in [1, \infty)$. It is thus reasonable to postulate that

$$f \in L^1_{\text{loc}}([0,\infty), X_{-1}).$$

We now formulate a first solution concept.

Definition 3.1. A function $z : [0,\infty) \to X$ is called *classical solution of* (3.1) *in* X if $z \in C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$ and (3.1) is satisfied for every $t \geq 0$.

We point out that in evolution equations one usually looks for classical solutions in X_1, that is functions $z \in C([0,\infty), X_1) \cap C^1([0,\infty), X)$ where X_1 is $\mathrm{D}(A)$ with a norm equivalent to the graph norm (again see Appendix A). Since B maps into X_{-1} we can not work with such solutions.

Unfortunately, due to the low regularity of f, we can not expect that classical solutions of (3.1) in X exist. In fact, if $z \in C^1([0,\infty), X_{-1})$ is a classical solution then f has to be continuous in X_{-1} as the identity $f = z' - Az$ shows. This setting is not suitable for us. Therefore we need a somewhat weaker notion of solution.

Definition 3.2. A function $z : [0,\infty) \to X$ is called *strong solution of* (3.1) *in* X if it belongs to $C([0,\infty), X)$ and satisfies the 'integrated equation'

$$z(t) - x_0 = \int_0^t (Az(s) + f(s))\,\mathrm{d}s \quad \text{for all } t \geq 0. \tag{3.2}$$

Note that the left-hand side of (3.2) belongs to X. Therefore also the value of the integral on the right-hand side has to lie in X. However, it is calculated in X_{-1} because under the given assumptions the functions Az and f belong to $L^1([0,t], X_{-1})$.

It is clear that classical solutions of (3.1) in X are strong solutions of (3.1) in X. The fact that Az and f are locally integrable with respect to $\|\boldsymbol{.}\|_{-1}$ implies that a strong solution of (3.1) in X as a function $z : [0,\infty) \to X_{-1}$ is absolutely continuous and differentiable almost everywhere. In particular we have $z'(t) = Az(t) + f(t)$ for almost every $t \in [0,\infty)$, c.f. Proposition 1.2.2 in [5].

Uniqueness is a crucial property of strong solutions. To be more precise, if there is a strong solution (or even a classical solution) of (3.1), then it is given by the 'variation of constants formula' (3.3). This is the claim of Proposition 3.4.

Definition 3.3. The *mild solution of* (3.1) is the function $z \in C([0,\infty), X_{-1})$ given by

$$z(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s. \tag{3.3}$$

The existence of the integral as well as the continuity of the function $[0,\infty) \to X_{-1}$; $t \mapsto \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s$ are shown in Proposition 1.3.4 of [5].

Proposition 3.4. *A strong solution z of* (3.1) *in X is a mild solution. In particular for each initial value $x_0 \in X$, there is at most one strong solution of* (3.1) *in X.*

Proof. Let $z \in C([0,\infty), X)$ be a strong solution. We have to show that (3.3) holds for all $t \geq 0$. For $t = 0$ the claim is trivial. Thus let $t > 0$. Consider the function $g : [0,t] \to X_{-1}$ defined by $g(s) = \mathbb{T}_{t-s} z(s)$. With the proof of the product rule as well as the boundedness of $\|\mathbb{T}_\sigma\|$ for $\sigma \in [0,t]$, one shows that g is differentiable almost everywhere on $[0,t]$. In fact, if z is differentiable at some $s \in [0,t]$, then g has the derivative

$$g'(s) = -A\mathbb{T}_{t-s} z(s) + \mathbb{T}_{t-s} z'(s) = \mathbb{T}_{t-s} f(s)$$

at s. Moreover, g is absolutely continuous: We use the notation $m_{\mathbb{T},t} = \sup_{\sigma \in [0,t]} \|\mathbb{T}_\sigma\| \geq 1$. For points $0 \leq a_1 \leq b_1 \leq \ldots \leq a_m \leq b_m \leq t$ we can estimate

$$\begin{aligned}
\sum_{k=1}^m \|g(b_k) - g(a_k)\|_{-1} &\leq \sum_{k=1}^m \|\mathbb{T}_{t-b_k}(z(b_k) - z(a_k))\|_{-1} + \sum_{k=1}^m \|\mathbb{T}_{t-b_k} z(a_k) - \mathbb{T}_{t-a_k} z(a_k)\|_{-1} \\
&\leq \sum_{k=1}^m m_{\mathbb{T},t} \|z(b_k) - z(a_k)\|_{-1} + \sum_{k=1}^m \left\| \int_{t-a_k}^{t-b_k} A\mathbb{T}_\sigma z(a_k)\,\mathrm{d}\sigma \right\|_{-1} \\
&\leq \sum_{k=1}^m m_{\mathbb{T},t} \|z(b_k) - z(a_k)\|_{-1} + \sum_{k=1}^m \|A\|_{\mathcal{L}(X,X_{-1})}\, m_{\mathbb{T},t}\, \|z(a_k)\|_X (b_k - a_k).
\end{aligned}$$

Let $\varepsilon > 0$. Using the absolute continuity of z as well as its boundedness on $[0,t]$ it is easy to find a number $\delta > 0$ such that $\sum_{k=1}^m (b_k - a_k) \leq \delta$ implies that the right-hand side is less or equal ε.

Since g is absolutely continuous and differentiable almost everywhere, it is the antiderivative of g' (see Proposition B.4). Therefore we have

$$z(t) - \mathbb{T}_t x_0 = g(t) - g(0) = \int_0^t g'(s)\,\mathrm{d}s = \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s. \qquad \square$$

We have seen that the mild solution is the only candidate for a strong solution (or even a classical one). On the other hand, the mild solution clearly exists for all $x_0 \in X$ and every $f \in L^1_{\mathrm{loc}}([0,\infty), X_{-1})$. We are thus looking for a condition under which the mild solution is of a stronger type.

Unfortunately, there are examples for mild solutions that are not strong solutions. Assume there is an $x \in X \setminus X_1$ and some $t_0 > 0$ with $\mathbb{T}_t x \notin X_1$ for all $t \in [0, t_0]$. Take $\lambda \in \rho(A)$ and set $x_{-1} := (\lambda - A)x$. Then $\mathbb{T}_t x_{-1} \in X_{-1} \setminus X$ for all $t \in [0, t_0]$. Consider $f : [0,\infty) \to X_{-1}$ defined as $f(t) = \mathbb{T}_t x_{-1}$. Due to continuity, f is measurable and locally integrable. The mild solution of (3.1) with this f and $x_0 = 0$ is given by

$$z(t) = \mathbb{T}_t 0 + \int_0^t \mathbb{T}_{t-s}\mathbb{T}_s x_{-1}\,\mathrm{d}s = \mathbb{T}_t x_{-1} \int_0^t 1\,\mathrm{d}s = t\mathbb{T}_t x_{-1}$$

for $t \geq 0$. Since $z(t) \notin X$ for $t \in [0, t_0]$, it can not be a strong solution. The next result characterizes strong solutions. It is proved below.

Theorem 3.5. *Let $x_0 \in X$, $f \in L^1_{\mathrm{loc}}([0,\infty), X_{-1})$ and let $z \in C([0,\infty), X_{-1})$ be the mild solution of (3.1). Then z is the strong solution of (3.1) in X if and only if $z \in C([0,\infty), X)$.*

One implication is trivial. The crucial step in the proof of the other is to approximate f by functions in $C^1([0,\infty), X_{-1})$. Therefore we first show that if f has higher regularity, then also the mild solution has better properties.

The 'homogeneous part' $z_{\mathrm{h}} : [0,\infty) \to X;\ t \mapsto \mathbb{T}_t x_0$ clearly belongs to $C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$. We thus focus on the 'inhomogeneous part' of the mild solution, that is

$$z_{\mathrm{ih}} : [0,\infty) \to X_{-1};\ z_{\mathrm{ih}}(t) = \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s.$$

Clearly, for the mild solution to be a strong solution, $z_{\mathrm{ih}}(t)$ has to belong to X for all $t \geq 0$. This is the case if and only if for all $t \geq 0$ the term $\frac{1}{\tau}(\mathbb{T}_\tau z_{\mathrm{ih}}(t) - z_{\mathrm{ih}}(t))$ converges with respect to $\|\,\mathbf{.}\,\|_{-1}$ as $\tau \to 0^+$. If $t \geq 0$ and $\tau > 0$, some transformations including a change of variables lead to

$$\begin{aligned} &\tfrac{1}{\tau}(\mathbb{T}_\tau z_{\mathrm{ih}}(t) - z_{\mathrm{ih}}(t)) \\ &\quad = \mathbb{T}_t \tfrac{1}{\tau}\int_0^\tau \mathbb{T}_{\tau-s} f(s)\,\mathrm{d}s + \int_0^t \mathbb{T}_{t-s}\tfrac{1}{\tau}(f(s+\tau) - f(s))\,\mathrm{d}s - \tfrac{1}{\tau}\int_0^\tau \mathbb{T}_{\tau-s} f(t+s)\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

The convergence of two of the summands on the right-hand side can easily be discussed. We do this in a lemma. In the proof of Proposition 3.7 we then only have to treat the third.

Lemma 3.6. *Let $f \in L^1_{\mathrm{loc}}([0,\infty), X_{-1})$. Assume that f has a Lebesgue point at $t \geq 0$. We then obtain*

$$\left\| \tfrac{1}{\tau}\int_0^\tau \mathbb{T}_{\tau-s} f(t+s)\,\mathrm{d}s - f(t) \right\|_{-1} \to 0 \quad \text{as } \tau \to 0^+.$$

Hence this is true for almost every $t \geq 0$. In particular, if f has a Lebesgue point at 0, then

$$\left\| \mathbb{T}_t \tfrac{1}{\tau} \int_0^\tau \mathbb{T}_{\tau-s} f(s)\,\mathrm{d}s - \mathbb{T}_t f(0) \right\|_{-1} \to 0 \quad \textit{as } \tau \to 0^+$$

for all $t \geq 0$. If f is continuous, then both is true for all $t \geq 0$.

Proof. We only have to prove the first claim. Let $t \geq 0$. For all $\tau \in (0,1]$ we have

$$\begin{aligned}
&\left\| \tfrac{1}{\tau} \int_0^\tau \mathbb{T}_{\tau-s} f(t+s)\,\mathrm{d}s - f(t) \right\| \\
&\quad \leq \left\| \tfrac{1}{\tau} \int_0^\tau \mathbb{T}_{\tau-s} (f(t+s) - f(t))\,\mathrm{d}s \right\|_{-1} + \left\| \tfrac{1}{\tau} \int_0^\tau (\mathbb{T}_{\tau-s} f(t) - f(t))\,\mathrm{d}s \right\|_{-1} \\
&\quad \leq m_{\mathbb{T},1}\, \tfrac{1}{\tau} \int_0^\tau \|f(t+s) - f(t)\|_{-1}\,\mathrm{d}s + \tfrac{1}{\tau} \int_0^\tau \|\mathbb{T}_s f(t) - f(t)\|_{-1}\,\mathrm{d}s.
\end{aligned}$$

The right-hand side converges to zero as $\tau \to 0^+$ for almost every $t \geq 0$ due to Lebesgue's differentiation theorem (see Theorem B.2) and the strong continuity of $\mathbb{T}$. □

We use these facts in the proof of the next proposition.

Proposition 3.7. *Assume that $f \in C^1([0,\infty), X_{-1})$. Then $z_{\mathrm{ih}} \in C([0,\infty), X)$ and it satisfies*

$$A z_{\mathrm{ih}}(t) = \mathbb{T}_t f(0) + \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s - f(t) \quad \textit{for all } t \geq 0.$$

Proof. In a first step we show that $z_{\mathrm{ih}}(t)$ fulfills the asserted identity for all $t \geq 0$. From formula (3.4) we deduce the estimate

$$\begin{aligned}
&\left\| \tfrac{1}{\tau}(\mathbb{T}_\tau z_{\mathrm{ih}}(t) - z_{\mathrm{ih}}(t)) - \mathbb{T}_t f(0) - \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s + f(t) \right\|_{-1} \\
&\quad \leq \left\| \tfrac{1}{\tau} \int_0^\tau \mathbb{T}_{\tau-s} f(t+s)\,\mathrm{d}s - f(t) \right\|_{-1} + \left\| \mathbb{T}_t \tfrac{1}{\tau} \int_0^\tau \mathbb{T}_{\tau-s} f(s)\,\mathrm{d}s - \mathbb{T}_t f(0) \right\|_{-1} \\
&\qquad + \left\| \int_0^t \mathbb{T}_{t-s} \tfrac{1}{\tau}(f(s+\tau) - f(s))\,\mathrm{d}s - \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s \right\|_{-1}
\end{aligned}$$

for $t \in [0,\infty)$ and $\tau \in (0,1]$. Because of Lemma 3.6 and the continuity of f it remains to prove that the term

$$\begin{aligned}
&\left\| \int_0^t \mathbb{T}_{t-s} \tfrac{1}{\tau}(f(s+\tau) - f(s))\,\mathrm{d}s - \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s \right\|_{-1} \\
&\qquad = \left\| \int_0^t \mathbb{T}_{t-s} [\tfrac{1}{\tau}(f(s+\tau) - f(s)) - f'(s)]\,\mathrm{d}s \right\|_{-1}
\end{aligned}$$

converges to zero as $\tau \to 0^+$. This follows from the dominated convergence theorem. Indeed, since $f \in C^1([0,\infty), X_{-1})$, the integrand converges to zero pointwise for all $s \in [0,t]$ as $\tau \to 0^+$. Moreover, for all $s \in [0,t]$ the fundamental theorem yields

$$\begin{aligned}
\|\mathbb{T}_{t-s}[\tfrac{1}{\tau}(f(s+\tau) - f(s)) - f'(s)]\|_{-1} &= \left\| \mathbb{T}_{t-s} \tfrac{1}{\tau} \int_0^\tau (f'(s+\sigma) - f'(s))\,\mathrm{d}\sigma \right\|_{-1} \\
&\leq 2 m_{\mathbb{T},t} \sup_{\sigma \in [0,t+1]} \|f'(\sigma)\|_{-1}.
\end{aligned}$$

Because the area of integration $[0,t]$ has finite measure, the first step of the proof is finished, and we obtain

$$Az_{\text{ih}}(t) = \mathbb{T}_t f(0) - f(t) + \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s.$$

Clearly, the right-hand side is continuous in X_{-1} as a function of t, i.e., $Az_{\text{ih}} \in C([0,\infty), X_{-1})$. Recall that $z_{\text{ih}} \in C([0,\infty), X_{-1})$ and that $R(\lambda, A) \in \mathcal{L}(X_{-1}, X)$ where $\lambda \in \rho(A)$. Hence the first claim follows from the identity

$$z_{\text{ih}} = R(\lambda, A) \circ (\lambda z_{\text{ih}} - Az_{\text{ih}}). \qquad \square$$

We will see below that $f \in C^1([0,\infty), X)$ is a sufficient conditions for a mild solution to be a classical solution in X. This is also true, if we merely assume that f is continuous and at the same time z_{ih} is continuous, see Corollary 3.9.

Since classical solutions are differentiable, we look at the difference quotient for z_{ih}. By a straightforward calculation one can verify the equations

$$\tfrac{1}{\tau}(\mathbb{T}_\tau z_{\text{ih}}(t) - z_{\text{ih}}(t)) = \tfrac{1}{\tau}(z_{\text{ih}}(t+\tau) - z_{\text{ih}}(t)) - \tfrac{1}{\tau}\int_0^\tau \mathbb{T}_{\tau-s} f(t+s)\,\mathrm{d}s, \tag{3.5}$$

$$\tfrac{1}{\tau}(z_{\text{ih}}(t) - z_{\text{ih}}(t-\tau)) = \tfrac{1}{\tau}(\mathbb{T}_\tau z_{\text{ih}}(t-\tau) - z_{\text{ih}}(t-\tau)) + \tfrac{1}{\tau}\int_{t-\tau}^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s \tag{3.6}$$

for $t \geq 0, \tau > 0$ and $t \geq \tau > 0$ respectively. Very similarly to Lemma 3.6 one checks that

$$\left\| \tfrac{1}{\tau}\int_{t-\tau}^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s - f(t) \right\|_{-1} \to 0 \quad \text{as } \tau \to 0^+ \tag{3.7}$$

for almost every $t > 0$. Again, for f in $C([0,\infty), X_{-1})$ this is true for all $t > 0$.

Proposition 3.8. *Let $f \in C([0,\infty), X_{-1})$ and assume that $z_{\text{ih}} \in C([0,\infty), X)$. Then z_{ih} belongs to $C^1([0,\infty), X_{-1})$ with $z'_{\text{ih}}(t) = Az_{\text{ih}}(t) + f(t)$ for all $t \geq 0$. That is, z_{ih} satisfies (3.1) with $x_0 = 0$ for all $t \geq 0$.*

Proof. Fix $t \geq 0$. From (3.5) for every $\tau > 0$ we obtain

$$\begin{aligned} &\|\tfrac{1}{\tau}(z_{\text{ih}}(t+\tau) - z_{\text{ih}}(t)) - Az_{\text{ih}}(t) - f(t)\|_{-1} \\ &\qquad \leq \|\tfrac{1}{\tau}(\mathbb{T}_\tau z_{\text{ih}}(t) - z_{\text{ih}}(t)) - Az_{\text{ih}}(t)\|_{-1} + \left\| \tfrac{1}{\tau}\int_t^{t+\tau} \mathbb{T}_{t+\tau-s} f(s)\,\mathrm{d}s - f(t) \right\|_{-1}. \end{aligned}$$

Since $z_{\text{ih}}(t) \in X$, the first term right-hand side converges to zero as $\tau \to 0^+$. The other can be treated as in Lemma 3.6.

On the other hand, for $t > 0$ and $\tau \in (0,t]$ equation (3.6) implies

$$\begin{aligned} &\|\tfrac{1}{\tau}(z_{\text{ih}}(t) - z_{\text{ih}}(t-\tau)) - Az_{\text{ih}}(t) - f(t)\|_{-1} \\ &\qquad \leq \|\tfrac{1}{\tau}(\mathbb{T}_\tau z_{\text{ih}}(t-\tau) - z_{\text{ih}}(t-\tau)) - Az_{\text{ih}}(t)\|_{-1} + \left\| \tfrac{1}{\tau}\int_{t-\tau}^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s - f(t) \right\|_{-1}. \end{aligned}$$

Again we have to show that the right-hand side converges to zero as $\tau \to 0^+$. For the second summand one can use (3.7). Moreover, an elementary lemma of semigroup theory (see Lemma B.14) yields $\mathbb{T}_\tau z_{\mathrm{ih}}(t-\tau) - z_{\mathrm{ih}}(t-\tau) = \int_0^\tau \mathbb{T}_\sigma A z_{\mathrm{ih}}(t-\tau)\,\mathrm{d}\sigma$. Because of the continuity of z_{ih} and the boundedness of $\|\mathbb{T}_\sigma\|$ for σ in compact sets, we have

$$\begin{aligned}\|\tfrac{1}{\tau}(\mathbb{T}_\tau z_{\mathrm{ih}}(t-\tau) - z_{\mathrm{ih}}(t-\tau)) - Az_{\mathrm{ih}}(t)\|_{-1} &= \left\|\tfrac{1}{\tau}\int_0^\tau \mathbb{T}_\sigma A z_{\mathrm{ih}}(t-\tau)\,\mathrm{d}\sigma - Az_{\mathrm{ih}}(t)\right\|_{-1}\\ &\le \left\|\tfrac{1}{\tau}\int_0^\tau \mathbb{T}_\sigma (Az_{\mathrm{ih}}(t-\tau) - Az_{\mathrm{ih}}(t))\,\mathrm{d}\sigma\right\|_{-1} + \left\|\tfrac{1}{\tau}\int_0^\tau (\mathbb{T}_\sigma A z_{\mathrm{ih}}(t) - Az_{\mathrm{ih}}(t))\,\mathrm{d}\sigma\right\|_{-1}\\ &\le m_{\mathbb{T},t}\,\|A\|_{\mathcal{L}(X,X_{-1})}\|z_{\mathrm{ih}}(t-\tau) - z_{\mathrm{ih}}(t)\|_X + \tfrac{1}{\tau}\int_0^\tau \|\mathbb{T}_\sigma A z_{\mathrm{ih}}(t) - Az_{\mathrm{ih}}(t)\|_{-1}\,\mathrm{d}\sigma.\end{aligned}$$

The right-hand side converges to zero as $\tau \to 0^+$. This shows that z_{ih} is differentiable on $[0,\infty)$ with $z_{\mathrm{ih}}'(t) = Az_{\mathrm{ih}}(t) + f(t)$ for all $t \ge 0$. From the last equation it is clear, that z_{ih}' belongs to $C([0,\infty), X_{-1})$. Hence z_{ih} is continuously differentiable. □

Remark. Without any further assumption on $f \in L^1_{\mathrm{loc}}([0,\infty), X_{-1})$ and with nearly the same proof (mainly replace $\|\cdot\|_{-1}$ by $\|\cdot\|_{-2}$), one can show that the mild solution z is differentiable and satisfies (3.1) almost everywhere on $[0,\infty)$ as a function $z : [0,\infty) \to X_{-2}$. ◊

The last two propositions at hand, one can easily deduce the following result.

Corollary 3.9. *Let $z \in C([0,\infty), X_{-1})$ be the mild solution of (3.1). If $f \in C^1([0,\infty), X_{-1})$ or if $f \in C([0,\infty), X_{-1})$ and $z \in C([0,\infty), X)$, then z is the classical solution of (3.1) in X. In case $f \in C^1([0,\infty), X_{-1})$, the derivative z' also is the mild solution of the problem*

$$w'(t) = Aw(t) + f'(t); \quad w(0) = Ax_0 + f(0).$$

We thus have

$$z'(t) = \mathbb{T}_t(Ax_0 + f(0)) + \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s \quad \textit{for } t \ge 0.$$

Proof. Let $z \in C([0,\infty), X_{-1})$ be the mild solution of (3.1). Then $z(t) = z_{\mathrm{h}}(t) + z_{\mathrm{ih}}(t)$, where as before $z_{\mathrm{ih}}(t) = \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s$ and $z_{\mathrm{h}}(t) = \mathbb{T}_t x_0$ for $t \ge 0$. Recall that z_{h} belongs to $C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$ and its derivative is given by $z_{\mathrm{h}}'(t) = A\mathbb{T}_t x_0 = Az_{\mathrm{h}}(t) = \mathbb{T}_t A x_0$ for $t \ge 0$.

First, assume that $f \in C([0,\infty), X_{-1})$ and $z \in C([0,\infty), X)$. Then $z_{\mathrm{ih}} = z - z_{\mathrm{h}}$ is contained in $C([0,\infty), X)$. Proposition 3.8 thus implies that $z_{\mathrm{ih}} \in C^1([0,\infty), X_{-1})$ with $z_{\mathrm{ih}}'(t) = Az_{\mathrm{ih}}(t) + f(t)$ for all $t \ge 0$. As a consequence, also $z = z_{\mathrm{h}} + z_{\mathrm{ih}}$ belongs to $C^1([0,\infty), X_{-1})$ and satisfies $z'(t) = Az_{\mathrm{h}}(t) + Az_{\mathrm{ih}}(t) + f(t) = Az(t) + f(t)$ for all $t \ge 0$. This means that z is the classical solution of (3.1) in X.

Second, let $f \in C^1([0,\infty), X_{-1})$. Then Proposition 3.7 yields $z_{\mathrm{ih}} \in C([0,\infty), X)$. Since in particular $f \in C([0,\infty), X_{-1})$, we are again in the first situation. Proposition 3.7 additionally implies the equation

$$z'(t) = z_{\mathrm{h}}'(t) + Az_{\mathrm{ih}}(t) + f(t) = \mathbb{T}_t A x_0 + \mathbb{T}_t f(0) - f(t) + \int_0^t \mathbb{T}_{t-s} f'(s)\,\mathrm{d}s + f(t),$$

so that the last assertion is true. □

Finally we are able to prove Theorem 3.5. It says that the mild solution is the strong solution if and only it is a continuous function with values in X.

Proof of Theorem 3.5. Let $z \in C([0,\infty), X_{-1})$ be the mild solution of (3.1) and assume that $z \in C([0,\infty), X)$. We have to verify that z satisfies (3.2), i.e.,

$$z(t) - x_0 = \int_0^t (Az(s) + f(s))\,\mathrm{d}s \quad \text{for all } t \geq 0.$$

The claim is trivial for $t = 0$. Thus let $t > 0$. Moreover, take a sequence (f_n) in $C^1([0,t], X_{-1})$ with $\|f_n - f\|_{L^1([0,t],X_{-1})} \to 0$ as $n \to \infty$. For brevity, we write $\|\,\boldsymbol{\cdot}\,\|_{L^1}$ instead of $\|\,\boldsymbol{\cdot}\,\|_{L^1([0,t],X_{-1})}$ in this proof.

For $n \in \mathbb{N}$ let z_n be the mild solution of (3.1) with the forcing term f_n, i.e., $z_n(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} f_n(s)\,\mathrm{d}s$. By Corollary 3.9 these are classical solutions, so that

$$z_n(t) - x_0 = \int_0^t (Az_n(s) + f_n(s))\,\mathrm{d}s.$$

for all $n \in \mathbb{N}$. The vectors $z_n(s)$ approximate $z(s) = \mathbb{T}_s x_0 + \int_0^s \mathbb{T}_{s-\sigma} f(\sigma)\,\mathrm{d}\sigma$ uniformly for $s \in [0,t]$, because

$$\|z_n(s) - z(s)\|_{-1} \leq \int_0^s \|\mathbb{T}_{s-\sigma}(f_n(\sigma) - f(\sigma))\|_{-1}\,\mathrm{d}\sigma \leq m_{\mathbb{T},t}\|f_n - f\|_{L^1} \to 0 \quad \text{as } n \to \infty.$$

Since $\|\,\boldsymbol{\cdot}\,\|_{-1}$ is stronger than $\|\,\boldsymbol{\cdot}\,\|_{-2}$, it follows that $\|z_n(t) - z(t)\|_{-2} \to 0$ as $n \to \infty$. Using the above estimate once more together with $A \in \mathcal{L}(X_{-1}, X_{-2})$, we further get

$$\begin{aligned}
\left\|z_n(t) - x_0 - \int_0^t (Az(s) + f(s))\,\mathrm{d}s\right\|_{-2} &= \left\|\int_0^t A(z_n(s) - z(s))\,\mathrm{d}s + \int_0^t f_n(s) - f(s)\,\mathrm{d}s\right\|_{-2} \\
&\leq \|A\|_{\mathcal{L}(X_{-1},X_{-2})} \int_0^t \|z_n(s) - z(s)\|_{-1}\,\mathrm{d}s + \int_0^t \|f_n(s) - f(s)\|_{-2}\,\mathrm{d}s \\
&\leq \|A\|_{\mathcal{L}(X_{-1},X_{-2})}\, m_{\mathbb{T},t}\, t\, \|f_n - f\|_{L^1} + c\|f_n - f\|_{L^1} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

for a constant $c \geq 0$ with $\|x\|_{-2} \leq c\|x\|_{-1}$ for all $x \in X_{-1}$. Combining both, we end up with

$$\begin{aligned}
&\left\|z(t) - x_0 - \int_0^t (Az(s) + f(s))\,\mathrm{d}s\right\|_{-2} \\
&\qquad \leq \|z(t) - z_n(t)\|_{-2} + \left\|z_n(t) - x_0 - \int_0^t (Az(s) + f(s))\,\mathrm{d}s\right\|_{-2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This shows the claimed identity for z, at first as an equation in X_{-2}. But since $z(t) - x_0$ is contained in X for all $t \geq 0$ the proof is finished. ☐

3.2 Perturbed Cauchy problems

In Chapter 5 we investigate perturbations $F : X \to X$ of the Cauchy problem (3.1). That is, we consider the problem

$$z'(t) = Az(t) + F(z(t)) + f(t); \quad z(0) = x_0. \tag{3.8}$$

Let us first discuss the notions of solution of this problem. We assume that F is continuous. Then, up to a certain point, we can proceed as in the first part.

Let $J = [0,T]$ or $J = [0,T)$ for some $T > 0$, or let $J = [0,\infty)$. Similar as above, any function $z \in C(J,X) \cap C^1(J,X_{-1})$ that satisfies (3.8) for every $t \in J$ is called *classical solution of* (3.8) *on* J. A *strong solution of* (3.8) *on* J is a function $z \in C(J,X)$ with

$$z(t) - x_0 = \int_0^t \big(A(z(s)) + F(z(s)) + f(s)\big)\,\mathrm{d}s \quad \text{for all } t \in J.$$

As before we see that classical solutions are strong solutions. Again, from Proposition 1.2.2 in [5] we deduce that any strong solution $z \in C(J,X)$ as a function $z : J \to X_{-1}$ is absolutely continuous, differentiable almost everywhere on J and satisfies (3.8) for almost every $t \in J$. Additional to the argumentation after Definition 3.2, we use that $F \circ z$ is continuous and therefore locally integrable as a function from J to X_{-1}.

A *mild solution of* (3.8) *on* J is a function $z \in C(J,X_{-1})$ with $z(t) \in X$ for almost all $t \in J$ that satisfies the fixed-point equation

$$z(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} F(z(s))\,\mathrm{d}s + \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s. \tag{3.9}$$

In the proof of Proposition 3.4 we solely used that strong solutions of (3.1) are absolutely continuous and differentiable satisfying (3.1) almost everywhere on J. Hence with the same proof we can show that any strong solution of (3.8) on J is a mild solution of (3.8) on J.

We saw that even in the simplest case $F = 0$ we can not expect to find classical solutions. If $F \neq 0$, in general the situation is even worse. Since by (3.9) mild solution are fixed-points, at first it is not clear if any such solution exists. Moreover, at this point we do not know if mild solutions are unique. Hence we can not deduce uniqueness for strong and classical solutions. In Chapter 5 we establish these properties under certain mild assumptions.

3.3 The solution space

We come back to special case of linear control systems. There is a sufficient condition for the existence of classical solutions (in X). In short it states that if the input is 'smooth' and if the initial state and the first input value satisfy a certain 'compatibility condition' then the strong solution of Proposition 2.10 actually is a classical solution. The precise formulation is given in Proposition 3.13.

The following is well known for Hilbert spaces and can be found e.g. in Section 2 of [51]. Most of these results are also true in a Banach space setting. Since we have not found a reference for that, we decided to include it in this work.

Let X and U be Banach spaces, $p \in [1,\infty)$ and let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Further let $B \in \mathcal{L}(U,X_{-1})$ be an L^p–admissible control operator for $\mathbb{T}$. Denote the corresponding control system on X and $L^p([0,\infty),U)$ by $(\mathbb{T},\Phi)$. For the whole section fix any $\lambda \in \rho(A)$.

Due to the causality $\Phi_t u = \Phi_t P_t u$ for $t \geq 0$ and $u \in L^p([0,\infty),U)$, the operators Φ_t posses an obvious extension to $L^p_{\mathrm{loc}}([0,\infty),U)$. We write χ_v for the constant function equal to $v \in U$ on all of $[0,\infty)$. Then $\Phi_t \chi_v$ is defined as $\Phi_t P_t \chi_v$.

Definition 3.10. The *solution space for* $(\mathbb{T},\Phi)$ is the vector space

$$Z := X_1 + R(\lambda,A)B(U) = R(\lambda,A)(X + B(U)).$$

It is easy to see that this definition is independent of the choice of $\lambda \in \rho(A)$. For a proof we refer to Lemma 2.2 of [51]. Obviously Z is a subspace of X. Since $B0 = 0$ we also have $X_1 \subseteq Z$. Consequently, Z is dense in X. We shall define a norm on Z such that the embeddings

$$X_1 \hookrightarrow Z \hookrightarrow X$$

are continuous. To this end, we identify Z with a factor space of $X \times U$. The latter is a Banach space when it is equipped with the norm given by

$$\|(x,v)\|_{X\times U} := \left(\|x\|_X^2 + \|v\|_U^2\right)^{1/2} \quad \text{for } (x,v) \in X \times U.$$

Consider the set

$$N = \{(x,v) \in X \times U \mid R(\lambda,A)(x+Bv) = 0\} = \{(x,v) \in X \times U \mid x + Bv = 0\}.$$

This is the kernel of $\iota \in \mathcal{L}(X \times U, X)$ given by $\iota(x,v) = R(\lambda,A)(x+Bv)$. Let us check that ι is actually bounded. For $(x,v) \in X \times U$ we have

$$\|\iota(x,v)\|_X \le \|R(\lambda,A)\|\|x+Bv\|_{-1} \lesssim \|x\|_X + \|B\|\|v\|_U \lesssim \left(\|x\|_X^2 + \|v\|_U^2\right)^{1/2},$$

where we write $\lesssim$ if there exists an (unspecified) number $c \ge 0$ such that the left-hand side is less or equal to c times the right-hand side for all parameters appearing in this equation.

Hence $N = \ker(\iota)$ is a closed subspace of $X \times U$. Note that $Z = \operatorname{Ran}(\iota)$. It follows that

$$X\times U/N = \{[(x,v)]_N \mid (x,v) \in X \times U\}$$

is a Banach space with the norm given by

$$\|[(x,v)]_N\|_{X\times U/N} = \inf\left\{\left(\|x+\tilde{x}\|_X^2 + \|v+\tilde{v}\|_U^2\right)^{1/2} \,\middle|\, (\tilde{x},\tilde{v}) \in N\right\}.$$

Here $[(x,v)]_N = (x,v) + N$ denotes the equivalence class of $(x,v) \in X \times U$. Since $0 \in N$ we have $\|[(x,v)]_N\|_{X\times U/N} \le \|(x,v)\|_{X\times U}$ for $(x,v) \in X \times U$. In particular the linear mapping

$$\pi : X \times U \to X\times U/N; \quad (x,v) \mapsto [(x,v)]_{X\times U/N}$$

is bounded. We identify Z with $X\times U/N$ via

$$\tilde{\iota} : X\times U/N \to Z; \quad \tilde{\iota}[(x,v)]_N = \iota(x,v) = R(\lambda,A)(x+Bv).$$

Because $\iota : X \times U \to X$ is onto and we factorized its kernel, $\tilde{\iota}$ is an isomorphism. The norm "transported" to Z by $\tilde{\iota}$ is given by

$$\|w\|_Z = \|\tilde{\iota}^{-1}w\|_{X\times U/N} = \inf\left\{\left(\|x\|_X^2 + \|v\|_U^2\right)^{1/2} \,\middle|\, (x,v) \in X \times U : \ R(\lambda,A)(x+Bv) = w\right\}.$$

We still have to show that the embeddings $X_1 \hookrightarrow Z$ and $Z \hookrightarrow X$ are continuous. First let $x \in X_1$. Then $x = R(\lambda,A)(\lambda - A)x$ and consequently

$$\|x\|_Z \le \left(\|(\lambda-A)x\|_X^2 + \|0\|_U^2\right)^{1/2} = \|(\lambda-A)x\|_X \lesssim \|x\|_1. \tag{3.10}$$

Now let $w \in Z$. For every pair $(x,v) \in X \times U$ with $\iota(x,v) = w$ the boundedness of ι yields the estimate

$$\|w\|_Z = \|\iota(x,v)\|_X \le \|\iota\|\left(\|x\|_X^2 + \|v\|_U^2\right)^{1/2}.$$

Unless $X = \{0\}$, we have $\|\iota\| > 0$ (e.g. $\iota(x,0) \neq 0$ for $x \in X \setminus \{0\}$). Therefore $\frac{1}{\|\iota\|}\|w\|_X$ is a lower bound for $\{\|(x,v)\|_{X\times U} \mid (x,v) \in X \times U :\ R(\lambda,A)(x+Bv) = w\}$ and it follows

$$\|w\|_X \lesssim \inf\{\|(x,v)\|_{X\times U} \mid (x,v) \in X \times U :\ R(\lambda,A)(x+Bv) = w\} = \|w\|_Z.$$

We introduced the solution space, because every classical solution with continuous inputs automatically maps to Z.

Remark 3.11. Let $x_0 \in X$ and let $u \in L^p_{\mathrm{loc}}([0,\infty),U)$ be continuous. Assume that $z \in C^1([0,\infty),X)$ satisfies $z'(t) = Az(t) + Bu(t)$; $z(0) = x_0$ for all $t \geq 0$. Then $z \in C([0,\infty),Z)$. Indeed, we have

$$(\lambda - A)z(t) = \lambda z(t) - z'(t) + Bu(t) \quad\iff\quad z(t) = R(\lambda,A)(\lambda z(t) - z'(t) + Bu(t)). \tag{3.11}$$

Due to the assumption, the map $f : [0,\infty) \to X \times U$; $f(t) = (\lambda z(t) - z'(t), u(t))$ is continuous. Hence also $z = \tilde{\iota} \circ \pi \circ f : [0,\infty) \to Z$ is continuous. ♢

We first formulate a simple version of the sufficient condition and later reduce the general case to that situation. To this end, we need the vector-valued Sobolev space $W^{1,p}([0,\infty),U)$. It consists of those absolutely continuous and almost everywhere differentiable functions $u \in L^p([0,\infty),U)$ for which also $u' \in L^p([0,\infty),U)$. We write $\dot{u}$ instead of u'. For an element $u \in W^{1,p}([0,\infty),U)$ we thus have

$$u(t) = u(0) + \int_0^t \dot{u}(s)\,\mathrm{d}s \quad \text{for all } t \geq 0. \tag{3.12}$$

The space $W^{1,p}_{\mathrm{loc}}([0,\infty),U)$ is the subspace of $L^p_{\mathrm{loc}}([0,\infty),U)$ where all this holds locally. More precisely $u \in W^{1,p}_{\mathrm{loc}}([0,\infty),U)$ if and only if u belongs to $L^p_{\mathrm{loc}}([0,\infty),U)$ is absolutely continuous and differentiable almost everywhere on $[0,\infty)$ and the derivative $\dot{u}$ lies in $L^p_{\mathrm{loc}}([0,\infty),U)$. Identity (3.12) is still valid.

Lemma 3.12. *Let $u \in W^{1,p}_{\mathrm{loc}}([0,\infty),U)$ with $u(0) = 0$. Then the strong solution $z \in C([0,\infty),X)$ of $z'(t) = Az(t) + Bu(t)$; $z(0) = 0$ actually is a classical solution (in X). It even belongs to $C([0,\infty),Z) \cap C^1([0,\infty),X)$.*

Proof. Take $\dot{u} \in L^p([0,\infty),U)$ with $u(s) = \int_0^s \dot{u}(\sigma)\,\mathrm{d}\sigma$ for all $s \geq 0$ and set $w(t) = \Phi_t \dot{u}$ for $t \geq 0$. Then $w \in C([0,\infty),X)$ is the strong solution of $w'(t) = Aw(t) + B\dot{u}(t)$; $w(0) = 0$. Further set

$$z(t) = \int_0^t w(s)\,\mathrm{d}s = \int_0^t \int_0^s \mathbb{T}_{s-\sigma} B\dot{u}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \quad \text{for } t \geq 0.$$

This clearly defines a function $z \in C^1([0,\infty),X)$ with $z(0) = 0$. Fubini's theorem (integration in X_{-1}) yields

$$\begin{aligned} z(t) &= \int_0^t \int_0^s \mathbb{T}_\sigma B\dot{u}(s-\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s = \int_0^t \int_\sigma^t \mathbb{T}_\sigma B\dot{u}(s-\sigma)\,\mathrm{d}s\,\mathrm{d}\sigma \\ &= \int_0^t \mathbb{T}_\sigma \int_0^{t-\sigma} B\dot{u}(s)\,\mathrm{d}s\,\mathrm{d}\sigma = \int_0^t \mathbb{T}_\sigma Bu(t-\sigma)\,\mathrm{d}\sigma = \Phi_t u. \end{aligned}$$

This shows that z is the strong solution of $z'(t) = Az(t) + Bu(t)$; $z(0) = 0$. We even have

$$z'(t) = w(t) = \int_0^t (Aw(s) + B\dot{u}(s))\,\mathrm{d}s = A\int_0^t w(s)\,\mathrm{d}s + B\int_0^t \dot{u}(s)\,\mathrm{d}s = Az(t) + Bu(t)$$

for all $t \geq 0$. Hence z is a classical solution. The last claim now follows from Remark 3.11. □

We can now prove the main result of this section. This is the Banach space version of Proposition 4.2.10 in [49]. Where it is important, we write $A|_m$ for the extension or restriction of A to X_m, cf. Appendix A. We do this only occasionally, because otherwise many formulas would blow up unnecessarily.

Proposition 3.13. *Let $u \in W^{1,p}_{\mathrm{loc}}([0,\infty), U)$ and $x_0 \in X$ satisfy $(A|_0)x_0 + Bu(0) \in X$. Then the strong solution $z \in C([0,\infty), X)$ of $z'(t) = Az(t) + Bu(t)$; $z(0) = x_0$ is indeed a solution of this equation in the classical sense. It belongs to $C([0,\infty), Z) \cap C^1([0,\infty), X)$.*

Proof. Take $\dot{u} \in L^p([0,\infty), U)$ with $u(s) = u(0) + \int_0^s \dot{u}(\sigma)\,\mathrm{d}\sigma$ for all $s \geq 0$ and set $w(t) = \Phi_t \dot{u}$ for $t \geq 0$. We consider $\tilde{u} \in W^{1,p}_{\mathrm{loc}}([0,\infty), U)$ given by $\tilde{u}(t) = u(t) - u(0)$, so that $\tilde{u}(0) = 0$. The linearity of Φ_t implies that the strong solution z from the claim has the decomposition

$$z(t) = \mathbb{T}_t x_0 + \Phi_t \chi_{u(0)} + \Phi_t \tilde{u} \quad \text{for } t \geq 0.$$

Set $z_{\mathrm{c}}(t) = \mathbb{T}_t x_0 + \Phi_t \chi_{u(0)}$ and $z_{\mathrm{n}}(t) = \Phi_t \tilde{u}$ for $t \geq 0$. From Lemma 3.12 we infer that $z_{\mathrm{n}} \in C([0,\infty), Z) \cap C^1([0,\infty), X)$ is the classical solution of $z_{\mathrm{n}}'(t) = Az_{\mathrm{n}}(t) + B\tilde{u}(t)$; $z_{\mathrm{n}}(0) = 0$. For the other part z_{c} we clearly have

$$z_{\mathrm{c}}(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_s Bu(0)\,\mathrm{d}s \quad \text{for all } t \geq 0.$$

We see that z_{c} is differentiable with respect to $\|\,\boldsymbol{.}\,\|_{-1}$ and that

$$z_{\mathrm{c}}'(t) = (A|_0)\mathbb{T}_t x_0 + \mathbb{T}_t Bu(0) = \mathbb{T}_t((A|_0)x_0 + Bu(0))$$

for $t \geq 0$. Due to the assumption $(A|_0)x_0 + Bu(0)$ belongs to X. It follows that z_{c}' is contained in $C([0,\infty), X)$ and consequently $z_{\mathrm{c}} \in C^1([0,\infty), X)$ since

$$z_{\mathrm{c}}(t) = x_0 + \int_0^t z_{\mathrm{c}}'(s)\,\mathrm{d}s.$$

Using Lemma B.14, we derive

$$\begin{aligned} Az_{\mathrm{c}}(t) + Bu(t) &= Ax_0 + A\int_0^t z_{\mathrm{c}}'(s)\,\mathrm{d}s + Bu(t) = Ax_0 + A\int_0^t \mathbb{T}_s(Ax_0 + Bu(0))\,\mathrm{d}s + Bu(t) \\ &= Ax_0 + \mathbb{T}_t(Ax_0 + Bu(t)) - Ax_0 - Bu(t) + Bu(t) \\ &= \mathbb{T}_t(Ax_0 + Bu(t)) = z_{\mathrm{c}}'(t), \end{aligned}$$

showing that z_{c} is a classical solution. We conclude that $z = z_{\mathrm{c}} + z_{\mathrm{n}} \in C^1([0,\infty), X)$ is a classical solution. Remark 3.11 yields that $z \in C([0,\infty), Z)$. □

Remark 3.14. In Lemma 3.12 we proved that Φ_t maps

$$W^{1,p}_L([0,\infty), U) = \{u \in W^{1,p}([0,\infty), U) \mid u(0) = 0\}$$

to Z. We shall show that the restriction of this linear operator is also bounded. We equip $W^{1,p}_L([0,\infty), U)$ with the norm given by $\|u\|_{W^{1,p}} = \left(\|u\|^p_{L^p} + \|\dot{u}\|^p_{L^p}\right)^{1/p}$. Let $T > 0$ and $u \in W^{1,p}_L([0,\infty), U)$. Then for all $t \in [0,T]$ we have

$$\|u(t)\|_U \leq \int_0^t \|\dot{u}(s)\|_U\,\mathrm{d}s = \|\dot{u}\|_{L^1([0,t],U)}.$$

If $p = 1$ the right-hand side is less or equal to $\|u\|_{W^{1,1}}$, else we continue the estimate

$$\|u(t)\|_U \leq t^{1/p'} \|\dot{u}\|_{L^p([0,t],U)} \leq T^{1/p'} \|\dot{u}\|_{L^p([0,\infty),U)} \leq T^{1/p'} \|u\|_{W^{1,p}}.$$

This means that $\|u\|_{L^\infty([0,T],U)} \leq T^{1/p'} \|u\|_{W^{1,p}}$ for all $u \in W_L^{1,p}([0,\infty),U)$. It is now easy to see that $W_L^{1,p}([0,\infty),U)$ is complete.

Fix $t \geq 0$. In the proof of Lemma 3.12 we saw that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t u = \Phi_t \dot{u}$. Combined with (3.11) this yields $\Phi_t u = R(\lambda, A)(\lambda \Phi_t u - \Phi_t \dot{u} + Bu(t))$. Therefore we have

$$\|\Phi_t u\|_Z \leq \left(\|\lambda \Phi_t u - \Phi_t \dot{u}\|_X^2 + \|u(t)\|_U^2\right)^{1/2}.$$

We estimate both summands on the right-hand side by $\|u\|_{W^{1,p}}$. Above we already showed that $\|u(t)\|_U \leq \|u\|_{L^\infty([0,t],U)} \leq t^{1/p'} \|u\|_{W^{1,p}}$. For the other one recall that Φ_t is a bounded linear operator between $\mathcal{L}(L^p([0,\infty),U)$ and X and hence

$$\begin{aligned}\|\lambda \Phi_t u - \Phi_t \dot{u}\|_X &\leq |\lambda| \|\Phi_t\| \|u\|_{L^p} + \|\Phi_t\| \|\dot{u}\|_{L^p} \leq (|\lambda| + 1) \|\Phi_t\| (\|u\|_{L^p} + \|\dot{u}\|_{L^p}) \\ &\lesssim (|\lambda| + 1) \|\Phi_t\| \left(\|u\|_{L^p}^p + \|\dot{u}\|_{L^p}^p\right)^{1/p} = (|\lambda| + 1) \|\Phi_t\| \|u\|_{W^{1,p}}.\end{aligned}$$

We have shown that there is a constant $c_t \geq 0$ such that $\|\Phi_t\|_Z \leq c_t \|u\|_{W^{1,p}}$ for all $u \in W_L^{1,p}([0,\infty),U)$. We stress that $c_{t_1} \leq c_{t_2}$ for $t_1 \leq t_2$. $\Diamond$

Chapter 4

Additive control systems

In this chapter let X and U be Banach spaces. Let us recall some notation. The symbol $U^{[0,\infty)}$ stands for the family of maps $u : [0,\infty) \to U$. We write χ_v for the constant function equal to $v \in U$ on all of $[0,\infty)$. For $\tau \geq 0$ we have the left shift operator S_τ^*, the right shift operator S_τ and truncation operator P_τ.

The vector space of piecewise constant and right continuous functions from $[0,\infty)$ to U is denoted by Ω_0, so a map $u : [0,\infty) \to U$ belongs to Ω_0 if there is an $m \in \mathbb{N}_0$ as well as $0 = t_0 < \ldots < t_m < \infty$ and $v_1, \ldots, v_m \in U$ such that

$$u = \sum_{k=1}^{m} \mathbb{1}_{[t_{k-1},t_k)} v_k.$$

If $m = 0$, then u is the zero function. This representation is unique if we additionally assume that $v_k \neq v_{k+1}$ for $k = 1, \ldots, m-1$ and $v_m \neq 0$. However, sometimes a representation with $v_m = 0$ is useful. In particular if we consider $P_t u$ where $t > t_m$.

Definition 4.1. A *Fréchet domain* is a subset $\Omega \subseteq U^{[0,\infty)}$ (or a set of equivalence classes of such functions) containing Ω_0 such that for all $u \in \Omega$ and every $\tau \geq 0$ we also have $S_\tau^* u, P_\tau u \in \Omega$.

Examples for Fréchet domains are Ω_0 itself, the set of piecewise continuous functions and $L^p([0,\infty), U)$ for some $p \in [1,\infty]$, but also $L^p_{\mathrm{loc}}([0,\infty), U)$.

Definition 4.2. Let $\Omega \subseteq U^{[0,\infty)}$ be a Fréchet domain and let $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ be a (strongly continuous) semigroup on X. A family $\Phi = (\Phi_t)_{t\geq 0}$ of maps $\Phi_t : \Omega \to X$ satisfies the *composition property for* $\mathbb{T}$ if

$$\Phi_{t+\tau}(u) = \mathbb{T}_t \Phi_\tau(u) + \Phi_t(S_\tau^* u) \quad \text{for all } t, \tau \geq 0 \text{ and } u \in \Omega. \tag{4.1}$$

We say that Φ is *causal* if $\Phi_t(u) = \Phi_t(P_t u)$ for all $t \geq 0$, $u \in \Omega$.

Let Φ be as in the above definition. Then from the composition property (4.1) with $t = \tau = 0$ we deduce that $\Phi_0(u) = \mathbb{T}_0 \Phi_0(u) + \Phi_0(S_0^* u) = 2\Phi_0(u)$ for all $u \in \Omega$ and therefore $\Phi_0 = 0$. Moreover, causality implies the equation

$$\Phi_t(P_s \chi_v) = \Phi_t(P_t P_s \chi_v) = \Phi_t(P_t \chi_v) \quad \text{for } t < s.$$

Hence we write $\Phi_t(\chi_v) := \Phi_t(P_t \chi_v)$ even though χ_v might not belong to Ω.

Note that we have not introduced a topology on Fréchet domains. Starting from Section 4.2 we will work with continuous Φ_t on the Fréchet domain $L^p([0,\infty), U)$. For now we only pose regularity assumptions on the map

$$\varphi : [0,\infty) \times U \to X; \ \varphi(t,v) = \Phi_t(\chi_v).$$

We mention that $\varphi(0,v) = \Phi_0(\chi_v) = 0$ for all $v \in U$.

Definition 4.3. Let X be a Banach space and let Ω be a Fréchet domain. An *additive control system on X and Ω* is a pair $(\mathbb{T}, \Phi)$ consisting of a strongly continuous semigroup $\mathbb{T}$ on X and a causal family $\Phi = (\Phi_t)_{t\geq 0}$ of maps $\Phi_t : \Omega \to X$ satisfying the composition property for $\mathbb{T}$ as well as the following conditions.

(i) For all $v \in U$ the function $\varphi(\,.\,, v) : [0,\infty) \to X$ is continuous at 0.

(ii) For all $T > 0$ the family $\{\varphi(\sigma, \,.\,) : U \to X \mid \sigma \in [0,T]\}$ is equicontinuous.

The operators Φ_t are called *input maps* of $(\mathbb{T}, \Phi)$.

Clearly, condition (ii) implies that for every compact subset K of $[0,\infty)$ the family $\{\varphi(\sigma, \,.\,) : U \to X \mid \sigma \in K\}$ is equicontinuous.

4.1 Representation of additive control systems

For the time being let Ω be a Fréchet domain and $(\mathbb{T}, \Phi)$ an additive control system on X and Ω. We first discuss several basic properties of this system. These facts will then lead to a representation of $(\mathbb{T}, \Phi)$ by a control operator $B : U \to X_{-1}$ as in Theorem 2.7.

Since $S_\tau^*\chi_v = \chi_v$ for all $v \in U$ and $\tau \geq 0$, the composition property for φ has the form

$$\varphi(t+\tau, v) = \mathbb{T}_t\varphi(\tau, v) + \varphi(t,v) \quad \text{for } t,\tau \geq 0 \text{ and } v \in U. \tag{4.2}$$

As a first application of this equation, we show that $\varphi(\,.\,, v)$ is continuous on $[0,\infty)$.

Lemma 4.4. *For every $v \in U$ the function $\varphi(\,.\,, v) : [0,\infty) \to X$ is continuous.*

Proof. First let $t, \tau \geq 0$. The composition property (4.2) yields

$$\|\varphi(t+\tau, v) - \varphi(t,v)\|_X = \|\mathbb{T}_t\varphi(\tau,v) + \varphi(t,v) - \varphi(t,v)\|_X \leq \|\mathbb{T}_t\|\|\varphi(\tau,v)\|_X.$$

Since $\varphi(0,v) = 0$, the right-hand side converges to 0 as $\tau \to 0$ by condition (i) in Definition 4.3. If $t > 0$ and $\tau \in [0,t]$, then $\varphi(t,v) = \varphi(t-\tau+\tau, v) = \mathbb{T}_{t-\tau}\varphi(\tau,v) + \varphi(t-\tau, v)$ and thus

$$\begin{aligned}\|\varphi(t,v) - \varphi(t-\tau,v)\|_X &= \|\mathbb{T}_{t-\tau}\varphi(\tau,v) + \varphi(t-\tau,v) - \varphi(t-\tau,v)\|_X \\ &\leq \|\mathbb{T}_{t-\tau}\|\|\varphi(\tau,v)\|_X \leq \sup_{s\in[0,t]}\|\mathbb{T}_s\|\|\varphi(\tau,v)\|_X.\end{aligned}$$

Again the right-hand side converges to 0 as $\tau \to 0$. □

Corollary 4.5. *The mapping $\varphi : [0,\infty) \times U \to X$ is continuous.*

Proof. Fix $t \geq 0$, $v \in U$ and let $\varepsilon > 0$. First choose a radius $\delta_1 > 0$ such that for $s \in [t-\delta_1, t+\delta_1] \cap [0,\infty)$ we have $\|\varphi(t,v) - \varphi(s,v)\|_X \leq \frac{\varepsilon}{2}$. Clearly $[t-\delta_1, t+\delta_1] \cap [0,\infty)$ is compact. Thus by condition (ii) in Definition 4.3 there is a $\delta_2 > 0$ with the property that $\|v - \tilde{v}\|_U \leq \delta_2$ implies $\|\varphi(s,v) - \varphi(s,\tilde{v})\|_X \leq \frac{\varepsilon}{2}$ for all $s \in [t-\delta_1, t+\delta_1] \cap [0,\infty)$ and $\tilde{v} \in U$.

Set $\delta := \min\{\delta_1, \delta_2\}$. Let $s \geq 0$ and $\tilde{v} \in U$ be such that $|t-s| + \|v - \tilde{v}\|_U \leq \delta$. Then obviously $|t-s| \leq \delta_1$ and $\|v-\tilde{v}\|_U \leq \delta_2$, so that

$$\|\varphi(t,v) - \varphi(s,\tilde{v})\|_X \leq \|\varphi(t,v) - \varphi(s,v)\|_X + \|\varphi(s,v) - \varphi(s,\tilde{v})\|_X \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \qquad \square$$

In the next lemma we prove exponential boundedness for $\varphi(\,\cdot\,, v)$ for fixed $v \in U$. This guarantees the existence of the Laplace transform of $\varphi(\,\cdot\,, v)$. For the needed facts on the Laplace transform of vector-valued functions we refer to Appendix B. We first state straightforward consequences of the above corollary.

Remark 4.6. Due to the continuity of φ, for all $v, \tilde{v} \in U$ the sets $\{\|\varphi(\sigma,v)\| \,|\, \sigma \in [0,1]\}$ and $\{\|\varphi(\sigma,v) - \varphi(\sigma,\tilde{v})\| \,|\, \sigma \in [0,1]\}$ are bounded. Hence the constants $L(v)$ and $L(v,\tilde{v})$ in the following lemma are finite. The continuity of φ also implies that the set $\{L(v) \,|\, v \in K\}$ is bounded for every compact subset $K \subseteq U$. Finally, from condition (ii) of Definition 4.3 we deduce that $L(v,\tilde{v}) \to 0$ as $\|v - \tilde{v}\|_U \to 0$. ♢

Lemma 4.7. *Let $\omega > 0$ and $M \geq 1$ be such that $\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. Then for all $v, \tilde{v} \in U$ and $t \geq 0$ we have*

$$\|\varphi(t,v)\|_X \leq L(v)\mathrm{e}^{\omega t} \qquad \textit{and} \qquad \|\varphi(t,v) - \varphi(t,\tilde{v})\|_X \leq L(v,\tilde{v})\mathrm{e}^{\omega t}$$

with constants (both depending on ω and M)

$$\begin{aligned} L(v) &:= M\Big(\sup_{\sigma\in[0,1]} \|\varphi(\sigma,v)\|_X + \|\varphi(1,v)\|_X \frac{\mathrm{e}^{\omega}}{\mathrm{e}^{\omega}-1}\Big) \;\leq\; M\frac{2\mathrm{e}^{\omega}-1}{\mathrm{e}^{\omega}-1} \sup_{\sigma\in[0,1]} \|\varphi(\sigma,v)\|_X, \\ L(v,\tilde{v}) &:= M\Big(\sup_{\sigma\in[0,1]} \|\varphi(\sigma,v) - \varphi(\sigma,\tilde{v})\|_X + \|\varphi(1,v) - \varphi(1,\tilde{v})\|_X \frac{\mathrm{e}^{\omega}}{\mathrm{e}^{\omega}-1}\Big) \\ &\leq M\frac{2\mathrm{e}^{\omega}-1}{\mathrm{e}^{\omega}-1} \sup_{\sigma\in[0,1]} \|\varphi(\sigma,v) - \varphi(\sigma,\tilde{v})\|_X. \end{aligned}$$

Proof. We only prove the second estimate, as the first can be shown analogously. Let $v, \tilde{v} \in U$. Using (4.2) in an easy induction, we derive

$$\begin{aligned} \varphi(n,v) - \varphi(n,\tilde{v}) &= \mathbb{T}_{n-1}(\varphi(1,v) - \varphi(1,\tilde{v})) + \varphi(n-1,v) - \varphi(n-1,\tilde{v}) \\ &= \sum_{k=1}^{n} \mathbb{T}_{n-k}(\varphi(1,v) - \varphi(1,\tilde{v})) = \sum_{k=0}^{n-1} \mathbb{T}_{k}(\varphi(1,v) - \varphi(1,\tilde{v})) \end{aligned}$$

for all $n \in \mathbb{N}$. The exponential boundedness of $\|\mathbb{T}_t\|$ thus yields

$$\begin{aligned} \|\varphi(n,v) - \varphi(n,\tilde{v})\|_X &\leq M\Big(\sum_{k=0}^{n-1} \mathrm{e}^{\omega k}\Big)\|\varphi(1,v) - \varphi(1,\tilde{v})\|_X = M\frac{\mathrm{e}^{\omega n}-1}{\mathrm{e}^{\omega}-1}\|\varphi(1,v) - \varphi(1,\tilde{v})\|_X \\ &\leq M\frac{\mathrm{e}^{\omega}}{\mathrm{e}^{\omega}-1}\|\varphi(1,v) - \varphi(1,\tilde{v})\|_X\, \mathrm{e}^{\omega(n-1)}. \end{aligned}$$

Let $t \in [0, \infty)$ and take the integer $n \in \mathbb{N}$ with $t \in [n-1, n)$. Equation (4.2) then yields

$$\varphi(n, v) - \varphi(n, \tilde{v}) = \mathbb{T}_t \varphi(n-t, v) + \varphi(t, v) - \mathbb{T}_t \varphi(n-t, \tilde{v}) - \varphi(t, \tilde{v}),$$

or equivalently

$$\varphi(t, v) - \varphi(t, \tilde{v}) = \mathbb{T}_t(\varphi(n-t, \tilde{v}) - \varphi(n-t, v)) + \varphi(n, v) - \varphi(n, \tilde{v}).$$

Since $n - t \in (0, 1]$, we now obtain the claimed estimate

$$\begin{aligned} \|\varphi(t, v) - \varphi(t, \tilde{v})\|_X &\leq \|\mathbb{T}_t\| \|\varphi(n-t, \tilde{v}) - \varphi(n-t, v)\|_X + \|\varphi(n, v) - \varphi(n, \tilde{v})\|_X \\ &\leq M e^{\omega t} \sup_{\sigma \in [0,1]} \|\varphi(\sigma, v) - \varphi(\sigma, \tilde{v})\|_X + \frac{M e^{\omega}}{e^{\omega} - 1} \|\varphi(1, v) - \varphi(1, \tilde{v})\|_X e^{\omega(n-1)} \\ &\leq L(v, \tilde{v}) e^{\omega t}. \end{aligned}$$

□

From the last result we deduce that for each $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \max\{\omega_0(\mathbb{T}), 0\}$ and every $v \in U$ the Laplace transform

$$\widehat{\varphi}(\lambda, v) := (\varphi(\,\cdot\,, v))\widehat{\ }(\lambda)$$

converges absolutely. On the other hand, all complex numbers λ with $\operatorname{Re} \lambda > \max\{\omega_0(\mathbb{T}), 0\}$ belong to $\rho(A)$, where A is the generator of $\mathbb{T}$. Recall from Appendix A that we may assume that $\lambda - A \in \mathcal{L}(X, X_{-1})$ is isometric for any fixed $\lambda \in \rho(A)$.

Proposition 4.8. *The function $\varphi(\,\cdot\,, v) : [0, \infty) \to X$ is continuously differentiable with respect to $\|\,\cdot\,\|_{-1}$ on X. The derivative is given by*

$$\partial_1 \varphi(t, v) = \mathbb{T}_t \partial_1 \varphi(0, v) = \mathbb{T}_t \lambda(\lambda - A) \widehat{\varphi}(\lambda, v), \tag{4.3}$$

where $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \max\{\omega_0(\mathbb{T}), 0\}$ can be chosen freely.

Proof. Let $v \in U$. Take $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \max\{\omega_0(\mathbb{T}), 0\}$. We write (4.2) in the form

$$\tfrac{1}{\tau}(\varphi(t + \tau, v) - \varphi(t, v)) = \mathbb{T}_t \tfrac{1}{\tau} \varphi(\tau, v) \quad \text{for } t \geq 0, \tau > 0. \tag{4.4}$$

On this equation we apply the Laplace transform with respect to t. The operational properties Lemma B.8 and Proposition B.15 yield

$$\begin{aligned} R(\lambda, A) \tfrac{1}{\tau} \varphi(\tau, v) &= \tfrac{1}{\tau} e^{\lambda \tau} \widehat{\varphi}(\lambda, v) - e^{\lambda \tau} \frac{1}{\tau} \int_0^{\tau} e^{\lambda s} \varphi(s, v) \, ds - \tfrac{1}{\tau} \widehat{\varphi}(\lambda, v) \\ &= \tfrac{1}{\tau}(e^{\lambda \tau} - 1) \widehat{\varphi}(\lambda, v) - e^{\lambda \tau} \frac{1}{\tau} \int_0^{\tau} e^{\lambda s} \varphi(s, v) \, ds\,. \end{aligned}$$

Because $\varphi(\,\cdot\,, v)$ is continuous at 0 and $\varphi(0, v) = 0$, the right-hand side converges to

$$\lambda \widehat{\varphi}(\lambda, v) - e^{\lambda 0} \varphi(0, v) = \lambda \widehat{\varphi}(\lambda, v)$$

with respect to $\|\,\cdot\,\|_X$ as $\tau \to 0^+$. Thus also the left-hand side converges as $\tau \to 0^+$. We obtain

$$\begin{aligned} \|R(\lambda, A) \tfrac{1}{\tau} \varphi(\tau, v) - \lambda \widehat{\varphi}(\lambda, v)\|_X &= \|R(\lambda, A)\big(\tfrac{1}{\tau} \varphi(\tau, v) - \lambda(\lambda - A) \widehat{\varphi}(\lambda, v)\big)\|_X \\ &\lesssim \|\tfrac{1}{\tau} \varphi(\tau, v) - \lambda(\lambda - A) \widehat{\varphi}(\lambda, v)\|_{-1} \to 0, \quad \text{as } \tau \to 0^+. \end{aligned}$$

This shows that $\varphi(\,\cdot\,,v)$ is differentiable at $t=0$ with respect to $\|\,\cdot\,\|_{-1}$ on X, and that $\partial_1\varphi(0,v)=\lambda(\lambda-A)\widehat{\varphi}(\lambda,v)$. From equation (4.4) we deduce that $\varphi(\,\cdot\,,v)$ is differentiable from the right at every $t\geq 0$ (with respect to $\|\,\cdot\,\|_{-1}$ on X) and that (4.3) is valid. The differentiability from the left at $t>0$ can be seen by the inequality

$$\begin{aligned}&\|\tfrac{1}{\tau}(\varphi(t,v)-\varphi(t-\tau,v))-\mathbb{T}_t\partial_1\varphi(0,v)\|_{-1}\\&\quad=\|\tfrac{1}{\tau}\mathbb{T}_{t-\tau}\varphi(\tau,v)+\tfrac{1}{\tau}\varphi(t-\tau,v)-\tfrac{1}{\tau}\varphi(t-\tau,v)-\mathbb{T}_t\partial_1\varphi(0,v)\|_{-1}\\&\quad\leq\|\mathbb{T}_{t-\tau}\|\|\tfrac{1}{\tau}\varphi(\tau,v)-\partial_1\varphi(0,v)\|_{-1}+\|\mathbb{T}_{t-\tau}\partial_1\varphi(0,v)-\mathbb{T}_t\partial_1\varphi(0,v)\|_{-1},\end{aligned}$$

where we used (4.2). The right-hand side converges to 0 as $\tau\to 0^+$. □

Since $\partial_1\varphi(0,v)$ exists in X_{-1} for every $v\in U$, it defines a map

$$B:U\to X_{-1};\quad B(v)=\partial_1\varphi(0,v). \tag{4.5}$$

The following theorem states (among other things) that B represents Φ. It is the main result of this section.

Theorem 4.9. *Let X and U be Banach spaces and Ω a Fréchet domain. Let $(\mathbb{T},\Phi)$ be an additive control system on X and Ω. Then there is a unique continuous map $B:U\to X_{-1}$ such that for all step function $u\in\Omega_0$ we have*

$$\Phi_t(u)=\int_0^t\mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s\quad\textit{for all } t\geq 0. \tag{4.6}$$

Proof. As indicated, $B:U\to X_{-1}$ is the map defined in (4.5). The first step is to prove that B is continuous. Fix $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\max\{\omega_0(\mathbb{T}),0\}$. Proposition 4.8 then yields $B(v)=\lambda(\lambda-A)\widehat{\varphi}(\lambda,v)$ for $v\in U$ and therefore

$$\|B(v)-B(\tilde{v})\|_{-1}\leq|\lambda|\|\lambda-A\|_{\mathcal{L}(X,X_{-1})}\|\widehat{\varphi}(\lambda,v)-\widehat{\varphi}(\lambda,\tilde{v})\|_X\quad\text{for all } v,\tilde{v}\in U.$$

Hence it suffices to show that the map $U\to X;\ v\mapsto\widehat{\varphi}(\lambda,v)$ is continuous.

Let $\varepsilon>0$. Choose $\omega>0$ and $M\geq 1$ with $\|\mathbb{T}_t\|\leq M\mathrm{e}^{\omega t}$ for all $t\geq 0$. Let $v,\tilde{v}\in U$. With Lemma 4.7 we infer

$$\begin{aligned}\|\widehat{\varphi}(\lambda,v)-\widehat{\varphi}(\lambda,\tilde{v})\|_X&\leq\int_0^\infty\mathrm{e}^{-\operatorname{Re}\lambda t}\|\varphi(t,v)-\varphi(t,\tilde{v})\|_X\,\mathrm{d}t\\&\leq\int_0^\infty\mathrm{e}^{-(\operatorname{Re}\lambda-\omega)t}\,\mathrm{d}t\,M\,\frac{2\mathrm{e}^\omega-1}{\mathrm{e}^\omega-1}\sup_{\sigma\in[0,1]}\|\varphi(\sigma,v)-\varphi(\sigma,\tilde{v})\|_X\\&=\frac{M}{\operatorname{Re}\lambda-\omega}\,\frac{2\mathrm{e}^\omega-1}{\mathrm{e}^\omega-1}\sup_{\sigma\in[0,1]}\|\varphi(\sigma,v)-\varphi(\sigma,\tilde{v})\|_X\,.\end{aligned}$$

By condition (ii) in Definition 4.3 there is a $\delta>0$ such that

$$\sup_{\sigma\in[0,1]}\|\varphi(\sigma,v)-\varphi(\sigma,\tilde{v})\|_X\leq\varepsilon\frac{\operatorname{Re}\lambda-\omega}{M}\,\frac{\mathrm{e}^\omega-1}{2\mathrm{e}^\omega-1},$$

provided that $\|v-\tilde{v}\|_U\leq\delta$. It thus follows that $B:U\to X_{-1}$ is continuous, since

$$\|\widehat{\varphi}(\lambda,v)-\widehat{\varphi}(\lambda,\tilde{v})\|_X\leq\varepsilon\quad\text{for all } v,\tilde{v}\in U \text{ with } \|v-\tilde{v}\|_U\leq\delta.$$

Next we prove (4.6) for step functions u. First recall from Proposition 4.8 that $\varphi(\,\cdot\,, v)$ is continuously differentiable for all $v \in U$. From the equations (4.3) and (4.5) we deduce

$$\varphi(t,v) = \varphi(t,v) - \varphi(0,v) = \int_0^t \partial_1\varphi(s,v)\,\mathrm{d}s = \int_0^t \mathbb{T}_s B(v)\,\mathrm{d}s \quad \text{for all } t \geq 0.$$

Let $v \in U$. The causality of Φ implies that

$$\Phi_t(\chi_v) = \Phi_t(P_t\chi_v) = \varphi(t,v) = \int_0^t \mathbb{T}_s B(v)\,\mathrm{d}s = \int_0^t \mathbb{T}_{t-s}B(\chi_v(s))\,\mathrm{d}s \quad \text{for all } t \geq 0. \tag{4.7}$$

Let $u \in \Omega_0$ be a step function. It has the form $u = \sum_{k=1}^m \mathbb{1}_{[t_{k-1},t_k)} v_k$, for some $m \in \mathbb{N}$, $0 = t_0 < \ldots < t_m < \infty$ and $v_1, \ldots, v_m \in U$. The representation (4.6) is proved by an induction over m.

