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A unified error analysis for spatial discretizations of wave-type equations with applications to dynamic boundary conditions

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Abstract

This thesis provides a unified framework for the error analysis of non-conforming space discretizations of linear wave equations in time-domain, which can be cast as symmetric hyperbolic systems or second-order wave equations. Such problems can be written as first-order evolution equations in Hilbert spaces with linear monotone operators. We employ semigroup theory for the well-posedness analysis and to obtain stability estimates for the space discretizations. To compare the finite dimensional approximations with the original solution, we use the concept of a lift from the discrete to the continuous space. Time integration with the Crank–Nicolson method is analyzed.

In this framework, we derive a priori error bounds for the abstract space semi-discretization in terms of interpolation and discretization errors. These error bounds yield previously unkown convergence rates for isoparametric finite element discretizations of wave equations with dynamic boundary conditions in smooth domains. Moreover, our results allow to consider already investigated space discretizations in a unified way. Here it successfully reproduces known error bounds. Among the examples which we dicuss in this thesis are discontinuous Galerkin discretizations of Maxwell's equations and finite elements with mass lumping for the scalar wave equation.

Acknowledgements

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I would like to take this opportunity to express my deepest gratitude to my wife Susanne. There are no words to describe the joy she brought to my life. Her love, patience and support during the final phase of this thesis were invaluable to me. Finally, I thank my family for their unconditional love and support.

David Hipp, 29. Mai 2017

Contents

Introduction

Motivation

The original goal of this thesis was to analyze the numerical discretization of wave equations with dynamic boundary conditions. Such boundary conditions account for the momentum of the wave on the boundary and are given by ordinary differential equations or even evolution equations on the boundary. One of the oldest examples is the acoustic boundary condition. Its formulation in in time domain was introduced by [Beale and Rosencrans, 1974] and is still a subject of current research, cf. [Graber, 2012] and [Vedurmudi et al., 2016]. A different class of dynamic boundary conditions for wave equations are kinetic boundary conditions, which model wave propagation along the boundary, cf. [Vitillaro, 2013], [Graber and Lasiecka, 2014], and [Lescarret and Zuazua, 2015]. The analysis of wave equations with dynamic boundary conditions has recently been developed further to include non-linear problems [Vitillaro, 2015] and control theory [Gal and Tebou, 2017].

However, to our knowledge, the analysis of numerical methods for wave equations with dynamic boundary conditions has not yet been considered. In addition, the numerical approximation of other, non-hyperbolic partial differential equations with dynamic boundary conditions has been investigated over the last years: In [Elliott and Ranner, 2013] an isoparametric finite element method for a coupled bulk-surface partial differential equation of elliptic type is proposed. [Kashiwabara et al., 2015] consider a finite element discretization of the Poisson equation with generalized Robin boundary conditions. Our CRC-project partners [Kovács and Lubich, 2016] analyze finite element discretizations of parabolic problems with dynamic boundary conditions.

In the course of analyzing finite element discretizations of wave equations with dynamic boundary conditions, we found that they can be treated as a special case of a much more general class of non-conforming space discretizations. This was particularly attractive, as such space discretizations appear in different contexts. For instance

- finite element methods on smooth domains,
- discontinuous Galerkin methods,
- mass lumping.

Achievements of the unified error analysis

We now give a short overview of the features of our unified error analysis.

Variational framework with access to semigroup theory

Linear Cauchy problems in a Gelfand triple of Hilbert spaces with quasi-monotone operators cover two important kinds of mathematical models for wave phenomena: second-order evolution equations as in [Showalter, 1994, Ch. VI.] and symmetric hyperbolic systems as in [Benzoni-Gavage and Serre, 2007]. Both references provide well-posedness results in the pivot space by using strongly continuous semigroups and unbounded operators. However, working with the variational formulation in the Gelfand triple fits much better to space discretizations as the finite element method.

Treatment in the thesis The general continuous problem is introduced and analyzed in Sections 2.1 and 2.2. Symmetric hyperbolic systems are discussed in Section 3.1. The analysis of second-order evolution equations is presented in Sections 4.1 and 4.2.

General error bounds for Cauchy problems with quasi-monotone operators

Given some basic stability properties of the space discretization, we prove a priori error bounds for general finite dimensional approximations of Cauchy problems with quasi-monotone operators. This part of our error analysis is kept very general by only assuming the existence of suitable operators which map between the continuous and the discrete space. In the tradition of the Lax-Equivalence principle, we introduce notions of "stability" and "consistency" for general non-conforming space discretizations, which are sufficient for the convergence of lifted approximations from finite dimensional spaces.

Previous state of research The Lax-Equivalence Principle and the Trotter–Kato Theorem are the classical approximation results for evolution equations, cf. [Ito and Kappel, 2002], [Guidetti et al., 2004], and [Banks, 2012, Ch. 12]. However, we believe our a priori error bounds are the first ones that can be used to prove convergence rates for our examples of interest.

Treatment in the thesis General non-conforming space discretizations are introduced in Section 2.3. We then show a priori error bounds in Section 2.5. Our convergence result is stated in Section 2.6.

Combined error analysis of time integration schemes

As a proof of concept, we consider time integration of the semi-discrete problem with the Crank–Nicolson method. Using the ideas from [Sturm, 2017], we derive error bounds for the full discretization. These error bounds yield quadratic convergence in the time step size and preserve the approximation properties of the space discretization. As these error estimates hold for Cauchy problems with monotone operators, they apply to symmetric hyperbolic systems as well as second-order wave equations. We believe that similar results hold for general Runge–Kutta and multistep methods.

Treatment in the thesis Time integration for Cauchy problems with monotone operators are discussed in Section 2.8. In Section 5.4, we apply these general results to second-order wave equations.

A priori error bounds for symmetric hyperbolic systems

We exploit the special structure of symmetric hyperbolic systems to show two different a priori error bounds: The first error bound applies to space discretizations of finite element type. The second error bound is tailored for discontinuous Galerkin methods. Both error bounds are competitive in the sense that they reproduce known convergence results which were derived specifically for the method in question.

Previous state of research We are not aware of similar results in the literature. Related publications consider stabilized finite element discretizations which lie outside our scope, cf. [Burman et al., 2010], or only provide results for specific applications, cf. [Cohen and Pernet, 2016].

Treatment in the thesis The error bound for finite element type methods is shown in Section 3.2.1. We treat discontinuous Galerkin methods in Section 3.2.2. The corresponding examples are discussed in Section 3.3.

A priori error bounds for second-order wave-type problems

The focus of this thesis is on second-order wave equations and suitable non-conforming space discretizations. To apply the results for Cauchy problems with monotone operators, we consider first-order in time formulations. Using the additional structure of the operator matrices, we then derive an a priori error bound in terms of data errors, interpolation errors and discretization errors.

Previous state of research To our knowledge, these results are new. Previous works in this direction considered conforming discretizations [Fujita et al., 2001, Sect. 2.8] or specific applications as, e.g., [Baker and Dougalis, 1976] or [Lubich and Mansour, 2015].

Treatment in the thesis We present and investigate non-conforming space discretizations of abstract second-order wave equations in Chapter 5.

Numerical analysis of wave equations with dynamic boundary conditions

Our considerations for abstract second-order wave-type problems have proven to be particularly fruitful to achieve our original objective: the analysis of finite element discretizations of wave equations with dynamic boundary conditions. First, we can prove well-posedness for wave equations with kinetic and with acoustic boundary conditions using the abstract results. Concerning the space discretization of such problems, we consider a isoparametric finite element method which was proposed in [Elliott and Ranner, 2013]. From our abstract a priori error bounds and the approximation properties of the method, we readily obtain convergence rates for these examples.

Treatment in the thesis The well-posedness analysis for wave equations with dynamic boundary conditions is presented in Chapter 6. In Chapter 7, we derive convergence rates for an isoparametric finite element discretization and time integration with the Crank–Nicolson method for two particular examples.

Previous state of research The convergence results for finite element discretizations of wave equations with dynamic boundary conditions are new. While the well-posedness results we discuss are known, our approach via degenerate coefficients allows us to analyse wave equations with Neumann, Robin, and kinetic boundary conditions (partially) in a unified way.

Conclusion

In summary, our unified error analysis has the following benefits:

Unification It supplies a common ground for the error analysis of symmetric hyperbolic systems and second-order wave equations.

A priori error bounds It provides a priori error bounds in terms of interpolation and discretization errors for both kinds of problems.

Accessibility Its results are formulated in a way that allows to easily infer convergence results from the abstract error bounds and information of the space discretization.

Further merits of our unified error analysis are the combined treatment of full discretizations for symmetric hyperbolic systems and second-order wave equations, and the categorization of convergence proofs. We refer to Section 2.7 for an overview of the examples which we discuss in this thesis.

Notation

In this chapter, we introduce some basic notation and conventions.

Hilbert spaces Let X, Y be two real Hilbert spaces with corresponding norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and let $\mathcal{L}(X, Y)$ be the space of bounded linear operators from X to Y. We endow $\mathcal{L}(X, Y)$ with the operator norm

$$\|\varphi\|_{Y\leftarrow X} \coloneqq \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|\varphi(x)\|_Y}{\|x\|_X} = \sup_{\substack{x\in X\\ \|x\|_X=1}} \|\varphi(x)\|_Y, \qquad \varphi \in \mathcal{L}(X, Y).$$

We define $X = (X, p)$ to be the Hilbert space X equipped with the inner product p. If $|||\cdot|||$ is an equivalent norm on X, we write $\|\cdot\|_X \sim |||\cdot|||$. Moreover, $X \simeq Y$ denotes that X and Y are isomorphic spaces.

Dual spaces If $Y = \mathbb{R}$, then $X^* \coloneqq \mathcal{L}(X, \mathbb{R})$ is the dual space of X and $\|\cdot\|_{X^*} \coloneqq \|\cdot\|_{\mathbb{R}\leftarrow X}$. Moreover,

$$\langle \varphi, y\rangle_X \coloneqq \varphi(x), \qquad \varphi \in X^*, x \in X,$$

denotes the duality pairing between X^* and X. Let $b\colon Y \times X \to \mathbb{R}$ be a continuous bilinear form. Fixing the first argument yields an operator in X^*. We denote the norm of this functional by

$$\|b(y)\|_{X^*} \coloneqq \|b(y, \cdot)\|_{X^*} = \sup_{\|x\|_X=1} |b(y, x)|, \qquad y \in Y. \tag{1}$$

Linear operators Let $A\colon D(A) \to X$ a linear operator defined on the linear subspace $D(A)$ of X. Then we denote by $[D(A)]$ the space $D(A)$ equipped with the graph norm of A (which is a Banach space if A is closed).

Cartesian product of Hilbert spaces Let $u, v \in X$. Then we write

$$\vec{u} = \begin{bmatrix}u, v\end{bmatrix}^\mathsf{T} \coloneqq \begin{bmatrix}u\\ v\end{bmatrix} \in X^2 \coloneqq X \times X.$$

For operators $A_1\colon X \to Y_1$ and $A_2\colon X \to Y_2$, we define $(A_1, A_2) : X^2 \to Y_1 \times Y_2$ by

$$(A_1, A_2)\begin{bmatrix}u\\ v\end{bmatrix} \coloneqq \begin{bmatrix}A_1 u\\ A_2 v\end{bmatrix}, \qquad u, v \in X.$$

Conventions for partial differential equations We consider evolution equations on the compact time interval $[0, T]$ for some $T > 0$. Our examples are partial differential equations on an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. We assume that it has a Lipschitz boundary $\Gamma = \partial\Omega$ and denote its outer unit normal by $n\colon \Gamma \to \mathbb{R}^d$. For the normal derivative of a function $f\colon \Omega \to \mathbb{R}$, we write $\partial_n f \coloneqq n \cdot \nabla f$. Surface integrals as

$$\int_\Gamma f \,\mathrm{d}s$$

are defined w.r.t. to the surface measure $\mathrm{d}s$.

Operations on vectors We write $\mathrm{x}\cdot\mathrm{y}$ for the scalar product of $\mathrm{x} \in \mathbb{R}^d$ with $\mathrm{y} \in \mathbb{R}^d$ and $|x| \coloneqq \sqrt{\mathrm{x}\cdot\mathrm{x}}$ denotes the Euclidean norm. For matrices $A \in \mathbb{R}^{d\times d}$, we use $|A|$ for the operator norm induced by the Euclidean norms in $\mathbb{R}^d$.

Mesh based discretizations We use $\mathcal{P}_k$ for the space of polynomials of maximal order k. If not specified differently, we consider space discretizations based on an admissible mesh sequence $\mathcal{T}_{\mathcal{H}} = \{\mathcal{T}_h \mid h \in \mathcal{H}\}$ of a polygonal domain Ω where h in $\mathcal{T}_h$ denotes the maximal diameter of all the elements $K \in \mathcal{T}_h$ and $\Omega = \cup_{K\in\mathcal{T}_h} K$. An admissible mesh sequence is shape-regular, contact-regular and satisfies an optimal polynomial approximation property, cf. [Di Pietro and Ern, 2012, Def. 1.57]. We assume that $\mathcal{T}_h$ consists of triangles or tetrahedra for $d = 2$ or $d = 3$, respectively.

Lebesgue spaces We denote by $L^p(U)$, $p \in [1,\infty]$ the space of measurable real-valued functions defined on the measurable open set $U \subset \mathbb{R}^d$ with

$$\|f\|_{L^p(U)} \coloneqq \Big(\int_U |f|^p \,\mathrm{dx}\Big)^{1/p} < \infty, \qquad p \in [1,\infty)$$

and $\|f\|_{L^\infty(U)} \coloneqq \operatorname{ess\,sup}_{x\in U}|f(\mathrm{x})|$. For vector-valued functions $\vec{v}\colon U \to \mathbb{R}^d$, $d \in \mathbb{N}$ we define

$$\|\vec{v}\|_{L^p(U)} \coloneqq \big\||\vec{v}|\big\|_{L^p(U)}, \qquad p \in [1,\infty].$$

Analogously, we write $\|A\|_{L^p(U)} \coloneqq \||A|\|_{L^p(U)}$ for matrix-valued functions $A\colon U \to \mathbb{R}^{d\times d}$. By $L^p((0,T);X)$ we denote the space of X-valued functions $f\colon (0,T) \to X$ with

$$\|f\|_{L^p(0,T;X)} \coloneqq \big\|\|f(t)\|_X\big\|_{L^p(0,T)} < \infty, \qquad p \in [1,\infty].$$

Since almost all Hilbert space-valued functions in this thesis are defined on the time interval $[0,T]$, we abbreviate $L^\infty(X) \coloneqq L^\infty((0,T);X)$ and use the short notation

$$\|f\|_{L^\infty(X)} \coloneqq \|f\|_{L^\infty((0,T);X)}$$

for the corresponding norm.

Space of bounded functions Let $U \subset \mathbb{R}^d$ be a non-empty set. We define the space of bounded functions by

$$B\big(U;X\big) \coloneqq \Big\{f\colon U \to X \mid \|f\|_{\infty,U\to X} \coloneqq \sup_{\mathrm{x}\in U}\|f(\mathrm{x})\|_X\Big\}.$$

Again, we use the short notation $B\big(X\big) \coloneqq B\big([0,T];X\big)$ with $\|f\|_{\infty,X} \coloneqq \|f\|_{\infty,[0,T]\to X}$.

Sobolev spaces Let $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$ be a multindex and define $\partial_\alpha f \coloneqq \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d} f$. For $k \in \mathbb{N}_0$, the Sobolev space of order k is given by

$$H^k(U) \coloneqq \big\{f\colon U \to \mathbb{R} \mid \partial_\alpha f \in L^2(U),\ |\alpha| \le k\big\}, \qquad |\alpha| \coloneqq \sum_{i=1}^d \alpha_i,$$

where the derivatives are understood in a weak sense. Note that $H^k(U)$ is a Hilbert space w.r.t. to the inner product

$$\big(f|g\big)_{H^k(U)} \coloneqq \sum_{|\alpha|\le k}\int_U \partial_\alpha f \partial_\alpha g \,\mathrm{dx}.$$

Hence the canonical norm on $H^k(U)$ is $\|f\|^2_{H^k(U)} \coloneqq \big(f|f\big)_{H^k(U)}$. Moreover, we introduce the seminorm

$$|f|^2_{H^k(U)} \coloneqq \sum_{|\alpha|=k}\|\partial_\alpha f\|^2_{L^2(U)}.$$

Divergence and curl operators Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open domain. We use standard weak definitions for the curl and div operators and the corresponding Sobolev spaces

$$H(\operatorname{div}, \Omega) \coloneqq \{v \in L^2(\Omega)^d \mid \operatorname{div} v \in L^2(\Omega)\},$$
$$H(\operatorname{curl}, \Omega) \coloneqq \{v \in L^2(\Omega)^d \mid \operatorname{curl} v \in L^2(\Omega)^d\}.$$

For more details about these spaces we refer to [Monk, 2003] and [Cohen and Pernet, 2016].

Analytical tools for boundary conditions

Space of continuous functions on closed sets Let $U \subset \mathbb{R}^N$, $N \geq 1$, be an open domain. Then we define

$$C^k(\overline{U}) \coloneqq \left\{ v|_U \mid v \in C_c^k(\mathbb{R}^N) \right\}, \qquad k \in \mathbb{N} \cup \{\infty\}.$$

Sobolev spaces on boundaries Let $k \geq 0$ be an integer and K be the cylinder

$$K = \{\xi = [\xi', \xi_d]^\intercal \in \mathbb{R}^d \mid \xi' \in \mathbb{R}^{d-1},\ |\xi'| < 1,\ \xi_d \in \mathbb{R}\}.$$

If Γ is a C^k (resp. a Lipschitz) boundary, then at each $\mathrm{x} \in \Gamma$ there exists a neighborhood U and a map $\phi\colon U \to K$ s.t. ϕ is a C^k (resp. Lipschitz continuous) diffeomorphism and

$$\phi(\Omega \cap U) = K \cap \{\xi_d > 0\},$$
$$\phi(\Gamma \cap U) = K \cap \{\xi_d = 0\}.$$

Since Ω is bounded, the boundary Γ can be covered by finitely many of these neighborhoods U_r, $r = 1, \ldots, r_0$ with corresponding maps $\phi_r\colon U_r \to K$. For $0 \leq s \leq k$ (resp. $0 \leq s \leq 1$), we define the Sobolev space

$$H^s(\Gamma) \coloneqq \left\{ f \in L^2(\Gamma) \mid f \circ \phi_r^{-1} \in H^s\big(K \cap \{\xi_d = 0\}\big),\ r = 1, \ldots, r_0 \right\}$$

and denote its dual by

$$H^{-s}(\Gamma) \coloneqq \big(H^s(\Gamma)\big)^*.$$

For more details we refer to [Grisvard, 2011].

Dirichlet trace We denote the Dirichlet trace operator by $\gamma\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ and note that $\gamma(u) = u\big|_\Gamma$ for $u \in C(\overline{\Omega})$, cf. [Han and Atkinson, 2009, Thm. 7.3.11]. Moreover, we define

$$H_0^1(\Omega) \coloneqq \{f \in H^1(\Omega) \mid \gamma(f) = 0\}.$$

Neumann trace We define the Neumann trace operator by $\gamma_{\partial n}(u) \coloneqq n \cdot \nabla u\big|_\Gamma = \partial_n u$ for $u \in C^1(\overline{\Omega})$ and remark that one can extend $\gamma_{\partial n}$ continuously to all $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, cf. [Tucsnak and Weiss, 2009, p. 107].

Dirichlet-type trace To cope with varying wave speeds, we introduce $\gamma_n(v) \coloneqq n \cdot v|_\Gamma$ for vector-valued functions $v \in C^1(\overline{\Omega})^d$. A weak definition of the Dirichlet-type trace operator leads to $\gamma_n \in \mathcal{L}(H(\operatorname{div}, \Omega), H^{-1/2}(\Gamma))$ which satisfies

$$\int_\Omega v \cdot \nabla\varphi + \operatorname{div}(v)\varphi \,\mathrm{dx} = \langle \gamma_n(v),\ \gamma(\varphi) \rangle_{H^{1/2}(\Gamma)} \tag{2}$$

for all $v \in H(\operatorname{div})$ and $\varphi \in H^1(\Omega)$, cf. [Schnaubelt and Weiss, 2010, Sect. 5]. For $H^1(\Omega)^d$-functions v, it satisfies $\gamma_n(v) = n \cdot \gamma(v)$ and belongs to $\gamma_n \in \mathcal{L}(H^1(\Omega)^d, H^{1/2}(\Gamma))$.

Surface differential operators The surface gradient of $u \in C^1(\overline{\Omega})$ is defined by

$$\nabla_\Gamma u \coloneqq (\partial_{i,\Gamma} u)_{i=1}^d \coloneqq (\mathsf{I} - nn^\mathsf{T})\nabla u$$

and the surface divergence of $v = (v_i)_{i=1}^d \in C^1(\overline{\Omega})^d$ by

$$\operatorname{div}_\Gamma v \coloneqq \sum_{i=1}^d \partial_{i,\Gamma} v_i.$$

Both operators admit a definition in terms of surface functions, i.e., $\nabla_\Gamma u = \nabla_\Gamma u|_\Gamma$ and $\operatorname{div}_\Gamma v = \operatorname{div}_\Gamma v|_\Gamma$. For details we refer to [Gilbarg and Trudinger, 2001]. Definitions in terms of local coordinates can be found in [Disser et al., 2015] and [Kashiwabara et al., 2015].

Integration by parts on surfaces Gauss' Theorem on the smooth surface $\Gamma \in C^2$ yields for sufficiently smooth functions $v\colon \Gamma \to \mathbb{R}^d$ and $\varphi\colon \Gamma \to \mathbb{R}$

$$-\int_\Gamma \operatorname{div}_\Gamma(v)\varphi \,\mathrm{d}s = \int_\Gamma v \cdot \nabla_\Gamma \varphi - \operatorname{div}_\Gamma(n)(v \cdot n)\varphi \,\mathrm{d}s, \tag{3}$$

cf. [Kashiwabara et al., 2015, (3.1)]. Inserting $v = \nabla_\Gamma u$ and using $n \cdot \nabla_\Gamma u = 0$, it follows that

$$-\int_\Gamma \operatorname{div}_\Gamma(\nabla_\Gamma v)\,\varphi \,\mathrm{d}s = \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \varphi \,\mathrm{d}s. \tag{4}$$

Bulk-surface Sobolev spaces For the analysis of wave equations with dynamic boundary conditions, we introduce the spaces

$$\begin{aligned}
\mathbb{H}^0 &\coloneqq L^2(\Omega) \times L^2(\Gamma) \\
\mathbb{H}^k &\coloneqq H^k(\Omega) \times H^k(\Gamma), \qquad k \in \mathbb{N} \\
\mathbb{H}^{-1} &\coloneqq \big(H^1(\Omega)\big)^* \times H^{-1}(\Gamma), \\
H^k(\Omega;\Gamma) &\coloneqq \Big\{ v \in H^k(\Omega) \mid \gamma(v) \in H^k(\Gamma) \Big\}, \qquad k \geq 1.
\end{aligned}$$

where we equip $\mathbb{H}^k$, $k \geq -1$ with its canonical inner product and $H^k(\Omega;\Gamma)$ with the inner product associated to

$$\|v\|_{H^k(\Omega;\Gamma)}^2 \coloneqq \|v\|_{H^k(\Omega)}^2 + \|\,\gamma(v)\|_{H^k(\Gamma)}^2.$$

[Kashiwabara et al., 2015, Lem. 2.5] shows that $H^k(\Omega;\Gamma)$ is a Hilbert space w.r.t. this norm.

Conventions In the following, all derivatives are understood in the sense of distributions. Furthermore, evaluation of functions on Γ and normal derivatives are defined via trace operators, even if they do no appear explicitly.

Chapter 1

Non-trivial boundary conditions for wave equations

Outline In this chapter, we show how boundary conditions emerge as an intrinsic part of models for different types of wave phenomena in bounded domains. First, we show how to use the principle of stationary action for the derivation of wave equations in Section 1.1. Classical options for boundary conditions for wave equations are then presented in Section 1.1.3. After that, in Section 1.2, we characterize the term "dynamic boundary condition". Last, we introduce the two main examples which motivated this work in Sections 1.2.1 and 1.2.2.

For better readability, we keep the following exposition on an informal level and assume that domains and functions are such that all derivatives and integrals exist.

1.1 Derivation of wave equations

In this thesis, we consider physical systems of wave phenomena in open and bounded domains $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma := \partial\Omega$. The scalar wave equation with homogeneous Neumann boundary conditions is a prototype for the equations of motion of such a wave phenomenon. It concerns a function $u\colon [0,T] \times \Omega \to \mathbb{R}$ s.t.

$$u_{tt}(t,\mathrm{x}) - c_\Omega \Delta u(t,\mathrm{x}) = 0, \qquad (t,\mathrm{x}) \in (0,T) \times \Omega, \tag{1.1a}$$

$$\partial_n u(t,\mathrm{x}) = 0, \qquad (t,\mathrm{x}) \in (0,T) \times \Gamma, \tag{1.1b}$$

$$u(0,\mathrm{x}) = u_0(\mathrm{x}), \quad u_t(0,\mathrm{x}) = v_0(\mathrm{x}), \qquad \mathrm{x} \in \Omega, \tag{1.1c}$$

where $T > 0$, $c_\Omega > 0$ is the wave speed, and $\partial_n u$ denotes the normal derivative of u on Γ. The three components of the mathematical problem (1.1) are referred to as

- the partial differential equation (1.1a), which describes the wave propagation in Ω
- the boundary condition (1.1b), which characterizes the behavior of u on the boundary Γ, and
- the initial values (1.1c), which specify the initial state and velocity at time $t = 0$.

Equations of motion describe the dynamics of physical systems in terms of mathematical functions, as (1.1a) and (1.1b). One particular approach to derive equations of motion for a wide range of applications is the principle of stationary action (sometimes also called principle of minimal action or Hamilton's principle).

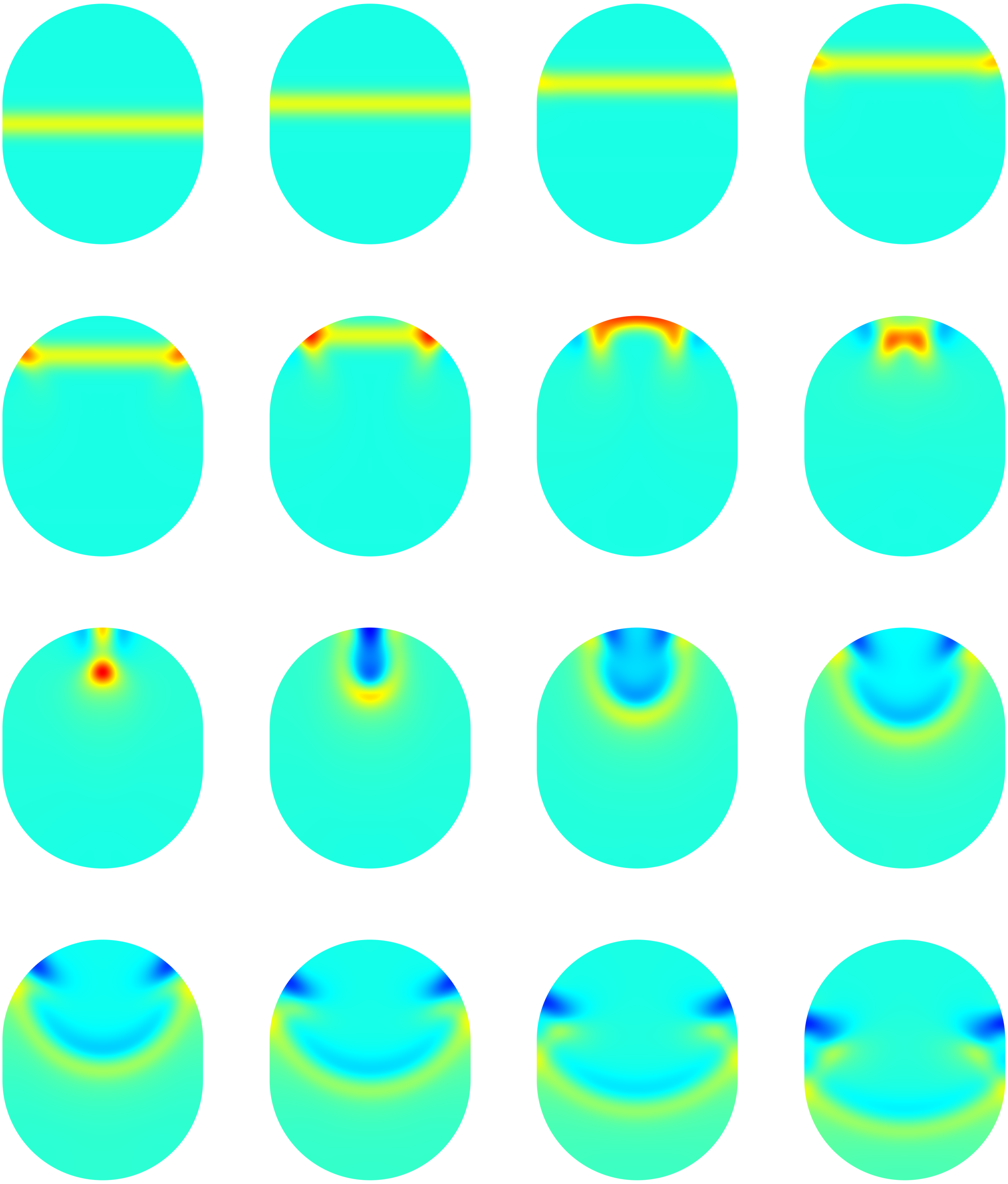


Figure 1.1: An example for a solution of the wave equation with homogeneous Neumann boundary conditions with $c_\Omega = 1$. The snapshots show the solution u at times $t = 0.2 \cdot k$, $k = 0, \ldots, 15$.

1.1.1 The principle of stationary action

The principle of stationary action states that:

> The transition of a physical system from an initial state at $t = 0$ to a final state at $t = T$ is the one for which the associated action is stationary to first order.

A mathematical description Assume that $x(t)$ describes the state of a physical system at time t. Then the trajectory $t \mapsto x(t)$ (or just x) describes the transition of the system from $t = 0$ to $t = T$.

Further assume that the action of the transition x of the physical system is given by the real valued function

$$\mathbf{S}(x).$$

Now let y be a perturbation of x such that for all sufficiently small $\varepsilon > 0$

$$t \mapsto x(t) + \varepsilon y(t)$$

is still an admissible trajectory of the physical system from $x(0)$ to $x(T)$. In particular, this implies $y(0) = y(T) = 0$. According to the principle of stationary action, the physical trajectory from $x(0)$ to $x(T)$ satisfies

$$\mathbf{S}(x + \varepsilon y) - \mathbf{S}(x) \overset{!}{=} o(\varepsilon) \tag{1.2}$$

for $\varepsilon > 0$ and all admissible perturbations y. Thus all terms in $\mathcal{O}(\varepsilon)$ vanish.

We now apply the principle of stationary action to the wave phenomenon "vibrating membrane" and show how (1.2) leads to the corresponding equations of motion.

1.1.2 The vibrating membrane

We are interested in the transverse motion of a vibrating membrane represented by $\Omega \subset \mathbb{R}^2$ with C^1-boundary Γ. Our approach on this problem is inspired by the calculations and results of [Goldstein, 2006].

The model Let $u(t, \mathrm{x})$ be the vertical displacement of the membrane at point x and time t. Then the action of the physical system "membrane" can be modeled by

$$\mathbf{S}(u) := \int_0^T \mathbf{K}(u_t(t)) - \mathbf{V}(u(t))\,\mathrm{d}t, \tag{1.3}$$

where the kinetic energy $\mathbf{K}$ and the potential energy $\mathbf{V}$ are given by

$$\mathbf{K}(w) := \frac{1}{2}\int_\Omega w^2\,\mathrm{dx} + \frac{1}{2}\int_\Gamma \mu w^2\,\mathrm{d}s, \tag{1.4a}$$

$$\mathbf{V}(w) := \frac{1}{2}\int_\Omega c_\Omega |\nabla w|^2\,\mathrm{dx} + \frac{1}{2}\int_\Gamma a_\Gamma w^2\,\mathrm{d}s \tag{1.4b}$$

with constants $\mu, a_\Gamma \geq 0$ and $c_\Omega > 0$.

Interpretation Physically, $\mathbf{K}$ describes the mass distribution of the membrane in Ω and on Γ, respectively. The first term in $\mathbf{V}$ reflects the amount of work needed to deform the membrane and the wave speed c_Ω depends on the material properties of the membrane. The second term in $\mathbf{V}$ characterizes the work needed to displace the membrane on the boundary from equilibrium position $u = 0$ on Γ. Such an action functional models a situation where the membrane is attached to infinitesimally small springs on Γ which oscillate vertically with spring constant a_Γ.

Equations of motion To derive the equations of motion from the principle of stationary action, we use the calculus of variations as in [Goldstein, 2006]. Let $w\colon [0, T] \times \Omega \to \mathbb{R}$ with $w(0) = w(T) = 0$ be sufficiently smooth and consider

$$\begin{aligned}
&\mathbf{S}(u + \varepsilon w) - \mathbf{S}(u) \\
&\quad = \varepsilon\left(\int_0^T \int_\Omega u_t w_t - c_\Omega \nabla u \cdot \nabla w \,\mathrm{dx}\,\mathrm{d}t + \int_0^T \int_\Gamma \mu u_t w_t - a_\Gamma u w \,\mathrm{d}s\,\mathrm{d}t\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Using integration by parts in space, i.e. Gauss' Theorem, we obtain

$$\begin{aligned}
&\mathbf{S}(u+\varepsilon w)-\mathbf{S}(u)\\
&= \varepsilon\Bigg(\int_0^T\int_\Omega u_t w_t + c_\Omega \Delta u w \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Gamma c_\Omega \partial_n u w \,\mathrm{d}s\,\mathrm{d}t\\
&\quad + \int_0^T\int_\Gamma \mu u_t w_t - a_\Gamma u w \,\mathrm{d}s\,\mathrm{d}t\Bigg) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Now we exchange the order of integration by Fubini's theorem and integrate by parts in time:

$$\begin{aligned}
&\mathbf{S}(u+\varepsilon w)-\mathbf{S}(u)\\
&= \varepsilon\Bigg(\int_\Omega\int_0^T \Big(-u_{tt}+c_\Omega\Delta u\Big) w\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega \Big[u_t w\Big]_{t=0}^T \mathrm{d}x\\
&\quad + \int_\Gamma\int_0^T \Big(-\mu u_{tt} - a_\Gamma u - c_\Omega \partial_n u\Big) w\,\mathrm{d}t\,\mathrm{d}s + \int_\Gamma \Big[u_t w\Big]_{t=0}^T \mathrm{d}s\Bigg) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Since $w(0)=w(T)=0$ by assumption, the second and fourth summands vanish, which yields

$$\begin{aligned}
&\mathbf{S}(u+\varepsilon w)-\mathbf{S}(u)\\
&= \varepsilon\Bigg(\int_0^T\int_\Omega \Big(-u_{tt}+c_\Omega\Delta u\Big) w\,\mathrm{d}x + \int_\Gamma \Big(-\mu u_{tt} - a_\Gamma u - c_\Omega\partial_n u\Big) w\,\mathrm{d}s\Bigg)\mathrm{d}t + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

According to (1.2), all terms in $\mathcal{O}(\varepsilon)$ have to vanish for all admissible perturbations w. This implies that the equations of motion for the vibrating membrane are given by

$$u_{tt}(t,\mathrm{x}) - c_\Omega \Delta u(t,\mathrm{x}) = 0, \qquad (t,\mathrm{x}) \in (0,T)\times\Omega, \tag{1.5a}$$
$$\mu u_{tt}(t,\mathrm{x}) + a_\Gamma u(t,\mathrm{x}) + c_\Omega \partial_n u(t,\mathrm{x}) = 0, \qquad (t,\mathrm{x}) \in (0,T)\times\Gamma. \tag{1.5b}$$

We emphasize that boundary conditions arise as an intrinsic part of the equations of motion which are determined by the physical system, respectively, its action.

1.1.3 Dirichlet boundary conditions and source terms

Now we assume that the membrane is partially clamped to the boundary. Therefore, we impose Dirichlet boundary conditions $u(t) = f_\mathsf{D}$ on Γ_D, where $f_\mathsf{D}\colon \Gamma_\mathsf{D} \to \mathbb{R}$ and Γ_D is a connected subset of Γ. We denote the other part of the boundary by $\Gamma_\mathsf{N} \coloneqq \Gamma\setminus\Gamma_\mathsf{D}$ and consider a time dependent version of the action functional (1.3) that incorporates external forces via $f_\Omega\colon [0,T]\times\Omega\to\mathbb{R}$ and $f_\Gamma\colon [0,T]\times\Gamma_\mathsf{N}\to\mathbb{R}$, and consists of

$$\begin{aligned}
\mathbf{K}(w) &\coloneqq \int_\Omega w^2\,\mathrm{d}x + \int_{\Gamma_\mathsf{N}} \mu w^2\,\mathrm{d}s,\\
\mathbf{V}(w,t) &\coloneqq \int_\Omega c_\Omega |\nabla w|^2\,\mathrm{d}x + \int_{\Gamma_\mathsf{N}} a_\Gamma w^2\,\mathrm{d}s - \int_\Omega w f_\Omega(t)\,\mathrm{d}x - \int_{\Gamma_\mathsf{N}} w f_\Gamma(t)\,\mathrm{d}s.
\end{aligned}$$

Then the principle of stationary action leads to inhomogeneous equations of motion

$$u_{tt}(t,\mathrm{x}) - c_\Omega\Delta u(t,\mathrm{x}) = f_\Omega(t,\mathrm{x}), \qquad (t,\mathrm{x})\in(0,T)\times\Omega, \tag{1.6a}$$
$$\mu u_{tt}(t,\mathrm{x}) + a_\Gamma u(t,\mathrm{x}) + c_\Omega\partial_n u(t,\mathrm{x}) = f_\Gamma(t,\mathrm{x}), \qquad (t,\mathrm{x})\in(0,T)\times\Gamma_\mathsf{N}, \tag{1.6b}$$
$$u(t,\mathrm{x}) = f_\mathsf{D}(\mathrm{x}), \qquad (t,\mathrm{x})\in(0,T)\times\Gamma_\mathsf{D}, \tag{1.6c}$$

as can be seen from [Goldstein, 2006, eq. (5.12)]. Note again that the boundary conditions are an integral part of the equations of motion.

