# Regular Random Field Solutions for Stochastic Evolution Equations

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*"Imagination is more important than knowledge.*
*For knowledge is limited, whereas imagination*
*embraces the entire world."*

- Albert Einstein, 1931

# Abstract


In this thesis we investigate stochastic evolution equations of the form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t, X(t))\,\mathrm{d}t + \sum_{n=1}^{\infty} B_n(t, X(t))\,\mathrm{d}\beta_n(t)$$

for random fields $X\colon \Omega \times [0,T] \times U \to \mathbb{R}$, where $[0,T]$ is a time interval, $(\Omega, \mathcal{F}, \mathbb{P})$ a measure space representing the randomness of the system, and $U$ is typically a domain in $\mathbb{R}^d$ (or again a measure space). More precisely, we concentrate on the parabolic situation where $A$ is the generator of an analytic semigroup on $L^p(U)$. We look for mild solutions so that $X(\omega, \cdot, \cdot)$ has values in $L^p(U; L^q[0,T])$ for almost all $\omega \in \Omega$ under appropriate Lipschitz and linear growth conditions on the nonlinearities $F$ and $B_n$, $n \in \mathbb{N}$. Compared to the classical semigroup approach, which gives $X(\omega, \cdot, \cdot) \in L^q([0,T]; L^p(U))$, the order of integration is reversed. We show that this new approach together with a strong Doob and Burkholder-Davis-Gundy inequality leads to strong regularity results in particular for the time variable of the random field $X(\omega, t, u)$, e.g. pointwise Hölder estimates for the paths $t \mapsto X(\omega, t, u)$, $\mathbb{P}$-almost surely. For less-optimal regularity estimates we only need the relatively mild assumption that the resolvents of $A$ extend uniformly to $L^p(U; L^q[0,T])$. However, in the maximal regularity case the difficulty of the reversed order of integration in time and space makes extended functional calculi results necessary. As a consequence, we obtain suitable estimates for deterministic and stochastic convolutions. Using Sobolev embedding theorems, we obtain solutions in $L^r(\Omega; L^p(U; C^\alpha[0,T]))$. In several applications where $A$ is an elliptic operator on a domain in $\mathbb{R}^d$ we show that for concrete examples of stochastic partial differential equations our theory leads to stronger results as known in the literature.

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Markus Antoni, December 21, 2016

# Contents

# Introduction

### A historic sketch

The theory of stochastic (ordinary) differential equations started with the works of Itô in 1946 (cf. [47, 48]). His short paper [47] of four pages already contained many of the main ideas future researchers would apply in generalizing his results. The Itô stochastic calculus was born and it inspired the field of stochastic analysis in a fruitful way. Around 30 years later, researchers looked beyond stochastic odinary differential equtations to stochastic partial differential equations. In the fields of physics, biology, or control theory many models were best described by stochastic *evolution* partial differential equations. A vivid example was described by John B. Walsh in [86]:

*"The general problem is this. Suppose one is given a physical system governed by a partial differential equation. Suppose that the system is then perturbed randomly, perhaps by some sort of a white noise. How does it evolve in time? Think for example of a guitar carelessly left outdoors. If $u(x,t)$ is the position of one of the strings at the point $x$ and time $t$, then in calm air $u(x,t)$ would satisfy the wave equation $\partial_{tt}u = \partial_{xx}u$. However, if a sandstorm should blow up, the string would be bombarded by a succession of sand grains. Let $\dot{W}(x,t)$ represent the intensity of the bombardment at the point $x$ and time $t$. The number of grains hitting the string at a given point and time will be largely independent of the number hitting at another point and time, so that, after subtracting a mean intensity, $W$ may be approximated by a white noise, and the final equation is*

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t) + \dot{W}(x,t)$$

*where $\dot{W}$ is a white noise in both time and space, or, in other words, a two-parameter white noise. One peculiarity of this equation - not surprising in view of the behavior of ordinary stochastic differential equations - is that none of the partial derivatives in it exist. However, one may rewrite it as an integral equation, and then show that in this form there is a solution which is a continuous, though non-differentiable, function.*

Similar models were applied to other equations in physics, biology, or most notably mathematical finance. As the terminology suggests, the theory of stochastic partial differential equations lies in the intersection of two fields: stochastic processes and partial differential equations. Therefore, several approaches to these equations emerged. In particular for filtering equations (see e.g. [51]), the stochastic evolution equation can be regarded as an 'ordinary' Itô equation

$$\mathrm{d}X(t) = A(t, X(t))\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}\boldsymbol{\beta}(t)$$

for processes $X$ taking values in a function space. Depending on the dimension of this space we could interpret this equation as a (finite or infinite) *system* of one-dimensional ordinary stochastic differential equations. An introduction to this approach can be found in [70]. There, the focus lies on deducing a counterpart to the scalar-valued Itô theory for the function space-valued case. One should remark that by considering an infinite set of independent Brownian motions $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$, this also covers equations driven by space-time white noise (see [54, Section 8.3]).

Another famous approach is that of Walsh [86]. He considered the solution of stochastic partial differential equations as random variables, or more precisely, as *random fields*, because the solution depends on more than one independent variable. The focus of this theory lies on scalar-valued techniques and measures on infinite dimensional function spaces regarding both space and time (see also [38, 69]).

As the title of this thesis suggests, we will somehow mix these approaches to create a new one. More precisely, we will investigate the *stochastic evolution equation*

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t, X(t))\,\mathrm{d}t + \sum_{n=1}^{\infty} B_n(t, X(t))\,\mathrm{d}\beta_n(t), \quad X(0) = x_0 \in L^p(U),$$

in $L^p$ spaces for $p \in (1, \infty)$, still thinking of the solution $X$ as a *random field*, i.e. a function $X\colon \Omega \times [0, T] \times U \to \mathbb{R}$, and concentrate on the regularity of the process $X(t, u)$.

Before explaining this in more detail, we give a short historical background on the development of the theory of stochastic evolution equations. Early on, many of Itô's results could be generalized to the Hilbert space case using the fact that the norm comes from an inner product (see [22, 20]). In particular, the Itô isometry is still valid in the way we would expect it:

$$\mathbb{E}\Big\| \int_0^T \phi(s)\,\mathrm{d}\beta(s) \Big\|_H^2 = \mathbb{E}\int_0^T \|\phi(s)\|_H^2\,\mathrm{d}s \tag{1}$$

for any Hilbert space $(H, \|\cdot\|_H)$ and each adapted process $\phi\colon \Omega \times [0, T] \to H$. Having a well-defined stochastic integral is the starting point of a reasonable theory. However, in this situation examples of stochastic evolution equations arose with very low regularity. One famous result is the heat equation on $\mathbb{R}^d$ with a stochastic disturbance of gradient type, more precisely

$$\mathrm{d}X(t) = \tfrac{1}{2}\Delta X(t) + \sum_{n=1}^{d} \partial_{x_n} X(t)\,\mathrm{d}\beta_n(t), \quad X(0) = x_0.$$

Using the calculus of Itô, we can verify that for each $x_0 \in W^{1,2}(\mathbb{R}^d)$ the function

$$X(t,u) = x_0(u + \boldsymbol{\beta}_t), \quad t \in [0,T],\, u \in \mathbb{R}^d,$$

is the unique (weak or mild) solution in $L^2([0,T]\times\mathbb{R}^d)$, but, in general, $X$ is not continuous for $d \geq 2$. Therefore, better Sobolev embedding theorems for large $p$ points towards an $L^p$ theory for $p \neq 2$ and turn our attention to a Banach space-valued approach. However, already in the construction of a stochastic integral difficulties appear which were not present in the Hilbert space case. It turns out that $L^2([0,T];E)$, where $E$ is a general Banach space, does not lead to two-sided estimates in (1). This means that $L^2([0,T];E)$ does not characterize the space of stochastically integrable functions. However, assuming additional geometric properties of $E$, one gets at least one-sided estimates. For the case of stochastic partial differential equations in M-type 2 Banach spaces, where the latter include all $L^p$ spaces for $p \geq 2$, this was done by Brzeźniak in [12] (see also [10, 30]). Regarding stochastic integration theory in general Banach spaces, van Neerven and Weis characterized in [84] the space of reasonable integrands via $\gamma$-radonifying operators. Intuitively speaking, since $L^2([0,T];E)$ does not do the job, they considered the space with reversed order of norms '$E(L^2[0,T])$', which is of course not defined in general. However, in the case of an $L^p$ space this turns out to be the right choice. The generalization of the 'square function norm' in '$E(L^2[0,T])$' finally leads to the space of $\gamma$-radonifying operators $\gamma(L^2[0,T];E)$. In [80] van Neerven, Veraar, and Weis extended this to processes with values in a UMD Banach space via a decoupling technique. These results were then used in a sequence of papers (see [81, 83, 82]) to study stochastic evolution equations and their regularity in UMD spaces.
As indicated above, one can avoid $\gamma$-radonifying operators in the special case of an $L^p(U)$ space $E$, since $\gamma(L^2[0,T];E)$ is isomorphic to the Bochner space $L^p(U;L^2[0,T])$. The stochastic integration theory was then investigated in [3], where the author tries to exploit the structural advantages one has compared to the abstract setting of UMD spaces. One of the main results in [3] is a stronger version of Doob's maximal inequality stating that

$$\mathbb{E}\Big\|\max_{n=1}^{N} |M_n|\Big\|_{L^p}^r \leq C\mathbb{E}\|M_N\|_{L^p}^r$$

for an $L^p$-valued $L^r$ martingale $(M_n)_{n=1}^N$, $p, r \in (1,\infty)$. This leads to a stronger version of the Burkholder-Davis-Gundy inequality for stochastic integrals:

$$\mathbb{E}\Big\|\sup_{t\in[0,T]}\Big|\int_0^t f(s)\,\mathrm{d}\beta(s)\Big|\Big\|_{L^p}^r \eqsim \mathbb{E}\Big\|\Big(\int_0^T |f(s)|^2\,\mathrm{d}s\Big)^{1/2}\Big\|_{L^p}^r.$$

This then in turn gives rise to the question if such stronger regularity properties do not only hold for stochastic Itô integrals in $L^p$ spaces, but also for solutions of stochastic evolution equations. This is the topic of this thesis.

### Our approach in a nutshell

In many results regarding regularity of stochastic evolution equations of the form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t,X(t))\,\mathrm{d}t + \sum_{n=1}^{\infty} B_n(t,X(t))\,\mathrm{d}\beta_n(t), \quad X(0) = x_0,$$

where $-A$ is supposed to be the generator of an analytic semigroup $(e^{-tA})_{t\geq 0}$, the solution is almost surely an element of spaces like

$$L^q([0,T];E), \qquad W^{\sigma,q}([0,T];E), \qquad \text{or} \qquad C([0,T];E) \quad \text{etc.}$$

(see [21, 81, 82] and the references therein). A typical way to deal with existence and uniqueness for these equations is to consider *mild* solutions $X$, which are defined as functions satisfying a fixed point equation such as

$$X(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F(s,X(s))\,\mathrm{d}s + \sum_{n=1}^{\infty}\int_0^t e^{-(t-s)A}B_n(s,X(s))\,\mathrm{d}\beta_n(s),$$

where the spaces mentioned above serve as fixed point spaces. In order to get for such evolution equations with $E = L^p(U)$ better regularity estimates in time, it is of advantage (as suggested in the motivation above) to interchange the order of integration and choose the spaces

$$L^p(U;L^q[0,T]), \qquad L^p(U;W^{\sigma,q}[0,T]), \qquad \text{or} \qquad L^p(U;C[0,T])$$

as fixed point spaces. Since the norm with respect to time is now *inside* of the space-norm, we get *pointwise* for each $u \in U$ knowledge about the temporal behavior of the solution process $t \mapsto X(t,u)$. In particular, in regard of continuity results, these stated regularity results are stronger than what we have known before. In order to apply Banach's fixed point theorem in $L^r(\Omega;L^p(U;L^q[0,T]))$, it is necessary to investigate the following three maps in this space:

1) the *orbit map* $t \mapsto e^{-tA}x$, $x \in L^p(U)$;

2) the *deterministic convolution* $t \mapsto \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s$, $\phi \in L^p(U;L^q[0,T])$;

3) the *stochastic convolution* $t \mapsto \sum_{n=1}^{\infty}\int_0^t e^{-(t-s)A}\phi_n(s)\,\mathrm{d}\beta_n(s)$, $\phi \in L^p(U;L^q([0,T];\ell^2))$.

Using estimates of these maps, existence and uniqueness of mild solutions can be proven. Once this is done, we turn to the study of more involved regularity results. However, even in the case of orbit maps as in 1) one should notice that the usual estimates for analytic semigroups used in the Banach space-valued case are not applicable now since we *first* have to estimate with respect to time and *then* with respect to the space variable in $U$. This brings up the question: Is there a way to bypass this obstacle?

We make the following observation: Since $-A$ is the generator of an analytic semigroup, we have the representation

$$e^{-tA}x = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda} R(\lambda, A)x \,\mathrm{d}\lambda$$

for each fixed $t \in [0,T]$ and $x \in L^p(U)$. Here, $\Gamma = \partial(\Sigma_\alpha \cup B(0,\delta))$ for a suitable angle $\alpha$ and radius $\delta > 0$ (see e.g. [59, Illustration 9.9]). In particular, the product structure inside of the complex line integral makes estimates in $L^p(U; L^q[0,T])$ accessible since we can handle time and space separately. In the case of convolutions this is completely different. Here,

$$e^{-(t-s)A}\phi(s) = \frac{1}{2\pi i}\int_\Gamma e^{-(t-s)\lambda} R(\lambda, A)\phi(s) \,\mathrm{d}\lambda$$

for a function $\phi\colon [0,T] \to L^p(U)$ and every fixed $s \in [0,t]$. The separation of time and space is no longer existent which makes additional requirements for the resolvents of $A$ necessary. In order to figure out these properties, observe that

$$e^{-(t-s)\lambda} R(\lambda, A)\phi(s) = R(\lambda, A)\big(e^{-(t-s)\lambda}\phi(s)\big).$$

Since we want to apply an $L^p(U; L^q[0,T])$ norm to this integral, it is quite natural to assume that the resolvents should extend to $L^p(U; L^q[0,T])$ having similar norm estimates as before (by this we mean that the set $\{\lambda R(\lambda, A)\colon \lambda \in \Sigma_{\alpha'}\}$ is still bounded in $L^p(U; L^q[0,T])$). For simple functions $f = \sum_{n=1}^N \mathbb{1}_{[t_{n-1},t_n]} x_n \in L^p(U; L^q[0,T])$, where $0 = t_0 < \ldots < t_N = T$ with $t_n - t_{n-1} = \delta$ and $(x_n)_{n=1}^N \subseteq L^p(U)$, such a condition reads as

$$\begin{aligned}\|\lambda R(\lambda, A)f\|_{L^p(U;L^q[0,T])} &= \delta \Big\| \Big(\sum_{n=1}^N |\lambda R(\lambda, A)x_n|^q\Big)^{1/q} \Big\|_{L^p(U)} \\ &\le C\delta \Big\| \Big(\sum_{n=1}^N |x_n|^q\Big)^{1/q} \Big\|_{L^p(U)} = C\|f\|_{L^p(U;L^q[0,T])}.\end{aligned}$$

This leads us to the notion of $\ell^q$- and $\mathcal{R}_q$-sectorial operators. The second terminology was first introduced by Weis in [87], and was further elaborated in [57, 79]. Estimating the convolution terms in 2) and 3) now reduces to the estimation of the 'scalar' convolutions

$$\int_0^t e^{-(t-s)\lambda}\phi(s)\,\mathrm{d}s \quad \text{and} \quad \sum_{n=1}^\infty \int_0^t e^{-(t-s)\lambda}\phi_n(s)\,\mathrm{d}\beta_n(s),$$

provided that the complex line integral still converges (which will always be the case in this setting). However, in order to investigate further regularity properties, this will not suffice. We are also interested in estimates of

$$A^\alpha \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s \quad \text{and} \quad A^\beta \sum_{n=1}^\infty \int_0^t e^{-(t-s)A}\phi_n(s)\,\mathrm{d}\beta_n(s)$$

for certain values of $\alpha$ and $\beta$. If $A$ is $\ell^q$-sectorial, we obtain estimates of the form

$$\mathbb{E}\Big\| A^\alpha \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s \Big\|^r_{L^p(U;L^q[0,T])} \le CT^{(1-\alpha)r}\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])},$$
$$\mathbb{E}\Big\| A^\beta \sum_{n=1}^\infty \int_0^t e^{-(t-s)A}\phi_n(s)\,\mathrm{d}\beta_n(s) \Big\|^r_{L^p(U;L^q[0,T])} \le CT^{(1/2-\beta)r}\mathbb{E}\|\phi\|^r_{L^p(U;L^q([0,T];\ell^2))},$$

for $\alpha < 1$ and $\beta < 1/2$. Note that the restriction for $\beta$ is forced by the $L^2[0,T]$ norm in the Itô isomorphism. The borderline cases $\alpha = 1$ and $\beta = 1/2$ are in general false. Therefore, we again need to impose additional assumptions on $A$, namely that the extension of $A$ to $L^p(U;L^q[0,T])$ has a *bounded $H^\infty$ calculus.*

This concept was introduced by McIntosh in [65] and gives an answer to the question whether the functional calculus for sectorial operators $A$ is bounded for functions $f \in H^\infty$. Following the ideas of the Dunford calculus, this calculus is defined by

$$\varphi(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \varphi(\lambda)R(\lambda,A)\,\mathrm{d}\lambda,$$

where $\varphi \in H_0^\infty$, i.e. $\varphi$ is a bounded and holomorphic function decaying polynomially to 0 as $\lambda$ tends to 0 and to $\infty$. For these functions the expression above is well-defined as a bounded operator since the integral is absolutely convergent. However, for functions $f \in H^\infty$ this leads, in general, only to unbounded operators. Thus, a sectorial operator $A$ is said to have a bounded $H^\infty$ calculus if $f(A)$ defines a bounded operator for each $f \in H^\infty$.

Assuming that the extension of $A$ has a bounded $H^\infty$ calculus on $L^p(U;L^q[0,T])$, we can also treat the cases $\alpha = 1$ and $\beta = 1/2$ by applying the following trick: We define the analytic families of bounded operators

$$(K_\lambda\phi)(t) := \int_0^t \lambda e^{-(t-s)\lambda}\phi(s)\,\mathrm{d}s, \quad \lambda \in \Sigma_\omega,\ \phi \in L^r(\Omega;L^p(U;L^q[0,T])),$$
$$(L_\lambda\phi)(t) := \sum_{n=1}^\infty \int_0^t \lambda^{1/2} e^{-(t-s)\lambda}\phi_n(s)\,\mathrm{d}\beta_n(s), \quad \lambda \in \Sigma_\omega,\ \phi \in L^r(\Omega;L^p(U;L^q([0,T];\ell^2))).$$

It is tempting to plug in $A$ for $\lambda$ in these formulas and hope to obtain operators $K_A$ and $L_A$ which are still bounded on $L^r(\Omega;L^p(U;L^q[0,T]))$ and $L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))$, respectively. This procedure can indeed be justified by the methods of the $H^\infty$ calculus. It requires certain randomization properties of the families $(K_\lambda)_{\lambda\in\Sigma_\omega}$ and $(L_\lambda)_{\lambda\in\Sigma_\omega}$, and the notion of an operator-valued functional calculus which will be explained in Section 2.2. Writing out the boundedness of $K_A$ and $L_A$ leads to

$$\mathbb{E}\Big\| A \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s \Big\|^r_{L^p(U;L^q[0,T])} \le C\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])},$$
$$\mathbb{E}\Big\| A^{1/2} \sum_{n=1}^\infty \int_0^t e^{-(t-s)A}\phi_n(s)\,\mathrm{d}\beta_n(s) \Big\|^r_{L^p(U;L^q[0,T])} \le C\mathbb{E}\|\phi\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

This case is often considered as the maximal or optimal regularity case. Furthermore, we also consider some 'regularity swapping' results. More precisely, we show that we can give up space regularity (which is encoded in the domains of the fractional powers of $A$) to obtain more time regularity, i.e.

$$\mathbb{E}\Big\| A^{1-\sigma}\int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s\Big\|^r_{L^p(U;W^{\sigma,q}[0,T])} \le C\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}, \quad \sigma\in[0,1),$$

$$\mathbb{E}\Big\| A^{1/2-\sigma}\sum_{n=1}^\infty\int_0^t e^{-(t-s)A}\phi_n(s)\,\mathrm{d}\beta_n(s)\Big\|^r_{L^p(U;W^{\sigma,q}[0,T])} \le C\mathbb{E}\|\phi\|^r_{L^p(U;L^q([0,T];\ell^2))}, \quad \sigma\in[0,1/2).$$

The very important question as to which operators $A$ have such an extension with a bounded $H^\infty$ calculus is treated in Chapter 2. It is shown that many of the common partial differential operators considered in applications do have this property.

These estimates are at the heart of our regularity theory. They also enable us to apply Banach's fixed point theorem to get unique mild solutions in $L^r(\Omega; L^p(U; L^q[0,T]))$, assuming certain Lipschitz conditions on $F$ and $\boldsymbol{B} = (B_n)_{n\in\mathbb{N}}$. We emphasize that the regularity with respect to time benefits from the 'swapping mechanism'. As a consequence, we get those strong results announced in the beginning. For a complete presentation of this approach we refer to Chapter 3.

## Outline of this thesis

The thesis is organized as follows. In **Chapter 1** we lay the foundation of the stochastic integration theory in mixed $L^p$ spaces. This is a continuation of [3], where the case of one $L^p$ space was considered. Although in the subsequent chapters no mixed $L^p$ spaces explicitly appear in the main results, we will be reliant on these results in proofs of Section 3.4. In Section 1.1 we start with the investigation of the stochastic integral for integrable processes in $L^p(U; L^q(V))$ with respect to *one* Brownian motion, where $(U, \Sigma, \mu)$ and $(V, \Xi, \nu)$ are $\sigma$-finite measure spaces. Besides giving meaning to the expression $\int_0^T f(s)\,\mathrm{d}\beta(s)$ for adapted processes $f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and deducing many properties of this integral, one of the main results of this section includes an extension of the stronger versions of Doob's maximal inequality and the Burkholder-Davis-Gundy inequality for mixed $L^p$ spaces. In contrast to [3] the latter now also includes the case $r = 1$. In Section 1.2 we proceed in a classical way by extending the stochastic integral to processes without integrability assumptions with respect to $\Omega$. This is done by a localization argument involving stopping time techniques. In particular, the Burkholder-Davis-Gundy inequality for $r = 1$ will lead to a general version of the stochastic Fubini theorem under minimal assumptions on the process considered. In Section 1.3 we extend the results both of Section 1.1 and 1.2 to a stochastic integral with respect to an infinite sequence of independent Brownian motions, i.e. to an integral of the form

$$\int_0^t \boldsymbol{b}(s)\,\mathrm{d}\boldsymbol{\beta}(s) = \sum_{n=1}^{\infty} \int_0^t b_n(s)\,\mathrm{d}\beta_n(s),$$

and formulate an appropriate version of Itô's formula. In the final Section 1.4 we briefly illustrate how we can use the stochastic integration theory developed for mixed $L^p$ spaces to obtain a stochastic integration theory in Sobolev and Besov spaces.

In **Chapter 2** we shortly leave the stochastic territory and turn to some spectral theory which will be important for Chapter 3. Section 2.1 provides a systematic introduction to the concepts of $\mathcal{R}_q$-boundedness and $\mathcal{R}_q$-sectoriality. In Section 2.2 we will continue with a short overview of the bounded $H^\infty$ and $\mathcal{R}H^\infty$ calculus. Based on these ideas, Section 2.3 is devoted solely to the concept of an $\mathcal{R}_q$-bounded $H^\infty$ calculus. Here we will use a result of Kunstmann and Ullmann (cf. [58]) to collect several examples of differential operators in divergence and non-divergence form having such a functional calculus. In Section 2.4 we establish the important connection between $A$ having an $\mathcal{R}_q$-bounded $H^\infty$ calculus and having an extension on $L^p(U; L^q[0,T])$ with a bounded $H^\infty$ calculus. Finally, in Section 2.5 we introduce a new family of interpolation spaces obtained by the $\ell^q$ interpolation method. As it turns out, these spaces are suitable for estimating orbit maps in $L^p(U; L^q[0,T])$. Therefore, this section is closely connected to Section 3.2.

In **Chapter 3** we will use the techniques announced and explained in the previous section to treat stochastic evolution equations in $L^p$ spaces. After a short motivation in Section 3.1, Section 3.2 provides a systematic treatment of the orbit map $t \mapsto e^{-tA}x$ in the spaces $L^p(U; L^q[0,T])$ and $L^p(U; W^{\sigma,q}[0,T])$. In Sections 3.3 and 3.4 we turn to the study of deterministic and stochastic convolutions, respectively. Here, many results of the previous chapters come together to produce some of the main results of this thesis. In Section 3.5 we apply the tools of the previous three sections in a fixed point argument to obtain unique mild solutions of abstract stochastic evolution equations in $L^p$ spaces, i.e. equations of the form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t, X(t))\,\mathrm{d}t + \boldsymbol{B}(t, X(t))\,\mathrm{d}\boldsymbol{\beta}(t), \quad X(0) = x_0.$$

Here, we will assume global Lipschitz and linear growth conditions for $F$ and $\boldsymbol{B}$ adjusted to the fixed point space $L^r(\Omega; L^p(U; L^q[0,T]))$. Moreover, the connection to strong and weak solutions is considered. Subsequently, we treat the non-autonomous and locally Lipschitz case in Subsections 3.5.3 and 3.5.4.

In the final **Chapter 4** we will apply the abstract theory to several stochastic partial differential equations. Here, we highly benefit from the fact that many differential operators have an $\mathcal{R}_q$-bounded $H^\infty$ calculus for all $q \in (1, \infty)$. In cases where the domain of the operator coincides with a Sobolev space, we show that our regularity results imply Hölder regularity both in space and time. This will be compared to existing results in the literature and reveals that in many situations our theory leads to stronger regularity results, which then give a new insight into such equations. As one example, we consider the stochastic heat equation

$$\begin{aligned} \mathrm{d}X(t,u) - \kappa\Delta_p X(t,u)\,\mathrm{d}t = f(t,u,X(t,u))\,\mathrm{d}t + \sum_{n=1}^{\infty} b_n(t,u,X(t,u))\,\mathrm{d}\beta_n(t), \\ X(t,u) = 0, \quad u \in \partial U,\ t \in [0,T], \\ X(0,u) = x_0(u), \quad u \in U. \end{aligned}$$

In this example we already see how our theory improves the results of others. On the whole range of $r, p \in (1,\infty)$ and for each $q \in (2,\infty)$ and $\gamma \in [0, 1/2)$ we obtain the regularity result

$$X \in L^r_{\mathbb{F}}(\Omega; H^{2(\gamma-\sigma),p}(U; C^{\sigma-1/q}[0,T])), \quad \sigma \in (1/q, \gamma].$$

This means that even the path $t \mapsto X(t,u)$ is $\alpha$-Hölder continuous for almost all $u \in U$ and each $\alpha \in (0, 1/2)$.

# Notational Conventions

In this introductory section we want to fix some notions and expressions used throughout this thesis.

**Miscellaneous**

- If not otherwise stated, the number $T > 0$ always stands for a fixed finite time, and $N \in \mathbb{N}$ for a fixed arbitrary integer.
- If $a \le C(q)\, b$ for non-negative numbers $a$ and $b$ and a constant $C(q) > 0$ depending only on the variable $q$, we write $a \lesssim_q b$. Additionally, we write $a \eqsim_q b$ if $a \lesssim_q b$ and $b \lesssim_q a$.
- For real numbers $x$ and $y$, we define $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

**Probabilistic setting**

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ always be a complete probability space equipped with a normal filtration $\mathbb{F} := (\mathcal{F}_t)_{t\ge 0}$, i.e. $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.
- Let $(\beta(t))_{t\ge 0}$ be a Brownian motion adapted to this filtration in the following way

  1) $\beta(t)$ is $\mathcal{F}_t$-measurable for all $t \ge 0$;
  2) $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s$ for all $s < t$.

  As an example, we may choose the Brownian filtration $\mathbb{F}^\beta = (\mathcal{F}_t^\beta)_{t\ge 0}$ given by

  $$\mathcal{F}_t^\beta := \sigma\big(\{\beta(s) \colon s \le t\}\big), \quad t \ge 0.$$

- By $(\beta_n)_{n\in\mathbb{N}}$ we denote a sequence of independent Brownian motions such that each $\beta_n$ is adapted to $\mathbb{F}$ in the way described above.
- We call a random variable $r \colon \Omega \to \{-1, 1\}$ satisfying

  $$\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = \tfrac{1}{2}$$

  a *Rademacher (random) variable.*

- A random variable $\gamma\colon \Omega \to \mathbb{R}$ will be called *standard Gaussian* if its distribution has density

$$f_\gamma\colon \mathbb{R} \to \mathbb{R}, \quad f_\gamma(t) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-\tfrac{1}{2}t^2\bigr),$$

with respect to the Lebesgue measure on $\mathbb{R}$.

**Normed spaces and linear operators**

- For two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we denote by $\mathcal{B}(E,F)$ the set of all linear and bounded operators $T\colon E \to F$ equipped with the norm

$$\|T\| := \sup_{\|x\|_E \le 1} \|Tx\|_F = \sup_{\|x\|_E = 1} \|Tx\|_F = \sup_{\|x\|_E \ne 1} \frac{\|Tx\|_F}{\|x\|_E}.$$

If $E = F$, then we let $\mathcal{B}(E) := \mathcal{B}(E,E)$. In the case $F = \mathbb{K}$, we define $E' := \mathcal{B}(E,\mathbb{K})$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $x' \in E'$ we use the notation

$$\langle x, x' \rangle := x'(x), \quad x \in E.$$

- We call a Banach space $(E, \|\cdot\|_E)$ a *UMD space*, if for some (equivalently for all) $r \in (1,\infty)$ there exists a real constant $C = C(r,E)$ such that for all $E$-valued $L^r$ martingales $(M_n)_{n=1}^N$ and any sequence of signs $(\varepsilon_n)_{n=1}^N$ we have

$$\mathbb{E}\Bigl\| \sum_{n=1}^N \varepsilon_n (M_n - M_{n-1}) \Bigr\|_E^r \le C\mathbb{E}\Bigl\| \sum_{n=1}^N (M_n - M_{n-1}) \Bigr\|_E^r = \mathbb{E}\|M_N\|_E^r.$$

The expression UMD is an abbreviation for *unconditional martingale differences*. Important examples of UMD spaces are Hilbert spaces and $L^p$ spaces for $p \in (1,\infty)$.

**Function spaces**

- If not otherwise stated, $(U, \Sigma, \mu)$ and $(V, \Xi, \nu)$ are always $\sigma$-finite measure spaces. For $p \in [1,\infty)$ and any Banach space $E$ we denote by $L^p(U;E)$ the space of all (equivalence classes of) strongly measurable functions $f\colon U \to E$ such that

$$\|f\|_{L^p(U,E)} := \Bigl( \int_U \|f(u)\|_E^p \,\mathrm{d}\mu(u) \Bigr)^{1/p} < \infty.$$

For $p = \infty$ we let $L^\infty(U;E)$ be the space of all (equivalence classes of) strongly measurable functions $f\colon U \to E$ for which we have a number $r \ge 0$ such that $\mu\bigl(u \in U\colon \|f(u)\|_E > r\bigr) = 0$. Then we endow $L^\infty(U;E)$ with the norm

$$\|f\|_{L^\infty(U;E)} := \inf\bigl\{ r \ge 0\colon \mu\bigl(u \in U\colon \|f(u)\|_E > r\bigr) = 0 \bigr\}.$$

- If $(U_n, \Sigma_n, \mu_n)$, $n \in \{1, \ldots, N\}$, is a sequence of $\sigma$-finite measure spaces and $E = L^{p_1}(U_1; L^{p_2}(U_2; \ldots L^{p_N}(U_N))$ is a mixed $L^p$ space, then for any Banach space $F$ we let

$$E(F) := L^{p_1}(U_1; L^{p_2}(U_2; \ldots L^{p_N}(U_N; F)),$$

  i.e. $F$ represents the innermost norm in the norm of $E(F)$.

- For any $r \in [1, \infty]$ we define the Hölder conjugate of $r$ by $r' := \frac{r}{r-1}$ (with $1' := \infty$ and $\infty' := 1$). Moreover, for $1 \leq r < \infty$, we identify the duality space $L^r(U)'$ with the space $L^{r'}(U)$ via $T_g(f) := T(f) = \int_U fg \, \mathrm{d}\mu$ for any $T \in L^r(U)'$. Thus, by

$$\langle f, g \rangle := \langle f, T_g \rangle = T_g(f) = \int_U fg \, \mathrm{d}\mu$$

  we denote the duality pairing of the elements $f \in L^r(U)$ and $g \in L^{r'}(U)$.

- For any function $f \in L^p(U)$ we introduce the notion

$$\|f\|_{L^p(U)} := \|f(t)\|_{L^p_{(t)}(U)}.$$

  In the presence of a mixed $L^p$ space norm this terminology helps us to maintain an overview which norm is taken with respect to which variable.

# Chapter 1

# Stochastic Integration in Mixed $L^p$ Spaces

In this chapter, we develop the stochastic integration theory in mixed $L^p$ spaces $E$. We closely follow the approach of [3]. More precisely, we start to define a stochastic integral first for *integrable* adapted processes $f \in L^r(\Omega; E)$ with respect to one Brownian motion $(\beta(t))_{t\in[0,T]}$. This will be further extended to *measurable* adapted processes $f \in L^0(\Omega; E)$. Both of these integrals will then be generalized to an integral with respect to an *infinite sequence* of independent Brownian motions. Central part in every section is the Itô isomorphism. Once this is available, properties of stochastic integrals will follow. Many underlying concepts of this chapter were first introduced in [80, 84].

## 1.1 Basic Theory

In this section, we discuss the stochastic integration theory for processes with values in mixed $L^p$ spaces like $L^p(U; L^q(V))$ for $p, q \in (1, \infty)$ and $\sigma$-finite measure spaces $(U, \Sigma, \mu)$ and $(V, \Xi, \nu)$. Mostly, this will be an enhancement of the $L^p(U)$-valued case, which was presented in more detail in [3].

In order to develop a meaningful integration theory, the first task is to figure out the correct space of integrands. Before turning to general processes $f\colon \Omega \times [0,T] \to L^p(U; L^q(V))$ we start with the simplest case and observe first 'step processes'.

**Definition 1.1.1.** Let $p, q \in (1, \infty)$. A function $\phi\colon \Omega \times [0,T] \to L^p(U; L^q(V))$ is called an *adapted step process* with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ if it is of the form

$$f(\omega, t) = \sum_{n=1}^{N} \mathbb{1}_{(t_{n-1},t_n]}(t) \sum_{k=1}^{K_n} \mathbb{1}_{A_{k,n}}(\omega) x_{k,n}, \quad (\omega, t) \in \Omega \times [0,T],$$

where $0 = t_0 < \ldots < t_N = T$, $x_{1,n}, \ldots, x_{K_n,n} \in L^p(U; L^q(V))$, and $A_{1,n}, \ldots, A_{K_n,n} \in \mathcal{F}_{t_{n-1}}$ for all $n \in \{1, \ldots, N\}$.

**Remark 1.1.2.** In many proofs, we will abbreviate the 'stochastic' sums in the definition above as

$$v_n(\omega) := \sum_{k=1}^{K_n} \mathbb{1}_{A_{k,n}}(\omega) x_{k,n}, \quad \omega \in \Omega.$$

So $v_n \colon \Omega \to L^p(U; L^q(V))$ becomes an $\mathcal{F}_{t_{n-1}}$-measurable simple process satisfying $v_n \in L^r(\Omega; L^p(U; L^q(V)))$ for any $r \in (1, \infty)$. Also, we can think of the step process $f = \sum_{n=1}^N \mathbb{1}_{(t_{n-1}, t_n]} v_n$ as a step function with values in $L^r(\Omega; L^p(U; L^q(V)))$. We should always think about these sums in this way because it makes the presentation of many results less complicated. The reason why we have chosen the sums as we did in Definition 1.1.1 is simply the fact that these simple processes are dense in $L^r(\Omega; L^p(U; L^q(V)))$ for every $r \in (1, \infty)$.

For these basic processes we can define a stochastic integral very similar to the scalar case.

**Definition 1.1.3.** Let $f$ be an adapted step process with respect to the filtration $\mathbb{F}$ as in Definition 1.1.1. Then we define the *stochastic integral* of $f$ with respect to the Brownian motion $(\beta(t))_{t \in [0,T]}$ by

$$\begin{aligned} \int_0^T f \, \mathrm{d}\beta(\omega) &:= \sum_{n=1}^N \sum_{k=1}^{K_n} \mathbb{1}_{A_{k,n}}(\omega) x_{k,n} \big(\beta(\omega, t_n) - \beta(\omega, t_{n-1})\big) \\ &= \sum_{n=1}^N v_n(\omega) \big(\beta(\omega, t_n) - \beta(\omega, t_{n-1})\big). \end{aligned}$$

In order to find the correct space of integrands, we first need the following lemma about Gaussian sums in mixed $L^p$ spaces.

**Lemma 1.1.4 (Kahane).** *Let $p, q, r \in [1, \infty)$, $(x_n)_{n=1}^N \subseteq L^p(U; L^q(V))$, $(r_n)_{n=1}^N$ be a sequence of independent Rademacher variables, and $(\gamma_n)_{n=1}^N$ be a sequence of independent standard Gaussian variables. Then*

$$\mathbb{E}\Big\| \sum_{n=1}^N r_n x_n \Big\|_{L^p(U;L^q(V))}^r \eqsim_C \Big\| \Big( \sum_{n=1}^N |x_n|^2 \Big)^{1/2} \Big\|_{L^p(U;L^q(V))}^r$$

*and*

$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n \Big\|_{L^p(U;L^q(V))}^r \eqsim_{C'} \Big\| \Big( \sum_{n=1}^N |x_n|^2 \Big)^{1/2} \Big\|_{L^p(U;L^q(V))}^r,$$

*where $C$ and $C'$ only depend on the maximum of $p, q$ and $r$.*

The statement of this lemma can be deduced as in [3, Theorem 1.4] using the result for $\mathbb{R}$-valued Gaussian and Rademacher sums and the $p\vee q$ concavity of the space $L^p(U;L^q(V))$ (or Minkowski's integral inequality twice). We leave the easy calculations to the reader.

If a step process $f$ were independent of $\Omega$, i.e. if it were just a step *function* $f\colon [0,T]\to L^p(U;L^q(V))$, $f=\sum_{n=1}^N \mathbb{1}_{(t_{n-1},t_n]}x_n$, then the stochastic integral of $f$ would be nothing more than a Gaussian sum as in the previous lemma. Indeed, for any partition $0=t_0<\ldots<t_N=T$, the random variables

$$\gamma_n := \frac{1}{\sqrt{t_n-t_{n-1}}}\big(\beta(t_n)-\beta(t_{n-1})\big),\quad n\in\{1,\ldots,N\},$$

define a sequence of independent standard Gaussian variables and

$$\int_0^T f\,\mathrm{d}\beta = \sum_{n=1}^N \gamma_n\sqrt{t_n-t_{n-1}}x_n.$$

Kahane's inequality now leads to the estimate

$$\begin{aligned}\mathbb{E}\Big\|\int_0^T f\,\mathrm{d}\beta\Big\|^r_{L^p(U;L^q(V))} &\eqsim_C \Big\|\Big(\sum_{n=1}^N (t_n-t_{n-1})|x_n|^2\Big)^{1/2}\Big\|^r_{L^p(U;L^q(V))}\\ &= \Big\|\Big(\int_0^T |f(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big\|^r_{L^p(U;L^q(V))}.\end{aligned}$$

Using the decoupling property of the UMD space $L^p(U;L^q(V))$ (see [3, Theorem 2.23 and Corollary 2.24]) we get the following result for step processes $f\colon\Omega\times[0,T]\to L^p(U;L^q(V))$ (see [3, Lemma 3.18]).

**Proposition 1.1.5 (Itô isomorphism for step processes).** *For $p,q,r\in(1,\infty)$ and every adapted step process $f\colon\Omega\times[0,T]\to L^p(U;L^q(V))$ we have*

$$\mathbb{E}\Big\|\int_0^T f\,\mathrm{d}\beta\Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\|\Big(\int_0^T |f(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big\|^r_{L^p(U;L^q(V))}.$$

In this proposition we can see that the space for reasonable integrands is at most isomorphic to

$$L^r(\Omega;L^p(U;L^q(V;L^2[0,T]))),$$

i.e. not a space of $L^p(U;L^q(V))$-valued processes! For the moment, this may be unusual and we should always be aware of this surprising fact. Although it is a little bit incorrect, we will still call an $L^p(U;L^q(V;L^2[0,T]))$-valued 'random variable' a *process* to signify the time-dependence.

**Remark 1.1.6.** If we did not have any adaptedness assumptions on our step processes with respect to a filtration, then the space above would indeed be the correct one. But by taking the closure of all adapted step processes in $L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$ we only get a closed subspace of it. We recall that a function $f\colon \Omega \times [0,T] \to L^p(U; L^q(V))$ is *adapted* to a filtration $\mathbb{F}$ if $f(t)\colon \Omega \to L^p(U; L^q(V))$ is strongly $\mathcal{F}_t$-measurable for all $t \in [0,T]$. For a process $f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$ we can not define adaptedness in this way, since in general $(u,v) \mapsto f(u,v,t) \notin L^p(U; L^q(V))$ for any fixed $t \in [0,T]$. To bypass this problem we note that at least $\langle f, h\rangle \in L^p(U; L^q(V))$ for every $h \in L^2[0,T]$.

This then leads to the following definition of *adaptedness*.

**Definition 1.1.7.** Let $p, q, r \in (1, \infty)$ and let $f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then we call $f$ an *adapted $L^r$ process* with respect to a filtration $\mathbb{F}$ if

$$\langle f, \mathbb{1}_{[0,t]}\rangle_{L^2} = \int_0^t f(s)\,\mathrm{d}s\colon \Omega \to L^p(U; L^q(V))$$

is strongly $\mathcal{F}_t$-measurable for every $t \in [0,T]$. We denote by $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ the closed subspace of $L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$ of all $\mathbb{F}$-adapted elements.

**Remark 1.1.8.** If we assume that a function $f\colon \Omega \times [0,T] \times U \times V \to \mathbb{R}$ is $(\mathcal{A} \otimes \mathcal{B}_{[0,T]} \otimes \Sigma \otimes \Xi)$-measurable such that additionally $f(t,u,v)\colon \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$ and

$$(+) \qquad \mathbb{E}\Big\| \Big(\int_0^T |f(t)|^2\,\mathrm{d}t\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} < \infty,$$

then $\langle f, \mathbb{1}_{[0,t]}\rangle_{L^2}\colon \Omega \to L^p(U; L^q(V))$ is well-defined for any $t \in [0,T]$ and by Fubini's theorem

$$\langle\langle f, \mathbb{1}_{[0,t]}\rangle_{L^2}, g\rangle_{L^p(U;L^q(V))} = \int_{[0,t]\times U\times V} f(s,\cdot) g\,\mathrm{d}(s \otimes \mu \otimes \nu)$$

is $\mathcal{F}_t$-measurable. Thus, the Pettis measurability theorem yields the strong measurability of $\langle f, \mathbb{1}_{[0,t]}\rangle_{L^2}$. This means that measurable functions which are adapted to a filtration $\mathbb{F}$ in the classical way and fulfill (+) are elements of $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$.

Moreover, if we define the linear space

$$D_{\mathbb{F}} := \{f\colon \Omega \times [0,T] \to L^p(U; L^q(V))\colon f \text{ is an adapted step process}\},$$

then we have the following density result.

**Proposition 1.1.9.** *Let $p, q, r \in (1, \infty)$. Then the closure of $D_{\mathbb{F}}$ with respect to the norm of $L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$ is equal to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$.*

**Proof.** 1) We start with some preliminary remarks. For $\delta > 0$ we define the shift operator

$$S_\delta\colon L^2[0,T] \to L^2[0,T], \qquad (S_\delta h)(t) := \begin{cases} h(t-\delta), & \text{for } t \in (\delta, T], \\ 0, & \text{for } t \in [0,\delta]. \end{cases}$$

Then $\|S_\delta h - h\|_{L^2[0,T]} \to 0$ as $\delta \to 0$. Now let $f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then, by the dominated convergence theorem (using the pointwise estimate $\|S_\delta f\|_{L^2[0,T]} \leq \|f\|_{L^2[0,T]}$) we get

$$\|S_\delta f - f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} \to 0 \quad \text{as } \delta \to 0.$$

Moreover, for $N \in \mathbb{N}$, we let $P_N$ be the orthogonal projection on $\operatorname{lin}\{\mathbb{1}_{(0,\frac{T}{N}]}, \dots, \mathbb{1}_{((N-1)\frac{T}{N},T]}\}$. Then $\|P_N h - h\|_{L^2[0,T]} \to 0$ as $N \to \infty$ and each $h \in L^2[0,T]$. Similar to the first case, the dominated convergence theorem yields the convergence

$$\|P_N f - f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} \to 0 \quad \text{as } N \to \infty$$

for any $f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))$.

2) Now let $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, and set $t_n := n\frac{T}{N}$ for $n \in \{0, 1, \dots, N\}$. Then we define

$$\begin{aligned} g_{N,\delta}(\omega,t) := (P_N S_\delta f)(\omega,t) &= \sum_{n=1}^{N} \tfrac{N}{T} \int_{t_{n-1}}^{t_n} (S_\delta f(\omega))(s)\,\mathrm{d}s\; \mathbb{1}_{(t_{n-1},t_n]}(t) \\ &= \sum_{n=1}^{N} \tfrac{N}{T} \int_{(t_{n-1}-\delta)\vee 0}^{(t_n-\delta)\vee 0} f(\omega,s)\,\mathrm{d}s\; \mathbb{1}_{(t_{n-1},t_n]}(t). \end{aligned}$$

Let $\varepsilon > 0$. First, choose $\delta_\varepsilon > 0$ so small that

$$\|f - S_{\delta_\varepsilon} f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} < \frac{\varepsilon}{3}.$$

Then, take $N_\varepsilon^{(1)} \in \mathbb{N}$ such that

$$\|S_{\delta_\varepsilon} f - P_N S_{\delta_\varepsilon} f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} < \frac{\varepsilon}{3} \qquad \text{for } N \geq N_\varepsilon^{(1)}.$$

Next, take $N_\varepsilon^{(2)} \in \mathbb{N}$ such that $\frac{T}{N_\varepsilon^{(2)}} < \delta_\varepsilon$. Now fix any $N \geq N_\varepsilon := \max\{N_\varepsilon^{(1)}, N_\varepsilon^{(2)}\}$, then $t_n - \delta_\varepsilon < t_{n-1}$ for $n = 1, \dots, N$, i.e. the random variable

$$X_n(\omega) := \int_{t_{n-1}}^{t_n} (S_{\delta_\varepsilon} f(\omega))(s)\,\mathrm{d}s = \int_{(t_{n-1}-\delta_\varepsilon)\vee 0}^{(t_n-\delta_\varepsilon)\vee 0} f(\omega,s)\,\mathrm{d}s$$

is strongly $\mathcal{F}_{t_{n-1}}$-measurable, by definition. As a consequence, the random variables $X_n$

are elements of $L^r(\Omega, \mathcal{F}_{t_{n-1}}; L^p(U; L^q(V)))$, $n \in \{1, \dots, N\}$. Then, however, we can choose $\mathcal{F}_{t_{n-1}}$-measurable simple random variables $Y_n \colon \Omega \to L^p(U; L^q(V))$ such that

$$\|Y_n - X_n\|_{L^r(\Omega; L^p(U; L^q(V)))} < \frac{\varepsilon}{3N(\frac{N}{T})^{1/2}}, \quad n \in \{1, \dots, N\}.$$

Finally, define $f_N := \sum_{n=1}^N \mathbb{1}_{(t_{n-1}, t_n]} \frac{N}{T} Y_n$. By construction, $f_N$ is an adapted step process satisfying

$$\begin{aligned} \|g_{N,\delta_\varepsilon} - f_N\|_{L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))} &\le \sum_{n=1}^N \tfrac{N}{T} \|\mathbb{1}_{(t_{n-1}, t_n]}\|_{L^2[0,T]} \|Y_n - X_n\|_{L^r(\Omega; L^p(U; L^q(V)))} \\ &= \sum_{n=1}^N \big(\tfrac{N}{T}\big)^{1/2} \|Y_n - X_n\|_{L^r(\Omega; L^p(U; L^q(V)))} < \frac{\varepsilon}{3}. \end{aligned}$$

Collecting all estimates, we obtain for $N \ge N_\varepsilon$

$$\|f - f_N\|_{L^r(\Omega; L^p(U; L^q(V; L^2[0,T])))} < \varepsilon. \qquad \square$$

**Remark 1.1.10.** If we take a closer look at the previous proof, we see that the constructed sequence $(f_N)_{N \in \mathbb{N}}$ of adapted step processes also converges almost surely and in $L^1(\Omega; L^p(U; L^q(V; L^2[0,T])))$ to $f$. In particular, this implies that $\lim_{N \to \infty} f_N = f$ in $L^p(U; L^q(V; L^2[0,T]))$ in probability.

Using Proposition 1.1.9, we finally obtain the following extension to Proposition 1.1.5.

**Definition/Theroem 1.1.11 (Itô isomorphism for adapted $L^r$ processes).** For every $p, q, r \in (1, \infty)$ the stochastic integral defined in Definition 1.1.3 extends uniquely to a bounded linear operator

$$I_{L^r} \colon L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \to L^r(\Omega, \mathcal{F}_T; L^p(U; L^q(V))),$$

which is an isomorphism onto its range and satisfies

$$\mathbb{E}\big\|I_{L^r}(f)\big\|^r_{L^p(U; L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big(\int_0^T |f(t)|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|^r_{L^p(U; L^q(V))}.$$

For a process $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ we then define the stochastic integral of $f$ by

$$\int_0^T f \,\mathrm{d}\beta := I_{L^r}(f)$$

and say that $f$ is *$L^r$-stochastically integrable.*

**Remark 1.1.12.** Observe that the map $I_{L^r}$ depends on the Brownian motion and the chosen filtration we fixed in the beginning. For example, if we choose the Brownian filtration $\mathbb{F}^\beta$ we get representation results for $L^p(U;L^q(V))$-valued random variables $X$ as in the scalar case (see [3, Theorem 3.27]), i.e.

$$X = \mathbb{E}X + \int_0^T f \,\mathrm{d}\beta$$

for a unique $f \in L^r_{\mathbb{F}^\beta}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. The map $I_{L^r}$ then leads to an isomorphism of Banach spaces

$$L^r_{\mathbb{F}^\beta}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \simeq L^r_0(\Omega, \mathcal{F}^\beta_T; L^p(U; L^q(V))),$$

where $L^r_0(\Omega, \mathcal{F}^\beta_T; L^p(U; L^q(V)))$ is the closed subspace of $L^r(\Omega, \mathcal{F}^\beta_T; L^p(U; L^q(V)))$ consisting of all elements with mean 0.

Having now finished the construction process of the stochastic integral for general processes $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, we collect some more or less elementary properties of it.

**Proposition 1.1.13 (Properties of the Itô integral).** *Let $p, q, r \in (1, \infty)$ and $f, g \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then the following properties hold:*

*a) The stochastic integral is linear, i.e. for $a, b \in \mathbb{R}$ we have*

$$\int_0^T af + bg \,\mathrm{d}\beta = a\int_0^T f \,\mathrm{d}\beta + b\int_0^T g \,\mathrm{d}\beta.$$

*b) $\int_0^T f \,\mathrm{d}\beta$ is $\mathcal{F}_T$-measurable and the expected value satisfies $\mathbb{E}\int_0^T f \,\mathrm{d}\beta = 0$.*

*c) For $S \in \mathcal{B}\big(L^p(U; L^q(V))\big)$, let $S^{L^2}$ be the bounded extension of $S$ on the space $L^p(U; L^q(V; L^2[0,T]))$ (see Remark 2.4.1). Then, $S^{L^2} f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and*

$$\int_0^T S^{L^2} f \,\mathrm{d}\beta = S\int_0^T f \,\mathrm{d}\beta.$$

*d) For every $s, t \in [0,T]$ with $s < t$ it holds that*

$$\int_s^t f \,\mathrm{d}\beta = \int_0^T \mathbb{1}_{[s,t]} f \,\mathrm{d}\beta.$$

*e) There exists a $\mu$-null set $N_\mu \in \Sigma$ such that $f(u)$ is $L^{p\wedge r}$-stochastically integrable, i.e. $f(u) \in L^{p\wedge r}_{\mathbb{F}}(\Omega; L^q(V; L^2[0,T]))$, and*

$$\int_0^T f(u) \,\mathrm{d}\beta = \Big(\int_0^T f \,\mathrm{d}\beta\Big)(u) \quad \text{for each } u \in U \setminus N_\mu.$$

*f) There exists a $\mu \otimes \nu$-null set $N \in \Sigma \otimes \Xi$ such that $f(u,v)$ is $L^{p\wedge q\wedge r}$-stochastically integrable, i.e. $f(u,v) \in L^{p\wedge q\wedge r}_{\mathbb{F}}(\Omega; L^2[0,T])$, and*

$$\int_0^T f(u,v)\,\mathrm{d}\beta = \Big(\int_0^T f\,\mathrm{d}\beta\Big)(u,v) \quad \textit{for each } (u,v) \in (U\times V)\setminus N.$$

**Proof.** a) is trivial. For adapted step processes the proof of b) for the $L^p$ case can be found in [3, Proposition 3.17] and can be done for the mixed $L^p$ case in the same way. The general case then follows by approximation. For part d) see [3, Corollary 3.25].

c) If $f$ is an adapted step process, then $Sf$ is obviously an adapted step process, too, and

$$\int_0^T Sf\,\mathrm{d}\beta = S\int_0^T f\,\mathrm{d}\beta.$$

Now let $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then there exists a sequence of adapted step processes $(f_n)_{n\in\mathbb{N}}$ such that

$$\|f - f_n\|_{L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))} \to 0 \qquad \text{as } n\to\infty.$$

Since $S^{L^2}$ is continuous from $L^p(U; L^q(V; L^2[0,T]))$ to $L^{\widetilde{p}}(\widetilde{U}; L^{\widetilde{q}}(\widetilde{V}; L^2[0,T]))$ we immediately obtain

$$\|S^{L^2}f - Sf_n\|_{L^r_{\mathbb{F}}(\Omega;L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V};L^2[0,T])))} = \|S^{L^2}(f-f_n)\|_{L^r_{\mathbb{F}}(\Omega;L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V};L^2[0,T])))} \to 0 \quad \text{as } n\to\infty.$$

Thus, $S^{L^2}f \in L^r_{\mathbb{F}}(\Omega; L^{\widetilde{p}}(\widetilde{U}; L^{\widetilde{q}}(\widetilde{V}; L^2[0,T])))$ and

$$\int_0^T S^{L^2}f\,\mathrm{d}\beta = \lim_{n\to\infty}\int_0^T Sf_n\,\mathrm{d}\beta = \lim_{n\to\infty} S\int_0^T f_n\,\mathrm{d}\beta = S\int_0^T f\,\mathrm{d}\beta$$

using Theorem 1.1.11, the estimate for adapted step processes, and the continuity of $S$.

e) If $f$ is an adapted step process, then by definition

$$\Big(\int_0^T f\,\mathrm{d}\beta\Big)(\omega,u) = \sum_{n=1}^N v_n(\omega,u)\big(\beta(\omega,t_n) - \beta(\omega,t_{n-1})\big) = \Big(\int_0^T f(u)\,\mathrm{d}\beta\Big)(\omega)$$

for all $\omega\in\Omega$ and $u\in U$. Moreover, by Minkowski's integral inequality ($p \ge r$) or Hölder's inequality ($p<r$) we have

$$f \in L^r(\Omega; L^p(U; L^q(V; L^2[0,T]))) \subseteq L^p(U; L^{p\wedge r}(\Omega; L^q(V; L^2[0,T])))$$

and there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of adapted step processes such that

$$\|f_n - f\|_{L^p(U;L^{p\wedge r}(\Omega;L^q(V;L^2[0,T])))} \le \|f_n - f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} \to 0$$

as $n \to \infty$. In any case, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a $\mu$-null set $N_0 \in \Sigma$ such that

$$\|f_{n_k}(u) - f(u)\|_{L^{p\wedge r}(\Omega;L^q(V;L^2[0,T]))} \to 0 \qquad \text{as } k \to \infty$$

for every $u \notin N_0$. In particular, $f(u)$ is adapted to the filtration $\mathbb{F}$, i.e. $f(u) \in L^{p\wedge r}_{\mathbb{F}}(\Omega; L^q(V; L^2[0,T]))$ for every $u \notin N_0$. As a consequence of this pointwise convergence we obtain

$$\Big\| \int_0^T f_{n_k}(u)\,\mathrm{d}\beta - \int_0^T f(u)\,\mathrm{d}\beta \Big\|_{L^{p\wedge r}(\Omega;L^q(V))} \to 0 \qquad (k \to \infty)$$

by Theorem 1.1.11. The same argument as above yields a $\mu$-null set $N_1 \in \Sigma$ such that

$$\Big\| \Big(\int_0^T f_{n_{k_j}}\,\mathrm{d}\beta\Big)(u) - \Big(\int_0^T f\,\mathrm{d}\beta\Big)(u) \Big\|_{L^{p\wedge r}(\Omega;L^q(V))} \to 0 \qquad (j \to \infty)$$

for every $u \notin N_1$. Combining now the estimate for adapted step processes with these convergence results finally leads to

$$\Big(\int_0^T f\,\mathrm{d}\beta\Big)(u) = \int_0^T f(u)\,\mathrm{d}\beta \qquad \text{in } L^{p\wedge r}(\Omega; L^q(V))$$

for every $u \in U \setminus (N_0 \cup N_1)$.

f) The proof here is done in nearly the same way as e). First, observe that, without loss of generality, we can assume that $\mu(U), \nu(V) < \infty$. Then by Hölder's and/or Minkowski's inequality

$$L^r(\Omega; L^p(U; L^q(V; L^2[0,T]))) \subseteq L^{p\wedge q}(U \times V; L^{p\wedge q\wedge r}(\Omega; L^2[0,T]))$$

with corresponding norm estimates, i.e. we can follow the lines of the proof of e). □

**REMARK 1.1.14.**

a) In part c) of the previous proposition we also could have considered a bounded operator mapping from one mixed $L^p$ space to another. More precisely, if $\widetilde{p}, \widetilde{q} \in (1,\infty)$ and $(\widetilde{U}, \widetilde{\Sigma}, \widetilde{\mu})$ and $(\widetilde{V}, \widetilde{\Xi}, \widetilde{\nu})$ are $\sigma$-finite measure spaces, operators like

$$S \in \mathcal{B}\big(L^p(U; L^q(V)), L^{\widetilde{p}}(\widetilde{U}; L^{\widetilde{q}}(\widetilde{V}))\big) \quad \text{or} \quad S \in \mathcal{B}\big(L^p(U; L^q(V)), L^{\widetilde{p}}(\widetilde{U})\big)$$

or other combinations can be considered.

b) If $T\colon L^p(U; L^q(V)) \to \mathbb{C}$ is linear and bounded, then the Riesz representation theorem gives a $g \in L^{p'}(U; L^{q'}(V))$ such that

$$Tf = \int_{U\times V} fg\,\mathrm{d}(\mu\otimes\nu), \quad f \in L^p(U; L^q(V)).$$

For every Hilbert space $H$ we also have the extension $T^H\colon L^p(U;L^q(V;H)) \to H$ which is now again given by the function $g$ above, i.e.

$$T^H f = \int_{U\times V} fg \,\mathrm{d}(\mu\otimes\nu), \quad f\in L^p(U;L^q(V;H)),$$

where now the integral takes values in $H$. This can be seen directly by computing $T^H f$ for $f=\sum_{n=1}^N f_n\otimes h_n \in L^p(U;L^q(V))\otimes H$ and finally using that these functions are dense in $L^p(U;L^q(V;H))$. In particular, we get

$$\Big\langle \int_0^T f\,\mathrm{d}\beta, g\Big\rangle_{L^p(U;L^q(V))} = \int_0^T \langle f,g\rangle^{L^2}_{L^p(U;L^q(V))}\,\mathrm{d}\beta,$$

where $\langle f,g\rangle^{L^2}_{L^p(U;L^q(V))} := \int_{U\times V} fg\,\mathrm{d}(\mu\otimes\nu)$ is now $L^2[0,T]$-valued.

c) If $f\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))$ such that $(u,v)\mapsto f(u,v,t)\in L^p(U;L^q(V))$ almost surely for each $t\in[0,T]$, then in Proposition 1.1.13 c) we have $(S^{L^2}f)(t) = S(f(t))$ and the assertion there reads as

$$\int_0^T Sf\,\mathrm{d}\beta = S\int_0^T f\,\mathrm{d}\beta.$$

In particular, we have for every $g\in L^{p'}(U;L^{q'}(V))$

$$\Big\langle \int_0^T f\,\mathrm{d}\beta, g\Big\rangle_{L^p(U;L^q(V))} = \int_0^T \langle f,g\rangle_{L^p(U;L^q(V))}\,\mathrm{d}\beta.$$

d) In part e) and f) of Proposition 1.1.13 we have seen how the $L^p(U;L^q(V))$-valued integral behaves in comparison to the $L^q(V)$-valued and $\mathbb{R}$-valued case. In the future we will also be interested in the connection to the $L^p(U)$-valued integral. Here, the question is if there might exist a $\nu$-null set $N_\nu$ such that $f(v)\in L^{\widetilde{r}}_{\mathbb{F}}(\Omega;L^p(U;L^2[0,T]))$ for some $\widetilde{r}\in(1,\infty)$ and $v\notin N_\nu$. In general, the answer here is no. Nevertheless, there still exist positive results. E.g. if $q\geq p$, then $L^p(U;L^q(V))\subseteq L^q(V;L^p(U))$ by Minkowski's inequality which leads to

$$L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))\subseteq L^q(V;L^{q\wedge r}(\Omega;L^p(U;L^2[0,T])))$$

and we can continue as in the proof of part e).

e) Another way to get the same result exists if we have more knowledge about the process $f$. For example, if we assume that $f\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))$ such that $f(v)\in L^{\widetilde{r}}_{\mathbb{F}}(\Omega;L^p(U;L^2[0,T]))$ for some $\widetilde{r}\in(1,\infty)$ and $\nu$-almost every $v\in V$, then we have

$$\int_0^T f(v)\,\mathrm{d}\beta = \Big(\int_0^T f\,\mathrm{d}\beta\Big)(v) \quad \text{for } \nu\text{-almost every } v\in V.$$

This follows from Proposition 1.1.13 e) and f), since we can find a $\nu$-nullset $N_\nu$ and for each $v \notin N_\nu$ and $\mu$-null set $N_{\mu,v}$ such that for each fixed $v \notin N_\nu$ we have

$$\Big(\int_0^T f(v)\,\mathrm{d}\beta\Big)(u) = \int_0^T f(u,v)\,\mathrm{d}\beta = \Big(\int_0^T f\,\mathrm{d}\beta\Big)(u,v)$$

for $u \notin N_{\mu,v}$, where equality holds in $L^{\overline{r}}(\Omega)$ for $\overline{r} = \widetilde{r} \wedge r \wedge p \wedge q$. This implies that $\int_0^T f(v)\,\mathrm{d}\beta = \big(\int_0^T f\,\mathrm{d}\beta\big)(v)$ for $v \notin N_\nu$ with equality in $L^{\widetilde{r}}(\Omega; L^p(U))$.

A basic tool in the deterministic integration theory is to interchange the order of integration. In the next theorem we will show under which condition we can interchange a stochastic integral and a Lebesgue integral. Note that this condition is quite strong. In the next section, we will see a beautiful generalization of this result using localization techniques.

**THEOREM 1.1.15 (Stochastic Fubini theorem I).** *Let $p, q, r \in (1,\infty)$, $(K, \mathcal{K}, \theta)$ be a $\sigma$-finite measure space, and let $f \in L^1(K; L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))))$. Then*

$$\int_K \int_0^T f(x,s)\,\mathrm{d}\beta(s)\,\mathrm{d}\theta(x) = \int_0^T \int_K f(x,s)\,\mathrm{d}\theta(x)\,\mathrm{d}\beta(s).$$

**PROOF.** By assumption, $f(x) \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for almost every $x \in K$ and by Theorem 1.1.11

$$x \mapsto \int_0^T f(x,s)\,\mathrm{d}\beta(s) \in L^1(K; L^r(\Omega; L^p(U; L^q(V)))).$$

By Minkowski's integral inequality (and Fubini's theorem for adaptedness) we also have $\int_K f(x)\,\mathrm{d}\theta(x) \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Now the estimate trivially follows from the continuity of the stochastic integral operator, i.e.

$$\begin{aligned}\int_K \int_0^T f(x,s)\,\mathrm{d}\beta(s)\,\mathrm{d}\theta(x) &= \int_K I_{L^r} f(x)\,\mathrm{d}\theta(x) = I_{L^r} \int_K f(x)\,\mathrm{d}\theta(x)\\ &= \int_0^T \int_K f(x,s)\,\mathrm{d}\theta(x)\,\mathrm{d}\beta(s).\end{aligned}$$ □

In the last part of this section we want to collect properties of the stochastic integral process $t \mapsto \int_0^t f\,\mathrm{d}\beta$. For this reason we will need maximal inequalities for our stochastic integral. In order to get these estimates we will use maximal inequalities arising from martingale theory.

**THEOREM 1.1.16 (Strong Doob inequality).** *Let $p, q, r \in (1,\infty)$ and $(M_n)_{n=1}^N$ be an $L^p(U; L^q(V))$-valued $L^r$ martingale with respect to $\mathbb{F}$. Then we have*

$$\mathbb{E}\big\|\max_{n=1}^{N} |M_n|\big\|^r_{L^p(U;L^q(V))} \lesssim_{p,q,r} \mathbb{E}\|M_N\|^r_{L^p(U;L^q(V))}.$$

**Proof.** The $L^q(V)$-valued case was treated in [3, Section 2.2]. To extend this to the $L^p(U;L^q(V))$-valued case we proceed 'inductively' and very similarly to the $L^q(V)$-valued case. The proof of this estimate consists of two steps. The first one is a reduction procedure showing that it suffices to proof the estimate for a special class of martingales, so called *Haar martingales*. The second step is then to show the estimate for these martingales.

1) The reduction process itself consists of three steps and can be done in exactly the same way for the $L^p(U;L^q(V))$-valued case as for the $L^q(V)$-valued case (cf. [3, Section 2.2.1]). In the first step we show that we can limit ourselves to *divisible* probability spaces $(\Omega,\mathcal{F},\mathbb{P})$, where divisible means that for all $A\in\mathcal{F}$ and $s\in(0,1)$ we can find sets $A_1,A_2\in\mathcal{F}$ such that $A=A_1\cup A_2$ and

$$\mathbb{P}(A_1)=s\mathbb{P}(A),\quad \mathbb{P}(A_2)=(1-s)\mathbb{P}(A).$$

The second and third step consist of reducing the assumptions on our filtration $(\mathcal{F}_n)_{n=1}^N$ which will lead to a special structure of the considered martingale. We first look at *dyadic* $\sigma$-algebras $(\mathcal{F}_n)_{n=1}^N$, i.e. each $\sigma$-algebra $\mathcal{F}_n$ is generated by $2^{m_n}$ disjoints sets of measure $2^{-m_n}$ for some integer $m_n\in\mathbb{N}$. In the final step we reduce this further to the class of *Haar filtrations*. This is a filtration $(\mathcal{F}_n)_{n=1}^N$ where $\mathcal{F}_1=\{\emptyset,\Omega\}$ and for $n\in\mathbb{N}$ each $\mathcal{F}_n$ is created from $\mathcal{F}_{n-1}$ by dividing precisely one atom of $\mathcal{F}_{n-1}$ of maximal measure into two sets of equal measure. By construction, each $\mathcal{F}_n$ is then generated by $n$ atoms of measure $2^{-k-1}$ or $2^{-k}$, where $k$ is the unique integer such that $2^{k-1}<n\leq 2^k$. The main advantage of a Haar martingale $(M_n)_{n=1}^N$ (i.e. a martingale with respect to a Haar filtration) is that $\|M_{n+1}-M_n\|_{L^p(U;L^q(V))}$ is $\mathcal{F}_n$-measurable. This predictability condition will imply that a special martingale transform is again a martingale.

2) By the reduction procedure it is sufficient to consider an $L^p(U;L^q(V))$-valued $L^r$ martingale $(M_n)_{n=1}^N$ with respect to a Haar filtration $(\mathcal{F}_n)_{n=1}^N$. Then we define

$$M^*(\omega):=\max_{n=1}^N\|M_n(\omega)\|_{L^p(U;L^q(V))},\quad \widetilde{M}^*(\omega):=\big\|\max_{n=1}^N|M_n(\omega)|\big\|_{L^p(U;L^q(V))}.$$

For the moment let $(V_n)_{n=1}^N$ be an arbitrary $L^p(U;L^q(V))$-valued $L^p$ martingale. Then $(V_n(u))_{n=1}^N$ is an $L^q(V)$-valued martingale for $\mu$-almost every $u\in U$. Thus, by the strong Doob inequality for the $L^q(V)$-valued case, we obtain

$$\mathbb{E}\big\|\max_{n=1}^N|V_n(u)|\big\|^p_{L^q(V)}\leq c^p_{p,q}\mathbb{E}\|V_N(u)\|^p_{L^q(V)}.$$

Then, by Fubini's theorem

$$\begin{aligned}\lambda^p\mathbb{P}\big(\widetilde{V}^*>\lambda\big)\leq\mathbb{E}(\widetilde{V}^*)^p&=\int_U\mathbb{E}\big\|\max_{n=1}^N|V_n(u)|\big\|^p_{L^q(V)}\,\mathrm{d}\mu(u)\\&\leq c^p_{p,q}\int_U\mathbb{E}\|V_N(u)\|^p_{L^q(V)}=\mathbb{E}\|V_N\|^p_{L^p(U;L^q(V))}\end{aligned}$$

for each $\lambda > 0$. This weak estimate plays a central role in the proof of the following good-$\lambda$-inequality: For all $\delta > 0$, $\beta > 2\delta + 1$, and all $\lambda > 0$ we have

$$\mathbb{P}\big(\widetilde{M}^* > \beta\lambda, M^* \le \delta\lambda\big) \le \alpha(\delta)^p \mathbb{P}\big(\widetilde{M}^* > \lambda\big),$$

where $\alpha(\delta) := c_{p,q}\frac{4\delta}{\beta-2\delta-1} \to 0$ as $\delta \to 0$. This estimate is the heart of the proof of the strong Doob inequality and can be shown in the same way as for the $L^q(V)$-valued case (see [3, Lemma 2.19]). The main idea is to construct a martingale transform $(V_n)_{n=1}^N$, which is again a martingale since we work with a Haar filtration, and using the estimate above. Note that until this point everything we have proved is independent of $r$. In the final step we bring this back into play. Using

$$\begin{aligned}\mathbb{P}\big(\widetilde{M}^* > \beta\lambda\big) &\le \mathbb{P}\big(\widetilde{M}^* > \beta\lambda,\, M^* \le \delta\lambda\big) + \mathbb{P}(M^* > \delta\lambda)\\ &\le \alpha(\delta)^p \mathbb{P}\big(\widetilde{M}^* > \lambda\big) + \mathbb{P}(M^* > \delta\lambda),\end{aligned}$$

we obtain by Doob's inequality

$$\begin{aligned}\mathbb{E}|\widetilde{M}^*|^r &= \int_0^\infty r\lambda^{r-1}\mathbb{P}\big(\widetilde{M}^* > \lambda\big)\,\mathrm{d}\lambda\\ &= \beta^r \int_0^\infty r\lambda^{r-1}\mathbb{P}\big(\widetilde{M}^* > \beta\lambda\big)\,\mathrm{d}\lambda\\ &\le \alpha(\delta)^p\beta^r \int_0^\infty r\lambda^{r-1}\mathbb{P}\big(\widetilde{M}^* > \lambda\big)\,\mathrm{d}\lambda + \beta^r \int_0^\infty r\lambda^{r-1}\mathbb{P}(M^* > \delta\lambda)\,\mathrm{d}\lambda\\ &= \alpha(\delta)^p\beta^r\mathbb{E}|\widetilde{M}^*|^r + \frac{\beta^r}{\delta^r}\mathbb{E}|M^*|^r\\ &\le \alpha(\delta)^p\beta^r\mathbb{E}|\widetilde{M}^*|^r + \frac{\beta^r}{\delta^r}\Big(\frac{r}{r-1}\Big)^r\mathbb{E}\|M_N\|^r_{L^p(U;L^q(V))}.\end{aligned}$$

Since $\lim_{\delta\to 0}\alpha(\delta) = 0$, we may take $\delta > 0$ small enough such that $\alpha(\delta)^p\beta^r < 1$. By recalling that $(M_n)_{n=1}^N$ is an $L^r$ martingale, we note that $\mathbb{E}|\widetilde{M}^*|^r < \infty$. Then we get

$$\mathbb{E}|\widetilde{M}^*|^r \le \frac{\beta^r\big(\frac{r}{r-1}\big)^r}{(1-\alpha(\delta)^p\beta^r)\delta^r}\mathbb{E}\|M_N\|^r_{L^p(U;L^q(V))}. \qquad \square$$

Using similar techniques we also obtain the following stronger version of the Burkholder-Davis-Gundy inequality.

**Theorem 1.1.17 (Strong Burkholder-Davis-Gundy inequality).** *Let $p, q \in (1,\infty)$, $r \in [1,\infty)$, and $(M_n)_{n=1}^N$ be an $L^p(U;L^q(V))$-valued $L^r$ martingale with respect to $\mathbb{F}$. Then we have*

$$\mathbb{E}\Big\|\max_{n=1}^N |M_n|\Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\|\Big(\sum_{n=1}^N |M_n - M_{n-1}|^2\Big)^{1/2}\Big\|^r_{L^p(U;L^q(V))}.$$

**Proof.** For the case $r \in (1,\infty)$ this estimate is a consequence of the strong Doob inequality. In fact, using Kahane's inequality for Rademacher sums as well as the UMD property of the space $L^p(U; L^q(V))$, we obtain

$$\begin{aligned}\mathbb{E}\Big\| \Big(\sum_{n=1}^N |M_n - M_{n-1}|^2\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} &\eqsim_{p,q,r} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \sum_{n=1}^N \widetilde{r}_n (M_n - M_{n-1}) \Big\|^r_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,r} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \sum_{n=1}^N (M_n - M_{n-1}) \Big\|^r_{L^p(U;L^q(V))} \\ &= \mathbb{E}\|M_N\|^r_{L^p(U;L^q(V))},\end{aligned}$$

and Theorem 1.1.16 yields the claim. So we only have to take a closer look on the case $r=1$. Here we will proceed similarly to the proof of Theorem 1.1.16.

1) The reduction to Haar martingales can be done almost exactly as in the previous proof. Only in the transition from dyadic filtrations to Haar filtrations we have to be a little bit more careful. In this case we have to examine the structure of Haar martingales more closely in order to prove the statement.

