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The β-function of Quantum Chromodynamics and the effective Higgs-gluon-gluon coupling in five-loop order

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We present the result of the analytical computation of the five-loop term in the beta function which governs the running of α_s — the quark-gluon coupling constant in QCD. Using the result, a low-energy theorem and the four-loop QCD decoupling formula, we derive the effective Higgs-gluon-gluon coupling constant in five-loop approximation.

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Asymptotic freedom, manifest by a coupling constant decreasing with increasing energy, is one of the basic predictions of nonabelian gauge theories. It is quantitatively encoded in the behaviour of the β-function of Quantum Chromodynamics (QCD):

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 0} \beta_i a_s^{i+2} \tag{1}$$

which describes the running of the quark-gluon coupling constant $a_s \equiv \alpha_s/\pi$ as a function of the renormalization scale μ. The four lowest terms were calculated in Refs. [1–8] with the result (we use the $\overline{\text{MS}}$ renormalization scheme throughout the paper):

$$\begin{aligned}
\beta_0 &= \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \\
\beta_1 &= \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \\
\beta_2 &= \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\}, \\
\beta_3 &= \frac{1}{4^4}\left\{\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right] n_f + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right] n_f^2 + \frac{1093}{729}n_f^3\right\},
\end{aligned}$$

where n_f denotes the number of active quark flavors. These results have moved the theory from qualitative agreement with experiment to precise quantitative predictions, covering a wide kinematical range, from τ-lepton decays up to LHC results.

Recently, the five-loop term has been evaluated with the result:

$$\begin{aligned}
\beta_4 = \frac{1}{4^5}\Bigg\{ & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+\ n_f & \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\
+\ n_f^2 & \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\
+\ n_f^3 & \left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right] \\
+\ n_f^4 & \left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]\Bigg\}.
\end{aligned} \tag{2}$$

As expected from the three and four-loop results, β_4 does not contain the higher transcendentalities ζ_3^2, ζ_6 and ζ_7 which do appear in individual diagrams. Note that the n_f^4 term is in full agreement with the result of [9] derived for a generic gauge group long time ago with a very different technique. Furthermore, quite recently, the n_f^3 term has been confirmed and even extended for a generic gauge group in [10].

In numerical form the coefficients $\beta_0 - \beta_4$ read

$$
\begin{aligned}
\beta_0 &\approx 2.75 - 0.166667\, n_f, \\
\beta_1 &\approx 6.375 - 0.791667\, n_f, \\
\beta_2 &\approx 22.3203 - 4.36892\, n_f + 0.0940394\, n_f^2, \\
\beta_3 &\approx 114.23 - 27.1339\, n_f + 1.58238\, n_f^2 + 0.0058567\, n_f^3, \\
\beta_4 &\approx 524.558 - 181.799\, n_f + 17.156\, n_f^2 - 0.22586\, n_f^3 - 0.0017993\, n_f^4.
\end{aligned}
\tag{3}
$$

For the particular cases of $n_f = 3, 4, 5$ and 6 we get:

$$
\begin{aligned}
\overline{\beta}(n_f = 3) &= 1 + 1.78\, a_s + 4.47\, a_s^2 + 20.99\, a_s^3 + 56.59\, a_s^4, \\
\overline{\beta}(n_f = 4) &= 1 + 1.54\, a_s + 3.05\, a_s^2 + 15.07\, a_s^3 + 27.33\, a_s^4, \\
\overline{\beta}(n_f = 5) &= 1 + 1.26\, a_s + 1.47\, a_s^2 + 9.83\, a_s^3 + 7.88\, a_s^4, \\
\overline{\beta}(n_f = 6) &= 1 + 0.93\, a_s - 0.29\, a_s^2 + 5.52\, a_s^3 + 0.15\, a_s^4,
\end{aligned}
\tag{4}
$$

where $\overline{\beta} \equiv \frac{\beta(a_s)}{-\beta_0 a_s^2} = 1 + \sum_{i \geq 1} \bar{\beta}_i a_s^i$. A very modest growth of the coefficients is observed and the (apparent) convergence is better than one would expect from comparison with other examples. The smallness of the five-loop contributions in eqs. (4) implies that the final result is hardly effected when running α_s from the scale of the τ-lepton mass m_τ to the mass of the Z-boson.