Naming the different boundary conditions Clearly, (1.6c) is a Dirichlet boundary condition, but (1.6b) has different names depending on the choice of the parameters.

- If $\mu = a_\Gamma = 0$, then (1.6b) is called Neumann boundary condition:

$$c_\Omega \partial_n u(t, \mathrm{x}) = f_\Gamma(t, \mathrm{x}), \qquad (t, \mathrm{x}) \in (0, T) \times \Gamma_\mathsf{N}.$$

- If $\mu = 0$ and $a_\Gamma > 0$, then (1.6b) is called Robin boundary condition:

$$a_\Gamma u(t, \mathrm{x}) + c_\Omega \partial_n u(t, \mathrm{x}) = f_\Gamma(t, \mathrm{x}), \qquad (t, \mathrm{x}) \in (0, T) \times \Gamma_\mathsf{N}.$$

- If $\mu, a_\Gamma > 0$, then (1.6b) is called kinetic boundary condition:

$$\mu u_{tt}(t, \mathrm{x}) + a_\Gamma u(t, \mathrm{x}) + c_\Omega \partial_n u(t, \mathrm{x}) = f_\Gamma(t, \mathrm{x}), \qquad (t, \mathrm{x}) \in (0, T) \times \Gamma_\mathsf{N}.$$

- If $\Gamma_\mathsf{D} \neq \emptyset$ and $\Gamma_\mathsf{N} \neq \emptyset$, then (1.6b) and (1.6c) are called mixed boundary conditions.

Note that kinetic boundary conditions are equivalent to so-called Wentzell boundary conditions for sufficiently smooth data, cf. [Mugnolo and Romanelli, 2006].

1.2 Dynamic boundary conditions

Neumann, Robin, or Dirichlet boundary conditions neglect the momentum of the wave on the boundary. Dynamic boundary conditions are means to account for this momentum.

Definition. *We call a boundary condition for a wave equation dynamic if it arises from an action functional with a kinetic energy $\mathbf{K}(w)$ that depends on the values of w on Γ.*

For $\mu > 0$, the kinetic boundary condition (1.5b) is an example of a dynamic boundary condition, since the kinetic energy (1.4a) depends on the values of w on Γ. In all examples we know of, dynamic boundary conditions are differential equations on the boundary. Or, in the terminology of [Elliott and Ranner, 2013], incorporating kinetic effects on the boundary leads to an evolution equation in the "bulk" Ω which is coupled to a differential equation on the "surface" Γ.

We now present the two main examples of dynamic boundary conditions which we consider in this thesis.

1.2.1 The wave equation with acoustic boundary conditions

A famous dynamic boundary condition for the wave equation is the acoustic boundary condition. It was introduced in the first edition of [Morse and Ingard, 1987] from 1968. The transient formulation was given in [Beale and Rosencrans, 1974] and the first well-posedness and spectral analysis of the wave equation with acoustic boundary conditions in dimension $d = 3$ was provided in the original paper [Beale, 1976]. It was recently reconsidered and generalized in different directions by [Gal et al., 2003], [Mugnolo, 2006a], [Frota et al., 2011], [Graber, 2012], [Vedurmudi et al., 2016] and others.

Original problem Let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain. The wave equation with acoustic boundary conditions describes the dynamics of the functions $u\colon [0,T]\times\Omega \to \mathbb{R}$ and $\delta\colon [0,T]\times\Gamma \to \mathbb{R}$ by

$$u_{tt} - c_\Omega \Delta u = 0 \qquad \text{in } \Omega, \tag{1.7a}$$

$$m_\Gamma \delta_{tt} + \alpha_\Gamma \delta_t + k_\Gamma \delta + c_\Omega u_t = 0 \qquad \text{on } \Gamma, \tag{1.7b}$$

$$\delta_t = \partial_n u \qquad \text{on } \Gamma, \tag{1.7c}$$

where $m_\Gamma, k_\Gamma, c_\Omega > 0$ and $\alpha_\Gamma \geq 0$ are constants.

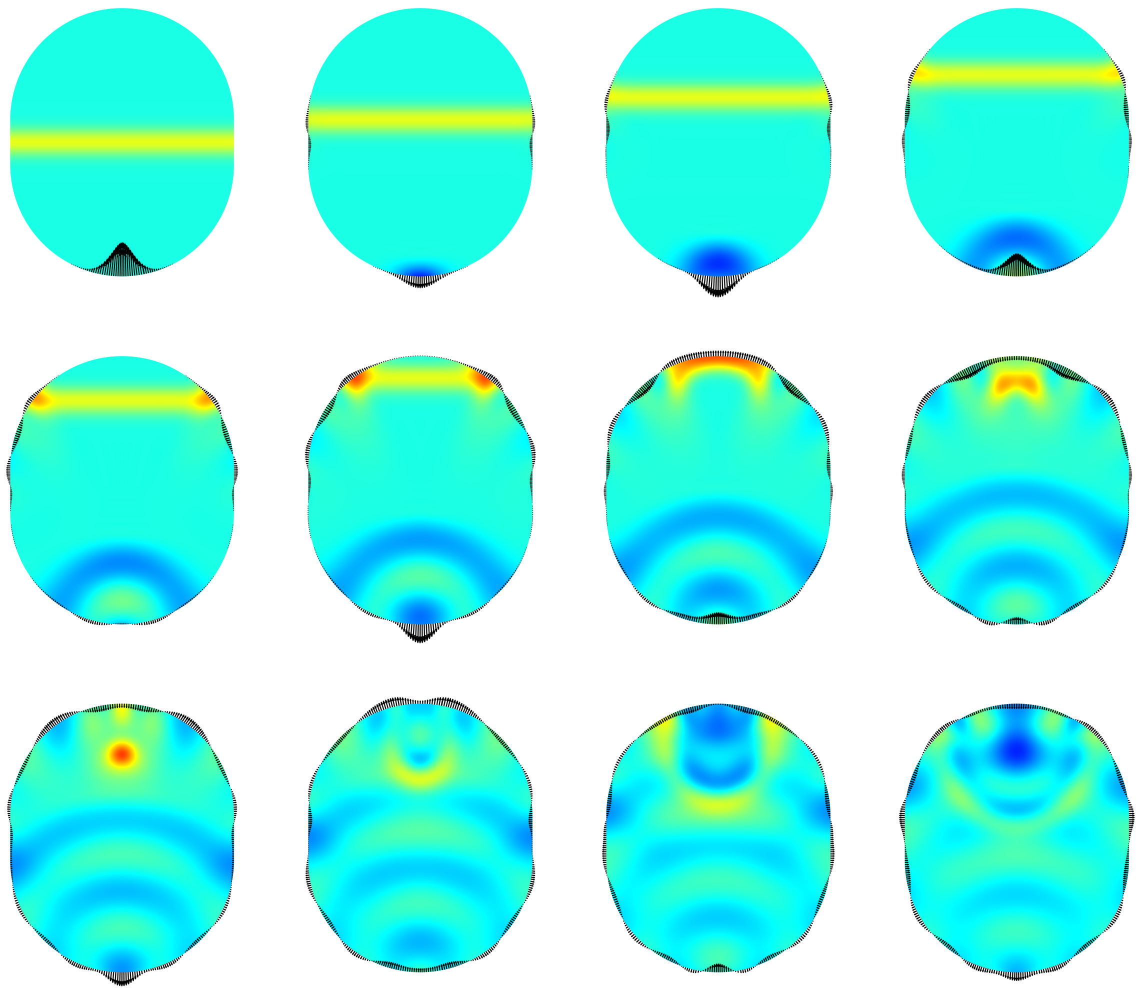


Figure 1.2: An example for a solution of the wave equation with acoustic boundary conditions with $c_\Omega = m_\Gamma = 1$, $k_\Gamma = 32\pi$ and $\alpha_\Gamma = 0$. The snapshots show the solution u at times $t = 0.2 \cdot k$, $k = 0, \ldots, 11$. The black arrows on the boundary visualize the function δ.

Physical system The evolution equation (1.7a) models the propagation of sound waves in a fluid at rest filling $\Omega \subset \mathbb{R}^3$. Equally important, (1.7b) describes small oscillations of the walls on Γ around the fluid in normal direction. In such a situation, the bulk function u corresponds to the acoustic potential and the surface function δ accounts for the infinitesmally small displacement of the wall. Further, u is coupled to δ via (1.7c). Thus $c_\Omega^{1/2}$ is the speed of sound and each point of the wall Γ has mass $m_\Gamma > 0$ and is attached to a resistive harmonic oscillator with stiffness $k_\Gamma > 0$. The oscillators react to the excess pressure of the sound wave independently of each other and $\alpha_\Gamma \geq 0$ describes the damping of the oscillations.

Derivation via the principle of stationary action We now show that, for $m_\Gamma = c_\Omega = 1$ and $\alpha_\Gamma = 0$, the action functional associated to the physical system is

$$\mathbf{S}(u, \delta) := \int_0^T \mathbf{K}\left(u_t(t), \delta_t(t)\right) - \mathbf{V}\left(u(t), \delta(t), u_t(t)\right) \mathrm{d}t, \tag{1.8}$$

where

$$\begin{aligned}\mathbf{K}\left(w,\eta\right) &:= \frac{1}{2}\int_\Omega w^2\,\mathrm{dx} + \frac{1}{2}\int_\Gamma \eta^2\,\mathrm{d}s,\\ \mathbf{V}\left(w,v,\eta\right) &:= \frac{1}{2}\int_\Omega |\nabla w|^2\,\mathrm{dx} + \frac{1}{2}\int_\Gamma k_\Gamma \eta^2\,\mathrm{d}s + \int_\Gamma v\eta\,\mathrm{d}s.\end{aligned}$$

As the kinetic energy $\mathbf{K}\left(u_t(t),\delta_t(t)\right)$ depends on δ_t on the boundary, acoustic boundary conditions are dynamic boundary conditions by definition. We now employ the principle of stationary action and the calculus of variations to find equations of motion for u and δ: Let $w\colon [0,T]\times\Omega\to\mathbb{R}$ and $\eta\colon [0,T]\times\Gamma\to\mathbb{R}$ with $w(0)=w(T)=0$ and $\eta(0)=\eta(T)=0$ be variations of u and δ. We expand the quadratic terms in

$$\begin{aligned}\mathbf{S}\left(u+\varepsilon w,\delta+\varepsilon\eta\right) =& \frac{1}{2}\int_0^T\int_\Omega (u+\varepsilon w)_t^2 - |\nabla(u+\varepsilon w)|^2\,\mathrm{dx}\,\mathrm{d}t\\ &+\frac{1}{2}\int_0^T\int_\Gamma (\delta+\varepsilon\eta)_t^2 - k_\Gamma(\delta+\varepsilon\eta)^2 - 2(u+\varepsilon w)_t(\delta+\varepsilon\eta)\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

and compute

$$\begin{aligned}\mathbf{S}\left(u+\varepsilon w,\delta+\varepsilon\eta\right) - \mathbf{S}\left(u,\delta\right) = \;& \varepsilon\Bigg(\int_0^T\int_\Omega u_t w_t - \nabla u\cdot\nabla w\,\mathrm{dx}\,\mathrm{d}t\\ &+\int_0^T\int_\Gamma \delta_t\eta_t - k_\Gamma\delta\eta - u_t\eta - \delta w_t\,\mathrm{d}s\,\mathrm{d}t\Bigg) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Integrating by parts in space, exchanging the order of integration and then integrating by parts in time yields

$$\begin{aligned}&\mathbf{S}\left(u+\varepsilon w,\delta+\varepsilon\eta\right) - \mathbf{S}\left(u,\delta\right)\\ &\quad= \varepsilon\Bigg(\int_0^T\left(\int_\Omega u_t w_t + \Delta u w\,\mathrm{dx} - \int_\Gamma \partial_n u w\,\mathrm{d}s\right)\mathrm{d}t\\ &\qquad+\int_0^T\int_\Gamma \delta_t\eta_t - k_\Gamma\delta\eta - u_t\eta - \delta w_t\,\mathrm{d}s\,\mathrm{d}t\Bigg) + \mathcal{O}(\varepsilon^2)\\ &\quad= \varepsilon\Bigg(\int_\Omega\left(\int_0^T \big(-u_{tt}+\Delta u\big) w\,\mathrm{d}t\right) + \big[u_t w\big]_{t=0}^T\,\mathrm{dx} - \int_0^T\int_\Gamma \partial_n u w\,\mathrm{d}s\,\mathrm{d}t\\ &\qquad+\int_\Gamma\left(\int_0^T\big(-\delta_{tt} - k_\Gamma\delta - u_t\big)\eta\,\mathrm{d}t\right) + \big[\delta_t\eta\big]_{t=0}^T\,\mathrm{d}s\\ &\qquad+\int_\Gamma\left(\int_0^T \delta_t w\,\mathrm{d}t\right) - \big[\delta w\big]_{t=0}^T\,\mathrm{d}s\Bigg) + \mathcal{O}(\varepsilon^2)\\ &\quad= \varepsilon\Bigg(\int_0^T\int_\Omega\big(-u_{tt}+\Delta u\big)w\,\mathrm{dx}\,\mathrm{d}t + \int_0^T\int_\Gamma\big(\delta_t - \partial_n u\big)w\,\mathrm{d}s\,\mathrm{d}t\\ &\qquad+\int_0^T\int_\Gamma\big(-\delta_{tt} - k_\Gamma\delta - u_t\big)\eta\,\mathrm{d}s\,\mathrm{d}t\Bigg) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

The principle of stationary action implies that all terms in $\mathcal{O}(\varepsilon)$ vanish for arbitrary perturbations w and η. Therefore (1.8) leads to the equations of motion (1.7) with our choice of parameters.

Comment on related models The physical situation behind this model is referred to as an "acoustic-elastic coupling" or "fluid-structure interaction". For a full model, one would consider the walls as elastic bodies which are coupled to the fluid in the bulk. Acoustic boundary conditions, however, account for the elasticity of the walls by effective properties of the surface Γ. This fits to the rule of thumb that boundary conditions model

"...the physics outside of Ω which we do not want to model." (Dan Givoli)

For further references on the effective models behind boundary conditions, cf. [Zhang et al., 2004], [Nobile and Vergara, 2008] and [Nicaise, 2017].

1.2.2 Non-locally reacting kinetic boundary conditions

Up to now, we only encountered boundary conditions where u (or δ) on the different parts of Γ do not influence each other directly, without the bulk function. Such boundary conditions are called locally reacting, cf. [Beale, 1976].

The model Consider the action (1.3) with kinetic energy (1.4a) and potential energy

$$\mathbf{V}(u) = \frac{1}{2}\int_\Omega c_\Omega |\nabla u|^2 \,\mathrm{dx} + \frac{1}{2}\int_\Gamma a_\Gamma u^2 + c_\Gamma |\nabla_\Gamma u|^2 \,\mathrm{d}s$$

where $c_\Omega, a_\Gamma, c_\Gamma > 0$ are constants and $\nabla_\Gamma u$ denotes the tangential gradient of u along Γ. Actions which contain tangential derivatives on Γ and their associated boundary conditions are called "non-locally reacting", since they model the propagation of waves on and along the surface.

Equations of motion In contrast to the locally reacting case, the action (1.3) contains an additional difference of surface integrals. Using integration by parts on the closed surface Γ, this difference is

$$\begin{aligned} -\int_0^T \int_\Gamma c_\Gamma \nabla_\Gamma(u+\varepsilon w)\cdot\nabla_\Gamma(u+\varepsilon w) - c_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma u \,\mathrm{d}s\,\mathrm{d}t \\ = -\varepsilon \int_0^T \int_\Gamma c_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma w \,\mathrm{d}s\,\mathrm{d}t + \mathcal{O}(\varepsilon^2) \\ = \varepsilon \int_0^T \int_\Gamma \mathrm{div}_\Gamma(c_\Gamma \nabla_\Gamma u) w \,\mathrm{d}s\,\mathrm{d}t + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Considering the complete action functional reveals, that the equations of motion for u obtained by the principle of stationary action are (1.6a) supplemented by

$$\mu u_{tt} + a_\Gamma u - c_\Gamma \Delta_\Gamma u + c_\Omega \partial_n u = 0 \qquad \text{on } \Gamma, \tag{1.9}$$

where $\Delta_\Gamma u := \mathrm{div}_\Gamma(\nabla_\Gamma u)$ denotes the Laplace-Beltrami operator. Altogether, u satisfies the wave equation (1.6a) in the bulk and the wave equation (1.9) on the surface. The coupling between those two is on the one hand implemented by u itself and on the other hand by the normal derivative $-c_\Omega \partial_n u$, which acts as a source on the surface.

Interpretation Thinking of u as the displacement of the vibrating membrane, the surface gradient in $\mathbf{V}$ reflects the amount of work necessary to deform u on the surface. Thus this action functional models Γ as a second wave medium which propagates waves along the surface. Such boundary conditions, mixed with Dirichlet boundary conditions, are used to model vibrations of the membrane of a bass drum, cf. [Vitillaro, 2015].

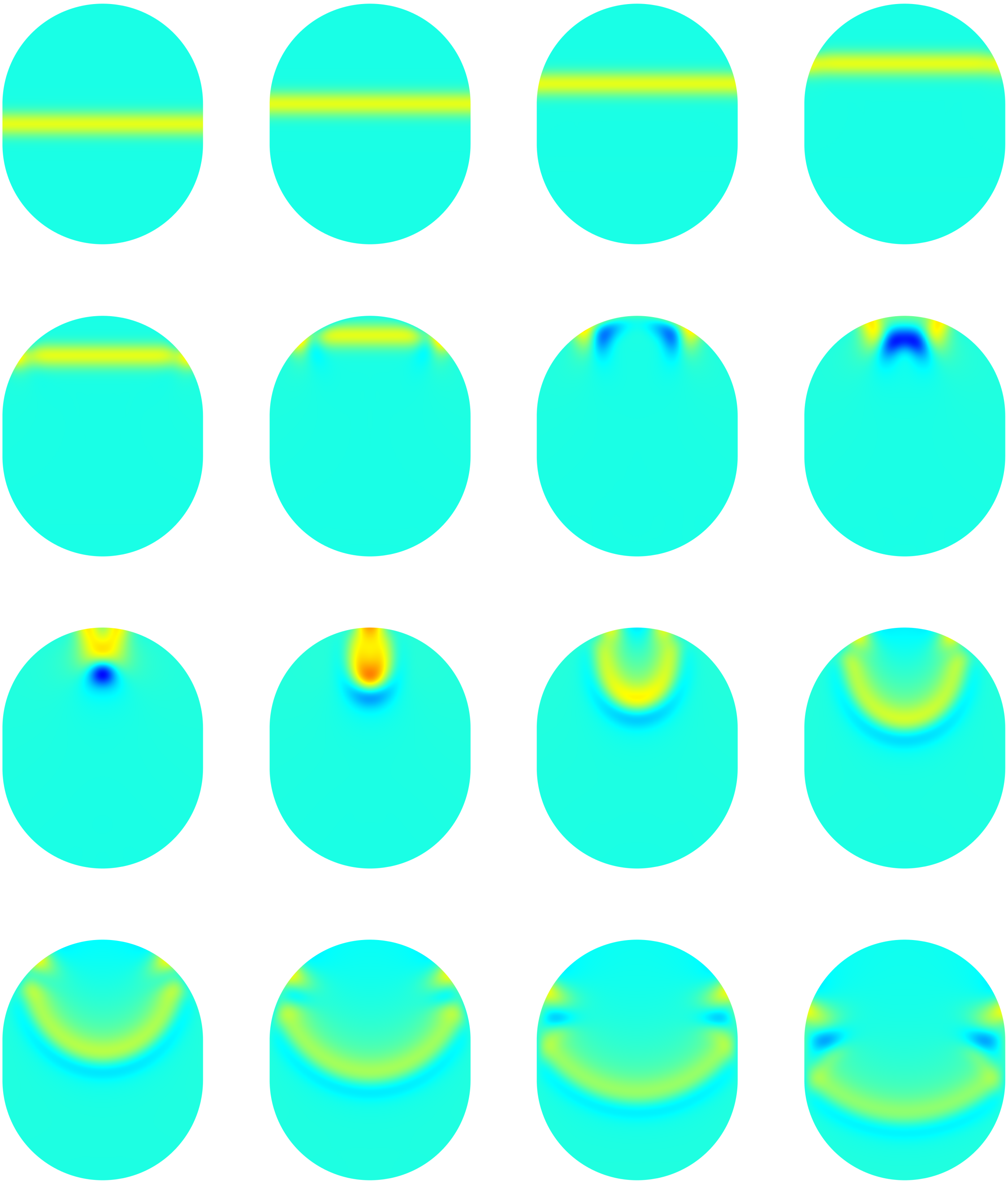


Figure 1.3: An example for a solution of the wave equation with non-locally reacting kinetic boundary conditions with $c_\Omega = \mu = c_\Gamma = 1$ and $a_\Gamma = 0$. The snapshots show the solution u at times $t = 0.2 \cdot k$, $k = 0, \ldots, 15$.

1.3 Further topics and literature

Derivation of boundary conditions Our exposition of boundary conditions and their derivation by the principle of stationary action is mainly inspired by [Goldstein, 2006]. Another approach has been suggested by [Figotin and Reyes, 2015]. They study boundary conditions as a surface-system which is coupled to a bulk-system via an interaction Lagrangian.

Necessity of boundary conditions The question "How many boundary conditions are necessary for a hyperbolic problem to be well-posed?" is still a topic of research. We refer to [Guaily and Epstein, 2013] and references therein.

Theories for wave equations with dynamic boundary conditions Various theories have been developed for the analysis of problems with dynamic boundary conditions. [Trostorff, 2014] and [Picard et al., 2014] consider boundary conditions in an abstract non-linear setting. They also provide examples with dynamic and frictional boundary conditions. [Nickel, 2004] and [Mugnolo, 2006a] approach dynamic boundary conditions with the theory of operator matrices. We further mention [Mugnolo, 2011] (for damped wave equations), [Xiao and Liang, 2004], and [Vitillaro, 2016] which directly consider evolution equations of second-order in time.

Artificial boundary conditions Boundary conditions play an important role in the design of numerical methods: To decrease the computational effort in the simulation of waves in large or infinite domains, the computational domain is often truncated. Artificial boundary conditions are then imposed on the articifial boundary to complete the statement of the problem. Ideally, they do not alter the solution of the original problem and produce no spurious wave reflections. The following literature covers some approaches. [Hagstrom and Lau, 2007], [Hagstrom et al., 2008], [Antoine et al., 2008], [Grote and Sim, 2011], [Barucq, Helene et al., 2012], [Joly, 2012] treat absorbing boundary conditions and [Banjai et al., 2015] use an FEM-BEM ansatz to implement transparent boundary conditions. Domain decomposition methods face similar challenges, because the solutions in the subdomains need to be "connected" with boundary conditions which are as "permeable" as possible. We refer to [Gander, 2015] and references therein.

Chapter 2

Error analysis for linear Cauchy problems with monotone operators

In this chapter, we present an abstract framework for wave equations and the unified error analysis of non-conforming space discretizations thereof. The main results in this chapter are the a priori error bound for such space discretizations and the a priori error bound for their time integration with the Crank–Nicolson method.

Outline In Section 2.1 and 2.2, we introduce and analyze the abstract Cauchy problem with a linear monotone operator. We define general non-conforming space discretizations and the tools for the error analysis in Sections 2.3 and 2.4. Then we proceed to show an a priori bound in Section 2.5. Using this error bound, we establish notions of "stability" and "consistency" in Section 2.6, which are sufficient for the "convergence" of the space discretization. A short overview of different applications for our theory is given in Section 2.7. Finally, in Section 2.8, we analyze a full discretization obtained by a method of lines approach and the Crank–Nicolson method.

Related literature The abstract Cauchy problem is motivated by [Showalter, 1994] and [Showalter, 2013]. Our formulation of the general non-conforming space discretization is similar to what is considered in [Zeidler, 1990b, Ch. 34], [Sanz-Serna and Palencia, 1985] for stationary problems and in [Pazy, 1992, Sect. 3.6] for evolution equations. Moreover, the variational nature of our error bounds is inspired by the so-called Strang Lemmas which apply to elliptic problems, cf. [Ciarlet, 2002].

2.1 Description of the continuous problem

For a given linear operator $\mathcal{S} \in \mathcal{L}(Y, Y^*)$ and a given function $g\colon [0,T] \to Y^*$, we seek a solution $x\colon [0,T] \to Y$ of

$$x'(t) + \mathcal{S}x(t) = g(t) \qquad \text{for } t \in [0,T], \tag{2.1a}$$

$$x(0) = x^0. \tag{2.1b}$$

This problem is not well-posed without further assumptions on $\mathcal{S}$, x^0 and g.

In this thesis, we consider (2.1) in the following setting of a Gelfand triple of Hilbert spaces

$$Y \overset{d}{\hookrightarrow} X \simeq X^* \overset{d}{\hookrightarrow} Y^* \tag{2.2}$$

with dense and continuous embeddings. By $p\colon X \times X \to \mathbb{R}$, we denote the inner product on X which induces the norm $\|\cdot\|_X$. Since $Y \overset{d}{\hookrightarrow} X$, there exists a constant $C_{X,Y} > 0$ such that

$$\|x\|_X \le C_{X,Y} \|x\|_Y \quad \forall x \in Y. \tag{2.3}$$

As an immediate consequence of the identification $X \simeq X^*$, we have

$$p(\varphi, y) = \varphi(y) = \langle \varphi, y \rangle_Y \qquad \forall \varphi \in X,\ y \in Y,$$

where $\langle \cdot, \cdot \rangle_Y$ denotes the duality pairing between Y^* and Y. The bilinear form associated to the linear operator $\mathcal{S}$ is denoted by

$$s(x, y) := \langle \mathcal{S}x, y \rangle_Y, \qquad \forall x, y \in Y. \tag{2.4}$$

Definition 2.1 (Maximal and linear quasi-monotone operators). *Let X and Y form a Gelfand triple of Hilbert spaces as in* (2.2) *and $W = Y^*$ or $W = X$.*

(i) *An operator $\mathcal{S} \in \mathcal{L}(Y, W)$ is called quasi-monotone iff there is a constant $c_{\mathsf{qm}} \geq 0$ s.t.*

$$\langle \mathcal{S}y, y \rangle_Y + c_{\mathsf{qm}} \|y\|_X^2 = s(y, y) + c_{\mathsf{qm}} p(y, y) \geq 0, \qquad \forall y \in Y. \tag{2.5}$$

(ii) *A quasi-monotone operator $\mathcal{S} \in \mathcal{L}(Y, W)$ is called maximal w.r.t. W, iff there exists a $\lambda > c_{\mathsf{qm}}$ s.t.* range$(\lambda + \mathcal{S}) = W$.

Remark 2.2. In the literature, monotone operators are mostly used in non-linear functional analysis. However, we feel that the term "quasi-monotone" is suitable in our (linear) context, cf. also [Showalter, 2013] and [Zeidler, 1990a]. A related notion can be found in [ter Elst et al., 2015]. Note also, that linear monotone operators ($c_{\mathsf{qm}} = 0$) on finite dimensional spaces are usually called positive semi-definite.

Remark 2.3. A skew-symmetric operator $\mathcal{S} \in \mathcal{L}(Y, W)$ is maximal, if range$(I \pm \mathcal{S}) = W$.

Next, we restrict the operator $\mathcal{S}$ to the Hilbert space X. The part of $\mathcal{S} \in \mathcal{L}(Y, Y^*)$ in X is denoted by $S = \mathcal{S}_{|X}$, cf. [Engel et al., 1999]. More precisely, this means

$$S\colon D(S) \subset Y \to X, \qquad y \mapsto Sy = \mathcal{S}y \qquad \text{on} \quad D(S) = \{ y \in Y \mid \mathcal{S}y \in X \}. \tag{2.6}$$

The following result can be found in [Zeidler, 1990b, Sect. 31.4].

Lemma 2.4. *Let $\mathcal{S} \in \mathcal{L}(Y, Y^*)$ and S be defined in* (2.6).

(i) *If $\mathcal{S}$ is skew-symmetric, then S is skew-symmetric.*

(ii) *If $\mathcal{S}$ is quasi-monotone, then $S + c_{\mathsf{qm}}$ is accretive (i.e. $-(S + c_{\mathsf{qm}})$ is dissipative).*

(iii) *If $\mathcal{S}$ is quasi-monotone and maximal w.r.t. Y^*, then* range$(\lambda + S) = X$ *for all $\lambda > c_{\mathsf{qm}}$ and $D(S)$ is dense in X.*

Proof. We only prove (iii), since (i) and (ii) are obvious.

Let $f \in X$ be arbitrary. Since $X \overset{d}{\hookrightarrow} Y^*$ the maximality of $\mathcal{S}$ ensures the existence of some $\lambda_0 > c_{\mathsf{qm}}$ and $y \in Y$ s.t. $(\lambda_0 + \mathcal{S})y = f$. Hence we have $\mathcal{S}y = f - \lambda_0 y \in X$, so that $y \in D(S)$ with $(\lambda_0 + S)y = f$.

The surjectivity of $\lambda + S$ for all $\lambda > c_{\mathsf{qm}}$ and the density of $D(S)$ follow from [Showalter, 2013, Prop. I.4.2]. □

From now on we consider the abstract Cauchy problem

$$x'(t) + Sx(t) = g(t) \qquad \text{for } t \in [0, T], \tag{2.7a}$$

$$x(0) = x^0 \in D(S), \tag{2.7b}$$

with

$$g \in C\big([0, T]; [D(S)]\big) + C^1([0, T]; X). \tag{2.7c}$$

2.2 Well-posedness of the continuous problem

With Lemma 2.4, the well-posedness of (2.7) can be easily shown via semigroup theory.

THEOREM 2.5. *Let $W = Y^*$ or $W = X$. Moreover, assume that $\mathcal{S} \in \mathcal{L}(Y, W)$ is quasi-monotone and maximal w.r.t. W. Then $-S$ defined in* (2.6) *generates a C_0-semigroup $\left(e^{-tS}\right)_{t \geq 0}$ and* (2.7) *has a unique solution $x \in C^1([0,T]; X) \cap C([0,T]; [D(S)])$ given by Duhamel's formula*

$$x(t) = e^{-tS} x^0 + \int_0^t e^{-(t-s)S} g(s) \, \mathrm{d}s.$$

Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. Then x satisfies the stability bound

$$\|x(t)\|_X \leq e^{c_{\mathsf{qm}} t} \Big(\|x^0\|_X + t^{1/p} \|g\|_{L^q(0,t;X)} \Big), \qquad t \in [0, T]. \tag{2.8}$$

Proof. $-(S + c_{\mathsf{qm}} \mathsf{I})$ generates a contraction semigroup due to the Lumer-Philipps theorem [Pazy, 1992, Sect. 1.3]. This implies

$$\left\| e^{-tS} \right\|_{X \leftarrow X} \leq e^{c_{\mathsf{qm}} t}.$$

For Duhamel's formula and the assumptions on f we refer to [Pazy, 1992, Sect. 4.2]. The stability estimate then follows from

$$\|x(t)\| \leq e^{c_{\mathsf{qm}} t} \|x^0\|_X + \int_0^t e^{c_{\mathsf{qm}}(t-s)} \|g(s)\|_X \, \mathrm{d}s \leq e^{c_{\mathsf{qm}} t} \left(\|x^0\|_X + \int_0^t 1 \cdot \|g(s)\|_X \, \mathrm{d}s \right)$$

and the Hölder inequality applied to the integral. □

2.3 Space discretization

This section is dedicated to general non-conforming space discretizations of (2.7) in the sense of [Ciarlet, 2002, Chap. 4]. Space discretizations seek to approximate the solution $x \in X$ in a finite dimensional Hilbert space X_h with inner product $p_h(\cdot, \cdot)$ and induced norm $\|\cdot\|_{X_h}$. The parameter $h > 0$ corresponds to a discretization parameter (e.g. the maximal diameter of the elements of a mesh). Since we are interested in non-conforming discretizations, we do *not* assume that X_h is a subspace of X.

Next, let $S_h \in \mathcal{L}(X_h, X_h)$ be a given discretization of S, e.g. resulting from a finite element or dG method. Since S_h corresponds to a discretized differential operator, it is not uniformly bounded in h. Similar to (2.4) for the continuous case, let the associated bilinear form s_h be defined by

$$s_h(x_h, y_h) = p_h(S_h x_h, y_h), \qquad \forall x_h, y_h \in X_h.$$

Moreover, let $g_h \colon [0, T] \to X_h$ be an approximation of g. Then we consider the semi-discrete problem: seek a solution $x_h \colon [0, T] \to X_h$ of

$$x_h'(t) + S_h x_h(t) = g_h(t) \qquad \text{for } t \in [0, T], \tag{2.9a}$$

$$x_h(0) = x_h^0 \in X_h. \tag{2.9b}$$

Similar to [Ciarlet, 2002, Chap. 4], we define conforming discretization methods as follows.

Definition 2.6. *The discretization* (2.9) *of the abstract Cauchy problem* (2.7) *is conforming, if the following three conditions are satisfied*

(i) $X_h \subset Y$,

(ii) $p(x_h, y_h) = p_h(x_h, y_h)$ *for all* $x_h, y_h \in X_h$,

(iii) $s(x_h, y_h) = s_h(x_h, y_h)$ *for all* $x_h, y_h \in X_h$.
Discretizations which violate at least one of these conditions are called non-conforming.

Note that these conditions are not completely independent of each other: $X_h \subset Y$ implies that $X_h \subset X$ which is necessary for the second and third conditions.

Example 2.7. *To illustrate our exposition we consider the advection equation as a model problem, see e.g. [Di Pietro and Ern, 2012, Chap. 2]. Let* Ω *be a bounded, polygonal, convex domain* $\Omega \subset \mathbb{R}^d$ *and consider*

$$u_t + \beta \cdot \nabla u + \mu u = f \qquad (0,T) \times \Omega, \tag{2.10a}$$

$$u = 0 \qquad (0,T) \times \Gamma^-, \tag{2.10b}$$

$$u(0) = u^0 \qquad \text{in } \Omega. \tag{2.10c}$$

Here, $\beta = (\beta_1, \ldots, \beta_d)^T \in \mathbb{R}^d$, $\nabla u = (\partial_{\mathrm{x}_1} u, \partial_{\mathrm{x}_2} u, \ldots, \partial_{\mathrm{x}_d} u)^T$ *denotes the gradient of* $u : \Omega \to \mathbb{R}$, *and*

$$\Gamma^- = \{\mathrm{x} \in \Gamma \mid \beta \cdot n(\mathrm{x}) < 0\} \tag{2.11}$$

denotes the inflow part of the boundary Γ.

For this problem, we choose $X = L^2(\Omega)$, p *as the* $L^2(\Omega)$ *inner product, and* Y *as the graph space*

$$Y = \{v \in L^2(\Omega) \mid \beta \cdot \nabla v \in L^2(\Omega),\ v|_{\Gamma^-} = 0\} \tag{2.12}$$

equipped with the graph norm

$$\|v\|_Y^2 = p(v, v) + p(\beta \cdot \nabla v, \beta \cdot \nabla v). \tag{2.13}$$

Obviously, this leads to a Gelfand triple (2.2)*. On the graph space* Y *we define the bilinear form*

$$s(u, v) = \int_\Omega \mu uv + (\beta \cdot \nabla u) v \,\mathrm{dx}, \qquad u, v \in Y. \tag{2.14}$$

The associated operator $\mathcal{S} \in \mathcal{L}(Y, X)$ *is monotone (i.e.* $c_{\mathsf{qm}} = 0$*) and maximal w.r.t.* X*, see, e.g., [Di Pietro and Ern, 2012, Theorem 2.9]. Thus, the problem is well-posed due to Theorem 2.5 for suitable initial values and source terms.*

For the discrete space X_h *we choose the space of piecewise linear functions defined on a triangulation* $\mathcal{T}_h$ *of* Ω *and equip it with the inner product* $p_h \coloneqq p$*. Since we also set* $s_h \coloneqq s$ *, this example corresponds to a conforming method, cf. Definition 2.6.* ◇

2.4 Notation for spaces and operators

In a non-conformal setting (where X_h need not be a subspace of X), the semi-discrete solution $x_h \in X_h$ cannot be compared directly to the solution $x \in X$. Consider for example finite element methods for smooth domains Ω where the computational domain $\Omega_h \approx \Omega$ is only an approximation of Ω. In such a situation, the finite element functions in X_h are defined in Ω_h and not in Ω. To deal with this issue, we use the linear and injective lifting operator Q_h which lifts the approximation in X_h to the continuous space

$$Q_h \colon X_h \to X, \qquad X_h^\ell \coloneqq Q_h(X_h) \subset X, \tag{2.15}$$

see, e.g., [Elliott and Ranner, 2013], [Ciarlet, 2002, Chap. 4] and [Cockburn et al., 2014]. For conforming methods, the lift operator can be chosen as $Q_h = \mathsf{I}$ which implies $X_h^\ell = X_h$.