2) Having finished the reduction procedure, we let $(M_n)_{n=1}^N$ be an $L^p(U; L^q(V))$-valued $L^r$ martingale with respect to a Haar filtration $(\mathcal{F}_n)_{n=1}^N$. Similar to the proof of Theorem 1.1.16 we obtain for $M^*$ and $\widetilde{M}^*$ the same good-$\lambda$-inequality as before, i.e.

$$\mathbb{P}\big(\widetilde{M}^* > \beta\lambda, M^* \le \delta\lambda\big) \le \alpha(\delta)^p \mathbb{P}\big(\widetilde{M}^* > \lambda\big)$$

for all $\delta > 0$, $\beta > 2\delta + 1$, and all $\lambda > 0$, where $\alpha(\delta) := c_{p,q} \frac{4\delta}{\beta - 2\delta - 1}$. Note that up to now everything was independent of $r$. In the proof of Theorem 1.1.16 this inequality and Doob's maximal inequality yielded the claim. In the case $r=1$ Doob's inequality is no longer available and we replace it with the Burkholder-Davis-Gundy inequality, i.e. we use

$$\mathbb{E} \max_{n=1}^N \|M_n\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big(\sum_{n=1}^N |M_n - M_{n-1}|^2\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))},$$

where this is true for any $r \in [1,\infty)$ (see [67, Poposition 5.36]). Using this, we then obtain

$$\begin{aligned}\mathbb{E}|\widetilde{M}^*| &= \int_0^\infty \mathbb{P}\big(\widetilde{M}^* > \lambda\big)\,\mathrm{d}\lambda = \beta \int_0^\infty \mathbb{P}\big(\widetilde{M}^* > \beta\lambda\big)\,\mathrm{d}\lambda \\ &\le \alpha(\delta)^p \beta \int_0^\infty \mathbb{P}\big(\widetilde{M}^* > \lambda\big)\,\mathrm{d}\lambda + \beta \int_0^\infty \mathbb{P}(M^* > \delta\lambda)\,\mathrm{d}\lambda \\ &= \alpha(\delta)^p \beta \mathbb{E}|\widetilde{M}^*|^r + \frac{\beta}{\delta}\mathbb{E}|M^*| \\ &\le \alpha(\delta)^p \beta \mathbb{E}|\widetilde{M}^*|^r + \frac{\beta}{\delta} c_{p,q,1} \mathbb{E}\Big\| \Big(\sum_{n=1}^N |M_n - M_{n-1}|^2\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}.\end{aligned}$$

Again, we choose $\delta > 0$ small enough such that $\alpha(\delta)^p\beta < 1$. We then finally get

$$\mathbb{E}|\widetilde{M}^*| \leq \frac{\beta c_{p,q,1}}{(1-\alpha(\delta)^p\beta)\delta}\mathbb{E}\Big\| \Big(\sum_{n=1}^N |M_n - M_{n-1}|^2\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}. \qquad \square$$

Having these inequalities at hand, we obtain the following regularity results for the stochastic integral process $t \mapsto \int_0^t f \, \mathrm{d}\beta$.

**THEOREM 1.1.18 (Properties of the integral process).** *Let $p, q, r \in (1, \infty)$ and $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then the following properties hold:*

*a)* ***Martingale property.*** *The integral process $(\int_0^t f \, \mathrm{d}\beta)_{t\in[0,T]}$ is an $L^r$ martingale with respect to the filtration $\mathbb{F}$.*

*b)* ***Continuity.*** *The integral process $(\int_0^t f \, \mathrm{d}\beta)_{t\in[0,T]}$ has a continuous version satisfying the maximal inequality*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \, \mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \lesssim_{p,q,r} \mathbb{E}\Big\| \int_0^T f \, \mathrm{d}\beta \Big\|^r_{L^p(U;L^q(V))}.$$

*c)* ***Burkholder-Davis-Gundy inequality.*** *As a consequence of b) and Theorem 1.1.11 we have*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \, \mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big( \int_0^T |f(t)|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

*Moreover, this estimate also holds in the case $r = 1$. In particular, the process $X(t) := \int_0^t f \, \mathrm{d}\beta$, $t \in [0,T]$, is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\Big\| \Big( \int_0^T |X(t)|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} \lesssim_{p,q,r} T^{1/2}\mathbb{E}\Big\| \Big( \int_0^T |f(t)|^2 \, \mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

**PROOF.** For the proofs of a) and b) see Proposition 3.30 and Theorem 3.31 in [3], and observe that the $L^p(U; L^q(V))$-valued case can be treated in the exact same way, now using Theorem 1.1.16 for part b) instead of the Strong Doob inequality for the $L^q(V)$-valued case. The first part of c) is an easy consequence of b) and Itô's isomorphism, and the last part follows by an application of Hölder's inequality. So the only thing left to prove is the Burkholder-Davis-Gundy inequality in the case $r = 1$. We first do this for an adapted step process $f = \sum_{n=1}^N \mathbb{1}_{(t_{n-1},t_n]} v_n$, where $0 = t_0 < \ldots < t_N = T$ and $v_n$ are $\mathcal{F}_{t_{n-1}}$-measurable simple random variables in $L^1(\Omega; L^p(U; L^q(V)))$, $n \in \{1, \ldots, N\}$. Before turning to the estimate, we add an important remark. On the product space $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$ we define the processes

$$\beta_t^{(1)}(\omega, \omega') := \beta_t(\omega), \quad \beta_t^{(2)}(\omega, \omega') := \beta_t(\omega'), \quad t \in [0,T].$$

Then $\beta^{(1)}$ and $\beta^{(2)}$ are Brownian motions adapted to the filtrations

$$\mathcal{F}_t^{(1)} := \mathcal{F}_t \otimes \{\emptyset, \Omega\}, \quad \mathcal{F}_t^{(2)} := \{\emptyset, \Omega\} \otimes \mathcal{F}_t, \quad t \in [0,T],$$

and $\beta^{(1)}$ is an independent copy of $\beta^{(2)}$, in particular it is independent of $\sigma(\mathcal{F}_t^{(2)}, t \in [0,T])$. We also identify the predictable sequence $(v_n)_{n=1}^N$ with the random variables $v_n(\omega,\omega') = v_n(\omega)$, $n \in \{1,\dots,N\}$. Then by [19, Proposition 2 and Example 1] we have

$$\begin{aligned}\mathbb{E}\Big\| \sum_{n=1}^N v_n\big(\beta(t_n) - \beta(t_{n-1})\big) \Big\|_{L^p(U;L^q(V))} &= \mathbb{E}\mathbb{E}'\Big\| \sum_{n=1}^N v_n\big(\beta^{(1)}(t_n) - \beta^{(1)}(t_{n-1})\big) \Big\|_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,1} \mathbb{E}\mathbb{E}'\Big\| \sum_{n=1}^N v_n\big(\beta^{(2)}(t_n) - \beta^{(2)}(t_{n-1})\big) \Big\|_{L^p(U;L^q(V))},\end{aligned}$$

since $L^p(U;L^q(V))$ is a UMD space. Observe that in the last line of this estimate the random variables $v_n$ and the process $\beta^{(2)}$ actually live on different probability spaces. This *decoupling* plays an important role in this proof.

We let $X(t) := \int_0^t f \,\mathrm{d}\beta$, $t \in [0,T]$. By a), $X_n := X(t_n)$, $n \in \{1,\dots,N\}$, is a martingale with respect to the filtraion $\mathcal{F}_n := \mathcal{F}_{t_n}$, $n \in \{1,\dots,N\}$. Moreover, we have

$$X_n - X_{n-1} = v_n\big(\beta(t_n) - \beta(t_{n-1})\big),$$

and $\gamma_n' := \frac{1}{\sqrt{t_n - t_{n-1}}}\big(\beta^{(2)}(t_n) - \beta^{(2)}(t_{n-1})\big)$, $n \in \{1,\dots,N\}$, defines a sequence of independent standard Gaussian variables. Now the strong Burkholder-Davis-Gundy inequality, Kahane's inequality for Rademacher sums, and the decoupling property above lead to

$$\begin{aligned}\mathbb{E}\big\|\max_{n=1}^N |X_n|\big\|_{L^p(U;L^q(V))} &\eqsim_{p,q,1} \mathbb{E}\Big\| \Big(\sum_{n=1}^N |X_n - X_{n-1}|^2\Big)^{1/2} \Big\|_{L^p(U;L^q(V))} \\ &= \mathbb{E}\Big\| \Big(\sum_{n=1}^N \big|v_n\big(\beta(t_n) - \beta(t_{n-1})\big)\big|^2\Big)^{1/2} \Big\|_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,1} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \sum_{n=1}^N \widetilde{r}_n v_n\big(\beta(t_n) - \beta(t_{n-1})\big) \Big\|_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,1} \widetilde{\mathbb{E}}\mathbb{E}\mathbb{E}'\Big\| \sum_{n=1}^N \widetilde{r}_n v_n\big(\beta^{(2)}(t_n) - \beta^{(2)}(t_{n-1})\big) \Big\|_{L^p(U;L^q(V))} \\ &= \widetilde{\mathbb{E}}\mathbb{E}\mathbb{E}'\Big\| \sum_{n=1}^N \widetilde{r}_n v_n (t_n - t_{n-1})^{1/2}\gamma_n' \Big\|_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,1} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \Big(\sum_{n=1}^N |\widetilde{r}_n v_n|^2 (t_n - t_{n-1})\Big)^{1/2} \Big\|_{L^p(U;L^q(V))} \\ &= \mathbb{E}\Big\| \Big(\int_0^T |f|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}.\end{aligned}$$

Now let $0 = s_0 < \ldots < s_M = T$ be any partition of $[0, T]$. Then, by the estimate above,

$$\mathbb{E}\Big\|\max_{m=1}^{M} |X(s_m)|\Big\|_{L^p(U;L^q(V))} \eqsim_{p,q,1} \mathbb{E}\Big\| \Big(\int_0^T |f|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}.$$

The pathwise continuity and the monotone convergence theorem now imply

$$\mathbb{E}\Big\|\sup_{t\in[0,T]} |X(t)|\Big\|_{L^p(U;L^q(V))} \eqsim_{p,q,1} \mathbb{E}\Big\| \Big(\int_0^T |f|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}.$$

Finally, let $f \in L^1_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \subseteq L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. In the next section we will see that the stochastic integral $X$ of $f$ is well-defined as an element of $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$. By Remark 1.1.10 we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of adapted step processes converging to $f$ in $L^1_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Therefore, by the estimate above, the sequence $(\int_0^{(\cdot)} f_n \,\mathrm{d}\beta)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega; L^p(U; L^q(V; C[0,T])))$, and the limit $\widetilde{X}$ equals $X$ almost surely. Hence, we arrive at

$$\begin{aligned}\mathbb{E}\Big\|\sup_{t\in[0,T]} |X(t)|\Big\|_{L^p(U;L^q(V))} &= \lim_{n\to\infty} \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big|\int_0^t f_n \,\mathrm{d}\beta\Big| \Big\|_{L^p(U;L^q(V))} \\ &\eqsim_{p,q,1} \lim_{n\to\infty} \mathbb{E}\Big\| \Big(\int_0^T |f_n|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|_{L^p(U;L^q(V))} \\ &= \mathbb{E}\Big\| \Big(\int_0^T |f|^2 \,\mathrm{d}t\Big)^{1/2} \Big\|_{L^p(U;L^q(V))}. \end{aligned}$$ □

We want to stress that these results are much stronger than in the usual Banach space setting: here the supremum can be taken pointwise for each $(u, v) \in U \times V$. Basically, these results were the starting point for a new regularity theory for stochastic evolution equations.

## 1.2 Stopping Times and Localization

The Itô integral itself has beautiful properties and many estimates from the scalar stochastic integration theory can be generalized to the $L^p(U)$ or $L^p(U; L^q(V))$-valued setting without getting too technical. One of the main problems of this integral (both in the scalar and vector-valued case) is the strong integrability condition we demand on our 'stochastically integrable' functions $f$. The thing is that even many continuous functions do not fulfill this property. The usual way to bypass this problem is to stop those 'bad' processes when they get 'too big' and somehow try to define a stochastic integral in this *localized way*.

As motivated above we cannot avoid stopping times in this construction procedure.

**DEFINITION 1.2.1.** Let $I \subseteq [0, \infty)$. A random variable $\tau : \Omega \to I \cup \{\infty\}$ is called a *stopping time* with respect to a filtration $(\mathcal{G}_i)_{i\in I}$ if

$$\{\tau \le i\} \in \mathcal{G}_i \qquad \text{for all } i \in I.$$

In a first step we want to investigate how stopping times behave in the integral we already know. Although the following proposition seems very natural, it is highly nontrivial to prove (see [3, Proposition 3.35] for the $L^p$-valued case and note that the proof can be done in exactly the same way for the mixed case).

**Proposition 1.2.2 (Itô integral and stopping times I).** *Let $p, q, r \in (1, \infty)$ and $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0, T])))$. Then for every stopping time $\tau\colon \Omega \to [0, T]$ with respect to $\mathbb{F}$ we have $\mathbb{1}_{[0,\tau]} f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0, T])))$, and for the continuous version of the integral process it holds that*

$$\int_0^\tau f \,\mathrm{d}\beta = \int_0^T \mathbb{1}_{[0,\tau]} f \,\mathrm{d}\beta \qquad \text{almost surely.}$$

In the theory of stochastic integration, especially in the context of stochastic convolutions, we are also interested in the way how stopping times behave in integral maps of the form

$$J\colon [0, T] \to L^r(\Omega; L^p(U; L^q(V))), \quad J(t) := \int_0^t f(t) \,\mathrm{d}\beta = \int_0^t f(t, s) \,\mathrm{d}\beta(s),$$

where $f\colon [0, T] \times \Omega \to L^p(U; L^q(V; L^2[0, T]))$ has the property that $f(t)$ is $L^r$-stochastically integrable for each $t \in [0, T]$ and some $p, q, r \in (1, \infty)$. In this situation it seems natural to write

$$J(t \wedge \tau) = \int_0^{t\wedge\tau} f(t \wedge \tau) \,\mathrm{d}\beta = \int_0^{t\wedge\tau} f(t \wedge \tau, s) \,\mathrm{d}\beta(s)$$

for a stopping time $\tau\colon \Omega \to [0, T]$. However, the expression on the right-hand side is meaningless since the integrand is in general not adapted, and therefore the stochastic integral is not well-defined. To cope with this inconvenience we consider the process $J_\tau$ defined by

$$J_\tau(t) := \int_0^t \mathbb{1}_{[0,\tau]} f(t) \,\mathrm{d}\beta = \int_0^t \mathbb{1}_{[0,\tau]}(s) f(t, s) \,\mathrm{d}\beta(s).$$

**Proposition 1.2.3 (Itô integral and stopping times II).** *Let $p, q, r \in (1, \infty)$ and $\tau\colon \Omega \to [0, T]$ be a stopping time with respect to $\mathbb{F}$. Let $f\colon [0, T] \to L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0, T])))$ be such that*

*i)* $t \mapsto f(t)\colon [0, T] \to L^r(\Omega; L^p(U; L^q(V; L^2[0, T])))$ *is continuous and*

*ii)* $J$ *and* $J_\tau$ *have continuous versions.*

*Then the processes $J$ and $J_\tau$ defined above satisfy almost surely*

$$J(t \wedge \tau) = J_\tau(t \wedge \tau) \qquad \text{for } t \in [0, T].$$

*In particular, we almost surely have*

$$\mathbb{1}_{[0,\tau]}(t)\int_0^t f(t,s)\,\mathrm{d}\beta(s) = \mathbb{1}_{[0,\tau]}(t)\int_0^t \mathbb{1}_{[0,\tau]}(s) f(t,s)\,\mathrm{d}\beta(s).$$

**Proof.** By the previous proposition, $\mathbb{1}_{[0,\tau]}f(t) \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, so $J(t)$ and $J_\tau(t)$ are well-defined for each $t \in [0,T]$. Thus, let us turn to the interesting part of proving the stated equality. We first prove it for a finitely-valued stopping time. Let $0 = t_0 < \ldots < t_N = T$ be a partition of the interval $[0,T]$ and $\tau_0 \colon \Omega \to \{t_0, \ldots, t_N\}$ be a stopping time. For any fixed $t \in [0,T]$ and $n \in \{0,\ldots,N\}$ we either have $t \geq t_n$ or $t < t_n$. In the first case we obtain

$$J(t\wedge t_n) = \int_0^{t_n} f(t_n)\,\mathrm{d}\beta = \int_0^{t_n} \mathbb{1}_{[0,t_n]} f(t_n)\,\mathrm{d}\beta = \int_0^{t\wedge t_n} \mathbb{1}_{[0,t_n]} f(t\wedge t_n)\,\mathrm{d}\beta = J_{t_n}(t\wedge t_n),$$

and in the second case we have

$$J(t\wedge t_n) = \int_0^{t} f(t)\,\mathrm{d}\beta = \int_0^{t} \mathbb{1}_{[0,t_n]} f(t)\,\mathrm{d}\beta = \int_0^{t\wedge t_n} \mathbb{1}_{[0,t_n]} f(t\wedge t_n)\,\mathrm{d}\beta = J_{t_n}(t\wedge t_n).$$

Observe that by Proposition 1.2.2

$$J_{\tau_0}(t) = \sum_{n=0}^{N} \mathbb{1}_{\{\tau_0 = t_n\}} J_{t_n}(t) \qquad \text{for all } t \in [0,T].$$

This leads to

$$J(t\wedge\tau_0) = \sum_{n=1}^{N} \mathbb{1}_{\{\tau_0 = t_n\}} J(t\wedge t_n) = \sum_{n=1}^{N} \mathbb{1}_{\{\tau_0 = t_n\}} J_{t_n}(t\wedge t_n) = J_{\tau_0}(t\wedge\tau_0).$$

Consider now for each $k \in \mathbb{N}$ the time steps $t_{n,k} := \frac{nT}{2^k}$, $n = 1,\ldots,2^k$, and the sequence of stopping times $(\tau_k)_{k\in\mathbb{N}}$ constructed by

$$\tau_k(\omega) := \min\bigl\{t \in \{t_{0,k},\ldots,t_{2^k,k}\} \colon t \geq \tau(\omega)\bigr\}, \qquad k \in \mathbb{N}.$$

Then $\lim_{k\to\infty}\tau_k = \tau$ almost surely and $\tau_k(\omega) \geq \tau_{k+1}(\omega) \geq \tau(\omega)$ for all $k \in \mathbb{N}$, $\omega \in \Omega$. Next, for $k \in \mathbb{N}$, define the real-valued functions $h_k \colon [0,T] \to \mathbb{R}$ by

$$\begin{aligned} h_k(t) &:= \|\mathbb{1}_{[0,\tau_k]} f(t) - \mathbb{1}_{[0,\tau]} f(t)\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))} \\ &= \|\mathbb{1}_{(\tau,\tau_k]} f(t)\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))}. \end{aligned}$$

Then $h_k$ is continuous by assumption (i), and $\lim_{k\to\infty} h_k(t) = 0$ for each fixed $t \in [0,T]$ by the dominated convergence theorem. Since the map $\delta \mapsto \|\mathbb{1}_{[0,\delta]} f\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))}$ is monotonically increasing, we have $h_k(t) \geq h_{k+1}(t)$ for all $k \in \mathbb{N}$ and any $t \in [0,T]$. Dini's theorem now yields the uniform convergence of the sequence $(h_k)_{k\in\mathbb{N}}$, i.e.,

$$\lim_{k\to\infty}\sup_{t'\in[0,T]}\|\mathbb{1}_{[0,\tau_k]}f(t')-\mathbb{1}_{[0,\tau]}f(t')\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))}=0.$$

As a consequence, we obtain for $t\in[0,T]$

$$\begin{aligned}
&\lim_{k\to\infty}\|J_{\tau_k}(t\wedge\tau_k)-J_\tau(t\wedge\tau_k)\|_{L^r(\Omega;L^p(U;L^q(V)))}\\
&\quad\le\lim_{k\to\infty}\sup_{t'\in[0,T]}\|J_{\tau_k}(t')-J_\tau(t')\|_{L^r(\Omega;L^p(U;L^q(V)))}\\
&\quad\eqsim_{p,q,r}\lim_{k\to\infty}\sup_{t'\in[0,T]}\|\mathbb{1}_{[0,\tau_k]}f(t')-\mathbb{1}_{[0,\tau]}f(t')\|_{L^r(\Omega;L^p(U;L^q(V;L^2[0,T])))}=0.
\end{aligned}$$

Finally, using this and the continuity of $J_\tau$ together with the dominated convergence theorem, we obtain

$$\begin{aligned}
&\|J_{\tau_k}(t\wedge\tau_k)-J_\tau(t\wedge\tau)\|_{L^r(\Omega;L^p(U;L^q(V)))}\\
&\quad\le\|J_{\tau_k}(t\wedge\tau_k)-J_\tau(t\wedge\tau_k)\|_{L^r(\Omega;L^p(U;L^q(V)))}+\|J_\tau(t\wedge\tau_k)-J_\tau(t\wedge\tau)\|_{L^r(\Omega;L^p(U;L^q(V)))}
\end{aligned}$$

which converges to 0 as $k\to\infty$. The claim now follows from the continuity of $J$ and the already proven equality for each finitely-valued stopping time $\tau_k$. ☐

**REMARK 1.2.4.**

a) In general, we do not have continuous versions of $J$ and $J_\tau$. In [14], Brzeźniak et al. have proved that the class of continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ with period 1 such that the stochastic convolution $t\mapsto\int_0^t f(t-s)\,\mathrm{d}\beta(s)$, $t\in[0,1]$, does not have a continuous version is of the second Baire category.

b) However, there are many situations where $J$ and $J_\tau$ have continuous (or even $\alpha$-Hölder continuous) versions. For example, if $A$ is the generator of an analytic semigroup on $L^2(U)$, then the stochastic convolution $t\mapsto\int_0^t e^{(t-s)A}\,\mathrm{d}\beta(s)$ has a version with $\alpha$-Hölder continuous paths for $\alpha<1/2$ (see [21, Theorems 5.14, 5.20 and 5.22]). In case $A$ is the generator of a contraction semigroup, and $\phi\in L^p_{\mathbb{F}}(\Omega;L^2(U\times[0,T]))$, the stochastic convolution $t\mapsto\int_0^t e^{(t-s)A}\phi(s)\,\mathrm{d}\beta(s)$ has a continuous version (see [21, Theorem 6.10], and [21, Propositions 6.13 and 7.3] for the case of an analytic generator and different integrability conditions for $\phi$). See also the appendix of [13].

c) If $J$ has a continuous version, then by assumption i) and the Burkholder-Davis-Gundy inequality $J_\tau$ also has a continuous version.

d) In case $A$ is $\ell^q$- or $\mathcal{R}_q$-sectorial on $L^p(U)$ and $\phi\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$ for some $q>2$, the function $f\colon[0,T]\to L^r_{\mathbb{F}}(\Omega;L^p(U;L^2[0,T]))$, $f(t)=\mathbb{1}_{[0,t]}e^{-(t-(\cdot))A}\phi$, is continuous. See Definition 2.1.8 and Remark 3.2.4 d).

The natural, but surprisingly non-trivial fact presented in Proposition 1.2.2 allows us to enhance the stochastic integral to processes $f\colon [0,T]\times\Omega \to L^p(U;L^q(V))$ which only satisfy the condition $f(\omega) \in L^p(U;L^q(V;L^2[0,T]))$ for $\mathbb{P}$-almost all $\omega \in \Omega$, i.e.,

$$\mathbb{P}\big(f \in L^p(U;L^q(V;L^2[0,T]))\big) = 1.$$

Since the enhancement process is rather technical, we give a short sketch of it. In the mixed $L^p$ space setting the idea is basically the same as in the scalar case: We want to stop the process $f$ such that the stopped process $\mathbb{1}_{[0,\tau]}f\colon [0,T]\times\Omega \to L^p(U;L^q(V))$ is $L^r$-stochastically integrable for some $r \in (1,\infty)$. More precisely, we do not only want to do this with one, but with an increasing sequence of stopping times in order to get a 'good' approximation of $f$. The preferred *localizing sequence* $(\tau_n)_{n\in\mathbb{N}}$ is of course

$$\tau_n(\omega) := T \wedge \inf\{t \in [0,T]\colon \|\mathbb{1}_{[0,t]}f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \geq n\}, \quad \omega \in \Omega,$$

having the properties $\mathbb{1}_{[0,\tau_n]}f \in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))$ for each $n \in \mathbb{N}$ as well as $\lim_{n\to\infty}\tau_n = T$ almost surely. The principle idea is now to just define the localized stochastic integral as

$$\int_0^T f\,\mathrm{d}\beta := \lim_{n\to\infty}\int_0^T \mathbb{1}_{[0,\tau_n]}f\,\mathrm{d}\beta,$$

where the convergence holds almost surely in $L^p(U;L^q(V))$. However, we still have to clarify how this limit is actually defined. In Section 1.1 this was done via Itô's isomorphism. Since this is no longer available, we would like to have a localized analogue of that granting the well-definedness of the localized stochastic integral. Replacing the space $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))$ of integrable functions with the vector space of strongly measurable functions $f\colon \Omega \to L^p(U;L^q(V;L^2[0,T]))$ leads to the search of a suitable metric. Since almost sure convergence fails to coincide with any metric, we have to consider a close relative of that convergence which does the job.

**Definition/Proposition 1.2.5.** For any Banach space $F$ we denote by $L^0(\Omega;F)$ the vector space of all equivalence classes of strongly measurable functions on $(\Omega,\mathcal{F},\mathbb{P})$ with values in $F$ which are identical almost surely. Together with the metric

$$d_{\mathbb{P}}\colon L^0(\Omega;F)\times L^0(\Omega;F) \to [0,\infty), \quad d_{\mathbb{P}}(X,Y) := \mathbb{E}(\|X-Y\|_F \wedge 1)$$

$L^0(\Omega;F)$ turns into a complete metric space, and convergence with respect to this metric coincides with convergence in probability (for this statement we refer to [50, Chapter 3], see also [31, Theorems 9.2.2 and 9.2.3] or [3, Proposition 3.37]). For $F = L^p(U;L^q(V;L^2[0,T]))$ we let $L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2[0,T])))$ be the subspace of $L^0(\Omega;L^p(U;L^q(V;L^2[0,T])))$ consisting of all elements which are adapted to the filtration $\mathbb{F}$.

One of the disadvantages of this enhancement process is the loss of the Itô isomorphism, but a somewhat weaker Itô-isomorphism-type estimate still holds. The first step is to prove it for processes in $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, $r \in (1,\infty)$.

**Lemma 1.2.6.** *Let $p, q, r \in (1,\infty)$ and $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then for each $\delta > 0$ and $\varepsilon > 0$ we have for the continuous version of $\int_0^t f \,\mathrm{d}\beta$ the estimates*

$$\mathbb{P}\Big(\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|_{L^p(U;L^q(V))} > \varepsilon\Big) \le \frac{C^r \delta^r}{\varepsilon^r} + \mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \ge \delta\big)$$

*and*

$$\mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} > \varepsilon\big) \le \frac{C^r \delta^r}{\varepsilon^r} + \mathbb{P}\Big(\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|_{L^p(U;L^q(V))} \ge \delta\Big),$$

*where $C > 0$ is the constant appearing in the Itô isomorphism.*

If we carefully take a look on the estimates above we can interpret these inequalities as an extension of the Burkholder-Davis-Gundy inequality from Theorem 1.1.18 for '$L^0$-norms'.

**Proof.** The proof closely follows the lines of [3, Lemma 3.36]. Since we will face this type of proof again later, we include the details. By Proposition 1.2.2 and the Burkholder-Davis-Gundy inequality, there exists a constant $C > 0$ such that

$$\begin{aligned}\mathbb{E}\Big\| \sup_{t\in[0,\tau]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} &= \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau} f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \\ &\eqsim_C \mathbb{E}\|\mathbb{1}_{[0,\tau]} f\|^r_{L^p(U;L^q(V;L^2[0,T]))}\end{aligned}$$

for each stopping time $\tau \colon \Omega \to [0,T]$ with respect to $\mathbb{F}$. For $\delta, \varepsilon > 0$ fixed we define the stopping times

$$\begin{aligned}\tau^{(1)} &:= T \wedge \inf\Big\{t \in [0,T] \colon \Big\| \sup_{s\in[0,t]} \Big| \int_0^s f \,\mathrm{d}\beta \Big| \Big\|_{L^p(U;L^q(V))} \ge \varepsilon\Big\}, \\ \tau^{(2)} &:= T \wedge \inf\big\{t \in [0,T] \colon \|\mathbb{1}_{[0,t]} f\|_{L^p(U;L^q(V;L^2[0,T]))} \ge \delta\big\}.\end{aligned}$$

Now take $\tau := \tau^{(1)} \wedge \tau^{(2)}$. Then $\tau$ is a stopping time with respect to $\mathbb{F}$ and

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau} f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \le \varepsilon^r, \qquad \mathbb{E}\|\mathbb{1}_{[0,\tau]} f\|^r_{L^p(U;L^q(V;L^2[0,T]))} \le \delta^r,$$

since $t \mapsto \mathbb{1}_{[0,t]} f$ and $t \mapsto \sup_{s\in[0,t]} \big|\int_0^s f \,\mathrm{d}\beta\big|$ have continuous paths starting at zero. Now observe that

$$\Big\{\Big\| \sup_{t\in[0,\tau^{(1)}]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|_{L^p(U;L^q(V))} < \varepsilon\Big\} \subseteq \Big\{\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|_{L^p(U;L^q(V))} \le \varepsilon\Big\}$$

by the definition of $\tau^{(1)}$, and on the set

$$\Big\{\Big\| \sup_{t\in[0,T]}\Big|\int_0^t f \,\mathrm{d}\beta\Big|\,\Big\|_{L^p(U;L^q(V))} > \varepsilon, \big\|\sup_{t\in[0,T]} \mathbb{1}_{[0,t]}|f|\big\|_{L^p(U;L^q(V;L^2[0,T]))} < \delta\Big\}$$

we have $\tau^{(2)} = T$ and therefore $\tau = \tau^{(1)}$. Together with Markov's inequality this leads to

$$\begin{aligned}\mathbb{P}\Big(\Big\| \sup_{t\in[0,T]}\Big|\int_0^t f \,\mathrm{d}\beta\Big|\,\Big\|_{L^p(U;L^q(V))} > \varepsilon, \big\|\sup_{t\in[0,T]} \mathbb{1}_{[0,t]}|f|\big\|_{L^p(U;L^q(V;L^2[0,T]))} < \delta\Big)\\ \le \mathbb{P}\Big(\Big\| \sup_{t\in[0,\tau]}\Big|\int_0^t f \,\mathrm{d}\beta\,\Big|\,\Big\|_{L^p(U;L^q(V))} \ge \varepsilon\Big) \le \tfrac{1}{\varepsilon^r}\mathbb{E}\Big\| \sup_{t\in[0,\tau]}\Big|\int_0^t f \,\mathrm{d}\beta\,\Big|\,\Big\|^r_{L^p(U;L^q(V))}\\ \le C^r\tfrac{1}{\varepsilon^r}\mathbb{E}\|\mathbb{1}_{[0,\tau]}f\|^r_{L^p(U;L^q(V;L^2[0,T]))} \le C^r\tfrac{\delta^r}{\varepsilon^r}.\end{aligned}$$

Using this estimate, we finally obtain

$$\begin{aligned}\mathbb{P}\Big(\Big\| \sup_{t\in[0,T]}\Big|\int_0^t f \,\mathrm{d}\beta\Big|\,\Big\|_{L^p(U;L^q(V))} > \varepsilon\Big) &\le C^r\tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\big(\big\|\sup_{t\in[0,T]} \mathbb{1}_{[0,t]}|f|\big\|_{L^p(U;L^q(V;L^2[0,T]))} \ge \delta\big)\\ &= C^r\tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \ge \delta\big),\end{aligned}$$

where the last equality follows from the fact that

$$\|f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} = \big\|\sup_{t\in[0,T]} \mathbb{1}_{[0,t]}|f(\omega)|\big\|_{L^p(U;L^q(V;L^2[0,T]))}$$

for each $\omega \in \Omega$. The second stated inequality is shown in exactly the same way by interchanging the two processes. □

As mentioned above, for the general case we need a sequence of stopping times to extend Lemma 1.2.6.

**DEFINITION 1.2.7.** Let $p, q \in (1,\infty)$, $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with respect to $\mathbb{F}$ and with values in $[0,T]$. Then we call the sequence $(\tau_n)_{n\in\mathbb{N}}$ a *localizing sequence for $f$* if

a) for all $\omega \in \Omega$ there exists an index $N(\omega) \in \mathbb{N}$ such that $\tau_n(\omega) = T$ for all $n \ge N(\omega)$, and

b) $\mathbb{1}_{[0,\tau_n]}f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for all $n \in \mathbb{N}$ and some $r \in (1,\infty)$.

**REMARK 1.2.8.** For every $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ is given by

$$\tau_n(\omega) := T \wedge \inf\{t \in [0,T] \colon \|\mathbb{1}_{[0,t]}f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \ge n\}, \quad \omega \in \Omega.$$

Let us prove this assertion. We first remark that $\tau_n$ is a stopping time with respect to

$\mathbb{F}$ for all $n \in \mathbb{N}$. To see that, note that the function $t \mapsto \|\mathbb{1}_{[0,t]} f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))}$ is continuous and increasing for each $\omega \in \Omega$, leading to

$$\{\tau_n < t\} = \{\|\mathbb{1}_{[0,t]} f\|_{L^p(U;L^q(V;L^2[0,T]))} > n\} \in \mathcal{F}_t, \quad t \in [0,T],$$

and by the usual conditions of $\mathbb{F}$ this is equivalent to $\{\tau_n \leq t\} \in \mathcal{F}_t$, $t \in [0,T]$. Additionally, for any fixed $\omega \in \Omega$ and each $t \in [0,T]$ we have

$$\|\mathbb{1}_{[0,t]} f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \leq \|f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} < \infty.$$

Therefore, there exists an integer $N(\omega) \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|\mathbb{1}_{[0,t]} f(\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \leq N(\omega).$$

But this just means that $\tau_n(\omega) = T$ for $n \geq N(\omega)$. Moreover, we have the estimate

$$\mathbb{E}\|\mathbb{1}_{[0,\tau_n]} f\|^r_{L^p(U;L^q(V;L^2[0,T]))} \leq n^r$$

for any $r \in (1,\infty)$ which combined with the fact that $\mathbb{1}_{[0,\tau_n]} f$ is adapted to $\mathbb{F}$ (see Proposition 1.2.2) concludes the proof.

Using localizing sequences we can extend the results from Lemma 1.2.6 to measurable processes $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$.

**Theorem 1.2.9 (Itô homeomorphism).** *Let $p, q, r \in (1,\infty)$. Then the Itô isomorphism $I_{L^r} \colon L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \to L^r(\Omega; L^p(U; L^q(V)))$ has a unique extension to a linear mapping*

$$I_{L^0} \colon L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \to L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$$

*which is a homeomorphism onto its closed range. Moreover, the estimates from Lemma 1.2.6 extend to arbitrary processes $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$.*

**Proof.** Let $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $f$. Then $f_n := \mathbb{1}_{[0,\tau_n]} f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for some $r \in (1,\infty)$ and by Theorem 1.1.18 there exists a version of $I_{L^r}(f_n) = \int_0^{(\cdot)} f_n \, \mathrm{d}\beta$ such that

$$\int_0^{(\cdot)} f_n \, \mathrm{d}\beta \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T]))) \subseteq L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f_n = f$ almost surely in $L^p(U; L^q(V; L^2[0,T]))$, the sequence $(f_n)_{n\in\mathbb{N}}$ is 'Cauchy in probability' and by Lemma 1.2.6 we deduce that $(I_{L^r}(f_n))_{n\in\mathbb{N}}$ is a

Cauchy sequence in $L^0(\Omega; L^p(U; L^q(V; C[0,T])))$. Since this is a complete metric space, there exists a limit $X \in L^0(\Omega; L^p(U; L^q(V; C[0,T])))$. Now define

$$I_{L^0}(f) := X = \lim_{n\to\infty} I_{L^r}(f_n) = \lim_{n\to\infty} \int_0^{(\cdot)} f_n \, \mathrm{d}\beta \quad \text{in } L^0(\Omega; L^p(U; L^q(V; C[0,T]))).$$

Then $I_{L^0}$ is well-defined and linear. To extend the estimates from Lemma 1.2.6, note that the convergence in $L^0(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and $L^0(\Omega; L^p(U; L^q(V; C[0,T])))$ implies that

$$\lim_{n\to\infty} \mathbb{P}\big(\|I_{L^r}(f_n) - I_{L^0}(f)\|_{L^p(U;L^q(V;C[0,T]))} \geq \rho\big) = 0$$

and

$$\lim_{n\to\infty} \mathbb{P}\big(\|f_n - f\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \rho\big) = 0$$

for each $\rho > 0$. Now let $\varepsilon, \delta > 0$ and $(\rho_k)_{k\in\mathbb{N}} \subseteq (0, \frac{1}{2}(\varepsilon \wedge \delta))$ be a decreasing null sequence. Then by Lemma 1.2.6

$$\begin{aligned}
&\mathbb{P}\big(\big\| \sup_{t\in[0,T]} |I_{L^0}(f)(t)|\big\|_{L^p(U;L^q(V))} > \varepsilon\big) \\
&\quad\leq \mathbb{P}\big(\|I_{L^r}(f_n) - I_{L^0}(f)\|_{L^p(U;L^q(V;C[0,T]))} \geq \rho_k\big) + \mathbb{P}\big(\|I_{L^r}(f_n)\|_{L^p(U;L^q(V;C[0,T]))} \geq \varepsilon - \rho_k\big) \\
&\quad\leq \mathbb{P}\big(\|I_{L^r}(f_n) - I_{L^0}(f)\|_{L^p(U;L^q(V;C[0,T])))} \geq \rho_k\big) + C^r \frac{(\delta-\rho_k)^r}{(\varepsilon-\rho_k)^r} \\
&\qquad + \mathbb{P}\big(\|f_n\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \delta - \rho_k\big) \\
&\quad\leq \mathbb{P}\big(\|I_{L^r}(f_n) - I_{L^0}(f)\|_{L^p(U;L^q(V;C[0,T]))} \geq \rho_k\big) + \mathbb{P}\big(\|f_n - f\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \rho_k\big) \\
&\qquad + C^r \frac{(\delta-\rho_k)^r}{(\varepsilon-\rho_k)^r} + \mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \delta - 2\rho_k\big).
\end{aligned}$$

If we take the limit $n \to \infty$ in this estimate for each $k \in \mathbb{N}$, we obtain

$$\mathbb{P}\big(\big\| \sup_{t\in[0,T]} |I_{L^0}(f)(t)|\big\|_{L^p(U;L^q(V))} > \varepsilon\big) \leq C^r \frac{(\delta-\rho_k)^r}{(\varepsilon-\rho_k)^r} + \mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \delta - 2\rho_k\big).$$

Using now the $\sigma$-continuity of the probability measure $\mathbb{P}$, we arrive at

$$\mathbb{P}\big(\big\| \sup_{t\in[0,T]} |I_{L^0}(f)(t)|\big\|_{L^p(U;L^q(V))} > \varepsilon\big) \leq C^r \frac{\delta^r}{\varepsilon^r} + \mathbb{P}\big(\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \geq \delta\big)$$

by letting $k \to \infty$. The other inequality in Lemma 1.2.6 can be extended in exactly the same way by interchanging the two processes. From this we infer that $I_{L^0}$ is continuous and has a continuous inverse. This also shows that the mapping $I_{L^0}$ has a closed range in $L^0(\Omega; L^p(U; L^q(V; C[0,T])))$. □

Similar to the integrable case we can now define the localized stochastic integral as the limit of stochastic integrals we already know.

**Definition 1.2.10.** Let $p, q, r \in (1, \infty)$, $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, and let $(f_n)_{n\in\mathbb{N}} \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ be an approximating sequence for $f$. Then we define the stochastic integral of $f$ by

$$\int_0^{(\cdot)} f \, \mathrm{d}\beta := \int_0^{(\cdot)} f(s) \, \mathrm{d}\beta(s) := I_{L^0}(f) = \lim_{n\to\infty} I_{L^r}(f_n),$$

where convergence holds in $L^0(\Omega; L^p(U; L^q(V; C[0,T])))$. In this case, we call $f$ $L^0$-*stochastically integrable.*

**Remark 1.2.11.**

a) The $L^0$-stochastic integral is well-defined in the sense that it is independent of the approximating sequence. Moreover, the localized integral has by definition continuous paths and is again $L^0$-stochastically integrable.

b) If $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for some $r \in (1, \infty)$, then the integral process which arises from the Itô integral coincides almost surely with the localized integral process. From that point of view, the localized integral provides a true enhancement of the Itô integral.

c) The new stochastic integral is in general no longer a martingale, but as in the scalar case a *local martingale* (see Theorem 1.2.15 below).

d) If we take a closer look on the proof of Theorem 1.2.9, we see that

$$\int_0^T f \, \mathrm{d}\beta = \lim_{n\to\infty} \int_0^T \mathbb{1}_{[0,\tau_n]} f \, \mathrm{d}\beta \quad \text{in probability,}$$

where the Itô homeomorphism guarantees that this limit actually exists. This also implies that $\int_0^T f \, \mathrm{d}\beta = \lim_{k\to\infty} \int_0^T \mathbb{1}_{[0,\tau_{n_k}]} f \, \mathrm{d}\beta$ almost surely for an appropriate subsequence as was indicated in the motivation. In order to just define the stochastic integral we also could have imitated the proof often appearing in the scalar-valued case via a sequential consistency of the sequence $\left(\int_0^T \mathbb{1}_{[0,\tau_n]} f \, \mathrm{d}\beta\right)_{n\in\mathbb{N}}$, with the disadvantage of not having the Itô homeomorphism.

We finally collect some properties of the new integral, where we mostly try to extend the results we already know from the Itô integral.

**Proposition 1.2.12 (Properties of the localized integral).** *Let $p, q \in (1, \infty)$ and $f, g \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then the following properties hold:*

a) *The stochastic integral is linear, i.e. for* $a, b \in \mathbb{R}$ *we have*

$$\int_0^{(\cdot)} af + bg \,\mathrm{d}\beta = a \int_0^{(\cdot)} f \,\mathrm{d}\beta + b \int_0^{(\cdot)} g \,\mathrm{d}\beta.$$

b) *The process* $\big(\int_0^t f \,\mathrm{d}\beta\big)_{t\in[0,T]}$ *is adapted to* $\mathbb{F}$.

c) *For* $S \in \mathcal{B}\big(L^p(U; L^q(V))\big)$, *let* $S^{L^2}$ *be the bounded extension of* $S$ *on the space* $L^p(U; L^q(V; L^2[0,T]))$ *(see Remark 2.4.1). Then,* $S^{L^2} f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ *and*

$$\int_0^t S^{L^2} f \,\mathrm{d}\beta = S \int_0^t f \,\mathrm{d}\beta \qquad \text{for all } t \in [0,T].$$

d) *For every* $s, t \in [0,T]$ *with* $s < t$ *it holds that*

$$\int_s^t f \,\mathrm{d}\beta = \int_0^T \mathbb{1}_{[s,t]} f \,\mathrm{d}\beta.$$

e) *There exists a* $\mu$*-null set* $N_\mu \in \Sigma$ *such that* $f(u)$ *is* $L^0$*-stochastically integrable, i.e.* $f(u) \in L^0_{\mathbb{F}}(\Omega; L^q(V; L^2[0,T]))$, *and*

$$\int_0^{(\cdot)} f(u) \,\mathrm{d}\beta = \Big(\int_0^{(\cdot)} f \,\mathrm{d}\beta\Big)(u) \quad \text{for each } u \in U \setminus N_\mu.$$

f) *There exists a* $\mu\otimes\nu$*-null set* $N \in \Sigma\otimes\Xi$ *such that* $f(u,v)$ *is* $L^0$*-stochastically integrable, i.e.* $f(u,v) \in L^0_{\mathbb{F}}(\Omega; L^2[0,T])$, *and*

$$\int_0^{(\cdot)} f(u,v) \,\mathrm{d}\beta = \Big(\int_0^{(\cdot)} f \,\mathrm{d}\beta\Big)(u,v) \quad \text{for each } (u,v) \in (U \times V) \setminus N.$$

**Proof.** a) and b) follow immediately from the definition.

c) Since $S^{L^2}$ is bounded, it follows that $S^{L^2} f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Now take an approximating sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for some $r \in (1,\infty)$. Then by Proposition 1.1.13 c) it holds that

$$\int_0^t S^{L^2} f_n \,\mathrm{d}\beta = S \int_0^t f_n \,\mathrm{d}\beta.$$

The boundedness of $S^{L^2}$ also implies that

$$\mathbb{E}\big(\|S^{L^2} f_n - S^{L^2} f\|_{L^p(U;L^q(V;L^2[0,T]))} \wedge 1\big) \le \big(\|S\| \vee 1\big)\mathbb{E}\big(\|f_n - f\|_{L^p(U;L^q(V;L^2[0,T]))} \wedge 1\big) \to 0$$

as $n \to \infty$, i.e. $\lim_{n\to\infty} S^{L^2} f_n = S^{L^2} f$ in $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Similarly we obtain

$$\lim_{n\to\infty} S\int_0^t f_n \,\mathrm{d}\beta = S\int_0^t f \,\mathrm{d}\beta \qquad \text{in } L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V))).$$

The Itô homeomorphism then yields

$$\int_0^t S^{L^2} f \,\mathrm{d}\beta = \lim_{n\to\infty} \int_0^t S^{L^2} f_n \,\mathrm{d}\beta = \lim_{n\to\infty} S\int_0^t f_n \,\mathrm{d}\beta = S\int_0^t f \,\mathrm{d}\beta$$

with convergence in $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V)))$.

d) This property follows from Proposition 1.1.13 d) and approximation.

e) Let $(f_n)_{n\in\mathbb{N}} \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ be an approximating sequence for $f$ for some $r \in (1,\infty)$. We first want to show that $\lim_{k\to\infty} f_{n_k}(u) = f(u)$ in $L^0(\Omega; L^q(V; L^2[0,T]))$ for $\mu$-almost every $u \in U$ and an appropriate subsequence $(f_{n_k})_{k\in\mathbb{N}}$. This then also implies the assertion about the adaptedness of $f(u)$. In order to do that, we can assume that $\mu(U) < \infty$ using the $\sigma$-finiteness of $(U,\Sigma,\mu)$. Since $\lim_{n\to\infty} f_n = f$ in $L^p(U; L^q(V; L^2[0,T]))$ in probability we obtain by Proposition 1.2.5

$$\begin{aligned}
\big\|\mathbb{E}(\|f_n - f\|_{L^q(V;L^2[0,T])} \wedge 1)\big\|_{L^p(U)} &\le \mathbb{E}\big\|\|f_n - f\|_{L^q(V;L^2[0,T])} \wedge 1\big\|_{L^p(U)} \\
&\le \mathbb{E}\big(\|f_n - f\|_{L^p(U;L^q(V;L^2[0,T]))} \wedge \mu(U)^{1/p}\big) \\
&= \mu(U)^{1/p}\mathbb{E}\big(\|\mu(U)^{-1/p}(f_n - f)\|_{L^p(U;L^q(V;L^2[0,T]))} \wedge 1\big) \to 0 \qquad \text{as } n\to\infty,
\end{aligned}$$

using the fact that the integral of the minimum of two functions is less or equal than the minimum of the integrals of these functions. Therefore, we can choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges $\mu$-almost everywhere to $f$ in $L^0(\Omega; L^q(V; L^2[0,T]))$. Similarly to above we may choose another subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\Big(\int_0^t f_{n_{k_j}} \,\mathrm{d}\beta\Big)(u) = \Big(\int_0^t f \,\mathrm{d}\beta\Big)(u) \qquad \text{in } L^0(\Omega; L^q(V))$$

for $\mu$-almost every $u \in U$. Using now Proposition 1.1.13 for every $f_{n_{k_j}}$, we obtain the desired result.

f) The proof here is done similarly to part e). Assuming that $\mu(U), \nu(V) < \infty$ we deduce that for $C := \mu(U)^{1/p\wedge q}\nu(V)^{1/p\wedge q}$

$$\begin{aligned}
\big\|\mathbb{E}(\|f_n - f\|_{L^2[0,T]} \wedge 1)\big\|_{L^{p\wedge q}(U\times V)} &\le C\,\mathbb{E}\big(\|\tfrac{1}{C}(f_n - f)\|_{L^{p\wedge q}(U\times V;L^2[0,T])} \wedge 1\big) \\
&\le C\,\mathbb{E}\big(\|\tfrac{1}{C}(f_n - f)\|_{L^p(U;L^q(V;L^2[0,T]))} \wedge 1\big) \to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Now the proof can be finished as in e). $\square$

**Remark 1.2.13.**

a) Observe that every property from Proposition 1.1.13 still holds except for the estimate of the expected value. The reason for that is that we can not assume that this even exists because of the missing integrability condition of $f$ with respect to $\Omega$.

b) With the same arguments, the results of Remark 1.1.14 a)-e) are still valid for the case $r = 0$.

The behavior of stopping times in stochastic integrals we proved in the beginning of this section enabled us to enlarge the class of possible integrands. In the next step we want to extend these results to the localized case. For this purpose, let $J$ and $J_\tau$ be defined as before Proposition 1.2.3.

**Proposition 1.2.14 (Localized integral and stopping times).** *Let $p, q \in (1, \infty)$ and $\tau\colon \Omega \to [0, T]$ be a stopping time with respect to $\mathbb{F}$.*

*a) Let $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then $\mathbb{1}_{[0,\tau]} f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ and for every $t \in [0, T]$ it holds that*

$$\int_0^{t\wedge\tau} f \,\mathrm{d}\beta = \int_0^t \mathbb{1}_{[0,\tau]} f \,\mathrm{d}\beta \qquad \textit{almost surely.}$$

*b) Let $f\colon [0,T] \to L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ be such that*

*i) $t \mapsto f(t)\colon [0,T] \to L^0(\Omega; L^p(U; L^q(V; L^2[0,T])))$ is continuous and*

*ii) $J$ and $J_\tau$ have continuous versions.*

*Then the processes $J$ and $J_\tau$ satisfy almost surely*

$$J(t\wedge\tau) = J_\tau(t\wedge\tau) \qquad \textit{for } t \in [0,T].$$

*In particular, we almost surely have*

$$\mathbb{1}_{[0,\tau]}(t) \int_0^t f(t,s)\,\mathrm{d}\beta(s) = \mathbb{1}_{[0,\tau]}(t) \int_0^t \mathbb{1}_{[0,\tau]}(s) f(t,s)\,\mathrm{d}\beta(s).$$

**Proof.** The proof of a) can be done as in [3, Proposition 3.43]. To prove b) we define for $n \in \mathbb{N}$ the stopping time

$$\tau_n(\omega) := T \wedge \inf\big\{s \in [0,T]\colon \sup_{t\in[0,T]} \|\mathbb{1}_{[0,s]} f(t,\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \geq n\big\}, \quad \omega \in \Omega.$$

Then $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for each $f(t)$, $t \in [0,T]$. In particular, for $f_n(t) := \mathbb{1}_{[0,\tau_n]} f(t)$, we have $\mathbb{E}\|f_n(t)\|^r_{L^p(U;L^q(V;L^2[0,T]))} \leq n^r < \infty$ as well as

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{and} \quad \lim_{n\to\infty} \mathbb{1}_{[0,\tau]} f_n(t) = \mathbb{1}_{[0,\tau]} f(t) \qquad \text{in } L^0(\Omega; L^p(U; L^q(V; L^2[0,T])))$$

for all $t \in [0,T]$. The Itô homeomorphism implies that

$$\lim_{n\to\infty} \int_0^t f_n(t)\,\mathrm{d}\beta = \int_0^t f(t)\,\mathrm{d}\beta \quad \text{and} \quad \lim_{n\to\infty} \int_0^t \mathbb{1}_{[0,\tau]} f_n(t)\,\mathrm{d}\beta = \int_0^t \mathbb{1}_{[0,\tau]} f(t)\,\mathrm{d}\beta$$

both in $L^0(\Omega; L^p(U; L^q(V)))$ and for each $t \in [0,T]$. It remains to show that $f_n$ fulfills the requirements of Proposition 1.2.3. Let $n \in \mathbb{N}$ be fixed. Above, we have seen that $f_n\colon [0,T] \to L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for some $r \in (1,\infty)$. Moreover, by the definition of the localizing sequence, we have for every $\omega \in \Omega$

$$\sup_{t\in[0,T]} \|f_n(t,\omega)\|_{L^p(U;L^q(V;L^2[0,T]))} \le n,$$

which is integrable with respect to $\Omega$. Now let $(h_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be a null sequence. Then, by the continuity assumption of $f$ we can choose a subsequence $(h_{k_j})_{j\in\mathbb{N}}$ such that

$$f_n(t+h_{k_j}) \to f_n(t) \qquad \text{almost surely in } L^p(U; L^q(V; L^2[0,T])) \text{ as } j \to \infty.$$

Now the dominated convergence theorem yields the continuity of $f_n$. Since $\tau \wedge \tau_n$ is also a stopping time, the assumption about the continuous versions follows from the continuity of $J$, part a), b) i), and the Itô homeomorphism. This concludes the proof. □

Having these results, we can show nearly the same properties for the localized stochastic integral process as we did for the Itô integral process.

**Theorem 1.2.15 (Properties of the localized integral process).** *Let $p, q \in (1,\infty)$, $r \in [1,\infty)$, and $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then the following properties hold:*

a) ***Local martingale property.*** *The integral process $(\int_0^t f\,\mathrm{d}\beta)_{t\in[0,T]}$ is a local martingale with respect to the filtration $\mathbb{F}$.*

b) ***Continuity and Burkholder-Davis-Gundy inequality.*** *The integral process $(\int_0^t f\,\mathrm{d}\beta)_{t\in[0,T]}$ is almost surely continuous satisfying the maximal inequality*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f\,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big( \int_0^T |f(t)|^2\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))},$$

*where this is understood in the sense that the left-hand side is finite if and only if the right-hand side is finite. If one of these cases holds, then the process $X(t) := \int_0^t f\,\mathrm{d}\beta$, $t \in [0,T]$, is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\Big\| \Big( \int_0^T |X(t)|^2\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} \lesssim_{p,r} T^{1/2} \mathbb{E}\Big\| \Big( \int_0^T |f(t)|^2\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

**Proof.** a) Let $(\tau_n)_{n\in\mathbb{N}}$ be the localizing sequence from Remark 1.2.8. Then $\tau_n$ is a stopping time with respect to $\mathbb{F}$, $\tau_n \le \tau_{n+1}$ and $\lim_{n\to\infty} \tau_n = T$ almost surely. Moreover, by Proposition 1.2.14 a) and Theorem 1.1.18 a) the process

$$\int_0^{t\wedge\tau_n} f \,\mathrm{d}\beta = \int_0^t \mathbb{1}_{[0,\tau_n]} f \,\mathrm{d}\beta, \quad t \in [0,T],$$

is a martingale with respect to $\mathbb{F}$. Therefore, $(\int_0^t f \,\mathrm{d}\beta)_{t\in[0,T]}$ is a local martingale with respect to $\mathbb{F}$.

b) The continuity assumption follows by definition. Assume first that the right-hand side is finite. Then the assertion is trivial and follows from Theorem 1.1.18 c). If the left-hand side is finite, we let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $f$, and define

$$f_n := \mathbb{1}_{[0,\tau_n]} f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))).$$

Then $\lim_{n\to\infty} f_n = f$ almost surely. Thus, Fatou's lemma, Theorem 1.1.18 c), and Proposition 1.2.14 a) yield

$$\begin{aligned}
\mathbb{E}\Big\| \Big( \int_0^T |f|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}} \Big\|^r_{L^p(U;L^q(V))} &\le \liminf_{n\to\infty} \mathbb{E}\Big\| \Big( \int_0^T |f_n|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}} \Big\|^r_{L^p(U;L^q(V))} \\
&\eqsim_{p,q,r} \lim_{n\to\infty} \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \mathbb{1}_{[0,\tau_n]} f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \\
&= \lim_{n\to\infty} \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau_n} f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))} \\
&\le \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t f \,\mathrm{d}\beta \Big| \Big\|^r_{L^p(U;L^q(V))}.
\end{aligned}$$

This shows that $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, and the result again follows from Theorem 1.1.18 c). □

In the last part of this section we want to give a beautiful generalization of the stochastic Fubini Theorem 1.1.15. We closely follow the proof of [85], where this was elaborated for the scalar-valued case and for stochastic integrals with respect to continuous semimartingales.

**Theorem 1.2.16 (Stochastic Fubini theorem II).** *Let $p, q \in (1,\infty)$, $(K, \mathcal{K}, \theta)$ be a $\sigma$-finite measure space, and $f\colon K \times \Omega \to L^p(U; L^q(V; L^2[0,T]))$ be strongly measurable such that*

$$\begin{aligned}
&f(\cdot,\omega) \in L^1(K; L^p(U; L^q(V; L^2[0,T]))) \quad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega, \\
&f(x,\cdot) \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T]))) \quad \text{for } \theta\text{-almost all } x \in K.
\end{aligned}$$

*Then the following assertions hold:*

a) *For $\theta$-almost all $x \in K$, $f(x,\cdot)$ is $L^0$-stochastically integrable, the process*

$$\xi(x,\omega,t) := \Big(\int_0^t f(x,s)\,\mathrm{d}\beta(s)\Big)(\omega)$$

*is measurable, and almost surely,*

$$\int_K \Big\| \sup_{t\in[0,T]} |\xi(x,t)| \Big\|_{L^p(U;L^q(V))} \,\mathrm{d}\theta(x) < \infty.$$

b) *For almost all $(\omega,t,u,v) \in \Omega \times [0,T] \times U \times V$ the function $x \mapsto f(x,\omega,t,u,v)$ is integrable and the process*

$$\eta(\omega,t) := \int_K f(x,\omega,t)\,\mathrm{d}\theta(x)$$

*is $L^0$-stochastically integrable.*

c) *Almost surely, we have*

$$\int_K \xi(x,t)\,\mathrm{d}\theta(x) = \int_0^t \eta(s)\,\mathrm{d}\beta(s), \quad t \in [0,T].$$

**Proof.** a) By assumption, $f(x,\cdot)$ is stochastically integrable for almost all $x \in K$, i.e. $\xi$ is well-defined. To show the additional property of $\xi$, we first assume that $f \in L^1(K) \otimes D_{\mathbb{F}}$ (in particular, Theorem 1.1.15 is valid for such $f$). Then by Fubini's theorem and the strong Burkholder-Davis-Gundy inequality for $r = 1$ we obtain

$$\begin{aligned}\mathbb{E}\Big(\int_K \Big\| \sup_{t\in[0,T]} |\xi(t)| \Big\|_{L^p(U;L^q(V))} \mathrm{d}\theta\Big) &= \int_K \mathbb{E}\Big\| \sup_{t\in[0,T]} |\xi(t)| \Big\|_{L^p(U;L^q(V))} \mathrm{d}\theta \\ &\le C_{p,q} \int_K \mathbb{E}\|f\|_{L^p(U;L^q(V;L^2[0,T]))} \,\mathrm{d}\theta \\ &= C_{p,q}\mathbb{E}\Big(\int_K \|f\|_{L^p(U;L^q(V;L^2[0,T]))} \,\mathrm{d}\theta\Big)\end{aligned}$$

for some constant $C_{p,q} > 0$. In particular, we have

$$\mathbb{E}\int_K \big\|\xi(\tau)\big\|_{L^p(U;L^q(V))} \,\mathrm{d}\theta \le C_{p,q}\mathbb{E}\Big(\int_K \|\mathbb{1}_{[0,\tau]} f\|_{L^p(U;L^q(V;L^2[0,T]))} \,\mathrm{d}\theta\Big)$$

by Proposition 1.2.2 for any stopping time $\tau\colon \Omega \to [0,T]$. Applying now the same technique as in the proof of Lemma 1.2.6 (just replace the processes $\big\|\sup_{s\in[0,t]}\big|\int_0^s f\,\mathrm{d}\beta\big|\big\|_{L^p(U;L^q(V))}$ by $\|\sup_{s\in[0,t]} |\xi(s)|\|_{L^1(K;L^p(U;L^q(V)))}$ and $\|\mathbb{1}_{[0,t]} f\|_{L^p(U;L^q(V;L^2[0,T]))}$ by $\|\mathbb{1}_{[0,t]} f\|_{L^1(K;L^p(U;L^q(V;L^2[0,T])))}$) we arrive at

$$\mathbb{P}\Big(\int_K \Big\| \sup_{t\in[0,T]} |\xi(t)| \Big\|_{L^p(U;L^q(V))} \mathrm{d}\theta > \varepsilon\Big) \le \frac{C_{p,q}\delta}{\varepsilon} + \mathbb{P}\big(\|f\|_{L^1(K;L^p(U;L^q(V;L^2[0,T])))} \ge \delta\big)$$

for any $\varepsilon, \delta > 0$. Now take any $f$ as stated in the Theorem. By Remark 1.1.10 we can find a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^1(K) \otimes D_{\mathbb{F}}$ such that almost surely $\lim_{n\to\infty} f_n = f$ in $L^1(K; L^p(U; L^q(V; L^2[0,T])))$, in particular the sequence also converges in probability to $f$. Define

$$\xi_n(x,\omega,t) := \Big(\int_0^t f_n(x,s)\,\mathrm{d}\beta(s)\Big)(\omega), \quad n \in \mathbb{N}.$$

By the remark above, $(\xi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^0(\Omega; L^1(K; L^p(U; L^q(V; C[0,T]))))$, i.e. there exists a limit $\widetilde{\xi}$ in this space. By considering a sufficient subsequence, we obtain on the one hand $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ in $L^p(U; L^q(V; L^2[0,T]))$ almost surely and for almost all $x \in K$, which implies

$$\lim_{k\to\infty} \int_0^{(\cdot)} f_{n_k}(x)\,\mathrm{d}\beta = \int_0^{(\cdot)} f(x)\,\mathrm{d}\beta = \xi(x,\cdot)$$

in $L^p(U; L^q(V; C[0,T]))$ by Itô's homeomorphism. On the other hand,

$$\lim_{k\to\infty} \int_0^{(\cdot)} f_{n_k}(x)\,\mathrm{d}\beta = \lim_{k\to\infty} \xi_{n_k}(x,\cdot) = \widetilde{\xi}(x,\cdot)$$

in $L^p(U; L^q(V; C[0,T]))$ almost surely and for almost all $x \in K$. This implies that $\xi(x,\cdot) = \widetilde{\xi}(x,\cdot)$ in $L^p(U; L^q(V; C[0,T]))$. In particular, $\xi$ has the property stated in the Theorem and

$$\lim_{n\to\infty} \int_K \xi_n\,\mathrm{d}\theta = \int_K \xi\,\mathrm{d}\theta \qquad \text{in } L^0(\Omega; L^p(U; L^q(V; C[0,T]))).$$

b) The first statement follows by the triangle inequality. By the same argument, $\eta \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$, i.e. $\eta$ is $L^0$-stochastically integrable.

c) It remains to prove the integral equality. Let $(f_n)_{n\in\mathbb{N}}$ be the approximating sequence of part a) and define

$$\eta_n(\omega,t) := \int_K f_n(x,\omega,t)\,\mathrm{d}\theta(x), \quad n \in \mathbb{N}.$$

Then

$$\|\eta_n - \eta\|_{L^p(U;L^q(V;L^2[0,T]))} \le \|f_n - f\|_{L^1(K;L^p(U;L^q(V;L^2[0,T])))},$$

which converges to 0 almost surely as $n \to \infty$. By the Itô homeomorphism we obtain

$$\lim_{n\to\infty} \int_0^{(\cdot)} \eta_n\,\mathrm{d}\beta = \int_0^{(\cdot)} \eta\,\mathrm{d}\beta \qquad \text{in } L^0(\Omega; L^p(U; L^q(V; C[0,T]))).$$

Now the statement follows since $\int_0^t \eta_n\,\mathrm{d}\beta = \int_K \xi_n(t)\,\mathrm{d}\theta$ by Theorem 1.1.15. □

## 1.3 Itô Processes and Itô's Formula

In the previous two sections we have already familiarized ourselves with the stochastic integration theory with respect to a single Brownian motion. In this section we will extend the theory given there to a 'stochastic integral' with respect to an independent family of Brownian motions. The main motivation for doing this is to have a more general approach when applying this theory to stochastic partial differential equations of the form

$$\mathrm{d}X(t) = F(t, X(t))\,\mathrm{d}t + \sum_{n=1}^{\infty} B_n(t, X(t))\,\mathrm{d}\beta_n(t), \qquad X(0) = X_0,$$

which is defined as the integral equation

$$X(t) = X(0) + \int_0^t F(s, X(s))\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^t B_n(s, X(s))\,\mathrm{d}\beta_n(s).$$

Here and in the following we will assume that $(\beta_n(t))_{t\in[0,T]}$, $n \in \mathbb{N}$, is a sequence of independent Brownian motions *adapted* to $\mathbb{F}$, i.e. each $\beta_n(t)$ is $\mathcal{F}_t$-measurable and $\beta_n(t) - \beta_n(s)$ is independent of $\mathcal{F}_s$ for $t > s$ and $n \in \mathbb{N}$.

To get solutions in spaces like $L^r(\Omega; L^p(U; L^q[0,T]))$ or in $L^r(\Omega; L^p(U; C[0,T]))$ the minimal requirements will be

$$\begin{aligned} f &:= F(\cdot, X(\cdot)) \in L^r(\Omega; L^p(U; L^1[0,T])), \\ b_n &:= B_n(\cdot, X(\cdot)) \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^2[0,T])), \quad n \in \mathbb{N}, \end{aligned}$$

and the series $\sum_{n=1}^{\infty} \int_0^t b_n \,\mathrm{d}\beta_n$ should converge in one of the spaces above. This is the reason why we want to study processes of the form

$$X(t) = X(0) + \int_0^t f(s)\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^t b_n(s)\,\mathrm{d}\beta_n(s)$$

where we assume that $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^1[0,T]))$ and $b_n \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^2[0,T]))$ for every $n \in \mathbb{N}$.

**Definition 1.3.1.** Let $p, q, r \in (1, \infty)$ and let $X_0 \in L^r(\Omega, \mathcal{F}_0; L^p(U; L^q(V)))$, $f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^1[0,T])))$, and $b_n \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for every $n \in \mathbb{N}$. If the series

$$X(t) = X_0 + \int_0^t f(s)\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^t b_n \,\mathrm{d}\beta_n, \quad t \in [0,T],$$

converges in $L^r(\Omega; L^p(U; L^q(V)))$, then $X\colon \Omega \times [0,T] \to L^p(U; L^q(V))$ is called an $L^r$ *Itô process* with respect to $\mathbb{F}$ and $(\beta_n)_{n\in\mathbb{N}}$. The integral $\int_0^t f(s)\,\mathrm{d}s$ is called the *deterministic part* and $\sum_{n=1}^{\infty} \int_0^t b_n \,\mathrm{d}\beta_n$ the *stochastic part* of the Itô process $X$.

**Remark 1.3.2.** For $\boldsymbol{b} := (b_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} := (\beta_n)_{n\in\mathbb{N}}$ as in the previous definition we will often use the shorthand notation

$$\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} := \sum_{n=1}^{\infty} \int_0^t b_n\,\mathrm{d}\beta_n$$

or symbolically

$$\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} := \sum_{n=1}^{\infty} b_n\,\mathrm{d}\beta_n \quad \text{and} \quad \mathrm{d}X = f\,\mathrm{d}t + \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

The interesting question here is of course under which condition the series in the stochastic part of an Itô process converges. The answer to that is given in the following theorem.

**Theorem 1.3.3 (Itô isomorphism for Itô processes).** *Let $p, q, r \in (1, \infty)$, $t \in [0, T]$, and $b_n \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0, T])))$ for every $n \in \mathbb{N}$. Then the series*

$$\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \sum_{n=1}^{\infty} \int_0^t b_n\,\mathrm{d}\beta_n$$

*converges in $L^r(\Omega, \mathcal{F}_t; L^p(U; L^q(V)))$ if and only if $\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0, t] \times \mathbb{N}))))$, i.e.*

$$\mathbb{E}\Big\| \Big(\int_0^t \sum_{n=1}^{\infty} |b_n(s)|^2\,\mathrm{d}s\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} = \mathbb{E}\Big\| \Big(\int_0^t \|\boldsymbol{b}(s)\|^2_{\ell^2}\,\mathrm{d}s\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} < \infty.$$

*In this case we have*

$$\mathbb{E}\Big\| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big(\int_0^t \|\boldsymbol{b}(s)\|^2_{\ell^2}\,\mathrm{d}s\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

For the proof of this theorem we need an 'Itô isomorphism' for finite sums. This is the content of the next lemma. The proof can be done exactly as in [3, Lemma 4.3], where the $L^p(U)$-valued case is treated. We only need the UMD property of the space $L^p(U; L^q(V))$ and Kahane's inequalities for Gaussian sums.

**Lemma 1.3.4.** *Let $p, q, r \in (1, \infty)$ and $(b_n)_{n=1}^N \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0, T])))$. Then it holds*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \int_0^t b_n\,\mathrm{d}\beta_n \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big(\int_0^t \sum_{n=1}^{N} |b_n|^2\,\mathrm{d}t\Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}$$

*for each $t \in [0, T]$.*

**Proof (of Theorem 1.3.3).** If $\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$, then each $b_n$ is $L^r$-stochastically integrable, i.e. the random variables

$$X_N(t) := \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n, \quad N \in \mathbb{N},\ t \in [0,T],$$

are well-defined. By Lemma 1.3.4, the sequence $(X_N(t))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^r(\Omega, \mathcal{F}_t; L^p(U; L^q(V)))$, which gives the desired convergence result. Another application of Lemma 1.3.4 and the dominated convergence theorem lead to

$$\begin{aligned}
\mathbb{E}\Big\| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big\|^r_{L^p(U;L^q(V))} &= \lim_{N\to\infty} \mathbb{E}\Big\| \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n \Big\|^r_{L^p(U;L^q(V))} \\
&\eqsim_{p,q,r} \lim_{N\to\infty} \mathbb{E}\Big\| \Big( \int_0^t \sum_{n=1}^{N} |b_n|^2 \,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} \\
&= \mathbb{E}\Big\| \Big( \int_0^t \|\boldsymbol{b}(s)\|^2_{\ell^2} \,\mathrm{d}s \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.
\end{aligned}$$

Now assume the converse. In this case we have

$$\lim_{N\to\infty} \mathbb{E}\Big\| \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n \Big\|^r_{L^p(U;L^q(V))} = \mathbb{E}\Big\| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big\|^r_{L^p(U;L^q(V))} < \infty.$$

An application of Fatou's lemma and Lemma 1.3.4 then yields

$$\begin{aligned}
\mathbb{E}\Big\| \Big( \int_0^t \|\boldsymbol{b}(s)\|^2_{\ell^2} \,\mathrm{d}s \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} &\le \liminf_{N\to\infty} \mathbb{E}\Big\| \Big( \int_0^t \sum_{n=1}^{N} |b_n(s)|^2 \,\mathrm{d}s \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} \\
&\eqsim_{p,q,r} \liminf_{N\to\infty} \mathbb{E}\Big\| \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n \Big\|^r_{L^p(U;L^q(V))} \\
&= \mathbb{E}\Big\| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big\|^r_{L^p(U;L^q(V))} < \infty.
\end{aligned}$$

□

As a consequence of this theorem, the correct assumptions in Definition 1.3.1 for an Itô process $dX = f\,\mathrm{d}t + \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ to be well-defined are

$$f \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^1[0,T]))) \quad \text{and} \quad \boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N})))).$$

In other words, if we say that $dX = f\,\mathrm{d}t + \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is an $L^r$ Itô process we will always assume these conditions.

The following properties of the stochastic part of an Itô process are now mostly immediate consequences of Proposition 1.1.13 and Theorem 1.3.3.

**PROPOSITION 1.3.5 (Properties of $L^r$ Itô processes).** *Let $p, q, r \in (1, \infty)$, $t \in [0, T]$, and $\boldsymbol{b}, \boldsymbol{c} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0, T] \times \mathbb{N}))))$. Then the following properties hold:*

a) *For $a, b \in \mathbb{R}$ we have*

$$(a\boldsymbol{b} + b\boldsymbol{c})\,\mathrm{d}\boldsymbol{\beta} = a(\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}) + b(\boldsymbol{c}\,\mathrm{d}\boldsymbol{\beta}).$$

b) *$\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is adapted to $\mathbb{F}$ and*

$$\mathbb{E}\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = 0.$$

c) *For $S \in \mathcal{B}\big(L^p(U; L^q(V))\big)$, let $S^{L^2}$ be the bounded extension of $S$ on the space $L^p(U; L^q(V; L^2([0, T] \times \mathbb{N})))$. Then, $S^{L^2}\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0, T] \times \mathbb{N}))))$ and*

$$\int_0^t S^{L^2}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = S\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

d) *For every $s, t \in [0, T]$ with $s < t$ it holds that*

$$\int_s^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \int_0^T \mathbb{1}_{[s,t]}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

e) *There exists a $\mu$-null set $N_\mu \in \Sigma$ such that $\boldsymbol{b}(u) \in L^{p\wedge r}_{\mathbb{F}}(\Omega; L^q(V; L^2([0, T] \times \mathbb{N})))$, and*

$$\int_0^t \boldsymbol{b}(u)\,\mathrm{d}\boldsymbol{\beta} = \Big(\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big)(u) \quad \text{for each } u \in U \setminus N_\mu.$$

f) *There exists a $\mu \otimes \nu$-null set $N \in \Sigma \otimes \Xi$ such that $\boldsymbol{b}(u, v) \in L^{p\wedge q\wedge r}_{\mathbb{F}}(\Omega; L^2[0, T])$, and*

$$\int_0^t \boldsymbol{b}(u, v)\,\mathrm{d}\boldsymbol{\beta} = \Big(\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big)(u, v) \quad \text{for each } (u, v) \in (U \times V) \setminus N.$$

**PROOF.** a) The linearity follows from the convergence in Theorem 1.3.3 and the linearity of the Itô integral.

b) Adaptedness follows from Theorem 1.3.3. Moreover, since $\mathbb{E}\sum_{n=1}^N \int_0^t b_n\,\mathrm{d}\beta_n = 0$ by Proposition 1.1.13 b), we obtain

$$\begin{aligned}\Big\|\mathbb{E}\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|_{L^p(U;L^q(V))} &= \Big\|\mathbb{E}\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} - \mathbb{E}\sum_{n=1}^N \int_0^t b_n\,\mathrm{d}\beta_n\Big\|_{L^p(U;L^q(V))} \\ &\le \Big(\mathbb{E}\Big\|\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} - \sum_{n=1}^N \int_0^t b_n\,\mathrm{d}\beta_n\Big\|^r_{L^p(U;L^q(V))}\Big)^{1/r} \to 0 \quad \text{as } N \to \infty,\end{aligned}$$

which implies the claim.

c) For finite sums we have

$$\sum_{n=1}^{N} \int_0^t S^{L^2} b_n \,\mathrm{d}\beta_n = S \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n$$

by Proposition 1.1.13 c). Moreover, we trivially have $S^{L^2}\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$. Hence, Theorem 1.3.3 and the continuity of $S$ imply

$$\int_0^t S^{L^2}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \lim_{N\to\infty} \sum_{n=1}^{N} \int_0^t S^{L^2} b_n \,\mathrm{d}\beta_n = \lim_{N\to\infty} S \sum_{n=1}^{N} \int_0^t b_n \,\mathrm{d}\beta_n = S \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta},$$

where the limits take place in $L^r(\Omega; L^p(U; L^q(V)))$.

For the proof of d) and e), note that the estimates for finite sums again follow from Proposition 1.1.13 d) and e). Then the proof can be concluded in the same way as in the proof of this proposition by approximation and Theorem 1.3.3. □

**Remark 1.3.6.** If we compare Proposition 1.1.13 and Proposition 1.3.5 we see that we transferred every property from there to the Itô process case. The only additional thing we actually needed was the convergence of the series in the stochastic part of the Itô process. As long as the property we demand of the Itô process gets not destroyed by this convergence, everything carries over. In particular, the statements of Remark 1.1.14 still hold true.