The combined uncertainty in $\alpha_s^{(5)}(M_Z)$ induced by running and matching from m_τ to M_Z can be conservatively estimated by the shift in $\alpha_s^{(5)}(M_Z)$ produced by the use of five-loop running (and, consequently) four-loop matching instead of four-loop running (and three-loop matching). It amounts to a $8 \cdot 10^{-5}$ which is by a factor of three less than the similar shift made by the use of four-loop running instead of the the three-loop one. The final value of $\alpha_s^{(5)}(M_Z) = 0.1143 \pm 0.0015$ which follows from $\alpha_s^{(3)}(M_\tau)$ is in remarkably good agreement with the fit to electroweak precision data ([11]) $\alpha_s^{(5)}(M_Z) = 0.1197 \pm 0.0028$.

Our result for the β-function leads to a determination of the effective Higgs-gluon-gluon coupling, improved by one more loop. In the heavy top limit the Higgs boson couples directly to gluons via the effective Lagrangian of the form [12–15]

$$
\mathscr{L}_{eff} = -2^{1/4} G_F^{1/2} H C_1(\mu^2/m_t^2, a_s(\mu))\, G^a_{\nu\rho} G^a_{\nu\rho}. \tag{5}
$$

The effective coupling constant $C_1(\mu^2/m_t^2, a_s(\mu))$ appears as a common factor in two quantities important for Higgs physics processes, namely, Higgs decay into gluons (one of the main decay channels for the Standard Model Higgs boson) and Higgs production via gluon fusion (the main Higgs production mode on LHC). It is expressible through massive tadpoles and was computed at four loops in 1997 [16] (long before the direct calculation of four-loop generic massive tadpoles started to be technically feasible). This happened to be possible due to a low energy theorem [1] (exact in all orders) [16]

$$
C_1 = -\frac{1}{2} m_h^2 \frac{\partial}{\partial m_h^2} \ln \zeta_g^2, \tag{6}
$$

[1]Very recently the theorem has been redirived with a different technique [17] and even extended on the case of multi-Higgs couplings to gluons in Ref. [18].

which connects C_1 with the corresponding "decoupling" constant ζ_g defined as a conversion factor in the relation

$$\alpha_s' = \alpha_s'(\mu, \alpha_s, m_h) = \zeta_g^2(\mu, \alpha_s, m_h)\, \alpha_s(\mu), \qquad \zeta_g^2 = 1 + \sum_{i\geq 1} a_s^i(\mu)\, d_{\mathrm{MS,i}},$$

where α_s' denotes the coupling constant in the effective QCD with the heavy quark, integrated out and n_l is the number of remaining active quark flavours.

Eq. (6) assumes that we are dealing with QCD with one heavy quark h and n_l massless ones and that Higgs field is coupled with the scalar heavy quark current via

$$-\frac{H}{v}\, m_h \bar{h} h.$$

The total number of active flavours is $n_f = n_l + 1$. For the case of the SM Higgs one should set $m_h = m_t$ and $n_f = 6$, $n_l = 5$. A more convenient form of the rhs of eq. (6) was found in Ref. [16]

$$C_1 = \frac{\pi}{2\alpha_s'[1 - 2\gamma_m(\alpha_s)]} \left[\beta'(\alpha_s') - \beta(\alpha_s) \frac{\partial \alpha_s'}{\partial \alpha_s} \right]. \tag{7}$$

By counting the powers of α_s in the rhs of eq. (7) one can easily see that, say, the $(L+1)$-loop contribution to C_1 is constructed from the $(L+1)$ loop β-function as well as from the quark mass anomalous dimension γ_m and the decoupling constant ζ_g^2, with the latter two quantities being sufficient to be known at L loops *only*.

The decoupling constant ζ_g^2 is currently known at four loops; the corresponding coefficients d_i read[2]

$$d_{\mathrm{MS},1} = -\frac{1}{6}\, \ell_{\mu m}, \tag{8}$$

$$d_{\mathrm{MS},2} = \frac{11}{72} - \frac{11}{24}\, \ell_{\mu m} + \frac{1}{36}\, \ell_{\mu m}^2, \tag{9}$$