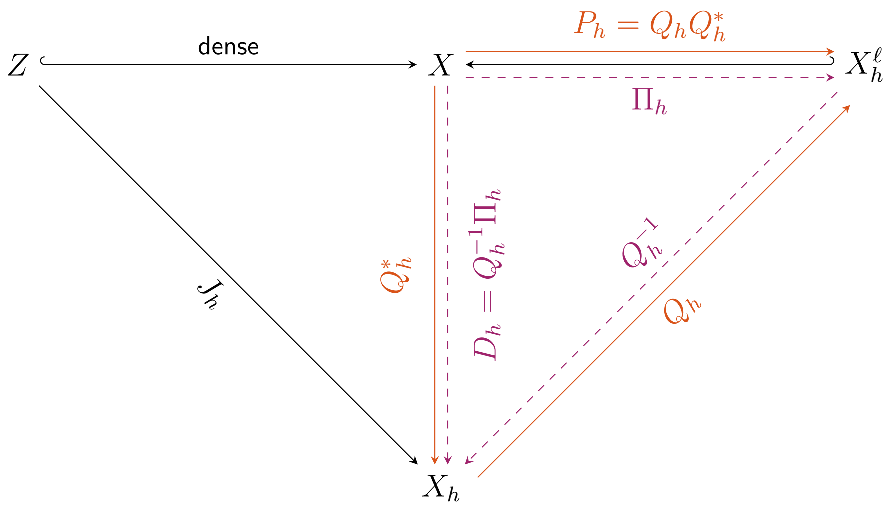


Figure 2.1: Overview of spaces and operators

Next we introduce a variety of different mappings between the spaces X, X_h, and X_h^ℓ, which we illustrate in Figure 2.1. The diagram also contains a dense subspace $Z \overset{d}{\hookrightarrow} X$, which is typically a higher order (broken) Sobolev space. This ensures that an interpolation operator

$$I_h \colon Z \to X_h \tag{2.16}$$

is well-defined with interpolation errors being of optimal order.

Let $J_h \in \mathcal{L}(Z, X_h)$ be a continuous linear operator. We call J_h the reference operators, since our error bounds are based on the following splitting of the error the following parts:

$$\|Q_h x_h - x\|_X \le \|Q_h(x_h - J_h x)\|_X + \|(Q_h J_h - \mathsf{I})x\|_X.$$

To obtain optimal convergence rates, the choice of J_h has to fit to the applications. For conforming methods, we choose the standard orthogonal projection onto X_h (w.r.t. p). However, for non-conforming methods, we will see below that an interpolation operator has to be used to prove optimal rates.

The X-orthogonal projection onto the lifted discrete space X_h^ℓ is denoted by

$$\Pi_h \colon X \to X_h^\ell, \qquad p\big((\mathsf{I} - \Pi_h)x, Q_h y_h\big) = 0 \qquad \forall x \in X, y_h \in X_h. \tag{2.17a}$$

By definition, $Q_h \colon X_h \to X_h^\ell$ is bijective, which allows us to define the operator

$$D_h := Q_h^{-1} \circ \Pi_h \colon X \to X_h. \tag{2.17b}$$

Alternatively, we can introduce the adjoint lift Q_h^* to map between these spaces

$$Q_h^* \colon X \to X_h, \qquad p_h\big(Q_h^* x, y_h\big) = p\big(x, Q_h y_h\big) \qquad \forall x \in X, y_h \in X_h, \tag{2.17c}$$

and further set

$$P_h := Q_h Q_h^* \colon X \to X_h^\ell. \tag{2.17d}$$

In a conforming method, where $Q_h = \mathsf{I}$ and $X_h^\ell = X_h$, we can omit many of these operators, since $P_h = \Pi_h = D_h = Q_h^*$ is just the p-orthogonal projection of X onto X_h.

Our error bounds will be given in terms of a remainder operator

$$R_h := Q_h^* S - S_h J_h \colon D(S) \cap Z \to X_h \tag{2.18a}$$

and in terms of errors in the discretized bilinear forms

$$\Delta p\big(x_h, y_h\big) := p\big(Q_h x_h, Q_h y_h\big) - p_h\big(x_h, y_h\big), \qquad x_h, y_h \in X_h \tag{2.18b}$$

$$\Delta s\big(x_h, y_h\big) := s\big(Q_h x_h, Q_h y_h\big) - s_h\big(x_h, y_h\big), \qquad x_h, y_h \in X_h, \tag{2.18c}$$

where for the latter definition we assume $X_h^\ell \subset Y$.

2.5 A priori error bounds

In the subsequent analysis, we always assume the semi-discretization to be stable in the following sense.

Assumption 2.8 (Stability)

(i) The discrete operator $S_h \in \mathcal{L}(X_h, X_h)$ is quasi-monotone in X_h with

$$p_h\big(S_h x_h, x_h\big) + \widehat{c}_{\mathsf{qm}} \|x_h\|_{X_h}^2 \geq 0, \qquad \forall x_h \in X_h.$$

(ii) There are constants $C_X > c_X > 0$ independent of h s.t.

$$c_X \|Q_h x_h\|_X \leq \|x_h\|_{X_h} \leq C_X \|Q_h x_h\|_X, \qquad \forall x_h \in X_h.$$

Under this assumption, Theorem 2.5 (with X replaced by X_h) shows that there exists a unique solution x_h with

$$\|x_h(t)\|_{X_h} \leq e^{\widehat{c}_{\mathsf{qm}} t}\big(\|x_h^0\|_{X_h} + t\|g_h\|_{L^\infty(0,t;X_h)}\big). \tag{2.19}$$

For the error analysis, we will only use (2.19) and not Assumption 2.8 (i). We now state our abstract error bound in the most general case.

Theorem 2.9. *Let the assumptions of Theorem 2.5 and Assumption 2.8 be fulfilled, and let x be the unique solution of* (2.7) *with $x \in C^1([0,T];Z)$. Furthermore, let x_h be the solution of* (2.9). *Then the lifted semi-discrete solution $Q_h x_h$ suffices the following error bound*

$$\|Q_h x_h(t) - x(t)\|_X \leq C e^{\widehat{c}_{\mathsf{qm}} t}\Big(\|x_h^0 - J_h x^0\|_{X_h} + t\|g_h - Q_h^* g\|_{L^\infty(0,t;X_h)} \tag{2.20a}$$

$$+ t\|(Q_h^* - J_h)x'\|_{L^\infty(0,t;X_h)} + t\|R_h x\|_{L^\infty(0,t;X_h)}\Big) \tag{2.20b}$$

$$+ \|(\mathsf{I} - Q_h J_h)x(t)\|_X \tag{2.20c}$$

for $t \in [0,T]$ and C independent of h and t.

Proof. Let $e_h := x_h - J_h x$ denote the discrete error. By Assumption 2.8 (ii), we find that

$$\|Q_h x_h - x\|_X \leq \|Q_h e_h\|_X + \|(Q_h J_h - \mathsf{I})x\|_X \leq \frac{1}{c_X}\|e_h\|_{X_h} + \|(Q_h J_h - \mathsf{I})x\|_X, \tag{2.21}$$

and so only the discrete error needs to be bounded.

Since $X_h^\ell \subset X$, the solution x of (2.7) satisfies

$$p\big(x', Q_h y_h\big) + s\big(x, Q_h y_h\big) = p\big(g, Q_h y_h\big) \qquad \forall y_h \in X_h$$

or, equivalently

$$p_h\big(Q_h^* x', y_h\big) + p_h\big(Q_h^* S x, y_h\big) = p_h\big(Q_h^* g, y_h\big) \qquad \forall y_h \in X_h.$$

Thus the reference solution $J_h x$ fulfills

$$\begin{aligned} p_h\big(J_h x', y_h\big) + s_h\big(J_h x, y_h\big) = \; & p_h\big(Q_h^* g, y_h\big) + p_h\big((J_h - Q_h^*)x', y_h\big) \\ & + p_h\big((S_h J_h - Q_h^* S)x, y_h\big) \qquad \forall y_h \in X_h. \end{aligned}$$

Subtracting this from the semi-discrete problem (2.9a), we find that the discrete error satisfies

$$\begin{aligned} p_h\big(e_h', y_h\big) + s_h\big(e_h, y_h\big) = \; & p_h\big(g_h - Q_h^* g, y_h\big) - p_h\big((J_h - Q_h^*)x', y_h\big) \\ & - p_h\big((S_h J_h - Q_h^* S)x, y_h\big) \qquad \forall y_h \in X_h. \end{aligned}$$

Here we used that $J_h \in \mathcal{L}(Z, X_h)$ and $x \in C^1([0,T]; Z)$, which implies $e_h \in C^1([0,T]; X_h)$ and, in particular, $e_h' = x_h' - J_h x'$. The discrete stability estimate (2.19) therefore yields an upper bound for the discrete error

$$\|e_h(t)\|_{X_h} \le e^{\widehat{c}_{\mathsf{qm}} t}\Big(\|e_h(0)\|_{X_h} + t\big(\|g_h - Q_h^* g\|_{L^\infty(0,t;X_h)} \\ + \|(Q_h^* - J_h)x'\|_{L^\infty(0,t;X_h)} + \|R_h x\|_{L^\infty(0,t;X_h)}\big)\Big).$$

Using this estimate in (2.21) completes the proof. □

Remark to conforming methods. For conforming methods, the stability assumptions follow directly from the monotonicity of $\mathcal{S}$ with $\widehat{c}_{\mathsf{qm}} = c_{\mathsf{qm}}$. Moreover, since the inner products of X and X_h coincide and $Q_h = \mathsf{I}$, we have $c_X = C_X = 1$. Then, for $J_h = \Pi_h = P_h = Q_h^*$ being the p-orthogonal projector onto X_h, $x_h^0 = \Pi_h x^0$, and $g_h = \Pi_h g$, the error bound (2.20) simplifies to

$$\|x_h(t) - x(t)\|_X \le e^{c_{\mathsf{qm}} t} t \|R_h x\|_{L^\infty(0,t;X_h)} + \|(\mathsf{I} - \Pi_h)x(t)\|_X, \qquad t \in [0,T], \tag{2.22}$$

with $R_h = \Pi_h S - S_h \Pi_h$. ◦

2.6 Convergence

A convergence result based on the error bound in Theorem 2.9 can be obtained if the following consistency assumptions are fulfilled.

Assumption 2.10 (Consistency)

(i) For all $x_h \in X_h$, we have $\|\Delta p(x_h)\|_{X_h^*} \to 0$, $h \to 0$.

(ii) For all $x \in D(S) \cap Z$, the operator R_h satisfies $\|R_h x\|_{X_h} \to 0$, $h \to 0$.

(iii) For all $x \in Z$, we have $\|(\mathsf{I} - Q_h J_h)x\|_X \to 0$, $h \to 0$.

For conforming methods, the first condition is trivially fulfilled with $\Delta p \equiv 0$.

Example 2.7 (continued). *In the advection example, we have $X = L^2(\Omega)$ and $\Delta p = 0$. If we choose $J_h = \Pi_h$ with $Z = H^2(\Omega)$, then Assumption 2.10 (iii) follows from to the approximation property of the nodal interpolation operator $I_h \colon Z \to X_h$ by*

$$\|x - \Pi_h x\|_X \le \|x - I_h x\|_{L^2(\Omega)} \le C h^2 |x|_{H^2(\Omega)}, \qquad x \in H^2(\Omega), \tag{2.23}$$

for suitable triangulations, cf. [Brenner and Scott, 2008, Sect. 4.4]. So only Assumption 2.10 (ii) still needs to be checked. ◇

The following lemma shows that the operator P_h is quasi-optimal. Related results can be found in [Elliott and Ranner, 2013] and [Kovács and Lubich, 2016].

Lemma 2.11 (Quasi-optimality of P_h). *Let Assumption 2.8 (ii) be satisfied. Then we have*

$$\|(P_h - \Pi_h)x\|_X \le C_X \|\Delta p(Q_h^* x)\|_{X_h^*} \qquad \forall x \in X. \tag{2.24}$$

Proof. We use (2.17a) to show that

$$\begin{aligned} \|(P_h - \Pi_h)x\|_X^2 &= p\big((P_h - \Pi_h)x, Q_h(Q_h^* - D_h)x\big) \\ &= p\big((P_h - \mathsf{I})x, Q_h(Q_h^* - D_h)x\big) + p\big((\mathsf{I} - \Pi_h)x, Q_h(Q_h^* - D_h)x\big) \\ &= p\big(P_h x, Q_h(Q_h^* - D_h)x\big) - p\big(x, Q_h(Q_h^* - D_h)x\big) \\ &= p\big(Q_h Q_h^* x, Q_h(Q_h^* - D_h)x\big) - p_h\big(Q_h^* x, (Q_h^* - D_h)x\big) \\ &= \Delta p\big(Q_h^* x, (Q_h^* - D_h)x\big). \end{aligned}$$

On the other hand, we have by Assumption 2.8 (ii) that

$$\|(P_h - \Pi_h)x\|_X = \|Q_h(Q_h^* - D_h)x\|_X \geq \frac{1}{C_X}\|(Q_h^* - D_h)x\|_{X_h}.$$

Altogether we find

$$\frac{\|(Q_h^* - D_h)x\|_{X_h}}{C_X}\|(P_h - \Pi_h)x\|_X \leq \|(P_h - \Pi_h)x\|_X^2 = \Delta p(Q_h^* x, (Q_h^* - D_h)x).$$

Division by $\|(Q_h^* - D_h)x\|_{X_h}$ gives

$$\begin{aligned}\|(P_h - \Pi_h)x\|_X &\leq C_X \frac{\Delta p\big(Q_h^* x, (Q_h^* - D_h)x)\big)}{\|(Q_h^* - D_h)x\|_{X_h}} \\ &\leq C_X \max_{\|y_h\|_{X_h}=1} |\Delta p\big(Q_h^* x, y_h\big)| \\ &= C_X \|\Delta p(Q_h^* x)\|_{X_h^*}.\end{aligned}$$

This proves the claim. □

REMARK 2.12. Note that (2.24) implies

$$\begin{aligned}\|(\mathsf{I} - P_h)x\|_X &\leq \|(\mathsf{I} - \Pi_h)x\|_X + \|(\Pi_h - P_h)x\|_X \\ &\leq \|(\mathsf{I} - \Pi_h)x\|_X + C_X \|\Delta p(Q_h^* x)\|_{X_h^*}.\end{aligned} \tag{2.25}$$

Therefore, the error $(\mathsf{I} - P_h)x$ corresponds to the best approximation error of x in X_h^ℓ up to the error in the inner product. As a consequence of the best approximation property of Π_h, it follows

$$\|(\mathsf{I} - P_h)z\|_X \leq \|(\mathsf{I} - Q_h J_h)z\|_X + C_X \|\Delta p(Q_h^* z)\|_{X_h^*}, \qquad z \in Z \tag{2.26a}$$

$$\|(\mathsf{I} - P_h)z\|_X \leq \|(\mathsf{I} - Q_h I_h)z\|_X + C_X \|\Delta p(Q_h^* z)\|_{X_h^*}, \qquad z \in Z. \tag{2.26b}$$

Therefore, Assumptions 2.10 (i) and 2.10 (iii) imply that we have $\|(\mathsf{I} - P_h)z\|_X \to 0$ and $\|(\mathsf{I} - \Pi_h)z\|_X \to 0$ as $h \to 0$ for $z \in Z$. ◦

COROLLARY 2.13. *Let the assumptions of Theorem 2.5 be satisfied and further let $g(t) \in Z$, $t \in [0, T]$. If the space discretization is stable and consistent in the sense of Assumptions 2.8 and 2.10, and if*

$$\|x_h^0 - J_h x^0\|_{X_h} \to 0 \qquad \text{and} \qquad \|g_h - J_h g\|_{L^\infty(0,T;X_h)} \to 0 \quad \text{as} \quad h \to 0$$

then the lifted semi-discrete solution converges, i.e.,

$$\|Q_h x_h(t) - x(t)\|_X \to 0,$$

for $t \in [0, T]$ as $h \to 0$.

Proof. We see directly from Assumption 2.10, that we only have to investigate $\|(Q_h^* - J_h)x'\|_{L^\infty(0,t;X_h)}$ and $\|g_h - Q_h^* g\|_{L^\infty(0,t;X_h)}$

For the first term we find with (2.26a) and Assumption 2.10 (iii) for $z \in Z$ and $h \to 0$

$$\begin{aligned}\|(Q_h^* - J_h)z\|_{X_h} &\leq C_X \Big(\|(P_h - \mathsf{I})z\|_X + \|(\mathsf{I} - Q_h J_h)z\|_X\Big) \\ &\leq C_X \Big(2\|(\mathsf{I} - Q_h J_h)z\|_X + C_X \|\Delta p(Q_h^* z)\|_{X_h^*}\Big) \to 0\end{aligned} \tag{2.27}$$

This implies $\|(Q_h^* - J_h)x'\|_{X_h} \to 0$ as $h \to 0$, since $x'(t) \in Z$ by assumption.

The second term also converges since

$$\|g_h(t) - Q_h^* g(t)\|_{X_h} \leq \|g_h(t) - J_h g(t)\|_{X_h} + \|(J_h - Q_h^*)g(t)\|_{X_h} \to 0, \qquad t \in [0, T],$$

as $h \to 0$ where we used that $g(t) \in Z$. □

Table 2.1: Overview and classification of the examples

	$X_h \subset Y$	$p = p_h$ on $X_h \times X_h$	$s = s_h$ on $X_h \times X_h$	Discussed in Section
Maxwell's eq. with Nédélec elements	✓	✓	✓	3.3.1
Maxwell's eq. with discontinuous Galerkin	✗	✓	✗	3.3.2
Wave eq. with Lagrange elements (exact integration)	✓	✓	✓	5.3
Wave eq. with Lagrange elements and quadrature	✓	✗	✓	5.3
Wave eq. with kinetic bc. in smooth domains	✗	✗	✗	7.2
Wave eq. with acoustic bc. in smooth domains	✗	✗	✗	7.3
HMM for the wave equation	✓	✓	✗	-

EXAMPLE 2.7 (continued). *Let* $u_h(0) = \Pi_h u^0$ *and* $g_h = \Pi_h g$. *Then* (2.22) *yields the error bound*

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} \le Ct\|R_h u\|_{L^\infty(0,t;L^2(\Omega))} + Ch^2|u(t)|_{H^2(\Omega)}$$

where we already used (2.23). *We will show* (3.2) *which bounds the remainder term by*

$$\|R_h u\|_{X_h} \le \|\mathcal{S}\|_{X \leftarrow Y}\|(\mathsf{I} - \Pi_h)u\|_Y.$$

Since for $u \in H^1(\Omega)$ *we have* $\|u\|_Y \le C\|u\|_{H^1(\Omega)}$ *it follows with* H^1 *error estimates for the* L^2*-orthogonal projection that*

$$\|(\mathsf{I} - \Pi_h)u\|_Y \le Ch|u|_{H^2(\Omega)}, \qquad u \in H^2(\Omega). \tag{2.28}$$

Altogether we obtain the known estimate

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} \le C(1+t)h\|u\|_{L^\infty(0,t;H^2(\Omega))}$$

for unstabilized and unfiltered Galerkin solutions of the advection equation, cf. [Layton, 1983] and [Dunca, 2017]. ◇

2.7 Overview of examples

For specific applications, the general error result of Theorem 2.9 has to be complemented with a bound on the remainder term $\|R_h x\|_{X_h}$. In Chapters 3 and 4, we will show such bounds for symmetric hyperbolic systems and for second-order wave-type problems. Actually, the analysis of different settings and discretizations of these two classes of problems inspired our work. An overview and classification of examples which fit into our general theory is given in Table 2.1. The table shows in which sense the conformity is violated.

An important application which is not discussed in this thesis are heterogeneous multiscale methods. Results as in [Abdulle and Grote, 2011] or [Hochbruck and Stohrer, 2016] can be reproduced using our results from Chapter 4.

2.8 Time integration with the Crank–Nicolson method

In this section we will derive error bounds for the Crank–Nicolson method. We first discuss the error of the time stepping of the continuous problem (2.7) and then of the semi-discrete problem (2.9). These results shall serve as a proof of concept that full discretization error bounds can be shown in our unified framework. We are confident that these proofs can be generalized to apply to suitable Runge–Kutta methods of higher order, cf. [Hochbruck and Pažur, 2015].

The scheme The Crank–Nicolson method or implicit trapezoidal rule [Hairer et al., 2010, Section II.1.1], [Hairer and Wanner, 2010, Section IV.3] applied to (2.7) yields

$$x^{n+1} = x^n - \frac{\tau}{2} S\big(x^{n+1} + x^n\big) + \frac{\tau}{2}\big(g^{n+1} + g^n\big), \qquad n \geq 0. \tag{2.29}$$

as an approximation to $x(t_{n+1})$. Here $\tau > 0$ denotes the time step size and $g^n = g(t_n)$ for $t_n = n\tau$. Note that we start with the exact initial value $x(0) = x^0 \in D(S)$.

Stability The stability of the Crank–Nicolson method is guaranteed by the following Lemma. Our proof relies on [Sturm, 2017] where evolution equations with dissipative operators are considered.

LEMMA 2.14. *Let $\tau c_{\mathsf{qm}} < 2$ and let the assumptions of Theorem 2.5 be fulfilled. Then the approximation x^n given by (2.29) satisfies $x^n \in D(S)$ and*

$$\|x^n\|_X \leq e^{t_n c_{\mathsf{qm}}} \Big(\|x^0\|_X + t_n \max_{m=0}^{n} \|g^m\|_X \Big), \qquad n \geq 0. \tag{2.30}$$

Proof. Using $R_+ := \mathsf{I} + \frac{\tau}{2} S$ and $R_- := \mathsf{I} - \frac{\tau}{2} S$, we can write the scheme (2.29) equivalently as

$$R_+ x^{n+1} = R_- x^n + \frac{\tau}{2}\big(g^{n+1} + g^n\big), \qquad n \geq 0.$$

We now show that R_+ is invertible for $\tau c_{\mathsf{qm}} < 2$. First note that $S_{\mathsf{m}} = \frac{\tau}{2}\left(S + c_{\mathsf{qm}} \mathsf{I}\right)$ is accretive, since $\tau > 0$. Then by [Showalter, 2013, Lem. I.4.1] we have for all $\lambda > 0$ that

$$\|(\lambda + S_{\mathsf{m}})^{-1} x\|_X \leq \|x\|_X, \qquad x \in X. \tag{2.31}$$

Since $\mathsf{I} + \frac{\tau}{2} S = \mu + S_{\mathsf{m}}$ for $\mu = 1 - \frac{\tau}{2} c_{\mathsf{qm}}$ and by assumption $\mu > 0$, this implies that

$$\|R_+^{-1}\|_{X \leftarrow X} \leq 1. \tag{2.32}$$

Thus the iteration matrix $R := R_+^{-1} R_-$ is well-defined and by construction we have $Rx \in D(S)$ for $x \in D(S)$. Hence the recursion is solved by

$$x^{n+1} = R^{n+1} x^0 + \frac{\tau}{2} \sum_{m=0}^{n} R^{n-m} R_+^{-1}\big(g^{m+1} + g^m\big), \qquad n \geq 0, \tag{2.33}$$

and we further find $x^{n+1} \in D(S)$ for $n \geq 0$.

Moreover, from

$$\begin{aligned} \|(\lambda + S_{\mathsf{m}})x\|_X^2 &= \lambda^2 \|x\|_X^2 + \|S_{\mathsf{m}} x\|_X^2 + 2p\big(S_{\mathsf{m}} x, x\big) \\ &\geq \lambda^2 \|x\|_X^2 + \|S_{\mathsf{m}} x\|_X^2 - 2p\big(S_{\mathsf{m}} x, x\big) \\ &= \|(\lambda - S_{\mathsf{m}})x\|_X^2 \end{aligned}$$

we conclude that

$$\|(\lambda + S_{\mathsf{m}})^{-1}(\lambda - S_{\mathsf{m}})x\|_X \leq \|x\|_X, \qquad x \in D(S). \tag{2.34}$$

Since $\mu > 0$ and

$$R = \left(\mathsf{I} + \tfrac{\tau}{2}S\right)^{-1}\left(\mathsf{I} - \tfrac{\tau}{2}S\right) = \left(\mu + S_{\mathsf{m}}\right)^{-1}\left(\mu + \tau c_{\mathsf{qm}} - S_{\mathsf{m}}\right)$$
$$= \left(\mu + S_{\mathsf{m}}\right)^{-1}\left(\mu - S_{\mathsf{m}}\right) + \tau c_{\mathsf{qm}}\left(\mu + S_{\mathsf{m}}\right)^{-1},$$

the estimates (2.31) and (2.34) yield

$$\|Rx\|_X \le (1 + \tau c_{\mathsf{qm}})\|x\|_X, \qquad x \in D(S).$$

By induction we thus have

$$\|R^n x\|_X \le (1 + \tau c_{\mathsf{qm}})^n \|x\|_X \le e^{n\tau c_{\mathsf{qm}}}\|x\|_X, \qquad n \ge 0,\ x \in D(S). \tag{2.35}$$

Taking the X norm in (2.33) and using (2.35), (2.32), we find

$$\|x^{n+1}\|_X \le e^{(n+1)\tau c_{\mathsf{qm}}}\|x^0\|_X + \frac{\tau}{2}\sum_{m=0}^{n} e^{(n-m)\tau c_{\mathsf{qm}}}\|g^{m+1} + g^m\|_X, \qquad n \ge 0. \tag{2.36}$$

The claim then follows from

$$\frac{\tau}{2}\sum_{m=0}^{n} e^{(n-m)\tau c_{\mathsf{qm}}}\|g^{m+1} + g^m\|_X \le \frac{\tau}{2}(n+1)e^{n\tau c_{\mathsf{qm}}}\max_{m=0}^{n}\|g^{m+1} + g^m\|_X$$
$$\le t_{n+1}e^{t_{n+1}c_{\mathsf{qm}}}\max_{m=0}^{n+1}\|g^m\|_X,$$

where we used the triangle inequality and $t_{n+1} = (n+1)\tau$ in the last step. □

Consistency To study the consistency of the Crank–Nicolson method, we insert the exact solution $\widetilde{x}^n = x(t_n)$ into the scheme (2.29). This determines the defect δ^{n+1} via

$$\widetilde{x}^{n+1} = \widetilde{x}^n - \frac{\tau}{2}S\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) + \frac{\tau}{2}\left(g^{n+1} + g^n\right) + \delta^{n+1}, \qquad n \ge 0. \tag{2.37}$$

In the following, we will write

$$x^{(k)}(t) := \frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t)$$

for the k-th temporal derivative of x.

LEMMA 2.15. *Let $x \in C^3([0,T];X)$ be the solution of (2.7). Then*

$$\left\|\delta^{n+1}\right\|_X \le C\tau^3\|x^{(3)}\|_{L^\infty(t_n,t_{n+1};X)}, \qquad n \ge 0.$$

Proof. With (2.37) and (2.7) we find

$$\delta^{n+1} = \widetilde{x}^{n+1} - \widetilde{x}^n - \frac{\tau}{2}\left(x'(t_{n+1}) + x'(t_n)\right)$$
$$= \int_{t_n}^{t_{n+1}} x'(t)\,\mathrm{d}t - \frac{\tau}{2}\left(x'(t_{n+1}) + x'(t_n)\right).$$

Hence δ^{n+1} is the quadrature error of the trapezoidal rule applied to x'. It can be given in terms of the Peano kernel $\kappa(s) := (1-s)s/2$ as

$$\delta^{n+1} = \int_{t_n}^{t_{n+1}} \frac{(t-t_n)(t_{n+1}-t)}{2}x^{(3)}(t)\,\mathrm{d}t = \tau^3\int_0^1 \kappa(s)x^{(3)}(t_n + \tau s)\,\mathrm{d}s. \tag{2.38}$$

This proves the claim. □

Convergence The two previous lemmas are the basic ingredients for the convergence of the scheme.

THEOREM 2.16. *Let the assumptions of Theorem 2.5 be fulfilled and let* $x \in C^3([0,T];X)$ *be the solution of* (2.7)*. Then the approximation* x^n *obtained by the Crank–Nicolson scheme* (2.29) *satisfies*

$$\|x^n - x(t_n)\|_X \le C e^{c_{\mathsf{qm}} t_n} t_n \tau^2 \|x^{(3)}\|^2_{L^\infty(0,t_n;X)}, \qquad n \ge 0.$$

Proof. Subtracting (2.37) from (2.29), we find the error $e^n = x^n - \widetilde{x}^n$ satisfies

$$e^{n+1} = e^n - \frac{\tau}{2} S\left(e^{n+1} + e^n\right) - \delta^{n+1}, \qquad n \ge 0.$$

Applying the stability estimate (2.36) to the error recursion, we find with $e^0 = 0$ that

$$\|e^n\|_X \le \tau \sum_{m=0}^{n} e^{(n-m)\tau c_{\mathsf{qm}}} \|\delta^{m+1}\|_X \le e^{c_{\mathsf{qm}} t_n} t_n \max_{m=0}^{n} \left\|\tau^{-1}\delta^{m+1}\right\|_X.$$

Finally, we use Lemma 2.15 to obtain the desired estimate. □

The fully discrete scheme We now apply the Crank–Nicolson method to the space discretization (2.9), this leads to x_h^{n+1} with $Q_h x_h^{n+1} \approx x(t_{n+1})$ given by

$$x_h^{n+1} = x_h^n - \frac{\tau}{2} S_h\left(x_h^{n+1} + x_h^n\right) + \frac{\tau}{2}\left(g_h^{n+1} + g_h^n\right), \qquad n \ge 0, \tag{2.39}$$

with the initial value given in (2.9b) and $g_h^n = g_h(t_n)$.

Stability of the full discretization If the spatial discretization (2.9) is stable, then the Crank–Nicolson scheme (2.39) is stable.

COROLLARY 2.17. *Let* $\tau \widehat{c}_{\mathsf{qm}} < 2$ *and the Assumption 2.8 be fulfilled. Then* x_h^n *given by* (2.29) *satisfies*

$$\|x_h^n\|_{X_h} \le e^{t_n \widehat{c}_{\mathsf{qm}}} \left(\|x_h^0\|_{X_h} + t_n \max_{m=0}^{n} \|g_h^m\|_{X_h}\right), \qquad n \ge 0. \tag{2.40}$$

Proof. The estimate can be shown as in Lemma 2.14. □

Convergence of the full discretization Finally, we provide an error estimate for the time discretization of the general spatial discretization (2.9) with the Crank–Nicolson method.

THEOREM 2.18. *Let the assumptions of Theorem 2.5 be fulfilled and let* x *be the unique solution of* (2.7) *which satisfies* $x \in C^1([0,T];Z)$ *and* $x \in C^3([0,T];X)$*. Furthermore, let the space discretization satisfy Assumption 2.8 and let* x_h^n, $n \ge 0$ *be given by* (2.39)*. Then the lifted approximation* $Q_h x_h^n$ *satisfies*

$$\begin{aligned}\|Q_h x_h^n - x(t_n)\|_X \le{}& \|(Q_h J_h - \mathsf{I})x(t_n)\|_X + C e^{t_n \widehat{c}_{\mathsf{qm}}} t_n \tau^2 \|x^{(3)}\|_{L^\infty(0,t_n;X)} \\ &+ e^{t_n \widehat{c}_{\mathsf{qm}}} \Bigg(\|x_h^0 - J_h x^0\|_{X_h} + t_n \|(J_h - Q_h^*)x'\|_{L^\infty(0,t_n;X_h)} \\ &+ t_n \max_{m=0}^{n} \left(\|R_h x(t_m)\|_{X_h} + \|g_h(t_m) - Q_h^* g(t_m)\|_{X_h}\right)\Bigg)\end{aligned}$$

for $\tau \widehat{c}_{\mathsf{qm}} < 2$ *and* $t_n \in [0,T]$.

Proof. The proof is done analogously to the proof of Theorem 2.9. We split the error into

$$e^n = Q_h x_h^n - \widetilde{x}^n = Q_h e_h^n + (Q_h J_h - \mathrm{I})\widetilde{x}^n, \qquad e_h^n := x_h^n - J_h \widetilde{x}^n.$$

Thus we have by Assumption 2.8 (ii) that

$$\|e^n\|_X \leq \frac{1}{c_X}\|e_h^n\|_{X_h} + \|(Q_h J_h - \mathrm{I})\widetilde{x}^n\|_X. \tag{2.41}$$

So it remains to bound $\|e_h^n\|_{X_h}$. To use the stability estimate, we formulate the recursion for e_h^n as a perturbed Crank–Nicolson scheme: Apply J_h to (2.37) and subtract it from (2.39). This yields

$$\begin{aligned}
e_h^{n+1} - e_h^n &= \frac{\tau}{2} J_h S\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) - \frac{\tau}{2} S_h\left(x_h^{n+1} + x_h^n\right) \\
&\quad + \frac{\tau}{2}\big(g_h^{n+1} - J_h g^{n+1} + g_h^n - J_h g^n\big) - J_h \delta^{n+1} \\
&= -\frac{\tau}{2} S_h\left(e_h^{n+1} + e_h^n\right) + \frac{\tau}{2}(J_h S - S_h J_h)\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) \\
&\quad + \frac{\tau}{2}\big(g_h^{n+1} - J_h g^{n+1} + g_h^n - J_h g^n\big) - J_h \delta^{n+1}.
\end{aligned}$$

We saw in the previous sections that the error in the differential operator is best studied in terms of the remainder term $R_h = Q_h^* S - S_h J_h$. Therefore we insert $Q_h^* S - Q_h^* S$ and use (2.37) to write the above as

$$\begin{aligned}
e_h^{n+1} - e_h^n + \frac{\tau}{2} S_h\left(e_h^{n+1} + e_h^n\right) &= \frac{\tau}{2}(J_h - Q_h^*) S\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) + \frac{\tau}{2} R_h\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) \\
&\quad + \frac{\tau}{2}\big(g_h^{n+1} - J_h g^{n+1} + g_h^n - J_h g^n\big) - J_h \delta^{n+1} \\
&= -(J_h - Q_h^*)\left(\widetilde{x}^{n+1} - \widetilde{x}^n\right) + \frac{\tau}{2} R_h\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right) \\
&\quad + \frac{\tau}{2}\big(g_h^{n+1} - Q_h^* g^{n+1} + g_h^n - Q_h^* g^n\big) - Q_h^* \delta^{n+1} \\
&=: d_{h,\tau}^{n+1}.
\end{aligned}$$

Applying the stability estimate (2.40) yields

$$\|e_h^{n+1}\|_{X_h} \leq e^{t_{n+1}\widehat{c}_{\mathsf{qm}}}\left(\|e_h^0\|_{X_h} + t_{n+1}\max_{m=0}^{n}\big\|\tau^{-1} d_{h,\tau}^m\big\|_{X_h}\right).$$

We now use

$$\begin{aligned}
\big\|(J_h - Q_h^*)\left(\widetilde{x}^{n+1} - \widetilde{x}^n\right)\big\|_{X_h} &= \|(J_h - Q_h^*)\int_{t_n}^{t_{n+1}} x'(t)\,\mathrm{d}t\|_{X_h} \\
&\leq \int_{t_n}^{t_{n+1}} \big\|(J_h - Q_h^*)x'(t)\big\|_{X_h}\,\mathrm{d}t \\
&\leq \tau\|(J_h - Q_h^*)x'\|_{L^\infty(t_n,t_{n+1};X_h)},
\end{aligned}$$

$\|Q_h^* x\|_{X_h} \leq c_X^{-1}\|x\|_X$ and Lemma 2.15 to bound the fully discrete defect by

$$\begin{aligned}
\|\tau^{-1} d_{h,\tau}^{n+1}\|_{X_h} &\leq \|(J_h - Q_h^*)x'\|_{L^\infty(t_n,t_{n+1};X_h)} + \frac{1}{2}\|R_h\left(\widetilde{x}^{n+1} + \widetilde{x}^n\right)\|_{X_h} \\
&\quad + \frac{1}{2}\|\big(g_h^{n+1} - Q_h^* g^{n+1} + g_h^n - Q_h^* g^n\big)\|_{X_h} + \|\tau^{-1} Q_h^* \delta^n\|_{X_h} \\
&\leq \|(J_h - Q_h^*)x'\|_{L^\infty(0,t_{n+1};X_h)} + (\tau c_X)^{-1}\|\delta^n\|_X \\
&\quad + \max_{m=n}^{n+1}\left(\|R_h \widetilde{x}^m\|_{X_h} + \|g_h^m - Q_h^* g^m\|_{X_h}\right) \\
&\leq \|(J_h - Q_h^*)x'\|_{L^\infty(0,t_{n+1};X_h)} + C\tau^2\|x^{(3)}\|_{L^\infty(0,t_{n+1};X)} \\
&\quad + \max_{m=0}^{n+1}\left(\|R_h \widetilde{x}^m\|_{X_h} + \|g_h^m - Q_h^* g^m\|_{X_h}\right).
\end{aligned}$$

The final estimate then follows from (2.41). □

Chapter 3

Error analysis for symmetric hyperbolic systems

In this chapter, we derive a priori error bounds for general non-conforming space discretizations of symmetric hyperbolic systems. For this purpose, we use the error bound from Theorem 2.9 and exploit the structural properties of the problem and its space discretization.

