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Neutrino mass sum rules and symmetries of the mass matrix

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Abstract Neutrino mass sum rules have recently gained again more attention as a powerful tool to discriminate and test various flavour models in the near future. A related question which has not yet been discussed fully satisfactorily was the origin of these sum rules and if they are related to any residual or accidental symmetry. We will address this open issue here systematically and find previous statements confirmed. Namely, the sum rules are not related to any enhanced symmetry of the Lagrangian after family symmetry breaking but they are simply the result of a reduction of free parameters due to skillful model building.

1 Introduction

The origin of flavour is still an open issue in the Standard Model of particle physics (SM) and most of its extensions. In the recent past a very popular approach is based on (discrete) family symmetries which can easily explain the number of generations and gives a very good leading order description of the neutrino mixing angles; for recent reviews, see, e.g., [1–5].

One particular prediction in plenty of these models is a so-called neutrino mass sum rule, which relates the three complex neutrino eigenvalues with each other. That means that the three masses can be described by two complex parameters only.[1] In Table 1 we have given a list of all 12 sum rules known in the literature. For recent detailed phenomenological studies of the mass sum rules, see [6–9]. All mass sum rules can be parametrised according to [6,7] as

$$s(m_1, m_2, m_3, \phi_1, \phi_2; c_1, c_2, d, \Delta\chi_{13}, \Delta\chi_{23})$$
$$\equiv c_1(m_1 \mathrm{e}^{-\mathrm{i}\phi_1})^d \mathrm{e}^{\mathrm{i}\Delta\chi_{13}} + c_2(m_2 \mathrm{e}^{-\mathrm{i}\phi_2})^d \mathrm{e}^{\mathrm{i}\Delta\chi_{23}} + m_3^d = 0, \quad (1.1)$$

where ϕ_1 and ϕ_2 are the Majorana phases. The quantities c_1, c_2, d, $\Delta\chi_{13}$, and $\Delta\chi_{23}$ are parameters which characterise the sum rule.

Since in these models we have less free parameters than observables one might wonder if there is some underlying symmetry behind the mass sum rules. This is particularly tempting since they emerge usually in models which have much more symmetry than the SM including neutrino masses so that it could well be that the full symmetry of the Lagrangian is actually not completely broken. A residual (or accidental) symmetry could then be responsible for the reduction of free parameters in the mass matrix and result in a sum rule.[2]

On the other hand this intriguing idea is challenged by the fact that the same sum rule emerges in models with different symmetries. For instance, sum rule 6 from Table 1 with $1/\tilde{m}_1 + 1/\tilde{m}_2 - 1/\tilde{m}_3 = 0$, where $\tilde{m}_i$ are the three complex neutrino masses, is realised in models with A_4 [21], S_4 [13] and A_5 [49–51] symmetry. Furthermore, in [7] we have already tried to argue against some more fundamental principle behind the mass sum rules by emphasizing that the only common feature of all this models is a reduction of parameters in the neutrino mass matrix. To be more precise the reduction of free parameters comes from an interplay of the choice of the family symmetry, the choice of particle representations under this symmetry, and the way the family symmetry is broken. Nevertheless, there is no general com-

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[1] In principle, one could imagine that the three masses are described by one complex parameter only, but we are not aware of any such model with phenomenologically viable predictions.

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[2] In this letter we only discuss mass sum rules. There are also the well-known mixing sum rules which originate from an additional breaking of the residual symmetries in the charged lepton or neutrino sector. For a recent very detailed account of all the possibilities, see, e.g., [76] and references therein.