$$\begin{aligned} d_{\mathrm{MS},3} \;=\; & \frac{564731}{124416} - \frac{82043}{27648}\, \zeta_3 - \frac{955}{576}\, \ell_{\mu m} + \frac{53}{576}\, \ell_{\mu m}^2 - \frac{1}{216}\, \ell_{\mu m}^3 \\ & + \; n_l \left[-\frac{2633}{31104} + \frac{67}{576}\, \ell_{\mu m} - \frac{1}{36}\, \ell_{\mu m}^2 \right], \end{aligned} \tag{10}$$

$$\begin{aligned} d_{\mathrm{MS},4} \;=\; & \Delta_{\mathrm{MS},4} + \frac{7391699}{746496}\, \ell_{\mu m} - \frac{2529743}{165888}\, \zeta_3\, \ell_{\mu m} + \frac{2177}{3456}\, \ell_{\mu m}^2 - \frac{1883}{10368}\, \ell_{\mu m}^3 + \frac{1}{1296}\, \ell_{\mu m}^4 \\ & + \; n_l \left[-\frac{110341}{373248}\, \ell_{\mu m} + \frac{110779}{82944}\, \zeta_3\, \ell_{\mu m} - \frac{1483}{10368}\, \ell_{\mu m}^2 - \frac{127}{5184}\, \ell_{\mu m}^3 \right] \\ & + \; n_l^2 \left[\frac{6865}{186624}\, \ell_{\mu m} - \frac{77}{20736}\, \ell_{\mu m}^2 + \frac{1}{324}\, \ell_{\mu m}^3 \right], \end{aligned} \tag{11}$$

[2]The original result for ζ_g^2 as obtained in Refs. [19, 20] was not completely analytical as a pair of relevant master integrals was then known only numerically. In the formula below we have used finding of Refs. [21, 22] to display the $\Delta_{\mathrm{MS},4}$ in a completely analytical form (it was also done in Ref. [23]).

$$
\begin{aligned}
\Delta_{\mathrm{MS},4} = {} & \frac{291716893}{6123600} - \frac{76940219}{195955200}\pi^4 - \frac{2362581983}{87091200}\zeta_3 - \frac{12057583}{483840}\zeta_5 \\
& + \frac{9318467}{32659200}\pi^4\ln 2 - \frac{3031309}{1306368}\pi^2\ln^2 2 + \frac{340853}{816480}\pi^2\ln^3 2 + \frac{3031309}{1306368}\ln^4 2 \\
& - \frac{340853}{1360800}\ln^5 2 + \frac{3031309}{54432}a_4 + \frac{340853}{11340}a_5 \\
& + n_l\left[-\frac{4770941}{2239488} - \frac{541549}{14929920}\pi^4 + \frac{3645913}{995328}\zeta_3 + \frac{115}{576}\zeta_5 \right. \\
& \quad \left. - \frac{685}{124416}\pi^2\ln^2 2 + \frac{685}{124416}\ln^4 2 + \frac{685}{5184}a_4\right] \\
& + n_l^2\left[-\frac{271883}{4478976} + \frac{167}{5184}\zeta_3\right]. \qquad (12)
\end{aligned}
$$

Here $\ell_{\mu m} = \ln\frac{\mu^2}{m^2(\mu)}$, $m(\mu)$ is the (running) heavy quark mass and $a_n = \mathrm{Li}_n(1/2) = \sum_{i=1}^{\infty} 1/(2^i i^n)$.

The quark mass anomalous dimenson γ_m is known at four loops since long [24, 25]. Finally, using our result for the β-function, ζ_g and γ_m we arrive at (μ_h is defined as $\mu_h = m_h(\mu_h)$):