Outline We define symmetric hyperbolic systems in Section 3.1. Then we prove two error bounds: For the error bound in Section 3.2.1, we assume $X_h^\ell \subset X$. The second error bound in Section 3.2.2 applies to the discontinuous Galerkin discretizations. In Section 3.3, we present two space discretizations of Maxwell's equation, which apply to these results: Nédélec finite elements and the discontinuous Galerkin method.

3.1 Description of the continuous problem

Following [Benzoni-Gavage and Serre, 2007] or [Burazin and Erceg, 2016], we use the following definition of symmetric hyperbolic systems.

Definition 3.1. *We call $\mathcal{S} \in \mathcal{L}(Y, Y^*)$ symmetric hyperbolic if it is quasi-monotone, $D(S) = Y$, and maximal w.r.t. X.*

If $\mathcal{S}$ is a symmetric hyperbolic operator, then we call (2.7) a symmetric hyperbolic system. Since by definition $\mathcal{S} \in \mathcal{L}(Y, X)$, we have here $S = \mathcal{S}$ and we can extend the associated bilinear form s defined in (2.4) to $Y \times X$. Then, s is bounded by

$$|s(y, x)| \leq \|S\|_{X \leftarrow Y} \|y\|_Y \|x\|_X, \quad y \in Y,\ x \in X.$$

By Theorem 2.5, a symmetric hyperbolic system has a unique solution $x \in C^1([0, T]; X) \cap C([0, T]; Y)$.

3.2 A priori error bounds

Consider space discretizations of symmetric hyperbolic systems (2.9) which are stable in the sense of Assumption 2.8. To obtain practicable error bounds from Theorem 2.9, we further need to estimate (2.20) in terms of interpolation errors, Δp, and Δs. As announced, there are two types of space discretizations which we analyze separately. In any of these two non-conforming space discretizations of symmetric hyperbolic systems, $J_h = I_h$ leads to optimal convergence rates.

3.2.1 The case $X_h^\ell \subset Y$

First we discuss problems, where X_h^ℓ is not only contained in X but also in the smaller subspace Y.

THEOREM 3.2. *Let $X_h^\ell \subset Y$ and assume that the unique solution x of the symmetric hyperbolic system (2.7) satisfies $x \in C^1([0,T];Z)$. Furthermore, let x_h be the solution of the semi-discrete problem (2.9) with $x_h^0 = I_h x^0$ and $g_h = Q_h^* g$. Then the lifted semi-discrete solution satisfies*

$$\begin{aligned}\|Q_h x_h(t) - x(t)\|_X \leq & C e^{\widehat{c}_{\mathsf{qm}} t} t \Big(\|(\mathsf{I} - Q_h I_h)x'\|_{L^\infty(X)} + \|\Delta p(Q_h^* x')\|_{L^\infty(X_h^*)} \\ & \qquad + \|(\mathsf{I} - Q_h I_h)x\|_{L^\infty(Y)} + \|\Delta s(I_h x)\|_{L^\infty(X_h^*)} \Big) \\ & + \|(Q_h I_h - \mathsf{I})x(t)\|_X\end{aligned}$$

for $t \in [0,T]$ where $C > 0$ independent of h and t.

Proof. Since Theorem 2.9 applies, the claim follows from (2.20). We first show an estimate of the remainder term. Note that

$$\|R_h y\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h(R_h y, y_h) \qquad \text{and} \qquad R_h = Q_h^* S - S_h J_h.$$

Now let $y \in Y$ and $y_h \in X_h$ with $\|y_h\|_{X_h} = 1$. Then we obtain from $X_h^\ell \subset Y$ and $\mathcal{S} \in \mathcal{L}(Y,X)$

$$\begin{aligned}p_h(R_h y, y_h) &= p_h\big((Q_h^* S - S_h J_h)y, y_h\big) \\ &= p\big(Sy, Q_h y_h\big) - p_h\big(S_h J_h y, y_h\big) \\ &= s\big((\mathsf{I} - Q_h J_h)y, Q_h y_h\big) + s\big(Q_h(J_h y), Q_h y_h\big) - s_h\big(J_h y, y_h\big) \\ &\leq \|S\|_{X \leftarrow Y} \|(\mathsf{I} - Q_h J_h)y\|_Y \|Q_h y_h\|_X + \Delta s(J_h y, y_h) \\ &\leq \|S\|_{X \leftarrow Y} \|(\mathsf{I} - Q_h J_h)y\|_Y c_X^{-1} + \Delta s(J_h y, y_h),\end{aligned}$$

which implies

$$\|R_h x\|_{X_h} \leq C\Big(\|(\mathsf{I} - Q_h J_h)x\|_Y + \|\Delta s(J_h x)\|_{X_h^*} \Big). \tag{3.1}$$

For the final estimate, we choose $J_h = I_h$ in (2.20). The errors in the data (2.20a) vanish by assumption. The first term in (2.20b) can be bounded as in (2.27) and (3.1) yields a bound for the remainder term. At last, (2.20c) is another interpolation error. □

REMARK TO CONFORMING METHODS. For conforming methods, it is possible to eliminate all terms in the error bound of the previous theorem which contain the derivative x'. More precisely, by choosing $J_h = \Pi_h$ we obtain from (3.1) and $\Delta s \equiv 0$ that

$$\|R_h y\|_{X_h} \leq C \|(\mathsf{I} - \Pi_h)y\|_Y, \qquad y \in Y. \tag{3.2}$$

Since $X_h \subset Y$ and since every two norms on the finite dimensional space X_h are equivalent, there exist $\delta_h > 0$ s.t. a so-called inverse estimate $\delta_h \|x_h\|_Y \leq \|x_h\|_X$, $x_h \in X_h$ holds. In general, we thus have $\delta_h \to 0$ as $h \to 0$. We obtain

$$\begin{aligned}\|(\mathsf{I} - \Pi_h)x\|_Y &\leq \|(\mathsf{I} - I_h)x\|_Y + \|(I_h - \Pi_h)x\|_Y \\ &\leq \|(\mathsf{I} - I_h)x\|_Y + \delta_h^{-1} \|(I_h - \Pi_h)x\|_X \\ &\leq \|(\mathsf{I} - I_h)x\|_Y + \delta_h^{-1} \Big(\|(I_h - \mathsf{I})x\|_X + \|(\mathsf{I} - \Pi_h)x\|_X \Big), \\ &\leq \|(\mathsf{I} - I_h)x\|_Y + 2\delta_h^{-1} \|(\mathsf{I} - I_h)x\|_X,\end{aligned} \tag{3.3}$$

where the last inequality follows from Π_h being the orthogonal projector onto X_h. For $x_h^0 = \Pi_h x^0$ and $g_h = \Pi_h g$, (2.22) shows

$$\|x_h(t) - x(t)\|_X \leq C(1 + e^{\widehat{c}_{\mathsf{qm}} t} t)\big(\delta_h^{-1} \|(\mathsf{I} - I_h)x\|_{\infty,X} + \|(\mathsf{I} - I_h)x\|_{L^\infty(Y)} \big) \tag{3.4}$$

for $t \in [0,T]$ and C independent of h and t. ◦

3.2.2 Error analysis for discontinuous Galerkin methods

Discontinuous Galerkin (dG) methods approximate the solution by an elementwise defined function on a given mesh $\mathcal{T}_h$ of Ω. This leads to space discretizations where

$$X_h \subset X, \qquad \text{but} \qquad X_h \not\subset Y.$$

A typical example for $X = L^2(\Omega)$ and $Y = H^1(\Omega)$ is the broken polynomial space $X_h = \mathcal{P}_k(\mathcal{T}_h)$ consisting of piecewise polynomials of degree at most k. Since $X_h \subset X$, we can choose the trivial lift operator $Q_h = \mathsf{I}$.

Our error analysis of the dG method requires the evaluation of s_h not only on $X_h \times X_h$ but in the first argument also on the solution x or on approximation errors. Since the evaluation of the bilinear form involves traces of x on the faces of the elements, we assume that there is a space $Y(\mathcal{T}_h)$ of piecewise smooth functions, e.g., the broken Sobolev space $H^1(\mathcal{T}_h)$, with

$$X_h \subset Y(\mathcal{T}_h) \overset{d}{\hookrightarrow} X \tag{3.5}$$

such that s_h can be extended to $Y(\mathcal{T}_h) \times X_h$. We adapt the notation of [Di Pietro and Ern, 2012] and write

$$Y_{\star,h} = Y_\star + X_h, \qquad Y_\star := Y \cap Y(\mathcal{T}_h),$$

then we can extend s_h to $Y_{\star,h} \times X_h$ and Δs to $Y_\star \times X_h$. Note that in standard applications, where integrals are evaluated exactly, $\Delta s \equiv 0$ on $Y_\star \times X_h$ due to the consistency of the method.

With these structural assumptions we can show the following error bound.

THEOREM 3.3. *Let $X_h \subset Y(\mathcal{T}_h) \overset{d}{\hookrightarrow} X$, $Q_h = \mathsf{I}$ and assume that the unique solution of the symmetric hyperbolic system* (2.7) *satisfies $x \in C^1([0,T];Z)$ and $x(t) \in Z \cap Y(\mathcal{T}_h)$, $t \in [0,T]$. Furthermore, let x_h be the solution of* (2.9) *and $x_h^0 = I_h x^0$ and $g_h = Q_h^* g$. Then there exists a constant C independent of h and t such that the semi-discrete dG solution x_h satisfies*

$$\begin{aligned}\|x_h(t) - x(t)\|_X \le\; & C e^{\widehat{c}_{\mathsf{qm}} t} t \Big(\|(\mathsf{I} - I_h)x'\|_{L^\infty(X)} + \|\Delta p(Q_h^* x')\|_{L^\infty(X_h^*)} \\ & \qquad + \|s_h\big((\mathsf{I} - I_h)x\big)\|_{L^\infty(X_h^*)} + \|\Delta s(x)\|_{L^\infty(X_h^*)} \Big) \\ & + \|(\mathsf{I} - I_h)x(t)\|_X\end{aligned}$$

for $t \in [0,T]$.

Proof. The proof starts from the bound (2.20). Let $y \in Y_\star$ and $y_h \in X_h$ with $\|y_h\|_{X_h} = 1$. By definition (2.18a) of R_h we find

$$\begin{aligned} p_h\big(R_h y, y_h\big) &= p\big(Sy, y_h\big) - p_h\big(S_h J_h y, y_h\big) \\ &= s\big(y, y_h\big) - s_h\big(J_h y, y_h\big) \\ &= \Delta s\big(y, y_h\big) + s_h\big((\mathsf{I} - J_h)y, y_h\big). \end{aligned}$$

Taking the maximum over all y_h gives the bound

$$\|R_h y\|_{X_h} \le \|s_h\big((\mathsf{I} - J_h)y\big)\|_{X_h^*} + \|\Delta s(y)\|_{X_h^*}, \qquad y \in Y_\star.$$

Moreover, setting $J_h = I_h$ and using (2.27) implies

$$\|(Q_h^* - I_h)x'\|_X \le C\big(\|(\mathsf{I} - I_h)x'\|_X + \|\Delta p(Q_h^* x')\|_{X_h^*}\big).$$

The claim now follows from (2.20). □

REMARK 3.4. If $\Delta p \equiv 0$ on $X_h \times X_h$ and $\Delta s \equiv 0$ on $Y_\star \times X_h$, and if the assumptions of Theorem 3.3 are satisfied, then similar arguments with $J_h = \Pi_h$ show

$$\|x_h(t) - x(t)\|_X \le\; C e^{\widehat{c}_{\mathsf{qm}} t} t \|s_h\big((\mathsf{I} - \Pi_h)x'\big)\|_{L^\infty(X_h^*)} + \|(\mathsf{I} - I_h)x(t)\|_X. \tag{3.6}$$

3.3 Examples: Maxwell's equations

As the prototype of a symmetric hyperbolic system we consider Maxwell's equations for linear isotropic materials with perfectly conducting boundary conditions, cf. [Kirsch and Hettlich, 2014].

Let $E\colon [0,T] \times \Omega \to \mathbb{R}^3$ be the electric field and $H\colon [0,T] \times \Omega \to \mathbb{R}^3$ be the magnetic field and assume that the permittivity ε and the permeability μ are piecewise constant and uniformly positive. The suitable functional analytic setting for $x = [H, E]^T$ is given by the Hilbert space $X := L^2(\Omega)^6$ endowed with a weighted inner product and $Y = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega) \overset{d}{\hookrightarrow} X$. Maxwell's equations are a symmetric hyperbolic system since the Maxwell operator $\mathcal{S} \in \mathcal{L}(Y, X)$ is maximal skew-symmetric, cf., e.g., [Hochbruck et al., 2014, Sect. 3.2]. Hence Maxwell's equations are well-posed in the sense of Theorem 2.5.

3.3.1 Edge element discretizations

For this example, we assume that $\mathcal{T}_h$ is quasi-uniform. We choose the discrete space as $X_h = V_h(\mathrm{curl}) \times V_{h,0}(\mathrm{curl})$ where

$$V_h(\mathrm{curl}) = \Big\{U_h \in H(\mathrm{curl}, \Omega) \mid U_h\big|_K \in (\mathcal{P}_1)^3 \text{ for } K \in \mathcal{T}_h\Big\},$$
$$V_{h,0}(\mathrm{curl}) = \{U_h \in V_h(\mathrm{curl}) \mid \nu \times U_h = 0 \text{ on } \Gamma\},$$

are first order curl-conforming elements of Nédélec's second family, cf. [Nédélec, 1986]. Since $X_h \subset Y$, we are in the situation of Section 3.2.1 and we can choose $p_h := p$ and $s_h := s$. Moreover, there exists an interpolation operator $I_h\colon Z \to X_h$, $Z = H^2(\Omega)^6$ s.t.

$$\|x - I_h x\|_X + h\|x - I_h x\|_Y \le Ch^2 \|x\|_{H^2(\Omega)^6}, \qquad x \in H^2(\Omega)^6,$$

cf. [Nédélec, 1986, Prop. 3].

Hence if the solution $x = \big[H, E\big]^{\mathsf{T}}$ of Maxwell's equations belongs to $C\big([0,T]; H^2(\Omega)^6\big)$, it follows from (3.4) that the semi-discrete solution $x_h = \big[H_h, E_h\big]^{\mathsf{T}}$ satisfies

$$\|x_h(t) - x(t)\|_{L^2(\Omega)^6} \le C(t)h.$$

Here we used that $\delta_h^{-1} \le Ch^{-1}$ if $0 < h \le 1$ due to the inverse estimate between $L^2(\Omega)$ and $H^1(\Omega)$. A similar convergence result for edge elements of Nédélec's first family can be found in [Zhao, 2004, Thm. 4.1].

If we use quadrature formulas to approximate the integrals in the definition of p and s, we have $\Delta p \ne 0$ and $\Delta s \ne 0$. However, if x and x' are sufficiently smooth, one can show that

$$\|\Delta p(Q_h^* x')\|_{L^\infty(X_h^*)} + \|\Delta s(I_h x)\|_{L^\infty(X_h^*)} \le Ch$$

if the quadrature formulas have sufficiently high order, cf. [Ciarlet, 2002] and Section 5.3. For such cases Theorem 3.2 shows, that the convergence order of $\|x(t) - x_h(t)\|_X$ is preserved.

3.3.2 Discontinuous Galerkin discretizations

We seek approximations in the set of piecewise polynomials on a given mesh, i.e., $X_h = \mathcal{P}_k(\mathcal{T}_h)^6$, $k \ge 0$, and consider a central (also centered) fluxes dG discretization of the Maxwell operator given by s_h. For details we refer to [Di Pietro and Ern, 2012]. By construction s_h is consistent in the sense that $\Delta s \equiv 0$ on $Y_\star \times X_h$ where $Y_\star = Y \cap H^1(\mathcal{T}_h)^6$.

By [Hochbruck and Sturm, 2016, (5.3) and (5.5)] we have

$$\|(\mathsf{I} - I_h)x\|_X + h\|s_h((\mathsf{I} - P_h)x)\|_{X_h^*} \le C \left(\sum_{K \in \mathcal{T}_h} h_K^{2k+2} |x|^2_{H^{k+1}(K)^6} \right)^{1/2},$$

where $I_h\colon H^{k+1}(\mathcal{T}_h)^6 \to X_h$ is the piecewise interpolation operator and $|x|_{H^{k+1}(K)^6}$ denotes the $H^{k+1}(K)$ semi-norm of x. The convergence result then follows from (3.6). If the solution x of Maxwell's equations is in $C\big([0,T]; H^{k+1}(\mathcal{T}_h)^6\big)$, then the dG approximation $x_h = \big[E_h, H_h\big]^\mathsf{T}$ satisfies

$$\|x_h(t) - x(t)\|_{L^2(\Omega)^6} \leq C(t)h^k.$$

Such results are for example shown in [Pazur, 2013].

Chapter 4

Second-order wave-type problems

In this chapter, we investigate the well-posedness of wave equations formulated as abstract second-order evolution equations. In addition to the main well-posedness result, we also show the unique existence of solutions under weaker conditions and derive the corresponding stability estimates.

Outline Sections 4.1 and 4.2 are devoted to second-order wave-type problems and their well-posedness in the framework of monotone operators. After that, in Section 4.3, we discuss a situation where a different stability estimate with favourable properties can be proven. We then move on to study weak well-posedness of second-order wave-type problems in Section 4.4, where we focus on stability estimates in weaker norms.

Related literature The well-posedness analysis of abstract second-order wave equations from Section 4.2 can be found in [Showalter, 1994, Ch. VI].

4.1 Description of the continuous problem

Let H and V be two Hilbert spaces with $V \overset{d}{\hookrightarrow} H$, i.e., there is a constant $C_{H,V} > 0$ s.t.

$$\|\varphi\|_m \leq C_{H,V} \|\varphi\|_V, \qquad \varphi \in V,$$

where $\|\cdot\|_m$ denotes the norm on H which is induced by the inner product $m\colon H \times H \to \mathbb{R}$. Moreover, we identify $H \simeq H^*$ and form the Gelfand triple

$$V \overset{d}{\hookrightarrow} H \simeq H^* \overset{d}{\hookrightarrow} V^*. \tag{4.1}$$

The second-order wave-type problem then reads: Find $u\colon [0,T] \to V$ s.t.

$$\langle u''(t), \varphi\rangle_V + b\big(u'(t), \varphi\big) + a\big(u(t), \varphi\big) = \langle f(t), \varphi\rangle_V \qquad \forall \varphi \in V, \tag{4.2a}$$

$$u(0) = u^0, \quad u'(0) = v^0, \tag{4.2b}$$

where $f\colon [0,T] \to V^*$ is a given function and the bilinear forms a and b satisfy the following assumption.

Assumption 4.1

(i) The bilinear form $a\colon V \times V \to \mathbb{R}$ is continuous, symmetric, and satisfies the Garding inequality

$$a\big(u,u\big) + c_G \|u\|_m^2 \geq \alpha \|u\|_V^2, \qquad u \in V \tag{4.3}$$

for constants $c_G \geq 0$ and $\alpha > 0$.

(ii) The bilinear form $b\colon V\times V\to\mathbb{R}$ is continuous and $b+\beta_{\mathsf{qm}}m$ is monotone for some $\beta_{\mathsf{qm}}\geq 0$, i.e.,

$$b\big(v,v\big)+\beta_{\mathsf{qm}}\|v\|_m^2\geq 0,\qquad v\in V.$$

Due to the continuity of the bilinear forms, a and b induce operators $\mathcal{A},\mathcal{B}\in\mathcal{L}(V,V^*)$, respectively. We can thus write (4.2) equivalently as the evolution equation

$$u''+\mathcal{B}u'+\mathcal{A}u=f\qquad\text{in } V^* \tag{4.4}$$

supplemented by the initial conditions $u(0)=u^0$ and $u'(0)=v^0$.

Furthermore, we introduce the bilinear form

$$\widetilde{a}\big(u,v\big):=a\big(u,v\big)+c_G\,m\big(u,v\big),\qquad u,v\in V, \tag{4.5}$$

which is coercive on $V\times V$ due to (4.3), and define $\widetilde{V}=(V,\widetilde{a})$ to be the Hilbert space equipped with $\widetilde{a}$. Note that the Garding inequality implies

$$\|\varphi\|_m\leq C_{H,V}\|\varphi\|_V\leq C_{H,V}\alpha^{-1/2}\|\varphi\|_{\widetilde{a}},\qquad \varphi\in\widetilde{V}. \tag{4.6}$$

The well-posedness result for the evolution equation (4.4) is a variant of [Showalter, 1994, Thm. VI.2.1]. Nevertheless, we give the complete proof as we need the connection to the framework of monotone operators from Chapter 2 for our error analysis.

4.2 Well-posedness of the continuous problem

To write (4.4) as the first-order in time problem (2.1), we introduce the velocity $v=u'$ and use

$$x(t)=\begin{bmatrix}u(t)\\v(t)\end{bmatrix},\quad \mathcal{S}=\begin{bmatrix}0&-\mathsf{I}\\\mathcal{A}&\mathcal{B}\end{bmatrix},\quad g(t)=\begin{bmatrix}0\\f(t)\end{bmatrix},\quad x^0=\begin{bmatrix}u^0\\v^0\end{bmatrix}. \tag{4.7}$$

For the Gelfand triple (2.2), we choose

$$Y=\widetilde{V}\times V\quad\text{and}\quad X=\widetilde{V}\times H,$$

equipped with their canonical inner products. In the following, we will refer to this problem as the first-order in time formulation, or just the first-order in time formulation, of the second-order wave-type problem (4.4).

Variants of the following results can be found in the proof of [Showalter, 1994, Thm. VI.2.1].

Lemma 4.2. *Let Assumption 4.1 be satisfied. Then $\mathcal{S}\in\mathcal{L}(Y,Y^*)$ is quasi-monotone with*

$$c_{\mathsf{qm}}=\tfrac{1}{2}c_GC_{H,V}\alpha^{-1/2}+\beta_{\mathsf{qm}} \tag{4.8}$$

and maximal w.r.t. Y^*.

Proof. Note that the first component of the identification $X\simeq X^*$ yields $\widetilde{V}\simeq\widetilde{V}^*$. Thus we have $Y^*\simeq\widetilde{V}^*\times V^*\simeq\widetilde{V}\times V^*$. This gives the first assertion, since $\mathcal{S}$ belongs to $\mathcal{L}(Y,\widetilde{V}\times V^*)$ which is a consquence of $\mathcal{A},\mathcal{B}\in\mathcal{L}(V,V^*)$.

For $x=\big[u,v\big]^{\mathsf{T}}\in Y$ we have

$$\begin{aligned}\langle\mathcal{S}x,x\rangle_Y=-\widetilde{a}\big(v,u\big)+\langle\mathcal{A}u+\mathcal{B}v,v\rangle_V&=-a\big(v,u\big)-c_Gm\big(v,u\big)+a\big(u,v\big)+b\big(v,v\big)\\&\geq -c_G\|v\|_mC_{H,V}\alpha^{-1/2}\|u\|_{\widetilde{a}}-\beta_{\mathsf{qm}}\|v\|_m^2\\&\geq -c_{\mathsf{qm}}\left(\|u\|_{\widetilde{a}}^2+\|v\|_m^2\right),\end{aligned}$$

where we first used the Cauchy–Schwarz inequality for m together with (4.6), and then applied Young's inequality for the last estimate. Hence $\mathcal{S}$ is quasi-monotone with constant c_{qm}.

Finally, $\mathcal{S}$ is maximal w.r.t. Y^*, if range$(\lambda + \mathcal{S}) = Y^*$ for some $\lambda > c_{\mathsf{qm}}$. Thus we have to show that for each $f = \big[f_1, f_2\big]^\mathsf{T} \in Y^*$ there is an $x = \big[u, v\big]^\mathsf{T} \in Y$ s.t.

$$(\lambda + \mathcal{S})x = f \qquad \iff \qquad \begin{bmatrix} \lambda u - v \\ \lambda v + \mathcal{A}u + \mathcal{B}v \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

From the system of equations we see that it is sufficient to find one $u \in \widetilde{V}$ s.t.

$$\lambda^2 u + \lambda \mathcal{B}u + \mathcal{A}u = f_2 + \lambda f_1 + \mathcal{B}f_1 \qquad \text{in } V^*,$$

since then $(\lambda + \mathcal{S})x = f$ for $x = \big[u, \lambda u - f_1\big]^\mathsf{T} \in Y$. The above problem is equivalent to

$$\begin{aligned} \Lambda\big(u, \psi\big) &:= \lambda^2 \langle u, \psi\rangle_V + \lambda b\big(u, \psi\big) + a\big(u, \psi\big) \\ &= \langle f_2, \psi\rangle_V + \lambda \langle f_1, \psi\rangle_V + b\big(f_1, \psi\big) =: \ell(\psi) \qquad \forall \psi \in V. \end{aligned}$$

and has a unique solution $u \in V$ for sufficiently large $\lambda > 0$ by the Lax-Milgram theorem: First, ℓ is in V^* since $f_1 \in \widetilde{V}$, $f_2 \in V^*$ and $\mathcal{B}f_1 \in V^*$. Second, Λ is continuous on $V \times V$ for every $\lambda \in \mathbb{R}$. Third, Λ is coercive on $V \times V$ for $\lambda \geq \beta_{\mathsf{qm}}/2 + \big(\beta_{\mathsf{qm}}^2/4 + c_G\big)^{1/2}$, since

$$\Lambda\big(u, u\big) = \lambda^2 \|u\|_m^2 + \lambda b\big(u, u\big) + a\big(u, u\big) \geq (\lambda^2 - \lambda\beta_{\mathsf{qm}} - c_G)\|u\|_m^2 + \alpha\|u\|_V^2, \qquad u \in V.$$

Thus we can apply the Lax-Milgram theorem which implies that, for sufficiently large λ, there exists a unique $u \in V$ s.t. $\Lambda\big(u, \psi\big) = \ell(\psi)$ for all $\psi \in V$. This finishes the proof. □

Expressing Theorem 2.5 in terms of (4.7) gives the following result.

THEOREM 4.3. *Let Assumption* 4.1 *be satisfied and assume that* $u^0, v^0 \in V$ *satisfy* $\mathcal{A}u^0 + \mathcal{B}v^0 \in H$, *and that* $f \in C^1([0,T]; H)$ *or* $\big[f, \mathcal{B}f\big]^\mathsf{T} \in C([0,T]; V \times H)$. *Then* (4.4) *has a unique solution* $u \in C^2([0,T]; H) \cap C^1([0,T]; V)$ *which satisfies* $\mathcal{A}u + \mathcal{B}u' \in C([0,T]; H)$ *and*

$$\Big(\|u(t)\|_{\tilde{a}}^2 + \|u'(t)\|_m^2\Big)^{1/2} \leq e^{c_{\mathsf{qm}} t}\left(\Big(\|u^0\|_{\tilde{a}}^2 + \|v^0\|_m^2\Big)^{1/2} + t\|f\|_{L^\infty(0,t;H)}\right), \qquad t \in [0,T],$$

for c_{qm} *from* (4.8).

If further $\mathcal{B} \in \mathcal{L}(V, H)$, *then we have* $u \in C^2([0,T]; H) \cap C^1([0,T]; V) \cap C([0,T]; [D(A)])$ *where* $D(A) = \{v \in V \mid \mathcal{A}v \in H\}$ *and* $A = \mathcal{A}_{|H}$.

Proof. The assumptions guarantee for the first-order in time formulation that $\mathcal{S}$ is quasi-monotone due to Lemma 4.2, that $x^0 = \big[u^0, v^0\big]^\mathsf{T} \in D(S)$, and, that $g \in C^1([0,T]; X)$ or $Sg \in C([0,T]; X)$. Using $v = u'$, we infer from Theorem 2.5 that (4.2) has the unique solution

$$\big[u, u'\big]^\mathsf{T} = x \in C([0,T]; [D(S)]) \cap C^1([0,T]; \widetilde{V} \times H),$$

where

$$D(S) = \big\{x \in Y \mid \mathcal{S}x \in X\big\} = \big\{\big[u, v\big]^\mathsf{T} \in \widetilde{V} \times V \mid \mathcal{A}u + \mathcal{B}v \in H\big\}.$$

Therefore the first claim follows from $u'' = v' \in C([0,T]; H)$ and the stability estimate from (2.8). Moreover, since $Sx \in C([0,T]; X)$, we obtain $t \mapsto \mathcal{A}u(t) + \mathcal{B}u'(t) \in C([0,T]; H)$.

If $\mathcal{B} \in \mathcal{L}(V, H)$, then $D(S) = D(A) \times V$ and therefore $u \in C([0,T]; [D(A)])$, which gives the second claim. □

4.3 Energy spaces

The (physical) energy corresponding to the second-order wave-type problem (4.2) at time t is given by

$$\mathbf{E}\left(u(t), u'(t)\right) := \frac{1}{2}\Big(a\big(u(t), u(t)\big) + m\big(u'(t), u'(t)\big)\Big). \tag{4.9}$$

We follow [Gal et al., 2003], [Mugnolo, 2006b] and [Graber and Lasiecka, 2014] and introduce the notion of an energy space.

Definition. *We call $\|\cdot\|_X$ the energy norm and X the energy space of the problem if $\|[w, v]^\intercal\|_X^2$ is proportional to the energy $\mathbf{E}(w, v)$.*

If $c_G = 0$, then $\|\cdot\|_X$ is an energy norm since

$$\mathbf{E}\left(u, u'\right) = \frac{1}{2}\|[u, u']^\intercal\|_X^2.$$

Thus Theorem 4.3 implies that solutions of homogeneous second-order wave-type problems with $c_G = 0$ have non-increasing energy if $\beta_{\mathsf{qm}} = 0$. For $c_G > 0$ the X-norm is not an energy norm.

Motivation Some dissipative wave phenomena lead to second-order wave-type problems with $c_G > 0$. Then Theorem 4.3 yields stability estimates that grow exponentially fast in time. This is undesirable for a dissipative wave phenomenon and we provide a non-increasing stability estimate the energy norm, to get rid of this growth.

Presentation of the problem We consider the following situation.

Assumption 4.4

(i) Let a and b satisfy Assumption 4.1 with $c_G > 0$ and $\beta_{\mathsf{qm}} = 0$.

(ii) M is a closed subspace of V s.t. a is continuous and coercive on M with

$$a\big(u, u\big) \geq \alpha\|u\|_V^2 \qquad \forall u \in M.$$

(iii) There is a continuous projection

$$P_M \colon V \to M \qquad \text{with} \qquad P_M v = v, \quad v \in M$$

which is invariant under a in the sense that

$$a\big(u, \varphi\big) = a\big(P_M u, \varphi\big), \qquad u, \varphi \in V.$$

Example 4.5. *The variational formulation of the wave equation with homogeneous Neumann boundary conditions* (1.1) *fulfills the assumption:*

Assumption 4.4 (i) holds since $H = L^2(\Omega)$ is the pivot space and the bilinear form

$$a\big(u, \varphi\big) := c_\Omega \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{dx}$$

is defined on $V = H^1(\Omega)$. Thus a is not coercive, but satisfies (4.3) *with $c_G = \alpha = c_\Omega$.*

This lack of coercivity is already an issue for the variational solution of the Poisson equation with Neumann boundary conditions. A workaround there is to solve the Poisson equation in the space of $H^1(\Omega)$ functions with zero mean

$$H^1_{\mathrm{mv}}(\Omega) := \big\{v \in H^1(\Omega) \mid \mathrm{mv}\,(v) = 0\big\}, \qquad \mathrm{mv}\,(v) = \tfrac{1}{|\Omega|} m\big(v, \mathbb{1}\big),$$

where $\mathbb{1}(x) = 1$ *for* $x \in \Omega$*, cf. [Han and Atkinson, 2009, Sect. 8.4.3]. Note that* $H^1_{\mathrm{mv}}(\Omega)$ *is a closed subspace of* V*, since* $\mathrm{mv} \in \mathcal{L}(V, \mathbb{R})$ *and we further find by [Han and Atkinson, 2009, Thm. 7.3.12] that on* $H^1_{\mathrm{mv}}(\Omega)$

$$|\cdot|_{H^1(\Omega)} \quad \sim \quad \|\cdot\|_{H^1(\Omega)}.$$

This implies that a *is coercive on* $M = H^1_{\mathrm{mv}}(\Omega)$ *and hence Assumption 4.4 (ii) holds.*

For the projection of V *to* M*, we choose*

$$P_M \colon V \to M, \qquad P_M v := v - \mathrm{mv}\,(v)\,\mathbb{1}.$$

Obviously, $P_M v = v$ *for* $v \in M$ *and an easy calculation shows that*

$$a\big(P_M u, \varphi\big) = \int_\Omega \nabla\big(u - \mathrm{mv}\,(u)\,\mathbb{1}\big) \cdot \nabla\varphi \,\mathrm{dx} = \int_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{dx} = a\big(u, \varphi\big),$$

since $\nabla\mathbb{1} = 0$*. Therefore Assumption 4.4 (iii) is also fulfilled.*

Outline of proof Let u be the solution of (4.4) and Assumption 4.4 be fulfilled. To show that $x = \big[u, u'\big]^{\mathsf{T}}$ satisfies a stability estimate in an energy norm, we first study the well-posedness of a first-order in time formulation of (4.4) in the energy space

$$X_{\mathsf{E}} := M_a \times H, \qquad M_a := \big(M, a(\cdot,\cdot)\big).$$

In a second step, we then prove that $\big[P_M u, u'\big]^{\mathsf{T}}$ coincides with the solution of the problem in X_{E}.

Alternative first-order in time formulation For an alternative first-order in time formulation of (4.4), we consider $\widetilde{u}(t) = P_M u(t)$ and $v(t) = u'(t)$. From Assumption 4.4 (iii), it follows that $\mathcal{A}P_M = \mathcal{A}$ and therefore

$$\widetilde{u}'(t) = P_M v(t), \tag{4.10a}$$
$$v'(t) + \mathcal{B}v(t) + \mathcal{A}\widetilde{u}(t) = f(t). \tag{4.10b}$$

The corresponding first-order in time formulation is

$$x_{\mathsf{E}}'(t) + \mathcal{S}_{\mathsf{E}} x_{\mathsf{E}}(t) = g(t), \qquad x_{\mathsf{E}}(0) = x_{\mathsf{E}}^0, \tag{4.11}$$

where

$$x_{\mathsf{E}} = \begin{bmatrix} \widetilde{u}(t) \\ v(t) \end{bmatrix}, \quad \mathcal{S}_{\mathsf{E}} = \begin{bmatrix} 0 & -P_M \\ \mathcal{A} & \mathcal{B} \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \quad x_{\mathsf{E}}^0 = \begin{bmatrix} P_M u^0 \\ v^0 \end{bmatrix}.$$

We consider (4.11) in the Gelfand triple $Y_{\mathsf{E}} \overset{d}{\hookrightarrow} X_{\mathsf{E}} \simeq X_{\mathsf{E}}^* \overset{d}{\hookrightarrow} Y_{\mathsf{E}}^*$ where

$$Y_{\mathsf{E}} = M_a \times V \qquad \text{and} \qquad Y_{\mathsf{E}}^* = M_a \times V^*,$$

and further define the projection

$$P_{\mathsf{E}} := \begin{bmatrix} P_M & 0 \\ 0 & \mathsf{I} \end{bmatrix} : Y^* \to Y_{\mathsf{E}}^*.$$

Then $\mathcal{S}_{\mathsf{E}}$ and $\mathcal{S}$ are related via the following result.

LEMMA 4.6. *Let Assumption 4.4 be fulfilled. Then* $\mathcal{S}_E \in \mathcal{L}(Y_E, Y_E^*)$ *is monotone, maximal, and satisfies*

$$\mathcal{S}_E P_E = P_E \mathcal{S}. \tag{4.12}$$

Proof. The continuity of $\mathcal{S}_{\mathsf{E}} \in \mathcal{L}(Y_{\mathsf{E}}, Y_{\mathsf{E}}^*)$ follows from the continuity of its components $\mathcal{A}, \mathcal{B}, P_M$.

To see that is $\mathcal{S}_{\mathsf{E}}$ monotone, observe that by Assumptions 4.4 (i) and 4.4 (iii) for $x \in Y_{\mathsf{E}}$

$$\begin{aligned}\langle \mathcal{S}_{\mathsf{E}} x, x\rangle_{Y_{\mathsf{E}}} = \Big\langle \begin{bmatrix} -P_M v \\ \mathcal{A}\widetilde{u} + \mathcal{B}v \end{bmatrix}, \begin{bmatrix} \widetilde{u} \\ v \end{bmatrix} \Big\rangle_{Y_{\mathsf{E}}} &= -a\big(P_M v, \widetilde{u}\big) + \langle \mathcal{A}\widetilde{u}, v\rangle_V + \langle \mathcal{B}v, v\rangle_V \\ &\geq -a\big(v, \widetilde{u}\big) + a\big(\widetilde{u}, v\big) \\ &= 0.\end{aligned}$$

Relation (4.12) is fulfilled, since we have for $x = \big[u, v\big]^{\mathsf{T}} \in Y$ and $y = \big[\widetilde{\varphi}, \psi\big]^{\mathsf{T}} \in Y_{\mathsf{E}}$

$$\langle \mathcal{S}_{\mathsf{E}} P_{\mathsf{E}} x, y\rangle_{Y_{\mathsf{E}}} = \Big\langle \begin{bmatrix} -P_M v \\ \mathcal{A}P_M u + \mathcal{B}v \end{bmatrix}, \begin{bmatrix} \widetilde{\varphi} \\ \psi \end{bmatrix} \Big\rangle_{Y_{\mathsf{E}}} = \Big\langle \begin{bmatrix} -P_M v \\ \mathcal{A}u + \mathcal{B}v \end{bmatrix}, \begin{bmatrix} \widetilde{\varphi} \\ \psi \end{bmatrix} \Big\rangle_{Y_{\mathsf{E}}} = \langle P_{\mathsf{E}} \mathcal{S} x, y\rangle_{Y_{\mathsf{E}}}.$$

Finally, we show that $\mathcal{S}_{\mathsf{E}}$ is maximal w.r.t. Y_{E}^*. Let $\lambda > 0$ and $g \in Y_{\mathsf{E}}^* \subset Y^*$. Since $\mathcal{S}$ is maximal w.r.t. Y^* by Lemma 4.2, we infer that for each sufficiently large λ there is a $y \in Y$ s.t.

$$(\lambda + \mathcal{S})y = g.$$

Now set $\widetilde{y} = P_{\mathsf{E}} y \in Y_{\mathsf{E}}$. Then we find with (4.12) that

$$(\lambda + \mathcal{S}_{\mathsf{E}})\widetilde{y} = (\lambda + \mathcal{S}_{\mathsf{E}})P_{\mathsf{E}} y = P_{\mathsf{E}}(\lambda + \mathcal{S})y = P_{\mathsf{E}} g = g,$$

where we used $P_{\mathsf{E}} x = x$ for $x \in Y_{\mathsf{E}}^*$ in the last step. Hence $\mathcal{S}_{\mathsf{E}}$ is maximal w.r.t. Y_{E}. □

The previous Lemma and Theorem 2.5 show that the part of $\mathcal{S}_{\mathsf{E}}$ in X_{E} generates a contraction semigroup on X_{E}. Together with (4.12), this implies that the solution $\big[u, u'\big]^{\mathsf{T}}$ of the original problem (4.2) has non-increasing X_{E}-norm.