Table 1 Summary table of the sum rules present in the literature. The parameters $c_1, c_2, d, \Delta\chi_{13}$, and $\Delta\chi_{23}$, which characterise them, are defined in Eq. (1.1). In sum rules 9 and 10 two possible signs appear which lead to two possible values of $\Delta\chi_{i3}$

Sum rule	References	c_1	c_2	d	$\Delta\chi_{13}$	$\Delta\chi_{23}$
1	[10–21]	1	1	1	π	π
2	[22]	1	2	1	π	π
3	[15–21,23–43]	1	2	1	π	0
4	[44–46]	1/2	1/2	1	π	π
5	[47]	$\frac{2}{\sqrt{3}+1}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	1	0	π
6	[13,14,21,48–51]	1	1	−1	π	π
7	[21,41–43,52–66]	1	2	−1	π	0
8	[67–70]	1	2	−1	0	π
9	[71]	1	2	−1	π	$\pi/2, 3\pi/2$
10	[72,73]	1	2	1/2	$\pi, 0, \pi/2$	$0, \pi, \pi/2$
11	[74]	1/3	1	1/2	π	0
12	[75]	1/2	1/2	−1/2	π	π

mon recipe simply due to the fact that there is no underlying symmetry argument as we will show in the following. In a sense mass sum rules are a mere result of skillful model building. Note that this implies that they can appear in direct, semi-direct and indirect models (for this classification see, e.g., [5]) since they can never be mapped to any subgroup of the family symmetry.

In this short letter we want to extend this previous discussion by giving more formal arguments to show that the symmetry of the Lagrangian is not enhanced by a neutrino mass sum rule after symmetry breaking and that the presence of a neutrino mass sum rule cannot be directly related to any symmetry. This is different from the case of the mixing angle predictions where the Klein symmetry of the neutrino mass matrix can be identified with some of the generators of the family symmetry, for explanations and references, see [1–4]. Hence, the claim that neutrino mass sum rules have no deeper reason than sophisticated model building is confirmed.

2 Symmetries of Majorana mass terms

We will start by considering a Majorana mass term in the Lagrangian for the neutrinos

$$\mathcal{L}_\nu \supset \frac{1}{2}\nu_L C^{-1} M_M \nu_L + \text{H.c.},\qquad(2.1)$$

where ν_L contains the three left-handed neutrino fields, C is the charge conjugation matrix and M_M is the symmetric, complex neutrino Majorana mass matrix. The recent success of flavour model building with (discrete) symmetries is based on the assumption that the Klein symmetry of the neutrino Majorana mass term is the remnant of a bigger family symmetry; see, e.g., [1–4]. The generators of the residual symmetry G of the mass matrix fulfill the symmetry condition

$$S^T M_M S = M_M,\qquad(2.2)$$

with unitary matrices $S \in G$. Note that we restrict ourselves here to unitary representations to keep the kinetic term canonical. The question is now, if there are additional possibilities for S if a mass sum rule is present which enhances the symmetry G. In the following we will work in a basis, where M_M is diagonal since an enhanced symmetry should be present in any basis and we are only interested in the masses here. The advantage is that in this basis the sum rule is most simple and obvious. Note also that the characteristic polynomial of the mass matrix is the same in the flavour and the mass basis.

We will begin our considerations with a more intuitive perturbative approach and later discuss a general calculation.

In our setup G could be maximally $U(3)$ and we can write $S = \exp(i \alpha_i T_i)$, $i = 1, \ldots, 9$ with the common eight generators T_a ($a = 1, \ldots, 8$) of $SU(3)$ and T_9 the generator of $U(1)$ [77].

If there is any continuous symmetry G this would be expressed in conditions on the generators T_i. For a continuous Lie group we can expand Eq. (2.2) in α_i to get up to $\mathcal{O}(\alpha_i^2)$

$$i \alpha_i ((T_i)^T M_M + M_M T_i) = 0 \text{ for } i = 1, \ldots, 9.\qquad(2.3)$$