First, let $m = 1$, i.e., $u = P_{t_1}\chi_{v_1}$. We write $\tau := t_1$ and $v := v_1$. For $t \leq \tau$ the causality of Φ implies that $\Phi_t(u) = \Phi_t(P_t P_\tau \chi_v) = \Phi_t(P_t\chi_v) = \varphi(t,v)$. Thus (4.6) for $0 \leq t \leq \tau$ follows from (4.7). Let $t > \tau$. Note that $S_\tau^* P_\tau \chi_v = \chi_0$. The composition property (4.1), equation (4.7) and a change of variables yield

$$\begin{aligned}
\Phi_t(u) &= \mathbb{T}_{t-\tau}\Phi_\tau(P_\tau\chi_v) + \Phi_{t-\tau}(S_\tau^* P_\tau \chi_v) = \mathbb{T}_{t-\tau}\varphi(\tau,v) + \varphi(t-\tau,0) \\
&= \mathbb{T}_{t-\tau}\int_0^\tau \mathbb{T}_{\tau-s}B(u(s))\,\mathrm{d}s + \int_0^{t-\tau} \mathbb{T}_{t-(\tau+s)}B(u(\tau+s))\,\mathrm{d}s \\
&= \int_0^\tau \mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s + \int_\tau^t \mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s = \int_0^t \mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s.
\end{aligned}$$

The reduction from $m+1$ to m works pretty similar. Let $u = \sum_{k=1}^{m+1} \mathbb{1}_{[t_{k-1},t_k)} v_k$ and set $\tilde{u} = \sum_{k=1}^m \mathbb{1}_{[t_{k-1},t_k)} v_k$. If $t \leq t_m$, then $P_t u = P_t \tilde{u}$, so we are in the case m.

If $t > t_m$, the composition property (4.1) and the causality lead to the equation $\Phi_t(u) = \mathbb{T}_{t-t_m}\Phi_{t_m}(P_{t_m}u) + \Phi_{t-t_m}(S_{t_m}^* u)$. Since $P_{t_m}u = P_{t_m}\tilde{u}$ and $S_{t_m}^* u$ has only one "step", from the induction hypothesis we infer

$$\mathbb{T}_{t-t_m}\Phi_{t_m}(u) = \mathbb{T}_{t-t_m}\int_0^{t_m} \mathbb{T}_{t_m-s}B(u(s))\,\mathrm{d}s = \int_0^{t_m} \mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s$$

as well as

$$\Phi_{t-t_m}(u) = \int_0^{t-t_m} \mathbb{T}_{t-t_m-s}B(u(s+t_m))\,\mathrm{d}s = \int_{t_m}^t \mathbb{T}_{t-s}B(u(s))\,\mathrm{d}s.$$

The claimed identity for $\Phi_t(u)$ now is obvious.

Finally, B is unique due to the fact that $B(v)$ is the derivative at $t = 0$ (in X_{-1}) of $\Phi_t(\chi_v) = \int_0^t \mathbb{T}_s B_0(v)\,\mathrm{d}s = \varphi(t,v)$ seen as a function of t. □

Definition 4.10. Let X and U be Banach spaces and let Ω be a Fréchet domain. Moreover, let $(\mathbb{T}, \Phi)$ be an additive control system on X and Ω. The map $B : U \to X_{-1}$ from Theorem 4.9 is the *control operator associated to* $(\mathbb{T}, \Phi)$.

Remark 4.11. Let $(\mathbb{T}, \Phi)$ and B be as in the last definition.

(a) It is easy to see that B is linear if Φ_t is linear for every $t \geq 0$. Indeed, this follows from the fact that for $v \in U$ the value $B(v)$ is the limit of $\frac{1}{\tau}\Phi_\tau\chi_v$ as $\tau \to 0^+$. Because B is continuous, it belongs to $\mathcal{L}(U, X_{-1})$ in this case.

(b) Let $(\mathbb{T}, \Phi^{(1)})$ and $(\mathbb{T}, \Phi^{(2)})$ be two additive control systems on X and Ω with the associated control operators $B_1, B_2 : U \to X_{-1}$. For fixed $\alpha_1, \alpha_2 \in \mathbb{C}$ and all $t \geq 0$ set

$$\Phi_t := \alpha_1 \Phi_t^{(1)} + \alpha_2 \Phi_t^{(2)}.$$

By a straightforward calculation one verifies that $(\mathbb{T}, (\Phi_t)_{t\geq 0})$ is an additive control systems on X and Ω and that the associated control operator is $\alpha_1 B_1 + \alpha_2 B_2$. $\Diamond$

4.2 Admissible control operators

Let $(\mathbb{T}, \Phi)$ be an additive control system on X and a Fréchet domain Ω. Further let A be the generator of $\mathbb{T}$ and let B be the control operator associated to $(\mathbb{T}, \Phi)$. For $u \in \Omega_0$ and $x_0 \in X$ consider the function $z : [0, \infty) \to X$ given by

$$z(t) := \mathbb{T}_t x_0 + \Phi_t(u) \quad \text{for } t \geq 0. \tag{4.8}$$

Obviously Bu is a piecewise constant function with values in X_{-1}. Thus it is locally integrable. Using representation formula (4.6), we see that z is the mild solution of

$$z'(t) = Az(t) + B(u(t)); \quad z(0) = x_0. \tag{4.9}$$

Actually, we want z to be the strong solution in X of (4.9). Moreover, we want this to be true not only for step functions. To this end, a topology on Ω is needed. In this section and the following, we work with the Fréchet domain $\Omega = L^p([0, \infty), U)$ where $p \in [1, \infty)$. We exclude $p = \infty$ because Ω_0 is not dense in $L^\infty([0, \infty), U)$. The density of Ω_0 in $L^p([0, \infty), U)$ is a crucial point in the proof of Proposition 4.17. We abbreviate $\|\,\cdot\,\|_{L^p} := \|\,\cdot\,\|_{L^p([0,\infty),U)}$.

The linear operator $S_\tau : L^p([0, \infty), U) \to L^p([0, \infty), U)$ is isometric. From the dominated convergence theorem one infers that for all $u \in L^p([0, \infty), U)$ we have

$$\|P_\tau u\|_{L^p} \to 0 \;\text{ and }\; \|S_\tau^* u - u\|_{L^p} \to 0 \quad \text{as } \tau \to 0^+.$$

These facts are needed in Lemma 4.14 below. Observe that $\|S_\tau^* u - u\|_{L^\infty} \to 0$ as $\tau \to 0^+$ if and only if u is uniformly continuous. This is another reason for not working with $\|\,\cdot\,\|_{L^\infty}$.

We have to add another condition to Definition 4.3. In the remark below we make clear that this is the concept which should be compared with the linear theory.

Definition 4.12. Let X and U be Banach spaces and $p \in [1, \infty)$. A *continuous additive control system on X and $L^p([0, \infty), U)$* is an additive control system $(\mathbb{T}, \Phi)$ on X and $L^p([0, \infty), U)$ with the additional property that $\Phi_t : L^p([0, \infty), U) \to X$ is continuous for every $t \geq 0$.

An additive control system on X and $L^p([0, \infty), U)$ is called *equicontinuous* if for all $T > 0$ the family $\{\Phi_t : L^p([0, \infty), U) \to X \mid t \in [0, T]\}$ is equicontinuous. It is called *Lipschitz* (*on bounded sets*) if Φ_t is Lipschitz (on bounded sets) for every $t \geq 0$.

Remark 4.13. Let us check that this definition is compatible with the definition of a linear control systems given by G. Weiss in Definition 2.1 of [52], also see page 17 of this thesis. Let $(\mathbb{T}, \Phi)$ be a linear control system on X and $L^p([0, \infty), U)$. In Proposition 2.3 of [52] it was shown that the mapping $[0, \infty) \times L^p([0, \infty), U) \to X$; $(t, u) \mapsto \Phi_t u$ is continuous. From the composition property (4.1) it follows easily that the operator norms $\|\Phi_t\|$ are non-decreasing in t. We deduce properties (i) and (ii) of Definition 4.3 and that $(\mathbb{T}, \Phi)$ is equicontinuous.

Conversely let $(\mathbb{T},\Phi)$ be a continuous additive control systems on X and $L^p([0,\infty),U)$ and assume that the input maps Φ_t are linear for all $t \geq 0$. Because the input maps are continuous we have $\Phi_t \in \mathcal{L}(L^p([0,\infty),U),X)$. Hence $(\mathbb{T},\Phi)$ satisfies Definition 2.1 of [52]. $\Diamond$

We consider z from (4.8) but with general $u \in L^p([0,\infty),U)$. In order to be the strong solution of (4.9) in X, the function z has to belong to $C([0,\infty),X)$, see Theorem 3.5. Since $[0,\infty) \to X$; $t \mapsto \mathbb{T}_t x_0$ is continuous, we only have to check that $t \mapsto \Phi_t(u)$ lies in $C([0,\infty),X)$.

Lemma 4.14. *Let $(\mathbb{T},\Phi)$ be a continuous additive control system on X and $L^p([0,\infty),U)$. Then for all $u \in L^p([0,\infty),U)$ the map $[0,\infty) \to X; t \mapsto \Phi_t(u)$ is continuous.*

Proof. Let $u \in L^p([0,\infty),U)$. We first prove right continuity of $t \mapsto \Phi_t(u)$ at $t = 0$. Take $\tau \in [0,1]$. From the composition property (4.1) we derive

$$\Phi_1(S_{1-\tau}u) = \mathbb{T}_\tau \Phi_{1-\tau}(P_{1-\tau}S_{1-\tau}u) + \Phi_\tau(S^*_{1-\tau}S_{1-\tau}u) = \mathbb{T}_\tau\Phi_{1-\tau}(\chi_0) + \Phi_\tau(u).$$

Therefore we can estimate

$$\begin{aligned}\|\Phi_\tau(u)\|_X &= \|\Phi_1(S_{1-\tau}u) - \mathbb{T}_\tau\Phi_{1-\tau}(\chi_0)\|_X = \|\Phi_1(P_1S_{1-\tau}u) - \mathbb{T}_\tau\Phi_{1-\tau}(\chi_0)\|_X \\ &\leq \|\Phi_1(P_1S_{1-\tau}u) - \Phi_1(\chi_0)\|_X + \|\mathbb{T}_\tau\Phi_{1-\tau}(\chi_0) - \Phi_1(\chi_0)\|_X.\end{aligned}$$

One easily sees the identity $P_1S_{1-\tau}u = S_{1-\tau}P_\tau u$. Since $S_{1-\tau}$ is isometric, the norms $\|S_{1-\tau}P_\tau u\|_{L^p} = \|P_\tau u\|_{L^p}$ converge to zero as $\tau \to 0^+$. Thus the continuity of Φ_1 implies

$$\|\Phi_1(P_1S_{1-\tau}u) - \Phi_1(\chi_0)\|_X \to 0 \quad \text{as } \tau \to 0^+.$$

Recall from Lemma 4.4 that $[0,\infty) \to X$; $s \mapsto \Phi_s(\chi_0)$ is continuous. From the boundedness of $\|\mathbb{T}_\sigma\|$ for $\sigma \in [0,1]$ we then deduce that

$$\begin{aligned}\|\mathbb{T}_\tau\Phi_{1-\tau}(\chi_0) - \Phi_1(\chi_0)\|_X &\leq \|\mathbb{T}_\tau\|\|\Phi_{1-\tau}(\chi_0) - \Phi_1(\chi_0)\|_X + \|\mathbb{T}_\tau\Phi_1(\chi_0) - \Phi_1(\chi_0)\|_X \\ &\leq m_{\mathbb{T},1}\,\|\Phi_{1-\tau}(\chi_0) - \Phi_1(\chi_0)\|_X + \|\mathbb{T}_\tau\Phi_1(\chi_0) - \Phi_1(\chi_0)\|_X\end{aligned}$$

converges to zero as $\tau \to 0^+$.

Next we show right- and left continuity. To this end, let $t > 0$. Using the composition property (4.1), we write

$$\Phi_{t+\tau}(u) - \Phi_t(u) = \mathbb{T}_\tau\Phi_t(u) - \Phi_t(u) + \Phi_\tau(S^*_t u)$$

for arbitrary $\tau \geq 0$. The strong continuity of $\mathbb{T}$ implies that the term $\|\mathbb{T}_\tau\Phi_t(u) - \Phi_t(u)\|_X$ converges to zero as $\tau \to 0^+$. The first step yields $\|\Phi_\tau(S^*_t u)\|_X \to 0$ as $\tau \to 0^+$.

For $\tau \in [0,t]$ from (4.1) we derive $\Phi_t(S_\tau u) = \mathbb{T}_{t-\tau}\Phi_\tau(\chi_0) + \Phi_{t-\tau}(u)$. It follows that

$$\begin{aligned}\|\Phi_t(u) - \Phi_{t-\tau}(u)\|_X &\leq \|\Phi_t(u) - \Phi_t(S_\tau u)\|_X + \|\mathbb{T}_{t-\tau}\Phi_\tau(\chi_0)\|_X \\ &\leq \|\Phi_t(u) - \Phi_t(S_\tau u)\|_X + m_{\mathbb{T},t}\|\Phi_\tau(\chi_0)\|_X.\end{aligned}$$

As argued before, the expression $\|\Phi_\tau(\chi_0)\|_X$ tends to zero as $\tau \to 0^+$. Since S_τ is isometric, we further have

$$\|u - S_\tau u\|_{L^p} = \|P_\tau u + S_\tau S^*_\tau u - S_\tau u\|_{L^p} \leq \|P_\tau u\|_{L^p} + \|S^*_\tau u - u\|_{L^p} \to 0 \quad \text{as } \tau \to 0^+.$$

Using the continuity of Φ_t we thus derive $\|\Phi_t(u) - \Phi_t(S_\tau u)\|_X \to 0$ as $\tau \to 0^+$. $\square$

Analog to Corollary 4.5 we see that $(t,u) \mapsto \Phi_t(u)$ is continuous. Since the proof is very similar, we omit it.

Corollary 4.15. *To the conditions of Lemma 4.14 add the assumption that $(\mathbb{T},\Phi)$ is equicontinuous. Then the mapping $[0,\infty) \times L^p([0,\infty),U) \to X$; $(t,u) \mapsto \Phi_t(u)$ is continuous.*

As intended, we can now check that $\mathbb{T}$ and Φ are the solution operators for the problem (4.9) determined by the generator A and the control operator B.

Proposition 4.16. *Assume that $(\mathbb{T},\Phi)$ is a continuous additive control system on X and $L^p([0,\infty),U)$. Further let $u \in L^p([0,\infty),U)$ satisfy (4.6) and let $x_0 \in X$. Then the function $z : [0,\infty) \to X$; $z(t) = \mathbb{T}_t x_0 + \Phi_t(u)$ is the strong solution in X of (4.9).*

Proof. Let $u \in L^p([0,\infty),U)$ and $x_0 \in X$ be as in the claim. Since (4.6) holds for u, the function z is the mild solution of (4.9). From Lemma 4.14 we deduce that z belongs to $C([0,\infty),X)$. Theorem 3.5 yields that z is the strong solution of (4.9) in X. □

Thanks to Theorem 4.9 we can apply the above result to step functions $u \in \Omega_0$. Next we show that polynomial boundedness of the control operator B (with an exponent $\eta \in [1,p]$) is a sufficient condition for (4.6) to hold for all $u \in L^p([0,\infty),U)$. But first recall from Proposition 1.3.4 in [5] that, if $Bu \in L^1_{\mathrm{loc}}([0,\infty),X_{-1})$ for some $u \in L^p([0,\infty),U)$, then the integral

$$\int_0^t \mathbb{T}_{t-s} B(u(s))\,\mathrm{d}s$$

exists for all $t \geq 0$ and defines a continuous function in t with values in X_{-1}.

Proposition 4.17. *Let $B \in C(U,X_{-1})$ be the control operator of the continuous additive control system $(\mathbb{T},\Phi)$ on X and $L^p([0,\infty),U)$. Moreover, assume that there are $\eta \in [1,p]$ and $c \geq 0$ such that $\|B(v)\|_{-1} \leq c(1+\|v\|_U^\eta)$ for all $v \in U$. Then (4.6) is satisfied for all $u \in L^p([0,\infty),U)$.*

Proof. In a *first step* we prove that $Bu \in L^1_{\mathrm{loc}}([0,\infty),X_{-1})$ for all $u \in L^p([0,\infty),U)$. Thus let $u \in L^p([0,\infty),U)$. Since $B : U \to X_{-1}$ is continuous and $u : [0,\infty) \to U$ is measurable, also the map $Bu : [0,\infty) \to X_{-1}$ is measurable. Further, the growth bound of B yields

$$\|B(u(s))\|_{-1}^{\frac{p}{\eta}} \leq c^{\frac{p}{\eta}}(1+\|u(s)\|_U^\eta)^{\frac{p}{\eta}} \leq (2c)^{\frac{p}{\eta}} \max\{1,\|u(s)\|_U^p\} \quad \text{for all } s \in [0,\infty).$$

Clearly, for fixed $t \geq 0$ the map $[0,t] \to \mathbb{R}$; $s \mapsto \max\{1,\|u(s)\|_U^p\}$ belongs to $L^1([0,t],\mathbb{R})$. With Bochner's theorem we derive that $s \mapsto B(u(s)) \in L^{\frac{p}{\eta}}([0,t],X_{-1})$ for every $t \geq 0$. This means that Bu lies in $L^1_{\mathrm{loc}}([0,\infty),X_{-1})$.

The *second step* is to actually verify (4.6). Since $Bu \in L^1_{\mathrm{loc}}([0,\infty),X_{-1})$ the integral

$$\int_0^t \mathbb{T}_{t-s} B(u(s))\,\mathrm{d}s$$

exists for all $t \geq 0$ and we have to show that it equals $\Phi_t(u)$. To this end, take a sequence (u_n) in Ω_0 with $\|u_n - u\|_{L^p} \to 0$ as $n \to \infty$. After passing to a subsequence, we may assume that (u_n) converges to u pointwise almost everywhere on $[0,\infty)$ and that we can find a function $g \in L^p([0,\infty),\mathbb{R})$ with $\|u_n(s)\|_U \leq g(s)$ for almost all $s \in [0,\infty)$ and $n \in \mathbb{N}$. This fact is a corollary to Theorem VI.5.2 of [25]. Again fix $t \in [0, \infty)$. If we can show that the right-hand side of

$$\left\| \Phi_t(u) - \int_0^t \mathbb{T}_{t-s} B(u(s))\, \mathrm{d}s \right\|_{-1} \le \|\Phi_t(u_n) - \Phi_t(u)\|_{-1} + \left\| \Phi_t(u_n) - \int_0^t \mathbb{T}_{t-s} B(u(s))\, \mathrm{d}s \right\|_{-1}$$

can be made arbitrarily small, then the claim is shown.

On the one hand, the continuity of Φ_t implies that $\|\Phi_t(u_n) - \Phi_t(u)\|_X \to 0$ and thus also $\|\Phi_t(u_n) - \Phi_t(u)\|_{-1} \to 0$ as $n \to \infty$. On the other hand (4.6) is valid for the inputs u_n, because they are piecewise constant. Therefore we can estimate

$$\begin{aligned} \left\| \Phi_t(u_n) - \int_0^t \mathbb{T}_{t-s} B(u(s))\, \mathrm{d}s \right\|_{-1} &= \left\| \int_0^t \mathbb{T}_{t-s} \big(B(u_n(s)) - B(u(s))\big)\, \mathrm{d}s \right\|_{-1} \\ &\le m_{\mathbb{T},t} \int_0^t \|B(u_n(s)) - B(u(s))\|_{-1}\, \mathrm{d}s \end{aligned}$$

for all $n \in \mathbb{N}$. With the help of the dominated convergence theorem, we check that the right-hand side converges to zero as $n \to \infty$. First the integrand $\|B(u_n(s)) - B(u(s))\|_{-1}$ converges to zero as $n \to \infty$ for almost all $s \in [0, t]$. This is true because B is continuous and the functions u_n converge to u pointwise almost everywhere on $[0, t]$. Moreover, we have

$$\|B(u_n(s)) - B(u(s))\|_{-1} \le c(1 + \|u_n(s)\|_U^\eta) + \|B(u(s))\|_{-1} \le c + c(g(s))^\eta + \|B(u(s))\|_{-1}$$

for $s \in [0, t]$. Clearly Hölder's inequality implies that $s \mapsto g(s)$ belongs to $L^\eta([0, t], \mathbb{R}) \subseteq L^1([0, t], \mathbb{R})$. In the first step we already showed that $s \mapsto \|B(u(s))\|_{-1} \in L^1([0, t], \mathbb{R})$. Hence the right-hand side above lies in $L^1([0, t], \mathbb{R})$. □

We can characterize polynomial boundedness of B in terms of an estimate for φ.

Lemma 4.18. *As in Proposition 4.17, let $B \in C(U, X_{-1})$ be the control operator of the continuous additive control system $(\mathbb{T}, \Phi)$ on X and $L^p([0, \infty), U)$. Let $\omega \in \mathbb{R}$ and $M \ge 1$ be such that $\|\mathbb{T}_t\| \le M e^{\omega t}$ for all $t \ge 0$. Then for $\eta \ge 1$ the following assertions are equivalent*

(i) $\exists c \ge 0\ \forall v \in U: \ \|B(v)\|_{-1} \le c(1 + \|v\|_U^\eta)$.

(ii) $\exists c \ge 0\ \forall v \in U, t \ge 0: \ \|\varphi(t, v)\|_{-1} \le c\frac{M}{\omega}(e^{\omega t} - 1)(1 + \|v\|_U^\eta)$.

Proof. Assume that (i) holds. Let $v \in U$ and $t \ge 0$. Since $\varphi(t, v) = \int_0^t \mathbb{T}_s B(v)\, \mathrm{d}s$ and $\|\mathbb{T}_s B(v)\|_{-1} \le M e^{\omega s} \|B(v)\|_{-1}$ for all $s \in [0, \infty)$, we obtain

$$\|\varphi(t, v)\|_{-1} \le M \int_0^t e^{\omega s}\, \mathrm{d}s\, \|B(v)\|_{-1} = \tfrac{M}{\omega}(e^{\omega t} - 1)\|B(v)\|_{-1} \le c\tfrac{M}{\omega}(e^{\omega t} - 1)(1 + \|v\|_U^\eta).$$

On the other hand assume that (ii) is satisfied. Again let $v \in U$. Because $B(v)$ is the derivative of $\varphi(\,\cdot\,, v)$ at 0 with respect to $\|\,\cdot\,\|_{-1}$, we have

$$\|B(v)\|_{-1} = \lim_{\tau \to 0^+} \tfrac{1}{\tau} \|\varphi(\tau, v)\|_{-1} \le c\tfrac{M}{\omega}\Big(\lim_{\tau \to 0^+} \tfrac{1}{\tau}(e^{\omega\tau} - 1)\Big)(1 + \|v\|_U^\eta) = cM(1 + \|v\|_U^\eta). \quad □$$

The preceding results now lead to the definition of an 'admissible control operator' B. For such an operator we expect that for every input u – in $L^p([0, \infty), U)$ say – the differential equation (4.9) has a unique strong solution in X. We saw that it is crucial that B is the control operator of a continuous additive control system on X and $L^p([0, \infty), U)$. Hence the informal definition above can be turned into real assumptions on B as follows.

Definition 4.19. Let X and U be Banach spaces and $p \in [1, \infty)$. Further let $\mathbb{T}$ be a strongly continuous semigroup on X. A continuous map $B : U \to X_{-1}$ is called *L^p–admissible control operator for* $\mathbb{T}$ (or shortly L^p–admissible for $\mathbb{T}$) if $Bu \in L^1_{\mathrm{loc}}([0, \infty), X_{-1})$ for all $u \in L^p([0, \infty), U)$ and the family $\Phi = (\Phi_t)_{t\geq 0}$ of *input maps* $\Phi_t : L^p([0, \infty), U) \to X_{-1}$ given by

$$\Phi_t(u) := \int_0^t \mathbb{T}_{t-s} B(u(s))\, \mathrm{d}s \qquad \text{for } t \geq 0$$

yields a continuous additive control system $(\mathbb{T}, \Phi)$ on X and $L^p([0, \infty), U)$.

We mention that the family $\Phi = (\Phi_t)_{t\geq 0}$ defined above is always causal and satisfies the composition property for $\mathbb{T}$. The first claim is obvious. To verify the second let $t, \tau \geq 0$ and $u \in L^p([0, \infty), U)$. By splitting the integral at τ and a change of variables we obtain

$$\begin{aligned}
\Phi_{t+\tau}(u) &= \int_0^{t+\tau} \mathbb{T}_{t+\tau-s} B(u(s))\, \mathrm{d}s \\
&= \int_0^{\tau} \mathbb{T}_t \mathbb{T}_{\tau-s} B(u(s))\, \mathrm{d}s + \int_\tau^{t+\tau} \mathbb{T}_{t-(s-\tau)} B(u(s-\tau+\tau))\, \mathrm{d}s \\
&= \mathbb{T}_t \int_0^{\tau} \mathbb{T}_{\tau-s} B(u(s))\, \mathrm{d}s + \int_0^{t} \mathbb{T}_{t-s} B(u(s+\tau))\, \mathrm{d}s = \mathbb{T}_t \Phi_\tau(u) + \Phi_t(S^*_\tau u).
\end{aligned}$$

Note that we only used that $Bu : [0, \infty) \to X_{-1}$ is locally integrable for all $u \in L^p([0, \infty), U)$.

Remark 4.20. (a) A linear operator $B \in \mathcal{L}(U, X_{-1})$ is L^p–admissible for $\mathbb{T}$ if and only it is 'p-admissible' as in Definition 4.1 of [52]. This follows from Remark 4.13.

(b) Let $(\mathbb{T}, \Phi)$ be a continuous additive control system on X and $L^p([0, \infty), U)$ and let $B : U \to X_{-1}$ be the associated control operator obtained in Theorem 4.9. Then B is L^p–admissible if $Bu \in L^1_{\mathrm{loc}}([0, \infty), X_{-1})$ for all $u \in L^p([0, \infty), U)$.

(c) It is easy to see that negatives and sums of L^p–admissible operators for the same space U and semigroup $\mathbb{T}$ are again L^p–admissible. Compare this to Remark 4.11 (b). ♢

Properties of continuous additive control systems

At the end of this section we gather additional results. Throughout let $(\mathbb{T}, \Phi)$ be a continuous additive control system on X and $L^p([0, \infty), U)$ for an exponent $p \in [1, \infty)$.

It is a consequence of the composition property that certain assumptions on Φ_t automatically hold uniformly for t in compact subsets of $[0, \infty)$. We say that Φ_t is Lipschitz on bounded sets for all $t \geq 0$ if

$$\begin{aligned}
&\forall \rho > 0, t > 0\ \exists M_{\rho,t} > 0\ \forall u_1, u_2 \in L^p([0, \infty), U) : \\
&\qquad \|u_1\|_{L^p}, \|u_2\|_{L^p} \leq \rho \implies \|\Phi_t(u_1) - \Phi_t(u_2)\|_X \leq M_{\rho,t} \|u_1 - u_2\|_{L^p}.
\end{aligned}$$

Lemma 4.21. *Let $(\mathbb{T}, \Phi)$ be a continuous additive control system on X and $L^p([0, \infty), U)$ and assume that Φ_t is Lipschitz on bounded sets for every $t \geq 0$. Then Φ_t is Lipschitz on bounded sets uniformly for t in compact subsets of $[0, \infty)$, i.e.,*

$$\begin{aligned}
&\forall T > 0, \rho > 0\ \exists M_{\rho,T} > 0\ \forall u_1, u_2 \in L^p([0, \infty), U), t \in [0, T] : \\
&\qquad \|u_1\|_{L^p}, \|u_2\|_{L^p} \leq \rho \implies \|\Phi_t(u_1) - \Phi_t(u_2)\|_X \leq M_{\rho,T} \|u_1 - u_2\|_{L^p}.
\end{aligned}$$

Further the control operator $B \in C(U, X_{-1})$ associated to $(\mathbb{T}, \Phi)$ is Lipschitz on bounded sets.

Proof. Let $T > 0$ and $\rho > 0$ and fix $t \in [0, T]$. For better readability we set $\tau := T - t$, so that $T = t + \tau$. Let $u_1, u_2 \in L^p([0, \infty), U)$. As in the proof of Lemma 4.14, the composition property (4.1) yields

$$\begin{aligned}\Phi_{t+\tau}(S_\tau u_1) - \Phi_{t+\tau}(S_\tau u_2) &= \mathbb{T}_t \Phi_\tau(S_\tau u_1) - \mathbb{T}_t \Phi_\tau(S_\tau u_2) + \Phi_t(u_1) - \Phi_t(u_2)\\ &= \mathbb{T}_t \Phi_\tau(\chi_0) - \mathbb{T}_t \Phi_\tau(\chi_0) + \Phi_t(u_1) - \Phi_t(u_2)\\ &= \Phi_t(u_1) - \Phi_t(u_2).\end{aligned}$$

The linear operator $S_\tau : L^p([0, \infty), U) \to L^p([0, \infty), U)$ is isometric. Hence $\|S_\tau u_j\|_{L^p} \le \rho$ if $\|u_j\|_{L^p} \le \rho$ for $j = 1, 2$ and $\|S_\tau u_1 - S_\tau u_2\|_{L^p} = \|u_1 - u_2\|_{L^p}$. We deduce that

$$\begin{aligned}\|\Phi_t(u_1) - \Phi_t(u_2)\|_X &= \|\Phi_T(S_\tau u_1) - \Phi_T(S_\tau u_2)\|_X\\ &\le M_{\rho,T}\|S_\tau u_1 - S_\tau u_2\|_{L^p} = M_{\rho,T}\|u_1 - u_2\|_{L^p}.\end{aligned}$$

To prove the second statement, take $\omega > \max\{0, \omega_0(\mathbb{T})\}$ and choose $M \ge 1$ such that $\|\mathbb{T}_t\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$. Further let $\lambda > \omega$. We may assume that $\|x\|_{-1} = \|R(\lambda, A)x\|_X$ for $x \in X$. The function $\varphi : [0, \infty) \times U \to X$ was given by $\varphi(t, v) = \Phi_t(P_t\chi_v)$. Recall that the control operator B associated to $(\mathbb{T}, \Phi)$ was defined as

$$B(v) = \lambda(\lambda - A)\widehat{\varphi}(\lambda, v) \quad \text{for all } v \in U.$$

For $v_1, v_2 \in \overline{B}(0, \rho) \subseteq U$ we have $\|P_1\chi_{v_j}\|_{L^p} = \|v_j\|_U$ for $j = 1, 2$. Lemma 4.7 and the first part then yield

$$\begin{aligned}\|\varphi(t, v_1) - \varphi(t, v_2)\|_X &\le \frac{M(2\mathrm{e}^\omega - 1)}{\mathrm{e}^\omega - 1} \sup_{\sigma\in[0,1]} \|\varphi(\sigma, v_1) - \varphi(\sigma, v_2)\|_X \mathrm{e}^{\omega t}\\ &\le \frac{M(2\mathrm{e}^\omega - 1)}{\mathrm{e}^\omega - 1} M_{\rho,1}\|P_1\chi_{v_1} - P_1\chi_{v_2}\|_{L^p}\mathrm{e}^{\omega t}\\ &\le \frac{M(2\mathrm{e}^\omega - 1)}{\mathrm{e}^\omega - 1} M_{\rho,1}\|v_1 - v_2\|_U\mathrm{e}^{\omega t} \quad \text{for all } t \ge 0.\end{aligned}$$

As a consequence, we can estimate

$$\begin{aligned}\|B(v_1) - B(v_2)\|_{-1} &= \lambda\|R(\lambda, A)(\lambda - A)(\varphi(\,\cdot\,, v_1) - \varphi(\,\cdot\,, v_2))\widehat{\ }(\lambda)\|_X\\ &\le \lambda \int_0^\infty \mathrm{e}^{-\lambda t}\|\varphi(t, v_1) - \varphi(t, v_2)\|_X \,\mathrm{d}t\\ &\le \frac{M\lambda(2\mathrm{e}^\omega - 1)}{\mathrm{e}^\omega - 1} M_{\rho,1} \int_0^\infty \mathrm{e}^{(\omega-\lambda)t} \,\mathrm{d}t \; \|v_1 - v_2\|_U.\end{aligned}$$

□

Corollary 4.22. *Under the assumption of the last lemma, the representation* (4.6) *is satisfied for all* $u \in L^p([0, \infty), U) \cap L^\infty_{\mathrm{loc}}([0, \infty), U)$.

Proof. Let $u \in L^p([0, \infty), U) \cap L^\infty_{\mathrm{loc}}([0, \infty), U)$ and let $t \ge 0$. Then there is some $\rho > 0$ with $\|u(s)\|_U \le \rho$ for almost all $s \in [0, t]$. It follows

$$\|B(u(s))\|_{-1} \le \|B(u(s)) - B(0)\|_{-1} + \|B(0)\|_{-1} \le N_\rho\|u(s)\|_U + \|B(0)\|_{-1}$$

for almost all $s \in [0, t]$, where $N_\rho \ge 0$ is the Lipschitz constant of B on $\overline{B}(0, \rho)$. As in the proof of Proposition 4.17 we see that Bu belongs to $L^1([0, t], U)$ and is therefore locally integrable. The claim follows by repeating the steps of the mentioned proof with the map

$$L^p([0, \infty), U) \cap L^\infty_{\mathrm{loc}}([0, \infty), U) \to X_{-1}; \quad u \mapsto \int_0^t \mathbb{T}_{t-s}B(u(s)) \,\mathrm{d}s$$

and some minor modifications. □

We say that Φ_t is bounded on bounded sets for every $t \geq 0$ if for every $\rho > 0$ and each $t \geq 0$ there is a constant $b_{\rho,t} \geq 0$ such that for all $u \in L^p([0,\infty), U)$ with $\|u\|_{L^p} \leq \rho$ we have $\|\Phi_t(u)\|_X \leq b_{\rho,t}$. Obviously this property is weaker than the Lipschitz property stated above.

The following lemma is verified in the same fashion as the last one, so we omit the proof.

Lemma 4.23. *Assume that Φ_t is bounded on bounded sets for every $t \geq 0$. Then Φ_t is bounded on bounded sets uniformly for t in compact subsets of $[0,\infty)$. This means*

$$\forall T > 0, \rho > 0 \ \exists c_{\rho,T} > 0 \ \forall u \in L^p([0,\infty), U), t \in [0,T] : \|u\|_{L^p} \leq \rho \implies \|\Phi_t(u)\|_X \leq c_{\rho,T}.$$

Further the control operator $B \in C(U, X_{-1})$ associated to $(\mathbb{T}, \Phi)$ is bounded on bounded sets.

Remark 4.24. In Lemmata 4.21 and 4.23 assume the respective properties hold globally, so that one can delete the amendment "on bounded sets" everywhere. From the proofs it is clear that all statements remain true, there simply is no ρ. ♢

In the next section we will assume that $\Phi_t : L^p([0,\infty), U) \to X$ is differentiable for all $t \geq 0$. This is a property that is hard to achieve if the underlying spaces U and X are complex vector spaces. However it is sufficient to assume that Φ_t is '$\mathbb{R}$–differentiable'.

Let V, W be Banach spaces and $O \subseteq V$ an open subset. Then a map $F : O \to W$ is called $\mathbb{R}$-*differentiable at* $x \in O$ if there exists a $F'(x) \in \mathcal{L}_{\mathbb{R}}(V,W)$ such that for all $\varepsilon > 0$ we find a $\delta > 0$ with

$$\|F(x+h) - F(x) - F'(x)h\|_W \leq \varepsilon \|h\|_V$$

for all $h \in V$ with $\|h\|_V \leq \delta$. Here

$$\mathcal{L}_{\mathbb{R}}(V,W) = \{T : V \to W \,|\, W \text{ is } \mathbb{R}\text{–linear and } \exists c \geq 0 \ \forall x \in V : \|Tx\|_W \leq c\|x\|_V\}.$$

The map F is called $\mathbb{R}$*–differentiable* if F is $\mathbb{R}$–differentiable at every point $x \in O$.

Clearly $\mathcal{L}(V,W) \subseteq \mathcal{L}_{\mathbb{R}}(V,W)$. Thus the difference is that $F'(x)$ lies in the larger $\mathcal{L}_{\mathbb{R}}(V,W)$. If F is $\mathbb{R}$–differentiable and we know that $F'(x) \in \mathcal{L}(V,W)$, then F is differentiable. In the very same way as for linear operators one shows that $\mathcal{L}_{\mathbb{R}}(V,W)$ is a vector space and that a norm on $\mathcal{L}_{\mathbb{R}}(V,W)$ is given by

$$\|P\|_{\mathcal{L}_{\mathbb{R}}(V,W)} = \inf\{c \geq 0 \,|\, \|Px\|_W \leq c\|x\|_V \text{ for all } x \in V\}.$$

The usual rules of differentiation are valid. For example, let us recall the fundamental theorem. Let $F : O \subseteq W$ be $\mathbb{R}$–differentiable and assume that $F' : O \to \mathcal{L}_{\mathbb{R}}(V,W)$ is continuous. Then we have

$$F(x) - F(w) = \int_0^1 F'(w + \sigma(x-w))(x-w)\,\mathrm{d}\sigma$$

for all $x, w \in O$ with $\{w + \sigma(x-w) \,|\, \sigma \in [0,1]\} \subseteq O$.

All the following statements remain true if we replace every occurrence of the word "differentiable" by "$\mathbb{R}$–differentiable". This fact will be used in our examples. However, in order to keep the exposition clear, we decided not to present the details. We just mention that most operator norms $\|\,\cdot\,\|_{\mathcal{L}(V,W)}$ have to replaced by $\|\,\cdot\,\|_{\mathcal{L}_{\mathbb{R}}(V,W)}$ and that the Laplace transform e.g. in Proposition 4.30 can only be defined for real λ.

Remark 4.25. Let $\Phi_t : L^p([0,\infty),U) \to X$ be differentiable for all $t \geq 0$. Then the derivative is causal and satisfies a composition property. More precisely, for all $u, w \in L^p([0,\infty),U)$ and $t, \tau \geq 0$ we have

$$\Phi_t'(u)w = \Phi_t'(P_t u)P_t w \qquad \text{and} \qquad \Phi_{t+\tau}'(u)w = \mathbb{T}_t \Phi_\tau'(u)w + \Phi_t'(S_\tau^* u)S_\tau^* w. \tag{4.10}$$

As before we write $\Phi_t'(\chi_v) := \Phi_t'(P_t\chi_v)$. To verify (4.10), let $u \in L^p([0,\infty),U)$, $t \geq 0$ and $\varepsilon > 0$. The causality of Φ yields

$$\|\Phi_t(u + h) - \Phi_t(u) - \Phi_t'(P_t u)P_t h\|_X = \|\Phi_t(P_t u + P_t h) - \Phi_t(P_t u) - \Phi_t'(P_t u)P_t h\|_X$$

for all $h \in L^p([0,\infty),U)$. We find a radius $\delta > 0$ such that for all $h \in L^p([0,\infty),U)$ with $\|P_t h\|_{L^p} \leq \delta$ the right-hand side is smaller than $\varepsilon\|P_t h\|_{L^p}$. Since $\|P_t h\|_{L^p} \leq \|h\|_{L^p}$, the uniqueness of the derivative implies the first part of (4.10).

For the proof of the second part let $\tau \geq 0$. For $\varepsilon > 0$ choose $\delta > 0$ such that for $h \in L^p([0,\infty),U)$ with $\|S_\tau^* h\|_{L^p} \leq \delta$ we have

$$\|\Phi_\tau(u + h) - \Phi_\tau(u) - \Phi_\tau'(u)h\|_X \leq \frac{\varepsilon}{2m_{\mathbb{T},t}}\|h\|_{L^p}.$$

On the other hand we may assume that from $\|h\|_{L^p} \leq \delta$ it follows that

$$\|\Phi_t(S_\tau^* u + S_\tau^* h) - \Phi_t(S_\tau^* u) - \Phi_t'(S_\tau^* u)S_\tau^* h\|_X \leq \frac{\varepsilon}{2}\|S_\tau^* h\|_{L^p}.$$

Using (4.1) and the fact that $\|S_\tau^* h\|_{L^p} \leq \|h\|_{L^p}$ we obtain

$$\begin{aligned}
&\|\Phi_{t+\tau}(u + h) - \Phi_{t+\tau}(u) - \mathbb{T}_t\Phi_\tau'(u)h - \Phi_t'(S_\tau^* u)S_\tau^* h\|_X \\
&\quad \leq \|\mathbb{T}_t(\Phi_\tau(u + h) - \Phi_\tau(u) - \Phi_\tau'(u)h)\|_X + \|\Phi_t(S_\tau^*(u + h)) - \Phi_t(S_\tau^* u) - \Phi_t'(S_\tau^* u)S_\tau^* h\|_X \\
&\quad \leq \frac{\varepsilon}{2m_{\mathbb{T},t}} m_{\mathbb{T},t}\|h\|_{L^p} + \frac{\varepsilon}{2}\|S_\tau^* h\|_{L^p} \leq \varepsilon\|h\|_{L^p}
\end{aligned}$$

for all $h \in L^p([0,\infty),U)$ with $\|h\|_{L^p} \leq \delta$ and the claim is shown. ◊

Again we have a result of the type of Lemmata 4.21 and 4.23.

Lemma 4.26. *Assume that $\Phi_t : L^p([0,\infty),U) \to X$ is continuously differentiable for every $t \geq 0$. Then Φ_t' is continuous at every χ_v uniformly for t in compact subsets of $[0,\infty)$ in the following sense*

$$\begin{aligned}
&\forall v \in U,\, T > 0,\, \varepsilon > 0\ \exists \delta > 0\ \forall t \in [0,T],\, \overline{u} \in L^p([0,\infty),U) \\
&\qquad\qquad \|\overline{u}\|_{L^p} \leq \delta \implies \|\Phi_t'(\chi_v + \overline{u}) - \Phi_t'(\chi_v)\|_{\mathcal{L}(L^p,X)} \leq \varepsilon.
\end{aligned}$$

Proof. Fix $v \in U$. Clearly $S_\tau^*\chi_v = \chi_v$ for all $\tau \geq 0$. Let $T > 0$ and $\varepsilon > 0$. Since by assumption the derivative $\Phi_T' : L^p([0,\infty),U) \to \mathcal{L}(L^p([0,\infty),U),X)$ is continuous at $P_T\chi_v$, we find a radius $\delta > 0$ with

$$\sup_{\overline{u} \in \overline{B}(0,\delta)} \|\Phi_T'(\chi_v + \overline{u}) - \Phi_T'(\chi_v)\|_{\mathcal{L}(L^p,X)} \leq \varepsilon.$$

Using the composition property and the causality from (4.10) for $t \in [0,T]$ and $\tau := T-t \geq 0$, we infer the identity

$$\begin{aligned}
&\Phi_T'(\chi_v + S_\tau \widetilde{u})S_\tau w - \Phi_T'(\chi_v)S_\tau w = \Phi_{t+\tau}'(\chi_v + S_\tau \widetilde{u})S_\tau w - \Phi_{t+\tau}'(\chi_v)S_\tau w \\
&\quad = \mathbb{T}_t \Phi_\tau'(\chi_v + S_\tau \widetilde{u})S_\tau w + \Phi_t'(S_\tau^* \chi_v + S_\tau^* S_\tau \widetilde{u})S_\tau^* S_\tau w - \mathbb{T}_t \Phi_\tau'(\chi_v)S_\tau w - \Phi_t'(S_\tau^*\chi_v)S_\tau^* S_\tau w \\
&\quad = \Phi_t'(\chi_v + \widetilde{u})w - \Phi_t'(\chi_v)w + \mathbb{T}_t\Phi_\tau'(P_\tau(\chi_v + S_\tau\widetilde{u}))\chi_0 - \mathbb{T}_t\Phi_\tau'(P_\tau\chi_v)\chi_0 \\
&\quad = \Phi_t'(\chi_v + \widetilde{u})w - \Phi_t'(\chi_v)w
\end{aligned}$$

for all $\widetilde{u}, w \in L^p([0,\infty),U)$. Let $\|\widetilde{u}\|_{L^p} \leq \delta$. Then $\|S_\tau\widetilde{u}\|_{L^p} = \|\widetilde{u}\|_{L^p} \leq \delta$ due to the fact that S_τ is isometric on $L^p([0,\infty),U)$. We conclude that

$$\begin{aligned}
\|\Phi_t'(\chi_v + \widetilde{u}) - \Phi_t'(\chi_v)\|_{\mathcal{L}(L^p,X)} &= \sup_{\|w\|=1} \|\Phi_T'(\chi_v + S_\tau\widetilde{u})S_\tau w - \Phi_T'(\chi_v)S_\tau w\|_X \\
&\leq \sup_{\|\overline{w}\|=1} \|\Phi_T'(\chi_v + S_\tau\widetilde{u})\overline{w} - \Phi_T'(\chi_v)\overline{w}\|_X \\
&= \|\Phi_T'(\chi_v + S_\tau\widetilde{u}) - \Phi_T'(\chi_v)\|_{\mathcal{L}(L^p,X)} \leq \varepsilon. \qquad \square
\end{aligned}$$

4.3 Linearization

Let $A : \mathrm{D}(A) \to X$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Further let $B \in \mathcal{L}(U, X_{-1})$ be an L^p–admissible control operator for $\mathbb{T}$. As usual $(\mathbb{T}, \Phi)$ denotes the corresponding additive control system on X and $L^p([0,\infty),U)$.

We now investigate the dependence of the strong solution of (4.9) on the data. For convenience we repeat equation (4.9):

$$z'(t) = Az(t) + B(u(t)); \quad z(0) = x_0. \tag{4.11}$$

Given $x_0 \in X$ and $u \in L^p([0,\infty),U)$, we write $z(\,\cdot\,, x_0, u) \in C([0,\infty),U)$ for the strong solution of (4.11), that is $z(t,x_0,u) = \mathbb{T}_t x_0 + \Phi_t(u)$.

A pair $(x_*, v_*) \in X \times U$ is called *equilibrium point* of (4.11) if $Ax_* + B(v_*) = 0$. We then also call (x_*, v_*) an equilibrium point of the system $(\mathbb{T}, \Phi)$. Set $u_* := \chi_{v_*}$. Clearly $z(\,\cdot\,, x_*, u_*)$ is the constant function equal to x_*. Since $z(\,\cdot\,, x_*, u_*) \in C^1([0,\infty),X)$, it is the classical solution in X of (4.11).

If additionally B is differentiable at v_*, we consider the so called *linearized problem at* (x_*, u_*) given by the linear inhomogeneous Cauchy problem

$$z_l'(t) = Az_l(t) + B'(v_*)\widetilde{u}(t); \quad z_l(0) = \widetilde{x}_0, \tag{4.12}$$

where $\widetilde{x}_0 \in X$ and $\widetilde{u} \in L^p([0,\infty),U)$. The aim of this section is to show that the following linearization principle is valid, which we state in a simplified form. Surely the conditions given so far are not strong enough.

A) *The operator $B'(v_*) \in \mathcal{L}(U, X_{-1})$ is L^p–admissible for $\mathbb{T}$.*

Hence problem (4.12) is well-posed, meaning that it has a unique strong solution $z_l(\,\cdot\,, \widetilde{x}_0, \widetilde{u})$ for all $\widetilde{x}_0 \in X$ and $\widetilde{u} \in L^p([0,\infty),U)$.

B) *For any fixed $T > 0$ the map*

$$X \times L^p([0,\infty),U) \to C([0,T],X); \quad (x_0, u) \mapsto z(\,\cdot\,, x_0, u)|_{[0,T]}$$

is differentiable at (x_*, u_*) *with derivative given by* $(\widetilde{x}_0, \widetilde{u}) \mapsto z_l(\,\boldsymbol{.}\,, \widetilde{x}_0, \widetilde{u})|_{[0,T]}$.

Since the derivative is close to the map in a small neighborhood of (x_*, u_*) we will be able to show that $z(T, x_0, u)$ can be steered to any state near x_* provided the linearized problem is controllable in a certain sense, see Definitions 4.35 and 4.36.

C) *Finally, if problem* (4.12) *is controllable, then problem* (4.11) *is controllable for data close to* (x_*, u_*).

Actually the equilibrium point (x_*, v_*) is only used ind part C).

Since we are dealing with unbounded operators A and B, we have to pose our conditions on the more regular object Φ. In fact, if $B : U \to X_{-1}$ is differentiable, we can derive that $\varphi(t, \,\boldsymbol{.}\,)$ is differentiable for all $t \geq 0$, but with respect to $\|\,\boldsymbol{.}\,\|_{-1}$, so that $\partial_2\varphi(t, v) \in \mathcal{L}(U, X_{-1})$. In order to show that $\varphi(t, v) \in \mathcal{L}(U, X)$, we need to assume $B'(v) \in \mathcal{L}(U, X_{-1})$ is admissible for all $v \in U$. Actually, we also need that the map $v \mapsto \Phi_t^l(v)$ is continuous where $\Phi_t^l(v)$ are the input maps associated to $B'(v)$. In this case we have $\partial_2\varphi(t, v) = \Phi_t^l(v)$.

We proceed as in Section 4.1 where we gave a minimal set of assumptions to derive a representation result (Theorem 4.9). For fixed $v \in U$ we will first obtain an operator $B^l(v) \in \mathcal{L}(U, X_{-1})$ which then turns out to be the derivative $B'(v)$. Recall the notation

$$\varphi : [0, \infty) \times U \to X; \;\; \varphi(t, v) = \Phi_t(P_t\chi_v) = \Phi_t(\chi_v).$$

We are working with the **standing assumption** stated next: Let $(\mathbb{T}, \Phi)$ be a continuous additive control system and assume that there is an open set $O \subseteq U$ with the following properties.

(H0) For all $t \geq 0$ the function $\varphi(t, \,\boldsymbol{.}\,) : U \to X$ is differentiable on O.

(H1) For all $v \in O$ the family $\partial_2\varphi(\,\boldsymbol{.}\,, v) : [0, \infty) \to \mathcal{L}(U, X)$ is strongly continuous at 0.

(H2) For all $w \in U$ and every $s \in [0, \infty)$ the map $\partial_2\varphi(s, \,\boldsymbol{.}\,)w : O \to X$ is continuous, equicontinuous for s in compact subsets of $[0, \infty)$.

Note that for every $v \in O$ we have $\partial_2\varphi(0, v) = 0 \in \mathcal{L}(U, X)$, because $\varphi(0, \tilde{v}) = 0$ for all $\tilde{v} \in U$.

We can treat the derivative $\partial_2\varphi$ very similar to the way we treated φ in the last section. The reason is that $\partial_2\varphi$ satisfies a composition property analog to (4.2). As a special case of (4.10) in Remark 4.25 we have

$$\partial_2\varphi(t + \tau, v) = \mathbb{T}_t\partial_2\varphi(\tau, v) + \partial_2\varphi(t, v) \quad \text{for all } t, \tau \geq 0, v \in O. \tag{4.13}$$

This equation can also be derived by the chain rule directly from (4.2). Properties (H1) and (H2) are pretty much the same for $\partial_2\varphi$ as (i) and (ii) of Definition 4.3 for φ. Therefore many results of the preceding section can be transferred to $\partial_2\varphi$.

Lemma 4.27. *Under the assumptions (H0) – (H2) the following assertions are valid.*

(a) For all $v \in O$ *the family* $\partial_2\varphi(\,\boldsymbol{.}\,, v) : [0, \infty) \to \mathcal{L}(U, X)$ *is strongly continuous.*

(b) The function $\partial_2\varphi : [0, \infty) \times O \to \mathcal{L}(U, X)$ *is strongly continuous, that is, the map* $(t, v) \mapsto \partial_2\varphi(t, v)w$ *is continuous for every* $w \in U$.

The proof is analogous to those of Lemma 4.4 and Corollary 4.5 and we thus skip it.

Let $[a,b] \subseteq [0,\infty)$ and $K \subseteq O$ both be compact. Then, due to the continuity of $\partial_2\varphi(\,\boldsymbol{.}\,,\,\boldsymbol{.}\,)w$ for every $w \in U$ the set $\{\|\partial_2\varphi(t,v)w\|_X \,|\, t \in [a,b], v \in K\}$ is bounded. From the uniform boundedness principle we infer that the set

$$\{\|\partial_2\varphi(t,v)\| \,|\, t \in [a,b], v \in K\} \quad \text{is bounded.}$$

This fact at hand, we can repeat the proof of Lemma 4.7 to derive exponential boundedness of $\|\partial_2\varphi(\,\boldsymbol{.}\,,v)\|$ for $v \in O$.

Lemma 4.28. *Assume that hypotheses (H0) – (H2) hold. Let $\omega > 0$ and $M \geq 1$ be such that $\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. Then for all $v, \tilde{v} \in O$ we have*

$$\begin{aligned} \|\partial_2\varphi(t,v)\|_{\mathcal{L}(U,X)} &\leq L_2(v)\mathrm{e}^{\omega t} \qquad \text{for all } t \geq 0, \\ \|\partial_2\varphi(t,v) - \partial_2\varphi(t,\tilde{v})\|_{\mathcal{L}(U,X)} &\leq L_2(v,\tilde{v})\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0 \end{aligned}$$

with constants (both depending on ω and M)

$$\begin{aligned} L_2(v) &= M\Big(\sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma,v)\| + \|\partial_2\varphi(1,v)\|\frac{\mathrm{e}^{\omega}}{\mathrm{e}^{\omega}-1}\Big) \leq M\frac{2\mathrm{e}^{\omega}-1}{\mathrm{e}^{\omega}-1}\sup_{\sigma\in[0,1]}\|\partial_2\varphi(\sigma,v)\| \\ L_2(v,\tilde{v}) &= M\Big(\sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma,v) - \partial_2\varphi(\sigma,\tilde{v})\| + \|\varphi(1,v) - \partial_2\varphi(1,\tilde{v})\|\frac{\mathrm{e}^{\omega}}{\mathrm{e}^{\omega}-1}\Big) \\ &\leq M\frac{2\mathrm{e}^{\omega}-1}{\mathrm{e}^{\omega}-1}\sup_{\sigma\in[0,1]}\|\partial_2\varphi(\sigma,v) - \partial_2\varphi(\sigma,\tilde{v})\|. \end{aligned}$$

Remark 4.29. If $K \subseteq O$ is compact, then $\{L_2(v) \,|\, v \in K\}$ is bounded. This is just another formulation of what was said before the last lemma. ◊

Fix $v \in O$. As an exponentially bounded strongly continuous family, $\partial_2\varphi(\,\boldsymbol{.}\,,v)$ has a Laplace transform $\widehat{\partial_2\varphi}(\lambda,v) := (\partial_2\varphi(\,\boldsymbol{.}\,,v))\widehat{\ }(\lambda) \in \mathcal{L}(U,X)$. It exists at least for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max\{\omega_0(\mathbb{T}),0\}$ and it is given by

$$\widehat{\partial_2\varphi}(\lambda,v)w = \int_0^\infty \mathrm{e}^{-\lambda t}\partial_2\varphi(t,v)w\,\mathrm{d}t \quad \text{for } w \in U.$$

For more details we refer to Appendix B.

Below we say '$\partial_2\varphi(\,\boldsymbol{.}\,,v)$ is strongly differentiable with respect to $\|\,\boldsymbol{.}\,\|_{-1}$ on X'. This means that for arbitrary $w \in U$ the mapping $[0,\infty) \to X;\ t \mapsto \partial_2\varphi(t,v)w$ is differentiable with respect to $\|\,\boldsymbol{.}\,\|_{-1}$ on X. Thus the derivative $\frac{\mathrm{d}}{\mathrm{d}t}[\partial_2\varphi(t,v)w] =: \partial_1\partial_2\varphi(t,v)w$ defines a linear map $\partial_1\partial_2\varphi(t,v) : U \to X_{-1}$. The representation (4.14) below shows that this operator is bounded.

Proposition 4.30. *Let (H0) – (H2) be satisfied. Then for all $v \in O$ the function $\partial_2\varphi(\,\boldsymbol{.}\,,v) : [0,\infty) \to \mathcal{L}(U,X)$ is strongly differentiable with respect to $\|\,\boldsymbol{.}\,\|_{-1}$ on X. Its derivative $\partial_1\partial_2\varphi(t,v) \in \mathcal{L}(U,X_{-1})$ is given by*

$$\partial_1\partial_2\varphi(t,v) = \mathbb{T}_t\partial_1\partial_2\varphi(0,v) = \mathbb{T}_t\lambda(\lambda-A)\widehat{\partial_2\varphi}(\lambda,v) \quad \text{for all } t \geq 0, \tag{4.14}$$

where $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max\{\omega_0(\mathbb{T}),0\}$ can be chosen freely. We see that $\partial_1\partial_2\varphi(\,\boldsymbol{.}\,,v)$ is strongly continuous.

Proof. For $v \in O$ and $w \in U$ one can repeat the proof of Proposition 4.8 with $\partial_2\varphi(\,\boldsymbol{.}\,, v)w$ in place of $\varphi(\,\boldsymbol{.}\,, v)$. $\square$

For the time being, fix some $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max\{\omega_0(\mathbb{T}), 0\}$. Let $v \in O$. We set

$$B^l(v) := \partial_1\partial_2\varphi(0, v) = \lambda(\lambda - A)\widehat{\partial_2\varphi}(\lambda, v) \in \mathcal{L}(U, X_{-1}).$$

Then equation (4.14) yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}[\partial_2\varphi(t, v)w] = \mathbb{T}_t B^l(v)w \quad \text{for } t \geq 0, v \in O \text{ and } w \in U.$$

Recall that $\partial_2\varphi(0, v) = 0$. As in the proof of Theorem 4.9 we use the fundamental theorem of calculus to derive the identity

$$\partial_2\varphi(t, v)w = \int_0^t \mathbb{T}_s B^l(v)w \,\mathrm{d}s = \int_0^t \mathbb{T}_{t-s} B^l(v)\chi_w(s)\,\mathrm{d}s. \tag{4.15}$$

We want to show that $B^l(v)$ is L^p–admissible for $\mathbb{T}$. To this end, we consider the family $\Phi^l(v) = (\Phi^l_t(v))_{t\geq 0}$ of operators $\Phi^l_t(v) \in \mathcal{L}(L^p([0, \infty), U), X_{-1})$ defined through

$$\Phi^l_t(v)u := \int_0^t \mathbb{T}_{t-s} B^l(v)u(s)\,\mathrm{d}s \tag{4.16}$$

for arbitrary $u \in L^p([0, \infty), U)$. Then obviously $\Phi^l_t(v)\chi_w = \partial_2\varphi(t, v)w$. Due to Proposition 4.2 in [52] and Remark 4.20 it suffices to find one $t > 0$ such that Φ^l_t maps $L^p([0, \infty), U)$ to X. In Proposition 4.33 we give a sufficient condition for this to be true.

First we want to identify $B^l(v)$ with $B'(v)$. In particular, $\partial_1\varphi(t, \,\boldsymbol{.}\,)$ has to be differentiable for every $t \geq 0$. To achieve this, it seems we have to replace condition (H2) by the following somewhat stronger assumption.

(H2') For every $s \in [0, \infty)$ the map $\partial_2\varphi(s, \,\boldsymbol{.}\,) : O \to \mathcal{L}(U, X)$ is continuous, equicontinuous for s in compact subsets of $[0, \infty)$.

Proposition 4.31. *Under the conditions (H0), (H1) and (H2') we have the following. For every $t \in [0, \infty)$ the map $\partial_1\varphi(t, \,\boldsymbol{.}\,) : U \to X_{-1}$ is continuously differentiable on O and*

$$\partial_2\partial_1\varphi(t, v) = \partial_1\partial_2\varphi(t, v) = \mathbb{T}_t\lambda(\lambda - A)\widehat{\partial_2\varphi}(\lambda, v) = \mathbb{T}_t B^l(v).$$

Moreover, $B : U \to X_{-1}$ is continuously differentiable on O with $B'(v) = B^l(v)$ for $v \in O$.

Proof. Let $t \geq 0$ and $v \in O$. There is a radius $r > 0$ with $B(v, r) \subseteq O$. Let $h \in B(0, r)$. Using (4.3), i.e., the identity $\partial_1\varphi(t, w) = \lambda(\lambda - A)\mathbb{T}_t\widehat{\varphi}(\lambda, w)$ for every $w \in U$, we obtain

$$\begin{aligned}
&\|\partial_1\varphi(t, v + h) - \partial_1\varphi(t, v) - \mathbb{T}_t B^l(v)h\|_{-1} \\
&\quad = \|\lambda(\lambda - A)\mathbb{T}_t(\widehat{\varphi}(\lambda, v + h) - \widehat{\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda, v)h)\|_{-1} \\
&\quad \leq |\lambda|\|\lambda - A\|_{\mathcal{L}(X, X_{-1})}\|\mathbb{T}_t\|\,\|\widehat{\varphi}(\lambda, v + h) - \widehat{\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda, v)h\|_X.
\end{aligned}$$

Let $\varepsilon > 0$. We show that the right-hand side can be estimated by $\varepsilon\|h\|_U$ provided $\|h\|_U$ is "small". It suffices to estimate the last factor. From the fundamental theorem we infer

$$\begin{aligned} &\|\widehat{\varphi}(\lambda, v+h) - \widehat{\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda, v)h\|_X \\ &\quad = \left\| \int_0^\infty \mathrm{e}^{-\lambda s}(\varphi(s, v+h) - \varphi(s,v) - \partial_2\varphi(s,v)h)\,\mathrm{d}s \right\|_X \\ &\quad \le \int_0^\infty \left\| \mathrm{e}^{-\lambda s} \int_0^1 (\partial_2\varphi(s, v+\tau h) - \partial_2\varphi(s,v))h\,\mathrm{d}\tau \right\|_X \mathrm{d}s. \end{aligned} \tag{4.17}$$

Take $\omega > 0$ and $M \ge 1$ such that $\|\mathbb{T}_\sigma\| \le M\mathrm{e}^{\omega\sigma}$ for all $\sigma \ge 0$. Lemma 4.28 yields

$$\|(\partial_2\varphi(s, v+\tau h) - \partial_2\varphi(s,v))h\|_X \le \mathrm{e}^{\omega s} M \frac{2\mathrm{e}^\omega - 1}{\mathrm{e}^\omega - 1} \sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma, v+\tau h) - \partial_2\varphi(\sigma, v)\|_{\mathcal{L}(U,X)}\|h\|_U.$$

for all $\tau \in [0,1]$ and $s \in [0,\infty)$, Moreover, by the equicontinuity of $\partial_2\varphi(s, \,\boldsymbol{.}\,) : O \to \mathcal{L}(U,X)$ there is some $\delta > 0$ such that for all $h \in B(0,r)$ with $\|v+\tau h - v\|_U \le \|h\|_U \le \delta$ we have

$$\sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma, v+\tau h) - \partial_2\varphi(\sigma, v)\|_{\mathcal{L}(U,X)} \le (\operatorname{Re}\lambda - \omega)\frac{\mathrm{e}^\omega - 1}{M(2\mathrm{e}^\omega - 1)}\varepsilon.$$

For arbitrary $s \in [0,\infty)$ and $h \in B(0,r)$ with $\|h\| \le \delta$ we thus obtain

$$\begin{aligned} &\left\| \mathrm{e}^{-\lambda s} \int_0^1 \partial_2\varphi(s, v+\tau h)h - \partial_2\varphi(s,v)h\,\mathrm{d}\tau \right\|_X \\ &\quad \le \mathrm{e}^{-\operatorname{Re}\lambda s}\mathrm{e}^{\omega s} M \frac{2\mathrm{e}^\omega - 1}{\mathrm{e}^\omega - 1} \sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma, v+\tau h) - \partial_2\varphi(\sigma, v)\|_{\mathcal{L}(U,X)}\|h\|_U \\ &\quad \le (\operatorname{Re}\lambda - \omega)\mathrm{e}^{-(\operatorname{Re}\lambda-\omega)s}\,\varepsilon\|h\|_U. \end{aligned}$$

Continuing estimate (4.17), we infer

$$\|\widehat{\varphi}(\lambda, v+h) - \widehat{\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda, v)h\|_X \le \varepsilon \int_0^\infty (\operatorname{Re}\lambda - \omega)\mathrm{e}^{-(\operatorname{Re}\lambda-\omega)s}\,\mathrm{d}s\|h\|_U = \varepsilon\|h\|_U.$$

It remains to show that $\partial_2\partial_1\varphi(t, \,\boldsymbol{.}\,)$ is continuous. This can be done similar to the proof of Theorem 4.9. Let $t \ge 0$. For $v, \tilde{v} \in U$ as above we derive

$$\|\partial_2\partial_1\varphi(t,v) - \partial_2\partial_1\varphi(t,\tilde{v})\|_{\mathcal{L}(U,X_{-1})} \le |\lambda|\|\lambda - A\|_{\mathcal{L}(X,X_{-1})}\|\mathbb{T}_t\|\|\widehat{\partial_2\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda,\tilde{v})\|_{\mathcal{L}(U,X)}\,.$$

Hence, it suffices to prove that $\|\widehat{\partial_2\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda,\tilde{v})\|_{\mathcal{L}(U,X)} \to 0$ as $\|v - \tilde{v}\|_U \to 0$. By means of Lemma 4.28, one directly verifies that

$$\begin{aligned} \|\widehat{\partial_2\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda,\tilde{v})\|_{\mathcal{L}(U,X)} &\le \int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda s}\|\partial_2\varphi(s,v) - \partial_2\varphi(s,\tilde{v})\|_{\mathcal{L}(U,X)}\,\mathrm{d}s \\ &\le \int_0^\infty \mathrm{e}^{-(\operatorname{Re}\lambda-\omega)s}\,\mathrm{d}s\, M\frac{2\mathrm{e}^\omega - 1}{\mathrm{e}^\omega - 1} \sup_{\sigma\in[0,1]} \|\partial_2\varphi(\sigma, v) - \partial_2\varphi(\sigma,\tilde{v})\|_{\mathcal{L}(U,X)}. \end{aligned}$$

For given $\varepsilon > 0$ and the same corresponding $\delta > 0$ as above, we conclude that

$$\|\widehat{\partial_2\varphi}(\lambda, v) - \widehat{\partial_2\varphi}(\lambda,\tilde{v})\|_{\mathcal{L}(U,X)} \le \varepsilon,$$

if $\|v - \tilde{v}\|_U \le \delta$. □

Remark 4.32. The following modification is immediate. Let conditions (H0) – (H2) be satisfied. Further assume that the family

$$\{\partial_2\varphi(s,\,\boldsymbol{.}\,) : O \to \mathcal{L}(U,X) \mid s \in [0,1]\}$$

is equicontinuous at some $v_* \in U$, i.e., for all $\varepsilon > 0$ exists some $\delta > 0$ such that $\overline{B}(v_*,\delta) \subseteq O$ and for all $h \in \overline{B}(0,\delta)$ we have

$$\sup_{s\in[0,1]} \|\partial_2\varphi(s, v_* + h) - \partial_2\varphi(s, v_*)\|_{\mathcal{L}(U,X)} \leq \varepsilon.$$

Then $B : U \to X_{-1}$ is differentiable at v_* with derivative $B^l(v_*)$. ◊

We next assume that $\Phi_t : L^p([0,\infty),U) \to X$ is differentiable for every $t \geq 0$. For fixed $v \in U$ consider the family $\Phi'(\chi_v) := (\Phi'_t(\chi_v))_{t\geq 0}$ of bounded linear operators $\Phi'_t(\chi_v)$ from $L^p([0,\infty),U)$ to X. In Remark 4.25 we have seen that $\Phi'(\chi_v)$ is causal and satisfies the composition property

$$\Phi'_{t+\tau}(\chi_v)u = \mathbb{T}_t\Phi'_\tau(\chi_v)u + \Phi'_t(\chi_v)S^*_\tau u \tag{4.18}$$

for all $t,\tau \geq 0$ and $u \in L^p([0,\infty),U)$. Hence $(\mathbb{T},\Phi'(\chi_v))$ satisfies Definition 2.1 of [52]. Consequently it is a continuous additive control system, see also Remark 4.13.

We are now able to prove the first part of the linearization principle stated at the beginning of this section on page 51.

Proposition 4.33. *Let $A : \mathrm{D}(A) \to X$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B : U \to X_{-1}$ be an L^p–admissible control operator for $\mathbb{T}$. As usual the corresponding input maps are denoted Φ_t for $t \geq 0$.*

Assume that $\Phi_t : L^p([0,\infty),U) \to X$ is continuously differentiable for every $t \geq 0$. Then B is continuously differentiable and $B'(v)$ is L^p–admissible for $\mathbb{T}$ for each $v \in U$. More precisely, $B'(v)$ is the control operator associated to $(\mathbb{T},\Phi'(\chi_v))$.

Proof. Recall that we use the abbreviation $\|\,\boldsymbol{.}\,\|_{L^p} := \|\,\boldsymbol{.}\,\|_{L^p([0,\infty),U)}$. We first check conditions (H0), (H1) and (H2') with $O = U$. To this end, let $t \geq 0$ and $v \in U$.

(H0): Clearly $\varphi(0,\,\boldsymbol{.}\,) = 0$ is differentiable and we may assume that $t > 0$. Let $\varepsilon > 0$. Because Φ_t is differentiable at $P_t\chi_v$ there is a $\tilde{\delta} > 0$ such that for all $u \in L^p([0,\infty),U) \setminus \{0\}$ with $\|u\|_{L^p} \leq \tilde{\delta}$ it follows that

$$\frac{1}{\|u\|_{L^p}}\|\Phi_t(P_t\chi_v + u) - \Phi_t(P_t\chi_v) - \Phi'_t(P_t\chi_v)u\|_X \leq \varepsilon t^{-1/p}.$$

For $h \in U \setminus \{0\}$ with $\|h\|_U \leq t^{-1/p}\tilde{\delta}$ we have $0 < \|P_t\chi_h\|_{L^p} = \|h\|_U\, t^{1/p} \leq \tilde{\delta}$. For convenience we write $u_0 := P_t\chi_v$ and $u := P_t\chi_h$. Clearly $u_0 + u = P_t(\chi_v + \chi_h) = P_t\chi_{v+h}$ so that $\Phi_t(u_0 + u) = \varphi(t, v + h)$. Under these assumptions we infer

$$\begin{aligned}
&\frac{1}{\|h\|_U}\|\varphi(t,v+h) - \varphi(t,v) - \Phi'_t(\chi_v)P_t\chi_h\|_X \\
&\quad= \frac{t^{1/p}}{\|h\|_U\, t^{1/p}}\|\Phi_t(u_0+u) - \Phi_t(u_0) - \Phi'_t(u_0)u\|_X \\
&\quad= \frac{1}{\|u\|_{L^p}}\|\Phi_t(u_0+u) - \Phi_t(u_0) - \Phi'_t(u_0)u\|_X\, t^{1/p} \leq \varepsilon\frac{t^{1/p}}{t^{1/p}} = \varepsilon.
\end{aligned}$$

Hence $\varphi(t,\,\boldsymbol{.}\,) : U \to X$ is differentiable and its derivative is given by

$$\partial_2\varphi(t,v)w = \Phi'_t(P_t\chi_v)P_t\chi_w = \Phi'_t(\chi_v)\chi_w \quad \text{for all } t \geq 0 \text{ and } v,w \in U. \tag{4.19}$$

(H1): We have to prove strong continuity at 0 for the map $\partial_2\varphi(\,\boldsymbol{.}\,,v) : [0,\infty) \to \mathcal{L}(U,X)$. We use that the operator norms $\|\Phi'_t(\chi_v)\|$ are non-decreasing in t. This is true because $\Phi'(\chi_v)$ is the family of input maps of a linear control system, see Proposition 2.3 in [52]. For completeness we repeat the reasoning.