Corollary 4.7. *Let Assumption 4.4 and the assumptions of Theorem 4.3 be fulfilled. Then the solution u of* (4.2) *satisfies*

$$\Big(|u(t)|_a^2 + \|u'(t)\|_m^2\Big)^{1/2} \leq \Big(|u^0|_a^2 + \|v^0\|_m^2\Big)^{1/2} + t\|f\|_{L^\infty(0,t;H)}$$

for $|\cdot|_a^2 := a\big(\cdot, \cdot\big)$ *and* $t \in [0, T]$.

Proof. Now let $S_{\mathsf{E}} \colon D(S_{\mathsf{E}}) \to X_{\mathsf{E}}$ be the part of $\mathcal{S}_{\mathsf{E}}$ in X_{E} with $D(S_{\mathsf{E}}) = \big\{x \in Y_{\mathsf{E}} \mid \mathcal{S}_{\mathsf{E}} x \in X_{\mathsf{E}}\big\}$. Since $\mathcal{S}_{\mathsf{E}} P_{\mathsf{E}} x = P_{\mathsf{E}} \mathcal{S} x \in X_{\mathsf{E}}$ for $x \in D(S)$ by (4.12), it follows that $P_{\mathsf{E}} D(S) \subset D(S_{\mathsf{E}})$ and

$$P_{\mathsf{E}} S x = P_{\mathsf{E}} \mathcal{S} x = \mathcal{S}_{\mathsf{E}}(P_{\mathsf{E}} x) = S_{\mathsf{E}} P_{\mathsf{E}} x, \qquad x \in D(S). \tag{4.13}$$

In particular, we find that $x_{\mathsf{E}}^0 = P_{\mathsf{E}} x^0 \in D(S_{\mathsf{E}})$ and $g \in C^1([0, T]; X_{\mathsf{E}})$ or $g \in C([0, T]; [D(S_{\mathsf{E}})])$. Theorem 2.5 then states that (4.11) has a unique solution $x_{\mathsf{E}} = \big[\widetilde{u}, v\big]^{\mathsf{T}}$ which satisfies

$$\Big(|\widetilde{u}(t)|_a^2 + \|v(t)\|_m^2\Big)^{1/2} \leq \Big(|P_M u^0|_a^2 + \|v^0\|_m^2\Big)^{1/2} + t\|f\|_{L^\infty(0,t;H)}. \tag{4.14}$$

Now let $x = \big[u, v\big]^{\mathsf{T}}$ be the solution of the first-order in time formulation (2.7). To see that $P_{\mathsf{E}} x$ also solves (4.11), we use that $P_{\mathsf{E}} \in \mathcal{L}(X, X_{\mathsf{E}})$ and apply (4.13)

$$(P_{\mathsf{E}} x)'(t) + S_{\mathsf{E}}(P_{\mathsf{E}} x)(t) = P_{\mathsf{E}}\big(x'(t) + Sx(t)\big) = P_{\mathsf{E}} g(t) = g(t),$$

where $P_{\mathsf{E}} g(t) = g(t)$ since $g = \big[0, f\big]^{\mathsf{T}}$. We conclude that $P_{\mathsf{E}} x = x_{\mathsf{E}}$ is the unique solution of (4.11) and note that this is equivalent to $\widetilde{u} = P_M u$ and $v = u'$. Finally, we obtain from Assumption 4.4 (iii)

$$|\widetilde{u}|_a^2 = |P_M u|_a^2 = a\big(P_M u, P_M u\big) = a\big(P_M u, u\big) = a\big(u, u\big) = |u|_a^2. \tag{4.15}$$

To derive the desired stability estimate, we then insert $v = u'$ and (4.15) into (4.14). □

Example 4.8. *Let u be the solution of the wave equation with homogeneous Neumann boundary conditions* (1.1)*. Then the stability estimate from Theorem 4.3 grows exponentially fast in time with $c_{\mathsf{qm}} = c_\Omega^{1/2}/2$, while Corollary 4.7 implies that*

$$c_\Omega \|\nabla u(t)\|^2_{L^2(\Omega)} + \|u'(t)\|^2_{L^2(\Omega)} \leq c_\Omega \|\nabla u^0\|^2_{L^2(\Omega)} + \|v^0\|^2_{L^2(\Omega)}, \qquad t \geq 0$$

for sufficiently smooth initial values.

4.4 Weak solutions

Motivation We expect that the error of a spatial semi-discretization converges with a higher rate if it is measured in a weaker norm. Consider for example a smooth solution u of the scalar acoustic wave equation with homogeneous Dirichlet boundary conditions. It is known that the finite element solution u_h with piecewise polynomials of degree $p \geq 1$ approximates u with

$$\|u - u_h\|_{H^1(\Omega)} = \mathcal{O}(h^p) \quad \text{and} \quad \|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{p+1}),$$

uniformly in $t \in [0, T]$, cf. [Fujita et al., 2001, Sect. 2.8] for both estimates and [Dupont, 1973], [Baker and Bramble, 1979] for L^2-estimates. While the first estimate typically follows from an error bound in the X-norm, the L^2-estimate requires different stability estimate.

Goal In this section, we derive stability estimates for second-order wave-type problems in weaker norms by using Sobolev towers. These stability estimates are necessary to derive error bounds in the L^2-norm. The corresponding error analysis is the content of ongoing work.

Intermezzo: Sobolev towers

A Sobolev tower is a sequence of densely embedded spaces $(X_n)_{n\in\mathbb{Z}}$, $X_n \overset{d}{\hookrightarrow} X_{n-1}$. It is constructed via an invertible operator $S\colon D(S) \subset X \to X$ such that a restriction (or extension) of S to X_n generates a C_0-semigroup on X_n, if S generates a C_0-semigroup in $X_0 := X$.

We define the "next weaker" space X_{-1} to X, cf. [Engel et al., 1999, Def. II.5.4].

Definition 4.9. *Let $S\colon D(S) \to X$ be a densely defined, invertible linear operator and*

$$\|x\|_{-1} := \|S^{-1}x\|_X, \qquad x \in X.$$

The space X_{-1} is defined by

$$X_{-1} := \big(X, \|\cdot\|_{-1}\big)^{\sim},$$

where the superscript $(\cdot)^{\sim}$ denotes the completion of the space.

Remark 4.10. Note that the completion of X is only unique up to an isometric isomorphism, cf. [Amann, 1995, V.1.3] (where Sobolev towers are called Banach scales).

Since $\|Sx\|_{-1} = \|x\|_X$ for $x \in D(S)$ and $D(S)$ is dense in X, there exists a unique continuous extension $S_{-1} \in \mathcal{L}(X, X_{-1})$ of S. We use the same notation for the unbounded operator on X_{-1}

$$S_{-1}\colon D(S_{-1}) \subset X_{-1} \to X_{-1}, \qquad D(S_{-1}) := X.$$

The next result follows from [Engel et al., 1999, Thm. II.5.5] and it is the reason why we use Sobolev towers.

Theorem 4.11. *Let $S\colon D(S) \to X$ be invertible and the infinitesimal generator of a contraction semigroup on X. Then the operator $S_{-1}\colon D(S_{-1}) \to X_{-1}$, $D(S_{-1}) = X$ is the infinitesimal generator of a contraction semigroup on X_{-1}.*

Weak solutions of second-order wave-type problems

We now apply the theory of Sobolev towers to the second-order wave-type problem (4.2). To benefit from Theorem 4.11, we first relate X_{-1} to the Hilbert space

$$W := (H \times \widetilde{V}^*, \|\cdot\|_W), \qquad \|x\|_W^2 := \|u\|_m^2 + \|v\|_{\widetilde{V}^*}^2. \tag{4.16}$$

LEMMA 4.12. *Let Assumption 4.1 be satisfied with $c_G = 0$. Then $S = \mathcal{S}_{|X}$ is invertible and there exists a continuous isometry $\widetilde{T} \in \mathcal{L}(X_{-1}, W)$ s.t.*

$$\|x\|_{-1}^2 = \|\widetilde{T}x\|_W^2 = \|u\|_m^2 + \|\mathcal{B}u + v\|_{\widetilde{V}^*}^2, \qquad x = \big[u, v\big]^\mathsf{T} \in X. \tag{4.17}$$

Before we turn to the proof note that a is coercive and that $\widetilde{V}$ is equipped with $\widetilde{a} = a$, since $c_G = 0$. Hence we have $\langle \mathcal{A}u, v\rangle_V = a\big(u, v\big) = \widetilde{a}\big(u, v\big)$ for $u, v \in V$ which implies that $\mathcal{A} \colon \widetilde{V} \to \widetilde{V}^*$ is the Riesz isomorphism with

$$\|v\|_{\widetilde{V}^*} = \big\|\mathcal{A}^{-1}v\big\|_{\widetilde{a}}, \qquad v \in \widetilde{V}^*.$$

Proof. To prove that S is invertible, we use that $\mathcal{A}$ is invertible with $\mathcal{A}^{-1} \in \mathcal{L}(V^*, V)$. Hence the inverse of $\mathcal{S} \in \mathcal{L}(Y, Y^*)$ is given by

$$\mathcal{S}^{-1} = \begin{bmatrix} 0 & -\mathsf{I} \\ \mathcal{A} & \mathcal{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{A}^{-1}\mathcal{B} & \mathcal{A}^{-1} \\ -\mathsf{I} & 0 \end{bmatrix} = \mathcal{L}(Y^*, Y). \tag{4.18}$$

Therefore $S^{-1} = \mathcal{S}^{-1}\big|_X$, since

$$\begin{aligned} &\text{for } x \in X \subset Y^* && x = \mathcal{S}\mathcal{S}^{-1}x = S\mathcal{S}^{-1}x \\ &\text{and for } x \in D(S) \subset X && x = \mathcal{S}^{-1}\mathcal{S}x = \mathcal{S}^{-1}Sx. \end{aligned}$$

To verify (4.17), we use Definition 4.9 and find that for $x = \big[u, v\big]^\mathsf{T} \in X$

$$\|x\|_{-1}^2 = \|S^{-1}x\|_X^2 = \|\mathcal{S}^{-1}x\|_X^2 = \|\mathcal{A}^{-1}(\mathcal{B}u + v)\|_{\widetilde{a}}^2 + \|u\|_m^2 = \|\mathcal{B}u + v\|_{\widetilde{V}^*}^2 + \|u\|_m^2$$

where we used that $\mathcal{A} \colon \widetilde{V} \to \widetilde{V}^*$ is the Riesz isomorphism. The operator

$$T := \begin{bmatrix} \mathsf{I} & 0 \\ \mathcal{B} & \mathsf{I} \end{bmatrix} : X \to W$$

allows us to write this equivalently as $\|x\|_{-1}^2 = \|Tx\|_W^2$ for $x \in X$. This yields (4.17). Finally there exists a unique isometric extension $\widetilde{T} \in \mathcal{L}(X_{-1}, W)$ of T, since X is dense in X_{-1}. □

The general results for second-order wave-type problems Having this Lemma at our disposal, we are now able to characterize second-order wave-type problems in X_{-1}.

THEOREM 4.13. *Let Assumption 4.1 be satisfied with $c_G = \beta_{\mathsf{qm}} = 0$. For initial values $u^0 \in V$, $v^0 \in H$ and source term $f \in C^1([0, T]; V^*) + C([0, T]; H)$*

$$\big(u'(t) + \mathcal{B}u(t)\big)' + \mathcal{A}u(t) = f(t) \quad \text{for } t \in [0, T], \tag{4.19a}$$

$$u(0) = u^0,\ u'(0) = v^0 \tag{4.19b}$$

has a unique solution $u \in C^1([0, T]; H) \cap C([0, T]; V)$ which satisfies $u' + \mathcal{B}u \in C^1([0, T]; V^*)$ and

$$\Big(\|u(t)\|_m^2 + \|u'(t) + \mathcal{B}u(t)\|_{\widetilde{V}^*}^2\Big)^{1/2} \le \Big(\|u^0\|_m^2 + \|v^0 + \mathcal{B}u^0\|_{\widetilde{V}^*}^2\Big)^{1/2} + t\|f\|_{L^\infty(0,t;\widetilde{V}^*)} \tag{4.20}$$

for $t \in [0, T]$.

Proof. There exists a unique weak solution $w \in C^1([0,T];X_{-1}) \cap C([0,T];X)$ of

$$w'(t) + S_{-1}w(t) = g(t), \qquad w(0) = x^0, \tag{4.21}$$

since standard results, cf. [Pazy, 1992, Sect. 4.2], from semigroup theory apply: First, S_{-1} is the generator of a contraction semigroup on X_{-1} by Theorem 4.11, since S is monotone by Lemma 4.2 and therefore generates a contraction semigroup on X. Second, $x^0 \in X = D(S_{-1})$ by assumption. Third, $g = \big[0, f\big]^\intercal$ satisfies

$$\begin{aligned} f \in C^1([0,T];\widetilde{V}^*) \quad &\iff \quad g \in C^1([0,T];W) \\ &\iff \quad g = \widetilde{T}g \in C^1([0,T];X_{-1}) \end{aligned}$$

or

$$f \in C([0,T];H) \quad \iff \quad g \in C([0,T];X).$$

Next, we characterize the evolution equation behind (4.21). Applying $\widetilde{T}$ to (4.21), we obtain from $\widetilde{T}w'(t) = (Tw)'(t)$ that

$$(Tw)'(t) + \widetilde{T}S_{-1}w(t) = g(t). \tag{4.22}$$

To see that (4.22) corresponds to (4.19a), we study the terms on the left hand side. First, a short computation yields that for $y = \big[\varphi, \psi\big]^\intercal \in D(S)$

$$\widetilde{T}S_{-1}y = TSy = \begin{bmatrix} \mathsf{I} & 0 \\ \mathcal{B} & \mathsf{I} \end{bmatrix} \begin{bmatrix} 0 & -\mathsf{I} \\ \mathcal{A} & \mathcal{B} \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & -\mathsf{I} \\ \mathcal{A} & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ \mathcal{A}\varphi \end{bmatrix}. \tag{4.23}$$

Thus we find that $\widetilde{T}S_{-1}w = \big[-v, \mathcal{A}u\big]^\intercal$ for $\big[u, v\big]^\intercal = w \in X$ by continuity. Second, we have that $Tw = \big[u, \mathcal{B}u + v\big]^\intercal$. Thus the first equation of (4.22) gives $v = u'$ and the second equation of (4.22) gives (4.19a). Finally, u solves (4.19), since $u \in C([0,T];V)$, $u' = v \in C([0,T];H)$ follows from $w \in C([0,T];X)$ and $u' + \mathcal{B}u \in C^1([0,T];\widetilde{V}^*)$ follows from $\widetilde{T}w \in C^1([0,T];W)$.

For the proof of the stability estimate (4.20), observe that w satisfies (2.8) with $\|\cdot\|_{-1}$ instead of $\|\cdot\|_X$ and $c_{\mathsf{qm}} = 0$. Using $w(t) \in X$ and (4.17), we can write this as

$$\|Tw(t)\|_W \le \|Tx^0\|_W + t\|g\|_{L^\infty(0,t;W)}.$$

To obtain (4.20), we then insert $w = \big[u, u'\big]^\intercal$, $Tw = \big[u, u' + \mathcal{B}u\big]^\intercal$ and $g = \big[0, f\big]^\intercal$.

Finally, to prove uniqueness of the solution u of (4.19), it suffices to argue that the solution corresponding to $u^0 = v^0 = 0$, $f = 0$ is identically zero. To that end, we show that $x(t) := \big[u(t), u'(t)\big]^\intercal$ solves (4.21):

First, note that then $Tx \in C^1([0,T];W)$ and $x \in C([0,T];X)$ by assumption and that $\widetilde{T}(X_{-1})$ is a closed subspace of W due to T being isometric. Hence

$$W_0 := \Big(\widetilde{T}(X_{-1}), \|\cdot\|_W\Big)$$

is a Hilbert space and T is invertible on W_0 with $\widetilde{T}^{-1} \in \mathcal{L}(W_0, X_{-1})$. Since $Tx \in C([0,T];W_0)$ by definition and $Tx \in C^1([0,T];W)$ by assumption, it follows that $Tx \in C^1([0,T];W_0)$. Applying $\widetilde{T}^{-1}$, we see that x belongs to $C^1([0,T];X_{-1})$.

Second, note that $\big(Tx\big)' = \big(\widetilde{T}x\big)' = \widetilde{T}x'$, since $x \in C^1([0,T];X_{-1}) \cap C([0,T];X)$ and $\widetilde{T} \in \mathcal{L}(X_{-1}, W_0)$. Thus it follows from (4.19a), (4.23), and $x(t) \in X$, that Tx satisfies

$$\widetilde{T}x'(t) = \big(Tx\big)'(t) = \begin{bmatrix} u'(t) \\ \big(u'(t) + \mathcal{B}u(t)\big)' \end{bmatrix} = \begin{bmatrix} u'(t) \\ -\mathcal{A}u(t) \end{bmatrix} = -\widetilde{T}S_{-1}x(t). \tag{4.24}$$

Applying $\widetilde{T}^{-1}$ to (4.24), reveals that x solves (4.21) with $g = 0$ and $x^0 = 0$ which implies that $x(t) = 0$ thus also $u(t) = 0$. This was the claim.

□

A direct estimate in the W-norm A direct stability estimate for $\|x\|_W$ does, presumably, not exist for general second-order wave-type problems. However, if we assume that $\mathcal{B} \in \mathcal{L}(H, V^*)$, i.e.,

$$b(u, \varphi) \leq \|\mathcal{B}\|_{V^* \leftarrow H} \|u\|_m \|\varphi\|_V, \qquad u \in H,\ \varphi \in V,$$

then the following result yields a stability estimate which directly bounds $\|x\|_W$.

COROLLARY 4.14. *Let Assumption 4.1 be satisfied with $c_G = \beta_{\mathsf{qm}} = 0$ and $\mathcal{B} \in \mathcal{L}(H, V^*)$. For initial values $u^0 \in V$, $v^0 \in H$ and source term $f \in C^1([0,T]; V^*) + C([0,T]; H)$, the second-order wave-type problem* (4.2) *has a unique solution $u \in C^2([0,T]; \widetilde{V}^*) \cap C^1([0,T]; H) \cap C([0,T]; V)$ which satisfies*

$$\Big(\|u(t)\|_m^2 + \|u'(t)\|_{\widetilde{V}^*}^2\Big)^{1/2} \leq C \left(\Big(\|u^0\|_m^2 + \|v^0\|_{\widetilde{V}^*}^2\Big)^{1/2} + t\|f\|_{L^\infty(0,t;\widetilde{V}^*)} \right) \tag{4.25}$$

for $t \in [0, T]$.

Proof. Theorem 4.13 shows that there exists a unique solution $u \in C^1([0,T]; H) \cap C([0,T]; V)$ of (4.19). Furthermore, we have by $u' + \mathcal{B}u \in C^1([0,T]; V^*)$ that

$$u' = \big(u' + \mathcal{B}u\big) - \mathcal{B}u \in C^1([0,T]; V^*)$$

and thus u belongs to $C^2([0,T]; \widetilde{V}^*)$. Furthermore, u satisfies (4.19a) and therefore also (4.2), since

$$u''(t) + \mathcal{A}u(t) + \mathcal{B}u'(t) = \big(u'(t) + \mathcal{B}u(t)\big)' + \mathcal{A}u(t) = f(t), \qquad t \in [0, T].$$

Finally, if $X_{-1} \simeq W$, then (4.25) follows from the stability estimate in X_{-1}.

Therefore it remains to show $X_{-1} \simeq W$. Let $x = \big[u, v\big]^\mathsf{T} \in X$. Then we have by (4.17) that

$$\|x\|_{-1}^2 = \|\mathcal{B}u + v\|_{\widetilde{V}^*}^2 + \|u\|_m^2 \leq \Big(\|\mathcal{B}\|_{\widetilde{V}^* \leftarrow H}^2 + 1\Big) \left(\|u\|_m^2 + \|v\|_{\widetilde{V}^*}^2\right) \leq C\|x\|_W^2.$$

On the other hand, $\mathcal{S} \in \mathcal{L}(W, X)$, since

$$\begin{aligned} \|\mathcal{S}x\|_W^2 = \|v\|_m^2 + \|\mathcal{A}u + \mathcal{B}v\|_{\widetilde{V}^*}^2 &\leq \|v\|_m^2 + 2\|\mathcal{A}u\|_{\widetilde{V}^*}^2 + 2\|\mathcal{B}v\|_{\widetilde{V}^*}^2 \\ &\leq \|v\|_m^2 + 2\|u\|_{\widetilde{a}}^2 + 2\|\mathcal{B}\|_{\widetilde{V}^* \leftarrow H}^2 \|v\|_m^2 \\ &\leq C\|x\|_X^2, \end{aligned}$$

where we used that $\mathcal{A}\colon \widetilde{V} \to \widetilde{V}^*$ is the Riesz isomorphism and $\mathcal{B} \in \mathcal{L}(H, V^*)$. Together with $\mathcal{S}^{-1}x = S^{-1}x$, this yields

$$\|x\|_W = \|\mathcal{S}\mathcal{S}^{-1}x\|_W \leq C\|S^{-1}x\|_X = C\|x\|_{-1}.$$

Hence we showed that there exist constants $c_{W,X_{-1}}, C_{W,X_{-1}} > 0$ s.t. for each $x \in X$

$$c_{W,X_{-1}}\|x\|_W \leq \|x\|_{-1} \leq C_{W,X_{-1}}\|x\|_W.$$

Therefore $X_{-1} \simeq W$, since X is densely and continuously embedded in both, X_{-1} and W. □

REMARK 4.15. In situations where $\mathcal{B} \in \mathcal{L}(V, H)$, the bilinear form b often admits a strong and a weak formulation. Then $\widetilde{\mathcal{B}} \in \mathcal{L}(V, H)$ is a continuous extension of $\mathcal{B}$ which associated to the weak formulation of b. In this sense, the weak-wellposedness from Corollary 4.14 corresponds to the sub-case in Theorem 4.3.

4.5 Further topics and literature

Theory of linear second-order evolution equations Our well-posedness result Theorem 4.3 is based on semigroup theory. Alternatively, one can show the existence of unique solutions with a Faedo-Galerkin approach, cf. [Lions and Magenes, 1972, Thm. 3.8.1] for $b = 0$ and [Zeidler, 1990b, Thm. 33.A] for non-linear and non-autonomous problems. Note that for autonomous problems with smooth coefficients, mild solutions (from semigroup theory) coincide with Faedo-Galerkin solutions, cf. [Banks, 2012, Thm. 7.5]. Pure second-order problems can also be treated with so-called cosine operator functions. The seminal work in this direction is [Fattorini, 1985] which was already published in 1985. For a more recent contribution, cf. [Arendt et al., 2011, Sect. 3.14].

Theory of non-linear second-order evolution equations We hope that this work will be the basis for a more general theory that considers also non-linear second-order evolution equations. The issue of well-posedness of such problems is tackled in [Zeidler, 1990b], [Emmrich and Thalhammer, 2010] and [Roubíček, 2013]. A theory for non-linear acoustic boundary conditions is provided by [Graber, 2012].

Weak solutions with semigroups [Kato, 1985, Sect. II.10.2] consider weak solution of second-order evolution equation in a setting similar to the one we used. An approach to negative norm estimates for space-time FEM can be found in [Bales and Lasiecka, 1995].

Identification of boundary conditions [Showalter, 1994, Thm. VI.2.4] and [Graber, 2012, Appendix] use trace operators, defined via an abstract Green's theorem, to extract the bulk-pde and the boundary condition from the abstract problem. This is the abstract version of what we do in Chapter 6 to prove that a variational solution solves the corresponding partial differential equation.

Chapter 5

Error analysis for second-order wave-type problems

In this chapter, we analyze general non-conforming space discretizations of second-order wave-type problems. To apply Theorem 2.9, we formulate them as a space discretization of the first-order Cauchy problem corresponding to the second-order wave-type problem. We then use the structure of the operator matrix to derive a priori errror bounds in terms of data and approximation errors.

Outline We describe the semi-discrete problem in Section 5.1 and prove a prior error bounds in Section 5.2. As shown in Section 5.3, these results readily lead to convergence rates for Lagrange finite elements with mass lumping. Last, we investigate the convergence of the full discretization with the Crank–Nicolson method in Section 5.4.

5.1 Space discretization

This chapter is dedicated to general non-conforming space discretizations of (4.2) that yield an approximation $u_h\colon [0,T] \to V_h$ in the finite dimensional vector space V_h determined by

$$m_h\big(u_h''(t), \varphi_h\big) + b_h\big(u_h'(t), \varphi_h\big) + a_h\big(u_h(t), \varphi_h\big) = m_h\big(f_h(t), \varphi_h\big) \qquad \forall \varphi_h \in V_h, \tag{5.1a}$$

$$u_h(0) = u_h^0, \qquad u_h'(0) = v_h^0. \tag{5.1b}$$

Here $u_h^0, v_h^0 \in V_h$, $f_h\colon [0,T] \to V_h$, and m_h, a_h, $b_h\colon V_h \times V_h \to \mathbb{R}$ are the discrete counterparts of u_h^0, v_h^0, f, and m, a, b, respectively. This ansatz covers a wide range of non-conforming space discretizations, since we do *not* assume that V_h is a subspace of V. Analogously to Section 2.4, we assume that there exists a lift operator

$$Q_h^V\colon V_h \to V,$$

which yields an approximation $u \approx Q_h^V u_h \in V$ of the exact solution u of (4.2).

Formulation as differential equation Let $m_h\colon V_h \times V_h \to \mathbb{R}$ be the inner product of the Hilbert space $H_h \coloneqq (V_h, m_h)$ and denote its norm by $\|\cdot\|_{m_h}$. Using the operators

$$A_h\colon H_h \to H_h, \qquad m_h\big(A_h u_h, \varphi_h\big) = a_h\big(u_h, \varphi_h\big), \qquad u_h,\ \varphi_h \in V_h,$$

and

$$B_h\colon H_h \to H_h, \qquad m_h\big(B_h u_h, \varphi_h\big) = b_h\big(u_h, \varphi_h\big), \qquad u_h,\ \varphi_h \in V_h,$$

we can express the variational problem (5.1) as the second-order differential equation

$$u_h''(t) + B_h u_h'(t) + A_h u_h(t) = f_h(t), \qquad u_h(0) = u_h^0, \quad u_h'(0) = v_h^0.$$

Stability For the error analysis of the semi-discretization (5.1), we additionally assume it is stable in the following sense.

ASSUMPTION 5.1 (Stability) The following conditions hold for $u_h, \varphi_h \in V_h$.

(i) The bilinear form a_h is monotone and we write $\|\varphi_h\|_{a_h} := a_h(\varphi_h, \varphi_h)^{1/2}$ for the induced semi-norm.

(ii) There is a constant $\widetilde{c}_G \geq 0$ s.t.

$$\widetilde{a}_h(u_h, \varphi_h) := a_h(u_h, \varphi_h) + \widetilde{c}_G m_h(u_h, \varphi_h) \tag{5.2a}$$

defines an inner product on V_h and we write $\widetilde{V}_h := (V_h, \widetilde{a}_h)$.

(iii) There is a constant $C_{m_h,\widetilde{a}_h} > 0$ independent of h s.t. $\|\varphi_h\|_{m_h} \leq C_{m_h,\widetilde{a}_h} \|\varphi_h\|_{\widetilde{a}_h}$.

(iv) There is a constant $\widetilde{\beta}_{\mathsf{qm}} \geq 0$ s.t. the bilinear form $b_h + \widetilde{\beta}_{\mathsf{qm}} m_h$ is monotone.

(v) There are constants $C_H \geq c_H > 0$ independent of h s.t.

$$c_H \|Q_h^V \varphi_h\|_m \leq \|\varphi_h\|_{m_h} \leq C_H \|Q_h^V \varphi_h\|_m. \tag{5.2b}$$

(vi) There are constants $C_V \geq c_V > 0$ independent of h s.t.

$$c_V \|Q_h^V \varphi_h\|_{\widetilde{a}} \leq \|\varphi_h\|_{\widetilde{a}_h} \leq C_V \|Q_h^V \varphi_h\|_{\widetilde{a}}. \tag{5.2c}$$

REMARK 5.2. Assumption 5.1 (ii) with $\widetilde{c}_G = 1$ and Assumption 5.1 (iii) with $C_{m_h,\widetilde{a}_h} = 1$ already follow from Assumption 5.1 (i).

Formulation in the framework of monotone operators Analogous to the continuous case, we write (5.1) as the first-order differential equation (2.9) with

$$x_h = \begin{bmatrix} u_h \\ u_h' \end{bmatrix}, \quad S_h = \begin{bmatrix} 0 & -\mathsf{I}_{V_h} \\ A_h & B_h \end{bmatrix}, \quad g_h = \begin{bmatrix} 0 \\ f_h(t) \end{bmatrix}. \tag{5.3}$$

in the Hilbert space $X_h = \widetilde{V}_h \times H_h$ endowed with the inner product

$$p_h([w_h, v_h]^\mathsf{T}, [\varphi_h, \psi_h]^\mathsf{T}) := \widetilde{a}_h(w_h, \varphi_h) + m_h(v_h, \psi_h). \tag{5.4a}$$

and norm

$$\|[w_h, v_h]^\mathsf{T}\|_{X_h}^2 := \widetilde{a}_h(w_h, w_h) + m_h(v_h, v_h). \tag{5.4b}$$

To compare the approximation with the exact solution, we define the lift operator $Q_h \colon X_h \to X$ as

$$Q_h \begin{bmatrix} w_h \\ v_h \end{bmatrix} := \begin{bmatrix} Q_h^V w_h \\ Q_h^V v_h \end{bmatrix}.$$

Note that one can also choose two different lifts for the components w_h and v_h. For the ease of presentation, we refrain from investigating this here.

Notation Before we turn to the error analysis, we introduce the necessary notation, cf. Section 2.4. Let the reference operator be given by $J_h = \big(J_h^V, J_h^H\big)$ and $I_h\colon Z^V \to V_h$ a continuous interpolation operator defined on the dense subspace Z^V of V. The adjoint lifts are characterized by

$$\begin{aligned} &Q_h^{H*}\colon H \to H_h, \qquad m_h\big(Q_h^{H*}u, \varphi_h\big) = m\big(u, Q_h^V \varphi_h\big), \qquad u \in H,\ \varphi_h \in V_h,\\ \text{and}\quad &Q_h^{V*}\colon V \to \widetilde{V}_h, \qquad \widetilde{a}_h\big(Q_h^{V*}u, \varphi_h\big) = \widetilde{a}\big(u, Q_h^V \varphi_h\big), \qquad u \in V,\ \varphi_h \in V_h, \end{aligned}$$

and we set $P_h^H \coloneqq Q_h^V Q_h^{H*}$, $P_h^V \coloneqq Q_h^V Q_h^{V*}$. For the orthogonal projections, we use the notation

$$\begin{aligned} &m\big((\mathsf{I} - \Pi_h^H)u, Q_h^V \varphi_h\big) = 0, \qquad u \in H,\ \varphi_h \in V_h,\\ \text{and}\quad &\widetilde{a}\big((\mathsf{I} - \Pi_h^V)u, Q_h^V \varphi_h\big) = 0, \qquad u \in V,\ \varphi_h \in V_h. \end{aligned}$$

From (2.25) with $X = \widetilde{V}$ and $X = H$, respectively, these operators satisfy the following error bounds

$$\|(\mathsf{I} - P_h^V)w\|_{\widetilde{a}} \le \|(\mathsf{I} - \Pi_h^V)w\|_{\widetilde{a}} + C_V \|\Delta\widetilde{a}(Q_h^{V*}w)\|_{\widetilde{V}_h^*}, \tag{5.5a}$$

$$\text{and}\quad \|(\mathsf{I} - P_h^H)w\|_m \le \|(\mathsf{I} - \Pi_h^H)w\|_m + C_H \|\Delta m(Q_h^{H*}w)\|_{H_h^*} \tag{5.5b}$$

for all $w \in Z^V$, where

$$\begin{aligned} &\Delta m\big(u_h, \varphi_h\big) \coloneqq m\big(Q_h^V u_h, Q_h^V \varphi_h\big) - m_h\big(u_h, \varphi_h\big)\\ \text{and}\quad &\Delta\widetilde{a}\big(u_h, \varphi_h\big) \coloneqq \widetilde{a}\big(Q_h^V u_h, Q_h^V \varphi_h\big) - \widetilde{a}_h\big(u_h, \varphi_h\big). \end{aligned}$$

Finally, since norms on finite vector dimensional spaces are equivalent, there is an $\varepsilon_h > 0$ s.t.

$$\varepsilon_h \|\varphi_h\|_{\widetilde{a}_h} \le \|\varphi_h\|_{m_h}, \qquad \varphi_h \in V_h. \tag{5.6}$$

REMARK 5.3. Let X_h be a function space which is based on a mesh $\mathcal{T}_h$ of Ω and consider a conforming finite element disretization with $\|\cdot\|_{m_h} \sim \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{\widetilde{a}_h} \sim \|\cdot\|_{H^1(\Omega)}$. The we have $\varepsilon_h \le Ch$ due to the inverse estimate from [Brenner and Scott, 2008, Lem. 4.5.3].

5.2 A priori error bounds

A bound of the remainder operator As a first step towards an a priori error estimate, we prove a bound for the remainder term $R_h = Q_h^* S - S_h J_h$.

LEMMA 5.4. *Let Assumption 5.1 be satisfied and* $x = \big[u, v\big]^\mathsf{T} \in Z \cap V \times V$ *s.t.* $\mathcal{A}u + \mathcal{B}v \in H$. *Then*

$$\begin{aligned} \|R_h x\|_{X_h} \le C\Big(&\|\Delta\widetilde{a}(Q_h^{V*}v)\|_{\widetilde{V}_h^*} + \|\Delta\widetilde{a}(Q_h^{V*}u)\|_{\widetilde{V}_h^*} + \|\Delta m(Q_h^{H*}u)\|_{H_h^*}\\ &+ \|(\mathsf{I} - \Pi_h^V)v\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^V)u\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^H)u\|_m\\ &+ \|(\mathsf{I} - Q_h^V J_h^H)v\|_{\widetilde{a}} + \varepsilon_h^{-1} \|(Q_h^{V*} - J_h^V)u\|_{\widetilde{a}_h}\\ &+ \max_{\|\psi_h\|_{m_h}=1} |b\big(v, Q_h^V \psi_h\big) - b_h\big(J_h^H v, \psi_h\big)|\Big). \end{aligned}$$

Proof. Recall that p and p_h denote the inner products on X and X_h respectively, and observe that we have

$$R_h \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -(Q_h^{V*} - J_h^H)v \\ Q_h^{H*}(\mathcal{A}u + \mathcal{B}v) - (A_h J_h^V u + B_h J_h^H v) \end{bmatrix}$$

by definition of $\mathcal{S}$, S_h and $R_h = Q_h^* S - S_h J_h$. To prove an estimate for

$$\|R_h x\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h\big(R_h x, y_h\big),$$

let $y_h = \big[\varphi_h, \psi_h\big]^\mathsf{T} \in X_h$ with $\|y_h\|_{X_h}^2 = \|\varphi_h\|_{\widetilde{a}_h}^2 + \|\psi_h\|_{m_h}^2 = 1$ and study

$$\begin{aligned} p_h\big(R_h x, y_h\big) &= -\widetilde{a}_h\big((Q_h^{V*} - J_h^H)v, \varphi_h\big) + m_h\big(Q_h^{H*}(\mathcal{A}u + \mathcal{B}v) - (A_h J_h^V u + B_h J_h^H v), \psi_h\big) \\ &= -\widetilde{a}_h\big((Q_h^{V*} - J_h^H)v, \varphi_h\big) + a\big(u, Q_h^V \psi_h\big) - a_h\big(J_h^V u, \psi_h\big) \\ &\quad + b\big(v, Q_h^V \psi_h\big) - b_h\big(J_h^H v, \psi_h\big). \end{aligned}$$

For the first term, we obtain with $\|\varphi_h\|_{\widetilde{a}_h} \le 1$ and $P_h^V = Q_h^V Q_h^{V*}$

$$\begin{aligned} \widetilde{a}_h\big((Q_h^{V*} - J_h^H)v, \varphi_h\big) &\le \|(Q_h^{V*} - J_h^H)v\|_{\widetilde{a}_h} \|\varphi_h\|_{\widetilde{a}_h} \\ &\le C_V\Big(\|(P_h^V - \mathsf{I})v\|_{\widetilde{a}} + \|(\mathsf{I} - Q_h^V J_h^H)v\|_{\widetilde{a}}\Big) \\ &\le C\Big(\|(\mathsf{I} - \Pi_h^V)v\|_{\widetilde{a}} + \|\Delta\widetilde{a}(Q_h^{V*}v)\|_{\widetilde{V}_h^*} + \|(\mathsf{I} - Q_h^V J_h^H)v\|_{\widetilde{a}}\Big), \end{aligned}$$

where we used the Cauchy–Schwarz inequality in $\widetilde{a}_h$, (5.2c), and (5.5a).