Using the explicit forms of the generators and $M = \text{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$ we obtain the following conditions:

$$i\alpha_1(\tilde{m}_1 + \tilde{m}_2) + \alpha_2(\tilde{m}_1 - \tilde{m}_2) = 0,\qquad(2.4)$$
$$i\alpha_4(\tilde{m}_1 + \tilde{m}_3) + \alpha_5(\tilde{m}_1 - \tilde{m}_3) = 0,\qquad(2.5)$$
$$i\alpha_6(\tilde{m}_2 + \tilde{m}_3) + \alpha_7(\tilde{m}_2 - \tilde{m}_3) = 0,\qquad(2.6)$$
$$2 i \tilde{m}_1 \left(\alpha_3 + \frac{\alpha_8}{\sqrt{3}} + \alpha_9\right) = 0,\qquad(2.7)$$

$$2\,\mathrm{i}\,\tilde{m}_2\left(-\alpha_3+\frac{\alpha_8}{\sqrt{3}}+\alpha_9\right)=0, \tag{2.8}$$

$$2\,\mathrm{i}\,\tilde{m}_3\left(-\frac{2\alpha_8}{\sqrt{3}}+\alpha_9\right)=0. \tag{2.9}$$

Before we discuss the sum rule case we briefly want to review some well-known cases. In the case of three distinct, independent mass eigenvalues we get from Eq. (2.3) $\alpha_i=0$ and we have no continuous symmetry apart from the trivial one in this case.

If one of the masses vanishes while the two other are independent and non-zero we obtain a relation between the diagonal generators which leads to a $U(1)$ symmetry for the massless state. For example, if $\tilde{m}_3=0$ we have the enhanced symmetry $G=\exp(\mathrm{i}\,\alpha\,T)$ with $T=\mathrm{Diag}(0,0,1)$ as expected.

In the case of two equal masses (and the third different and non-zero) we obtain, for instance, if $\tilde{m}_2=\tilde{m}_3$ only the $SO(2)$ generator in the 1–2 block as again expected

Now for the interesting case, that $\tilde{m}_3$ is a function depending on the two other masses we have explicitly checked that for all 12 sum rules in Table 1 all $\alpha_i=0$ as in the case for three distinct, independent masses.

Another approach is to start now from a point in the parameter space which has a well-known enhanced symmetry. If there would be an enhanced symmetry in the case of a mass sum rule it should still be there after a small perturbation. For instance, setting $\tilde{m}_3\equiv\tilde{m}_2$ will fix $\tilde{m}_1$ for a given sum rule. But for this particular point we have a $SO(2)$ symmetry. If there is any non-trivial residual symmetry G for a small perturbation of the symmetric points it must be related to a small perturbation to the elements of $SO(2)$.

In the concrete case of $\tilde{m}_3=\tilde{m}_1+2\tilde{m}_2$ we take $\tilde{m}_3=\tilde{m}_2(1+\epsilon)$ with ϵ a small perturbation from the enhanced symmetry point. The mass matrix is then

$$M_{\mathrm{M}}=\mathrm{Diag}(-\tilde{m}_2(1-\epsilon),\tilde{m}_2,\tilde{m}_2(1+\epsilon)). \tag{2.10}$$

Now we can expand in all α_i, $i\neq6$, in Eq. (2.3) and set the α_i to be of $\mathcal{O}(\epsilon)$. The only solution to this equation is again $\alpha_i=0, i=1,\ldots9$. This is also true in the case of other sum rules with different coefficients as can easily be understood from considering only the 2–3 block of M_{M} which exhibits a $SO(2)$ symmetry for $\epsilon=0$. In the case of $\epsilon\neq0$ the eigenvalues are different and we find no symmetry anymore.

We have also expanded around the other symmetry points for all sum rules and around the well-known non-trivial $\mathbb{Z}_2^3$ symmetry of the Majorana mass matrix (which corresponds to expanding α_3, α_8 and α_9 around π) but never found any non-trivial solution for the α_i.

Up to now we have only considered continuous symmetries where we can expand in small α_i around the elements of $\mathbb{Z}_2^3$ and concluded that the presence of a sum rule does not enhance the symmetry of the mass matrix. One might wonder

if the residual symmetry we are looking for is not anywhere near these points – which would be surprising but cannot be completely ruled out at this point.