Let $\tau \geq 0$ and $u \in L^p([0,\infty),U)$. The composition property (4.18) and causality yieldy

$$\Phi'_{t+\tau}(\chi_v)S_\tau u = \mathbb{T}_t\Phi'_\tau(\chi_v)P_\tau S_\tau u + \Phi'_t(\chi_v)S^*_\tau S_\tau u = \Phi'_t(\chi_v)u.$$

Therefore we have $\|\Phi'_t(\chi_v)u\|_X \leq \|\Phi'_{t+\tau}(\chi_v)\|\|S_\tau u\|_{L^p} = \|\Phi'_{t+\tau}(\chi_v)\|\|u\|_{L^p}$ and consequently

$$\|\Phi'_t(\chi_v)\| \leq \|\Phi'_{t+\tau}(\chi_v)\|.$$

Now (H1) follows easily. Let $w \in U$ and recall that $\partial_2\varphi(0,v)w = 0$. For $\delta \in [0,1]$ we compute

$$\begin{aligned}\|\partial_2\varphi(\delta,v)w\|_X &= \|\Phi'_\delta(P_\delta\chi_v)P_\delta\chi_w\|_X \leq \|\Phi'_\delta(P_\delta\chi_v)\|\|P_\delta\chi_w\|_{L^p} \\ &\leq \|\Phi'_1(P_1\chi_v)\|\|P_\delta\chi_w\|_{L^p} \to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

(H2'): Finally let $T > 0$. We check that the family $\{\partial_2\varphi(t,\,\boldsymbol{.}\,) : U \to \mathcal{L}(U,X) \,|\, t \in [0,T]\}$ is equicontinuous. Take $t \in [0,T]$ and $v,\tilde{v} \in U$. As above one sees that $\|\Phi'_t(\chi_v) - \Phi'_t(\chi_{\tilde{v}})\| \leq \|\Phi'_T(\chi_v) - \Phi'_T(\chi_{\tilde{v}})\|$. Consequently,

$$\|\partial_2\varphi(t,v)w - \partial_2\varphi(t,\tilde{v})w\|_X \leq \|\Phi'_t(\chi_v) - \Phi'_t(\chi_{\tilde{v}})\|\|P_t\chi_w\|_{L^p} \leq t^{1/p}\|\Phi'_T(\chi_v) - \Phi'_T(\chi_{\tilde{v}})\|\|w\|_U$$

for all $w \in U$. Hence $\|\partial_2\varphi(t,v) - \partial_2\varphi(t,\tilde{v})\| \leq t^{1/p}\|\Phi'_T(\chi_v) - \Phi'_T(\chi_{\tilde{v}})\|$.

Let $\varepsilon > 0$. Due to the continuity of Φ'_T there is some $\tilde{\delta} > 0$ such that for all $u,\tilde{u} \in L^p([0,\infty),U)$ with $\|u - \tilde{u}\|_{L^p} \leq \tilde{\delta}$ we have $\|\Phi'_T(u) - \Phi'_T(\tilde{u})\| \leq \varepsilon T^{-1/p}$. If v and $\tilde{v}$ satisfy $\|v - \tilde{v}\|_U \leq \tilde{\delta}T^{1/p}$, then $\|P_T\chi_v - P_T\chi_{\tilde{v}}\|_{L^p} \leq \tilde{\delta}$ and thus

$$\|\partial_2\varphi(t,v) - \partial_2\varphi(t,\tilde{v})\| \leq t^{1/p}\|\Phi'_t(P_t\chi_v) - \Phi'_t(P_t\chi_{\tilde{v}})\| \leq T^{1/p}\|\Phi'_T(P_T\chi_v) - \Phi'_T(P_T\chi_{\tilde{v}})\| \leq \varepsilon.$$

This means that $\partial_2\varphi(t,\,\boldsymbol{.}\,) : U \to \mathcal{L}(U,X)$ is equicontinuous for $t \in [0,T]$.

The last step is to show that $\Phi'_t(\chi_v)$ and $\Phi^l_t(v)$ coincide. Using (4.15), (4.16) and (4.19) we already have

$$\Phi'_t(\chi_v)\chi_w = \partial_2\varphi(t,v)w = \Phi^l_t(v)\chi_w \quad \text{for } t \geq 0 \text{ and } v,w \in U.$$

By linearity we infer that $\Phi'_t(\chi_v)u = \Phi^l_t(v)u$ for all piecewise constant functions $u \in \Omega_0$. Since $\Phi'_t(\chi_v)$ and $\Phi^l_t(v)$ both belong to $\mathcal{L}(L^p([0,\infty),U),X_{-1})$ and Ω_0 is dense in $L^p([0,\infty),U)$ the operators $\Phi'_t(\chi_v)$ and $\Phi^l_t(v)$ are equal. From Proposition 4.31 we deduce

$$\Phi'_t(\chi_v)u = \Phi^l_t(v)u = \int_0^t \mathbb{T}_{t-s}B^l(v)u(s)\,\mathrm{d}s = \int_0^t \mathbb{T}_{t-s}B'(v)u(s)\,\mathrm{d}s \tag{4.20}$$

for all $t \geq 0$ and $u \in L^p([0,\infty),U)$. Due to uniqueness, $B'(v)$ is the control operator associated to $(\mathbb{T},\Phi'(\chi_v))$. In particular, this means that $B'(v) \in \mathcal{L}(U,X_{-1})$ is L^p–admissible for $\mathbb{T}$. □

We mention that under the conditions of the last proposition using (4.10) in the same induction as in the proof of Theorem 4.9 we obtain

$$\Phi_t'(u)w = \int_0^t \mathbb{T}_{t-s} B'(u(s))w(s)\,\mathrm{d}s$$

for each step function $u \in \Omega_0$ as well as all $w \in L^p([0,\infty),U)$ and $t \geq 0$.

It is now easy to verify also part B) of the linearization principle. Recall that for $x_0 \in X$ and $u \in L^p([0,\infty),U)$ we denoted the strong solution of (4.11) by $z(\,\cdot\,,x_0,u)$.

Corollary 4.34. *To the conditions of Proposition 4.33 add the existence of an equilibrium point $(x_*,v_*) \in X \times U$. We write $u_* = \chi_{v_*}$. Then for every $\widetilde{x}_0 \in X$ and $\widetilde{u} \in L^p([0,\infty),U)$ the strong solution $z_l(\,\cdot\,,\widetilde{x}_0,\widetilde{u}) \in C([0,\infty),X)$ of the linearized problem (4.12) is given by*

$$z_l(t,\widetilde{x}_0,\widetilde{u}) = \mathbb{T}_t x_0 + \Phi_t'(\chi_{v_*})\widetilde{u}.$$

Moreover, for $T > 0$ the map $X \times L^p([0,\infty),U) \to C([0,T],X)$; $(x_0,u) \to z(\,\cdot\,,x_0,u)|_{[0,T]}$ is differentiable at (x_*,u_*) with derivative given by $(\widetilde{x}_0,\widetilde{u}) \mapsto z_l(\,\cdot\,,\widetilde{x}_0,\widetilde{u})$.

Proof. The first part is a direct consequence of Proposition 4.16 and (4.20). Let $T > 0$. Take $\widetilde{x}_0 \in X$ and $\widetilde{u} \in L^p([0,\infty),U)$. Plugging in the definition of z and z_l, we obtain

$$\begin{aligned}
&\|z(t,x_*+\widetilde{x}_0,u_*+\widetilde{u}) - z(t,x_*,u_*) - z_l(t,\widetilde{x}_0,\widetilde{u})\|_X \\
&\qquad = \|\mathbb{T}_t x_* + \mathbb{T}_t\widetilde{x}_0 + \Phi_t(u_*+\widetilde{u}) - \mathbb{T}_t x_* - \Phi_t(u_*) - \mathbb{T}_t\widetilde{x}_0 - \Phi_t'(u_*)\widetilde{u}\|_X \\
&\qquad = \|\Phi_t(u_*+\widetilde{u}) - \Phi_t(u_*) - \Phi_t'(u_*)\widetilde{u}\|_X \quad \text{for all } t \in [0,T].
\end{aligned}$$

Let $\varepsilon > 0$. Lemma 4.26 yields some $\delta > 0$ with $\|\Phi_t'(\chi_{v_*}+\overline{u}) - \Phi_t'(\chi_{v_*})\|_{\mathcal{L}(L^p,X)} \leq \varepsilon$ for all $t \in [0,T]$ provided $\overline{u} \in \overline{B}(0,\delta) \subseteq L^p([0,\infty),U)$. With the fundamental theorem we derive

$$\begin{aligned}
&\|z(\,\cdot\,,x_*+\widetilde{x}_0,u_*+\widetilde{u}) - z(\,\cdot\,,x_*,u_*) - z_l(\,\cdot\,,\widetilde{x}_0,\widetilde{u})\|_{L^\infty([0,T],X)} \\
&\quad = \sup_{t\in[0,T]} \|\Phi_t(u_*+\widetilde{u}) - \Phi_t(u_*) - \Phi_t'(u_*)\widetilde{u}\|_X = \sup_{t\in[0,T]} \left\| \int_0^1 (\Phi_t'(u_*+\sigma\widetilde{u}) - \Phi_t'(u_*))\widetilde{u}\,\mathrm{d}\sigma \right\|_X \\
&\quad \leq \sup_{t\in[0,T]} \sup_{\overline{u}\in\overline{B}(0,\delta)} \|\Phi_t'(u_*+\overline{u}) - \Phi_t'(u_*)\|_{\mathcal{L}(L^p,X)} \|\widetilde{u}\|_{L^p} \leq \varepsilon\|\widetilde{u}\|_{L^p} \leq \varepsilon(\|\widetilde{u}\|_{L^p} + \|\widetilde{x}_0\|_X)
\end{aligned}$$

for all $\widetilde{x}_0 \in X$ and $\widetilde{u} \in L^p([0,\infty),U)$ with $\|\widetilde{x}_0\|_X + \|\widetilde{u}\|_{L^p} \leq \delta$. □

Before we can show the last part of the linearization principle, we have to introduce some notions. There are several controllability concepts in the literature. See e.g. Chapter 11 of [49] for the linear case. We follow the definitions in Section 3.1 of [11] which treats finite dimensional systems. The lineariziation principle relates the global property 'exact controllability' of the linearized system to the 'local controllability' of the original problem. It relies on the general fact that derivatives describe the local behavior of functions.

Definition 4.35. A *linear* control system $(\mathbb{T},\Phi)$ on X and $L^p([0,\infty),U)$ is called *exactly controllable in time $T > 0$* if $\operatorname{Ran}\Phi_T = X$.

In the situation of this definition, for all $x_1, x_2 \in X$ we find an input $u \in L^p([0,\infty),U)$ such that $\Phi_T u = x_1 - \mathbb{T}_T x_0$ and therefore $\mathbb{T}_T x_0 + \Phi_T u = x_1$. This means that the state can be steered from arbitrary initial states to arbitrary final states. The last definition has particular importance if U is a Hilbert space and $p = 2$, since then $L^p([0,\infty),U)$ is a Hilbert space. In this case $\Phi_T \in \mathcal{L}(L^p([0,\infty),U),X)$ has a bounded right inverse $\Phi_T^{\#} \in \mathcal{L}(X, L^p([0,\infty),U))$. In fact, let $P \in \mathcal{L}(H,X)$ be a linear operator mapping a Hilbert space H to the Banach space E. Assume that P is onto, i.e., $\operatorname{Ran} P = E$. Then there exists an operator $P^{\#} \in \mathcal{L}(E,H)$ with $PP^{\#}x = x$ for all $x \in E$. Such an operator is called *bounded right inverse of P*. See Appendix C for more details. This is the reason we need a Hilbert space in the theorem below, which is the main result of the section.

Definition 4.36. Let $(\mathbb{T},\Phi)$ be an additive control system on X and $L^p([0,\infty),U)$. Assume that $(\mathbb{T},\Phi)$ has an equilibrium point $(x_*, v_*) \in X \times U$. We write $u_* = \chi_{v_*}$. Then $(\mathbb{T},\Phi)$ is called *locally controllable at (x_*, u_*) in time $T > 0$* if for every $R > 0$ there are radii $r_1, r_2 \in (0,R]$ such that for all $x_0 \in \overline{B}(x_*, r_1)$ and $x_1 \in \overline{B}(x_*, r_2)$ we find an input $u \in L^p([0,\infty),U)$ with $z(T, x_0, u) = x_1$ and $\|z(t,x_0,u) - x_*\|_X \leq R$ for all $t \in [0,T]$.

Letting R go to zero, we see that we can consider local controllability only at equilibrium points $(x_*, v_*) \in X \times U$.

Theorem 4.37. *Let X be a Banach space and let U be a Hilbert space. Assume that $(\mathbb{T},\Phi)$ is an additive control system on X and $L^2([0,\infty),U)$ which has an equilibrium point $(x_*, v_*) \in X \times U$. We write $u_* := \chi_{v_*}$. Further let $\Phi_t : L^2([0,\infty),U) \to X$ be continuously differentiable for every $t \geq 0$. Finally, assume that $(\mathbb{T}, \Phi'(\chi_{v_*}))$ is exactly controllable in some time $T > 0$. Then $(\mathbb{T},\Phi)$ is locally controllable at (x_*, u_*) in time $T > 0$.*

Proof. As a preparation, we first transform the problem. For that we need the 'remainder' $\Phi_t^{\mathrm{rem}} : L^2([0,\infty),U) \to X$ given by

$$\Phi_t^{\mathrm{rem}}(\widetilde{u}) = \Phi_t(u_* + \widetilde{u}) - \Phi_t(u_*) - \Phi_t'(u_*)\widetilde{u}$$

for $t \geq 0$. By assumption, Φ_t^{rem} is continuously differentiable with derivative

$$(\Phi_t^{\mathrm{rem}})'(w) = \Phi_t'(u_* + w) - \Phi_t'(u_*) \quad \text{for } w \in L^2([0,\infty),U).$$

For $x_0, x_1 \in X$ and $u \in L^2([0,\infty),U)$ set $\widetilde{x}_j := x_j - x_*$ for $j = 0,1$ and $\xi := \widetilde{x}_1 - \mathbb{T}_T \widetilde{x}_0$ as well as $\widetilde{u} := u - u_*$. The identity $x_1 = z(T, x_0, u)$ is equivalent to the identity

$$\begin{aligned}\widetilde{x}_1 &= z(T, x_* + \widetilde{x}_0, u_* + \widetilde{u}) - z(T, x_*, u_*) = \mathbb{T}_T(\widetilde{x}_0 + x_*) + \Phi_T(u_* + \widetilde{u}) - \mathbb{T}_T x_* - \Phi_T(\widetilde{u}) \\ &= \mathbb{T}_T \widetilde{x}_0 + \Phi_T^{\mathrm{rem}}(\widetilde{u}) + \Phi_T'(u_*)\widetilde{u}, \qquad (4.21)\end{aligned}$$

and thus to the equation

$$\Phi_T'(u_*)\widetilde{u} = \widetilde{x}_1 - \mathbb{T}_T \widetilde{x}_0 - \Phi_t^{\mathrm{rem}}(\widetilde{u}) = \xi - \Phi_t^{\mathrm{rem}}(\widetilde{u}). \qquad (4.22)$$

Let $Q := (\Phi_T'(u_*))^{\#} \in \mathcal{L}(X, L^2([0,\infty),U))$ be a bounded right inverse of $\Phi_T'(u_*)$, cf. Corollary C.11. Surely $Q \neq 0$. If we find a fixed-point $\widetilde{u} \in L^2([0,\infty),U)$ of

$$\widetilde{u} = Q(\xi - \Phi_T^{\mathrm{rem}}(\widetilde{u})) = Q\xi - Q\Phi_T^{\mathrm{rem}}(\widetilde{u}) =: \mathcal{C}(\widetilde{u}),$$

then (4.22) is satisfied. Clearly an operator $\mathcal{C} : L^2([0,\infty),U) \to L^2([0,\infty),U)$ is defined by the above equation. Hence the next step is to check that $\mathcal{C}$ is strictly contractive on a ball $\overline{B}(0,\rho) \subseteq L^2([0,\infty),U)$ for some $\rho > 0$.

Let $R > 0$. Lemma 4.26 yields a number $\rho_0 > 0$ with

$$\|\Phi_t'(u_* + \overline{u}) - \Phi_t'(u_*)\| \leq \frac{1}{2\|Q\|}$$

for all $\overline{u} \in L^2([0,\infty),U)$ with $\|\overline{u}\|_{L^2} \leq \rho_0$ and each $t \in [0,T]$. On the one hand, we thus have

$$\begin{aligned}\|\Phi_t^{\mathrm{rem}}(\widetilde{u})\|_X &= \left\|\int_0^1 (\Phi_t'(u_* + \sigma\widetilde{u}) - \Phi_t'(u_*))\widetilde{u}\,\mathrm{d}\sigma\right\|_X \\ &\leq \sup_{\overline{u}\in\overline{B}(0,\rho_0)} \|\Phi_t'(u_* + \overline{u}) - \Phi_t'(u_*)\|\|\widetilde{u}\|_{L^2} \leq \frac{1}{2\|Q\|}\|\widetilde{u}\|_{L^2} \leq \frac{1}{2\|Q\|}\rho_0. \end{aligned} \tag{4.23}$$

for all $t \in [0,T]$ and every $\widetilde{u} \in L^2([0,\infty),U)$ with $\|\widetilde{u}\|_{L^2} \leq \rho_0$. On the other hand, it follows that

$$\|(\Phi_T^{\mathrm{rem}})'(w)\| \leq \|\Phi_T'(u_* + w) - \Phi_T'(u_*)\| \leq \frac{1}{2\|Q\|} \tag{4.24}$$

for all $w \in L^2([0,\infty),U)$ with $\|w\|_{L^2} \leq \rho_0$. We can now fix the constants

$$\rho = \min\left\{\rho_0, R\left(\frac{3}{4\|Q\|} + \|\Phi_T'(u_*)\|\right)^{-1}\right\}, \qquad r_0 = \frac{\rho}{4\|Q\|m_{\mathbb{T},T}} \qquad \text{and} \qquad r_1 = \frac{\rho}{4\|Q\|}.$$

Note that $r_0 \leq r_1$. Let $\widetilde{x}_0 \in \overline{B}(0,r_0)$ and $\widetilde{x}_1 \in \overline{B}(0,r_1)$. The choice of r_0 and r_1 yields

$$\|\xi\|_X \leq \|\widetilde{x}_1\|_X + \|\mathbb{T}_T\|\|\widetilde{x}_0\|_X \leq \frac{\rho}{4\|Q\|} + \frac{\|\mathbb{T}_T\|}{m_{\mathbb{T},T}}\frac{\rho}{4\|Q\|} \leq \frac{\rho}{2\|Q\|}. \tag{4.25}$$

By the chain rule, $\mathcal{C}$ is continuously differentiable with $\mathcal{C}'(w) = -Q(\Phi_T^{\mathrm{rem}})'(w)$ for all $w \in L^2([0,\infty),U)$. From (4.24) we deduce

$$\|\mathcal{C}'(w)\| \leq \|Q\|\|(\Phi_T^{\mathrm{rem}})'(w)\| \leq \tfrac{1}{2} < 1$$

if $\|w\|_{L^2} \leq \rho_0$. For $\widetilde{u}_1, \widetilde{u}_2 \in \overline{B}(0,\rho)$ we thus have

$$\begin{aligned}\|\mathcal{C}(\widetilde{u}_1) - \mathcal{C}(\widetilde{u}_2)\|_{L^2} &= \left\|\int_0^1 \mathcal{C}'(\widetilde{u}_2 + \sigma(\widetilde{u}_1 - \widetilde{u}_2)) \cdot (\widetilde{u}_1 - \widetilde{u}_2)\,\mathrm{d}\sigma\right\|_{L^2} \\ &\leq \sup_{w\in\overline{B}(0,\rho)} \|\mathcal{C}'(w)\|\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^2} \leq \tfrac{1}{2}\|\widetilde{u}_1 - \widetilde{u}_2\|_{L^2}.\end{aligned}$$

Further, the estimate (4.23) with $t = T$ and (4.25) lead to the bound

$$\|\mathcal{C}(\widetilde{u})\|_{L^2} \leq \|Q\|(\|\xi\|_X + \|\Phi_T^{\mathrm{rem}}(\widetilde{u})\|) \leq \|Q\|\frac{\rho}{2\|Q\|} + \|Q\|\frac{\rho}{2\|Q\|} = \rho.$$

Of course, $\overline{B}(0,\rho)$ as a closed subset of $L^2([0,\infty),U)$ is a complete nonempty metric space. The contraction mapping principle yields the existence of a fixed-point $\widetilde{u}$ of $\mathcal{C}$ in $\overline{B}(0,\rho)$.

For $x_0 \in \overline{B}(x_*, r_0)$ and $x_1 \in \overline{B}(x_*, r_1)$, as above let $\widetilde{x}_j = x_j - x_*$ for $j = 1, 2$. The fixed point $\widetilde{u} \in \overline{B}(0, \rho)$ found above fulfills equation (4.22) and therefore also $z(T, x_0, u_* + \widetilde{u}) = x_1$. Moreover, formulas (4.21) and (4.23) imply that

$$\begin{aligned}\|z(t, x_* + \widetilde{x}_0, u_* + \widetilde{u}) - z(t, x_*, u_*)\|_X &\leq \|\mathbb{T}_t\| \|\widetilde{x}_0\|_X + \|\Phi_t^{\text{rem}}(\widetilde{u})\|_X + \|\Phi_t'(u_*)\| \|\widetilde{u}\|_{L^2} \\ &\leq m_{\mathbb{T},T} \|\widetilde{x}_0\| + \frac{\rho}{2\|Q\|} + \|\Phi_T'(u_*)\| \rho \leq \left(\frac{3}{4\|Q\|} + \|\Phi_T'(u_*)\| \right) \rho \leq R\end{aligned}$$

for all $t \in [0, T]$ due to the choice of r_0 and ρ. Here we also used that $\|\Phi_t'(u_*)\|$ is non decreasing in t. □

Remark 4.38. The proof of the last theorem works if $\Phi_T'(u_*)$ has a bounded right inverse Q. If we know that such Q exists then we do not need that $(\mathbb{T}, \Phi)$ is a control system on $L^2([0, \infty), U)$, we can take any other exponent $p \geq 1$ instead. However, in general it is not easy to find a bounded right inverse in the non–Hilbert case. ◊

4.4 Applications

Let us start with a description of the structure of the examples in this section. Take Banach spaces X, U_l and let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Further let $B^l \in \mathcal{L}(U_l, X_{-1})$ be an L^p-admissible control operator for $\mathbb{T}$. We denote the corresponding input maps by $\Phi_t^l \in \mathcal{L}(L^p([0, \infty), U_l), X)$.

Let U be another Banach space. We introduce a measurable map $M : U \to U_l$ and aim to replace the input maps Φ_t^l by the operators

$$\Phi_t(u) = \int_0^t \mathbb{T}_{t-s} B^l M(u(s)) \,\mathrm{d}s.$$

Clearly they correspond to the control operator $B : U \to X_{-1}$; $B(v) = B^l M(v)$. In order to really obtain an additive control system $(\mathbb{T}, \Phi)$, we need further assumptions. Let M be continuous and assume that there is a constant $c \geq 0$ and an exponent $\eta \geq \frac{1}{p}$ such that

$$\|M(v)\|_{U_l} \leq c(\|v\|_U^\eta + 1) \quad \text{for all } v \in U. \tag{4.26}$$

Note that if this is true for one $\eta > 0$, then it also holds for all larger expontents $\tilde{\eta} > \eta$. We will now frequently use the following type of estimate without further comments

$$(a + b)^e \leq (2 \max\{a, b\})^e = 2^e \max\{a^e, b^e\} \leq 2^e (a^e + b^e) \quad \text{for } a, b, e \geq 0.$$

For brevity, as before, we write $\|\cdot\|_{L^{p\eta}} := \|\cdot\|_{L^{p\eta}([0,\infty),U)}$ and $\|\cdot\|_{L^p} := \|\cdot\|_{L^p([0,\infty),U_l)}$.

Fix any $u \in L^{p\eta}([0, \infty), U)$. We first show that $M \circ u$ lies in $L^p_{\text{loc}}([0, \infty), U_l)$. To this end, take $t > 0$. From (4.26) we infer that

$$\begin{aligned}\left(\int_0^t \|M(u(s))\|_{U_l}^p \,\mathrm{d}s \right)^{1/p} &\leq \left(\int_0^t c^p (\|u(s)\|_U + 1)^{p\eta} \,\mathrm{d}s \right)^{1/p} = c \| \|u(\cdot)\|_U + 1 \|_{L^{p\eta}([0,t],\mathbb{R})}^\eta \\ &\leq c(\|u\|_{L^{p\eta}} + t^{\frac{1}{p\eta}})^\eta \leq c 2^\eta (\|u\|_{L^{p\eta}}^\eta + t^{\frac{1}{p}}) \leq c 2^\eta (t^{\frac{1}{p}} + 1)(\|u\|_{L^{p\eta}}^\eta + 1).\end{aligned}$$

This means that $M \circ u \in L^p([0,t],U_l)$. We define $N : L^{p\eta}([0,\infty),U) \to L^p_{\mathrm{loc}}([0,\infty),U_l)$; $N(u) = M \circ u$. The last estimate then reads

$$\|N(u)\|_{L^p([0,t],U_l)} \le c_t(\|u\|^\eta_{L^{p\eta}} + 1) \quad \text{for all } t > 0, u \in L^{p\eta}([0,\infty),U), \tag{4.27}$$

where $c_t = c2^\eta(t^{\frac{1}{p}} + 1)$. As already announced above, for every $t \ge 0$ we now define $\Phi_t : L^{p\eta}([0,\infty),U) \to X$ through

$$\Phi_t(u) = \Phi^l_t N(u) = \int_0^t \mathbb{T}_{t-s} B^l M(u(s))\,\mathrm{d}s.$$

We shall prove that $(\mathbb{T},\Phi)$ is a continuous additive control system on X and $L^{p\eta}([0,\infty),U)$, where $\Phi = (\Phi_t)_{t\ge 0}$. Since $B \circ u \in L^1_{\mathrm{loc}}([0,\infty),X_{-1})$ for all $u \in L^{p\eta}([0,\infty),U)$, it then follows that B is $L^{p\eta}$–admissible for $\mathbb{T}$.

As argued after Definition 4.19, the family Φ is causal and satisfies the composition property for $\mathbb{T}$. To verify condition (i) of Definition 4.3, take $v \in U$. Because $(\mathbb{T},\Phi^l)$ is a linear control system, we obtain

$$\|\Phi_t(\chi_v)\|_X = \left\| \int_0^t \mathbb{T}_{t-s} B^l M(v)\,\mathrm{d}s \right\|_X = \|\Phi^l_t \chi_{M(v)}\|_X \to 0 \quad \text{as } t \to 0^+,$$

see Proposition 2.3 of [52]. Also condition (ii) of Definition 4.3 is satisfied. We easily deduce this from the fact that M is continuous. Indeed, let $v \in U$ and $T > 0$. Then for all $t \in [0,T]$ and $\tilde{v} \in U$ we have

$$\begin{aligned}
\|\varphi(t,v) - \varphi(t,\tilde{v})\|_X &\le \|\Phi^l_t\|_{\mathcal{L}(L^p,X)} \|P_t N(\chi_v) - P_t N(\chi_{\tilde{v}})\|_{L^p} \\
&\le \|\Phi^l_T\|_{\mathcal{L}(L^p,X)} \|\chi_{M(v)-M(\tilde{v})}\|_{L^p([0,T],U_l)} \\
&\le \|\Phi^l_T\|_{\mathcal{L}(L^p,X)} T^{\frac{1}{p}} \|M(v) - M(\tilde{v})\|_{U_l}.
\end{aligned}$$

Obviously for $\varepsilon > 0$ there is a $\delta > 0$ such that $\|M(v) - M(\tilde{v})\|_{U_l} \le \varepsilon(\|\Phi^l_T\|_{\mathcal{L}(L^p,X)} T^{1/p})^{-1}$.

We next show that the family $\{\Phi_t : L^{p\eta}([0,\infty),U) \to X \mid t \in [0,T]\}$ is equicontinuous for each $T > 0$. Let $u, h \in L^{p\eta}([0,\infty),U)$, $T > 0$ and $t \in [0,T]$. We then compute

$$\begin{aligned}
\|\Phi_t(u+h) - \Phi_t(u)\|_X &\le \|\Phi^l_t\| \|P_t N(u+h) - P_t N(u)\|_{L^p} \\
&\le \|\Phi^l_T\| \|M(u(\cdot) + h(\cdot)) - M(u(\cdot))\|_{L^p([0,T],U_l)}.
\end{aligned} \tag{4.28}$$

To estimate the right-hand side we need the following lemma. This type of expression will now appear several times here and in Sections 5.3 and 6.3.

Lemma 4.39. *Let V and W be Banach spaces and $p \in [1,\infty)$. Assume that $G : V \to W$ is continuous and satisfies*

$$\|G(v)\|_W \le c_1 \|v\|^\eta_V + c_2 \quad \textit{for all } v \in V, \tag{4.29}$$

where $c_1, c_2 \ge 0$ and $\eta \ge \frac{1}{p}$ are fixed numbers. Set $q := p\eta$. Further let (J,μ) be a measure space. If $c_2 \ne 0$ require that J is finite.

Then $G \circ f \in L^p_{\mathrm{loc}}(J,W)$ for every $f \in L^q(J,V)$, Moreover, for all $f \in L^q(J,V)$ and $\varepsilon > 0$ there is a number $\delta > 0$ such that for $h \in L^q(J,V)$ with $\|h\|_{L^q(J,V)} \le \delta$ we have

$$\|G(f(\cdot) + h(\cdot)) - G(f(\cdot))\|_{L^p(J,V)} \le \varepsilon.$$

Proof. The first claim can be shown as in the preceding text. To verify the second, let $f \in L^q(J,V)$. Let (h_k) be any sequence in $L^q(J,V)$ with $\|h_k\|_{L^q} \to 0$ as $k \to \infty$. Then we find a subsequence (h_{k_l}) with $\|h_{k_l}(t)\|_V \to 0$ for almost all $t \in J$. The subsequence can further be chosen such that there is a function $g \in L^q(J,\mathbb{R})$ with $\|h_{k_l}(t)\| \le g(t)$ for almost every $t \in J$. For simplicity let us write (h_k) again. Observe that

$$\|G(f(\,\boldsymbol{.}\,)+h_k(\,\boldsymbol{.}\,)) - G(f(\,\boldsymbol{.}\,))\|_{L^p(J,V)}^p = \int_J \|G(f(t)+h_k(t)) - G(f(t))\|_W^p \,\mathrm{d}t.$$

Using the continuity of G, we infer that the integrand converges to zero pointwise almost everywhere on J. On the other hand we have

$$\begin{aligned}\|G(f(t)+h_k(t)) - G(f(t))\|_W &\le c_1\|f(t)+h_k(t)\|_V^\eta + c_1\|f(t)\|_V^\eta + 2c_2\\ &\le c_1(2^\eta+1)\|f(t)\|_V^\eta + 2^\eta c_1\|h_k(t)\|_V^\eta + 2c_2\\ &\le c_1(2^\eta+1)\|f(t)\|_V^\eta + 2^\eta c_1 g(t)^\eta + 2c_2.\end{aligned}$$

The right-hand side as a function of t is p–integrable (recall that $\mu(J) < \infty$ if $c_2 \neq 0$). By the dominated convergence theorem we have

$$\|G(f(\,\boldsymbol{.}\,)+h_k(\,\boldsymbol{.}\,)) - G(f(\,\boldsymbol{.}\,))\|_{L^p(J,V)}^p \to 0 \quad \text{as } k \to \infty.$$

Now assume the claim is false. Then there is an $\varepsilon_0 > 0$ and for every $k \in \mathbb{N}$ a function $h_k \in L^q(J,V)$ with $\|h_k\|_{L^q(J,V)} \le \frac{1}{k}$ and $\|G(f(\,\boldsymbol{.}\,)+h_k(\,\boldsymbol{.}\,)) - G(f(\,\boldsymbol{.}\,))\|_{L^p(J,V)}^p \ge \varepsilon_0^p$. But this contradicts the existence of the subsequence of (h_k) constructed above. □

Applying this lemma to (4.28), we finish the proof of the fact that $(\mathbb{T},\Phi)$ is an equicontinuous additive control system on X and $L^{p\eta}([0\infty),U)$. As we already said, by Remark 4.20 (b) the map B is an $L^{p\eta}$–admissible control operator for $\mathbb{T}$ because $B \circ u = B^l M(u(\,\boldsymbol{.}\,))$ is locally integrable.

In the following let $\eta > 1$. We add the condition that M is continuously $\mathbb{R}$–differentiable and satisfies the estimate

$$\|M'(v)\|_{\mathcal{L}_\mathbb{R}(U,U_l)} \le c(\|v\|_U^{\eta-1} + 1) \quad \text{for all } v \in U, \tag{4.30}$$

where $c \ge 0$. The value of c is not important, so we took the constant from (4.26). Again we drop the $\mathbb{R}$ in $\mathcal{L}_\mathbb{R}(U,U_l)$ to keep the formulas simple. In the same way as above we infer that

$$M'(u(\,\boldsymbol{.}\,)) \in L_{\text{loc}}^{\frac{p\eta}{\eta-1}}([0,\infty),\mathcal{L}(U,U_l))$$

for every $u \in L^{p\eta}([0,\infty),U)$. Using Hölder's inequality, we deduce that $M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$ is contained in $L^p([0,T],U_l)$ for all $u,w \in L^{p\eta}([0,\infty),U)$ and $T > 0$, and that

$$\begin{aligned}\|M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)\|_{L^p([0,T],U_l)} &\le \|M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,T],\mathcal{L}(U,U_l))}\|w\|_{L^{p\eta}([0,T],U)}\\ &\le \|M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,T],\mathcal{L}(U,U_l))}\|w\|_{L^{p\eta}}.\end{aligned}$$

Hence for fixed $u \in L^{p\eta}([0,\infty),U)$ the linear mapping $w \mapsto M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$ from $L^{p\eta}([0,\infty),U)$ to $L^p([0,T],U_l)$ is bounded.

We can now prove that $P_tN : L^{p\eta}([0,\infty),U) \to L^p([0,t],U_l)$ is $\mathbb{R}$–differentiable for each $t > 0$ with derivative given by $(P_tN)'(u)w = M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$. To this end, fix $u \in L^{p\eta}([0,\infty),U)$ and $t > 0$. Using the fundamental theorem, we then write

$$\begin{aligned}
&\|N(u+h) - N(u) - M'(u(\,\boldsymbol{.}\,))h(\,\boldsymbol{.}\,)\|_{L^p([0,t],U_l)} \\
&\qquad = \Big(\int_0^t \|M(u(s)+h(s)) - M(u(s)) - M'(u(s))h(s)\|_{U_l}^p \,\mathrm{d}s\Big)^{1/p} \\
&\qquad \le \Big(\int_0^t \Big(\int_0^1 \|M'(u(s)+\sigma h(s)) - M'(u(s))\|_{\mathcal{L}(U,U_l)} \|h(s)\|_U \,\mathrm{d}\sigma\Big)^p \mathrm{d}s\Big)^{1/p}
\end{aligned}$$

for $h \in L^{p\eta}([0,\infty),U)$. Minkowski's and Hölder's inequalities yield

$$\begin{aligned}
&\Big(\int_0^t \Big(\int_0^1 \|M'(u(s)+\sigma h(s)) - M'(u(s))\|_{\mathcal{L}(U,U_l)} \|h(s)\|_U \,\mathrm{d}\sigma\Big)^p \mathrm{d}s\Big)^{1/p} \\
&\qquad \le \int_0^1 \Big(\int_0^t \|M'(u(s)+\sigma h(s)) - M'(u(s))\|_{\mathcal{L}(U,U_l)}^p \|h(s)\|_U^p \,\mathrm{d}s\Big)^{1/p} \mathrm{d}\sigma \\
&\qquad \le \int_0^1 \|M'(u(\,\boldsymbol{.}\,)+\sigma h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,t],\mathcal{L}(U,U_l))} \,\mathrm{d}\sigma\, \|h\|_{L^{p\eta}([0,t],U)} \\
&\qquad \le \int_0^1 \|M'(u(\,\boldsymbol{.}\,)+\sigma h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,t],\mathcal{L}(U,U_l))} \,\mathrm{d}\sigma\, \|h\|_{L^{p\eta}}.
\end{aligned}$$

Let $\varepsilon > 0$. We apply Lemma 4.39 (with $\eta - 1$ instead of η and $\frac{p\eta}{\eta-1}$ instead of p) to the term under the integral on the right-hand side to obtain a number $\delta > 0$ with

$$\|M'(u(\,\boldsymbol{.}\,)+\sigma h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,t],\mathcal{L}(U,U_l))} \le \varepsilon$$

for all $\sigma \in [0,1]$ and $h \in L^{p\eta}([0,\infty),U)$ with $\|\sigma h\|_{L^{p\eta}} \le \|h\|_{L^{p\eta}} \le \delta$. Hence we have proved the claimed differentiability of P_tN. Finally with $\sigma = 1$ it also follows that $(P_tN)'$ is continuous. Indeed, for $u,h \in L^{p\eta}([0,\infty),U)$ and $\varepsilon,\delta > 0$ as above as well as arbitrary $w \in L^{p\eta}([0,\infty),U)$ we have

$$\begin{aligned}
\|(M'(u+h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,)))w(\,\boldsymbol{.}\,)\|_{L^p} &\le \|M'(u+h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^{\frac{p\eta}{\eta-1}}([0,\infty),\mathcal{L}(U,U_l))} \|w\|_{L^{p\eta}} \\
&\le \varepsilon \|w\|_{L^{p\eta}},
\end{aligned}$$

showing that the map P_tN is even continuously $\mathbb{R}$–differentiable. As a consequence, Φ_t is continuously $\mathbb{R}$–differentiable for every $t \ge 0$. In fact, from the linearity of Φ_t^l we derive

$$\begin{aligned}
&\|\Phi_t(u+h) - \Phi_t(u) - \Phi_t^l (P_tN)'(u)h\|_X \\
&\qquad = \|\Phi_t^l(P_tN(u+h) - P_tN(u) - (P_tN)'(u)h)\|_X \\
&\qquad \le \|\Phi_t^l\|_{\mathcal{L}(L^p,X)} \|P_tN(u+h) - P_tN(u) - (P_tN)'(u)h\|_{L^p} \\
&\qquad \le \|\Phi_t^l\|_{\mathcal{L}(L^p,X)} \|N(u+h) - N(u) - (P_tN)'(u)h\|_{L^p([0,t],U_l)}
\end{aligned}$$

for $u,h \in L^{p\eta}([0,\infty),U)$. Thus the derivative of Φ_t at u is given by $\Phi_t'(u)w = \Phi_t^l M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$ for $w \in L^{p\eta}([0,\infty),U)$.

We summarize the preceding results in a lemma.

Lemma 4.40. *Let X, U_l and U be Banach spaces. Assume that $(\mathbb{T}, \Phi^l)$ is a linear control system on X and $L^p([0,\infty), U_l)$ for an exponent $p \in [1,\infty)$. Denote by $B^l \in \mathcal{L}(U_l, X_{-1})$ the control operator of $(\mathbb{T}, \Phi^l)$. Let $M : U \to U_l$ be a continuous map satisfying (4.26), i.e., assume that there exist constants $\eta \geq \frac{1}{p}$ and $c \geq 0$ such that*

$$\|M(v)\|_{U_l} \leq c(\|v\|_U^\eta + 1) \quad \text{for all } v \in U.$$

Then the family $\Phi = (\Phi_t)_{t\geq 0}$ of maps $\Phi_t : L^{p\eta}([0,\infty), U) \to X$ given by

$$\Phi_t(u) = \int_0^t \mathbb{T}_{t-s} B^l M(u(s))\, \mathrm{d}s$$

yield a equicontinuous additive control system $(\mathbb{T}, \Phi)$ on X and $L^{p\eta}([0,\infty), U)$.

Further assume that $\eta > 1$, that M is continuously $\mathbb{R}$–differentiable and that the derivative satisfies the growth bound

$$\|M'(v)\|_{\mathcal{L}_\mathbb{R}(U,U_l)} \leq c(\|v\|_U^{\eta-1} + 1) \quad \text{for all } v \in U.$$

Then Φ_t is continuously $\mathbb{R}$–differentiable for every $t \geq 0$ with derivative given by $\Phi_t'(u)w = \Phi_t^l M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$ for $u, w \in L^{p\eta}([0,\infty), U)$.

Let $(x_*, v_*) \in X \times U$ be an equilibrium point of $(\mathbb{T}, \Phi)$. Then by a little abuse of notation we write $\Phi_t'(\chi_{v_*}) = \Phi_t^l M'(v_*)$. Whenever the linear control system $(\mathbb{T}, \Phi^l)$ is exactly controllable in some time $T > 0$ and we find a bounded right inverse for $M'(v_*) \in \mathcal{L}(U, U_l)$, we can apply Theorem 4.37, also see Remark 4.38. This requires that $M'(v_*)$ is onto. In view of Proposition C.10 it is rather promising to look for invertible derivatives $M'(v_*)$.

The case $p = 2$ and $\eta = 1$

Obviously the above argumentation is not valid for $\eta = 1$. Note that for our linearization result Theorem 4.37 we need $p\eta = 2$ which is equivalent to $p = \frac{2}{\eta}$. Moreover, I am only aware of linear L^2–admissible control operators B^l that yield exactly controllable systems for infinite dimensional X and U_l. Thus we also have to take $p = 2$ which results in $\eta = 1$.

Let $U_l = U$ be a Hilbert space. So far we have not used the special form of Φ_t^l in this section. This is mainly because B^l is possibly unbounded. However, if $B^l \in \mathcal{L}(U, X)$ is bounded, we can directly verify that Φ_t is $\mathbb{R}$–differentiable, even if N is not differentiable. It helps that we can pull the norm $\|\,\boldsymbol{.}\,\|_X$ under the integral.

Lemma 4.41. *Let X be a Banach space and U be a Hilbert space. Assume that $(\mathbb{T}, \Phi^l)$ is a linear control system on X and $L^2([0,\infty), U)$. Denote by $B^l \in \mathcal{L}(U, X)$ the control operator of $(\mathbb{T}, \Phi^l)$. Assume that $M : U \to U$ is Lipschitz, i.e., assume that there is a constant $c \geq 0$ such that*

$$\|M(v) - M(\tilde{v})\|_U \leq c\|v - \tilde{v}\|_U \quad \text{for all } v, \tilde{v} \in U.$$

Further, let M be continuously $\mathbb{R}$–differentiable with $\|M'(v) - M'(\tilde{v})\|_{\mathcal{L}_\mathbb{R}(U)} \leq c\|v - \tilde{v}\|_U$ for all $v, \tilde{v} \in U$. Then for every $t \geq 0$ the map $\Phi_t : L^2([0,\infty), U) \to X$ given by

$$\Phi_t(u) = \int_0^t \mathbb{T}_{t-s} B^l M(u(s))\, \mathrm{d}s$$

is continuously $\mathbb{R}$–differentiable with derivative given by $\Phi_t'(u)w = \Phi_t^l M'(u(\,\boldsymbol{.}\,))w(\,\boldsymbol{.}\,)$ for $u, w \in L^2([0,\infty), U)$. Moreover, the derivative is Lipschitz.

Proof. Clearly (4.26) is satisfied with $\eta = 1$. In the last section we saw that $M \circ u : [0, \infty) \to U$ is locally square integrable for all $u \in L^2([0, \infty), U)$. Moreover, in the same way we derive that $M' \circ u \in L^2_{\text{loc}}([0, \infty), \mathcal{L}_{\mathbb{R}}(U))$ as well as that

$$\begin{aligned}\|(M'(\tilde{u}(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,)))w(\,\boldsymbol{.}\,)\|_{L^1} &\leq \|M'(\tilde{u}(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^2([0,\infty),\mathcal{L}_{\mathbb{R}}(U))}\|w\|_{L^2} \\ &\leq c\|\tilde{u} - u\|_{L^2}\|w\|_{L^2}\end{aligned}$$

for all $u, \tilde{u} \in L^2([0, \infty), U)$. Let $t \geq 0$, $u \in L^2([0, \infty), U)$ and $\varepsilon > 0$. For all $h \in L^2([0, \infty), U)$ with $\|h\|_{L^2} \leq 2(cm_{\mathbb{T},t}\|B\|)^{-1}\varepsilon$ we can estimate

$$\begin{aligned}&\left\|\Phi_t(u + h) - \Phi_t(u) - \int_0^t \mathbb{T}_{t-s}B^l M'(u(s))h(s)\,\mathrm{d}s\right\|_X \\ &\quad\leq \int_0^t \int_0^1 \|\mathbb{T}_{t-s}B^l(M'(u(s) + \sigma h(s)) - M'(u(s)))h(s)\|_X \,\mathrm{d}\sigma\,\mathrm{d}s \\ &\quad\leq m_{\mathbb{T},t}\|B^l\|_{\mathcal{L}(U,X)} \int_0^1 \int_0^t \|M'(u(s) + \sigma h(s)) - M'(u(s))\|_{\mathcal{L}_{\mathbb{R}}(U)}\|h(s)\|_U \,\mathrm{d}s\,\mathrm{d}\sigma \\ &\quad\leq m_{\mathbb{T},t}\|B^l\|_{\mathcal{L}(U,X)} \int_0^1 \|M'(u(\,\boldsymbol{.}\,) + \sigma h(\,\boldsymbol{.}\,)) - M'(u(\,\boldsymbol{.}\,))\|_{L^2([0,\infty),U)}\,\mathrm{d}\sigma\|h\|_{L^2} \\ &\quad\leq cm_{\mathbb{T},t}\|B^l\|_{\mathcal{L}(U,X)}\|h\|^2_{L^2} \leq \varepsilon\|h\|_{L^2},\end{aligned}$$

showing that Φ_t is $\mathbb{R}$–differentiable at u. Very similar on sees that the derivative is Lipschitz and thus Φ_t is continuously $\mathbb{R}$–differentiable. □

The last paragraph of the previous subsection holds accordingly. More precisely, if $(x_*, v_*) \in X \times U$ is an equilibrium point of $(\mathbb{T}, \Phi)$ and $M'(v_*) \in \mathcal{L}_{\mathbb{R}}(U)$ is invertible, then exact controllability of the underlying linear system $(\mathbb{T}, \Phi^l)$ passes over to $(\mathbb{T}, \Phi)$.

In case B^l is not bounded, we consider "smooth" inputs. We mention that U as a Hilbert space satisfies the Radon-Nikodym property. This means that every absolutely continuous function $u : [0, \infty) \to U$ is differentiable almost everywhere. The vector-valued Sobolev space

$$H^1([0, \infty), U) = \{u \in L^2([0, \infty), U) \mid u \text{ is absolutely continuous and } u' \in L^2([0, \infty), U)\}.$$

is a Hilbert space. The closed subspace $H^1_{\mathrm{L}}([0, \infty), U)$ consists of those $u \in H^1([0, \infty), U)$ with $u(0) = 0$. In contrast to $L^2([0, \infty), U)$ this Sobolev space is continuously embedded in $L^\infty([0, \infty), U)$. This means that there is a constant $d \geq 0$ with $\|h\|_{L^\infty} \leq d\|h\|_{H^1}$ for $h \in H^1_{\mathrm{L}}([0, \infty), U)$. We will drop the constant and write $\|\,\boldsymbol{.}\,\|_{L^\infty} \lesssim \|\,\boldsymbol{.}\,\|_{H^1}$.

Let Z be the solution space for $(\mathbb{T}, \Phi^l)$ introduced in Section 3.3. From Theorem 11.3.6 in [49] (see also Section 2 of [51], especially Theorem 2.5) we know that

$$\Phi^l_T(H^1_{\mathrm{L}}([0, \infty), U)) = Z$$

provided that $(\mathbb{T}, \Phi^l)$ is exactly controllable in time $T > 0$. In the following we want to replace $L^2([0, \infty), U)$ in our reasoning by $H^1_{\mathrm{L}}([0, \infty), U)$. Consequently, X has to be replaced by Z. In Remark 3.14 we derived that $\Phi^l_t \in \mathcal{L}(H^1_{\mathrm{L}}([0, \infty), U), Z)$. Unfortunately, Z might not be invariant under $\mathbb{T}$, see e.g. Example 10.1.9 of [49]. Hence we can not directly translate Theorem 4.37 to this situation. However we have the following result.

Proposition 4.42. *Let U be a Hilbert space and $B \in C(U, X_{-1})$ be an L^2–admissible control operator for $\mathbb{T}$ with the corresponding input maps $\Phi_t : L^2([0,\infty), U) \to X$. Assume that $B(0) = 0$. Moreover, let the restrictions $\Phi_t : H^1_{\mathrm{L}}([0,\infty), U) \to Z$ be continuously $\mathbb{R}$–differentiable for all $t \geq 0$. Finally, assume that $\Phi'_T(\chi_0)(H^1_{\mathrm{L}}([0,\infty), U)) = Z$ for some $T > 0$. Then we find radii $r_1, r_2 > 0$ such that for all $x_0 \in X_1$, $x_1 \in Z$ with $\|x_0\|_1 \leq r_1$ and $\|x_1\|_Z \leq r_2$ there is an input $u \in H^1_{\mathrm{L}}([0,\infty), U)$ with*

$$x_1 = \mathbb{T}_T x_0 + \Phi_T(u).$$

Since χ_v does not belong to $H^1_{\mathrm{L}}([0,\infty), U)$ (not even locally) for $v \neq 0$, we are committed to the equilibrium point $(0, \chi_0)$. The proof the above proposition is similar to the one of Theorem 4.37. So we think an outline is sufficient.

Proof. Note that $B(0) = 0$ implies that $(0, \chi_0) \in X \times U$ is an equilibrium point of $z'(t) = Az(t) + Bu(t)$. Moreover, in this case $\Phi_t(\chi_0) = 0$ for all $t \geq 0$. Again one considers the remainder $\Phi^{\mathrm{rem}}_T : H^1_{\mathrm{L}}([0,\infty), U) \to Z$. In this situation it is given by

$$\Phi^{\mathrm{rem}}_T(u) = \Phi_T(u) - \Phi_T(\chi_0) - \Phi'_T(\chi_0)u = \Phi_T(u) - \Phi'_T(\chi_0)u,$$

so that $\Phi_T(u) = \Phi'_T(\chi_0)u + \Phi^{\mathrm{rem}}_T(u)$ for $u \in H^1_{\mathrm{L}}([0,\infty), U)$. By our assumptions, Φ^{rem}_T is continuously differentiable with derivative $(\Phi^{\mathrm{rem}}_T)'(u) = \Phi'_T(u) - \Phi'_T(\chi_0)$ for $u \in H^1_{\mathrm{L}}([0,\infty), U)$. In particular, we have $(\Phi^{\mathrm{rem}}_T)'(\chi_0) = 0$.

Let $Q \in \mathcal{L}(Z, H^1_{\mathrm{L}}([0,\infty), U))$ be a bounded right inverse of $\Phi'_T(\chi_0) \in \mathcal{L}(H^1_{\mathrm{L}}([0,\infty), U), Z)$. Take $x_0 \in X_1$ and $x_1 \in Z$. If we find an input $u \in H^1_{\mathrm{L}}([0,\infty), U)$ satisfying

$$u = Q(x_1 - \mathbb{T}_T x_0 - \Phi^{\mathrm{rem}}_T(u)) =: \mathcal{C}(u),$$

then we get $\Phi'_T(\chi_0)u = x_1 - \mathbb{T}_T x_0 - \Phi^{\mathrm{rem}}_T(u)$ or equivalently

$$x_1 = \mathbb{T}_T x_0 + \Phi'_T(\chi_0)u + \Phi^{\mathrm{rem}}_T(u) = \mathbb{T}_T x_0 + \Phi_T(u).$$

The existence of such an element u can be shown with the contraction mapping principle applied to the restriction of $\mathcal{C} : H^1_{\mathrm{L}}([0,\infty), U) \to H^1_{\mathrm{L}}([0,\infty), U)$ to a certain ball. The following estimates are essential. Let $u_1, u_2 \in H^1_{\mathrm{L}}([0,\infty), U)$. We then obtain

$$\begin{aligned}
\|\mathcal{C}(u_1) - \mathcal{C}(u_2)\|_{H^1_{\mathrm{L}}} &\leq \|Q\| \|\Phi^{\mathrm{rem}}_T(u_1) - \Phi^{\mathrm{rem}}_T(u_2)\|_Z \\
&\leq \|Q\| \int_0^1 \|(\Phi^{\mathrm{rem}}_T)'(u_2 + \sigma(u_1 - u_2))(u_1 - u_2)\|_Z \,\mathrm{d}\sigma \\
&\leq \|Q\| \sup_{\tilde{u} \in \overline{B}(0,\rho)} \|(\Phi^{\mathrm{rem}}_T)'(\tilde{u})\|_{\mathcal{L}(H^1_{\mathrm{L}}, Z)} \|u_1 - u_2\|_{H^1}.
\end{aligned}$$

Due to the continuity of $(\Phi^{\mathrm{rem}}_T)'$ the factor $\sup_{\tilde{u} \in \overline{B}(0,\rho)} \|(\Phi^{\mathrm{rem}}_T)'(\tilde{u})\|_{\mathcal{L}(H^1_{\mathrm{L}}, Z)}$ can be made small by choosing $\rho > 0$ small. On the other hand, for $u \in H^1_{\mathrm{L}}([0,\infty), U)$ we compute

$$\begin{aligned}
\|\mathcal{C}(u)\|_{H^1_{\mathrm{L}}} &\leq \|Q\| (\|x_1 - \mathbb{T}_T x_0\|_Z + \|\Phi^{\mathrm{rem}}_T(u)\|_Z) \\
&\leq \|Q\| (\|x_1\|_Z + \|\mathbb{T}_T x_0\|_1 + \|\Phi^{\mathrm{rem}}_T(u)\|_Z) \\
&\lesssim \|Q\| (\|x_1\|_Z + \|\mathbb{T}_T\| \|x_0\|_1 + \|\Phi^{\mathrm{rem}}_T(u)\|_Z),
\end{aligned}$$

by means of the embedding $X_1 \hookrightarrow Z$ from (3.10). The right-hand side can be controlled using that $\|\Phi^{\mathrm{rem}}_T(u)\|_Z$ tends to zero as $\|u\|_{H^1} \to 0$, and by keeping $\|x_1\|_Z$ and $\|x_0\|_1$ small, which yields the radii $r_1, r_2 > 0$. □

We are now looking for assumptions under which N maps $H^1_{\mathrm{L}}([0,T],U)$ to itself for all $T>0$ and the restriction is continuously $\mathbb{R}$–differentiable. If this is the case, Φ_t is $\mathbb{R}$–differentiable on $H^1_{\mathrm{L}}([0,\infty),U)$ for all $t \geq 0$ with $\Phi_t'(u) = \Phi_t^l N'(u)$ for $u \in H^1_{\mathrm{L}}([0,\infty),U)$. Then the differentiability conditions of Proposition 4.42 are satisfied. In contrast to Lemma 4.41 we do not need global Lipschitz bounds here.

Lemma 4.43. *Assume that $M \in C^2_{\mathbb{R}}(U,U)$ and let M as well as M' be Lipschitz on bounded sets, i.e., for every $\rho > 0$ there is a constant $c_\rho > 0$ such that for all $v, \tilde{v} \in U$ with $\|v\|_U, \|\tilde{v}\|_U \leq \rho$ the inequalities*

$$\|M(v) - M(\tilde{v})\|_U \leq c_\rho \|v - \tilde{v}\|_U \quad \text{and} \quad \|M'(v) - M'(\tilde{v})\|_{\mathcal{L}_{\mathbb{R}}(U)} \leq c_\rho \|v - \tilde{v}\|_U.$$

hold. Further assume that $M(0) = 0$. Then N maps $H^1_{\mathrm{L}}([0,\infty),U))$ to itself. Moreover, for every $T > 0$ the restriction $N : H^1_{\mathrm{L}}([0,T],U) \to H^1_{\mathrm{L}}([0,T],U)$ is continuously $\mathbb{R}$–differentiable.

For a clear notation we again suppress the $\mathbb{R}$ in $\mathcal{L}_{\mathbb{R}}(U)$ and $\mathcal{L}_{\mathbb{R}}(U, \mathcal{L}_{\mathbb{R}}(U))$.

Proof. Note that the derivative of a Lipschitz and differentiable map is bounded. We infer the bounds $\|M'(v)\|_{\mathcal{L}(U)} \leq c_\rho$ and $\|M''(v)\|_{\mathcal{L}(U,\mathcal{L}(U))} \leq c_\rho$ for all $v \in U$ with $\|v\|_U < \rho$.

Fix any $u \in H^1_{\mathrm{L}}([0,\infty),U)$ and take $\rho > \|u\|_{L^\infty}$ so that $u(t) \in B(0,\rho) \subseteq U$ for all $t \geq 0$. From the assumption we derive

$$\|M(u(t))\|_U = \|M(u(t)) - M(0)\|_U \leq c_\rho \|u(t)\|_U$$

for almost all $t \geq 0$. This shows that $N(u)$ lies in $L^2([0,\infty),U)$.

Because M is Lipschitz on $B(0,\rho)$, the function $M \circ u$ is still absolutely continuous. Indeed, for arbitrary $m \in \mathbb{N}$ and $0 \leq a_1 \leq b_1 \leq \ldots \leq a_m \leq b_m$ we can estimate

$$\sum_{k=1}^m \|M(u(b_k)) - M(u(a_k))\|_U \leq c_\rho \sum_{k=1}^m \|u(b_k) - u(a_k)\|_U.$$

Using that u is absolutely continuous, for every $\varepsilon > 0$ we find a $\delta > 0$ such that the right-hand side is less or equal ε provided that $\sum_{k=1}^m (b_k - a_k) \leq \delta$.

If u is differentiable at $t \geq 0$, then by the chain rule $(M \circ u)'(t) = M'(u(t))u'(t)$. Hence this is true almost everywhere on $[0,\infty)$. The boundedness of $\|M'(v)\|_{\mathcal{L}(U)}$ for $v \in B(0,\rho)$ implies that

$$\|M'(u(t))u'(t)\|_U \leq \|M'(u(t))\|_{\mathcal{L}(U)} \|u'(t)\|_U \leq c_\rho \|u'(t)\|_U$$

for almost all $t \geq 0$ so that $M'(u(\,\cdot\,))u'(\,\cdot\,)$ belongs to $L^2([0,\infty),U)$. Since finally $(M \circ u)(0) = M(0) = 0$, we deduce that $M \circ u$ lies in $H^1_{\mathrm{L}}([0,\infty),U)$.

Next, let $h \in H^1_{\mathrm{L}}([0,\infty),U)$. We claim that the truncation of $M'(u(\,\cdot\,))h(\,\cdot\,)$ to $[0,T]$ is the derivative of $P_T N$ at u applied to h. Let us first show that $M'(u(\,\cdot\,))h(\,\cdot\,)$ belongs to $H^1_{\mathrm{L}}([0,\infty),U)$. We can clearly bound

$$\|M'(u(\,\cdot\,))h(\,\cdot\,)\|_{L^2} \leq \|M'(u(\,\cdot\,))\|_{L^\infty([0,\infty),\mathcal{L}(U))} \|h\|_{L^2} \leq c_\rho \|h\|_{H^1},$$

and linearity implies that $M'(u(0))h(0) = M'(0)0 = 0$.

Again we have to make sure that $M'(u(\,\cdot\,))h(\,\cdot\,)$ is absolutely continuous. Similar to the above procedure this is derived from the fact that M' is Lipschitz on $B(0,\rho)$. The central estimate is

$$\begin{aligned}
&\sum_{k=1}^{m}\|M'(u(b_k))h(b_k)-M'(u(a_k))h(a_k)\|_U\\
&\quad\le \sum_{k=1}^{m}\|M'(u(b_k))\|_{\mathcal{L}(U)}\|h(b_k)-h(a_k)\|_U+\sum_{k=1}^{m}\|M'(u(b_k))-M'(u(a_k))\|_{\mathcal{L}(U)}\|h(a_k)\|_U\\
&\quad\le c_\rho\sum_{k=1}^{m}\|u(b_k)-u(a_k)\|_U+c_\rho\|h\|_{L^\infty}\sum_{k=1}^{m}\|u(b_k)-u(a_k)\|_U.
\end{aligned}$$

Now we use that u, h are absolutely continuous and that h is bounded. Applications of the product rule as well as the chain rule next yield that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}[M'(u(t))h(t)]=[M''(u(t))u'(t)]h(t)+M'(u(t))h'(t),$$

whenever u and h are differentiable at $t\ge 0$. For almost all $t\ge 0$ we have

$$\begin{aligned}
&\|[M''(u(t))u'(t)]h(t)+M'(u(t))h'(t)\|_U\\
&\qquad\le \|M''(u(t))\|_{\mathcal{L}(U,\mathcal{L}(U))}\|h\|_{L^\infty}\|u(t)\|_U+\|M'(u(t))\|_{\mathcal{L}(U)}\|h'(t)\|_U\\
&\qquad\le c_\rho\|h\|_{L^\infty}\|u(t)\|_U+c_\rho\|h'(t)\|_U.
\end{aligned}$$

Since the right-hand side is square integrable as a function of t, we deduce that the function $[M''(u(\,\cdot\,))u'(\,\cdot\,)]h(\,\cdot\,)+M'(u(\,\cdot\,))h'(\,\cdot\,)$ belongs to $L^2([0,\infty),U)$. This in turn means that $M'(u(\,\cdot\,))h(\,\cdot\,)$ is contained in $H^1_{\mathrm{L}}([0,\infty),U)$. Moreover, we obtain

$$\begin{aligned}
\|[M''(u(\,\cdot\,))u'(\,\cdot\,)]h(\,\cdot\,)+M'(u(\,\cdot\,))h'(\,\cdot\,)\|_{L^2}&\le c_\rho\|h\|_{L^\infty}\|u'\|_{L^2}+c_\rho\|h'\|_{L^2}\\
&\lesssim (c_\rho\|u'\|_{L^2}+1)\|h\|_{H^1}.
\end{aligned}$$

Note that we have also shown that $\|M'(u(\,\cdot\,))h(\,\cdot\,)\|_{H^1}$ is bounded by a constant times $\|h\|_{H^1}$ and consequently the linear operator $H^1_{\mathrm{L}}([0,\infty),U)\to H^1_{\mathrm{L}}([0,\infty),U)$; $h\mapsto M'(u(\,\cdot\,))h(\,\cdot\,)$ is bounded for fixed $u\in H^1_{\mathrm{L}}([0,\infty),U)$.

Now let $T>0$. We still have to prove that $P_TM'(u(\,\cdot\,))h(\,\cdot\,)$ yields the derivative of P_TN at u. To keep the following estimates as simple as possible, here we deviate from our habit to abbreviate $\|\,\cdot\,\|_{L^2([0,\infty),U)}$ by $\|\,\cdot\,\|_{L^2}$. Instead, for the rest of the section we write $\|\,\cdot\,\|_{L^2}:=\|\,\cdot\,\|_{L^2([0,T],U)}$. Similarly let $\|\,\cdot\,\|_{H^1}$ denote $\|\,\cdot\,\|_{H^1([0,T],U)}$. We can then also drop the truncation P_T.

Clearly, the expression $\|N(u+h)-N(u)-M'(u(\,\cdot\,))h(\,\cdot\,)\|_{H^1}$ is less or equal $\sqrt{2}$ times

$$\begin{aligned}
\|M'(u(\,\cdot\,)+h(\,\cdot\,))(u'(\,\cdot\,)+h'(\,\cdot\,))-M'(u(\,\cdot\,))u'(\,\cdot\,)-[M''(u(\,\cdot\,))u'(\,\cdot\,)]h(\,\cdot\,)-M'(u(\,\cdot\,))h'(\,\cdot\,)\|_{L^2}&\\
+\|M(u(\,\cdot\,)+h(\,\cdot\,))-M(u(\,\cdot\,))-M'(u(\,\cdot\,))h(\,\cdot\,)\|_{L^2}.&\qquad(4.31)
\end{aligned}$$

We treat both summands separately. The reasoning is analog to what we did before, so we decided to be rather brief. The second summand is the easy part. With Minkowski's and

Hölder's inequalities we derive that

$$
\begin{aligned}
&\|M(u(\cdot)+h(\cdot))-M(u(\cdot))-M'(u(\cdot))h(\cdot)\|_{L^2}\\
&\qquad=\left\|\int_0^1 (M'(u(\cdot)+\sigma h(\cdot))-M'(u(\cdot)))h(\cdot)\,\mathrm{d}\sigma\right\|_{L^2}\\
&\qquad\le\int_0^1 \left\|(M'(u(\cdot)+\sigma h(\cdot))-M'(u(\cdot)))h(\cdot)\right\|_{L^2}\,\mathrm{d}\sigma\\
&\qquad\le\int_0^1 \|M'(u(\cdot)+\sigma h(\cdot))-M'(u(\cdot))\|_{L^2([0,T],\mathcal{L}(U))}\,\mathrm{d}\sigma\,\|h\|_{L^\infty}\\
&\qquad\lesssim\int_0^1 \|M'(u(\cdot)+\sigma h(\cdot))-M'(u(\cdot))\|_{L^2([0,T],\mathcal{L}(U))}\,\mathrm{d}\sigma\,\|h\|_{H^1}.
\end{aligned}
$$

Let $\varepsilon>0$ and $\sigma\in[0,1]$. Surely for $\delta\le\varepsilon\, c_{\rho+1}^{-1}$ and all $h\in H^1_{\mathrm{L}}([0,\infty),U)$ with $\|h\|_{H^1}\le\delta$ we have $\|h\|_{L^2}\le\|h\|_{H^1}\le\delta$. On the other hand we can choose δ so small that $\|h\|_{L^\infty}<1$ provided that $\|h\|_{H^1}\le\delta$. Then $u(t)+\sigma h(t)\in B(0,\rho+1)\subseteq U$ for all $t\ge0$. We can thus estimate

$$
\begin{aligned}
\|M'(u(\cdot)+\sigma h(\cdot))-M'(u(\cdot))\|^2_{L^2([0,T],\mathcal{L}(U))}&=\int_0^T\|M'(u(t)+\sigma h(t))-M'(u(t))\|^2_{\mathcal{L}(U)}\,\mathrm{d}t\\
&\le c_{\rho+1}^2\int_0^T\sigma^2\|h(t)\|_U^2\,\mathrm{d}t\le c_{\rho+1}^2\|h\|^2_{L^2}\le\varepsilon^2.
\end{aligned}
$$

Now we consider the other summand in (4.31). The same techniques as above yield that for all $h\in H^1_{\mathrm{L}}([0,\infty),U)$ with $\|h\|_{H^1}<1$ we have

$$
\begin{aligned}
&\|M'(u(\cdot)+h(\cdot))(u'(\cdot)+h'(\cdot))-M'(u(\cdot))u'(\cdot)-[M''(u(\cdot))u'(\cdot)]h(\cdot)-M'(u(\cdot))h'(\cdot)\|_{L^2}\\
&\quad\le\|M'(u(\cdot)+h(\cdot))h'(\cdot)-M'(u(\cdot))h'(\cdot)\|_{L^2}\\
&\qquad\quad+\|M'(u(\cdot)+h(\cdot))u'(\cdot)-M'(u(\cdot))u'(\cdot)-[M''(u(\cdot))u'(\cdot)]h(\cdot)\|_{L^2}\\
&\quad\le\int_0^1\|[M''(u(\cdot)+\sigma h(\cdot))h(\cdot)]h'(\cdot)\|_{L^2}\,\mathrm{d}\sigma\\
&\qquad\quad+\int_0^1\|[M''(u(\cdot)+\sigma h(\cdot))u'(\cdot)-M''(u(\cdot))u'(\cdot)]h(\cdot)\|_{L^2}\,\mathrm{d}\sigma\\
&\quad\le\int_0^1\|[M''(u(\cdot)+\sigma h(\cdot))\|_{\mathcal{L}(U,\mathcal{L}(U))}\,\mathrm{d}\sigma\,\|h\|_{L^\infty}\|h'\|_{L^2}\\
&\qquad\quad+\int_0^1\|[M''(u(\cdot)+\sigma h(\cdot))u'(\cdot)-M''(u(\cdot))u'(\cdot)]h(\cdot)\|_{L^2}\,\mathrm{d}\sigma\\
&\quad\lesssim c_{\rho+1}\|h\|^2_{H^1}\\
&\qquad\quad+\int_0^1\|[M''(u(\cdot)+\sigma h(\cdot))u'(\cdot)-M''(u(\cdot))u'(\cdot)]\|_{L^2([0,T],\mathcal{L}(U))}\,\mathrm{d}\sigma\,\|h\|_{L^\infty}.
\end{aligned}
$$

To treat the second part of the right-hand side let (h_k) be a sequence in $H^1_{\mathrm{L}}([0,\infty),U)$ with $\|h_k\|_{H^1}\to0$ as $k\to\infty$. Note that $\|h_k(t)\|_U\le\|h_k\|_{L^\infty}\lesssim\|h_k\|_{H^1}$ converges to zero for all $t\ge0$. Choosing a subsequence (again denoted (h_k)) we can assume that $\|h_k\|_{L^\infty}<1$ for all $k\in\mathbb{N}$. We shall show that by the dominated convergence theorem

$$
\begin{aligned}
&\|[M''(u(\cdot)+\sigma h_k(\cdot))u'(\cdot)-M''(u(\cdot))u'(\cdot)]\|^2_{L^2([0,T],\mathcal{L}(U))}\\
&\qquad\qquad=\int_0^T\|[M''(u(t)+\sigma h_k(t))u'(t)-M''(u(t))u'(t)]\|^2_{\mathcal{L}(U)}\,\mathrm{d}t
\end{aligned}
$$

converges to zero as $k \to \infty$. Clearly, the integrand tends to zero for almost all $t \geq 0$ as $k \to \infty$. On the other hand, it can be estimated by $2c_{\rho+1}^2 \|u'(t)\|_U^2$ for almost every $t \geq 0$. As in the proof of Lemma 4.39 we conclude that for arbitrary $\varepsilon > 0$ we find a radius $\delta > 0$ such that for all $h \in H^1_{\mathrm{L}}([0,\infty), U)$ with $\|h\|_{H^1} < \delta$ the expression in the last display is less or equal ε^2 for all $\sigma \in [0,1]$.