For the second term, we estimate with $\|\psi_h\|_{m_h} \le 1$

$$\begin{aligned} a\big(u, Q_h^V \psi_h\big) - a_h\big(J_h^V u, \psi_h\big) &\le |a\big(u, Q_h^V \psi_h\big) - a_h\big(Q_h^{V*} u, \psi_h\big)| + |a_h\big((Q_h^{V*} - J_h^V)u, \psi_h\big)| \\ &\le |\widetilde{a}\big(u, Q_h^V \psi_h\big) - \widetilde{a}_h\big(Q_h^{V*} u, \psi_h\big) - \big(c_G m\big(u, Q_h^V \psi_h\big) - \widetilde{c}_G m_h\big(Q_h^{V*} u, \psi_h\big)\big)| \\ &\quad + \|(Q_h^{V*} - J_h^V)u\|_{a_h} \|\psi_h\|_{a_h} \\ &\le \max\{c_G, \widetilde{c}_G\} |m_h\big((Q_h^{V*} - Q_h^{H*})u, \psi_h\big)| + \|(Q_h^{V*} - J_h^V)u\|_{\widetilde{a}_h} \|\psi_h\|_{\widetilde{a}_h} \\ &\le \max\{c_G, \widetilde{c}_G\} \|(Q_h^{V*} - Q_h^{H*})u\|_{m_h} + \varepsilon_h^{-1} \|(Q_h^{V*} - J_h^V)u\|_{\widetilde{a}_h}. \end{aligned}$$

where we used the definitions (4.5) and (5.2a) for $\widetilde{a}$ and $\widetilde{a}_h$, respectively, applied the Cauchy–Schwarz inequality for a_h together with $\|u\|_{a_h} \le \|u\|_{\widetilde{a}_h}$, and employed (5.6). We further estimate

$$\begin{aligned} \|(Q_h^{V*} - Q_h^{H*})u\|_{m_h} &\le C_H \|(P_h^V - P_h^H)u\|_m \\ &\le C_H\Big(C_{H,V}\alpha^{-1/2}\|(P_h^V - \mathsf{I})u\|_{\widetilde{a}} + \|(\mathsf{I} - P_h^H)u\|_m\Big) \\ &\le C_H C_{H,V}\alpha^{-1/2}\Big(\|(\mathsf{I} - \Pi_h^V)u\|_{\widetilde{a}} + C_V\|\Delta\widetilde{a}(Q_h^{V*}u)\|_{\widetilde{V}_h^*}\Big) \\ &\quad + C_H\Big(\|(\mathsf{I} - \Pi_h^H)u\|_m + C_H\|\Delta m(Q_h^{H*}u)\|_{H_h^*}\Big) \end{aligned}$$

with (5.2b) in the first, (4.6) in the second, and (5.5a) and (5.5b) in the third estimate.

Collecting these estimates yields the desired bound. □

We now state our a priori error bound for general non-conforming space discretizations of second-order wave-type problems. To prove it, we will express the results from Theorem 2.9 in terms of second-order wave-type problems and use the previous lemma.

THEOREM 5.5. *Let the assumptions of Theorem 4.3 be fulfilled and let u be the unique solution of* (4.2) *with $u \in C^2\big([0, T]; Z^V\big)$. Furthermore, let Assumption 5.1 be satisfied and let x_h be the solution of the semi-discrete problem* (5.1). *Then the lifted semi-discrete solution $Q_h^V u$ satisfies*

$$\|Q_h^V u_h(t) - u(t)\|_{\widetilde{a}} + \|Q_h^V u_h'(t) - u'(t)\|_m \le C e^{\widehat{c}_{\mathsf{qm}} t}\big(1 + t\big) \sum_{i=1}^{4} E_i$$

for $t \in [0,T]$*, where* C *is independent of* h *and* t*,* $\widehat{c}_{\mathsf{qm}} = \widetilde{c}_G C_{m_h,\widetilde{a}_h}/2 + \widetilde{\beta}_{\mathsf{qm}}$*, and*

$$E_1 := \|u_h^0 - Q_h^{V*}u^0\|_{\widetilde{a}_h} + \|v_h^0 - I_h v^0\|_{m_h} + \|f_h - Q_h^{H*}f\|_{L^\infty(H_h)}, \tag{5.7a}$$

$$E_2 := \|(\mathsf{I} - Q_h^V I_h)u\|_{\infty,\widetilde{V}} + \|(\mathsf{I} - Q_h^V I_h)u'\|_{\infty,\widetilde{V}} + \|(\mathsf{I} - Q_h^V I_h)u''\|_{L^\infty(H)}, \tag{5.7b}$$

$$E_3 := \|\Delta\widetilde{a}(Q_h^{V*}u)\|_{\infty,\widetilde{V}_h^*} + \|\Delta m(Q_h^{H*}u)\|_{L^\infty(H_h^*)} + \|\Delta\widetilde{a}(Q_h^{V*}u')\|_{L^\infty(\widetilde{V}_h^*)} + \|\Delta m(Q_h^{H*}u'')\|_{L^\infty(H_h^*)}, \tag{5.7c}$$

$$E_4 := \Big\| \max_{\|\psi_h\|_{m_h}=1} |b(u', Q_h^V\psi_h) - b_h(I_h u', \psi_h)| \Big\|_{L^\infty(0,T)}. \tag{5.7d}$$

Proof. Theorem 2.9 applies since (2.9) with (5.3) is stable on $X_h = \widetilde{V}_h \times H_h$ in the sense of Assumption 2.8: By Assumption 5.1 m_h, b_h, and a_h have the same properties as their continuous counterparts, and it follows as in the proof of Lemma 4.2 that S_h is maximal and quasi-monotone with $\widehat{c}_{\mathsf{qm}} = \widetilde{c}_G C_{m_h,\widetilde{a}_h}/2 + \widetilde{\beta}_{\mathsf{qm}}$. Moreover, Assumptions 5.1 (v) and 5.1 (vi) imply that the lift is stable in the sense of 2.8 (ii).

Thus the general first order error bound from Theorem 2.9 holds for $x = [u, u']^\mathsf{T}$ and $x_h = [u_h, u_h']^\mathsf{T}$ where u is the solution of (4.2) and u_h is the solution of (5.1). Since

$$\|Q_h^V u_h - u\|_{\widetilde{a}} + \|Q_h^V u_h' - u'\|_m \le \sqrt{2}\|Q_h x_h - x\|_X,$$

it remains to bound the single terms in (2.20).

Starting from (2.20), we choose the reference operator $J_h = (Q_h^{V*}, I_h) \in \mathcal{L}(V \times Z^V, V_h \times V_h)$. Then the errors in the initial values and in the source term (2.20a) are bounded by $\sqrt{2}E_1$.

Due to $J_h^V = Q_h^{V*}$, the first term in (2.20b) is bounded from above by

$$\begin{aligned}
\|(Q_h^* - J_h)x'\|_{X_h} &= \|(Q_h^{H*} - I_h)u''\|_{m_h} \\
&\le C_H\Big(\|(P_h^H - \mathsf{I})u''\|_m + \|(\mathsf{I} - Q_h^V I_h)u''\|_m\Big) \\
&\le C_H\Big(\|(\mathsf{I} - \Pi_h^H)u''\|_m + C_H\|\Delta m(Q_h^{H*}u'')\|_{H_h^*} + \|(\mathsf{I} - Q_h^V I_h)u''\|_m\Big),
\end{aligned}$$

where we used (5.2b), $P_h^H = Q_h^V Q_h^{H*}$, and (5.5b). To bound $\|R_h x\|_{X_h}$, we use the estimate from Lemma 5.4 which simplifies to

$$\begin{aligned}
\|R_h x\|_{X_h} \le C\Big(&\|\Delta\widetilde{a}(Q_h^{V*}u')\|_{\widetilde{V}_h^*} + \|\Delta\widetilde{a}(Q_h^{V*}u)\|_{\widetilde{V}_h^*} + \|\Delta m(Q_h^{H*}u)\|_{H_h^*} \\
&+ \|(\mathsf{I} - \Pi_h^V)u'\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^V)u\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^H)u\|_m + \|(\mathsf{I} - Q_h^V I_h)u'\|_{\widetilde{a}} \\
&+ \max_{\|\psi_h\|_{m_h}=1} |b(u', Q_h^V\psi_h) - b_h(I_h u', \psi_h)|\Big).
\end{aligned}$$

Finally, (5.5a) and $P_h^V = Q_h^V Q_h^{V*}$ yield that the reference error (2.20c) satisfies

$$\begin{aligned}
\|(\mathsf{I} - Q_h J_h)x\|_X &\le \|(\mathsf{I} - P_h^V)u\|_{\widetilde{a}} + \|(\mathsf{I} - Q_h^V I_h)u'\|_m Big) \\
&\le \|(\mathsf{I} - \Pi_h^V)u\|_{\widetilde{a}} + C_V\|\Delta\widetilde{a}(Q_h^{V*}u)\|_{\widetilde{V}_h^*} + \|(\mathsf{I} - Q_h^V I_h)u'\|_m.
\end{aligned}$$

The final estimate then follows from bounding the orthogonal projections errors by interpolation errors of $Q_h^V I_h$ and collecting terms. □

REMARK 5.6. Some terms in the error bound can be further estimated.

(i) Each term in data error E_1 can be estimated against an interpolation error. For the first term, we find with (5.2c)

$$\begin{aligned}
\|u_h^0 - Q_h^{V*}u^0\|_{\widetilde{a}_h} &\le \|u_h^0 - I_h u^0\|_{\widetilde{a}_h} + C_V\|Q_h^V(I_h - Q_h^{V*})u^0\|_{\widetilde{a}} \\
&\le \|u_h^0 - I_h u^0\|_{\widetilde{a}_h} + C_V\|(Q_h^V I_h - \mathsf{I})u^0\|_{\widetilde{a}} + C_V\|(\mathsf{I} - P_h^V)u^0\|_{\widetilde{a}}.
\end{aligned}$$

Using (5.5a) and the best approximation property of Π_h^V, we further estimate

$$\begin{aligned}\|(\mathsf{I} - P_h^V)u^0\|_{\widetilde{a}} &\le \|(\mathsf{I} - \Pi_h^V)u^0\|_{\widetilde{a}} + C_V\|\Delta\widetilde{a}(Q_h^{V*}u^0)\|_{\widetilde{V}_h^*} \\ &\le \|(\mathsf{I} - Q_h^V I_h)u^0\|_{\widetilde{a}} + C_V\|\Delta\widetilde{a}(Q_h^{V*}u^0)\|_{\widetilde{V}_h^*}.\end{aligned}$$

Altogether, we obtain

$$\|u_h^0 - Q_h^{V*}u^0\|_{\widetilde{a}_h} \le \|u_h^0 - I_h u^0\|_{\widetilde{a}_h} + 2C_V\|(\mathsf{I} - Q_h^V I_h)u^0\|_{\widetilde{a}} + C_V^2\|\Delta\widetilde{a}(Q_h^{V*}u^0)\|_{\widetilde{V}_h^*},$$

and, analogously,

$$\|f_h - Q_h^{H*}f\|_{m_h} \le \|f_h - I_h f\|_{m_h} + 2C_H\|(\mathsf{I} - Q_h^V I_h)f\|_m + C_H^2\|\Delta m(Q_h^{H*}f)\|_{H_h^*},$$

if $f \in Z^V$.

(ii) If $\mathcal{B} \in \mathcal{L}(\widetilde{V}, H)$, then (5.2b) yields for $v \in \widetilde{V}$ and $\psi_h \in V_h$ with $\|\psi_h\|_{m_h} = 1$

$$\begin{aligned}|b(v, Q_h^V\psi_h) - b_h(I_h v, \psi_h)| &\le |b((\mathsf{I} - Q_h^V I_h)v, Q_h^V\psi_h)| + |b(I_h v, Q_h^V\psi_h) - b_h(I_h v, \psi_h)| \\ &\le |\langle \mathcal{B}(\mathsf{I} - Q_h^V I_h)v, Q_h^V\psi_h\rangle_V| + |\Delta b(I_h v, \psi_h)| \\ &\le \|\mathcal{B}\|_{H\leftarrow\widetilde{V}}\|(\mathsf{I} - Q_h^V I_h)v\|_{\widetilde{a}}\|Q_h^V\psi_h\|_m + |\Delta b(I_h v, \psi_h)| \\ &\le \|\mathcal{B}\|_{H\leftarrow\widetilde{V}}\|(\mathsf{I} - Q_h^V I_h)v\|_{\widetilde{a}}c_H^{-1} + |\Delta b(I_h v, \psi_h)|.\end{aligned}$$

Therefore, E_4 is bounded by

$$E_4 \le C\Big(\|(\mathsf{I} - Q_h^V I_h)u'\|_{L^\infty(\widetilde{V})} + \|\Delta b(I_h u')\|_{L^\infty(H_h^*)}\Big).$$

We call the discretization (5.1) conforming, if

$$V_h \subset V, \qquad Q_h^V = \mathsf{I}, \qquad \Delta m = 0, \qquad \Delta a = 0.$$

For conforming discretizations we state an error bound which is independent of u''.

Corollary 5.7. *Let* (5.1) *be a conforming discretization and consider the situation from Theorem 5.5. Then the semi-discrete solution u_h satisfies*

$$\begin{aligned}&\|u_h(t) - u(t)\|_{\widetilde{a}} + \|u_h'(t) - u'(t)\|_m \\ &\quad\le Ce^{c_{\mathsf{qm}}t}(1+t)\Big(\|u_h^0 - \Pi_h^V u^0\|_{\widetilde{a}_h} + \|v_h^0 - \Pi_h^H v^0\|_{m_h} + \|f_h - \Pi_h^H f\|_{L^\infty(H_h)} \\ &\qquad + \|(\mathsf{I} - I_h)u\|_{\infty,\widetilde{V}} + \|(\mathsf{I} - I_h)u'\|_{\infty,\widetilde{V}} + \varepsilon_h^{-1}\|(\mathsf{I} - I_h)u'\|_{L^\infty(H)} \\ &\qquad + \Big\|\max_{\|\psi_h\|_{m_h}=1}|b(u', \psi_h) - b_h(\Pi_h^H u', \psi_h)|\Big\|_{L^\infty(0,t)}\Big).\end{aligned}$$

for $t \in [0, T]$, where C is independent of h and t, and $c_{\mathsf{qm}} = \widetilde{c}_G C_{m_h,\widetilde{a}_h}/2 + \widetilde{\beta}_{\mathsf{qm}}$.

Proof. In comparison to the previous proof, there are only three changes:

First, $\widehat{c}_{\mathsf{qm}}$ is equal to c_{qm} from Lemma 4.2, since the discrete bilinear forms obey the same constants as their continuous counterparts. Second, we choose $J_h = \Pi_h = P_h$. Therefore the first term in (2.20b) completely vanishes. Third, since $Q_h^{H*} = \Pi_h^H$ and $Q_h^{V*} = \Pi_h^V$ for conforming methods, the estimate from Lemma 5.4 reads

$$\begin{aligned}\|R_h x\|_{X_h} \le C\Big(&\|(\mathsf{I} - \Pi_h^V)u'\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^V)u\|_{\widetilde{a}} + \|(\mathsf{I} - \Pi_h^H)u\|_m + \|(\mathsf{I} - \Pi_h^H)u'\|_{\widetilde{a}} \\ &+ \max_{\|\psi_h\|_{m_h}=1}|b(u', \psi_h) - b_h(\Pi_h^H u', \psi_h)|\Big).\end{aligned}$$

To bound the H-orthogonal projection error in the $\widetilde{V}$-norm, we then use (3.3) with $X = H$ and $Y = \widetilde{V}$ which gives

$$\|(\mathsf{I} - \Pi_h^H)u'\|_{\widetilde{a}} \le \|(\mathsf{I} - I_h)u'\|_{\widetilde{a}} + 2\varepsilon_h^{-1}\|(\mathsf{I} - I_h)u'\|_m.$$

For the final bound, we collect terms and estimate orthogonal projections errors by interpolation errors. □

5.3 Example: Finite elements for the acoustic wave equation

In this section, we show how the results from the previous section can be used to show convergence rates for a specific example: the acoustic wave equation and its space discretization with linear Lagrange finite elements.

We seek the solution $u : [0, T] \times \Omega \to \mathbb{R}$ of

$$u_{tt} - \operatorname{div}(c_\Omega \nabla u) = f \qquad \text{in } \Omega, \tag{5.8a}$$
$$u(t) = 0 \qquad \text{on } \Gamma, \tag{5.8b}$$
$$u(0) = u^0, \quad u_t(0) = v^0 \qquad \text{in } \Omega. \tag{5.8c}$$

Here, f is a given source term and $c_\Omega \in L^\infty(\Omega)^{d\times d}$ models the wave speed. We assume that $c_\Omega(\mathrm{x})$, $\mathrm{x} \in \Omega$ is be symmetric, and that there are $c_\Omega^+ \ge c_\Omega^- > 0$ s.t.

$$c_\Omega^- \|\xi\|^2 \le c_\Omega(\mathrm{x})\xi \cdot \xi \le c_\Omega^+ \|\xi\|^2 \qquad \text{for a.e. } \mathrm{x} \in \Omega \text{ and all } \xi \in \mathbb{R}^d.$$

We can write the variational formulation of (5.8) in the form of (4.2) making the following identifications. For the functional spaces we set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. As usual, the bilinear form a is given by

$$a(u, \varphi) := \int_\Omega c_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{dx} \qquad u, \varphi \in V$$

and b vanishes. Thus, Assumption 4.1 holds with $c_G = 0$ and we have $\widetilde{a} = a$. Hence we can apply Theorem 4.3 and get for suitable u^0, v^0 and f that there exists a unique solution of (5.8) with

$$u \in C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_0^1(\Omega))) \cap C([0, T]; [D(A)]),$$

where

$$D(A) = \Big\{u \in H_0^1(\Omega) \mid \operatorname{div}(c_\Omega \nabla u) \in L^2(\Omega)\Big\}.$$

The associated operator of the first-order in time formulation is skew-symmetric and the energy (4.9) is conserved, cf. [Engel et al., 1999, Thm. II.3.24].

For the spatial discretization we restrict us to linear finite elements for this exposition. Note, however, that higher order elements, could be handled as well. Assume that the mesh $\mathcal{T}_h$ is a triangulation of Ω. Let V_h be the space of linear finite elements on $\mathcal{T}_h$. Since V_h is a subspace of V, the lift operator $Q_h^V = \mathsf{I}$ is trivial.

First, we study finite elements with exact intetration. This means that we choose $m_h = m$ as the standard $L^2(\Omega)$ inner product and $a_h = a$. For this choice Assumption 5.1 holds trivially since $\Delta m = \Delta \widetilde{a} = 0$.

We can use the error bound from Corollary 5.7 to find

$$\|u_h(t) - u(t)\|_{\widetilde{a}} + \|u_h'(t) - u'(t)\|_m$$
$$\le C(1+t) \sup_{\tau\in[0,t]} \Big(\|(\mathsf{I} - I_h)u(\tau)\|_{\widetilde{a}} + \|(\mathsf{I} - I_h)u'(\tau)\|_{\widetilde{a}} + \varepsilon_h^{-1}\|(\mathsf{I} - I_h)u'(\tau)\|_m \Big).$$

For the standard nodal interpolation operator I_h it is known that

$$\|(\mathsf{I}-I_h)\varphi\|_m + h\|(\mathsf{I}-I_h)\varphi\|_{\widetilde{a}} \le Ch^2|\varphi|_{H^2(\Omega)}, \qquad \varphi \in H^2(\Omega)$$

cf. [Brenner and Scott, 2008, Sect. 4.4]. Overall, we find that the difference in the energy norm between the exact solution u of (5.8) and its corresponding FEM approximation u_h scales like h.

We next study the effect of numerical integration. The main difference to exact integration is, that now m_h and a_h differ from m and a, respectively. Applying Theorem 5.5 in this case, we have

$$\begin{aligned} &\|u_h(t)-u(t)\|_{\widetilde{a}} + \|u_h'(t)-u'(t)\|_m \\ &\quad\le C(1+t)\sup_{\tau\in[0,t]}\Big(\|u(\tau)-I_hu(\tau)\|_{\widetilde{a}} + \|u'(\tau)-I_hu'(\tau)\|_{\widetilde{a}} + \|u''(\tau)-I_hu''(\tau)\|_m + \\ &\qquad \|\Delta\widetilde{a}(Q_h^{V*}u(\tau))\|_{\widetilde{V}_h^*} + \|\Delta m(Q_h^{H*}u(\tau))\|_{H_h^*} + \|\Delta\widetilde{a}(Q_h^{V*}u'(\tau))\|_{\widetilde{V}_h^*} + \\ &\qquad \|\Delta m(Q_h^{H*}u'(\tau))\|_{H_h^*} + \|\Delta m(Q_h^{H*}u''(\tau))\|_{H_h^*}\Big). \end{aligned}$$

Hence we need to quantify the differences in the bilinear form. For example in the above setting one can use the d-dimensional trapezoidal rule to approximate the integrals. More precisely, let $\{\mathrm{x}_{K,j}\}_{j=1}^{d+1}$ be vertices of the element $K \in \mathfrak{T}_h$. Then m_h and a_h are given by the quadrature formulas

$$m_h(v,w) = \sum_{K\in\mathfrak{T}_h}\sum_{j=1}^{d+1}\frac{|K|}{d+1}v(\mathrm{x}_{K,j})w(\mathrm{x}_{K,j})$$

and respectively

$$\widetilde{a}_h(v,w) = \sum_{K\in\mathfrak{T}_h}\sum_{j=1}^{d+1}\frac{|K|}{d+1}c_\Omega(\mathrm{x}_{K,j})\nabla v(\mathrm{x}_{K,j})\cdot\nabla w(\mathrm{x}_{K,j}).$$

Under appropriate regularity assumption on c it is well-know that $\|\Delta\widetilde{a}(v_h)\|_{\widetilde{V}_h^*} \in \mathcal{O}(h)$ and $\|\Delta m(v_h)\|_{H_h^*} \in \mathcal{O}(h)$ for all $v_h \in V_h$, see e.g. [Ciarlet, 2002, Section 4.1]. Inserting this into the a priori bound above shows that there is no order reduction due to use of numerical quadrature.

REMARK 5.8. These results correspond to those of [Dupont, 1973], [Baker, 1976], and [Baker and Dougalis, 1976]. Numerical quadrature was only taken into acount in the latter references. The choice of the trapezoidal rule for linear finite elements leads to diagonal mass matrix. This is known as *mass lumping*. For further reference, see [Cohen, 2002, Chapters 11–13].

5.4 Full discretization with the Crank–Nicolson method

In this section, we consider the time integration of general non-conforming space discretizations with the Crank–Nicolson method. To obtain the fully discrete scheme, we apply (2.39) to the first-order in time formulation of (5.1). With $x_h^n := \left[u_h^n, v_h^n\right]^\mathsf{T}$ and S_h, g_h defined in (5.3), this leads to

$$\begin{bmatrix} u_h^{n+1} \\ v_h^{n+1}\end{bmatrix} = \begin{bmatrix} u_h^{n} \\ v_h^{n}\end{bmatrix} - \frac{\tau}{2}\begin{bmatrix} 0 & -\mathsf{I}_{V_h} \\ A_h & B_h\end{bmatrix}\begin{bmatrix} u_h^{n+1}+u_h^n \\ v_h^{n+1}+v_h^n\end{bmatrix} + \frac{\tau}{2}\begin{bmatrix} 0 \\ f_h^{n+1}+f_h^n\end{bmatrix}, \qquad n \ge 0. \tag{5.9}$$

The convergence result for $Q_h^V u_h^n \approx u(t_n)$, $t_n = n\tau$ is a direct consequence of Theorem 2.18 and our considerations in Theorem 5.5.

COROLLARY 5.9. *Let the assumptions of Theorem 5.5 be fulfilled and* $\tau\widehat{c}_{\mathsf{qm}} < 2$. *If the solution* u *suffices* $u \in C^4([0,T];H)\cap C^3([0,T];\widetilde{V})$, *then the lifted approximations* $Q_h^V u_h^n$ *and* $Q_h^V v_h^n$ *given by* (5.9) *satisfy*

$$\Big(\|Q_h^V u_h^n - u(t_n)\|_{\widetilde{a}}^2 + \|Q_h^V v_h^n - u'(t_n)\|_m^2\Big)^{1/2} \le Ce^{t_n\widehat{c}_{\mathsf{qm}}}t_n\tau^2E_5 + C(1+t_ne^{\widehat{c}_{\mathsf{qm}}t_n})\sum_{i=1}^4 E_i$$

for $t_n \in [0,T]$, where C is independent of h and t, $\widehat{c}_{\mathsf{qm}}$ and E_i, $i = 1,2,3,4$ are defined in (5.7)*, and*

$$E_5 := \|u^{(3)}\|_{L^\infty(\widetilde{V})} + \|u^{(4)}\|_{L^\infty(H)}.$$

Proof. Let $x := \left[u, u'\right]^\mathsf{T}$. Then x solves (2.7) in $X = \widetilde{V} \times H$ with $\mathcal{S}$ and g from (4.7). Furthermore, $u \in C^4([0,T];H) \cap C^3([0,T];\widetilde{V})$ implies $x \in C^3([0,T];X)$. By definition, x_h^n satisfies (2.39) with S_h and g_h defined in (5.3). Therefore Theorem 2.18 yields an error bound for $x_h^n = \left[u_h^n, v_h^n\right]^\mathsf{T}$. We derive the desired upper bound from

$$\|x^{(3)}\|_{L^\infty(0,t_n;X)} \leq \sqrt{2}\big(\|u^{(3)}\|_{L^\infty(0,t_n;\widetilde{V})} + \|u^{(4)}\|_{L^\infty(0,t_n;H)}\big)$$

and the estimates in the proof of Theorem 5.5 by choosing $J_h = \left(Q_h^{V*}, I_h\right)$. There we showed that each term in the bound for $\|Q_h x_h^n - x(t_n)\|_X$ is smaller than $E_1 + E_2 + E_3 + E_4$. □

Chapter 6

Analysis of wave equations with dynamic boundary conditions

In this chapter, we demonstrate how the theory for second-order wave-type problems provides well-posedness results for specific wave equations. We are particularly interested in the results for our two main examples: the wave equations with kinetic and acoustic boundary conditions.

A road-map Let us shortly sketch how we prove the well-posedness of a given wave equation. To formulate the partial differential equation as a second-order wave-type problem,

- we derive a variational formulation,
- we define the spaces H,V, the bilinear forms m, a, b, the source term f, and
- we show that Assumption 4.1 is satisfied.

This classifies the variational formulation as a second-order wave-type problem which is well-posed by Theorem 4.3 provided that

$$u^0, v^0 \in V \quad \text{s.t.} \quad \mathcal{A}u^0 + \mathcal{B}v^0 \in H \tag{6.1a}$$

$$\text{and} \quad f \in C^1\big([0,T];H\big) \quad \text{or} \quad \big[f, \mathcal{B}f\big]^\mathsf{T} \in C\big([0,T]; V \times H\big). \tag{6.1b}$$

In the second part of the proof,

- we verify if and in which sense the solution satisfies the original partial differential equation,
- we characterize the assumptions on the data (6.1) in terms of Sobolev spaces, and,
- we derive weak stability estimates by using the considerations from Section 4.4.

Outline In Section 6.1, we consider the wave equation with degenerate non-locally reacting kinetic boundary conditions and show that its variational formulation is a second-order wave-type problem. We use these preparatory results to show specific well-posedness results for the wave equation with Robin boundary conditions and the wave equation with kinetic boundary conditions in Section 6.2. We conclude this chapter in Section 6.3 with an analysis of the wave equation with acoustic boundary conditions.

6.1 Degenerate non-locally reacting kinetic boundary conditions

In this section, we consider a problem which covers a wide range of linear wave equations with various boundary conditions. By analyzing this problem, we can give substantial parts of the well-posedness proofs for these examples in a unified way. In comparison to the non-locally reacting kinetic boundary conditions from Section 1.2.2, the considered model also accounts for damping and advection effects. In addition, all coefficients but c_Ω may be degenerate, i.e., not uniformly positive.

The partial differential equation We seek a function $u\colon [0,T]\times\overline{\Omega}\to\mathbb{R}$ such that

$$u_{tt} + \big(\alpha_\Omega + \beta_\Omega\cdot\nabla\big)u_t - \operatorname{div}(\gamma_\Omega\nabla u_t) + a_\Omega u - \operatorname{div}(c_\Omega\nabla u) = f_\Omega \qquad \text{in } \Omega, \tag{6.2a}$$

$$\begin{aligned}\mu u_{tt} + \big(\alpha_\Gamma + \beta_\Gamma\cdot\nabla_\Gamma\big)u_t - \operatorname{div}_\Gamma(\gamma_\Gamma\nabla_\Gamma u_t) + a_\Gamma u - \operatorname{div}_\Gamma(c_\Gamma\nabla_\Gamma u)&\\ = -n\cdot c_\Omega\nabla u - \gamma_\Omega\partial_n u_t + f_\Gamma& \qquad \text{on } \Gamma,\end{aligned} \tag{6.2b}$$

where we assume the following.

Assumption 6.1

(i) The wave speed $c_\Omega \in L^\infty(\Omega)^{d\times d}$ is symmetric and uniformly elliptic, i.e., there exist $c_\Omega^+, c_\Omega^- > 0$ s.t.

$$c_\Omega^-|\xi|^2 \le c_\Omega(\mathrm{x})\xi\cdot\xi \le c_\Omega^+|\xi|^2 \qquad \text{for a.e. } \mathrm{x}\in\overline{\Omega} \text{ and all } \xi\in\mathbb{R}^d \tag{6.3}$$

(ii) $\alpha_\Omega, \gamma_\Omega, a_\Omega \in L^\infty(\Omega)$ are non-negative.

(iii) $\mu, \alpha_\Gamma, \gamma_\Gamma, c_\Gamma, a_\Gamma \in L^\infty(\Gamma)$ are non-negative,

(iv) $\beta_\Omega \in L^\infty(\Omega)^d$ is a vector field in Ω with $\operatorname{div}\beta_\Omega \in L^\infty(\Omega)$.

(v) $\beta_\Gamma \in L^\infty(\Gamma)^d$ is a vector field on Γ with $\operatorname{div}_\Gamma\beta_\Gamma \in L^\infty(\Gamma)$.

(vi) The source terms $f_\Omega\colon [0,T]\times\Omega\to\mathbb{R}$, $f_\Gamma\colon [0,T]\times\Gamma\to\mathbb{R}$ are functions.

Interpretation The coefficients α_Ω and α_Γ describe viscous damping and the coefficients γ_Ω and γ_Γ describe strong damping in the bulk and on the surface, respectively. Strong damping effects are important for applications as they can be used to approximate the behavior of non-Hookean materials under high strains. Moreover, advective (also convective) flows in the volume and the surface are modeled by the vector fields β_Ω and β_Γ, respectively, cf. [Campos, 2007, (W4)].

Degenerate coefficients Since all coefficients but c_Ω are only required to be non-negative, effects as wave propagation on the surface or damping may be absent. In particular, coefficients can vanish on parts of their domain. This allows us for example to impose Robin and kinetic boundary conditions on different parts of Γ.

Simplifying assumptions We make the following simplifying assumptions for this section to keep this discussion at a reasonable length.

Assumption 6.2 Let

(i) $\gamma_\Gamma, c_\Gamma \in W^{1,\infty}(\Gamma)$, $\gamma_\Omega \in W^{1,\infty}(\Omega)$, $\beta_\Omega \in W^{1,\infty}(\Omega)^d$ and $c_\Omega \in W^{1,\infty}(\Omega)^{d\times d}$,

(ii) Γ be a closed surface with $\Gamma \in C^2$,

(iii) $\beta_\Gamma(\mathrm{x})\cdot n(\mathrm{x}) = 0$ for $\mathrm{x}\in\Gamma$, and,

(iv) $\big(\operatorname{supp}\beta_\Gamma \cup \operatorname{supp}\gamma_\Gamma\big) \subset \operatorname{supp} c_\Gamma$.

In Remark 6.4, we discuss under which conditions these assumptions can be weakened.

Related literature Our approach on dynamic boundary conditions with degenerate coefficients is inspired by [Disser et al., 2015]. Kinetic boundary conditions with strong damping are for example considered in [Graber and Lasiecka, 2014] and [Graber and Shomberg, 2016].

Following our user's guide, we start with the derivation of a variational formulation of (6.2).

The variational formulation

Let u be a sufficiently smooth solution of (6.2). We multiply (6.2a) with a function $\varphi \in C^\infty(\overline{\Omega})$ and integrate over Ω. Applying Gauss' Theorem twice, we end up with

$$\int_\Omega u_{tt}\varphi \,\mathrm{dx} + \int_\Omega \alpha_\Omega u_t\varphi + \big(\beta_\Omega \cdot \nabla u_t\big)\varphi + \gamma_\Omega \nabla u_t \cdot \nabla\varphi \,\mathrm{dx} + \int_\Omega a_\Omega u\varphi + c_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{dx}$$
$$= \int_\Omega f_\Omega \varphi \,\mathrm{dx} + \int_\Gamma \big(n \cdot c_\Omega \nabla u + \gamma_\Omega \partial_n u_t\big)\varphi \,\mathrm{d}s.$$

The last term can be rewritten using the boundary condition (6.2b)

$$\int_\Gamma \big(n \cdot c_\Omega \nabla u + \gamma_\Omega \partial_n u_t\big)\varphi \,\mathrm{d}s$$
$$= -\int_\Gamma \Big(\mu u_{tt} + (\alpha_\Gamma + \beta_\Gamma \cdot \nabla_\Gamma)u_t - \mathrm{div}_\Gamma(\gamma_\Gamma \nabla_\Gamma u_t) + a_\Gamma u - \mathrm{div}_\Gamma(c_\Gamma \nabla_\Gamma u) - f_\Gamma\Big)\varphi \,\mathrm{d}s.$$

Finally, it follows from (4) that

$$-\int_\Gamma \mathrm{div}_\Gamma(\gamma_\Gamma \nabla_\Gamma u_t)\varphi + \mathrm{div}_\Gamma(c_\Gamma \nabla_\Gamma u)\varphi \,\mathrm{d}s = \int_\Gamma \gamma_\Gamma \nabla_\Gamma u_t \cdot \nabla_\Gamma \varphi + c_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \,\mathrm{d}s.$$

Putting all pieces together, we find

$$\int_\Omega u_{tt}\varphi \,\mathrm{dx} + \int_\Gamma \mu u_{tt}\varphi \,\mathrm{d}s$$
$$+ \int_\Omega \big(\alpha_\Omega u_t + \beta_\Omega \cdot \nabla u_t\big)\varphi + \gamma_\Omega \nabla u_t \cdot \nabla\varphi \,\mathrm{dx} + \int_\Gamma \big(\alpha_\Gamma u_t + \beta_\Gamma \cdot \nabla_\Gamma u_t\big)\varphi + \gamma_\Gamma \nabla_\Gamma u_t \cdot \nabla_\Gamma \varphi \,\mathrm{d}s$$
$$+ \int_\Omega a_\Omega u\varphi + c_\Omega \nabla u \cdot \nabla\varphi \,\mathrm{dx} + \int_\Gamma a_\Gamma u\varphi + c_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \,\mathrm{d}s$$
$$= \int_\Omega f_\Omega \varphi \,\mathrm{dx} + \int_\Gamma f_\Gamma \varphi \,\mathrm{d}s \qquad \forall \varphi \in C^\infty(\overline{\Omega}).$$

Hence each classical solution $u \in C^2\big(\overline{\Omega} \times [0,T]\big)$ of (6.2) satisfies the variational formulation

$$m\big(u_{tt}(t,\cdot),\varphi\big) + b\big(u_t(t,\cdot),\varphi\big) + a\big(u(t,\cdot),\varphi\big) = \langle f(t),\varphi\rangle \qquad \forall \varphi \in C^\infty(\overline{\Omega}) \tag{6.4}$$

where

$$m\big(w,\varphi\big) := \int_\Omega w\varphi \,\mathrm{dx} + \int_\Gamma \mu w\varphi \,\mathrm{d}s, \tag{6.5a}$$
$$b\big(w,\varphi\big) := \int_\Omega \big(\alpha_\Omega w + \beta_\Omega \cdot \nabla w\big)\varphi + \gamma_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} \tag{6.5b}$$
$$+ \int_\Gamma \big(\alpha_\Gamma w + \beta_\Gamma \cdot \nabla_\Gamma w\big)\varphi + \gamma_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \,\mathrm{d}s, \tag{6.5c}$$
$$a\big(w,\varphi\big) := \int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} + \int_\Gamma a_\Gamma w\varphi + c_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \,\mathrm{d}s, \tag{6.5d}$$
$$\langle f(t),\varphi\rangle := \int_\Omega f_\Omega(t)\varphi \,\mathrm{dx} + \int_\Gamma f_\Gamma(t)\varphi \,\mathrm{d}s, \qquad t \in [0,T]. \tag{6.5e}$$

According to (4.9), the physical energy corresponding to (6.2) is given by

$$\mathbf{E}\,(u,u_t) = \frac{1}{2}\left(\int_\Omega u_t^2 \,\mathrm{dx} + \int_\Gamma \mu u_t^2 \,\mathrm{d}s + \int_\Omega a_\Omega u^2 + c_\Omega \nabla u \cdot \nabla u \,\mathrm{dx} + \int_\Gamma a_\Gamma u^2 + c_\Gamma |\nabla_\Gamma u|^2 \,\mathrm{d}s\right).$$

The second-order wave-type problem

To formulate (6.4) as an abstract second-order wave-type problem, we require the appropriate functional analytic framework.