Similar to the Itô integral process, the stochastic part of an Itô process has some useful regularity properties which we collect in the next theorem.

**Theorem 1.3.7 (More properties of $L^r$ Itô processes).** *Let $p, q, r \in (1,\infty)$ and $\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$. Then the following properties hold:*

a) ***Martingale property.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is a martingale with respect to the filtration $\mathbb{F}$.*

b) ***Continuity.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ has a continuous version satisfying the maximal inequality*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|^r_{L^p(U;L^q(V))} \lesssim_{p,q,r} \mathbb{E}\Big\| \int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big\|^r_{L^p(U;L^q(V))}.$$

c) ***Burkholder-Davis-Gundy inequality.*** *As a consequence of* b) *and Theorem 1.3.3 we have*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big( \int_0^T \|\boldsymbol{b}(t)\|^2_{\ell^2}\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

*Moreover, this estimate also holds true for $r = 1$. In particular, the Itô process $X(t) := \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$, $t \in [0,T]$, is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\Big\| \Big( \int_0^T |X(t)|^2 \,\mathrm{d}t \Big)^{1/2} \Big\|_{L^p(U;L^q(V))}^r \lesssim_{p,q,r} T^{1/2} \mathbb{E}\Big\| \Big( \int_0^T \|\boldsymbol{b}(t)\|_{\ell^2}^2 \,\mathrm{d}t \Big)^{1/2} \Big\|_{L^p(U;L^q(V))}^r .$$

**Proof.** For the proof of a) and b) we can proceed analogously to [3, Proposition 4.5]. In this case the martingale property carries over since the conditional expectation operator is continuous in $L^r(\Omega; L^p(U; L^q(V)))$. However, once we have a), part b) follows in the same way as in Theorem 1.1.18 using the strong Doob inequality.

c) For the case $r \in (1, \infty)$ there is nothing left to prove. If $r = 1$, we proceed similarly to the proof of Theorem 1.1.18 c). We can use the same decoupling technique to show that

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \sum_{n=1}^N \int_0^t b_n \,\mathrm{d}\beta_n \Big| \Big\|_{L^p(U;L^q(V))} \eqsim_{p,q,1} \mathbb{E}\Big\| \Big( \int_0^T \sum_{n=1}^N |b_n|^2 \,\mathrm{d}t \Big)^{1/2} \Big\|_{L^p(U;L^q(V))},$$

first for adapted step processes and then for arbitrary $b_n \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ by approximation. Especially for the first part, the independence of the Brownian motions is important (see also the proof of [3, Lemma 4.3]).

Here again, we have to anticipate some results for the localized case. Since $\boldsymbol{b}$ is an element of $L^1_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T]\times\mathbb{N}))))$ we will see that $\int_0^{(\cdot)} \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is well-defined, at least as an element of $L^0(\Omega; L^p(U; L^q(V; C[0,T])))$. Additionally, by the estimate above, the sequence $\big(\sum_{n=1}^N \int_0^{(\cdot)} b_n \,\mathrm{d}\beta_n\big)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega; L^p(U; L^q(V; C[0,T])))$. Hence, there exists a limit $\widetilde{X} \in L^1(\Omega; L^p(U; L^q(V; C[0,T])))$, and by considering subsequences we can easily verify that $\widetilde{X}(t)$ almost surely coincides with $\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$. This finally leads to

$$\begin{aligned}
\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|_{L^p(U;L^q(V))} &= \lim_{N\to\infty} \mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \sum_{n=1}^N \int_0^t b_n \,\mathrm{d}\beta_n \Big| \Big\|_{L^p(U;L^q(V))} \\
&\eqsim_{p,q,1} \lim_{N\to\infty} \mathbb{E}\Big\| \Big( \int_0^T \sum_{n=1}^N |b_n|^2 \,\mathrm{d}t \Big)^{1/2} \Big\|_{L^p(U;L^q(V))} \\
&= \mathbb{E}\Big\| \Big( \int_0^T \|\boldsymbol{b}\|_{\ell^2}^2 \,\mathrm{d}t \Big)^{1/2} \Big\|_{L^p(U;L^q(V))}. \qquad \square
\end{aligned}$$

Analogously to Section 1.2, we want to extend Itô processes to the localized case, i.e. we want to get rid of the integrability condition with respect to $\Omega$. In particular in regard to an $L^p(U; L^q(V))$-valued analogue of Itô's formula this is of huge interest.

**Definition 1.3.8.** Let $p, q \in (1, \infty)$ and let $X_0 \in L^0(\Omega, \mathcal{F}_0; L^p(U; L^q(V)))$, $f \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^1[0,T])))$, and $\boldsymbol{b} \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$. Then we call the process $X\colon \Omega\times[0,T]\to L^p(U;L^q(V))$ given by

$$X(t) = X_0 + \int_0^t f(s)\,\mathrm{d}s + \sum_{n=1}^{\infty}\int_0^t b_n\,\mathrm{d}\beta_n = X_0 + \int_0^t f(s)\,\mathrm{d}s + \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$$

an $L^0$ *Itô process* with respect to $\mathbb{F}$ and $(\beta_n)_{n\in\mathbb{N}}$.

Looking at the first half of this section it should not be a big surprise that this definition is indeed well-defined. Similar to $L^r$ Itô processes, this follows from an extension of the Itô homeomorphism. However, in this case we have to be careful since we now work in a metric space. One problem in this setting is that in general summation is no longer continuous and in many cases not even defined. In our case, the space $L^0_{\mathbb{F}}(\Omega;E)$ (where $E$ is any Banach space appearing here) is a vector space and, luckily, the metric on it is translation invariant. These two facts suffice to obtain the following result.

**THEOREM 1.3.9 (Itô homeomorphism for Itô processes).** *Let $p,q\in(1,\infty)$ and let $\boldsymbol{b}\in L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$. Then the process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is well-defined as an element of $L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;C[0,T])))$. Moreover, we have for all $\delta>0$ and $\varepsilon>0$ the estimates*

$$\mathbb{P}\Big(\Big\|\sup_{t\in[0,T]}\Big|\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big|\Big\|_{L^p(U;L^q(V))}>\varepsilon\Big)\le C^r\tfrac{\delta^r}{\varepsilon^r}+\mathbb{P}\big(\|\boldsymbol{b}\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))}\ge\delta\big)$$

*and*

$$\mathbb{P}\big(\|\boldsymbol{b}\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))}>\varepsilon\big)\le C^r\tfrac{\delta^r}{\varepsilon^r}+\mathbb{P}\Big(\Big\|\sup_{t\in[0,T]}\Big|\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big|\Big\|_{L^p(U;L^q(V))}\ge\delta\Big)$$

*for some $r\in(1,\infty)$ and the constant $C>0$ appearing in Theorem 1.3.3.*

**REMARK 1.3.10.**

a) Observe that the statement of Proposition 1.2.2 about stopping times in Itô integrals carries over to the $L^r$ Itô process case without any problems. Indeed, for any stopping time $\tau\colon\Omega\to[0,T]$ with respect to $\mathbb{F}$ and some $\boldsymbol{b}\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$ we of course have $\mathbb{1}_{[0,\tau]}\boldsymbol{b}\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$, and Proposition 1.2.2 almost surely implies

$$\int_0^\tau \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}=\sum_{n=1}^{\infty}\int_0^\tau b_n\,\mathrm{d}\beta_n=\sum_{n=1}^{\infty}\int_0^T \mathbb{1}_{[0,\tau]}b_n\,\mathrm{d}\beta_n=\int_0^T \mathbb{1}_{[0,\tau]}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

b) Part a) and Theorem 1.3.7 can now be used to show that

$$\mathbb{E}\Big\|\sup_{t\in[0,\tau]}\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|^r_{L^p(U;L^q(V))}\eqsim_C \mathbb{E}\|\mathbb{1}_{[0,\tau]}\boldsymbol{b}\|^r_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))}$$

for some constant $C > 0$ and $r \in (1, \infty)$. This fact extends Lemma 1.2.6 for processes $\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$ using the same stopping time argument as in the proof of this lemma. This means we obtain for all $\delta > 0$ and $\varepsilon > 0$ the estimates

$$\mathbb{P}\Big(\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|_{L^p(U;L^q(V))} > \varepsilon\Big) \le C^r \tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\big(\|\boldsymbol{b}\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} \ge \delta\big)$$

and

$$\mathbb{P}\big(\|\boldsymbol{b}\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} > \varepsilon\big) \le C^r \tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\Big(\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|_{L^p(U;L^q(V))} \ge \delta\Big).$$

**Proof (of Theorem 1.3.9).** For each $n \in \mathbb{N}$ let $(\tau_{n,k})_{k\in\mathbb{N}}$ be a localizing sequence for $b_n \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$. Then $b_{n,k} := \mathbb{1}_{[0,\tau_{n,k}]} b_n \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for some $r \in (1,\infty)$, and $\lim_{k\to\infty} b_{n,k} = b_n$ in $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2[0,T])))$ for all $n \in \mathbb{N}$. By Theorem 1.2.9 we have

$$\lim_{k\to\infty} \int_0^{\cdot} b_{n,k}\,\mathrm{d}\beta_n = \int_0^{\cdot} b_n\,\mathrm{d}\beta_n \qquad \text{in } L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T]))),$$

and since the metric of $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$ is translation-invariant, we obtain

$$\lim_{k\to\infty} \sum_{n=1}^N \int_0^{\cdot} b_{n,k}\,\mathrm{d}\beta_n = \sum_{n=1}^N \int_0^{\cdot} b_n\,\mathrm{d}\beta_n \qquad \text{in } L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$$

for each $N \in \mathbb{N}$. Using now Remark 1.3.10 b) similarly to the proof of Theorem 1.2.9, we arrive at

$$\mathbb{P}\Big(\Big\| \sum_{n=1}^N \int_0^{\cdot} b_n\,\mathrm{d}\beta_n \Big\|_{L^p(U;L^q(V;C[0,T]))} > \varepsilon\Big) \le C^r \tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\big(\|(b_n)_{n=1}^N\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} \ge \delta\big)$$

and

$$\mathbb{P}\big(\|(b_n)_{n=1}^N\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} \ge \delta\big) \le C^r \tfrac{\delta^r}{\varepsilon^r} + \mathbb{P}\Big(\Big\| \sum_{n=1}^N \int_0^{\cdot} b_n\,\mathrm{d}\beta_n \Big\|_{L^p(U;L^q(V;C[0,T]))} > \varepsilon\Big)$$

for each $\varepsilon > 0$ and $\delta > 0$. Using now the assumption $\boldsymbol{b} \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$, we see that

$$\lim_{M,N\to\infty} \mathbb{P}\big(\|(b_n)_{n=M}^N\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} \ge \delta\big) = 0.$$

Thus, by the previous estimate, $\big(\sum_{n=1}^N \int_0^{\cdot} b_n\,\mathrm{d}\beta_n\big)_{N\in\mathbb{N}}$ is a Cauchy sequence in the space $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; C[0,T])))$. By the completeness of this space we now obtain the convergence of the series and the well-definedness of the process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$. The extension of Remark 1.3.10 b) to $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ follows by a limiting argument similar to the proof of Theorem 1.2.9. □

Using this Theorem together with Proposition 1.2.12, we can derive the following list of properties by arguing similarly to the proof of Proposition 1.3.5.

**Proposition 1.3.11 (Properties of $L^0$ Itô processes).** *Let $p, q \in (1,\infty)$, $t \in [0,T]$, and $\boldsymbol{b}, \boldsymbol{c} \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$. Then the following properties hold:*

a) *For $a, b \in \mathbb{R}$ we have*

$$(a\boldsymbol{b} + b\boldsymbol{c})\,\mathrm{d}\boldsymbol{\beta} = a(\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}) + b(\boldsymbol{c}\,\mathrm{d}\boldsymbol{\beta}).$$

b) *$\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is adapted to $\mathbb{F}$.*

c) *For $S \in \mathcal{B}\big(L^p(U; L^q(V))\big)$, let $S^{L^2}$ be the bounded extension of $S$ on the space $L^p(U; L^q(V; L^2([0,T] \times \mathbb{N})))$. Then, $S^{L^2}\boldsymbol{b} \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$ and*

$$\int_0^t S^{L^2}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = S\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

d) *For every $s, t \in [0,T]$ with $s < t$ it holds that*

$$\int_s^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \int_0^T \mathbb{1}_{[s,t]}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

e) *There exists a $\mu$-null set $N_\mu \in \Sigma$ such that $\boldsymbol{b}(u) \in L^0_{\mathbb{F}}(\Omega; L^q(V; L^2([0,T] \times \mathbb{N})))$, and*

$$\int_0^t \boldsymbol{b}(u)\,\mathrm{d}\boldsymbol{\beta} = \Big(\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big)(u) \quad \text{for each } u \in U \setminus N_\mu.$$

f) *There exists a $\mu \otimes \nu$-null set $N \in \Sigma \otimes \Xi$ such that $\boldsymbol{b}(u,v) \in L^0_{\mathbb{F}}(\Omega; L^2[0,T])$, and*

$$\int_0^t \boldsymbol{b}(u,v)\,\mathrm{d}\boldsymbol{\beta} = \Big(\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big)(u,v) \quad \text{for each } (u,v) \in (U \times V) \setminus N.$$

**Remark 1.3.12.** As for the localized Itô integral, we generally can not say anything about the expected value of $\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$. However, the results of Remark 1.1.14 adjusted to the Itô process setting in the obvious way are still valid.

In Section 1.2 we proved several results regarding the behavior of stopping times in stochastic integrals. Later, when dealing with existence and uniqueness results for stochastic evolution equations, Itô processes like $\int_0^t \boldsymbol{b}(t)\,\mathrm{d}\boldsymbol{\beta}$ appear. Especially in the uniqueness part for measurable initial values, we rely on these results since we will apply stopping times. Let us recall (and slightly modify) the definition of $J$ and $J_\tau$ from the previous section. For any function $\boldsymbol{b}\colon [0,T] \to L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T] \times \mathbb{N}))))$ and any stopping time

$\tau\colon\Omega\to[0,T]$ let

$$J^{(\boldsymbol{b})}(t) := \int_0^t \boldsymbol{b}(t)\,\mathrm{d}\boldsymbol{\beta} = \int_0^t \boldsymbol{b}(t,s)\,\mathrm{d}\boldsymbol{\beta}(s)$$

and

$$J_\tau^{(\boldsymbol{b})}(t) := \int_0^t \mathbb{1}_{[0,\tau]}\boldsymbol{b}(t)\,\mathrm{d}\boldsymbol{\beta} = \int_0^t \mathbb{1}_{[0,\tau]}(s)\boldsymbol{b}(t,s)\,\mathrm{d}\boldsymbol{\beta}(s).$$

Then we get the following results.

**Proposition 1.3.13 (Itô processes and stopping times).** *Let $p,q\in(1,\infty)$ and $\tau\colon\Omega\to[0,T]$ be a stopping time with respect to $\mathbb{F}$.*

*a) Let $\boldsymbol{b}\in L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$. Then $\mathbb{1}_{[0,\tau]}\boldsymbol{b}\in L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$ and for every $t\in[0,T]$ it holds that*

$$\int_0^{t\wedge\tau} \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \int_0^t \mathbb{1}_{[0,\tau]}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \qquad \text{almost surely.}$$

*b) Let $\boldsymbol{b}\colon[0,T]\to L^0_{\mathbb{F}}(\Omega;L^p(U;L^q(V;L^2([0,T]\times\mathbb{N}))))$ be such that*

*i) $t\mapsto b_n(t)\colon[0,T]\to L^0(\Omega;L^p(U;L^q(V;L^2[0,T])))$ is continuous for each $n\in\mathbb{N}$ and*

*ii) $J$ and $J_\tau$ have continuous versions.*

*Then the processes $J$ and $J_\tau$ satisfy almost surely*

$$J(t\wedge\tau) = J_\tau(t\wedge\tau) \qquad \text{for } t\in[0,T].$$

*In particular, we almost surely have*

$$\mathbb{1}_{[0,\tau]}(t)\int_0^t \boldsymbol{b}(t,s)\,\mathrm{d}\boldsymbol{\beta}(s) = \mathbb{1}_{[0,\tau]}(t)\int_0^t \mathbb{1}_{[0,\tau]}(s)\boldsymbol{b}(t,s)\,\mathrm{d}\boldsymbol{\beta}(s).$$

**Proof.** The proof of a) follows immediately from Proposition 1.2.14, similarly to Remark 1.3.10. For part b), we remark that by Proposition 1.2.14 we almost surely have

$$J^{(b_n)}(t\wedge\tau) = J_\tau^{(b_n)}(t\wedge\tau)$$

for each fixed $n\in\mathbb{N}$. Now the claim follows from the observation

$$J^{(\boldsymbol{b})}(t) = \int_0^t \boldsymbol{b}(t)\,\mathrm{d}\boldsymbol{\beta} = \sum_{n=1}^\infty \int_0^t b_n(t)\,\mathrm{d}\beta_n(t) = \sum_{n=1}^\infty J^{(b_n)}(t),$$

and similarly $J_\tau^{(\boldsymbol{b})}(t) = \sum_{n=1}^\infty J_\tau^{(b_n)}(t)$ for each $t\in[0,T]$. □

In the same manner as before, we turn to regularity properties of the localized Itô process.

**THEOREM 1.3.14 (More properties of $L^0$ Itô processes).** *Let $p, q \in (1,\infty)$, $r \in [1,\infty)$, and $\boldsymbol{b} \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T]\times\mathbb{N}))))$. Then the following properties hold:*

a) ***Local martingale property.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is a local martingale with respect to the filtration $\mathbb{F}$.*

b) ***Continuity and Burkholder-Davis-Gundy inequality.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is almost surely continuous satisfying the maximal inequality*

$$\mathbb{E}\Big\| \sup_{t\in[0,T]} \Big| \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} \Big| \Big\|^r_{L^p(U;L^q(V))} \eqsim_{p,q,r} \mathbb{E}\Big\| \Big( \int_0^T \|\boldsymbol{b}(t)\|^2_{\ell^2}\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))},$$

*where this is understood in the sense that the left-hand side is finite if and only if the right-hand side is finite. If one of these cases hold, then the Itô process $X(t) := \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\Big\| \Big( \int_0^T |X(t)|^2\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))} \lesssim_{p,q,r} T^{1/2}\mathbb{E}\Big\| \Big( \int_0^T \|\boldsymbol{b}(t)\|^2_{\ell^2}\,\mathrm{d}t \Big)^{1/2} \Big\|^r_{L^p(U;L^q(V))}.$$

**PROOF.** Let $(\tau_k)_{k\in\mathbb{N}}$ be defined by

$$\tau_k(\omega) := T \wedge \inf\big\{ t\in[0,T] \colon \|\mathbb{1}_{[0,t]}\boldsymbol{b}(\omega)\|_{L^p(U;L^q(V;L^2([0,T]\times\mathbb{N})))} \geq k \big\}, \quad \omega\in\Omega.$$

As in Remark 1.2.8 we can show that $\tau_k$ is a stopping time with respect to $\mathbb{F}$ satisfying $\tau_k \leq \tau_{k+1}$, $\lim_{k\to\infty}\tau_k = T$ almost surely, and $\boldsymbol{b}_k := \mathbb{1}_{[0,\tau_k]}\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T]\times\mathbb{N}))))$ for each $k\in\mathbb{N}$ and some $r\in(1,\infty)$.

Now the proof of a) and b) can be done by following the lines of the proof of Theorem 1.2.15, using Theorem 1.3.7 c) and Proposition 1.3.13 a). □

With very little effort we can now even prove a generalization of the stochastic Fubini Theorem 1.2.16.

**THEOREM 1.3.15 (Stochastic Fubini theorem for Itô processes).** *Let $p, q \in (1,\infty)$, $(K,\mathcal{K},\theta)$ be a $\sigma$-finite measure space, and $\boldsymbol{b}\colon K\times\Omega \to L^p(U; L^q(V; L^2([0,T]\times\mathbb{N})))$ be strongly measurable such that*

$$\begin{aligned} &\boldsymbol{b}(\cdot,\omega) \in L^1(K; L^p(U; L^q(V; L^2([0,T]\times\mathbb{N})))) \quad \textit{for } \mathbb{P}\textit{-almost all } \omega\in\Omega, \\ &\boldsymbol{b}(x,\cdot) \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q(V; L^2([0,T]\times\mathbb{N})))) \quad \textit{for } \theta\textit{-almost all } x\in K. \end{aligned}$$

*Then the following assertions hold:*

a) *For $\theta$-almost all $x \in K$, $\boldsymbol{b}(x,\cdot)\,\mathrm{d}\boldsymbol{\beta}$ is an $L^0$-Itô process and*

$$\xi(x,\omega,t) := \Big(\int_0^t \boldsymbol{b}(x,s)\,\mathrm{d}\boldsymbol{\beta}(s)\Big)(\omega)$$

*is measurable satisfying almost surely*

$$\int_K \Big\| \sup_{t\in[0,T]} |\xi(x,t)| \Big\|_{L^p(U;L^q(V))}\,\mathrm{d}\theta(x) < \infty.$$

b) *For almost all $(\omega,t,u,v) \in \Omega \times [0,T] \times U \times V$ the functions $x \mapsto b_n(x,\omega,t,u,v)$ are integrable for all $n \in \mathbb{N}$ and for*

$$\boldsymbol{\eta}(\omega,t) := \int_K \boldsymbol{b}(x,\omega,t)\,\mathrm{d}\theta(x)$$

*the process $\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{\beta}$ is an $L^0$-Itô process.*

c) *Almost surely, we have*

$$\int_K \xi(x,t)\,\mathrm{d}\theta(x) = \int_0^t \boldsymbol{\eta}(s)\,\mathrm{d}\boldsymbol{\beta}(s), \quad t \in [0,T].$$

**Proof.** By using the strong Burkholder-Davis-Gundy inequality from Theorem 1.3.7, part a) can be shown in the same way as in the proof of Theorem 1.2.16. The statements of b) and c) follow in the same way. □

As already announced earlier, we finally show Itô's formula, which can be thought of as a counterpart of the chain rule in stochastic calculus. More precisely, we want to determine a 'Taylor expansion' of the process $\Phi(\cdot,X)\colon \Omega\times[0,T] \to L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V}))$, where $\Phi\colon [0,T]\times L^p(U;L^q(V)) \to L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V}))$ is a sufficiently differentiable function, $X$ is an $L^0$ Itô process and $(\widetilde{U},\widetilde{\Sigma},\widetilde{\mu})$ and $(\widetilde{V},\widetilde{\Xi},\widetilde{\nu})$ are $\sigma$-finite measure spaces.

**Theorem 1.3.16 (Itô's formula).** *Let $p,q,\widetilde{p},\widetilde{q} \in (1,\infty)$, $\Phi\colon [0,T]\times L^p(U;L^q(V)) \to L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V}))$ be an element of $C^{1,2}\big([0,T]\times L^p(U;L^q(V)); L^{\widetilde{p}}(\widetilde{U};L^{\widetilde{q}}(\widetilde{V}))\big)$, $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent Brownian motions, and $X$ be an $L^p(U;L^q(V))$-valued Itô process given by $\mathrm{d}X = f\,\mathrm{d}t + \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$. Further, let $\boldsymbol{b} \in L^0(\Omega;L^2([0,T]\times\mathbb{N};L^p(U;L^q(V))))$. Then, almost surely for all $t\in[0,T]$ we have*

$$\begin{aligned}\Phi\big(t,X(t)\big) = \Phi\big(0,X(0)\big) &+ \int_0^t \partial_t\Phi\big(s,X(s)\big)\,\mathrm{d}s + \int_0^t D_2\Phi\big(s,X(s)\big)f(s)\,\mathrm{d}s \\ &+ \sum_{n=1}^\infty \int_0^t D_2\Phi\big(s,X(s)\big)b_n(s)\,\mathrm{d}\beta_n(s) \\ &+ \frac{1}{2}\int_0^t \sum_{n=1}^\infty \Big(D_2^2\Phi\big(s,X(s)\big)b_n(s)\Big)b_n(s)\,\mathrm{d}s.\end{aligned}$$

For the proof of this statement see [15, Theorem 2.4] or [3, Theorem 4.16]. As an immediate consequence of this formula, we obtain the following product rule for Itô processes. For the proof we refer to [15, Corollary 2.6] (see also [3, Corollary 4.18]).

**Corollary 1.3.17 (Product rule).** *Let $p, q \in (1, \infty)$, $X$ be an $L^p(U; L^q(V))$-valued and $Y$ be an $L^{p'}(U; L^{q'}(V))$-valued Itô process given by $\mathrm{d}X = f\,\mathrm{d}t + \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ and $\mathrm{d}Y = g\,\mathrm{d}t + \boldsymbol{c}\,\mathrm{d}\boldsymbol{\beta}$, respectively. Let $X$ and $Y$ satisfy the assumptions of Theorem 1.3.16. Then, almost surely for all $t \in [0, T]$ we have*

$$\begin{aligned}\langle X(t), Y(t)\rangle = \langle X(0), Y(0)\rangle &+ \int_0^t \langle X(s), g(s)\rangle + \langle f(s), Y(s)\rangle \,\mathrm{d}s \\ &+ \sum_{n=1}^{\infty} \int_0^t \langle X(s), c_n(s)\rangle + \langle b_n(s), Y(s)\rangle \,\mathrm{d}\beta_n(s) \\ &+ \int_0^t \sum_{n=1}^{\infty} \langle b_n(s), c_n(s)\rangle \,\mathrm{d}s.\end{aligned}$$

## 1.4 Stochastic Integration in Sobolev and Besov Spaces

When taking a closer look at Sections 1.1, 1.2, and 1.3, it is straightforward to show the same results for other mixed $L^p$ spaces like

$$E = L^{p_1}(U_1; L^{p_2}(U_2; \ldots L^{p_N}(U_N)) \ldots)$$

by induction. The key to everything is the integrability condition

$$\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; E(L^2([0, T] \times \mathbb{N})))$$

for some $r \in \{0\} \cap (1, \infty)$ and with the $L^2([0, T] \times \mathbb{N})$ norm *inside* of the norm in $E$. This is the reason that makes stochastic integration theory in $L^p$ spaces or more generally in Banach spaces not as easy as deterministic integration theory.

Employing these results, we can treat the stochastic integration theory in (mixed) Sobolev and Besov spaces very easily. We do not want to consider this in too much detail here. However, we want to give an overview of how the integration theory in mixed $L^p$ spaces can be used to characterize the integration theory in such spaces. Let $U \subseteq \mathbb{R}^d$ be an open set (with possibly non-smooth boundary), $s > 0$ and $p \in [1, \infty)$. For the case $s \in (0, 1)$ we recall that a function $f \in L^p(U)$ is in the Sobolev-Slobodeckij space $W^{s,p}(U)$ if and only if the function $d_{W^{s,p}}[f]$ given by

$$d_{W^{s,p}}[f](x, y) := \frac{1}{|x - y|^{d/p+s}} \big(f(x) - f(y)\big)$$

is an element of $L^p(U\times U)$, and $W^{s,p}(U)$ is a Banach space with respect to the norm

$$\|f\|_{W^{s,p}(U)} = \big(\|f\|_{L^p(U)}^p + \|d_{W^{s,p}}[f]\|_{L^p(U\times U)}^p\big)^{1/p}.$$

In the case of a Banach space-valued Sobolev space $W^{s,p}(U;E)$ the norms are given by

$$\|f\|_{W^{s,p}(U;E)} = \big(\|f\|_{L^p(U;E)}^p + \|d_{W^{s,p}}[f]\|_{L^p(U\times U;E)}^p\big)^{1/p}.$$

If $s\in\mathbb{N}$, the space $W^{s,p}(U)$ is of course the usual Sobolev space, i.e. the space of all functions $f\in L^p(U)$ having $L^p$-integrable weak derivatives up to order $s$; in other words, $D^\alpha f$ exists in the weak sense and $D^\alpha f\in L^p(U)$ for all $|\alpha|\le s$. If $s>1$, then we can find an integer $m\in\mathbb{N}$ and $\sigma\in(0,1)$ such that $s=m+\sigma$. Here a function $f\in W^{m,p}(U)$ is in $W^{s,p}(U)$ if and only if $D^\alpha f\in W^{\sigma,p}(U)$ for each $\alpha\in\mathbb{N}_0^d$ satisfying $|\alpha|=m$. In this case we have the norm

$$\|f\|_{W^{s,p}(U)} = \Big(\|f\|_{W^{m,p}(U)}^p + \sum_{|\alpha|=m}\|D^\alpha f\|_{W^{\sigma,p}(U)}^p\Big)^{1/p}.$$

The first thing we want to know is the correct space of integrands. Looking at the previous sections, it is no surprise that it is given by $L^r_{\mathbb{F}}(\Omega;W^{s,p}(U;L^2([0,T]\times\mathbb{N})))$, again with the $L^2([0,T]\times\mathbb{N})$ norm *inside* of the Sobolev norm. To prove that, we first consider the case $s\in(0,1)$. If we want to estimate the integral $\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ for '$W^{s,p}(U)$-valued' $\boldsymbol{b}$ we have to estimate it in the $L^p(U)$ norm and $d_{W^{s,p}}[\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}]$ in the $L^p(U\times U)$ norm. Observe that the stochastic integral is well-defined since $W^{s,p}(U)\subseteq L^p(U)$. In view of $d_{W^{s,p}}[\boldsymbol{b}] = (d_{W^{s,p}}[b_n])_{n\in\mathbb{N}}\in L^r_{\mathbb{F}}(\Omega;L^p(U\times U;L^2([0,T]\times\mathbb{N})))$, Proposition 1.3.11 e) yields

$$d_{W^{s,p}}\Big[\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big] = \int_0^T d_{W^{s,p}}[\boldsymbol{b}]\,\mathrm{d}\boldsymbol{\beta}.$$

Using the Itô isomorphism for the $L^p$ case we obtain the following Itô isomorphism for Sobolev spaces:

$$\begin{aligned}\mathbb{E}\Big\|\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|_{W^{s,p}(U)}^r &\eqsim_{p,r} \mathbb{E}\Big\|\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|_{L^p(U)}^r + \mathbb{E}\Big\|\int_0^T d_{W^{s,p}}[\boldsymbol{b}]\,\mathrm{d}\boldsymbol{\beta}\Big\|_{L^p(U\times U)}^r\\ &\eqsim_{p,r} \mathbb{E}\|\boldsymbol{b}\|_{L^p(U;L^2([0,T]\times\mathbb{N}))}^r + \mathbb{E}\|d_{W^{s,p}}[\boldsymbol{b}]\|_{L^p(U\times U;L^2([0,T]\times\mathbb{N}))}^r\\ &\eqsim_{p,r} \mathbb{E}\big(\|\boldsymbol{b}\|_{L^p(U;L^2([0,T]\times\mathbb{N}))}^p + \|d_{W^{s,p}}[\boldsymbol{b}]\|_{L^p(U\times U;L^2([0,T]\times\mathbb{N}))}^p\big)^{r/p}\\ &= \mathbb{E}\|\boldsymbol{b}\|_{W^{s,p}(U;L^2([0,T]\times\mathbb{N}))}^r.\end{aligned}$$

One interesting fact is that the constants appearing here are independent of $s$. In the case $s\in\mathbb{N}$ or $s>1$, we can see very similarly that the same Itô isomorphism holds. Here we additionally use that

$$D^\alpha\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \int_0^T D^\alpha\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$$

for any multi-index $\alpha \in \mathbb{N}_0^d$. To see this, we use Proposition 1.3.11 c) and obtain for any $\phi \in C_c^\infty(U)$

$$\int_U \int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\, D^\alpha \phi \,\mathrm{d}x = \Big\langle \int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{b}, D^\alpha \phi \Big\rangle_{L^p(U)} = \int_0^T \langle \boldsymbol{b}, D^\alpha \phi\rangle_{L^p(U)}^{L^2} \,\mathrm{d}\boldsymbol{\beta}$$
$$= (-1)^{|\alpha|} \int_0^T \langle D^\alpha \boldsymbol{b}, \phi\rangle_{L^p(U)}^{L^2} \,\mathrm{d}\boldsymbol{\beta} = (-1)^{|\alpha|} \int_U \int_0^T D^\alpha \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\,\phi\,\mathrm{d}x.$$

Now the theory goes through without any problems. As a first result we obtain:

**Proposition 1.4.1 (Properties of the Sobolev space-valued integral).** *Let $s > 0$, $p \in (1,\infty)$, $r \in \{0\} \cup (1,\infty)$, and $\boldsymbol{b}, \widetilde{\boldsymbol{b}} \in L_{\mathbb{F}}^r(\Omega; W^{s,p}(U; L^2([0,T] \times \mathbb{N})))$. Then the following properties hold:*

a) *The stochastic integral is linear, i.e. for $a, b \in \mathbb{R}$ we have*

$$\int_0^T a\boldsymbol{b} + b\widetilde{\boldsymbol{b}}\,\mathrm{d}\boldsymbol{\beta} = a\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} + b\int_0^T \widetilde{\boldsymbol{b}}\,\mathrm{d}\boldsymbol{\beta}.$$

b) *$\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is $\mathcal{F}_T$-measurable and, if $r \in (1,\infty)$, the expected value satisfies $\mathbb{E}\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = 0$.*

c) *For $S \in \mathcal{B}\big(W^{s,p}(U)\big)$, let $S^{L^2}$ be the bounded extension of $S$ on $W^{s,p}(U; L^2([0,T] \times \mathbb{N}))$. Then, $S^{L^2}\boldsymbol{b} \in L_{\mathbb{F}}^r(\Omega; W^{s,p}(U; L^2([0,T] \times \mathbb{N})))$ and*

$$\int_0^T S^{L^2}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = S\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

d) *For every $\widetilde{s}, t \in [0,T]$ with $\widetilde{s} < t$ it holds that*

$$\int_{\widetilde{s}}^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta} = \int_0^T \mathbb{1}_{[\widetilde{s},t]}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}.$$

e) *There exists a null set $N \in \mathcal{B}_U$ such that $\boldsymbol{b}(u) \in L_{\mathbb{F}}^{p\wedge r}(\Omega; L^2[0,T])$, and*

$$\int_0^T \boldsymbol{b}(u)\,\mathrm{d}\boldsymbol{\beta} = \Big(\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big)(u) \quad \textit{for each } u \in U \setminus N.$$

The next step is to investigate the Sobolev space-valued integral process $t \mapsto \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$. One crucial property we needed in that part of the theory is the martingale property and the strong Doob and Burkholder-Davis-Gundy inequalities. In order to extend the results from the $L^p$-valued case, we have to carefully check if the differences and derivatives appearing in the Sobolev norms do not destroy any martingale structure. For this reason we need to prove the following lemma.

**LEMMA 1.4.2.** *Let $s > 0$, $p, r \in [1, \infty)$, and $(M_n)_{n=1}^N$ be a $W^{s,p}(U)$-valued $L^r$ martingale with respect to the filtration $(\mathcal{F}_n)_{n=1}^N$.*

*a) If $s \in (0, 1)$, then $(d_{W^{s,p}}[M_n])_{n=1}^N$ is an $L^p(U \times U)$-valued $L^r$ martingale with respect to $(\mathcal{F}_n)_{n=1}^N$.*

*b) If $s \in \mathbb{N}$, then $(D^\alpha M_n)_{n=1}^N$ is an $L^p(U)$-valued $L^r$ martingale with respect to $(\mathcal{F}_n)_{n=1}^N$ for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq s$. In particular, $(M_n)_{n=1}^N$ is an $L^p(U)$-valued $L^r$ martingale.*

**PROOF.** The strong measurability of $d_{W^{s,p}}[M_n]$ and $D^\alpha M_n$ with respect to $\mathcal{F}_n$ is trivial. Also, the integrability condition follows immediately from the assumption. Hence, the only thing left to check is the projection property. In the following let $n > m$.

a) Since $\mathbb{E}[M_n|\mathcal{F}_m] = M_m$ in $W^{s,p}(U)$ almost surely, we obtain that $\mathbb{E}[M_n|\mathcal{F}_m] = M_m$ in $L^p(U)$ and $d_{W^{s,p}}\big[\mathbb{E}[M_n|\mathcal{F}_m]\big] = d_{W^{s,p}}[M_m]$ in $L^p(U \times U)$ almost surely. The result now follows from

$$d_{W^{s,p}}\big[\mathbb{E}[M_n|\mathcal{F}_m]\big](x, y) = \mathbb{E}\big[d_{W^{s,p}}[M_n]\big|\mathcal{F}_m\big](x, y)$$

for almost every $(x, y) \in U \times U$.

b) By assumption we have $D^\alpha \mathbb{E}[M_n|\mathcal{F}_m] = D^\alpha M_m$ in $L^p(U)$ almost surely. Additionally, for $A \in \mathcal{F}_m$ and $\phi \in C_c^\infty(U)$ we have

$$\begin{aligned}\Big\langle \int_A D^\alpha M_n \,\mathrm{d}\mathbb{P}, \phi \Big\rangle_{L^p} &= \int_A \langle D^\alpha M_n, \phi\rangle_{L^p} \,\mathrm{d}\mathbb{P} = (-1)^{|\alpha|} \int_A \langle M_n, D^\alpha \phi\rangle_{L^p} \,\mathrm{d}\mathbb{P} \\ &= (-1)^{|\alpha|} \Big\langle \int_A \mathbb{E}[M_n|\mathcal{F}_m] \,\mathrm{d}\mathbb{P}, D^\alpha \phi \Big\rangle_{L^p} \\ &= \Big\langle \int_A D^\alpha \mathbb{E}[M_n|\mathcal{F}_m] \,\mathrm{d}\mathbb{P}, \phi \Big\rangle_{L^p},\end{aligned}$$

where we used that $M_n, \mathbb{E}[M_n|\mathcal{F}_m] \in W^{s,p}(U)$. Thus, $\mathbb{E}[D^\alpha M_n|\mathcal{F}_m] = D^\alpha \mathbb{E}[M_n|\mathcal{F}_m] = D^\alpha M_m$ almost surely. ☐

As a consequence we obtain, among other results, the following version of Doob's maximal inequality.

**THEOREM 1.4.3 (Strong Doob inequality, II).** *Let $s > 0$, $p, r \in (1, \infty)$, and $(M_n)_{n=1}^N$ be an $W^{s,p}(U)$-valued $L^r$ martingale with respect to the filtration $(\mathcal{F}_n)_{n=1}^N$. Then we have*

$$\mathbb{E}\big\|(M_n)_{n=1}^N\big\|_{W^{s,p}(U;\ell^\infty)}^r \lesssim_{p,r} \mathbb{E}\|M_N\|_{W^{s,p}(U)}^r.$$

*In particular, $\mathbb{E}\big\|\max_{n=1}^N |M_n|\big\|_{W^{s,p}(U)}^r \lesssim_{p,r} \mathbb{E}\|M_N\|_{W^{s,p}(U)}^r$.*

**Proof.** We first take a look at the case $s \in (0,1)$. Lemma 1.4.2 yields that $(d_{W^{s,p}}[M_n])_{n=1}^N$ and $(M_n)_{n=1}^N$ are $L^p$-valued $L^r$ martingales with respect to $(\mathcal{F}_n)_{n=1}^N$. The Strong Doob inequality now leads to

$$\begin{aligned}\mathbb{E}\big\|(M_n)_{n=1}^N\big\|_{W^{s,p}(U;\ell^\infty)}^r &\eqsim_{p,r} \mathbb{E}\big\|\max_{n=1}^N |M_n|\big\|_{L^p(U)}^r + \mathbb{E}\big\|\max_{n=1}^N |d_{W^{s,p}}[M_n]|\big\|_{L^p(U\times U)}^r \\ &\lesssim_{p,r} \mathbb{E}\|M_N\|_{L^p(U)}^r + \mathbb{E}\|d_{W^{s,p}}[M_N]\|_{L^p(U\times U)}^r \\ &\eqsim_{p,r} \mathbb{E}\|M_N\|_{W^{s,p}(U)}^r.\end{aligned}$$

The case $s \geq 1$ now follows in the same way using that $(D^\alpha M_n)_{n=1}^N$ and (if $s \notin \mathbb{N}$) $(d_{W^{s,p}}[D^\alpha M_n])_{n=1}^N$ are $L^p$-valued $L^r$ martingales with respect to $(\mathcal{F}_n)_{n=1}^N$ by the previous lemma. □

With nearly the same methods we obtain:

**Theorem 1.4.4 (Strong Burkholder-Davis-Gundy inequality, II).** *Let $s > 0$, $p \in (1,\infty)$, $r \in [1,\infty)$, and $(M_n)_{n=1}^N$ be an $W^{s,p}(U)$-valued $L^r$ martingale with respect to the filtration $(\mathcal{F}_n)_{n=1}^N$. Then we have*

$$\mathbb{E}\big\|(M_n)_{n=1}^N\big\|_{W^{s,p}(U;\ell^\infty)}^r \eqsim_{p,r} \mathbb{E}\|(M_n - M_{n-1})_{n=1}^N\|_{W^{s,p}(U;\ell^2)}^r.$$

*In particular,* $\mathbb{E}\big\|\max_{n=1}^N |M_n|\big\|_{W^{s,p}(U)}^r \eqsim_{p,r} \mathbb{E}\|(M_n - M_{n-1})_{n=1}^N\|_{W^{s,p}(U;\ell^2)}^r$.

These maximal inequalities were the heart of the regularity results for stochastic integrals. Analogously to the previous sections, these results may now be used to obtain corresponding properties for the Sobolev space-valued integral process. Alternatively, we can also use the results of the $L^p$-valued case.

**Theorem 1.4.5 (Properties of the integral processes).** *Let $s > 0$, $p \in (1,\infty)$, $r \in \{0\} \cup (1,\infty)$, and $\boldsymbol{b} \in L^r_{\mathbb{F}}(\Omega; W^{s,p}(U; L^2([0,T]\times\mathbb{N})))$. Then the following properties hold:*

1) *In the case $r \in (1,\infty)$:*

   a) ***Martingale property.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is a martingale with respect to the filtration $\mathbb{F}$.*

   b) ***Continuity.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ has a continuous version satisfying the maximal inequality*

$$\mathbb{E}\Big\|t \mapsto \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|_{W^{s,p}(U;C[0,T])}^r \lesssim_{p,r} \mathbb{E}\Big\|\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|_{W^{s,p}(U)}^r.$$

   *In particular,* $\mathbb{E}\big\|\sup_{t\in[0,T]}\big|\int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\big|\big\|_{W^{s,p}(U)}^r \lesssim_{p,r} \mathbb{E}\big\|\int_0^T \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\big\|_{W^{s,p}(U)}^r$.

c) ***Burkholder-Davis-Gundy inequality.*** *As a consequence of b) and the Itô isomorphism we have*

$$\mathbb{E}\Big\| t \mapsto \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|^r_{W^{s,p}(U;C[0,T])} \eqsim_{p,r} \mathbb{E}\|\boldsymbol{b}\|^r_{W^{s,p}(U;L^2([0,T]\times\mathbb{N}))},$$

*where this also holds for $r = 1$. In particular, the process $X(t) := \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$, $t \in [0,T]$, is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\|X\|^r_{W^{s,p}(U;L^2[0,T])} \lesssim_{p,r} T^{1/2}\mathbb{E}\|\boldsymbol{b}\|^r_{W^{s,p}(U;L^2([0,T]\times\mathbb{N}))}.$$

2) *In the case $r = 0$:*

a) ***Local martingale property.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is a local martingale with respect to the filtration $\mathbb{F}$.*

b) ***Continuity and Burkholder-Gundy inequality.*** *The Itô process $\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is almost surely continuous satisfying the maximal inequality*

$$\mathbb{E}\Big\| t \mapsto \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\Big\|^r_{W^{s,p}(U;C[0,T])} \eqsim_{p,r} \mathbb{E}\|\boldsymbol{b}\|^r_{W^{s,p}(U;L^2([0,T]\times\mathbb{N}))}.$$

*where this is understood in the sense that the left-hand side is finite if and only if the right-hand side is finite. If one of these cases holds, then the process $X(t) := \int_0^t \boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}$ is again $L^r$-stochastically integrable satisfying*

$$\mathbb{E}\|X\|^r_{W^{s,p}(U;L^2[0,T])} \lesssim_{p,r} T^{1/2}\mathbb{E}\|\boldsymbol{b}\|^r_{W^{s,p}(U;L^2([0,T]\times\mathbb{N}))}.$$

**Proof.** These results follow by applying the results of Theorems 1.3.7 and 1.3.14 to $\boldsymbol{b}$ (as an element of $L^p(U;L^2([0,T]\times\mathbb{N}))$) and $d_{W^{s,p}}[\boldsymbol{b}]$ separately, similarly to the calculations of the Itô isomorphism above. □

At this point we conclude the discussion about Sobolev space-valued stochastic integration theory, and turn to the case of Besov spaces. Here we define for $l \in \mathbb{N}_0$ and $h \in \mathbb{R}^d$ the set

$$U_{h,l} := \bigcap_{j=0}^{l} \{x \in U : x + jh \in U\} \subseteq U$$

and the difference operator

$$(\Delta_h^l f)(x) := f(x + lh) - f(x + (l-1)h).$$

Let $s > 0$ and $p, q \in (1,\infty)$. Choose $k, l \in \mathbb{N}_0$ such that $k < s$ and $l > s - k$ (e.g. if $s \in \mathbb{R}_+ \setminus \mathbb{N}$, then we could take $k = [s] < s$ and $l = 1$, and if $s \in \mathbb{N}$ then $k = s - 1$ and

$l = 2$ would suffice). Then we define the function $d_{B^{s,p}_q}[f]$ by

$$d_{B^{s,p}_q}[f](h,x) := \mathbb{1}_{U_{h,l}}(x)|h|^{-d/q-(s-k)}(\Delta^l_h f)(x).$$

Therewith we define the space $B^{s,p}_q(U)$ as the set of all functions $f \in L^p(U)$ such that $d_{B^{s,p}_q}[D^\alpha f] \in L^q(\mathbb{R}^d; L^p(U))$ for each $\alpha \in \mathbb{N}^d_0$ with $|\alpha| \le k$. Then $B^{s,p}_q(U)$ is a Banach space with respect to the norm

$$\|f\|_{B^{s,p}_q(U)} = \Big(\|f\|^p_{L^p(U)} + \sum_{|\alpha|\le k} \|d_{B^{s,p}_q}[D^\alpha f]\|^p_{L^q(\mathbb{R}^d;L^p(U))}\Big)^{1/p}.$$

We now get *exactly* the same results for Besov spaces as for Sobolev spaces by replacing $d_{W^{s,p}}[\cdot]$ with $d_{B^{s,p}_q}[\cdot]$ and $L^p(U\times U)$ by the mixed $L^p$ space $L^q(\mathbb{R}^d; L^p(U))$.

As a consequence of the remark given in the beginning of this section, we can also extend this theory to mixed Besov and/or Sobolev spaces. The only thing we have to remind ourselves of is that the $L^2([0,T]\times\mathbb{N})$ norm is always inside of the mixed space in order to have a well-defined stochastic integral.

This theory is now perfect to study time regularity for stochastic convolutions. Until this point we have only discussed regularity of the stochastic integral process

$$t \mapsto \int_0^t \boldsymbol{b}(s)\,\mathrm{d}\boldsymbol{\beta}(s)$$

of $L^p$-valued processes $f$. As in the deterministic case, integrals of the form

$$t \mapsto \int_0^t e^{-(t-s)A}\boldsymbol{b}(s)\,\mathrm{d}\boldsymbol{\beta}(s)$$

will appear in the formulation of mild solutions for stochastic evolution equations, where $(-A)$ is the generator of an analytic semigroup. Since

$$t \mapsto \int_0^t e^{-(t-s)A}\boldsymbol{b}(s)\,\mathrm{d}\boldsymbol{\beta}(s) = \int_0^T \mathbb{1}_{[0,t]}(s)e^{-(t-s)A}\boldsymbol{b}(s)\,\mathrm{d}\boldsymbol{\beta}(s)$$

by Proposition 1.4.1, investigating regularity of stochastic convolutions in $L^p(U; L^q[0,T])$ or $L^p(U; W^{s,q}[0,T])$ reduces to the estimation of the function

$$\mathbb{1}_{[0,t]}(s)e^{-(t-s)A}\boldsymbol{b}(s)$$

in $L^p(U; L^q_{(t)}([0,T]; L^2_{(s)}[0,T]))$ or $L^p(U; W^{s,q}_{(t)}([0,T]; L^2_{(s)}[0,T]))$, respectively.

# Chapter 2

# Functional Analytic Operator Properties

In Chapter 3 we want to use functional calculi results to deduce regularity properties of deterministic and stochastic convolutions. These in turn will lead to new regularity results for stochastic evolution equations. In the following sections we introduce several notions which appear in this context. The basic question here is: How can we define the expression $f(A)$ for a linear operator $A$ and some function $f$? And which conditions do we have to impose on $A$ or $f$ to get nice properties of $f(A)$?

## 2.1 $\mathcal{R}_q$-boundedness and $\mathcal{R}_q$-sectorial Operators

In this section we concentrate on basic notions coming into focus when dealing with functional calculi results. To give a short motivation, we recall Cauchy's integral formula for holomorphic functions $f$, stating that

$$f(\lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-\lambda} \, \mathrm{d}z,$$

where $\Gamma$ is a closed path around the singularity $\lambda$. If we 'plug in' an operator $A$ in this equation, we would end up with

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A) \, \mathrm{d}z,$$

where now $\Gamma$ should circumvent the 'singularity' of $R(z, A)$, i.e. the spectrum $\sigma(A)$. Of course, this is just a motivation. In the next section we will give a reasonable definition of this idea. However, this already indicates the necessity of some characteristic features the resolvent function of $A$ should have.

Before turning to that, we start with a special randomization property for a set of bounded operators.

**Definition 2.1.1.** For any Banach spaces $E$ and $F$ we call a set of operators $\mathcal{T} \subseteq \mathcal{B}(E,F)$ *$\mathcal{R}$-bounded* if

$$\widetilde{\mathbb{E}}\Big\| \sum_{n=1}^{N} \widetilde{r}_n T_n x_n \Big\|_F \lesssim_{E,F,\mathcal{T}} \widetilde{\mathbb{E}}\Big\| \sum_{n=1}^{N} \widetilde{r}_n x_n \Big\|_E$$

for each finite sequences $(x_n)_{n=1}^N \subseteq E$, $(T_n)_{n=1}^N \subseteq \mathcal{T}$, and each Rademacher sequence $(\widetilde{r}_n)_{n=1}^N$ on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

$\mathcal{R}$-boundedness is a generalization of a square function estimate. In the special case of a mixed $L^p$ space $E$, like $E = L^r(\Omega; L^p(U; L^q(V)))$, this is particularly obvious since we have here the following characterization.

**Proposition 2.1.2.** *Let $E$ and $F$ be two mixed $L^p$ spaces. Then $\mathcal{T} \subseteq \mathcal{B}(E,F)$ is $\mathcal{R}$-bounded if and only if*

$$\Big\| \Big( \sum_{n=1}^{N} |T_n f_n|^2 \Big)^{1/2} \Big\|_F \lesssim_{E,F,\mathcal{T}} \Big\| \Big( \sum_{n=1}^{N} |f_n|^2 \Big)^{1/2} \Big\|_E$$

*for each $(f_n)_{n=1}^N \subseteq E$ and $(T_n)_{n=1}^N \subset \mathcal{T}$.*

**Proof.** This is a consequence of the special form Kahane's inequality has in this particular case. Let $G \in \{E, F\}$. Using the estimate for $\mathbb{K}$-valued Rademacher sums, i.e.

$$\widetilde{\mathbb{E}}\Big| \sum_{n=1}^{N} \alpha_n \widetilde{r}_n \Big|^{\widetilde{p}} \eqsim_{\widetilde{p}} \Big( \sum_{n=1}^{N} |\alpha_n|^2 \Big)^{\widetilde{p}/2}$$

for any $\widetilde{p} \in [1,\infty)$ and $(\alpha_n)_{n=1}^N \subseteq \mathbb{K}$, as well as the $q$-concavity of the space $G$ for some $q \in [1,\infty)$, we obtain

$$\widetilde{\mathbb{E}}\Big\| \sum_{n=1}^{N} \widetilde{r}_n g_n \Big\|_G \eqsim \Big\| \Big( \sum_{n=1}^{N} |g_n|^2 \Big)^{1/2} \Big\|_G$$

for any sequence $(g_n)_{n=1}^N \subseteq G$. □

In his paper [87] Lutz Weis extended the concept of $\mathcal{R}$-boundedness and introduced the notion of *$\mathcal{R}_q$-boundedness* in the special case of $L^p$ spaces. In [79] this was elaborated in detail in the setting of Banach function spaces (see also [57]).

**Definition 2.1.3.** For any mixed $L^p$ spaces $E$ and $F$ we call a set of operators $\mathcal{T} \subseteq \mathcal{B}(E,F)$ $\mathcal{R}_q$*-bounded* for some $q \in [1,\infty]$ if

$$\Big\| \Big(\sum_{n=1}^{N} |T_n f_n|^q\Big)^{1/q} \Big\|_F \lesssim_{E,F,\mathcal{T},q} \Big\| \Big(\sum_{n=1}^{N} |f_n|^q\Big)^{1/q} \Big\|_E$$

for each finite sequences $(f_n)_{n=1}^N \subseteq E$ and $(T_n)_{n=1}^N \subseteq \mathcal{T}$ (with the obvious modification for $q=\infty$). We call a single operator $T \in \mathcal{B}(E,F)$ $\mathcal{R}_q$*-bounded* if $\{T\}$ is $\mathcal{R}_q$-bounded.

**Remark 2.1.4.**

a) The boundedness assumption $\mathcal{T} \subseteq \mathcal{B}(E,F)$ is not necessary since any linear operator in an $\mathcal{R}_q$-bounded set $\mathcal{T}$ is automatically bounded. This can easily be seen by taking $N=1$ in the definition.

b) By Proposition 2.1.2 $\mathcal{R}$-boundedness is equivalent to $\mathcal{R}_2$-boundedness in the case of mixed $L^p$ spaces. In particular, this implies that every *single* bounded operator is automatically $\mathcal{R}_2$-bounded. For $q \neq 2$, this is in general not the case (see [32, Chapter 8]).

c) By Fatou's lemma, one can replace the finite sums in the definition by infinite series. In particular, a single operator $T \in \mathcal{B}(E,F)$ is $\mathcal{R}_q$-bounded if and only if the diagonal operator

$$\widetilde{T}\colon E(\ell^q) \to F(\ell^q), \quad \widetilde{T}(x_n)_n = (Tx_n)_n,$$

defines a bounded operator.

There also exists a continuous version of $\mathcal{R}_q$-boundedness (cf. Lemma 4 a) in [87] and in particular Proposition 2.12 in [57]).

**Proposition 2.1.5.** *Let $E, F$ be mixed $L^p$ spaces, $q \in [1,\infty)$, $(V,\Xi,\nu)$ be a $\sigma$-finite measure space, and $S\colon V \to \mathcal{B}(E,F)$ be strongly measurable such that $S(V)$ is $\mathcal{R}_q$-bounded. Then for all measurable $f\colon V \to E$ we have*

$$\Big\| \Big(\int_V |S(v)f(v)|^q \,\mathrm{d}\nu(v)\Big)^{1/q} \Big\|_F \leq C \Big\| \Big(\int_V |f(v)|^q \,\mathrm{d}\nu(v)\Big)^{1/q} \Big\|_E$$

*for a constant $C = C(E,F,S,q) > 0$.*

The last comment in Remark 2.1.4 already indicates the connection to classical harmonic analysis, where the terminology of $\mathcal{R}_q$-boundedness is mostly replaced by $\ell^q$ *extensions* or $\ell^q$*-valued estimates*. Nevertheless, there are many classical results for special classes of operators showing $\mathcal{R}_q$-boundedness. See e.g. the monographs [36] or [37] for Banach space-valued singular integral operators. Famous examples which happen to be $\mathcal{R}_q$-bounded include the Hilbert transform and the Riesz transform on $L^p$ for $p, q \in (1, \infty)$ (see [11] or [39, Corollary 5.6.3]). Another famous result is the Fefferman-Stein-inequality for the (uncentered) Hardy-Littlewood maximal function

$$(Mf)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f| \,\mathrm{d}\mu, \quad f \in L^q_{\mathrm{loc}}(\mathbb{R}^d), x \in \mathbb{R}^d,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^d$ containing $x$. Here, $|B|$ denotes the Lebesgue measure of $B$. For the proof of the following result see [35] or [39, Theorem 5.6.6].

**Theorem 2.1.6 (Fefferman-Stein).** *Let $p, q \in (1, \infty)$. Then the Hardy-Littlewood maximal operator $M$ is $\mathcal{R}_q$-bounded on $L^p(\mathbb{R}^d)$.*

**Remark 2.1.7.** This result is also true if we replace $\mathbb{R}^d$ by a metric measure space $(U, d, \mu)$ of homogeneous type (cf. [40]), i.e. $(U, d)$ is a metric space and $\mu$ is a $\sigma$-finite regular Borel measure on $U$ with the doubling property which in turn means that there exists a constant $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad x \in U, r > 0,$$

where $B(x, r)$ denotes the ball with center $x$ and radius $r$.

With the concepts of $\mathcal{R}$-boundedness and $\mathcal{R}_q$-boundedness we can now focus on some notions for resolvents as indicated in the beginning. For this purpose we need open sectors in $\mathbb{C}$, which we abbreviate as

$$\Sigma_\sigma := \{z \in \mathbb{C} \setminus \{0\} \colon |\arg(z)| < \sigma\}, \quad \sigma \in (0, \pi],$$

and $\Sigma_0 := (0, \infty)$.

**Definition 2.1.8.** Let $E$ be a Banach space and let $A \colon D(A) \subseteq E \to E$ be a closed linear operator.

a) $A$ is called a *sectorial operator of angle* $\alpha \in [0, \pi)$ if its spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\alpha$ and there exists a constant $C_\alpha > 0$ such that

$$\|\lambda R(\lambda, A)\|_E \leq C_\alpha \qquad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\alpha.$$

The infimum over all such $\alpha$ is denoted by $\omega(A)$.

b) $A$ is called an $\mathcal{R}$*-sectorial operator of angle* $\alpha \in [0, \pi)$ if its spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\alpha$ and the set $\{\lambda R(\lambda, A) \colon \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\alpha\}$ is $\mathcal{R}$-bounded, i.e. there exists a constant $C_\alpha > 0$ such that

$$\widetilde{\mathbb{E}}\Big\| \sum_{n=1}^{N} \widetilde{r}_n \lambda_n R(\lambda_n, A) x_n \Big\|_E \leq C_\alpha \, \widetilde{\mathbb{E}}\Big\| \sum_{n=1}^{N} \widetilde{r}_n x_n \Big\|_E$$

for each finite sequence $(\lambda_n)_{n=1}^N \subseteq \mathbb{C} \setminus \overline{\Sigma}_\alpha$, $(x_n)_{n=1}^N \subseteq E$, and each Rademacher sequence $(\widetilde{r}_n)_{n=1}^N$ on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. In this case, we denote by $\omega_{\mathcal{R}}(A)$ the infimum over all such $\alpha$.

c) Let $E$ be a mixed $L^p$ space and $q \in [1, \infty]$. Then we call $A$ an *$\ell^q$-sectorial operator of angle* $\alpha \in [0, \pi)$ if its spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\alpha$ and there exists a constant $C_\alpha > 0$ such that

$$\Big\| \Big( \sum_{n=1}^{N} |\lambda R(\lambda, A) x_n|^q \Big)^{1/q} \Big\|_E \leq C_\alpha \Big\| \Big( \sum_{n=1}^{N} |x_n|^q \Big)^{1/q} \Big\|_E \quad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\alpha$$

and each finite sequence $(x_n)_{n=1}^N \subseteq E$ (with the obvious modification for $q = \infty$). The infimum over all such $\alpha$ is denoted by $\omega_{\ell^q}(A)$.

d) Let $E$ be a mixed $L^p$ space and $q \in [1, \infty]$. Then we call $A$ an *$\mathcal{R}_q$-sectorial operator of angle* $\alpha \in [0, \pi)$ if its spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\alpha$ and the set $\{\lambda R(\lambda, A) \colon \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\alpha\}$ is $\mathcal{R}_q$-bounded, i.e. there exists a constant $C_\alpha > 0$ such that

$$\Big\| \Big( \sum_{n=1}^{N} |\lambda_n R(\lambda_n, A) x_n|^q \Big)^{1/q} \Big\|_E \leq C_\alpha \Big\| \Big( \sum_{n=1}^{N} |x_n|^q \Big)^{1/q} \Big\|_E$$

for each finite sequence $(\lambda_n)_{n=1}^N \subseteq \mathbb{C} \setminus \overline{\Sigma}_\alpha$ and $(x_n)_{n=1}^N \subseteq E$ (with the obvious modification for $q = \infty$). The infimum over all such $\alpha$ is denoted by $\omega_{\mathcal{R}_q}(A)$.

**Remark 2.1.9.**

a) In the case of mixed $L^p$ spaces, Proposition 2.1.2 directly yields that $\mathcal{R}$-sectoriality is equivalent to $\mathcal{R}_2$-sectoriality.

b) The difference between part c) and d) is the following: If $A$ is and $\ell^q$-sectorial operator, then every *single* operator set $\{\lambda R(\lambda, A)\}$, $\lambda \in \Sigma_\alpha$, is $\mathcal{R}_q$-bounded with a uniform constant $C_\alpha$. In particular, every $\mathcal{R}_q$-sectorial operator is $\ell^q$-sectorial.

Remark 2.1.4 already indicates the connection to a diagonal operator (see [57, Proposition 3.2]). Looking closely at the proof of this statement, one sees that we only need $\ell^q$-sectoriality to get the following result.

**Proposition 2.1.10.** *Let $q \in [1,\infty]$, $E$ be a mixed $L^p$ space, and $A$ be $\ell^q$-sectorial. Then we define*

$$D(\widetilde{A}) := \{(x_n)_{n\in\mathbb{N}} \in E(\ell^q) \colon x_n \in D(A) \text{ for all } n \in \mathbb{N} \text{ and } (Ax_n)_{n\in\mathbb{N}} \in E(\ell^q)\}$$

*and $\widetilde{A}x = (Ax_n)_{n\in\mathbb{N}}$, for $x \in D(\widetilde{A})$. Then $\widetilde{A}$ is a sectorial operator with $\omega(\widetilde{A}) \le \omega_{\ell^q}(A)$ and*

$$R(\lambda,\widetilde{A})x = (R(\lambda,A)x_n)_{n\in\mathbb{N}} \quad \text{for } \lambda \notin \overline{\Sigma}_{\omega_{\ell^q}(A)} \text{ and } x \in E(\ell^q).$$

In Section 2.3 we will see more results on the connection of these notions.

## 2.2 $H^\infty$ and $\mathcal{R}H^\infty$ Calculus

Using the terminology of the previous section, we can define a functional calculus for sectorial operators. In this section we will always assume that $A\colon D(A) \to E$ is a closed operator on some Banach space $E$ with dense domain and dense range. By the sectoriality, $A$ is then already injective (cf. [43, Proposition 2.11]). This assumption is not really restrictive, since we mostly work in $L^p$ spaces for $p \in (1,\infty)$, i.e in the reflexive case. In this situation $A$ always has dense domain, and the injectivity is equivalent to $A$ having dense range.

Let in the following be $\alpha \in (\omega(A),\pi]$. For functions $f\colon \Sigma_\alpha \to \mathbb{C}$ we define the norm

$$\|f\|_{\infty,\alpha} := \sup_{\lambda\in\Sigma_\alpha} |f(\lambda)|$$

and the space

$$H^\infty(\Sigma_\alpha) := \big\{f\colon \Sigma_\alpha \to \mathbb{C}\colon f \text{ is analytic and } \|f\|_{\infty,\alpha} < \infty\big\},$$

as well as

$$H_0^\infty(\Sigma_\alpha) := \Big\{f \in H^\infty(\Sigma_\alpha)\colon \sup_{\lambda\in\Sigma_\alpha} \big(|\lambda|^\varepsilon \vee |\lambda|^{-\varepsilon}\big)|f(\lambda)| < \infty \text{ for some } \varepsilon > 0\Big\},$$

Now let $\sigma \in (\omega(A),\alpha)$. Then we define the path

$$\Gamma(\sigma) := \big\{\lambda \in \mathbb{C}\colon \lambda = \gamma(t) = |t|e^{-i\operatorname{sign}(t)\sigma}, t \in \mathbb{R}\big\}.$$

As indicated in the beginning of the previous section, we define for functions $\varphi \in H_0^\infty(\Sigma_\alpha)$ the expression $\varphi(A)$ as the integral

$$\varphi(A) := \frac{1}{2\pi i}\int_{\Gamma(\sigma)} \varphi(\lambda)R(\lambda,A)\,\mathrm{d}\lambda,$$

which is well-defined as a Bochner integral in $\mathcal{B}(E)$ since $\|\varphi(\gamma(\cdot))R(\gamma(\cdot),A)\|_{\mathcal{B}(E)}$ is integrable in $\mathbb{R}$. Note that the algebra homomorphism

$$\varphi \mapsto \varphi(A) \colon H_0^\infty(\Sigma_\alpha) \to \mathcal{B}(E)$$

is independent of $\sigma \in (\omega(A), \alpha)$ by Cauchy's integral formula. Following [59], we can extend this functional calculus to functions $f \in H^\infty(\Sigma_\alpha)$ and even larger classes of functions (for more details in this direction see e.g. [43, Section 2.2]). However, without any additional assumptions on $A$, these extended functional calculi only yield closed operators.

One of the most important features of this calculus is the following convergence property (see [43, Proposition 5.1.4]).

**Proposition 2.2.1.** *Let $(f_n)_{n\geq 1} \subseteq H^\infty(\Sigma_\alpha)$ with the following properties*

*a)* $\exists\, f_0(\lambda) := \lim_{n\to\infty} f_n(\lambda)$ *for all* $\lambda \in \Sigma_\alpha$*;*

*b)* $\sup_{n\in\mathbb{N}} \|f_n\|_{\infty,\alpha} < \infty$*;*

*c)* $f_n(A) \in \mathcal{B}(E)$ *for all* $n \in \mathbb{N}$ *and* $M := \sup_{n\in\mathbb{N}} \|f_n(A)\| < \infty$*.*

*Then* $f_0 \in H^\infty(\Sigma_\alpha)$ *and* $f_0(A) \in \mathcal{B}(E)$*, satisfying* $\|f_0(A)\| \leq M$*. Moreover,*

$$\lim_{n\to\infty} f_n(A)x = f_0(A)x, \quad x \in E.$$

Now let us proceed to the definition of a bounded $H^\infty(\Sigma_\alpha)$ calculus.

**Definition 2.2.2.** Let $\alpha \in (\omega(A), \pi]$. Then $A$ has a *bounded $H^\infty(\Sigma_\alpha)$ calculus* if there is a constant $C_\alpha < \infty$ such that

$$\|f(A)\| \leq C_\alpha \|f\|_{\infty,\alpha} \quad \text{for all } f \in H^\infty(\Sigma_\alpha).$$

In this case we define

$$\omega_{H^\infty}(A) := \inf\{\alpha \in (\omega(A), \pi] \colon A \text{ has a bounded } H^\infty(\Sigma_\alpha) \text{ calculus}\}.$$

Following [59, Remark 9.11] or [43, Proposition 5.3.4], the convergence property and the closed graph theorem imply a slightly different characterization of a bounded $H^\infty$ calculus.

**Corollary 2.2.3.** *The operator $A$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus if and only if there is a constant $C_\alpha > 0$ such that*

$$\|\varphi(A)\| \leq C_\alpha \|\varphi\|_{\infty,\alpha} \quad \textit{for all } \varphi \in H_0^\infty(\Sigma_\alpha).$$

In [52] the authors extended this calculus to operator-valued functions with $\mathcal{R}$-bounded range, the so called $\mathcal{R}H^\infty$-functional calculus. Under some geometric assumptions on the underlying Banach space they proved that this calculus is again $\mathcal{R}$-bounded. The following notions are taken from [52] and [59, Chapter 12]. We denote by

$$\mathcal{A} := \{B \in \mathcal{B}(E) \colon B \text{ commutes with the resolvents of } A\},$$

and for $\alpha \in (\omega(A), \pi]$ the set

$$\mathcal{R}H^\infty(\Sigma_\alpha) := \{F \colon \Sigma_\alpha \to \mathcal{A} \colon F \text{ is analytic and } F(\Sigma_\alpha) \text{ is } \mathcal{R}\text{-bounded}\}$$

as well as

$$\mathcal{R}H_0^\infty(\Sigma_\alpha) := \{F \in \mathcal{R}H^\infty(\Sigma_\alpha) \colon \sup_{\lambda \in \Sigma_\alpha} \big(|\lambda|^\varepsilon \vee |\lambda|^{-\varepsilon}\big)\|F(\lambda)\| < \infty \text{ for some } \varepsilon > 0\}.$$

In the same way as above we can define for $\sigma \in (\omega(A), \alpha)$ and $F \in \mathcal{R}H_0^\infty(\Sigma_\alpha)$ the integral

$$F(A) := \frac{1}{2\pi i} \int_{\Gamma(\sigma)} F(\lambda) R(\lambda, A) \,\mathrm{d}\lambda$$

as a Bochner integral in $\mathcal{B}(E)$. The mapping

$$\Phi_A \colon \mathcal{R}H_0^\infty(\Sigma_\alpha) \to \mathcal{B}(E)$$

defines a functional calculus which can be extended to

$$\Phi_A \colon \mathcal{R}H^\infty(\Sigma_{\alpha'}) \to \mathcal{B}(E)$$

for some $\alpha' > \alpha$ if $A$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus (see [52, Theorem 4.4] or [59, Theorem 12.7]). If $E$ has additional geometric properties, namely *Pisier's property* $(\alpha)$, this can be used to self-improve the $H^\infty$ calculus.

**Definition 2.2.4.** Let $(r_n)_{n\geq 1}$ and $(\widetilde{r}_n)_{n\geq 1}$ be two independent Rademacher sequences. Then $E$ has *property* $(\alpha)$ if there is a constant $C < \infty$ such that for all $N \in \mathbb{N}$, $(\alpha_{j,k})_{j,k=1}^N \subseteq \{+1, -1\}$, and all $(x_{j,k})_{j,k=1}^N \subseteq E$ we have

$$\mathbb{E}\widetilde{\mathbb{E}}\Big\| \sum_{j,k=1}^N \alpha_{j,k} r_j \widetilde{r}_k x_{j,k} \Big\|_E \leq C \mathbb{E}\widetilde{\mathbb{E}}\Big\| \sum_{j,k=1}^N r_j \widetilde{r}_k x_{j,k} \Big\|_E.$$

As an example, $q$-concave Banach function spaces possess this property. Therefore, especially $L^p$ spaces do have the property $(\alpha)$. Putting these facts together, we obtain the following remarkable result (see [52, Theorem 5.3 and Corollary 5.4] or [59, Theorem 12.8 and Remark 12.10]).

**COROLLARY 2.2.5.** *Assume that $E$ has property $(\alpha)$ and $A$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus. Then for each $\alpha' > \alpha$ it holds that*

$$\{f(A)\colon \|f\|_{\infty,\alpha'} \leq 1\} \quad \text{is } \mathcal{R}\text{-bounded},$$

*i.e. $A$ has an $\mathcal{R}$-bounded $H^\infty(\Sigma_{\alpha'})$ calculus. Moreover, also the set*

$$\{F(A)\colon F \in \mathcal{R}H^\infty(\Sigma_{\alpha'}), \|F\|_{\mathcal{R}H^\infty(\Sigma_{\alpha'})} \leq 1\}$$

*is $\mathcal{R}$-bounded, i.e. $A$ even has an $\mathcal{R}$-bounded $\mathcal{R}H^\infty(\Sigma_{\alpha'})$ calculus.*

In particular, if $A$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus, then $A$ is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(A) \leq \omega_{H^\infty}(A)$ (for this assertion see also [52], where this was proved under much weaker conditions on $E$).

Looking now at the previous section again, we have seen that in mixed $L^p$ spaces, $\mathcal{R}$-boundedness is equivalent to $\mathcal{R}_2$-boundedness. Therefore, it is quite natural to ask which operators have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus for some $q \in [1,\infty]$. This is the content of the next section.

## 2.3 $\mathcal{R}_q$-bounded $H^\infty$ Calculus

In the following let $E$ be any mixed $L^p$ space with exponents $p \in [1,\infty)$ and $A\colon D(A) \to E$ be a sectorial operator.

**DEFINITION 2.3.1.** Let $\alpha \in (\omega(A), \pi]$. Then $A$ has an *$\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus* if the set

$$\{f(A)\colon f \in H^\infty(\Sigma_\alpha), \|f\|_{\infty,\alpha} \leq 1\}$$

is $\mathcal{R}_q$-bounded, which is equivalent to the existence of a constant $C > 0$ such that

$$\Big\| \Big(\sum_{n=1}^N |f_n(A)x_n|^q\Big)^{1/q} \Big\|_E \leq C \max_{n=1}^N \|f_n\|_{\infty,\alpha} \Big\| \Big(\sum_{n=1}^N |x_n|^q\Big)^{1/q} \Big\|_E$$

is valid for each sequence $(f_n)_{n=1}^N \subseteq H^\infty(\Sigma_\alpha)$ and $(x_n)_{n=1}^N \subseteq E$. In this case, we define

$$\omega_{\mathcal{R}_q^\infty}(A) := \inf\{\alpha \in (\omega(A), \pi]\colon A \text{ has an } \mathcal{R}_q\text{-bounded } H^\infty(\Sigma_\alpha) \text{ calculus}\}.$$

**REMARK 2.3.2.** Trivially, any sectorial operator with an $\mathcal{R}_q$-bounded $H^\infty$-calculus has automatically a bounded $H^\infty$ calculus. In the special case of $q = 2$ the converse was proven in Corollary 2.2.5. Moreover, if $A$ has an $\mathcal{R}_q$-bounded $H^\infty$-calculus, then $A$ is also $\mathcal{R}_q$-sectorial with $\omega_{\mathcal{R}_q}(A) \leq \omega_{\mathcal{R}_q^\infty}(A)$.

Analogously to Section 2.1 we emphasize the connection between $A$ and its diagonal operator $\widetilde{A}$ as defined in Proposition 2.1.10. The next result is taken from [57, Lemma 3.20].

**LEMMA 2.3.3.** *Let $q \in [1,\infty]$, $A$ be an $\mathcal{R}_q$-sectorial operator, and $\alpha \in (\omega_{\mathcal{R}_q}(A), \pi]$. Then the following conditions are equivalent:*

a) *For each $f \in H^\infty(\Sigma_\alpha)$ the operator $f(A)$ is $\mathcal{R}_q$-bounded.*

b) *The diagonal operator $\widetilde{A}$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus in $E(\ell^q)$.*

In [57, Theorem 3.21] it was also proven that the statement a) in the previous lemma, i.e. the $\mathcal{R}_q$-boundedness of each single operator $f(A)$, already implies an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_{\alpha'})$ calculus for all $\alpha' > \alpha$.