It remains to prove that N' is continuous. Once more we use that M' is Lipschitz on bounded sets. Let $u, h, w \in H^1_{\mathrm{L}}([0,\infty), U)$ and choose $\rho > \max\{\|u+h\|_{L^\infty}, \|u\|_{L^\infty}\}$. We infer that

$$\begin{aligned}
(\sqrt{2})^{-1}&\|M'(u(\cdot)+h(\cdot))w(\cdot) - M'(u(\cdot))w(\cdot)\|_{H^1} \\
&\leq \|[M'(u(\cdot)+h(\cdot)) - M'(u(\cdot))]w(\cdot)\|_{L^2} \\
&\qquad + \|[M''(u(\cdot)+h(\cdot))(u+h)'(\cdot) - M''(u(\cdot))u'(\cdot)]w(\cdot)\|_{L^2} \\
&\qquad + \|[M'(u(\cdot)+h(\cdot)) - M'(u(\cdot))]w'(\cdot)\|_{L^2} \\
&\leq c_\rho \|u+h-u\|_{L^\infty}(\|w\|_{L^2} + \|w'\|_{L^2}) + \|[M''(u+h(\cdot))h'(\cdot)]w(\cdot)\|_{L^2} \\
&\qquad + \|M''(u+h(\cdot))u'(\cdot) - M''(u(\cdot))u'(\cdot)\|_{L^2([0,T],\mathcal{L}(U))}\|w\|_{L^\infty} \\
&\leq c_\rho \|h\|_{L^\infty}(\|w\|_{L^2} + \|w'\|_{L^2}) + c_\rho \|h'\|_{L^2}\|w\|_{L^\infty} \\
&\qquad + \|M''(u+h(\cdot))u'(\cdot) - M''(u(\cdot))u'(\cdot)\|_{L^2([0,T],\mathcal{L}(U))}\|w\|_{L^\infty}.
\end{aligned}$$

The term $\|M''(u+h(\cdot))u'(\cdot) - M''(u(\cdot))u'(\cdot)\|_{L^2([0,T],\mathcal{L}(U))}$ is of the same type as the one the last step. Hence the right-hand side tends to zero as $\|h\|_{L^2} \leq \|h\|_{H^1} \to 0$. □

4.4.1 The Dirichlet Laplacian

To prepare our examples with the wave equation, we collect some known facts on Sobolev spaces and the Dirichlet Laplacian. In this paragraph we follow Sections 3.6 and 3.7 of [49]. We shall also frequently use notation and results from Appendix A.

Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open and bounded set. Every $f \in L^1_{\mathrm{loc}}(\mathcal{O})$ can be considered as the distribution (a "continuous" linear functional on $C_c^\infty(\mathcal{O})$) via

$$C_c^\infty(\mathcal{O}) \to \mathbb{C}; \quad h \mapsto \langle h, f \rangle := \int_{\mathcal{O}} h(x) f(x)\,\mathrm{d}x.$$

For $\alpha \in \mathbb{N}_0^n$, the distributional derivative of f is the distribution given by

$$\langle h, \partial^\alpha f \rangle := (-1)^{|\alpha|}\langle \partial^\alpha h, f \rangle = (-1)^{|\alpha|} \int_{\mathcal{O}} \partial^\alpha h(x) f(x)\,\mathrm{d}x.$$

By saying e.g. $\partial^\alpha f \in L^2(\mathcal{O})$, we mean that there is another function $g \in L^2(\mathcal{O})$ such that the distribution determined by g equals $\partial^\alpha f$, i.e.,

$$\langle h, g \rangle = \int_{\mathcal{O}} g(x) h(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathcal{O}} f(x) \partial^\alpha h(x)\,\mathrm{d}x = \langle h, \partial^\alpha f \rangle$$

for every $h \in C_c^\infty(\mathcal{O})$. Let us collect some facts on the Sobolev spaces

$$H^m(\mathcal{O}) = \{f \in L^1_{\mathrm{loc}}(\mathcal{O}) \,|\, \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq m : \ \partial^\alpha f \in L^2(\mathcal{O})\}.$$

where $m \in \mathbb{N}_0$. Choosing $\alpha = 0$, we see that $H^m(\mathcal{O})$ is a subset of $L^2(\mathcal{O})$. Actually it is a dense subset, because $C_c^\infty(\mathcal{O})$ is contained in $H^m(\mathcal{O})$. This follows from the fact that if $f \in L^1_{\mathrm{loc}}(\mathcal{O})$ is continuously differentiable in the classical sense the distribution defined by the classical derivative coincides with the distributional derivative. More precisely, for $f \in C^{|\alpha|}(\mathcal{O})$ we have $\partial^\alpha f \in C(\mathcal{O}) \subseteq L^1_{\mathrm{loc}}(\mathcal{O})$ and partial integration yields

$$\int_{\mathcal{O}} h(x)\partial^\alpha f(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_{\mathcal{O}} \partial^\alpha h(x) f(x)\,\mathrm{d}x = \langle h, \partial^\alpha f\rangle \quad \text{for all } h \in C_c^\infty(\mathcal{O}).$$

The Sobolev space $H^m(\mathcal{O})$ is a Hilbert space, when equipped with the norm given by

$$\|f\|^2_{H^m(\mathcal{O})} = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le m}} \|\partial^\alpha f\|^2_{L^2(\mathcal{O})}.$$

For $m = 1$ this reads

$$\|f\|^2_{H^1(\mathcal{O})} = \|f\|^2_{L^2(\mathcal{O})} + \sum_{j=1}^n \|\partial_j f\|^2_{L^2(\mathcal{O})}.$$

Where no confusion is to be expected, we write $\|\cdot\|_{H^m}$ instead of $\|\cdot\|_{H^m(\mathcal{O})}$ and similar with $\|\cdot\|_{L^2}$. We shall further need the completion of $C_c^\infty(\mathcal{O})$ with respect to $\|\cdot\|_{H^1}$, that is

$$H_0^1(\mathcal{O}) := \overline{C_c^\infty(\mathcal{O})}^{\|\cdot\|_{H^1}} \subseteq H^1(\mathcal{O}).$$

Finally, $H^{-1}(\mathcal{O})$ is defined as the dual space of $H_0^1(\mathcal{O})$.

Let $f \in L^1_{\mathrm{loc}}(\mathcal{O})$. Then Δf denotes the distribution defined by

$$\langle h, \Delta f\rangle := \sum_{j=1}^n \langle h, \partial_j^2 f\rangle = \sum_{j=1}^n (-1)^2\langle \partial_j^2 h, f\rangle = \sum_{j=1}^n \int_{\mathcal{O}} \partial_j^2 h(x) f(x)\,\mathrm{d}x \quad \text{for } h \in C_c^\infty(\mathcal{O}).$$

Moreover, the symbol ∇f stands for the "vector of distributions" $\nabla f = (\partial_1 f, \ldots, \partial_n f)$.

Assume that $f \in H^1(\mathcal{O})$ and let $j \in \{1, \ldots, n\}$. Since then $\partial_j f$ belongs to $L^2(\mathcal{O})$ the distributional derivative $\partial_j^2 f = \partial_j(\partial_j f)$ is given by

$$\langle h, \partial_j^2 f\rangle = -\langle \partial_j h, \partial_j f\rangle = -\int_{\mathcal{O}} \partial_j f(x)\partial_j h(x)\,\mathrm{d}x = -(\partial_j f \mid \partial_j \overline{h})_{L^2}$$

for $h \in C_c^\infty(\mathcal{O})$, where $(\cdot \mid \cdot)_{L^2}$ denotes the inner product on $L^2(\mathcal{O})$. Note that in this situation $\nabla f \in (L^2(\mathcal{O}))^n$. It follows that

$$\langle h, \Delta f\rangle = -\sum_{j=1}^n (\partial_j f \mid \partial_j \overline{h})_{L^2} = -(\nabla f \mid \nabla \overline{h})_{(L^2)^n} \tag{4.32}$$

for $f \in H^1(\mathcal{O})$ and $h \in C_c^\infty(\mathcal{O})$.

Next we show that $f \in H^1(\mathcal{O})$ with $\Delta f \in L^2(\mathcal{O})$ and $g \in H_0^1(\mathcal{O})$ satisfy

$$(\Delta f \mid g)_{L^2} = -(\nabla f \mid \nabla g)_{(L^2)^n}. \tag{4.33}$$

In fact, we find a sequence (h_k) in $C_c^\infty(\mathcal{O})$ with $\|g - h_k\|_{H^1} \to 0$ as $k \to \infty$. Using Hölder's inequality, the claim follows from

$$\begin{aligned}
\left|(\Delta f \mid g)_{L^2} - (-1)(\nabla f \mid \nabla g)_{(L^2)^n}\right| &\le \left|(\Delta f \mid g)_{L^2} - \langle \overline{h_k}, \Delta f\rangle\right| + \left|(\nabla f \mid \nabla g)_{(L^2)^n} + \langle \overline{h_k}, \Delta f\rangle\right| \\
&= |(\Delta f \mid g - h_k)_{L^2}| + \left|(\nabla f \mid \nabla g)_{(L^2)^n} - (\nabla f \mid \nabla h_k)_{(L^2)^n}\right| \\
&\le \|\Delta f\|_{L^2}\|g - h_k\|_{L^2} + \sum_{j=1}^n \|\partial_j f\|_{L^2}\|\partial_j g - \partial_j h_k\|_{L^2} \\
&\le (\|\Delta f\|_{L^2} + \|f\|_{H^1})\,\|g - h_k\|_{H^1} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

The *Dirichlet Laplacian on $\mathcal{O}$* is $-A_0$, where A_0 is the linear operator in $L^2(\mathcal{O})$ with domain $\mathrm{D}(A_0) = \{f \in H_0^1(\mathcal{O}) \,|\, \Delta f \in L^2(\mathcal{O})\}$ defined as $A_0 f = -\Delta f$. Hence the graph is

$$\{(f,g) \in L^2(\mathcal{O}) \times L^2(\mathcal{O}) \,|\, f \in H_0^1(\mathcal{O}) \text{ and } -\Delta f = g\}.$$

Equation (4.33) yields

$$(A_0 f \,|\, g)_{L^2} = -(\Delta f \,|\, g)_{L^2} = (\nabla f \,|\, \nabla g)_{(L^2)^n} = (f \,|\, A_0 g)_{L^2} \tag{4.34}$$

for $f, g \in \mathrm{D}(A_0)$, showing that A_0 is symmetric. To prove that A_0 is self-adjoint and even strictly positive, we need Poincaré's inequality.

Theorem 4.44 (Poincaré inequality)**.** *Let $\mathcal{O} \subseteq \mathbb{R}^n$ open and bounded. Then there is a constant $c > 0$ such that for all $f \in H_0^1(\mathcal{O})$ we have*

$$\|f\|_{L^2(\mathcal{O})} \le c\|\nabla f\|_{(L^2(\mathcal{O}))^n}.$$

For a proof we refer to Proposition 13.4.10 of [49]. See Theorem 3 in Section 5.6.1 of [16] for a version of this result on general Sobolev spaces $W_0^{1,p}(\mathcal{O})$. Note that the special version for $L^2(\mathcal{O})$ has an elementary proof. Further note that this result also holds for some unbounded sets $\mathcal{O}$. For example it suffices that $\mathcal{O}$ is bounded in one direction, actually Proposition 13.4.10 cited above only assumes this.

The first consequence of Poincaré's inequality is that $\|\,.\,\|_{H^1}$ on $H_0^1(\mathcal{O})$ is equivalent to the norm given by $\|\nabla f\|_{(L^2)^n}$, since we have

$$\|\nabla f\|_{(L^2)^n} \le \|f\|_{H^1} = \left(\|f\|_{L^2}^2 + \|\nabla f\|_{(L^2)^n}^2\right)^{1/2} \le \sqrt{c^2+1}\|\nabla f\|_{(L^2)^n}$$

for all $f \in H_0^1(\mathcal{O})$. We mention that this norm corresponds to the scalar product defined by $(\nabla f \,|\, \nabla g)_{(L^2)^n}$ for $f, g \in H_0^1(\mathcal{O})$, which we already encountered.

If we can prove that A_0 is onto, then it follows that $0 \in \rho(A_0)$ and that A_0 is self-adjoint (see Lemma A.7). To this end, let $g \in L^2(\mathcal{O})$. From the Cauchy-Schwarz inequality we derive

$$|(g \,|\, \overline{f})_{L^2}| \le \|g\|_{L^2}\|f\|_{L^2} \le \|g\|_{L^2}\|f\|_{H^1} \lesssim \|g\|_{L^2}\|\nabla f\|_{(L^2)^n} \quad \text{for all } f \in H_0^1(\mathcal{O}).$$

Hence $f \mapsto (g \,|\, \overline{f})_{L^2}$ is a bounded functional on $H_0^1(\mathcal{O})$. Therefore the Riesz representation theorem yields a function $f_0 \in H_0^1(\mathcal{O})$ with $(g \,|\, \overline{f})_{L^2} = (\nabla f_0 \,|\, \nabla \overline{f})_{(L^2)^n}$ for every $f \in H_0^1(\mathcal{O})$. Using (4.33) we conclude

$$\langle h, g\rangle = (g \,|\, \overline{h})_{L^2} = (\nabla f_0 \,|\, \nabla \overline{h})_{(L^2)^n} = (-1)\langle h, \Delta f_0\rangle \quad \text{for every } h \in C_c^\infty(\mathcal{O}).$$

This means $-\Delta f_0 = g \in L^2(\mathcal{O})$ and thus $f_0 \in \mathrm{D}(A_0)$ as well as $A_0 f_0 = g$.

The second consequence of Poincaré's inequality is that A_0 is strictly positive. Indeed, set $m = c^{-1} > 0$ where $c > 0$ is from Theorem 4.44. Then we have

$$(A_0 f \,|\, f)_{L^2} = (\nabla f \,|\, \nabla f)_{(L^2)^n} \ge m\|f\|_{L^2}^2 \quad \text{for all } f \in \mathrm{D}(A_0).$$

We emphasize that from (4.33) it follows that $\|\,.\,\|_{1/2}$ actually equals $\|\nabla(\,.\,)\|_{(L^2)^n}$, see (A.6). Clearly $C_c^\infty(\mathcal{O})$ is a subspace of $\mathrm{D}(A_0) \subseteq H_0^1(\mathcal{O})$. On the other hand $C_c^\infty(\mathcal{O})$ is dense in $H_0^1(\mathcal{O})$. Since $[\mathrm{D}(A_0)]_{1/2}$ is the closure of $\mathrm{D}(A_0)$ with respect to $\|\,.\,\|_{1/2}$ we get

$$[\mathrm{D}(A_0)]_{1/2} = H_0^1(\mathcal{O}).$$

4.4.2 A wave equation with distributed control

This example is based on Example 11.2.2 of [49]. Assume that $\mathcal{O} \subseteq \mathbb{R}^n$ is open and bounded with boundary $\partial\mathcal{O}$ of class C^2. It can be shown that in this case $\mathrm{D}(A_0) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, see Theorem 3.6.2 in [49]. Further let $\mathcal{O}_\mathrm{c} \subseteq \mathcal{O}$ be open and nonempty. A linear wave equation with inner control and zero Dirichlet boundary condition is

$$\begin{aligned} \partial_t^2\omega(t,\xi) &= \Delta\omega(t,\xi) + \mu(t,\xi), && (t,\xi) \in (0,\infty)\times\mathcal{O} \\ \omega(t,\xi) &= 0, && (t,\xi) \in (0,\infty)\times\partial\mathcal{O} \\ \omega(0,\xi) &= f_0(\xi), \quad \partial_t\omega(0,\xi) = g_0(\xi), && \xi \in \mathcal{O} \end{aligned} \tag{4.35}$$

for some initial values $f_0, g_0 : \mathcal{O} \to \mathbb{C}$ and a control $\mu : [0,\infty)\times\mathcal{O} \to \mathbb{C}$. To model that the control μ only acts on $\mathcal{O}_\mathrm{c}$, let $\mu(t,\xi) = 0$ for almost all $\xi \in \mathcal{O}\setminus\mathcal{O}_\mathrm{c}$ and every $t \geq 0$.

Via $z(t) = (\omega(t,\,\cdot\,), \partial_t\omega(t,\,\cdot\,))$, $x_0 = (f,g)$ and $u(t) = \mu(t,\,\cdot\,)$ for $t \geq 0$ this problem can be written formally equivalent in the form (4.9), that is

$$z'(t) = Az(t) + B^l u(t); \quad z(0) = x_0. \tag{4.36}$$

More precisely, to satisfy the boundary condition (see below) set $X = H_0^1(\mathcal{O}) \times L^2(\mathcal{O}) = [\mathrm{D}(A_0)]_{1/2} \times [\mathrm{D}(A_0)]_0$. Further let

$$A : \mathrm{D}(A) \to X; \quad A(f,g) = (g, -A_0 f),$$

where $\mathrm{D}(A) = [\mathrm{D}(A_0)]_1 \times [\mathrm{D}(A_0)]_{1/2} = (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})) \times H_0^1(\mathcal{O})$. Finally set $U = L^2(\mathcal{O}_\mathrm{c})$ and define the linear operator

$$B^l : U \to X; \quad B^l v = (0, v).$$

In this we consider $L^2(\mathcal{O}_\mathrm{c})$ as a subset of $L^2(\mathcal{O})$ where $v(\xi) = 0$ for almost every $\xi \in \mathcal{O}\setminus\mathcal{O}_\mathrm{c}$ for elements $v \in L^2(\mathcal{O}_\mathrm{c})$. Obviously B^l is bounded, i.e., $B^l \in \mathcal{L}(U,X)$. Thus this operator is L^2–admissible for every strongly continuous semigroup on X.

We shall see that A is the generator of a unitary group $\mathbb{T}$ on X. Indeed, one easily checks that A is skew-symmetric using that X is a Hilbert space with inner product given by

$$\Big((f_1,g_1)\,\Big|\,(f_2,g_2)\Big)_X = \Big(A_0^{1/2} f_1 \,\Big|\, A_0^{1/2} f_2\Big)_{L^2} + (g_1 \,|\, g_2)_{L^2}$$

for $(f_1,g_1), (f_2,g_2) \in X$. The calculation is omitted because we do a similar calculation in the example of Subsection 4.4.4. Using that $-A_0 : \mathrm{D}(A_0) \to L^2(\mathcal{O})$ is onto we deduce that A is onto. It follows that $0 \in \rho(A)$ and that $-A$ is skew-adjoint. Stone's theorem (see e.g. Theorem II.3.24 in [15]) yields the claim.

As usual, let $\mathbb{T}$ be the group generated by A and let Φ^l be the family of input maps corresponding to B^l. In Example 11.2.2 of [49] the authors provide sufficient conditions under which $(\mathbb{T}, \Phi^l)$ is exactly controllable in some time specified below. They read as follows. If there exists a reference point $\xi_0 \in \mathbb{R}^n$ and a radius $\varepsilon > 0$ such that

$$\{\xi \in \mathcal{O} \,|\, \mathrm{d}(\xi, \Gamma_{\xi_0}) < \varepsilon\} \subseteq \overline{\mathcal{O}_\mathrm{c}} \qquad \text{where} \qquad \Gamma_{\xi_0} = \{\xi \in \partial\mathcal{O} \,|\, (\xi - \xi_0)\cdot\nu(\xi) > 0\},$$

then $(\mathbb{T}, \Phi^l)$ is exactly controllable in any time $T > 2\sup\{|\xi - \xi_0| \,|\, \xi \in \mathcal{O}\}$.

We take the system (4.36) as our starting point. In this example, let us explain the connection between the abstract problem (4.36) and the partial differential equations (4.35). Let $z \in C([0,\infty), X)$ be a strong solution of (4.36) and for $t \geq 0$ let $\omega(t, \boldsymbol{.})$ be the first component of $z(t)$. Then $\omega(t, \boldsymbol{.})$ lies in $H_0^1(\mathcal{O})$ for each $t \geq 0$, which means that it satisfies the Dirichlet boundary conditions of (4.35) in the sense of traces. From the corresponding properties of z it follows easily that $[0,\infty) \to L^2(\mathcal{O})$; $t \mapsto \omega(t, \boldsymbol{.})$ is absolutely continuous and differentiable almost everywhere on $[0,\infty)$, see the text after Definition 3.2. Since (4.36) holds for almost all $t \geq 0$ we can further infer the equations

$$\partial_t \omega(t, \boldsymbol{.}) = g(t) \qquad \text{and} \qquad \partial_t g(t) = -A_0 \omega(t, \boldsymbol{.}) + u(t)$$

for almost every $t \geq 0$, where $g(t)$ denotes the second component of $z(t)$. From the first equation we derive that $t \mapsto \omega(t, \boldsymbol{.})$ belongs to the Soblev space $H^2_{\mathrm{loc}}([0,\infty), H^{-1}(\mathcal{O}))$. By the second equation, we then see that ω satisfies also the first line of (4.35) in a generalized sense, that is $\partial_t^2 \omega(t, \boldsymbol{.}) = -A_0 \omega(t, \boldsymbol{.}) + u(t)$ holds in $H^{-1}(\mathcal{O})$ for almost all $t \geq 0$.

Take a function $m \in C^2(\mathbb{R}, \mathbb{R})$ with $m(0) \neq 0$ and $m(a) = 0$ for all $a \in \mathbb{R}$ with $|a| > R$ for some radius $R > 0$. We set $M(v) = m(\|v\|_U^2)v$ for $v \in U$. Since m is bounded on $\mathbb{R}$, this defines a map $M : U \to U$.

One could think of M being a fuse that blows if the energy $\|u\|_U^2$ gets to large. In this case m would be something like a cut-off function, that is constant on a neighborhood of the origin and then rapidly decreasing to zero near R. However, this is only an interpretation.

Using that $U \to \mathbb{R}$; $v \mapsto \|v\|_U^2 = (v \,|\, v)$ is $\mathbb{R}$–differentiable, with the product rule we infer that M is $\mathbb{R}$–differentiable. The derivative is given by

$$M'(v)w = m(\|v\|_U^2)w + m'(\|v\|_U^2) 2 \operatorname{Re}(v \,|\, w) v \quad \text{for } v, w \in U.$$

Clearly M' is continuous. From the fact that $m(a) = m'(a) = 0$ for $|a| > R$ we further deduce that it is bounded. Indeed,

$$\begin{aligned} \|M'(v)\|_{\mathcal{L}_{\mathbb{R}}(U)} &\leq |m(\|v\|_U^2)| + 2|m'(\|v\|_U^2)| \, \|v\|^2 \\ &\leq \begin{cases} \|m\|_{L^\infty(\mathbb{R})} + \|m'\|_{L^\infty(\mathbb{R})} R, & \|v\|_U^2 \leq R \\ 0, & \|v\|_U^2 > R \end{cases} \end{aligned} \tag{4.37}$$

for all $v \in U$. It follows that M is Lipschitz. Applying the product rule once more, we see that M is two times $\mathbb{R}$–differentiable with

$$\begin{aligned} [M''(v)w]h = {} & 2m'(\|v\|_U^2) \operatorname{Re}(v \,|\, h) w + 2m'(\|v\|_U^2) \operatorname{Re}(v \,|\, w) h \\ & + 2m'(\|v\|_U^2) \operatorname{Re}(h \,|\, w) v + 4m''(\|v\|_U^2) \operatorname{Re}(v \,|\, h) \operatorname{Re}(v \,|\, w) v. \end{aligned}$$

As before we infer that M'' is bounded and conclude that M' is Lipschitz.

Note that $M(0) = m(0)0 = 0$. Therefore $(0,0) \in X \times U$ is an equilibrium point of $(\mathbb{T}, \Phi)$. We further have

$$M'(0)w = m(0)w + 2m'(0) \operatorname{Re}(0 \,|\, w) v = m(0)w.$$

Because $m(0) \neq 0$ the operator $M'(0) \in \mathcal{L}_{\mathbb{R}}(U)$ is invertible and we may apply Theorem 4.37 provided that the linear system is exactly controllable. For every $R > 0$ it then yields radii

$r_1, r_2 > 0$ such that for all $x_0, x_1 \in X$ with $\|x_0\|_X \le r_1$ and $\|x_1\|_X \le r_2$ we find a function $u \in L^2([0,\infty), U)$ with

$$x_1 = \mathbb{T}_T x_0 + \Phi_T u \qquad \text{as well as} \qquad \|\mathbb{T}_t x_0 + \Phi_t u\|_X \le R \quad \text{for all } t \in [0,T].$$

Using that $0 \in \rho(A)$, it is easy to construct other equilibrium points $(x_*, v_*) \in X \times U$. Indeed, take any $v_* \in U$. Then for $x_* \in X$ the identity $Ax_* + B(v_*)$ is equivalent to $x_* = -A^{-1}B(v_*) = -A^{-1}B^l M(v_*)$. Due to the interpretation, we should then consider $M(v) = m(\|v - v_*\|_U^2)v$.

We further remark that we can replace M with the map given by $M(v) = M_0(v)v$, where $M_0 \in C^2(U, \mathcal{L}_\mathbb{R}(U))$ has bounded support and $M_0(0) \in \mathcal{L}_\mathbb{R}(U)$ is invertible. More generally, instead of the bounded support it suffices to assume that the operator norm $\|M_0(v)\|$ is bounded and that $\|M_0'(v)\|$ as well as $\|M_0''(v)\|$ decay sufficiently as $\|v\| \to \infty$.

4.4.3 An example with the transport equation

One of the most elementary linear control systems with an unbounded control operator is the one dimensional transport equation with input on the left boundary. To get an exactly controllable system, the space domain has to be bounded. We choose the interval $[0,1]$. Then the system is described by

$$\begin{aligned} \partial_t \omega(t,\xi) &= -\tfrac{\mathrm{d}}{\mathrm{d}\xi}\omega(t,\xi), && t \ge 0,\ \xi \in [0,1] \\ \omega(t,0) &= u(t), && t \ge 0 \\ \omega(0,\xi) &= x_0(\xi), && \xi \in [0,1]. \end{aligned} \tag{4.38}$$

This problem but with infinite space domain $[0,\infty)$ can be found in Example 10.1.9 of [49]. We proceed analogously and formulate (4.38) as 'boundary control system'. For this concept see Appendix C. We have to check the conditions of Definition C.2.

Set $X = L^2[0,1]$, $U = \mathbb{C}$ and $Z = H^1(0,1)$. Clearly Z is continuously embedded in X. Further set $Gx = x(0)$ and $Lx = -x'$ for $x \in Z$. Since $|Gx| \le \|x\|_{L^\infty} \lesssim \|x\|_{H^1}$ and $\|Lx\|_{L^2} = \|x'\|_{L^2} \le \|x\|_{H^1}$ this defines bounded operators $G \in \mathcal{L}(Z,U)$ and $L \in \mathcal{L}(Z,X)$. Formally system (4.38) is equivalent to

$$\begin{aligned} z'(t) &= Lz(t); \qquad z(0) = x_0, \\ Gz(t) &= u(t). \end{aligned}$$

Obviously $\ker G = H^1_\mathrm{L}(0,1) = \{f \in H^1(0,1) \,|\, f(0) = 0\}$ is dense in $L^2[0,1]$ and has a bounded right inverse. Hence we consider the restriction of L to $H^1_L(0,1)$, that is $Ax = -x'$ with domain $\mathrm{D}(A) = H^1_\mathrm{L}(0,1)$. It is well-known that A generates the strongly continuous semigroup $\mathbb{T}$ on X given by

$$(\mathbb{T}_t f)(s) = \begin{cases} 0, & s \in [0,t) \\ f(s-t), & s \in [t,1], \end{cases}$$

which is called the *vanishing right shift semigroup*. So (L,G) is a boundary control system. Further we know that the dual semigroup $\mathbb{T}^*$ is formed by the *vanishing left shift operators*

$$(\mathbb{T}_t^* f)(s) = \begin{cases} f(s+t), & s \in [0,1-t) \\ 0, & s \in [1-t,1] \end{cases}$$

for $t \in [0,1)$ and $\mathbb{T}_t^* = 0$ for $t \geq 1$. It is generated by $A^* f = f'$ with domain

$$\mathrm{D}(A^*) = H^1_{\mathrm{R}}(0,1) = \{f \in H^1(0,1) \,|\, f(1) = 0\} = X_{1,d}.$$

See e.g. Examples 2.3.7 and 2.8.7 in [49]. In particular we can apply Proposition C.5 and obtain an operator $B^l \in \mathcal{L}(U, X_{-1})$ with $Lx = (A|_0)x + B^l G x$ for $x \in Z$. For a better description of B^l we identify $X_{-1} = (X_{1,d})^*$. Then for $x \in Z$ and $f \in H^1_{\mathrm{R}}(0,1)$ we have

$$\begin{aligned} -\int_0^1 f(\xi) x'(\xi)\,\mathrm{d}\xi - \int_0^1 f'(\xi) x(\xi)\,\mathrm{d}\xi &= \langle f, Lx\rangle - \langle A^* f, x\rangle = \langle f, Lx - (A|_0)x\rangle_{X_{1,d}} \\ &= \langle f, B^l G x\rangle_{\mathbb{C}} = \langle (B^l)^* f, Gx\rangle = x(0)(B^l)^* f. \end{aligned}$$

Integration by parts applied to the first integral on the left-hand side now yields that

$$\begin{aligned} x(0)(B^l)^* f &= -f(1)x(1) + f(0)x(0) + \int_0^1 f'(\xi)x(\xi)\,\mathrm{d}\xi - \int_0^1 f'(\xi)x(\xi)\,\mathrm{d}\xi \\ &= f(0)x(0). \end{aligned}$$

We obtain the identity $(B^l)^* f = f(0)$ for $f \in H^1_{\mathrm{R}}(0,1)$. Note that $B^l \in \mathcal{L}(\mathbb{C}, X_{-1})$ is determined by $B^l 1$, so $\langle f, B^l 1\rangle = \langle (B^l)^* f, 1\rangle = (B^l)^* f = f(0)$. This means that $B^l v = v\delta_0$ with the delta functional $\delta_0 \in (H^1_{\mathrm{R}}(0,1))^*$. By an easy calculation one verifies that $\langle f, \mathbb{T}_t \delta_0\rangle = \langle f, \delta_t\rangle = f(t)$ for $f \in H^1_{\mathrm{R}}(0,1)$. Let $u \in L^2([0,\infty)$. It is still simple but tedious to check that the input maps defined by $\mathbb{T}$ and B^l can be expressed as

$$(\Phi_t^l u)(s) = \begin{cases} u(t-s), & s \in [0,t) \\ 0, & s \in [t,1] \end{cases}$$

if $t < 1$ and $(\Phi_t^l u)(s) = u(t-s)$ for $s \in [0,1]$ if $t \geq 1$. We see that $\Phi_t^l u \in X$ for all $t \geq 0$ and so B^l is L^2–admissible for $\mathbb{T}$. Further note that Φ_1^l equals the time-reflection operator $Я_1$ from (2.10). We conclude that the control system $(\mathbb{T}, \Phi^l)$ is exactly controllable in time 1. In fact, for $x \in L^2[0,1]$ we have $\Phi_1^l Я_1 x = P_1 x = x$ and thus Φ_1^l is onto.

Let us finally introduce the map $M : U \to U$. To this end, choose a non-negative function $\dot{m} \in C_c^1(\mathbb{R})$ with $\dot{m}(0) > 0$, $\operatorname{supp} \dot{m} = [-1,1]$ and $\dot{m}(-a) = \dot{m}(a)$ for all $a \in \mathbb{R}$. For simplicity assume that $\int_{-1}^1 \dot{m}(s)\,\mathrm{d}s = 1$. We define $m \in C^2(\mathbb{R})$ by

$$m(a) = 2\int_{-\infty}^a \dot{m}(b)\,\mathrm{d}b - 1 \quad \text{for } a \in \mathbb{R}.$$

Then m is increasing and we have $m(a) = -1$ for $a \in (-\infty,-1)$ as well as $m(a) = 1$ for $a \in (1,\infty)$. Clearly $m' = 2\dot{m}$ and $m(0) = 2\int_{-1}^0 \dot{m}(b)\,\mathrm{d}b - 1 = 0$. For $v \in U = \mathbb{C}$ we define

$$M(v) = \big(m(\operatorname{Re} v), m(\operatorname{Im} v)\big).$$

More precisely $M(v) = \iota \circ M_0 \circ \iota^{-1} v$ where $\iota : \mathbb{R}^2 \to \mathbb{C}$ is the isometric $\mathbb{R}$–linear invertible map given by $(a,b) \mapsto a + ib$ and $M_0 : \mathbb{R}^2 \to \mathbb{R}^2$; $M_0(a,b) = (m(a), m(b))$. However, here and in the following we suppress ι and its inverse. Obviously M_0 is two-times continuously (partially) differentiable. It follows that M is two-times continuously $\mathbb{R}$–differentiable. We identify $M'(v)$ with

$$M_0'(v) = \begin{pmatrix} m'(\operatorname{Re} v) & 0 \\ 0 & m'(\operatorname{Im} v) \end{pmatrix}.$$

Note that $M'(0)$ is invertible. We shall now prove that M and M' are Lipschitz. Let $v, \tilde{v} \in \mathbb{C}$. By the mean value theorem there are $a, b \in \mathbb{R}$ with

$$\begin{aligned}|M(v) - M(\tilde{v})| &= \sqrt{(m(\operatorname{Re} v) - m(\operatorname{Re} \tilde{v}))^2 + (m(\operatorname{Im} v) - m(\operatorname{Im} \tilde{v}))^2} \\ &= \sqrt{(m'(a))^2(\operatorname{Re} v - \operatorname{Re} \tilde{v})^2 + (m'(b))^2(\operatorname{Im} v - \operatorname{Im} \tilde{v})^2} \\ &\le \|m'\|_{L^\infty(\mathbb{R})}|v - \tilde{v}|.\end{aligned}$$

In the same way, using equivalent matrix norms, we get

$$\begin{aligned}\|M'(v) - M'(\tilde{v})\|_{\mathcal{L}(U)} &\lesssim \left\| \begin{pmatrix} m'(\operatorname{Re} v) - m'(\operatorname{Re} \tilde{v}) & 0 \\ 0 & m'(\operatorname{Im} v) - m'(\operatorname{Im} \tilde{v}) \end{pmatrix} \right\|_{\mathcal{L}(\mathbb{R}^2)} \\ &= |m'(\operatorname{Re} v) - m'(\operatorname{Re} \tilde{v})| + |m'(\operatorname{Im} v) - m'(\operatorname{Im} \tilde{v})| \\ &\lesssim \|m''\|_{L^\infty(\mathbb{R})}|v - \tilde{v}|.\end{aligned}$$

Recall that $B(v) = B^l M(v)$ and $\Phi_t(u) = \Phi_t^l M \circ u$. Since $M(0) = 0$, it follows that $(0, 0) \in X \times U$ is an equilibrium point for $(\mathbb{T}, \Phi)$. Thus the conditions of Proposition 4.42 are satisfied. This means that we find radii $r_1, r_2 > 0$ such that for all $x_0 \in \overline{B}(0, r_1) \subseteq X_1$ and $x_1 \in \overline{B}(0, r_2) \subseteq Z$ we find an input $u \in H_L^1([0, \infty), U)$ with

$$x_1 = \mathbb{T}_T x_0 + \Phi_T u.$$

We mention that we can get rid of some simplifying assumptions. Instead of the special m constructed above, we can take any function $m \in C^2(\mathbb{R}, \mathbb{R})$ with the property that $m(0) = 0$, m' and m'' are bounded and $m'(0) \neq 0$. More generally we can also take two different such functions m_1, m_2 acting on the real and imaginary part.

4.4.4 A wave equation with boundary control

We continue to use the notation of Subsection 4.4.1. In this example we discuss the wave equation with a forcing term on the boundary. To this end, assume that $\mathcal{O} \subseteq \mathbb{R}^n$ is a bounded domain with boundary $\partial\mathcal{O}$ of class C^2. Consider an open subset $\Gamma \subset \partial\mathcal{O}$ of the boundary (open relative to $\partial\mathcal{O}$). Denote by σ the surface measure on $\partial\mathcal{O}$. We remark that $\sigma(\partial\mathcal{O}) < \infty$ due to the regularity of $\partial\mathcal{O}$ and the fact that $\mathcal{O}$ is bounded.

Following Section 10.9 of [49], we repeat some known linear results. A linear wave equation with Dirichlet boundary control is

$$\begin{aligned} \partial_t^2 \omega(t, \xi) &= \Delta\omega(t, \xi), && (t, \xi) \in (0, \infty) \times \mathcal{O} \\ \omega(t, \xi) &= 0, && (t, \xi) \in (0, \infty) \times \partial\mathcal{O} \setminus \Gamma \\ \omega(t, \xi) &= \mu(t, \xi), && (t, \xi) \in (0, \infty) \times \Gamma \\ \omega(0, \xi) &= f_0(\xi), \quad \partial_t\omega(0, \xi) = g_0(\xi), && \xi \in \mathcal{O}. \end{aligned} \tag{4.39}$$

Here $f_0, g_0 : \Omega \to \mathbb{C}$ are (probably given) initial values and $\mu : [0, \infty) \times \Gamma \to \mathbb{C}$ is the input.

We can think of ω being the displacement of an object over $\mathcal{O}$. Then $\partial_t\omega$ and $\partial_t^2\omega$ are velocity and acceleration, respectively. The differential operator Δ acts only on the space variable ξ. The object is kept in place on one part of the boundary – namely $\partial\mathcal{O} \setminus \Gamma$ – while it is forced to a displaced position μ on the other part Γ.

Equation (4.39) is in the form of a 'boundary control system'. In Appendix C we described a transformation of such systems to a control problem of the form (4.9), i.e,

$$z'(t) = Az(t) + B^l u(t); \quad z(0) = x_0. \tag{4.40}$$

As to be expected, we then have $z(t) = (\omega(t, \cdot), \omega'(t, \cdot))$, $x_0 = (f_0, g_0)$ and $u(t) = \mu(t, \cdot)$. The transformation is carried out in Section 10.9 of [49]. Let us depict the outcome of this procedure, the operators A and B^l. We mention that under natural conditions, solutions of problem (4.40) yield solutions of (4.39). See Proposition C.7 for more details.

Set $X := [\mathrm{D}(A_0)]_0 \times [\mathrm{D}(A_0)]_{-1/2} = L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$. In the last identity we used Remark A.12 to identify $[\mathrm{D}(A_0)]_{-1/2}$ with $([\mathrm{D}(A_0)]_{1/2})^*$. Note that X is a Hilbert space with inner product given by

$$\Big((f_1, g_1) \,\Big|\, (f_2, g_2)\Big)_X = (f_1 \,|\, f_2)_{L^2} + \Big(A_0^{-1/2} g_1 \,\Big|\, A_0^{-1/2} g_2\Big)_{L^2}.$$

Further consider the subspace $\mathrm{D}(A) := [\mathrm{D}(A_0)]_{1/2} \times [\mathrm{D}(A_0)]_0 = H_0^1(\mathcal{O}) \times L^2(\mathcal{O})$ as well as the linear operator

$$A : \mathrm{D}(A) \to X; \quad A(f, g) = (g, -A_0 f).$$

The extension $A_0 : H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O})$ is onto because $0 \in \rho(A_0)$, see (A.4). It follows that A is onto. Moreover, this map is skew-symmetric. Indeed, for $(f_1, g_1), (f_2, g_2) \in \mathrm{D}(A)$ we have

$$\begin{aligned}
\Big(A(f_1, g_1) \,\Big|\, (f_2, g_2)\Big)_X &= (g_1 \,|\, f_2)_{L^2} + \Big(A_0^{-1/2}(-A_0) f_1 \,\Big|\, A_0^{-1/2} g_2\Big)_{L^2} \\
&= (-1)\Big(g_1 \,\Big|\, (-1) A_0^{-1} A_0 f_2\Big)_{L^2} - \Big(A_0^{-1/2} A_0^{-1/2} A_0 f_1 \,\Big|\, g_2\Big)_{L^2} \\
&= (-1)\Big[\Big(A_0^{-1/2} g_1 \,\Big|\, A_0^{-1/2}(-A_0) f_2\Big)_{L^2} + (f_1 \,|\, g_2)_{L^2}\Big] \\
&= (-1)\Big((f_1, g_1) \,\Big|\, A(f_2, g_2)\Big)_X.
\end{aligned}$$

This shows that $0 \in \rho(A)$ and that A is skew-adjoint. As in the example of Subsection 4.4.2, from Stone's theorem we infer that A generates a unitary group $\mathbb{T}$ on X.

Set $U := L^2(\Gamma)$. As usual, elements $v \in L^2(\Gamma)$ are seen as members of $L^2(\partial\mathcal{O})$ via zero extension on $\partial\mathcal{O} \setminus \Gamma$. It is shown in Section 10.9 of [49] that

$$B^l : U \to X_{-1}; \quad B^l v = (0, A_0 D v)$$

is an L^2–admissible control operator for $\mathbb{T}$. Here A_0 stands for the extension to $L^2(\mathcal{O}) = [\mathrm{D}(A_0)]_0$ and $D \in \mathcal{L}(L^2(\partial\mathcal{O}), L^2(\mathcal{O}))$ is the Dirichlet map for $\mathcal{O}$. The latter is constructed in Section 10.6 of [49]. This operator yields the unique solution $z = Dv$ of the Dirichlet problem $\Delta z = 0$, $\operatorname{tr} z = v$ for every $v \in L^2(\partial\mathcal{O})$. We remark that tr is an extension of the common trace operator $\operatorname{tr} \in \mathcal{L}(H^1(\mathcal{O}), L^2(\partial\mathcal{O}))$. For more details we refer to the reference above.

In Corollary 11.6.4 of [49] we find sufficient conditions for exact controllability of the linear control system $(\mathbb{T}, \Phi^l)$ associated to (4.39) which we repeat here. Let $\nu \in L^\infty(\partial\mathcal{O}, \mathbb{R}^n)$ be the outward normal vector field on $\partial\mathcal{O}$. If there is a point $\xi_0 \in \mathbb{R}^n$ such that

$$\{\xi \in \partial\mathcal{O} \,|\, (\xi - \xi_0) \cdot \nu(\xi) > 0\} \subseteq \Gamma,$$

then $(\mathbb{T}, \Phi^l)$ is exactly controllable in any time $T > 2 \sup\{|\xi - \xi_0| \,|\, \xi \in \mathcal{O}\}$. We remark that this follows by duality from Theorem 7.2.4 in [49], also see Theorem 11.2.1 therein.

For the nonlinearity M we choose the same map as in the example with the distributed control, see Subsection 4.4.2, namely

$$M : U \to U; \quad M(v) = m(\|v\|_U^2)v,$$

where $m \in C^2(\mathbb{R}, \mathbb{R})$ was a function with $m(0) \neq 0$ and $m(a) = 0$ for $|a|$ greater than a number $R > 0$. It makes no difference that there we had $U = L^2(\mathcal{O}_c)$ for a subset $\mathcal{O}_c \subseteq \mathcal{O}$ and here we have $U = L^2(\Gamma)$. We saw that in order to apply Proposition 4.42 we need that $M \in C^2_{\mathbb{R}}(U, U)$, $M(0) = 0$ and that M as well as M' are Lipschitz. All this conditions were verified in Subsection 4.4.2. The conclusion of the previous example holds accordingly.

We shall introduce another interesting nonlinearity M. Take any measurable function $b : \Gamma \times \Gamma \times \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $b(\cdot, \cdot, 0) \in L^2(\Gamma \times \Gamma)$. Moreover, let there be a nullset $\mathcal{N} \subseteq \Gamma$ and function $\kappa \in L^2(\Gamma \times \Gamma, [0, \infty))$ with

$$\forall \xi, \zeta \in \Gamma \setminus \mathcal{N} \ \forall \alpha, \beta \in \mathbb{C} : \ |b(\xi, \zeta, \alpha) - b(\xi, \zeta, \beta)| \leq \kappa(\xi, \zeta)|\alpha - \beta|. \tag{4.41}$$

We aim to define a nonlinear operator $M : L^2(\Gamma) \to L^2(\Gamma)$ via

$$M(v)(\xi) := \int_\Gamma b(\xi, \zeta, v(\zeta)) \, \mathrm{d}\sigma(\zeta). \tag{4.42}$$

where $v \in L^2(\Gamma)$ and $\xi \in \Gamma$. We first check that for fixed $\xi \in \Gamma \setminus \mathcal{N}$ and every $v \in L^2(\Gamma)$ the function $\Gamma \to \mathbb{C}$; $\zeta \mapsto b(\xi, \zeta, v(\zeta))$ belongs to $L^1(\Gamma)$. This follows from the estimate

$$\begin{aligned} |b(\xi, \zeta, v(\zeta))| &\leq |b(\xi, \zeta, v(\zeta)) - b(\xi, \zeta, 0)| + |b(\xi, \zeta, 0)| \\ &\leq \kappa(\xi, \zeta)|v(\zeta)| + |b(\xi, \zeta, 0)| \quad \text{for } \zeta \in \Gamma \setminus \mathcal{N}, \end{aligned} \tag{4.43}$$

since the right-hand side is integrable as a function of ζ. If κ is also bounded, then the map $\Gamma \times \Gamma \to \mathbb{C}$; $(\xi, \zeta) \to b(\xi, \zeta, v(\zeta))$ belongs to $L^2(\Gamma \times \Gamma)$.

Next we prove that $M(v)$ lies in $L^2(\Gamma)$ for $v \in L^2(\Gamma)$. Using that Γ has finite measure, the case $v = 0$ follows from standard estimates. For arbitrary $v, \tilde{v} \in L^2(\Gamma)$ we have

$$\begin{aligned} \int_\Gamma |M(v)(\xi) - M(\tilde{v})(\xi)|^2 \, \mathrm{d}\sigma(\xi) &\leq \int_\Gamma \left(\int_\Gamma |b(\xi, \zeta, v(\zeta)) - b(\xi, \zeta, \tilde{v}(\zeta))| \, \mathrm{d}\sigma(\zeta) \right)^2 \mathrm{d}\sigma(\xi) \\ &\leq \int_\Gamma \left(\int_\Gamma \kappa(\xi, \zeta)|v(\zeta) - \tilde{v}(\zeta)| \, \mathrm{d}\sigma(\zeta) \right)^2 \mathrm{d}\sigma(\xi) \leq \|\kappa\|^2_{L^2(\Gamma \times \Gamma)} \|v - \tilde{v}\|^2_{L^2(\Gamma)}. \end{aligned}$$

This shows that $M(v) - M(\tilde{v})$ is contained in $L^2(\Gamma)$ for all $v, \tilde{v} \in L^2(\Gamma)$. Consequently $M(v) = (M(v) - M(0)) + M(0)$ belongs to $L^2(\Gamma)$ for every $v \in L^2(\Gamma)$. On the other hand we see that M is Lipschitz, more precisely we obtain

$$\|M(v) - M(\tilde{v})\|_{L^2(\Gamma)} \leq \|\kappa\|_{L^2(\Gamma \times \Gamma)} \|v - \tilde{v}\|_{L^2(\Gamma)} \quad \text{for all } v, \tilde{v} \in L^2(\Gamma). \tag{4.44}$$

Hence (4.26) is satisfied with $c = \|\kappa\|_{L^2(\Gamma \times \Gamma)}$ and $\eta = 1$.

Via $B(v) = B^l M(v)$ for $v \in U$, this "convolution map" M yields an example for an L^2–admissible control operator $B : U \to X_{-1}$ for $\mathbb{T}$.

Chapter 5

Control for semilinear state equations

In this chapter let X and U be Banach spaces and let $\mathbb{T}$ be a strongly continuous semigroup on X with generator A. Further let $p \in [1, \infty)$. Again, we use the abbreviation $\|\,\cdot\,\|_{L^p} := \|\,\cdot\,\|_{L^p([0,\infty),U)}$. Later we will focus on the case where U is a Hilbert space and consider only $p = 2$ so that $L^2([0, \infty), U)$ is a Hilbert space.

Given maps $B : U \to X_{-1}$ and $F : X \to X$, we are now looking at the problem

$$z'(t) = Az(t) + F(z(t)) + B(u(t)); \qquad z(0) = x_0, \tag{5.1}$$

where $u \in L^p([0, \infty), U)$ and $x_0 \in X$. As it was discussed in Section 3.2, every strong solution $z \in C(J, X)$ of (5.1) satisfies the fixed-point equation

$$z(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} F(z(s))\,\mathrm{d}s + \int_0^t \mathbb{T}_{t-s} B(u(s))\,\mathrm{d}s \quad \text{for } t \in J. \tag{5.2}$$

Here $J \subseteq [0, \infty)$ is an interval with $\min J = 0$. Other than in the case $F = 0$ it is not clear whether such 'mild solutions' exist, even for small times. For $B = 0$ such existence results are well-known, see Section 6.1 in [33]. In the first part of this chapter we extend this theory to infinite dimensional control theory with unbounded operators.

5.1 Existence and uniqueness of mild solutions

Throughout assume that $B \in C(U, X_{-1})$ is L^p–admissible for $\mathbb{T}$. Among other things, this means that $Bu \in L^1_{\mathrm{loc}}([0, \infty), X_{-1})$ for all $u \in L^p([0, \infty), U)$ and for each $t \geq 0$ the integral

$$\Phi_t(u) = \int_0^t \mathbb{T}_{t-s} B(u(s))\,\mathrm{d}s \quad \text{for } u \in L^p([0, \infty), U)$$

defines a continuous map $\Phi_t : L^p([0, \infty), U) \to X$. In Lemma 4.14 we saw that for every fixed $u \in L^p([0, \infty), U)$ the map $t \mapsto \Phi_t(u)$ belongs to $C([0, \infty), X)$.

The following result guarantees the existence of mild solutions for every pair $x_0 \in X$ and $u \in L^p([0, \infty), U)$. In addition to that, it yields a minimal existence time $T > 0$ which is uniform for data in closed balls around the origin. For technical reasons we take $T \leq 1$.

Lemma 5.1. *Assume that $F : X \to X$ is Lipschitz on bounded sets. Further assume that Φ_t is bounded on bounded sets for each $t \geq 0$. Then for all $r, \rho > 0$ there is a time $T > 0$ (see (5.3) below) such that for all initial values $x_0 \in \overline{B}(0, r) \subseteq X$ and all inputs $u \in \overline{B}(0, \rho) \subseteq L^p([0, \infty), U)$ there exists a function $z \in C([0, T], X)$ satisfying*

$$z(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} F(z(s))\, \mathrm{d}s + \Phi_t(u) \quad \text{for } t \in [0, T],$$

i.e., z is a mild solution of (5.1) on $[0, T]$.

Proof. Due to Lemma 4.23, Φ_t is bounded on bounded sets uniformly for $t \in [0, 1]$, that is

$$\forall \rho > 0\ \exists c_{\rho,1} > 0\ \forall u \in L^p([0, \infty), U),\ t \in [0, 1] :\ \|u\|_{L^p} \leq \rho \implies \|\Phi_t(u)\|_X \leq c_{\rho,1}.$$

The assumption on F means that for each $R > 0$ there is a Lipschitz constant $L(R) > 0$ such that for all $x_1, x_2 \in X$ with $\|x_1\|_X, \|x_2\|_X \leq R$ we have

$$\|F(x_1) - F(x_2)\|_X \leq L(R)\|x_1 - x_2\|_X.$$

Let $r, \rho > 0$. Take $x_0 \in \overline{B}(0, r) \subseteq X$ and $u \in \overline{B}(0, \rho) \subseteq L^p([0, \infty), U)$. For $T \in (0, 1]$ and $z \in C([0, T], X)$ we consider the function $\mathcal{C}_{x_0,u}(z) \in X^{[0,\infty)}$ given by

$$\mathcal{C}_{x_0,u}(z)(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} F(z(s))\, \mathrm{d}s + \Phi_t(u) \quad \text{for } t \geq 0.$$

One easily sees that this defines a map $\mathcal{C}_{x_0,u} : C([0, T], X) \to C([0, T], X)$. Clearly, if z is a fixed-point of $\mathcal{C}_{x_0,u}$ then it is a mild solution. We will show that $\mathcal{C}_{x_0,u}$ maps the closed ball $\overline{B}(0, R) \subseteq C([0, T], X)$ contractively to itself, provided that $R > 0$ is chosen large enough and $T > 0$ is small enough.

Recall that we wrote $m_{\mathbb{T},t} = \sup_{\sigma \in [0,t]} \|\mathbb{T}_\sigma\|$ for $t \geq 0$. The continuity of the map

$$[0, \infty) \to \mathbb{R};\ t \mapsto \left\| \int_0^t \mathbb{T}_s F(0)\, \mathrm{d}s \right\|_X$$

implies that it has a maximum $c \geq 0$ on $[0, 1]$. We set

$$R := \max\{4c_{\rho,1}, 4c, 4m_{\mathbb{T},1} r\} \qquad \text{and} \qquad T := \min\left\{1, \frac{1}{4m_{\mathbb{T},1} L(R)}\right\}. \tag{5.3}$$

Then for all $z \in \overline{B}(0, R)$ we estimate

$$\begin{aligned}
\|\mathcal{C}_{x_0,u}(z)(t)\|_X &\leq m_{\mathbb{T},t}\|x_0\|_X + \int_0^t m_{\mathbb{T},t}\|F(z(s)) - F(0)\|_X\, \mathrm{d}s + \|\Phi_t(u)\|_X + \left\| \int_0^t \mathbb{T}_s F(0)\, \mathrm{d}s \right\|_X \\
&\leq m_{\mathbb{T},1} r + m_{\mathbb{T},1} L(R) \int_0^t \|z(s)\|_X\, \mathrm{d}s + c_{\rho,1} + c \\
&\leq \frac{R}{4} + m_{\mathbb{T},1} L(R) T R + \frac{R}{4} + \frac{R}{4} \leq R \quad \text{for all } t \in [0, T].
\end{aligned}$$

This means that $\mathcal{C}_{x_0,u}(z)$ lies in $\overline{B}(0, R)$. On the other hand, for $z_1, z_2 \in \overline{B}(0, R)$ we obtain

$$\begin{aligned}
\|\mathcal{C}_{x_0,u}(z_1)(t) - \mathcal{C}_{x_0,u}(z_2)(t)\|_X &= \left\| \int_0^t \mathbb{T}_{t-s}\big(F(z_1(s)) - F(z_2(s))\big)\, \mathrm{d}s \right\|_X \\
&\leq m_{\mathbb{T},1} L(R) T \|z_1 - z_2\|_{L^\infty([0,T],X)} \leq \tfrac{1}{4}\|z_1 - z_2\|_{L^\infty([0,T],X)}
\end{aligned}$$

for all $t \in [0, T]$. The contraction mapping principle yields a fixed-point $z \in \overline{B}(0, R)$ of $\mathcal{C}_{x_0,u}$. As argued above, z is a mild solution of (5.1). □

We point out that for all $x_0 \in X$ and $u \in L^p([0,\infty), U)$ there is a mild solution of (5.1) and it is even unique in a specific ball $\overline{B}(0,R)$. In the following lemma we show a stronger form of uniqueness, namely that whenever two solutions exist, then they are equal on their common domain.

Lemma 5.2. *Let $F : X \to X$ be Lipschitz on bounded sets as in Lemma 5.1. Further let $x_0 \in X$ and $u \in L^p([0,\infty), U)$. Assume there are functions $z_1 \in C(J_1, X)$ and $z_2 \in C(J_2, X)$ (where $J_1, J_2 \subseteq [0,\infty)$ are intervals with $\min J_1 = \min J_2 = 0$) satisfying*

$$z_j(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} F(z_j(s))\, \mathrm{d}s + \Phi_t(u) \quad \text{for } t \in J_j \qquad (j = 1,2).$$

Then we have $z_1(t) = z_2(t)$ for all $t \in J_1 \cap J_2$.

Proof. Assume $\mathcal{T} = \{t \in J_1 \cap J_2 \mid z_1(t) \neq z_2(t)\}$ was not empty and set $t_0 = \inf \mathcal{T} \geq 0$. By definition of t_0 we have $z_1(t) = z_2(t)$ for all $t \in [0, t_0)$ provided that $t_0 > 0$. Since z_1 and z_2 are continuous, it follows that $z_1 = z_2$ on $[0, t_0]$. The latter is also true if $t_0 = 0$, because $z_1(0) = z_2(0) = x_0$.

The assumption $\mathcal{T} \neq \emptyset$ implies that $[0, t_0] \neq J_1 \cap J_2$. Hence there is a $\delta > 0$ with $[0, t_0 + \delta] \subseteq J_1 \cap J_2$. We find a radius $R \geq 0$ such that $\|z_1(t)\|_X, \|z_2(t)\|_X \leq R$ for all $t \in [0, t_0 + \delta]$. We can thus estimate

$$\begin{aligned} \|z_1(t) - z_2(t)\|_X &= \left\| \int_0^t \mathbb{T}_{t-s}(F(z_1(s)) - F(z_2(s)))\, \mathrm{d}s \right\|_X \\ &\leq m_{\mathbb{T}, t_0+\delta} L(R) \int_0^t \|z_1(s) - z_2(s)\|_X \, \mathrm{d}s \quad \text{for } t \in [0, t_0 + \delta]. \end{aligned}$$

Gronwall's inequality (see e.g. Lemma 6.1 and Corollary 6.2 in [3]) now yields the bound

$$\|z_1(t) - z_2(t)\|_X \leq 0 \exp(m_{\mathbb{T}, t_0+\delta} L(R) t) = 0 \quad \text{for } t \in [0, t_0 + \delta].$$

This means that $z_1 = z_2$ on $[0, t_0 + \delta]$ and thus contradicts $t_0 = \inf \mathcal{T}$. □

Standing assumption: Let $F \in C(X, X)$ be Lipschitz on bounded sets and assume that Φ_t is bounded on bounded sets for every $t \geq 0$.

For $x_0 \in X$ and $u \in L^p([0,\infty), U)$ define the *maximal existence time*

$$t_\infty(x_0, u) = \sup\{T > 0 \mid \exists z \in C([0,T], X) \text{ mild solution of (5.1)}\}.$$

Due to Lemma 5.1 we have $t_\infty(x_0, u) > 0$ for all $x_0 \in X$ and $u \in L^p([0,\infty), U)$. It might be that $t_\infty(x_0, u) = \infty$, e.g. for $x_0 = x_*$ and $u = \chi_{v_*}$ with $Ax_* + F(x_*) + B(v_*) = 0$.

Next, for all $x_0 \in X$ and $u \in L^p([0,\infty), U)$ we construct a mild solution $z(\,\cdot\,, x_0, u)$ on $[0, t_\infty(x_0, u))$. Here we shortly write $t_\infty := t_\infty(x_0, u)$.

For $t \in [0, t_\infty)$ take any $T \in (t, t_\infty)$. Then there is a solution $z_T \in C([0,T], X)$ of (5.1). Set $z(t, x_0, u) = z_T(t)$. Lemma 5.2 implies that $z(\,\cdot\,, x_0, u)$ is well-defined and that $z(s, x_0, u) = z_T(s)$ for all $s \in [0, T]$. Hence $z(\,\cdot\,, x_0, u)$ is continuous and it satisfies (5.2) for every $t \in [0, t_\infty)$. This means that $z(\,\cdot\,, x_0, u)$ is a mild solution of (5.1) on $[0, t_\infty)$.

We call $z(\,\cdot\,, x_0, u)$ the *maximal mild solution* of (5.1) (for the data x_0 and u). It is maximal in the sense that for every mild solution $\tilde{z} \in C(J, X)$ for the data x_0 and u we have $J \subseteq [0, t_\infty)$ and $\tilde{z} = z(\,\cdot\,, x_0, u)$ on J.

The uniqueness shown above also implies that the maximal mild solution satisfies a composition property as in the next lemma.

Lemma 5.3. *Assume that the conditions of Lemma 5.1 are satisfied. Let $u \in L^p([0,\infty), U)$ and $x_0 \in X$. Then for all $\tau \in [0, t_\infty(x_0,u))$ we have*

$$t_\infty(z(\tau, x_0, u), S^*_\tau u) = t_\infty(x_0, u) - \tau \qquad \textit{and} \qquad S^*_\tau z(\,\cdot\,, x_0, u) = z(\,\cdot\,, z(\tau, x_0, u), S^*_\tau u).$$

This means that the shifted function $S^*_\tau z(\,\cdot\,, x_0, u) \in C([0, t_\infty(x_0,u) - \tau), X)$ is the maximal mild solution of

$$z'(t) = Az(t) + F(z(t)) + B((S^*_\tau u)(t)); \quad z(0) = z(\tau, x_0, u).$$

Proof. Abbreviate $t_\infty := t_\infty(x_0, u)$ and $x_\tau := z(\tau, x_0, u)$. For $t \in [0, t_\infty - \tau)$ we compute

$$\begin{aligned} S^*_\tau z(\,\cdot\,, x_0, u)(t) &= z(\tau + t, x_0, u) = \mathbb{T}_{\tau+t} x_0 + \int_0^{\tau+t} \mathbb{T}_{\tau+t-s} F(z(s, x_0, u))\,\mathrm{d}s + \Phi_{\tau+t} u \\ &= \mathbb{T}_t \mathbb{T}_\tau x_0 + \int_0^\tau \mathbb{T}_t \mathbb{T}_{\tau-s} F(z(s, x_0, u))\,\mathrm{d}s \\ &\qquad\qquad + \int_\tau^{t+\tau} \mathbb{T}_{t-(s-\tau)} F(z(s, x_0, u))\,\mathrm{d}s + \mathbb{T}_t \Phi_\tau u + \Phi_t S^*_\tau u \\ &= \mathbb{T}_t \left(\mathbb{T}_\tau x_0 + \int_0^\tau \mathbb{T}_{\tau-s} F(z(s, x_0, u))\,\mathrm{d}s + \Phi_\tau u \right) \\ &\qquad\qquad + \int_0^t \mathbb{T}_{t-s} F(z(s+\tau, x_0, u))\,\mathrm{d}s + \Phi_t S^*_\tau u \\ &= \mathbb{T}_t z(\tau, x_0, u) + \int_0^t \mathbb{T}_{t-s} F((S^*_\tau z(\,\cdot\,, x_0, u))(s))\,\mathrm{d}s + \Phi_t S^*_\tau u. \end{aligned}$$

Hence $S^*_\tau z(\,\cdot\,, x_0, u)$ is a mild solution of (5.1) for the data x_τ and $S^*_\tau u$ on $[0, t_\infty - \tau)$. It follows that $t_\infty(x_\tau, S^*_\tau u) \geq t_\infty - \tau$ and $S^*_\tau z(\,\cdot\,, x_0, u) = z(\,\cdot\,, x_\tau, S^*_\tau u)$ on $[0, t_\infty - \tau)$. Reading the above identity backwards also yields $t_\infty(x_\tau, S^*_\tau u) \leq t_\infty - \tau$. We thus have $t_\infty(x_\tau, S^*_\tau u) = t_\infty - \tau$. ☐

The maximality of $z(\,\cdot\,, x_0, u)$ is equivalent to a 'non-extendability' expressed in the following statement.

Lemma 5.4. *Under the assumptions of Lemma 5.1 we can show that if $t_\infty := t_\infty(x_0, u) < \infty$ for some $x_0 \in X$ and $u \in L^p([0,\infty), U)$, then $\|z(t, x_0, u)\|_X \to \infty$ as $t \to {t_\infty}^-$.*

Proof. Assume that this was not the case. Then we find a sequence (t_n) in $[0, t_\infty)$ with $t_n \to t_\infty$ as $n \to \infty$ and some $r > 0$ with $\|z(t_n, x_0, u)\|_X \leq r$ for all $n \in \mathbb{N}$.

We set $x_n := z(t_n, x_0, u)$ for $n \in \mathbb{N}$. Note that $\|S^*_{t_n} u\|_{L^p} \leq \|u\|_{L^p} =: \rho$ for every $n \in \mathbb{N}$. Lemma 5.3 yields that $t_\infty(x_n, S^*_{t_n} u) = t_\infty - t_n \to 0$ as $n \to \infty$. This contradicts the fact that by Lemma 5.1 there is some $T > 0$ (depending only on r and ρ) such that $t_\infty(x_n, S^*_{t_n} u) > T$ for all $n \in \mathbb{N}$. ☐

The inversion of this "blow-up result" yields a condition for global existence. In short, if the norms $\|z(t, x_0, u)\|_X$ can be bounded for all $t \geq 0$, then the maximal existence time $t_\infty(x_0, u)$ can not be finite. A simple example is the following corollary. Note that it can be applied especially if F is (globally) Lipschitz.

Corollary 5.5. *In addition to the conditions of Lemma 5.1 assume that there is a constant $c \geq 0$ with $\|F(x)\|_X \leq c(\|x\|_X + 1)$ for all $x \in X$. Then $t(x_0, u) = \infty$ for all $x_0 \in X$ and $u \in L^p([0,\infty), U)$.*

Proof. The claim follows by a contradiction argument. Assume we had $t_\infty := t_\infty(x_0, u) < \infty$ for a some data $x_0 \in X$ and $u \in L^p([0,\infty), U)$. Since $(\mathbb{T}, \Phi)$ is a continuous additive control system, the function $[0,\infty) \to X$; $t \mapsto \mathbb{T}_t x_0 + \Phi_t(u)$ is continuous by Lemma 4.14. Consequently the term $\|\mathbb{T}_t x_0 + \Phi_t(u)\|_X$ is bounded for $t \in [0, t_\infty]$. We can thus estimate

$$\begin{aligned}\|z(t,x_0,u)\|_X &\le \|\mathbb{T}_t x_0 + \Phi_t(u)\|_X + \int_0^t \|\mathbb{T}_{t-s} F(z(s,x_0,u))\|_X \,\mathrm{d}s \\ &\le \sup_{s\in[0,t_\infty]} \|\mathbb{T}_s x_0 + \Phi_s(u)\|_X + c m_{\mathbb{T},t_\infty} t_\infty + c\int_0^t \|z(s,x_0,u)\|_X \,\mathrm{d}s\end{aligned}$$

for all $t \in [0, t_\infty]$. Set $c_1 := \sup_{s\in[0,t_\infty]} \|\mathbb{T}_s x_0 + \Phi_s(u)\|_X + c m_{\mathbb{T},t_\infty} t_\infty$. Then Gronwall's inequality yields

$$\|z(t,x_0,u)\|_X \le c_1 \exp(ct) \quad \text{for all } t \in [0, t_\infty].$$

This contradicts the assumption that $\|z(t,x_0,u)\|_X \to \infty$ for $t \to t_\infty{}^-$. □

In the addendum of the next result we will assume that Φ_t is Lipschitz on bounded sets for every $t \ge 0$. Due to Lemma 4.21 we can choose a uniform Lipschitz constant for t in compact subsets of $[0,\infty)$. It is clear that in this situation Φ_t is bounded on bounded sets for every $t \ge 0$ and that $(\mathbb{T}, \Phi)$ is equicontinuous. Recall that Φ_t is globally Lipschitz if

$$\exists M_t > 0 \;\forall u, \tilde{u} \in L^p([0,\infty), U): \;\; \|\Phi_t(u) - \Phi_t(\tilde{u})\|_X \le M_t \|u - \tilde{u}\|_{L^p}.$$

We mention that one can choose a uniform Lipschitz constant for t in compact sets, see Lemma 4.21 and Remark 4.24.

Let us assume that we know the existence time $t_1 := t_\infty(x_*, u_*)$ of some data $x_* \in X$ and $u_* \in L^p([0,\infty), U)$. We shall need that for any $\tau \in [0, t_1)$ the maximal existence time of data "close to" x_* and u_* can be bounded from below by τ. We also show continuous dependence on the data.

Lemma 5.6. *As in Lemma 5.1 assume that F is Lipschitz on bounded sets and that Φ_t is bounded on bounded sets for all $t \ge 0$. Further let $(\mathbb{T}, \Phi)$ be equicontinuous. Then for all $x_* \in X$, $u_* \in L^p([0,\infty), U)$ and every $\tau \in [0, t_\infty(x_*, u_*))$ there are radii $r, \rho > 0$ such that for all $x_0 \in \overline{B}(x_*, r)$ and $u \in \overline{B}(u_*, \rho)$ we have $t_\infty(x_0, u) > \tau$.*

Let Φ_t be Lipschitz on bounded sets for all $t \ge 0$. Then there is a constant $K_\tau > 0$ such that for all $x, \tilde{x} \in \overline{B}(x_*, r)$ and $u, \tilde{u} \in \overline{B}(u_*, \rho)$ we have

$$\|z(\,\cdot\,, x, u) - z(\,\cdot\,, \tilde{x}, \tilde{u})\|_{L^\infty([0,\tau],X)} \le K_\tau (\|x - \tilde{x}\|_X + \|u - \tilde{u}\|_{L^p}).$$

If Φ_t is globally Lipschitz for all $t \ge 0$, then ρ can be chosen independently of x_* and u_*.

Proof. 1) Let $\tau \in [0, t_\infty(x_*, u_*))$. Due to continuity, $z(\,\cdot\,, x_*, u_*)$ is bounded on $[0,\tau]$. Set $c := \max_{t\in[0,\tau]} \|z(t, x_*, u_*)\|_X$ exists. Consider the radii

$$r_0 := 2 + c \qquad \text{and} \qquad \rho_0 := \|u_*\|_{L^p} + 1.$$

Lemma 5.1 yields a time $T \in (0,1]$ such that $t_\infty(\tilde{x}, \tilde{u}) > T$ for all $\tilde{x} \in \overline{B}(0, r_0)$, $\tilde{u} \in \overline{B}(0, \rho_0)$. Let $\tilde{x}, \overline{x} \in \overline{B}(0, r_0)$ and $\tilde{u}, \overline{u} \in \overline{B}(0, \rho_0)$. Recall from the proof of Lemma 5.1 that $z(\,\cdot\,, \tilde{x}, \tilde{u})|_{[0,T]}$

and $z(\,\cdot\,,\overline{x},\overline{u})|_{[0,T]}$ are fixed-points of the contractive maps $\mathcal{C}_{\tilde{x},\tilde{u}}$ and $\mathcal{C}_{\overline{x},\overline{u}}$ respectively. The Lipschitz constant was $\frac{1}{4}$. We obtain the estimate

$$\begin{aligned}
&\|z(t,\tilde{x},\tilde{u})-z(t,\overline{x},\overline{u})\|_X=\|\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\tilde{x},\tilde{u}))(t)-\mathcal{C}_{\overline{x},\overline{u}}(z(\,\cdot\,,\overline{x},\overline{u}))(t)\|_X\\
&\quad\le\|\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\tilde{x},\tilde{u}))(t)-\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\overline{x},\overline{u}))(t)\|_X+\|\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\overline{x},\overline{u}))(t)-\mathcal{C}_{\overline{x},\overline{u}}(z(\,\cdot\,,\overline{x},\overline{u}))(t)\|_X\\
&\quad\le\|\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\tilde{x},\tilde{u}))-\mathcal{C}_{\tilde{x},\tilde{u}}(z(\,\cdot\,,\overline{x},\overline{u}))\|_{L^\infty([0,T],X)}+\|\mathbb{T}_t\|\|\tilde{x}-\overline{x}\|_X+\|\Phi_t(\tilde{u})-\Phi_t(\overline{u})\|_X\\
&\quad\le\tfrac{1}{4}\|z(\,\cdot\,,\tilde{x},\tilde{u})-z(\,\cdot\,,\overline{x},\overline{u})\|_{L^\infty([0,T],X)}+m_{\mathbb{T},1}\|\tilde{x}-\overline{x}\|_X+\sup_{\sigma\in[0,1]}\|\Phi_\sigma(\tilde{u})-\Phi_\sigma(\overline{u})\|_X
\end{aligned}$$

for all $t\in[0,T]$. It follows that

$$\|z(\,\cdot\,,\tilde{x},\tilde{u})-z(\,\cdot\,,\overline{x},\overline{u})\|_{L^\infty([0,T],X)}\le 2m_{\mathbb{T},1}\|\tilde{x}-\overline{x}\|_X+2\sup_{\sigma\in[0,1]}\|\Phi_\sigma(\tilde{u})-\Phi_\sigma(\overline{u})\|_X \tag{5.4}$$

for all $\tilde{x},\overline{x}\in\overline{B}(0,r_0)$ and $\tilde{u},\overline{u}\in\overline{B}(0,\rho_0)$, where for convenience we estimated $\frac{4}{3}\le 2$. The supremum exists because $t\mapsto\Phi_t(u)$ is continuous for $u\in L^p([0,\infty),U)$.