Hilbert spaces We choose the Hilbert spaces H and V as

$$\begin{aligned} H &:= \text{completion of } C^\infty(\overline{\Omega}) \text{ w.r.t. } \|\cdot\|_m, & \|u\|_m^2 &:= m(u,u), \\ V &:= \text{completion of } C^\infty(\overline{\Omega}) \text{ w.r.t. } \|\cdot\|_V, & \|u\|_V^2 &:= m(u,u) + a(u,u). \end{aligned}$$

Note that by construction $\|v\|_m \leq \|v\|_V$ for $v \in C^\infty(\overline{\Omega})$. We extend this embedding continuously from $C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ to $V \overset{d}{\hookrightarrow} H$ which allows us to form the Gelfand triple (4.1).

Bilinear forms Let $u, \varphi \in C^\infty(\overline{\Omega})$. By the Cauchy–Schwarz inequality it follows that

$$m(u,\varphi) \leq \|u\|_m \|\varphi\|_m \quad \text{and} \quad a(u,\varphi) \leq \|u\|_V \|\varphi\|_V,$$

so that both bilinear forms extend continuously from $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ to $H \times H$ and $V \times V$, respectively. Moreover Assumption 6.2 (iv) ensures that there exists a constant $C_b > 0$ s.t.

$$|b(u,\varphi)| \leq C_b \|u\|_V \|\varphi\|_V.$$

Hence b also extends continuously to $V \times V$.

Source term Finally, it follows from

$$\begin{aligned} |\langle f(t), \varphi\rangle| &\leq \|f_\Omega(t)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + \|f_\Gamma(t)\|_{L^2(\Gamma)} \|\,\gamma(\varphi)\|_{L^2(\Gamma)} \\ &\leq \Big(\|f_\Omega(t)\|_{L^2(\Omega)} + \|\gamma\|_{L^2(\Gamma)\leftarrow H^1(\Omega)} \|f_\Gamma(t)\|_{L^2(\Gamma)} \Big) \|\varphi\|_{H^1(\Omega)} \\ &\leq C(f_\Omega(t), f_\Gamma(t), \gamma, a_\Omega, c_\Omega) \|\varphi\|_V, \end{aligned}$$

that $f(t)\colon C^\infty(\overline{\Omega}) \to \mathbb{R}$ has a continuous extension $f(t) \in V^*$ for all $t \in [0,T]$ if $f_\Omega(t) \in L^2(\Omega)$ and $f_\Gamma(t) \in L^2(\Gamma)$.

Assumptions on the bilinear forms We now study which coefficient constellations are sufficient for Assumption 4.1. Note that a satisfies a Garding inequality with $c_G = \alpha = 1$ by construction. So it only remains to check for which coefficient constellations a is coercive and b is monotone.

LEMMA 6.3.

(i) *If* $\int_\Omega a_\Omega \,\mathrm{dx} + \int_\Gamma a_\Gamma \,\mathrm{d}s > 0$, *then* a *is coercive on* $V \times V$.

(ii) *If* $\alpha_\Omega(\mathrm{x}) - \frac{1}{2}\operatorname{div}\beta_\Omega(\mathrm{x}) \geq 0$ *for* $\mathrm{x} \in \Omega$ *and* $\alpha_\Gamma(\mathrm{x}) + \frac{1}{2}\big(\beta_\Omega(\mathrm{x}) \cdot n(\mathrm{x}) - \operatorname{div}_\Gamma \beta_\Gamma(\mathrm{x})\big) \geq 0$ *for* $\mathrm{x} \in \Gamma$, *then* b *is monotone on* $V \times V$.

Proof. Note that it is sufficient to show the respective estimates on $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

(i) Note that we have by assumption $\int_\Gamma a_\Gamma \,\mathrm{d}s > 0$ or $\int_\Omega a_\Omega \,\mathrm{dx} > 0$ (or both). First assume that $\int_\Gamma a_\Gamma \,\mathrm{d}s > 0$. Then $w(u) := (\int_\Gamma a_\Gamma u^2 \,\mathrm{d}s)^{1/2}$ defines a semi-norm on $H^1(\Omega)$ which satisfies $0 \leq w(u) \leq \|u\|_{H^1(\Omega)}$ and is positive for constant functions $p \in \mathcal{P}_0$, i.e.,

$$w(p) = \int_\Gamma a_\Gamma p^2 \,\mathrm{d}s = p^2 \int_\Gamma a_\Gamma \,\mathrm{d}s \overset{!}{=} 0 \qquad \Longleftrightarrow \qquad p = 0.$$

Thus [Han and Atkinson, 2009, Thm. 7.3.12] implies that there exists some $c_{a_\Gamma} > 0$ s.t.

$$\int_\Gamma a_\Gamma u^2 \,\mathrm{d}s + \int_\Omega c_\Omega \nabla u \cdot \nabla u \,\mathrm{dx} \geq \int_\Gamma a_\Gamma u^2 \,\mathrm{d}s + \int_\Omega c_\Omega^- |\nabla u|^2 \,\mathrm{dx} \geq c_{a_\Gamma} \|u\|^2_{H^1(\Omega)},$$

where we used the uniform positivity of c_Ω for the first estimate. Together with

$$\begin{aligned} \|u\|_m^2 = \|u\|^2_{L^2(\Omega)} + \int_\Gamma \mu u^2 \,\mathrm{d}s &\leq \|u\|^2_{L^2(\Omega)} + \|\mu\|_{L^\infty(\Gamma)} \| \gamma(u)\|^2_{L^2(\Gamma)} \\ &\leq \big(1 + \|\mu\|_{L^\infty(\Gamma)} \|\gamma\|^2_{L^2(\Gamma)\leftarrow H^1(\Omega)}\big)\|u\|^2_{H^1(\Omega)}, \end{aligned}$$

this shows that there is a constant $\alpha_H > 0$ s.t. $a\big(u,u\big) \geq \alpha_H m\big(u,u\big)$. The claim now follows with $\alpha = \frac{1}{2}\min\{1, \alpha_H\}$ from

$$a\big(u,u\big) = \frac{1}{2} a\big(u,u\big) + \frac{1}{2} a\big(u,u\big) \geq \frac{1}{2}\Big(a\big(u,u\big) + \alpha_H m\big(u,u\big)\Big) \geq \alpha \|u\|_V^2.$$

The claim for $\int_\Omega a_\Omega \,\mathrm{dx} > 0$ can be shown analogously by considering $w(u) \coloneqq (\int_\Omega a_\Omega u^2 \,\mathrm{dx})^{1/2}$.

(ii) Let $u, \varphi \in C^\infty\big(\overline{\Omega}\big)$ and consider

$$\begin{aligned} b\big(u,u\big) = &\int_\Omega \alpha_\Omega u^2 + \big(\beta_\Omega \cdot \nabla u\big) u + \gamma_\Omega |\nabla u|^2 \,\mathrm{dx} \\ &+ \int_\Gamma \alpha_\Gamma u^2 + \big(\beta_\Gamma \cdot \nabla_\Gamma u\big) u + \gamma_\Gamma |\nabla_\Gamma u|^2 \,\mathrm{d}s. \end{aligned} \tag{6.6}$$

Therefore it only remains to study the advection terms as all other terms are non-negative. We use Gauss' Theorem to rewrite the bulk advection term as

$$\begin{aligned} \int_\Omega \nabla u \cdot (\varphi \beta_\Omega) \,\mathrm{dx} &= -\int_\Omega u \operatorname{div}(\varphi \beta_\Omega) \,\mathrm{dx} + \int_\Gamma (\beta_\Omega \cdot n) u \varphi \,\mathrm{d}s \\ &= -\int_\Omega u\big(\operatorname{div}(\beta_\Omega)\varphi + \beta_\Omega \cdot \nabla \varphi\big) \,\mathrm{dx} + \int_\Gamma (\beta_\Omega \cdot n) u \varphi \,\mathrm{d}s. \end{aligned}$$

Next, we set $\varphi = u$ to obtain

$$\int_\Omega \nabla u \cdot (u \beta_\Omega) \,\mathrm{dx} = -\frac{1}{2} \int_\Omega \operatorname{div}(\beta_\Omega) u^2 \,\mathrm{dx} + \frac{1}{2} \int_\Gamma (\beta_\Omega \cdot n) u^2 \,\mathrm{d}s. \tag{6.7}$$

Applying (3) to the surface advection term, we find

$$\begin{aligned} \int_\Gamma \nabla_\Gamma u \cdot (\varphi \beta_\Gamma) \,\mathrm{d}s &= \int_\Gamma -u \operatorname{div}_\Gamma(\varphi \beta_\Gamma) + \operatorname{div}_\Gamma(n)(\beta_\Gamma \cdot n) u \varphi \,\mathrm{d}s \\ &= -\int_\Gamma u\big(\operatorname{div}_\Gamma(\beta_\Gamma)\varphi + \beta_\Gamma \cdot \nabla_\Gamma \varphi\big) \,\mathrm{d}s, \end{aligned}$$

where we used Assumption 6.2 (iii) and [Kashiwabara et al., 2015, Lem. 2.3 (iii)] for the second equality. Setting $\varphi = u$ here yields

$$\int_\Gamma \nabla_\Gamma u \cdot (u \beta_\Gamma) \,\mathrm{d}s = -\frac{1}{2} \int_\Gamma \operatorname{div}_\Gamma(\beta_\Gamma) u^2 \,\mathrm{d}s. \tag{6.8}$$

Finally, to see that b is monotone, we insert (6.7) and (6.8) into (6.6).

□

The Hilbert space $\widetilde{V}$ Recall that we defined $\widetilde{V}$ to be the Hilbert space V equipped with the inner product $\widetilde{a} = a + c_G m$. If a is coercive, then we set $c_G = 0$ and $\widetilde{a} = a$ is the inner product of $\widetilde{V}$.

Sufficient conditions for flow fields If

$$\operatorname{div} \beta_\Omega \leq 0 \qquad \text{and} \qquad \beta_\Omega \cdot n \geq \operatorname{div}_\Gamma \beta_\Gamma,$$

then b is monotone by the previous Lemma. These two conditions are physically justified: The first one rules out sources in β_Ω and the second one requires that any out- or inflow of β_Ω over the boundary Γ is compensated by sinks or sources of β_Γ, respectively.

Well-posedness of the abstract second-order wave-type problem

Let the conditions in Lemma 6.3 (ii) be fulfilled, then Assumption 4.1 holds true. If the source terms and initial values $u(0) = u^0$, $u'(0) = v^0$ satisfy (6.1), then Theorem 4.3 states that the second-order wave-type problem associated with (6.4) has a unique solution $u \in C^2([0,T]; H) \cap C^1([0,T]; V)$ with $\mathcal{A}u + \mathcal{B}u' \in C([0,T]; H)$ and which satisfies

$$\begin{aligned}\Bigg(\int_\Omega (c_G + a_\Omega) u(t)^2 + c_\Omega \nabla u(t) \cdot \nabla u(t) + u'(t)^2 \,\mathrm{dx} \\ + \int_\Gamma (c_G \mu + a_\Gamma) u(t)^2 + c_\Gamma |\nabla_\Gamma u(t)|^2 + \mu u'(t)^2 \,\mathrm{d}s \Bigg)^{1/2} \\ \leq e^{\frac{c_G}{2} t} \Bigg(\Big(\|u^0\|_{\widetilde{a}}^2 + \|v^0\|_m^2 \Big)^{1/2} + t \sup_{\tau \in (0,t)} \Big(\int_\Omega f_\Omega(\tau)^2 \,\mathrm{dx} + \int_\Gamma \mu f_\Gamma(\tau)^2 \,\mathrm{d}s \Big)^{1/2} \Bigg)\end{aligned} \tag{6.9}$$

for $t \in [0,T]$. If a is coercive, then (6.9) is a stability estimate in the energy norm.

REMARK 6.4. We remark that the above considerations can be generalized in several directions:

(i) As in Section 1.1.3, we can consider (6.2a) with mixed boundary conditions: Let Γ be disjointly decomposed into a closed Dirichlet part Γ_D and a Neumann part $\Gamma_\mathsf{N} := \Gamma \backslash \Gamma_\mathsf{D}$. On the Dirichlet part we impose

$$u = f_\mathsf{D} \quad \text{on } \Gamma_\mathsf{D} \subset \Gamma$$

while we demand that the solution satisfies (6.2b) on Γ_N. The spaces H and V for such problems are then completions of

$$C_\mathsf{D}^\infty(\overline{\Omega}) = \{\varphi|_\Omega \mid C_c^\infty(\mathbb{R}^d),\ \operatorname{supp}(\varphi) \cap \Gamma_\mathsf{D} = \emptyset\}.$$

If (6.2b) is non-locally reacting, i.e., $\gamma_\Gamma \neq 0$ or $\beta_\Gamma \neq 0$ or $c_\Gamma \neq 0$, then additional boundary conditions on $\partial\Gamma_\mathsf{N}$ need to be assigned, cf. [Disser et al., 2015]. Note that the variational formulation (6.4) with Γ_N and $C_\mathsf{D}^\infty(\overline{\Omega})$ instead of Γ and $C^\infty(\overline{\Omega})$ enforces homogeneous Neumann on $\partial\Gamma_\mathsf{N}$.

(ii) Assumption 6.2 (i) can be weakened to allow coefficient functions which are only smooth on parts of their domain: For example consider a partition of $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$ into two open sub-domains Ω_1 and Ω_2 which are divided by the Lipschitz interface $\Sigma := \overline{\Omega_1} \cap \overline{\Omega_2}$. Further let, $c_{\Omega_1} := c_\Omega|_{\Omega_1} \in W^{1,\infty}(\Omega_1; \mathbb{R}^{d\times d})$ and $c_{\Omega_2} := c_\Omega|_{\Omega_2} \in W^{1,\infty}(\Omega_2; \mathbb{R}^{d\times d})$. Although the variational problem (6.4) still determines a unique solution u in this case, u does not solve the original problem (6.2). Rather $u_1 := u|_{\Omega_1}$ and $u_2 := u|_{\Omega_2}$ solve the partial differential equation (6.2a) in Ω_1 and Ω_2 respectively, suffice the boundary conditions (6.2b) on $\Gamma_1 := \partial\Omega_1 \setminus \Sigma$ and $\Gamma_2 := \partial\Omega_2 \setminus \Sigma$ respectively, and, additionally satisfy the transmission conditions

$$n_\Sigma \cdot c_{\Omega_1} \nabla u_1 = n_\Sigma \cdot c_{\Omega_2} \nabla u_2 \qquad \text{on } \Sigma.$$

Here $n_\Sigma \colon \Sigma \to \mathbb{R}^d$ denotes the unit normal vector on Σ, cf. [Cohen and Pernet, 2016, Thm. 2].

(iii) Assumption 6.2 (ii) is not fulfilled for domains Ω with piecewise C^2-boundary Γ. Then (4) does not hold globally on Γ, since the boundary terms between the C^2-parts of Γ do not cancel out, cf. [Kashiwabara et al., 2015, Rmk. 3.1]. Thus (6.2) with piecewise smooth boundary Γ needs to be supplemented by additional conditions on the boundary between the boundary parts.

(iv) The expression $\beta_\Gamma \cdot \nabla_\Gamma \varphi$ is independent of the normal part of β_Γ, since $n \cdot \nabla_\Gamma \varphi = 0$ for every sufficiently smooth function φ. Therefore Assumption 6.2 (iii) holds without loss of generality.

(v) Assumption 6.2 (iv) is not satisfied if there is advection $\beta_\Gamma > 0$ or strong damping $\gamma_\Gamma > 0$ on a part of the surface where there is no wave propagation $c_\Gamma = 0$. Then $\mathcal{B}\colon D(\mathcal{B}) \to V^*$ is only defined on the domain $D(\mathcal{B}) \subset V$ so Theorem 4.3 does not apply. The well-posedness results in [Showalter, 1994, Sect. VI.2], however, cover this case.

6.2 Analysis of specific examples

In this section, we derive concrete well-posedness results for two examples of (6.2). If (6.1) is satisfied, then our previous considerations show that there is a $u \in C^2([0,T];H)$ which solves the second-order wave-type problem corresponding to (6.2). Recall that we defined H and V as completions of $C^\infty(\overline{\Omega})$ w.r.t. norms that depend on the coefficients. Hence it still remains to identify the type of functions contained in H and V for both examples. After that we may continue with the second part of our user's guide, i.e., we

- verify if and in which sense the solution u satisfies the original partial differential equation,
- characterize (6.1) in terms of Sobolev spaces, and,
- derive weak stability estimates by using the considerations from Section 4.4.

In the following, we assume that the coefficient functions are as described for (6.2) and that the sought solution originates from initial values $u(0) = u^0$ and $u_t(0) = v^0$.

6.2.1 Robin type boundary conditions

We seek the solution $u\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ of

$$u_{tt} - \operatorname{div}(c_\Omega \nabla u) = f_\Omega \qquad \text{in } \Omega, \tag{6.10a}$$
$$\alpha_\Gamma u_t + a_\Gamma u = -n \cdot c_\Omega \nabla u \qquad \text{on } \Gamma, \tag{6.10b}$$

where $\alpha_\Gamma, a_\Gamma \in L^\infty(\Gamma)$, $\int_\Gamma a_\Gamma \,\mathrm{d}s > 0$ and $c_\Omega \in W^{1,\infty}(\Omega)^{d\times d}$ is uniformly elliptic.

Corollary 6.5. *Let the coefficients be as described above.*

(i) *If* $u^0, v^0 \in H^1(\Omega)$ *satisfy* $\operatorname{div}(c_\Omega \nabla u^0) \in L^2(\Omega)$ *and* $\alpha_\Gamma v^0 + a_\Gamma u^0 + n \cdot c_\Omega \nabla u^0 = 0$ *on* Γ*, and* $f_\Omega \in C^1\big([0,T];L^2(\Omega)\big)$ *or* $f_\Omega \in C\big([0,T];H^1(\Omega)\big)$ *with* $\gamma(f_\Omega) = 0$*, then* (6.10) *has a unique solution*

$$u \in C^2\big([0,T];L^2(\Omega)\big) \cap C^1\big([0,T];H^1(\Omega)\big), \quad \operatorname{div}(c_\Omega \nabla u) \in C\big([0,T];L^2(\Omega)\big),$$

which satisfies the stability estimate (6.9) *with* $c_{\mathsf{qm}} = 0$.

(ii) *If* $u^0 \in H^1(\Omega)$*,* $v^0 \in L^2(\Omega)$*, and* $f_\Omega \in C^1\left([0,T];H^1(\Omega)^*\right) + C\left([0,T];L^2(\Omega)\right)$*, then there exists a unique weak solution* $u \in C^1\big([0,T];L^2(\Omega)\big) \cap C\big([0,T];H^1(\Omega)\big)$ *of* (6.10) *which satisfies*

$$\|u(t)\|_{L^2(\Omega)} \le \Big(\big\|u^0\big\|^2_{L^2(\Omega)} + \big\|v^0\big\|^2_{H^1(\Omega)^*} + C\big\|u^0\big\|^2_{L^2(\Gamma)}\Big)^{1/2} + t\|f_\Omega\|_{L^\infty(0,t;H^1(\Omega)^*)}$$

for $t \in [0,T]$.

Proof. Before we prove the well-posedness results, we identify H and V in part (A), and characterize (6.1a) in part (B). For the convenience of the reader, we give the definitions (6.5) for the coefficient constellation of (6.10)

$$\begin{aligned} m(w,\varphi) &:= \int_\Omega w\varphi \,\mathrm{dx}, \\ b(w,\varphi) &:= \int_\Gamma \alpha_\Gamma w\varphi \,\mathrm{d}s, \\ a(w,\varphi) &:= \int_\Omega c_\Omega \nabla w\cdot\nabla\varphi \,\mathrm{dx} + \int_\Gamma a_\Gamma w\varphi\,\mathrm{d}s, \\ \langle f(t),\varphi\rangle &:= \int_\Omega f_\Omega(t)\varphi\,\mathrm{dx}, \qquad t\in[0,T]. \end{aligned}$$

(A) First note that $C^\infty(\overline{\Omega})$ is a dense subspace of $H^k(\Omega)$, $k\geq 0$ by [Han and Atkinson, 2009, Thm. 7.3.2]. Thus $H\simeq L^2(\Omega)$, since $C^\infty(\overline{\Omega})$ is dense in H and $L^2(\Omega)$, and m coincides with the $L^2(\Omega)$-inner product. Furthermore, $V\simeq H^1(\Omega)$. Again, this follows from the density of $C^\infty(\overline{\Omega})$ in V and $H^1(\Omega)$, and $\|\cdot\|_{H^1(\Omega)}\sim\|\cdot\|_V$. To see the latter part, let $\varphi\in C^\infty(\overline{\Omega})$. Then we obtain from (6.3) and the continuity of the trace opterator

$$\|\varphi\|^2_{H^1(\Omega)} \leq \max\{1,1/c_\Omega^-\}\left(\int_\Omega \varphi^2 + c_\Omega\nabla\varphi\cdot\nabla\varphi\,\mathrm{dx} + \int_\Gamma a_\Gamma\varphi^2\,\mathrm{d}s\right) = C(c_\Omega^-)\|\varphi\|_V^2,$$

and

$$\|\varphi\|_V^2 = \int_\Omega \varphi^2 + c_\Omega\nabla\varphi\cdot\nabla\varphi\,\mathrm{dx} + \int_\Gamma a_\Gamma\varphi^2\,\mathrm{d}s \leq C(\gamma, c_\Omega^+)\|\varphi\|^2_{H^1(\Omega)}.$$

(B) We claim that for $w,v\in V=H^1(\Omega)$

$$\mathcal{A}w+\mathcal{B}v\in H \iff \operatorname{div}(c_\Omega\nabla w)\in L^2(\Omega) \text{ and } \alpha_\Gamma\,\gamma(v)+a_\Gamma\,\gamma(w)+\gamma_n(c_\Omega\nabla w)=0. \tag{6.11}$$

First assume that $\mathcal{A}w+\mathcal{B}v\in H$. Then we have for $\varphi\in H^1(\Omega)$

$$\langle\mathcal{A}w+\mathcal{B}v,\varphi\rangle_V = \int_\Omega c_\Omega\nabla w\cdot\nabla\varphi\,\mathrm{dx} + \int_\Gamma \big(a_\Gamma\,\gamma(w)+\alpha_\Gamma\,\gamma(v)\big)\,\gamma(\varphi)\,\mathrm{d}s \leq C(w,v)\|\varphi\|_{L^2(\Omega)}.$$

To show that $\operatorname{div}(c_\Omega\nabla w)\in L^2(\Omega)$, we insert $\varphi\in C_c^\infty(\Omega)$ and deduce $c_\Omega\nabla w\in H(\operatorname{div},\Omega)$, since the surface integral vanishes. For the trace identity, form the Gelfand triple

$$H^{1/2}(\Gamma)\overset{d}{\hookrightarrow} L^2(\Gamma)\simeq L^2(\Gamma)^*\overset{d}{\hookrightarrow} H^{-1/2}(\Gamma).$$

Thus we may replace the surface integral in $\langle\mathcal{A}w+\mathcal{B}v,\varphi\rangle_V$ with the duality pairing in $H^{1/2}(\Gamma)$. We further insert (2) in the form of

$$\int_\Omega c_\Omega\nabla w\cdot\nabla\varphi\,\mathrm{d}s = -\int_\Omega \operatorname{div}(c_\Omega\nabla w)\varphi\,\mathrm{dx} + \langle\gamma_n(c_\Omega\nabla w),\,\gamma(\varphi)\rangle_{H^{1/2}(\Gamma)}. \tag{6.12}$$

and obtain

$$\langle\mathcal{A}w+\mathcal{B}v,\varphi\rangle_V = -\int_\Omega \operatorname{div}(c_\Omega\nabla w)\varphi\,\mathrm{dx} + \langle\gamma_n(c_\Omega\nabla w)+a_\Gamma\,\gamma(w)+\alpha_\Gamma\,\gamma(v),\,\gamma(\varphi)\rangle_{H^{1/2}(\Gamma)} \tag{6.13}$$

for all $\varphi\in H^1(\Omega)$. Now let $\varphi_\Gamma\in H^{1/2}(\Gamma)$. Then there exists a sequence $(\varphi_k)_{k\geq 0}\subset H^1(\Omega)$ s.t. $\gamma(\varphi_k)=\varphi_\Gamma$, $k\geq 0$ and $\|\varphi_k\|_{L^2(\Omega)}\to 0$, $k\to\infty$ as shown in [Schnaubelt and Weiss, 2010, (4) in proof of Thm. 5.1]. We insert this sequence into (6.13) which yields

$$\begin{aligned} |\langle\gamma_n(c_\Omega\nabla w)+a_\Gamma\,\gamma(w)+\alpha_\Gamma\,\gamma(v),\varphi_\Gamma\rangle_{H^{1/2}(\Gamma)}| &\leq |\langle\mathcal{A}w+\mathcal{B}v,\varphi_k\rangle_V| + |\int_\Omega \operatorname{div}(c_\Omega\nabla w)\varphi_k\,\mathrm{dx}| \\ &\leq C\|\varphi_k\|_{L^2(\Omega)}\to 0, \qquad k\to\infty. \end{aligned}$$

Thus $\gamma_n(c_\Omega \nabla w) + a_\Gamma\, \gamma(w) + \alpha_\Gamma\, \gamma(v)$ is identically zero, which finishes the proof of the first implication in (6.11). For the second implication assume that $w, v \in H^1(\Omega)$ satisfy $\operatorname{div}(c_\Omega \nabla w) \in L^2(\Omega)$ and $\gamma_n(c_\Omega \nabla w) + a_\Gamma\, \gamma(w) + \alpha_\Gamma\, \gamma(v) = 0$. Then $\mathcal{A}w + \mathcal{B}v \in H$ follows from (6.13).

After these preparations, we now show the well-posedness result.

(i) Note that a is coercive and b is monotone due to Lemma 6.3 (ii). So it remains to check (6.1), before we apply Theorem 4.3. It follows from (6.11) that the initial values u^0 and v^0 suffice (6.1a). To verify (6.1b), note that $f = f_\Omega$. Thus we have by assumption $f \in C^1\big([0,T]; H\big)$ or $f \in C\big([0,T]; V\big)$ with $\gamma(f_\Omega) = 0$. Hence it remains to check if $\mathcal{B}f \in C\big([0,T]; H\big)$ in the latter case. In fact $\mathcal{B}f(t) = 0$, since

$$\langle \mathcal{B}f(t), \varphi \rangle_V = \int_\Gamma \alpha_\Gamma\, \gamma(f_\Omega(t))\, \gamma(\varphi)\, \mathrm{d}s = 0, \qquad \varphi \in H^1(\Omega).$$

Thus, by Theorem 4.3, there exists a unique solution $u \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^1(\Omega))$ of the second-order wave-type problem (4.2), which satisfies $\mathcal{A}u + \mathcal{B}u' \in C([0,T]; L^2(\Omega))$.

To see that u solves (6.10a), we apply $\varphi \in C_c^\infty(\Omega)$ to the second-order wave-type problem (4.2). Then we find with (6.13)

$$\begin{aligned} \int_\Omega u''(t)\varphi - \operatorname{div}(c_\Omega \nabla u)\varphi \,\mathrm{dx} &= \langle u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t), \varphi \rangle_V \\ &= \langle f(t), \varphi \rangle_V \\ &= \int_\Omega f_\Omega(t) \varphi \,\mathrm{dx}, \end{aligned}$$

which shows that u solves (6.10a). Using (6.11), we infer from $u \in C^1([0,T]; H^1(\Omega))$ and $\mathcal{A}u(t) + \mathcal{B}u'(t) \in L^2(\Omega)$ that u also satisfies the boundary condition (6.10b) .

Finally, suppose $\widetilde{u}$ is a solution of (6.10) with the stated properties. By the same computation, which we used to derive the variational formulation (6.4), it follows that $\widetilde{u}$ also solves the corresponding second-order wave-type problem. Since second-order wave-type problems are uniquely solvable, we find $\widetilde{u} = u$. Therefore (6.10) has a unique solution.

(ii) Our considerations from Section 4.4 apply with $\widetilde{V}^* \simeq H^1(\Omega)^*$: We have $c_G = \beta_{\mathsf{qm}} = 0$ by Lemma 6.3 (ii), and the data meets the assumptions of Theorem 4.13. Therefore (4.20) gives an upper bound for $\|u(t)\|_{L^2(\Omega)}$. To obtain the desired stability estimate, we apply

$$\begin{aligned} \|v^0 + \mathcal{B}u^0\|_{\widetilde{V}^*} \le \|v^0\|_{\widetilde{V}^*} + \|\mathcal{B}u^0\|_{\widetilde{V}^*} &\le \|v^0\|_{\widetilde{V}^*} + \sup_{\|\varphi\|_{\widetilde{a}}=1} \int_\Omega \alpha_\Gamma\, \gamma(u^0)\, \gamma(\varphi)\, \mathrm{d}s \\ &\le \|v^0\|_{H^1(\Omega)^*} + C(\alpha_\Gamma, \gamma) \|\, \gamma(u^0)\|_{L^2(\Gamma)}. \end{aligned}$$

Here, we used the continuity of the trace operator $\gamma\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ and $V \simeq H^1(\Omega)$. □

Remark 6.6. The wave equation with homogeneous Neumann boundary conditions is contained in (6.10) with $a_\Gamma = \alpha_\Gamma = 0$. However the stability estimate (6.9) grows exponentially fast in time. In this situation Corollary 4.7 provides a different stability estimate in an energy norm. The estimate for the homogeneous problem with $f_\Omega = 0$ was given in Example 4.8.

6.2.2 Kinetic boundary conditions

We seek the solution $u\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ of

$$u_{tt} + \big(\alpha_\Omega + \beta_\Omega \cdot \nabla\big)u_t + a_\Omega u - \operatorname{div}(c_\Omega \nabla u) = f_\Omega \qquad \text{in } \Omega, \qquad (6.14a)$$

$$\mu u_{tt} + \big(\alpha_\Gamma + \beta_\Gamma \cdot \nabla_\Gamma\big)u_t + a_\Gamma u - c_\Gamma \Delta_\Gamma u = -n \cdot c_\Omega \nabla u + f_\Gamma \qquad \text{on } \Gamma, \qquad (6.14b)$$

where we assume that

(i) the wave speed $c_\Omega \in W^{1,\infty}(\Omega)$ is scalar (and hence isotropic) and uniformly positive,

(ii) $a_\Gamma \geq 0$ and $c_\Gamma > 0$ are constants,

(iii) $\mu \in L^\infty(\Gamma)$ is uniformly positive, and

(iv) $\alpha_\Omega, a_\Omega \in L^\infty(\Omega)$, $\beta_\Omega \in W^{1,\infty}(\Omega)^d$,$\alpha_\Gamma \in L^\infty(\Gamma)$, $\beta_\Gamma \in W^{1,\infty}(\Gamma)^d$,

(v) the conditions of Lemma 6.3 (ii) are satisfied.

Recall that we defined

$$\begin{aligned}
\mathbb{H}^0 &:= L^2(\Omega) \times L^2(\Gamma) \\
\mathbb{H}^k &:= H^k(\Omega) \times H^k(\Gamma), \qquad k \in \mathbb{N} \\
\mathbb{H}^{-1} &:= \big(H^1(\Omega)\big)^* \times H^{-1}(\Gamma), \\
H^k(\Omega;\Gamma) &:= \Big\{ v \in H^k(\Omega) \mid \gamma(v) \in H^k(\Gamma) \Big\}, \qquad k \geq 1.
\end{aligned}$$

Corollary 6.7. *Let Γ be a C^2-boundary and let the coefficients be as described above.*

(i) *If $u^0, v^0 \in H^1(\Omega;\Gamma)$ satisfy $\big[\operatorname{div}(c_\Omega \nabla u^0), \Delta_\Gamma u^0\big]^\mathsf{T} \in \mathbb{H}^0$, and $\big[f_\Omega, f_\Gamma\big]^\mathsf{T} \in C^1([0,T];\mathbb{H}^0)$ or $f_\Omega \in C\big([0,T];H^1(\Omega;\Gamma)\big)$ with $f_\Gamma = \mu\,\gamma(f_\Omega)$, then* (6.14) *has a unique solution*

$$u \in C^2\big([0,T];\mathbb{H}^0\big) \cap C^1\big([0,T];H^1(\Omega;\Gamma)\big), \quad \big[\operatorname{div}(c_\Omega \nabla u), \Delta_\Gamma u\big]^\mathsf{T} \in C\big([0,T];\mathbb{H}^0\big),$$

which satisfies the stability estimate (6.9).

(ii) *Further, assume that $\int_\Omega a_\Omega \,\mathrm{dx} + \int_\Gamma a_\Gamma \,\mathrm{d}s > 0$ and $\beta_\Omega \in W^{1,\infty}(\Omega)^d$. For initial values $u^0 \in H^1(\Omega;\Gamma)$, $v^0 \in \mathbb{H}^0$ and source terms $\big[f_\Omega, f_\Gamma\big]^\mathsf{T} \in C^1\left([0,T];\mathbb{H}^{-1}\right) + C\left([0,T];\mathbb{H}^0\right)$, there exists a unique weak solution $u \in C^1\big([0,T];\mathbb{H}^0\big) \cap C\big([0,T];H^1(\Omega;\Gamma)\big)$ of* (6.14) *which satisfies*

$$\|u(t)\|_{\mathbb{H}^0} \leq C\left(\left\|u^0\right\|_{\mathbb{H}^0}^2 + \left\|v^0\right\|_{\mathbb{H}^{-1}} + t\|\left[f_\Omega, f_\Gamma\right]^\mathsf{T}\|_{L^\infty(0,t;\mathbb{H}^{-1})}\right)$$

for $t \in [0,T]$.

Proof. Again, we begin by identifying H and V in part (A) of this proof. In part (B), we characterize (6.1a) proceeding analogously to [Vitillaro, 2013, Lem. 2]. For (6.14), the definitions from (6.5) read

$$\begin{aligned}
m(w,\varphi) &:= \int_\Omega w\varphi \,\mathrm{dx} + \int_\Gamma \mu w \varphi \,\mathrm{d}s, \\
b(w,\varphi) &:= \int_\Omega \big(\alpha_\Omega w + \beta_\Omega \cdot \nabla w\big)\varphi \,\mathrm{dx} + \int_\Gamma \big(\alpha_\Gamma w + \beta_\Gamma \cdot \nabla_\Gamma w\big)\varphi \,\mathrm{d}s, \\
a(w,\varphi) &:= \int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla \varphi \,\mathrm{dx} + \int_\Gamma a_\Gamma w \varphi + c_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \,\mathrm{d}s, \\
\langle f(t), \varphi\rangle &:= \int_\Omega f_\Omega(t)\varphi \,\mathrm{dx} + \int_\Gamma f_\Gamma(t)\varphi \,\mathrm{d}s, \qquad t \in [0,T].
\end{aligned}$$

(A) To identify the pivot space H, note that its norm is equivalent ot the $\mathbb{H}^0$-norm for functions from $C^\infty\big(\overline{\Omega}\big)$. Therefore we have $H \simeq \mathbb{H}^0$ since [ter Elst et al., 2012, Lem. 2.10] implies $C^\infty\big(\overline{\Omega}\big) \overset{d}{\hookrightarrow} \mathbb{H}^0$ via $\varphi \mapsto \big[\varphi, \varphi|_\Gamma\big]^\mathsf{T}$ and $C^\infty\big(\overline{\Omega}\big) \overset{d}{\hookrightarrow} H$ by definition. Moreover, we have $V \simeq H^1(\Omega;\Gamma)$ by the same arguments. The norm equivalence $\|\cdot\|_{H^1(\Omega;\Gamma)} \sim \|\cdot\|_{\widetilde{a}}$ is a consequence of easy uniform postivity and boundedness of μ, c_Ω and c_Γ. The assertion the follows from $C^\infty\big(\overline{\Omega}\big)$ being dense in $\widetilde{V}$ and $H^1(\Omega;\Gamma)$, which is shown in [Ben Belgacem et al., 1997].