**THEOREM 2.3.4.** *Let $q \in [1,\infty]$, $A$ be an $\mathcal{R}_q$-sectorial operator, and $\alpha, \alpha' \in (\omega_{\mathcal{R}_q}(A), \pi]$. Consider the following assertions:*

a) *$A$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_{\alpha'})$ calculus.*

b) *For each $f \in H^\infty(\Sigma_\alpha)$ the operator $f(A)$ is $\mathcal{R}_q$-bounded.*

c) *For each $\varphi \in H_0^\infty(\Sigma_\alpha)$ the operator $\varphi(A)$ is $\mathcal{R}_q$-bounded, and there is a constant $C > 0$, independent of $\varphi$, such that*

$$\Big\| \Big( \sum_{n=1}^{N} |\varphi(A) x_n|^q \Big)^{1/q} \Big\|_E \leq C \|\varphi\|_{\infty,\alpha} \Big\| \Big( \sum_{n=1}^{N} |x_n|^q \Big)^{1/q} \Big\|_E$$

*for each $(x_n)_{n=1}^N \subseteq E$.*

*Then a) $\Rightarrow$ c) $\Rightarrow$ b) if $\alpha \geq \alpha'$ and b) $\Rightarrow$ a) if $\alpha' > \alpha$.*

Combining Lemma 2.3.3 and Theorem 2.3.4 (and slightly neglecting the angles) we see that $A$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $E$ if and only if the diagonal operator $\widetilde{A}$ has a bounded $H^\infty$ calculus on $E(\ell^q)$. Since this *extension result* is quite important for our purposes, we will return to this property again in the next section.

The standard example of an operator having an $\mathcal{R}_q$-bounded $H^\infty$ calculus is the Laplace operator on $\mathbb{R}^d$ (see [57, Proposition 3.22]).

**EXAMPLE 2.3.5.** Let $d, m \in \mathbb{N}$ and $p, q \in (1,\infty)$. Then the Laplace operator $A := (-\Delta)^m$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus in $L^p(\mathbb{R}^d)$ with $\omega_{\mathcal{R}_q^\infty}(A) = 0$.

Actually, many operators have an $\mathcal{R}_q$-bounded $H^\infty$ calculus. For some elliptic operators in divergence and non-divergence form as well as Schrödinger operators with singular potentials this was elaborated in [58]. To establish this property the authors used *(generalized)*

*Gaussian estimates* of the corresponding operators. Below we will recall and expand the existing list using the same tools they did. To formulate the main result we have to introduce some notions. Let in the following be $(U, d)$ be a metric space and $\mu$ be a $\sigma$-finite regular Borel measure on $U$ such that $(U, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss (see [17], [18]), i.e. there exists a constant $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \quad x \in U,\, r > 0,$$

where $B(x, r)$ denotes the ball with center $x$ and radius $r$. This then implies the existence of constants $D > 0$ and $C_D \geq 1$ such that

$$\mu(B(x, \lambda r)) \leq C_D \lambda^D \mu(B(x, r)), \quad x \in U,\, r > 0,\, \lambda \geq 1.$$

We also define the annulus

$$A_k(x, r) := B(x, (k+1)r) \setminus B(x, kr), \quad x \in U,\, r > 0,\, k \in \mathbb{N}.$$

The main result then reads as follows (see [58, Theorem 2.3]).

**THEOREM 2.3.6.** *Let $1 \leq p_0 < 2 < p_1 \leq \infty$ and $\omega_0 \in (0, \pi/2)$. Let $A$ be a sectorial operator in $L^2(U)$ such that $A$ has a bounded $H^\infty$ calculus in $L^2(U)$ with $\omega_{H^\infty}(A) \leq \omega_0$. Assume that the generated semigroup $T(\lambda) := e^{-\lambda A}$ satisfies the following weighted norm estimates for each $\theta > \omega_0$:*

$$\left\| \mathbb{1}_{A_k(x,|\lambda|^{1/m})} T(\lambda) \mathbb{1}_{B(x,|\lambda|^{1/m})} \right\|_{\mathcal{B}(L^{p_0}(U), L^{p_1}(U))} \leq C_\theta \mu(B(x, |\lambda|^{1/m}))^{\frac{1}{p_1} - \frac{1}{p_0}} (1+k)^{-\kappa_\theta},$$
$$\left\| \mathbb{1}_{B(x,|\lambda|^{1/m})} T(\lambda) \mathbb{1}_{A_k(x,|\lambda|^{1/m})} \right\|_{\mathcal{B}(L^{p_0}(U), L^{p_1}(U))} \leq C_\theta \mu(B(x, |\lambda|^{1/m}))^{\frac{1}{p_1} - \frac{1}{p_0}} (1+k)^{-\kappa_\theta},$$

*for all $x \in U$, $k \in \mathbb{N}_0$, $\lambda \in \Sigma_{\pi/2-\theta}$, and some constants $m > 0$, $\kappa_\theta > \max\{\frac{1}{p_0} + \frac{D}{p_1'}, \frac{1}{p_1'} + \frac{D}{p_0}\}$ and $C_\theta > 0$. Then for all $p, q \in (p_0, p_1)$ and $\alpha > \omega_0$ the operator $A$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus in $L^p(U)$.*

This statement should be understood in the way that the semigroup $T$ induces a *consistent* $C_0$-semigroup $T_p$ on $L^p(U)$ with generator $(-A_p)$ and for all $q \in (p_0, p_1)$ the operator $A_p$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus with $\omega_{\mathcal{R}_q^\infty}(A) \leq \omega_0$.

**REMARK 2.3.7.**

a) The assertion of Theorem 2.3.6 is still true if we replace $L^2(U)$ by a general $L^p(U)$ space where $1 \leq p_0 < p < p_1 \leq \infty$ (see [58, Remark 2.4]).

b) Note that the off-diagonal estimates of Theorem 2.3.6 are equivalent to classical pointwise kernel estimates if the operators $T(t)$ are integral operators with operator-valued kernels and $p_0 := 1$, $p_1 := \infty$ (see [58, Lemma 2.2] and [59, Lemma 8.5]).

In the next part of this section we collect examples of operators having an $\mathcal{R}_q$-bounded $H^\infty$ calculus. More precisely, we consider elliptic operators in divergence and non-divergence form. Some of these cases were already mentioned in [58].

**Example A: Elliptic operators in divergence form**
Let $U \subseteq \mathbb{R}^d$ be an arbitrary open set. Then we shall consider elliptic operators in divergence form given formally by

$$\mathcal{A}f := \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha\big(a_{\alpha,\beta} D^\beta f\big),$$

with coefficients $a_{\alpha,\beta} \in L^\infty(U,\mathbb{C})$. Since we want to apply Theorem 2.3.6, we have to check two properties. Firstly that there is a realization of $\mathcal{A}$ in $L^2(U)$ having a bounded $H^\infty$ calculus, and secondly that the semigroup generated by this realization satisfies the off-diagonal estimates of Theorem 2.3.6.

We define the realization $A_2$ of the operator $\mathcal{A}$ in $L^2(U)$ as the operator associated to the form

$$a(f,g) := \int_U \sum_{|\alpha|,|\beta|\leq m} a_{\alpha,\beta}(x) D^\beta f(x) \overline{D^\alpha g(x)} \, \mathrm{d}x.$$

The natural domain $V$ of this form of course depends on $U$ and the boundary conditions. Here, we will consider two different situations:

1) $U \subseteq \mathbb{R}^d$ is an arbitrary domain and we impose Dirichlet boundary conditions on $\mathcal{A}$: Here we take $V := W_0^{m,2}(U)$.

2) $U \subseteq \mathbb{R}^d$ is an arbitrary domain and we consider Neumann boundary conditions for $\mathcal{A}$: Then we let $V := W^{m,2}(U)$.

In all situations we assume that the form $a$ is sectorial, i.e. there exists an $\omega \in [0, \pi/2)$ such that

$$|\operatorname{Im} a(f,f)| \leq \tan(\omega) \operatorname{Re} a(f,f) \quad \text{for } f \in V.$$

Moreover, we require the following ellipticity condition/Gårding's inequality for $a$ to hold:

$$\operatorname{Re} a(f,f) \geq \alpha_0 \big\|(-\Delta)^{m/2} f\big\|_{L^2(U)}^2 \quad \text{for } f \in V$$

and some $\alpha_0 > 0$. Note that in the case of $m = 1$ both of these conditions are a consequence of the following uniform strong ellipticity condition:

$$\operatorname{Re} \sum_{j,k=1}^d a_{e_j,e_k}(x) \xi_j \overline{\xi}_k \geq \alpha_0 |\xi|^2, \quad \text{for all } \xi \in \mathbb{C}^d \text{ and } x \in U.$$

With these assumptions the operator $A_2$ associated to the form $a$ is sectorial and has a bounded $H^\infty$ calculus with $\omega_{H^\infty}(A_2) \leq \alpha_0$ (see [59, Chapter 11]). To show the off-diagonal estimates we make the following distinctions:

a) Let $U \subseteq \mathbb{R}^d$ be an arbitrary domain, $m = 1$, and consider $\mathcal{A}$ with Dirichlet boundary conditions. If the coefficients $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$ are real-valued, then by [23, Theorem 6.1] the semigroup generated by $A_2$ has a kernel $k_t$ which satisfies classical Gaussian bounds, i.e. there exist $\omega_1 \geq 0$, $\omega_2 > 0$ such that for all $\varepsilon \in (0,1]$ there is a constant $C_\varepsilon > 0$ satisfying

$$|k_t(x,y)| \leq C_\varepsilon t^{-d/2} e^{\omega_1(1+\varepsilon)t} \exp\Big(-\frac{|x-y|^2}{4t\omega_2(1+\varepsilon)}\Big) \quad \text{for all } x, y \in U,\, t > 0.$$

Therefore, the operator $\omega_1(1+\varepsilon) + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(U)$ for all $p, q \in (1,\infty)$. If we do not have any lower order terms (i.e. if $a_{\alpha,\beta} = 0$ for $|\alpha| + |\beta| < 2$), then we can set $\omega_1 = 0$. In the symmetric case without lower order coefficients this can also be found in [24, Corollary 3.2.8]. In [4] similar results where shown under stronger conditions. However, in the case $(a_{\alpha,\beta})_{|\alpha|,|\beta|=1} \subseteq W^{1,\infty}(U)$ the authors included complex-valued lower order terms.

b) Let $U \subseteq \mathbb{R}^d$ be a (bounded or unbounded) domain satisfying an *interior cone condition* (see [1, Definition 4.6]), let $m = 1$, and assume Neumann boundary conditions. In the case of real-valued coefficients $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$ [23, Theorem 6.1] implies the same Gaussian estimate as in a), with the difference that we have to take $\omega_1 = \alpha_0$ in the absence of lower order terms (for this case see also [24, Theorem 3.2.9]). In particular, the operator $\omega_1(1+\varepsilon) + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(U)$ for all $p, q \in (1,\infty)$.

Note that in [23] also the time-dependent case and Robin boundary conditions were studied. For complex-valued coefficients the situation is very different.

c) Consider first $U = \mathbb{R}^d$, $m = 1$, and let $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$ be complex-valued. In dimension $d = 1$ and $d = 2$ Theorems 2.36 and 3.11 in [6] imply the existence of constants $C, \beta, \omega_1 > 0$ such that the kernel $k_t$ of the semigroup of $A_2$ satisfies

$$|k_t(x,y)| \leq C t^{-d/2} e^{\omega_1 t} \exp\Big(-\frac{\beta|x-y|^2}{t}\Big) \quad \text{for all } x, y \in \mathbb{R}^d,\, t > 0.$$

According to [6, Theorems 2.21 and 3.5], we can choose $\omega_1 = 0$ if we do not have any lower order terms. This means that $\omega_1 + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(U)$ for all $p, q \in (1,\infty)$. For $d \geq 3$ there are examples of operators failing to have pointwise Gaussian bounds (see [44, Corollary 2.19]). In this case there are only positive results if we have additional assumptions on the coefficients. Moreover, even in the absence of lower order terms we have to consider $\nu + A$ for some $\nu > 0$ to obtain Gaussian estimates. This was done in [5, Theorem 4.8] for uniformly continuous coefficients $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$. In this case, $\nu + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(U)$ for all $p, q \in (1, \infty)$.

d) In [8] similar results as in c) were obtained by considering Lipschitz domains $U \subseteq \mathbb{R}^d$ where the Lipschitz constant is small enough (see [8, Theorem 7]). If the Lipschitz constant is too large, $\mathcal{A}$ might fail to have Gaussian bounds even in the case of constant coefficients (see [8, Proposition 6]).

e) Let $U \subseteq \mathbb{R}^d$ be an arbitrary domain, $m = 1$, and let $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$ be complex-valued. Consider $\mathcal{A}$ with Dirichlet boundary conditions. In this case we get Gaussian estimates for $\mathcal{A}$ under further assumptions on the imaginary part of the coefficients. More precisely, if

$$\sum_{j=1}^{d} D_j \mathrm{Im}\, a_{e_k,e_j} \in L^\infty(U) \quad \text{and} \quad \mathrm{Im}\,(a_{e_k,e_j} + a_{e_j,e_k}) = 0 \quad \text{for } 1 \leq j,k \leq d,$$

then the semigroup of $A_2$ is given by a kernel $k_t$ which satisfies the Gaussian bound

$$|k_t(x,y)| \leq C t^{-d/2} e^{\delta_1 t} \exp\Big(-\frac{|x-y|^2}{4\delta_2 t}\Big) \quad \text{for all } x, y \in U,\ t > 0,$$

and some constants $\delta_1, \delta_2 > 0$ (see [66, Theorem 6.10]), i.e. $\delta_1 + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(U)$ for all $p, q \in (1, \infty)$.

f) Let $U \subseteq \mathbb{R}^d$ be a domain having the extension property (i.e. there exists a bounded linear operator $P\colon W^{1,2}(U) \to W^{1,2}(\mathbb{R}^d)$ such that $Pf$ is an extension of $f$ from $U$ to $\mathbb{R}^d$), $m = 1$, and let $(a_{\alpha,\beta})_{|\alpha|,|\beta|\leq 1}$ be complex-valued. Consider now $\mathcal{A}$ with Neumann boundary conditions. Under the same assumption on the coefficients as in part e) [66, Theorem 6.10] implies the same Gaussian bound, leading also to an $\mathcal{R}_q$-bounded $H^\infty$ calculus of $\delta_1 + A_2$ on $L^p(U)$ for all $p, q \in (1, \infty)$.

g) In the general case $m \in \mathbb{N}$ we make the following distinction: If $d \leq 2m$ then we define $p_1 := \infty$, and if $d > 2m$ we let $p_1 := \frac{2d}{d-2m}$. Then by [59, Remark 8.23] (see also [25], [27], and [7]) we obtain a $\nu \geq 0$ such that the semigroup $T$ generated by $-(\nu + A_2)$ in $L^2(\mathbb{R}^d)$ satisfies Gaussian bounds of the form

$$\big\|\mathbb{1}_{B(x,|\lambda|^{1/2m})} T(\lambda) \mathbb{1}_{B(y,|\lambda|^{1/2m})}\big\|_{\mathcal{B}(L^{p_0}(U),L^{p_1}(U))} \leq C|\lambda|^{-\frac{d}{2m}(\frac{1}{p_1}-\frac{1}{p_0})} \exp\Big(-b\big(\tfrac{|x-y|^{2m}}{|\lambda|}\big)^{\frac{1}{2m-1}}\Big)$$

for all $\lambda \in \Sigma_\delta$, for some constants $C, b, \delta > 0$, and for $p_0 := p_1'$. In particular, the estimates of Theorem 2.3.6 hold for all $\kappa_\theta > 0$ and some $\theta \in (0, \pi/2)$. This then implies that $\nu + A_2$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(\mathbb{R}^d)$ for all $p, q \in (p_0, p_1)$. In [26] it is shown that the range for $p$ here is optimal. More precisely, for each $p \notin [p_0, p_1]$ we can find an operator $\mathcal{A}$ of the form above such that the generated semigroup does not extend to $L^p(\mathbb{R}^d)$.

**Example B: Elliptic operators in non-divergence form**
We only consider the case $U = \mathbb{R}^d$. Let $m \in \mathbb{N}$, $D(A_p) := W^{2m,p}(\mathbb{R}^d)$, and $A_p$ be the realization in $L^p(\mathbb{R}^d)$ of the elliptic differential operator

$$\mathcal{A}f := \sum_{|\alpha|\leq 2m} a_\alpha D^\alpha f,$$

where $a_\alpha \in L^\infty(\mathbb{R}^d, \mathbb{C})$ for each $|\alpha| \leq 2m$. As in the case of elliptic operators in divergence form, we first have to check that $A_p$ has a bounded $H^\infty$ calculus and then that the generated semigroup satisfies (generalized) Gaussian estimates. For this purpose we assume that there exist $\sigma \in (0, \pi/2)$ and $\delta > 0$ such that

$$\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \in \Sigma_\sigma \quad \text{and} \quad \Big| \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \Big| \geq \delta |\xi|^{2m}$$

for all $x, \xi \in \mathbb{R}^d$. To proceed further, we will make the following distinction:

a) Assume that the coefficients of the principal part are bounded and uniformly continuous, i.e. $a_\alpha \in BUC(\mathbb{R}^d, \mathbb{C})$ for $|\alpha| = 2m$. Then [33, Theorem 6.1] implies that $\nu + A_2$ has a bounded $H^\infty$ calculus for some $\nu \geq 0$. Moreover, by [55, Theorem 6.1] there exists an $\nu \in (0, \pi/2)$ such that $-(\nu + A_2)$ generates an analytic semigroup $(T(z))_{z\in\Sigma_\omega}$ satisfying the estimates of Theorem 2.3.6 for any $p_0 > 1$ and $p_1 := \infty$.

b) Let $m = 1$, $a_\alpha = 0$ for $|\alpha| < 2$ and for $|\alpha| = 2$ let $a_\alpha$ be of vanishing mean oscillation, i.e. $a_\alpha \in VMO(\mathbb{R}^d, \mathbb{C})$. Then by [34] there is a $\nu \geq 0$ such that $\nu + A_2$ has a bounded $H^\infty$ calculus. And by [55, Section 6.1] (here we do not need the restriction $m = 1$ and $a_\alpha = 0$ for $|\alpha| < 2$) there is an $\omega \in (0, \pi/2)$ such that $-(\nu + A_2)$ generates an analytic semigroup $(T(z))_{z\in\Sigma_\omega}$ satisfying the estimates of Theorem 2.3.6 for any $p_0 > 1$ and $p_1 := \infty$.

In both cases Theorem 2.3.6 yields that $\nu + \mathcal{A}$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus for all $p, q \in (p_0, \infty)$.

## 2.4 Extension Properties

In this section we deal with the problem of extending a bounded or unbounded operator $A$ on $L^p(U)$ to the Banach space-valued $L^p$ space $L^p(U; E)$ for some Banach space $E$.

In the following let $p, q \in [1, \infty)$ and $E$ be a Banach space. For any function $f : U \to \mathbb{C}$ and any $x \in E$ we define the function

$$f \otimes x : U \to E \quad \text{by} \quad (f \otimes x)(u) = f(u)x.$$

For any linear subspace $D_p \subseteq L^p(U)$ we let

$$D_p \otimes E := \Big\{ \sum_{n=1}^{N} f_n \otimes x_n \colon (f_n)_{n=1}^N \subseteq D_p, (x_n)_{n=1}^N \subseteq E, N \in \mathbb{N} \Big\}.$$

Note that $D_p \otimes E$ is dense in $L^p(U;E)$ if $D_p$ is dense in $L^p(U)$.

For any closed linear operator $T \colon D(T) \subseteq L^p(U) \to L^q(V)$ we now define

$$T \otimes I_E \colon D(T) \otimes E \to L^q(V;E), \quad (T \otimes I_E)\Big( \sum_{n=1}^{N} f_n \otimes x_n \Big) = \sum_{n=1}^{N} T f_n \otimes x_n.$$

In the special case of $T \in \mathcal{B}(L^p(U), L^q(V))$ we want to know if $T \otimes I_E$ can be extended to a bounded operator in $\mathcal{B}(L^p(U;E); L^q(V;E))$. In general this is not the case. A prominent example is the Hilbert transform, which is bounded on $L^p(\mathbb{R})$, but only has a vector-valued bounded extension on $L^p(\mathbb{R};E)$ if $E$ is a UMD space. For more counterexamples see [60, Theorem 6.1 and 6.2].

On the other hand, there are a few notable positive results.

**Remark 2.4.1.**

a) If $T \in \mathcal{B}(L^p(U))$ and $E = L^p(V)$, then $T \otimes I_E$ always has a bounded extension on $L^p(U; L^p(V))$ by Fubini's theorem.

b) If $T \in \mathcal{B}(L^p(U), L^q(V))$ is *positive* (i.e. $Tf \geq 0$ almost everywhere if $f \geq 0$ almost everywhere), then $T \otimes I_E$ always extends to a bounded linear operator from $L^p(U;E)$ to $L^q(V;E)$ for every Banach space $E$ (see [39, Proposition 5.5.10]).

c) If $E$ is a *Hilbert space*, then every bounded operator $T \in \mathcal{B}(L^p(U), L^q(V))$ extends to a bounded operator from $L^p(U;E)$ to $L^q(V;E)$ (see [39, Theorem 5.5.1]).

Next we will turn to the definition of an $E$-valued extension of a closed linear operator $A \colon D(A) \subseteq L^p(U) \to L^p(U)$. In this setting, we define for $f, g \in L^p(U;E)$

$$f \in D(A^E) \text{ with } A^E f = g \iff \langle f, x' \rangle \in D(A) \text{ and } A\langle f, x' \rangle = \langle g, x' \rangle \quad \forall x' \in E'.$$

Then $A^E$ is well-defined, and moreover we have the following properties:

**Proposition 2.4.2.** *The following assertions are true:*

*a) The operator $A^E$ is closed and $A \otimes I_E \subseteq A^E$, i.e. $A \otimes I_E$ is closable.*

*b) If $A$ is densely defined, then $A^E$ is also densely defined.*

c) *Let $\lambda \in \mathbb{C}$, then*

$$\lambda \in \rho(A^E) \iff \lambda \in \rho(A) \text{ and } R(\lambda, A)^E \in \mathcal{B}(L^p(U;X)),$$

*and in this case* $R(\lambda, A)^E = \overline{R(\lambda, A) \otimes I_E} = R(\lambda, A^E)$.

d) *If $\rho(A^E) \neq \emptyset$, then $A^E = \overline{A \otimes I_E}$. In particular, if $D \subseteq D(A)$ is a core for $A$, then $D \otimes E$ is a core for $A^E$.*

e) *If $E := L^q(V)$ and $f\colon V \to D(A)$ satisfies $f, Af \in L^p(U; L^q(V))$, then $f \in D(A^E)$ and $(A^E f)(v) = Af(v)$ for almost every $v \in V$.*

**Proof.** For the proof of a)-d) see [78, Propositions 5.1.2 and 5.2.1]. To show e), take any $h \in L^{q'}(V)$. Then

$$\langle f, h\rangle = \int_V f(v)h(v)\,\mathrm{d}\nu(v) \in D(A),$$

since $A$ is closed, i.e. $f \in D(A^E)$. Moreover,

$$\langle A^E f, h\rangle = A\langle f, h\rangle = A\int_V f(v)h(v)\,\mathrm{d}\nu(v) = \int_V Af(v)h(v)\,\mathrm{d}\nu(v) = \langle Af, h\rangle,$$

which implies the claim. $\square$

**Remark 2.4.3.** If we define the set

$$\mathcal{D} := \{f\colon V \to D(A)\colon f, Af \in L^p(U; L^q(V))\},$$

then Proposition 2.4.2 e) implies that $\mathcal{D} \subseteq D(A^{L^q})$. Moreover, since $D(A) \otimes L^q(V) \subseteq \mathcal{D}$, part d) of Proposition 2.4.2 yields that $\mathcal{D}$ is a core for $A^{L^q}$. In the case that $q \geq p$ we even obtain

$$\mathcal{D} = D(A^{L^q}).$$

In fact, since $q \geq p$, we know by Minkowski's integral inequality that

$$L^p(U; L^q(V)) \subseteq L^q(V; L^p(U)).$$

Hence, each function $f \in D(A^{L^q})$ is actually a function $f\colon V \to L^p(U)$ such that $f \in L^p(U; L^q(V))$. Since $\mathcal{D}$ is a core for $A^{L^q}$, the closedness of $A$ finally yields $f(v) \in D(A)$ and $Af(v) = A^{L^q} f(v)$ for $\nu$-almost every $v \in V$, which means that $f \in \mathcal{D}$.

For the special case that $A$ is sectorial, Proposition 2.4.2 implies that $A^E$ is densely defined, and if $A^E$ is also sectorial, then part c) yields for $\sigma > \omega(A) \vee \omega(A^E)$ the identity

$$\varphi(A^E) = \overline{\varphi(A) \otimes I_E} = \varphi(A)^E \quad \text{for } \varphi \in H_0^\infty(\Sigma_\sigma).$$

A proof of this result for a larger class of functions $\varphi$ can be found in [78, Theorem 5.2.2]. If we additionally assume that $A$ is $\ell^q$-sectorial and $E = \ell^q$, then Remark 3.2.4 in [79] says that $A^{\ell^q} = \widetilde{A}$, where $\widetilde{A}$ is the diagonal operator from Proposition 2.1.10. Note that the assumption of $\mathcal{R}_q$-sectoriality in [79] can be weakened to $\ell^q$-sectoriality. Using the same proposition we derive that $A^{\ell^q}$ is a sectorial extension of $A$ on $L^p(U;\ell^q)$. Now Proposition 2.4.2 immediately yields the following results (see also [79, Corollary 3.2.5]).

**Corollary 2.4.4.** *Let $A$ be an $\ell^q$-sectorial operator on $L^p(U)$. Then we have*

a) $\widetilde{A} = \overline{A \otimes I_{\ell^q}}$.

b) *If $\lambda \notin \overline{\Sigma}_{\omega_{\ell^q}(A)}$, then* $R(\lambda, \widetilde{A}) = \overline{R(\lambda, A) \otimes I_{\ell^q}} = \widetilde{R(\lambda, A)}$.

c) *For $\sigma > \omega_{\ell^q}(A)$ and $f \in H_0^\infty(\Sigma_\sigma)$ we have* $f(\widetilde{A}) = \overline{f(A) \otimes I_{\ell^q}} = \widetilde{f(A)}$.

The main result of this section is now the following generalization of this corollary to the space $L^q(V)$.

**Theorem 2.4.5.** *Let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be a closed operator.*

a) *If $A$ is $\ell^q$-sectorial, then the extension $A^{L^q}$ on $L^p(U;L^q(V))$ is sectorial with $\omega(A^{L^q}) \leq \omega_{\ell^q}(A)$.*

b) *If $A$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus on $L^p(U)$ for some $\alpha \in (\omega_{\mathcal{R}_q^\infty}(A), \pi]$, then the extension $A^{L^q}$ has a bounded $H^\infty(\Sigma_{\alpha'})$ calculus on $L^p(U;L^q(V))$ for each $\alpha' \geq \alpha$.*

**Proof.** a) Let $f = \sum_{n=1}^N \mathbb{1}_{A_n} x_n \in L^p(U;L^q(V))$, where $x_n \in L^p(U)$ and $A_n \in \Xi$ are pairwise disjoint with finite measure. Such functions are dense in $L^p(U;L^q(V))$, and for these functions we obtain

$$\begin{aligned}
\|\lambda R(\lambda, A)^{L^q} f\|_{L^p(U;L^q(V))} &= \Big\| \sum_{n=1}^N \mathbb{1}_{A_n} \lambda R(\lambda, A) x_n \Big\|_{L^p(U;L^q(V))} \\
&= \Big\| \Big( \sum_{n=1}^N \nu(A_n) \big|\lambda R(\lambda, A) x_n\big|^q \Big)^{1/q} \Big\|_{L^p(U)} \\
&\leq C \Big\| \Big( \sum_{n=1}^N \nu(A_n) |x_n|^q \Big)^{1/q} \Big\|_{L^p(U)} = C\|f\|_{L^p(U;L^q(V))}.
\end{aligned}$$

This means that $R(\lambda, A)^{L^q} \in \mathcal{B}\big(L^p(U;L^q(V))\big)$. Now Proposition 2.4.2 implies that $\rho(A) = \rho(A^{L^q})$ and $R(\lambda, A^{L^q}) = R(\lambda, A)^{L^q}$. The estimate above finally concludes the proof of a).

b) By part a), $A^{L^q}$ is sectorial. The remark after Proposition 2.4.2 then leads to

$$\varphi(A^{L^q}) = \varphi(A)^{L^q} \quad \text{for each } \varphi \in H_0^\infty(\Sigma_\alpha).$$

Applying this in the same manner as in part a), we obtain for simple functions $f$ the estimate

$$\begin{aligned}\|\varphi(A^{L^q})f\|_{L^p(U;L^q(V))} &= \Big\| \Big(\sum_{n=1}^N \nu(A_n)|\varphi(A)x_n|^q\Big)^{1/q} \Big\|_{L^p(U)} \\ &\le C\|\varphi\|_{\infty,\alpha}\Big\| \Big(\sum_{n=1}^N \nu(A_n)|x_n|^q\Big)^{1/q} \Big\|_{L^p(U)} \\ &= C\|\varphi\|_{\infty,\alpha}\|f\|_{L^p(U;L^q(V))}.\end{aligned}$$

□

**REMARK 2.4.6.** Similar to Proposition 2.4.2 e) we obtain for any function $g\colon V \to L^p(U)$ satisfying $g \in L^p(U;L^q(V))$ the identity

$$\big(f(A^{L^q})g\big)(t) = f(A)g(t)$$

for each $f \in H^\infty(\Sigma_\alpha)$.

**EXAMPLE 2.4.7.** Let $\beta \in \mathbb{N}_0^d$, $U \subseteq \mathbb{R}^d$ be open, $A = D^\beta$ be a differential operator of order $k = |\beta|$ with domain $D(A) = W^{k,p}(U)$, and $B = D^\beta$ be the vector-valued differential operator of order $k$ with domain $D(B) = W^{k,p}(U;E)$ (please note that these operators are in general not closed). Then $B = A^E$, in particular $D(A^E) = W^{k,p}(U;E)$.

In fact, if $f \in D(B)$, then $g := Bf = D^\beta f \in L^p(U;E)$ and $\langle f, x'\rangle \in W^{k,p}(U)$. Moreover,

$$\begin{aligned}\int_U \langle g, x'\rangle \phi \,\mathrm{d}u &= \Big\langle \int_U g\phi \,\mathrm{d}u, x' \Big\rangle = \Big\langle (-1)^{|\beta|} \int_U f D^\beta \phi \,\mathrm{d}u, x'\Big\rangle \\ &= (-1)^{|\beta|} \int_U \langle f, x'\rangle D^\beta \phi \,\mathrm{d}u.\end{aligned}$$

for each $\phi \in C_c^\infty(U)$ and $x' \in E'$. Hence, $A\langle f, x'\rangle = D^\beta\langle f, x'\rangle = \langle g, x'\rangle$ for each $x' \in E'$, and $f \in D(A^E)$. Conversely, assume that $f \in D(A^E)$. Then for any $x' \in E'$ we have $\langle f, x'\rangle \in W^{k,p}(U)$ and $\langle A^E f, x'\rangle = A\langle f, x'\rangle$. Therefore,

$$\begin{aligned}\Big\langle \int_U A^E f\phi \,\mathrm{d}u, x'\Big\rangle &= \int_U \langle A^E f, x'\rangle \phi \,\mathrm{d}u = \int_U D^\beta \langle f, x'\rangle \phi \,\mathrm{d}u \\ &= (-1)^{|\beta|} \int_U \langle f, x'\rangle D^\beta \phi \,\mathrm{d}u \\ &= \Big\langle (-1)^{|\beta|} \int_U f D^\beta \phi \,\mathrm{d}u, x'\Big\rangle.\end{aligned}$$

Since this holds for each $x' \in E'$, we infer that $f \in D(B)$ and $A^E f = D^\beta f = Bf$.

## 2.5 $\ell^q$ Interpolation Method

In the subsequent chapter we will be faced with the question about the 'correct' space of initial values for stochastic evolution equations, and in this context real interpolation spaces come into focus in a natural way. In our setting, we will need a new family of interpolation spaces obtained by the so called $\ell^q$ *interpolation method* first introduced by Kunstmann in [56] for closed subspaces of Banach function spaces. In this section we concentrate on the $\ell^q$ interpolation of an $L^p$ space and the domain $D(A)$ of a closed operator $A\colon D(A) \subseteq L^p(U) \to L^p(U)$.

Let in the following

$$L^q_*(a,b) := L^q((a,b), \tfrac{\mathrm{d}t}{t}) \quad \text{and} \quad L^q_* = L^q_*(0,\infty)$$

for $0 \le a < b \le \infty$, $q \in [1,\infty)$.

**Definition 2.5.1.** Let $\theta \in (0,1)$, $p,q \in [1,\infty)$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be a closed operator. Then we let

$$\begin{aligned}\|x\|_{\theta,\ell^q} &:= \|x\|_{(L^p(U),D(A))_{\theta,\ell^q}}\\ &:= \inf\big\{\|t^{-\theta}u(t)\|_{L^p(U;L^q_{*(t)})} + \|t^{1-\theta}v(t)\|_{L^p(U;L^q_{*(t)})} + \|t^{1-\theta}Av(t)\|_{L^p(U;L^q_{*(t)})}\colon\\ &\qquad x = u(t)+v(t), t \ge 0, u(t) \in L^p(U), v(t) \in D(A)\big\}\end{aligned}$$

for $x \in L^p(U)$ and define

$$(L^p(U), D(A))_{\theta,\ell^q} := \{x \in L^p(U)\colon \|x\|_{\theta,\ell^q} < \infty\}.$$

**Remark 2.5.2.**

a) It is now straightforward to show that $(L^p(U), D(A))_{\theta,\ell^q}$ is a Banach space (see also [56, Proposition 2.10]).

b) In the definition above we may replace the half-line $(0,\infty)$ by any interval $(0,T)$, $T > 0$. To see this, let

$$\begin{aligned}\|x\|^{(0,T)}_{\theta,\ell^q} &:= \|x\|^{(0,T)}_{(L^p(U),D(A))_{\theta,\ell^q}}\\ &:= \inf\big\{\|t^{-\theta}u(t)\|_{L^p(U;L^q_{*(t)}(0,T))} + \|t^{1-\theta}v(t)\|_{L^p(U;L^q_{*(t)}(0,T))} + \|t^{1-\theta}Av(t)\|_{L^p(U;L^q_{*(t)}(0,T))}\colon\\ &\qquad x = u(t)+v(t), t \in (0,T), u(t) \in L^p(U), v(t) \in D(A)\big\}\end{aligned}$$

First observe that

$$1 = \big(\theta q(b^{\theta q} - a^{\theta q})(ab)^{-\theta q}\big)^{1/q}\Big(\int_a^b t^{-\theta q-1}\,\mathrm{d}t\Big)^{1/q}$$

for $0 < a < b < \infty$. For any $T > 0$ this then leads to

$$\|x\|_{L^p(U)} = \inf_{x=a+b} \|a\|_{L^p(U)} = C_{\theta,q} T^{-\theta} \inf_{x=a+b} \Big\| \Big( \int_{T/2}^{T} t^{-\theta q - 1} \,\mathrm{d}t \Big)^{1/q} a \Big\|_{L^p(U)}$$
$$\leq C_{\theta,q} T^{-\theta} \|x\|_{\theta,\ell^q}^{(0,T)},$$

where $C_{\theta,q} := (\theta q(2^{\theta q} - 1))^{1/q}$. Using this inequality, we arrive at

$$\|x\|_{\theta,\ell^q}^{(0,T)} \leq \|x\|_{\theta,\ell^q}^{(0,\infty)} \leq \|x\|_{\theta,\ell^q}^{(0,T)} + \|t^{-\theta}\|_{L^q_{*(t)}(0,T)} \|x\|_{L^p(U)}$$
$$\leq \big(1 + (2^{\theta q} - 1)^{1/q} T^{-2\theta}\big) \|x\|_{\theta,\ell^q}^{(0,T)},$$

i.e. $\|\cdot\|_{\theta,\ell^q}^{(0,T)}$ is an equivalent norm in $(L^p(U), D(A))_{\theta,\ell^q}$.

To obtain a different characterization of these spaces, we define

$$V_{\theta,\ell^q}([0,T], D(A)) := \big\{ w\colon [0,T] \to D(A) \colon w \in L^p(U; W^{1,q}[0,T]) \text{ and } [w]_{\theta,\ell^q} < \infty \big\},$$

where

$$[w]_{\theta,\ell^q} := \|t^{1-\theta} w'(t)\|_{L^p(U; L^q_{*,(t)}(0,T))} + \|t^{1-\theta} w(t)\|_{L^p(U; L^q_{*,(t)}(0,T))} + \|t^{1-\theta} A w(t)\|_{L^p(U; L^q_{*,(t)}(0,T))}.$$

With these notions we let

$$\|x\|_{\theta,\ell^q}^{\mathrm{Tr}} := \|x\|_{(L^p(U),D(A))_{\theta,\ell^q}}^{\mathrm{Tr}} := \inf_{w, w(0)=x} [w]_{\theta,\ell^q},$$

and define

$$(L^p(U), D(A))_{\theta,\ell^q}^{\mathrm{Tr}} := \{ x \in L^p(U) \colon \exists w \in V_{\theta,\ell^q}([0,T], D(A)) \text{ with } w(0) = x \}.$$

With these notion we obtain the following connection between $\ell^q$ interpolation theory and trace theory.

**Proposition 2.5.3 (Trace method).** *Let $\theta \in (0,1)$, $p, q \in [1,\infty)$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be a closed operator. Then*

$$(L^p(U), D(A))_{\theta,\ell^q} = (L^p(U), D(A))_{\theta,\ell^q}^{\mathrm{Tr}}$$

*with equivalent norms. More precisely, we have*

$$\|\cdot\|_{\theta,\ell^q}^{Tr} \leq 4(1 + \tfrac{2}{\theta}) \|\cdot\|_{\theta,\ell^q} \leq \tfrac{4}{\theta}(1 + \tfrac{2}{\theta}) \|\cdot\|_{\theta,\ell^q}^{Tr}.$$

**Proof.** Here we closely follow the lines of Proposition 1.13 in [63].

First let $x \in (L^p(U), D(A))^{\mathrm{Tr}}_{\theta,\ell^q}$ and $w \in V_{\theta,\ell^q}([0,T], D(A))$ satisfying $w(0) = x$. Then

$$x = x - w(t) + w(t) = -\int_0^t w'(s)\,\mathrm{d}s + w(t), \quad t \in [0,T].$$

By Remark 2.5.2 and Hardy's inequality (see e.g. [63, Corollary A.13]) we obtain

$$\begin{aligned}\|x\|_{\theta,\ell^q} &\lesssim_{\theta,q,T} \Big\| t^{1-\theta}\tfrac{1}{t}\int_0^t w'(s)\,\mathrm{d}s \Big\|_{L^p(U,L^q_{*(t)}(0,T))} + \big\|t^{1-\theta} w(t)\big\|_{L^p(U;L^q_{*(t)}(0,T))} \\ &\quad + \big\|t^{1-\theta} Aw(t)\big\|_{L^p(U;L^q_{*(t)}(0,T))} \\ &\leq \tfrac{1}{\theta}\|t^{1-\theta}w'(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \big\|t^{1-\theta} w(t)\big\|_{L^p(U;L^q_{*(t)}(0,T))} \\ &\quad + \|t^{1-\theta} Aw(t)\|_{L^p(U;L^q_{*,(t)}(0,T))}.\end{aligned}$$

Taking now the infimum over all such $w \in V_{\theta,\ell^q}([0,T], D(A))$ we arrive at

$$\|x\|_{\theta,\ell^q} \leq \tfrac{1}{\theta}\|x\|^{\mathrm{Tr}}_{\theta,\ell^q}.$$

Now let $x \in (L^p(U), D(A))_{\theta,\ell^q}$. For $t \geq 0$ and $x \in L^p(U)$ we define the function

$$K(t,x,u) := \inf_{x=a+b, a\in L^p, b\in D(A)} |a(u)| + |tb(u)| + |t(Ab)(u)|, \quad u \in U.$$

Then, by definition,

$$\|t^{-\theta}K(t,x)\|_{L^p(U;L^q_{*(t)}(0,T))} \leq \|x\|_{\theta,\ell^q}.$$

Now choose for each $n \in \mathbb{N}$ elements $a_n \in L^p(U)$ and $b_n \in D(A)$ such that $a_n + b_n = x$ and

$$|a_n(u)| + \tfrac{1}{n}|b_n(u)| + \tfrac{1}{n}|Ab_n(u)| \leq 2K(\tfrac{1}{n}, x, u).$$

Since

$$t^{-\theta}K(t,x,u) \leq (\theta q)^{1/q}\|t^{-\theta}K(t,x,u)\|_{L^q_{*(t)}} \quad \text{and} \quad \lim_{t\to 0} t^{-\theta}K(t,x,u) = 0$$

by [63, (1.7)], it holds that $\lim_{n\to\infty}|a_n(u)| = 0$. Now define

$$\begin{aligned} v(t,u) &:= \sum_{n=1}^{\infty} b_{n+1}(u)\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]}(t) = \sum_{n=1}^{\infty}\big(x(u) - a_{n+1}(u)\big)\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]}(t) \quad \text{and} \\ w(t,u) &:= \frac{1}{t}\int_0^t v(s,u)\,\mathrm{d}s = x - \frac{1}{t}\int_0^t \sum_{n=1}^{\infty} a_{n+1}(u)\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]}(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\lim_{n\to\infty}|x(u) - v(\tfrac{1}{n},u)| = \lim_{n\to\infty}|a_{n+1}(u)| = 0,$$

which means that $v(0) = w(0) = x$. Moreover, by [63, (1.26) and (1.27)]

$$|t^{1-\theta}w'(t,u)| \le 4t^{-\theta}K(t,x,u) \quad \text{and} \quad |t^{1-\theta}v(t,u)| + |t^{1-\theta}Av(t,u)| \le 4t^{-\theta}K(t,x,u).$$

These estimates and Hardy's inequality imply that

$$\begin{aligned}
\|x\|^{\mathrm{Tr}}_{\theta,\ell^q} &\le \|t^{1-\theta}w'(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \|t^{1-\theta}w(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \|t^{1-\theta}Aw(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} \\
&\le \|t^{1-\theta}w'(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \tfrac{1}{\theta}\|t^{1-\theta}v(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \tfrac{1}{\theta}\|t^{1-\theta}Av(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} \\
&\le 4(1+\tfrac{2}{\theta})\|t^{-\theta}K(t,x)\|_{L^p(U;L^q_{*(t)}(0,T))} \\
&\le 4(1+\tfrac{2}{\theta})\|x\|_{\theta,\ell^q}.
\end{aligned}$$

□

In the case of an $\mathcal{R}_q$-sectorial operator we have the following additional results.

**Theorem 2.5.4.** *Let $\theta \in (0,1)$, $p,q \in [1,\infty)$, $\alpha \ge 0$, and $A$ be an $\mathcal{R}_q$-sectorial operator. We let*

$$\begin{aligned}
X^1_{\theta,\ell^q,\alpha} &:= \big\{x \in L^p(U)\colon [x]^1_{\theta,\ell^q,\alpha} := \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})} < \infty\big\}, \\
X^2_{\theta,\ell^q,\alpha} &:= \big\{x \in L^p(U)\colon [x]^2_{\theta,\ell^q,\alpha} := \big\|\lambda^{\theta\alpha}\big[A(\lambda+A)^{-1}\big]^\alpha x\big\|_{L^p(U;L^q_{*(\lambda)})} < \infty\big\}, \\
X^3_{\theta,\ell^q} &:= \big\{x \in L^p(U)\colon [x]^3_{\theta,\ell^q} := \big\|t^{-\theta}\big(e^{-tA}x - x\big)\big\|_{L^p(U;L^q_{*(t)})} < \infty\big\}.
\end{aligned}$$

*Then*

$$(L^p(U), D(A^\alpha))_{\theta,\ell^q} = X^1_{\theta,\ell^q,\alpha} = X^2_{\theta,\ell^q,\alpha},$$

*and $\|\cdot\|_{(L^p(U),D(A^\alpha))_{\theta,\ell^q}}$, $\|\cdot\|^1_{\theta,\ell^q,\alpha} := \|\cdot\|_{L^p(U)} + [\cdot]^1_{\theta,\ell^q,\alpha}$, and $\|\cdot\|^2_{\theta,\ell^q,\alpha} := \|\cdot\|_{L^p(U)} + [\cdot]^2_{\theta,\ell^q,\alpha}$ are equivalent norms.*

*Additionally, if $\alpha = 1$, we have*

$$(L^p(U), D(A))_{\theta,\ell^q} = X^3_{\theta,\ell^q},$$

*and $\|\cdot\|_{(L^p(U),D(A))_{\theta,\ell^q}}$ and $\|\cdot\|^3_{\theta,\ell^q} := \|\cdot\|_{L^p(U)} + [\cdot]^3_{\theta,\ell^q}$ are equivalent.*

**Remark 2.5.5.**

a) For sectorial operators $A$ and interchanged $L^p$ and $L^q_*$ norms these results are well-known (see e.g. [53, 42] or [64, Section 11.3]). Since we first apply the norm with respect to time and then with respect to space, the assumption of sectoriality is now replaced by $\mathcal{R}_q$-sectoriality. The latter property deals with this new situation in order to obtain the results we would expect from the reversed situation.

b) In the definition of $[\cdot]^1_{\theta,\ell^q,\alpha}$ we can replace $L^q_*$ by $L^q_*(0,T)$ for any $T > 0$. In fact, using $\mathcal{R}_q$-boundedness of the set $\{(tA)^\alpha e^{-tA} \colon t>0\}$ (see Corollary 3.7 in [57]) we obtain by Proposition 2.1.5

$$\begin{aligned}
[x]^1_{\theta,\ell^q,\alpha} &\le \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)}(0,T))} + \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)}(T,\infty))}\\
&\lesssim \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)}(0,T))} + \|t^{-\theta\alpha}x\|_{L^p(U;L^q_{*(t)}(T,\infty))}\\
&= \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)}(0,T))} + (\theta\alpha q)^{-1/q}T^{-\alpha\theta}\|x\|_{L^p(U)}.
\end{aligned}$$

This implies that

$$X^1_{\theta,\ell^q,\alpha} = \big\{x\in L^p(U)\colon [x]^1_{\theta,\ell^q,\alpha,T} := \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)}(0,T))} < \infty\big\}$$

and $\|\cdot\|^1_{\theta,\ell^q,\alpha}$ and $\|\cdot\|^1_{\theta,\ell^q,\alpha,T} := \|\cdot\|_{L^p(U)} + [\cdot]^1_{\theta,\ell^q,\alpha,T}$ are equivalent norms. Similarly we obtain

$$\begin{aligned}
X^2_{\theta,\ell^q,\alpha} &= \big\{x\in L^p(U)\colon [x]^2_{\theta,\ell^q,\alpha,T} := \big\|\lambda^{\theta\alpha}\big[A(\lambda+A)^{-1}\big]^\alpha x\big\|_{L^p(U;L^q_{*(\lambda)}(T,\infty))} < \infty\big\},\\
X^3_{\theta,\ell^q} &= \big\{x\in L^p(U)\colon [x]^3_{\theta,\ell^q,T} := \big\|t^{-\theta}\big(e^{-tA}x-x\big)\big\|_{L^p(U;L^q_{*(t)}(0,T))} < \infty\big\},
\end{aligned}$$

with corresponding equivalent norms.

**Proof (of Theorem 2.5.4).** We show that

$$(L^p(U),D(A^\alpha))_{\theta,\ell^q} \subseteq X^1_{\theta,\ell^q,\alpha} \subseteq X^2_{\theta,\ell^q,\alpha} \subseteq (L^p(U),D(A^\alpha))_{\theta,\ell^q}.$$

First let $x\in (L^p(U),D(A^\alpha))_{\theta,\ell^q}$, and $u(t)\in L^p(U)$ and $v(t)\in D(A)$ such that $x = u(t)+v(t)$ for all $t\ge 0$. Then we also have $x = u(t^{1/\alpha}) + v(t^{1/\alpha})$, $t\ge 0$. Moreover, since $A$ is $\mathcal{R}_q$-sectorial, Corollary 3.7 in [57] implies the $\mathcal{R}_q$-boundedness of the set

$$\{t^\beta A^\beta e^{-tA}\colon t>0\},\quad \beta\ge 0.$$

Therefore, by Proposition 2.1.5 we obtain constants $C_0>0$ and $C_\alpha>0$ such that

$$\begin{aligned}
&\big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})}\\
&\quad\le \big\|t^{-\theta\alpha}t^\alpha A^\alpha e^{-tA}u(t)\big\|_{L^p(U;L^q_{*(t)})} + \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}v(t)\big\|_{L^p(U;L^q_{*(t)})}\\
&\quad\le C_\alpha\big\|t^{-\theta\alpha}u(t)\big\|_{L^p(U;L^q_{*(t)})} + C_0\big\|t^{(1-\theta)\alpha}A^\alpha v(t)\big\|_{L^p(U;L^q_{*(t)})}\\
&\quad= C_\alpha\alpha^{-1/q}\|t^{-\theta}u(t^{1/\alpha})\|_{L^p(U;L^q_{*(t)})} + C_0\alpha^{-1/q}\|t^{1-\theta}A^\alpha v(t^{1/\alpha})\|_{L^p(U;L^q_{*(t)})}.
\end{aligned}$$

Taking now the infimum over all such $u$ and $v$, we arrive at

$$[x]^1_{\theta,\ell^q,\alpha} = \big\|t^{(1-\theta)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})} \le (C_\alpha\vee C_0)\alpha^{-1/q}\|x\|_{(L^p(U),D(A^\alpha))_{\theta,\ell^q}}.$$

Now let $x \in X^1_{\theta,\ell^q,\alpha}$. We use the representation

$$\big[A(\lambda+A)^{-1}\big]^{\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t\lambda}A^{\alpha}e^{-tA}x\,\mathrm{d}t.$$

Additionally, observe that

$$\begin{aligned}\int_0^\infty \lambda^{\theta\alpha}t^{\alpha-1}e^{-t\lambda}A^{\alpha}e^{-tA}x\,\mathrm{d}t &= \int_0^\infty \big[(t\lambda)^{\theta\alpha}e^{-t\lambda}\big]\big[t^{(1-\theta)\alpha}A^{\alpha}e^{-tA}x\big]\frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \big(s^{\theta\alpha}e^{-s}\big)\big[\big(\tfrac{s}{\lambda}\big)^{(1-\theta)\alpha}A^{\alpha}e^{-\frac{s}{\lambda}A}x\big]\frac{\mathrm{d}s}{s}.\end{aligned}$$

Applying norms on both sides and using triangle inequality lead to

$$\begin{aligned}[x]^2_{\theta,\ell^q,\alpha} &= \big\|\lambda^{\theta\alpha}\big[A(\lambda+A)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(\lambda)})}\\ &= \frac{1}{\Gamma(\alpha)}\Big\|\int_0^\infty \lambda^{\theta\alpha}t^{\alpha-1}e^{-t\lambda}A^{\alpha}e^{-tA}x\,\mathrm{d}t\Big\|_{L^p(U;L^q_{*(t)})}\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\theta\alpha}e^{-s}\big\|\big(\tfrac{s}{\lambda}\big)^{(1-\theta)\alpha}A^{\alpha}e^{-\frac{s}{\lambda}A}x\big\|_{L^p(U;L^q_{*(\lambda)})}\frac{\mathrm{d}s}{s}\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\theta\alpha-1}e^{-s}\,\mathrm{d}s\,\big\|\mu^{(1-\theta)\alpha}A^{\alpha}e^{-\mu A}x\big\|_{L^p(U;L^q_{*(\mu)})}\\ &= \frac{\Gamma(\theta\alpha)}{\Gamma(\alpha)}\big\|\mu^{(1-\theta)\alpha}A^{\alpha}e^{-\mu A}x\big\|_{L^p(U;L^q_{*(\mu)})} = \frac{\Gamma(\theta\alpha)}{\Gamma(\alpha)}[x]^1_{\theta,\ell^q,\alpha},\end{aligned}$$

which is finite by assumption.

In the next step, we assume that $x \in X^2_{\theta,\ell^q,\alpha}$. We first remark that

$$\big\|\lambda^{\theta\alpha}\big[A(\lambda+A)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(\lambda)})} = \big\|t^{(1-\theta)\alpha}\big[A(1+tA)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(t)})}.$$

Let

$$w(t) := (1+t^{1/\alpha}A)^{-\alpha}x, \quad t \ge 0.$$

Then $w(0) = x$ and $w'(t) = -t^{1/\alpha-1}A(1+t^{1/\alpha}A)^{-\alpha-1}x$. Proposition 2.5.3 and Proposition 2.1.5 (together with Corollary 3.7 in [57], similarly as in the first part) imply that

$$\begin{aligned}&\|x\|_{(L^p(U),D(A^{\alpha}))_{\theta,\ell^q}}\\ &\quad\lesssim \|t^{1-\theta}w'(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \|t^{1-\theta}w(t)\|_{L^p(U;L^q_{*,(t)}(0,T))} + \|t^{1-\theta}A^{\alpha}w(t)\|_{L^p(U;L^q_{*,(t)}(0,T))}\\ &\quad= \alpha^{1/q}\big\|\big(t^{1-\alpha}A^{1-\alpha}(1+tA)^{-1}\big)\big(t^{(1-\theta)\alpha}\big[A(1+tA)^{-1}\big]^{\alpha}x\big)\big\|_{L^p(U;L^q_{*(t)}(0,T))}\\ &\qquad\quad + \alpha^{1/q}\big\|t^{(1-\theta)\alpha}\big[(1+tA)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(t)}(0,T))}\\ &\qquad\quad + \alpha^{1/q}\big\|t^{(1-\theta)\alpha}\big[A(1+tA)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(t)}(0,T))}\\ &\quad\le \alpha^{1/q}(1+C_{\alpha})\big\|t^{(1-\theta)\alpha}\big[A(1+tA)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(t)})} + C'_{\alpha,T}\|t^{(1-\theta)\alpha}\|_{L^q_{*(t)}(0,T)}\|x\|_{L^p(U)}.\\ &\quad= \alpha^{1/q}(1+C_{\alpha})\big\|\lambda^{\theta\alpha}\big[A(\lambda+A)^{-1}\big]^{\alpha}x\big\|_{L^p(U;L^q_{*(\lambda)})} + C'_{\alpha,T}\big((1-\theta)\alpha q\big)^{-1/q}T^{(1-\theta)\alpha}\|x\|_{L^p(U)}.\end{aligned}$$

To prove the last part of this proposition, we will show that $X^1_{\theta,\ell^q,1} \subseteq X^3_{\theta,\ell^q} \subseteq X^2_{\theta,\ell^q,1}$. For the moment assume that $x \in X^1_{\theta,\ell^q,1}$. Then on $U$ we have

$$\begin{aligned}\big\|t^{-\theta}\big(e^{-tA}x - x\big)\big\|^q_{L^q_{*(t)}} &= \int_0^\infty \Big|t^{-\theta}\int_0^t Ae^{-sA}\,\mathrm{d}s\Big|^q \frac{\mathrm{d}t}{t} \\ &= \int_{\mathbb{R}}\Big|\int_{-\infty}^s e^{-\theta(s-v)}e^{(1-\theta)v}Ae^{-e^vA}x\,\mathrm{d}v\Big|^q\,\mathrm{d}s \\ &= \Big\|\int_{-\infty}^s e^{-\theta(s-v)}e^{(1-\theta)v}Ae^{-e^vA}x\,\mathrm{d}v\Big\|^q_{L^q_{(s)}(\mathbb{R})}.\end{aligned}$$

Now Young's inequality yields

$$\begin{aligned}\big\|t^{-\theta}\big(e^{-tA}x - x\big)\big\|_{L^q_{*(t)}} &\le \|e^{-\theta(\cdot)}\|_{L^1(0,\infty)}\big\|e^{(1-\theta)(\cdot)}Ae^{-e^{(\cdot)}A}x\big\|_{L^q(\mathbb{R})} \\ &= \frac{1}{\theta}\big\|t^{1-\theta}Ae^{-tA}x\big\|_{L^q_{*(t)}}.\end{aligned}$$

Applying $L^p$ norms on both sides leads to $[x]^3_{\theta,\ell^q} \le \frac{1}{\theta}[x]^1_{\theta,\ell^q,1}$.

Finally, we assume that $x \in X^3_{\theta,\ell^q}$. Using that

$$A(\lambda + A)^{-1} = x - \lambda(\lambda + A)^{-1}x = \int_0^\infty \lambda e^{-\lambda t}(x - e^{-tA}x)\,\mathrm{d}t,$$

we derive

$$\begin{aligned}\big\|\lambda^\theta A(\lambda + A)^{-1}x\big\|_{L^p(U;L^q_{*(\lambda)})} &= \Big\|\int_0^\infty \lambda^{\theta+1}e^{-\lambda t}(e^{-tA}x - x)\,\mathrm{d}t\Big\|_{L^p(U;L^q_{*(\lambda)})} \\ &= \Big\|\int_0^\infty s^\theta e^{-s}\Big(\big(\tfrac{s}{\lambda}\big)^{-\theta}\big(e^{-\frac{s}{\lambda}A}x - x\big)\Big)\,\mathrm{d}s\Big\|_{L^p(U;L^q_{*(\lambda)})} \\ &\le \int_0^\infty s^\theta e^{-s}\big\|\big(\tfrac{s}{\lambda}\big)^{-\theta}\big(e^{-\frac{s}{\lambda}A}x - x\big)\big\|_{L^p(U;L^q_{*(\lambda)})}\,\mathrm{d}s \\ &= \int_0^\infty s^\theta e^{-s}\,\mathrm{d}s\,\big\|\mu^{-\theta}(e^{-\mu A}x - x)\big\|_{L^p(U;L^q_{*(\mu)})} \\ &= \Gamma(\theta+1)\big\|\mu^{-\theta}(e^{-\mu A}x - x)\big\|_{L^p(U;L^q_{*(\mu)})},\end{aligned}$$

i.e. $[x]^2_{\theta,\ell^q,1} \le \Gamma(\theta+1)[x]^3_{\theta,\ell^q}$. □

The $q$-power function norms appearing in the previous proposition were already investigated by Kunstmann and Ullmann in [57]. As an application of their results we obtain that these spaces are in fact intermediate spaces in the classical sense.

**Proposition 2.5.6.** *Let $\theta \in (0,1)$, $p, q_1, q_2 \in [1,\infty)$ with $q_1 \le q_2$, and $A$ be an $\mathcal{R}_{q_1}$- and $\mathcal{R}_{q_2}$-sectorial operator. Then*

$$(L^p(U), D(A))_{\theta,1} \hookrightarrow (L^p(U), D(A))_{\theta,\ell^{q_1}} \hookrightarrow (L^p(U), D(A))_{\theta,\ell^{q_2}} \hookrightarrow (L^p(U), D(A))_{\theta,\infty}.$$

**Proof.** By Theorem 2.5.4 and [57, Proposition 4.2] (see also [57, Example 3.13]) we have

$$\begin{aligned}\|x\|_{\theta,\ell^{q_1}} &\eqsim \|x\|_{L^p(U)} + \big\|t^{(1-\theta)}Ae^{-tA}x\big\|_{L^p(U;L^{q_1}_{*(t)})}\\ &\eqsim \|x\|_{L^p(U)} + \Big\| \Big(\sum_{j\in\mathbb{Z}}|2^{(1-\theta)j}Ae^{-2^jA}x|^{q_1}\Big)^{1/q_1}\Big\|_{L^p(U)}.\end{aligned}$$

Using that

$$\|x\|_{\theta,r} \eqsim \|x\|_{L^p(U)} + \big\|2^{(1-\theta)j}Ae^{-2^jA}\big\|_{\ell^r_{(j)}(\mathbb{Z};L^p(U))}, \quad r\in[1,\infty],$$

(see e.g. the proof of [57, Proposition 4.16]) and $\ell^1 \hookrightarrow \ell^{q_1} \hookrightarrow \ell^{q_2} \hookrightarrow \ell^\infty$, we obtain

$$\begin{aligned}\|x\|_{\theta,\infty} &\lesssim \|x\|_{L^p(U)} + \big\|\sup_{j\in\mathbb{Z}}|2^{(1-\theta)j}Ae^{-2^jA}|\big\|_{L^p(U)}\\ &\le \|x\|_{L^p(U)} + \Big\| \Big(\sum_{j\in\mathbb{Z}}|2^{(1-\theta)j}Ae^{-2^jA}x|^{q_1}\Big)^{1/q_1}\Big\|_{L^p(U)}\\ &\le \|x\|_{L^p(U)} + \Big\| \Big(\sum_{j\in\mathbb{Z}}|2^{(1-\theta)j}Ae^{-2^jA}x|^{q_2}\Big)^{1/q_2}\Big\|_{L^p(U)}\\ &\le \|x\|_{L^p(U)} + \Big\| \sum_{j\in\mathbb{Z}}|2^{(1-\theta)j}Ae^{-2^jA}x|\Big\|_{L^p(U)} \lesssim \|x\|_{\theta,1}.\end{aligned}$$

□

In the proof of the preceding proposition we can see why it is reasonable to call the spaces $(L^p(U), D(A))_{\theta,\ell^q}$ *$\ell^q$ interpolation spaces.* Another interesting application is the following.

**Proposition 2.5.7.** *Let $\theta\in(0,1)$, $\alpha>\theta$, $p,q\in[1,\infty)$, and $A$ be an $\mathcal{R}_q$-sectorial operator. Then*

$$(L^p(U), D(A^\alpha))_{\theta/\alpha,\ell^q} = (L^p(U), D(A))_{\theta,\ell^q}.$$

**Proof.** Since $\alpha>\theta$, [57, Proposition 4.2] implies that

$$\big\|t^{-\theta}(tA)^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})} \eqsim \big\|t^{-\theta}tAe^{-tA}x\big\|_{L^p(U;L^q_{*(t)})}.$$

Now the result follows from Theorem 2.5.4, because

$$\begin{aligned}\|x\|_{(L^p(U),D(A^\alpha))_{\theta/\alpha,\ell^q}} &\eqsim \|x\|_{L^p(U)} + \big\|t^{(1-\theta/\alpha)\alpha}A^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})}\\ &= \|x\|_{L^p(U)} + \big\|t^{-\theta}(tA)^\alpha e^{-tA}x\big\|_{L^p(U;L^q_{*(t)})}\\ &\eqsim \|x\|_{L^p(U)} + \big\|t^{-\theta}tAe^{-tA}x\big\|_{L^p(U;L^q_{*(t)})}\\ &= \|x\|_{L^p(U)} + \big\|t^{(1-\theta)}Ae^{-tA}x\big\|_{L^p(U;L^q_{*(t)})} \eqsim \|x\|_{(L^p(U),D(A))_{\theta,\ell^q}}.\end{aligned}$$

□

If we interchange the $L^p$ and $L^q_*$ norm in the definition of the $\ell^q$ interpolation spaces, we get the usual real interpolation spaces as we have seen in the proof of Proposition 2.5.6. If we take as an example the Laplace operator $A = (-\Delta)$ on $L^p(\mathbb{R}^d)$, then the real interpolation space $(L^p(\mathbb{R}^d), D(A))_{\theta,q}$ is the Besov space $B^{2\theta,p}_q(\mathbb{R}^d)$ (see e.g. [63, Example 1.10]). To characterize the spaces $(L^p(\mathbb{R}^d), D(A))_{\theta,\ell^q}$ in this particular case, we would expect to obtain those spaces we get by interchanging the $L^p$ and $L^q$ norm in the definition of the Besov space norm, and this turns our attention to *Triebel-Lizorkin spaces.*

**Example 2.5.8.** We first give a short introduction of Triebel-Lizorkin spaces. There are, of course, many ways to characterize them (see e.g. [76]). Analogously as for Besov spaces in Section 1.4 we will define them via differences (see [76, Section 2.5.10]). Let $s > 0$ and $M > s$. We let

$$d_{F^{s,p}_q}[f](h,x) := d_{B^{s,p}_q}[f](h,x) = |h|^{-d/q-s}(\Delta^M_h f)(x),$$

where we have chosen $k = 0$ in the definition of $d_{B^{s,p}_q}[f]$ (see Section 1.4). Then $F^{s,p}_q := F^{s,p}_q(\mathbb{R}^d)$ is the set of all functions $f \in L^p(\mathbb{R}^d)$ such that $d_{F^{s,p}_q}[f] \in L^p(\mathbb{R}^d; L^q(\mathbb{R}^d))$, and $F^{s,p}_q$ is a Banach space with respect to the norm

$$\|f\|_{F^{s,p}_q} := \|f\|_{L^p(\mathbb{R}^d)} + \|d_{F^{s,p}_q}[f]\|_{L^p(\mathbb{R}^d;L^q(\mathbb{R}^d))}.$$

Moreover, we let $\dot{F}^{s,p}_q$ be the homogeneous counterpart of $F^{s,p}_q$, i.e. the completion of $F^{s,p}_q$ with respect to the norm

$$\|f\|_{\dot{F}^{s,p}_q} := \|d_{F^{s,p}_q}[f]\|_{L^p(\mathbb{R}^d;L^q(\mathbb{R}^d))}.$$

a) Let $\theta \in (0,1)$ and $A = (-\Delta)$ on $L^p(\mathbb{R}^d)$ with $D(A) = W^{2,p}(\mathbb{R}^d)$. Then by [75, Corollary 1 in Section 3.3] we have

$$\|f\|_{\dot{F}^{2\theta,p}_q} \eqsim \left\|t^{(1-\theta)}Ae^{-tA}f\right\|_{L^p(U;L^q_{*(t)})}.$$

Hence, by Theorem 2.5.4 we obtain

$$(L^p(U), D(A))_{\theta,\ell^q} = F^{2\theta,p}_q.$$

b) Let $m \in \mathbb{N}$, $A = (-\Delta)^m$, and $D(A) = W^{2m,p}(\mathbb{R}^d)$. Then Theorem 2.5.4 and [57, Proposition 4.13] show that

$$(L^p(U), D(A))_{\theta,\ell^q} = X^1_{\theta,\ell^q,1} = F^{2m\theta,p}_q.$$

c) Similarly, if $m \in \mathbb{N}$, $D(A_p) = W^{2m,p}(\mathbb{R}^d)$, and $A_p$ is the realization of an elliptic differential operator $\mathcal{A}$ in non-divergence form as considered in Example B of Section

2.3, then

$$(L^p(U), D(A_p))_{\theta,\ell^q} = F_q^{2m\theta,p}.$$

The equality $X^1_{\theta,\ell^q,1} = F_q^{2m\theta,p}$ was also shown in Theorem 3.6.3 of [79].

d) In [79, Section 3.6.2] Ullmann also treated elliptic differential operators $\mathcal{A}$ of second order in divergence form. If $A_2$ is the operator associated to the form of $\mathcal{A}$, then he proved that

$$X^1_{\theta,\ell^q,1} = F_q^{2\theta,p}.$$

This also follows from part a) and Theorem 2.5.4, since

$$(L^p(U), D(A_2))_{\theta,\ell^q} = (L^p(U), D(-\Delta))_{\theta,\ell^q} = F_q^{2\theta,p}.$$

To have a more sophisticated formulation of the next result we introduce a new space. Let here $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be a closed operator. Then we define

$$D(A; L^q[0,T]) := \{v\colon [0,T] \to D(A)\colon v, Av \in L^p(U; L^q[0,T])\}$$

and equip it with the norm

$$\|v\|_{D(A;L^q[0,T])} := \|v\|_{L^p(U;L^q[0,T])} + \|Av\|_{L^p(U;L^q[0,T])}, \quad v \in D(A; L^q[0,T]).$$

**THEOREM 2.5.9.** *Let $p, q \in [1,\infty)$, $\alpha \in (1/q, 1+1/q)$, and $A$ be an $\mathcal{R}_q$-sectorial operator. Then we have the continuous embedding*

$$D(A^\alpha, L^q[0,T]) \cap L^p(U; W^{\alpha,q}[0,T]) \hookrightarrow C\big([0,T]; (L^p(U), D(A))_{\alpha-1/q,\ell^q}\big).$$

**PROOF.** Let $v \in D(A^\alpha, L^q[0,T]) \cap L^p(U; W^{\alpha,q}[0,T])$. Central to this proof is the representation

$$v(0) = t^{-1}\int_0^t v(\tau)\,\mathrm{d}\tau - \int_0^t \tau^{-2}\int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu\,\mathrm{d}\tau.$$

Then

$$\begin{aligned}\big\|t^{1-\alpha+1/q}Ae^{-tA}v(0)\big\|_{L^p(U;L^q_{*(t)}(0,T))} &\le \Big\|t^{-\alpha}Ae^{-tA}\int_0^t v(\tau)\,\mathrm{d}\tau\Big\|_{L^p(U;L^q_{(t)}(0,T))}\\ &+\Big\|t^{1-\alpha}Ae^{-tA}\int_0^t \tau^{-2}\int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu\,\mathrm{d}\tau\Big\|_{L^p(U;L^q_{(t)}(0,T))}\end{aligned}$$

We estimate the first summand. Using Proposition 2.1.5 and Corollary 3.7 in [57], as well as Hardy's inequality we obtain

$$\begin{aligned}
\Big\| t^{-\alpha} A e^{-tA} \int_0^t v(\tau)\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} &= \Big\| (tA)^{1-\alpha} e^{-tA} t^{-1} \int_0^t A^\alpha v(\tau)\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} \\
&\lesssim \Big\| t^{-1} \int_0^t A^\alpha v(\tau)\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} \\
&= \Big\| t^{-1+1/q} \int_0^t \tau A^\alpha v(\tau) \frac{\mathrm{d}\tau}{\tau} \Big\|_{L^p(U;L^q_{*(t)}(0,T))} \\
&\le \frac{1}{1-1/q} \big\| s^{-1+1/q+1} A^\alpha v(s) \big\|_{L^p(U;L^q_{*(s)}(0,T))} \\
&= \frac{1}{1-1/q} \|A^\alpha v\|_{L^p(U;L^q[0,T])}.
\end{aligned}$$

For the second summand we first remark that

$$\|d_{W^{\alpha,q}}[v]\|^q_{L^q([0,T]\times[0,T])} = 2 \int_0^T \int_0^s \frac{|v(t)-v(s)|^q}{(s-t)^{\alpha q+1}}\,\mathrm{d}t\,\mathrm{d}s.$$

Then we again use Proposition 2.1.5 together with Corollary 3.7 of [57] and Hardy's inequality, as well as Hölder's inequality to get

$$\begin{aligned}
&\Big\| t^{1-\alpha} A e^{-tA} \int_0^t \tau^{-2} \int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} \\
&\quad= \Big\| t^{-\alpha} (tA) e^{-tA} \int_0^t \tau^{-2} \int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} \\
&\quad\lesssim \Big\| t^{-\alpha} \int_0^t \tau^{-2} \int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu\,\mathrm{d}\tau \Big\|_{L^p(U;L^q_{(t)}(0,T))} \\
&\quad= \Big\| t^{-\alpha+1/q} \int_0^t \tau^{-1} \int_0^\tau \big(v(\tau)-v(\mu)\big)\,\mathrm{d}\mu \frac{\mathrm{d}\tau}{\tau} \Big\|_{L^p(U;L^q_{*(t)}(0,T))} \\
&\quad\le \frac{1}{\alpha-1/q} \Big\| s^{-\alpha+1/q-1} \int_0^s \big(v(s)-v(\mu)\big)\,\mathrm{d}\mu \Big\|_{L^p(U;L^q_{*(s)}(0,T))} \\
&\quad= \frac{1}{\alpha-1/q} \Big\| \Big( \int_0^T s^{-(1+\alpha)q} \Big| \int_0^s \big(v(s)-v(\mu)\big)\,\mathrm{d}\mu \Big|^q \mathrm{d}s \Big)^{1/q} \Big\|_{L^p(U)} \\
&\quad\le \frac{1}{\alpha-1/q} \Big\| \Big( \int_0^T s^{-\alpha q-1} \int_0^s \big|v(s)-v(\mu)\big|^q \,\mathrm{d}\mu\,\mathrm{d}s \Big)^{1/q} \Big\|_{L^p(U)} \\
&\quad\le \frac{1}{\alpha-/q} \|v\|_{L^p(U;W^{\alpha,q}[0,T])}.
\end{aligned}$$

Now Theorem 2.5.4, Remark 2.5.5 b), and Sobolev's embedding theorem imply

$$\|v(0)\|_{\alpha-1/q,\ell^q} \lesssim \|A^\alpha v\|_{L^p(U;L^q[0,T])} + \|v\|_{L^p(U;W^{\alpha,q}[0,T])}.$$

The claim now follows by the strong continuity of the translation group. □

# Chapter 3

# Stochastic Evolution Equations

In this chapter we are concerned with 'abstract' results regarding stochastic evolution equations in $L^p$ spaces with an emphasis on existence, uniqueness, and regularity of their solutions. Here, we pursue a completely new approach by interchanging the usual order of integration. Since we want to apply a fixed point argument, we first study mild solutions, i.e. functions of the form

$$X(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F(s, X(s))\, \mathrm{d}s + \int_0^t e^{-(t-s)A}\boldsymbol{B}(s, X(s))\, \mathrm{d}\boldsymbol{\beta}(s).$$

Therefore, we study orbit maps, deterministic convolutions, and stochastic convolutions. In the final section, we apply these results via a fixed point argument. As a consequence, existence, uniqueness, and regularity results follow.

## 3.1 Motivation

To motivate the study of stochastic evolution equations we present some examples arising from from physics or other applied sciences.

**Stochastic population growth**

Assume that $X(t, u)$ models the population of a species in a random environment on an island $U \subseteq \mathbb{R}^2$ with finite resources. Then the competition between the members of that species will limit the population growth. Modeling this scenario leads to the following reaction diffusion equation

$$\begin{aligned} \partial_t X(t, u) &= \nu \Delta X(t, u) + X(t, u)\big(\alpha \dot{W}(t, u) - \beta X(t, u)\big), \\ \partial_\nu X|_{\partial U} &= 0, \\ X(0, u) &= x_0(u), \end{aligned}$$

wher $\alpha, \beta \geq 0$ and $\dot{W}(t, u)$ is a white noise.

**Turbulence**

Let $X(t,u)$ be the concentration of a substance, $\nu$ be the diffusion coefficient and $V = (V_n)_{n=1}^3$ be the random turbulent velocity field. Then the mixing of the substance forced through the turbulence in a domain $U \subset \mathbb{R}^3$ can be described as

$$\begin{aligned} \partial_t X(t,u) &= \nu \Delta X(t,u) - \sum_{n=1}^{3} V_n(\omega,t,u)\partial_n X(t,u) + q(t,u), \\ \partial_\nu X|_{\partial U} &= 0, \\ X(0,u) &= x_0(u), \end{aligned}$$

where $q$ is given. If the random field $V$ fluctuates rapidly, it may be approximated by a Gaussian white noise.

In both examples, the white noise could be modeled as

$$W(t,u) = \sum_{n=1}^{d} h_n(u)\beta_n(t), \quad u \in U,\, t \in [0,T],$$

where $(\beta_1,\dots,\beta_n)$ is an $\mathbb{R}^d$-valued Brownian motion. For more examples, see e.g. [16, 21].

Motivated by these examples, we want to investigate 'abstract' stochastic evolution equations in $L^p$ spaces, i.e. equations of the form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t,X(t))\,\mathrm{d}t + \boldsymbol{B}(t,X(t))\,\mathrm{d}\boldsymbol{\beta}(t), \quad X_0 = x_0, \tag{3.1}$$

which is the shorthand notation for the integral equation

$$X(t) + \int_0^t AX(s)\,\mathrm{d}s = x_0 + \int_0^t F(s,X(s))\,\mathrm{d}s + \sum_{n=1}^{\infty}\int_0^t B_n(s,X(s))\,\mathrm{d}\beta_n(s).$$

Here, $(-A)$ is the generator of an analytic semigroup $(e^{-tA})_{t\geq 0}$ on some space $L^p(U)$ and $F\colon \Omega \times [0,T] \times L^p(U) \to L^p(U)$ and $\boldsymbol{B}\colon \Omega \times [0,T] \times \mathbb{N} \times L^p(U) \to L^p(U)$ are functions defined in such a way that at least everything in (3.1) is well-defined. In this case, we can choose a number $\nu \geq 0$ such that the semigroup generated by $-A_\nu := -(\nu + A)$ is uniformly exponentially stable and $0 \in \rho(A_\nu)$. In particular, the fractional powers $A_\nu^\gamma$, $\gamma \in \mathbb{R}$, are well-defined and the space

$$D_\gamma := D(A_\nu^\gamma)$$

is a Banach space with respect to the norm $\|x\|_{D_\gamma} := \|A_\nu^\gamma x\|_{L^p(U)}$. Up to an equivalent norm, this definition is independent of $\nu$. In this context, we use the following notions for solutions.