2) In case $T>\tau$ the proof of the first part is finished. Else take (the minimal) $N\in\mathbb{N}$ with $NT\ge\tau$. Set $t_j=j\frac{\tau}{N}$ and abbreviate $u_{*,j}:=S^*_{t_j}u_*$ for $j=0,\dots,N$.

For each $j\in\{0,\dots,N-1\}$ the equicontinuity of Φ_t yields a number $l_j>0$ with

$$\sup_{\sigma\in[0,1]}\|\Phi_\sigma(\tilde{u})-\Phi_\sigma(u_{*,j})\|_X\le\varepsilon:=\left(2\sum_{k=0}^{N-1}(2m_{\mathbb{T},1})^k\right)^{-1}\quad\text{for all }\tilde{u}\in\overline{B}(u_{*,j},l_j). \tag{5.5}$$

Now set

$$r:=(2m_{\mathbb{T},1})^{-N}>0\qquad\text{and}\qquad\rho:=\min\{1,l_0,\dots,l_{N-1}\}>0.$$

Since $r,\rho\le 1$ it follows that $r_0\ge\|x_*\|_X+r$ and $\rho_0=\|u_*\|_{L^p}+1\ge\|u_*\|_{L^p}+\rho$. We thus have the inclusions $\overline{B}(x_*,r)\subseteq\overline{B}(0,r_0)$ and $\overline{B}(u_*,\rho)\subseteq\overline{B}(0,\rho_0)$.

Let $x_0\in\overline{B}(x_*,r)$ and $u\in\overline{B}(u_*,\rho)$. We inductively show that for all $j\in\{0,\dots,N\}$ we obtain $t_\infty(x_0,u)>t_j$ as well as

$$\|z(t,x_0,u)-z(t,x_*,u_*)\|_X\le(2m_{\mathbb{T},1})^j\|x_0-x_*\|_X+2\left(\sum_{k=0}^{j-1}(2m_{\mathbb{T},1})^k\right)\varepsilon$$

for all $t\in[0,t_j]$. For $j=0$ this is trivial, since $\|z(0,x_0,u)-z(0,x_*,u_*)\|_X=\|x_0-x_*\|$, and $t_\infty(x_0,u)>t_0=0$ by Lemma 5.1.

Assume that the claim is true for some $j\in\{0,\dots,N-1\}$. Using the choice of r and ρ, we calculate

$$\begin{aligned}
\|z(t_j,x_0,u)\|_X&\le\|z(t_j,x_0,u)-z(t_j,x_*,u_*)\|_X+\|z(t_j,x_*,u_*)\|_X\\
&\le(2m_{\mathbb{T},1})^j\|x_0-x_*\|_X+2\left(\sum_{k=0}^{j-1}(2m_{\mathbb{T},1})^k\right)\varepsilon+c\\
&\le 1+1+c=r_0.
\end{aligned}$$

Observe that $\|S^*_{t_j}u\|_{L^p}\le\|u\|_{L^p}\le\rho_0$ and in the same way $\|u_{*,j}\|_{L^p}\le\|u_*\|\le\rho_0$. Lemma 5.3 and the first step then imply the inequality

$$t_\infty(z(t_j,x_0,u),S^*_{t_j}u)=t_\infty(x_0,u)-t_j>T\ge\frac{\tau}{N},$$

which means that $t_\infty(x_0,u) > t_j + \frac{\tau}{N} = t_{j+1}$. On the other hand we clearly have

$$\|z(t_j,x_*,u_*)\|_X \le c \le r_0 \qquad \text{and} \qquad \|S_{t_j}u - u_{*,j}\|_{L^p} \le \|u-u_*\|_{L^p} \le \rho \le l_j.$$

From Lemma 5.3, (5.4) and the induction hypotheses we deduce

$$\begin{aligned}\|z(t_j+s,x_0,u) - z(t_j+s,x_*,u_*)\|_X &= \|z(s,z(t_j,x_0,u),S^*_{t_j}u) - z(s,z(t_j,x_*,u_*),S^*_{t_j}u_*)\|_X \\ &\le 2m_{\mathbb{T},1}\|z(t_j,x_0,u) - z(t_j,x_0,u)\|_X + 2\sup_{\sigma\in[0,1]}\|\Phi_\sigma(S^*_{t_j}u) - \Phi_\sigma(u_{*,j})\|_{L^p} \\ &\le 2m_{\mathbb{T},1}\left((2m_{\mathbb{T},1})^j\|x_0-x_*\|_X + 2\left(\sum_{k=0}^{j-1}(2m_{\mathbb{T},1})^k\right)\varepsilon\right) + 2\varepsilon \\ &= (2m_{\mathbb{T},1})^{j+1}\|x_0-x_*\|_X + 2\left(\sum_{k=1}^{j}(2m_{\mathbb{T},1})^k\right)\varepsilon + 2\varepsilon\end{aligned}$$

for all $s\in[0,\frac{\tau}{N}]$. The induction hypotheses further implies the estimate

$$\begin{aligned}\|z(t,x_0,u) - z(t,x_*,u_*)\|_X &\le (2m_{\mathbb{T},1})^j\|x_0-x_*\|_X + 2\left(\sum_{k=0}^{j-1}(2m_{\mathbb{T},1})^k\right)\varepsilon \\ &\le (2m_{\mathbb{T},1})^{j+1}\|x_0-x_*\|_X + 2\left(\sum_{k=0}^{j}(2m_{\mathbb{T},1})^k\right)\varepsilon.\end{aligned}$$

for $t\in[0,t_j]$. In particular we have shown that $t_\infty(x_0,u) > t_N = \tau$.

3) We still have to prove the addenda. Assume that Φ_t is Lipschitz on bounded sets for every $t\ge 0$. Then automatically Φ_t is Lipschitz on bounded sets uniformly for t in compact sets by Lemma 4.21. We thus find a constant $M_{\rho_0,\tau} > 0$ such that

$$\|\Phi_t(u) - \Phi_t(\tilde u)\|_X \le M_{\rho_0,\tau}\|u-\tilde u\|_{L^p} \quad \text{for all } u,\tilde u\in\overline{B}(0,\rho_0),\ t\in[0,\tau]. \tag{5.6}$$

Let $x,\tilde x\in\overline{B}(x_*,r)$ and $u,\tilde u\in\overline{B}(u_*,\rho)\subseteq\overline{B}(0,\rho_0)$. Above we have shown that $\|z(t,x,u)\|_X$ and $\|z(t,\tilde x,\tilde u)\|_X$ are less or equal to r_0 for all $t\in[0,\tau]$. The assumption on F yields a number $L(r_0)\ge 0$ with

$$\begin{aligned}&\|z(t,x,u) - z(t,\tilde x,\tilde u)\|_X \\ &\quad\le m_{\mathbb{T},\tau}\|x-\tilde x\|_X + \int_0^t m_{\mathbb{T},\tau}\|F(z(s,x,u)) - F(z(s,\tilde x,\tilde u))\|_X\,\mathrm{d}s + \|\Phi_t(u)-\Phi_t(\tilde u)\|_X \\ &\quad\le m_{\mathbb{T},\tau}\|x-\tilde x\|_X + M_{\rho_0,\tau}\|u-\tilde u\|_{L^p} + m_{\mathbb{T},\tau}L(r_0)\int_0^t\|z(s,x,u)-z(s,\tilde x,\tilde u)\|_X\,\mathrm{d}s\end{aligned}$$

for every $t\in[0,\tau]$, where we also used (5.6). With Gronwall's inequality, we derive

$$\begin{aligned}\|z(t,x,u) - z(t,\tilde x,\tilde u)\|_X &\le (m_{\mathbb{T},\tau}\|x-\tilde x\|_X + M_{\rho_0,\tau}\|u-\tilde u\|_{L^p})\exp(m_{\mathbb{T},\tau}L(r_0)t) \\ &\le \max\{m_{\mathbb{T},\tau},M_{\rho_0,\tau}\}(\|x-\tilde x\|_X + \|u-\tilde u\|_{L^p})\exp(m_{\mathbb{T},\tau}L(r_0)\tau),\end{aligned}$$

for all $t\in[0,\tau]$. This means that

$$\|z(\cdot,x,u) - z(\cdot,\tilde x,\tilde u)\|_{L^\infty} \le K_\tau(\|x-\tilde x\|_X + \|u-\tilde u\|_{L^p}),$$

where $K_\tau = \max\{m_{\mathbb{T},\tau}, M_{\rho_0,\tau}\} \exp(\tau m_{\mathbb{T},\tau} L(r_0))$.

4) In case Φ_t is globally Lipschitz for all $t \in [0,1]$, as before deduce that the property holds uniformly for $t \in [0,1]$ (see Remark 4.24). Estimate (5.4) then directly implies

$$\begin{aligned} \|z(\,\cdot\,,\tilde{x},\tilde{u}) - z(\,\cdot\,,\overline{x},\overline{u})\|_{L^\infty([0,T],X)} &\le 2m_{\mathbb{T},1}\|\tilde{x}-\overline{x}\|_X + 2\sup_{\sigma\in[0,1]}\|\Phi_\sigma(\tilde{u}) - \Phi_\sigma(\overline{u})\|_X \\ &\le 2m_{\mathbb{T},1}\|\tilde{x}-\overline{x}\|_X + 2M_1\|\tilde{u}-\overline{u}\|_{L^p} \end{aligned} \tag{5.7}$$

for all $\tilde{x},\overline{x} \in \overline{B}(0,r_0)$ and $\tilde{u},\overline{u} \in \overline{B}(0,\rho_0)$. Choosing $\rho = \left(2M_1 \sum_{k=0}^{N-1}(2m_{\mathbb{T},1})^k\right)^{-1}$, the induction in step 2) works in the same way if we replace ε with $M_1\|\tilde{u}-\overline{u}\|_{L^p}$ and use (5.7) instead of (5.4). □

Remark 5.7. Let the conditions of the first part of Lemma 5.6 be satisfied. A close inspection of step 2) in the proof shows that the map

$$\overline{B}(x_*,r) \times \overline{B}(u_*,\rho) \to C([0,\tau],X); \quad (x_0,u) \to z(\,\cdot\,,x_0,u)$$

is continuous at (x_*,u_*). To see this, one only has to replace ε in (5.5) by an appropriate smaller number. ◊

We summarize the results of this section in a theorem.

Theorem 5.8. *Let X and U be Banach spaces and let $\mathbb{T}$ be a strongly continuous semigroup on X with generator A. Assume that $B \in C(U,X_{-1})$ is an L^p–admissible control operator for $\mathbb{T}$ with $p \in [1,\infty)$ and denote the corresponding input maps by Φ_t for $t \ge 0$. Let $F : X \to X$ be Lipschitz on bounded sets and assume that Φ_t is bounded on bounded sets for each $t \ge 0$.*

Then for every pair $(x_0,u) \in X \times L^p([0,\infty),U)$ there exists a maximal existence time $t_\infty(x_0,u) \in (0,\infty]$ and a unique maximal mild solution

$$z(\,\cdot\,,x_0,u) \in C([0,t_\infty(x_0,u)),X)$$

of (5.1). If $t_\infty(x_0,u)$ is finite, then $\|z(t,x_0,u)\|_X \to \infty$ as $t \to t_\infty(x_0,u)$.

Additionally assume that $(\mathbb{T},\Phi)$ is equicontinuous. Then the maximal existence time is lower semi-continuous in the sense that for every pair $(x_*,u_*) \in X \times L^p([0,\infty),U)$ and each time $\tau \in [0,t_\infty(x_*,u_*))$ there are radii $r,\rho > 0$ such that for all $(x_0,u) \in \overline{B}(x_*,r) \times \overline{B}(u_*,\rho)$ we have $t_\infty(x_0,u) > \tau$. Moreover, the map

$$\overline{B}(x_*,r) \times \overline{B}(u_*,\rho) \to C([0,\tau],X); \quad (x_0,u) \to z(\,\cdot\,,x_0,u)$$

is continuous at (x_*,u_*). Finally, this map is Lipschitz if Φ_t is Lipschitz on bounded sets for every $t \ge 0$.

5.2 Linearization

For convenience, we repeat problem (5.1). Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B \in C(U,X_{-1})$ be L^p–admissible for $\mathbb{T}$. As usual, the corresponding input maps are denoted by $\Phi_t : L^p([0,\infty),U) \to X$. Further let $F : X \to X$ be Lipschitz on bounded sets. We are looking at the inhomogeneous Cauchy problem

$$z'(t) = Az(t) + F(z(t)) + B(u(t)); \qquad z(0) = x_0. \tag{5.8}$$

Assume that Φ_t is bounded on bounded sets for all $t \geq 0$. Then Lemma 5.1 yields the existence of the unique maximal mild solution $z(\,\centerdot\,, x_0, u)$ for all $x_0 \in X$ and $u \in L^p([0,\infty), U)$.

We next derive a linearization principle very similar to the one in Section 4.3. To this end, assume that (5.8) has an equilibrium point $(x_*, v_*) \in X \times U$, i.e.,

$$Ax_* + F(x_*) + B(v_*) = 0.$$

As before, we abbreviate $u_* := \chi_{v_*}$. The maximal mild solution corresponding to the data (x_*, u_*) is given by $z(t, x_*, u_*) = x_*$ for all $t \geq 0$.

Moreover, we assume that F and B are differentiable at x_* and v_* respectively. The linearized problem of (5.8) at (x_*, u_*) then reads

$$z_l'(t) = (A + F'(x_*))z_l(t) + B'(v_*)\tilde{u}; \quad z_l(0) = \tilde{x}_0. \tag{5.9}$$

As in part A) of the linearization principle on page 51 of the thesis we claim that this linear problem is well-posed. We have to show that $B'(v_*)$ is L^p–admissible for the semigroup generated by $A + F'(x_*)$. Indeed, due to Proposition 4.33 the linear operator $B'(v_*)$ is L^p–admissible for $\mathbb{T}$. Corollary 5.5.1 in [49] (a far more general result) then yields the claim.

We shall establish a perturbation theory for our nonlinear problem. To this end, we first collect some properties that the "perturbed control system" (5.8) inherits from the unperturbed one, namely (5.8) with $F = 0$. For the time being we replace the perturbation $F'(x_*)$ by an arbitrary bounded operator $P \in \mathcal{L}(X)$. Other properties of $F'(x_*)$ are not important here.

To have a proper spectral theory at hand, vector spaces are mostly assumed to be complex. In the text before Remark 4.25 we already mentioned that this seems to cause problems in applications: Sometimes nonlinear terms are not differentiable in the common sense, but merely $\mathbb{R}$–differentiable. Clearly the generator of a semigroup $\mathbb{T}$ in $\mathcal{L}(X)$ has to be $\mathbb{C}$–linear. The perturbation results refereed to below are based on Hille–Yosida generation theorem, see Theorem II.3.8 in [15]. The latter, although formulated for complex vector spaces, is valid if we consider $\mathbb{T}$ as a family in $\mathcal{L}_{\mathbb{R}}(X)$. We simply have to exclude part (c) of the cited theorem. Thus again every appearance of the word "differentiable" can be replaced by "$\mathbb{R}$–differentiable".

Admissibility under bounded perturbations of the generator

It is well known that $A + P : \mathrm{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{S}$ on X. For every $\tau \geq 0$ the operator $\mathbb{S}_\tau \in \mathcal{L}(X)$ satisfies the equations

$$\mathbb{S}_\tau x = \mathbb{T}_\tau x + \int_0^\tau \mathbb{S}_\sigma P \mathbb{T}_{\tau-\sigma} x \,\mathrm{d}\sigma = \mathbb{T}_\tau x + \int_0^\tau \mathbb{S}_{\tau-\sigma} P \mathbb{T}_\sigma x \,\mathrm{d}\sigma. \tag{5.10}$$

Moreover, the norms on X given by $\|R(\lambda, A)x\|_X$ and $\|R(\lambda, A+P)x\|_X$ are equivalent if $\lambda > \omega + 2M\|P\|$. Here, as usual, $\omega \in \mathbb{R}$ and $M \geq 1$ are numbers with $\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. As a consequence, we can say that "the spaces X_{-1} for both A and $A + P$ coincide". With our notation from Appendix A this means $[\mathrm{D}(A)]_{-1} = [\mathrm{D}(A+P)]_{-1}$. For a proof of these statements we refer to Theorem III.1.3, Corollary III.1.4 and Corollary III.1.7 of [15]. As for $\mathbb{T}$ we use the symbol $m_{\mathbb{S},t} := \sup_{\sigma \in [0,t]} \|\mathbb{S}_\sigma\|$.

Proposition 5.9. *Let $B \in C(U, X_{-1})$ be L^p–admissible for $\mathbb{T}$ and assume that Φ_t is bounded on bounded sets for all $t \geq 0$. Further let $P \in \mathcal{L}(X)$. Then B is also L^p–admissible for $\mathbb{S}$. The corresponding input maps $\Phi_t, \Phi_t^P \in C(L^p([0,\infty), U), X)$ are related by*

$$\Phi_t^P(u) = \Phi_t(u) + \int_0^t \mathbb{S}_{t-s} P \Phi_s(u)\,\mathrm{d}s \quad \text{for all } u \in L^p([0,\infty), U) \text{ and } t \geq 0. \tag{5.11}$$

Proof. Let $u \in L^p([0,\infty), U)$. Since B is L^p–admissible for $\mathbb{T}$, the function $Bu : [0,\infty) \to X_{-1}$ is locally integrable. Thus for $t \geq 0$ and $u \in L^p([0,\infty), U)$ we may define

$$\Phi_t^P : L^p([0,\infty), U) \to X_{-1}; \quad \Phi_t^P(u) = \int_0^t \mathbb{S}_{t-s} B(u(s))\,\mathrm{d}s.$$

We have to show that $(\mathbb{S}, \Phi^P)$ is a continuous additive control system on X and $L^p([0,\infty), U)$. The needed properties can be derived from (5.11). Hence, we first verify this equation. Let $u \in L^p([0,\infty), U)$. Using the representation (5.10) and Fubini's theorem, we infer

$$\begin{aligned}
\Phi_t^P(u) &= \int_0^t \mathbb{S}_{t-s} B(u(s))\,\mathrm{d}s = \int_0^t \left[\mathbb{T}_{t-s} B(u(s)) + \int_0^{t-s} \mathbb{S}_\sigma P \mathbb{T}_{t-s-\sigma} B(u(s))\,\mathrm{d}\sigma\right] \mathrm{d}s \\
&= \Phi_t(u) + \int_0^t \mathbb{S}_\sigma P \int_0^{t-\sigma} \mathbb{T}_{t-\sigma-s} B(u(s))\,\mathrm{d}s\,\mathrm{d}\sigma = \Phi_t(u) + \int_0^t \mathbb{S}_\sigma P \Phi_{t-\sigma}(u)\,\mathrm{d}\sigma \\
&= \Phi_t(u) + \int_0^t \mathbb{S}_{t-s} P \Phi_s(u)\,\mathrm{d}s
\end{aligned}$$

for all $t \geq 0$. Note that by Lemma 4.14 the map $[0,\infty) \to X;\ s \mapsto \Phi_s(u)$ is continuous. Therefore the last integral exists in X and consequently $\Phi_t^P(u) \in X$ for all $t \geq 0$. Proposition 1.3.4 in [5] further yields that

$$[0,\infty) \to X; \quad t \mapsto \int_0^t \mathbb{S}_{t-s} P \Phi_s(u)\,\mathrm{d}s$$

is continuous. We deduce that $[0,\infty) \to X; t \mapsto \Phi_t^P(u)$ is continuous for all $u \in L^p([0,\infty), U)$. Similarly one checks this properties for $u = \chi_v$ where $v \in U$ is arbitrary.

Let $T > 0$. We next show that the family $\{U \to X; v \mapsto \Phi_t^P(\chi_v) \,|\, t \in [0,T]\}$ is equicontinuous. To this end, let $v_0 \in U$ and $\varepsilon > 0$. Because $\{U \to X; v \mapsto \Phi_t(\chi_v) \,|\, t \in [0,T]\}$ is equicontinuous, we find a number $\delta > 0$ such that

$$\|\Phi_t(\chi_v) - \Phi_t(\chi_{v_0})\|_X \leq \varepsilon_1 := \frac{\varepsilon}{1 + m_{\mathbb{S},T}\|P\|T}$$

for all $v \in U$ with $\|v - v_0\|_U \leq \delta$ and every $t \in [0,T]$. For such v and t it follows that

$$\begin{aligned}
\|\Phi_t^P(\chi_v) - \Phi_t^P(\chi_{v_0})\|_X &\leq \|\Phi_t(\chi_v) - \Phi_t(\chi_{v_0})\|_X + \int_0^t \|\mathbb{S}_{t-s} P(\Phi_s(\chi_v) - \Phi_s(\chi_{v_0}))\|_X\,\mathrm{d}s \\
&\leq \varepsilon_1(1 + m_{\mathbb{S},T}\|P\|T) = \varepsilon.
\end{aligned}$$

It remains to verify that $\Phi_t^P : L^p([0,\infty), U) \to X$ is continuous for all $t \geq 0$. Let $t \geq 0$ and $u_0 \in L^p([0,\infty), U)$. For $h \in L^p([0,\infty), U)$ consider

$$\Phi_t(u_0 + h) - \Phi_t(u_0) = \Phi_t(u_0 + h) - \Phi_t(u_0) + \int_0^t \mathbb{S}_{t-s} P\big(\Phi_s(u_0 + h) - \Phi_s(u_0)\big)\,\mathrm{d}s.$$

Due to the continuity of Φ_t, we only have to show that the integral on the right-hand side converges to zero as $\|h\|_{L^p} \to 0$. Let (h_k) be any sequence in $L^p([0,\infty),U)$ with $\|h_k\|_{L^p} \to 0$ as $k \to \infty$. Taking a subsequence, we may assume that $\|h_k\|_{L^p} \le 1$ for all $k \in \mathbb{N}$. Choose $\rho := \|u_0\|_{L^p} + 1$, so that $\|u_0 + h_k\|_{L^p} \le \rho$ for all $k \in \mathbb{N}$.

Again the continuity of Φ_s yields that $\|\mathbb{S}_{t-s}P(\Phi_s(u_0+h_k) - \Phi_s(u_0))\|_X \to 0$ as $k \to \infty$ for each $s \in [0,t]$. On the other hand, Lemma 4.23 yields a constant $c_{\rho,t} > 0$ with $\|\Phi_s(u)\|_X \le c_{\rho,t}$ for all $u \in L^p([0,\infty),U)$ with $\|u\|_{L^p} \le \rho$. Thus we obtain the estimate

$$\|\mathbb{S}_{t-s}P(\Phi_s(u_0 + h_k) - \Phi_s(u_0))\|_X \le m_{\mathbb{S},t}\|P\| 2c_{\rho,t}$$

for all $k \in \mathbb{N}$. The claim now follows from the dominated convergence theorem. □

Assuming that $\Phi_t : L^p([0,\infty),U) \to X$ is continuously differentiable, we can now show the well-posedness of (5.9) without referring to Corollary 5.5.1 of [49]. In fact, if B is L^p–admissible for $\mathbb{T}$ then it is L^p–admissible for $\mathbb{S}$ by the last proposition. Proposition 4.33 then shows that $B'(v_*)$ is L^p–admissible for $\mathbb{S}$.

Remark 5.10. The last step of the preceding proof simplifies considerable if we assume that the additive control system $(\mathbb{T},\Phi)$ is equicontinuous. It is easy to see that in this case also $(\mathbb{S},\Phi^P)$ is equicontinuous. ♢

In the addendum of Lemma 5.6 the condition appeares that Φ_t is Lipschitz on bounded sets. This is another property that translates over to Φ_t^P.

Lemma 5.11. *Let B be L^p–admissible for $\mathbb{T}$ and let the input maps $\Phi_t : L^p([0,\infty),U) \to X$ be Lipschitz on bounded sets for all $t \ge 0$. Then $\Phi_t^P : L^p([0,\infty),U) \to X$ is Lipschitz on bounded sets uniformly for t in compact subsets of* $[0,\infty)$.

Proof. Take any $T > 0$ and $\rho > 0$. From Lemma 4.21 we know that the Φ_t are Lipschitz on bounded sets uniformly for t in compact subsets of $[0,\infty)$. We use the notation introduced in this lemma. Let $t \in [0,T]$ and $u_1, u_2 \in L^p([0,\infty),U)$ with $\|u_1\|_{L^p}, \|u_2\|_{L^p} \le \rho$. Once more using (5.11), we then obtain

$$\begin{aligned}\|\Phi_t^P(u_1) - \Phi_t^P(u_2)\|_X &\le \|\Phi_t(u_1) - \Phi_t(u_2)\|_X + \left\| \int_0^t \mathbb{S}_{t-s}P(\Phi_s(u_1) - \Phi_s(u_2))\,\mathrm{d}s \right\|_X \\ &\le M_{T,\rho}(1 + m_{\mathbb{S},T}\|P\|T)\|u_1 - u_2\|_{L^p}.\end{aligned}$$

□

Assume that Φ_t is continuously differentiable for all $t \ge 0$. Then Φ_t^P is differentiable at constant functions. More importantly, we can express the derivative of Φ_t^P at such points through the derivative of Φ_t.

Lemma 5.12. *Let B be L^p–admissible for $\mathbb{T}$. Further assume that the corresponding input maps $\Phi_t : L^p([0,\infty),U) \to X$ are continuously differentiable for all $t \ge 0$. Then for every $v \in U$ the perturbed input map $\Phi_t^P : L^p([0,\infty),U) \to X$ is differentiable at χ_v for all $t \ge 0$. The derivative is given by*

$$(\Phi_t^P)'(\chi_v)\tilde{u} = \Phi_t'(\chi_v)\tilde{u} + \int_0^t \mathbb{S}_{t-s}P\Phi_s'(\chi_v)\tilde{u}\,\mathrm{d}s \quad \text{for } \tilde{u} \in L^p([0,\infty),U). \tag{5.12}$$

Proof. Let $v \in U$, $t \ge 0$ and $\varepsilon > 0$. Lemma 4.26 yields a radius $\delta > 0$ such that

$$\|\Phi_s'(\chi_v + \overline{u}) - \Phi_s'(\chi_v)\|_{\mathcal{L}(L^p,X)} \le \varepsilon(1 + m_{\mathbb{S},t}\|P\|t)^{-1}.$$

for all $\overline{u} \in \overline{B}(0,\delta)$ and every $s \in [0,t]$. For $\tilde{u} \in \overline{B}(0,\delta) \subseteq L^p([0,\infty),U)$ formula (5.11) yields

$$\begin{aligned}
&\left\|\Phi_t^P(\chi_v+\tilde{u}) - \Phi_t^P(\chi_v) - \Phi_t'(\chi_v)\tilde{u} - \int_0^t \mathbb{S}_{t-s} P \Phi_s'(\chi_v)\tilde{u}\,\mathrm{d}s\right\|_X \\
&\qquad \le \|\Phi_t(\chi_v+\tilde{u}) - \Phi_t(\chi_v) - \Phi_t'(\chi_v)\tilde{u}\|_X \\
&\qquad\qquad + \left\|\int_0^t \mathbb{S}_{t-s} P\big(\Phi_s(\chi_v+\tilde{u}) - \Phi_s(\chi_v) - \Phi_s'(\chi_v)\tilde{u}\big)\,\mathrm{d}s\right\|_X \\
&\qquad \le \left\|\int_0^1 \big(\Phi_t'(\chi_v+\sigma\tilde{u})\tilde{u} - \Phi_t'(\chi_v)\tilde{u}\big)\,\mathrm{d}\sigma\right\|_X \\
&\qquad\qquad + m_{\mathbb{S},t}\|P\| t \sup_{s\in[0,t]} \left\|\int_0^1 \big(\Phi_s'(\chi_v+\sigma\tilde{u})\tilde{u} - \Phi_s'(\chi_v)\tilde{u}\big)\,\mathrm{d}\sigma\right\|_X \\
&\qquad \le (1+m_{\mathbb{S},t}\|P\|t) \sup_{s\in[0,t]} \sup_{\overline{u}\in\overline{B}(0,\delta)} \|\Phi_s'(\chi_v+\overline{u}) - \Phi_s'(\chi_v)\|_{\mathcal{L}(L^p,X)} \|\tilde{u}\|_{L^p} \le \varepsilon \|\tilde{u}\|_{L^p}.
\end{aligned}$$

Hence the claim is shown. ☐

Remark 5.13. Under the conditions of the last lemma, for $v \in U$ and $t \ge 0$ we define the maps $\Phi_t^{\mathrm{rem}} : L^p([0,\infty),U) \to X$ and $(\Phi_t^P)^{\mathrm{rem}} : L^p([0,\infty),U) \to X$ by

$$\begin{aligned}
\Phi_t^{\mathrm{rem}}(\tilde{u}) &= \Phi_t(\chi_v+\tilde{u}) - \Phi_t(\chi_v) - \Phi_t'(\chi_v)\tilde{u}, \\
(\Phi_t^P)^{\mathrm{rem}}(\tilde{u}) &= \Phi_t^P(\chi_v+\tilde{u}) - \Phi_t^P(\chi_v) - (\Phi_t^P)'(\chi_v)\tilde{u}.
\end{aligned}$$

Using (5.11) and Lemma 5.12 we easily deduce the identity

$$(\Phi_t^P)^{\mathrm{rem}}(\tilde{u}) = \Phi_t^{\mathrm{rem}}(\tilde{u}) + \int_0^t \mathbb{S}_{t-s} P \Phi_s^{\mathrm{rem}}(\tilde{u})\,\mathrm{d}s \quad \text{for } \tilde{u} \in L^p([0,\infty),U). \qquad \Diamond$$

Remark 5.14. To the assumptions of Lemma 5.12 we add that $(\mathbb{T}, \Phi'(\chi_v))$ is exactly controllable in some time $T > 0$, see Definition 4.35. Then $(\mathbb{T}, (\Phi^P)'(\chi_v))$ is also exactly controllable in time $T > 0$ provided $\|P\|_{\mathcal{L}(X)}$ is "small enough". Indeed, this follows from Remark C.12 and equation (5.12) since

$$\left\|\int_0^T \mathbb{S}_{T-s} P \Phi_s'(\chi_v)\,\mathrm{d}s\right\|_{\mathcal{L}(L^2,X)} \le T m_{\mathbb{S},T} \|\Phi_T'(\chi_v)\|_{\mathcal{L}(L^2,U)} \|P\|_{\mathcal{L}(X)}.$$

Note that the smallness condition in Remark C.12 involves the norm of a right inverse of the unperturbed operator $\Phi_T'(\chi_v)$. $\Diamond$

We come back to problem (5.8). Recall that (x_*, v_*) is an equilibrium point for this equation. We assumed that $F : X \to X$ is differentiable at x_*. Consider the remainder $F^{\mathrm{rem}} : X \to X$ given by

$$F^{\mathrm{rem}}(\tilde{x}) = F(x_*+\tilde{x}) - F(x_*) - F'(x_*)\tilde{x}.$$

Clearly, if F is differentiable at $x_* + \tilde{x}$, then F^{rem} is differentiable at $\tilde{x} \in X$ with derivative

$$(F^{\mathrm{rem}})'(\tilde{x}) = F'(x_*+\tilde{x}) - F'(x_*).$$

In particular this implies that $(F^{\mathrm{rem}})'(0) = 0$. We shall see that a mild solution of

$$\tilde{z}\,'(t) = (A + F'(x_*))\tilde{z}(t) + F^{\mathrm{rem}}(\tilde{z}(t)) + B(u_*+\tilde{u}) - B(u_*); \quad \tilde{z}(0) = \tilde{x}_0. \tag{5.13}$$

is given by $\tilde{z} := z(\,\boldsymbol{.}\,, x_* + \tilde{x}_0, u_* + \tilde{u}) - z(\,\boldsymbol{.}\,, x_*, u_*)$. Let $\mathbb{S}$ be the semigroup generated by $A + F'(x_*)$. For the sake of a short notation we denote $P := F'(x_*) \in \mathcal{L}(X)$. By Proposition 5.9 the control operator B is admissible for $\mathbb{S}$. Let $\Phi_t^P : L^p([0,\infty), U) \to X$ for $t \geq 0$ be the corresponding input maps. Further let $\tilde{x}_0 \in X$, $\tilde{u} \in L^p([0,\infty), U)$ and $t \in [0, t_\infty)$ where $t_\infty := t_\infty(x_* + \tilde{x}_0, u_* + \tilde{u})$. We first mention that $z(s, x_* + \tilde{x}_0, u_* + \tilde{u}) = x_* + \tilde{z}(s)$ for all $s \in [0, t_\infty)$ since $z(\,\boldsymbol{.}\,, x_*, u_*) = x_*$. On the other hand, we may write

$$\begin{aligned}
\tilde{z}(t) &= z(t, x_* + \tilde{x}_0, u_* + \tilde{u}) - z(t, x_*, u_*) \\
&= \mathbb{T}_t x_* + \mathbb{T}_t \tilde{x}_0 + \int_0^t \mathbb{T}_{t-s} F(x_* + \tilde{z}(s))\,\mathrm{d}s + \Phi_t(u_* + \tilde{u}) \\
&\qquad - \mathbb{T}_t x_* - \int_0^t \mathbb{T}_{t-s} F(x_*)\,\mathrm{d}s - \Phi_t(u_*) \\
&= \mathbb{T}_t \tilde{x}_0 + \int_0^t \mathbb{T}_{t-s} F'(x_*)\tilde{z}(s)\,\mathrm{d}s + \int_0^t \mathbb{T}_{t-s} F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s \\
&\qquad + \Phi_t(u_* + \tilde{u}) - \Phi_t(u_*) \quad \text{for } t \in [0, t_\infty).
\end{aligned} \tag{5.14}$$

We emphasize that this is a fixed-point equation for $\tilde{z}$. From (5.10) we easily obtain the following identities for the two integral terms on the right-hand side.

$$\begin{aligned}
\int_0^t \mathbb{T}_{t-s} F'(x_*)\tilde{z}(s)\,\mathrm{d}s &= \int_0^t \mathbb{S}_{t-s} F'(x_*)\tilde{z}(s)\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s} F'(x_*) \int_0^s \mathbb{T}_{s-\sigma} F'(x_*)\tilde{z}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s, \\
\int_0^t \mathbb{T}_{t-s} F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s &= \int_0^t \mathbb{S}_{t-s} F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s} F'(x_*) \int_0^s \mathbb{T}_{s-\sigma} F^{\mathrm{rem}}(\tilde{z}(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}$$

Plugging them into the last identity for $\tilde{z}$ and using once more (5.10) together with (5.11), by a tedious but easy calculation, we derive

$$\tilde{z}(t) = \mathbb{S}_t \tilde{x}_0 + \int_0^t \mathbb{S}_{t-s} F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s + \Phi_t^P(u_* + \tilde{u}) - \Phi_t^P(u_*) \quad \text{for all } t \in [0, t_\infty). \tag{5.15}$$

The calculation is carried out at the end of this section on page 96. Equation (5.15) means that $\tilde{z}$ is a mild solution of (5.13). We additionally assume that Φ_t is continuously differentiable for every $t \geq 0$. Then by Lemma 5.12 the derivative $(\Phi_t^P)'(u_*)$ exists, and with the operator $(\Phi_t^P)^{\mathrm{rem}}$ defined in Remark 5.13 we may write

$$\tilde{z}(t) = \mathbb{S}_t \tilde{x}_0 + \int_0^t \mathbb{S}_{t-s} F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s + (\Phi_t^P)^{\mathrm{rem}}(\tilde{u}) + (\Phi_t^P)'(u_*)\tilde{u}. \tag{5.16}$$

We nee the following adaption of Definition 4.36. The system (5.8) is called *locally controllable at* (x_*, u_*) if

$$\begin{aligned}
&\forall R > 0\ \exists r_0, r_1 \in (0, R]\ \forall x_0, x_1 \in X \text{ with } \|x_0\|_X \leq r_0, \|x_1\| \leq r_1\ \exists u \in L^2([0,\infty), U): \\
&\qquad\qquad z(T, x_0, u) = x_1 \quad \text{and} \quad \|z(t, x_0, u) - x_*\| < R \quad \text{for all } t \in [0, T].
\end{aligned}$$

The main result of this section is the following linearization theorem. In short, it says that (5.8) is locally controllable near the equilibrium point if the linearized problem (5.9) is exactly controllable. The latter notion was introduced in Definitions 4.35.

Theorem 5.15. *Let X be a Banach space and let U be a Hilbert space. Assume that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B : U \to X_{-1}$ be an L^2–admissible control operator for $\mathbb{T}$. Denote by $\Phi_t : L^2([0,\infty),U) \to X$, $t \geq 0$ the corresponding input maps. Let $F : X \to X$ be Lipschitz on bounded sets. Assume that we have an equilibrium point $(x_*, v_*) \in X \times U$ satisfying*

$$Ax_* + F(x_*) + B(v_*) = 0.$$

Write $u_* := \chi_{v_*}$. Assume that Φ_t is continuously differentiable and Lipschitz on bounded sets for every $t \geq 0$. Let F be continuously differentiable on a neighborhood $O \subseteq X$ of x_* and set $P := F'(x_*)$. Let $\mathbb{S}$ be the strongly continuous semigroup generated by $A+P$ and let Φ_t^P be the perturbed input maps from (5.11). Finally, assume that the linearized system $(\mathbb{S}, (\Phi^P)'(u_*))$ is exactly controllable in time $T > 0$. Then the system (5.8) is locally controllable at (x_*, u_*).

Proof. From Lemma 5.12 we infer that Φ_T^P is differentiable at u_*. Thus the last condition makes sense. Since $L^2([0,\infty),U)$ is a Hilbert space and by assumption the bounded linear operator $(\Phi_T^P)'(u_*)$ is onto, it has a bounded right inverse

$$Q := \left((\Phi_T^P)'(u_*)\right)^{\#} \in \mathcal{L}(X, L^2([0,\infty),U)).$$

We use the notation F^{rem}, Φ_t^{rem} and $(\Phi_t^P)^{\text{rem}}$ introduced above. As remarked there, F^{rem} is continuously differentiable on $\tilde{O} := O - x_*$.

Lemma 5.6 yields radii $r > 0$ and $\rho > 0$ such that $t_\infty(x_0, u) > T$ for all $x_0 \in \overline{B}(x_*, r) \subseteq X$ and $u \in \overline{B}(u_*, \rho) \subseteq L^p([0,\infty),U)$. We may assume that $\overline{B}(x_*, r) \subseteq \tilde{O}$. Moreover, there is a constant $K_T > 0$ with

$$\|z(\,\cdot\,, x_1, u_1) - z(\,\cdot\,, x_2, u_2)\|_{L^\infty([0,T],X)} \leq K_T(\|x_1 - x_2\|_X + \|u_1 - u_2\|_{L^2})$$

for all $x_1, x_2 \in \overline{B}(x_*, r)$ and all $u_1, u_2 \in \overline{B}(u_*, \rho)$. Let $\tilde{x}_0 \in \overline{B}(0, r)$ and $\tilde{u} \in \overline{B}(0, \rho)$. Instead of the mild solution z, we first treat the shifted function $\tilde{z}(\,\cdot\,, \tilde{x}_0, \tilde{u}) \in C([0,T],X)$ given by

$$\tilde{z}(\,\cdot\,, \tilde{x}_0, \tilde{u}) = z(\,\cdot\,, x_* + \tilde{x}_0, u_* + \tilde{u}) - z(\,\cdot\,, x_*, u_*).$$

Observe that $\tilde{z}(\,\cdot\,, 0, 0) = 0$. From the above Lipschitz estimate we obtain

$$\|\tilde{z}(\,\cdot\,, \tilde{x}_1, \tilde{u}_1) - \tilde{z}(\,\cdot\,, \tilde{x}_2, \tilde{u}_2)\|_{L^\infty([0,T],X)} \leq K_T(\|\tilde{x}_1 - \tilde{x}_2\|_X + \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}) \tag{5.17}$$

for all $\tilde{x}_1, \tilde{x}_2 \in \overline{B}(0, r)$ and $\tilde{u}_1, \tilde{u}_2 \in \overline{B}(0, \rho)$. In particular, we have

$$\|\tilde{z}(t, \tilde{x}_0, \tilde{u})\|_X \leq K_T(\|\tilde{x}_0\|_X + \|\tilde{u}\|_{L^2})$$

for all $t \in [0,T]$. As a consequence, we can choose $r > 0$ and $\rho > 0$ so small that $\tilde{z}(t, \tilde{x}, \tilde{u}) \in \tilde{O}$ for all $t \in [0,T]$. In (5.16) we saw that $\tilde{z}$ satisfies the equation

$$\tilde{z}(T, \tilde{x}_0, \tilde{u}) = \mathbb{S}_T \tilde{x}_0 + \int_0^T \mathbb{S}_{T-s} F^{\text{rem}}(\tilde{z}(s, \tilde{x}_0, \tilde{u}))\,\mathrm{d}s + (\Phi_T^P)^{\text{rem}}(\tilde{u}) + (\Phi_T^P)'(u_*)\tilde{u}.$$

Let $\tilde{x}_1 \in X$ and set $\xi := \tilde{x}_1 - \mathbb{S}_T \tilde{x}_0$. If we find an input $\tilde{u} \in \overline{B}(0, \rho)$ with

$$\tilde{u} = Q\left(\xi - \int_0^T \mathbb{S}_{T-s} (F^P)^{\text{rem}}(\tilde{z}(s, \tilde{x}_0, \tilde{u}))\,\mathrm{d}s - (\Phi_T^P)^{\text{rem}}(\tilde{u})\right) =: \mathcal{C}(\tilde{u}),$$

then we obtain

$$\tilde{z}(T,\tilde{x}_0,\tilde{u}) = \mathbb{S}_T\tilde{x}_0 + \int_0^T \mathbb{S}_{T-s}F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s + (\Phi_T^P)^{\mathrm{rem}}(\tilde{u}) + (\Phi_T^P)'(u_*)\tilde{u} = \tilde{x}_1.$$

Hence we are looking for a fixed-point of the map $\mathcal{C} : \overline{B}(0,\rho_0) \to \overline{B}(0,\rho_0)$ defined through the equation above. We will find a radius $\rho_0 \in (0,\rho]$ such that $\mathcal{C}$ is strictly contractive on $\overline{B}(0,\rho_0)$ provided $\tilde{x}_0 \in \overline{B}(0,r_0)$ and $\tilde{x}_1 \in \overline{B}(0,r_1)$ for some $r_0,r_1 > 0$. To this end, we need several estimates.

Let $R > 0$. Lemma 4.26 yields a radius $\rho_1 > 0$ with

$$\|(\Phi_t^{\mathrm{rem}})'(\overline{u})\|_{\mathcal{L}(X,L^2)} = \|\Phi_t'(u_* + \overline{u}) - \Phi_t'(u_*)\|_{\mathcal{L}(X,L^2)} \le \frac{1}{4\|Q\|}(1 + m_{\mathbb{S},T}\|P\|T)^{-1}$$

for all $\overline{u} \in L^2([0,\infty),U)$ with $\|\overline{u}\|_{L^2} \le \rho_1$ and each $t \in [0,T]$. Using Remark 5.13, for $\tilde{u}_1,\tilde{u}_2 \in L^2([0,\infty),U)$ with $\|\tilde{u}_1\|_{L^2}, \|\tilde{u}_2\|_{L^2} \le \rho_1$ we estimate

$$\begin{aligned}
&\|(\Phi_T^P)^{\mathrm{rem}}(\tilde{u}_1) - (\Phi_T^P)^{\mathrm{rem}}(\tilde{u}_2)\|_X \\
&\quad\le \|\Phi_T^{\mathrm{rem}}(\tilde{u}_1) - \Phi_T^{\mathrm{rem}}(\tilde{u}_2)\|_X + m_{\mathbb{S},T}\|P\|T \sup_{t\in[0,T]} \|\Phi_t^{\mathrm{rem}}(\tilde{u}_1) - \Phi_t^{\mathrm{rem}}(\tilde{u}_2)\|_X \\
&\quad\le (1 + m_{\mathbb{S},T}\|P\|T) \sup_{t\in[0,T]} \left\| \int_0^1 (\Phi_t^{\mathrm{rem}})'(\tilde{u}_2 - \sigma(\tilde{u}_1 - \tilde{u}_2))(\tilde{u}_1 - \tilde{u}_2)\,\mathrm{d}\sigma \right\|_X \\
&\quad\le (1 + m_{\mathbb{S},T}\|P\|T) \sup_{t\in[0,T]} \sup_{\overline{u}\in\overline{B}(0,\rho_1)} \|(\Phi_t^{\mathrm{rem}})'(\overline{u})\|_{\mathcal{L}(L^2,X)} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}.
\end{aligned}$$

For such $\tilde{u}_1$ and $\tilde{u}_2$ we thus obtain

$$\|(\Phi_T^P)^{\mathrm{rem}}(\tilde{u}_1) - (\Phi_T^P)^{\mathrm{rem}}(\tilde{u}_2)\|_X \le \frac{1}{4\|Q\|}\|\tilde{u}_1 - \tilde{u}_2\|_{L^2}. \tag{5.18}$$

Since $(\Phi_T^P)^{\mathrm{rem}}(0) = 0$ for $\tilde{u} \in \overline{B}(0,\rho_1)$, it follows

$$\|(\Phi_T^P)^{\mathrm{rem}}(\tilde{u})\|_X \le \frac{1}{4\|Q\|}\|\tilde{u}\|_{L^2}. \tag{5.19}$$

Further, because F is differentiable at x_*, we find a number $R_1 > 0$ such that

$$\|F^{\mathrm{rem}}(w)\|_X = \|F(x_* + w) - F(x_*) - F'(x_*)w\|_X \le \frac{1}{8\|Q\| m_{\mathbb{S},T} T K_T}\|w\|_X \tag{5.20}$$

for all $w \in \overline{B}(0,R_1) \subseteq X$. We may assume that $R_1 \le R$. Moreover, as F' is continuous, we may choose R_1 so small that $w \in \tilde{O}$ and

$$\|(F^{\mathrm{rem}})'(w)\|_{\mathcal{L}(X)} = \|F'(x_* + w) - F'(x_*)\|_{\mathcal{L}(X)} \le \frac{1}{4\|Q\| m_{\mathbb{S},T} T K_T} \tag{5.21}$$

for all $w \in X$ with $\|w\|_X \le R_1$. We can now fix the constants. Set

$$\rho_0 = \min\left\{\rho, \rho_1, \frac{R_1}{2K_T}\right\}, \quad r_0 = \min\left\{r, \rho_0, \frac{\rho_0}{4\|Q\| m_{\mathbb{S},T}}, \frac{R_1}{2K_T}\right\} \quad \text{and} \quad r_1 = \frac{\rho_0}{4\|Q\|}.$$

Let $\tilde{x}_0 \in \overline{B}(0, r_0)$ and $\tilde{x}_1 \in \overline{B}(0, r_1)$. For all $\tilde{u} \in \overline{B}(0, \rho_0)$ from (5.17) we infer

$$\|\tilde{z}(t, \tilde{x}_0, \tilde{u})\|_X \leq K_T(\|\tilde{x}_0\|_X + \|\tilde{u}\|_{L^2}) \leq R_1 \leq R \quad \text{for all } t \in [0, T]. \tag{5.22}$$

Therefore we can use (5.19) and (5.20) to deduce

$$\begin{aligned}
\|\mathcal{C}(\tilde{u})\|_{L^2} &\leq \|Q\| \left(\|\tilde{x}_1\| + \|\mathbb{S}_T\| \|\tilde{x}_0\|_X + \int_0^T \|\mathbb{S}_{T-s} F^{\text{rem}}(\tilde{z}(s, \tilde{x}_0, \tilde{u}))\|_X \, \mathrm{d}s + \|(\Phi_T^P)^{\text{rem}}(\tilde{u})\|_X \right) \\
&\leq \frac{\rho_0}{4} + \frac{\rho_0}{4} + \|Q\| m_{\mathbb{S},T} \int_0^T \|F^{\text{rem}}(\tilde{z}(s, \tilde{x}_0, \tilde{u}))\|_X \, \mathrm{d}s + \frac{1}{4} \|\tilde{u}\|_{L^2} \\
&\leq \frac{\rho_0}{2} + \frac{\|Q\| m_{\mathbb{S},T}}{8 \|Q\| m_{\mathbb{S},T} T K_T} \int_0^T \|\tilde{z}(s, \tilde{x}_0, \tilde{u})\|_X \, \mathrm{d}s + \frac{\rho_0}{4} \\
&\leq \frac{3\rho_0}{4} + \frac{T}{8 T K_T} \|\tilde{z}(\,\cdot\,, \tilde{x}_0, \tilde{u})\|_{L^\infty([0,T],X)} \leq \frac{3\rho_0}{4} + \frac{1}{8}(\|\tilde{x}_0\|_X + \|\tilde{u}\|_{L^2}) \leq \rho_0
\end{aligned}$$

for all $\tilde{u} \in \overline{B}(0, \rho_0)$. As a result, $\mathcal{C}$ maps $\overline{B}(0, \rho_0)$ to itself. To show that $\mathcal{C}$ is strictly contractive on this ball, take $\tilde{u}_1, \tilde{u}_2 \in \overline{B}(0, \rho_0)$. We abbreviate $\tilde{z}_j := \tilde{z}(\,\cdot\,, \tilde{x}_0, \tilde{u}_j)$ for $j = 1, 2$. Employing (5.18) and (5.21), we derive

$$\begin{aligned}
\|\mathcal{C}(\tilde{u}_1) - \mathcal{C}(\tilde{u}_2)\|_{L^2} &\leq \|Q\| \int_0^T \left\| \mathbb{S}_{T-s}(F^{\text{rem}}(\tilde{z}_1(s)) - F^{\text{rem}}(\tilde{z}_2(s))) \right\|_X \, \mathrm{d}s \\
&\qquad + \|Q\| \|(\Phi_T^P)^{\text{rem}}(\tilde{u}_1) - (\Phi_T^P)^{\text{rem}}(\tilde{u}_2)\|_X \\
&\leq \|Q\| m_{\mathbb{S},T} \int_0^T \int_0^1 \|(F^{\text{rem}})'(\tilde{z}_2(s) + \sigma[\tilde{z}_1(s) - \tilde{z}_2(s)])(\tilde{z}_1(s) - \tilde{z}_2(s))\|_X \, \mathrm{d}\sigma \, \mathrm{d}s \\
&\qquad + \frac{1}{4} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2} \\
&\leq \frac{\|Q\| m_{\mathbb{S},T}}{4 \|Q\| m_{\mathbb{S},T} K_T T} \int_0^T \|\tilde{z}_1(s) - \tilde{z}_2(s)\|_X \, \mathrm{d}s + \frac{1}{4} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2} \\
&\leq \frac{K_T T}{4 K_T T} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2} + \frac{1}{4} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2} = \frac{1}{2} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}.
\end{aligned}$$

Of course, $\overline{B}(0, \rho)$ as a closed subset of $L^2([0, \infty), U)$ is a complete nonempty metrical space. The contraction mapping principle yields the existence of a fixed-point $\tilde{u}$ of $\mathcal{C}$ in $\overline{B}(0, \rho)$.

Finally we translate the results back to the original problem. Let $x_0 \in \overline{B}(x_*, r_0)$ and $x_1 \in \overline{B}(x_*, r_1)$ and set $\tilde{x}_j := x_j - x_*$ for $j = 1, 2$. Take the corresponding fixed point $\tilde{u} \in L^2([0, \infty), U)$ obtained above. Then for $u := u_* + \tilde{u}$ we infer the identity

$$z(T, x_0, u) = \tilde{z}(T, \tilde{x}_0, \tilde{u}) + x_* = \tilde{x}_1 + x_* = x_1.$$

Moreover, in (5.22) we saw that $\|z(t, x_0, u) - x_*\|_X = \|\tilde{z}(t, x_0 - x_*, u - u_*)\|_X \leq R$ for all $t \in [0, T]$. Thus the proof is finished. □

Proof of equation (5.15)

Using (5.14), we have to verify the identity

$$\begin{aligned}
\mathbb{T}_t \tilde{x}_0 + \int_0^t \mathbb{T}_{t-s} F'(x_*) \tilde{z}(s) \, \mathrm{d}s + \int_0^t \mathbb{T}_{t-s} F^{\text{rem}}(\tilde{z}(s)) \, \mathrm{d}s + \Phi_t(u_* + \tilde{u}) - \Phi_t(u_*) \\
= \mathbb{S}_t \tilde{x}_0 + \int_0^t \mathbb{S}_{t-s} F^{\text{rem}}(\tilde{z}(s)) \, \mathrm{d}s + \Phi_t^P(u_* + \tilde{u}) - \Phi_t^P(u_*).
\end{aligned}$$

As announced in the text above, on the left-hand side we plug in

$$\int_0^t \mathbb{T}_{t-s}F'(x_*)\tilde{z}(s)\,\mathrm{d}s = \int_0^t \mathbb{S}_{t-s}F'(x_*)\tilde{z}(s)\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s}F'(x_*)\int_0^s \mathbb{T}_{s-\sigma}F'(x_*)\tilde{z}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s,$$
$$\int_0^t \mathbb{T}_{t-s}F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s = \int_0^t \mathbb{S}_{t-s}F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s}F'(x_*)\int_0^s \mathbb{T}_{s-\sigma}F^{\mathrm{rem}}(\tilde{z}(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s.$$

On the right-hand side we use (5.10) in the form $\mathbb{S}_t\tilde{x}_0 = \mathbb{T}_t\tilde{x}_0 + \int_0^t \mathbb{S}_{t-s}F'(x_*)\mathbb{T}_s\tilde{x}_0\,\mathrm{d}s$ as well as (5.11), more precisely

$$\Phi_t^P(u_* + \tilde{u}) = \Phi_t(u_* + \tilde{u}) + \int_0^t \mathbb{S}_{t-s}F'(x_*)\Phi_s(u_* + \tilde{u})\,\mathrm{d}s,$$
$$\Phi_t^P(u_*) = \Phi_t(u_*) + \int_0^t \mathbb{S}_{t-s}F'(x_*)\Phi_s(u_*)\,\mathrm{d}s.$$

We end up with the equation

$$\begin{aligned}
&\mathbb{T}_t\tilde{x}_0 + \int_0^t \mathbb{S}_{t-s}F'(x_*)\tilde{z}(s)\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s}F'(x_*)\int_0^s \mathbb{T}_{s-\sigma}F'(x_*)\tilde{z}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\quad + \int_0^t \mathbb{S}_{t-s}F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s}F'(x_*)\int_0^s \mathbb{T}_{s-\sigma}F^{\mathrm{rem}}(\tilde{z}(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\quad + \Phi_t(u_* + \tilde{u}) - \Phi_t(u_*)\\
&\qquad = \mathbb{T}_t\tilde{x}_0 + \int_0^t \mathbb{S}_{t-s}F'(x_*)\mathbb{T}_s\tilde{x}_0\,\mathrm{d}s + \int_0^t \mathbb{S}_{t-s}F^{\mathrm{rem}}(\tilde{z}(s))\,\mathrm{d}s\\
&\qquad\quad + \int_0^t \mathbb{S}_{t-s}F'(x_*)\Phi_s(u_* + \tilde{u})\,\mathrm{d}s - \int_0^t \mathbb{S}_{t-s}F'(x_*)\Phi_s(u_*)\,\mathrm{d}s\\
&\qquad\quad + \Phi_t(u_* + \tilde{u}) - \Phi_t(u_*).
\end{aligned}$$

After deleting all terms that are equal on both sides, this is

$$\begin{aligned}
&\int_0^t \mathbb{S}_{t-s}F'(x_*)\left[\tilde{z}(s) - \int_0^s \mathbb{T}_{s-\sigma}F'(x_*)\tilde{z}(\sigma)\,\mathrm{d}\sigma - \int_0^s \mathbb{T}_{s-\sigma}F^{\mathrm{rem}}(\tilde{z}(\sigma))\,\mathrm{d}\sigma\right]\,\mathrm{d}s\\
&\qquad = \int_0^t \mathbb{S}_{t-s}F'(x_*)\left[\mathbb{T}_s\tilde{x}_0 + \Phi_s(u_* + \tilde{u}) - \Phi_s(u_*)\right]\,\mathrm{d}s.
\end{aligned}$$

It remains to show that the brackets under the integral are equal, that means

$$\tilde{z}(s) - \int_0^s \mathbb{T}_{s-\sigma}F'(x_*)\tilde{z}(\sigma)\,\mathrm{d}\sigma - \int_0^s \mathbb{T}_{s-\sigma}F^{\mathrm{rem}}(\tilde{z}(\sigma))\,\mathrm{d}\sigma - \mathbb{T}_s\tilde{x}_0 - \Phi_s(u_* + \tilde{u}) + \Phi_s(u_*) = 0.$$

But this is just (5.14) again.

5.3 Applications

Due to the structure of the space $X = [\mathrm{D}(A_0)]_0 \times [\mathrm{D}(A_0)]_{-1/2}$ the wave equation behaves relatively well under polynomially bounded perturbations of the state equation. We shall see this in two examples.

5.3.1 The wave equation with Dirichlet boundary control

As in the example of Subsection 4.4.4 we consider the wave equation with Dirichlet boundary control. Here we leave the input unmodified but we add a nonlinear term to the state equation. First recall the following notation and facts. Let $n \in \mathbb{N}$ and let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded open domain with boundary $\partial\mathcal{O}$ of class C^2. The Dirichlet Laplacian on $\mathcal{O}$ was $-A_0$, where A_0 is the strictly positive operator in $L^2(\mathcal{O})$ given by

$$\mathrm{D}(A_0) = \{f \in H_0^1(\mathcal{O}) \,|\, \Delta f \in L^2(\mathcal{O})\} \quad \text{and} \quad A_0 f = -\Delta f.$$

We had $X = L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$ and $\mathrm{D}(A) = H_0^1(\mathcal{O}) \times L^2(\mathcal{O}) \subseteq X$. The skew-adjoint operator $A : \mathrm{D}(A) \to X$ was defined as

$$A(f, g) = (g, -A_0 f).$$

It is the generator of a unitary group $\mathbb{T}$ on X. Further we fixed a relatively open part $\Gamma \subseteq \partial\mathcal{O}$ of the boundary and set $U = L^2(\Gamma)$. The linear operator $B^l \in \mathcal{L}(U, X_{-1})$ given by

$$B^l(v) = (0, A_0 D v)$$

is L^2–admissible for $\mathbb{T}$. Here $D \in \mathcal{L}(L^2(\partial\mathcal{O}), L^2(\mathcal{O}))$ is the Dirichlet map for $\mathcal{O}$. See Section 10.6 of [49] for details. We treat the nonlinear wave equation

$$\begin{aligned}
\partial_t^2 \omega(t,\xi) &= \Delta\omega(t,\xi) + F_0(\omega(t,\xi)), && (t,\xi) \in (0,\infty) \times \mathcal{O} \\
\omega(t,\xi) &= 0, && (t,\xi) \in (0,\infty) \times \partial\mathcal{O} \setminus \Gamma \\
\omega(t,\xi) &= \mu(t,\xi), && (t,\xi) \in (0,\infty) \times \Gamma \\
\omega(0,\xi) &= f_0(\xi), \quad \partial_t\omega(0,\xi) = g_0(\xi), && \xi \in \mathcal{O}
\end{aligned} \tag{5.23}$$

In Subsection 4.4.4 we discussed that the linear version (4.39) of this equation (with $F_0 = 0$) can be transformed to the control problem

$$z'(t) = Az(t) + B^l u(t); \quad z(0) = x_0.$$

with $z(t) = (\omega(t, \cdot), \omega'(t, \cdot))$, $u(t) = \mu(t, \cdot)$ and $x_0 = (f_0, g_0)$. We do not deal with the question in which sense the solutions this problem or our modified problem yield solutions of (5.23). However, in Subsection 4.4.2 we made comments on that.

Adding $F_0(\omega(t,\xi))$ to the state equation in (4.39) clearly corresponds to adding a term $F(z(t))$ to the last problem, where we put $F(f,g) = (0, F_0 \circ f)$ for $(f,g) \in X$. Therefore (5.23) corresponds to the system

$$z'(t) = Az(t) + F(z(t)) + B^l u(t); \quad z(0) = x_0. \tag{5.24}$$

We shall show that we can apply the theory developed in the previous sections. In the following we assume that $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. Moreover, let there be a number $\alpha \in (1, 2]$ with

$$|F_0(\mathbf{a})| \lesssim |\mathbf{a}|^\alpha \quad \text{and} \quad |F_0'(\mathbf{a})|_{\mathcal{L}(\mathbb{R}^2)} \lesssim |\mathbf{a}|^{\alpha-1} \quad \text{for all } \mathbf{a} \in \mathbb{R}^2. \tag{5.25}$$

We emphasize that in this case $F_0(0,0) = (0,0)$ and $F_0'(0,0) \in \mathcal{L}(\mathbb{R}^2)$ is the zero operator.

Remark. It is only a matter of calculation to see that conditions (5.25) are satisfied for the standard nonlinearity $F_0(\mathbf{a}) = |\mathbf{a}|^{\alpha-1}\mathbf{a}$. Actually, in this situation we have

$$F_0'(\mathbf{a}) = \begin{pmatrix} |\mathbf{a}|^{\alpha-1} + (\alpha-1)|\mathbf{a}|^{\alpha-3}a^2 & (\alpha-1)|\mathbf{a}|^{\alpha-3}ab \\ (\alpha-1)|\mathbf{a}|^{\alpha-3}ab & |\mathbf{a}|^{\alpha-1} + (\alpha-1)|\mathbf{a}|^{\alpha-3}b^2 \end{pmatrix}$$

for $\mathbf{a} = (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. To verify the bound for $F_0'(\mathbf{a})$ use that $\mathcal{L}(\mathbb{R}^2)$ can equivalently be seen as $\mathbb{R}^4$ with the maximum norm and estimate each entry separately. We will not use this special form. Note that $F_0'(\mathbf{a})$ belongs to $\mathcal{L}(\mathbb{C})$ if and only if $\mathbf{a} = (0,0)$, since only then the entries on the anti-diagonal are additive inverse to each other. ◊

Recall that writing $F_0 \circ f$ for a function $f : \mathcal{O} \to \mathbb{C}$ we actually mean $\iota \circ F_0 \circ \iota^{-1} \circ f$ where $\iota : \mathbb{R}^2 \to \mathbb{C}$ is the isometric $\mathbb{R}$–linear invertible map given by $(a,b) \mapsto a + \mathrm{i}b$. However, for the sake of a simple notation we suppress ι and its inverse.

It is easy to verify that $F_0 \circ f$ belongs to $L^{2/\alpha}(\mathcal{O})$ for all $f \in L^2(\mathcal{O})$. The arguments were given in Section 4.4. Indeed, $F_0 \circ f$ is measurable due to the continuity of F_0. So choosing a representative $f : \mathcal{O} \to \mathbb{C}$, the claim follows from the estimate $|F_0(f(\xi))|^{2/\alpha} \lesssim |f(\xi)|^2$ for almost all $\xi \in \mathcal{O}$. We also see that

$$\|F_0 \circ f\|_{L^{\frac{2}{\alpha}}(\mathcal{O})} \lesssim \|f\|_{L^2(\mathcal{O})} \quad \text{for every } f \in L^2(\mathcal{O}).$$

This in turn shows that the map $G : L^2(\mathcal{O}) \to L^{2/\alpha}(\mathcal{O})$; $f \mapsto F_0 \circ f$ is bounded on bounded sets. Next we prove that G is $\mathbb{R}$–differentiable with derivative given by

$$G'(f)g = F_0'(f(\,\cdot\,))g(\,\cdot\,) \quad \text{for } f,g \in L^2(\mathcal{O}).$$

It is obvious that this defines a $\mathbb{R}$–linear map $G'(f)$ on $L^2(\mathcal{O})$ for every $f \in L^2(\mathcal{O})$. In order to see that it is also bounded with values in $L^{2/\alpha}(\mathcal{O})$, let $f \in L^2(\mathcal{O})$ and note that due to the growth bound (5.25) we have

$$F_0' \circ f \in L^{\frac{2}{(\alpha-1)}}(\mathcal{O}, \mathcal{L}(\mathbb{R}^2)) \quad \text{and} \quad \|F_0' \circ f\|_{L^{\frac{2}{(\alpha-1)}}(\mathcal{O},\mathcal{L}(\mathbb{R}^2))} \lesssim \|f\|_{L^2(\mathcal{O})}^{\alpha-1}.$$

Now using Hölder's inequality with exponents $\alpha > 1$ and $\alpha' = \frac{\alpha}{\alpha-1}$, we obtain

$$\begin{aligned}\|F_0'(f(\,\cdot\,))g(\,\cdot\,)\|_{L^{\frac{2}{\alpha}}(\mathcal{O})} &\le \left(\int_{\mathcal{O}} \|F_0'(f(\xi))\|_{\mathcal{L}(\mathbb{R}^2)}^{\frac{2}{\alpha}} |g(\xi)|^{\frac{2}{\alpha}} \,\mathrm{d}\xi\right)^{\alpha/2} \\ &\le \|F_0' \circ f\|_{L^{\frac{2}{\alpha-1}}(\mathcal{O},\mathcal{L}(\mathbb{R}^2))} \|g\|_{L^2(\mathcal{O})} \lesssim \|f\|_{L^2(\mathcal{O})}^{\alpha-1} \|g\|_{L^2(\mathcal{O})}.\end{aligned}$$

Therefore $G'(f)$ belongs to $\mathcal{L}_{\mathbb{R}}(L^2(\mathcal{O}), L^{2/\alpha}(\mathcal{O}))$ and satisfies

$$\|G'(f)\|_{\mathcal{L}_{\mathbb{R}}(L^2(\mathcal{O}),L^{\frac{2}{\alpha}}(\mathcal{O}))} \lesssim \|f\|_{L^2(\mathcal{O})}^{\alpha-1}. \tag{5.26}$$

Again with Hölder's inequality and using also Minkowski's inequality, we derive the estimate

$$\begin{aligned}
&\Big(\int_{\mathcal{O}} |F_0(f(\xi)+h(\xi)) - F_0(f(\xi)) - F_0'(f(\xi))h(\xi)|^{\frac{2}{\alpha}}\,\mathrm{d}\xi\Big)^{\alpha/2} \\
&\qquad \le \Big(\int_{\mathcal{O}} \Big|\int_0^1 (F_0'(f(\xi)+rh(\xi)) - F_0'(f(\xi)))h(\xi)\,\mathrm{d}r\Big|^{2/\alpha}\mathrm{d}\xi\Big)^{\alpha/2} \\
&\qquad \le \int_0^1 \Big(\int_{\mathcal{O}} |F_0'(f(\xi)+rh(\xi)) - F_0'(f(\xi))|^{\frac{2}{\alpha}}_{\mathcal{L}(\mathbb{R}^2)} |h(\xi)|^{\frac{2}{\alpha}}\,\mathrm{d}\xi\Big)^{\alpha/2}\mathrm{d}r \\
&\qquad \le \Big(\int_0^1 \|F_0'(f(\,\cdot\,)+rh(\,\cdot\,)) - F_0'(f(\,\cdot\,))\|_{L^{\frac{2}{\alpha-1}}(\mathcal{O},\mathcal{L}(\mathbb{R}^2))}\,\mathrm{d}r\Big)\|h\|_{L^2(\mathcal{O})}
\end{aligned}$$

for all $h \in L^2(\mathcal{O})$. For $\varepsilon > 0$ Lemma 4.39 (with to $J = \mathcal{O}$) yields a number $\delta > 0$ such that for $h \in L^2(\mathcal{O})$ with $\|h\|_{L^2(\mathcal{O})} < \delta$ the right-hand side is less or equal to $\varepsilon\|h\|_{L^2(\mathcal{O})}$. In the very same way we show that G' is continuous. In fact,

$$\|G'(f) - G'(\tilde{f})\|_{\mathcal{L}_{\mathbb{R}}(L^2(\mathcal{O}),L^{\frac{2}{\alpha}}(\mathcal{O}))} \le \|F_0' \circ f - F_0' \circ \tilde{f}\|_{L^{\frac{2}{\alpha-1}}(\mathcal{O},\mathcal{L}(\mathbb{R}^2))} \quad \text{for } f, \tilde{f} \in L^2(\mathcal{O}).$$

We have shown that G is continuously $\mathbb{R}$–differentiable. From (5.26) we easily deduce that G' is bounded on bounded sets. As a consequence, G is Lipschitz on bounded sets. Indeed, let $r > 0$. Then for all $f, \tilde{f} \in L^2(\mathcal{O})$ with $\|f\|_{L^2(\mathcal{O})}, \|\tilde{f}\|_{L^2(\mathcal{O})} \le r$ we have

$$\|G(f)-G(\tilde{f})\|_{L^{\frac{2}{\alpha}}(\mathcal{O})} \le \int_0^1 \|G'(\tilde{f}+s(f-\tilde{f}))\|_{\mathcal{L}_{\mathbb{R}}(\dots)}\|f-\tilde{f}\|_{L^2(\mathcal{O})}\,\mathrm{d}s \lesssim r^{\alpha-1}\|f-\tilde{f}\|_{L^2(\mathcal{O})}. \tag{5.27}$$

As announced we now define $F(f,g) = (0, F_0 \circ f) = (0, G(f))$ for $(f,g) \in X$. In order to check that $F(f,g)$ is contained in X for every pair $(f,g) \in X$, we need an embedding of $L^{2/\alpha}(\mathcal{O})$ into $H^{-1}(\mathcal{O})$. Since the existence of such embeddings depends on the dimension n of $\mathcal{O}$ we distinguish the cases $n \in \{1,2\}$ and $n \ge 3$. The standard Sobolev embedding theorem (see Theorem 4.12 in [2]) yields[1]

$$H_0^1(\mathcal{O}) \hookrightarrow L^p(\mathcal{O}) \quad \text{for all } p \in \begin{cases} [2,\infty), & n \in \{1,2\} \\ [2, \frac{2n}{n-2}], & n \ge 3. \end{cases}$$

Thus by duality we have the embedding

$$L^q(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O}) = \big(H_0^1(\mathcal{O})\big)^* \quad \text{for all } q = p' \in \begin{cases} (1,2], & n \in \{1,2\} \\ [\frac{2n}{n+2}, 2], & n \ge 3. \end{cases} \tag{5.28}$$

Plugging in $q = \frac{2}{\alpha}$, we see that F maps X to itself if $\alpha \in (1,2]$ in case $n \in \{1,2\}$ and if $\alpha \in (1, \frac{n+2}{n}]$ in case $n \ge 3$. For the remainder of the example let us assume that these conditions are fulfilled.