$$
\begin{aligned}
C_1 = {} & -\frac{1}{12}a_s(\mu_h)\left\{1 + \frac{11}{4}a_s(\mu_h) + a_s^2(\mu_h)\left[\frac{2821}{288} + n_l\left(-\frac{67}{96}\right)\right]\right. \\
& + a_s^3(\mu_h)\left[-\frac{4004351}{62208} + \frac{1305893}{13824}\zeta(3) + n_l\left(\frac{115607}{62208} - \frac{110779}{13824}\zeta(3)\right) + n_l^2\left(-\frac{6865}{31104}\right)\right] \\
& + a_s^4(\mu_h)\left[\left(-\frac{348837110927}{223948800} + \frac{1959268669}{111974400}\pi^4 + \frac{35074684379}{24883200}\zeta_3 + \frac{127050929}{138240}\zeta_5\right.\right. \\
& \quad - \frac{121140071}{9331200}\pi^4\ln 2 + \frac{39407017}{373248}\pi^2\ln^2 2 - \frac{4431089}{233280}\pi^2\ln^3 2 - \frac{39407017}{373248}\ln^4 2 \\
& \quad \left. + \frac{4431089}{388800}\ln^5 2 - \frac{39407017}{15552}a_4 - \frac{4431089}{3240}a_5\right) \\
& + n_l\left(\frac{223712155249}{1567641600} + \frac{190184767}{348364800}\pi^4 - \frac{108268867471}{348364800}\zeta_3 - \frac{22007549}{483840}\zeta_5\right. \\
& \quad + \frac{9318467}{10886400}\pi^4\ln 2 - \frac{3896297}{580608}\pi^2\ln^2 2 + \frac{340853}{272160}\pi^2\ln^3 2 + \frac{3896297}{580608}\ln^4 2 \\
& \quad \left. - \frac{340853}{453600}\ln^5 2 + \frac{3896297}{24192}a_4 + \frac{340853}{3780}a_5\right) \\
& + n_l^2\left(-\frac{8059709}{4478976} - \frac{576757}{4976640}\pi^4 + \frac{455777}{36864}\zeta_3 - \frac{115}{384}\zeta_5 - \frac{685}{41472}\pi^2\ln^2 2 + \frac{685}{41472}\ln^4 2 + \frac{685}{1728}a_4\right) \\
& \left.\left. + n_l^3\left(-\frac{270407}{1492992} + \frac{211}{1728}\zeta_3\right)\right]\right\},
\end{aligned}
$$

or, numerically,

$$
\begin{aligned}
C_1 \approx -\frac{1}{12}a_s(\mu_h)\Big[1 & + 2.75\,a_s(\mu_h) + (9.795 - 0.698\,n_l)\,a_s^2(\mu_h) \\
& + \left(49.183 - 7.774\,n_l - 0.221\,n_l^2\right)a_s^3(\mu_h) \\
& + \left(326.716 - 64.163\,n_l + 1.594\,n_l^2 - 0.0343\,n_l^3\right)a_s^4(\mu_h)\Big]. \qquad (13)
\end{aligned}
$$

The last expession should be modified if one want to use the pole quark mass M_h instead of the running $\overline{\text{MS}}$ one. Using the corresponding conversion formula from [26–28] we arrive at

$$C_1 \approx -\frac{1}{12} a_s(M_h) \Big[1 \, + \, 2.75 \, a_s(M_h) + (9.351 - 0.698 \, n_l) \, a_s^2(M_h) \\ + \left(43.609 - 6.538 \, n_l - 0.221 \, n_l^2\right) a_s^3(M_h) \\ + \left(240.383 - 30.231 \, n_l + 0.8001 \, n_l^2 - 0.0343 \, n_l^3\right) a_s^4(M_h) \Big]. \tag{14}$$

For the case of the SM Higgs we obtain:

$$C_1 = -\frac{1}{12} a_s(\mu_t) \left[1 + 2.750 \, a_s(\mu_t) + 6.306 \, a_s^2(\mu_t) + 4.794 \, a_s^3(\mu_t) + 41.447 \, a_s^4(\mu_t) \right]. \tag{15}$$

Note that the contribution due to β_4 to the last coefficient (boxed below) is significant, namely,

$$41.447 = -47.611 + \boxed{89.058}. \tag{16}$$

After transformation to the pole top quark mass eq. (15) reads

$$C_1 = -\frac{1}{12} a_s(M_t) \left[(1 + 2.750 \, a_s(M_t) + 5.861 \, a_s^2(M_t) + 5.3996 \, a_s^3(M_t) + 54.951 \, a_s^4(M_t) \right], \tag{17}$$

with the last term decomposition assuming the form:

$$54.951 = -34.107 + \boxed{89.058}. \tag{18}$$

Let us close this paper with a brief discussion of its technical details[3]. The total number of five-loop diagrams contributing to the β-function (as generated by QGRAF [30]) amounts to about one and a half million. Every power of n_f in (2) was computed separately with the help of the FORM [31, 32] program BAICER, implementing the algorithm of works [33–35].

Our analytical results are also available in computer readable form under the URL
`http://www.ttp.kit.edu/Progdata/ttp16/ttp16-032`

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