**Definition 3.1.1.** Let $\gamma \geq 0$, $r \in \{0\} \cup (1, \infty)$ and $p, q \in (1, \infty)$.

1) We call a process $X \colon \Omega \times [0,T] \to D_\gamma$ a *strong* $(r,p,q)$ *solution* of (3.1) with respect to the filtration $\mathbb{F}$ if

   a) $X$ is measurable and $A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$;

   b) almost surely, $X \in D(A)$, $AX \in L^p(U; L^1[0,T])$, and $F(\cdot, X(\cdot)) \in L^p(U; L^1[0,T])$, and $\boldsymbol{B}(\cdot, X(\cdot)) \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^2([0,T] \times \mathbb{N})))$, i.e. everything in (3.1) is well-defined;

   c) almost surely, $X$ solves the equation

$$X(t) + \int_0^t AX(s)\,\mathrm{d}s = x_0 + \int_0^t F(s, X(s))\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^t B_n(s, X(s))\,\mathrm{d}\beta_n(s).$$

2) We call a process $X \colon \Omega \times [0,T] \to D_\gamma$ a *weak* $(r,p,q)$ *solution* of (3.1) with respect to the filtration $\mathbb{F}$ if

   a) $X$ is measurable and $A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$;

   b) $\langle F(\cdot, X(\cdot)), \psi \rangle \in L^1[0,T]$ almost surely, and $\langle \boldsymbol{B}(\cdot, X(\cdot)), \psi \rangle \in L^0_{\mathbb{F}}(\Omega; L^2([0,T] \times \mathbb{N}))$ for each $\psi \in D(A')$;

   c) almost surely and for all $\psi \in D(A')$, $X$ solves the equation

$$\begin{aligned}\langle X(t), \psi \rangle + \int_0^t \langle X(s), A'\psi \rangle\,\mathrm{d}s = x_0 &+ \int_0^t \langle F(s, X(s)), \psi \rangle\,\mathrm{d}s \\ &+ \sum_{n=1}^{\infty} \int_0^t \langle B_n(s, X(s)), \psi \rangle^{L^2}\,\mathrm{d}\beta_n(s).\end{aligned}$$

3) We call a process $X \colon \Omega \times [0,T] \to D_\gamma$ a *mild* $(r,p,q)$ *solution* of (3.1) with respect to the filtration $\mathbb{F}$ if

   a) $X$ is measurable and $A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$;

   b) $e^{-(t-(\cdot))A} F(\cdot, X(\cdot)) \in L^p(U; L^1[0,t])$ almost surely and $e^{-(t-(\cdot))A} \boldsymbol{B}(\cdot, X(\cdot)) \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^2([0,t] \times \mathbb{N})))$ for every $t \in [0,T]$;

   c) almost surely, $X$ solves the equation

$$X(t) = e^{-tA} x_0 + \int_0^t e^{-(t-s)A} F(s, X(s))\,\mathrm{d}s + \sum_{n=1}^{\infty} \int_0^t e^{-(t-s)A} B_n(s, X(s))\,\mathrm{d}\beta_n(s).$$

**Remark 3.1.2.**

a) Looking at the assumptions in the definition of solutions, we see that we do not consider the deterministic integrals as $L^p(U)$-valued Bochner integrals. Instead we assume that the integrands are integrable for almost every $u \in U$. The integrals are then still elements of $L^p(U)$.

b) One should also notice, that in the case of a mild and weak solutions we do not assume that $X \in D(A)$. In some situations this will be true, but in general it is not reasonable to assume that.

We will see in Section 3.5 that under some assumptions on the operator $A$ and the nonlinearities $F$ and $\boldsymbol{B}$ these definitions are equivalent. This is the reason why we have a huge interest in estimates for deterministic and stochastic convolutions (see Sections 3.3 and 3.4). Before turning to that we investigate the regularity of the orbit map $t \mapsto e^{-tA}x_0$ for suitable generators $(-A)$ and initial values $x_0 \in L^p(U)$.

## 3.2 Orbit Maps

We start with a lemma, which also implies well-definedness results for deterministic and stochastic convolutions. At the moment this does not seem to be relevant for orbit maps. However, this already includes some ideas and problems of this approach.

**Lemma 3.2.1.** *Let $p, q \in (1,\infty)$, $\widetilde{q} \in (1,q)$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\ell^q$-sectorial of angle $\omega_{\ell^q} < \pi/2$ with $0 \in \rho(A)$. Then for any $\beta < \frac{q-\widetilde{q}}{q\widetilde{q}}$ we have*

$$\|A^\beta e^{-(\cdot)A}\boldsymbol{\phi}(\cdot)\|_{L^p(U;L^{\widetilde{q}}([0,T];\ell^2))} \leq CT^{\frac{q-\widetilde{q}}{q\widetilde{q}}-\beta}\|\boldsymbol{\phi}\|_{L^p(U;L^q([0,T];\ell^2))},$$

*for $\boldsymbol{\phi} \in L^p(U;L^q([0,T];\ell^2))$, where $C = C(\beta)$ and $\lim_{\beta\to\frac{q-\widetilde{q}}{q\widetilde{q}}} C(\beta) = \infty$. In particular,*

$$\|e^{-(t-s)A}\boldsymbol{\phi}(s)\|_{L^p(U;L^{\widetilde{q}}_{(s)}([0,t];\ell^2))} \leq CT^{\frac{q-\widetilde{q}}{q\widetilde{q}}}\|\boldsymbol{\phi}\|_{L^p(U;L^q([0,t];\ell^2))}$$

*for each $\boldsymbol{\phi} \in L^p(U;L^q([0,t];\ell^2))$.*

**Proof.** First let $\phi \in L^p(U;L^q[0,T])$ and $\beta > 0$. For $\theta \in (\omega_{\ell^q}(A), \pi/2)$ we define the path $\Gamma(\theta) := \{\gamma(\rho) := |\rho|e^{-i\operatorname{sign}(\rho)\theta} \colon \rho \in \mathbb{R}\} = \partial\Sigma_\theta$. Then, by the functional calculus for sectorial operators we have

$$A^\beta e^{-tA}\phi(t) = \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta e^{-t\lambda}R(\lambda,A)\phi(t)\,\mathrm{d}\lambda, \quad t \in [0,T],$$

where the representation is independent of $\theta$. Now observe that for each $\lambda \in \Gamma(\theta)$ we have $\operatorname{Re}\lambda = \cos(\theta)|\lambda|$. For $r \in [1,\infty)$ we therefore obtain

$$\|e^{-(\cdot)\operatorname{Re}\lambda}\|_{L^r[0,T]} = \Big(\frac{1}{r\operatorname{Re}\lambda}(1-e^{-Tr\operatorname{Re}\lambda})\Big)^{1/r} \leq \Big(T \wedge \frac{1}{r\operatorname{Re}\lambda}\Big)^{1/r} = \Big(T \wedge \frac{1}{r\cos(\theta)|\lambda|}\Big)^{1/r}.$$

Now choose $r$ such that $\frac{1}{\widetilde{q}} = \frac{1}{r} + \frac{1}{q}$ (i.e. $r = \frac{q\widetilde{q}}{q-\widetilde{q}}$). Hölder's inequality and the $\ell^q$-sectoriality then lead to

$$\begin{aligned}\big\|\lambda^\beta e^{-(\cdot)\lambda}R(\lambda,A)\phi\big\|_{L^p(U;L^{\widetilde{q}}[0,T]}&\leq C_\theta|\lambda|^\beta\|e^{-(\cdot)\lambda}\|_{L^r[0,T]}\|R(\lambda,A)\phi\|_{L^p(U;L^q[0,T])}\\&\leq C_\theta|\lambda|^{\beta-1}\Big(T\wedge\frac{1}{r\cos(\theta)|\lambda|}\Big)^{1/r}\|\phi\|_{L^p(U;L^q[0,T])}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|A^\beta e^{-(\cdot)A}\phi(\cdot)\|_{L^p(U;L^{\widetilde{q}}[0,T])}&\leq\frac{2C_\theta}{2\pi}\int_0^\infty\rho^{\beta-1}\Big(T\wedge\frac{1}{r\cos(\theta)\rho}\Big)^{1/r}\,\mathrm{d}\rho\;\|\phi\|_{L^p(U;L^q[0,T])}\\&=\frac{C_\theta}{\pi}\Big(\int_0^{\frac{1}{r\cos(\theta)T}}\rho^{\beta-1}T^{1/r}\,\mathrm{d}\rho+\frac{1}{r^{1/r}\cos(\theta)^{1/r}}\int_{\frac{1}{r\cos(\theta)T}}^\infty\rho^{\beta-1/r-1}\,\mathrm{d}\rho\Big)\|\phi\|_{L^p(U;L^q[0,T])}\\&=\frac{C_\theta}{\pi(r\cos(\theta))^\beta}\frac{1}{\beta(1-r\beta)}T^{1/r-\beta}\|\phi\|_{L^p(U;L^q[0,T])}.\end{aligned}$$

If $\beta=0$, we have to add a circle around 0 in the path $\Gamma(\theta)$ (see also Example 9.8 in [59]). Here we take $\Gamma'(\theta):=\partial\big(\Sigma_\theta\cup B(0,\frac{1}{T})\big)$. Similar calculations as above then lead to

$$\begin{aligned}&\|e^{-(\cdot)A}\phi(\cdot)\|_{L^p(U;L^{\widetilde{q}}[0,T])}\\&\quad\leq\Big(\frac{2C_\theta}{2\pi}\int_{1/T}^\infty\big(\tfrac{1}{r\cos(\theta)\rho}\big)^{1/r}\rho^{-1}\,\mathrm{d}\rho+\frac{C_\theta}{2\pi}\int_\theta^{2\pi-\theta}\big(\tfrac{1}{r\cos(\theta)\frac{1}{T}}\big)^{1/r}\,\mathrm{d}\alpha\Big)\|\phi\|_{L^p(U;L^q[0,T])}\\&\quad=\Big(\frac{C_\theta}{r^{1/r}\cos(\theta)^{1/r}}\frac{r}{\pi}T^{1/r}+\frac{C_\theta}{r^{1/r}\cos(\theta)^{1/r}}\frac{2\pi-2\theta}{2\pi}T^{1/r}\Big)\|\phi\|_{L^p(U;L^q[0,T])}\\&\quad\leq\frac{C_\theta}{r^{1/r}\cos(\theta)^{1/r}}\big(\tfrac{r}{\pi}+1\big)T^{1/r}\|\phi\|_{L^p(U;L^q[0,T])}.\end{aligned}$$

For the general case $\boldsymbol{\phi}\in L^p(U;L^q([0,T];\ell^2))$, we use Kahane's inequality and the estimate above to deduce

$$\begin{aligned}\|A^\beta e^{-(\cdot)A}\boldsymbol{\phi}\|_{L^p(U;L^q([0,T];\ell^2))}&=\Big\|\Big(\sum_{n\geq1}\big|A^\beta e^{-(\cdot)A}\phi_n\big|^2\Big)^{1/2}\Big\|_{L^p(U;L^q[0,T])}\\&\eqsim_{p,q}\widetilde{\mathbb{E}}\Big\|\sum_{n\geq1}\widetilde{r}_nA^\beta e^{-(\cdot)A}\phi_n\Big\|_{L^p(U;L^q[0,T])}\\&\lesssim_C T^{1/r-\beta}\widetilde{\mathbb{E}}\Big\|\sum_{n\geq1}\widetilde{r}_n\phi_n\Big\|_{L^p(U;L^q[0,T])}\\&\eqsim_{p,q}T^{1/r-\beta}\|\boldsymbol{\phi}\|_{L^p(U;L^q([0,T];\ell^2))},\end{aligned}$$

where $(\widetilde{r}_n)_{n\in\mathbb{N}}$ is a Rademacher sequence on some probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$. Since $r=\frac{q\widetilde{q}}{q-\widetilde{q}}$, the claim follows. □

**REMARK 3.2.2.** If we assume that $A$ is $\ell^{\widetilde{q}}$-sectorial in the previous lemma, then we obtain the same result by interchanging the application of Hölder's inequality and the estimate of the $\ell^q$-sectoriality.

If we assume $\mathcal{R}_q$-sectoriality of $A$ instead of $\ell^q$-sectoriality, we obtain the following result.

**Lemma 3.2.3.** *Let $p, q \in [1, \infty)$, $\beta \geq 0$, $t \in [0, T]$ be fixed, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\mathcal{R}_q$-sectorial of angle $\omega_{\mathcal{R}_q}(A) < \pi/2$ with $0 \in \rho(A)$. Then there exists a constant $C > 0$ such that*

$$\|(\cdot A)^\beta e^{-(\cdot)A} \boldsymbol{\phi}\|_{L^p(U;L^q([0,T];\ell^2))} \leq C \|\boldsymbol{\phi}\|_{L^p(U;L^q([0,T];\ell^2))}$$

*for $\boldsymbol{\phi} \in L^p(U; L^q([0,t]; \ell^2))$. In particular,*

$$\|e^{-(t-s)A} \boldsymbol{\phi}(s)\|_{L^p(U;L^q_{(s)}([0,t];\ell^2))} \leq C \|\boldsymbol{\phi}\|_{L^p(U;L^q([0,t];\ell^2))}$$

*for each $\boldsymbol{\phi} \in L^p(U; L^q([0,t]; \ell^2))$.*

**Proof.** Let $\phi \in L^p(U; L^q[0,T])$. The general case follows by an application of Kahane's inequality as in Lemma 3.2.1. By Corollary 3.7 of [57], the $\mathcal{R}_q$-sectoriality of $A$ implies the $\mathcal{R}_q$-boundedness of the set $\{(sA)^\beta e^{-sA} \colon s > 0\}$. Therefore, also the set $\{(sA)^\beta e^{-sA} \colon s \in [0, T]\}$ is $\mathcal{R}_q$-bounded as a subset of the first one. By Proposition 2.1.5 we obtain a constant $C > 0$ such that

$$\Big\| \Big( \int_0^T \big|(sA)^\beta e^{-sA} \phi(s)\big|^q \,\mathrm{d}s \Big)^{1/q} \Big\|_{L^p(U)} \leq C \Big\| \Big( \int_0^T \big|\phi(s)\big|^q \,\mathrm{d}s \Big)^{1/q} \Big\|_{L^p(U)}.$$

□

**Remark 3.2.4.**

a) A comparison of Lemma 3.2.1 and Lemma 3.2.3 shows that $\mathcal{R}_q$-sectoriality might be needed if one wants to stay on the same function space.

b) Note that the assumption $0 \in \rho(A)$ is only required such that the fractional powers of $A$ are well-defined. For any result without these fractional powers we can ignore this assumption.

c) In particular, if $A$ is $\ell^q$-sectorial and $q > 2$, then for any $\boldsymbol{\phi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,t]; \ell^2)))$ the process

$$s \mapsto e^{-(t-s)A} \boldsymbol{\phi}(s), \quad s \in [0, T],$$

is deterministically *and* stochastically integrable, since, by Hölder's inequality, we have

$$\begin{aligned} \big\|e^{-(t-s)A} \boldsymbol{\phi}(s)\big\|_{L^r(\Omega;L^p(U;L^1_{(s)}([0,t];\ell^2)))} &\leq T^{1-1/\tilde{q}} \big\|e^{-(t-s)A} \boldsymbol{\phi}(s)\big\|_{L^r(\Omega;L^p(U;L^{\tilde{q}}_{(s)}([0,t];\ell^2)))} \\ &\leq C_T T^{1-1/\tilde{q}} \|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,t];\ell^2)))} \end{aligned}$$

and

$$\big\|e^{-(t-s)A}\phi(s)\big\|_{L^r(\Omega;L^p(U;L^2_{(s)}([0,t]\times\mathbb{N})))} \le T^{1/2-1/\widetilde{q}}\big\|e^{-(t-s)A}\phi(s)\big\|_{L^r(\Omega;L^p(U;L^{\widetilde{q}}_{(s)}([0,t];\ell^2)))}$$
$$\le C_T T^{1/2-1/\widetilde{q}}\|\phi\|_{L^r(\Omega;L^p(U;L^q([0,t];\ell^2)))},$$

where $2 \le \widetilde{q} < q$ and $C_T$ is the constant from Lemma 3.2.1. If $A$ happens to be $\mathcal{R}_2$-sectorial (i.e. $\mathcal{R}$-sectorial), then we obtain a similar result also for $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^2([0,t];\ell^2)))$.

d) If $A$ is $\ell^q$-sectorial and $q > 2$, then the function $f\colon [0,T] \to L^r_{\mathbb{F}}(\Omega; L^p(U; L^2([0,T];\ell^2)))$, $f(t) = \mathbb{1}_{(0,t]}e^{-(t-(\cdot))A}\phi$, is continuous for any $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$. Let us prove this. For any $s,t \in [0,T]$, $s < t$, we have for $2 \le \widetilde{q} < q$

$$\|f(t) - f(s)\|_{L^r(\Omega;L^p(U;L^2([0,T];\ell^2)))}$$
$$= \big\|\mathbb{1}_{(s,t]}e^{-(t-(\cdot))A}\phi + \mathbb{1}_{(0,s]}\big(e^{-(t-(\cdot))A} - e^{-(s-(\cdot))A}\big)\phi\big\|_{L^r(\Omega;L^p(U;L^2([0,T];\ell^2)))}$$
$$\le T^{1/2-1/\widetilde{q}}\|e^{-(t-(\cdot))A}(\mathbb{1}_{(s,t]}\phi)\|_{L^r(\Omega;L^p(U;L^q([0,t];\ell^2)))}$$
$$+ T^{1/2-1/\widetilde{q}}\big\|e^{-(s-(\cdot))A}\big(e^{-(t-s)A} - I\big)\phi\big\|_{L^r(\Omega;L^p(U;L^q([0,s];\ell^2)))}$$
$$\le T^{1/2-1/\widetilde{q}}C\|\mathbb{1}_{(s,t]}\phi\|_{L^r(\Omega;L^p(U;L^q([0,t];\ell^2)))}$$
$$+ T^{1/2-1/\widetilde{q}}C\big\|\big(e^{-(t-s)A} - I\big)\phi\big\|_{L^r(\Omega;L^p(U;L^q([0,s];\ell^2)))}.$$

Now, if $s \to t$, the first part converges to 0 by the dominated convergence theorem. Since the semigroup $e^{-tA}$ can be extended to a strongly continuous semigroup on $L^p(U; L^q([0,T];\ell^2))$, the second summand also converges to 0 for $s \to t$. This proves the claim. As in part b), $\mathcal{R}_2$-sectoriality of $A$ would include the case $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^2([0,T];\ell^2)))$.

After this short excursion, we turn to the actual topic of this section and start to investigate the orbit map $t \mapsto e^{-tA}x$, $x \in L^p(U)$. We start with an elementary observation using the same technique as in Lemma 3.2.1.

**Proposition 3.2.5.** *Let $p,q \in [1,\infty)$, $\beta \in [0, 1/q)$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be sectorial of angle $\omega(A) < \pi/2$ with $0 \in \rho(A)$. Then there exists a constant $C > 0$ such that*

$$\|A^\beta e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])} \le CT^{1/q-\beta}\|x_0\|_{L^p(U)}$$

*for $x_0 \in L^p(U)$.*

**PROOF.** Observe that

$$\begin{aligned}\big\|\lambda^\beta e^{-(\cdot)\lambda}R(\lambda,A)x_0\big\|_{L^p(U;L^q[0,T])} &= |\lambda|^\beta\|R(\lambda,A)x_0\|_{L^p(U)}\|e^{-(\cdot)\lambda}\|_{L^q[0,T]}\\ &\le C_\theta|\lambda|^{\beta-1}\Big(T\wedge\frac{1}{q\cos(\theta)|\lambda|}\Big)^{1/q}\|x_0\|_{L^p(U)}.\end{aligned}$$

Now the result can be deduced in the same way as in Lemma 3.2.1. □

As indicated in the beginning of Section 2.5, $\ell^q$ interpolation spaces will play an important role in connection with initial values for stochastic evolution equations, and by the reformulation as a fixed point equation also for orbit maps. The key result is Theorem 2.5.4.

**LEMMA 3.2.6.** *Let $p,q\in[1,\infty)$, $1/q<\alpha<\beta<\gamma$, and $A\colon D(A)\subseteq L^p(U)\to L^p(U)$ be $\mathcal{R}_q$-sectorial of angle $\omega_{\mathcal{R}_q}(A)<\pi/2$ with $0\in\rho(A)$. Then there exists a constant $C>0$ such that*

$$\|A^{\alpha-1/q}x_0\|_{L^p(U)}\le C(T^{\alpha-1/q}+T^{\alpha-\beta})\|x_0\|^2_{\beta-1/q,\ell^q,1,T}$$

*for each $x_0\in(L^p(U),D(A))_{\beta-1/q,\ell^q}$ and*

$$[x_0]^1_{1-1/\beta q,\ell^q,\beta,T}=\|A^\beta e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])}\le CT^{\gamma-\beta}\|A^{\gamma-1/q}x_0\|_{L^p(U)}$$

*for $x_0\in D(A^{\gamma-1/q})$. In particular, we have*

$$\begin{aligned}\|x_0\|_{L^p(U)}+\|A^{\alpha-1/q}x_0\|_{L^p(U)}&\lesssim\|x_0\|_{L^p(U)}+\|A^\beta e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])}\eqsim\|x_0\|_{\beta-1/q,\ell^q}\\ &\lesssim\|x_0\|_{L^p(U)}+\|A^{\gamma-1/q}x_0\|_{L^p(U)}.\end{aligned}$$

**PROOF.** To show the first result we use the representation formula

$$A^{\alpha-1/q}x_0=\frac{\sin(\pi(\alpha-1/q))}{\pi}A\int_0^\infty\lambda^{\alpha-1/q-1}(\lambda+A)^{-1}x_0\,\mathrm{d}\lambda.$$

Then, by the sectoriality of $A$ and Hölder's inequality we have

$$\begin{aligned}&\|A^{\alpha-1/q}x_0\|_{L^p(U)}\\ &\lesssim_{\alpha,q}\Big\|\int_0^T\lambda^{\alpha-1/q}A(\lambda+A)^{-1}x_0\frac{\mathrm{d}\lambda}{\lambda}\Big\|_{L^p(U)}+\Big\|\int_T^\infty\lambda^{\alpha-\beta}\lambda^{\beta-1/q}A(\lambda+A)^{-1}x_0\frac{\mathrm{d}\lambda}{\lambda}\Big\|_{L^p(U)}\\ &\lesssim\int_0^T\lambda^{\alpha-1/q-1}\,\mathrm{d}\lambda\,\|x_0\|_{L^p(U)}+\|\lambda^{\alpha-\beta}\|_{L^{q'}_{*(\lambda)}(T,\infty)}\big\|\lambda^{\beta-1/q}A(\lambda+A)^{-1}x_0\big\|_{L^p(U;L^q_{*(\lambda)}(T,\infty))}\\ &=\frac{1}{\alpha-1/q}T^{\alpha-1/q}\|x_0\|_{L^p(U)}+\frac{1}{((\beta-\alpha)q')^{1/q'}}T^{\alpha-\beta}[x_0]^2_{\beta-1/q,\ell^q,1,T}.\end{aligned}$$

The second estimate follows from the definition of $[\cdot]^1_{1-1/\beta q,\ell^q,\beta,T}$ and Proposition 3.2.5.

Finally, the last assertion follows from the estimates above, Proposition 2.5.7, and Theorem 2.5.4. □

The next result is an immediate consequence of Lemma 3.2.6.

**COROLLARY 3.2.7.** *Let $p, q \in [1,\infty)$, $0 < \alpha < \beta < \gamma < 1$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\mathcal{R}_q$-sectorial of angle $\omega_{\mathcal{R}_q}(A) < \pi/2$ with $0 \in \rho(A)$. Then*

$$D(A^\gamma) \hookrightarrow (L^p(U), D(A))_{\beta,\ell^q} \hookrightarrow D(A^\alpha).$$

**REMARK 3.2.8.** Corollary 3.2.7 can also be deduced by using results of Section 2.5 and real interpolation theory. Applying [63, Proposition 1.3], Proposition 2.5.6, Proposition 2.5.7, again Proposition 2.5.6, [63, Proposition 1.4] and [63, Proposition 4.7] we obtain

$$\begin{aligned} D(A^\gamma) &\hookrightarrow (L^p(U), D(A^\gamma))_{\beta/\gamma,1} \hookrightarrow (L^p(U), D(A^\gamma))_{\beta/\gamma,\ell^q} \hookrightarrow (L^p(U), D(A))_{\beta,\ell^q} \\ &\hookrightarrow (L^p(U), D(A))_{\beta,\infty} \hookrightarrow (L^p(U), D(A))_{\alpha,1} \hookrightarrow D(A^\alpha). \end{aligned}$$

In Lemma 3.2.6 we are also interested in the case $\beta = \gamma$ which corresponds to $\beta = 1/q$ in Proposition 3.2.5. Unfortunately, this is, generally, not correct. To bypass this problem, we need to assume more on the operator $A$. However, even if we assume that $A$ is $\mathcal{R}_q$-sectorial in Proposition 3.2.5, then Lemma 3.2.3 leads to

$$\begin{aligned} \|A^\beta e^{-(\cdot)A} x_0\|_{L^p(U;L^q[0,T])} &= \|(tA)^\beta e^{-tA}(t^{-\beta} x_0)\|_{L^p(U;L^q_{(t)}[0,T])} \\ &\le C\|t^{-\beta} x_0\|_{L^p(U;L^q_{(t)}[0,T])} \\ &= \frac{C}{(1-\beta q)^{1/q}} T^{1/q-\beta} \|x_0\|_{L^p(U)} \end{aligned}$$

for $\beta < 1/q$. This suggests that we have to assume even more.

**THEOREM 3.2.9.** *Let $p \in [1,\infty)$, $q \in [2,\infty)$, and let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus of angle $\alpha \in (\omega_{\mathcal{R}_q^\infty}(A), \pi/2)$ with $0 \in \rho(A)$. Then there exists a constant $C > 0$ such that*

$$\|A^{1/q} e^{-(\cdot)A} x_0\|_{L^p(U;L^q[0,T])} \le C\|x_0\|_{L^p(U)}$$

*for $x_0 \in L^p(U)$. In particular, if $\beta \in \mathbb{R}$,*

$$\|A^\beta e^{-(\cdot)A} x_0\|_{L^p(U;L^q[0,T])} \le C\|A^{\beta-1/q} x_0\|_{L^p(U)}$$

*for $x_0 \in D(A^{\beta-1/q})$.*

**Proof.** Let $\nu \in (\alpha, \pi/2)$ and define the multiplication operator

$$M_\lambda \colon L^p(U) \to L^p(U; L^q[0,T]), \quad (M_\lambda x)(t) := \lambda^{1/q} e^{-t\lambda} x,$$

for $\lambda \in \Sigma_\nu$. Then

$$\begin{aligned} \|\lambda^{1/q} e^{-(\cdot)\lambda}\|_{L^q[0,T]} &= |\lambda|^{1/q} \big( \tfrac{1}{q \mathrm{Re}\, \lambda} - \tfrac{1}{q\mathrm{Re}\,\lambda} e^{-Tq \mathrm{Re}\,\lambda} \big)^{1/q} \\ &\le \tfrac{1}{q^{1/q}} \big( \tfrac{|\lambda|}{\mathrm{Re}\,\lambda} \big)^{1/q} \le \tfrac{1}{(q\cos(\nu))^{1/q}} =: C \end{aligned}$$

for each $\lambda \in \Sigma_\nu$. Using this, we want to show that $\{M_\lambda \colon \lambda \in \Sigma_\nu\}$ is $\mathcal{R}$-bounded. For this purpose, let $(\lambda_n)_{n=1}^N \subseteq \Sigma_\nu$, $(x_n)_{n=1}^N \subseteq L^p(U)$, and let $(\widetilde{r}_n)_{n=1}^N$ be a Rademacher sequence on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Then

$$\begin{aligned} \widetilde{\mathbb{E}} \Big\| \sum_{n=1}^N \widetilde{r}_n M_{\lambda_n} x_n \Big\|_{L^p(U;L^q[0,T])} &\eqsim_{p,q} \Big\| \Big( \sum_{n=1}^N |M_{\lambda_n} x_n|^2 \Big)^{1/2} \Big\|_{L^p(U;L^q[0,T])} \\ &\le \Big\| \Big( \sum_{n=1}^N \|M_{\lambda_n} x_n\|_{L^q[0,T]}^2 \Big)^{1/2} \Big\|_{L^p(U)} \\ &\lesssim_C \Big\| \Big( \sum_{n=1}^N |x_n|^2 \Big)^{1/2} \Big\|_{L^p(U)} \\ &\eqsim_p \widetilde{\mathbb{E}} \Big\| \sum_{n=1}^N \widetilde{r}_n x_n \Big\|_{L^p(U)}. \end{aligned}$$

Now define the operator $M_{\lambda,J} \colon L^p(U; L^q[0,T]) \to L^p(U; L^q[0,T])$ by $M_{\lambda,J}\phi := M_\lambda J \phi$, where

$$J \colon L^p(U; L^q[0,T]) \to L^p(U), \quad J\phi = \frac{1}{T} \int_0^T \phi(t)\, \mathrm{d}t.$$

Since, by Hölder's inequality,

$$\|J\phi\|_{L^p(U)} \le T^{-1/q} \|\phi\|_{L^p(U;L^q[0,T])},$$

it is easy to see that the operator family $\{M_{\lambda,J} \colon \lambda \in \Sigma_\nu\}$ is $\mathcal{R}$-bounded on $L^p(U; L^q[0,T])$ with constant $CT^{-1/q}$. Moreover, by Theorem 2.4.5 the operator $A$ has an extension $A^{L^q}$ on $L^p(U; L^q[0,T])$ such that $A^{L^q}$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus on $L^p(U; L^q[0,T])$. Since each operator $M_{\lambda,J}$ obviously commutes with $R(\lambda, A^{L^q})$, Theorem 4.4 of [52] implies that

$$(+) \qquad \phi \mapsto \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\lambda, A^{L^q}) M_{\lambda,J} \phi \, \mathrm{d}\lambda$$

defines a bounded operator on $L^p(U; L^q[0,T])$ for $\alpha' \in (\alpha, \nu)$. For any $x_0 \in L^p(U)$ let $\phi = \mathbb{1}_{[0,T]} x_0$. Then $J\phi = x_0$ and $\|\phi\|_{L^p(U;L^q[0,T])} = T^{1/q} \|x_0\|_{L^p(U)}$. Using the boundedness of (+), this leads to

$$\Big\| \frac{1}{2\pi i} \int_{\partial\Sigma_{\alpha'}} R(\lambda, A^{L^q}) M_\lambda x_0 \,\mathrm{d}\lambda \Big\|_{L^p(U;L^q[0,T])} = \Big\| \frac{1}{2\pi i} \int_{\partial\Sigma_{\alpha'}} R(\lambda, A^{L^q}) M_{\lambda,J}\phi \,\mathrm{d}\lambda \Big\|_{L^p(U;L^q[0,T])}$$
$$\le CT^{-1/q}\|\phi\|_{L^p(U;L^q[0,T])} = C\|x_0\|_{L^p(U)}.$$

Observe that $A$ is by definition also sectorial, and that $t \mapsto f_t(\lambda) := \lambda^{1/q}e^{-t\lambda} \in H_0^\infty(\Sigma_\nu)$ for $t > 0$. For $t$ fixed, the functional calculus for sectorial operators implies

$$A^{1/q}e^{-tA}x_0 = \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} f_t(\lambda)R(\lambda,A)x_0 \,\mathrm{d}\lambda = \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\lambda,A)(M_\lambda x_0)(t)\,\mathrm{d}\lambda.$$

Together with the boundedness result above this concludes the proof. □

**Corollary 3.2.10.** *Let $p \in [1,\infty)$, $q \in (1,\infty)$, $\beta \in (0,1)$, and let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus of angle $\alpha \in (\omega_{\mathcal{R}_q^\infty}(A), \pi/2)$ with $0 \in \rho(A)$. Then*

$$D(A^\beta) \hookrightarrow (L^p(U), D(A))_{\beta,\ell^q}, \qquad \text{if } q \ge 2,$$

*and*

$$(L^p(U), D(A))_{\beta,\ell^q} \hookrightarrow D(A^\beta), \qquad \text{if } q \le 2.$$

**Proof.** The first embedding follows from Theorem 3.2.9. To show the second estimate we use a duality argument. First observe that $A'\colon D(A') \subseteq L^{p'}(U) \to L^{p'}(U)$ has an $\mathcal{R}_{q'}$-bounded $H^\infty$ calculus and that Theorem 3.2.9 also holds for $T = \infty$ because the constant $C$ is independent of $T$. Then for any $y \in L^{p'}(U)$ Hölder's inequality and Theorem 3.2.9 imply

$$\Big|\Big\langle \int_0^\infty Ae^{-tA}x_0 \,\mathrm{d}t, y\Big\rangle_{L^p(U)}\Big| = \Big|\int_0^\infty \langle A^{1/q}e^{-\frac{1}{2}tA}x_0, (A')^{1/q'}e^{-\frac{1}{2}tA'}y\rangle_{L^p(U)}\,\mathrm{d}t\Big|$$
$$\le 2\int_U\int_0^\infty |A^{1/q}e^{-tA}x_0(A')^{1/q'}e^{-tA'}y|\,\mathrm{d}t\,\mathrm{d}\mu$$
$$\le 2\|A^{1/q}e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,\infty))}\|(A')^{1/q'}e^{-(\cdot)A'}y\|_{L^{p'}(U;L^{q'}[0,\infty))}$$
$$\le 2C\|A^{1/q}e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,\infty))}\|y\|_{L^{p'}(U)}.$$

Now we use that $x_0 = \int_0^\infty Ae^{-tA}x_0\,\mathrm{d}t$ to obtain

$$\|x_0\|_{L^p(U)} \le 2C\|A^{1/q}e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,\infty))}.$$

This concludes the proof. □

For the rest of this section we want to study Sobolev regularity in time. Again we start with an elementary estimate using a similar approach as in Proposition 3.2.5.

**PROPOSITION 3.2.11.** *Let $p, q \in [1, \infty)$, $\sigma \in (0,1)$, $\beta \in \mathbb{R}$ such that $0 \leq \beta + \sigma < 1/q$, and $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be sectorial of angle $\omega(A) < \pi/2$ with $0 \in \rho(A)$. Then there exists a constant $C > 0$ such that*

$$\|A^\beta e^{-(\cdot)A} x_0\|_{L^p(U;W^{\sigma,q}[0,T])} \leq C(T^{1/q-\beta} + T^{1/q-\beta-\sigma})\|x_0\|_{L^p(U)}$$

*for $x_0 \in L^p(U)$.*

**PROOF.** We first prove it for $\beta \geq 0$. If we take any $t \in [0,T]$, then the functional calculus for sectorial operators implies

$$A^\beta e^{-tA} x_0 = \frac{1}{2\pi i} \int_{\partial\Sigma_{\alpha'}} \lambda^\beta e^{-t\lambda} R(\lambda, A) x_0 \,\mathrm{d}\lambda,$$

for some $\alpha' \in (\omega(A), \pi/2)$, in particular,

$$d_{W^{\sigma,q}}\big[A^\beta e^{-(\cdot)A} x_0\big](h,t) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\alpha'}} \lambda^\beta d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}](h,t) R(\lambda, A) x_0 \,\mathrm{d}\lambda.$$

Therefore, we first compute $\|d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}]\|_{L^q[0,T]^2}$. Since

$$d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}](h,t) = \mathbb{1}_{[0,T-h]}(t) \frac{1}{h^{1/q+\sigma}} e^{-t\lambda}(e^{-h\lambda} - 1),$$

we have

$$\|d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}]\|_{L^q[0,T]^2} \leq \|e^{-(\cdot)\lambda}\|_{L^q[0,T]} \big\|\tfrac{1}{h^{1/q+\sigma}}(e^{-h\lambda} - 1)\big\|_{L^q_{(h)}[0,T]}$$

We take $c = c_{q,\alpha'} := \max\{2, \frac{1}{q\cos(\alpha')}\}$. Using that $|e^{-h\lambda} - 1| \leq c \wedge |h\lambda|$, we estimate

$$\begin{aligned}\int_0^T \frac{1}{h^{1+\sigma q}} |e^{-h\lambda} - 1|^q \,\mathrm{d}h &\leq \int_0^{T\wedge\frac{c}{|\lambda|}} h^{(1-\sigma)q-1} |\lambda|^q \,\mathrm{d}h + \int_{T\wedge\frac{c}{|\lambda|}}^T h^{-\sigma q-1} c^q \,\mathrm{d}h \\ &= \frac{1}{(1-\sigma)q} |\lambda|^q (T \wedge \tfrac{c}{|\lambda|})^{(1-\sigma)q} + \frac{c^q}{\sigma q}\big((T \wedge \tfrac{c}{|\lambda|})^{-\sigma q} - T^{-\sigma q}\big).\end{aligned}$$

Moreover, we have

$$\|e^{-(\cdot)\lambda}\|^q_{L^q[0,T]} = \big(\tfrac{1}{q\mathrm{Re}\,\lambda} - \tfrac{1}{q\mathrm{Re}\,\lambda} e^{-T\mathrm{Re}\,\lambda q}\big) \leq (T \wedge \tfrac{1}{q\mathrm{Re}\,\lambda}) = (T \wedge \tfrac{1}{q\cos(\alpha')|\lambda|}) \leq (T \wedge \tfrac{c}{|\lambda|}),$$

where we used that $\mathrm{Re}\,\lambda = \cos(\alpha')|\lambda|$ if $\lambda \in \partial\Sigma_{\alpha'}$. For $|\lambda| \geq \frac{c}{T}$ the calculations above yield

$$\|d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}]\|_{L^q[0,T]^2} \leq \frac{c^{1+1/q}}{((1-\sigma)\sigma q)^{1/q}} |\lambda|^{\sigma-1/q}.$$

And for $|\lambda| \le \frac{c}{T}$ we obtain

$$\|d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}]\|_{L^q[0,T]^2} \le \frac{c}{((1-\sigma)q)^{1/q}} T^{1/q-\sigma}.$$

Using the parametrization $\partial\Sigma_{\alpha'} = \{|\rho| e^{-i\operatorname{sign}(t)\alpha'} \colon \rho \in \mathbb{R}\}$ we finally get

$$\begin{aligned}
&\|d_{W^{\sigma,p}}\big[A^\beta e^{-(\cdot)A} x_0\big]\|_{L^p(U;L^q[0,T]^2)} \\
&\lesssim_{\alpha',\sigma,q} \frac{2}{2\pi} T^{1/q-\sigma} \int_0^{c/T} \rho^{\beta-1} \|x_0\|_{L^p(U)} \,\mathrm{d}\rho + \frac{2}{2\pi} \int_{c/T}^\infty \rho^{\beta+\sigma-1/q-1} \|x_0\|_{L^p(U)} \,\mathrm{d}\rho \\
&= \Big( \frac{c^\beta}{\pi\beta} T^{1/q-\beta-\sigma} + \frac{c^{\beta+\sigma-1/q}}{1/q-\beta-\sigma} T^{1/q-\beta-\sigma} \Big) \|x_0\|_{L^p(U)}.
\end{aligned}$$

Together with Proposition 3.2.5 this leads to the claim.

If $\beta < 0$ we cannot use the representation formula of the functional calculus. Instead we define the paths

$$\begin{aligned}
\Gamma_1(R,\theta) &:= \{\lambda \in \mathbb{C} \colon \lambda = \gamma_1(\rho) = -\rho e^{i\theta}, \rho \in (-\infty, -R)\}, \\
\Gamma_2(R,\theta) &:= \{\lambda \in \mathbb{C} \colon \lambda = \gamma_2(\varphi) = R e^{-i\varphi}, \varphi \in (-\theta, \theta)\}, \\
\Gamma_3(R,\theta) &:= \{\lambda \in \mathbb{C} \colon \lambda = \gamma_3(\rho) = \rho e^{-i\theta}, \rho \in (R, \infty)\},
\end{aligned}$$

and $\Gamma(R,\theta) := \Gamma_1(R,\theta) + \Gamma_2(R,\theta) + \Gamma_3(R,\theta)$. Then, by Example 9.8 in [59] we have

$$e^{-tA} x_0 = \frac{1}{2\pi i} \int_{\Gamma(R,\theta)} e^{-t\lambda} R(\lambda, A) x_0 \,\mathrm{d}\lambda, \quad t > 0,$$

as long as $R$ is small enough. Moreover, the representation is independent of $R$ and $\theta \in (\omega(A), \pi/2)$. We choose $R = \frac{\varepsilon}{T}$, for $\varepsilon > 0$ sufficiently small. Then

$$\begin{aligned}
A^\beta e^{-tA} x_0 &= \frac{1}{\Gamma(-\beta)} \int_0^\infty s^{-\beta-1} e^{-sA} e^{-tA} x_0 \,\mathrm{d}s \\
&= \frac{1}{\Gamma(-\beta)} \int_0^\infty s^{-\beta-1} \frac{1}{2\pi i} \int_{\Gamma(\frac{\varepsilon}{T},\theta)} e^{-(s+t)\lambda} R(\lambda, A) x_0 \,\mathrm{d}\lambda \,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&d_{W^{\sigma,q}}\big[A^\beta e^{-(\cdot)A} x_0\big](h,t) \\
&\quad = \frac{1}{\Gamma(-\beta)} \int_0^\infty s^{-\beta-1} \frac{1}{2\pi i} \int_{\Gamma(\frac{\varepsilon}{T},\theta)} e^{-s\lambda} d_{W^{\sigma,q}}[e^{-(\cdot)\lambda}](h,t) R(\lambda, A) x_0 \,\mathrm{d}\lambda \,\mathrm{d}s.
\end{aligned}$$

Using the same computation as above as well as

$$\frac{1}{\Gamma(-\beta)} \int_0^\infty s^{-\beta-1} e^{-s\operatorname{Re}\lambda} \,\mathrm{d}s = (\operatorname{Re}\lambda)^\beta = \cos(\arg(\lambda))^\beta |\lambda|^\beta,$$

we arrive at

$$
\begin{aligned}
&\|d_{W^{\sigma,q}}\big[A^{\beta}e^{-(\cdot)A}x_0\big]\|_{L^p(U;L^q[0,T]^2)} \lesssim_\theta \frac{2}{2\pi}\int_{\frac{\varepsilon}{T}}^{\infty} \rho^{\beta-1}\|d_{W^{\sigma,q}}[e^{-(\cdot)\gamma_1(\rho)}]\|_{L^q[0,T]^2}\,\mathrm{d}\rho\ \|x_0\|_{L^p(U)}\\
&\quad + \frac{1}{2\pi}\int_{-\theta}^{\theta}\big(\tfrac{\varepsilon}{T}\big)^{\beta-1}\|d_{W^{\sigma,q}}[e^{-(\cdot)\gamma_2(\varphi)}]\|_{L^q[0,T]^2}\tfrac{\varepsilon}{T}\,\mathrm{d}\varphi\ \|x_0\|_{L^p(U)}\\
&\lesssim_{\sigma,q} \frac{1}{\pi}\int_{\frac{\varepsilon}{T}}^{\infty}\rho^{\beta+\sigma-1/q-1}\,\mathrm{d}\rho\ \|x_0\|_{L^p(U)} + \frac{1}{2\pi}\int_{-\theta}^{\theta}\varepsilon^{\beta}T^{1/q-\sigma-\beta}\,\mathrm{d}\varphi\ \|x_0\|_{L^p(U)}\\
&\le \Big(\frac{1}{\pi}\frac{\varepsilon^{\beta+\sigma-1/q}}{1/q-\beta-\sigma}+\varepsilon^{\beta}\Big)T^{1/q-\sigma-\beta}\|x_0\|_{L^p(U)}. \qquad \square
\end{aligned}
$$

If we assume slightly more on the operator $A$ than sectoriality, we obtain stronger results similar to Lemma 3.2.6. In the following we will say that a sectorial operator $A$ has *bounded imaginary powers* or property *BIP*, if $A^{it}$, $t\in\mathbb{R}$, are bounded operators and there are constants $c,\omega>0$ such that

$$\|A^{it}\| \le ce^{\omega|t|},\quad t\in\mathbb{R}.$$

Operators having this property are, for example, operators with a bounded $H^\infty$ functional calculus.

**Proposition 3.2.12.** *Let $p,q\in[1,\infty)$, $\beta>1/q$, $\sigma\in(0,1)$, and $A\colon D(A)\subseteq L^p(U)\to L^p(U)$ be $\mathcal{R}_q$-sectorial of angle $\omega_{\mathcal{R}_q}(A)<\pi/2$ with $0\in\rho(A)$ and such that $A^{L^q}$ has BIP. Then*

$$\|A^{\beta-\sigma}e^{-(\cdot)A}x_0\|_{L^p(U;W^{\sigma,q}[0,T])} \lesssim \|A^{\beta}e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])} \lesssim \|x_0\|_{\beta-1/q,\ell^q}$$

*for each $x_0\in(L^p(U),D(A))_{\beta-1/q,\ell^q}$.*

**Proof.** On $L^p(U;L^q[0,T])$ we define $B=\frac{\mathrm{d}}{\mathrm{d}t}$ with $D(B)=L^p(U;W^{1,q}[0,T]$. Then $B$ has property BIP (this follows e.g. from [63, Proposition 4.23]). Now [68, Theorem 2.1], the mixed derivative theorem due to Sobolevskiĭ (see [73]), and Remark 2.4.6 imply

$$
\begin{aligned}
\|A^{\beta-\sigma}e^{-(\cdot)A}x_0\|_{L^p(U;W^{\sigma,q}[0,T])} &= \|A^{1-\sigma}B^{\sigma}\big(A^{\beta-1}e^{-(\cdot)A}x_0\big)\|_{L^p(U;L^q[0,T])}\\
&\le C\|A^{\beta}e^{-(\cdot)A}x_0 + A^{\beta-1}Be^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])}\\
&\le 2C\|A^{\beta}e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])}.
\end{aligned}
$$

The final estimate follows from Lemma 3.2.6. $\square$

As a consequence of Proposition 3.2.12 and Theorem 3.2.9 we obtain even stronger results assuming an $\mathcal{R}_q$-bounded $H^\infty$ calculus.

**THEOREM 3.2.13.** *Let $p \in [1,\infty)$, $q \in [2,\infty)$, $\beta \geq 1/q$, $\sigma \in (0,1)$, and let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus of angle $\alpha \in (\omega_{\mathcal{R}_q}(A), \pi/2)$ with $0 \in \rho(A)$. Then*

$$\|A^{\beta-\sigma}e^{-(\cdot)A}x_0\|_{L^p(U;W^{\sigma,q}[0,T])} \lesssim \|A^\beta e^{-(\cdot)A}x_0\|_{L^p(U;L^q[0,T])} \lesssim \|A^{\beta-1/q}x_0\|_{L^p(U)}$$

*for each $x_0 \in D(A^{\beta-1/q})$.*

**PROOF.** Since $A^{L^q}$ has a bounded $H^\infty$ calculus by Theorem 2.4.5, $A^{L^q}$ also has property BIP. Therefore, the first estimate follows from Proposition 3.2.12, and the second one from Theorem 3.2.9. □

## 3.3 Deterministic Convolutions

We start with a more or less easy result, but this already indicates the problems we face by giving strong estimates for convolution terms.

**PROPOSITION 3.3.1.** *Let $p, q, r \in [1,\infty)$ and $\beta \in [0,1)$. Let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\ell^q$-sectorial of angle $\omega_{\ell^q}(A) < \pi/2$ with $0 \in \rho(A)$, and $\phi\colon \Omega \times [0,T] \to L^p(U)$ be such that $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. Then the convolution process*

$$\Phi(t) := \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

*is well-defined, takes values in $D(A^\beta)$ almost surely and*

$$\mathbb{E}\|A^\beta \Phi\|^r_{L^p(U;L^q[0,T])} \leq C^r T^{(1-\beta)r}\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])},$$

*where $C = C(\beta)$ and $\lim_{\beta\to 1} C(\beta) = \infty$.*

**PROOF.** We define for $\theta \in (\omega_{\ell^q}(A), \pi/2)$ the path $\Gamma(\theta) := \{\gamma(\rho) := |\rho|e^{-i\operatorname{sign}(\rho)\theta}\colon \rho \in \mathbb{R}\} = \partial\Sigma_\theta$. We only show the case $\beta \in (0,1)$. For $\beta = 0$ we proceed similarly to Lemma 3.2.1 by using the path $\Gamma'(\theta) := \partial\big(\Sigma_\theta \cup B(0,\frac{1}{T})\big)$ instead of $\Gamma(\theta)$. By the functional calculus for sectorial operators we have

$$A^\beta e^{-(t-s)A}\phi(s) = \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta e^{-(t-s)\lambda}R(\lambda,A)\phi(s)\,\mathrm{d}\lambda, \quad s \in [0,t],$$

where the representation is independent of $\theta$. Now observe that for each $\lambda \in \Gamma(\theta)$ we have $\operatorname{Re}\lambda = \cos(\theta)|\lambda|$ and therefore

$$\|e^{-(\cdot)\operatorname{Re}\lambda}\|_{L^1[0,T]} = \frac{1}{\operatorname{Re}\lambda}(1 - e^{-T\operatorname{Re}\lambda}) \leq T \wedge \frac{1}{\operatorname{Re}\lambda} = T \wedge \frac{1}{\cos(\theta)|\lambda|}.$$

We also have

$$\Big|\int_0^t A^\beta e^{-(t-s)A}\phi(s)\,\mathrm{d}s\Big| = \Big|\int_0^t \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta e^{-(t-s)\lambda}R(\lambda,A)\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}s\Big|$$
$$\leq \int_0^t \frac{2}{2\pi}\int_0^\infty |\gamma(\rho)|^\beta e^{-(t-s)\mathrm{Re}\,\gamma(\rho)}\big|R(\gamma(\rho),A)\phi(s)\big|\,\mathrm{d}\rho\,\mathrm{d}s$$
$$= \frac{1}{\pi}\int_0^\infty\int_0^t |\gamma(\rho)|^\beta e^{-(t-s)\mathrm{Re}\,\gamma(\rho)}\big|R(\gamma(\rho),A)\phi(s)\big|\,\mathrm{d}s\,\mathrm{d}\rho.$$

By Young's inequality we thus arrive at

$$\Big\|\int_0^t A^\beta e^{-(t-s)A}\phi(s)\,\mathrm{d}s\Big\|_{L^q_{(t)}[0,T]} = \Big\|\int_0^t \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta e^{-(t-s)\lambda}R(\lambda,A)\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}s\Big\|_{L^q_{(t)}[0,T]}$$
$$\leq \frac{1}{\pi}\int_0^\infty \rho^\beta\big\|e^{-(\cdot)\mathrm{Re}\,\gamma(\rho)}\big\|_{L^1[0,T]}\big\|R(\gamma(\rho),A)\phi\big\|_{L^q[0,T]}\,\mathrm{d}\rho$$
$$\leq \frac{1}{\pi}\int_0^\infty \rho^\beta\big(T\wedge \tfrac{1}{\cos(\theta)\rho}\big)\big\|R(\gamma(\rho),A)\phi\big\|_{L^q[0,T]}\,\mathrm{d}\rho.$$

Using now the $\ell^q$-sectoriality of $A$, we obtain

$$\Big\|\int_0^t A^\beta e^{-(t-s)A}\phi(s)\,\mathrm{d}s\Big\|_{L^p(U;L^q_{(t)}[0,T])}$$
$$= \Big\|\int_0^t \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta e^{-(t-s)\lambda}R(\lambda,A)\phi(s)\,\mathrm{d}\lambda\,\mathrm{d}s\Big\|_{L^p(U;L^q_{(t)}[0,T])}$$
$$\leq \Big(\frac{C_\theta}{\pi}\int_0^{\frac{1}{T\cos\theta}} \rho^{\beta-1}T\,\mathrm{d}\rho + \frac{C_\theta}{\pi\cos(\theta)}\int_{\frac{1}{T\cos\theta}}^\infty \rho^{\beta-2}\,\mathrm{d}\rho\Big)\|\phi\|_{L^p(U;L^q[0,T])}$$
$$= \frac{C_\theta}{\pi\cos(\theta)^\beta}\frac{1}{(1-\beta)\beta}T^{1-\beta}\|\phi\|_{L^p(U;L^q[0,T])}.$$

Applying these estimates pointwise for each $\omega\in\Omega$ we finally obtain a constant $C = C(\beta)$ such that

$$\mathbb{E}\|A^\beta\Phi\|^r_{L^p(U;L^q[0,T])} \leq C^r T^{(1-\beta)r}\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}. \qquad \square$$

In a similar way we deduce a Sobolev regularity result.

**Proposition 3.3.2.** *Let $p,q,r\in[1,\infty)$ and $\alpha,\beta\in[0,1)$ such that $\alpha+\beta<1$. Let $A\colon D(A)\subseteq L^p(U)\to L^p(U)$ be $\ell^q$-sectorial of angle $\omega_{\ell^q}(A)<\pi/2$ with $0\in\rho(A)$, and $\phi\colon\Omega\times[0,T]\to L^p(U)$ be such that $\phi\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$. Then the convolution process $\Phi$ of Proposition 3.3.1 has the following property:*

$$\mathbb{E}\|A^\beta\Phi\|^r_{L^p(U;W^{\alpha,q}[0,T])} \leq C^r(T^{1-\beta}+T^{1-\alpha-\beta})^r\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])},$$

*where $C = C(\alpha,\beta)>0$ and $\lim_{\alpha+\beta\to1}C(\alpha,\beta)=\infty$.*

**PROOF.** We use the same path $\Gamma(\theta)$ for some $\theta \in (\omega_{\ell^q}(A), \pi/2)$ as in the proof of Proposition 3.3.1. In particular, we have the same formula for $A^\beta e^{-(t-s)A}\phi(s)$. For the moment let $\psi \in L^q[0,T]$ be arbitrary (later it will be replaced by $R(\lambda, A)\phi$). Then

$$d_{W^{\alpha,q}}\Big[\int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big] = \frac{1}{h^{1/q+\alpha}}\Big(\int_0^{t+h} e^{-(t+h-s)\lambda}\psi(s)\,\mathrm{d}s - \int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big)$$
$$= \frac{1}{h^{1/q+\alpha}}\Big(\int_t^{t+h} e^{-(t+h-s)\lambda}\psi(s)\,\mathrm{d}s + \int_0^t \big(e^{-(t+h-s)\lambda} - e^{-(t-s)\lambda}\big)\psi(s)\,\mathrm{d}s\Big)$$
$$= \frac{1}{h^{1/q+\alpha}}\Big(e^{-h\lambda}\int_{\mathbb{R}} \mathbb{1}_{[-h,0]}(t-s)e^{-(t-s)\lambda}\mathbb{1}_{[0,T]}(s)\psi(s)\,\mathrm{d}s + (e^{-h\lambda}-1)\int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big).$$

An application of Young's inequality therefore gives

$$\Big\|d_{W^{\alpha,q}}\Big[\int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big]\Big\|_{L^q_{(t)}[0,T]}$$
$$\le \frac{1}{h^{1/q+\alpha}}\big(e^{-h\operatorname{Re}\lambda}\|e^{-(\cdot)\lambda}\|_{L^1[-h,0]} + |e^{-h\lambda}-1|\|e^{-(\cdot)\lambda}\|_{L^1[0,T]}\big)\|\psi\|_{L^q[0,T]}$$
$$= \frac{1}{h^{1/q+\alpha}}\frac{1}{\operatorname{Re}\lambda}\big((1-e^{-h\operatorname{Re}\lambda}) + |e^{-h\lambda}-1|(1-e^{-T\operatorname{Re}\lambda})\big)\|\psi\|_{L^q[0,T]}.$$

If we set $c := \max\{2, \frac{1}{\cos(\theta)}\}$, we use

$$\tfrac{1}{\operatorname{Re}\lambda}(1-e^{-h\operatorname{Re}\lambda}) \le h \wedge \tfrac{c}{|\lambda|}, \quad |e^{-h\lambda}-1| \le |h\lambda| \wedge c, \quad \tfrac{1}{\operatorname{Re}\lambda}(1-e^{-T\operatorname{Re}\lambda}) \le \tfrac{c}{|\lambda|} \wedge T,$$

for $\lambda \in \Gamma(\theta)$ to estimate the following integrals:

$$\int_0^T \tfrac{1}{\operatorname{Re}\lambda^q}(1-e^{-h\operatorname{Re}\lambda})^q h^{-1-\alpha q}\,\mathrm{d}h \le \int_0^{\frac{c}{|\lambda|}\wedge T} h^{q-1-\alpha q}\,\mathrm{d}h + \int_{\frac{c}{|\lambda|}\wedge T}^T \frac{c^q}{|\lambda|^q}h^{-1-\alpha q}\,\mathrm{d}h$$
$$= \frac{1}{(1-\alpha)q}\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(1-\alpha)q} + \frac{1}{\alpha q}\frac{c^q}{|\lambda|^q}\Big(\big(\tfrac{c}{|\lambda|}\wedge T\big)^{-\alpha q} - T^{-\alpha q}\Big),$$

and similarly

$$\int_0^T |e^{-h\lambda}-1|^q \tfrac{1}{\operatorname{Re}\lambda^q}(1-e^{-T\operatorname{Re}\lambda})^q h^{-1-\alpha q}\,\mathrm{d}h$$
$$\le \big(\tfrac{c}{|\lambda|}\wedge T\big)^q\Big(\int_0^{\frac{c}{|\lambda|}\wedge T} h^{q-1-\alpha q}|\lambda|^q\,\mathrm{d}h + \int_{\frac{c}{|\lambda|}\wedge T}^T c^q h^{-1-\alpha q}\,\mathrm{d}h\Big)$$
$$= \frac{1}{(1-\alpha)q}\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(2-\alpha)q}|\lambda|^q + \frac{c^q}{\alpha q}\Big(\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(1-\alpha)q} - \big(\tfrac{c}{|\lambda|}\wedge T\big)^q T^{-\alpha q}\Big).$$

With these calculations we obtain for $T \le \frac{c}{|\lambda|}$ the estimate

$$\Big\|d_{W^{\alpha,q}}\Big[\int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big]\Big\|_{L^q[0,T]^2} \le \frac{(1+c)c^{1-\alpha}}{((1-\alpha)q)^{1/q}}\big(\tfrac{T}{c}\big)^{1-\alpha}\|\psi\|_{L^q[0,T]},$$

and for $T \geq \frac{c}{|\lambda|}$

$$\Big\| d_{W^{\alpha,q}}\Big[\int_0^t e^{-(t-s)\lambda}\psi(s)\,\mathrm{d}s\Big]\Big\|_{L^q[0,T]^2} \leq \frac{(1+c)c^{1-\alpha}}{((1-\alpha)\alpha q)^{1/q}}|\lambda|^{-(1-\alpha)}\|\psi\|_{L^q[0,T]}.$$

Here we finish our preliminary calculations. Since

$$d_{W^{\alpha,q}}[A^\beta\Phi] = \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta d_{W^{\alpha,q}}\Big[\int_0^t e^{-(t-s)\lambda}R(\lambda,A)\phi(s)\,\mathrm{d}s\Big]\,\mathrm{d}\lambda,$$

the work above yields a constant $C_\alpha$ such that

$$\begin{aligned}\|d_{W^{\alpha,q}}[A^\beta\Phi]\|_{L^p(U;L^q[0,T]^2)} &\leq \frac{2}{2\pi}\int_0^\infty C_\alpha\rho^\beta\big(\tfrac{T}{c}\wedge\tfrac{1}{\rho}\big)^{1-\alpha}\|R(\gamma(\rho),A)\phi\|_{L^p(U;L^q[0,T])}\,\mathrm{d}\rho\\ &\leq \frac{C_\alpha C_\theta}{\pi}\Big(\int_0^{c/T} c^{\alpha-1}T^{1-\alpha}\rho^{\beta-1}\,\mathrm{d}\rho + \int_{c/T}^\infty \rho^{\alpha+\beta-2}\,\mathrm{d}\rho\Big)\|\phi\|_{L^p(U;L^q[0,T])}\\ &= \frac{C_\alpha C_\theta}{c^{1-\alpha-\beta}\pi}\big(\tfrac{1}{\beta}+\tfrac{1}{1-(\alpha+\beta)}\big)T^{1-\alpha-\beta}\|\phi\|_{L^p(U;L^q[0,T])}.\end{aligned}$$

Finally, applying this pointwise for each $\omega\in\Omega$ and using Proposition 3.3.1 we can choose a constant $C = C(\alpha,\beta) > 0$ such that

$$\begin{aligned}\|A^\beta\Phi\|_{L^r(\Omega;L^p(U;W^{\alpha,q}[0,T]))} &\leq \|A^\beta\Phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \|d_{W^{\alpha,q}}[A^\beta\Phi]\|_{L^r(U;L^p(U;L^q[0,T]^2))}\\ &\leq C(T^{1-\beta}+T^{1-\alpha-\beta})\|\phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))}.\end{aligned}$$ □

Using Sobolev embedding results (see e.g. [72, Corollary 26]), we obtain:

**COROLLARY 3.3.3 (Hölder regularity).** *Assume the assumptions of the previous proposition and let $\alpha\in(1/q,1)$, then there exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$\mathbb{E}\|A^\beta\Phi\|^r_{L^p(U;C^{\alpha-1/q}[0,T])} \leq C^r(T^{1-\beta}+T^{1-\alpha-\beta})^r\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

The next result is a consequence of Theorem 2.5.9.

**COROLLARY 3.3.4.** *In addition to the assumptions of the previous corollary, we assume that $A$ is $\mathcal{R}_q$-sectorial. Then there exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$\mathbb{E}\|A^\beta\Phi\|^r_{C([0,T];(L^p(U),D(A))_{\alpha-1/q})} \leq C^r(T^{1-\beta}+T^{1-\alpha-\beta})^r\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

**PROOF.** Since $\alpha+\beta<1$, Theorem 2.5.9, Proposition 3.3.1 and 3.3.2 imply

$$\begin{aligned}\mathbb{E}\|A^\beta\Phi\|^r_{C([0,T];(L^p(U),D(A))_{\alpha-1/q})} &\lesssim \mathbb{E}\|A^\beta\Phi\|^r_{L^p(U;W^{\alpha,q}[0,T])} + \mathbb{E}\|A^{\alpha+\beta}\Phi\|^r_{L^p(U;L^q[0,T])}\\ &\lesssim (T^{1-\beta}+T^{1-\alpha-\beta})^r\mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.\end{aligned}$$ □

To obtain stronger estimates for deterministic convolutions (i.e. the borderline cases $\beta = 1$ or $\alpha + \beta = 1$, respectively) we therefore have to approach in a different way. In doing this, the following lemma will play the central role.

**Lemma 3.3.5.** *Let $q \in (1,\infty)$, $\sigma \in (0,1)$, and $(\delta_n)_{n=1}^\infty \subseteq (0,\infty)$. Then the following assertions hold:*

*a) The operator*

$$A_\delta \colon L^q([0,T];\ell^2) \to L^q([0,T];\ell^2), \quad (A_\delta f)(t,n) = \frac{1}{\delta_n} \int_{(t-\delta_n)\vee 0}^t f_n \,\mathrm{d}s$$

*is well-defined and*

$$\|A_\delta f\|_{L^q([0,T];\ell^2)} \lesssim_q \|f\|_{L^q([0,T];\ell^2)}.$$

*b) Let $q \geq 2$. The operator $B_\delta^\sigma \colon L^q([0,T];\ell^2) \to L^q([0,T]^2;\ell^2)$ given by*

$$(B_\delta^\sigma f)(h,t,n) = \mathbb{1}_{[0,T-h]}(t) \frac{1}{\delta_n^{1-\sigma}} \frac{1}{h^{1/q+\sigma}} \int_0^T \left|\mathbb{1}_{[(t+h-\delta_n)\vee 0, t+h]} - \mathbb{1}_{[(t-\delta_n)\vee 0,t]}\right| f_n \,\mathrm{d}s$$

*is well-defined and*

$$\|B_\delta^\sigma f\|_{L^q([0,T]^2;\ell^2)} \lesssim_{q,\sigma} \|f\|_{L^q([0,T];\ell^2)}.$$

**Proof.** Let us start with a small remark. If we define by

$$M \colon L^q(\mathbb{R}) \to L^q(\mathbb{R}), \quad Mg(t) := \sup_{B \ni t} \frac{1}{|B|} \int_B g(s)\,\mathrm{d}s,$$

the Hardy-Littlewood maximal operator, then $M$ is bounded (see e.g. [74, Theorem 4.1]). By Theorem 2.1.6, $M$ is also $\mathcal{R}_2$-bounded which implies that $M$ has a bounded extension on $L^q(\mathbb{R};\ell^2)$, $q \in (1,\infty)$. Using this powerful tool, there is nearly nothing to prove.

a) We have for any $f \in L^q([0,T];\ell^2)$

$$\begin{aligned} \|A_\delta f\|_{L^q([0,T];\ell^2)} &= \Big\| \Big( \sum_{n=1}^\infty \Big| \frac{1}{\delta_n} \int_{(t-\delta_n)\vee 0}^t f_n \,\mathrm{d}s \Big|^2 \Big)^{1/2} \Big\|_{L^q[0,T]} \\ &\leq 2 \Big\| \Big( \sum_{n=1}^\infty \Big( \sup_{I_\delta \ni t} \frac{1}{2\delta} \int_{I_\delta} \mathbb{1}_{[0,T]} |f_n| \,\mathrm{d}s \Big)^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R})} \\ &= 2\|M(\mathbb{1}_{[0,T]} f)\|_{L^q(\mathbb{R};\ell^2)} \leq 2C_q \|f\|_{L^q([0,T];\ell^2)}, \end{aligned}$$

for some constant $C_q > 0$ only depending on $q$. Here, the supremum is taken over all intervals $I_\delta$ of length $2\delta$ containig $t$.

b) In this case we first observe that

$$\big|\mathbb{1}_{[t+h-\delta_n,t+h]} - \mathbb{1}_{[t-\delta_n,t]}\big| = \begin{cases} \mathbb{1}_{[t+h-\delta_n,t+h]} + \mathbb{1}_{[t-\delta_n,t]}, & \text{if } h > \delta_n, \\ \mathbb{1}_{[t-\delta_n,t-\delta_n+h]} + \mathbb{1}_{[t,t+h]}, & \text{if } h \le \delta_n. \end{cases}$$

Using this, we obtain pointwise on $[0,T]^2 \times \mathbb{N}$

$$\begin{aligned}\big|(B_\delta^\sigma f)(h,t,n)\big| &= \frac{1}{\delta_n^{1-\sigma}} \mathbb{1}_{[0,\delta_n]}(h) \frac{1}{h^{1/q+\sigma}} \Big( \int_{t-\delta_n}^{t-\delta_n+h} |f_n(s)|\,\mathrm{d}s + \int_t^{t+h} |f_n(s)|\,\mathrm{d}s \Big) \\ &\quad + \frac{1}{\delta_n^{1-\sigma}} \mathbb{1}_{[\delta_n,T]}(h) \frac{1}{h^{1/q+\sigma}} \Big( \int_{t+h-\delta_n}^{t+h} |f_n(s)|\,\mathrm{d}s + \int_{t-\delta_n}^{t} |f_n(s)|\,\mathrm{d}s \Big) \\ &\le 4\delta_n^{-1+\sigma} \mathbb{1}_{[0,\delta_n]}(h) h^{1-1/q-\sigma} (M|f_n|)(t) + 4\delta_n^\sigma \mathbb{1}_{[\delta_n,T]}(h) h^{-1/q-\sigma} (M|f_n|)(t).\end{aligned}$$

Applying now the $L^q[0,T]$ norm with respect to $h$, we have

$$\begin{aligned}\|(B_\delta^\sigma f)(\cdot,t,n)\|_{L^q[0,T]} &\le 4\delta_n^{-1+\sigma} (M|f_n|)(t) \Big( \int_0^{\delta_n} h^{q-1-\sigma q}\,\mathrm{d}h \Big)^{1/q} + 4\delta_n^\sigma (M|f_n|)(t) \Big( \int_{\delta_n}^T h^{-1-\sigma q}\,\mathrm{d}h \Big)^{1/q} \\ &\le \frac{4}{((1-\sigma)q)^{1/q}} (M|f_n|)(t) + \frac{4}{(\sigma q)^{1/q}} (M|f_n|)(t).\end{aligned}$$

We now use the same argument for $M$ as in the first case. This finally leads to

$$\begin{aligned}\|B_\delta^\sigma f\|_{L^q([0,T]^2;\ell^2)} &\le \big\| \|(B_\delta^\sigma f)(h,\cdot,\cdot)\|_{L^q_{(h)}[0,T]} \big\|_{L^q([0,T];\ell^2)} \\ &\le \Big( \frac{4}{((1-\sigma)q)^{1/q}} + \frac{4}{(\sigma q)^{1/q}} \Big) \|M|f_n|\|_{L^q([0,T];\ell^2)} \\ &\le C_q \Big( \frac{4}{((1-\sigma)q)^{1/q}} + \frac{4}{(\sigma q)^{1/q}} \Big) \|f\|_{L^q([0,T];\ell^2)}. \qquad \square\end{aligned}$$

The next step is now to use these bounded operators to show an $\mathcal{R}$-boundedness result for the following (deterministic) operator families

$$(D_\delta \phi)(t) := \frac{1}{\delta} \int_{(t-\delta)\vee 0}^t \phi\,\mathrm{d}s, \quad t \in [0,T],\ \delta > 0,$$

$$(D_\delta^\sigma \phi)(h,t) := \mathbb{1}_{[0,T-h]}(t) \delta^\sigma \frac{1}{h^{1/q+\sigma}} \big( (D_\delta\phi)(t+h) - (D_\delta\phi)(t) \big), \quad (h,t) \in [0,T]^2,\ \delta > 0,$$

where $\sigma \in (0,1)$.

**Proposition 3.3.6.** *For $q \in (1,\infty)$, $p, r \in [1,\infty)$, and $\sigma \in (0,1)$ the following assertions hold:*

a) *The operator family $(D_\delta)_{\delta>0}$ is $\mathcal{R}$-bounded on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$.*

b) *For $q \ge 2$, the operator family $(D_\delta^\sigma)_{\delta>0}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$.*

**Proof.** Let $(\delta_n)_{n=1}^N \subseteq (0,\infty)$, $(\phi_n)_{n=1}^N \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$, $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be a probability space, and $(\widetilde{r}_n)_{n=1}^N$ be a Rademacher sequence defined on this space.

a) Applying Lemma 3.3.5 a), we arrive at

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big\| \sum_{n=1}^N \widetilde{r}_n D_{\delta_n} \phi_n \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} &\eqsim_{p,q,r} \Big\| \Big( \sum_{n=1}^N |D_{\delta_n}\phi_n|^2 \Big)^{1/2} \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \\
&= \big\| A_\delta (\phi_n)_{n=1}^N \big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))} \\
&\lesssim_q \big\| (\phi_n)_{n=1}^N \big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))} \\
&= \Big\| \Big( \sum_{n=1}^N |\phi_n|^2 \Big)^{1/2} \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \\
&\eqsim_{p,q,r} \widetilde{\mathbb{E}}\Big\| \sum_{n=1}^N \widetilde{r}_n \phi_n \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}
\end{aligned}$$

b) Using now Lemma 3.3.5 b) we obtain

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big\| \sum_{n=1}^N \widetilde{r}_n D^\sigma_{\delta_n} \phi_n \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} &\eqsim_{p,q,r} \Big\| \Big( \sum_{n=1}^N |D^\sigma_{\delta_n}\phi_n|^2 \Big)^{1/2} \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} \\
&\leq \big\| B^\sigma_\delta (\phi_n)_{n=1}^N \big\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;\ell^2)))} \\
&\lesssim_{q,\sigma} \big\| (\phi_n)_{n=1}^N \big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))} \\
&= \Big\| \Big( \sum_{n=1}^N |\phi_n|^2 \Big)^{1/2} \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \\
&\eqsim_{p,q,r} \widetilde{\mathbb{E}}\Big\| \sum_{n=1}^N \widetilde{r}_n \phi_n \Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}
\end{aligned}$$

□

Using this proposition we can use functional calculi results to deduce estimates for deterministic convolutions. We first prove it for the scalar-valued case. For this purpose we define for $\sigma \in [0,1)$ the set

$$\mathcal{A}_\sigma := \Big\{ f\colon [0,\infty) \to \mathbb{C} \colon f \text{ is abs. continuous}, \lim_{t\to\infty} f(t) = 0, \text{ and } \int_0^\infty t^{1-\sigma} |f'(t)| \,\mathrm{d}t \leq 1 \Big\}.$$

In particular, we have $f(t) = -\int_t^\infty f'(s)\,\mathrm{d}s$ for each $f \in \mathcal{A}_\sigma$.