To see that F is Lipschitz on bounded sets, we take $r > 0$ and $(f,g), (\tilde{f},\tilde{g}) \in X$ with $\|(f,g)\|_X, \|(\tilde{f},\tilde{g})\|_X \le r$. Then $\|f\|_{L^2(\mathcal{O})}, \|\tilde{f}\|_{L^2(\mathcal{O})} \le r$ and (5.27) together with the embedding (5.28) yield

$$\begin{aligned}
\|F(f,g) - F(\tilde{f},\tilde{g})\|_X = \|G(f) - G(\tilde{f})\|_{H^{-1}} &\lesssim \|G(f) - G(\tilde{f})\|_{L^{\frac{2}{\alpha}}(\mathcal{O})} \\
&\lesssim r^{\alpha-1}\|f - \tilde{f}\|_{L^2(\mathcal{O})} \le r^{\alpha-1}\|(f,g) - (\tilde{f},\tilde{g})\|_X.
\end{aligned}$$

[1] Indeed, we have $H_0^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ in case $n = 1$. However this doesn't make a difference in the following.

Since G is differentiable, it follows that F is continuously $\mathbb{R}$–differentiable with derivative

$$F'(f,g) = (0, G'(f)) \quad \text{for } (f,g) \in X.$$

To verify this, take $(f,g),(h,o) \in X$ and let $\varepsilon > 0$. There is a radius $\delta > 0$ such that $\|G(f+h) - G(f) - G'(f)h\|_{L^{2/\alpha}(\mathcal{O})} \le \varepsilon \|h\|_{L^2(\mathcal{O})}$ provided $\|h\|_{L^2(\mathcal{O})} \le \delta$. Let $\|(h,o)\|_X \le \delta$. Since $\|h\|_{L^2(\mathcal{O})} \le \|(h,o)\|_X$ we can deduce

$$\begin{aligned}\|F(f+h,g+o) - F(f,g) - (0,G'(f)h)\|_X &= \|G(f+h) - G(f) - G'(f)h\|_{H^{-1}(\mathcal{O})} \\ &\lesssim \|G(f+h) - G(f) - G'(f)\|_{L^{\frac{2}{\alpha}}(\mathcal{O})} \le \varepsilon \|h\|_{L^2(\mathcal{O})} \le \varepsilon \|(h,o)\|_X.\end{aligned}$$

Clearly, for each fixed $(f,g) \in X$ the mapping $X \to X$; $(h,o) \mapsto (0, G(f)h)$ is $\mathbb{R}$–linear and bounded. We skip the elementary proof which as a side product also gives

$$\|F'(f,g)\|_{\mathcal{L}_\mathbb{R}(X)} \lesssim \|G'(f)\|_{\mathcal{L}_\mathbb{R}(L^2(\mathcal{O}),L^{\frac{2}{\alpha}}(\mathcal{O}))} \lesssim \|f\|_{L^2(\mathcal{O})}^{\alpha-1} \le \|(f,g)\|_X^{\alpha-1} \tag{5.29}$$

for all $(f,g) \in X$, see (5.26).

Since $F(0,0) = (0,0)$, with the linearity of A and B^l we conclude that an equilibrium point of (5.24) is given by

$$x_* = (0,0), \quad v_* = 0.$$

Because $F'(x_*)$ is the zero operator, it trivially belongs to $\mathcal{L}(X)$. The theory of the previous sections can be applied. In this example we have $P = 0$. Therefore the semigroup $\mathbb{S}$ generated by $A + P$ from (5.10) equals $\mathbb{T}$. Consequently also the perturbed input maps Φ^P from (5.11) coincide with the unperturbed input maps Φ^l. Thus whenever $(\mathbb{T}, \Phi^l)$ is exactly controllable in some time $T > 0$, then the assumptions of Theorem 5.15 are satisfied. We gave a sufficient condition for that in Subsection 4.4.4. In this case, for every $R > 0$ there are radii $r_1, r_2 > 0$ such that for all $x_0 \in \overline{B}(0,r_1) \subseteq X$ and $x_1 \in \overline{B}(0,r_2) \subseteq X$ we find an input $u \in L^2([0,\infty),U)$ with

$$x_1 = z(T,x_0,u) \qquad \text{as well as} \qquad \|z(t,x_0,u)\|_X \le R \quad \text{for all } t \in [0,T].$$

5.3.2 The wave equation with mixed boundary control

As a second example we study the wave equation with mixed boundary control. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\mathcal{O}$. Take nonempty relatively open disjoint subsets $\Gamma_0, \Gamma_1 \subseteq \partial\mathcal{O}$ of the boundary such that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \partial\mathcal{O}$. Further assume that $\partial\Gamma_0 = \overline{\Gamma}_0 \setminus \Gamma_0$ and $\partial\Gamma_1$ are nullsets. Note that then $\partial\mathcal{O} \setminus (\Gamma_0 \cup \Gamma_1) = \partial\Gamma_0 \cup \partial\Gamma_1$ is a nullset. According to Section 7 in [47] the system (for simplicity we chose $b \equiv 1$)

$$\begin{aligned} \partial_t^2 \omega(t,\xi) &= \Delta\omega(t,\xi), && (t,\xi) \in (0,\infty) \times \mathcal{O} \\ \omega(t,\xi) &= 0, && (t,\xi) \in (0,\infty) \times \Gamma_0 \\ \frac{\partial}{\partial\nu}\omega(t,\xi) + \partial_t\omega(t,x) &= \sqrt{2}\mu(t,\xi), && (t,\xi) \in (0,\infty) \times \Gamma_1 \\ \omega(0,\xi) = f_0(\xi), \quad \partial_t\omega(0,\xi) &= g_0(\xi), && \xi \in \mathcal{O}. \end{aligned} \tag{5.30}$$

can be formulated as control system of the form (4.9). The operators A and B are not easy to grasp. Therefore we repeat their construction in detail.

We can see $L^2(\Gamma_1)$ is a subspace of $L^2(\partial\mathcal{O})$ as follows

$$\begin{aligned} L^2(\Gamma_1) &= \{h \in L^2(\partial\mathcal{O}) \,|\, h = 0 \text{ almost everywhere on } \partial\mathcal{O} \setminus \Gamma_1\} \\ &= \{h \in L^2(\partial\mathcal{O}) \,|\, h = 0 \text{ almost everywhere on } \Gamma_0\}, \end{aligned}$$

where we used that $\partial\mathcal{O} \setminus (\Gamma_0 \cup \Gamma_1)$ is a nullset for the second equation. With the Dirichlet trace operator $\operatorname{tr} \in \mathcal{L}(H^1(\mathcal{O}), L^2(\partial\mathcal{O}))$ we define

$$H^1_{\Gamma_0}(\mathcal{O}) = \{f \in H^1(\mathcal{O}) \,|\, \operatorname{tr} f = 0 \text{ almost everywhere on } \Gamma_0\}.$$

It is easy to see that $H^1_{\Gamma_0}(\mathcal{O})$ is the closed subspace of $H^1(\mathcal{O})$. We mention that $H^1_0(\mathcal{O}) = H^1_{\partial\mathcal{O}}(\mathcal{O})$. Clearly we have $\operatorname{tr}(H^1_{\Gamma_0}(\mathcal{O})) \subseteq L^2(\Gamma_1)$. Remark 13.6.14 in [49] says that $\operatorname{tr}(H^1_{\Gamma_0}(\mathcal{O}))$ is dense in $L^2(\Gamma_1)$. For our interpretation of $\frac{\partial}{\partial\nu}$ we recall Green's formula

$$(\tfrac{\partial}{\partial\nu} f \,|\, \operatorname{tr} g)_{L^2(\partial\mathcal{O})} = (\Delta f \,|\, g)_{L^2(\mathcal{O})} + (\nabla f \,|\, \nabla g)_{(L^2(\mathcal{O}))^n} \tag{5.31}$$

for $f \in H^2(\mathcal{O})$ and $g \in H^1(\mathcal{O})$, see Lemma 1.5.3.7 in [17]. Here $\frac{\partial}{\partial\nu} \in \mathcal{L}(H^2(\mathcal{O}), L^2(\partial\mathcal{O}))$ is the continuous extension of $\frac{\partial}{\partial\nu} f(\xi) = \nabla f(\xi) \cdot \nu(\xi)$ defined on $C^2(\overline{\mathcal{O}})$. We shall now extend this definition. To this end, let $f \in H^1_{\Gamma_0}(\mathcal{O})$ with $\Delta f \in L^2(\mathcal{O})$. We say that $\frac{\partial}{\partial\nu} f|_{\Gamma_1}$ exists[2] in $L^2(\Gamma_1)$ if there is a function $h \in L^2(\Gamma_1)$ with

$$(h \,|\, \operatorname{tr} g)_{L^2(\Gamma_1)} = (\Delta f \,|\, g)_{L^2(\mathcal{O})} + (\nabla f \,|\, \nabla g)_{(L^2(\mathcal{O}))^n} \quad \text{for all } g \in H^1_{\Gamma_0}(\mathcal{O}).$$

Due to the fact that $\operatorname{tr}(H^1_{\Gamma_0}(\mathcal{O}))$ is dense in $L^2(\Gamma_1)$, there is at most one such $h \in L^2(\Gamma_1)$. We can thus define $\frac{\partial}{\partial\nu} f|_{\Gamma_1} := h$. Instead of saying $\frac{\partial}{\partial\nu} f|_{\Gamma_1}$ exists and equals $h \in L^2(\Gamma_1)$ we simply write $\frac{\partial}{\partial\nu} f|_{\Gamma_1} = h$. From (5.31) it is clear that $\frac{\partial}{\partial\nu} f|_{\Gamma_1} = \frac{\partial}{\partial\nu} f \cdot \mathbb{1}_{\Gamma_1}$ for $f \in H^2(\mathcal{O})$.

We now consider the operator A_1 in $L^2(\mathcal{O})$ defined as $A_1 f = -\Delta f$ with domain

$$\mathrm{D}(A_1) = \{f \in H^1_{\Gamma_0}(\mathcal{O}) \,|\, \Delta f \in L^2(\mathcal{O}) \text{ and } \tfrac{\partial}{\partial\nu} f|_{\Gamma_1} = 0\}.$$

We emphasize that the last condition $\frac{\partial}{\partial\nu} f|_{\Gamma_1} = 0$ ensures that (4.33) holds for all $f \in \mathrm{D}(A_1)$ and $g \in H^1_{\Gamma_0}(\mathcal{O})$, i.e.,

$$(\Delta f \,|\, g)_{L^2} = -(\nabla f \,|\, \nabla g)_{(L^2)^n}.$$

Moreover, it is proved in Theorem 13.6.9 that the Poincaré inequality is valid for $f \in H^1_{\Gamma_0}(\mathcal{O})$, meaning that there is a number $c_{\mathrm{p}} > 0$ such that

$$\|f\|_{L^2} \le c_{\mathrm{p}} \|\nabla f\|_{(L^2)^n} \quad \text{for all } f \in H^1_{\Gamma_0}(\mathcal{O}).$$

For this reason we can argue exactly as for the Dirichlet Laplacian in Section 4.4 and obtain that A_1 is strictly positive. Further $\|\,\cdot\,\|_{H^1}$ is equivalent to the norm $\|\nabla(\,\cdot\,)\|_{(L^2)^n}$ which is induced by the inner product given by $(\nabla f \,|\, \nabla g)_{(L^2)^n}$ for $f, g \in H^1_{\Gamma_0}(\mathcal{O})$.

Next we construct the so called *Neumann map* $N \in \mathcal{L}(L^2(\Gamma_1), H^1_{\Gamma_0}(\mathcal{O}))$. Let $v \in L^2(\Gamma_1)$. Then for all $g \in H^1_{\Gamma_0}(\mathcal{O})$ we have

$$|(v \,|\, \operatorname{tr} \overline{g})_{L^2(\Gamma_1)}| \le \|v\|_{L^2(\Gamma_1)} \|\operatorname{tr}\| \|g\|_{H^1} \le \|v\|_{L^2(\Gamma_1)} \|\operatorname{tr}\| \|\nabla g\|_{(L^2)^n},$$

[2] Actually $\frac{\partial}{\partial\nu} f|_{\Gamma_1}$ can be defined as a functional on $\operatorname{tr}(H^1_{\Gamma_0}(\mathcal{O}))$ for all $f \in H^1_{\Gamma_0}(\mathcal{O})$ with $\Delta f \in L^2(\mathcal{O})$ if the space $\operatorname{tr}(H^1_{\Gamma_0}(\mathcal{O}))$ is normed properly, but this is not important here.

showing that $(v \mid \operatorname{tr} \overline{(\cdot)})_{L^2(\Gamma_1)}$ is a bounded functional on $H^1_{\Gamma_0}(\mathcal{O})$. The Riesz representation theorem yields an element Nv of $H^1_{\Gamma_0}(\mathcal{O})$ with

$$(v \mid \operatorname{tr} g)_{L^2(\Gamma_1)} = (\nabla N v \mid \nabla g)_{(L^2)^n} \quad \text{for all } g \in H^1_{\Gamma_0}(\mathcal{O}). \tag{5.32}$$

Thus a map $N : L^2(\Gamma_1) \to H^1_{\Gamma_0}(\mathcal{O})$ is determined. We claim that $\Delta N v = 0$ for all $v \in L^2(\Gamma_1)$. In fact, for $g \in C_c^\infty(\mathcal{O})$ from the last equation, (4.32) and the fact that $\operatorname{tr} \overline{g} = 0$ we infer

$$\langle g, \Delta N v \rangle = (-1)(\nabla N v \mid \nabla \overline{g}) = (-1)(v \mid \operatorname{tr} \overline{g})_{L^2(\Gamma_1)} = 0.$$

From (5.32) we derive that $\frac{\partial}{\partial \nu} N v|_{\Gamma_1} = v \in L^2(\Gamma_1)$. We set $X = H^1_{\Gamma_0}(\mathcal{O}) \times L^2(\mathcal{O})$ and

$$\mathrm{D}(A) = \{(f, g) \in X \mid g \in H^1_{\Gamma_0}(\mathcal{O}), \Delta f \in L^2(\mathcal{O}), \tfrac{\partial}{\partial \nu} f|_{\Gamma_1} = -\operatorname{tr} g\}.$$

It is shown in Section 7 of [47] that the operator $A : \mathrm{D}(A) \to X$; $A(f, g) = (g, A_1 f)$ is m-dissipative and therefore generates a semigroup of contractions $\mathbb{T}$ on X. Further let $U = L^2(\Gamma_1)$. An L^2–admissible control operator $B^l \in \mathcal{L}(U, X_{-1})$ for $\mathbb{T}$ is given by $Bv = (0, -\sqrt{2} A_1 N v)$.

We add a nonlinear term $F_0(\omega(t, \xi))$ to the state equation in (5.30), i.e.,

$$\begin{aligned}
\partial_t^2 \omega(t, \xi) &= \Delta \omega(t, \xi) + F_0(\omega(t, \xi)), && (t, \xi) \in (0, \infty) \times \mathcal{O} \\
\omega(t, \xi) &= 0, && (t, \xi) \in (0, \infty) \times \Gamma_0 \\
\frac{\partial}{\partial \nu} \omega(t, \xi) + \partial_t \omega(t, x) &= \sqrt{2} \mu(t, \xi), && (t, \xi) \in (0, \infty) \times \Gamma_1 \\
\omega(0, \xi) = f_0(\xi), \quad \partial_t \omega(0, \xi) &= g_0(\xi), && \xi \in \mathcal{O}.
\end{aligned} \tag{5.33}$$

As in the previous example, this corresponds to the problem

$$z'(t) = A z(t) + F(z(t)) + B^l u(t); \quad z(0) = x_0,$$

where $F(f, g) = (0, F_0 \circ f)$ for $(f, g) \in X$. Under the assumption that $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies (5.25) we proceed as above and derive that $G(f) := F_0 \circ f$ belongs to $L^{p/\alpha}(\mathcal{O})$ for $f \in L^p(\mathcal{O})$. Due to the regularity of the boundary $\partial \mathcal{O}$ we have the Sobolev embedding

$$H^1(\mathcal{O}) \hookrightarrow L^p(\mathcal{O}) \quad \text{for all } p \in \begin{cases} [2, \infty), & n \in \{1, 2\} \\ [2, \frac{2n}{n-2}], & n \geq 3. \end{cases}$$

Recall that $\alpha > 1$ and choose $p = 2\alpha$. We infer that $G(H^1_\Gamma(\mathcal{O})) \subseteq G(L^{2\alpha}(\mathcal{O})) \subseteq L^2(\mathcal{O})$ if

$$\begin{cases} \alpha \in (1, \infty), & n \in \{1, 2\} \\ \alpha \in (1, \frac{n}{n-2}], & n \geq 3. \end{cases}$$

In this case F maps X to itself.

From Theorem 1.3 of [48] we know that the linear control system defined by A and B^l is exactly controllable if the semigroup $\mathbb{T}$ is exponentially stable. Sufficient conditions for that are given in Section 7.6 of [49], see especially Corollary 7.6.4. The assumptions are that the boundary $\partial \mathcal{O}$ is of class C^2 and that there is a reference point $\xi_0 \in \mathbb{R}^n$ such that Γ_0, Γ_1 can be represented as

$$\Gamma_0 = \{\xi \in \partial \mathcal{O} \mid (\xi - \xi_0) \cdot \nu(\xi) < 0\} \qquad \text{and} \qquad \Gamma_1 = \{\xi \in \partial \mathcal{O} \mid (\xi - \xi_0) \cdot \nu(\xi) > 0\}.$$

Further it is assumed that Γ_0, Γ_1 are nonempty, open and closed. Hence these conditions only allow a disconnected boundary $\partial\mathcal{O}$. The conclusion of the previous example holds accordingly.

In Theorem 1.2 of [29] the authors prove exponential boundedness under conditions including the case

$$\{\xi \in \partial\mathcal{O} \,|\, (\xi - \xi_0) \cdot \nu(\xi) > 0\} \subseteq \Gamma_1.$$

Here $\Gamma_0 = \emptyset$ is possible and $\partial\mathcal{O}$ must not be connected. However, they assume that the boundary is smooth.

Chapter 6

Observation systems

We now turn our attention to the output. The first three sections are devoted to nonlinear observation systems without inputs. In the final section we consider regular systems with both inputs and outputs.

In this chapter let X, U and Y be Banach spaces and $p \in [1, \infty]$ be fixed. The space U only appears in Section 6.4.

Definition 6.1. Let $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ be a strongly continuous semigroup on X and assume the map $\Psi_\infty : X \to L^p_{\mathrm{loc}}([0, \infty), Y)$ is continuous. Then the pair $(\mathbb{T}, \Psi_\infty)$ is called an *observation system* on X and $L^p_{\mathrm{loc}}([0, \infty), Y)$ if Ψ_∞ satisfies the composition property

$$S^*_\tau \Psi_\infty(x) = \Psi_\infty(\mathbb{T}_\tau x) \quad \text{for all } x \in X \text{ and } \tau \geq 0. \tag{6.1}$$

The operator Ψ_∞ is called *(extended) output map* of $(\mathbb{T}, \Phi)$.

Note that $L^p_{\mathrm{loc}}([0, \infty), Y)$ is a Fréchet space. Thus Ψ_∞ is continuous if and only if $\|P_t \Psi_\infty(x) - P_t \Psi_\infty(\tilde{x})\|_{L^p([0,\infty),Y)}$ converges to zero as $\|x - \tilde{x}\|_X \to 0$ for each $t \geq 0$. Equivalently one considers the family $(\Psi_t)_{t \geq 0}$ of operators $\Psi_t : X \to L^p([0, \infty), Y)$ defined by $\Psi_t(x) := P_t \Psi_\infty(x)$ for $t \geq 0$ and $x \in X$. Then one requires that Ψ_t is continuous for every $t \geq 0$. For this operators the composition property translates to

$$S^*_\tau \Psi_{t+\tau}(x) = \Psi_t(\mathbb{T}_\tau x) \quad \text{for all } x \in X \text{ and } t, \tau \geq 0. \tag{6.2}$$

We work with Ψ_∞ most of the time.

6.1 Representation of observation systems

Definition 6.2. Let $(\mathbb{T}, \Psi_\infty)$ be an observation system on X and $L^p_{\mathrm{loc}}([0, \infty), Y)$. The *Lebesgue extension* associated to the system $(\mathbb{T}, \Psi_\infty)$ is the operator C_L in X with the graph

$$\left\{ (x, w) \in X \times Y \,\middle|\, \forall \varepsilon > 0\ \exists \delta > 0\ \forall \tau \in (0, \delta] : \left\| w - \frac{1}{\tau} \int_0^\tau \Psi_\infty(x)(s)\, \mathrm{d}s \right\|_Y \leq \varepsilon \right\}.$$

By definition a vector $x \in X$ belongs to $\mathrm{D}(C_L)$ if and only if $\frac{1}{\tau} \int_0^\tau \Psi_\infty(x)(s)\, \mathrm{d}s$ has a limit in Y as τ tends to zero from above. In this case we define $C_L x$ as the limit. Thus $x \in \mathrm{D}(C_L)$ if $\Psi_\infty(x)$ has a right Lebesgue point at 0, cf. Appendix B.

Standing assumption: For the time being let $(\mathbb{T}, \Psi_\infty)$ be an observation system on X and $L^p_{\mathrm{loc}}([0, \infty), Y)$.

Proposition 6.3. *For $x \in X$ and $t \geq 0$ we have $\mathbb{T}_t x \in \mathrm{D}(C_L)$ if $\Psi_\infty(x)$ has a right Lebesgue point at t and then $C_L(\mathbb{T}_t x) = \Psi_\infty(x)(t)$. Hence for every $x \in X$ we have*

$$\Psi_\infty(x)(t) = C_L(\mathbb{T}_t x) \quad \textit{for almost all } t \geq 0.$$

Proof. The composition property (6.1) yields

$$\frac{1}{\tau}\int_t^{t+\tau} \Psi_\infty(x)(s)\,\mathrm{d}s = \frac{1}{\tau}\int_0^\tau \Psi_\infty(x)(t+s)\,\mathrm{d}s = \frac{1}{\tau}\int_0^\tau \Psi_\infty(\mathbb{T}_t x)(s)\,\mathrm{d}s\,.$$

This immediately proves the first claim. The last statement follows from Lebesgue's differentiation theorem, see Theorem B.2. □

Remark 6.4. From the last result we derive that $\mathrm{D}(C_L)$ is dense in X. Indeed, for each $x \in X$ we can find a sequence (t_n) in $[0,\infty)$ with $t_n \to 0$ as $n \to \infty$ and $\mathbb{T}_{t_n} x \in \mathrm{D}(C_L)$ for all $n \in \mathbb{N}$. Since $\mathbb{T}$ is strongly continuous, we deduce $\|\mathbb{T}_{t_n} x - x\|_X \to 0$ as $n \to \infty$. ♢

At this point it is not clear why C_L is called an 'extension'. Recall from Chapter 2 that in the linear theory the operator C_L is defined as the extension of an L^p–admissible observation operator $C \in \mathcal{L}(X_1, Y)$. In turn we shall find that the restriction of C_L to X_1 is continuous, see Theorem 6.8. To prove this result, we will add one more assumption on Ψ_∞. It is automatically satisfied if the system $(\mathbb{T}, \Psi_\infty)$ is linear, cf. Remark 6.13.

As in Section 4.1, a first step is to apply the Laplace transform. Of course have to make sure that $(\Psi_\infty(x))\widehat{\ }(\lambda)$ exists. In this regard we state a special case of Lemma B.8.

Lemma 6.5. *Let $x \in X$ and $\lambda \in \mathbb{C}$. Then $(\Psi_\infty(x))\widehat{\ }(\lambda)$ exists if and only if $(\Psi_\infty(\mathbb{T}_\tau x))\widehat{\ }(\lambda)$ exists for one and hence all $\tau \geq 0$. In this case we have*

$$(\Psi_\infty(x))\widehat{\ }(\lambda) = \int_0^\tau \mathrm{e}^{\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s + \mathrm{e}^{-\lambda\tau}(\Psi_\infty(\mathbb{T}_\tau x))\widehat{\ }(\lambda) \quad \textit{for all } \tau \geq 0. \tag{6.3}$$

Proof. Let $x \in X$, $\lambda \in \mathbb{C}$ and $\tau \geq 0$. Further let $N > \tau$. The composition property (6.1) and a change of variables imply

$$\begin{aligned}\int_0^N \mathrm{e}^{-\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s &= \int_0^\tau \mathrm{e}^{-\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s + \int_0^{N-\tau} \mathrm{e}^{-\lambda(s+\tau)}\Psi_\infty(x)(s+\tau)\,\mathrm{d}s \\ &= \int_0^\tau \mathrm{e}^{-\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s + \mathrm{e}^{-\lambda\tau}\int_0^{N-\tau} \mathrm{e}^{-\lambda s}\Psi_\infty(\mathbb{T}_\tau x)(s)\,\mathrm{d}s.\end{aligned}$$

The left hand side converges as $N \to \infty$ if and only if the right-hand side does. This calculation also shows the identity (6.3). □

As a consequence, we obtain a characterization of $\mathrm{D}(C_L)$, see Proposition 6.7. Let us first state a lemma that simplifies the proof of the latter.

Lemma 6.6. *Let $f \in L^1([0,1], Y)$ and $g \in C^1([0,1], \mathbb{C})$. Then we have*

$$\tfrac{1}{\tau}\int_0^\tau g(s)f(s)\,\mathrm{d}s - \tfrac{1}{\tau}\int_0^\tau g(0)f(s)\,\mathrm{d}s \to 0 \quad \textit{as } \tau \to 0^+.$$

Consequently, $\frac{1}{\tau}\int_0^\tau g(0)f(s)\,\mathrm{d}s$ converges if and only if $\frac{1}{\tau}\int_0^\tau g(s)f(s)\,\mathrm{d}s$ converges as $\tau \to 0^+$ and then the limits are equal.

Proof. Due to the mean value theorem, $\frac{1}{s}|g(s)-g(0)|$ is bounded by $\max_{\sigma\in[0,1]}|g'(\sigma)|$ for all $s\in(0,1]$. We thus obtain

$$\begin{aligned}\left\|\tfrac{1}{\tau}\int_0^\tau g(s)f(s)\,\mathrm{d}s-\tfrac{1}{\tau}\int_0^\tau g(0)f(s)\,\mathrm{d}s\right\|_Y &\le \int_0^\tau \tfrac{s}{\tau}\tfrac{1}{s}|g(s)-g(0)|\|f(s)\|_Y\,\mathrm{d}s\\ &\le \max_{\sigma\in[0,1]}|g'(\sigma)|\int_0^\tau \|f(s)\|_Y\,\mathrm{d}s.\end{aligned}$$

Since $f\in L^1([0,1],Y)$ the right-hand side converges to 0 as $\tau\to 0^+$. □

Proposition 6.7. *Let $x\in X$ and $\lambda\in\mathbb{C}$ be such that $(\Psi_\infty(x))\widehat{\ }(\lambda)$ exists. Then x belongs to* $\mathrm{D}(C_L)$ *if and only if the "difference quotient"*

$$\tfrac{1}{\tau}\big((\Psi_\infty(\mathbb{T}_\tau x))\widehat{\ }(\lambda)-(\Psi_\infty(x))\widehat{\ }(\lambda)\big)=\Big(\tfrac{1}{\tau}\big(\Psi_\infty(\mathbb{T}_\tau x)-\Psi_\infty(x)\big)\Big)\widehat{\ }(\lambda)$$

converges to some y in Y as $\tau\to 0^+$. In this case we have $C_L(x)=\lambda(\Psi_\infty(x))\widehat{\ }(\lambda)-y$.

Proof. From (6.3) we easily deduce

$$\tfrac{1}{\tau}\big((\Psi_\infty(\mathbb{T}_\tau x))\widehat{\ }(\lambda)-(\Psi_\infty(x))\widehat{\ }(\lambda)\big)=\tfrac{1}{\tau}(\mathrm{e}^{\lambda\tau}-1)(\Psi_\infty(x))\widehat{\ }(\lambda)-\mathrm{e}^{\lambda\tau}\frac{1}{\tau}\int_0^\tau \mathrm{e}^{-\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s. \quad (6.4)$$

Clearly $\lim_{\tau\to0^+}\frac{1}{\tau}(\mathrm{e}^{\lambda\tau}-1)=\lambda$ and $\lim_{\tau\to0^+}\mathrm{e}^{\lambda\tau}=1$. An application of the last lemma with $f=\Psi_\infty(x)$ and $g=\mathrm{e}^{-\lambda(\cdot)}$ yields that the limit of $\frac{1}{\tau}\int_0^\tau\Psi_\infty(x)(s)\,\mathrm{d}s$ as $\tau\to0^+$ exists if and only if it does for $\frac{1}{\tau}\int_0^\tau\mathrm{e}^{-\lambda s}\Psi_\infty(x)(s)\,\mathrm{d}s$. This means that $x\in\mathrm{D}(C_L)$ if and only if the right-hand side of (6.4) converges if and only if the left-hand side of (6.4) converges. □

We saw that $x\in X$ is contained in $\mathrm{D}(C_L)$ if and only if $\Psi_\infty(x)$ is "differentiable under the Laplace transform", provided $\mathrm{abs}(\Psi_\infty(x))<\infty$. To put this into a correct framework, we consider weighted L^p–spaces. In Appendix B on page 142 we introduce the space

$$L^p_\mu([0,\infty),Y):=\{f\in L^1_{\mathrm{loc}}([0,\infty),Y)\,|\,\mathrm{e}^{-\mu(\cdot)}f\in L^p([0,\infty),Y)\},$$

where $\mu\in\mathbb{R}$. It is a Banach space when equipped with the norm given by

$$\|f\|_{L^p_\mu}:=\|\mathrm{e}^{-\mu(\cdot)}f\|_{L^p([0,\infty),Y)}=\int_0^\infty\|\mathrm{e}^{-\mu s}f(s)\|_Y^p\,\mathrm{d}s.$$

If a locally integrable function $f:[0,\infty)\to Y$ belongs to $L^p_\mu([0,\infty),Y)$, then $\widehat{f}(\lambda)$ converges absolutely for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>\mu$. Moreover, we can estimate $\|\widehat{f}(\lambda)\|_Y\le\|\mathrm{e}^{-(\mathrm{Re}\,\lambda-\mu)(\cdot)}\|_{L^{p'}}\|f\|_{L^p_\mu}$. Obviously $\|\mathrm{e}^{-(\mathrm{Re}\,\lambda-\mu)(\cdot)}\|_{L^\infty}=1$. Further for all $\tau\ge0$ the left shift operator S^*_τ maps $L^p_\mu([0,\infty),Y)$ to itself. In fact, for every $f\in L^p_\mu([0,\infty),Y)$ we have

$$\|S^*_\tau f\|_{L^p_\mu}=\left(\int_\tau^\infty\mathrm{e}^{-\mu ps}\mathrm{e}^{\mu p\tau}\|f(s)\|_Y^p\,\mathrm{d}s\right)^{1/p}\le\mathrm{e}^{\mu\tau}\|f\|_{L^p_\mu}.$$

We can now state and prove the main result of this section. Recall that "differentiable" in applications often means "$\mathbb{R}$–differentiable". We made comments on that in the previous chapters, see e.g. the text before Remark 4.25.

Theorem 6.8. *Let X and Y be Banach spaces, $p \in [1,\infty]$ and let $(\mathbb{T},\Psi_\infty)$ be an observation system on X and $L^p_{\mathrm{loc}}([0,\infty),Y)$. Assume that we have $\Psi_\infty \in C^1(X, L^1_\mu([0,\infty),Y))$ for some $\mu \in \mathbb{R}$. Then $\mathrm{D}(A) \subseteq \mathrm{D}(C_L)$ and the restriction $C := C_L|_{\mathrm{D}(A)} : X_1 \to Y$ is continuous. For $x \in \mathrm{D}(A)$ we have the formula*

$$C(x) = \mu(\Psi_\infty(x))\widehat{\ }(\mu) - (\Psi_\infty'(x)Ax)\widehat{\ }(\mu).$$

Proof. The *first step* is to show the inclusion $\mathrm{D}(A) \subseteq \mathrm{D}(C_L)$. To this end, take $x \in \mathrm{D}(A)$. The orbit $[0,\infty) \to X;\ t \mapsto \mathbb{T}_t x$ is differentiable at $t = 0$. By the chain rule also the map $g : [0,\infty) \to L^1_\mu([0,\infty),Y);\ t \mapsto \Psi_\infty(\mathbb{T}_t x)$ is differentiable at $t = 0$ with derivative $\Psi_\infty'(x)Ax \in L^1_\mu([0,\infty),Y)$. Let $\varepsilon > 0$. Due to the differentiability of g, there is a number $\delta > 0$ such that for $\tau \in (0,\delta]$ we have

$$\begin{aligned}&\|\tfrac{1}{\tau}\big[(\Psi_\infty(\mathbb{T}_\tau x))\widehat{\ }(\mu) - (\Psi_\infty(x))\widehat{\ }(\mu)\big] - (\Psi_\infty'(x)Ax)\widehat{\ }(\mu)\|_Y \\ &\qquad\le \|\tfrac{1}{\tau}\big[\Psi_\infty(\mathbb{T}_\tau x) - \Psi_\infty(x)\big] - \Psi_\infty'(x)Ax\|_{L^1_\mu} = \|\tfrac{1}{\tau}(g(\tau)-g(0)) - g'(0)\|_{L^1_\mu} \le \varepsilon.\end{aligned}$$

Proposition 6.7 now yields that x is contained in $\mathrm{D}(C_L)$ and that

$$C(x) = C_L(x) = \mu(\Psi_\infty(x))\widehat{\ }(\mu) - (\Psi_\infty'(x)Ax)\widehat{\ }(\mu) \quad \text{for all } x \in X_1.$$

In the *second step* we now show that C is continuous. To this end, let $x \in X_1$. We may assume that $\mu > 0$. For all $\tilde{x} \in X_1$ with $\|x - \tilde{x}\|_1 \le 1$ we have $\|\tilde{x}\|_1 \le \|x\|_1 + 1$ and thus

$$\begin{aligned}\|C(x) - C(\tilde{x})\|_Y &\le \mu\|(\Psi_\infty(x))\widehat{\ }(\mu) - (\Psi_\infty(\tilde{x}))\widehat{\ }(\mu)\|_Y + \|(\Psi_\infty'(x)Ax)\widehat{\ }(\mu) - (\Psi_\infty'(\tilde{x})A\tilde{x})\widehat{\ }(\mu)\|_Y \\ &\le \mu\|\Psi_\infty(x) - \Psi_\infty(\tilde{x})\|_{L^1_\mu} + \|\Psi_\infty'(x)Ax - \Psi_\infty'(\tilde{x})A\tilde{x} \pm \Psi_\infty'(x)A\tilde{x}\|_{L^1_\mu} \\ &\lesssim \mu\|\Psi_\infty(x) - \Psi_\infty(\tilde{x})\|_{L^1_\mu} + \|\Psi_\infty'(x)\|\,\|x - \tilde{x}\|_1 + \|\Psi_\infty'(x) - \Psi_\infty'(\tilde{x})\|\,\|\tilde{x}\|_1 \\ &\le \mu\|\Psi_\infty(x) - \Psi_\infty(\tilde{x})\|_{L^1_\mu} + \|\Psi_\infty'(x)\|\,\|x - \tilde{x}\|_1 \\ &\qquad + \|\Psi_\infty'(x) - \Psi_\infty'(\tilde{x})\|\,(\|x\|_1 + 1).\end{aligned}$$

In the third line we used the equivalence of $\|\,\boldsymbol{.}\,\|_1$ and the graph norm of A. Let $\varepsilon > 0$. Since $\Psi_\infty : X \to L^1_\mu([0,\infty),Y)$ is continuously differentiable, we find a number $\delta_1 > 0$ such that for all $\tilde{x} \in X$ with $\|x - \tilde{x}\|_X \le \delta_1$ we have

$$\|\Psi_\infty(x) - \Psi_\infty(\tilde{x})\|_{L^1_\mu} \le \varepsilon \qquad \text{and} \qquad \|\Psi_\infty'(x) - \Psi_\infty'(\tilde{x})\|_{\mathcal{L}(X,L^1_\mu)} \le \varepsilon.$$

Recall that $\|x - \tilde{x}\|_X \le c\|x - \tilde{x}\|_1$ for some $c \ge 0$. Thus if $\|x - \tilde{x}\|_1 \le \min\{1, \frac{\delta_1}{c}, \varepsilon\}$ then

$$\|C(x) - C(\tilde{x})\|_Y \lesssim \mu\varepsilon + \|\Psi_\infty'(x)\|\varepsilon + \varepsilon(\|x\|_1 + 1) = \varepsilon(\mu + \|\Psi_\infty'(x)\| + \|x\|_1 + 1),$$

which implies that C is continuous at x and thus on X_1. □

Remark 6.9. The first step of the preceding proof works if we only assume that Ψ_∞ maps X to $L^1_\mu([0,\infty),Y)$ and that it is differentiable in $\mathrm{D}(A)$ equipped with $\|\,\boldsymbol{.}\,\|$. In fact, the chain rule can still be applied if we merely have

$$\begin{aligned}&\forall x \in \mathrm{D}(A)\ \exists \Psi_\infty'(x) \in \mathcal{L}(X, L^1_\mu([0,\infty),Y))\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathrm{D}(A): \\ &\qquad\qquad \|h\|_X \le \delta \implies \|\Psi_\infty(x+h) - \Psi_\infty(x) - \Psi_\infty'(x)h\|_{L^1_\mu} \le \varepsilon\|h\|_X.\end{aligned}$$

Hence also this weaker assumptions imply that $\mathrm{D}(A) \subseteq \mathrm{D}(C_L)$. Certainly, for the continuity of C it suffices that

$$\forall x \in \mathrm{D}(A), \varepsilon > 0\ \exists \delta > 0\ \forall \tilde{x} \in \mathrm{D}(A) : \|x - \tilde{x}\|_X \le \delta \implies \|\Psi_\infty'(x) - \Psi_\infty'(\tilde{x})\|_{\mathcal{L}(X,L^1_\mu)} \le \varepsilon. \quad \Diamond$$

Under the conditions of Theorem 6.8, for all $x \in X_1$ the mapping $[0, \infty) \to Y$; $t \mapsto C\mathbb{T}_t x$ is continuous and coincides with $\Psi_\infty(x)$ almost everywhere on $[0, \infty)$. Thus $\Psi_\infty(x)$ has a continuous representative.

Clearly C is linear if Ψ_∞ is linear. Then, due to continuity, we even have $C \in \mathcal{L}(X_1, Y)$.

Definition 6.10. Let $(\mathbb{T}, \Psi_\infty)$ be an observation system on X and $L^1_\mu([0, \infty), U)$ as in Theorem 6.8. The continuous map $C : X_1 \to Y$ is called the *observation operator associated to* $(\mathbb{T}, \Psi_\infty)$.

We shall give a sufficient condition for $\Psi_\infty(x) \in L^1_\mu([0, \infty), Y)$ for all $x \in X$ for some $\mu \in \mathbb{R}$. For better readability we write $\|\,\boldsymbol{\cdot}\,\|_{L^p}$ instead of $\|\,\boldsymbol{\cdot}\,\|_{L^p([0,\infty),Y)}$.

Lemma 6.11. *Let $\gamma > 0$ and $M \geq 1$ be such that $\|\mathbb{T}_t\| \leq M e^{\gamma t}$ for all $t \geq 0$. Further assume there exist $\eta > 0$ and $c \geq 0$ with*

$$\|\Psi_1(x)\|_{L^p} \leq c(\|x\|_X^\eta + 1) \quad \text{for all } x \in X. \tag{6.5}$$

Then we have

$$\|\Psi_t(x)\|_{L^p} \leq cM^\eta \frac{e^{\gamma\eta}}{e^{\gamma\eta} - 1} \|x\|_X^\eta \, e^{\gamma\eta t} + c(t + 1) \quad \text{for all } x \in X, t \geq 0.$$

Estimating $t + 1 \leq (1 + \frac{1}{\gamma\eta})e^{\gamma\eta t}$ for $t \geq 0$, we see that there is a constant $\tilde{c} \geq 0$ such that

$$\|\Psi_t(x)\|_{L^p} \leq \tilde{c}(\|x\|_X^\eta + 1)e^{\gamma\eta t} \quad \text{for all } x \in X, t \geq 0. \tag{6.6}$$

Proof. Let $x \in \mathrm{D}(A)$. In an easy induction we first show that for all $n \in \mathbb{N}$ we have

$$\Psi_n(x) = \sum_{k=0}^{n-1} S_k \Psi_1(\mathbb{T}_k x).$$

The case $n = 1$ is trivial. Assume the claim holds for some $n \in \mathbb{N}$. Then the composition property (6.2) and the induction hypothesis yield

$$\begin{aligned}\Psi_{n+1}(x) &= P_n \Psi_{n+1}(x) + S_n S_n^* \Psi_{1+n}(x) = \Psi_n(x) + S_n \Psi_1(\mathbb{T}_n x) \\ &= \sum_{k=0}^{n-1} S_k \Psi_1(\mathbb{T}_k x) + S_n \Psi_1(\mathbb{T}_n x) = \sum_{k=0}^{n} S_k \Psi_1(\mathbb{T}_k x).\end{aligned}$$

From the condition (6.5) we now derive

$$\begin{aligned}\|\Psi_n(x)\|_{L^p} &\leq \sum_{k=0}^{n-1} \|S_k \Psi_1(\mathbb{T}_k x)\|_{L^p} = \sum_{k=0}^{n-1} \|\Psi_1(\mathbb{T}_k x)\|_{L^p} \leq c \sum_{k=0}^{n-1} (1 + \|\mathbb{T}_k x\|_X^\eta) \\ &\leq cn + cM^\eta \sum_{k=0}^{n-1} e^{\gamma\eta k} \|x\|_X^\eta = cM^\eta \|x\|_X^\eta \frac{e^{\gamma\eta n} - 1}{e^{\gamma\eta} - 1} + cn \\ &\leq cM^\eta \frac{1}{e^{\gamma\eta} - 1} \|x\|_X^\eta e^{\gamma\eta n} + cn.\end{aligned}$$

Now let $t \geq 0$. Take $n \in \mathbb{N}$ with $t \in [n-1, n)$ and set $\tau = t - n + 1$. The above estimate and (6.2) imply

$$\begin{aligned}
\|\Psi_t(x)\|_{L^p} &\leq \|P_{n-1}\Psi_t(x)\|_{L^p} + \|S_{n-1}S_{n-1}^*\Psi_{\tau+n-1}(x)\|_{L^p} \leq \|\Psi_{n-1}(x)\|_{L^p} + \|\Psi_\tau(\mathbb{T}_{n-1}x)\|_{L^p} \\
&\leq \|\Psi_{n-1}(x)\|_{L^p} + \|\Psi_1(\mathbb{T}_{n-1}x)\|_{L^p} \\
&\leq cM^\eta \frac{1}{\mathrm{e}^{\gamma\eta} - 1} \|x\|_X^\eta \mathrm{e}^{\gamma\eta(n-1)} + c(n-1) + cM^\eta \|x\|_X^\eta \mathrm{e}^{\gamma\eta(n-1)} + c \\
&= cM^\eta \frac{\mathrm{e}^{\gamma\eta}}{\mathrm{e}^{\gamma\eta} - 1} \|x\|_X^\eta \mathrm{e}^{\gamma\eta(n-1)} + cn \leq cM^\eta \frac{\mathrm{e}^{\gamma\eta}}{\mathrm{e}^{\gamma\eta} - 1} \|x\|_X^\eta \mathrm{e}^{\gamma\eta t} + c(t+1). \qquad \square
\end{aligned}$$

Corollary 6.12. *Under the assumptions of the Lemma 6.11 for all $\omega > \max\{\omega_0(\mathbb{T}), 0\}$ the function $\Psi_\infty(x)$ belongs to $L^1_{\omega\eta}([0,\infty), Y) \cap L^p_{\omega\eta}([0,\infty), Y)$ for every $x \in X$. Moreover, there are constants $c_1, c_p \geq 0$ with $\|\Psi_\infty(x)\|_{L^q_{\omega\eta}} \leq c_q(\|x\|_X^\eta + 1)$ where $q \in \{1, p\}$.*

Proof. Let $x \in X$ and $\omega > \max\{\omega_0(\mathbb{T}), 0\} =: a$. Fix any $\gamma \in (a, \omega)$. For $n \in \mathbb{N}$, from (6.6) we derive the estimate

$$\begin{aligned}
\|\Psi_n(x)\|_{L^p_{\omega\eta}}^p &= \int_0^n \|\mathrm{e}^{-\omega\eta s}\Psi_\infty(x)(s)\|_Y^p \,\mathrm{d}s = \sum_{k=1}^n \int_{k-1}^k \mathrm{e}^{-\omega\eta p s}\|\Psi_\infty(x)(s)\|_Y^p \,\mathrm{d}s \\
&\leq \sum_{k=1}^n \mathrm{e}^{-\omega\eta p(k-1)} \|\Psi_k(x)\|_{L^p([k-1,k],Y)}^p \leq \mathrm{e}^{\omega\eta p} \sum_{k=1}^n \mathrm{e}^{-\omega\eta p k} \|\Psi_k(x)\|_{L^p}^p \\
&\leq \mathrm{e}^{\omega\eta p} \tilde{c}^p (\|x\|_X^\eta + 1)^p \sum_{k=0}^\infty \left(\mathrm{e}^{-(\omega-\gamma)\eta p}\right)^k.
\end{aligned}$$

The geometric series on the right-hand side converges. Hence Lemma B.9 yields that $\Psi_\infty(x)$ belongs to $L^p_{\omega\eta}([0,\infty), Y)$ as well as the claimed estimate for $\|\Psi_\infty(x)\|_{L^p_{\omega\eta}}$. On the other hand, using also Hölder's inequality, we obtain

$$\begin{aligned}
\|\Psi_n(x)\|_{L^1_{\omega\eta}} &= \int_0^n \|\mathrm{e}^{-\omega\eta s}\Psi_\infty(x)(s)\|_Y \,\mathrm{d}s = \sum_{k=1}^n \int_{k-1}^k \mathrm{e}^{-\omega\eta s}\|\Psi_\infty(x)(s)\|_Y \,\mathrm{d}s \\
&\leq \sum_{k=1}^n \mathrm{e}^{-\omega\eta(k-1)} \|\Psi_k(x)\|_{L^1([k-1,k],Y)} \leq \mathrm{e}^{\omega\eta} \sum_{k=1}^n \mathrm{e}^{-\omega\eta k} \|\Psi_k(x)\|_{L^p([k-1,k],Y)} \\
&\leq \mathrm{e}^{\omega\eta} \sum_{k=1}^n \mathrm{e}^{-\omega\eta k} \|\Psi_k(x)\|_{L^p} \leq \mathrm{e}^{\omega\eta} \tilde{c}(\|x\|_X^\eta + 1) \sum_{k=0}^\infty \left(\mathrm{e}^{(\gamma-\omega)\eta}\right)^k.
\end{aligned}$$

As above, it follows from Lemma B.9 that $\Psi_\infty(x)$ lies in $L^1_{\omega\eta}([0,\infty), Y)$. $\square$

Remark 6.13. Let $(\mathbb{T}, \Psi)$ be a linear observation system on X and $L^p([0,\infty), Y)$. Since $\Psi_1 \in \mathcal{L}(X, L^p([0,\infty), Y))$ the assumptions of Lemma 6.11 are satisfied with $\eta = 1$ and $c = \|\Psi_1\|_{\mathcal{L}(X,L^p)}$. Consequently $\Psi_\infty(x)$ is contained in $L^1_\omega([0,\infty), Y) \cap L^p_\omega([0,\infty), Y)$ for any $\omega > \omega_0(\mathbb{T})$. See also Proposition 2.3 in [53]. It is further easy to prove that we actually have

$$\Psi_\infty \in \mathcal{L}(X, L^1_\omega([0,\infty), Y)) \cap \mathcal{L}(X, L^p_\omega([0,\infty), Y)).$$

Because $\Psi_\infty : X \to L^1_\mu([0,\infty), Y)$ is linear, it clearly is continuously differentiable. Consequently also the assumptions of Theorem 6.8 are satisfied in this situation. The Lebesgue extension C_L is linear in this case. Therefore we have $C \in \mathcal{L}(X_1, Y)$. $\Diamond$

Remark 6.14. Let $\gamma > 0$ and $M \geq 1$ be as in Lemma 6.11. Further assume that $\|\Psi_1(x)\|_{L^p} \leq c$ for all $x \in X$. Then with the same technique we derive the estimate $\|\Psi_n(x)\|_{L^p} \leq cn$ for $n \in \mathbb{N}$. A reasoning analog to Corollary 6.12 yields $\Psi_\infty(x) \in L^1_\mu([0,\infty),Y) \cap L^p_\mu([0,\infty),Y)$ for all $\mu > 0$, since

$$\|\Psi_n(x)\|^p_{L^p_\mu} \leq c^p \sum_{k=1}^{n} k^p \mathrm{e}^{-\mu p(k-1)}, \qquad \|\Psi_n(x)\|_{L^1_\mu} \leq c \sum_{k=1}^{n} k \mathrm{e}^{-\mu(k-1)}.$$

Again the series on the right-hand sides converge. ◊

6.2 Linearization

As before let $(\mathbb{T},\Psi_\infty)$ be an observation system on X and $L^p_{\mathrm{loc}}([0,\infty),Y)$ for Banach spaces X, Y and some $p \in [1,\infty]$. Denote by A the generator of the semigroup $\mathbb{T}$.

Take $x_* \in X_1$ with $Ax_* = 0$, i.e., an equilibrium point of the problem $z'(t) = Az(t)$. Then $\mathbb{T}_t x_* = x_*$ for every $t \geq 0$. Assume that $\Psi_t : X \to L^p([0,\infty),Y)$ is differentiable at x_* for all $t \geq 0$. Let us check that the family $\Psi'(x_*) = (\Psi'_t(x_*))_{t\geq 0}$ of linear operators $\Psi'_t(x_*) \in \mathcal{L}(X, L^p([0,\infty),Y)$ together with $\mathbb{T}$ yield a linear control system on X and $L^p_{\mathrm{loc}}([0,\infty),Y)$.

To this end, for $t,\tau \geq 0$ consider the map $F_{t,\tau} : X \to L^p([0,\infty),Y)$; $F_{t,\tau}(x) = S^*_\tau \Psi_{t+\tau}(x) = \Psi_t(\mathbb{T}_\tau x)$. By the chain rule $F_{t,\tau}$ is differentiable at x^* and we get

$$S^*_\tau \Psi'_{t+\tau}(x_*) = F'_{t,\tau}(x_*) = \Psi'_t(\mathbb{T}_\tau x_*)\mathbb{T}_\tau = \Psi'_t(x_*)\mathbb{T}_\tau.$$

Very similarly one verifies that $P_\tau \Psi'_t(x_*) = \Psi'_\tau(x_*)$ for all $t \geq \tau \geq 0$.

As for controllability, there are several observability concepts. Some of the most important linear concepts are discussed in Section 6.1 of [49]. We repeat one of Definition 6.1.1 therein.

Definition 6.15. A *linear* observation system $(\mathbb{T},\Psi^l_\infty)$ on X and $L^p_{\mathrm{loc}}([0,\infty),Y)$ is called *exactly observable in time* $T > 0$, if $\Psi^l_T \in \mathcal{L}(X, L^p([0,\infty),Y)$ is bounded from below. This means that there is lower bound $k_T > 0$ such that

$$\|\Psi^l_T x\|_{L^p} \geq k_T \|x\|_X \quad \text{for all } x \in X. \tag{6.7}$$

For the moment, let X be reflexive and recall the notion of the dual system $(\mathbb{T}^*,\Phi^d)$ from the end of Section 2.2. We saw that $\Psi^*_\tau = \Phi^d_\tau Я_\tau$ for all $\tau \geq 0$. Standard arguments on dual operators yield that a linear observation system $(\mathbb{T},\Psi)$ is exactly observable if and only if $(\mathbb{T}^*,\Phi^d)$ is exactly controllable, see Theorem 11.2.1 of [49].

The following definition is taken from [8]. Obviously a linear observation system is locally exactly observable if and only if it is exactly observable.

Definition 6.16. An observation system $(\mathbb{T},\Psi_\infty)$ on X and $L^p_{\mathrm{loc}}([0,\infty),Y)$ is called *locally exactly observable around* x_* *in time* $T > 0$ if there is a radius $\rho > 0$ and a lower bound $l_T > 0$ such that

$$\|\Psi_T x_1 - \Psi_T x_2\|_{L^p} \geq l_T \|x_1 - x_2\|_X \quad \text{for all } x_1, x_2 \in \overline{B}(x_*,\rho).$$

It is clear that the linear system $(\mathbb{T},\Psi^l_\infty)$ from Definition 6.15 is exactly controllable in time $T > 0$ if and only if Ψ^l_T is one-to-one and the left inverse $Q : \mathrm{Ran}(\Psi^l_T) \to X$ is bounded. Moreover, $\mathrm{Ran}(\Psi^l_T)$ is a closed subspace of $L^p([0,\infty),Y)$.

Compared to this, from the condition in Definition 6.16 we can merely deduce that Ψ_T is one-to-one on $\overline{B}(x_*, \rho)$ and that the left inverse $Q : \Psi_T(\overline{B}(x_*, \rho)) \to \overline{B}(x_*, \rho)$ is Lipschitz. However, it is easy to see that a linear observation system is exactly observable in time $T > 0$ if and only if it is locally exactly observable around 0 in time T.

Once more the exact observability of the linearized system implies the exact local observability of the original system. Note that the proof of the following theorem only uses elementary arguments. Compared to that, the structure of the proof of Theorem 4.37 is notably more complicated.

Theorem 6.17. *Let X and Y be Banach spaces, $p \in [1, \infty]$ and let $(\mathbb{T}, \Psi_\infty)$ be an observation system on X and $L^p_{\text{loc}}([0, \infty), Y)$. Assume that we have $\Psi_t \in C^1(X, L^p([0, \infty), Y))$ for all $t \geq 0$. Moreover, let there be some $x_* \in \mathrm{D}(A)$ with $Ax_* = 0$ such that $(\mathbb{T}, \Psi'_\infty(x_*))$ is exactly observable in time $T > 0$. Then $(\mathbb{T}, \Psi_\infty)$ is locally exactly observable around x_* in time T.*

Proof. The assumption yields a lower bound $k_T > 0$ with

$$\|\Psi'_T(x_*)x\|_{L^p} \geq k_T \|x\|_X \quad \text{for all } x \in X.$$

Define $\Psi_T^{\text{rem}} : X \to L^p([0, \infty), Y)$; $\Psi_T^{\text{rem}}(x) = \Psi_T(x) - \Psi_T(x_*) - \Psi'_T(x_*)(x - x_*)$ so that

$$\Psi_T(x) = \Psi_T(x_*) + \Psi'_T(x_*)(x - x_*) + \Psi_T^{\text{rem}}(x) \quad \text{for all } x \in X. \tag{6.8}$$

It is clear that Ψ_T^{rem} is continuously differentiable with derivative given by $(\Psi_T^{\text{rem}})'(x) = \Psi'_T(x) - \Psi'_T(x_*)$. Note that we have $(\Psi_T^{\text{rem}})'(x_*) = 0$.

Since the map $X \mapsto \mathcal{L}(X, L^p([0, \infty), Y))$; $x \mapsto (\Psi_T^{\text{rem}})'(x)$ is continuous, we find a radius $\rho > 0$ with $\|(\Psi_T^{\text{rem}})'(x)\|_{\mathcal{L}(X, L^p)} \leq \frac{1}{2} k_T$ for all $x \in \overline{B}(x_*, \rho)$. Let $x_1, x_2 \in B(x_*, \rho)$. By the fundamental theorem we have

$$\begin{aligned}\Psi_T(x_1) - \Psi_T(x_2) &= \Psi'_T(x_*)(x_1 - x_2) + \Psi_T^{\text{rem}}(x_1) - \Psi_T^{\text{rem}}(x_2) \\ &= \Psi'_T(x_*)(x_1 - x_2) + \int_0^1 (\Psi_T^{\text{rem}})'(x_2 + \sigma(x_1 - x_2))(x_1 - x_2) \, \mathrm{d}\sigma.\end{aligned}$$

Since $x_\sigma := x_2 + \sigma(x_1 - x_2)$ is contained in the ball $\overline{B}(x_*, \rho)$ for all $\sigma \in [0, 1]$ we obtain

$$\begin{aligned}k_T\|x_1 - x_2\|_X &\leq \|\Psi'_T(x_*)(x_1 - x_2)\|_{L^p} \\ &\leq \|\Psi_T(x_1) - \Psi_T(x_2)\|_{L^p} + \int_0^1 \|(\Psi_T^{\text{rem}})'(x_\sigma)\|_{\mathcal{L}(X, L^p)} \, \mathrm{d}\sigma \|x_1 - x_2\|_X \\ &\leq \|\Psi_T(x_1) - \Psi_T(x_2)\|_{L^p} + \sup_{x \in \overline{B}(0,\rho)} \|(\Psi_T^{\text{rem}})'(x)\|_{\mathcal{L}(X, L^p)} \|x_1 - x_2\|_X \\ &\leq \|\Psi_T(x_1) - \Psi_T(x_2)\|_{L^p} + \tfrac{1}{2} k_T \|x_1 - x_2\|_X.\end{aligned} \tag{6.9}$$

Subtracting $\frac{1}{2} k_T \|x_1 - x_2\|_X$ the claim follows. □

6.3 Applications

Let X and Y_l be Banach spaces and let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Further assume that $C^l \in \mathcal{L}(X_1, Y_l)$ is an L^p–admissible linear control operator for $\mathbb{T}$. Denote by $\Psi^l_\infty : X \to L^p_{\text{loc}}([0, \infty), Y_l)$ the extended output map defined by C^l and $\mathbb{T}$.

Let Y be another Banach space. We are looking at a class of continuous maps $M : Y_l \to Y$ with the property that there are constants $\eta \in (0,p]$ and $c \geq 0$ such that

$$\|M(y)\|_Y \leq c(\|y\|_{Y_l}^\eta + 1) \quad \text{for all } y \in Y_l. \tag{6.10}$$

As in Section 4.4 we will frequently use the fact that for $a, b, e \geq 0$ we have

$$(a+b)^e \leq (2\max\{a,b\})^e = 2^e \max\{a^e, b^e\} \leq 2^e(a^e + b^e).$$

For $x \in X$ we set $\Psi_\infty(x) = M \circ \Psi_\infty^l x$. Then for almost every $t \geq 0$ we have

$$\|\Psi_\infty(x)(t)\|_Y^{\frac{p}{\eta}} = \|M(\Psi_\infty^l x(t))\|_Y^{\frac{p}{\eta}} \leq c^{\frac{p}{\eta}}(\|\Psi_\infty^l x(t)\|_{Y_l}^\eta + 1)^{\frac{p}{\eta}} \leq (2c)^{\frac{p}{\eta}}(\|\Psi_\infty^l x(t)\|_{Y_l}^p + 1).$$

The right-hand side as a function of t is locally integrable. Hence $\Psi_\infty(x)$ lies in $L^{p/\eta}_{\text{loc}}([0,\infty), Y)$ for all $x \in X$. We obtain a map

$$\Psi_\infty : X \to L^{\frac{p}{\eta}}_{\text{loc}}([0,\infty), Y).$$

We shall show that $(\mathbb{T}, \Psi_\infty)$ is an observation system on X and $L^{p/\eta}_{\text{loc}}([0,\infty), Y)$. First, let us verify the composition property (6.1). Take $\tau \geq 0$ and $x \in X$. Using (6.1) for Ψ_∞^l, we derive

$$S_\tau^* \Psi_\infty(x)(t) = M\big(\Psi_\infty^l x(t+\tau)\big) = M\big(\Psi_\infty^l \mathbb{T}_\tau x(t)\big) = \Psi_\infty(\mathbb{T}_\tau x)(t)$$

for almost all $t \geq 0$.

We next prove that $\Psi_\tau = P_\tau \Psi_\infty : X \to L^{p/\eta}([0,\infty), Y)$ is continuous. Note that $\Psi_\tau = P_\tau M \circ \Psi_\tau^l$. (Clearly the latter equals $M \circ \Psi_\tau^l$ if and only if $M(0) = 0$.) We proceed as in the second step of the proof of Proposition 4.17. Let (x_n) be a sequence in X with $x_n \to x$ as $n \to \infty$. We apply the dominated convergence theorem to

$$\|\Psi_\tau(x) - \Psi_\tau(x_n)\|_{L^{\frac{p}{\eta}}} = \left(\int_0^\tau \|M(\Psi_\tau^l x(t)) - M(\Psi_\tau^l x_n(t))\|_Y^{\frac{p}{\eta}}\right)^{\eta/p}.$$

Due to the boundedness of Ψ_τ^l, we then have $\|\Psi_\tau^l(x_n - x)\|_{L^p} \to 0$ as $n \to \infty$. Passing to a subsequence we may assume that $\Psi_\tau^l x_n$ converges to $\Psi_\tau^l x$ pointwise almost everywhere on $[0,\tau]$. Since M is continuous, it follows that $\|M(\Psi_\tau^l x(t)) - M(\Psi_\tau^l x_n(t))\|_Y \to 0$ as $n \to \infty$. Moreover, the subsequence can be chosen such that we find a function $f \in L^p([0,\tau], \mathbb{R})$ with $\|\Psi_\tau^l x_n(t)\|_{Y_l} \leq f(t)$ for almost all $t \in [0,\tau]$. By (6.10) we obtain

$$\begin{aligned}\|M(\Psi_\tau^l x(t)) - M(\Psi_\tau^l x_n(t))\|_Y &\leq \|M(\Psi_\tau^l x(t))\|_Y + \|M(\Psi_\tau^l x_n(t))\|_Y \\ &\leq c\|\Psi_\tau^l x(t)\|_{Y_l}^\eta + c\|\Psi_\tau^l x_n(t)\|_{Y_l}^\eta + 2c \leq c\|\Psi_\tau^l x(t)\|_{Y_l}^\eta + f(t)^\eta + 2c\end{aligned}$$

for almost every $t \in [0,\tau]$. Each of the summands on the right-hand side defines a function in $L^{p/\eta}([0,\tau], \mathbb{R})$. Thus the conditions of the dominated convergence theorem are satisfied. With a reasoning similar to the final step of Lemma 4.39 we conclude that Ψ_t is continuous.

It is easy to see that Ψ_∞ satisfies the assumption of Lemma 6.11. Indeed, let $x \in X$ and $\tau > 0$, then with (6.10) we can estimate

$$\begin{aligned}\|\Psi_\tau(x)\|_{L^{\frac{p}{\eta}}} &\leq \left(\int_0^\tau c^{\frac{p}{\eta}}(\|\Psi_\tau^l x(t)\|_{Y_l}^\eta + 1)^{\frac{p}{\eta}}\,\mathrm{d}t\right)^{\eta/p} \leq 2c\left(\int_0^\tau \max\{\|\Psi_\tau^l x(t)\|_{Y_l}, 1\}^p\,\mathrm{d}t\right)^{\eta/p} \\ &= 2c\|\max\{\|\Psi_\tau^l x\|_{Y_l}, 1\}\|_{L^p([0,\tau],\mathbb{R})}^\eta \leq 2^{\eta+1}c(\|\Psi_\tau^l x\|_{L^p}^\eta + \tau^\eta) \\ &\leq 2^{\eta+1}c\max\{\tau^\eta, 1\}(\|\Psi_\tau^l x\|_{L^p}^\eta + 1).\end{aligned}$$

Plugging in $\tau = 1$ we infer

$$\|\Psi_1(x)\|_{L^{\frac{p}{\eta}}} \leq 2^{\eta+1}c(\|\Psi_1^l\|_{\mathcal{L}(X,L^p)}^{\eta}\|x\|_X^{\eta} + 1) \leq 2^{\eta+1}c\max\{\|\Psi_1^l\|_{\mathcal{L}(X,L^p)}^{\eta}, 1\}(\|x\|_X^{\eta} + 1).$$

With Corollary 6.12 we conclude that $\Psi_\infty(x) \in L^1_{\omega\eta}([0,\infty), Y) \cap L^{p/\eta}_{\omega\eta}([0,\infty), Y)$, where $\omega > 0$ is such that $\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}$ for some $M \geq 1$ and all $t \geq 0$.

In our example below we have $p = 2$ so that $\eta \in (0, 2]$ and $L^1_{\omega\eta}([0,\infty), Y)$ is contained in $L^1_{2\omega}([0,\infty), Y)$. Hence, in the following we assume that (6.10) holds with $\eta = 2$.

Additionally, let M be continuously $\mathbb{R}$–differentiable and assume that the derivative M' satisfy the growth bound

$$\|M'(v)\|_{\mathcal{L}_{\mathbb{R}}(Y_l,Y)} \leq c(\|y\|_{Y_l} + 1) \quad \text{for all } y \in Y_l. \tag{6.11}$$

As before, we drop the $\mathbb{R}$. The constant $c \geq 0$ has no special importance and without loss of generality, we may assume it is as in (6.10). In the same way as before we deduce that $M' \circ \Psi_\infty^l x \in L^2([0,\infty), \mathcal{L}(Y_l, Y))$ as well as

$$M' \circ \Psi_\infty^l x \in L^2_\omega([0,\infty), \mathcal{L}(Y_l, Y))$$

for all $x \in X$. We briefly write $\|\,\boldsymbol{.}\,\|_{L^2_\omega,\mathcal{L}}$ for $\|\,\boldsymbol{.}\,\|_{L^2_\omega([0,\infty),\mathcal{L}(Y_l,Y))}$. Let $x, h \in X$ and recall from Remark 6.13 that $\Psi_\infty^l \in \mathcal{L}(X, L^2_\omega([0,\infty), Y))$. It follows that

$$\begin{aligned}
\|M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\Psi_\infty^l h(\,\boldsymbol{.}\,)\|_{L^1_{2\omega}} &\leq \int_0^\infty \mathrm{e}^{-\omega t}\|M'(\Psi_\infty^l x(t))\|_{\mathcal{L}(Y_l,Y)}\mathrm{e}^{-\omega t}\|\Psi_\infty^l h(t)\|_{Y_l}\,\mathrm{d}t \\
&\leq \|M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\|_{L^2_\omega,\mathcal{L}}\|\Psi_\infty^l h\|_{L^2_\omega} \\
&\leq c_2(\|x\|_X + 1)\|\Psi_\infty^l\|_{\mathcal{L}(X,L^2_\omega)}\|h\|_X.
\end{aligned}$$

Hence $M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\Psi_\infty^l h(\,\boldsymbol{.}\,)$ is contained in the space $L^1_{2\omega}([0,\infty), Y)$ and the linear map $X \to L^1_{2\omega}([0,\infty), Y)$; $h \mapsto M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\Psi_\infty^l h(\,\boldsymbol{.}\,)$ is bounded. We next prove that Ψ_∞ is continuously $\mathbb{R}$–differentiable with derivative given by

$$\Psi_\infty'(x)h = M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\Psi_\infty^l h(\,\boldsymbol{.}\,) \quad \text{for } x, h \in X.$$

We argue as in Section 5.3. Using Fubini's theorem and Hölder's inequality we infer the estimate

$$\begin{aligned}
&\|\Psi_\infty(x+h) - \Psi_\infty(x) - M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\Psi_\infty^l h(\,\boldsymbol{.}\,)\|_{L^1_{2\omega}} \\
&\quad = \int_0^\infty \mathrm{e}^{-2\omega t}\|M(\Psi_\infty^l(x+h)(t)) - M(\Psi_\infty^l x(t)) - M'(\Psi_\infty^l x(t))\Psi_\infty^l h(t)\|_Y\,\mathrm{d}t \\
&\quad \leq \int_0^\infty \mathrm{e}^{-\omega t}\mathrm{e}^{-\omega t}\int_0^1 \|M'(\Psi_\infty^l x(t) + r\Psi_\infty^l h(t)) - M'(\Psi_\infty^l x(t))\|_{\mathcal{L}(Y_l,Y)}\|\Psi_\infty^l h(t)\|_{Y_l}\,\mathrm{d}r\,\mathrm{d}t \\
&\quad \leq \int_0^1 \|M'(\Psi_\infty^l x(\,\boldsymbol{.}\,) + r\Psi_\infty^l h(\,\boldsymbol{.}\,)) - M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\|_{L^2_\omega,\mathcal{L}}\,\mathrm{d}r\|\Psi_\infty^l h\|_{L^2_\omega} \\
&\quad \leq \int_0^1 \|M'(\Psi_\infty^l x(\,\boldsymbol{.}\,) + r\Psi_\infty^l h(\,\boldsymbol{.}\,)) - M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\|_{L^2_\omega,\mathcal{L}}\,\mathrm{d}r\|\Psi_\infty^l\|_{\mathcal{L}(X,L^2_\omega)}\|h\|_X.
\end{aligned}$$

It remains to show that $\|M'(\Psi_\infty^l x(\,\boldsymbol{.}\,) + r\Psi_\infty^l h(\,\boldsymbol{.}\,)) - M'(\Psi_\infty^l x(\,\boldsymbol{.}\,))\|_{L^2_\omega,\mathcal{L}}$ can be made arbitrarily small for all $r \in [0,1]$ by choosing $\|h\|_X$ small. This mainly follows from Lemma 4.39 with $\eta = 1$ and $p = 2$ there. We have to make sure it can be applied.

Set $f := \Psi_\infty^l x$ and $f_h := \Psi_\infty^l h$. Then $\|f_h\|_{L^2_\omega} \le \|\Psi_\infty^l\|\|h\|_X$ decreases with $\|h\|_X$. On the other hand we have $\|f_h\|_{L^2_\omega} = \|e^{-\omega(\cdot)} f_h\|_{L^2}$. Minor modifications of the proof of Lemma 4.39 show that

$$\begin{aligned}\|M'(\Psi_\infty^l x(\cdot) + r\Psi_\infty^l h(\cdot)) - M'(\Psi_\infty^l x(\cdot))\|^2_{L^2_\omega,\mathcal{L}} \\ = \int_0^\infty e^{-2\omega t}\|M'(f(t) + rf_h(t)) - M'(f(t))\|^2_{\mathcal{L}(Y_l,Y)}\,dt\end{aligned}$$

can be controlled by $\|f_h\|_{L^2_\omega}$ and thus by $\|h\|_X$. Let us summarize these results in a lemma.

Lemma 6.18. *Let X, Y_l and Y be Banach spaces and let $(\mathbb{T}, \Psi_\infty^l)$ be a linear observation system on X and $L^p([0,\infty), Y_l)$. Denote by $C^l \in \mathcal{L}(X_1, Y_l)$ the observation operator of $(\mathbb{T}, \Psi_\infty^l)$. Let $M : Y_l \to Y$ be a continuous map satisfying (6.10). Then the map $\Psi_\infty : X \to L^{p/\eta}_{\text{loc}}([0,\infty), Y)$; $\Psi_\infty(x) = M \circ \Psi_\infty^l x$ yields an observation system $(\mathbb{T}, \Psi_\infty)$ on X and $L^{p/\eta}([0,\infty), Y)$. Moreover, Ψ_∞ maps X to $L^1_{\omega\eta}([0,\infty), Y) \cap L^{p/\eta}_{\omega\eta}([0,\infty), Y)$*

Additionally let $\eta = 2$ and assume that M is continuously $\mathbb{R}$–differentiable and that the derivative satisfies the growth bound (6.11). Then Ψ_∞ is continuously $\mathbb{R}$–differentiable as a map from X to $L^1_{2\omega}([0,\infty), Y)$ with derivative given by $\Psi_\infty'(x)h = M'(\Psi_\infty^l x(\cdot))\Psi_\infty^l h(\cdot)$ for $x, h \in X$.