To rule this out as well we also did the general calculation for an arbitrary $S\in U(3)$ which is nevertheless tedious and not very insightful compared to the perturbative approach. We will discuss here how to do this for sum rule 1, cf. Table 1, as an example. For the other sum rules similar calculations give the same result as we have checked. An element $S\in U(3)$ can be written as $S=P_1U_{23}U_{13}U_{12}$, for notation and conventions; see Appendix A of [8]. Since S is unitary we can rewrite Eq. (2.2)

$$M_{\mathrm{M}}S=S^*M_{\mathrm{M}}. \tag{2.11}$$

The 1–1 element of this equation reads

$$m_1\cos\theta_{12}\cos\theta_{13}\left(\mathrm{e}^{2\,\mathrm{i}\,\omega_1}-1\right)=0, \tag{2.12}$$

which has four possible solutions. Let us discuss first $\theta_{13}=\pi/2$ (note that in our conventions $\theta_{ij}\in[0,\pi]$). From the 1–3 element of Eq. (2.11) we find that

$$\delta_{13}=\omega_1\quad\text{and}\quad m_2=2\,m_1\cos(\phi_2-\phi_1), \tag{2.13}$$

which is in general not satisfied and we exclude the solution with $\theta_{13}=\pi/2$. For the same reason we have to discard $\theta_{12}=\pi/2$. From the remaining two solutions $\omega_2=0$ or π it is sufficient to discuss $\omega_2=0$. At this point they are related by a global sign.

From the 1–2 element of Eq. (2.11) we derive

$$(m_1\mathrm{e}^{\mathrm{i}\phi_2}-m_2\mathrm{e}^{\mathrm{i}(\phi_1+2\delta_{12})})\cos\theta_{13}\sin\theta_{12}=0. \tag{2.14}$$

As we have discussed above we have to discard the solution $\theta_{13}=\pi/2$ and the only two remaining solutions are $\theta_{12}=0$ or π. For simplicity, we will only discuss here $\theta_{12}=0$. From the 1–3 element of Eq. (2.11) we then find that

$$(m_1(\mathrm{e}^{2\,\mathrm{i}\,\delta_{13}}-1)-m_2\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2+2\delta_{13})})\sin\theta_{13}=0. \tag{2.15}$$

Again $\theta_{13}=0$ or π and we discuss only $\theta_{13}=0$. From the 3–2 element of Eq. (2.11) we find

$$(m_1\mathrm{e}^{\mathrm{i}(\phi_2+2\delta_{23}+2\omega_3)}+m_2\mathrm{e}^{\mathrm{i}(\phi_1+2\delta_{23}+2\omega_3)}-m_2\mathrm{e}^{\mathrm{i}\phi_1})\sin\theta_{23}=0. \tag{2.16}$$

Hence $\theta_{23}=0$ or π.

Now we know that S has to be diagonal, it is trivial to see that the remaining phases have to take trivial values. So we have shown without resorting to any expansion that S can be only an element of $\mathbb{Z}_2^3$, i.e. it must be a diagonal matrix with ±1 on the diagonal. For the other sum rules we have checked with similar calculations that S can be only an element of $\mathbb{Z}_2^3$.

Hence, we conclude that there is no particular residual (continuous or discrete) symmetry in the case of a neutrino mass sum rule. This is actually not surprising. Apart from the ubiquitous symmetric points where masses are equal or vanish even in the case of neutrino mass sum rules the three neutrino masses are different, which is known to have no other symmetry than the $\mathbb{Z}_2^3$ (physically the Klein symmetry $\mathbb{Z}_2^2$ after absorbing an unphysical sign corresponding to one of the $\mathbb{Z}_2$ factors).

3 Symmetries of Dirac mass terms

We turn now to the case of Dirac mass matrices which is nevertheless only realised in one of the known flavour models exhibiting a mass sum rule [78] in the literature (this sum rule can also be realised in models with Majorana neutrinos [79]).