**Proposition 3.3.7 (The scalar-valued case).** *Let $q \in (1,\infty)$, $p, r \in [1,\infty)$, and $\sigma \in (0,1)$. Then we have:*

*a) The operator family $(C_{\mathrm{det}}(f))_{f\in\mathcal{A}_0}$ given by*

$$\big[C_{\mathrm{det}}(f)\phi\big](t) := \int_0^t f(t-s)\phi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

*is $\mathcal{R}$-bounded on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$.*

b) *Let $q \geq 2$. The operator family $(C^\sigma_{\det}(f))_{f\in\mathcal{A}_\sigma}$ given by*

$$\big[C^\sigma_{\det}(f)\phi\big](h,t) := \mathbb{1}_{[0,T-h]}(t)\frac{1}{h^{1/q+\sigma}}\big(\big[C_{\det}(f)\phi\big](t+h)-\big[C_{\det}(f)\phi\big](t)\big), \quad (h,t)\in[0,T]^2,$$

*is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$ to $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]^2))$.*

**Proof.** By Proposition 3.3.6 the maps $\delta\mapsto D_\delta\colon(0,\infty)\to\mathcal{B}\big(L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))\big)$ and $\delta\mapsto D^\sigma_\delta\colon(0,\infty)\to\mathcal{B}\big(L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T])),L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]^2))\big)$ have an $\mathcal{R}$-bounded range. Corollary 2.14 of [59] now implies that the operator families $\{T_h\colon \|h\|_{L^1}\leq 1\}$ and $\{T^\sigma_h\colon\|h\|_{L^1}\leq 1\}$ defined by

$$T_h\phi := \int_0^\infty h(\delta)D_\delta\phi\,\mathrm{d}\delta, \quad \phi\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T])),$$
$$T^\sigma_h\phi := \int_0^\infty h(\delta)D^\sigma_\delta\phi\,\mathrm{d}\delta, \quad \phi\in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T])),$$

for $h\in L^1(0,\infty)$ are also $\mathcal{R}$-bounded. The results a) and b) finally follow from the observations

$$\begin{aligned}\big[C_{\det}(f)\phi\big](t) &= -\int_0^t\int_{t-s}^\infty f'(\delta)\phi(s)\,\mathrm{d}\delta\,\mathrm{d}s\\ &= -\int_0^\infty f'(\delta)\int_{(t-\delta)\vee 0}^t\phi(s)\,\mathrm{d}s\,\mathrm{d}\delta\\ &= -\int_0^\infty \delta f'(\delta)(D_\delta\phi)(t)\,\mathrm{d}\delta,\end{aligned}$$

and similarly

$$\begin{aligned}\big[C^\sigma_{\det}(f)\phi\big](h,t) &= \mathbb{1}_{[0,T-h]}(t)\frac{1}{h^{1/q+\sigma}}\big(\big[C_{\det}(f)\phi\big](t+h)-\big[C_{\det}(f)\phi\big](t)\big)\\ &= -\mathbb{1}_{[0,T-h]}(t)\frac{1}{h^{1/q+\sigma}}\int_0^\infty f'(\delta)\Big(\int_{(t+h-\delta)\vee 0}^{t+h}\phi(s)\,\mathrm{d}s-\int_{(t-\delta)\vee 0}^t\phi(s)\,\mathrm{d}s\Big)\,\mathrm{d}\delta\\ &= -\int_0^\infty\delta^{1-\sigma}f'(\delta)(D^\sigma_\delta\phi)(h,t)\,\mathrm{d}\delta.\end{aligned}$$

□

**Corollary 3.3.8.** *Let $q\in(1,\infty)$, $p,r\in[1,\infty)$, and $\nu\in(0,\pi/2)$. For $\sigma\in[0,1)$ and $\mu\in\Sigma_\nu$ we define the function*

$$f^\sigma_\mu\colon[0,\infty)\to\mathbb{C}, \quad f^\sigma_\mu(t) := \mu^{1-\sigma}e^{-\mu t}.$$

*Then $\frac{\cos(\nu)^{2-\sigma}}{\Gamma(2-\sigma)}f^\sigma_\mu\in\mathcal{A}_\sigma$. As a consequence, the set $\{K_\mu := C_{\det}(f^0_\mu)\colon\mu\in\Sigma_\nu\}$ is $\mathcal{R}$-bounded on $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$, and for $\sigma\in(0,1)$ and $q\geq 2$ the set $\{K^\sigma_\mu := C^\sigma_{\det}(f^\sigma_\mu)\colon\mu\in\Sigma_\nu\}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$ to $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]^2))$.*

**Proof.** Since $\operatorname{Re}\mu > 0$ for $\mu \in \Sigma_\nu$, we have

$$|f_\mu^\sigma(t)| = |\mu|^{1-\sigma} e^{-t\operatorname{Re}\mu} \to 0 \quad \text{for } t \to \infty.$$

Moreover,

$$\begin{aligned}\int_0^\infty t^{1-\sigma}\left|\tfrac{\mathrm{d}}{\mathrm{d}t} f_\mu^\sigma(t)\right| \mathrm{d}t &= \int_0^\infty t^{1-\sigma}|\mu|^{2-\sigma} e^{-t\operatorname{Re}\mu}\,\mathrm{d}t \\ &\le \frac{1}{\cos(\nu)^{2-\sigma}} \int_0^\infty (t\operatorname{Re}\mu)^{1-\sigma}\operatorname{Re}\mu\, e^{-t\operatorname{Re}\mu}\,\mathrm{d}t \\ &= \frac{1}{\cos(\nu)^{2-\sigma}} \int_0^\infty s^{1-\sigma} e^{-s}\,\mathrm{d}s = \frac{\Gamma(2-\sigma)}{\cos(\nu)^{2-\sigma}},\end{aligned}$$

where we used that $\operatorname{Re}\mu = \cos(\arg(\mu))|\mu| \ge \cos(\nu)|\mu|$ in the second line. This implies the first claim. The $\mathcal{R}$-boundedness results finally follow from Proposition 3.3.7. □

To extend these results to the operator-valued case (i.e. if the function $f$ in Proposition 3.3.7 is replaced by a semigroup), we need the assumption that $A$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus.

**Theorem 3.3.9 (Deterministic maximal regularity).** *Let $q \in (1,\infty)$, $p, r \in [1,\infty)$, and $\sigma \in (0,1)$. Let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus of angle $\alpha < \pi/2$ with $0 \in \rho(A)$, and let $\phi\colon \Omega \times [0,T] \to L^p(U)$ be such that $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. Then the process*

$$\Phi(t) := \int_0^t e^{-(t-s)A}\phi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

*is well-defined, takes values in $D(A)$ almost surely, and*

$$\mathbb{E}\|A\Phi\|^r_{L^p(U;L^q[0,T])} \lesssim_{p,q,r} \mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

*Moreover, for $q \ge 2$, we have the following Sobolev regularity result*

$$\mathbb{E}\|A^{1-\sigma}\Phi\|^r_{L^p(U;W^{\sigma,q}[0,T])} \lesssim_{p,q,r,\sigma} (1+T^\sigma)^r\, \mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

**Proof.** By Theorem 2.4.5 the extension $A^{L^q}$ of $A$ has a bounded $H^\infty$ calculus, and by Corollary 3.3.8 the function $\mu \mapsto K_\mu$ is analytic on $\Sigma_\nu$, $\nu \in (\alpha, \pi/2)$, has an $\mathcal{R}$-bounded range, and obviously commutes with $R(\mu, A^{L^q})$. By Theorem 4.4 of [52] the map

$$\phi \mapsto \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\mu, A^{L^q}) K_\mu \phi\,\mathrm{d}\mu$$

defines a bounded operator on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$ for $\alpha' \in (\alpha,\nu)$. For the moment let

$\phi(\omega) = \sum_{n=1}^{N} v_n(\omega) \otimes \psi_n(\omega) \in L^p(U) \otimes L^q[0,T]$. Then, by Fubini's theorem, we obtain

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\mu, A^{L^q})K_\mu \phi \,\mathrm{d}\mu &= \frac{1}{2\pi i}\sum_{n=1}^{N}\int_{\partial\Sigma_{\alpha'}} K_\mu \psi_n R(\mu, A) v_n \,\mathrm{d}\mu \\
&= \frac{1}{2\pi i}\sum_{n=1}^{N}\int_{\partial\Sigma_{\alpha'}}\int_0^t \mu e^{-\mu(t-s)}\psi_n(s) R(\mu, A) v_n \,\mathrm{d}s\,\mathrm{d}\mu \\
&= \sum_{n=1}^{N}\int_0^t \psi_n(s)\frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} \mu e^{-\mu(t-s)} R(\mu, A) v_n \,\mathrm{d}\mu\,\mathrm{d}s \\
&= \int_0^t \sum_{n=1}^{N}\psi_n(s) A e^{-(t-s)A} v_n \,\mathrm{d}s \\
&= \int_0^t A e^{-(t-s)A}\big(\phi(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Using the boundedness result we get

$$\mathbb{E}\Big\| t \mapsto \int_0^t A e^{-(t-s)A}\big(\phi(s)\big)\,\mathrm{d}s \Big\|^r_{L^p(U;L^q[0,T])} \lesssim_{p,q,r} \mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}$$

The general case then follows by approximation and the convergence property of the $\mathcal{R}H^\infty$ functional calculus.

For the second part of the theorem, we remark that $A$ can also be extended to an operator $A^{L^q}$ on $L^p(U; L^q[0,T]^2)$ such that $A^{L^q}$ has a bounded $H^\infty(\Sigma_\alpha)$ calculus. Moreover, we define on this space the following operator

$$(B^\sigma_\mu \psi)(h,t) := (K^\sigma_\mu J\psi)(h,t), \quad \psi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2)),$$

where $(J\psi)(s) := \frac{1}{T}\int_0^T \psi(\tau, s)\,\mathrm{d}\tau$. Observe that $J\psi \in L^r(\Omega; L^p(U; L^q[0,T]))$ for $\psi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$, since by Hölder's inequality we have

$$\|J\psi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \le T^{-1/q}\|\psi\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}.$$

Then, by Corollary 3.3.8 there exists a constant $C > 0$ such that

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n B^\sigma_{\mu_n}\psi_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} &= \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n K^\sigma_{\mu_n} J\psi_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} \\
&\le C\,\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n J\psi_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \\
&\le CT^{-1/q}\,\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n \psi_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))},
\end{aligned}$$

for each finite sequences $(\mu_n)_{n=1}^N \subseteq \Sigma_\nu$, $(\psi_n)_{n=1}^N \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$, and each Rademacher sequence $(\widetilde{r}_n)_{n=1}^N$. In other words, the set $\{B^\sigma_\mu \colon \mu \in \Sigma_\nu\}$ is also $\mathcal{R}$-bounded.

Again, by Theorem 4.4 of [52] the linear map

$$\psi \mapsto \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q}) B^\sigma_\mu \psi \, \mathrm{d}\mu$$

defines a bounded operator on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$. Now take $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$ such that $\phi(\omega) \in L^p(U) \otimes L^q[0,T]$, and let $\psi(\tau, s) := \mathbb{1}_{[0,T]}(\tau)\phi(s)$. Then, of course, $\psi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$ and $(J\psi)(s) = \phi(s)$. Looking at the equality proven in the first case, we see that the operators $B^\sigma_\mu$, $\mu \in \Sigma_\nu$, have been chosen in such a way that

$$d_{W^{\sigma,q}}[A^{1-\sigma}U] = \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q}) B^\sigma_\mu \psi \, \mathrm{d}\mu$$

Using now the boundedness result as well as Proposition 3.3.1 we arrive at

$$\begin{aligned}
&\|A^{1-\sigma}\Phi\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} \leq \|A^{1-\sigma}\Phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \|d_{W^{\sigma,q}}[A^{1-\sigma}\Phi]\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} \\
&\quad = \|A^{1-\sigma}\Phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \left\| \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q}) B^\sigma_\mu \psi \, \mathrm{d}\mu \right\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} \\
&\quad \lesssim_{p,q,r,\sigma} T^\sigma \|\phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + T^{-1/q}\|\psi\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} \\
&\quad = (1 + T^\sigma)\|\phi\|_{L^r(\Omega;L^p(U;L^q[0,T]))},
\end{aligned}$$

The general case again follows by approximation. □

As in Corollary 3.3.3 we can apply Sobolev's embedding theorem to obtain:

**Corollary 3.3.10 (Hölder regularity).** *Under the assumptions of the previous theorem, we obtain for each $\alpha \in (1/q, 1)$, $q \in [2, \infty)$, a constant $C = C(p, q, r, \alpha) > 0$ such that*

$$\mathbb{E}\|A^{1-\alpha}\Phi\|^r_{L^p(U;C^{\alpha-1/q}[0,T])} \leq C^r(1 + T^\alpha)^r \mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

By replacing Proposition 3.3.1 and 3.3.2 with the results of Theorem 3.3.9 in the proof of Corollary 3.3.4 we get the following result.

**Corollary 3.3.11.** *Under the assumptions of Theorem 3.3.9, we obtain for each $\alpha \in (1/q, 1]$, $q \in [2, \infty)$, a constant $C = C(p, q, r, \alpha) > 0$ such that*

$$\mathbb{E}\|A^{1-\alpha}\Phi\|^r_{C([0,T];(L^p(U),D(A))_{\alpha-1/q})} \leq C^r(1 + T^\alpha)^r \mathbb{E}\|\phi\|^r_{L^p(U;L^q[0,T])}.$$

**Remark 3.3.12.** The results concerning Sobolev regularity (especially the second part of Theorem 3.3.9) might also be true for $q \in (1, 2)$. The problem lies in the $\mathcal{R}$-boundedness of certain multiplication operators. Following [41, Satz 4.4.4], this could be further observed. Since we do not need the case $q \in (1, 2)$ in the following part, we do not pursue this any further.

## 3.4 Stochastic Convolutions

By investigating the time regularity of stochastic evolution equations we started to study stochastic convolutions first. The ideas for the proofs in the previous section actually arose from the stochastic part. Here we saw how we should compare these two convolutions and that they are *nearly* the same. For easier reading we of course wanted to start with the more common Lebesgue integral. The case of the stochastic convolution is now very similar to the proofs of the previous section, but the reader should be aware of the fact that we started with this part and transferred it to deterministic convolutions much later.

We start to prove regularity results assuming only $\ell^q$-sectoriality of the operator $A$. One advantage is that we can familiarize with the stochastic convolution and recognize the differences to the deterministic case. The basis of the following result is the Itô isomorphism for mixed $L^p$ spaces.

**Proposition 3.4.1.** *Let $p, r \in (1,\infty)$, $q \in (2,\infty)$, and $\beta \in [0, 1/2)$. Let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\ell^q$-sectorial of angle $\omega_{\ell^q}(A) < \pi/2$ with $0 \in \rho(A)$, and $\phi_n\colon \Omega \times [0,T] \to L^p(U)$, $n \in \mathbb{N}$, be such that $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$. Then the convolution process*

$$\Psi(t) := \int_0^t e^{-(t-s)A}\boldsymbol{\phi}(s)\,\mathrm{d}\boldsymbol{\beta}(s), \quad t \in [0,T],$$

*is well-defined, takes values in $D(A^\beta)$ almost surely and*

$$\mathbb{E}\|A^\beta\Psi\|^r_{L^p(U;L^q[0,T])} \le C^r T^{(1/2-\beta)r}\mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))},$$

*where $C = C(p,q,r,\beta)$ and $\lim_{\beta\to 1/2} C(p,q,r,\beta) = \infty$.*

**Remark 3.4.2.** In comparison to Proposition 3.3.1 we see that two changes have been made. More precisely, we have the restrictions $q > 2$ and $\beta < 1/2$. If we assume that $A$ is $\mathcal{R}_q$-sectorial, the previous results would stay true even for $q \ge 2$ by Remark 3.2.4. However, if $q \in (1,2)$ the stochastic integral is no longer well-defined by Itô's isomorphism. This makes the requirement for $q \ge 2$ necessary. In return, this condition is responsible that we can only assume $\beta < 1/2$ as we will see in the proof.

**Proof (of Proposition 3.4.1).** By Remark 3.2.4 the process $\Psi(t)$ is well-defined for each fixed $t \in [0,T]$. Moreover, by Itô's isomorphism for mixed $L^p$ spaces (see Theorem 1.3.3) we have

$$\begin{aligned}\mathbb{E}\|A^\beta\Psi\|^r_{L^p(U;L^q[0,T])} &= \mathbb{E}\Big\|\int_0^T \mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\,\mathrm{d}\boldsymbol{\beta}(s)\Big\|^r_{L^p(U;L^q_{(t)}[0,T])}\\ &\eqsim_{p,q,r} \mathbb{E}\big\|\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big\|^r_{L^p(U;L^q_{(t)}([0,T];L^2_{(s)}([0,T]\times\mathbb{N})))}.\end{aligned}$$

We now want to take a closer look on the innermost norm, i.e. the $L^2([0,T]\times\mathbb{N})$ norm with respect to $s$ and $n$. Assume that $\beta>0$ (the case $\beta=0$ can then be shown in the same way as in Lemma 3.2.1). We define for $\theta\in(\omega_{\ell^q}(A),\pi/2)$ the path $\Gamma(\theta)$ as in Proposition 3.3.1, and recall that by the functional calculus for sectorial operators we have

$$A^\beta e^{-(t-s)A}\phi_n(s)=\frac{1}{2\pi i}\int_{\Gamma(\theta)}\lambda^\beta e^{-(t-s)\lambda}R(\lambda,A)\phi_n(s)\,\mathrm{d}\lambda,\quad s\in[0,t],\ n\in\mathbb{N},$$

where the representation is independent of $\theta$. By Minkowski's inequality we deduce that

$$\begin{aligned}\big\|\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big\|_{L^2_{(s)}([0,T]\times\mathbb{N})}&=\Big(\sum_{n=1}^\infty\int_0^t|A^\beta e^{-(t-s)A}\phi_n(s)|^2\,\mathrm{d}s\Big)^{1/2}\\ &\le\frac{2}{2\pi}\int_0^\infty|\gamma(\rho)|^\beta\Big(\sum_{n=1}^\infty\int_0^t e^{-2(t-s)\mathrm{Re}\,\gamma(\rho)}\big|R(\gamma(\rho),A)\phi_n(s)\big|^2\,\mathrm{d}s\Big)^{1/2}\mathrm{d}\rho\\ &=\frac{1}{\pi}\int_0^\infty|\gamma(\rho)|^\beta\Big(\int_0^t e^{-2(t-s)\mathrm{Re}\,\gamma(\rho)}\sum_{n=1}^\infty\big|R(\gamma(\rho),A)\phi_n(s)\big|^2\,\mathrm{d}s\Big)^{1/2}\mathrm{d}\rho.\end{aligned}$$

Now we apply Minkowski's inequality again for the $L^q[0,T]$ norm and then Young's inequality to obtain

$$\begin{aligned}&\big\|\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big\|_{L^q_{(t)}([0,T];L^2_{(s)}([0,T]\times\mathbb{N}))}\\ &\quad\le\frac{1}{\pi}\int_0^\infty|\gamma(\rho)|^\beta\Big\|\int_0^t e^{-2(t-s)\mathrm{Re}\,\gamma(\rho)}\sum_{n=1}^\infty\big|R(\gamma(\rho),A)\phi_n(s)\big|^2\,\mathrm{d}s\Big\|^{1/2}_{L^{q/2}_{(t)}[0,T]}\mathrm{d}\rho\\ &\quad\le\frac{1}{\pi}\int_0^\infty|\gamma(\rho)|^\beta\big\|e^{-2(\cdot)\mathrm{Re}\,\gamma(\rho)}\big\|^{1/2}_{L^1[0,T]}\Big\|\sum_{n=1}^\infty\big|R(\gamma(\rho),A)\phi_n\big|^2\Big\|^{1/2}_{L^{q/2}[0,T]}\mathrm{d}\rho\\ &\quad=\frac{1}{\pi}\int_0^\infty|\gamma(\rho)|^\beta\big\|e^{-2(\cdot)\mathrm{Re}\,\gamma(\rho)}\big\|^{1/2}_{L^1[0,T]}\big\|R(\gamma(\rho),A)\boldsymbol{\phi}\big\|_{L^q([0,T];\ell^2)}\,\mathrm{d}\rho.\end{aligned}$$

Next, we apply the $L^r(\Omega;L^p(U))$ norm on both sides. But first we make two remarks.

1) Note that $A$ can be extended to a sectorial operator on $L^p(U;L^q[0,T])$ by Theorem 2.4.5. Since $\ell^2$ is a Hilbert space, we can extend it again to a sectorial operator on $L^p(U;L^q([0,T];\ell^2))$.
2) We compute

$$\big\|e^{-2(\cdot)\mathrm{Re}\,\gamma(\rho)}\big\|^{1/2}_{L^1[0,T]}=\big(\tfrac{1}{2\mathrm{Re}\,\gamma(\rho)}(1-e^{-2T\mathrm{Re}\,\gamma(\rho)})\big)^{1/2}\le\tfrac{1}{(2\mathrm{Re}\,\gamma(\rho))^{1/2}}\wedge T^{1/2}.$$

Applying these remarks we obtain

$$\begin{aligned}&\big\|\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big\|_{L^r(\Omega;L^p(U;L^q_{(t)}([0,T];L^2_{(s)}([0,T]\times\mathbb{N}))))}\\ &\le\frac{C_\theta}{\pi}\int_0^\infty|\gamma(\rho)|^{\beta-1}\big((2\mathrm{Re}\,\gamma(\rho))^{-1/2}\wedge T^{1/2}\big)\,\mathrm{d}\rho\,\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\\ &=\frac{C_\theta}{\pi}\Big(\int_0^{\frac{1}{2\cos(\theta)T}}\rho^{\beta-1}T^{1/2}\,\mathrm{d}\rho+\int_{\frac{1}{2\cos(\theta)T}}^\infty\frac{1}{\sqrt{2\cos(\theta)}}\rho^{\beta-3/2}\,\mathrm{d}\rho\Big)\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\\ &=\frac{C_\theta}{(2\cos(\theta))^\beta\pi}\frac{1}{2\beta(1/2-\beta)}T^{1/2-\beta}\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}.\end{aligned}$$

Together with the estimate in the beginning, we obtain a constant $C = C(p,q,r,\beta) > 0$ such that

$$\|A^\beta \Psi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \eqsim_{p,q,r} \big\|\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big\|_{L^r(\Omega;L^p(U;L^q_{(t)}([0,T];L^2_{(s)}([0,T]\times\mathbb{N}))))}$$
$$\leq CT^{1/2-\beta}\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}. \qquad \square$$

Of course, we also want to study the case of a Sobolev norm instead of an $L^q$ norm. If we compare the results of Proposition 3.3.1 and 3.4.1, and take again a look on Proposition 3.3.2 it is no surprise that in the case of stochastic convolutions the restriction on $\alpha$ and $\beta$ will be $\alpha + \beta < 1/2$.

**Proposition 3.4.3.** *Let $p, r \in (1,\infty)$, $q \in (2,\infty)$, and $\alpha, \beta \in [0, 1/2)$ such that $\alpha + \beta < 1/2$. Let $A\colon D(A) \subseteq L^p(U) \to L^p(U)$ be $\ell^q$-sectorial of angle $\omega_{\ell^q}(A) < \pi/2$ with $0 \in \rho(A)$, and $\phi_n\colon \Omega \times [0,T] \to L^p(U)$, $n \in \mathbb{N}$, be such that $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$. Then the convolution process $\Psi$ of Proposition 3.4.1 has the following property:*

$$\mathbb{E}\|A^\beta\Psi\|^r_{L^p(U;W^{\alpha,q}[0,T])} \leq C^r(T^{1/2-\beta} + T^{1/2-\alpha-\beta})^r\mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))},$$

*where $C = C(p,q,r,\alpha,\beta) > 0$ and $\lim_{\alpha+\beta\to 1} C(p,q,r,\alpha,\beta) = \infty$.*

**Proof.** Let $\Gamma(\theta)$ be the path of Proposition 3.3.1 for some $\theta \in (\omega_{\mathcal{R}_q}(A), \pi/2)$. Then

$$\begin{aligned} d_{W^{\alpha,q}}[A^\beta\Psi] &= \int_0^T d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,t]}(s)A^\beta e^{-(t-s)A}\boldsymbol{\phi}(s)\big]\,\mathrm{d}\boldsymbol{\beta}(s) \\ &= \int_0^T \frac{1}{2\pi i}\int_{\Gamma(\theta)} \lambda^\beta d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,t]}(s)e^{-(t-s)\lambda}R(\lambda,A)\boldsymbol{\phi}(s)\big]\,\mathrm{d}\lambda\,\mathrm{d}\boldsymbol{\beta}(s). \end{aligned}$$

For the moment imagine to take the $L^r(\Omega; L^p(U; L^q[0,T]^2))$ norm on both sides, then apply the Itô isomorphism and Minkowski's inequality on the last term. It will be natural to estimate the term

$$\big\|d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,(\cdot)]}(s)e^{-((\cdot)-s)\lambda}R(\lambda,A)\boldsymbol{\phi}(s)\big](h,t)\big\|_{L^r(\Omega;L^p(U;L^q_{(h,t)}([0,T]^2;L^2_{(s)}([0,T]\times\mathbb{N}))))}.$$

To keep this calculation simple, we let $\boldsymbol{\psi} := R(\lambda,A)\boldsymbol{\phi} \in L^r(\Omega; L^p(U; L^q([0,T];\ell^2)))$. Let us start with the innermost norm:

$$\begin{aligned} &\big\|\mathbb{1}_{[0,t+h]}(s)e^{-(t+h-s)\lambda}\boldsymbol{\psi}(s) - \mathbb{1}_{[0,t]}(s)e^{-(t-s)\lambda}\boldsymbol{\psi}(s)\big\|^2_{L^2_{(s)}([0,T]\times\mathbb{N})} \\ &\quad= \sum_{n=1}^\infty \int_0^T \big|\mathbb{1}_{[0,t+h]}(s)e^{-(t+h-s)\lambda} - \mathbb{1}_{[0,t]}(s)e^{-(t-s)\lambda}\big|^2|\psi_n(s)|^2\,\mathrm{d}s \\ &\quad\leq \int_0^T \Big(|\mathbb{1}_{[0,t+h]}(s) - \mathbb{1}_{[0,t]}(s)|e^{-(t+h-s)\mathrm{Re}\,\lambda} + \mathbb{1}_{[0,t]}(s)\big|e^{-(t+h-s)\lambda} - e^{-(t-s)\lambda}\big|\Big)^2\Big(\sum_{n=1}^\infty|\psi_n(s)|^2\Big)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+h} e^{-2(t+h-s)\operatorname{Re}\lambda} \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s + \int_0^t e^{-2(t-s)\operatorname{Re}\lambda} |e^{-h\lambda}-1|^2 \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s \\
&= e^{-2h\operatorname{Re}\lambda} \int_0^T \mathbb{1}_{[-h,0]}(t-s) e^{-2(t-s)\operatorname{Re}\lambda} \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s \\
&\qquad + |e^{-h\lambda}-1|^2 \int_0^t e^{-2(t-s)\operatorname{Re}\lambda} \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s.
\end{aligned}
$$

An application of Young's inequality now leads to

$$
\begin{aligned}
&\big\|\mathbb{1}_{[0,t+h]}(s) e^{-(t+h-s)\lambda} \boldsymbol{\psi}(s) - \mathbb{1}_{[0,t]}(s) e^{-(t-s)\lambda} \boldsymbol{\psi}(s)\big\|_{L^q_{(t)}([0,T];L^2_{(s)}([0,T]\times\mathbb{N}))}^2 \\
&\quad \le \left\| e^{-2h\operatorname{Re}\lambda} \int_0^T \mathbb{1}_{[-h,0]}(t-s) e^{-2(t-s)\operatorname{Re}\lambda} \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s \right\|_{L^{q/2}_{(t)}[0,T]} \\
&\qquad + \left\| |e^{-h\lambda}-1|^2 \int_0^t e^{-2(t-s)\operatorname{Re}\lambda} \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \,\mathrm{d}s \right\|_{L^{q/2}_{(t)}[0,T]} \\
&\quad \le e^{-2h\operatorname{Re}\lambda} \|e^{-2(\cdot)\operatorname{Re}\lambda}\|_{L^1[-h,0]} \big\| \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \big\|_{L^{q/2}[0,T]} \\
&\qquad + |e^{-h\lambda}-1|^2 \|e^{-2(\cdot)\operatorname{Re}\lambda}\|_{L^1[0,T]} \big\| \|\boldsymbol{\psi}(s)\|_{\ell^2}^2 \big\|_{L^{q/2}[0,T]} \\
&\quad = \Big( \tfrac{1}{2\operatorname{Re}\lambda} \big(1 - e^{-2h\operatorname{Re}\lambda}\big) + \tfrac{1}{2\operatorname{Re}\lambda} |e^{-h\lambda}-1|^2 \big(1 - e^{-2T\operatorname{Re}\lambda}\big) \Big) \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2.
\end{aligned}
$$

In the next step we apply the second $L^q[0,T]$ norm with respect to $h$. For this purpose we use $c := \max\{2, \frac{1}{2\cos(\theta)}\}$, and

$$
\tfrac{1}{2\operatorname{Re}\lambda}\big(1 - e^{-2h\operatorname{Re}\lambda}\big) \le \tfrac{c}{|\lambda|} \wedge h, \quad |e^{-h\lambda}-1| \le |h\lambda| \wedge c, \quad \tfrac{1}{2\operatorname{Re}\lambda}\big(1 - e^{-2T\operatorname{Re}\lambda}\big) \le \tfrac{c}{|\lambda|} \wedge T,
$$

for $\lambda \in \Gamma(\theta)$. Then we obtain

$$
\begin{aligned}
&\big\| d_{W^{\alpha,q}} \big[\mathbb{1}_{[0,(\cdot)]}(s) e^{-((\cdot)-s)\lambda} \boldsymbol{\psi}(s)\big](h,t) \big\|_{L^q_{(h,t)}([0,T]^2;L^2_{(s)}([0,T]\times\mathbb{N}))}^2 \\
&\quad \le \left\| h^{-1/q-\alpha} \Big( \tfrac{1}{2\operatorname{Re}\lambda}\big(1 - e^{-2h\operatorname{Re}\lambda}\big) + \tfrac{1}{2\operatorname{Re}\lambda} |e^{-h\lambda}-1|^2 \big(1 - e^{-2T\operatorname{Re}\lambda}\big) \Big)^{1/2} \right\|_{L^q_{(h)}[0,T]}^2 \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2 \\
&\quad = \left\| h^{-2/q-2\alpha} \Big( \tfrac{1}{2\operatorname{Re}\lambda}\big(1 - e^{-2h\operatorname{Re}\lambda}\big) + \tfrac{1}{2\operatorname{Re}\lambda} |e^{-h\lambda}-1|^2 \big(1 - e^{-2T\operatorname{Re}\lambda}\big) \Big) \right\|_{L^{q/2}_{(h)}[0,T]} \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2 \\
&\quad \le \big\| h^{-2/q-2\alpha} \big( (\tfrac{c}{|\lambda|} \wedge h) + (|h\lambda|^2 \wedge c^2)(\tfrac{c}{|\lambda|} \wedge T) \big) \big\|_{L^{q/2}_{(h)}[0,T]} \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2 \\
&\quad \le \Big( \frac{1}{(1/2-\alpha)q} \big(\tfrac{c}{|\lambda|} \wedge T\big)^{(1/2-\alpha)q} + \frac{1}{\alpha q} \frac{c^{q/2}}{|\lambda|^{q/2}} \Big( \big(\tfrac{c}{|\lambda|} \wedge T\big)^{-\alpha q} - T^{-\alpha q} \Big) \Big)^{2/q} \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2 \\
&\qquad + \Big( \frac{1}{(1-\alpha)q} |\lambda|^q \big(\tfrac{c}{|\lambda|} \wedge T\big)^{(3/2-\alpha)q} + \frac{c^q}{\alpha q} \Big( \big(\tfrac{c}{|\lambda|} \wedge T\big)^{(1/2-\alpha)q} - \big(\tfrac{c}{|\lambda|} \wedge T\big)^{q/2} T^{-\alpha q} \Big) \Big)^{2/q} \|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}^2.
\end{aligned}
$$

In the last inequality we used the computations

$$\big\|h^{-2/q-2\alpha}\big(\tfrac{c}{|\lambda|}\wedge h\big)\big\|_{L^{q/2}_{(h)}[0,T]} = \Big(\int_0^{\frac{c}{|\lambda|}\wedge T} h^{q/2-1-\alpha q}\,\mathrm{d}h + \int_{\frac{c}{|\lambda|}\wedge T}^{T} \frac{c^{q/2}}{|\lambda|^{q/2}} h^{-1-\alpha q}\,\mathrm{d}h\Big)^{2/q}$$
$$= \Big(\frac{1}{(1/2-\alpha)q}\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(1/2-\alpha)q} + \frac{1}{\alpha q}\frac{c^{q/2}}{|\lambda|^{q/2}}\Big(\big(\tfrac{c}{|\lambda|}\wedge T\big)^{-\alpha q} - T^{-\alpha q}\Big)\Big)^{2/q}$$

and

$$\big\|h^{-2/q-2\alpha}\big(|h\lambda|^2\wedge c^2\big)\big(\tfrac{c}{|\lambda|}\wedge T\big)\big\|_{L^{q/2}_{(h)}[0,T]}$$
$$= \big(\tfrac{c}{|\lambda|}\wedge T\big)\Big(\int_0^{\frac{c}{|\lambda|}\wedge T} |\lambda|^q h^{q-1-\alpha q}\,\mathrm{d}h + \int_{\frac{c}{|\lambda|}\wedge T}^{T} c^q h^{-1-\alpha q}\,\mathrm{d}h\Big)^{2/q}$$
$$= \Big(\frac{1}{(1-\alpha)q}|\lambda|^q\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(3/2-\alpha)q} + \frac{c^q}{\alpha q}\Big(\big(\tfrac{c}{|\lambda|}\wedge T\big)^{(1/2-\alpha)q} - \big(\tfrac{c}{|\lambda|}\wedge T\big)^{q/2}T^{-\alpha q}\Big)\Big)^{2/q}.$$

Since this looks a little bit overwhelming, we consider the cases $T \leq \frac{c}{|\lambda|}$ and $T \geq \frac{c}{|\lambda|}$ separately. If $T \leq \frac{c}{|\lambda|}$, then

$$\big\|d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,(\cdot)]}(s)e^{-((\cdot)-s)\lambda}\boldsymbol{\psi}(s)\big]\big\|_{L^q([0,T]^2;L^2_{(s)}([0,T]\times\mathbb{N}))} \leq \frac{1+c}{((1/2-\alpha)q)^{1/q}}T^{1/2-\alpha}\|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)},$$

and if $T \geq \frac{c}{|\lambda|}$ we have

$$\big\|d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,(\cdot)]}(s)e^{-((\cdot)-s)\lambda}\boldsymbol{\psi}(s)\big]\big\|_{L^q([0,T]^2;L^2_{(s)}([0,T]\times\mathbb{N}))} \leq \frac{1+c}{((1/2-\alpha)2\alpha q)^{1/q}}\big(\tfrac{c}{|\lambda|}\big)^{1/2-\alpha}\|\boldsymbol{\psi}\|_{L^q([0,T];\ell^2)}.$$

Now we can go back to the beginning. By using Itô's isomorphism, Minkowski's inequality as well as the calculations above and the sectoriality of $A$ in $L^p(U;L^q([0,T];\ell^2))$ we find constants $C_{\alpha,q}>0$ and $C_\theta>0$ such that

$$\|d_{W^{\alpha,q}}[A^\beta\Psi]\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}$$
$$\eqsim_{p,q,r} \Big\|\frac{1}{2\pi i}\int_{\Gamma(\theta)}\lambda^\beta d_{W^{\alpha,q}}\big[\mathbb{1}_{[0,(\cdot)]}(s)e^{-((\cdot)-s)\lambda}R(\lambda,A)\boldsymbol{\phi}(s)\big]\,\mathrm{d}\lambda\Big\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;L^2_{(s)}([0,T]\times\mathbb{N}))))}$$
$$\leq 2\frac{C_{\alpha,q}}{2\pi}\int_0^\infty \rho^\beta\big(T\wedge\tfrac{c}{\rho}\big)^{1/2-\alpha}\|R(\gamma(\rho),A)\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\,\mathrm{d}\rho$$
$$\leq \frac{C_{\alpha,q}C_\theta}{\pi}\Big(\int_0^{c/T}\rho^{\beta-1}T^{1/2-\alpha}\,\mathrm{d}\rho + \int_{c/T}^\infty c^{1/2-\alpha}\rho^{\beta+\alpha-3/2}\,\mathrm{d}\rho\Big)\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}$$
$$= \frac{C_{\alpha,q}C_\theta c^\beta}{\pi}\Big(\frac{1}{\beta}+\frac{1}{1/2-\alpha-\beta}\Big)T^{1/2-\alpha-\beta}\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}.$$

Together with Proposition 3.4.1 the claim follows for $\beta>0$. If $\beta=0$ take $\Gamma'(\theta) = \partial\big(\Sigma_\theta\cup B(0,\tfrac{1}{T})\big)$ instead of $\Gamma(\theta)$ and proceed similarly to Lemma 3.2.1. □

Using Sobolev embedding results, we obtain:

**Corollary 3.4.4 (Hölder regularity).** *Under the assumptions of the previous proposition, we obtain for each $\alpha \in (1/q, 1/2)$ a constant $C = C(r,p,q,\alpha,\beta) > 0$ such that*

$$\mathbb{E}\|A^{\beta}\Psi\|^r_{L^p(U;C^{\alpha-1/q}[0,T])} \leq C^r(T^{1/2-\beta} + T^{1/2-\alpha-\beta})^r \mathbb{E}\|\phi\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

Similarly as in Corollary 3.3.4, Propositions 3.4.1 and 3.4.3 together with Theorem 2.5.9 imply the following result.

**Corollary 3.4.5.** *In addition to the assumptions of the previous corollary, we assume that $A$ is $\mathcal{R}_q$-sectorial. Then there exists a constant $C = C(r,p,q,\alpha,\beta) > 0$ such that*

$$\mathbb{E}\|A^{\beta}\Psi\|^r_{C([0,T];(L^p(U),D(A))_{\alpha-1/q})} \leq C^r(T^{1/2-\beta} + T^{1/2-\alpha-\beta})^r \mathbb{E}\|\phi\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

As in the case of deterministic convolutions we want to improve these results for the cases $\beta = 1/2$ and $\alpha + \beta = 1/2$, respectively. By comparing Proposition 3.3.1 and 3.4.1 as well as Proposition 3.3.2 and 3.4.3 we see that the methods used there are very similar. Roughly speaking, the $L^1$ norm in time of the deterministic case is replaced by an $L^2$ norm in time in the stochastic setting. So the strategy to prove maximal regularity results will be again quite similar. Central to everything is the following lemma.

**Lemma 3.4.6.** *Let $q \in [1,\infty), \sigma \in (0,1)$, and $(\delta_n)_{n=1}^{\infty} \subseteq (0,\infty)$. Then the following assertions hold:*

*a)* *The operator*

$$A_\delta \colon L^q([0,T];\ell^1) \to L^q([0,T];\ell^1), \quad (A_\delta f)(t,n) = \frac{1}{\delta_n}\int_{(t-\delta_n)\vee 0}^{t} f_n \,\mathrm{d}s$$

*is well-defined and*

$$\|A_\delta f\|_{L^q([0,T];\ell^1)} \lesssim_q \|f\|_{L^q([0,T];\ell^1)}.$$

*b)* *The operator $B^\sigma_\delta \colon L^q([0,T];\ell^1) \to L^q([0,T]^2;\ell^1)$ given by*

$$(B^\sigma_\delta f)(h,t,n) = \mathbb{1}_{[0,T-h]}(t)\frac{1}{\delta_n^{1-\sigma}}\frac{1}{h^{1/q+\sigma}}\int_0^T \left|\mathbb{1}_{[(t+h-\delta_n)\vee 0,t+h]} - \mathbb{1}_{[(t-\delta_n)\vee 0,t]}\right| f_n \,\mathrm{d}s$$

*is well-defined and*

$$\|B^\sigma_\delta f\|_{L^q([0,T]^2;\ell^1)} \lesssim_{q,\sigma} \|f\|_{L^q([0,T];\ell^1)}.$$

**Proof.** To simplify the notion we will assume that any function defined on the interval $[0,T]$ is actually defined on $\mathbb{R}$ with the value 0 outside of $[0,T]$.

We start with the much simpler proof of a). It also gives some hint how to proceed in part b). Let $g \in L^{q'}([0,T],\ell^\infty)$. Then with Fubini's theorem

$$\begin{aligned}
\langle A_\delta f, g\rangle &= \int_0^T \sum_{n=1}^\infty \frac{1}{\delta_n} \int_{t-\delta_n}^t f_n(s)\,\mathrm{d}s\; g_n(t)\,\mathrm{d}t \\
&= \int_0^T \sum_{n=1}^\infty \frac{1}{\delta_n} \int_s^{s+\delta_n} g_n(t)\,\mathrm{d}t\; f_n(s)\,\mathrm{d}s \\
&= \langle f, A'_\delta g\rangle,
\end{aligned}$$

where $A'_\delta \colon L^{q'}([0,T],\ell^\infty) \to L^{q'}([0,T],\ell^\infty)$ is given by

$$(A'_\delta g)(s,n) = \frac{1}{\delta_n} \int_s^{s+\delta_n} g_n(t)\,\mathrm{d}t.$$

To conclude the proof of a), it certainly suffices to check the boundedness of $A'_\delta$. So, let $g \in L^{q'}([0,T],\ell^\infty)$, then we obtain for each fixed $s \in [0,T]$

$$\sup_{n\in\mathbb{N}} |(A'_\delta g)(s,n)| \le \sup_{n\in\mathbb{N}} \frac{1}{\delta_n} \int_s^{s+\delta_n} |g_n(t)|\,\mathrm{d}t \le 2 \sup_{\delta>0} \frac{1}{2\delta} \int_{s-\delta}^{s+\delta} \sup_{n\in\mathbb{N}} |g_n(t)|\,\mathrm{d}t.$$

Using that $\sup_{n\in\mathbb{N}} |g_n| \in L^{q'}(\mathbb{R})$ as well as the boundedness of the Hardy-Littlewood maximal operator, we obtain a constant $C_q > 0$ such that

$$\begin{aligned}
\|A'_\delta g\|_{L^{q'}([0,T];\ell^\infty)} &\le 2 \Big\| \sup_{\delta>0} \frac{1}{2\delta} \int_{s-\delta}^{s+\delta} \sup_{n\in\mathbb{N}} |g_n|\,\mathrm{d}t \Big\|_{L^{q'}(\mathbb{R})} \\
&\le 2C_q \big\| \sup_{n\in\mathbb{N}} |g_n| \big\|_{L^{q'}(\mathbb{R})} \\
&= 2C_q \|g\|_{L^{q'}([0,T];\ell^\infty)}.
\end{aligned}$$

b) Similar to the proof above we show the boundedness of the adjoint operator

$$(B^\sigma_\delta)' \colon L^{q'}([0,T]^2;\ell^\infty) \to L^{q'}([0,T];\ell^\infty),$$

which is given by

$$[(B^\sigma_\delta)' g](s,n) = \frac{1}{\delta_n^{1-\sigma}} \int_0^T \frac{1}{h^{1/q+\sigma}} \int_0^T \big| \mathbb{1}_{[s-h,s-h+\delta_n]}(t) - \mathbb{1}_{[s,s+\delta_n]}(t) \big| g_n(h,t)\,\mathrm{d}t\,\mathrm{d}h.$$

Observe that

$$\big| \mathbb{1}_{[s-h,s-h+\delta_n]} - \mathbb{1}_{[s,s+\delta_n]} \big| = \begin{cases} \mathbb{1}_{[s-h,s]} + \mathbb{1}_{[s-h+\delta_n,s+\delta_n]}, & \text{if } h \le \delta_n, \\ \mathbb{1}_{[s-h,s-h+\delta_n]} + \mathbb{1}_{[s,s+\delta_n]}, & \text{if } h > \delta_n, \end{cases}$$

such that in any case the intervals appearing on the right-hand side have the length $h \wedge \delta_n$.

Using this we obtain

$$\begin{aligned}\big|[(B_\delta^\sigma)'g](s,n)\big| &\le \frac{1}{\delta_n^{1-\sigma}}\int_0^{\delta_n}\frac{1}{h^{1/q+\sigma}}\Big(\int_{s-h}^{s}|g_n(h,t)|\,\mathrm{d}t+\int_{s-h+\delta_n}^{s+\delta_n}|g_n(h,t)|\,\mathrm{d}t\Big)\,\mathrm{d}h\\ &\quad+\frac{1}{\delta_n^{1-\sigma}}\int_{\delta_n}^{T}\frac{1}{h^{1/q+\sigma}}\Big(\int_{s-h}^{s-h+\delta_n}|g_n(h,t)|\,\mathrm{d}t+\int_{s}^{s+\delta_n}|g_n(h,t)|\,\mathrm{d}t\Big)\,\mathrm{d}h\\ &=: B_1(s,n)+B_2(s,n).\end{aligned}$$

We estimate each summand separately. First we remark that

$$\begin{aligned}&\int_{s-h}^{s}|g_n(h,t)|\,\mathrm{d}t+\int_{s-h+\delta_n}^{s+\delta_n}|g_n(h,t)|\,\mathrm{d}t\\ &\quad\le 2h\sup_{\varepsilon>0}\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}\|g(h,t)\|_{\ell^\infty}\,\mathrm{d}t+2h\sup_{\varepsilon>0}\frac{1}{2\varepsilon}\int_{s+\delta_n-\varepsilon}^{s+\delta_n+\varepsilon}\|g(h,t)\|_{\ell^\infty}\,\mathrm{d}s\\ &\quad\le 4h\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(h,t)\|_{\ell^\infty}\,\mathrm{d}t,\end{aligned}$$

where the supremum in the last line is taken over all intervals $I_\varepsilon\subset\mathbb{R}$ of length $2\varepsilon$ containing $s$. With this estimate Hölder's inequality leads to

$$\begin{aligned}B_1(s,n)&=\frac{1}{\delta_n^{1-\sigma}}\int_0^{\delta_n}h^{1-1/q-\sigma}\Big(\frac{1}{h}\int_{s-h}^{s}|g_n(h,t)|\,\mathrm{d}t+\frac{1}{h}\int_{s-h+\delta_n}^{s+\delta_n}|g_n(h,t)|\,\mathrm{d}t\Big)\,\mathrm{d}h\\ &\le\frac{1}{\delta_n^{1-\sigma}}\Big(\int_0^{\delta_n}h^{(1-\sigma)q-1}\,\mathrm{d}h\Big)^{1/q}\Big\|4\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}[0,T]}\\ &=4\big(\tfrac{1}{(1-\sigma)q}\big)^{1/q}\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}[0,T]}.\end{aligned}$$

Similarly we estimate the second summand. Using that

$$\int_{s-h}^{s-h+\delta_n}|g_n(h,t)|\,\mathrm{d}t+\int_{s}^{s+\delta_n}|g_n(h,t)|\,\mathrm{d}t\le 4\delta_n\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(h,t)\|_{\ell^\infty}\,\mathrm{d}t,$$

and Hölder's inequality, we obtain

$$\begin{aligned}B_2(s,n)&\le 4\delta_n^\sigma\Big(\int_{\delta_n}^{T}h^{-1-\sigma q}\,\mathrm{d}h\Big)^{1/q}\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}[0,T]}\\ &=4\delta_n^\sigma\big(\tfrac{1}{\sigma q}\delta_n^{-\sigma q}-\tfrac{1}{\sigma q}T^{-\sigma q}\big)^{1/q}\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}[0,T]}\\ &\le 4\big(\tfrac{1}{\sigma q}\big)^{1/q}\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}[0,T]}.\end{aligned}$$

In both cases the right-hand side is now independent of $n$. We set $C_{q,\sigma}:=4\big(\frac{1}{\sigma q}\big)^{1/q}+4\big(\frac{1}{(1-\sigma)q}\big)^{1/q}$.

Then Fubini's theorem and the boundedness of the Hardy-Littlewood maximal operator yield

$$\begin{aligned}\|(B_\delta^\sigma)'g\|_{L^{q'}([0,T];\ell^\infty)} &\le C_{q,\sigma}\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(h,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}_{(h,s)}[0,T]^2}\\ &= C_{q,\sigma}\Big\|\Big\|\sup_{I_\varepsilon\ni s}\frac{1}{2\varepsilon}\int_{I_\varepsilon}\|g(\cdot,t)\|_{\ell^\infty}\,\mathrm{d}t\Big\|_{L^{q'}_{(s)}[0,T]}\Big\|_{L^{q'}_{(h)}[0,T])}\\ &\le C_q C_{q,\sigma}\|g\|_{L^{q'}([0,T]^2;\ell^\infty)},\end{aligned}$$

which proves the desired estimate. □

As a first step we want to prove an $\mathcal{R}$-boundedness result for the following (stochastic) operator families

$$(S_\delta\boldsymbol{\phi})(t) := \frac{1}{\delta^{1/2}}\int_{(t-\delta)\vee 0}^{t}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{\beta},\quad t\in[0,T],\ \delta>0,$$

$$(S_\delta^\sigma\boldsymbol{\phi})(h,t) := \mathbb{1}_{[0,T-h]}(t)\frac{\delta^\sigma}{h^{1/q+\sigma}}\big((S_\delta\boldsymbol{\phi})(t+h)-(S_\delta\boldsymbol{\phi})(t)\big),\quad (h,t)\in[0,T]^2,\ \delta>0,$$

where $\sigma\in[0,1/2)$. Apart from the stochastic integral as one difference to the deterministic case, we also notice that the exponent of the fraction in front of it has changed from 1 to $1/2$ and from $1-\sigma$ to $1/2-\sigma$, respectively. This coincides with the changes made in the previous results.

**Proposition 3.4.7.** *For $q\in[2,\infty)$, $p,r\in(1,\infty)$, and $\sigma\in(0,1/2)$ the following assertions hold:*

a) *The set $(S_\delta)_{\delta>0}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$.*

b) *The set $(S_\delta^\sigma)_{\delta>0}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]^2))$.*

**Proof.** Let $(\delta_n)_{n=1}^N\subseteq(0,\infty)$, $(\boldsymbol{\phi}_n)_{n=1}^N\subseteq L^r_{\mathbb{F}}(\Omega;L^p(U;L^q([0,T];\ell^2)))$, and $(\widetilde{r}_n)_{n=1}^N$ be a Rademacher sequence defined on some probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$.

a) Define $\boldsymbol{\psi}_n(\omega,s,t,u) := \frac{1}{\delta_n^{1/2}}\mathbb{1}_{[(t-\delta_n)\vee 0,t]}(s)\boldsymbol{\phi}_n(\omega,s,u)$. Then by Proposition 1.3.5 d) and Itô's isomorphism for mixed $L^p$ spaces (see Theorem 1.3.3) we have

$$\begin{aligned}\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N\widetilde{r}_n S_{\delta_n}\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))} &= \widetilde{\mathbb{E}}\Big\|\int_0^T\sum_{n=1}^N\widetilde{r}_n\boldsymbol{\psi}_n\,\mathrm{d}\boldsymbol{\beta}\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\ &\eqsim_{p,q,r}\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N\widetilde{r}_n\boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];L^2([0,T]\times\mathbb{N}))))}.\end{aligned}$$

An application of Kahane's inequality and Lemma 3.4.6 now leads to

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n\boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];L^2([0,T]\times\mathbb{N}))))} &\eqsim_{p,q,r} \Big\|\Big(\sum_{n=1}^{N}|\boldsymbol{\psi}_n|^2\Big)^{1/2}\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];L^2([0,T]\times\mathbb{N}))))}\\
&= \Big\|\Big(\sum_{n=1}^{N}\frac{1}{\delta_n}\int_{(t-\delta_n)\vee 0}^{t}\|\boldsymbol{\phi}_n\|_{\ell^2}^2\,\mathrm{d}s\Big)^{1/2}\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\
&= \Big\|\,\|A_\delta(\|\boldsymbol{\phi}_n\|_{\ell^2}^2)_{n=1}^N\|_{L^{q/2}([0,T];\ell^1)}^{1/2}\Big\|_{L^r(\Omega;L^p(U))}\\
&\lesssim_q \Big\|\,\Big\|\sum_{n=1}^{N}\|\boldsymbol{\phi}_n\|_{\ell^2}^2\Big\|_{L^{q/2}[0,T]}^{1/2}\Big\|_{L^r(\Omega;L^p(U))}\\
&= \Big\|\Big(\sum_{n=1}^{N}|\boldsymbol{\phi}_n|^2\Big)^{1/2}\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\\
&\eqsim_{p,q,r} \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}.
\end{aligned}$$

b) As in part a) we obtain by Itô's isomorphism

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n S_{\delta_n}^{\sigma}\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}\\
&\quad= \widetilde{\mathbb{E}}\Big\|\int_0^T\sum_{n=1}^{N}\widetilde{r}_n\frac{1}{\delta_n^{1/2-\sigma}}d_{W^{\sigma,q}}[\mathbb{1}_{[(t-\delta_n)\vee 0,t]}]\boldsymbol{\phi}_n\,\mathrm{d}\boldsymbol{\beta}\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}\\
&\quad\eqsim_{p,q,r} \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n\frac{1}{\delta_n^{1/2-\sigma}}d_{W^{\sigma,q}}[\mathbb{1}_{[(t-\delta_n)\vee 0,t]}]\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;L^2([0,T]\times\mathbb{N}))))}.
\end{aligned}$$

By Kahane's inequality and Lemma 3.4.6 b) we finally arrive at

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n\frac{1}{\delta_n^{1/2-\sigma}}d_{W^{\sigma,q}}[\mathbb{1}_{[(t-\delta_n)\vee 0,t]}]\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;L^2([0,T]\times\mathbb{N}))))}\\
&\eqsim_{p,q,r} \Big\|\Big(\sum_{n=1}^{N}\frac{1}{h^{2/q+2\sigma}}\frac{1}{\delta_n^{1-2\sigma}}\big|\mathbb{1}_{[(t+h-\delta_n)\vee 0,t+h]}-\mathbb{1}_{[(t-\delta_n)\vee 0,t]}\big|\|\boldsymbol{\phi}_n\|_{\ell^2}^2\Big)^{1/2}\Big\|_{L^r(\Omega;L^p(U;L^q_{(h,t)}([0,T]^2;L^2_{(s)}[0,T])))}\\
&= \Big\|\,\|B_\delta^{2\sigma}(\|\boldsymbol{\phi}_n\|_{\ell^2}^2)_{n=1}^N\|_{L^{q/2}([0,T]^2;\ell^1)}^{1/2}\Big\|_{L^r(\Omega;L^p(U))}\\
&\lesssim_{q,\sigma} \Big\|\,\Big\|\sum_{n=1}^{N}\|\boldsymbol{\phi}_n\|_{\ell^2}^2\Big\|_{L^{q/2}[0,T]}^{1/2}\Big\|_{L^r(\Omega;L^p(U))}\\
&= \Big\|\Big(\sum_{n=1}^{N}|\boldsymbol{\phi}_n|^2\Big)^{1/2}\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\\
&\eqsim_{p,q,r} \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^{N}\widetilde{r}_n\boldsymbol{\phi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}.
\end{aligned}$$

□

As in the previous section we first prove an $\mathcal{R}$-boundedness result for the scalar-valued convolution. Here we need the set

$$\mathcal{B}_\sigma := \Big\{ f\colon [0,\infty) \to \mathbb{C}\colon f \text{ is abs. continuous}, \lim_{t\to\infty} f(t) = 0 \text{ and } \int_0^\infty t^{1/2-\sigma} |f'(t)| \,\mathrm{d}t \le 1 \Big\},$$

where $\sigma \in [0, 1/2)$. In particular, we again have $f(t) = -\int_t^\infty f'(s)\,\mathrm{d}s$ for each $f \in \mathcal{B}_\sigma$.

**Proposition 3.4.8 (The scalar-valued case).** *Let $q \in [2,\infty)$, $p, r \in (1,\infty)$, and $\sigma \in [0, 1/2)$. Then we have*

a) *The operator family $(C_{stoch}(f))_{f\in\mathcal{B}_0}$ given by*

$$\big[C_{stoch}(f)\boldsymbol{\phi}\big](t) := \int_0^t f(t-s)\boldsymbol{\phi}(s)\,\mathrm{d}\boldsymbol{\beta}(s), \quad t \in [0,T],$$

*is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$.*

b) *The operator family $(C^\sigma_{stoch}(f))_{f\in\mathcal{B}_\sigma}$ given by*

$$\big[C^\sigma_{stoch}(f)\boldsymbol{\phi}\big](h,t) := d_{W^{\sigma,q}}[C_{stoch}(f)\boldsymbol{\phi}](h,t), \quad (h,t) \in [0,T]^2$$

*is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$.*

**Proof.** By Proposition 3.4.7 the maps $\delta \mapsto S_\delta$ and $\delta \mapsto S^\sigma_\delta$ have an $\mathcal{R}$-bounded range. Corollary 2.14 of [59] now implies that the operator families $\{T_h\colon \|h\|_{L^1} \le 1\}$ and $\{T^\sigma_h\colon \|h\|_{L^1} \le 1\}$ defined by

$$T_h\boldsymbol{\phi} := \int_0^\infty h(\delta) S_\delta \boldsymbol{\phi}\,\mathrm{d}\delta, \quad \boldsymbol{\phi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2))),$$
$$T^\sigma_h\boldsymbol{\phi} := \int_0^\infty h(\delta) S^\sigma_\delta \boldsymbol{\phi}\,\mathrm{d}\delta, \quad \boldsymbol{\phi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2))),$$

for $h \in L^1(0,\infty)$ are also $\mathcal{R}$-bounded. The results finally follow from

$$\begin{aligned}\big[C_{\mathrm{stoch}}(f)\boldsymbol{\phi}\big](t) &= -\int_0^t \int_{t-s}^\infty f'(\delta)\boldsymbol{\phi}(s)\,\mathrm{d}\delta\,\mathrm{d}\boldsymbol{\beta}(s)\\ &= -\int_0^\infty f'(\delta) \int_{(t-\delta)\vee 0}^t \boldsymbol{\phi}(s)\,\mathrm{d}\boldsymbol{\beta}(s)\,\mathrm{d}\delta\\ &= -\int_0^\infty \delta^{1/2} f'(\delta)(S_\delta\boldsymbol{\phi})(t)\,\mathrm{d}\delta,\end{aligned}$$

and in the same way we can show that

$$\big[C^\sigma_{\mathrm{stoch}}(f)\boldsymbol{\phi}\big](h,t) = -\int_0^\infty \delta^{1/2-\sigma} f'(\delta)(S^\sigma_\delta\boldsymbol{\phi})(h,t)\,\mathrm{d}\delta. \qquad \square$$

**COROLLARY 3.4.9.** *Let $q \in [2,\infty)$, $p,r \in (1,\infty)$, and $\nu \in (0,\pi/2)$. For $\sigma \in [0,1/2)$ and $\mu \in \Sigma_\nu$ we define the function*

$$g_\mu^\sigma \colon [0,\infty) \to \mathbb{C}, \quad g_\mu^\sigma(t) := \mu^{1/2-\sigma} e^{-\mu t}.$$

*Then $\frac{\cos(\nu)^{3/2-\sigma}}{\Gamma(3/2-\sigma)} g_\mu^\sigma \in \mathcal{B}_\sigma$. As a consequence, the set $\{L_\mu := C_{stoch}(g_\mu^0) \colon \mu \in \Sigma_\nu\}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$, and for $\sigma \in (0,1/2)$ the set $\{L_\mu^\sigma := C^\sigma_{stoch}(g_\mu^\sigma) \colon \mu \in \Sigma_\nu\}$ is $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$.*

**PROOF.** Since $\operatorname{Re}\mu > 0$ for $\mu \in \Sigma_\nu$, we have

$$|g_\mu^\sigma(t)| = |\mu|^{1/2-\sigma} e^{-t\operatorname{Re}\mu} \to 0 \quad \text{for } t \to \infty.$$

Moreover,

$$\begin{aligned} \int_0^\infty t^{1/2-\sigma} \left|\tfrac{\mathrm{d}}{\mathrm{d}t} g_\mu^\sigma(t)\right| \mathrm{d}t &= \int_0^\infty t^{1/2-\sigma} |\mu|^{3/2-\sigma} e^{-t\operatorname{Re}\mu}\,\mathrm{d}t \\ &\le \frac{1}{\cos(\nu)^{3/2-\sigma}} \int_0^\infty (t\operatorname{Re}\mu)^{1/2-\sigma} \operatorname{Re}\mu\, e^{-t\operatorname{Re}\mu}\,\mathrm{d}t \\ &= \frac{1}{\cos(\nu)^{3/2-\sigma}} \int_0^\infty s^{1/2-\sigma} e^{-s}\,\mathrm{d}s = \frac{\Gamma(3/2-\sigma)}{\cos(\nu)^{3/2-\sigma}}, \end{aligned}$$

where we used that $\operatorname{Re}\mu = \cos(\arg(\mu))|\mu| \ge \cos(\nu)|\mu|$ in the second line. This implies the first claim. The $\mathcal{R}$-boundedness results finally follow from Proposition 3.4.8. □

In the final step we extend these results to the operator-valued case.

**THEOREM 3.4.10 (Stochastic maximal regularity).** *Let $q \in [2,\infty)$, $p,r \in (1,\infty)$, and $\sigma \in (0,1/2)$. Let $A \colon D(A) \subseteq L^p(U) \to L^p(U)$ have an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus of angle $\alpha \in (0,\pi/2)$ with $0 \in \rho(A)$, and let $\phi_n \colon \Omega \times [0,T] \to L^p(U)$, $n \in \mathbb{N}$, be such that $\boldsymbol{\phi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$. Then the process*

$$\Psi(t) := \int_0^t e^{-(t-s)A} \boldsymbol{\phi}(s)\,\mathrm{d}\boldsymbol{\beta}(s), \quad t \in [0,T],$$

*is well-defined, takes values in $D(A^{1/2})$ almost surely, and*

$$\mathbb{E}\|A^{1/2}\Psi\|^r_{L^p(U;L^q[0,T])} \lesssim_{p,q,r} \mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

*Moreover, we have the following Sobolev regularity result*

$$\mathbb{E}\|A^{1/2-\sigma}\Psi\|^r_{L^p(U;W^{\sigma,q}[0,T])} \lesssim_{p,q,r,\sigma} \mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

**Proof.** By Theorem 2.4.5 the extension $A^{L^q}$ of $A$ has a bounded $H^\infty$ calculus on $L^p(U;L^q[0,T])$. And since $\ell^2$ is a Hilbert space, we can extend $A^{L^q}$ one more time to an operator $A^{L^q(\ell^2)}$ which also has a bounded $H^\infty$ calculus (see Remark 2.4.1, and [60, Proposition 5.1, Theorem 5.2] as well as [61, Theorem 4] for more information on this topic). By Corollary 3.4.9 the function $\mu \mapsto L_\mu$ is analytic on $\Sigma_\nu$, $\nu \in (\alpha, \pi/2)$, and $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. To view this as an $\mathcal{R}$-bounded operator family on the space $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ we define

$$\widetilde{L}_\mu \boldsymbol{\phi} := (L_\mu \boldsymbol{\phi}, 0, 0, \ldots) \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2))).$$

Obviously, $\{\widetilde{L}_\mu \colon \mu \in \Sigma_\nu\}$ is now $\mathcal{R}$-bounded on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ and commutes with $R(\mu, A^{L^q(\ell^2)})$. By Theorem 4.4 of [52] the linear map

$$\phi \mapsto \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) \widetilde{L}_\mu \boldsymbol{\phi} \, \mathrm{d}\mu$$

defines a well-defined and bounded operator on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ for $\alpha' \in (\alpha, \nu)$. Then by Theorem 4.5 of [52] the operator

$$\phi \mapsto \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) L_\mu \boldsymbol{\phi} \, \mathrm{d}\mu$$

is also well-defined and bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. By the stochastic Fubini theorem we obtain

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) L_\mu \boldsymbol{\phi} \, \mathrm{d}\mu &= \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} \int_0^t \mu^{1/2} e^{-\mu(t-s)} R(\mu, A) \boldsymbol{\phi}(s) \, \mathrm{d}\boldsymbol{\beta}(s) \, \mathrm{d}\mu \\
&= \int_0^t \frac{1}{2\pi i} \int_{\partial \Sigma_{\alpha'}} \mu^{1/2} e^{-\mu(t-s)} R(\mu, A) \boldsymbol{\phi}(s) \, \mathrm{d}\mu \, \mathrm{d}\boldsymbol{\beta}(s) \\
&= \int_0^t A^{1/2} e^{-(t-s)A} \boldsymbol{\phi}(s) \, \mathrm{d}\boldsymbol{\beta}(s).
\end{aligned}$$

Using the boundedness result we get

$$\mathbb{E} \Big\| t \mapsto \int_0^t A^{1/2} e^{-(t-s)A} \boldsymbol{\phi}(s) \, \mathrm{d}\boldsymbol{\beta}(s) \Big\|^r_{L^p(U;L^q[0,T])} \lesssim_{p,q,r} \mathbb{E} \|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))}$$

For the second part of the theorem, we remark that, by assumption, $A$ can also be extended to an operator $A^{L^q(\ell^2)}$ on $L^p(U; L^q([0,T]^2;\ell^2))$ such that $A^{L^q(\ell^2)}$ has a bounded $H^\infty$ calculus. Moreover, we define on this space the following operator

$$(M^\sigma_\mu \boldsymbol{\psi})(t,h) := (L^\sigma_\mu J \boldsymbol{\psi})(h,t), \quad \boldsymbol{\psi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T]^2;\ell^2))),$$

where $(J\boldsymbol{\psi})_n(s) := \frac{1}{T} \int_0^T \psi_n(\tau, s) \, \mathrm{d}\tau$, $s \in [0,T]$, $n \in \mathbb{N}$. Observe that $J\boldsymbol{\psi} \in L^r(\Omega; L^p(U; L^q([0,T];\ell^2)))$

since by Hölder's inequality we have

$$\|J\boldsymbol{\psi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))} \le T^{-1/q}\|\boldsymbol{\psi}\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;\ell^2)))}.$$

Then, by Corollary 3.4.9 there exists a constant $C > 0$ such that

$$\begin{aligned}\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \widetilde{r}_n M_{\mu_n}^\sigma \boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))} &= \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \widetilde{r}_n L_{\mu_n}^\sigma J\boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}\\ &\le C\,\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \widetilde{r}_n J\boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}\\ &\le CT^{-1/q}\,\widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \widetilde{r}_n \boldsymbol{\psi}_n\Big\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;\ell^2)))},\end{aligned}$$

for each finite sequences $(\mu_n)_{n=1}^N \subseteq \Sigma_\nu$, $(\boldsymbol{\psi}_n)_{n=1}^N \subseteq L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T]^2;\ell^2)))$, and each Rademacher sequence $(\widetilde{r}_n)_{n=1}^N$. In other words, the set $\{M_\mu^\sigma : \mu \in \Sigma_\nu\}$ is also $\mathcal{R}$-bounded from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T]^2;\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$. Again, by Theorem 4.4 of [52] and a similar argument as in the first case the linear map

$$\boldsymbol{\psi} \mapsto \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) M_\mu^\sigma \boldsymbol{\psi}\,\mathrm{d}\mu$$

defines a bounded operator from $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T]^2;\ell^2)))$ to $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]^2))$. Now take any $\boldsymbol{\phi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T];\ell^2)))$, and let $\boldsymbol{\psi}(\tau, s) := \mathbb{1}_{[0,T]}(\tau)\boldsymbol{\phi}(s)$. Then, of course, we see that $\boldsymbol{\psi} \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q([0,T]^2;\ell^2)))$ and $(J\boldsymbol{\psi})(s) = \boldsymbol{\phi}(s)$. By the definition of the operators $M_\mu^\sigma$, $\mu \in \Sigma_\nu$, and the equality proven above we obtain

$$d_{W^{\sigma,q}}[A^{1/2-\sigma}\Psi] = \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) M_\mu^\sigma \boldsymbol{\psi}\,\mathrm{d}\mu$$

Using now the boundedness result as well as Proposition 3.4.1 we arrive at

$$\begin{aligned}&\|A^{1/2-\sigma}\Psi\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} \le \|A^{1/2-\sigma}\Psi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \|d_{W^{\sigma,q}}[A^{1/2-\sigma}\Psi]\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}\\ &\quad= \|A^{1/2-\sigma}\Psi\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \Big\|\frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha'}} R(\mu, A^{L^q(\ell^2)}) M_\mu^\sigma \boldsymbol{\psi}\,\mathrm{d}\mu\Big\|_{L^r(\Omega;L^p(U;L^q[0,T]^2))}\\ &\quad\lesssim_{p,q,r,\sigma} T^\sigma\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))} + T^{-1/q}\|\boldsymbol{\psi}\|_{L^r(\Omega;L^p(U;L^q([0,T]^2;\ell^2)))}\\ &\quad= (1+T^\sigma)\|\boldsymbol{\phi}\|_{L^r(\Omega;L^p(U;L^q([0,T];\ell^2)))}.\end{aligned}$$

□

In [83, Theorem 1.1 and 1.2] the authors investigate stochastic maximal regularity in the space $L^q(\Omega \times [0,T]; L^p(U))$ for $p \in [2,\infty)$ and $q \in (2,\infty)$ (where $q = 2$ is allowed if $p = 2$). In these spaces they obtain the corresponding result to Theorem 3.4.10 and present a counterexample for the case $q = 2$ (see Section 6 in [83]), which means that maximal regularity results in these spaces seem to have some unexpected limits. In our approach we can include all values $p \in (1,\infty)$ and $q \ge 2$.

Similar to the previous cases, an application of Sobolev's embedding theorem yields the following result:

**COROLLARY 3.4.11 (Hölder regularity).** *Under the assumptions of the previous theorem, we obtain for each* $\alpha \in (1/q, 1/2)$ *a constant* $C = C(p,q,r,\alpha) > 0$ *such that*

$$\mathbb{E}\|A^{1/2-\alpha}\Psi\|^r_{L^p(U;C^{\alpha-1/q}[0,T])} \leq C^r(1+T^\alpha)^r \mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))},$$

If we apply the results of Theorem 3.4.10 closely following the proof of Corollary 3.3.4 we obtain:

**COROLLARY 3.4.12.** *Under the assumptions of Theorem 3.4.10, we obtain for each* $\alpha \in (1/q, 1/2)$ *a constant* $C = C(p,q,r,\alpha) > 0$ *such that*

$$\mathbb{E}\|A^{1/2-\alpha}\Psi\|^r_{C([0,T];(L^p(U),D(A))_{\alpha-1/q})} \leq C^r(1+T^\alpha)^r \mathbb{E}\|\boldsymbol{\phi}\|^r_{L^p(U;L^q([0,T];\ell^2))}.$$

## 3.5 Existence and Uniqueness Results

The previous three sections form the basis to investigate the existence and uniqueness as well as the regularity of solutions for stochastic evolution equations in $L^p$ spaces. Before turning to that, we first give a short introduction of the Lipschitz notions we will need in this context.

### 3.5.1 Lipschitz Notions

This section is devoted to some preliminary notions which appear in the following sections of this chapter. In the usual theory of stochastic evolution equations in Banach spaces one assumes Lipschitz continuity of the nonlinearities involved (see (3.1)). The reason for that is the application of fixed point arguments in the proof of existence and uniqueness of mild solutions. In our case we need a different type of Lipschitz continuity.

**DEFINITION 3.5.1.** Let $p, q \in [1,\infty]$.

a) We call a function $\boldsymbol{B}\colon [0,T]\times\mathbb{N}\times L^p(U) \to L^p(U)$ *$L^q$-Lipschitz continuous* if

$$\big\|\boldsymbol{B}\big(\cdot,\phi(\cdot)\big) - \boldsymbol{B}\big(\cdot,\psi(\cdot)\big)\big\|_{L^p(U;L^q([0,T];\ell^2))} \leq L\|\phi-\psi\|_{L^p(U;L^q[0,T])}$$

for some constant $L > 0$ and all $\phi,\psi\colon [0,T] \to L^p(U)$ satisfying $\phi,\psi \in L^p(U;L^q[0,T])$.

b) We call a function $\boldsymbol{B}\colon [0,T]\times\mathbb{N}\times L^p(U)\to L^p(U)$ *locally $L^q$-Lipschitz continuous* if for each $R>0$ there exists a constant $L_R>0$ such that

$$\big\|\boldsymbol{B}\big(\cdot,\phi(\cdot)\big)-\boldsymbol{B}\big(\cdot,\psi(\cdot)\big)\big\|_{L^p(U;L^q([0,T];\ell^2))}\le L_R\|\phi-\psi\|_{L^p(U;L^q[0,T])}$$

for all $\phi,\psi\colon[0,T]\to L^p(U)$ satisfying $\|\phi\|_{L^p(U;L^q[0,T])},\|\psi\|_{L^p(U;L^q[0,T])}\le R$.

**Remark 3.5.2.** By Fubini's theorem every (locally) Lipschitz continuous function $B\colon L^p(U)\to L^p(U)$ is (locally) $L^p$-Lipschitz continuous.

**Example 3.5.3 (Nemytskii maps).** Let $b\colon\mathbb{R}\to\mathbb{R}$ be Lipschitz continous, and define

$$B\colon L^p(U;L^q[0,T])\to L^p(U;L^q[0,T])\quad\text{by}\quad B(\phi)(u,t):=b(\phi(u,t)).$$

Then $B$ is $L^q$-Lipschitz continuous with the same Lipschitz constant as $b$. This easily follows by estimating $B$ pointwise and by the monotonicity of the norms involved.

### 3.5.2 The Globally Lipschitz Case

Let us shortly recall the considered equation in this subsection. On the space $L^p(U)$ we want to investigate the equation

$$\mathrm{d}X(t)+AX(t)\,\mathrm{d}t=F(t,X(t))\,\mathrm{d}t+\boldsymbol{B}(t,X(t))\,\mathrm{d}\boldsymbol{\beta}(t),\quad X(0)=x_0,\tag{3.2}$$

and analyze the existence and uniqueness of solutions as well as their regularity. To do this we will assume the following hypothesis for the operator $A$, the nonlinearities $F$ and $\boldsymbol{B}$, and the random initial value $x_0$. For this we introduce the abbreviation

$$D_A^{\ell q}(\theta):=(L^p(U),D(A))_{\theta,\ell^q},\quad\theta\in(0,1).$$

**Hypothesis 3.5.4.** Let $r\in\{0\}\cup(1,\infty)$, $p\in(1,\infty)$, $q\in[2,\infty)$, and $\gamma,\gamma_F,\gamma_B\in\mathbb{R}$.

**(HA) Assumption on the operator $A$:** The linear operator $A\colon D(A)\subseteq L^p(U)\to L^p(U)$ is closed and there exists a $\nu>0$ such that $A_\nu:=\nu+A$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus for some $\alpha\in(0,\pi/2)$ with $0\in\rho(A_\nu)$.

**(HF) Assumption on the nonlinearity $F$:** The function $F\colon\Omega\times[0,T]\times D(A_\nu^\gamma)\to D(A_\nu^{-\gamma_F})$ is strongly measurable and

a) for all $t\in[0,T]$ and $x\in D(A_\nu^\gamma)$ the random variable $\omega\mapsto F(\omega,t,x)$ is strongly $\mathcal{F}_t$-measurable;

b) there exist constants $L_F$, $\widetilde{L}_F$, $C_F \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi\colon [0,T] \to D(A_\nu^\gamma)$ satisfying $A_\nu^\gamma \phi, A_\nu^\gamma \psi \in L^p(U; L^q[0,T])$,

$$\big\| A_\nu^{-\gamma_F} \big( F(\omega, \cdot, \phi) - F(\omega, \cdot, \psi) \big) \big\|_{L^p(U;L^q[0,T])} \leq L_F \big\| A_\nu^\gamma (\phi - \psi) \big\|_{L^p(U;L^q[0,T])} + \widetilde{L}_F \big\| A_\nu^{-\gamma_F} (\phi - \psi) \big\|_{L^p(U;L^q[0,T])}$$

and

$$\| A_\nu^{-\gamma_F} F(\omega, \cdot, \phi) \|_{L^p(U;L^q[0,T])} \leq C_F (1 + \| A_\nu^\gamma \phi \|_{L^p(U;L^q[0,T])}).$$

**(HB) Assumption on the nonlinearity $\boldsymbol{B}$:** The function $\boldsymbol{B}\colon \Omega \times [0,T] \times \mathbb{N} \times D(A_\nu^\gamma) \to D(A_\nu^{-\gamma_B})$ is strongly measurable and

a) for all $t \in [0,T]$, $n \in \mathbb{N}$, and $x \in D(A_\nu^\gamma)$ the random variable $\omega \mapsto B_n(\omega, t, x)$ is strongly $\mathcal{F}_t$-measurable;

b) there exist constants $L_B$, $\widetilde{L}_B$, $C_B \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi\colon [0,T] \to D(A_\nu^\gamma)$ satisfying $A_\nu^\gamma \phi, A_\nu^\gamma \psi \in L^p(U; L^q[0,T])$,

$$\big\| A_\nu^{-\gamma_B} \big( \boldsymbol{B}(\omega, \cdot, \phi) - \boldsymbol{B}(\omega, \cdot, \psi) \big) \big\|_{L^p(U;L^q([0,T];\ell^2))} \leq L_B \big\| A_\nu^\gamma (\phi - \psi) \big\|_{L^p(U;L^q[0,T])} + \widetilde{L}_B \big\| A_\nu^{-\gamma_B} (\phi - \psi) \big\|_{L^p(U;L^q[0,T])}$$

and

$$\| A_\nu^{-\gamma_B} \boldsymbol{B}(\omega, \cdot, \phi) \|_{L^p(U;L^q([0,T];\ell^2))} \leq C_B (1 + \| A_\nu^\gamma \phi \|_{L^p(U;L^q[0,T])}).$$

**(H$x_0$) Assumption on the initial value $x_0$:** The initial value $x_0\colon \Omega \to D_{A_\nu}^{\ell^q}(\gamma - 1/q)$ is strongly $\mathcal{F}_0$-measurable.