6.3.1 A wave equation with Neumann boundary observation

We repeat some known results from Section 7.1 of [49]. Let $\mathcal{O}$ be a bounded domain with boundary $\partial\mathcal{O}$ of class C^2. The surface measure on $\partial\mathcal{O}$ is denoted σ. Since $\mathcal{O}$ is bounded and $\partial\mathcal{O}$ is continuous, it is clear that $\sigma(\partial\mathcal{O}) < \infty$. In Subsection 4.4.2 we already encountered the linear wave equation

$$\begin{aligned} \partial_t^2\omega(t,\xi) &= \Delta\omega(t,\xi), && (t,\xi) \in (0,\infty)\times\mathcal{O} \\ \omega(t,\xi) &= 0, && (t,\xi) \in (0,\infty)\times\partial\mathcal{O} \\ \omega(0,\xi) &= f_0(\xi), \quad \partial_t\omega(0,\xi) = g_0(\xi), && \xi \in \mathcal{O}\end{aligned} \tag{6.12}$$

with initial values $f_0, g_0 : \Omega \to \mathbb{C}$, see (4.35). The difference is that there is no input μ. We saw that setting $z(t) = (\eta(t,\cdot), \partial_t\eta(t,\cdot))$ and $x_0 = (f_0, g_0)$ this problem can be written as

$$z'(t) = Az(t); \quad z(0) = x_0.$$

The state space was $X = H_0^1(\mathcal{O}) \times L^2(\mathcal{O})$ so that the boundary condition is satisfied in the sense of traces. Moreover, we had $\mathrm{D}(A) = (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})) \times H_0^1(\mathcal{O}) \subseteq X$ and

$$A : \mathrm{D}(A) \to X; \quad A(f,g) = (g, -A_0 f),$$

where $-A_0$ is the Dirichlet Laplacian also defined in Section 4.4. We argued that A is skew-adjoint and thus the generator of a unitary group $\mathbb{T}$ on X.

In Subsection 4.4.2 we discussed how the abstract equation and the partial differential equation are connected. We do not repeat the arguments because they work analogously in the present situation.

Let $\nu \in L^\infty(\partial\mathcal{O}, \mathbb{R}^n)$ be outward normal vector field on $\partial\mathcal{O}$. Recall that the normal derivative $\frac{\partial}{\partial\nu}$ is the continuous extension of the map

$$\tfrac{\partial}{\partial\nu} : C^2(\overline{\mathcal{O}}) \to L^2(\partial\mathcal{O}); \quad \tfrac{\partial}{\partial\nu} f(\xi) = \nabla f(\xi)\cdot\nu(\xi).$$

to an operator $\frac{\partial}{\partial\nu} \in \mathcal{L}(H^2(\mathcal{O}), L^2(\partial\mathcal{O}))$. Consider an open subset $\Gamma \subset \partial\mathcal{O}$ of the boundary. Set $Y = Y_l = L^2(\Gamma)$ and define $C^l \in \mathcal{L}(X_1, Y)$ by

$$C^l(f,g) := \tfrac{\partial}{\partial\nu} f|_\Gamma := \tfrac{\partial}{\partial\nu} f \cdot \mathbb{1}_\Gamma.$$

It is shown in Theorem 7.1.3 of [49] that C^l is an L^2–admissible observation operator for $\mathbb{T}$.

We can choose M as in the example of Subsection 4.4.2. More precisely, we define

$$M(y) = m(\|y\|_Y^2)y \quad \text{for } y \in Y,$$

where $m \in C^2(\mathbb{R}, \mathbb{R})$ was a function with $m(0) \neq 0$ and $m(a) = 0$ for $|a|$ greater than a number $R > 0$. We saw that M is two-times continuously $\mathbb{R}$–differentiable with bounded derivatives and thus M as well as M' are Lipschitz. As a consequence, (6.10) is satisfied with $\eta = 1$ and hence with $\eta = 2$. Also (6.11) holds. With Theorem 6.8 we infer that $\mathrm{D}(A) \subseteq \mathrm{D}(C_L)$ and that the restriction $C := C_L|_{\mathrm{D}(A)}$ is continuous as a map from X_1 to Y.

Let us briefly show that in this example Theorem 6.17 can be applied with $p = 2$ and the equilibrium point $x_* = 0$. However, we have to adapt the proof, more precisely, we have to replace the estimate (6.9). First note that with a reasoning analog to Section 4.4 one can show that $\Psi_t : X \to L^1([0,\infty), Y)$ is continuously $\mathbb{R}$–differentiable for all $t \geq 0$ with derivative given by

$$\Psi_t'(x)h = M'(\Psi_t^l x(\,\bullet\,))\Psi_t^l h(\,\bullet\,) \quad \text{for } x, h \in X.$$

Here we only use that $M : Y \to Y$ is continuously $\mathbb{R}$–differentiable and both M and M' are Lipschitz. Plugging in $x_* = 0$ we obtain

$$\Psi_t'(0)h = M'(0)\Psi_t^l h = m(0)\Psi_t^l h.$$

Recall that in the proof of Theorem 6.17 we used the identity

$$\Psi_T'(0)(x_1 - x_2) = \Psi_T(x_1) - \Psi_T(x_2) - \int_0^1 (\Psi_T^{\mathrm{rem}})'(x_\sigma)(x_1 - x_2)\,\mathrm{d}\sigma, \tag{6.13}$$

for $x_1, x_2 \in X$ where $x_\sigma = x_2 + \sigma(x_1 - x_2)$. Here we have

$$(\Psi_T^{\mathrm{rem}})'(x_\sigma)(x_1 - x_2) = [\Psi_t'(x_\sigma) - \Psi_t'(0)](x_1 - x_2) = [M'(\Psi_t^l x_\sigma(\,\bullet\,)) - M'(0)]\Psi_t^l(x_1 - x_2)(\,\bullet\,)$$

The left-hand side of (6.13) equals $m(0)\Psi_T^l(x_1 - x_2)$. Let us assume that $(\mathbb{T}, \Psi_\infty^l)$ is exactly observable in time $T > 0$. Theorem 7.2.4 of [49] yields that this is the case if there is a vector $\xi_0 \in \mathbb{R}^n$ such that $\{\xi \in \partial\mathcal{O} \,|\, (\xi - \xi_0) \cdot \nu(\xi) > 0\} \subseteq \Gamma$. Then we find a constant $k_T > 0$ with

$$\|m(0)\Psi_T^l(x_1 - x_2)\|_{L^2} \geq |m(0)| k_T \|x_1 - x_2\|_X.$$

As in (6.9) we want to bound the L^2–norm of the integral on the right-hand side of (6.13) by $q|m(0)|k_T\|x_1 - x_2\|_X$ for some number $q \in (0,1)$. Using (4.37) we can estimate

$$\begin{aligned}
&\left(\int_0^T \left(\int_0^1 \|(\Psi_T^{\mathrm{rem}})'(x_\sigma)(x_1 - x_2)\|_Y \,\mathrm{d}\sigma\right)^2 \mathrm{d}t\right)^{1/2} \\
&\qquad = \left(\int_0^T \left(\int_0^1 \|M'(\Psi_T^l x_\sigma(t)) - M'(0)\|_{\mathcal{L}_\mathbb{R}(Y)} \|\Psi_T^l(x_1 - x_2)(t)\|_Y \,\mathrm{d}\sigma\right)^2 \mathrm{d}t\right)^{1/2} \\
&\qquad \leq (\|m\|_{L^\infty} + \|m'\|_{L^\infty} R + |m(0)|)\|\Psi_T^l(x_1 - x_2)\|_{L^2([0,\infty),Y)} \\
&\qquad \leq (\|m\|_{L^\infty} + \|m'\|_{L^\infty} R + |m(0)|)\|\Psi_T^l\|_{\mathcal{L}(X,L^2)}\|x_1 - x_2\|_{L^2([0,\infty),Y)}.
\end{aligned}$$

Hence, the condition on m is that $\|m\|_{L^\infty} + \|m'\|_{L^\infty} R + |m(0)| < |m(0)| k_T$.

Above we made heavy use of the special structure. In view of the efforts made in Section 4.4 it becomes clear that it does not suffice to assume that $M : Y \to Y$ is continuously $\mathbb{R}$–differentiable and that M, M' are Lipschitz. We can then still prove differentiability of $\Psi_t : X \to L^1([0,\infty), Y)$, but we have to apply Theorem 6.17 with $p = 1$. However, even if $M'(0)$ is invertible, we can not derive the lower bound (6.7) for $\Psi_t'(0)$ with $p = 1$, because Ψ^l_∞ satisfies (6.7) with the stronger norm $\|\,\boldsymbol{.}\,\|_{L^2}$. We emphasize that there is no Hilbert space assumption in Theorem 6.17.

One alternative is to consider bounded observation operators $C^l \in \mathcal{L}(X, Y)$. In this case the output $P_t y = \Psi^l_t x = P_t C^l \mathbb{T}_{(\,\boldsymbol{.}\,)} x$ is a continuous function and therefore we can infer that Ψ_t is differentiable with values in $L^2([0,\infty), Y)$.

6.4 Regular additive well-posed systems

To complete the text we now discuss the dependence of the output on the input. We saw in Section 2.3 that the additive structure is only justifiable if the output map is linear. Hence, the nonlinearity resides in the input.

Definition 6.19. Let X, U and Y be Banach spaces and $p \in [1,\infty)$. An *additive well-posed system* on X, $L^p([0,\infty), U)$ and $L^p([0,\infty), Y)$ is a quadruple $\Sigma = (\mathbb{T}, \Phi, \Psi_\infty, F_\infty)$ consisting of

(i) a strongly continuous semigroup $\mathbb{T}$ on X,

(ii) a family $\Phi = (\Phi_t)_{t\geq 0}$ of maps $\Phi_t : L^p([0,\infty), U) \to X$ such that $(\mathbb{T}, \Phi)$ is a continuous additive control system on X and $L^p([0,\infty), U)$,

(iii) a linear operator $\Psi_\infty : X \to L^p_{\text{loc}}([0,\infty), Y)$ such that $(\mathbb{T}, \Psi_\infty)$ is a linear observation system on X and $L^p_{\text{loc}}([0,\infty), Y)$,

(iv) a map $F_\infty : L^p_{\text{loc}}([0,\infty), U) \to L^p_{\text{loc}}([0,\infty), Y)$ satisfying the composition property

$$S_t^* F_\infty(u) = \Psi_\infty \Phi_t(u) + F_\infty(S_t^* u) \tag{6.14}$$

as well as the causality $P_t F_\infty(u) = P_t F_\infty(P_t u)$ for all $u \in L^p_{\text{loc}}([0,\infty), U)$ and $t \geq 0$.

The operators Φ_t are called *output maps*, the map Ψ_∞ is the *(extended) output map* and we call F_∞ the *(extended) input-output map*.

Let $v \in U$. In the definition below we consider the output $F_\infty(\chi_v)$. It is called the *step response* of Σ corresponding to v. To provide a motivation, for the moment assume that Σ is a finite dimensional linear system given by matrices A, B, C and D. It is well-known that in this situation y_v can be represented

$$y_v(t) = C \int_0^t \mathrm{e}^{As} Bv \,\mathrm{d}s + Dv \quad \text{for } t \geq 0. \tag{6.15}$$

Obviously, this function is continuous and thus has a Lebesgue point at 0, namely $y_v(0) = Dv$.

Definition 6.20. An additive well-posed system $\Sigma = (\mathbb{T}, \Phi, \Psi, F_\infty)$ on X, $L^p([0,\infty), U)$ and $L^p([0,\infty), Y)$ is called *regular* if there exists a map $D : U \to Y$ such that for all $v \in U$ the integral

$$\tfrac{1}{\tau}\int_0^\tau F_\infty(\chi_v)(s)\,\mathrm{d}s$$

converges to $D(v)$ in Y as $\tau \to 0^+$. The operator D is called *feedthrough operator* of Σ.

We shall now see that for regular systems the analog of (6.15) is valid. In fact, also the output corresponding to inputs from Ω_0 can be represented accordingly. The space of step functions Ω_0 was introduced at the beginning of Chapter 4.

Theorem 6.21. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, F_\infty)$ be a regular additive well-posed system on the spaces X, $L^p([0,\infty), U)$ and $L^p([0,\infty), Y)$ for an exponent $p \in [1,\infty)$. Then for each $u \in \Omega_0$ and almost every $t \geq 0$ we have $\Phi_t(u) \in \mathrm{D}(C_L)$ as well as*

$$F_\infty(u)(t) = C_L \Phi_t(u) + D(u(t)). \tag{6.16}$$

Proof. For all $u \in L^p_{\mathrm{loc}}([0,\infty), U)$, $t \geq 0$ and $\tau > 0$ the composition property (6.14) yields

$$\begin{aligned}\tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s &= \tfrac{1}{\tau}\int_0^{\tau} F_\infty(u)(t+s)\,\mathrm{d}s = \int_0^{\tau} (S_t^* F_\infty(u))(s)\,\mathrm{d}s \\ &= \tfrac{1}{\tau}\int_0^{\tau} (\Psi_\infty \Phi_t(u))(s)\,\mathrm{d}s + \tfrac{1}{\tau}\int_0^{\tau} F_\infty(S_t^* u)(s)\,\mathrm{d}s.\end{aligned} \tag{6.17}$$

Fix $u \in \Omega_0 \subseteq L^p_{\mathrm{loc}}([0,\infty), U)$. There are $0 = t_0 < t_1 < \ldots < t_m$ and $v_1, \ldots, v_m \in U$ with

$$u = \sum_{k=1}^{m} \mathbb{1}_{[t_{k-1}, t_k)} v_k.$$

Since $F_\infty(u)$ belongs to $L^1_{\mathrm{loc}}([0,\infty), U)$, Theorem B.2 yields a nullset $\mathcal{N} \subseteq [0,\infty)$ such that for all $t \in [0,\infty) \setminus \mathcal{N}$ we have

$$\tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s \to F_\infty(u)(t) \quad \text{as } \tau \to 0^+$$

with convergence in Y.

Let $t \in [0,\infty) \setminus \mathcal{N}$. If $t < t_m$ there is exactly one $k \in \{1, \ldots, m\}$ with $t \in [t_{k-1}, t_k)$ and we set $\delta_0 := t_k - t$. If $t \geq t_m$ we set $\delta_0 = 1$. Then for all $s \in [0, \delta_0)$ it follows $S_t^*(u)(s) = u(t+s) = u(t)$. To express it differently, we have $P_{\delta_0} S_t^*(u) = P_{\delta_0} \chi_{u(t)}$.

Let $\varepsilon > 0$. There is some $\delta \in (0, \delta_0)$ such that for all $\tau \in (0, \delta]$ we have

$$\left\| \tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s - F_\infty(u)(t) \right\|_Y \leq \tfrac{\varepsilon}{2}.$$

Choosing δ small enough, causality and regularity yield

$$\begin{aligned}\left\| \tfrac{1}{\tau}\int_0^{\tau} F_\infty(S_t^* u)(s)\,\mathrm{d}s - D(u(t)) \right\|_Y &= \left\| \tfrac{1}{\tau}\int_0^{\tau} P_{\delta_0} F_\infty(S_t^* u)(s)\,\mathrm{d}s - D(u(t)) \right\|_Y \\ &= \left\| \tfrac{1}{\tau}\int_0^{\tau} P_{\delta_0} F_\infty(P_{\delta_0} S_t^* u)(s)\,\mathrm{d}s - D(u(t)) \right\|_Y \\ &= \left\| \tfrac{1}{\tau}\int_0^{\tau} F_\infty(\chi_{u(t)})(s)\,\mathrm{d}s - D(u(t)) \right\|_Y \leq \tfrac{\varepsilon}{2}.\end{aligned}$$

For sufficiently small $\tau > 0$ it follows

$$\begin{aligned}&\left\| \tfrac{1}{\tau}\int_0^{\tau} \Psi_\infty(\Phi_t(u))(s)\,\mathrm{d}s - F_\infty(u)(t) + D(u(t)) \right\|_X \\ &= \left\| \tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s - F_\infty(u)(t) - \left(\tfrac{1}{\tau}\int_0^{\tau} F_\infty(S_t^* u)(s)\,\mathrm{d}s - D(u(t)) \right) \right\|_Y \\ &\leq \left\| \tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s - F_\infty(u)(t) \right\|_Y + \left\| \tfrac{1}{\tau}\int_0^{\tau} F_\infty(S_t^* u)(s)\,\mathrm{d}s - D(u(t)) \right\|_Y \\ &\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This means that $\Phi_t(u) \in \mathrm{D}(C_L)$ and that $C_L(\Phi_t(u)) = F_\infty(u)(t) - D(u(t))$. $\square$

Imposing further continuity assumptions on F_∞ we can extend (6.16) to larger subspaces of $L^p_{\mathrm{loc}}([0,\infty),U)$. To this end, as for the output map Ψ_∞, we consider truncations of the input-output map. For $t \geq 0$ define the operator $F_t : L^p([0,\infty),U) \to L^p([0,\infty),U)$ by $F_t u := P_t F_\infty u$ for $u \in L^p([0,\infty),U)$.

We say that F_t is Lipschitz on bounded sets if for every radius $\rho > 0$ there is a constant $c_\rho \geq 0$ such that

$$\|F_t(u) - F_t(\tilde{u})\|_{L^p([0,\infty),Y)} \leq c_\rho \|u - \tilde{u}\|_{L^p([0,\infty),U)}$$

for all $u, \tilde{u} \in L^p([0,\infty),U)$ with $\|u\|_{L^p}, \|\tilde{u}\|_{L^p} \leq \rho$. If the set $\{c_\rho \,|\, \rho > 0\}$ is bounded, then we call F_t (globally) Lipschitz.

Of course we can plug $u \in L^p_{\mathrm{loc}}([0,\infty),U)$ into the definition $F_t(u) := P_t F_\infty(u)$. By that we obtain an extension of F_t to $L^p_{\mathrm{loc}}([0,\infty),U)$. We then have $F_t(u) = F_t(P_t u)$ due to causality. In case $F_t : L^p([0,\infty),U) \to L^p([0,\infty),U)$ is e.g. Lipschitz, it follows that

$$\|F_t(u) - F_t(\tilde{u})\|_{L^p([0,\infty),Y)} \leq c \|P_t u - P_t \tilde{u}\|_{L^p([0,\infty),U)}$$

for all $u, \tilde{u} \in L^p_{\mathrm{loc}}([0,\infty),U)$.

Without proof we repeat Lemma 6.1 of [44] which is a corollary of Lebesgue's differentiation theorem, see Theorem B.2. The generalization to $p \in [1,\infty)$ is immediate.

Lemma 6.22. *Let W be a Banach space, $p \in [1,\infty)$ and $u \in L^p_{\mathrm{loc}}([0,\infty),W)$. Then for almost every $t \geq 0$ we have*

$$\left(\tfrac{1}{\tau}\int_0^\tau \|u(t+s) - u(t)\|_W^p \,\mathrm{d}t\right)^{1/p} \to 0 \quad \text{as } \tau \to 0^+.$$

Theorem 6.23. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, F_\infty)$ be a regular semi-linear well-posed system on X, $L^p([0,\infty),U)$ and $L^p([0,\infty),Y)$ for an exponent $p \in [1,\infty)$. Further assume that F_{t_0} is Lipschitz on bounded sets for a time $t_0 > 0$. Then for each $u \in L^\infty([0,\infty),U)$ and almost every $t \geq 0$ we have $\Phi_t(u) \in \mathrm{D}(C_L)$ as well as*

$$F_\infty(u)(t) = C_L \Phi_t(u) + D(u(t)).$$

If F_{t_0} is Lipschitz, the assertion is true for all $u \in L^p_{\mathrm{loc}}([0,\infty),U)$.

Proof. We continue equation (6.17) from the last proof. For all $u \in L^p_{\mathrm{loc}}([0,\infty),U)$, $t \geq 0$ and $\tau > 0$ the composition property (6.14) yields

$$\begin{aligned}
\tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s &= \tfrac{1}{\tau}\int_0^\tau F_\infty(u)(t+s)\,\mathrm{d}s = \int_0^\tau (S_t^* F_\infty(u))(s)\,\mathrm{d}s \\
&= \tfrac{1}{\tau}\int_0^\tau \big(\Psi_\infty \Phi_t(u)\big)(s)\,\mathrm{d}s + \tfrac{1}{\tau}\int_0^\tau F_\infty(S_t^* u)(s)\,\mathrm{d}s \\
&= \tfrac{1}{\tau}\int_0^\tau \big(\Psi_\infty \Phi_t(u)\big)(s)\,\mathrm{d}s + \tfrac{1}{\tau}\int_0^\tau \left[F_\infty(S_t^* u)(s) - F_\infty(\chi_{u(t)})(s)\right]\mathrm{d}s \\
&\qquad + \tfrac{1}{\tau}\int_0^\tau F_\infty(\chi_{u(t)})(s)\,\mathrm{d}s.
\end{aligned} \tag{6.18}$$

At first, let F_{t_0} be Lipschitz on bounded sets. Take any $u \in L^\infty([0,\infty),U)$ and set $\rho := t_0^{1/p}\|u\|_{L^\infty}$. Then $\|P_{t_0}S_t^* u\|_{L^p([0,\infty),U)} = \|S_t^* u\|_{L^p([0,t_0],U)} \leq t_0^{1/p}\|u\|_{L^\infty} \leq \rho$ and clearly also $\|P_{t_0}\chi_{u(t)}\|_{L^p([0,\infty),U)} \leq t_0^{1/p}\|u\|_{L^\infty} \leq \rho$ for all $t \geq 0$.

We claim that the first summand on the right-hand side of (6.18) converges. To prove this we show the convergence of each of the other terms. For the last expression on the right-hand side this follows from the fact that Σ is regular, more precisely,

$$\left\| \tfrac{1}{\tau}\int_0^\tau F_\infty(\chi_{u(t)})(s)\,\mathrm{d}s - D(u(t))\right\|_Y \to 0 \quad \text{as } \tau \to 0^+.$$

Since $F_\infty(u)$ belongs to $L^1_{\mathrm{loc}}([0,\infty),U)$, Theorem B.2 yields a nullset $\mathcal{N}_1 \subseteq [0,\infty)$ such that for all $t \in [0,\infty) \setminus \mathcal{N}_1$ we have

$$\tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s \to F_\infty(u)(t) \quad \text{as } \tau \to 0^+$$

with convergence in Y. For the remaining term, assume that $\tau \leq t_0$. Using the Lipschitz property of F_∞ we then infer the estimate

$$\begin{aligned}
\left\|\tfrac{1}{\tau}\int_0^\tau \left[F_\infty(S_t^*u)(s) - F_\infty(\chi_{u(t)})(s)\right]\mathrm{d}s\right\|_Y &\leq \tfrac{1}{\tau}\int_0^\tau \|F_\infty(S_t^*u)(s) - F_\infty(\chi_{u(t)})(s)\|_Y\,\mathrm{d}s\\
&= \tfrac{1}{\tau}\,\tau^{\frac{1}{p'}}\left(\int_0^\tau \|F_\infty(S_t^*u)(s) - F_\infty(\chi_{u(t)})(s)\|_Y^p\,\mathrm{d}s\right)^{1/p}\\
&= \tau^{-\frac{1}{p}}\|P_\tau(F_{t_0}(S_t^*u) - F_{t_0}(\chi_{u(t)}))\|_{L^p([0,\infty),Y)}\\
&\leq c_\rho \tau^{-\frac{1}{p}}\|P_\tau(S_t^*u - \chi_{u(t)})\|_{L^p([0,\infty),U)}\\
&= c_\rho\left(\tfrac{1}{\tau}\int_0^\tau \|(S_t^*u)(s) - u(t)\|_U^p\,\mathrm{d}s\right)^{1/p}.
\end{aligned}$$

By Lemma 6.22 there is another nullset $\mathcal{N}_2 \subseteq [0,\infty)$ with

$$c_\rho\left(\tfrac{1}{\tau}\int_0^\tau \|(S_t^*u)(s) - u(t)\|_U^p\,\mathrm{d}s\right)^{1/p} \to 0, \quad \text{as } \tau \to 0^+$$

for all $t \in [0,\infty)\setminus\mathcal{N}_2$. Let $t \in [0,\infty)\setminus\mathcal{N}$, where $\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2$. From (6.18) we obtain

$$\begin{aligned}
\tfrac{1}{\tau}\int_0^\tau (\Psi_\infty\Phi_t(u))(s)\,\mathrm{d}s = \tfrac{1}{\tau}\int_t^{t+\tau} F_\infty(u)(s)\,\mathrm{d}s - \tfrac{1}{\tau}\int_0^\tau \left[F_\infty(S_t^*u)(s) - F_\infty(\chi_{u(t)})(s)\right]\mathrm{d}s\\
- \int_0^\tau F_\infty(\chi_{u(t)})(s)\,\mathrm{d}s \;\to\; F_\infty(u)(t) - D(u(t))
\end{aligned}$$

as $\tau \to 0^+$ for all with convergence in Y. By the definition of the Lebesgue extension C_L this means that $\Phi_t(u) \in \mathrm{D}(C_L)$ and that $C_L\Phi_t(u) = F_\infty(u)(t) - D(u(t))$.

If F is Lipschitz, then all arguments remain valid even for $u \in L^p_{\mathrm{loc}}([0,\infty),U)$ since then we do not need estimates for $\|P_{t_0}S_t^*u\|_{L^p([0,\infty),U)}$ and $\|P_{t_0}\chi_{u(t)}\|_{L^p([0,\infty),U)}$. □

The output of the system Σ corresponding to the initial state $x_0 \in X$ and the input $u \in L^p([0,\infty),U)$ is defined as

$$y := \Psi_\infty x_0 + F_\infty(u).$$

Combining the last two propositions with Proposition 6.3, we obtain that y is represented by the Lebesgue extension C_L and the feedthrough operator D.

Corollary 6.24. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, F_\infty)$ be a regular semi-linear well-posed system on X, $L^p([0,\infty), U)$ and $L^p([0,\infty), Y)$ for an exponent $p \in [1,\infty)$. Then for all $x_0 \in X$ and $u \in \Omega_0$ as well as almost all $t \geq 0$ we have the identity*

$$\begin{aligned}\Psi_\infty x_0(t) + F_\infty(u)(t) &= C_L \mathbb{T}_t x_0 + C_L \Phi_t(u) + D(u(t)) \\ &= C_L(\mathbb{T}_t x_0 + \Phi_t(u)) + D(u(t)).\end{aligned} \tag{6.19}$$

The nullset depends on x_0 and u.

If F_{t_0} is Lipschitz on bounded sets for a time $t_0 > 0$, then the assertion is true for all $u \in L^\infty([0,\infty), U)$. Finally, if F_{t_0} is Lipschitz, then (6.19) holds for all $u \in L^p_{\text{loc}}([0,\infty), U)$.

6.4.1 A wave equation with point control and point observation

In this example we revisit the linear wave equation with zero Dirchlet boundary conditions from Subsection 4.4.2, but with the one dimensional domain $(0,1)$. Moreover, we consider different control and observation operators, namely a 'point control' and a 'point velocity observation'. Formally the system can be described by

$$\begin{aligned}\partial_t^2 \omega(t,\xi) &= \partial_\xi^2 \omega(t,\xi) + \mu(t)\delta_{\xi_0}(\xi), && t \geq 0,\ \xi \in (0,1) \\ \omega(t,0) &= \omega(t,1) = 0, && t \geq 0 \\ \omega(0,\xi) &= f_0(\xi), \quad \partial_t \omega(0,\xi) = g_0(\xi), && \xi \in (0,1) \\ \gamma(t) &= -\partial_t \omega(t,\xi_0) && t \geq 0.\end{aligned} \tag{6.20}$$

Here $\xi_0 \in (0,1)$ is a fixed point in the domain, $\mu : [0,\infty) \to \mathbb{C}$ is the input and $\gamma : [0,\infty) \to \mathbb{C}$ is the output.

To show that these equations can be described by a well-posed linear system, we use the Dirichlet Laplacian A_0 defined in Subsection 4.4.1. Set $X := H_0^1(0,1) \times L^2[0,1]$ and $U := Y := \mathbb{C}$. Note that in then $X = [\mathrm{D}(A_0)]_{1/2} \times [\mathrm{D}(A_0)]_0$ with the notation introduced in Appendix A.

With the framework developed in [4] it can be seen, that a well-posed linear system on X, $L^2([0,\infty), U)$ and $L^2([0,\infty), Y)$ is determined by the operators A, B^l and C^l constructed as follows. As in the example of Subsection 4.4.2 the generator of a unitary group $\mathbb{T}$ on X is given by

$$A(f,g) = (g, -A_0 f)$$

with domain $\mathrm{D}(A) = H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$. Recall that $X_{-1} = L^2[0,1] \times H^{-1}(0,1) = [\mathrm{D}(A_0)]_0 \times [\mathrm{D}(A_0)]_{-1/2}$. The observation operator $C^l \in \mathcal{L}(X_1, Y)$ and the control operator $B^l \in \mathcal{L}(U, X_{-1})$ are defined as

$$C^l(f,g) = \delta_{\xi_0} g = g(\xi_0) \quad \text{and} \quad B^l(v) = (0, v\delta_{\xi_0}).$$

Here $\delta_{\xi_0} \in \mathcal{L}(H_0^1(0,1), Y)$ is the delta functional given by $\delta_{\xi_0}(h) = h(\xi_0)$. Its boundedness is clear due to the continuous embedding $H_0^1(0,1) \hookrightarrow C[0,1]$.

We mention that C^l and B^l can be identified with the operator matrices $C^l = [0, \delta_{\xi_0}]$ and $B^l = [0, \delta_{\xi_0}^*]^\top$. Formally problem (6.20) is equivalent to the system

$$\begin{aligned}z'(t) &= Az(t) + B^l u(t); \quad z(0) = x_0 \\ y(t) &= C^l z(t),\end{aligned} \tag{6.21}$$

where $z(t) = (\omega(t, \cdot), \partial_t \omega(t, \cdot))$, $x_0 = (f_0, g_0)$, $u(t) = \mu(t)$ and $y(t) = \gamma(t)$. As before we do not discuss the question in which sense solutions of the last equation are solutions of (6.20).

The regularity of the system under consideration is shown in Proposition 5.9 of [50]. It is remarkable that in this example a 'transfer function' can be determined explicitly. Let $\omega_0 := \omega_0(\mathbb{T})$ be the growth bound of $\mathbb{T}$. A *transfer function* is an analytic map $\mathbf{G} : \mathbb{C}_{\omega_0} \to \mathcal{L}(U, Y)$ linking the Laplace transform of the input with the Laplace transform of the output via the equation

$$\widehat{y}(s) = \mathbf{G}(s)\widehat{u}(s) \quad \text{for } s \in \mathbb{C}_{\omega_0}.$$

It is unique up to an additive constant. For more details on transfer functions we refer to [55]. In our case we have

$$\mathbf{G}(s) = \frac{\sinh(s\xi_0)\sinh(s(\xi_0 - 1))}{\sinh(s)} \quad \text{for } \operatorname{Re} s > 0.$$

Note that $\mathbf{G}(s) \to -\frac{1}{2}$ as $s \to \infty$. By Theorem 5.8 of [55] it follows that our system is regular with feedthrough operator $D = -\frac{1}{2} \in \mathcal{L}(U, Y) = \mathbb{C}$.

Denote the well-posed linear system on X, $L^2([0,\infty), U)$ and $L^2([0,\infty), Y)$ associated to (6.21) by $(\mathbb{T}, \Phi^l, \Psi^l_\infty, F^l_\infty)$. Let M be the nonlinear map introduced in Subsection 4.4.3, i.e., $M : \mathbb{C} \to \mathbb{C}$ given by

$$M(v) = \big(m(\operatorname{Re} v), m(\operatorname{Im} v)\big),$$

where m is a function in $C^2(\mathbb{R})$ with the properties that m' and m'' are bounded, $m(0) = 0$ and $m'(0) \neq 0$. Further recall that M is Lipschitz. Since $M(0) = 0$, it follows that the map N given by $N(u) = M \circ u$ maps the space $L^2([0,\infty), U)$ to itself and obviously also $L^2_{\text{loc}}([0,\infty), U)$. Moreover, N is Lipschitz.

As before we replace the input u by $N(u)$. More precisely, we consider the nonlinear maps $\Phi_t : L^2([0,\infty), U) \to X$ for $t \geq 0$ as well as $F_\infty : L^2_{\text{loc}}([0,\infty), U) \to L^2_{\text{loc}}([0,\infty), Y)$ given by $\Phi_t(u) = \Phi^l_t N(u)$ and $F_\infty(u) = F^l_\infty N(u)$ respectively.

In Section 4.4 we already saw that $(\mathbb{T}, \Phi)$ is a continuous additive control system on X and $L^2([0,\infty), U)$, where $\Phi = (\Phi_t)_{t \geq 0}$. Hence, in order to verify that the quadruple $(\mathbb{T}, \Phi, \Psi^l_\infty, F_\infty)$ is an additive well-posed system, it remains to check F_∞ is causal and satisfies the composition property (6.14). To this end, let $t \geq 0$ and $u \in L^2_{\text{loc}}([0,\infty), U)$. It is clear that $P_t M(u(\cdot)) = P_t M(P_t u(\cdot))$ and $S^*_t M(u(\cdot)) = M(S^*_t u(\cdot))$. The claim now follows from the fact that F^l_∞ is causal and fulfills (6.14). For convenience we carry out the calculation

$$\begin{aligned} P_t F_\infty(u) &= P_t F^l_\infty M(u(\cdot)) = P_t F^l_\infty P_t M(u(\cdot)) = P_t F^l_\infty P_t M(P_t u(\cdot)) \\ &= P_t F^l_\infty M(P_t u(\cdot)) = P_t F_\infty(P_t u), \\ S^*_t F_\infty(u) &= S^*_t F^l_\infty M(u(\cdot)) = \Psi^l_\infty \Phi^l_t M(u(\cdot)) + F^l_\infty S^*_t M(u(\cdot)) \\ &= \Psi^l_\infty \Phi_t(u) + F^l_\infty M(S^*_t u(\cdot)) = \Psi^l_\infty \Phi_t(u) + F_\infty(S^*_t u). \end{aligned}$$

Next, from the fact that $(\mathbb{T}, \Phi^l, \Psi^l_\infty, F^l_\infty)$ is regular we infer that $(\mathbb{T}, \Phi, \Psi^l_\infty, F_\infty)$ is regular with the feedthrough operator given by $D(v) = D^l M(v)$ for $v \in U$. Indeed, we have

$$\left| \frac{1}{\tau} \int_0^\tau (F_\infty(\chi_v))(s) \, \mathrm{d}s - D^l M(v) \right| = \left| \frac{1}{\tau} \int_0^\tau (F^l_\infty(\chi_{M(v)})(s) \, \mathrm{d}s - D^l M(v) \right| \to 0, \quad \text{as } \tau \to 0^+.$$

for all $v \in U$. Finally, note that F_t is Lipschitz for every $t \geq 0$ because N is Lipschitz.

Let $x_0 \in X$ and $u \in L^2_{\text{loc}}([0,\infty),U)$ and set $y := \Psi^l_\infty x_0 + F_\infty(u)$. Then, by Corollary 6.24

$$y(t) = C_L(\mathbb{T}_t x_0 + \Phi_t(u)) + D^l M(u(t))$$

holds for almost every $t \geq 0$, where C_L is the Lebesgue extension of C^l.

Appendix A

Extrapolation spaces

This chapter was written on basis of Paragraph 3 in [52] and Section 6 of [53]. Another source is Section II.5a of [15]. In this an the following appendices, we collect known facts from the literature and fix some notation. For the convenience of the reader we give several shorter proofs of statements that are crucial for the thesis.

Let X be a Banach space with norm $\|\cdot\|_X$. Further let $A : \mathrm{D}(A) \to X$ be a linear operator in X. Recall that the *graph norm* on $\mathrm{D}(A)$ is given by

$$\|x\|_A := \|x\|_X + \|Ax\|_X.$$

It is known that operator A is closed if and only if $(\mathrm{D}(A), \|\cdot\|_A)$ is a Banach space. In any case $A : \mathrm{D}(A) \to X$ is bounded when $\mathrm{D}(A)$ is equipped with the graph norm.

In the following we assume that A is closed and has nonempty resolvent set $\rho(A) \neq \emptyset$. For $\mu \in \rho(A)$ we use the abbreviation $R_\mu := (\mu - A)^{-1} \in \mathcal{L}(X)$. We fix some $\lambda \in \rho(A)$. If $0 \in \rho(A)$, the choice $\lambda = 0$ is convenient. Another norm $\|\cdot\|_1$ on $\mathrm{D}(A)$ is defined through

$$\|x\|_1 := \|x\|_{1,\lambda} := \|(\lambda - A)x\|_X.$$

The straightforward proof is omitted. We write X_1 for $\mathrm{D}(A)$ endowed with $\|\cdot\|_1$. Let us prove that $\|\cdot\|_1$ and $\|\cdot\|_A$ are equivalent. To this end, take $x \in \mathrm{D}(A)$. From

$$\|x\|_1 \le |\lambda|\|x\|_X + \|Ax\|_X \le (1 + |\lambda|)\|x\|_A$$

we see that $\|\cdot\|_1$ is dominated by $\|\cdot\|_A$. On the other hand we have

$$\begin{aligned}\|x\|_A &= \|x\|_X + \|(\lambda - A)x - \lambda x\|_X \le (1 + |\lambda|)\|x\|_X + \|(\lambda - A)x\|_X \\ &= (1 + |\lambda|)\|R_\lambda(\lambda - A)x\|_X + \|x\|_1 \le (1 + |\lambda|)\|R_\lambda\| \, \|x\|_1 + \|x\|_1.\end{aligned}$$

As a consequence, X_1 is a Banach space and we have

$$A \in \mathcal{L}(X_1, X).$$

Note that these properties are independent of the choice of $\lambda \in \rho(A)$. Let $x \in X_1$. From the estimate

$$\|x\|_X = \|R_\lambda(\lambda - A)x\|_X \le \|R_\lambda\| \, \|x\|_1 \quad \text{for } x \in X_1 \tag{A.1}$$

we infer that X_1 is continuously embedded in X. It is clear that $\lambda - A \in \mathcal{L}(X_1, X)$ and $R_\lambda \in \mathcal{L}(X, X_1)$ are isometric isomorphisms.

Set $\mathrm{D}(A^0) := X$. Then for $m \in \mathbb{N}$ the iterated domain $\mathrm{D}(A^m)$ is defined recursively by

$$\mathrm{D}(A^m) := \{x \in X \,|\, x \in \mathrm{D}(A) \text{ and } Ax \in \mathrm{D}(A^{m-1})\}.$$

Due to the linearity of A, this is a vector space. Obviously we have $\mathrm{D}(A^m) \subseteq \mathrm{D}(A^{m-1})$ for every $m \in \mathbb{N}$. As we shall see, it is consistent to denote $X_0 := X$ and $\|\cdot\|_0 := \|\cdot\|_X$. Unless we say otherwise, the following statements are verified by straightforward inductions.

For $\mu \in \rho(A)$ and $m \in \mathbb{N}$ the operator $\mu - A$ maps the space $\mathrm{D}(A^m)$ onto $\mathrm{D}(A^{m-1})$. Since clearly every restriction of $\mu - A$ is one-to-one it follows that

$$\mu - A|_{\mathrm{D}(A^m)} : \mathrm{D}(A^m) \to \mathrm{D}(A^{m-1})$$

is an isomorphism. Now for all $m \in \mathbb{N}$ we may recursively define the norms $\|\cdot\|_m = \|\cdot\|_{m,\lambda}$ on $\mathrm{D}(A^m)$ by

$$\|x\|_m := \|x\|_{m,\lambda} := \|(\lambda - A)x\|_{m-1}.$$

Where a distinction between the different restrictions is necessary or leads to a better understanding, we shortly write $A|_m := A|_{\mathrm{D}(A^m)}$. We further write X_m for $\mathrm{D}(A^m)$ equipped with $\|\cdot\|_m$.

Let $m \geq 2$. Due to the definition of $\|\cdot\|_m$, the operator $\lambda - A|_m : X_m \to X_{m-1}$ is an isometric isomorphism. Inductively it follows that X_m is a Banach space. Clearly A maps X_m to X_{m-1}. Using also that A commutes with $\lambda - A$, we infer that $A|_m \in \mathcal{L}(X_m, X_{m-1})$. In fact, using the definition of $\|\cdot\|_m$ we even have

$$\begin{aligned} \|A|_m\|_{\mathcal{L}(X_m, X_{m-1})} &= \sup_{\substack{x \in X_m,\\ \|x\|_m = 1}} \|(A|_m)x\|_{m-1} \\ &= \sup\Big\{\|(\lambda - A|_{m-1})(A|_m)x\|_{m-2} \,\Big|\, x \in X_m \text{ with } \|(\lambda - A|_m)x\|_{m-1} = 1\Big\} \\ &= \sup\Big\{\|(A|_{m-1})(\lambda - A|_m)x\|_{m-2} \,\Big|\, x \in X_m \text{ with } \|(\lambda - A|_m)x\|_{m-1} = 1\Big\} \\ &= \sup_{\substack{\xi \in X_{m-1},\\ \|\xi\|_{m-1} = 1}} \|A|_{m-1}\xi\|_{m-2} = \|A|_{m-1}\|_{\mathcal{L}(X_{m-1}, X_{m-2})} = \|A\|_{\mathcal{L}(X_1, X)}. \end{aligned} \tag{A.2}$$

From an estimate analog to (A.1), we infer that X_m is continuously embedded in X_{m-1}.

Again let $m \in \mathbb{N}$. We consider $B := A|_{m+1} : \mathrm{D}(A^{m+1}) \to X_m$ as an operator in X_m. The corresponding graph norm is given by $\|x\|_B = \|x\|_m + \|Bx\|_m$ for $x \in \mathrm{D}(A^{m+1})$. Essentially with the same proof as above we see that $\|\cdot\|_B$ and $\|\cdot\|_{m+1}$ are equivalent. It follows that B is closed. We already argued that for $\mu \in \rho(A)$ the map $\mu - B : \mathrm{D}(A^{m+1}) \to X_m$ is one-to-one and onto and hence $\mu \in \rho(B)$. But this means that for each $m \in \mathbb{N}$ we have

$$\rho(A) \subseteq \rho(A|_m).$$

Let $T \in \mathcal{L}(X)$ and assume that T leaves X_m invariant, that is $T(X_m) \subseteq X_m$. Using the completeness of X_m and the closed graph theorem we deduce that $T|_{X_m}$ belongs to $\mathcal{L}(X_m)$. If T even commutes with A, then as in (A.2) one derives the identity

$$\|T|_{X_m}\|_{\mathcal{L}(X_m)} = \|T\|_{\mathcal{L}(X)}.$$

We apply these results to the case where A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Let us write $\mathbb{T}_t|_m := \mathbb{T}_t|_{X_m}$ and $\mathbb{T}|_m := (\mathbb{T}_t|_m)_{t\geq 0}$.

Proposition A.1. *Let A be the generator of a strongly continuous semigroup on X. Then for all $m \in \mathbb{N}$ the family $\mathbb{T}|_m$ is a strongly continuous semigroup on X_m. Its generator is $A|_{m+1}$, the restriction of A to $\mathrm{D}(A^{m+1})$. Moreover, $\|\mathbb{T}_t|_m\|_{\mathcal{L}(X_m)} = \|\mathbb{T}_t\|_{\mathcal{L}(X)}$ for all $t \geq 0$.*

Proof. We only treat the case $m = 1$. The other cases follow inductively. Clearly, restricting semigroups to subspaces preserves the semigroup laws. The strong continuity of the family follows from the estimate

$$\|\mathbb{T}_h x - x\|_1 \leq |\lambda| \|\mathbb{T}_h x - x\|_X + \|\mathbb{T}_h Ax - Ax\|_X$$

and the strong continuity of $\mathbb{T}$. Denote the generator of $\mathbb{T}|_1$ by $B : \mathrm{D}(B) \to X_1$. Let $x \in \mathrm{D}(A^2)$. Then we have $w := (\lambda - A)x \in \mathrm{D}(A)$ and consequently

$$\left\|\tfrac{1}{h}(\mathbb{T}_h x - x) - Ax\right\|_1 = \left\|(\lambda - A)\tfrac{1}{h}(\mathbb{T}_h x - x) - (\lambda - A)Ax\right\|_X = \left\|\tfrac{1}{h}(\mathbb{T}_h w - w) - Aw\right\|_X$$

converges to zero as $h \to 0^+$. This means $A|_{\mathrm{D}(A^2)} \subseteq B$. It is standard to conclude that $A|_{\mathrm{D}(A^2)} = B$ from $\rho(A^2) \cap \rho(B) \supseteq \rho(A) \cap \rho(B) \neq \emptyset$. ☐

Now additionally let A be densely defined, meaning that $\mathrm{D}(A)$ is dense in X. Then for all $m \in \mathbb{N}$ the space $\mathrm{D}(A^m)$ is dense in X_{m-1}. Indeed, assume that the claim is true for some $m \in \mathbb{N}$ and take $x \in X_m$. Then for $w := (\lambda - A)x \in X_{m-1}$ there is a sequence (w_k) in $\mathrm{D}(A^m)$ converging to w with respect to $\|\cdot\|_{m-1}$. Setting $x_k := R_\lambda w_k$ for $k \in \mathbb{N}$ we obtain a sequence (x_k) in $\mathrm{D}(A^{m+1})$ with

$$\|x - x_k\|_m = \|(\lambda - A)(x - x_k)\|_{m-1} = \|w - w_k\|_{m-1} \to 0 \quad \text{as } k \to \infty.$$

We remark that $A|_m \in \mathcal{L}(X_m, X_{m-1})$ is the continuous extension of $A|_{m+1}$ with respect to $\|\cdot\|_m$ on X_{m+1} and $\|\cdot\|_{m-1}$ on X_m respectively.

Let $\mu \in \rho(A|_{m+1})$. Then $\mu - A|_m$ is the continuous extension of $\mu - A|_{m+1}$. Consequently $\mu - A|_m$ is one-to-one and onto, as it is easy to see using the equivalence of $\|\cdot\|_{m,\lambda}$ and $\|\cdot\|_{m,\mu}$. Therefore we have $\mu \in \rho(A|_m)$. Together with our previous results, we conclude that

$$\rho(A) = \rho(A|_m) \quad \text{for all } m \in \mathbb{N}. \tag{A.3}$$

The denseness of $\mathrm{D}(A^m)$ is equivalent to the fact that the space X_{m-1} is the completion of $\mathrm{D}(A^m)$ with respect to $\|\cdot\|_{m-1}$. Note that for $x \in \mathrm{D}(A^{m-1})$ we have $R_\lambda x \in \mathrm{D}(A^m)$ as well as $x = (\lambda - A)R_\lambda x$ and we thus derive that

$$\|x\|_{m-1} = \|(\lambda - A)R_\lambda x\|_{m-1} = \|R_\lambda x\|_m.$$

This motivates the construction of the spaces X_{-m} with negative index. Formally inserting $m = 0$, for $x \in X$ we set

$$\|x\|_{-1} := \|x\|_{-1,\lambda} := \|R_\lambda x\|_X.$$

Again it is easy to see that this yields a norm $\|\cdot\|_{-1}$ on X. We define X_{-1} as the completion of X with respect to $\|\cdot\|_{-1}$. Note that as *topological space* X_{-1} is independent of the choice of $\lambda \in \rho(A)$ since the norms $\|\,\mathbf{.}\,\|_{1,\lambda}$ and $\|\,\mathbf{.}\,\|_{1,\mu}$ are equivalent for each $\mu \in \rho(A)$. This follows immediately from the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \qquad \text{for } \mu \in \rho(A).$$

We shall see that X_{-1} has very similar properties as X_m for $m \in \mathbb{N}_0$. Indeed, in the following paragraphs we frequently refer to arguments given earlier.

First, X_{-1} is a Banach space by definition. Moreover, from the estimate

$$\|x\|_{-1} \le \|R_\lambda\|_{\mathcal{L}(X)} \|x\|_X \qquad \text{for all } x \in X.$$

we see that X is continuously embedded in X_{-1}. Next the operator $\lambda - A : \mathrm{D}(A) \to X$ has an isometric extension to X. For $x \in \mathrm{D}(A)$ we have $(\lambda - A)x \in X$ and the claim follows from

$$\|(\lambda - A)x\|_{-1} = \|R_\lambda(\lambda - A)x\|_X = \|x\|_X.$$

Extending the notation above, we write $\lambda - A|_0 \in \mathcal{L}(X, X_{-1})$. From the fact that $\lambda - A$ is onto we deduce that $\lambda - A|_0$ is onto and hence is an isometric isomorphism.

With the same arguments, using also the equivalence of $\|\,\mathbf{.}\,\|_{1,\lambda}$ and $\|\,\mathbf{.}\,\|_{1,\mu}$, we see that $\mu - A$ has a continuous extension $\mu - A|_0 \in \mathcal{L}(X, X_{-1})$ for any other $\mu \in \rho(A)$, which is an isomorphism between X and X_{-1}.

Also the resolvent $R_\lambda : X \to \mathrm{D}(A)$ has a continuous extension $R_\lambda|_{-1} \in \mathcal{L}(X_{-1}, X)$ which is an isometric isomorphism. This is clear, because $\|R_\lambda x\|_X = \|x\|_{-1}$ for $x \in X$. It is only a matter of calculation to verify that $\lambda - A|_0$ and $R_\lambda|_{-1}$ are inverse to each other. Since X is continuously embedded in X_{-1}, we may interpret $R_\lambda|_{-1}$ as an element of $\mathcal{L}(X_{-1})$. More precisely, for $x \in X_{-1}$ we have $(R_\lambda|_{-1})x \in X$ and thus

$$\|(R_\lambda|_{-1})x\|_{-1} = \|R_\lambda(R_\lambda|_{-1})x\|_X \le \|R_\lambda\|_{\mathcal{L}(X)} \|(R_\lambda|_{-1})x\|_X = \|R_\lambda\|_{\mathcal{L}(X)} \|x\|_{-1}.$$

Now successively for all $m \in \mathbb{N}$ we introduce a norm $\|\,\mathbf{.}\,\|_{-m}$ on X_{-m+1} via

$$\|x\|_{-m} := \|x\|_{-m,\lambda} := \|(R_\lambda|_{-m+1})x\|_{-m+1} \quad \text{for } x \in X_{-m+1}$$

Then we define the Banach space X_{-m} as the completion of X_{-m+1} with respect to $\|\,\mathbf{.}\,\|_{-m}$. As above, we see that X_{-m+1} is embedded continuously in X_{-m}. Further the maps $\lambda - A|_{-m+2}$ and $R_\lambda|_{-m+1}$ posses continuous extensions denoted

$$\lambda - A|_{-m+1} \in \mathcal{L}(X_{-m+1}, X_{-m}) \qquad \text{and} \qquad R_\lambda|_{-m} \in \mathcal{L}(X_{-m}, X_{-m+1}).$$

Actually these operators are isometric isomorphisms. Finally we may consider $R_\lambda|_{-m}$ as an element of $\mathcal{L}(X_{-m})$. Inductively one easily verifies that for $\mu \in \rho(A)$ the operators $\mu - A|_{-m} : X_{-m} \to X_{-m-1}$ are one-to-one and onto. We already discussed the case $m = 0$.

Let $m \in \mathbb{Z}$, $m \le 0$. Unless $0 \in \rho(A)$, we set

$$A|_m := \lambda - (\lambda - A|_m) \in \mathcal{L}(X_m, X_{m-1}).$$

With symbolically the same identity as (A.2) (there we had $m \ge 2$) we derive

$$\|A|_m\|_{\mathcal{L}(X_m, X_{m-1})} = \|A\|_{\mathcal{L}(X_1, X)}.$$

Combined with (A.2), we obtain this identity for every $m \in \mathbb{Z}$. Moreover, for all $k, l \in \mathbb{Z}$ we have $A|_k = (A|_l)|_{X_k}$ if $k \le l$. Hence the notation is justified. On the other hand $A|_k$ is the continuous extension of $A|_{k+1} : X_{k+1} \to X_k$.

Again let $m \in \mathbb{Z}$ with $m \le 0$ and consider $B := A|_m$ as an operator in X_{m-1}. Using that $\lambda - A|_{-m}$ isometrically maps X_{-m} to X_{-m-1} we see that the operator norm corresponding to B is equivalent to $\|\,\boldsymbol{.}\,\|_{-m}$. The completeness of X_{-m} implies that B is closed. As mentioned earlier, for $\mu \in \rho(A)$ the operator $\mu - B = \mu - A|_{-m}$ is one-to-one and onto. We conclude that the statement of (A.3) can be extended to

$$\rho(A) = \rho(A|_m) \quad \text{for all } m \in \mathbb{Z}. \tag{A.4}$$

Again assume that some $T \in \mathcal{L}(X)$ commutes with A. Then clearly T also commutes with R_λ. In particular, for $x \in X$ we have

$$\|Tx\|_{-1} = \|R_\lambda Tx\|_X = \|TR_\lambda x\|_X \le \|T\|_{\mathcal{L}(X)}\|x\|_{-1}.$$

We infer that T has a continuous extension $T|_{-1} \in \mathcal{L}(X_{-1})$ with $\|T|_{-1}\|_{\mathcal{L}(X_{-1})} \le \|T\|_{\mathcal{L}(X)}$. It is easy to see that $T|_{-1}$ commutes with $\lambda - A|_0$ and with $R_\lambda|_{-1}$. As in (A.2) we deduce the identity $\|T|_{-1}\|_{\mathcal{L}(X_{-1})} = \|T\|_{\mathcal{L}(X)}$. For convenience we give the proof. From the fact that $\lambda - A|_0 \in \mathcal{L}(X, X_{-1})$ is an isometric isomorphism we deduce

$$\begin{aligned}\|T\|_{\mathcal{L}(X)} &= \sup_{\substack{x\in X\\ \|x\|_X=1}} \|Tx\|_X = \sup_{\substack{x\in X\\ \|x\|_X=1}} \|(\lambda - A|_0)Tx\|_{-1} = \sup_{\substack{x\in X\\ \|x\|_X=1}} \|(T|_{-1})(\lambda - A|_0)x\|_{-1}\\ &= \sup_{\substack{\xi\in X_{-1}\\ \|\xi\|_{-1}=1}} \|(T|_{-1})\xi\|_{-1} = \|T|_{-1}\|_{\mathcal{L}(X_{-1})}.\end{aligned}$$

Iterating this procedure, for all $m \in \mathbb{N}$ we construct extensions $T|_{-m} \in \mathcal{L}(X_{-m})$ of $T|_{-m+1}$ and thus of T. Each extension satisfies

$$\|T|_{-m}\|_{\mathcal{L}(X_{-m})} = \|T\|_{\mathcal{L}(X)}.$$

Once more we apply the previous results to the case when A generates a strongly continuous semigroup $\mathbb{T}$ on X. Let $m \in \mathbb{N}$. For $t \ge 0$ let us denote the extension of $\mathbb{T}_t$ to X_{-m} by $\mathbb{T}_t|_{-m}$. Then the latter operators form a family $\mathbb{T}|_{-m} := (\mathbb{T}_t|_{-m})_{t\ge 0}$ in $\mathcal{L}(X_{-m})$.

Proposition A.2. *Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Then for all $m \in \mathbb{N}$ the family $\mathbb{T}|_{-m}$ in $\mathcal{L}(X_{-m})$ is a strongly continuous semigroup on X_{-m}. Its generator is $A|_{-m+1}$ seen as a densely defined operator in X_{-m}.*

Proof. As in Proposition A.1 we only treat the case $m = 1$ since the other cases follow inductively. The semigroup laws are verified in a straight forward calculation. Proposition I.5.3 in [15] yields the strong continuity of $\mathbb{T}|_{-1}$. Indeed, we only need that X is dense in X_{-1} and that $\mathbb{T}$ is the restriction of $\mathbb{T}|_{-1}$ to X.

Now let $B : \mathrm{D}(B) \to X_{-1}$ be the generator of $\mathbb{T}|_{-1}$. We shall show that $A|_0 \subseteq B$, that is $X \in \mathrm{D}(B)$ and $Bx = (A|_0)x$ for $x \in X$. To this end, take $x \in X$. Then $R_\lambda x \in \mathrm{D}(A)$. Because $R_\lambda|_{-1} \in \mathcal{L}(X_{-1}, X)$ is isometric and $\mathbb{T}_t$ commutes with R_λ for each $t \ge 0$, it follows

$$\begin{aligned}\left\|\tfrac{1}{h}(\mathbb{T}_h x - x) - (A|_0)x\right\|_{-1} &= \left\|R_\lambda \tfrac{1}{h}(\mathbb{T}_h x - x) - (R_\lambda|_{-1})(A|_0)x\right\|_X\\ &= \left\|\tfrac{1}{h}(\mathbb{T}_h R_\lambda x - R_\lambda x) - AR_\lambda x\right\|_X \to 0 \quad \text{as } h \to 0^+.\end{aligned}$$

Now the claim follows from the fact that $\rho(A|_0) \cap \rho(B) = \rho(A) \cap \rho(B) \ne \emptyset$. □

Note that from a theoretical viewpoint the spaces X_m together with the semigroup $\mathbb{T}$ are all equal in the sense that starting from X_{m_0} where $m_0 \in \mathbb{Z}$ is arbitrary, any other space and corresponding semigroup can be constructed. We can summarize this in a diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & X_{m+1} & \xrightarrow{\lambda-A} & X_m & \xrightarrow{\lambda-A} & X_{m-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mathbb{T}_t} & & \downarrow{\scriptstyle \mathbb{T}_t} & & \downarrow{\scriptstyle \mathbb{T}_t} & & \\
\dots & \longleftarrow & X_{m+1} & \xleftarrow[R_\lambda]{} & X_m & \xleftarrow[R_\lambda]{} & X_{m-1} & \longleftarrow & \dots
\end{array}$$

It is common to denote by $[\mathrm{D}(A)]$ the Banach space $\mathrm{D}(A)$ equipped with the graph norm. So, we sometimes use the notation $[\mathrm{D}(A)]_1$ for X_1 especially if the variable X is used otherwise. In generalization of that we define $[\mathrm{D}(A)]_m$ as X_m for all $m \in \mathbb{Z}$. In particular $[\mathrm{D}(A)]_0$ is X.

Let us now examine the dual space $X^* = \mathcal{L}(X, \mathbb{C})$ of X. Because A is densely defined, we have its dual operator A^*, which is closed. Further it is known that $\rho(A) = \rho(A^*)$ and $R(\mu, A^*) = R(\mu, A)^*$ for $\mu \in \rho(A)$. (See e.g. Theorem 2 of Section VIII.6 in [56])

Assume that A^* is densely defined. Thus A^* has all the properties needed to repeat the above construction. We obtain the spaces $(X^*)_k$ for $k \in \mathbb{Z}$. To avoid confusion we write $X_{k,d} := (X^*)_k$. The respective norms are denoted $\| \centerdot \|_{k,d}$. In particular we write $\| \centerdot \|_{0,d}$ for the norm on X^*.

Recall that $X_{1,d}$ is continuously embedded in X^*. It is easy to see that hence X^{**} is continuously embedded in $(X_{1,d})^*$.

With the following result we connect the extrapolation spaces to their counterparts on the side of the dual operator. Especially for reflexive spaces it yields a crucial identification to establish duality results.

Proposition A.3. *The canonical embedding $J : X \to X^{**};\ \ x \mapsto \langle x, \centerdot \rangle_X$ extends to an isometric operator $J : X_{-1} \to (X_{1,d})^*$. For $x \in X_{-1}$ it is given by*

$$\langle \varphi, Jx \rangle_{X_{1,d}} = \lim_{n\to\infty} \langle x_n, \varphi \rangle_X, \qquad \varphi \in X_{1,d}, \tag{A.5}$$

where (x_n) is any sequence in X with $\|x_n - x\|_{-1} \to 0$ as $n \to \infty$. If X is reflexive, the extension of J is an isometric isomorphism.

Proof. Fix $x \in X$. For $\varphi \in X_{1,d} \subseteq X^*$ we have $\varphi = R(\lambda, A^*)(\lambda - A^*)\varphi$. We infer the estimate

$$\begin{aligned}
|(Jx)(\varphi)| &= |\langle x, \varphi \rangle_X| = |\langle x, R(\lambda, A^*)(\lambda - A^*)\varphi \rangle_X| = |\langle R_\lambda x, (\lambda - A^*)\varphi \rangle_X| \\
&\le \|R_\lambda x\|_X \|(\lambda - A^*)\varphi\|_{X^*} = \|x\|_{-1} \|\varphi\|_{1,d} \,.
\end{aligned}$$

It follows that $Jx \in (X_{1,d})^*$ and $\|Jx\|_{(X_{1,d})^*} \le \|x\|_{-1}$. To see that actually $\|Jx\|_{(X_{1,d})^*}$ equals $\|x\|_{-1}$ take $y \in X^*$ such that $\|y\|_{X^*} = 1$ and $\|R_\lambda x\|_X = |\langle R_\lambda x, y \rangle|$. Since then $R(\lambda, A^*)y \in \mathrm{D}(A^*) = X_{1,d}$ and $\|R(\lambda, A^*)y\|_{1,d} = \|(\lambda - A^*)R(\lambda, A^*)y\|_{X^*} = \|y\|_{X^*} = 1$ we obtain

$$\|x\|_{-1} = \|R_\lambda x\|_X = |\langle R_\lambda x, y \rangle| = |\langle x, R_\lambda^* y \rangle| \le \sup_{\substack{\varphi \in X_{1,d}, \\ \|\varphi\|_{1,d} \le 1}} |(Jx)(\varphi)| = \|Jx\|_{(X_{1,d})^*} \,.$$

We have shown that $J : X \to (X_{1,d})^*$ is bounded where X is equipped with $\|\,\boldsymbol{.}\,\|_{-1}$. Because X is dense in X_{-1}, the claimed bounded extension $J : X_{-1} \to (X_{1,d})^*$ of J exists and satisfies (A.5) as well as

$$\|Jx\|_{(X_{1,d})^*} = \lim_{n\to\infty} \|Jx_n\|_{(X_{1,d})^*} = \lim_{n\to\infty} \|x_n\|_{-1} = \|x\|_{-1} \quad \text{for } x \in X_{-1},$$

where (x_n) is any sequence in X with $\|x_n - x\|_{-1} \to 0$ as $n \to \infty$.

Let us now assume that X is reflexive, i.e., $J(X) = X^{**}$. We have to prove that the extension $J : X_{-1} \to (X_{1,d})^*$ is onto. It suffices to show that X^{**} is dense in $(X_{1,d})^*$.

First recall that X is reflexive if and only if X^* is reflexive. Furthermore $X_{1,d}$ is reflexive, because $\lambda - A^* \in \mathcal{L}(X_{1,d}, X^*)$ has a bounded inverse. In fact, if $j_{1,d}$ and j_* are the canonical embeddings, the following diagram is commutative.[1]

$$\begin{array}{ccc} X_{1,d} & \xrightarrow{\lambda - A^*} & X^* \\ {\scriptstyle j_{1,d}}\downarrow & & \downarrow{\scriptstyle j_*} \\ (X_{1,d})^{**} & \xleftarrow[R_\lambda^{**}]{} & X^{***} \end{array}$$

Let $\Psi \in (X_{1,d})^{**}$ be such that $\langle \mathbf{x}, \Psi \rangle = 0$ for all $\mathbf{x} \in X^{**}$. There is some $\psi \in X_{1,d}$ representing Ψ, i.e., $\Psi = j_{1,d}\,\psi$. In particular, we have

$$0 = \langle \mathbf{x}, \Psi \rangle = \langle \psi, \mathbf{x} \rangle \quad \text{for all } \mathbf{x} \in X^{**}.$$

Thus $\psi = 0$ (as an element of $X^* \supseteq X_{1,d}$) and therefore $\Psi = 0$. This shows that X^{**} is dense in $(X_{1,d})^*$. $\square$

Let $k \in \mathbb{Z}$. Since $(\lambda - A)^k \in \mathcal{L}(X_k, X)$ is an isomorphisms, the space X_k are reflexive if and only if X is reflexive. Further X is reflexive if and only if X^* is reflexive. Therefore X is reflexive if and only if $X_{k,d}$ is reflexive.

Let X be reflexive. In the same way as above we can identify $X_{-1,d}$ with X_1^*. Shifting these results from the level $k = 1$, inductively we obtain

$$X_{-k} \cong X_{k,d}^* \quad \text{and} \quad X_{-k,d} \cong X_k^*$$

for all $k \in \mathbb{N}_0$. Finally, using that $X_{k,d}$ and X_k are reflexive we obtain the identifications for every $k \in \mathbb{Z}$.

In Corollary A.5 below we will assume that the family of duals $(\mathbb{T}_t^*)_{t\geq 0}$ is again a strongly continuous semigroup, which in general is not true. In any case we write $\mathbb{T}^* := (\mathbb{T}_t^*)_{t\geq 0}$. We repeat Corollary 1.10.6 of [33] without a proof.

Proposition A.4. *Let X be reflexive and let $\mathbb{T}$ be a strongly continuous semigroup on X. Then the family of duals $\mathbb{T}^*$ is a strongly continuous semigroup on X^*.*

[1] Let $P := \lambda - A^*$, then $(P^{**})^{-1} = (P^{-1})^{**} \in \mathcal{L}(X^{***}, (X_{1,d})^{**})$. For $\psi \in X_{1,d}$ and $\mathbf{w} \in (X_{1,d})^*$ we calculate $\langle \mathbf{w}, (P^{-1})^{**} j_*(P\psi) \rangle = \langle (P^{-1})^* \mathbf{w}, j_*(P\psi) \rangle = \langle P\psi, (P^{-1})^* \mathbf{w} \rangle = \langle P^{-1}P\psi, \mathbf{w} \rangle = \langle \psi, \mathbf{w} \rangle = \langle \mathbf{w}, j_{1,d}(\psi) \rangle$. Thus $j_{1,d} = (P^{**})^{-1} \circ j_* \circ P$. Because $(P^{**})^{-1}$, j_* and P are onto, so is $j_{1,d}$.

The dual operator A^* of A is the generator of $\mathbb{T}^*$. In fact, let $y \in X^*$ be such that $\frac{1}{h}(\mathbb{T}_h^* y - y)$ converges in X^* to some z as $h \to 0$. Then for all $x \in \mathrm{D}(A)$ the identity

$$\langle x, z \rangle = \lim_{h \to 0} \langle x, \tfrac{1}{h}(\mathbb{T}_h^* y - y) \rangle = \lim_{h \to 0} \langle \tfrac{1}{h}(\mathbb{T}_h x - x), y \rangle = \langle Ax, y \rangle$$

shows that $y \in \mathrm{D}(A^*)$ and that A^* is an extension of the generator of $\mathbb{T}^*$. Since a half-plane is included in the resolvent set of both of these operators, they actually coincide.

Corollary A.5. *Let $\mathbb{T}$ be a strongly continuous semigroup on X and assume that also $\mathbb{T}^*$ is strongly continuous on X^*. For $t \geq 0$ we set $\mathbb{S}_t = \mathbb{T}_t^*|_{X_{1,d}}$. Then $\mathbb{S}_t^*$ coincides with $\mathbb{T}_t$ on X_{-1} via*

$$J\mathbb{T}_t x = \mathbb{S}_t^* J x \qquad \textit{for } x \in X_{-1},$$

where $\mathbb{S}_t^*$ is the dual of $\mathbb{S}_t$ with respect to $\| \cdot \|_{1,d}$ and J is from Proposition A.3.

Proof. Let $x \in X$ and $\varphi \in X_{1,d}$. In view of the calculation

$$\begin{aligned} \langle \varphi, \mathbb{S}_t^* J x \rangle_{X_{1,d}} &= \langle \mathbb{S}_t \varphi, J x \rangle_{X_{1,d}} = \langle x, \mathbb{S}_t \varphi \rangle_X \\ &= \langle \mathbb{T}_t x, \varphi \rangle_X = \langle \varphi, J\mathbb{T}_t x \rangle_{X_{1,d}}, \end{aligned}$$

the assertion follows from the density of X in X_{-1} as well as the boundedness of the operators $\mathbb{T}_t \in \mathcal{L}(X_{-1})$, $J \in \mathcal{L}(X_{-1}, (X_{1,d})^*)$ and $\mathbb{S}_t^* \in \mathcal{L}((X_{1,d})^*)$. □

The intermediate space $X_{1/2}$

Now let X be a Hilbert space with inner product $(\cdot \mid \cdot)$. We still assume that A is a densely defined linear operator in X. Note that in this case one usually defines the graph norm via

$$\|x\|_{A,2}^2 = \|x\|_X^2 + \|Ax\|_X^2.$$

since then $(\mathrm{D}(A), \| \cdot \|_{A,2})$ is an inner product space too. However, $\| \cdot \|_{A,2}$ is equivalent to $\| \cdot \|_A$ and thus all statements in this chapter are also valid for $\| \cdot \|_{A,2}$. Recall that the relation

$$A' = \{(y, z) \in X \times X \mid \forall x \in \mathrm{D}(A) : \ (Ax \mid y) = (x \mid z)\}$$

defines an operator A' in X, the *adjoint of* A. We mention that A' is closed. The following well-known statement will be needed. For a proof see e.g. Proposition 2.8.4 in [49].

Lemma A.6. *Let B be a densely defined closed operator in X. For all $\lambda \in \rho(B)$ we have $\overline{\lambda} \in \rho(B')$ as well as*

$$(\overline{\lambda} - B')^{-1} = \big[(\lambda - B)^{-1}\big]'.$$

Assume that B is a *self-adjoint* operator in X, which means that it coincides with its adjoint, i.e., $B = B'$. The last lemma yields $\sigma(B) = \rho(B)^{\mathrm{c}} \subseteq \mathbb{R}$. Moreover, if $0 \in \rho(B)$ then B^{-1} is self-adjoint since then

$$(B^{-1})' = (B')^{-1} = B^{-1}.$$

A (densely defined) operator B in X is called *symmetric* if we have $B \subseteq B'$. In the following lemma we give a simple condition under which this weaker property implies self-adjointness. This is a special case of Proposition 3.2.4 in [49].

Lemma A.7. *Let B be a densely defined symmetric operator in X. Further assume that B is onto. Then $0 \in \rho(B)$ and B is self-adjoint.*

The space $X_{1/2}$ is defined as the domain of the 'square root' $A^{1/2}$ of A. In order to construct the operator $A^{1/2}$, it is necessary that A has the following property.

Definition A.8. A self-adjoint operator B in X is called *positive* if we have $(Bx \mid x) \geq 0$ for all $x \in \mathrm{D}(B)$. We then write $B \geq 0$. If there even is a number $m > 0$ with

$$(Bx \mid x) \geq m\|x\|_X^2 \quad \text{for all } x \in \mathrm{D}(B)$$

we say that B is *strictly positive* and we write $B > 0$.