One has to be careful since in the mass sum rule for Dirac neutrinos the Majorana phases are unphysical. Nevertheless, these sum rules still lead to one of the major predictions of mass sum rules, the lower bound on the lightest mass.

We will show that also in the case of Dirac neutrinos a mass sum rule does not lead to any particular symmetry of the mass matrix.

A Dirac mass term reads

$$\mathcal{L}_\nu \supset -\bar{\nu}_\mathrm{L} M_\mathrm{D} \nu_\mathrm{R} \tag{3.1}$$

with the left- and right-handed neutrino fields ν_L and ν_R.

For a Dirac mass matrix M_D the relation for an enhanced residual or accidental symmetry H is

$$R^\dagger M_\mathrm{D}^\dagger M_\mathrm{D} R = M_\mathrm{D}^\dagger M_\mathrm{D} \tag{3.2}$$

with unitary matrices $R \in H$. Again H can be maximally $U(3)$ and we set $R = \exp(\mathrm{i}\,\beta_i T_i)$, $i = 1, \ldots, 9$. Now we have to find the solution for

$$\mathrm{i}\,(-\beta_i (T_i)^\dagger M_\mathrm{D}^\dagger M_\mathrm{D} + M_\mathrm{D}^\dagger M_\mathrm{D} \beta_i T_i) = 0. \tag{3.3}$$

In the case of three distinct eigenvalues we see that β_3, β_8 and β_9 are undetermined, which leads to a $U(1)^3$ symmetry corresponding to individual neutrino flavour numbers. This also does not change for a vanishing mass this time.

For two equal masses we get additionally that β_1 and $\beta_2 \neq 0$ are undetermined for $\tilde{m}_1 = \tilde{m}_2$ which corresponds to $U(2) \times U(1)$ rotations.

In the case of a sum rule we find again that the symmetry group is not enhanced for phenomenologically relevant parameter points. This is true also in the vicinity of symmetry points as discussed above. And for definiteness we have also checked here our statement for an arbitrary element of $U(3)$. So again neutrino mass sum rules do not lead to any

particular residual symmetry of the Lagrangian also in the case of Dirac neutrino masses.

4 Summary and conclusions

Neutrino mass sum rules are a powerful way to test more than 60 flavour models. Although the phenomenology of these models has been studied already in great detail [6–9] the exact origin of the neutrino mass sum rules had not been addressed systematically yet.

Since they usually appear in the context of non-Abelian discrete family symmetries it is tempting to think of them in the same framework and try to connect the neutrino mass sum rules to any residual symmetry of the Lagrangian (the neutrino mass terms). We have demonstrated here that this is not the case. From the viewpoint of residual symmetries there is no difference between a mass matrix which fulfills a neutrino mass sum rule and a mass matrix which does not. In that sense mass sum rules present themselves as a model-building artifact found in many flavour models which have no one-to-one mapping to any definite property of the flavour model in the unbroken phase. Despite that they still offer robust predictions for the neutrino mass ordering and scale, for instance, which can be tested in the future.

Starting from the symmetry conditions, Eqs. (2.2) and (3.2), we have provided perturbative arguments and shown an explicit (non-perturbative) example calculation which prove our statement. We have checked that the given statements and calculations indeed extend to all sum rules and we conclude that a neutrino mass sum rule is not related to an enhanced or particular residual symmetry of the Lagrangian as long as all the masses are distinct. This is in complete agreement with the widely used claim that non-Abelian family symmetries cannot determine the neutrino masses. In fact, our proof is equally valid for the case without any mass sum rule (three completely independent masses). To our knowledge, this is the first formal proof of this widely held conviction in the literature.

These considerations clarify and settle an open question in the literature and prove that neutrino mass sum rules are simply related to some minimal breaking of the symmetries in flavour models in the sense that only the minimal set of parameters is introduced in the neutrino mass matrix to allow for three non-vanishing eigenvalues.

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