**Remark 3.5.5.**

a) Assuming this hypothesis and certain values of $\gamma, \gamma_F, \gamma_B$ we note that the definition of strong, weak, and mild $(r,p,q)$ solutions we made in Section 3.1 thins down a little bit. If $\gamma \geq 1$, then $\gamma_F \leq 0$, and (HF) implies that

$$\begin{aligned} \|F(\cdot, X(\cdot))\|_{L^p(U;L^1[0,T])} &\leq T^{1-1/q} \|F(\cdot, X(\cdot))\|_{L^p(U;L^q[0,T])} \\ &= T^{1-1/q} \|A_\nu^{\gamma_F} A_\nu^{-\gamma_F} F(\cdot, X(\cdot))\|_{L^p(U;L^q[0,T])} \\ &\leq C T^{1-1/q} \|A_\nu^{-\gamma_F} F(\cdot, X(\cdot))\|_{L^p(U;L^q[0,T])} \\ &\leq C C_F T^{1-1/q} (1 + \|A_\nu^\gamma X\|_{L^p(U;L^q[0,T])}) \end{aligned}$$

which means that $F(\cdot, X(\cdot)) \in L^p(U; L^1[0,T])$ almost surely. Similarly, one shows that $\boldsymbol{B}(\cdot, X(\cdot)) \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^2([0,T] \times \mathbb{N})))$. Moreover, $AX \in L^p(U; L^1[0,T])$ almost surely since $A$ has a closed extension on $L^p(U; L^q[0,T])$.

For weak solutions, we assume that $\gamma_F, \gamma_B \le 1$. Then

$$\begin{aligned}\big\|\langle F(\cdot,X(\cdot)),\psi\rangle\big\|_{L^1[0,T]} &= \int_0^T \big|\langle A_\nu^{-\gamma_F}F(t,X(t)),(A_\nu^{\gamma_F})'\psi\rangle\big|\,\mathrm{d}t\\ &\le \int_U\Big(\int_0^T |A_\nu^{-\gamma_F}F(t,X(t))|\,\mathrm{d}t\Big)|(A^{\gamma_F})'\psi|\,\mathrm{d}\mu\\ &\le \Big\|\int_0^T |A_\nu^{-\gamma_F}F(t,X(t))|\,\mathrm{d}t\Big\|_{L^p(U)}\|(A^{\gamma_F})'\psi\|_{L^{p'}(U)}\\ &\le T^{1-1/q}\|A_\nu^{-\gamma_F}F(\cdot,X(\cdot))\|_{L^p(U;L^q[0,T])}\|(A^{\gamma_F})'\psi\|_{L^{p'}(U)}\\ &\le C_F T^{1-1/q}(1+\|A_\nu^{\gamma}X\|_{L^p(U;L^q[0,T])})\|(A^{\gamma_F})'\psi\|_{L^{p'}(U)}\end{aligned}$$

for each $\psi \in D(A')$, i.e. $\langle F(\cdot,X(\cdot)),\psi\rangle \in L^1[0,T]$ almost surely. In the same way, it follows that $\langle \boldsymbol{B}(\cdot,X(\cdot)),\psi\rangle \in L^0_{\mathbb{F}}(\Omega;L^2([0,T]\times\mathbb{N}))$ for each $\psi\in D(A')$.

For mild solutions and $\gamma_F,\gamma_B \le 0$ we obtain

$$\begin{aligned}\big\|e^{-(t-(\cdot))A}F(\cdot,X(\cdot))\big\|_{L^p(U;L^1[0,t])} &= \big\|e^{(t-(\cdot))\nu}e^{-(t-(\cdot))A_\nu}F(\cdot,X(\cdot))\big\|_{L^p(U;L^1[0,t])}\\ &\le Ce^{\nu T}\big\|e^{-(t-(\cdot))A_\nu}A_\nu^{-\gamma_F}F(\cdot,X(\cdot))\big\|_{L^p(U;L^1[0,t])}\\ &\le CC_Te^{\nu T}\|A_\nu^{-\gamma_F}F(\cdot,X(\cdot))\|_{L^p(U;L^q[0,T])}\\ &\le CC_TC_Fe^{\nu T}(1+\|A_\nu^{\gamma}X\|_{L^p(U;L^q[0,T])})\end{aligned}$$

by Remark 3.2.4. Similarly, we have $e^{-(t-(\cdot))A}\boldsymbol{B}(\cdot,X(\cdot)) \in L^0_{\mathbb{F}}(\Omega : L^p(U;L^2([0,t]\times\mathbb{N}))$ for every $t\in[0,T]$.

b) Observe that

$$-AX(t)+F(t,X(t)) = -(\nu+A)X(t)+\big(\nu X(t)+F(t,X(t))\big).$$

Moreover, the function $F_\nu$ defined by

$$F_\nu(t,X(t)) := \nu X(t)+F(t,X(t))$$

satisfies assumption (HF) if and only $F$ satisfies (HF) with slightly modified Lipschitz and linear growth constants. Therefore, in the following we may replace $A$ and $F$ by $\nu+A$ and $F_\nu$ and assume, without loss of generality, that $\nu=0$ and $0\in\rho(A)$ in (HA).

**Proposition 3.5.6.** *a) If Hypothesis 3.5.4 is satisfied for some $\gamma \ge 1$, a process $X\colon\Omega\times[0,T]\to D(A^\gamma)$ is a strong $(r,p,q)$ solution of (3.2) if and only if it is a mild $(r,p,q)$ solution of (3.2).*

*b) If Hypothesis 3.5.4 is satisfied for some $\gamma_F,\gamma_B\le 0$, a process $X\colon\Omega\times[0,T]\to D(A^\gamma)$ is a weak $(r,p,q)$ solution of (3.2) if and only if it is a mild $(r,p,q)$ solution of (3.2).*

**Proof.** a) Let $X$ be a mild $(r,p,q)$ solution. Then by Theorems 3.3.9 and 3.4.10 the process $X$ takes values in $D(A)$ almost surely. Therefore, we have

$$
\begin{aligned}
&x_0 - \int_0^t AX(s) + F(s,X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s) \\
&\quad = x_0 - \int_0^t A\Big[e^{-sA}x_0 + \int_0^s e^{-(s-\tau)A}F(\tau,X(\tau))\,\mathrm{d}\tau + \int_0^s e^{-(s-\tau)A}\boldsymbol{B}(\tau,X(\tau))\,\mathrm{d}\boldsymbol{\beta}(\tau)\Big] \\
&\qquad + F(s,X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s) \\
&\quad = x_0 - \int_0^t Ae^{-sA}x_0\,\mathrm{d}s - \int_0^t\int_0^s Ae^{-(s-\tau)A}F(\tau,X(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad - \int_0^t\int_0^s Ae^{-(s-\tau)A}\boldsymbol{B}(\tau,X(\tau))\,\mathrm{d}\boldsymbol{\beta}(\tau)\,\mathrm{d}s + \int_0^t F(s,X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s) \\
&\quad = e^{-tA}x_0 - \int_0^t\int_\tau^t Ae^{-(s-\tau)A}F(\tau,X(\tau))\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t F(s,X(s))\,\mathrm{d}s \\
&\qquad - \int_0^t\int_\tau^t Ae^{-(s-\tau)A}\boldsymbol{B}(\tau,X(\tau))\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\beta}(\tau) + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s) \\
&\quad = e^{-tA}x_0 + \int_0^t e^{-(t-\tau)A}F(\tau,X(\tau))\,\mathrm{d}\tau + \int_0^t e^{-(t-\tau)A}\boldsymbol{B}(\tau,X(\tau))\,\mathrm{d}\boldsymbol{\beta}(\tau) = X(t),
\end{aligned}
$$

i.e. $X(t)$, $t\in[0,T]$, is a strong $(r,p,q)$ solution.

Now assume the converse. For any fixed $g\in D(A')$ we define the function

$$f\colon [0,t]\times L^p(U)\to\mathbb{C} \quad \text{as} \quad f(s,x) = \langle x, e^{-(t-s)A'}g\rangle.$$

Then $f\in C^{1,2}([0,t]\times L^p(U))$ and

$$\partial_s f(s,x) = \langle x, A'e^{-(t-s)A'}g\rangle, \quad \partial_x f(s,x) = \langle \cdot, e^{-(t-s)A'}g\rangle, \quad \partial_x^2 f(s,x) = 0.$$

By Itô's formula we obtain for the Itô process $X$

$$f(t,X(t)) - f(0,X(0)) = \int_0^t \partial_s f(s,X(s))\,\mathrm{d}s + \int_0^t \partial_x f(s,X(s))\,\mathrm{d}X(s).$$

In other words

$$
\begin{aligned}
\langle X(t),g\rangle - \langle e^{-tA}x_0,g\rangle &= \int_0^t \langle X(s), A'e^{-(t-s)A'}g\rangle + \int_0^t \langle -AX(s)+F(s,X(s)), e^{-(t-s)A'}g\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \langle \boldsymbol{B}(s,X(s)), e^{-(t-s)A'}g\rangle\,\mathrm{d}\boldsymbol{\beta}(s) \\
&= \int_0^t \langle e^{-(t-s)A}F(s,X(s)),g\rangle\,\mathrm{d}s + \int_0^t \langle e^{-(t-s)A}\boldsymbol{B}(s,X(s)),g\rangle\,\mathrm{d}\boldsymbol{\beta}(s).
\end{aligned}
$$

And therefore,

$$X(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F(s,X(s))\,\mathrm{d}s + \int_0^t e^{-(t-s)A}\boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s),$$

i.e. $X(t)$, $t\in[0,T]$, is a mild $(r,p,q)$ solution.

b) Let $X$ be a mild $(r,p,q)$ solution. Then we use the identities

$$e^{-tA'}\psi = -\int_0^t A'e^{-sA'}\psi\,\mathrm{d}s + \psi,$$
$$e^{-(t-s)A'}\psi = -\int_s^t A'e^{-(r-s)A'}\psi\,\mathrm{d}r + \psi,$$

for $\psi \in D(A')$. Then we obtain by the deterministic and stochastic Fubini theorem

$$\begin{aligned}
\langle X(t),\psi\rangle &= \langle e^{-tA}x_0,\psi\rangle + \int_0^t \langle e^{-(t-s)A}F(s,X(s)),\psi\rangle\,\mathrm{d}s + \int_0^t \langle e^{-(t-s)A}\boldsymbol{B}(s,X(s)),\psi\rangle^{L^2}\,\mathrm{d}\boldsymbol{\beta}(s)\\
&= \langle x_0,\psi\rangle - \int_0^t \langle x_0, A'e^{-sA'}\psi\rangle\,\mathrm{d}s\\
&\quad + \int_0^t \langle F(s,X(s)),\psi\rangle\,\mathrm{d}s - \int_0^t\int_s^t \langle F(s,X(s)), A'e^{-(r-s)A'}\psi\rangle\,\mathrm{d}r\,\mathrm{d}s\\
&\quad + \int_0^t \langle \boldsymbol{B}(s,X(s)),\psi\rangle^{L^2}\,\mathrm{d}\boldsymbol{\beta}(s) - \int_0^t\int_s^t \langle \boldsymbol{B}(s,X(s)), A'e^{-(r-s)A'}\psi\rangle^{L^2}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\beta}(s)\\
&= \langle x_0,\psi\rangle - \int_0^t \langle e^{-sA}x_0, A'\psi\rangle\,\mathrm{d}s\\
&\quad + \int_0^t \langle F(s,X(s)),\psi\rangle\,\mathrm{d}s - \int_0^t \Big\langle \int_0^r e^{-(r-s)A}F(s,X(s))\,\mathrm{d}s, A'\psi\Big\rangle\,\mathrm{d}r\\
&\quad + \int_0^t \langle \boldsymbol{B}(s,X(s)),\psi\rangle^{L^2}\,\mathrm{d}\boldsymbol{\beta}(s) - \int_0^t \Big\langle \int_0^r e^{-(r-s)A}\boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s), A'\psi\Big\rangle\,\mathrm{d}r\\
&= \langle x_0,\psi\rangle - \int_0^t \langle X(s), A'\psi\rangle\,\mathrm{d}s + \int_0^t \langle F(s,X(s)),\psi\rangle\,\mathrm{d}s + \int_0^t \langle \boldsymbol{B}(s,X(s)),\psi\rangle^{L^2}\,\mathrm{d}\boldsymbol{\beta}(s),
\end{aligned}$$

which means that $X$ is a weak $(r,p,q)$ solution.

Let $X$ be a weak $(r,p,q)$ solution and $z \in C^1([0,T];D(A'))$ of the form

$$z(t) = \varphi(t)\psi, \quad \varphi \in C^1[0,T],\ \psi \in D(A').$$

Using that $\varphi(t) = \varphi(0) + \int_0^t \varphi'(s)\,\mathrm{d}s$, we obtain by Itô's formula (see Corollary 1.3.17)

$$\begin{aligned}
\langle X(t), z(t)\rangle &= \varphi(t)\langle X(t),\psi\rangle\\
&= \varphi(0)\langle X(0),\psi\rangle + \int_0^t \big(\langle F(s,X(s)),\psi\rangle - \langle X(s),A'\psi\rangle\big)\varphi(s) + \varphi'(s)\langle X(s),\psi\rangle\,\mathrm{d}s\\
&\quad + \int_0^t \langle \boldsymbol{B}(s,X(s)),\psi\rangle\varphi(s)\,\mathrm{d}\boldsymbol{\beta}(s)\\
&= \langle X(0), z(0)\rangle + \int_0^t \langle X(s), z'(s) - A'z(s)\rangle + \langle F(s,X(s)), z(s)\rangle\,\mathrm{d}s\\
&\quad + \int_0^t \langle \boldsymbol{B}(s,X(s)), z(s)\rangle\,\mathrm{d}\boldsymbol{\beta}(s)
\end{aligned}$$

Since linear combinations of such functions are dense in $C^1([0,T];D(A'))$, this equality also holds for general $z \in C^1([0,T];D(A'))$. Now take

$$z(s) := e^{-(t-s)A'}\psi, \quad \psi \in D(A').$$

Then $z'(s) = A'z(s)$ and the identity above is equivalent to

$$\langle X(t), z(t)\rangle = \langle X(0), z(0)\rangle + \int_0^t \langle F(s, X(s)), z(s)\rangle \,\mathrm{d}s + \int_0^t \langle \boldsymbol{B}(s, X(s)), z(s)\rangle \,\mathrm{d}\boldsymbol{\beta}(s),$$

which in turn is

$$\langle X(t), \psi\rangle = \langle e^{-tA}x_0, \psi\rangle + \int_0^t \langle e^{-(t-s)A}F(s, X(s)), \psi\rangle \,\mathrm{d}s + \int_0^t \langle e^{-(t-s)A}\boldsymbol{B}(s, X(s)), \psi\rangle^{L^2} \,\mathrm{d}\boldsymbol{\beta}(s).$$

This implies that $X$ is a mild $(r, p, q)$ solution. □

To prepare the next results, let us denote by $K_{\text{det}} > 0$ and $K^{(\sigma)}_{\text{det}} > 0$, as well as $K_{\text{stoch}} > 0$ and $K^{(\sigma)}_{\text{stoch}} > 0$ the constants from Theorems 3.3.9 and 3.4.10 (where the index $\sigma$ refers to the Sobolev estimates). By proceeding as in Remark 3.5.5 one should note that these constants depend on $\nu$, in general.

**Theorem 3.5.7 (Existence and Uniqueness).** *Let Hypothesis 3.5.4 be satisfied, and $\gamma_F, \gamma_B \leq 0$ such that $\gamma + \gamma_F \in [0, 1]$ and $\gamma + \gamma_B \in [0, 1/2]$. If the Lipschitz constants $L_F$ and $L_B$ satisfy*

$$L_F K_{det} + L_B K_{stoch} < 1,$$

*in the case of $\gamma + \gamma_F = 1$ or $\gamma + \gamma_B = 1/2$, then the following assertions hold true:*

a) *If $x_0 \in L^r(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$, then (3.2) has a unique mild $(r, p, q)$ solution $X$ satisfying the a-priori estimate*

$$\|A^\gamma_\nu X\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \leq C(1 + \|x_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma-1/q))}).$$

b) *If $x_0 \in L^0(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$, then (3.2) has a unique mild $(0, p, q)$ solution $X$ satisfying $A^\gamma_\nu X \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0, T]))$.*

**Proof.** We split the proof in two parts, one for the maximal regularity case $\gamma + \gamma_F = 1$ and $\gamma + \gamma_B = 1/2$ and the other, if one of these conditions is not satisfied. As indicated in Remark 3.5.5 we will assume, without loss of generality, that $\nu = 0$. Moreover, we assume that $\widetilde{L}_F = \widetilde{L}_B = 0$, since it follows from a combination of part I and II below.

I.1) We start with the seemingly 'easier' case of maximal regularity. The proof here is a little bit shorter, but we need some smallness assumption on our constants. Moreover, we should not forget, that the hard work in this case was done in the previous sections. So let $\gamma + \gamma_F = 1$ and $\gamma + \gamma_B = 1/2$ and set $\theta := L_F K_{\text{det}} + L_B K_{\text{stoch}} \in [0, 1)$. Then we define the operators

$$L(X)(t) := e^{-tA}x_0 + \int_0^t e^{-(t-s)A} F(s, X(s))\, \mathrm{d}s + \int_0^t e^{-(t-s)A} \boldsymbol{B}(s, X(s))\, \mathrm{d}\boldsymbol{\beta}(s) \quad \text{and}$$
$$L^\gamma(Y)(t) := A^\gamma e^{-tA}x_0 + A^\gamma \int_0^t e^{-(t-s)A} F(s, A^{-\gamma}Y(s))\, \mathrm{d}s + \int_0^t e^{-(t-s)A} \boldsymbol{B}(s, A^{-\gamma}Y(s))\, \mathrm{d}\boldsymbol{\beta}(s).$$

We will now show that $L^\gamma$ is a well-defined contraction on the fixed point space $E := L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, T]))$. If $Y$ is the unique fixed point of $L^\gamma$, then $X := A^{-\gamma}Y$ is the unique fixed point of $L$, making $X$ the unique mild $(r, p, q)$ solution of (3.2).

By Proposition 3.2.12, Theorem 3.3.9 and Theorem 3.4.10, and our assumptions we have

$$\begin{aligned}
\big\|A^\gamma e^{-(\cdot)A}x_0\big\|_E &\le C\|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}, \\
\Big\| A^\gamma \int_0^{(\cdot)} e^{-((\cdot)-s)A} F(s, A^{-\gamma}Y(s))\, \mathrm{d}s \Big\|_E &\le K_{\text{det}} \|A^{-\gamma_F} F(\cdot, A^{-\gamma}Y(\cdot))\|_E \\
&\le K_{\text{det}} C_F (1 + \|Y\|_E), \\
\Big\| A^\gamma \int_0^{(\cdot)} e^{-((\cdot)-s)A} \boldsymbol{B}(s, A^{-\gamma}Y(s))\, \mathrm{d}\boldsymbol{\beta}(s) \Big\|_E &\le K_{\text{stoch}} \|A^{-\gamma_B} \boldsymbol{B}(\cdot, A^{-\gamma}Y(\cdot))\|_{E(\ell^2)} \\
&\le K_{\text{stoch}} C_B (1 + \|Y\|_E),
\end{aligned}$$

for some constant $C > 0$ and any $Y \in E$, so $L^\gamma \colon E \to E$ is well-defined. It is also a contraction since by Theorems 3.3.9 and 3.4.10, and the Lipschitz properties of $F$ and $\boldsymbol{B}$ we have

$$\begin{aligned}
\|L^\gamma(Y) - L^\gamma(Z)\|_E &\le K_{\text{det}} \big\|A^{-\gamma_F}\big(F(\cdot, A^{-\gamma}Y) - F(\cdot, A^{-\gamma}Z)\big)\big\|_E \\
&\quad + K_{\text{stoch}} \big\|A^{-\gamma_B}\big(\boldsymbol{B}(\cdot, A^{-\gamma}Y) - \boldsymbol{B}(\cdot, A^{-\gamma}Z)\big)\big\|_{E(\ell^2)} \\
&\le K_{\text{det}} L_F \|Y - Z\|_E + K_{\text{stoch}} L_B \|Y - Z\|_E \\
&= \theta \|Y - Z\|_E.
\end{aligned}$$

By the Banach fixed point theorem, $L^\gamma$ has a unique fixed point $Y \in E$, and as stated above $X := A^{-\gamma}Y$ is the unique mild $(r, p, q)$ solution we were looking for. To obtain the a-priori estimate we use the contractivity of $L^\gamma$. Then

$$\begin{aligned}
\|A^\gamma X\|_E = \|Y\|_E = \|L^\gamma(Y)\|_E &\le \|L^\gamma(Y) - L^\gamma(0)\|_E + \|L^\gamma(0)\|_E \\
&\le \theta\|Y\|_E + C\|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))} + K_{\text{det}} C_F + K_{\text{stoch}} C_B \\
&= \theta \|A^\gamma X\|_E + \widetilde{C}(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}).
\end{aligned}$$

Since $\theta \in [0, 1)$, this is equivalent to

$$\|A^\gamma X\|_E \le \frac{\widetilde{C}}{1 - \theta}(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}).$$

I.2) Now let $x_0 \in L^0(\Omega, \mathcal{F}_0; D_A^{\ell q}(\gamma - 1/q))$, and define the set

$$\Gamma_n := \{\|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))} \leq n\}, \quad n \in \mathbb{N},$$

as well as $x_{0,n} := \mathbb{1}_{\Gamma_n} x_0 \in L^r(\Omega, \mathcal{F}_0; D_A^{\ell q}(\gamma - 1/q))$. By the first step we obtain a unique mild $(r,p,q)$ solution $X_n$ of (3.2) such that $Y_n := A^\gamma X_n \in E$. Since $\Gamma_m \in \mathcal{F}_0$ we obtain for $m \leq n$

$$\begin{aligned}\big\|\mathbb{1}_{\Gamma_m}(Y_m - Y_n)\big\|_E = \big\|\mathbb{1}_{\Gamma_m}\big(L^\gamma(Y_m) - L^\gamma(Y_n)\big)\big\|_E &= \big\|\mathbb{1}_{\Gamma_m}\big(L^\gamma(\mathbb{1}_{\Gamma_m}Y_m) - L^\gamma(\mathbb{1}_{\Gamma_m}Y_n)\big)\big\|_E \\ &\leq \big\|L^\gamma(\mathbb{1}_{\Gamma_m}Y_m) - L^\gamma(\mathbb{1}_{\Gamma_m}Y_n)\big\|_E \\ &\leq \theta\big\|\mathbb{1}_{\Gamma_m}(Y_m - Y_n)\big\|_E.\end{aligned}$$

Hence, $Y_m = Y_n$ (and $X_m = X_n$) on $\Gamma_m$ for $m \leq n$. Now we define

$$X := X_n \quad \text{on } \Gamma_n.$$

Then $X$ is well-defined and $X = L(X)$ almost surely. Moreover, $X \in D(A^\gamma)$ almost surely and satisfies $A^\gamma X \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. This proves the existence of a mild $(0,p,q)$ solution. It remains to prove its uniqueness. So let $X$ and $Z$ be two solutions for the same initial value $x_0$. Then we define the stopping times

$$\begin{aligned}\tau_n^X(\omega) &:= T \wedge \inf\big\{t \in [0,T] \colon \|\mathbb{1}_{[0,t]}A^\gamma X(\omega)\|_{L^p(U;L^q[0,T])} \geq n\big\},\\ \tau_n^Z(\omega) &:= T \wedge \inf\big\{t \in [0,T] \colon \|\mathbb{1}_{[0,t]}A^\gamma Z(\omega)\|_{L^p(U;L^q[0,T])} \geq n\big\}\end{aligned}$$

and $\tau_n := \tau_n^X \wedge \tau_n^Z$. Then it suffices to show that the processes

$$U_n := \mathbb{1}_{[0,\tau_n]}A^\gamma X \quad \text{and} \quad V_n := \mathbb{1}_{[0,\tau_n]}A^\gamma Z$$

are equal almost surely for each $n \in \mathbb{N}$. By Proposition 1.3.13 we have

$$\begin{aligned}U_n(t) &= \mathbb{1}_{[0,\tau_n]}(t)A^\gamma X(t) = \mathbb{1}_{[0,\tau_n]}(t)A^\gamma L(X)(t)\\ &= \mathbb{1}_{[0,\tau_n]}(t)A^\gamma e^{-tA}x_0 + \mathbb{1}_{[0,\tau_n]}(t)A^\gamma\int_0^t \mathbb{1}_{[0,\tau_n]}(s)e^{-(t-s)A}F(s, A^{-\gamma}U_n(s))\,\mathrm{d}s\\ &\quad + \mathbb{1}_{[0,\tau_n]}(t)\int_0^t \mathbb{1}_{[0,\tau_n]}(s)e^{-(t-s)A}\boldsymbol{B}(s, A^{-\gamma}U_n(s))\,\mathrm{d}\boldsymbol{\beta}(s),\end{aligned}$$

and similarly for $V_n$. This implies that

$$\begin{aligned}\|U_n - V_n\|_E &\leq K_{\text{det}}\big\|\mathbb{1}_{[0,\tau_n]}A^{-\gamma_F}\big(F(\cdot, A^{-\gamma}U_n) - F(\cdot, A^{-\gamma}V_n)\big)\big\|_E\\ &\qquad + K_{\text{stoch}}\big\|\mathbb{1}_{[0,\tau_n]}A^{-\gamma_B}\big(\boldsymbol{B}(\cdot, A^{-\gamma}U_n) - \boldsymbol{B}(\cdot, A^{-\gamma}V_n)\big)\big\|_{E(\ell^2)}\\ &\leq \theta\|U_n - V_n\|_E.\end{aligned}$$

Since $\theta < 1$, this yields $U_n = V_n$ almost surely.

II.1) If $\gamma+\gamma_F<1$ or $\gamma+\gamma_B<1/2$ we get similar estimates as in the first case. Here we have the opportunity that the parameter $T$ is still in play. (However, observe that if one of the parameter sums satisfies $\gamma+\gamma_F=1$ or $\gamma+\gamma_B=1/2$, then we still need the smallness assumption on $L_FK_{\mathrm{det}}+L_BK_{\mathrm{stoch}}$, where these constants now might depend on $T$). Therefore, we only consider the case $\gamma+\gamma_F<1$ and $\gamma+\gamma_B<1/2$ here. Let $L$ and $L^\gamma$ be the same mappings as in I.1), but this time we define as the fixed point space

$$E_\kappa := L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,\kappa])), \quad \kappa\in[0,T].$$

Then, of course, $L^\gamma\colon E_\kappa\to E_\kappa$ is still well-defined by Proposition 3.2.12, and Propositions 3.3.1 and 3.4.1. And by these results we also obtain constants $c_{\mathrm{det}}, c_{\mathrm{stoch}}>0$ such that

$$\begin{aligned}\|L^\gamma(Y)-L^\gamma(Z)\|_{E_\kappa} &\le c_{\mathrm{det}}\kappa^{1-\gamma-\gamma_F}\big\|A^{-\gamma_F}\big(F(\cdot,A^{-\gamma}Y)-F(\cdot,A^{-\gamma}Z)\big)\big\|_{E_\kappa}\\ &\quad + c_{\mathrm{stoch}}\kappa^{1/2-\gamma-\gamma_B}\big\|A^{-\gamma_B}\big(\boldsymbol{B}(\cdot,A^{-\gamma}Y)-\boldsymbol{B}(\cdot,A^{-\gamma}Z)\big)\big\|_{E_\kappa(\ell^2)}\\ &\le c_{\mathrm{det}}L_F\kappa^{1-\gamma-\gamma_F}\|Y-Z\|_{E_\kappa}+c_{\mathrm{stoch}}L_B\kappa^{1/2-\gamma-\gamma_B}\|Y-Z\|_{E_\kappa}\\ &=\theta(\kappa)\|Y-Z\|_{E_\kappa},\end{aligned}$$

where $\theta(\kappa):=c_{\mathrm{det}}L_F\kappa^{1-\gamma-\gamma_F}+c_{\mathrm{stoch}}L_B\kappa^{1/2-\gamma-\gamma_B}$. In this case, we can choose $\kappa\in[0,T]$ small enough such that $\theta(\kappa)<1$, or in other words $L^\gamma\colon E_\kappa\to E_\kappa$ is a contraction. Then we get a unique fixed point $Y_1\in E_\kappa$, and $X_1:=A^{-\gamma}Y_1$ is the unique mild $(r,p,q)$ solution of (3.2) on $[0,\kappa]$. Restricted on this interval the first part of this proof can now be repeated in the exact same way giving us for one thing the a-priori estimate

$$\|A^\gamma X_1\|_{E_\kappa}\le C(\kappa)(1+\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}),$$

and $A^\gamma X_1\in L^0_{\mathbb{F}}(\Omega;L^p(U;L^q[0,\kappa]))$ constructed as in part I.2) is the unique mild $(0,p,q)$ solution if $x_0\in L^0(\Omega,\mathcal{F}_0;D_A^{\ell q}(\gamma-1/q))$. Before we continue, we want to remark that the constant $C(\kappa)$ of the a-priori estimate, derived as in the first part, depends on the Lipschitz constants $L_F$ and $L_B$. For later reference, we want to point out that this is not necessary in this case. By Proposition 3.2.12, and Propositions 3.3.1 and 3.4.1 we obtain a constant $C>0$ such that

$$\begin{aligned}\|A^\gamma X\|_{E_\kappa} &= \|L^\gamma(A^\gamma X)\|_{E_\kappa}\\ &\le C\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}+\big(K_{\mathrm{det}}C_F\kappa^{1-\gamma-\gamma_F}+K_{\mathrm{stoch}}C_B\kappa^{1/2-\gamma-\gamma_B}\big)(1+\|A^\gamma X\|_{E_\kappa})\\ &= C\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}+c(\kappa)+c(\kappa)\|A^\gamma X\|_{E_\kappa}.\end{aligned}$$

We may choose $\kappa$ even a little bit smaller as above, such that

$$c(\kappa):=K_{\mathrm{det}}C_F\kappa^{1-\gamma-\gamma_F}+K_{\mathrm{stoch}}C_B\kappa^{1/2-\gamma-\gamma_B}<1.$$

Then the estimate above leads to

$$\|A^\gamma X\|_{E_\kappa} \le \frac{C \vee c(\kappa)}{1-c(\kappa)}(1+\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}).$$

II.2) In the next step, we extend the solutions found in II.1) on some interval $[0,\kappa]$ to the next interval $[\kappa, 2\kappa]$. If we continue to do this procedure finitely many times, we will finally get a solution on the whole interval $[0,T]$. Let $x_{0,\kappa} := X_1(\kappa) \in L^0(\Omega, \mathcal{F}_\kappa; D_A^{\ell q}(\gamma - 1/q))$ and define

$$F^\kappa(s,\phi(s)) := F(s+\kappa,\phi(s)), \quad \boldsymbol{B}^{\boldsymbol{\kappa}}(s,\phi(s)) := \boldsymbol{B}(s+\kappa,\phi(s)),$$

as well as

$$\boldsymbol{\beta}^{\boldsymbol{\kappa}}(s) = (\beta_n^\kappa(s))_{n\in\mathbb{N}} := (\beta_n(s+\kappa)-\beta_n(\kappa))_{n\in\mathbb{N}}.$$

Then $F^\kappa$ and $\boldsymbol{B}^{\boldsymbol{\kappa}}$ still satisfy (HF) and (HB) on $[0,\kappa]$ with the same constants as before. Also, $\boldsymbol{\beta}^{\boldsymbol{\kappa}}$ is a sequence of independent Brownian motions adapted to the filtration $\mathbb{F}^\kappa = (\mathcal{F}_t^\kappa)_{t\ge 0} := (\mathcal{F}_{t+\kappa})_{t\ge 0}$. If we replace $x_0$ by $x_{0,\kappa}$, $F$ by $F^\kappa$, $\boldsymbol{B}$ by $\boldsymbol{B}^{\boldsymbol{\kappa}}$, $\boldsymbol{\beta}$ by $\boldsymbol{\beta}^{\boldsymbol{\kappa}}$, and $\mathbb{F}$ by $\mathbb{F}^\kappa$ in our fixed point operators $L$ and $L^\gamma$ we can construct an $(r,p,q)$ solution $\widetilde{X}$ on $[0,\kappa]$ as before. Then we define

$$\begin{aligned} X_2(t) &:= X_1(t) \quad \text{for } t\in[0,\kappa],\\ X_2(t) &:= \widetilde{X}(t-\kappa) \quad \text{for } t\in[\kappa,2\kappa]. \end{aligned}$$

The process $X_2$ is then an element of $E_{2\kappa}$, which also satisfies $X_2(t) = X_1(t) = L(X_1)(t) = L(X_2)(t)$ for $t\in[0,\kappa]$ and

$$\begin{aligned} X_2(t) &= \widetilde{X}(t-\kappa) = L(\widetilde{X})(t-\kappa)\\ &= e^{-(t-\kappa)A}x_{0,\kappa} + \int_0^{t-\kappa} e^{(t-\kappa-s)A}F^\kappa(s,\widetilde{X}(s))\,\mathrm{d}s + \int_0^{t-\kappa} e^{-(t-\kappa-s)A}\boldsymbol{B}^{\boldsymbol{\kappa}}(s,\widetilde{X}(s))\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{\kappa}}(s)\\ &= e^{-(t-\kappa)A}e^{-\kappa A}x_0 + e^{-(t-\kappa)A}\int_0^\kappa e^{-(\kappa-s)A}F(s,X_1(s))\,\mathrm{d}s\\ &\quad + e^{-(t-\kappa)A}\int_0^\kappa e^{-(\kappa-s)A}\boldsymbol{B}(s,X_1(s))\,\mathrm{d}\boldsymbol{\beta}(s) + \int_\kappa^t e^{(t-s)A}F(s,\widetilde{X}(s-\kappa))\,\mathrm{d}s\\ &\quad + \int_\kappa^t e^{-(t-s)A}\boldsymbol{B}(s,\widetilde{X}(s-\kappa))\,\mathrm{d}\boldsymbol{\beta}(s)\\ &= e^{-tA}x_0 + \int_0^\kappa e^{-(t-s)A}F(s,X_2(s))\,\mathrm{d}s + \int_0^\kappa e^{-(t-s)A}\boldsymbol{B}(s,X_2(s))\,\mathrm{d}\boldsymbol{\beta}(s)\\ &\quad + \int_\kappa^t e^{-(t-s)A}F(s,X_2(s))\,\mathrm{d}s + \int_\kappa^t e^{-(t-s)A}\boldsymbol{B}(s,X_2(s))\,\mathrm{d}\boldsymbol{\beta}(s)\\ &= L(X_2)(t), \qquad t\in[\kappa,2\kappa], \end{aligned}$$

i.e. $X_2$ is mild $(0,p,q)$ solution on $[0,2\kappa]$. Iterating this till we reach $[0,T]$ we get a mild $(0,p,q)$ solution $X$. In the next part of the proof we will show that this solution is indeed unique. So let $Z \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$ be another solution. By the uniqueness result on $[0,\kappa]$ we have $X|_{[0,\kappa]} = Z|_{[0,\kappa]}$, in particular $X(\kappa) = Z(\kappa)$ almost surely. By uniqueness on the interval $[\kappa, 2\kappa]$ we then obtain that the mild solutions $X|_{[\kappa,2\kappa]}$ and $Z|_{[\kappa,2\kappa]}$ are equal almost surely. Again, iterating this finitely many times, we get $X = Z$ on $[0,T]$. It remains to check the a-priori estimate in this case for the whole interval. We recall that by II.1) we have

$$\|A^\gamma X\|_{E_\kappa} \leq C(\kappa)(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}).$$

In the following theorem we will show that

$$\|X\|_{L^r(\Omega; C([0,T]; D_A^{\ell q}(\gamma - 1/q)))} \leq C(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}),$$

in particular,

$$\|X(\kappa)\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))} \leq C(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}).$$

Using $X(\kappa)$ as the new initial value for $X$ on $[\kappa, 2\kappa]$, we obtain in the same way

$$\|A^\gamma X\|_{E_{2\kappa}} \leq C(1 + \|X(\kappa)\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}) \leq \widetilde{C}(1 + \|x_0\|_{L^r(\Omega; D_A^{\ell q}(\gamma - 1/q))}).$$

Repeating this till we arrive at the interval $[0,T]$, this implies the claim. □

**REMARK 3.5.8.**

a) In the case $\gamma \geq 1$, in particular in the very important case $\gamma = 1$, Proposition 3.5.6 and Theorem 3.5.7 imply that (3.2) has a unique strong $(r,p,q)$ solution $X$ satisfying the estimates of Theorem 3.5.7.

b) We want to remark, that in the case of $\gamma + \gamma_F < 1$ and $\gamma + \gamma_B < 1/2$, $\ell^q$-sectoriality and a stronger assumption on the initial value (e.g. $x_0 \in D(A_\nu^\gamma)$ almost surely) would suffice to obtain similar results.

Now that we have found our solution we want to prove higher regularity properties. Here we benefit highly from the 'regularity swapping results' we proved for deterministic and stochastic convolutions.

**THEOREM 3.5.9 (Regularity).** *Under the assumptions of the previous theorem the mild $(r,p,q)$ solution $X$ of (3.2) satisfies the following regularity properties:*

$$A_\nu^{\gamma - \sigma} X \in L^0_{\mathbb{F}}(\Omega; L^p(U; W^{\sigma,q}[0,T])), \quad \sigma \in [0, 1/2),\ \sigma \leq \gamma.$$

*In particular,*

$$X \in L^0_{\mathbb{F}}(\Omega; C([0,T]; D^{\ell q}_{A_\nu}(\gamma - 1/q))),$$

*and if $q > 2$ we have*

$$A_\nu^{\gamma-\sigma} X \in L^0_{\mathbb{F}}(\Omega; L^p(U; C^{\sigma - 1/q}[0,T])), \quad \sigma \in (1/q, 1/2),\ \sigma \le \gamma.$$

*If $x_0 \in L^r(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$ for some $r \in (1,\infty)$, we find a constant $C > 0$ such that*

$$\|A_\nu^{\gamma-\sigma} X\|_{L^r(\Omega; L^p(U; W^{\sigma,q}[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma - 1/q))}), \quad \sigma \in [0, 1/2),\ \sigma \le \gamma,$$

$$\|X\|_{L^r(\Omega; C([0,T]; D^{\ell q}_{A_\nu}(\gamma - 1/q)))} \le C(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma - 1/q))}),$$

$$\|A_\nu^{\gamma-\sigma} X\|_{L^r(\Omega; L^p(U; C^{\sigma - 1/q}[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma - 1/q))}), \quad \sigma \in (1/q, 1/2),\ \sigma \le \gamma,\ q > 2.$$

*Moreover, in the case of $\gamma + \gamma_F < 1$ and $\gamma + \gamma_B < 1/2$ let $\varepsilon_\gamma := (1 - \gamma - \gamma_F) \wedge (1/2 - \gamma - \gamma_B)$. Then we have for each $\varepsilon \in [0, \varepsilon_\gamma)$ and $x_0 \in L^0(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma + \varepsilon - 1/q))$ the estimate*

$$A_\nu^{\gamma+\varepsilon-\sigma} X \in L^0_{\mathbb{F}}(\Omega; L^p(U; W^{\sigma,q}[0,T])), \quad \sigma \in [0, 1/2),\ \sigma \le \gamma + \varepsilon.$$

*In particular,*

$$X \in L^0_{\mathbb{F}}(\Omega; C([0,T]; D^{\ell q}_{A_\nu}(\gamma + \varepsilon - 1/q))),$$

$$A_\nu^{\gamma+\varepsilon-\sigma} X \in L^0_{\mathbb{F}}(\Omega; L^p(U; C^{\sigma - 1/q}[0,T])), \quad \sigma \in (1/q, 1/2),\ \sigma \le \gamma + \varepsilon,\ q > 2.$$

*And if $x_0 \in L^r(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma + \varepsilon - 1/q))$ for some $r \in (1,\infty)$ we have for $\sigma \le \gamma + \varepsilon$*

$$\|A_\nu^{\gamma+\varepsilon-\sigma} X\|_{L^r(\Omega; L^p(U; W^{\sigma,q}[0,T]))} \le C_T(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma+\varepsilon - 1/q))}), \quad \sigma \in [0, 1/2),$$

$$\|X\|_{L^r(\Omega; C([0,T]; D^{\ell q}_{A_\nu}(\gamma+\varepsilon - 1/q)))} \le C_T(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma+\varepsilon - 1/q))}),$$

$$\|A_\nu^{\gamma+\varepsilon-\sigma} X\|_{L^r(\Omega; L^p(U; C^{\sigma - 1/q}[0,T]))} \le C_T(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_{A_\nu}(\gamma+\varepsilon - 1/q))}), \quad \sigma \in (1/q, 1/2),\ q > 2.$$

**Proof.** Without loss of generality let $\nu = 0$, and let $X$ be the unique mild $(r,p,q)$ solution of (3.2). Then by Proposition 3.2.12, 3.3.9, 3.4.10 and the a-priori estimate of the previous theorem the following estimates hold

$$\begin{aligned}
\|A^{\gamma-\sigma} X\|_{L^r(\Omega; L^p(U; W^{\sigma,q}[0,T]))} &= \|A^{\gamma-\sigma} L(X)\|_{L^r(\Omega; L^p(U; W^{\sigma,q}[0,T]))} \\
&\le C\|x_0\|_{L^r(\Omega; D^{\ell q}_A(\gamma - 1/q))} + K^{(\sigma)}_{\text{det}} C_F(1 + \|A^\gamma X\|_{L^r(\Omega; L^p(U; L^q[0,T]))}) \\
&\quad + K^{(\sigma)}_{\text{stoch}} C_B(1 + \|A^\gamma X\|_{L^r(\Omega; L^p(U; L^q[0,T]))}) \\
&\le \widetilde{C}(1 + \|x_0\|_{L^r(\Omega; D^{\ell q}_A(\gamma - 1/q))}).
\end{aligned}$$

Now if $\gamma + \gamma_F < 1$ and $\gamma + \gamma_B < 1/2$, choose any $\varepsilon \in [0, \varepsilon_\gamma)$. Then similar calculations as above using Proposition 3.2.12, and Propositions 3.3.2 and 3.4.3 lead to

$$\begin{aligned}\|A^{\gamma+\varepsilon-\sigma}X\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} &= \|A^{\gamma+\varepsilon-\sigma}L(X)\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))}\\ &\le C\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma+\varepsilon-1/q))} + c_{\mathrm{det}}^{(\sigma)}C_F T^{1-\gamma-\gamma_F-\varepsilon}\big(1+\|A^\gamma X\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\big)\\ &\quad + c_{\mathrm{stoch}}^{(\sigma)}C_B T^{1/2-\gamma-\gamma_B-\varepsilon}\big(1+\|A^\gamma X\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\big)\\ &\le C_T\big(1+\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma+\varepsilon-1/q))}\big).\end{aligned}$$

If $\gamma < 1/2$, Theorem 2.5.9 implies that

$$\begin{aligned}\|X\|_{L^r(\Omega;C([0,T];D_A^{\ell q}(\gamma-1/q)))} &\lesssim \|A^\gamma X\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + \|X\|_{L^r(\Omega;L^p(U;W^{\gamma,q}[0,T]))}\\ &\le C\big(1+\|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}\big).\end{aligned}$$

If $\gamma \ge 1/2$ we use $\|\cdot\|_{D_A^{\ell q}(\gamma-1/q)} \eqsim \|A^{\gamma-\beta}\cdot\|_{D_A^{\ell q}(\beta-1/q)}$ for some $\beta < 1/2$ and the same argument as above. The Hölder regularity results are a direct consequence of the Sobolev regularity and the appropriate Sobolev embedding. The first statements for the cases $r = 0$ follow by applying these estimates to $X$ on each set $\Gamma_n$ as in the previous proof. □

**REMARK 3.5.10.**

a) As a consequence, if $\gamma + \gamma_F < 1$ and $\gamma + \gamma_B < 1/2$, besides losing the smallness condition on our constants we also get some additional regularity for our solutions. More precisely, for each $\varepsilon \in [0, \varepsilon_\gamma \wedge (1/2 - 1/q))$ we obtain

$$\|A_\nu^{\gamma-1/q}X\|_{L^r(\Omega;L^p(U;C^\varepsilon[0,T]))} \le C_T\big(1+\|x_0\|_{L^r(\Omega;D_{A_\nu}^{\ell q}(\gamma+\varepsilon-1/q))}\big).$$

Observe that this is always possible if $q > 2$.

b) In general, we cannot assume this type of continuity for the case $q = 2$. On $L^2(\mathbb{R}^d)$ consider the equation

$$\mathrm{d}X(t) = \tfrac{1}{2}\Delta X(t)\,\mathrm{d}t + \sum_{n=1}^{d}\partial_n X(t)\,\mathrm{d}\beta_n(t), \quad X(0) = x_0.$$

For $x_0 \in W^{1,2}(\mathbb{R}^d)$, the function

$$X(t,u) = x_0(u + \boldsymbol{\beta}(t)), \quad t \in [0,T],\ u \in U,$$

is the (unique) weak solution of this equation. To see this, let $\varphi \in D(\tfrac{1}{2}\Delta) = W^{2,2}(\mathbb{R}^d)$. Then, by Itô's formula

$$\varphi(u - \boldsymbol{\beta}(t)) = \varphi(u) - \sum_{n=1}^{d}\int_0^t \partial_n\varphi(u-\boldsymbol{\beta}(s))\,\mathrm{d}\beta_n(s) + \tfrac{1}{2}\int_0^t \Delta\varphi(u-\boldsymbol{\beta}(s))\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\langle X(t),\varphi\rangle &= \int_{\mathbb{R}^d} x_0(u)\varphi(u-\boldsymbol{\beta}(t))\,\mathrm{d}u\\ &= \langle x_0,\varphi\rangle - \sum_{n=1}^{d}\int_0^t \langle x_0, \partial_n\varphi(\cdot-\boldsymbol{\beta}(s))\rangle\,\mathrm{d}\beta_n(s) + \tfrac{1}{2}\int_0^t \langle x_0,(\Delta\varphi)(\cdot-\boldsymbol{\beta}(s))\rangle\,\mathrm{d}s\\ &= \langle x_0,\varphi\rangle + \sum_{n=1}^{d}\int_0^t \langle \partial_n X(s),\varphi\rangle\,\mathrm{d}\beta_n(s) + \tfrac{1}{2}\int_0^t \langle X(s),\Delta\varphi\rangle\,\mathrm{d}s.\end{aligned}$$

Moreover, in this situation one actually knows that $K_{\text{stoch}} = \frac{1}{\sqrt{2}}$ (see [82, Section 5.3]). Therefore, we can apply Theorem 3.5.7 to this equation for $\gamma = 1/2$ and $\gamma_B = 0$. Note that $(-\Delta)$ fulfills assumption (HA) of Hypothesis 3.5.4 by Section 2.3. In particular, this implies that our solution is unique. However, for $d \geq 2$ the function $X$ is in general not continuous.

Finally, we collect results regarding continuous dependence of the initial values.

**Theorem 3.5.11 (Continuous dependence of data).** *Under the assumptions of Theorem 3.5.7, we find a constant $C > 0$ such that for all $x_0, y_0 \in L^r(\Omega,\mathcal{F}_0; D_A^{\ell q}(\gamma - 1/q))$ and the corresponding solutions $X$ and $Y$ the following statements hold*

$$\begin{aligned}\|A_\nu^{\gamma}(X-Y)\|_{L^r(\Omega;L^p(U;L^q[0,T]))} &\leq C\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma-1/q))},\\ \|A_\nu^{\gamma-\sigma}(X-Y)\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} &\leq C\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma-1/q))}, \quad \sigma\in[0,1/2),\\ \|X-Y\|_{L^r(\Omega;C([0,T];D^{\ell q}_{A_\nu}(\gamma-1/q)))} &\leq C\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma-1/q))},\\ \|A_\nu^{\gamma-\sigma}(X-Y)\|_{L^r(\Omega;L^p(U;C^{\sigma-1/q}[0,T]))} &\leq C\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma-1/q))}, \quad \sigma\in(1/q,1/2),\end{aligned}$$

*for $\sigma \leq \gamma$ and $q > 2$ in the last estimate.*

**Proof.** Without loss of generality, let $\nu = 0$. By Theorem 3.5.7 and Theorem 3.2.9 we obtain for $L = L_{u_0}$ in the first case

$$\begin{aligned}\|A^{\gamma}(X-Y)\|_{L^r(\Omega;L^p(U;L^q[0,T]))} &= \|A^{\gamma}(L(X)-L(Y)) + A^{\gamma}e^{-(\cdot)A}(u_0-v_0)\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\ &\leq \theta\|A^{\gamma}(X-Y)\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + C\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A}(\gamma-1/q))},\end{aligned}$$

which is equivalent to

$$\|A^{\gamma}(X-Y)\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \leq \frac{C}{1-\theta}\|x_0-y_0\|_{L^r(\Omega;D^{\ell q}_{A}(\gamma-1/q))}.$$

In the second case we proceed similarly as in the previous theorem. By Proposition 3.2.12 and Theorems 3.3.9, 3.4.10 and the first result we have

$$\begin{aligned}
&\|A^{\gamma-\sigma}(X-Y)\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} \\
&\quad = \|A^{\gamma-\sigma}(L(X)-L(Y)) + A^{\gamma-\sigma}e^{-(\cdot)A}(u_0-v_0)\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} \\
&\quad \le C_\sigma \|A^\gamma(X-Y)\|_{L^r(\Omega;L^p(U;L^q[0,T]))} + C\|x_0-y_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))} \\
&\quad \le \widetilde{C}\|x_0-y_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}.
\end{aligned}$$

The last statements finally follow from the second one and Theorem 2.5.9 or Sobolev's embedding theorem, respectively. □

**REMARK 3.5.12.** In many applications it happens that the operator $A$ will depend on $\omega \in \Omega$. In this case, one has to adjust the assumption of $A$ in Hypothesis 3.5.4 appropriately. More precisely, we will assume that

**(HA($\omega$)) Assumption on the operator $A$:** Each operator $A(\omega)\colon D \subseteq L^p(U) \to L^p(U)$, defined on the same domain $D(A(\omega)) = D$ is closed. The operator function $A\colon \Omega \to \mathcal{B}(D, L^p(U))$ is strongly $\mathcal{F}_0$-measurable and there exists a $\nu > 0$ such that for each $\omega \in \Omega$ the operator $\nu + A(\omega)$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\alpha)$ calculus for some $\alpha \in (0, \pi/2)$, where $\alpha$ and $\nu$ are independent of $\omega \in \Omega$. Moreover, there is a constant $C > 0$ (independent of $\omega \in \Omega$) such that

$$\|f(\nu + A(\omega))\|_{\mathcal{B}(L^p(U;L^q[0,T]))} \le C\|f\|_{\infty,\alpha} \quad \text{for all } f \in H^\infty(\Sigma_\alpha).$$

Since the $\mathcal{R}_q$-bounded $H^\infty$ calculus is 'independent' of $\omega \in \Omega$, the relevant Theorems 2.5.9, 3.3.9, and 3.4.10, as well as Propositions 3.3.1, 3.3.2, 3.4.1, and 3.4.3 all remain true in this case. Therefore, the results of Theorems 3.5.7, 3.5.9, and 3.5.11 follow in exactly the same way.

### 3.5.3 The Time-dependent Case

In this subsection we consider the stochastic partial differential equation

$$\text{(3.3)} \qquad \mathrm{d}X_t + A(t)X_t\,\mathrm{d}t = F(t, X_t)\,\mathrm{d}t + \boldsymbol{B}(t, X_t)\,\mathrm{d}\boldsymbol{\beta}_t, \quad X_0 = x_0.$$

The difference to (3.2) is that we consider instead of the operator $A$ the operator family $(A(t))_{t\in[0,T]}$. In this case we will assume the following hypothesis.

**HYPOTHESIS 3.5.13.** Let $r \in \{0\} \cup (1,\infty)$, $p \in (1,\infty)$, $q \in [2,\infty)$, and $\gamma, \gamma_F, \gamma_B \in \mathbb{R}$. Let (HF), (HB) and (H$x_0$) from Hypothesis 3.5.4 be satisfied. Instead of (HA) we assume

**(HA(t)) Assumptions on the operator $A$:** The map $A\colon \Omega\times[0,T] \to \mathcal{B}(D(A(0)), L^p(U))$ is strongly measurable and adapted to $\mathbb{F}$. Each operator $A(\omega, t)\colon D(A(0)) \to L^p(U)$, defined on the same domain, is closed, invertible (i.e. $0 \in \rho(A(t,\omega))$) and has an $\mathcal{R}_q$-bounded

$H^\infty(\Sigma_\alpha)$ calculus for some $\alpha \in (0, \pi/2)$, where $\alpha$ is independent of $\omega$ and $t$. There is a constant $C > 0$ (independent of $\omega$ and $t$) such that

$$\|f(A(\omega,t))\|_{\mathcal{B}(L^p(U;L^q[0,T]))} \leq C\|f\|_{\infty,\alpha} \quad \text{for all } f \in H^\infty(\Sigma_\alpha).$$

Moreover, we assume the following continuity property: Let $0 = t_0 < \ldots < t_N = T$ such that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\omega \in \Omega$, $n \in \{1,\ldots,N\}$ and all $s,t \in [t_{n-1}, t_n]$ and $\phi\colon \Omega \times [0,T] \to D(A(0)^\gamma)$ satisfying $A(0)^\gamma\phi \in L^p(U; L^q[0,T])$ we have for $|t-s| < \delta$ the estimate

$$\big\|A(0)^{-\gamma_F}\big(A(\cdot)\phi(\cdot) - A(s)\phi(\cdot)\big)\big\|_{L^p(U;L^q[s,t])} < \varepsilon\|A(0)^\gamma\phi\|_{L^p(U;L^q[s,t])}.$$

In this setting it is not possible to define a mild solution of (3.3) since the evolution family $e^{-sA(t)}$ of $A(t)$ becomes $\mathcal{F}_t$-measurable and therefore $e^{-sA(t)}\boldsymbol{B}(s, X(s))$ is no longer $\mathcal{F}_s$-measurable for $s \in [0,T]$. Due to this loss of adaptedness we would need an *anticipating integral*, which we do not consider here (see [62] for more information in this direction). But we can extend the definition of a strong solution to this case.

**Definition 3.5.14.** Let Hypothesis 3.5.13 be satisfied. Then we call a process $X\colon \Omega\times [0,T] \to D(A(0)^\gamma)$ a *strong* $(r,p,q)$ *solution* of (3.3) with respect to the filtration $\mathbb{F}$ if

a) $X$ is measurable, $X \in D(A(0))$ almost surely, and $A(0)^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$;

b) $X$ solves the equation (3.3) almost surely, i.e.

$$X(t) + \int_0^t A(s)X(s)\,\mathrm{d}s = x_0 + \int_0^t F(s, X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s, X(s))\,\mathrm{d}\boldsymbol{\beta}(s).$$

In the statement of the main result in this section we need the following constants

$$K_{\mathrm{det}}^{\Omega\times[0,T]} := \sup_{(\omega,t)\in\Omega\times[0,T]} K_{\mathrm{det}}(\omega,t) \quad \text{and} \quad K_{\mathrm{stoch}}^{\Omega\times[0,T]} := \sup_{(\omega,t)\in\Omega\times[0,T]} K_{\mathrm{stoch}}(\omega,t).$$

where the constants $K_{\mathrm{det}}(\omega,t)$ and $K_{\mathrm{stoch}}(\omega,t)$ are from Theorems 3.3.9 and 3.4.10 with respect to $A(\omega,t)$ for any fixed $(\omega,t) \in \Omega\times[0,T]$. Then $K_{\mathrm{det}}^{\Omega\times[0,T]}$ and $K_{\mathrm{stoch}}^{\Omega\times[0,T]}$ are finite since we assumed that the constants appearing in the $H^\infty$ calculus were uniform with respect to $(\omega,t) \in \Omega\times[0,T]$.

**Theorem 3.5.15.** *Let Hypothesis 3.5.13 be satisfied, and $\gamma \geq 1$, $\gamma_F, \gamma_B \in \mathbb{R}$ such that $\gamma + \gamma_F \in [0,1]$ and $\gamma + \gamma_B \in [0, 1/2]$. If the constants $K_{\mathrm{det}}^{\Omega\times[0,T]}$ and $K_{\mathrm{stoch}}^{\Omega\times[0,T]}$ and the Lipschitz constants $L_F$ and $L_B$ satisfy*

$$L_F K_{\mathrm{det}}^{\Omega\times[0,T]} + L_B K_{\mathrm{stoch}}^{\Omega\times[0,T]} < 1,$$

*in the case of $\gamma + \gamma_F = 1$ or $\gamma + \gamma_B = 1/2$, then the assertions of Theorems 3.5.7, 3.5.9, and 3.5.11 remain true for (3.3).*

**Proof.** Let $\theta := L_F K_{\text{det}}^{\Omega\times[0,T]} + L_B K_{\text{stoch}}^{\Omega\times[0,T]} \in [0,1)$, and for $\varepsilon := \frac{\frac{1}{2}(1-\theta)}{K_{\text{det}}^{\Omega\times[0,T]}}$ we choose a $\delta > 0$ such that for all $n \in \{1,\dots,N\}$, all $s,t \in [t_{n-1},t_n]$, and all $\phi\colon \Omega\times[0,T] \to D(A(0)^\gamma)$ satisfying $A(0)^\gamma\phi \in L^p(U;L^q[0,T])$ we have

$$\big\|A(0)^{-\gamma_F}\big(A(\cdot)\phi(\cdot) - A(s)\phi(\cdot)\big)\big\|_{L^p(U;L^q[s,t])} < \varepsilon\|A(0)^\gamma\phi\|_{L^p(U;L^q[s,t])}.$$

if $|t-s| < \delta$. Then fix $0 = s_0 < \ldots < s_M = T$ such that $\{t_0,\dots,t_N\}$ is a subset of $\{s_0,\dots,s_M\}$ and $|s_m - s_{m-1}| < \delta$ for each $m \in \{1,\dots,M\}$. On $[0,s_1]$ we define the map $F_{A,0}\colon \Omega\times[0,T]\times D(A(0)^\gamma) \to D(A(0)^{-\gamma_F})$ by

$$F_{A,0}(t,\phi(t)) := F(t,\phi(t)) - A(t)\phi(t) + A(0)\phi(t).$$

Then

$$\begin{aligned}
&\big\|A(0)^{-\gamma_F}\big(F_{A,0}(\cdot,\phi) - F_{A,0}(\cdot,\psi)\big)\big\|_{L^p(U;L^q[0,s_1])}\\
&\quad\le \big\|A(0)^{-\gamma_F}\big(F(\cdot,\phi) - F(\cdot,\psi)\big)\big\|_{L^p(U;L^q[0,s_1])} + \big\|A(0)^{-\gamma_F}\big(A(\cdot) - A(0)\big)(\phi-\psi)\big\|_{L^p(U;L^q[0,s_1])}\\
&\quad\le L_F\|A(0)^\gamma(\phi-\psi)\|_{L^p(U;L^q[0,s_1])} + \varepsilon\|A(0)^\gamma(\phi-\psi)\|_{L^p(U;L^q[0,s_1])}\\
&\quad= (L_F+\varepsilon)\|A(0)^\gamma(\phi-\psi)\|_{L^p(U;L^q[0,s_1])},
\end{aligned}$$

i.e. the map $F_{A,0}$ satisfies hypothesis (HF) with $F$ replaced by $F_{A,0}$ and $L_F$ replaced by $L_{F_{A,0}} := L_F + \varepsilon$. Since $L_{F_{A,0}}$ satisfies

$$L_{F_{A,0}} K_{\text{det}}^{\Omega\times[0,T]} + L_B K_{\text{stoch}}^{\Omega\times[0,T]} = \theta + \varepsilon K_{\text{det}}^{\Omega\times[0,T]} = \tfrac{1}{2}(\theta+1) < 1,$$

we can now apply Theorem 3.5.7 to this case (with a particular attention to Remark 3.5.8 and Remark 3.5.12), and get a unique strong $(r,p,q)$ solution $X$ on $[0,s_1]$ of (3.2) satisfying $A(0)^\gamma X \in L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,s_1]))$, i.e.

$$X(t) + \int_0^t A(0)X(s)\,\mathrm{d}s = x_0 + \int_0^t F_{A,0}(s,X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s),$$

which is equivalent to

$$X(t) + \int_0^t A(s)X(s)\,\mathrm{d}s = x_0 + \int_0^t F(s,X(s))\,\mathrm{d}s + \int_0^t \boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s),$$

i.e. $X$ is the unique strong $(r,p,q)$ solution of (3.3) on $[0,s_1]$. Additionally, all the statements of Theorems 3.5.9 and 3.5.11 remain true for $X$ on the interval $[0,s_1]$.

Now we continue by induction. If the statements of Theorem 3.5.7, 3.5.9, and 3.5.11 are true for equation (3.3) on the interval $[0,s_m]$ for some $m \in \{1,\dots,M-1\}$, then we consider the problem

$$\mathrm{d}Y(t) + A(s_m)Y(t)\,\mathrm{d}t = F_{A,m}^{s_m}(t,Y(t))\,\mathrm{d}t + \boldsymbol{B}^{\boldsymbol{s_m}}(t,Y(t))\,\mathrm{d}\boldsymbol{\beta}_t^{\boldsymbol{s_m}}, \quad Y(0) = X(s_m),$$

on the interval $[0, s_{m+1} - s_m]$, where $F^{s_m}_{A,m}(t, \phi) := F(t + s_m, \phi) - A(t + s_m)\phi + A(s_m)\phi$, $\boldsymbol{B}^{\boldsymbol{s_m}}(s, \phi) = \boldsymbol{B}(s + s_m, \phi)$, and $\boldsymbol{\beta}^{\boldsymbol{s_m}}$ is the family of shifted Brownian motions adapted to the shifted filtration $\mathbb{F}^{s_m}$ as considered in the proof of part II.2) of Theorem 3.5.7. Exactly as before, we get a unique strong $(r, p, q)$ solution $Y \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, s_{m+1} - s_m]))$ of (3.3) having all the properties of Theorem 3.5.7, 3.5.9, and 3.5.11. Then we extend the solution $X$ on $[0, s_m]$ to the interval $[0, s_{m+1}]$ by taking

$$X(t) := Y(t - s_m), \quad t \in [s_m, s_{m+1}].$$

$X$ is then an element of $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, s_{m+1}]))$. Calculations similarly to the the proof of Theorem 3.5.7 and above, and the induction hypothesis imply that $X$ is a strong $(r, p, q)$ solution of (3.3) on $[0, s_{m+1}]$. Also the results of Theorems 3.5.9 and 3.5.11 are now true on the interval $[0, s_m]$ and $[s_m, s_{m+1}]$, and by the triangle inequality also on $[0, s_{m+1}]$. We continue by showing that $X$ is also the unique solution of (3.3) on $[0, s_{m+1}]$. For this let $Z \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, s_{m+1}]))$ be another strong $(r, p, q)$ solution of (3.3). The induction hypothesis then implies that $X = Z$ in $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, s_m]))$, especially $X(s_m) = Z(s_m)$. Since $Z$ is a strong solution, one can now easily show that

$$Z(t) = Z(s_m) - \int_{s_m}^t A(s)Z(s)\,\mathrm{d}s + \int_{s_m}^t F(s, Z(s))\,\mathrm{d}s + \int_{s_m}^t \boldsymbol{B}(s, Z(s))\,\mathrm{d}\boldsymbol{\beta}(s),$$

i.e. $Z$ is a strong solution on $[s_m, s_{m+1}]$ of (3.3) with initial value $Z(s_m) = X(s_m)$. Since the solution is also unique on $[s_m, s_{m+1}]$ by the construction process above, we obtain $X = Z$ in $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[s_m, s_{m+1}]))$. Together with the uniqueness on $[0, s_m]$ this implies $X = Z$ on $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, s_{m+1}]))$. □

### 3.5.4 The Locally Lipschitz Case

In this subsection we extend the results of the global Lipschitz case to the case where the nonlinearities $F$ and $\boldsymbol{B}$ only satisfy local Lipschitz conditions. Therefore, we change the assumptions of Hypothesis 3.5.4 to the following

**Hypothesis 3.5.16.** Let $r \in \{0\} \cup (1, \infty)$, $p \in (1, \infty)$, $q \in [2, \infty)$, $\varepsilon > 0$, and $\gamma, \gamma_F, \gamma_B \in \mathbb{R}$. Let (HA($\omega$)) and (H$x_0$) from Hypothesis 3.5.4 and Remark 3.5.12 be satisfied. Assumption (HF) and (HB) are replaced by

**(HF)$_{\text{loc}}$ Assumptions on the nonlinearity $F$:** The function $F\colon \Omega \times [0, T] \times D(A_\nu^\gamma) \to D(A_\nu^{-\gamma_F})$ given by $F = F_1 + F_2$ is strongly measurable. Moreover,

a) for all $t \in [0, T]$ and $x \in D(A_\nu^\gamma)$ the random variable $\omega \mapsto F_1(\omega, t, x)$ is strongly $\mathcal{F}_t$-measurable;

b) **(globally Lipschitz part)** there are constants $L_{F_1}$, $\widetilde{L}_{F_1}$, $C_{F_1} \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi \colon [0,T] \to D(A_\nu^\gamma)$ satisfying $A_\nu^\gamma \phi, A_\nu^\gamma \psi \in L^p(U; L^q[0,T])$,

$$\begin{aligned}\big\|A_\nu^{-\gamma_F}\big(F_1(\omega,\cdot,\phi) - F_1(\omega,\cdot,\psi)\big)\big\|_{L^p(U;L^q[0,T])} \leq L_{F_1}\big\|A_\nu^\gamma(\phi-\psi)\big\|_{L^p(U;L^q[0,T])}\\ + \widetilde{L}_{F_1}\big\|A_\nu^{-\gamma_F}(\phi-\psi)\big\|_{L^p(U;L^q[0,T])}\end{aligned}$$

and

$$\|A_\nu^{-\gamma_F} F(\omega,\cdot,\phi)\|_{L^p(U;L^q[0,T])} \leq C_{F_1}(1 + \|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}).$$

c) for all $t \in [0,T]$ and $x \in D(A_\nu^\gamma)$ the random variable $\omega \mapsto F_2(\omega,t,x)$ is strongly $\mathcal{F}_t$-measurable;

d) **(locally Lipschitz part)** for all $R > 0$ there is a constant $L_{F_2,R} > 0$ such that for all $\omega \in \Omega$ and $\phi, \psi \colon [0,T] \to D(A_\nu^\gamma)$ satisfying $\|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}, \|A_\nu^\gamma \psi\|_{L^p(U;L^q[0,T])} \leq R$ it holds that

$$\big\|A_\nu^{-\gamma_F+\varepsilon}\big(F_2(\omega,\cdot,\phi) - F_2(\omega,\cdot,\psi)\big)\big\|_{L^p(U;L^q[0,T])} \leq L_{F_2,R}\|A_\nu^\gamma(\phi-\psi)\|_{L^p(U;L^q[0,T])}.$$

Moreover, we assume that there is a constant $C_{F_2,0} > 0$ such that for all $\omega \in \Omega$ we have

$$\|A_\nu^{-\gamma_F+\varepsilon} F_2(\omega,\cdot,0)\|_{L^p(U;L^q[0,T])} \leq C_{F_2,0}.$$

**$(\mathrm{HB})_{\mathrm{loc}}$ Assumptions on the nonlinearity $\boldsymbol{B}$:** The function $\boldsymbol{B} \colon \Omega \times [0,T] \times \mathbb{N} \times D(A_\nu^\gamma) \to D(A_\nu^{-\gamma_B})$ given by $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$ is strongly measurable. Moreover,

a) for all $t \in [0,T]$ and $x \in D(A_\nu^\gamma)$ the random variable $\omega \mapsto \boldsymbol{B}_1(\omega,t,x)$ is strongly $\mathcal{F}_t$-measurable;

b) **(globally Lipschitz part)** there are constants $L_{B_1}$, $\widetilde{L}_{B_1}$, $C_{B_1} \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi \colon [0,T] \to D(A_\nu^\gamma)$ satisfying $A_\nu^\gamma \phi, A_\nu^\gamma \psi \in L^p(U; L^q[0,T])$ we have

$$\begin{aligned}\big\|A_\nu^{-\gamma_B}\big(\boldsymbol{B}_1(\omega,\cdot,\phi) - \boldsymbol{B}_1(\omega,\cdot,\psi)\big)\big\|_{L^p(U;L^q([0,T];\ell^2))} \leq L_{B_1}\big\|A_\nu^\gamma(\phi-\psi)\big\|_{L^p(U;L^q[0,T])}\\ + \widetilde{L}_{B_1}\big\|A_\nu^{-\gamma_B}(\phi-\psi)\big\|_{L^p(U;L^q[0,T])}\end{aligned}$$

and

$$\|A_\nu^{-\gamma_B} \boldsymbol{B}(\omega,\cdot,\phi)\|_{L^p(U;L^q([0,T];\ell^2))} \leq C_{B_1}(1 + \|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}).$$

c) for all $t \in [0,T]$ and $x \in D(A_\nu^\gamma)$ the random variable $\omega \mapsto \boldsymbol{B}_2(\omega,t,x)$ is strongly $\mathcal{F}_t$-measurable;

d) **(locally Lipschitz part)** for all $R > 0$ there is a constant $L_{B_2,R} > 0$ such that for all $\omega \in \Omega$ and $\phi, \psi \colon [0,T] \to D(A_\nu^\gamma)$ satisfying $\|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}, \|A_\nu^\gamma \psi\|_{L^p(U;L^q[0,T])} \le R$ it holds that

$$\big\|A_\nu^{-\gamma_B+\varepsilon}\big(\boldsymbol{B}_2(\omega,\cdot,\phi) - \boldsymbol{B}_2(\omega,\cdot,\psi)\big)\big\|_{L^p(U;L^q[0,T])} \le L_{B_2,R}\|A_\nu^\gamma(\phi-\psi)\|_{L^p(U;L^q[0,T])}.$$

Moreover, we assume that there is a constant $C_{B_2,0} > 0$ such that for all $\omega \in \Omega$ we have

$$\|A_\nu^{-\gamma_B+\varepsilon}\boldsymbol{B}_2(\omega,\cdot,0)\|_{L^p(U;L^q[0,T])} \le C_{B_2,0}.$$

**Remark 3.5.17.** We note that we assume here $F$ and $\boldsymbol{B}$ to be a little bit more regular in the locally Lipschitz case. The reason for that is that we can not assume any smallness condition for $K_{\text{det}}L_{F,R} + K_{\text{stoch}}L_{B,R}$ and simultaneously let $R \to \infty$. In most cases this will be not reasonable. We need another parameter making this constant small enough.

As we know from the deterministic case, locally Lipschitz conditions do, in general, not lead to global solutions, i.e. there is the possibility that the solution might only exist on some limited time interval. In the case of stochastic evolution equations this *explosion time* will depend on each $\omega \in \Omega$. Therefore, we introduce the following notion. If $\tau \colon \Omega \to [0,T]$ is a stopping time, then

$$\Omega \times [0,\tau) := \{(\omega,t) \in \Omega \times [0,T] \colon t \in [0,\tau(\omega))\},$$

and similarly

$$\Omega \times [0,\tau] := \{(\omega,t) \in \Omega \times [0,T] \colon t \in [0,\tau(\omega)]\}.$$

This leads to the following definition of local solutions.

**Definition 3.5.18.** Let Hypothesis 3.5.16 be satisfied and $\tau \colon \Omega \to [0,T]$ be a stopping time.

a) We call a process $X \colon \Omega \times [0,\tau) \to D(A_\nu^\gamma)$ a *local mild* $(r,p,q)$ *solution* of (3.2) with respect to the filtration $\mathbb{F}$ if there exists a sequence of increasing stopping times $\tau_n \colon \Omega \to [0,T]$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \tau_n = \tau$ almost surely, such that

   1) $X$ is measurable and $\mathbb{1}_{[0,\tau_n]}A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U;L^q[0,T]))$;
   2) $X$ solves the equation

$$X(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}F(s,X(s))\,\mathrm{d}s + \int_0^t e^{-(t-s)A}\boldsymbol{B}(s,X(s))\,\mathrm{d}\boldsymbol{\beta}(s)$$

   almost surely on $[0,\tau_n]$ for each $n \in \mathbb{N}$.

b) We call a process $X\colon \Omega \times [0,\tau) \to D(A_\nu^\gamma)$ a *local strong* $(r,p,q)$ *solution* of (3.2) with respect to the filtration $\mathbb{F}$ if there exists a sequence of increasing stopping times $\tau_n\colon \Omega \to [0,T]$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \tau_n = \tau$ almost surely, such that

   1) $X$ is measurable, $X(t) \in D(A)$ almost surely, and $\mathbb{1}_{[0,\tau_n]} A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$;

   2) $X$ solves the equation (3.2) almost surely on $[0,\tau_n]$ for each $n \in \mathbb{N}$.

c) We call a local solution $X\colon \Omega\times[0,\tau) \to D(A_\nu^\gamma)$ *maximal* on $[0,T]$ if for every stopping time $\tau'\colon \Omega \to [0,T]$ and every other local solution $V\colon \Omega \times [0,\tau') \to D(A_\nu^\gamma)$ we have $\tau \geq \tau'$ and $U = V$ on $[0,\tau')$.

d) We call a local solution $X\colon \Omega \times [0,\tau) \to D(A_\nu^\gamma)$ a *global* solution if $\tau = T$ almost surely and $A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$.

e) We say that $\tau$ is an *explosion time* if for almost all $\omega \in \Omega$ with $\tau(\omega) < T$ we have

$$\limsup_{t\to\tau(\omega)} \|\mathbb{1}_{[0,t]} A_\nu^\gamma X\|_{L^p(U;L^q[0,T])} = \infty.$$

   We should remark that $\tau(\omega) = T$ is an explosion time by definition. However, in this case the blow up condition does not have to be true.

Motivated by this definition, we define the space $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,\tau)))$ as the space of functions $\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$ for which we have an increasing sequence of stopping times $\tau_n\colon \Omega \to [0,T]$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \tau_n = \tau$ almost surely and $\mathbb{1}_{[0,\tau_n]}\phi \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. Similarly, we make the same definition for spaces like

$$L^r_{\mathbb{F}}(\Omega; L^p(U; W^{\sigma,q}[0,\tau))) \quad \text{and} \quad L^r_{\mathbb{F}}(\Omega; L^p(U; C^a[0,\tau))).$$

Note that, if $\tau_n(\omega) = T$ for almost all $\omega \in \Omega$ and $n$ large enough, then

$$L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,\tau))) = L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T])).$$

In the following we will only consider local and global mild $(r,p,q)$ solutions. It can be shown similarly to Proposition 3.5.6 that mild and strong solutions are still equivalent if we assume $\gamma \geq 1$. In this situation we have the following result.