It is possible to characterize the (strictly) positive operators among the self-adjoint in terms of the spectrum. We combined Proposition 3.3.3 and Remark 3.3.4 of [49].

Lemma A.9. *A self-adjoint operator B in X is positive if and only if $\sigma(B) \subseteq [0, \infty)$. It is strictly positive if and only if $\sigma(B) \subseteq (0, \infty)$.*

The construction of the square root of a strictly positive unbounded operator is based on the existence of the square root of a bounded positive operator. Without a proof we repeat Theorem 12.3.4 of [49].

Theorem A.10. *Let $T \in \mathcal{L}(X)$ be positive. Then there is exactly one positive operator $S \in \mathcal{L}(X)$ with $S^2 = T$.*

We write $T^{1/2} := S$ and call $T^{1/2}$ the *square root* of T. Note that for $x \in X$ with $T^{1/2}x = 0$ we also have $Tx = T^{1/2}T^{1/2}x = T^{1/2}0 = 0$ and hence $\ker(T^{1/2}) \subseteq \ker T$. Very similarly one sees that $\operatorname{Ran}(T^{1/2}) \supseteq \operatorname{Ran} T$.

Proposition A.11. *Let A be a strictly positive operator in X. Then there is exactly one strictly positive operator $A^{1/2}$ in X with $(A^{1/2})^2 = A$.*

Proof. Lemma A.9 yields that $0 \in \rho(A)$. We saw that $A^{-1} \in \mathcal{L}(X)$ is self-adjoint. For all $y \in X$ there is a vector $x \in \mathrm{D}(A)$ with $Ax = y$. We obtain the inequality

$$(A^{-1}y \mid y) = (A^{-1}Ax \mid Ax) = (x \mid Ax) = (Ax \mid x) \geq 0$$

showing that A^{-1} is positive. Theorem A.10 yields the square root $A^{-1/2} := (A^{-1})^{1/2} \in \mathcal{L}(X)$ of A^{-1}. From the inclusion

$$\ker(A^{-1/2}) \subseteq \ker(A^{-1}) = \{0\}$$

we derive that $A^{-1/2}$ has a (probably unbounded) inverse

$$A^{1/2} := (A^{-1/2})^{-1} : \operatorname{Ran}(A^{-1/2}) \to X.$$

Note that $A^{1/2}$ is onto. Further we have $\mathrm{D}(A) = \operatorname{Ran}(A^{-1}) \subseteq \operatorname{Ran}(A^{-1/2}) = \mathrm{D}(A^{1/2})$. Next we prove that

$$\mathrm{D}(A) = \mathrm{D}((A^{1/2})^2) := \{x \in X \mid x \in \mathrm{D}(A^{1/2}) \text{ and } A^{1/2}x \in \mathrm{D}(A^{1/2})\}.$$

Let $x \in \mathrm{D}(A)$. Then for $w := Ax \in X$ we have $x = A^{-1}w = A^{-1/2}A^{-1/2}w$ and thus

$$A^{1/2}x = A^{1/2}A^{-1/2}A^{-1/2}w = A^{-1/2}w \in \operatorname{Ran}(A^{-1/2}) = \mathrm{D}(A^{1/2}).$$

This fact also implies that $A^{1/2}A^{1/2}x = w = Ax$ and hence $A \subseteq (A^{1/2})^2$. Conversely let $x \in \mathrm{D}((A^{1/2})^2)$. Since then $A^{1/2}x$ belongs to $\mathrm{D}(A^{1/2}) = \mathrm{Ran}(A^{-1/2})$ we find some $w \in X$ with $A^{1/2}x = A^{-1/2}w$. Applying $A^{-1/2}$ to this equation yields

$$x = A^{-1/2}A^{-1/2}w = A^{-1}w \in \mathrm{Ran}(A^{-1}) = \mathrm{D}(A).$$

We have thus shown that $\mathrm{D}(A)$ and $\mathrm{D}((A^{1/2})^2)$ coincide.

To see that $A^{1/2}$ is symmetric, we take $x_1, x_2 \in \mathrm{D}(A^{1/2})$ and set $w_j := A^{1/2}x_j$. Then we have $x_j = A^{-1/2}w_j$ where $j \in \{1, 2\}$. Using that $A^{-1/2}$ is self-adjoint we deduce

$$(A^{1/2}x_1 \,|\, x_2) = (w_1 \,|\, A^{-1/2}w_2) = (A^{-1/2}w_1 \,|\, w_2) = (x_1 \,|\, A^{1/2}x_2).$$

As mentioned earlier, $A^{1/2}$ is onto. Therefore by Lemma A.9 the square root $A^{1/2}$ is self-adjoint and 0 lies in $\rho(A^{1/2})$. Taking $x_1 = x_2 = x$ in the last equation yields

$$(A^{1/2}x \,|\, x) = (A^{-1/2}w_1 \,|\, w_1) \geq 0.$$

This means that $A^{1/2}$ is positive. Since $0 \in \rho(A^{1/2})$, with Lemma A.9 we infer that $A^{1/2}$ is even strictly positive.

It remains to show the uniqueness of $A^{1/2}$. Assume B is another positive and particularly closed operator in X with $B^2 = A$. As above, from the facts that

$$\ker B \subseteq \ker A = \{0\} \qquad \text{and} \qquad \mathrm{Ran}\, B \supseteq \mathrm{Ran}\, A = X$$

it follows that 0 is contained in $\rho(B)$. Consequently B must be strictly positive and the inverse $B^{-1} \in \mathcal{L}(X)$ is positive. For $x \in X$ we further derive

$$B^{-1}B^{-1}x = B^{-1}B^{-1}AA^{-1}x = B^{-1}B^{-1}BBA^{-1}x = A^{-1}x.$$

Thus B^{-1} is the square root of A^{-1}. Due to uniqueness, we have $B^{-1} = A^{-1/2}$ and consequently also $B = A^{1/2}$. □

Let A be strictly positive. As in the bounded case, the operator $A^{1/2}$ constructed in the preceding theorem is called the *square root of* A. Note that due to the self-adjointness, A is densely defined and closed. Moreover, since A is strictly positive, at least $\mathbb{C} \setminus (0, \infty)$ belongs to its resolvent set $\rho(A)$. Hence we may apply the theory developed in the first part of the chapter. Let us choose $\lambda = 0$ so that

$$\|x\|_1 = \|Ax\|_X \quad \text{for } x \in \mathrm{D}(A).$$

Still X_1 denotes $\mathrm{D}(A)$ endowed with $\|\,\cdot\,\|_1$. It is obvious, that $\|\,\cdot\,\|_1$ is induced by the inner product defined via

$$(x \,|\, y)_1 = (Ax \,|\, Ay) \quad \text{for } x, y \in \mathrm{D}(A).$$

Also the square root $A^{1/2}$ is strictly positive. Thus we can construct corresponding interpolation and extrapolation spaces. We discuss in detail what that means. First,

$$\|x\|_{1/2} := \|A^{1/2}x\|_X \quad \text{for } x \in \mathrm{D}(A^{1/2})$$

defines a norm on $\mathrm{D}(A^{1/2})$ corresponding to the scalar product $(\,\cdot\,|\,\cdot\,)_{1/2}$ on $\mathrm{D}(A^{1/2})$ given by

$$(x \,|\, y)_{1/2} = (A^{1/2}x \,|\, A^{1/2}y) \quad \text{for } x, y \in \mathrm{D}(A^{1/2}).$$

We use the symbol $X_{1/2}$ for $\mathrm{D}(A^{1/2})$ equipped with $\|\,\cdot\,\|_{1/2}$. Hence $X_{1/2}$ is a Hilbert space, continuously embedded in X. On the other hand, it is dense in X since $\mathrm{D}(A) \subseteq X_{1/2}$.

Successively, for $m \in \mathbb{N}$ we obtain the spaces $X_{m/2}$ as $\mathrm{D}((A^{1/2})^m)$ endowed with the norm $\|\,\cdot\,\|_{m/2}$ given by

$$\|x\|_{m/2} := \|A^{1/2}x\|_{(m-1)/2} = \|(A^{1/2})^m x\|_X.$$

It becomes clear that $X_{2k/2} = X_k$. As a consequence, X is dense in $X_{1/2}$ with continuous embedding. We know that

$$A^{1/2} \in \mathcal{L}(X_{1/2}, X) \qquad \text{and} \qquad A^{1/2}|_1 := A^{1/2}|_X \in \mathcal{L}(X_1, X_{1/2}).$$

These operators are actually isometric isomorphisms.

Using the inverse $A^{-1/2} \in \mathcal{L}(X)$ we also define $X_{-1/2}$ as the completion of X with respect to the norm given by

$$\|x\|_{-1/2} := \|A^{-1/2}x\|_X \quad \text{for } x \in X.$$

As before we see that $X_{-1/2}$ is a Hilbert space and that X is dense in $X_{-1/2}$ with continuous embedding. Moreover, $A^{1/2}$ has a bounded extension

$$A^{1/2}|_0 \in \mathcal{L}(X, X_{-1/2}).$$

In fact, the latter is an isometric isomorphism.

The inverse is $A^{-1/2}|_{-1/2} \in \mathcal{L}(X_{-1/2}, X)$ the extension of $A^{-1/2}$ to $X_{-1/2}$. Inductively we define $X_{-m/2}$ as the completion of $X_{-(m-1)/2}$ with respect to the norm given by $\|x\|_{-m/2} = \|A^{-1/2}|_{-(m-1)/2}x\|_{-(m+1)/2}$. As above, we infer that $X_{-2k/2} = X_{-k}$ for $k \in \mathbb{N}$. Again, we shall sometimes write $[\mathrm{D}(A)]_{m/2}$ for $X_{m/2} = [\mathrm{D}(A^{1/2})]_m$, where $m \in \mathbb{Z}$. Summing things up we have the chain of continuous embeddings

$$X_1 \hookrightarrow X_{1/2} \hookrightarrow X \hookrightarrow X_{-1/2} \hookrightarrow X_{-1}.$$

Each of these space is dense in the subsequent.

There is a way to define $X_{1/2}$ and $X_{-1/2}$ without referencing $A^{1/2}$. Nevertheless, we still need that A is strictly positive. Since $A^{1/2}$ is self-adjoint, for $x \in \mathrm{D}(A)$ we have

$$\|x\|_{1/2} = \sqrt{(A^{1/2}x \mid A^{1/2}x)} = \sqrt{(Ax \mid x)}. \tag{A.6}$$

Thus $X_{1/2}$ is the completion of $\mathrm{D}(A)$ with respect to this norm. We can also take this property as the definition of $X_{1/2}$. Similarly, we have $\|x\|_{-1/2} = \sqrt{(A^{-1}x \mid x)}$ for $x \in X$.

This facts at hand it is easy to verify that A has continuous extension to $X_{1/2}$. Indeed, for $x \in \mathrm{D}(A)$ we have $Ax \in X$, and the claim follows from

$$\|Ax\|_{-1/2} = \sqrt{(A^{-1}Ax \mid Ax)} = \sqrt{(x \mid Ax)} = \sqrt{(Ax \mid x)} = \|x\|_{1/2}.$$

Let us denote this extension (which actually is an isometric isomorphism) by

$$A|_{1/2} \in \mathcal{L}(X_{1/2}, X_{-1/2}). \tag{A.7}$$

Remark A.12. In Proposition A.3 applied to $A^{1/2}$ we consider $([\mathrm{D}((A^{1/2})^*)]_1)^*$. Luckily we can forget about this nasty notation, because $A^{1/2}$ is self-adjoint and hence $([\mathrm{D}((A^{1/2})^*)]_1)^* = ([\mathrm{D}(A^{1/2})]_1)^* = X_{1/2}^*$. Since as a Hilbert spaces X is reflexive, we obtain an isometric isomorphism between $X_{-1/2}$ and $X_{1/2}^*$. ◊

Appendix B

Laplace transforms

In this section we recall the definition of the Laplace transform of vector-valued functions. We also repeat some of its main properties. For an extensive treatment of this topic see [5].

We assume that the reader is familiar with the theory of the Lebesgue-Bochner integral as it is introduced e.g. in [25]. However we shortly repeat some results on antiderivatives. Throughout, let V be a Banach space over the field $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $J \subseteq \mathbb{R}$ be an interval and $f : J \to V$ be a locally integrable function.

Antiderivatives and integration by parts

The classical rule of integration by parts is based on the fundamental theorem of calculus. In this subsection we repeat generalizations of these results. The following notion plays a crucial role. A function $F : J \to V$ is called *antiderivative of f*, if

$$F(t) = F(t_0) + \int_{t_0}^{t} f(s)\,\mathrm{d}s \quad \text{for all } t_0, t \in J.$$

In other words: the formula in the fundamental theorem of calculus holds for F and f.

Recall that a function $G : J \to V$ is called *absolutely continuous on* $[a, b] \subseteq J$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall m \in \mathbb{N},\ a \le a_1 \le b_1 \le \ldots \le a_m \le b_m \le b :$$
$$\sum_{k=1}^{m} (b_k - a_k) \le \delta \quad \Longrightarrow \quad \sum_{k=1}^{m} \|G(b_k) - G(a_k)\| \le \varepsilon.$$

The function G is called *absolutely continuous*, if it is absolutely continuous on every compact subinterval of J. Clearly absolutely continuous functions are continuous (take $m = 1$).

A scalar function is an antiderivative of a locally integrable function if and only if it is absolutely continuous, see Theorem 7.20 in [37]. Example 1.2.8 in [5] shows that this is not true for vector-valued functions. The validity depends on a property of V called the “Radon-Nikodym property”. We don’t go into details. Nevertheless we have the somewhat weaker statement Proposition B.4. We only comment on the proofs, because they would go beyond the scope of this text.

Lemma B.1. *Let $F : J \to V$ be an antiderivative of f. Then F is absolutely continuous.*

The proof is quite easy using the following fact. If $[a,b] \subseteq J$ is a compact subinterval, then $\|f\| : [a,b] \to \mathbb{R}$ is integrable and thus for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_M \|f(s)\| \, \mathrm{d}s \leq \varepsilon$ for all measurable sets $M \subseteq [a,b]$ with $|M| \leq \delta$.

Antiderivatives are differentiable almost everywhere on J with derivative f. This follows from Lebesgue's differentiation theorem repeated below. A point t in the domain of f is called *right-Lebesgue point of f* if

$$\tfrac{1}{\delta}\int_t^{t+\delta} \|f(s) - f(t)\| \, \mathrm{d}s \to 0 \qquad \text{as } \delta \to 0^+.$$

Similarly *left-Lebesgue points* are defined by integrating from $t - \delta$ to t. The point t is called *Lebesgue point of f* if it is a left- and a right-Lebesgue point of f. This is the case if and only if

$$\tfrac{1}{2\delta}\int_{t-\delta}^{t+\delta} \|f(s) - f(t)\| \, \mathrm{d}s = \tfrac{1}{2\delta}\int_t^{t+\delta} \|f(s) - f(t)\| \, \mathrm{d}s + \tfrac{1}{2\delta}\int_{t-\delta}^{t} \|f(s) - f(t)\| \, \mathrm{d}s \to 0 \qquad \text{as } \delta \to 0^+.$$

Using uniform continuity on compact sets we easily deduce that for continuous f every point of J is a Lebesgue point. This is part of the fundamental theorem of calculus. Astonishingly, for arbitrary locally integrable f almost the same holds. This is the claim of Lebesgue's differentiation theorem:

Theorem B.2 (Lebesgue)**.** *Let $f : J \to V$ be locally integrable. Then for almost all $t \in J$ we have*

$$\tfrac{1}{\delta}\int_t^{t+\delta} \|f(s) - f(t)\| \, \mathrm{d}s \to 0 \quad \textit{as } \delta \to 0.$$

In particular f has a right-Lebesgue point at almost every $t \in J$.

The addendum simply follows from the estimate

$$\left\| \tfrac{1}{\delta}\int_t^{t+\delta} f(s) \, \mathrm{d}s - f(t) \right\| = \left\| \tfrac{1}{\delta}\int_t^{t+\delta} (f(s) - f(t)) \, \mathrm{d}s \right\| \leq \tfrac{1}{\delta}\int_t^{t+\delta} \|f(s) - f(t)\| \, \mathrm{d}s$$

valid for all $t \in J$ and $\delta \in \mathbb{R} \setminus \{0\}$ such that $t + \delta \in J$. The rest is based on the scalar version of this result, see Theorem 1.4 in Chapter 3 of [46].

Note that $t \in J$ is a Lebesgue point of f if and only if some antiderivative of f is differentiable at t and its derivative is $f(t)$. As immediate consequence we obtain the following.

Corollary B.3. *Let $F : J \to V$ be an antiderivative of f. Then F is differentiable almost everywhere on J with derivative f.*

As already remarked, not every absolutely continuous function with values in V is an antiderivative unless V has the "Radon-Nikodym property". The following weaker implication is correct for every Banach space V.

Proposition B.4. *Let $G : J \to V$ be absolutely continuous and differentiable almost everywhere on J. Denote its derivative[1] by $G' : J \to V$. Then G' is locally integrable and G is an antiderivative of G'.*

[1] If G is not differentiable in $t \in J$, then $G'(t)$ can be defined arbitrarily

Since this is not easy as 1-2-3, we refer to Proposition 1.2.3 of [5]. The purpose of the next lemma is to prepare a technical detail in the proof of the rule of integration by parts.

Lemma B.5. *Let $F : J \to V$ and $g : J \to \mathbb{K}$ be absolutely continuous. Then $Fg : J \to V$ is absolutely continuous.*

Proof. Recall that absolutely continuous functions are continuous and therefore bounded on compact sets. Let $[a, b] \subseteq J$ be compact and set $m_F = \max_{t\in[a,b]}\|F(t)\|$, $m_g = \max_{t\in[a,b]}|g(t)|$. For $m \in \mathbb{N}$ and points $a \le a_1 \le b_1 \le \ldots \le a_m \le b_m \le b$ we then estimate

$$\begin{aligned}\sum_{k=1}^{m}\|g(b_k)F(b_k) - g(a_k)F(a_k)\| &\le \sum_{k=1}^{m}|g(b_k)|\,\|F(b_k) - F(a_k)\| + \sum_{k=1}^{m}\|F(b_k)\|\,|g(b_k) - g(a_k)| \\ &\le m_g \sum_{k=1}^{m}\|F(b_k) - F(a_k)\| + m_F \sum_{k=1}^{m}|g(b_k) - g(a_k)|.\end{aligned}$$

Let $\varepsilon > 0$. Using the assumptions, we can now choose a number $\delta > 0$ such that the inequality $\sum_{k=1}^{m}(b_k - a_k) \le \delta$ implies that the right-hand side of the above displayed estimate is less or equal ε. □

The rule of integration by parts is an essential tool for dealing with integrals over antiderivatives.

Proposition B.6 (integration by parts)**.** *Let $f : J \to V$ be locally integrable and $g : J \to \mathbb{K}$ absolutely continuous and differentiable almost everywhere. Moreover, let $F : J \to V$ be an antiderivative of f. Then gf as well as $g'F$ are locally integrable and for all $t_0, t \in J$ we have*

$$\int_{t_0}^{t} g(s)f(s)\,\mathrm{d}s = g(t)F(t) - g(t_0)F(t_0) - \int_{t_0}^{t} g'(s)F(s)\,\mathrm{d}s,$$

where $g' : J \to \mathbb{K}$ is the derivative of g.

Proof. The last lemma yields that the function $gF : J \to V$ is absolutely continuous. With the product rule we infer, that gF is differentiable almost everywhere with derivative $gf + g'F$. By Proposition B.4 the latter function is locally integrable and gF is one of its antiderivatives. In particular we have

$$g(t)F(t) = g(t_0)F(t_0) + \int_{t_0}^{t}(g(s)f(s) + g'(s)F(s))\,\mathrm{d}s \quad \text{for all } t_0, t \in J.$$

Proposition B.4 further yields, that g' is locally integrable. Because g and F are continuous, these functions are locally bounded. Hölder's inequality therefore implies that gf and $g'F$ are locally integrable. Thus we may split the integral in the last equality and reorganize it to obtain the claim. □

Definition and properties of the Laplace transform

In this section we consider locally integrable functions of the type $f : [0, \infty) \to V$.

The scalar function $e^{-\lambda(\cdot)}$ belongs to $L^\infty([0,N],\mathbb{K})$ for all $\lambda \in \mathbb{K}$ and $N > 0$. Hölder's inequality implies that $[0,N] \to V; t \mapsto e^{-\lambda t} f(t)$ is integrable. In case the family of integrals $\left(\int_0^N e^{-\lambda t} f(t)\,dt\right)$ has a limit

$$\widehat{f}(\lambda) := \lim_{N\to\infty} \int_0^N e^{-\lambda t} f(t)\,dt$$

in V, then $\widehat{f}(\lambda)$ is called the *Laplace transform of f at λ*. The *domain of convergence of $\widehat{f}$* is the set of all $\lambda \in \mathbb{K}$ for which $\widehat{f}(\lambda)$ exists. We say that f is *Laplace transformable* if this set is nonempty.

In the most comfortable situation the function $[0,\infty) \to V; t \mapsto e^{-\lambda t} f(t)$ is integrable for some $\lambda \in \mathbb{K}$. By means of the dominated convergence theorem we then infer that $\widehat{f}(\lambda)$ exists and equals the Bochner integral

$$\int_0^\infty e^{-\lambda t} f(t)\,dt.$$

We say that $\widehat{f}(\lambda)$ *converges absolutely.*

Due to Hölder's inequality, this is the case for all $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > 0$ if $f \in L^p([0,\infty),V)$ for some $p \in [1,\infty]$, because then $e^{-\lambda(\cdot)}$ lies in $L^{p'}([0,\infty),\mathbb{K})$. As usual $p' \in [1,\infty]$ is the dual exponent of p. If $f \in L^1([0,\infty),V)$, then $\widehat{f}(\lambda)$ converges absolutely whenever $\operatorname{Re}\lambda \geq 0$. Of course $\widehat{f}(\lambda)$ converges absolutely for all $\lambda \in \mathbb{K}$ if f has support in some compact interval $[0,T]$.

The most important condition for the existence of $\widehat{f}(\lambda)$, is 'exponential boundedness': Assume there are $\omega \in \mathbb{R}$ and $M \geq 0$ such that $\|f(t)\| \leq M e^{\omega t}$ for almost all $t \in [0,\infty)$. The dominated convergence theorem then yields that $\widehat{f}(\lambda)$ converges absolutely for every $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > w$. We define the *exponential growth bound of f* as

$$\omega_0(f) = \inf\{\omega \in \mathbb{R} \mid \exists M \geq 0\ \forall t \geq 0:\ \|f(t)\| \leq M e^{\omega t}\},$$

where by convention $\inf \emptyset = \infty$ and $\inf \mathbb{R} = -\infty$. See Example 1.4.4 in [5] for a function f with $\omega_0(f) = \infty$ and nonempty domain of convergence.

Lemma B.7. *Assume that $\widehat{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{K}$. Then $\widehat{f}(\lambda)$ exists for every $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$.*

In preparation of the proof, for $\mu \in \mathbb{K}$ let $G_\mu : [0,\infty) \to V$ be the antiderivative of $e^{-\mu(\cdot)} f$ with $G_\mu(0) = 0$. That is

$$G_\mu(t) = \int_0^t e^{-\mu s} f(s)\,ds \quad \text{for } t \geq 0.$$

As an antiderivative, G_μ is continuous. If $\widehat{f}(\mu)$ exists, then G_μ has the limit $\widehat{f}(\mu) = \lim_{N\to\infty} G_\mu(N)$, in particular G_μ is then bounded.

Proof. Let $\lambda \in \mathbb{K}$, $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$. Then for arbitrary $N > 0$ integration by parts yields

$$\begin{aligned}\int_0^N e^{-\lambda t} f(t)\,dt &= \int_0^N e^{-(\lambda-\lambda_0)t} e^{-\lambda_0 t} f(t)\,dt \\ &= e^{-(\lambda-\lambda_0)N} G_{\lambda_0}(N) + \int_0^N (\lambda-\lambda_0) e^{-(\lambda-\lambda_0)t} G_{\lambda_0}(t)\,dt.\end{aligned}$$

Because we assumed that $\widehat{f}(\lambda_0)$ exists, the antiderivative G_{λ_0} is bounded. On the other hand $\mathrm{e}^{-(\lambda-\lambda_0)(\cdot)}$ belongs to $L^1([0,\infty),\mathbb{K})$, since $\operatorname{Re}(\lambda-\lambda_0) = \operatorname{Re}\lambda - \operatorname{Re}\lambda_0 < 0$. Using Hölder's inequality and the dominated convergence theorem, we conclude that the limit of the above integrals as $N \to \infty$ exists and equals

$$\widehat{f}(\lambda) = (\lambda-\lambda_0)\int_0^\infty \mathrm{e}^{-(\lambda-\lambda_0)t} G_{\lambda_0}(t)\,\mathrm{d}t.$$

This completes the proof. □

We can now deduce that the domain of convergence is an open right half-plane together with some (possibly empty) subset of its boundary. Let

$$\operatorname{abs}(f) = \inf\{\operatorname{Re}\lambda \mid \lambda \in \mathbb{K} : \widehat{f}(\lambda) \text{ exists}\},$$

where we put $\inf\emptyset = \infty$ and $\inf\mathbb{R} = -\infty$. This quantity is called the *abscissa of convergence of* f. Assume that $\operatorname{abs}(f) \in \mathbb{R}$. Then clearly $\widehat{f}(\mu)$ does *not* exist if $\operatorname{Re}\mu < \operatorname{abs}(f)$. In turn, by Lemma B.7, the Laplace transform of f at λ exists for all $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \operatorname{abs}(f)$.

Operational properties

We are interested in the behavior of the Laplace transform under certain operations. Linearity of integral and limit imply that the Laplace transform is linear. To be more precise, let $f, g \in L^1_{\mathrm{loc}}([0,\infty),V)$ and $\alpha,\beta \in \mathbb{K}$. If $\widehat{f}(\lambda)$ and $\widehat{g}(\lambda)$ both exist for some $\lambda \in \mathbb{K}$, then also $(\alpha f + \beta g)\widehat{\ }(\lambda)$ exists and equals $\alpha\widehat{f}(\lambda) + \beta\widehat{g}(\lambda)$.

Let $\tau \geq 0$. Recall that left shift $S_\tau^* g \in V^{[0,\infty)}$ and right shift $S_\tau g \in V^{[0,\infty)}$ of some $g \in V^{[0,\infty)}$ where given by $(S_\tau^* g)(t) = g(t+\tau)$ for $t \geq 0$ as well as $(S_\tau g)(t) = 0$ for $t \in [0,\tau)$ and $(S_\tau g)(t) = g(t-\tau)$ for $t \in [\tau,\infty)$. Obviously S_τ^* and S_τ map $L^1_{\mathrm{loc}}([0,\infty),V)$ to itself.

Lemma B.8. *Let $f \in L^1_{\mathrm{loc}}([0,\infty),V)$ and $\tau \geq 0$. Then for all $\lambda \in \mathbb{K}$ the Laplace transform $\widehat{f}(\lambda)$ exists if and only if $(S_\tau^* f)\widehat{\ }(\lambda)$ exists. If this is the case, then*

$$(S_\tau^* f)\widehat{\ }(\lambda) = \mathrm{e}^{\lambda\tau}\widehat{f}(\lambda) - \mathrm{e}^{\lambda\tau}\int_0^\tau \mathrm{e}^{\lambda s} f(s)\,\mathrm{d}s. \tag{B.1}$$

Proof. Let $\lambda \in \mathbb{K}$ and $N > \tau$. A change of variables yields

$$\begin{aligned}\int_0^N \mathrm{e}^{-\lambda t} S_\tau^* f(t)\,\mathrm{d}t &= \int_0^N \mathrm{e}^{\lambda\tau}\mathrm{e}^{-\lambda(t+\tau)} f(t+\tau)\,\mathrm{d}t = \mathrm{e}^{\lambda\tau}\int_\tau^{N+\tau} \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s \\ &= \mathrm{e}^{\lambda\tau}\int_0^{N+\tau} \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s - \mathrm{e}^{\lambda\tau}\int_0^\tau \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s\,.\end{aligned}$$

The left-hand side has a limit as $N \to \infty$ if and only if the right-hand side converges. This fact shows the claim and also proves the given formula for $(S_\tau^* f)\widehat{\ }(\lambda)$. □

In case $\widehat{f}(\lambda)$ exists for some $\lambda \in \mathbb{K}$, by reorganizing (B.1) we obtain

$$\widehat{f}(\lambda) = \mathrm{e}^{-\lambda\tau}(S_\tau^* f)\widehat{\ }(\lambda) + \int_0^\tau \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s \quad \text{for all } \tau \geq 0.$$

Weighted Lebesgue spaces and absolute convergence of the Laplace transform

By definition the Laplace transform $\widehat{f}(\lambda)$ converges absolutely if and only if f belongs to a 'weighted L^1–space'. More generally, for $p \in [1,\infty)$ we set

$$L^p_\lambda([0,\infty),V) := \left\{ f \in L^1_{\mathrm{loc}}([0,\infty),V) \,\middle|\, \mathrm{e}^{-\lambda(\cdot)} f \in L^p([0,\infty),V) \right\}.$$

Clearly $L^p_\lambda([0,\infty),V)$ is a vector space and $\|f\|_{L^p_\lambda} := \|\mathrm{e}^{-\lambda(\cdot)} f\|_{L^p}$ defines a norm $\|\cdot\|_{L^p_\lambda}$ on it. It is further easy to see that $L^p_\lambda([0,\infty),V)$ equipped with this norm is complete. Bochner's theorem implies that $L^p_\lambda([0,\infty),V) = L^p_{\operatorname{Re}\lambda}([0,\infty),V)$. Hence it suffices to consider weights $\mu \in \mathbb{R}$. Hölder's inequality yields that the spaces are ordered as follows. For $\mu_1,\mu_2 \in \mathbb{R}$ with $\mu_1 \le \mu_2$ we have

$$L^p_{\mu_1}([0,\infty),V) \subseteq L^p_{\mu_2}([0,\infty),V) \qquad \text{and} \qquad \|f\|_{L^p_{\mu_2}} \le \|f\|_{L^p_{\mu_1}} \quad \text{for all } f \in L^p_{\mu_1}([0,\infty),V).$$

(Compare the case $p = 1$ to Lemma B.7.) Let $f \in L^p_\mu([0,\infty),V)$ and $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \mu$. Then $\widehat{f}(\lambda)$ converges absolutely and we obtain the estimate

$$\begin{aligned}\|\widehat{f}(\lambda)\|_V &\le \int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda t} \|f(t)\|_V \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(\operatorname{Re}\lambda - \mu)t} \mathrm{e}^{-\mu t} \|f(t)\|_V \,\mathrm{d}t \\ &\le \|\mathrm{e}^{-(\operatorname{Re}\lambda-\mu)(\cdot)}\|_{L^{p'}([0,\infty),\mathbb{R})} \|f\|_{L^p_\mu}.\end{aligned}$$

If $p = 1$ we might as well take $\operatorname{Re}\lambda \ge \mu$. Then we have $\|\widehat{f}(\lambda)\|_V \le \|f\|_{L^1_{\operatorname{Re}\lambda}} \le \|f\|_{L^1_\mu}$. At the beginning of this chapter we argued that $L^p([0,\infty),V) \subseteq L^1_\mu([0,\infty),V)$ for all $\mu > 0$. Now we have seen that $L^p([0,\infty),V) = L^p_0([0,\infty),V) \subseteq L^p_\mu([0,\infty),V)$ for $\mu \ge 0$.

Lemma B.9. *Let $f \in L^1_{\mathrm{loc}}([0,\infty),V)$ and $\mu \in \mathbb{R}$. Assume that the sequence of the norms $\|P_n f\|_{L^p_\mu}$ is bounded, i.e., $\|P_n f\|_{L^p_\mu} \le c$ for some $c \ge 0$ and all $n \in \mathbb{N}$, Then f belongs to $L^p_\mu([0,\infty),V)$ and we have $\|f\|_{L^1_\mu} \le c$.*

Proof. For simplicity we write $\|g\|_V$ for the function $[0,\infty) \to \mathbb{R}$; $t \mapsto \|g(t)\|_V$, where g is a map from $[0,\infty)$ to V.

Obviously the functions $P_n(\mathrm{e}^{-\mu(\cdot)} f) = \mathrm{e}^{-\mu(\cdot)} P_n f$ converge to $\mathrm{e}^{-\mu(\cdot)} f$ pointwise almost everywhere on $[0,\infty)$ as $n \to \infty$. Hence also $P_n \|\mathrm{e}^{-\mu(\cdot)} f\|_V^p \to \|\mathrm{e}^{-\mu(\cdot)} f\|_V^p$ as $n \to \infty$ pointwise for almost everywhere on $[0,\infty)$. The assumption further implies that

$$\left\| P_n \|\mathrm{e}^{-\mu(\cdot)} f\|_V^p \right\|_{L^1([0,\infty),\mathbb{R})} = \|P_n f\|_{L^p_\mu} \le c^p \quad \text{for all } n \in \mathbb{N}.$$

Corollary VI.5.10 in [25] yields that $\|\mathrm{e}^{-\mu(\cdot)} f\|_V^p$ belongs to $L^1([0,\infty),V)$ which in turn means that $f \in L^p_\mu([0,\infty),V)$. The last claim follows from the fact that $\|\|\mathrm{e}^{-\mu(\cdot)} f\|_V^p\|_{L^1} = \|f\|^p_{L^p_\mu}$ □

Laplace transforms of antiderivatives

As a continuous function an antiderivative F of f is locally integrable. Its exponential growth bound is closely related to the abscissa of convergence $\operatorname{abs}(f)$.

Preceding the proof of Lemma B.7, we defined G_λ as the antiderivative of $\mathrm{e}^{-\lambda(\cdot)} f$ with $G_\lambda(0) = 0$. For convenience set $F_0 = G_0$, i.e.,

$$F_0(t) = \int_0^t f(s)\,\mathrm{d}s \quad \text{for } t \ge 0.$$

Also recall that G_λ is bounded if $\widehat{f}(\lambda)$ exists. In this case let

$$C_\lambda = \sup_{t\in[0,\infty)} \|G_\lambda(t)\|.$$

Lemma B.10. *Assume that* $\omega_0(F_0) < \infty$. *Then* $\operatorname{abs}(f) \le \omega_0(F_0)$. *If this is the case, then*

$$\widehat{f}(\lambda) = \lambda\widehat{F_0}(\lambda) \quad \textit{for all } \lambda \in \mathbb{K} \textit{ with } \operatorname{Re}\lambda > \omega_0(F_0).$$

Proof. Let $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \omega_0(F_0)$. Then for every $N > 0$, integration by parts gives

$$\int_0^N \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t = \mathrm{e}^{-\lambda N} F_0(N) + \int_0^N \lambda \mathrm{e}^{-\lambda t} F_0(t)\,\mathrm{d}t.$$

Because $\widehat{F_0}(\lambda)$ exists and $\mathrm{e}^{-\lambda N}F_0(N) \to 0$ due to the exponential boundedness of F_0, the left-hand side converges as $N \to \infty$. Hence $\widehat{f}(\lambda)$ exists and satisfies the claimed identity. □

The next lemma shows, that the assumption $\omega_0(F_0) < \infty$ is not artificial.

Lemma B.11. *Assume that* $\operatorname{abs}(f) < \infty$. *Then* $\omega_0(F_0) \le \max\{0, \operatorname{abs}(f)\} < \infty$. *More precisely, this means that*

$$\forall \omega > \max\{0, \operatorname{abs}(f)\}\ \exists M = M_\omega \ge 0\ \forall t \ge 0 :\ \|F_0(t)\| \le M\mathrm{e}^{\omega t}.$$

Proof. Let $\omega > \operatorname{abs}(f)$ and additionally $\omega \ge 0$. Then $\widehat{f}(\omega)$ exists, and hence G_ω is bounded. The case $\omega = 0$ follows from the inequality

$$\|F_0(t)\| = \|G_0(t)\| \le C_0 = C_\omega \mathrm{e}^{\omega t} \quad \text{for all } t \ge 0.$$

Let $\omega > 0$. For every $t \ge 0$ an integration by parts yields

$$F_0(t) = \int_0^t \mathrm{e}^{\omega s}\mathrm{e}^{-\omega s} f(s)\,\mathrm{d}s = \mathrm{e}^{\omega t}G_\omega(t) - \int_0^t \omega \mathrm{e}^{\omega s} G_\omega(s)\,\mathrm{d}s$$

Obvious estimates lead to

$$\begin{aligned}\|F_0(t)\| &\le \mathrm{e}^{\omega t}\|G_\omega(t)\| + \int_0^t \omega\mathrm{e}^{\omega s}\|G_\omega(s)\|\,\mathrm{d}s \le C_\omega \mathrm{e}^{\omega t} + C_\omega \int_0^t \omega \mathrm{e}^{\omega s}\,\mathrm{d}s \\ &= C_\omega\mathrm{e}^{\omega t} + C_\omega(\mathrm{e}^{\omega t} - 1) \le 2C_\omega \mathrm{e}^{\omega t} \quad \text{for all } t \ge 0.\end{aligned}$$

□

Corollary B.12. *Let* $F : [0,\infty) \to V$ *be an antiderivative of* f. *If* $\operatorname{abs}(f) < \infty$, *then* $\widehat{F}(\lambda)$ *exists for every* $\lambda \in \mathbb{K}$ *with* $\operatorname{Re}\lambda > \max\{0, \operatorname{abs}(f)\}$ *and we have*

$$\widehat{F}(\lambda) = \tfrac{1}{\lambda}(\widehat{f}(\lambda) + F(0)).$$

Proof. Let λ be as in the claim and $N > 0$. Integration by parts yields

$$\begin{aligned}\int_0^N \mathrm{e}^{-\lambda t}F(t)\,\mathrm{d}t &= -\tfrac{1}{\lambda}\mathrm{e}^{-\lambda N}F(N) + \tfrac{1}{\lambda}F(0) + \int_0^N \tfrac{1}{\lambda}\mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t \\ &= -\tfrac{1}{\lambda}\mathrm{e}^{-\lambda N}F_0(N) - \tfrac{1}{\lambda}\mathrm{e}^{-\lambda N}F(0) + \tfrac{1}{\lambda}F(0) + \tfrac{1}{\lambda}\int_0^N \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t.\end{aligned}$$

From Lemma B.11 we derive that $\mathrm{e}^{-\lambda N}F_0(N) \to 0$ as $N \to \infty$. Moreover, $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\lambda > \operatorname{abs}(f)$ imply that $\mathrm{e}^{-\lambda N}F(0) \to 0$ as $N \to \infty$ and $\widehat{f}(\lambda)$ exists. Hence the left-hand side converges as $N \to \infty$, which means that $\widehat{F}(\lambda)$ exists. Also the claimed identity for $\widehat{F}(\lambda)$ now is obvious. □

We saw in Lemmas B.10 and B.11 that f is Laplace transformable if and only if F_0 is exponentially bounded, i.e., $\operatorname{abs}(f) < \infty$ if and only if $\operatorname{abs}(F_0) \le \omega_0(F_0) < \infty$. More precisely we know that

$$\operatorname{abs}(f) \le \omega_0(F_0) \le \max\{0, \operatorname{abs}(f)\}.$$

Thus, if $\operatorname{abs}(f) \ge 0$, then we have $\operatorname{abs}(f) = \omega_0(F_0)$.

In case $\operatorname{abs}(f) < 0$, the antiderivative F_0 has the limit $\widehat{f}(0) = \lim_{t\to\infty} F_0(t)$. Thus we can not expect that $\|F_0(t)\|$ decreases exponentially as $t \to \infty$. Instead we then have $\operatorname{abs}(f) = \omega_0(F_0 - \widehat{f}(0))$. We won't use this fact. For a proof we refer to Theorem 1.4.3 in [5].

Laplace transforms of strongly continuous operator families

Let W be another Banach spaces and let $(T_t)_{t\ge0}$ be a strongly continuous family of operators $T_t \in \mathcal{L}(V, W)$. In this subsection we define the Laplace transform $\widehat{T}(\lambda)$ of $(T_t)_{t\ge0}$.

Later we will focus on a strongly continuous semigroup $\mathbb{T}$ on a Banach space X, as this is our most important example. Moreover, we will see that its Laplace transform can be identified with the resolvent of its generator. Let us also start with this example to motivate the approach.

It is known that strongly continuous semigroups are exponentially bounded. Regarding $\mathbb{T}$ as a map $\mathbb{T} : [0,\infty) \to \mathcal{L}(X)$, one might expect that $\widehat{\mathbb{T}}(\lambda) \in \mathcal{L}(X)$ is obtained as before, at least for $\operatorname{Re}\lambda > \omega_0(\mathbb{T})$. However we do not know if $\mathbb{T} : [0,\infty) \to \mathcal{L}(X)$ is measurable. To avoid this problem, we make use of the strong continuity.

For $x \in V$ consider the function $f_x : [0,\infty) \to W;\ t \mapsto T_t x$. It is continuous and hence locally integrable. In case $\widehat{f_x}(\lambda)$ exists for some $\lambda \in \mathbb{K}$ and $x \in V$, we set

$$\widehat{T}(\lambda)x := \widehat{f_x}(\lambda) = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-\lambda t} T_t x \,\mathrm{d}t.$$

If $\lambda \in \mathbb{K}$ is such that $\widehat{T}(\lambda)x$ exists for all $x \in V$, then a mapping $\widehat{T}(\lambda) : V \to W$ is determined. We call $\widehat{T}(\lambda)$ the *Laplace transform of* $(T_t)_{t\ge0}$ *at* λ. As before, we define the quantity

$$\begin{aligned}\operatorname{abs}((T_t)_{t\ge0}) &= \inf\{\operatorname{Re}\lambda \mid \lambda \in \mathbb{K} : \widehat{T}(\lambda) \text{ exists}\} \\ &= \inf\{\operatorname{Re}\lambda \mid \lambda \in \mathbb{K} : \forall x \in V : \widehat{f_x}(\lambda) \text{ exists}\} = \sup_{x\in V} \operatorname{abs}(f_x).\end{aligned}$$

Lemma B.7 yields that $\widehat{T}(\lambda)$ exists for all $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \operatorname{abs}((T_t)_{t\ge0})$.

Assume that $\widehat{T}(\lambda)$ exists for some $\lambda \in \mathbb{K}$. We show that $\widehat{T}(\lambda) \in \mathcal{L}(V, W)$. Clearly, the linearity of T_t for every $t \ge 0$ implies that $\widehat{T}(\lambda)$ is linear. Let $N > 0$. Using the uniform boundedness principle, one easily sees that $\{\|T_t\| \mid t \in [0, N]\}$ is bounded. Therefore we have

$$\left\| \int_0^N \mathrm{e}^{-\lambda t} T_t x \,\mathrm{d}t \right\| \le \int_0^N \mathrm{e}^{-\operatorname{Re}\lambda t} \,\mathrm{d}t \sup_{t\in[0,N]} \|T_t\| \, \|x\| \,.$$

This means that the linear mapping $V \to W;\ x \mapsto \int_0^N \mathrm{e}^{-\lambda t} T_t x \,\mathrm{d}t$ is bounded for each $N > 0$. A corollary to the uniform boundedness principle then yields that $\widehat{T}(\lambda)$ as the pointwise limit of these bounded operators is itself bounded.

Let us now assume that $(T_t)_{t\geq 0}$ is exponentially bounded. Obviously the exponential bounds pass over to f_x, more precisely $\operatorname{abs}(f_x) \leq \omega_0(f_x) \leq \omega_0((T_t)_{t\geq 0})$. As expected, we can thus estimate

$$\operatorname{abs}((T_t)_{t\geq 0}) \leq \omega_0((T_t)_{t\geq 0}).$$

There is an estimate for $\|\widehat{T}(\lambda)\|$, which is of special importance in the case $(T_t)_{t\geq 0} = \mathbb{T}$.

Lemma B.13. *Let $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \omega_0((T_t)_{t\geq 0})$. Moreover, choose $\omega \in \mathbb{R}$, $M = M_\omega \geq 0$ with $\omega_0((T_t)_{t\geq 0}) < \omega < \operatorname{Re}\lambda$ and $\|T_t\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. Then we have*

$$\|\widehat{T}(\lambda)\| \leq \frac{M}{\operatorname{Re}\lambda - \omega}\,.$$

Proof. Let $x \in V$. Observe that $\mathrm{e}^{-\lambda(\boldsymbol{\cdot})} f_x \in L^1([0,\infty), W)$ and $\|\mathrm{e}^{-\lambda t} T_t x\| \leq M\mathrm{e}^{-(\operatorname{Re}\lambda-\omega)t}\|x\|$ for $t \geq 0$. It follows that

$$\|\widehat{T}(\lambda)x\| = \left\|\int_0^\infty \mathrm{e}^{-\lambda t} T_t x \,\mathrm{d}t\right\| \leq \int_0^\infty M\mathrm{e}^{-(\operatorname{Re}\lambda-\omega)t}\|x\|\,\mathrm{d}t = \frac{M}{\operatorname{Re}\lambda - \omega}\|x\|. \qquad \square$$

Let A be the generator of $\mathbb{T}$. To obtain an identification of $\widehat{\mathbb{T}}(\lambda)$ as the resolvent of A, we need the elementary (but important) lemma stated below. We further make use of the so called 'rescaled semigroup'. For $\mu \in \mathbb{K}$ and $t \geq 0$ consider $\mathbb{S}_t^\mu = \mathrm{e}^{\mu t}\mathbb{T}_t$. Simple calculations show that this defines a strongly continuous semigroup $\mathbb{S}^\mu$ on X with generator $\mu + A$.

Lemma B.14. *For all $x \in X$ and $N > 0$ we have*

$$\int_0^N \mathbb{T}_t x \,\mathrm{d}t \in \mathrm{D}(A) \qquad \textit{and} \qquad A\int_0^N \mathbb{T}_t x\,\mathrm{d}t = \mathbb{T}_N x - x.$$

If $x \in \mathrm{D}(A)$, we even have

$$\int_0^N \mathbb{T}_t Ax\,\mathrm{d}t = \mathbb{T}_N x - x = A\int_0^N \mathbb{T}_t x\,\mathrm{d}t.$$

For a proof see Lemma II.1.3 in [15]. We remark that the first part can be checked using the definition of the generator. The second part follows from the fundamental theorem of calculus.

Proposition B.15. *Let $\lambda \in \mathbb{K}$. Assume that $\widehat{\mathbb{T}}(\lambda)x$ converges absolutely for all $x \in X$. Then $\lambda \in \rho(A)$ and $R(\lambda, A) = \widehat{\mathbb{T}}(\lambda)$.*

Proof. Let $x \in X$. First we show that $\widehat{\mathbb{T}}(\lambda)x \in \mathrm{D}(A)$ and that $(\lambda - A)\widehat{\mathbb{T}}(\lambda)x = x$. Set $\mathbb{S}_t = \mathbb{S}_t^{-\lambda} = \mathrm{e}^{-\lambda t}\mathbb{T}_t$ for $t \geq 0$. Recall that $\mathbb{S}$ is a strongly continuous semigroup with generator $-\lambda + A$ and that $\mathrm{D}(-\lambda + A) = \mathrm{D}(A)$. For $\delta > 0$ a change of variables leads to

$$\begin{aligned}\tfrac{1}{\delta}(\mathbb{S}_\delta\widehat{\mathbb{T}}(\lambda)x - \widehat{\mathbb{T}}(\lambda)x) &= \tfrac{1}{\delta}\,\mathbb{S}_\delta\int_0^\infty \mathbb{S}_t x\,\mathrm{d}t - \tfrac{1}{\delta}\int_0^\infty \mathbb{S}_t x\,\mathrm{d}t \\ &= \tfrac{1}{\delta}\int_\delta^\infty \mathbb{S}_t x\,\mathrm{d}t - \tfrac{1}{\delta}\int_0^\infty \mathbb{S}_t x\,\mathrm{d}t = -\tfrac{1}{\delta}\int_0^\delta \mathbb{S}_t x\,\mathrm{d}t.\end{aligned}$$

Since the function $t \mapsto \mathbb{S}_t x$ is continuous, the right-hand side converges to $-x$ as $\delta \to \infty$. This shows that $\widehat{\mathbb{T}}(\lambda)x \in \mathrm{D}(A)$ and that $(-\lambda + A)\widehat{\mathbb{T}}(\lambda)x = -x$. Hence $(\lambda - A)\widehat{\mathbb{T}}(\lambda)x = x$.

Now let $x \in \mathrm{D}(A)$. We have to prove that $\widehat{\mathbb{T}}(\lambda)(\lambda - A)x = x$. Lemma B.14 yields

$$(-\lambda + A) \int_0^N \mathbb{S}_t x \, \mathrm{d}t = \int_0^N \mathbb{S}_t(-\lambda + A)x \, \mathrm{d}t$$

for every $N > 0$. By the assumption the right-hand side converges to $\widehat{\mathbb{T}}(\lambda)(-\lambda + A)x$ as $N \to \infty$. Because on the other hand $\int_0^N \mathbb{S}_t x \, \mathrm{d}t$ converges to $\widehat{\mathbb{T}}(\lambda)$ as $N \to \infty$ and $-\lambda + A$ is closed, we obtain $(-\lambda+A)\widehat{\mathbb{T}}(\lambda)x = \widehat{\mathbb{T}}(\lambda)(-\lambda+A)x$. The first steps now finishes the proof. □

Note that the assumption of Proposition B.15 is satisfied for $\lambda \in \mathbb{K}$ with $\operatorname{Re} \lambda > \omega_0(\mathbb{T})$. As a corollary we obtain the following basic result from semigroup theory.

Corollary B.16. *Let $\lambda \in \mathbb{K}$ with $\operatorname{Re} \lambda > \omega_0(\mathbb{T})$. Then $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re} \lambda - \omega}$ for every pair $\omega \in \mathbb{R}$, $M \geq 1$ such that $\omega_0(\mathbb{T}) < \omega < \operatorname{Re} \lambda$ and $\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$.*

Appendix C

Boundary control systems

The contents of this chapter are taken from Section 10.1 of [49] except for the last section which is standard. Some control problems obtained from partial differential equations such as the ones in Subsections 4.4.3 and 4.4.4 naturally lead to 'boundary control systems'. These are equations of the form

$$\begin{aligned} z'(t) &= Lz(t); \qquad z(0) = x_0, \\ Gz(t) &= u(t) \end{aligned} \tag{C.1}$$

Here $z(t)$ is the state of the system at time $t \geq 0$, x_0 is the initial state and u is the control. We shall see that the following solution concept for (C.1) is justified.

Definition C.1. Let X, U, Z be Banach spaces and let $Z \subseteq X$ be continuously embedded in X. Further let $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ as well as $x_0 \in X$ and $u \in L^1_{\text{loc}}([0, \infty), U)$. Then a *solution of* (C.1) is a function

$$z \in C([0, \infty), Z) \cap C^1([0, \infty), X)$$

which satisfies (C.1) in the classical sense.

Next we give conditions under which problem (C.1) can be transformed to control systems in the common form

$$z'(t) = Az(t) + Bu(t); \quad z(0) = x_0. \tag{C.2}$$

Definition C.2. Let X, U, Z be Banach spaces where $Z \subseteq X$ is continuously embedded in X. Further let $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$. We define the linear operator A in X by its graph

$$A = \{(x, y) \in X \times X \,|\, x \in Z \text{ and } Gx = 0 \text{ and } y = Lx\}.$$

Then the pair (L, G) is called *boundary control system on* X*,* U *and* Z if the following conditions are satisfied.

(i) The subspace $\ker G$ is dense in X.

(ii) The operator G has a bounded right inverse $G^{\#} \in \mathcal{L}(U, Z)$

(iii) The map A is closed and $\rho(A) \neq \emptyset$.

Observe that the operator A is the restriction of L to $\ker G$. It is densely defined due assumption (i). Hence A satisfies all conditions necessary for the construction of the spaces X_1 and X_{-1} as described in Appendix A and we have the extension $A|_0 \in \mathcal{L}(X, X_{-1})$.

We first show that $\|\,\boldsymbol{.}\,\|_Z$ is equivalent to $\|\,\boldsymbol{.}\,\|_1$ on $\mathrm{D}(A)$. As a closed subspace the domain $\mathrm{D}(A) = \ker G$ endowed with $\|\,\boldsymbol{.}\,\|_Z$ is complete. Let $x \in \mathrm{D}(A)$. Using the boundedness of L and the equation $Ax = Lx$ we obtain

$$\|x\|_1 \leq |\lambda|\|x\|_X + \|Lx\|_X \leq \big(c|\lambda| + \|L\|_{\mathcal{L}(Z,X)}\big)\|x\|_Z,$$

where $c \geq 0$ is such that $\|w\|_X \leq c\|w\|_Z$ for all $w \in Z$. Since also X_1 is complete, the open mapping theorem yields the claim.

Remark C.3. It is possible formulate Definition C.2 only in terms of L and G. To this end, replace condition (iii) by

(iii') $\exists \lambda \in \mathbb{C} : \lambda - L|_{\ker G} : \ker G \to X$ is one-to-one and onto.

It is clear that (iii) implies (iii'). To show the other implication, let $\lambda \in \mathbb{C}$ be as in (iii'). Using the equivalence of $\|(\lambda - A)\,\boldsymbol{.}\,\|_X$, $\|\,\boldsymbol{.}\,\|_1$ and $\|\,\boldsymbol{.}\,\|_Z$ on $\mathrm{D}(A)$ we get

$$\|(\lambda - A)x\|_X \lesssim \|x\|_1 \lesssim \|x\|_Z \quad \text{for all } x \in \mathrm{D}(A).$$

Thus the bounded inverse $(\lambda - A)^{-1} \in \mathcal{L}(X, (\mathrm{D}(A), \|\,\boldsymbol{.}\,\|_Z))$ exists. Because Z is continuously embedded in X, the operator $(\lambda - A)^{-1}$ belongs to $\mathcal{L}(X)$ which in turn means that A is closed as well as that $\lambda \in \rho(A)$. ♢

Remark C.4. Note that in order to satisfy (ii) it is necessary that G is onto. However, if Z is a Hilbert space then this is also sufficient. We have put this and related statements at the end of the chapter. ♢

Proposition C.5. *Let (L, G) be a boundary control system on X, U and Z. Then there is a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$Lw = (A|_0)w + BGw \quad \textit{for all } w \in Z.$$

For each $\lambda \in \rho(A)$ we further have $R(\lambda, A)B \in \mathcal{L}(U, Z)$ and $GR(\lambda, A)B = \mathrm{Id}_U$.

Proof. Let $G^{\#} \in \mathcal{L}(U, Z)$ be the bounded right inverse of G from (ii) of Definition C.2. We define a linear map $B : U \to X_{-1}$ via

$$Bv := (L - A|_0)G^{\#}v \quad \text{for } v \in U.$$

In order to prove that B is bounded, let $\lambda \in \rho(A)$. We may assume that $\|x\|_{-1}$ is given by $\|R(\lambda, A)x\|_X$ for $x \in X$. Now for all $v \in U$ we estimate

$$\begin{aligned}
\|Bv\|_{-1} &\leq \|LG^{\#}v\|_{-1} + \|(A|_0)G^{\#}v\|_{-1} \\
&\leq \|R(\lambda, A)\|_{\mathcal{L}(X)}\|LG^{\#}v\|_X + \|A|_0\|_{\mathcal{L}(X,X_{-1})}\|G^{\#}v\|_X \\
&\lesssim (\|L\|_{\mathcal{L}(Z,X)} + 1)\|G^{\#}v\|_Z \lesssim \|G\|_{\mathcal{L}(Z,U)}\|v\|_U.
\end{aligned}$$

This shows that B is bounded.

Let $w \in Z$. It is easy to see that $(\mathrm{Id}_Z - G^{\#}G)w \in \ker G$. Because L and $A|_0$ coincide on $\ker G$, we thus have

$$\begin{aligned} BGw &= (L - A|_0)G^{\#}Gw \pm (L - A|_0)w \\ &= -(L - A|_0)(\mathrm{Id}_Z - G^{\#}G)w + (L - A|_0)w = Lw - (A|_0)w. \end{aligned}$$

This however implies the first identity in the claim. Moreover, by inserting $w = G^{\#}v$ for $v \in U$ it becomes evident that B is unique. Now let $v \in U$. We compute

$$Bv = LG^{\#}v - (A|_0)G^{\#}v \pm \lambda G^{\#}v = (\lambda \,\mathrm{Id}_X - A|_0)G^{\#}v - (\lambda \,\mathrm{Id}_Z - L)G^{\#}v.$$

From the fact that $(\lambda - L)G^{\#}v$ belongs to X we infer

$$R(\lambda, A)Bv = G^{\#}v - R(\lambda, A)(\lambda - L)G^{\#}v \in Z. \tag{C.3}$$

This identity implies that $R(\lambda, A)B \in \mathcal{L}(U, Z)$. Since $R(\lambda, A)(\lambda - L)G^{\#}v$ lies in $X_1 = \ker G$, we finally obtain

$$GR(\lambda, A)Bv = GG^{\#}v - GR(\lambda, A)(\lambda - L)G^{\#}v = v. \qquad \square$$

Although the notation suggests that (A, B) defines a linear control system, it is in general not clear whether this is true. Nevertheless A and B are called *generator* and *control operator* corresponding to (L, G) respectively. Further Z is the *solution space* for (L, G). As usual X is called *state space* and U is called *input space*.

Take any $\lambda \in \rho(A)$. Then the solution space Z has a remarkable decomposition. For all $w \in Z$ there exist unique $x \in X_1$ and $v \in U$ with $w = x + R(\lambda, A)Bv$. That is

$$Z = X_1 + R(\lambda, A)B(U).$$

Indeed, X_1 is a subspace of Z. From (C.3) we know that $R(\lambda, A)Bv \in Z$ for all $v \in U$. On the other hand let $z \in Z$ and set $v = Gz$. Proposition C.5 yields that $GR(\lambda, A)B = \mathrm{Id}_U$ and therefore

$$Gz = GR(\lambda, A)Bv \quad \Longleftrightarrow \quad G(z - R(\lambda, A)Bv) = 0.$$

This means that $x := z - R(\lambda, A)Bv$ is contained in $\ker G = X_1$ and clearly $z = x + R(\lambda, A)Bv$. In order to show uniqueness, let $x_{\mathrm{a}}, x_{\mathrm{b}} \in X_1$ and $v_{\mathrm{a}}, v_{\mathrm{b}} \in U$ satisfy

$$x_{\mathrm{a}} + R(\lambda, A)Bv_{\mathrm{a}} = x_{\mathrm{b}} + R(\lambda, A)Bv_{\mathrm{b}} \quad \Longleftrightarrow \quad x_{\mathrm{a}} - x_{\mathrm{b}} = R(\lambda, A)B(v_{\mathrm{b}} - v_{\mathrm{a}})$$

Note that the left hand side of the last equation belongs to $\mathrm{D}(A) = \ker G$. Thus applying G and using $GR(\lambda, A)B = \mathrm{Id}_U$ again yields $0 = G(x_{\mathrm{a}} - x_{\mathrm{b}}) = v_{\mathrm{b}} - v_{\mathrm{a}}$, which in turn implies $x_{\mathrm{a}} - x_{\mathrm{b}} = R(\lambda, A)B0 = 0$.

We see that Z equals the solution space introduced in Section 3.3. There the decomposition $w = x + R(\lambda, A)Bv$ is not necessarily unique, so we are in a special situation here.

In Section 3.3 a norm on Z was given by $w = x + R(\lambda, A)Bv \mapsto (\|x\|_1^2 + \|v\|_U^2)^{1/2}$. We easily see that it is equivalent to $\|\,.\,\|_Z$. In fact, let $w = x + R(\lambda, A)Bv \in Z$. Then we have $x = w - R(\lambda, A)Bv$ as well as $v = Gw$. Hence, using the equivalence of $\|\,.\,\|_1$ and $\|\,.\,\|_Z$ on $\mathrm{D}(A)$ we derive

$$\begin{aligned} \|x\|_1^2 + \|v\|_U^2 &= \|w - R(\lambda, A)Bv\|_1^2 + \|Gw\|_U^2 \\ &\lesssim \|w - R(\lambda, A)BGw\|_Z^2 + \|G\|_{\mathcal{L}(Z,U)}^2 \|w\|_Z^2 \\ &\le \big(\|w\|_Z + \|R(\lambda, A)B\|_{\mathcal{L}(U,Z)} \|G\|_{\mathcal{L}(Z,U)} \|w\|_Z\big)^2 + \|G\|_{\mathcal{L}(Z,U)}^2 \|w\|_Z^2. \end{aligned}$$

The other way around we have

$$\begin{aligned}\|w\|_Z &= \|x + R(\lambda, A)Bv\|_Z \le \|x\|_Z + \|R(\lambda, A)Bv\|_Z \\ &\lesssim \|x\|_1 + \|R(\lambda, A)B\|_{\mathcal{L}(U,Z)}\|v\|_U \le 2(1 + \|R(\lambda, A)B\|)(\|x\|_1^2 + \|v\|_U^2)^{1/2}.\end{aligned}$$

Now assume that (A, B) define a linear control system on X and $L^p([0, \infty), U)$ for some $p \in [1, \infty)$. Recall that by Proposition 3.13 the strong solution $z \in C([0, \infty), X)$ of $z'(t) = Az(t) + Bu(t)$; $z(0) = x_0$ actually is the classical solution and even satisfies

$$z \in C([0, \infty), Z) \cap C^1([0, \infty), X).$$

provided that $u \in W^{1,p}_{\text{loc}}([0, \infty), U)$ and $x_0 \in X$ satisfy $Ax_0 + Bu(0) \in X$.

Note that leaving out the information that $z \in C([0, \infty), Z)$, this statement does not refer to Z. Nevertheless for $x_0 \in X$, $v \in U$ and $\lambda \in \rho(A)$ the following equivalences hold

$$\begin{aligned}Ax_0 + Bv \in X &\iff \lambda x_0 - Ax_0 - Bv \in X \iff x_0 - R(\lambda, A)Bv \in X_1 \\ &\iff \exists x \in X_1 : x_0 = x + R(\lambda, A)Bv \in Z.\end{aligned}$$

We emphasize that in this situation we have $Gx_0 = v$.

Definition C.6. Let (L, G) be a boundary control system on X, U and Z as in Definition C.2. Then (L, G) is called L^p*–well-posed* if the corresponding generator A and the control operator B define a linear control system on X and $L^p([0, \infty), U)$ for some $p \in [1, \infty)$.

Now we can state and prove the main result of the chapter.

Proposition C.7. *Let X, U and Z be Banach spaces where $Z \subseteq X$ is continuously embedded in X. Further let $p \in [1, \infty)$ and let (L, G) be an L^p–well-posed boundary control system on X, U and Z. Then for all $x_0 \in Z$ and $u \in W^{1,p}_{\text{loc}}([0, \infty), U)$ with $Gx_0 = u(0)$ problem (C.1) has a solution z. It is the classical solution of $z'(t) = Az(t) + Bu(t)$; $z(0) = x_0$, where A is the generator and B is the control operator corresponding to (L, G).*

Proof. In a first step we check that the conditions of Proposition 3.13 are satisfied. Since $x_0 \in Z$, there are $x \in X_1 = \ker G$ and $v \in U$ with $x_0 = x + R(\lambda, A)Bv$. Because $GR(\lambda, A)B = \mathrm{Id}_U$, it follows

$$u(0) = Gx_0 = Gx + GR(\lambda, A)Bv = v.$$

Therefore we have

$$\begin{aligned}Ax_0 + Bu(0) &= Ax \pm \lambda R(\lambda, A)Bu(0) + AR(\lambda, A)Bu(0) + Bu(0) \\ &= Ax + \lambda R(\lambda, A)Bu(0) - (\lambda - A)R(\lambda, A)Bu(0) + Bu(0) \\ &= Ax + \lambda R(\lambda, A)Bu(0) \in X.\end{aligned}$$

Now let $z \in C([0, \infty), Z) \cap C^1([0, \infty), X)$ be the classical solution of $z'(t) = Az(t) + Bu(t)$; $z(0) = x_0$ from Proposition 3.13. The differential equation is fulfilled pointwise and we derive

$$z(t) = R(\lambda, A)(\lambda - A|_0)z(t) = R(\lambda, A)\big(\lambda z(t) - z'(t) + Bu(t)\big) \quad \text{for all } t \ge 0.$$

Note that $R(\lambda, A)(\lambda z(t) - z'(t)) \in X_1 = \ker G$. Applying G we thus get

$$Gz(t) = GR(\lambda, A)\big(\lambda z(t) - z'(t)\big) + GR(\lambda, A)Bu(t) = u(t) \quad \text{for all } t \ge 0.$$

With the representation of L from Proposition C.5 we finally infer the equation

$$Lz(t) = (A|_0)z(t) + BGz(t) = (A|_0)z(t) + Bu(t) = z'(t) \quad \text{for all } t \ge 0.$$

This means that z is a solution of (C.1). ☐

Bounded right inverse

In this section let V, W be Banach spaces and let $T \in \mathcal{L}(V, W)$. A map $S : W \to V$ is a *right inverse of* T, if $TS = \mathrm{Id}_W$. In case S is linear and bounded, i.e., $S \in \mathcal{L}(W, V)$, it is called *bounded*.

Clearly, a right inverse exists if and only if T is onto. However, unless T is one-to-one and onto, right inverse are never unique. For example consider the map

$$T : \mathbb{C}^2 \to \mathbb{C}; \quad T(a, b) = a.$$

Here, for each $c \in \mathbb{C}$ a right inverse $S_c : \mathbb{C} \to \mathbb{C}^2$ is given by $S_c a = (a, c)$. Note that S_0 is bounded. Another bounded right inverse is given e.g. by $a \mapsto (a, a)$.

We shall characterize the existence of a bounded right inverse. To this end, we need the following notation. Let $V_0, V_1 \subseteq V$ two subspaces. Then we write

$$V = V_0 \oplus V_1$$

if $V_0 \cap V_1 = \{0\}$ and for all $x \in V$ there are $x_0 \in V_0$ and $x_1 \in V_1$ with $v = v_1 + v_2$. In this case V_0 is called a *complement* of V_1 in V (and vice versa).

Lemma C.8. *Let $T \in \mathcal{L}(V, W)$ and assume that T has a bounded right inverse $S \in \mathcal{L}(W, V)$. Then we have*

$$V_1 := \operatorname{Ran} S = \ker(\mathrm{Id}_V - ST),$$

so that V_1 is a closed subspace of V. Moreover, V can be decomposed into $V = V_1 \oplus \ker T$.

Proof. Let $x \in V$. Then $x - STx = 0$ is equivalent to $x = S(Tx)$. Since T is onto, we infer that $x \in \ker(\mathrm{Id}_V - ST)$ if and only if $x \in \operatorname{Ran} S$.

We set $x_1 = STx$ and $x_0 = x - x_1$. Clearly $x_1 \in V_1$ and $x = x_0 + x_1$. We further have

$$Tx_0 = Tx - TSTx = Tx - Tx = 0.$$

This means $x_0 \in \ker T$. For $x \in \ker T \cap V_1$ the first statement yields $0 = x - STx = x - S0 = x$. We conclude that $\ker T \cap V_1 = \{0\}$. □

Lemma C.9. *Assume that $T \in \mathcal{L}(V, W)$ is onto and that $\ker T$ has a closed complement V_1 in V. Then T has a bounded right inverse S with $\operatorname{Ran} S = V_1$.*

Proof. In the first step we show that $T|_{V_1}$ is onto and one-to-one. Let $y \in W$. Using that T is onto, we find a vector $x \in V$ with $Tx = y$. Moreover, there are $x_0 \in \ker T$ and $x_1 \in V_1$ with $x = x_0 + x_1$. It follows that

$$(T|_{V_1})(x_1) = Tx_1 = T(x_0 + x_1) = Tx = y.$$

Thus $T|_{V_1}$ is onto. Next take $x_1 \in V_1$ with $(T|_{V_1})(x_1) = Tx_1 = 0$, i.e., $x_1 \in \ker T$. Since $\ker T \cap V_1 = \{0\}$ by assumption, we deduce that $x_1 = 0$ and hence $T|_{V_1}$ is one-to-one. Therefore the open mapping theorem yields a bounded inverse $S \in \mathcal{L}(W, V_1)$ of $T|_{V_1}$. In particular we obtain

$$TSy = (T|_{V_1})(Sy) = y$$

for all $y \in W$. We consider S as an element of $\mathcal{L}(W, V)$. Then clearly $\operatorname{Ran} S = V_1$. □

Summing things up, we have shown the following result.

Proposition C.10. *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be onto. Then the following assertions are equivalent.*

(i) $\exists S \in \mathcal{L}(W, V)\ \forall y \in W: \ TSy = y$.

(ii) $\ker T$ *has a closed complement in* V.

Despite the fact that bounded right inverse of $T \in \mathcal{L}(V, W)$ are not unique we denote them by $T^{\#}$ whenever such operators exist.

In Hilbert spaces the orthogonal complement of any subspace is also a closed complement. This fact lets us simplify the last proposition.

Corollary C.11. *Let V be a Hilbert space and W be a Banach space. For $T \in \mathcal{L}(V, W)$ the following are equivalent*

(a) $\exists T^{\#} \in \mathcal{L}(W, V)\ \forall y \in W: \ TT^{\#}y = y$.

(b) T *is onto.*

Let V be a Hilbert space and assume that T is onto. Let $T^{\#}$ be the bounded right inverse of T constructed in Lemma C.9 corresponding to $V_1 = (\ker T)^{\perp}$, i.e., $T^{\#}$ is $(T|_{(\ker T)^{\perp}})^{-1}$ seen as an operator in $\mathcal{L}(W, V)$.

Take any other right inverse $S : W \to V$ of T. Using Pythagoras's theorem one easily deduces $\|T^{\#}y\|_V \leq \|Sy\|_V$ for all $y \in Y$. In case S is linear and bounded it follows that $\|T^{\#}\|_{\mathcal{L}(W,V)} \leq \|S\|_{\mathcal{L}(W,V)}$.

Remark C.12. Assume that $T \in \mathcal{L}(V, W)$ has a bounded right inverse $T^{\#} \in \mathcal{L}(W, V)$. Further let $R \in \mathcal{L}(V, W)$. Then $T + R \in \mathcal{L}(V, W)$ is onto if

$$\|R\|_{\mathcal{L}(V,W)} < \|T^{\#}\|^{-1}_{\mathcal{L}(W,V)}. \tag{C.4}$$

Indeed, in this case the Neumann series for $(T + R)T^{\#} = \mathrm{Id}_W + RT^{\#}$, since in particular $\|RT^{\#}\|_{\mathcal{L}(W)} < 1$. Denote the inverse of $(T+R)T^{\#}$ by $S \in \mathcal{L}(W)$, so that $(T+R)T^{\#}S = \mathrm{Id}_W$. It follows that $T^{\#}S \in \mathcal{L}(W, V)$ is a bounded right inverse of $T + R$.

With the notation of Lemma C.9 and the reasoning above, the largest possible bound for $\|R\|_{\mathcal{L}(V,W)}$ in (C.4) is $\|(T|_{V_1})^{-1}\|^{-1}_{\mathcal{L}(W,V_1)}$. ♢

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