**Theorem 3.5.19.** *Let Hypothesis 3.5.16 be satisfied and let* $\gamma_F, \gamma_B \leq 0$ *such that* $\gamma + \gamma_F \in [0,1]$ *and* $\gamma + \gamma_B \in [0, 1/2]$*. Further assume that*

$$L_{F_1} K_{det} + L_{B_1} K_{stoch} < 1,$$

*Then the following assertions hold true:*

a) *If* $x_0 \in L^0(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$ *then* (3.2) *has a unique maximal local mild* $(0, p, q)$ *solution* $(X(t))_{t\in[0,\tau)}$ *satisfying*

$$A_\nu^\gamma X \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0, \tau))).$$

b) *If additionally to a) we assume that* $F_2$ *and* $\boldsymbol{B}_2$ *satisfy linear growth conditions, i.e.*

$$\|A_\nu^{-\gamma_F+\varepsilon} F_2(\omega, \cdot, \phi)\|_{L^p(U;L^q[0,T])} \le C_{F_2}(1 + \|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}),$$
$$\|A_\nu^{-\gamma_B+\varepsilon} \boldsymbol{B}_2(\omega, \cdot, \phi)\|_{L^p(U;L^q([0,T];\ell^2))} \le C_{B_2}(1 + \|A_\nu^\gamma \phi\|_{L^p(U;L^q[0,T])}),$$

*for some constants* $C_{F_2}, C_{B_2} > 0$ *independent of* $\omega \in \Omega$*, then the solution* $X$ *in a) is a global mild* $(0, p, q)$ *solution.*

c) *If additionally to a) and b) we have* $x_0 \in L^r(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$ *for some* $r \in (1, \infty)$*, then the global solution* $X$ *of b) satisfies*

$$A_\nu^\gamma X \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, T]))$$

*and*

$$\|A_\nu^\gamma X\|_{L^r(\Omega;L^p(U;L^q[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega;D^{\ell q}_{A_\nu}(\gamma - 1/q))}).$$

Before turning to the proof of the theorem we will show the following lemma about local uniqueness.

**Lemma 3.5.20.** *Under the assumptions of Theorem 3.5.19 let* $X_1 : \Omega \times [0, \tau_1) \to D(A_\nu^\gamma)$ *and* $X_2 : \Omega \times [0, \tau_2) \to D(A_\nu^\gamma)$ *be local mild* $(0, p, q)$ *solutions of* (3.2) *with initial values* $x_{0,1}$ *and* $x_{0,2}$*. Then on the set* $\Omega_0 := \{x_{0,1} = x_{0,2}\}$ *we almost surely have*

$$X_1|_{[0,\tau_1\wedge\tau_2)} = X_2|_{[0,\tau_1\wedge\tau_2)}.$$

*Moreover, if* $\tau_1$ *is an explosion time for* $X_1$ *then almost surely on* $\Omega_0$ *we have* $\tau_1 \ge \tau_2$*. If both* $\tau_1$ *and* $\tau_2$ *are explosion times for* $X_1$ *and* $X_2$*, respectively, then almost surely on* $\Omega_0$ *we have* $\tau_1 = \tau_2$ *and* $X_1 = X_2$*.*

**Proof.** This is a light modification of [71, Lemma 5.3] (see also [81, Lemma 8.2]). Let $(\tau_{1,n})_{n\in\mathbb{N}}$ and $(\tau_{2,n})_{n\in\mathbb{N}}$ be the sequences of increasing stopping times for $\tau_1$ and $\tau_2$, respectively, as required in the definition. Then define

$$\rho_{1,n} := \tau_{1,n} \wedge \inf\{t \in [0, T] : \|\mathbb{1}_{[0,t]} A_\nu^\gamma X_1\|_{L^p(U;L^q[0,T])} \ge n\},$$
$$\rho_{2,n} := \tau_{2,n} \wedge \inf\{t \in [0, T] : \|\mathbb{1}_{[0,t]} A_\nu^\gamma X_2\|_{L^p(U;L^q[0,T])} \ge n\},$$

and $\rho_n := \rho_{1,n} \wedge \rho_{2,n}$, $n \in \mathbb{N}$.

Let $n$ be fixed for a while. Since the set $\Omega_0$ is $\mathcal{F}_0$ measurable and $\rho_n$ is adapted to $\mathbb{F}$, we have by Proposition 1.3.13

$$\begin{aligned}
&\big\|\mathbb{1}_{\Omega_0\times[0,\rho_n]}A_\nu^\gamma(X_1-X_2)\big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\
&\qquad=\big\|\mathbb{1}_{\Omega_0\times[0,\rho_n]}L^\gamma\big(\mathbb{1}_{\Omega_0\times[0,\rho_n]}\ A_\nu^\gamma(X_1-X_2)\big)\big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\
&\qquad\leq\big\|L^\gamma\big(\mathbb{1}_{\Omega_0\times[0,\rho_n]}\ A_\nu^\gamma(X_1-X_2)\big)\big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}\\
&\qquad\leq C(n,T)\big\|\mathbb{1}_{\Omega_0\times[0,\rho_n]}\ A_\nu^\gamma(X_1-X_2)\big\|_{L^r(\Omega;L^p(U;L^q[0,T]))}
\end{aligned}$$

for some constant $C(n,T)$ having the property $\lim_{T\to 0}C(n,T)<1$ (see the proof of Theorem 3.5.19 below). For $T'$ small enough, we obtain that $\mathbb{1}_{\Omega_0\times[0,\rho_n]}A_\nu^\gamma X_1=\mathbb{1}_{\Omega_0\times[0,\rho_n]}A_\nu^\gamma X_2$ in $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T']))$. Similar as in the proof of Theorem 3.5.7 we can extend this equality to the whole interval $[0,T]$ by induction. Therefore, we obtain

$$\mathbb{1}_{\Omega_0}X_1(t)=\mathbb{1}_{\Omega_0}X_2(t)$$

almost surely on the set $\{t\leq\rho_n\}$ for arbitrary $n\in\mathbb{N}$. By passing $n\to\infty$ we finally get

$$\mathbb{1}_{\Omega_0}X_1(t)=\mathbb{1}_{\Omega_0}X_2(t)$$

on the set $\{t<\tau_1\wedge\tau_2\}$.

Now let $\tau_1$ be an explosion time and assume that $\tau_1(\omega)<\tau_2(\omega)$ for some $\omega\in\Omega_0$. Then we can find an integer $n\in\mathbb{N}$ such that $\tau_1(\omega)<\rho_{2,n}(\omega)$, but $X_1(\omega,t)=X_2(\omega,t)$ for $0\leq t\leq\rho_{1,n+1}(\omega)\leq\tau_1(\omega)$. This implies

$$\begin{aligned}
n+1=\|\mathbb{1}_{[0,\rho_{n+1}]}A_\nu^\gamma X_1(\omega)\|_{L^p(U;L^q[0,T])}&=\|\mathbb{1}_{[0,\mu_{n+1}]}A_\nu^\gamma X_2(\omega)\|_{L^p(U;L^q[0,T])}\\
&\leq\|\mathbb{1}_{[0,\rho_{2,n}]}A_\nu^\gamma X_2(\omega)\|_{L^p(U;L^q[0,T])}=n,
\end{aligned}$$

which is a contradiction. If both stopping times are explosion times, we obtain by the previous part that $\tau_1=\tau_2$ almost surely on $\Omega_0$. Therefore, $X_1=X_2$ on $\Omega_0$. □

**Proof (of Theorem 3.5.19).** Without loss of generality, we assume that $\nu=0$. Moreover, for the sake of simplicity, we only consider the case $F=F_2$ and $\boldsymbol{B}=\boldsymbol{B}_2$. The general case then follows as a combination of this case and Theorem 3.5.7. The following proof contains ideas of [71].

a) Let $E:=L^p(U;L^q[0,T])$. We start with a small observation. For $N\in\mathbb{N}$ we define the function

$$R_N(\phi):=\begin{cases}\phi, & \text{if } \|A^\gamma\phi\|_E\leq N,\\ \frac{N\phi}{\|A^\gamma\phi\|_E}, & \text{if } \|A^\gamma\phi\|_E>N.\end{cases}$$

Then, for $\|A^\gamma\phi\|_E,\|A^\gamma\psi\|_E\leq N$ we trivially have $\|A^\gamma(R_N(\phi)-R_N(\psi))\|_E=\|A^\gamma(\phi-\psi)\|_E$.

If $\|A^\gamma \phi\|_E \le N$ and $\|A^\gamma \psi\|_E > N$ we use that

$$A^\gamma(R_N(\phi) - R_N(\psi)) = A^\gamma(\phi - \psi) + \big(1 - \tfrac{N}{\|A^\gamma\psi\|_E}\big)A^\gamma \psi$$

to obtain

$$\begin{aligned}\|A^\gamma(R_N(\phi) - R_N(\psi))\|_E &\le \|A^\gamma(\phi-\psi)\|_E + \big\|\big(1 - \tfrac{N}{\|A^\gamma\psi\|_E}\big)A^\gamma\psi\big\|_E \\ &= \|A^\gamma(\phi-\psi)\|_E + \|A^\gamma\psi\|_E - N \\ &\le 2\|A^\gamma(\phi-\psi)\|_E.\end{aligned}$$

Finally, in the case that $\|A^\gamma\phi\|_E > N$ and $\|A^\gamma\psi\|_E > N$ we have

$$\begin{aligned}R_N(\phi) - R_N(\psi) &= \frac{N}{\|A^\gamma\phi\|_E}(\phi-\psi) + \frac{N}{\|A^\gamma\phi\|_E}\psi - \frac{N}{\|A^\gamma\psi\|_E}\psi \\ &= \frac{N}{\|A^\gamma\phi\|_E}(\phi-\psi) + \frac{N(\|A^\gamma\psi\|_E - \|A^\gamma\phi\|_E)}{\|A^\gamma\phi\|_E\|A^\gamma\psi\|_E}\psi.\end{aligned}$$

This then leads to

$$\begin{aligned}\|A^\gamma(R_N(\phi) - R_N(\psi))\|_E &\le \|A^\gamma(\phi-\psi)\|_E + \big|\|A^\gamma\psi\|_E - \|A^\gamma\phi\|_E\big| \\ &\le 2\|A^\gamma(\phi-\psi)\|_E.\end{aligned}$$

Therefore, we obtain in any case $\|A^\gamma(R_N(\phi) - R_N(\psi))\|_E \le 2\|A^\gamma(\phi-\psi)\|_E$. Having this at hand, we define the functions

$$F_N(\omega,t,\phi) := F(\omega,t,R_N(\phi)) \quad \text{and} \quad \boldsymbol{B}_N(\omega,t,\phi) := \boldsymbol{B}(\omega,t,R_N(\phi)).$$

By assumptions $(\mathrm{HF})_{\mathrm{loc}}$ and $(\mathrm{HB})_{\mathrm{loc}}$ we then obtain

$$\begin{aligned}\big\|A^{-\gamma_F+\varepsilon}\big(F_N(\omega,\cdot,\phi) - F_N(\omega,\cdot,\psi)\big)\big\|_E &\le L_{F,N}\|A^\gamma(R_N(\phi) - R_N(\psi))\|_E \\ &\le 2L_{F,N}\|A^\gamma(\phi-\psi)\|_E,\end{aligned}$$

and similarly

$$\begin{aligned}\|A^{-\gamma_F+\varepsilon}F_N(\omega,\cdot,\phi)\|_E &\le \big\|A^{-\gamma_F+\varepsilon}\big(F_N(\omega,\cdot,\phi) - F_N(\omega,\cdot,0)\big)\big\|_E + \|A^{-\gamma_F+\varepsilon}F_N(\omega,\cdot,0)\|_E \\ &\le 2L_{F,N}\|A^\gamma\phi\|_E + C_{F,0}, \\ &\le \widetilde{C}_{F,N}(1 + \|A^\gamma\phi\|_E)\end{aligned}$$

with no restriction on the norms of $A^\gamma\phi$ or $A^\gamma\psi$. Similar results hold for $\boldsymbol{B}_N$ in place of $F_N$. Hence, $F_N$ and $\boldsymbol{B}_N$ satisfy the assumptions (HF) and (HB) of Hypothesis 3.5.4 with Lipschitz constants $\widetilde{L}_F = 2L_{F,N}$, $\widetilde{L}_B = 2L_{B,N}$, linear growth constants $\widetilde{C}_F = \widetilde{C}_{F,N}$, $\widetilde{C}_B = \widetilde{C}_{B,N}$, and $\widetilde{\gamma}_F = \gamma_F - \varepsilon$, $\widetilde{\gamma}_B = \gamma_B - \varepsilon$. Note that $\gamma + \widetilde{\gamma}_F < 1$ and $\gamma + \widetilde{\gamma}_B < 1/2$, so we do not need any smallness assumption on our constants. By Theorem 3.5.7 it follows that there exists a unique mild $(0, p, q)$ solution $X_N$ of the modified equation (3.2) (with nonlinearities $F_N$ and $\boldsymbol{B}_N$) satisfying

$$A^\gamma X_N \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T])).$$

In particular, $X_N$ is a solution of the original equation (3.2) on the restricted interval $[0, \tau_N]$, where

$$\tau_N(\omega) := T \wedge \inf\{t \in [0,T] \colon \|\mathbb{1}_{[0,t]} A^\gamma X_N(\omega)\|_E \geq N\}, \quad \omega \in \Omega.$$

By Lemma 3.5.20 we then have $X_N = X_M$ on $[0, \tau_N \wedge \tau_M]$ for $M \leq N$. In particular, $\tau_M \leq \tau_N$. Since $(\tau_N(\omega))_{N \in \mathbb{N}}$ is a bounded and increasing sequence, we can define

$$\tau(\omega) := \lim_{N \to \infty} \tau_N(\omega), \quad \omega \in \Omega,$$

and $X(\omega, t) := X_N(\omega, t)$ for $t \in [0, \tau_N(\omega)]$. By definition, $A^\gamma X \in L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0,\tau)))$, and $X$ is a local mild $(0, p, q)$ solution. Uniqueness follows in the same way as in the proof of Theorem 3.5.7, part I.2). $X$ is also maximal, since $\tau$ is an explosion time. In fact, if $\tau(\omega) < T$, then

$$\limsup_{t \to \tau(\omega)} \|\mathbb{1}_{[0,t]} A^\gamma X(\omega)\|_E \geq \limsup_{n \to \infty} \|\mathbb{1}_{[0,\tau_n(\omega)]} A^\gamma X(\omega)\| \leq \limsup_{n \to \infty} n = \infty.$$

b) We define for any fixed $\delta > 0$ the set

$$\Omega_\delta := \{\|x_0\|_{D^{\ell q}_A(\gamma - 1/q)} \leq \delta\},$$

and $x_{0,\delta} := \mathbb{1}_{\Omega_\delta} x_0$, which is an element of $L^r_{\mathbb{F}}(\Omega, \mathcal{F}_0; D^{\ell q}_{A_\nu}(\gamma - 1/q))$ for some $r \in (1, \infty)$. Similar as above, we obtain for each $\delta > 0$ a local mild $(r, p, q)$ solution $X^\delta$ satisfying $A^\gamma X^\delta \in L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0, \tau^\delta)))$ for some stopping time $\tau^\delta$. On the set $\Omega_\delta$, the uniqueness of the solution $X$ found in a) implies that $X^\delta = X$ and $\tau^\delta = \tau$ almost surely. Moreover, since we additionally assume linear growth conditions, the definition of $F_N$ and $\boldsymbol{B}_N$ implies

$$\sup_{N \in \mathbb{N}} \|A^{-\gamma_F + \varepsilon} F_N(\omega, \cdot, \phi)\|_E \leq C_F(1 + \|A^\gamma \phi\|_E),$$
$$\sup_{N \in \mathbb{N}} \|A^{-\gamma_B + \varepsilon} \boldsymbol{B}_N(\omega, \cdot, \phi)\|_E \leq C_B(1 + \|A^\gamma \phi\|_E).$$

Now observe that in the case of $\widetilde{\gamma}_F + \gamma < 1$ and $\widetilde{\gamma}_B + \gamma < 1/2$ the constant $C$ of the a-priori estimate can be chosen independent of the Lipschitz constants. In particular, by the property above, it is independent of $N$. Then we obtain for $(\tau_N)_{n \in \mathbb{N}}$ as in a)

$$\begin{aligned}\mathbb{P}(\{\tau_N < T\} \times \Omega_\delta) &= \mathbb{P}\big(\{\|A^\gamma X_N\|_E \geq N\} \times \Omega_\delta\big) \leq N^{-r} \mathbb{E}\|\mathbb{1}_{\Omega_\delta} A^\gamma X_N\|_E^r \\ &\leq N^{-r} C^r (1 + \|x_{0,\delta}\|_{L^r(\Omega; D^{\ell q}_A(\gamma - 1/q))})^r \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

This implies

$$\begin{aligned}\mathbb{P}(\{\tau < T\} \times \Omega_\delta) &= \mathbb{P}(\{\lim_{n\to\infty} \tau_n < T\} \times \Omega_\delta) = \mathbb{P}(\{\sup_{n\in\mathbb{N}} \tau_n < T\} \times \Omega_\delta) \\ &= \mathbb{P}\Big(\bigcap_{n\in\mathbb{N}} \{\tau_n < T\} \times \Omega_\delta\Big) = \lim_{n\to\infty} \mathbb{P}(\{\tau_n < T\} \times \Omega_\delta) = 0,\end{aligned}$$

i.e. $\tau = T$ almost surely on each set $\Omega_\delta$. This implies that $X$ is a global solution.

c) Moreover, if we have $x_0 \in L^r(\Omega, \mathcal{F}_0; D_A^{\ell q}(\gamma - 1/q))$, then we do not need any construction involving the sets $\Omega_\delta$ in part b). Then, the same a-priori estimate applied to each $X_N$ and Fatou's lemma yield

$$\|A^\gamma X\|_{L^r(\Omega;E)} \le \liminf_{N\to\infty} \|A^\gamma X_N\|_{L^r(\Omega;E)} \le C(1 + \|x_0\|_{L^r(\Omega;D_A^{\ell q}(\gamma-1/q))}). \qquad \square$$

**Remark 3.5.21.** By restricting the solution $X$ of the previous theorem on each interval $\mathbb{1}_{[0,\tau_N]}$, $N \in \mathbb{N}$, it immediately follows that the regularity results of Theorem 3.5.9 stay true for $r = 0$ up to the random time $\tau$. In particular, under the assumptions of b) ($r = 0$) and/or c) ($r \in (1, \infty)$) we obtain the corresponding results of Theorem 3.5.9 on the whole time interval $[0, T]$.

# Chapter 4

# Applications to Stochastic Partial Differential Equations

In this chapter we apply the theory developed in Chapter 3 to stochastic PDE's. In contrast to existing results, we achieve stronger regularity results with respect to time simply because the corresponding norms are now *inside* of the other norms. Although the assumptions we made in the abstract theory might be more restrictive than usual, we will see that in many concrete cases they still hold. The following examples are chosen to illustrate different aspects of our regularity theory. Other combinations of nonlinearities and operators are of course possible. We also would like to point out that the theory we developed is quite new. Since we change the 'space-time' order of the usual regularity theory, we do not have the extensive research basis of the existing literature, which connects the abstract theory to partial differential equations. Nevertheless, this also means that there is still a lot of potential for further research.

## 4.1 Bounded Generators

Let us start with the case of a bounded generator $A$ on $L^p(U)$, where $(U, \Sigma, \mu)$ is an arbitrary $\sigma$-finite measure space. Even in this case we have to make some additional assumptions on $A$. We consider the equation

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(t, X(t))\,\mathrm{d}t + \boldsymbol{B}(t, X(t))\,\mathrm{d}\boldsymbol{\beta}(t), \quad X_0 = x_0, \tag{4.1}$$

with the following assumptions for $A$, $F$, $\boldsymbol{B}$, and $x_0$.

**Hypothesis 4.1.1.** Let $r \in \{0\} \cup (1, \infty)$, $p \in (1, \infty)$, and $q \in [2, \infty)$.

**(HA) Assumptions on the operator:** The linear operator $A\colon L^p(U) \to L^p(U)$ is bounded and has a bounded extension $A^{L^q} \in \mathcal{B}\big(L^p(U; L^q[0, T])\big)$.

**(HF) Assumptions on the nonlinearity $F$:** The function $F\colon \Omega \times [0,T] \times L^p(U) \to L^p(U)$ is strongly measurable, adapted to $\mathbb{F}$, and is $L^q$-Lipschitz continuous and of linear growth, i.e. there exist constants $L_F$, $C_F \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi\colon [0,T] \to L^p(U)$ satisfying $\phi, \psi \in L^p(U; L^q[0,T])$, we have

$$\big\| F(\omega, \cdot, \phi) - F(\omega, \cdot, \psi) \big\|_{L^p(U;L^q[0,T])} \leq L_F \big\| \phi - \psi \big\|_{L^p(U;L^q[0,T])}$$

and

$$\| F(\omega, \cdot, \phi) \|_{L^p(U;L^q[0,T])} \leq C_F (1 + \|\phi\|_{L^p(U;L^q[0,T])}).$$

**(HB) Assumptions on the nonlinearity $\boldsymbol{B}$:** The function $\boldsymbol{B}\colon \Omega \times [0,T] \times \mathbb{N} \times L^p(U) \to L^p(U)$ is strongly measurable, adapted to $\mathbb{F}$, and is also $L^q$-Lipschitz continuous and of linear growth, i.e. there exist constants $L_B$, $C_B \geq 0$ such that for all $\omega \in \Omega$ and $\phi, \psi\colon [0,T] \to L^p(U)$ satisfying $\phi, \psi \in L^p(U; L^q[0,T])$,

$$\big\| \boldsymbol{B}(\omega, \cdot, \phi) - \boldsymbol{B}(\omega, \cdot, \psi) \big\|_{L^p(U;L^q([0,T];\ell^2))} \leq L_B \big\| \phi - \psi \big\|_{L^p(U;L^q[0,T])}$$

and

$$\| \boldsymbol{B}(\omega, \cdot, \phi) \|_{L^p(U;L^q([0,T];\ell^2))} \leq C_B (1 + \|\phi\|_{L^p(U;L^q[0,T])}).$$

**(H$x_0$) Assumptions on the initial value $x_0$:** The initial value $x_0\colon \Omega \to L^p(U)$ is strongly $\mathcal{F}_0$-measurable.

Then we obtain the following results.

**Theorem 4.1.2.** *Under the assumptions of Hypothesis 4.1.1, we obtain for each $x_0 \in L^r(\Omega, \mathcal{F}_0; L^p(U))$ a unique strong and mild $(r,p,q)$ solution $X\colon \Omega \times [0,T] \to L^p(U)$ of (4.1) in $L^r_{\mathbb{F}}(\Omega; L^p(U; L^q[0,T]))$. Moreover, $X$ has a version satisfying*

$$X \in L^r_{\mathbb{F}}(\Omega; L^p(U; W^{\sigma,q}[0,T])), \quad \sigma \in [0, 1/2),$$
$$X \in L^r_{\mathbb{F}}(\Omega; L^p(U; C^{\sigma - 1/q}[0,T])), \quad \sigma \in [1/q, 1/2), \text{ if } q > 2,$$

*with corresponding a-priori estimates*

$$\|X\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))} \leq C(1 + \|x_0\|_{L^r(\Omega;L^p(U))}), \quad \sigma \in [0, 1/2),$$
$$\|X\|_{L^r(\Omega;L^p(U;C^{\sigma-1/q}[0,T]))} \leq C(1 + \|x_0\|_{L^r(\Omega;L^p(U))}), \quad \sigma \in [1/q, 1/2),$$

*in the case $r \in (1, \infty)$.*

**Proof.** Since $A^{L^q}$ is bounded, we have $D(A^n) = L^p(U)$ for all $n \in \mathbb{N}$, and the function

$$\widetilde{F}\colon \Omega \times [0,T] \times L^p(U) \to L^p(U), \quad \widetilde{F}(\omega,t,\phi) = -A^{L^q}\phi + F(\omega,t,\phi),$$

is $L^q$ Lipschitz continuous. In particular, $\widetilde{F}$ and $\boldsymbol{B}$ satisfy assumptions (HF) and (HB) of Hypothesis 3.5.4. Therefore, it suffices to consider the case $A = 0$, which clearly satisfies Hypothesis (HA). Since we can choose $\gamma \in \mathbb{R}$ as large as we want to, Theorems 3.5.7, 3.5.9, and 3.5.11 (see also Proposition 3.5.6 and Remark 3.5.8), imply the stated results. □

**Remark 4.1.3.** There are several situations where $A$ has a bounded extension for *every* space $L^q[0,T]$, e.g. for $q \in [2,\infty)$ (or in a larger interval). One important example is the case of a positive operator $A$. Here, $A$ always has a bounded extension $A^{L^q}$ (see also Remark 2.4.1). Since we can choose $q$ arbitrarily large, this leads to solutions in $L^r_{\mathbb{F}}(\Omega; L^p(U; C^\lambda[0,T]))$ for all $\lambda \in [0, 1/2)$. Another example is the Hilbert transform $H$ on $L^p(\mathbb{R})$, which has a vector-valued bounded extension $H^E$ if and only if $E$ is a UMD space. In particular, this includes every $L^q[0,T]$ space for $q \in (1,\infty)$.

## 4.2 Stochastic Heat Equation

Let $U \subseteq \mathbb{R}^d$ be an open domain. Then we consider the stochastic heat equation with Dirichlet boundary conditions

$$\begin{aligned} \mathrm{d}X(t,u) - \kappa\Delta_p X(t,u)\,\mathrm{d}t &= f(t,u,X(t,u))\,\mathrm{d}t + \sum_{n=1}^{\infty} b_n(t,u,X(t,u))\,\mathrm{d}\beta_n(t), \\ X(t,u) &= 0, \quad u \in \partial U, t \in [0,T], \\ X(0,u) &= x_0(u), \quad u \in U. \end{aligned} \tag{4.2}$$

for some thermal diffusivity $\kappa > 0$. On the space $L^p(U)$ for some $p \in (1,\infty)$ we let $\Delta_p$ be the Dirichlet Laplacian with domain $D(\Delta_p)$. If, e.g., $U$ is a bounded domain with boundary $\partial U \in C^2$, then we can identify $D(\Delta_p) = W_0^{1,p}(U) \cap W^{2,p}(U)$ (cf. [20, (A.44)]). In this situation we make the following assumptions about $f$, $b_n$, $n \in \mathbb{N}$, and $x_0$.

**Hypothesis 4.2.1.** Let $r \in \{0\} \cup (1,\infty)$, $p \in (1,\infty)$, and $q \in (2,\infty)$.

**(Hfb) Assumptions on the nonlinearities $f, b_n$:** The functions $f, b_n\colon \Omega \times [0,T] \times U \times \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable, adapted to $\mathbb{F}$, and are globally Lipschitz continuous, i.e. there exist constants $L_f$, $L_{b_n} \geq 0$ such that for all $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, and $x, y \in \mathbb{R}$

we have

$$|f(\omega,t,u,x)-f(\omega,t,u,y)|\le L_f|x-y|,$$
$$|b_n(\omega,t,u,x)-b_n(\omega,t,u,y)|\le L_{b_n}|x-y|.$$

Moreover,

$$L_b:=\Big(\sum_{n=1}^{\infty}L_{b_n}^2\Big)^{1/2}<\infty,$$

and

$$\|f(\omega,t,u,0)\|_{L^p_{(u)}(U;L^q_{(t)}[0,T])}\le C_f,$$
$$\|b(\omega,t,u,0)\|_{L^p_{(u)}(U;L^q_{(t)}([0,T];\ell^2))}\le C_b,$$

for all $\omega\in\Omega$ and constants $C_f, C_b\ge 0$ independent of $\omega$.

**(H$x_0$) Assumptions on the initial value $x_0$:** The initial value $x_0\colon\Omega\to L^p(U)$ is strongly $\mathcal{F}_0$-measurable.

Under these assumptions the abstract regularity theory of Section 3.5 leads to the following results.

**THEOREM 4.2.2.** *Let Hypothesis 4.2.1 be satisfied, $U$ be an open domain in $\mathbb{R}^d$, and $\eta\in[0,1/2)$. For $x_0\in L^r(\Omega,\mathcal{F}_0;D^{\ell q}_{(-\Delta_p)}(\eta-1/q))$ there exists a unique mild $(r,p,q)$ solution $X\colon\Omega\times[0,T]\to D((-\Delta_p)^\eta)$ of (4.2) in $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$ satisfying*

$$(-\Delta_p)^{\eta-\sigma}X\in L^r_{\mathbb{F}}(\Omega;L^p(U;W^{\sigma,q}[0,T])),\quad \sigma\in[0,\eta],$$
$$X\in L^r_{\mathbb{F}}(\Omega;C([0,T];D^{\ell q}_{(-\Delta_p)}(\eta-1/q))),$$
$$(-\Delta_p)^{\eta-\sigma}X\in L^r_{\mathbb{F}}(\Omega;L^p(U;C^{\sigma-1/q}[0,T])),\quad \sigma\in(1/q,\eta],$$

*and having the following a-priori estimates*

$$\|(-\Delta_p)^{\eta-\sigma}X\|_{L^r(\Omega;L^p(U;W^{\sigma,q}[0,T]))}\le C(1+\|x_0\|_{L^r(\Omega;D^{\ell q}_{(-\Delta_p)}(\eta-1/q))}),\quad \sigma\in[0,\eta],$$
$$\|X\|_{L^r(\Omega;C([0,T];D^{\ell q}_{(-\Delta_p)}(\eta-1/q)))}\le C(1+\|x_0\|_{L^r(\Omega;D^{\ell q}_{(-\Delta_p)}(\eta-1/q))}),$$
$$\|(-\Delta_p)^{\eta-\sigma}X\|_{L^r(\Omega;L^p(U;C^{\sigma-1/q}[0,T]))}\le C(1+\|x_0\|_{L^r(\Omega;D^{\ell q}_{(-\Delta_p)}(\eta-1/q))}),\quad \sigma\in(1/q,\eta],$$

*in the case $r\in(1,\infty)$.*

**PROOF.** We check the assumptions of Hypothesis 3.5.4. By Section 2.3, the Dirichlet Laplacian $-\Delta_p$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus in $L^p(U)$ for all $p,q\in(1,\infty)$. This implies (HA). To model $f$ and $b_n$, $n\in\mathbb{N}$, we define for $\omega\in\Omega$, $t\in[0,T]$, $u\in U$, and

$\phi \in L^p(U; L^q[0,T])$

$$F(\omega, t, \phi(t))(u) := f(\omega, t, u, \phi(t)),$$
$$\boldsymbol{B}(\omega, t, n, \phi(t))(u) := b_n(\omega, t, u, \phi(t)), \quad n \in \mathbb{N}.$$

Then the pointwise estimates of Hypothesis 4.2.1 imply

$$\begin{aligned} \|F(\cdot,\phi) - F(\cdot,\psi)\|_{L^p(U;L^q[0,T])} &\le L_f \|\phi - \psi\|_{L^p(U;L^q[0,T])}, \\ \|\boldsymbol{B}(\cdot,\phi) - \boldsymbol{B}(\cdot,\psi)\|_{L^p(U;L^q([0,T];\ell^2))} &\le \Big\| \Big( \sum_{n=1}^{\infty} L_{b_n}^2 |\phi - \psi|^2 \Big)^{1/2} \Big\|_{L^p(U;L^q[0,T])} \\ &= L_b \|\phi - \psi\|_{L^p(U;L^q[0,T])}, \end{aligned}$$

as well as

$$\begin{aligned} \|F(\cdot,\phi)\|_{L^p(U;L^q[0,T])} &\le \|F(\cdot,\phi) - F(\cdot,0)\|_{L^p(U;L^q[0,T])} + \|F(\cdot,0)\|_{L^p(U;L^q[0,T])} \\ &\le L_f \|\phi\|_{L^p(U;L^q[0,T])} + C_f \\ &\le (L_f \vee C_f)\big(1 + \|\phi\|_{L^p(U;L^q[0,T])}\big), \\ \|\boldsymbol{B}(\cdot,\phi)\|_{L^p(U;L^q[0,T])} &\le \|\boldsymbol{B}(\cdot,\phi) - \boldsymbol{B}(\cdot,0)\|_{L^p(U;L^q([0,T];\ell^2))} + \|\boldsymbol{B}(\cdot,0)\|_{L^p(U;L^q([0,T];\ell^2))} \\ &\le L_b \|\phi\|_{L^p(U;L^q[0,T])} + C_b \\ &\le (L_b \vee C_b)\big(1 + \|\phi\|_{L^p(U;L^q[0,T])}\big), \end{aligned}$$

for all $\phi, \psi \in L^p(U; L^q[0,T])$. These calculations finally show (HF) and (HB) for $\gamma_F = \gamma_B = \gamma = 0$. Now the claim follows from Theorems 3.5.7 and 3.5.9, where in the latter we may choose $\varepsilon = \eta$. □

**REMARK 4.2.3.**

a) If we assume that the Lipschitz constants $L_f$ and $L_b$ are small enough, we can also include the maximal regularity case $\eta = 1/2$ by Theorem 3.5.7.

b) Assuming that $U \subseteq \mathbb{R}^d$ satisfies an interior cone condition (see [1, Definition 4.6]), Example A b) of Section 2.3 implies that the Laplace operator with Neumann boundary conditions has an $\mathcal{R}_q$-bounded $H^\infty$ calculus. Therefore, the results of Theorem 4.2.2 also hold for the Neumann Laplacian.

c) If $U \subseteq \mathbb{R}^d$ is bounded domain with $C^2$ boundary, then the estimates imply that

$$X \in L^r_{\mathbb{F}}(\Omega; H^{2(\eta-\sigma),p}(U; W^{\sigma,q}[0,T])), \quad \sigma \in [0,\eta],$$

where $H^{\alpha,p}(U; L^q[0,T])$, $\alpha > 0$, are the Bessel potential spaces (cf. [76]). To see this, observe that $(-\Delta_p^{L^q})$ has property BIP, which yields

$$D((-\Delta_p^{L^q})^{\eta-\sigma}) = [L^p(U; L^q[0,T]), D(-\Delta_p^{L^q})]_{\eta-\sigma}$$

by [77, Theorem 1.15.3]. Now Example 2.4.7 and [46, Theorem 5.93] further lead to

$$\big\|(-\Delta_p^{L^q})^{\eta-\sigma} f\big\|_{L^p(U;L^q[0,T])} \eqsim \|f\|_{H^{2(\eta-\sigma),p}(U;L^q[0,T])}, \quad f \in H^{2(\eta-\sigma),p}(U;L^q[0,T]).$$

We conclude with a comparison to other results in the literature.

**Discussion 4.2.4.** In [49], Jentzen and Röckner considered the same equation (4.2) in the Hilbert space setting $L^2(U)$ for $U = (0,1)^d$, and assuming a particular structure of the functions $b_n$, $n \in \mathbb{N}$ (see equation (32) in [49]). More precisely, they assumed that

$$b_n(\omega,t,u,x) = \sqrt{\mu_n} b(u,x) g_n(u), \quad \omega \in \Omega,\ t \in [0,T],\ u \in U,\ x \in \mathbb{R},$$

for a globally Lipschitz function $b\colon U \times \mathbb{R} \to \mathbb{R}$ (in both variables), and sequences $(\mu_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$, $(g_n)_{n\in\mathbb{N}} \subseteq L^2(U)$ satisfying

$$\sup_{n\in\mathbb{N}} \|g_n\|_{C(U)} < \infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} \mu_n \|g_n\|^2_{C^\delta(U)} < \infty, \quad \delta \in (0,1].$$

As a result they obtain for each initial value $x_0 \in C^2(U) \subsetneq D((-\Delta_p)^{\eta-1/q}) \subseteq D^{\ell^q}_{(-\Delta_p)}(\eta-1/q)$ a unique mild solution $X$ satisfying

$$X \in C^\sigma([0,T]; L^r(\Omega; W^{2(\eta-\sigma),2}(U))), \quad \sigma \in [0, \eta \wedge 1/2],$$

for $r \geq 2$ and $\eta \in [0, \frac{3\wedge(2\delta+2)}{4})$. This means that the regularity of the coefficients $(g_n)_{n\in\mathbb{N}}$ improves the regularity in space, at least for $\delta \in (0, 1/2]$.

In our case, we obtain for a bounded domain $U \subseteq \mathbb{R}^d$ with $C^2$ boundary and $\delta = 0$ (or, more generally, coefficients in $L^\infty(U)$) the estimate

$$X \in L^r_{\mathbb{F}}(\Omega; H^{2(\eta-\sigma)}(U; C^\sigma[0,T])), \quad \sigma \in [0,\eta],$$

for $\eta \in [0, 1/2)$, $p \in (1,\infty)$, and $r \in (1,\infty)$ by choosing $q$ sufficiently large. This means that our theory leads to *pointwise* Hölder continuity. More precisely, for allmost every (fixed) point in space, the path $t \mapsto X(t,u)$ is Hölder continuous. Besides having a stronger estimate on a general domain $U$ and for a larger class of initial values, we also include the cases $r \in (1,2)$ and $p \neq 2$. Note that Jentzen and Röckner can consider the borderline case $\sigma = 1/2$ because the Hölder regularity is true for the *moments* of the solution $X$ and not the solution itself (see also Remark 6.5 in [9]).

## 4.3 Parabolic Equations on $\mathbb{R}^d$

In this section we consider on $U = \mathbb{R}^d$ the equation

$$
\begin{aligned}
\mathrm{d}X(t,u) + \mathcal{A}(u)X(t,u)\,\mathrm{d}t &= f(t,u,X(t,u))\,\mathrm{d}t + \sum_{n=1}^{\infty} b_n(t,u,X(t,u))\,\mathrm{d}\beta_n(t), \\
X(0,u) &= x_0(u), \quad u \in \mathbb{R}^d,
\end{aligned}
\tag{4.3}
$$

where

$$\mathcal{A}(\omega,u) = \sum_{|\alpha|\le 2m} a_\alpha(\omega,u)D^\alpha$$

is an elliptic differential operator of order $2m$ in non-divergence form, $m \in \mathbb{N}$, and with bounded coefficients $a_\alpha \in L^\infty(\Omega \times \mathbb{R}^d, \mathbb{C})$ for $|\alpha| \le 2m$. Let $A_p$ be the realization of $\nu + \mathcal{A}$ in $L^p(\mathbb{R}^d)$ with domain $D(A_p) = W^{2m,p}(\mathbb{R}^d)$. The spectral shift $\nu > 0$ will be introduced later to guarantee that $A_p$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus. Then we make the following additional assumptions about the nonlinearities $f$ and $b_n$, and the initial value $x_0$.

**HYPOTHESIS 4.3.1.** Let $r \in \{0\} \cup (1,\infty)$, $p \in (1,\infty)$, and $q \in [2,\infty)$.

**(Ha) Assumptions on the coefficients:** Let $a_\alpha \colon \Omega \times \mathbb{R}^d \to \mathbb{C}$ be $\mathcal{F}_0 \otimes \mathcal{B}_{\mathbb{R}^d}$-measurable. Furthermore, let

$$
\begin{aligned}
a_\alpha &\in L^\infty(\Omega; BUC(\mathbb{R}^d)), \quad |\alpha| = 2m, \\
a_\alpha &\in L^\infty(\Omega \times \mathbb{R}^d), \quad |\alpha| < 2m,
\end{aligned}
$$

satisfying

$$\max_{|\alpha|=2m} \|a_\alpha(\omega,\cdot)\|_{C(h)} := \max_{|\alpha|=2m} \|a_\alpha(\omega,\cdot)\|_\infty + \sup_{u\ne v} \frac{|a_\alpha(\omega,u) - a_\alpha(\omega,v)|}{h(|u-v|)} \le M, \quad \omega \in \Omega,$$

where $M > 0$ is independent of $\omega \in \Omega$ and $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity. That is, an increasing function which is continuous in 0 with $h(0) = 0$ and $h(t) > 0$, and satisfies $h(2t) \le ch(t)$, $t > 0$ (see [2, Section 4]). As an example, this assumption is satisfied if the coefficients $a_\alpha$, $|\alpha| = 2m$, are Hölder continuous with uniform Hölder norm independent of $\omega \in \Omega$. We also assume that

$$\int_0^1 t^{-1}h(t)^{1/3}\,\mathrm{d}t < \infty,$$

and there exist an angle $\sigma \in (0, \pi/2)$ and $\delta > 0$ such that for all $\omega \in \Omega$

$$\sum_{|\alpha|=2m} a_\alpha(\omega,u)\xi^\alpha \in \Sigma_\sigma \quad \text{and} \quad \Big|\sum_{|\alpha|=2m} a_\alpha(\omega,u)\xi^\alpha\Big| \ge \delta|\xi|^{2m}$$

for all $u, \xi \in \mathbb{R}^d$.

**(Hf) Assumptions on the nonlinearity $f$:** The function $f\colon \Omega\times[0,T]\times\mathbb{R}^d\times W^{2m,p}(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ is measurable and adapted, and there exist constants $L_f$, $C_f\geq 0$ such that for all $\omega\in\Omega$ and $\phi,\psi\colon[0,T]\to W^{2m,p}(\mathbb{R}^d)$ satisfying $\phi,\psi\in W^{2m,p}(\mathbb{R}^d;L^q[0,T])$ we have

$$\big\|f(\omega,\cdot,\phi)-f(\omega,\cdot,\psi)\big\|_{L^p(\mathbb{R}^d;L^q[0,T])}\leq L_f\big\|\phi-\psi\big\|_{W^{2m,p}(\mathbb{R}^d;L^q[0,T])}$$

and

$$\|f(\omega,\cdot,\phi)\|_{L^p(\mathbb{R}^d;L^q[0,T])}\leq C_F(1+\|\phi\|_{W^{2m,p}(\mathbb{R}^d;L^q[0,T])}).$$

**(Hb) Assumptions on the nonlinearities $b_n$:** The function $b_n\colon\Omega\times[0,T]\times\mathbb{R}^d\times W^{2m,p}(\mathbb{R}^d)\to W^{m,p}(\mathbb{R}^d)$ is measurable and adapted for each $n\in\mathbb{N}$, and for $b:=(b_n)_{n\in\mathbb{N}}$ there exist constants $L_b$, $C_b\geq 0$ such that for all $\phi,\psi\colon[0,T]\to W^{2m,p}(\mathbb{R}^d)$ satisfying $\phi,\psi\in W^{2m,p}(\mathbb{R}^d;L^q[0,T])$ we have

$$\big\|b(\omega,\cdot,\phi)-b(\omega,\cdot,\psi)\big\|_{W^{m,p}(\mathbb{R}^d;L^q([0,T];\ell^2))}\leq L_b\big\|\phi-\psi\big\|_{W^{2m,p}(\mathbb{R}^d;L^q[0,T])}$$

and

$$\|b(\omega,\cdot,\phi)\|_{W^{m,p}(\mathbb{R}^d;L^q([0,T];\ell^2))}\leq C_b(1+\|\phi\|_{W^{2m,p}(\mathbb{R}^d;L^q[0,T])}).$$

**(H$x_0$) Assumptions on the initial value $x_0$:** Let $x_0\colon\Omega\to F_q^{2m-2m/q,p}(\mathbb{R}^d)$ be strongly $\mathcal{F}_0$-measurable.

Since the nonlinearity $f$ is allowed to lose regularity of order 1 and $b$ of order $1/2$, this is an example of the maximal regularity case. In particular, for $m=1$, this setting includes nonlinearities $b$ of gradient type. As a consequence of the abstract theory of Chapter 3 we have the following results.

**THEOREM 4.3.2.** *Assume Hypothesis 4.3.1 and*

$$L_fK_{det}+L_bK_{stoch}<1.$$

*Then for each initial value $x_0\in L^r(\Omega,\mathcal{F}_0;F_q^{2m-2m/q,p}(\mathbb{R}^d))$ equation (4.3) has a unique mild and strong $(r,p,q)$ solution $X\colon\Omega\times[0,T]\to W^{2m,p}(\mathbb{R}^d)$ in $L^r_{\mathbb{F}}(\Omega;L^p(U;L^q[0,T]))$. Additionally, the solution $X$ satisfies*

$$X\in L^r_{\mathbb{F}}(\Omega;H^{2m(1-\sigma)}(\mathbb{R}^d;W^{\sigma,q}[0,T])),\quad \sigma\in[0,1/2).$$

*In particular*

$$X\in L^r_{\mathbb{F}}(\Omega;C([0,T];F_q^{2m-2m/q,p}(\mathbb{R}^d))),$$
$$X\in L^r_{\mathbb{F}}(\Omega;H^{2m(1-\sigma)}(\mathbb{R}^d;C^{\sigma-1/q}[0,T])),\quad \sigma\in(1/q,1/2),\ \text{if } q>2.$$

*Moreover, if $r \in (1, \infty)$, the solution $X$ has the properties*

$$\|X\|_{L^r(\Omega;H^{2m(1-\sigma)}(\mathbb{R}^d;W^{\sigma,q}[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega;F_q^{2m-2m/q,p}(\mathbb{R}^d))}), \quad \sigma \in [0, 1/2),$$
$$\|X\|_{L^r(\Omega;C([0,T];F_q^{2m-2m/q,p}(\mathbb{R}^d)))} \le C(1 + \|x_0\|_{L^r(\Omega;F_q^{2m-2m/q,p}(\mathbb{R}^d))}),$$
$$\|X\|_{L^r(\Omega;H^{2m(1-\sigma)}(\mathbb{R}^d;C^{\sigma-1/q}[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega;F_q^{2m-2m/q,p}(\mathbb{R}^d))}), \quad \sigma \in (1/q, 1/2),\ q > 2.$$

**Proof.** We check the conditions of Hypothesis 3.5.4. By Example B of Section 2.3 there exist values $p_0 \in (1, p \wedge q)$ and $\nu \ge 0$ such that the differential operator $\nu + \mathcal{A}$ in non-divergence form has an $\mathcal{R}_{\widetilde{q}}$-bounded $H^\infty$ calculus for all $\widetilde{p}, \widetilde{q} \in (p_0, \infty)$. In particular, this is true for $\widetilde{p} = p$ and $\widetilde{q} = q$. The coefficiants of $\mathcal{A}$ are chosen in such a way that the constants of the $\mathcal{R}_q$-bounded $H^\infty$ calculus are independent of $\omega \in \Omega$ (see [56, Theorem 3.1] and in particular [2, Theorem 9.6]). Moreover, by Example 2.4.7 we have $D(A_p^{L^q}) = W^{2m,p}(\mathbb{R}^d; L^q[0,T])$. Hence, [77, Theorem 1.15.3] and [46, Theorem 5.93] imply

$$D((A_p^{L^q})^\theta) = [L^p(\mathbb{R}^d; L^q[0,T]), D(A_p^{L^q})]_\theta = [L^p(\mathbb{R}^d; L^q[0,T]), W^{2m,p}(\mathbb{R}^d; L^q[0,T])]_\theta$$
$$= H^{2m\theta,p}(\mathbb{R}^d; L^q[0,T]).$$

We also have $H^{k,p}(\mathbb{R}^d; L^q[0,T]) = W^{k,p}(\mathbb{R}^d; L^q[0,T])$, $k \in \mathbb{N}$ (see [46]), in particular it holds that $D((A_p^{L^q})^{1/2}) = W^{m,p}(\mathbb{R}^d; L^q[0,T])$. By Example 2.4.7 we additionally get

$$D_{A_p}^{\ell q}(1 - 1/q) = F_q^{2m-2m/q,p}(\mathbb{R}^d).$$

With these results in mind we define $F\colon \Omega \times [0,T] \times D(A_p) \to L^p(\mathbb{R}^d)$ and $\boldsymbol{B}\colon \Omega \times [0,T] \times \mathbb{N} \times D(A_p) \to D(A_p^{1/2})$ by

$$F(\omega, t, x)(u) := f(\omega, t, u, x) \quad \text{and} \quad \boldsymbol{B}(\omega, t, n, x)(u) := b_n(\omega, t, u, x)$$

for each $\omega \in \Omega$, $t \in [0,T]$, $u \in \mathbb{R}^d$, and $x \in D(A_p)$. Then $F$ and $\boldsymbol{B}$ clearly satisfy assumptions (HF) and (HB) for $\gamma = 1$, $\gamma_F = 0$, and $\gamma_B = -1/2$. Thus, the results finally follow from Theorems 3.5.7 and 3.5.9. □

**Remark 4.3.3.**

a) For $m = 1$, coefficients independent of $\Omega$, and without lower order terms, we also could have assumed that

$$a_\alpha \in VMO(\mathbb{R}^d), \quad |\alpha| = 2.$$

In this case, Example B of Section 2.3 implies that we can choose $p_0 \in (1, p \wedge q)$ and $\nu \ge 0$ such that $\nu + \mathcal{A}$ also has an $\mathcal{R}_{\widetilde{q}}$-bounded $H^\infty$ calculus on $L^{\widetilde{p}}(\mathbb{R}^d)$ for all $\widetilde{p}, \widetilde{q} \in (p_0, \infty)$.

b) Instead of elliptic operators in non-divergence form, also operators in divergence form could have been considered. In this case, the assumptions on the coefficients can be further weakened (see also Section 2.3 and 4.4).

With a slight modification of Hypothesis 4.3.1 the non-autonomous case can also be treated. More precisely, we consider the equation

$$\begin{aligned} \mathrm{d}X(t,u) + \mathcal{A}(t,u)X(t,u)\,\mathrm{d}t &= f(t,u,X(t,u))\,\mathrm{d}t + \sum_{n=1}^{\infty} b_n(t,u,X(t,u))\,\mathrm{d}\beta_n(t), \\ X(0,u) &= x_0(u), \quad u \in \mathbb{R}^d, \end{aligned} \tag{4.4}$$

for the differential operator

$$\mathcal{A}(\omega,t,u) = \sum_{|\alpha|\le 2m} a_\alpha(\omega,t,u)D^\alpha$$

with time-dependent coefficients $a_\alpha$, $|\alpha| \le 2m$, $m \in \mathbb{N}$. In this case we have to change (Ha) of Hypothesis 4.3.1 to the following:

**(Ha(t)) Assumptions on the coefficients:** Let $a_\alpha\colon \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{C}$ be measurable and adapted, and let

$$\begin{aligned} a_\alpha &\in L^\infty(\Omega; BUC(\mathbb{R}^d; C[0,T])), \quad |\alpha| = 2m, \\ a_\alpha &\in L^\infty(\Omega \times \mathbb{R}^d; C[0,T]), \quad |\alpha| < 2m. \end{aligned}$$

satisfying

$$\max_{|\alpha|=2m} \|a_\alpha(\omega,t,\cdot)\|_{C(h)} \le M, \quad \omega \in \Omega,\ t \in [0,T],$$

where $M > 0$ is independent of $(\omega,t) \in \Omega \times [0,T]$ and $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity with

$$\int_0^1 t^{-1} h(t)^{1/3}\,\mathrm{d}t < \infty.$$

We also assume that there exist $\sigma \in (0, \pi/2)$ and $\delta > 0$ such that for all $\omega \in \Omega$ and all $t \in [0,T]$

$$\sum_{|\alpha|=2m} a_\alpha(\omega,t,u)\xi^\alpha \in \Sigma_\sigma \quad \text{and} \quad \Big|\sum_{|\alpha|=2m} a_\alpha(\omega,t,u)\xi^\alpha\Big| \ge \delta|\xi|^{2m}$$

for all $u, \xi \in \mathbb{R}^d$.

Then the realization $A_p(t)$ of $\nu + \mathcal{A}(t)$ in $L^p(U)$ with time-independent domain $D(A_p(t)) = W^{2m,p}(\mathbb{R}^d)$ has the same properties as the operator $A_p$ in Theorem 4.3.2 for each fixed

$t \in [0,T]$. In particular, $A_p(t)$ has an $\mathcal{R}_q$-bounded $H^\infty$ calculus on $L^p(\mathbb{R}^d)$ for each $p,q \in (1,\infty)$ and the constants of the $\mathcal{R}_q$-boundend $H^\infty$ calculus are independent of $\omega \in \Omega$ and $t \in [0,T]$. To apply Theorem 3.5.15 instead of Theorem 3.5.7 we still have to show a continuity property of $A_p(\cdot)$. For this purpose let $\varepsilon > 0$. By assumption, the function

$$a\colon [0,T] \to \mathbb{C}, \quad a(t) = \sum_{|\alpha|\le 2m} a_\alpha(\omega,t,u)z = \sum_{|\alpha|\le 2m} \frac{a_\alpha(\omega,t,u)}{\nu + a_\alpha(\omega,0,u)}(\nu + a_\alpha(\omega,0,u))z,$$

is uniformly continuous for each fixed $\omega \in \Omega$, $u \in \mathbb{R}^d$, and $z \in \mathbb{C}$. Hence, we can find an $\eta > 0$ (independent of $\omega \in \Omega$, $u \in \mathbb{R}^d$, and $z \in \mathbb{C}$) such that for $s,t \in [0,T]$ with $|t-s| < \eta$ we obtain

$$\begin{aligned} |a(t) - a(s)| &= \Big| \sum_{|\alpha|\le 2m} \frac{a_\alpha(\omega,t,u) - a_\alpha(\omega,s,u)}{\nu + a_\alpha(\omega,0,u)}(\nu + a_\alpha(\omega,0,u))z \Big| \\ &< \varepsilon \Big| \sum_{|\alpha|\le 2m} (\nu + a_\alpha(\omega,0,u))z \Big|. \end{aligned}$$

This immediately implies the desired continuity, more precisely, for each $s,t \in [0,T]$ with $|t-s| < \eta$ and each $\phi\colon [0,T] \to D(A_p(0))$ we have

$$\big\| A_p(\cdot)\phi(\cdot) - A_p(s)\phi(\cdot) \big\|_{L^p(\mathbb{R}^d;L^q[s,t])} < \varepsilon \| A_p(0)\phi \|_{L^p(\mathbb{R}^d;L^q[0,T])}.$$

Then, (HA(t)) of Hypothesis 3.5.13 is satisfied, and by Theorem 3.5.15 we obtain a unique strong $(r,p,q)$ solution $X\colon \Omega \times [0,T] \to W^{2m,p}(\mathbb{R}^d)$, having the same properties as in Theorem 4.3.2.

**Discussion 4.3.4.** The same problem (4.4) was considered by van Neerven, Veraar, and Weis in [82, Section 6] (see also [54, Theorem 5.1]). Basically, they assumed the same assumptions for the differential operator $\mathcal{A}$, but slightly different Lipschitz and linear growth conditions of the nonlinearities $f$ and $b$. In contrast to our theory, they choose Lipschitz conditions with respect to the space norm only and with $t \in [0,T]$ fixed. Both in [82] and here, these conditions were chosen to fit the respective abstract theory. In [82, Theorem 6.3] the authors obtain a strong solution $X\colon \Omega \times [0,T] \to W^{2m,p}(\mathbb{R}^d)$ such that

$$X \in L^q_{\mathbb{F}}(\Omega \times [0,T]; W^{2m,p}(\mathbb{R}^d)).$$

Moreover, the solution has trajectories in $C([0,T]; B_q^{2m(1-1/q),p}(\mathbb{R}^d))$ for $r = q \in \{0\} \cup (2,\infty)$ and $p \ge 2$. In our situation, we obtain a strong $(r,p,q)$ solution $X\colon \Omega \times [0,T] \to W^{2m,p}(\mathbb{R}^d)$ satisfying

$$X \in L^r_{\mathbb{F}}(\Omega; W^{2m,p}(\mathbb{R}^d; L^q[0,T]))$$

for all $q \in [2,\infty)$ and $p,r \in (1,\infty)$ without any relation of $r$ and $q$. Without that connection of the exponents $r$ and $q$, we can choose $q$ larger to open more possibilities for the time regularity. In particular, we also have the continuity properties

$$\begin{aligned} &X \in L^r_{\mathbb{F}}(\Omega; C([0,T]; F_q^{2m(1-1/q),p}(\mathbb{R}^d))) \\ &X \in L^r_{\mathbb{F}}(\Omega; H^{2m(1-\sigma)}(\mathbb{R}^d; C^{\sigma-1/q}[0,T])), \quad \text{for } \sigma \in (1/q, 1/2),\ q > 2. \end{aligned}$$

The latter is stronger than the one above, since we have pointwise Hölder regularity. However, we also had to assume more restrictive Lipschitz and linear growth conditions.

## 4.4 Second Order Parabolic Equations on Domains

In this part we investigate regularity properties of second order elliptic equations on an open domain $U \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions. In contrast to the examples above, we also include the locally Lipschitz case. More precisely, we consider the problem

$$\begin{aligned} \mathrm{d}X(t,u) + \mathcal{A}(u)X(t,u)\,\mathrm{d}t &= f(t,u,X(t,u),\nabla X(t,u))\,\mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} b_n(t,u,X(t,u),\nabla X(t,u))\,\mathrm{d}\beta_n(t), \\ X(t,u) &= 0, \quad u \in \partial U,\ t \in [0,T], \\ X(0,u) &= x_0(u), \quad u \in U. \end{aligned} \tag{4.5}$$

Here, $\mathcal{A}(\omega,u)$ is a second order differential operator in divergence form, formally given by

$$\mathcal{A}(\omega,u) = -\sum_{i,j=1}^{d} D_i(a_{i,j}(\omega,u)D_j) + \sum_{i=1}^{d} a_i(\omega,u)D_i + a_0(\omega,u),$$

see also Section 2.3. To apply the abstract theory of Chapter 3 we will make the following assumptions.

**Hypothesis 4.4.1.** Let $r \in \{0\} \cup (1,\infty)$, $p \in (1,\infty)$, and $q \in [2,\infty)$.

**(Ha) Assumptions on the coefficients:** Let $a_\alpha \colon \Omega \times U \to \mathbb{R}$ be $\mathcal{F}_0 \otimes \mathcal{B}_U$-measurable. Furthermore, let

$$a_{i,j}, a_i, a_0 \in L^\infty(\Omega \times U, \mathbb{R}), \quad i,j \in \{1,\dots,d\},$$

and assume that the principal part of $\mathcal{A}$ satisfies the uniform strong ellipticity condition

$$\sum_{i,j=1}^{d} a_{i,j}(\omega,u)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,\ u \in U,\ \omega \in \Omega.$$

Denote by $A_p$ the realization of $\mathcal{A}$ in $L^p(U)$, where the domain is given by

$$D(A_p) = W_D^{2,p}(U) := \{f \in W^{2,p}(U) \colon f = 0 \text{ on } \partial U\}$$

assuming that the boundary of $U$ is smooth.

**(Hf) Assumptions on the nonlinearity $f$:** The function $f = f_1 + f_2$, where $f_1 \colon \Omega \times [0,T] \times U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $f_2 \colon \Omega \times [0,T] \times U \times \mathbb{R} \to \mathbb{R}$, is measurable and adapted. Moreover, $f_1$ is globally Lipschitz continuous and of linear growth, i.e. there exist constants $L_{f_1}, C_{f_1} \geq 0$ such that

$$\big|f_1(\omega,t,u,x,v) - f_1(\omega,t,u,y,w)\big| \leq L_{f_1}\big(|x-y| + |v-w|\big)$$

and

$$\big\|f_1(\omega,\cdot,\cdot\cdot,\phi)\big\|_{L^p(U;L^q[0,T])} \leq C_{f_1}(1 + \|\phi\|_{L^p(U;L^q[0,T])})$$

for all $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, $x,y \in \mathbb{R}$, $v,w \in \mathbb{R}^d$, and $\phi \in L^p(U;L^q[0,T])$. Regarding $f_2$ we assume a local Lipschitz condition as well as boundedness at 0. That means, there exists a constant $C_{f_2} \geq 0$, and for each $R > 0$ there is a constant $L_{f_2,R} \geq 0$ such that

$$\big|f_2(\omega,t,u,x) - f_2(\omega,t,u,y)\big| \leq L_{f_2,R}|x-y|$$

and

$$\big\|f_2(\omega,\cdot,\cdot\cdot,0)\big\|_{L^p(U;L^q[0,T])} \leq C_{f_2}$$

for all $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, and $x,y \in \mathbb{R}$ satisfying $|x|,|y| \leq R$.

**(Hb) Assumptions on the nonlinearities $b_n$:** For each $n \in \mathbb{N}$ let $b_n = b_{n,1} + b_{n,2}$, where $b_{n,1} \colon \Omega \times [0,T] \times U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $b_{n,2} \colon \Omega \times [0,T] \times U \times \mathbb{R} \to \mathbb{R}$ are measurable and adapted. We also assume that $b_{n,1}$ is globally Lipschitz continuous and of linear growth, i.e. there exist constants $L_{b_{n,1}}, C_{b_{n,1}} \geq 0$ such that

$$\big|b_{n,1}(\omega,t,u,x,v) - b_{n,1}(\omega,t,u,y,w)\big| \leq L_{b_{n,1}}\big(|x-y| + |v-w|\big)$$

and

$$\big\|b_{n,1}(\omega,\cdot,\cdot\cdot,\phi)\big\|_{L^p(U;L^q[0,T])} \leq C_{b_{n,1}}(1 + \|\phi\|_{L^p(U;L^q[0,T])})$$

for all $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, $x,y \in \mathbb{R}$, $v,w \in \mathbb{R}^d$, and $\phi \in L^p(U;L^q[0,T])$. The function $b_{n,2}$ is assumed to be locally Lipschitz continuous and bounded in 0, i.e. there exists a constant $C_{b_{n,2}} \geq 0$, and for each $R > 0$ there is a constant $L_{b_{n,2},R} \geq 0$ such that

$$\big|b_{n,2}(\omega,t,u,x) - b_{n,2}(\omega,t,u,y)\big| \leq L_{b_{n,2},R}|x-y|$$

and

$$\left\|b_{n,2}(\omega,\cdot,\cdot\cdot,0)\right\|_{L^p(U;L^q[0,T])} \le C_{b_{n,2}}$$

for all $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, and $x, y \in \mathbb{R}$ satisfying $|x|, |y| \le R$. For the sequences $(L_{b_{n,1}})_{n\in\mathbb{N}}$, $(C_{b_{n,1}})_{n\in\mathbb{N}}$, $(L_{b_{n,2},R})_{n\in\mathbb{N}}$, and $(C_{b_{n,2}})_{n\in\mathbb{N}}$ we assume that

$$L_{b_1} := \Big(\sum_{n=1}^{\infty} |L_{b_{n,1}}|^2\Big)^{1/2} < \infty, \quad C_{b_1} := \Big(\sum_{n=1}^{\infty} |C_{b_{n,1}}|^2\Big)^{1/2} < \infty,$$
$$L_{b_2,R} := \Big(\sum_{n=1}^{\infty} |L_{b_{n,2},R}|^2\Big)^{1/2} < \infty, \quad C_{b_2} := \Big(\sum_{n=1}^{\infty} |C_{b_{n,2}}|^2\Big)^{1/2} < \infty.$$

**(H$x_0$) Assumptions on the initial value $x_0$:** Let $x_0 \colon \Omega \to W^{1,p}_D(U)$ be strongly $\mathcal{F}_0$-measurable.

Then we obtain the following result.

**THEOREM 4.4.2.** *Under the assumption of Hypothesis 4.4.1 and*

$$L_{f_1} K_{det} + L_{b_1} K_{stoch} < 1,$$

*we obtain for each $x_0 \in L^0(\Omega, \mathcal{F}_0; W^{1,p}_D(U))$ a unique maximal local mild $(0,p,q)$ solution $X \colon \Omega \times [0,\tau) \to W^{1,p}(U)$ for (4.5) in $L^0_{\mathbb{F}}(\Omega; L^p(U; L^q[0,\tau)))$. Moreover, we have:*

1) *If we additionally assume that $f_2$ and $b_2 = (b_{n,2})_{n\in\mathbb{N}}$ satisfy the linear growth conditions*

$$\left\|f_2(\omega,\cdot,\cdot\cdot,\phi)\right\|_{L^p(U;L^q[0,T])} \le C_{f_2}(1 + \|\phi\|_{L^p(U;L^q[0,T])}),$$
$$\left\|b_2(\omega,\cdot,\cdot\cdot,\phi)\right\|_{L^p(U;L^q([0,T];\ell^2))} \le C_{b_2}(1 + \|\phi\|_{L^p(U;L^q[0,T])})$$

*for all $\phi \in L^p(U; L^q[0,T])$ and some constants $C_{f_2}, C_{b_2} > 0$ independent of $\omega \in \Omega$, then the solution $X$ above is a global mild $(0,p,q)$ solution.*

2) *If, in addition to that, we have $x_0 \in L^r(\Omega, \mathcal{F}_0; W^{1,p}_D(U))$ for some $r \in (1,\infty)$, then the global solution $X$ of part 1) satisfies*

$$X \in L^r_{\mathbb{F}}(\Omega; W^{1,p}_D(U; L^q[0,T]))$$

*and*

$$\|X\|_{L^r(\Omega;W^{1,p}(U;L^q[0,T]))} \le C(1 + \|x_0\|_{L^r(\Omega;W^{1,p}(U))}).$$

**PROOF.** We want to apply Theorem 3.5.19 and therefore have to check the conditions of Hypothesis 3.5.16. The assumption on $A_p$ is fulfilled by Section 2.3, see Example A. Note that the constants of the $\mathcal{R}_q$-bounded $H^\infty$ calculus only depend on $\alpha_0$ and $\max\{\|a_{i,j}\|_\infty, \|a_i\|_\infty, \|a_0\|_\infty \colon i,j \in \{1,\dots,d\}\}$. Moreover, we have by Example 2.4.7, [77, Theorem 1.15.3], and [46, Theorem 5.93]

$$D((A_p^{L^q})^{1/2}) = [L^p(U;L^q[0,T]), W_D^{2,p}(U;L^q[0,T])]_{1/2} = W_D^{1,p}(U;L^q[0,T]).$$

To model the nonlinearities $f$ and $b_n$, we let

$$\begin{aligned} F(\omega,t,\phi)(u) &:= F_1(\omega,t,\phi)(u) + F_2(\omega,t,\phi)(u) \\ &:= f_1(\omega,t,u,\phi,\nabla\phi) + f_2(\omega,t,u,\phi) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{B}(\omega,t,n,\phi)(u) &:= \boldsymbol{B}_1(\omega,t,n,\phi)(u) + \boldsymbol{B}_2(\omega,t,n,\phi)(u) \\ &:= b_{n,1}(\omega,t,u,\phi,\nabla\phi) + b_{n,2}(\omega,t,u,\phi) \end{aligned}$$

for $\omega \in \Omega$, $t \in [0,T]$, $u \in U$, $n \in \mathbb{N}$, and $\phi\colon [0,T] \to W^{1,p}(U)$. Then the remark above and the assumptions of $f_1$ and $f_2$ lead to

$$\begin{aligned} &\big\|F_1(\omega,\cdot,\phi) - F_1(\omega,\cdot,\psi)\big\|_{L^p(U;L^q[0,T])} \lesssim L_{f_1}\big(\|\phi-\psi\|_{L^p(U;L^q[0,T])} + \|A_p^{1/2}(\phi-\psi)\|_{L^p(U;L^q[0,T])}\big), \\ &\big\|F_2(\omega,\cdot,\phi) - F_2(\omega,\cdot,\psi)\big\|_{L^p(U;L^q[0,T])} \le L_{f_2,R}\|\phi-\psi\|_{L^p(U;L^q[0,T])}, \end{aligned}$$

and

$$\begin{aligned} &\|F_1(\omega,\cdot,\phi)\|_{L^p(U;L^q[0,T])} \le C_{f_1}(1 + \|\phi\|_{L^p(U;L^q[0,T])}), \\ &\|F_2(\omega,\cdot,0)\|_{L^p(U;L^q[0,T])} \le C_{f_2}. \end{aligned}$$

Therefore, $(\mathrm{HF})_{\mathrm{loc}}$ is satisfied for $\gamma = 1/2$ and $\gamma_F = 0$. In almost the same way we can verify $(\mathrm{HB})_{\mathrm{loc}}$ for $\boldsymbol{B}$ and $\gamma_B = 0$. Finally, since $x_0 \in W_D^{1,p}(U)$ almost surely, Corollary 3.2.10 implies that

$$x_0 \in D(A_p^{1/2}) \hookrightarrow D_{A_p}^{\ell q}(1/2 - 1/q)$$

Hence, the claim follows from Theorem 3.5.19. □

We finally compare these results to already existing results in the literature.

**Discussion 4.4.3.** Similar equations to (4.5) have been considered by many authors (see e.g. [9, 29, 28, 45, 81, 82]). In [9] Beck and Flandoli investigated the regularity of weak solutions of

$$\mathrm{d}X(t) = \operatorname{div}\big(a(u,t)DX(t)\big)\,\mathrm{d}t + \sum_{n=1}^{N} b_n(DX(t))\,\mathrm{d}\beta_n(t), \quad X(0) = x_0,$$

on a regular and bounded domain $U \subseteq \mathbb{R}^d$. They assumed globally Lipschitz continuity of $b = (b_n)_{n=1}^N$ with a sufficiently small Lipschitz constant and coefficients $a \in L^\infty([0,T]; C^1(U; \mathbb{R}^{d\times d}))$. For each $x_0 \in W^{1,p}(U)$, $p > d$, and every weak solution $X$, it was proved that $X \in C^\alpha(U \times [0,T])$ for some $\alpha > 0$ with probability 1 (see [9, Theorem 1.4]). Existence and uniqueness results were not considered. In the non-autonomous case, the results of Theorem 4.4.2 lead to a solution $X\colon \Omega \times [0,T] \to W_D^{1,p}(U)$ such that

$$X \in L^0_{\mathbb{F}}(\Omega; H^{1-2\sigma}(U; W^{\sigma,q}[0,T])), \quad \sigma \in [0, 1/2).$$

If we choose $\sigma \in (1/q, 1/2)$ and $p > \frac{d}{1-2\sigma}$ and use Sobolev's embedding theorem, we obtain

$$X \in L^0_{\mathbb{F}}(\Omega; C^{1-2\sigma-d/p}(U; C^{\sigma-1/q}[0,T])) \subseteq L^0_{\mathbb{F}}(\Omega; C^\alpha(U \times [0,T])),$$

where $\alpha = (1 - 2\sigma - d/p) \wedge (\sigma - 1/q) > 0$. Therefore, we arrive at the same regularity result as Beck and Flandoli. In particular, since this result is an *implication* of our theory, this means that the stated regularity of $X$ is indeed sharper.

We also want to emphasize that there are some limits of our theory. In [28] Denis, Matoussi, and Stoica considered equation (4.5) in $L^\infty(U)$ for an arbitrary open domain $U \subseteq \mathbb{R}^d$ of finite measure and initial values $x_0 \in L^\infty(U)$. This particular case can not be treated using our results since $L^\infty(U)$ is not a UMD space.

## 4.5 The Deterministic Case

In this section we shortly summarize the case if there are no stochastic terms in the abstract setting, i.e. if $\boldsymbol{B} = 0$. In this case we also get new results for the equation

$$X'(t) + AX(t) = F(t, X(t)), \quad X(0) = x_0. \tag{4.6}$$

Assuming the same assumptions as in Hypothesis 3.5.4 for the operator $A$ and the nonlinearity $F$, we obtain in the same way as in Section 3.5.2 the following theorem.

**Theorem 4.5.1 (Deterministic case).** *Let $p, q \in (1, \infty)$. Let (HA) and (HF) of Hypothesis 3.5.4 be satisfied, and $\gamma_F \le 0$ such that $\gamma + \gamma_F \in [0,1]$. In the case $\gamma + \gamma_F = 1$ we additionally assume that $L_F K_{\det} < 1$. Then the following assertions hold true:*

a) ***Existence and uniquenes:*** *If $x_0 \in D_A^{\ell q}(\gamma - 1/q)$, then (4.6) has a unique mild solution $X$ satisfying the a-priori estimate*

$$\|A^\gamma X\|_{L^p(U; L^q[0,T])} \le C(1 + \|x_0\|_{D_A^{\ell q}(\gamma - 1/q)}).$$

b) ***Regularity I:*** *For $q \geq 2$ the mild solution of a) has the following properties:*

$$A^{\gamma-\sigma}X \in L^p(U; W^{\sigma,q}[0,T]), \quad \sigma \in [0,1),\ \sigma \leq \gamma.$$

*In particular,*

$$\begin{aligned} X &\in C([0,T]; D_A^{\ell q}(\gamma - 1/q)), \\ A^{\gamma-\sigma}X &\in L^p(U; C^{\sigma-1/q}[0,T]), \quad \sigma \in (1/q, 1),\ \sigma \leq \gamma. \end{aligned}$$

*In addition to a) we have the following a-priori estimates*

$$\begin{aligned} \|A^{\gamma-\sigma}X\|_{L^p(U;W^{\sigma,q}[0,T])} &\leq C(1 + \|x_0\|_{D_A^{\ell q}(\gamma-1/q)}), \quad \sigma \in [0,1),\ \sigma \leq \gamma, \\ \|X\|_{C([0,T];D_A^{\ell q}(\gamma-1/q))} &\leq C(1 + \|x_0\|_{D_A^{\ell q}(\gamma-1/q)}), \\ \|A^{\gamma-\sigma}X\|_{L^p(U;C^{\sigma-1/q}[0,T])} &\leq C(1 + \|x_0\|_{D_A^{\ell q}(\gamma-1/q)}), \quad \sigma \in (1/q, 1),\ \sigma \leq \gamma. \end{aligned}$$

c) ***Regularity II:*** *If $\gamma + \gamma_F < 1$, we have for each $\varepsilon \in [0, 1 - \gamma - \gamma_F)$ and $x_0 \in D_A^{\ell q}(\gamma + \varepsilon - 1/q)$*

$$\begin{aligned} A^{\gamma+\varepsilon-\sigma}X &\in L^p(U; W^{\sigma,q}[0,T]), \quad \sigma \in [0,1),\ \sigma \leq \gamma + \varepsilon, \\ X &\in C([0,T]; D_A^{\ell q}(\gamma + \varepsilon - 1/q)), \end{aligned}$$

*and*

$$A^{\gamma+\varepsilon-\sigma}X \in L^p(U; C^{\sigma-1/q}[0,T]), \quad \sigma \in (1/q, 1),\ \sigma \leq \gamma + \varepsilon.$$

*satisfying*

$$\begin{aligned} \|A^{\gamma+\varepsilon-\sigma}X\|_{L^p(U;W^{\sigma,q}[0,T])} &\leq C_T(1 + \|x_0\|_{D_A^{\ell q}(\gamma+\varepsilon-1/q)}), \quad \sigma \in [0,1),\ \sigma \leq \gamma + \varepsilon, \\ \|X\|_{C([0,T];D_A^{\ell q}(\gamma+\varepsilon-1/q))} &\leq C_T(1 + \|x_0\|_{D_A^{\ell q}(\gamma+\varepsilon-1/q)}), \\ \|A^{\gamma+\varepsilon-\sigma}X\|_{L^p(U;C^{\sigma-1/q}[0,T])} &\leq C_T(1 + \|x_0\|_{D_A^{\ell q}(\gamma+\varepsilon-1/q)}), \quad \sigma \in (1/q, 1),\ \sigma \leq \gamma + \varepsilon. \end{aligned}$$

d) ***Continuous dependence of data:*** *For initial values $x_0, y_0 \in D_A^{\ell q}(\gamma - 1/q)$ and the corresponding solutions $X$ and $Y$ we have*

$$\begin{aligned} \|A^{\gamma}(X - Y)\|_{L^p(U;L^q[0,T])} &\leq C\|x_0 - y_0\|_{D_A^{\ell q}(\gamma-1/q)}, \\ \|A^{\gamma-\sigma}(X - Y)\|_{L^p(U;W^{\sigma,q}[0,T])} &\leq C\|x_0 - y_0\|_{D_A^{\ell q}(\gamma-1/q)}, \quad \sigma \in [0,1), \quad \sigma \leq \gamma, \\ \|X - Y\|_{C([0,T];D_A^{\ell q}(\gamma-1/q))} &\leq C\|x_0 - y_0\|_{D_A^{\ell q}(\gamma-1/q)}, \\ \|A^{\gamma-\sigma}(X - Y)\|_{L^p(U;C^{\sigma-1/q}[0,T])} &\leq C\|x_0 - y_0\|_{D_A^{\ell q}(\gamma-1/q)}, \quad \sigma \in (1/q, 1),\ \sigma \leq \gamma. \end{aligned}$$

Similarly, we obtain the corresponding versions for the time-dependent and locally Lipschitz case, and if $\gamma \geq 1$ we obtain strong solutions. One should note that the restrictions in the regularity theorems for values of $\sigma$, i.e. $\sigma < 1/2$ (see e.g. Theorem 3.5.9), is only attributed to the properties of the stochastic convolution, not the deterministic one. This improves all regularity results to the case $\sigma < 1$. In particular, we obtain a new regularity theory for deterministic evolution equations in $L^p$ spaces with stronger results regarding time regularity.

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