(B) We claim that for $w \in V = H^1(\Omega;\Gamma)$

$$\mathcal{A}w \in H \qquad \Longleftrightarrow \qquad \big[\operatorname{div}(c_\Omega \nabla w), \Delta_\Gamma w\big]^\intercal \in \mathbb{H}^0. \tag{6.15}$$

First assume that $g \coloneqq \mathcal{A}w \in H$. Then we have for $g = \big[g_\Omega, g_\Gamma\big]^\intercal$ and $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega c_\Omega \nabla w \cdot \nabla \varphi \,\mathrm{dx} = \langle \mathcal{A}w, \varphi\rangle_V - \int_\Omega a_\Omega w \varphi \,\mathrm{dx} = \int_\Omega \big(g_\Omega - a_\Omega w\big)\varphi \,\mathrm{dx}$$

which proves $c_\Omega \nabla w \in H(\operatorname{div}, \Omega)$ with $\operatorname{div}(c_\Omega \nabla w) = g_\Omega - a_\Omega w \in L^2(\Omega)$. So it remains to show $\Delta_\Gamma w \in L^2(\Gamma)$. First, we introduce $\widetilde{g} = \big[\widetilde{g}_\Omega, \widetilde{g}_\Gamma\big]^\intercal \coloneqq c_\Gamma^{-1}\mathcal{A}w + \big[0, \mu^{-1}(1 - a_\Gamma/c_\Gamma)\,\gamma(w)\big]^\intercal \in H$ s.t. for $\varphi \in H^1(\Omega;\Gamma)$

$$\begin{aligned}
\langle \widetilde{g}, \varphi\rangle_V &= \frac{1}{c_\Gamma}\int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} + \int_\Gamma \frac{a_\Gamma}{c_\Omega} w\varphi + \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \,\mathrm{d}s \\
&\quad + \int_\Gamma \mu\mu^{-1}\Big(1 - \frac{a_\Gamma}{c_\Omega}\Big)\,\gamma(w)\varphi \,\mathrm{d}s \\
&= \frac{1}{c_\Gamma}\int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} + \int_\Gamma w\varphi + \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \,\mathrm{d}s.
\end{aligned}$$

Obviously, $\widetilde{g}_\Omega = c_\Gamma^{-1} g_\Omega$ holds by construction and thus $c_\Gamma \widetilde{g}_\Omega - a_\Omega w = \operatorname{div}(c_\Omega \nabla w)$. Hence, we find for $\varphi \in H^1(\Omega;\Gamma)$

$$\begin{aligned}
\int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \varphi + w\varphi \,\mathrm{d}s &= \langle \widetilde{g}, \varphi\rangle_V - \frac{1}{c_\Gamma}\int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} \\
&= \int_\Gamma \mu \widetilde{g}_\Gamma \varphi \,\mathrm{d}s - \frac{1}{c_\Gamma}\int_\Omega \big(c_\Gamma \widetilde{g}_\Omega - a_\Omega w\big)\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} \\
&= \int_\Gamma \mu \widetilde{g}_\Gamma \varphi \,\mathrm{d}s - \frac{1}{c_\Gamma}\int_\Omega \operatorname{div}(c_\Omega \nabla w)\varphi + c_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{dx} \\
&= \int_\Gamma \mu \widetilde{g}_\Gamma \varphi \,\mathrm{d}s - \frac{1}{c_\Gamma}\langle \gamma_n(c_\Omega \nabla w), \varphi\rangle_{H^{1/2}(\Gamma)},
\end{aligned}$$

where we used (6.12) in the last equality. Then $(\Delta_\Gamma + 1)w \in H^{-1/2}(\Gamma)$, since the right hand side is bounded by $\|\varphi\|_{H^{1/2}(\Gamma)}$. We define $f_\mathsf{D} \coloneqq \gamma(w) \in H^{1/2}(\Gamma)$ and rewrite the above identity as

$$(\Delta_\Gamma + 1) f_\mathsf{D} = \widetilde{g}_\Gamma - \frac{1}{c_\Gamma}\gamma_n(c_\Omega \nabla w) \in H^{1/2}(\Gamma).$$

Since Γ is C^2, the operator $\Delta_\Gamma + 1$ has a continuous inverse $(\Delta_\Gamma + 1)^{-1}\colon H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$, cf. [Vitillaro, 2013, p. 299]. Therefore,

$$f_\mathsf{D} = (\Delta_\Gamma + 1)^{-1}\Big(\widetilde{g}_\Gamma - \frac{1}{c_\Gamma}\gamma_n(c_\Omega \nabla w)\Big) \in H^{3/2}(\Gamma).$$

Hence, w solves the elliptic problem $\operatorname{div}(c_\Omega \nabla w) + a_\Omega w = g_\Omega \in L^2(\Omega)$ in Ω with Dirichlet boundary condition $\gamma(w) = f_\mathsf{D} \in H^{3/2}(\Gamma)$ on the C^2-boundary Γ. Then elliptic theory implies that w actually belongs to $H^2(\Omega)$, cf. [Grisvard, 2011, Theorem 2.4.2.5]. Thus, $\gamma_n(c_\Omega \nabla w) \in L^2(\Gamma)$ exists in the trace sense and we have shown $\Delta_\Gamma w = \widetilde{g}_\Gamma - c_\Gamma^{-1}\gamma_n(c_\Omega \nabla w) - w \in L^2(\Gamma)$. This finishes the proof of the first direction in (6.15). For the second implication, insert $w \in H^1(\Omega;\Gamma)$ with $\big[\operatorname{div}(c_\Omega \nabla w), \Delta_\Gamma w\big]^\intercal \in \mathbb{H}^0$ into $\langle \mathcal{A}w, \varphi\rangle_V$, $\varphi \in H^1(\Omega;\Gamma)$. Applying Gauss' Theorem in the bulk and (4) on the surface, leads to

$$\langle \mathcal{A}w, \varphi\rangle_V = \int_\Omega \big(a_\Omega w - \operatorname{div}(c_\Omega \nabla w)\big)\varphi \,\mathrm{dx} + \langle \gamma_n(c_\Omega \nabla w), \varphi\rangle_{H^{1/2}(\Gamma)} + \int_\Gamma \big(a_\Gamma w - \Delta_\Gamma w\big)\varphi \,\mathrm{d}s. \tag{6.16}$$

Therefore, it remains to show that $\gamma_n(c_\Omega \nabla w) \in L^2(\Gamma)$. In a first step, we obtain $w \in H^2(\Gamma)$ from [Vitillaro, 2013, p. 299], since $(\Delta_\Gamma + 1)w \in L^2(\Gamma)$ and Γ is C^2 by assumption. Furthermore, w solves the Dirichlet problem $\operatorname{div}(c_\Omega \nabla w) \in L^2(\Omega)$ with smooth boundary values $\gamma(w) \in H^2(\Gamma)$. Again, [Grisvard, 2011, Theorem 2.4.2.5] yields $w \in H^2(\Omega)$ and hence also $\gamma_n(c_\Omega \nabla w) \in H^{1/2}(\Gamma)$. This shows the claim.

After these preparations, we now show the well-posedness result.

(i) Note that $c_G = 1$ and b is monotone due to Lemma 6.3 (ii). So it remains to check (6.1). The condition (6.1a) is satisfied due to standing assumptions on the initial values. Now note that $f = \big[f_\Omega, \mu^{-1} f_\Gamma\big]^\intercal$ if $f \in \mathbb{H}^0$. Therefore f satisfies the first option in (6.1b), if $\big[f_\Omega, f_\Gamma\big]^\intercal \in C^1([0,T]; \mathbb{H}^0)$. If $f_\Omega \in C\big([0,T]; H^1(\Omega;\Gamma)\big)$ with $f_\Gamma = \mu\, \gamma(f_\Omega)$, then $f \in C\big([0,T]; V\big)$. Therefore the second option in (6.1b) is satisfied, since clearly $\mathcal{B} \in \mathcal{L}(V,H)$ in this example. Thus, by Theorem 4.3, there exists a unique solution $u \in C^2([0,T]; \mathbb{H}^0) \cap C^1([0,T]; H^1(\Omega;\Gamma))$ of the second-order wave-type problem (4.2), which satisfies $u \in C([0,T]; [D(A)])$. Therefore, (6.15) gives $\big[\operatorname{div}(c_\Omega \nabla u), \Delta_\Gamma u\big]^\intercal \in C\big([0,T]; \mathbb{H}^0\big)$, since $u \in C^1([0,T]; V)$ and $\mathcal{A}u \in C([0,T]; H)$. As a consequence, we find for all $\varphi \in H^1(\Omega;\Gamma)$

$$
\begin{aligned}
0 &= \langle u'' + \mathcal{B}u' + \mathcal{A}u - f, \varphi \rangle_V \\
&= \int_\Omega \Big(u_{tt} + \big(\alpha_\Omega + \beta_\Omega \cdot \nabla\big) u_t + a_\Omega u - \operatorname{div}(c_\Omega \nabla u) - f_\Omega \Big) \varphi \,\mathrm{dx} \\
&\qquad + \int_\Gamma \Big(\mu u_{tt} + \big(\alpha_\Gamma + \beta_\Gamma \cdot \nabla_\Gamma\big) u_t + a_\Gamma u - c_\Gamma \Delta_\Gamma u + n \cdot c_\Omega \nabla u - f_\Gamma \Big) \gamma(\varphi) \,\mathrm{d}s,
\end{aligned}
$$

where we used (6.16) and $\gamma_n(c_\Omega \nabla u) \in L^2(\Gamma)$ by elliptic regularity. Since the right hand side can be bounded by $\|\varphi\|_{\mathbb{H}^0}$ and $H^1(\Omega;\Gamma) \overset{d}{\hookrightarrow} \mathbb{H}^0$, the equation continues to hold for $\varphi_\Omega \in L^2(\Omega)$ and $\varphi_\Gamma \in L^2(\Gamma)$ in place of φ and $\gamma(\varphi)$, respectively. Hence, choosing $\varphi_\Gamma = 0$ and $\varphi_\Omega = 0$, yields that u solves (6.14a) and (6.14b), respectively. Finally, u is the unique solution of (6.14), since each solution of (6.14) solves the corresponding second-order wave-type problem, cf. part (i) in the proof of Corollary 6.5.

(ii) We check the assumptions of Corollary 4.14: $\mathcal{B}$ belongs to $\mathcal{L}(H, V^*)$, since for $w, \varphi \in H^1(\Omega;\Gamma)$

$$
\begin{aligned}
b(w,\varphi) &= \int_\Omega \big(\alpha_\Omega w + \beta_\Omega \cdot \nabla w\big) \varphi \,\mathrm{dx} + \int_\Gamma \big(\alpha_\Gamma w + \beta_\Gamma \cdot \nabla_\Gamma w\big) \varphi \,\mathrm{d}s \\
&= \int_\Omega w\big((\alpha_\Omega - \operatorname{div} \beta_\Omega)\varphi - \beta_\Omega \cdot \nabla \varphi\big) \,\mathrm{dx} \\
&\qquad + \int_\Gamma w\big((n \cdot \beta_\Omega + \alpha_\Gamma - \operatorname{div}_\Gamma \beta_\Gamma)\varphi - \beta_\Gamma \cdot \nabla_\Gamma \varphi\big) \,\mathrm{d}s \\
&\le C(\alpha_\Omega, \beta_\Omega, \alpha_\Gamma, \beta_\Gamma) \|w\|_{\mathbb{H}^0} \|\varphi\|_{H^1(\Omega;\Gamma)},
\end{aligned}
$$

where we applied Gauss' Theorem in the bulk and (3). Furthermore, a is coercive by Lemma 6.3 (i) and the initial values satisfy the conditions of Corollary 4.14. To investigate the source terms, define the embedding

$$
J \colon \mathbb{H}^{-1} \to (H^1(\Omega;\Gamma))^*, \qquad \langle J\big[g_\Omega, g_\Gamma\big]^\intercal, \varphi \rangle_{H^1(\Omega;\Gamma)} := \langle g_\Omega, \varphi \rangle_{H^1(\Omega)} + \langle g_\Gamma, \gamma(\varphi) \rangle_{H^1(\Gamma)}
$$

and note that $f = J\big[f_\Omega, f_\Gamma\big]^\intercal$ by definition of f. Thus f belongs to $C^1\left([0,T]; V^*\right)$ or $C\left([0,T]; H\right)$ by assumption. Hence Corollary 4.14 yields the unique weak solution including stability estimate (4.25). To obtain the asserted estimate, we further use $Jw = w$, $w \in \mathbb{H}^0$ and $J \in \mathcal{L}(\mathbb{H}^{-1}, (H^1(\Omega;\Gamma))^*)$ to estimate

$$
\|f(t)\|_{(H^1(\Omega;\Gamma))^*} = \|Jf(t)\|_{(H^1(\Omega;\Gamma))^*} \le C \|f(t)\|_{\mathbb{H}^{-1}}.
$$

□

REMARK 6.8. Corollary 4.7 yields stability estimate in the energy norm of (6.14) even in the case where $\int_\Omega a_\Omega \,\mathrm{dx} + \int_\Gamma a_\Gamma \,\mathrm{d}s = 0$ and thus $c_G = 1$.

The analysis of wave equations with dynamic boundary conditions Comparing both results, we notice that the difference between the analytic frameworks for dynamic and non-dynamic boundary conditions is the pivot space H. It is $H \simeq \mathbb{H}^0$ in the framework for kinetic boundary conditions, while we have $H \simeq L^2(\Omega)$ in the non-dynamic case. Hence the framework for kinetic boundary conditions admits a rate of change $\left[u''_\Omega, u''_\Gamma\right]^\intercal := u'' \in \mathbb{H}^0 = L^2(\Omega) \times L^2(\Gamma)$, where $\gamma(u''_\Omega) \neq u''_\Gamma$ in general. Therefore $u|_\Gamma$ evolves differently than u, or, in other words, $u|_\Gamma$ has its own dynamic.

6.3 Non-locally reacting acoustic boundary conditions

In this section, we prove a well-posedness result for the wave equation with non-locally reacting acoustic boundary conditions. We restrict our discussion to the case of constant coefficients to improve the readability.

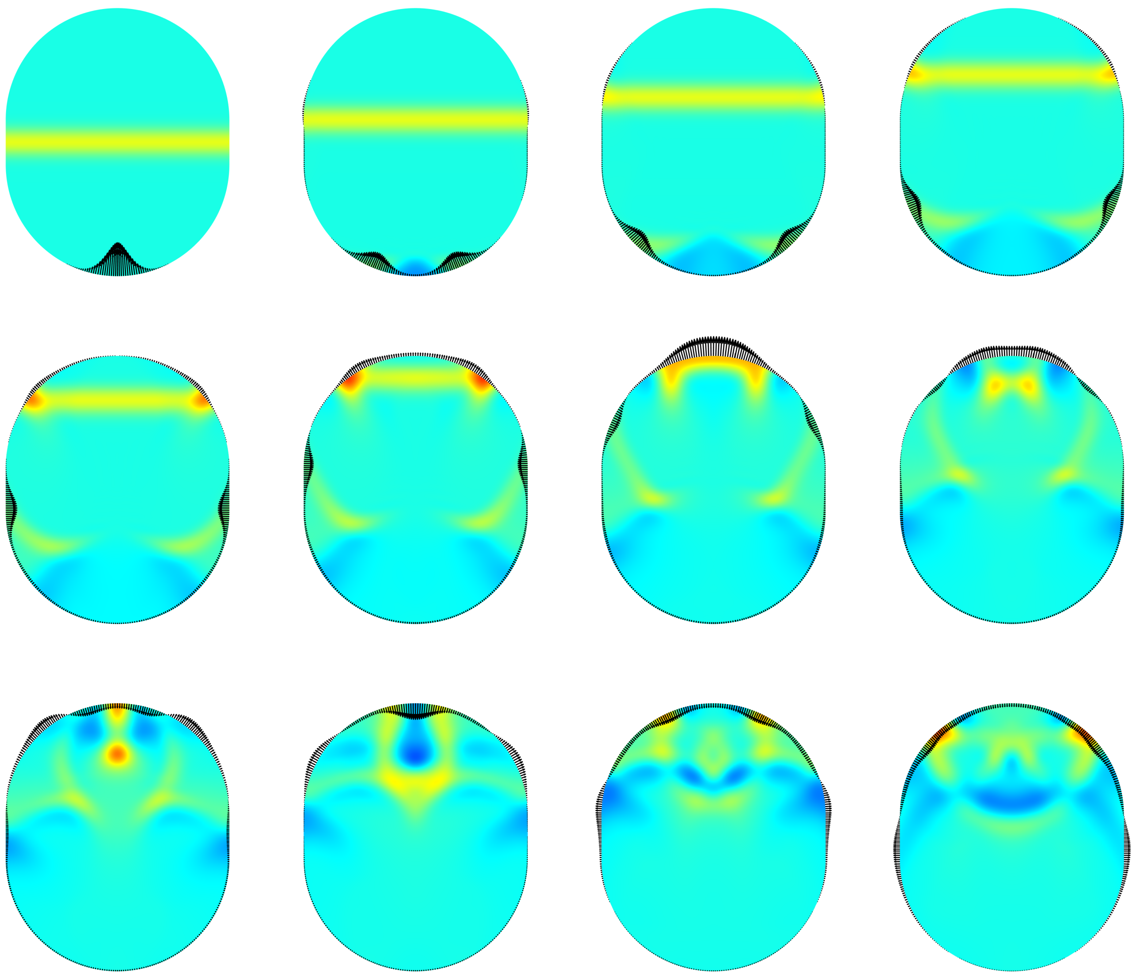


Figure 6.1: An example for a solution of the wave equation with non-locally reacting acoustic boundary conditions with $c_\Omega = m_\Gamma = 1$, $c_\Gamma = 4$ and $a_\Omega = f_\Omega = \alpha_\Gamma = k_\Gamma = f_\Gamma = 0$. The snapshots show the solution u at times $t = 0.2 \cdot k$, $k = 0, \ldots, 11$. The black arrows on the boundary visualize the function δ.

The partial differential equation We seek $u\colon [0,T]\times\Omega\to\mathbb{R}$ and $\delta\colon [0,T]\times\Gamma\to\mathbb{R}$ such that

$$u_{tt} + a_\Omega u - c_\Omega \Delta u = f_\Omega \qquad \text{in } \Omega, \tag{6.17a}$$

$$m_\Gamma \delta_{tt} + k_\Gamma \delta - c_\Gamma \Delta_\Gamma \delta + c_\Omega u_t = f_\Gamma \qquad \text{on } \Gamma, \tag{6.17b}$$

$$\delta_t = \partial_n u \qquad \text{on } \Gamma, \tag{6.17c}$$

where we assume that $c_\Gamma, c_\Omega, m_\Gamma > 0$ and $a_\Omega, k_\Gamma \geq 0$ are constants, that Γ is a C^2-boundary, and that u and δ take initial values $u(0)=u^0$, $u_t(0)=v^0$, $\delta(0)=\delta^0$, $\delta_t(0)=\vartheta^0$.

The variational formulation Following our user's guide, we first derive a variational formulation: Let u and δ be sufficiently smooth solutions of (6.17). Multiplying (6.17a) with $\varphi \in C^\infty(\overline{\Omega})$, integrating over Ω, applying Gauss' Theorem and inserting the boundary condition (6.17c) gives us

$$\int_\Omega u_{tt}\varphi + a_\Omega u\varphi + c_\Omega \nabla u\cdot\nabla\varphi \,\mathrm{dx} - \int_\Gamma c_\Omega \delta_t \varphi \,\mathrm{d}s = \int_\Omega f_\Omega \varphi \,\mathrm{dx}. \tag{6.18a}$$

Analogously, we multiply (6.17b) with $\psi \in C^2(\Gamma)$, integrate over Γ, and use (3) to find

$$\int_\Gamma m_\Gamma \delta_{tt}\psi + k_\Gamma \delta\psi + c_\Gamma \nabla_\Gamma \delta\cdot\nabla_\Gamma\psi + c_\Omega u_t \psi \,\mathrm{d}s = \int_\Gamma f_\Gamma \psi \,\mathrm{d}s. \tag{6.18b}$$

To obtain the complete variational problem for $\vec{u}(t) := \big[u(t),\delta(t)\big]^\mathsf{T}$, we add (6.18b) to (6.18a). Hence classical solutions $\vec{u} \in C^2(\overline{\Omega}\times[0,T]) \times C^2(\Gamma\times[0,T])$ of (6.17) satisfy

$$m\big(\vec{u}''(t),\vec{\varphi}\big) + b\big(\vec{u}'(t),\vec{\varphi}\big) + a\big(\vec{u}(t),\vec{\varphi}\big) = \langle f(t),\vec{\varphi}\rangle \tag{6.19}$$

for all $\vec{\varphi} = \big[\varphi,\psi\big]^\mathsf{T} \in C^\infty(\overline{\Omega})\times C^2(\Gamma)$, where for $\vec{w} = \big[w,\omega\big]^\mathsf{T}$ and $\vec{\varphi} = \big[\varphi,\psi\big]^\mathsf{T}$

$$m\big(\vec{w},\vec{\varphi}\big) := \int_\Omega w\varphi \,\mathrm{dx} + \int_\Gamma m_\Gamma \omega\psi \,\mathrm{d}s \tag{6.20a}$$

$$b\big(\vec{w},\vec{\varphi}\big) := c_\Omega \int_\Gamma w\psi - \omega\varphi \,\mathrm{d}s, \tag{6.20b}$$

$$a\big(\vec{w},\vec{\varphi}\big) := \int_\Omega a_\Omega w\varphi + c_\Omega \nabla w\cdot\nabla\varphi \,\mathrm{dx} + \int_\Gamma k_\Gamma \omega\psi + c_\Gamma \nabla_\Gamma\omega\cdot\nabla_\Gamma\psi \,\mathrm{d}s, \tag{6.20c}$$

$$\langle f(t),\vec{\varphi}\rangle := \int_\Omega f_\Omega(t)\varphi \,\mathrm{dx} + \int_\Gamma f_\Gamma(t)\psi \,\mathrm{d}s, \qquad t\in[0,T]. \tag{6.20d}$$

Well-posedness For the well-posedness result, we need to address the issues in the second part of our user's guide.

COROLLARY 6.9. *Let Γ be a C^2-boundary and the coefficients as described above.*

(i) *If the initial values $\big[u^0,\delta^0\big]^\mathsf{T}, \big[v^0,\vartheta^0\big]^\mathsf{T} \in \mathbb{H}^1$ satisfy $\big[\Delta u^0, \Delta_\Gamma \delta^0\big]^\mathsf{T} \in \mathbb{H}^0$ and $\vartheta^0 = \partial_n u^0$, and $\big[f_\Omega, f_\Gamma\big]^\mathsf{T} \in C^1([0,T];\mathbb{H}^0)$ or $f_\Omega \in C([0,T];H^1(\Omega))$ with $f_\Gamma = 0$, then (6.17) has a unique solution*

$$\big[u,\delta\big]^\mathsf{T} \in C^2\big([0,T];\mathbb{H}^0\big)\cap C^1\big([0,T];\mathbb{H}^1\big), \quad \big[\Delta u, \Delta_\Gamma\delta\big]^\mathsf{T} \in C\big([0,T];\mathbb{H}^0\big),$$

which satisfies

$$\Bigg(\int_\Omega (a_\Omega + c_G)u(t)^2 + c_\Omega|\nabla u(t)|^2 + u'(t)^2 \,\mathrm{d}s + \int_\Gamma (k_\Gamma + m_\Gamma c_G)\delta(t)^2 + c_\Gamma|\nabla_\Gamma \delta(t)|^2 + m_\Gamma \delta'(t)^2 \,\mathrm{d}s\Bigg)^{1/2}$$

$$\leq e^{c_{\mathsf{qm}} t}\Bigg(\Big(\big\|\big[u^0,\delta^0\big]^\mathsf{T}\big\|_{\tilde{a}}^2 + \big\|\big[v^0,\vartheta^0\big]^\mathsf{T}\big\|_m^2\Big)^{1/2} + t\big\|\big[f_\Omega,f_\Gamma\big]^\mathsf{T}\big\|_{L^\infty(0,t;H)}\Bigg)$$

for $t\in[0,T]$ and $c_{\mathsf{qm}} = (\min\{c_\Omega, c_\Gamma\})^{1/2}/2$.

(ii) *Assume that* $a_\Omega > 0$ *and* $k_\Gamma > 0$. *If* $\left[u^0, \delta^0\right]^\mathsf{T} \in \mathbb{H}^1$, $\left[v^0, \vartheta^0\right]^\mathsf{T} \in \mathbb{H}^0$, *and* $\left[f_\Omega, f_\Gamma\right]^\mathsf{T} \in C^1([0,T];\mathbb{H}^{-1}) + C([0,T];\mathbb{H}^0)$, *then there exists a unique weak solution* $u \in C^1([0,T];\mathbb{H}^0) \cap C([0,T];\mathbb{H}^1)$ *of* (6.17) *which satisfies*

$$\begin{aligned}\Big(\int_\Omega u(t)^2\,\mathrm{dx} + \int_\Gamma m_\Gamma \delta(t)^2\,\mathrm{d}s\Big)^{1/2}&\\ \leq C\Bigg(\big\|\left[u^0,\delta^0\right]^\mathsf{T}\big\|_{\mathbb{H}^0} + \big\|u^0\big\|_{L^2(\Gamma)} + \big\|\left[v^0,\vartheta^0\right]^\mathsf{T}\big\|_{\mathbb{H}^{-1}} + t\big\|\left[f_\Omega, f_\Gamma\right]^\mathsf{T}\big\|_{L^\infty(0,t;\mathbb{H}^{-1})}\Bigg)&\end{aligned}$$

for $t \in [0,T]$.

Proof. We choose $H = \mathbb{H}^0$ and $V = \mathbb{H}^1$ for the abstract second-order wave-type problem (4.2) associated with (6.20). For that purpose, we extend m to $H \times H$ and b, a to $V \times V$ continuously. Assumption 4.1 (i) is then satisfied with $c_G = \alpha = \min\{c_\Omega, c_\Gamma\} > 0$. Moreover, since b is skew-symmetric, Assumption 4.1 (ii) holds with $\beta_{\mathsf{qm}} = 0$. If $a_\Omega, k_\Gamma > 0$ as assumed in claim (ii), then a is coercive with $c_G = 0$ and $\alpha = \min\{c_\Omega, a_\Gamma, c_\Gamma, k_\Gamma\} > 0$. Provided (6.1) is fulfilled, Theorem 4.3 states that the second-order wave-type problem (4.2) associated to (6.17) has a unique solution $\vec{u}$. But before we continue with the proof of (i), we characterize (6.1a) in part (A).

(A) We claim that for $\vec{w} = \left[w, \omega\right]^\mathsf{T}, \vec{v} = \left[v, \vartheta\right]^\mathsf{T} \in V = \mathbb{H}^1$

$$\mathcal{A}\vec{w} + \mathcal{B}\vec{v} \in H \qquad \Longleftrightarrow \qquad \left[\Delta w, \Delta_\Gamma \omega\right]^\mathsf{T} \in \mathbb{H}^0 \text{ and } \gamma_n(\nabla w) = \vartheta. \tag{6.21}$$

First assume that $\mathcal{A}w + \mathcal{B}v \in H$. Applying $\left[\varphi, 0\right]^\mathsf{T}$, $\varphi \in H^1(\Omega)$ to $\mathcal{A}\vec{w} + \mathcal{B}\vec{v}$, gives

$$\langle \mathcal{A}\vec{w} + \mathcal{B}\vec{v}, \left[\varphi, 0\right]^\mathsf{T}\rangle_V = \int_\Omega a_\Omega w\varphi + c_\Omega \nabla w \cdot \nabla\varphi\,\mathrm{dx} - \int_\Gamma c_\Omega \vartheta\,\gamma(\varphi)\,\mathrm{d}s.$$

If also $\varphi \in C_c^\infty(\Omega)$, then the surface integral vanishes and we obtain $\nabla w \in H(\mathrm{div}, \Omega)$. Now let $\varphi_\Gamma \in H^{1/2}(\Gamma)$ and let $(\varphi_k) \subset H^1(\Omega)$ be the corresponding sequence from the proof of Corollary 6.5. Then we have for $\vec{\varphi}_k := \left[\varphi_k, 0\right]^\mathsf{T}$

$$\begin{aligned}|\langle c_\Omega\big(\gamma_n(\nabla w) - \vartheta\big), \varphi_\Gamma\rangle_{H^{1/2}(\Gamma)}| &\leq |\langle \mathcal{A}\vec{w} + \mathcal{B}\vec{v}, \vec{\varphi}_k\rangle_V| + \Big|\int_\Omega \big(a_\Omega w - c_\Omega \Delta w\big)\varphi_k\,\mathrm{dx}\Big|\\ &\leq C(\vec{w},\vec{v})\|\vec{\varphi}_k\|_{\mathbb{H}^0}\\ &= C(\vec{w},\vec{v})\|\varphi_k\|_{L^2(\Omega)} \to 0, \qquad k \to \infty.\end{aligned}$$

Hence $\gamma_n(\nabla w) = \vartheta$ holds. Finally, since the right hand side of

$$\int_\Gamma c_\Gamma \nabla_\Gamma \omega \cdot \nabla_\Gamma \psi\,\mathrm{d}s = \langle \mathcal{A}\vec{w} + \mathcal{B}\vec{v}, \left[0, \psi\right]^\mathsf{T}\rangle_V - \int_\Gamma k_\Gamma \omega\psi + c_\Omega\,\gamma(w)\psi\,\mathrm{d}s, \qquad \psi \in H^1(\Gamma),$$

is bounded by $\|\psi\|_{L^2(\Gamma)}$, it follows that $\Delta_\Gamma \omega = \mathrm{div}_\Gamma(\nabla_\Gamma \omega) \in L^2(\Gamma)$. In summary, we showed "$\Rightarrow$" in (6.21). For the other direction, assume that $\vec{w} = \left[w, \omega\right]^\mathsf{T}, \vec{v} = \left[v, \vartheta\right]^\mathsf{T} \in V = \mathbb{H}^1$ satisfy $\left[\Delta w, \Delta_\Gamma \omega\right]^\mathsf{T} \in \mathbb{H}^0$ and $\gamma_n(\nabla w) = \vartheta$. Applying Gauss' Theorem and (3), then yields

$$\begin{aligned}\langle \mathcal{A}\vec{w} + \mathcal{B}\vec{v}, \vec{\varphi}\rangle_V &= \int_\Omega \big(a_\Omega w - c_\Omega \Delta w\big)\varphi\,\mathrm{dx} - \int_\Gamma c_\Omega \gamma_n(\nabla w)\varphi\,\mathrm{d}s\\ &\quad + \int_\Gamma \big(k_\Gamma \omega + c_\Gamma \Delta_\Gamma \omega\big)\psi\,\mathrm{d}s - c_\Omega \int_\Gamma v\psi - \vartheta\varphi\,\mathrm{d}s\\ &= \int_\Omega \big(a_\Omega w - c_\Omega \Delta w\big)\varphi\,\mathrm{dx} + \int_\Gamma \big(k_\Gamma \omega - c_\Gamma \Delta_\Gamma \omega + c_\Omega v\big)\psi\,\mathrm{d}s\end{aligned}$$

where we used $\gamma_n(\nabla w) = \vartheta$ in the second equality. Since the expression on the right hand side is bounded by $C(\vec{w},\vec{v})\|\vec{\varphi}\|_{\mathbb{H}^0}$, this already finishes the proof of (6.21).

Having this auxiliary result at hand, we now show the two assertions from the statement.

(i) Let us now show that (6.1) is true under standing assumptions: To see that (6.1a) holds, we use (6.21). For (6.1b), first observe that $f = \big[f_\Omega, m_\Gamma^{-1} f_\Gamma\big]^\mathsf{T}$ if $f \in H$. Therefore, f satisfies $f \in C^1([0,T];H)$, or $f \in C([0,T];V)$ with $\mathcal{B}f \in C([0,T];H)$, since $f_\Gamma = 0$ and thus

$$b\big(f, \vec{\varphi}\big) = \int_\Gamma c_\Omega f_\Omega \psi \,\mathrm{d}s \le c_\Omega \|f_\Omega\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)} \le C(c_\Omega)\|\vec{\varphi}\|_{\mathbb{H}^0}.$$

Thus, by Theorem 4.3, there exists a unique solution $\vec{u} \in C^2([0,T];\mathbb{H}^0) \cap C^1([0,T];\mathbb{H}^1)$ of the second-order wave-type problem, which satisfies $\mathcal{A}\vec{u} + \mathcal{B}\vec{u}' \in C([0,T];\mathbb{H}^0)$. Analogously to Corollary 6.7, we can show that solution $\vec{u}$ solves (6.17): Observe that with (6.21)

$$\begin{aligned} 0 &= \langle u'' + \mathcal{B}u' + \mathcal{A}u - f, \vec{\varphi}\rangle_V \\ &= \int_\Omega \big(u_{tt} + a_\Omega u - c_\Omega \Delta u - f_\Omega\big)\varphi \,\mathrm{dx} + \int_\Gamma \big(m_\Gamma \delta_{tt} + k_\Gamma \delta - c_\Gamma \Delta_\Gamma \delta + c_\Omega u_t - f_\Gamma\big)\psi \,\mathrm{d}s \end{aligned}$$

for all $\vec{\varphi} = \big[\varphi, \psi\big]^\mathsf{T} \in \mathbb{H}^1$. Inserting $\vec{\varphi} = \big[\varphi, 0\big]^\mathsf{T}$ yields (6.17a), and inserting $\vec{\varphi} = \big[0, \psi\big]^\mathsf{T}$ yields (6.17b). The coupling condition (6.17b) follows directly from (6.21). Finally, $\vec{u}$ with the stated regularity is the unique solution of (6.17), since any such solution satisfies (6.19) and therefore also the associated second-order wave-type problem.

(ii) First note that $(\mathbb{H}^1)^* \simeq \mathbb{H}^{-1}$, since

$$J\colon \mathbb{H}^{-1} \to (\mathbb{H}^1)^*, \qquad \langle J\big[g_\Omega, g_\Gamma\big]^\mathsf{T}, \vec{\varphi}\rangle_{\mathbb{H}^1} \coloneqq \langle g_\Omega, \varphi\rangle_{H^1(\Omega)} + \langle g_\Gamma, \psi\rangle_{H^1(\Gamma)}$$

is continuous and continuously invertible with inverse

$$\widetilde{J}\colon (\mathbb{H}^1)^* \to \mathbb{H}^{-1}, \qquad \widetilde{J}g \coloneqq \big[\varphi \mapsto \langle g, \big[\varphi, 0\big]^\mathsf{T}\rangle_{\mathbb{H}^1}, \psi \mapsto \langle g, \big[0, \psi\big]^\mathsf{T}\rangle_{\mathbb{H}^1}\big]^\mathsf{T},$$

where $\vec{\varphi} = \big[\varphi, \psi\big]^\mathsf{T} \in \mathbb{H}^1$, $\big[g_\Omega, g_\Gamma\big]^\mathsf{T} \in \mathbb{H}^0$, $g \in (\mathbb{H}^1)^*$. Thus we can write the source term as $f = J\big[f_\Omega, f_\Gamma\big]^\mathsf{T}$. As arleady pointed out in the beginning of this proof, the second-order wave-type problem associated to (6.19) satisfies Assumption 4.1 with $c_G = \beta_{\mathsf{qm}} = 0$. It is easy to see that the initial values and the source term $f = J\big[f_\Omega, f_\Gamma\big]^\mathsf{T}$ suffice the conditions of Theorem 4.13. Hence there is a unique weak solution with the claimed regularity. Finally, we obtain the weak stability estimate from (4.25) by using $\|f\|_{(\mathbb{H}^1)^*} \le C\|\big[f_\Omega, f_\Gamma\big]^\mathsf{T}\|_{\mathbb{H}^{-1}}$ and

$$\begin{aligned} \|\mathcal{B}\vec{u}\|_{\widetilde{V}^*} &= \sup_{\|\vec{\varphi}\|_{\widetilde{a}}=1} \int_\Gamma c_\Omega\big(u\psi - \delta\varphi\big)\,\mathrm{d}s \\ &\le c_\Omega \sup_{\|\vec{\varphi}\|_{\widetilde{a}}=1} \|u\|_{L^2(\Gamma)}\|\psi\|_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)}\|\,\gamma(\varphi)\|_{L^2(\Gamma)} \\ &\le C(c_\Omega, \gamma, \alpha)\Big(\|u\|_{L^2(\Gamma)} + \|\delta\|_{L^2(\Gamma)}\Big). \end{aligned}$$

□

Remark 6.10.

(i) It is possible to extend to above considerations to problems with mixed boundary conditions and coefficient functions instead of constants.

(ii) The coupling condition (6.17c) can also be replaced by the elastic coupling

$$\beta\delta_t - \alpha u_t = \partial_n u \qquad \text{on } \Gamma, \qquad \alpha, \beta > 0,$$

which is inspired by [Elliott and Ranner, 2013], or porous couplings as discussed in [Graber, 2012].

(iii) If $c_\Gamma = 0$, then $V = H^1(\Omega) \times L^2(\Gamma)$ is the proper space for the second-order wave-type problem. In this case, Corollary 6.9 reproduces the original well-posedness result from [Beale, 1976]. Related results can be found in [Frota et al., 2011], [Mugnolo, 2006a] and [Gal et al., 2003].

Chapter 7

Numerics for wave equations with dynamic boundary conditions

In this chapter, we discuss the numerical solution of wave equations with dynamic boundary conditions. More precisely, we prove error bounds for isoparametric finite element discretizations of the wave equation with kinetic boundary conditions (6.2) and acoustic boundary conditions (6.17). These results are accompanied by error bounds for full discretizations with the Crank–Nicolson method.

Outline We start by introducing the bulk-surface finite element method for the spatial discretization of partial differential equations in smooth domains, while the remaining part of Section 7.1 is a collection of approximation properties. Using these approximation results in the error bounds of Theorem 5.5 and Theorem 2.18, we then show convergence results for kinetic boundary conditions in Section 7.2 and for acoustic boundary conditions in Section 7.3. We end this chapter with the discussion of some numerical experiments in Section 7.4.

Related works While an error analysis for finite element discretization of wave equations with dynamic boundary conditions seems not be covered by the literature, we mention two important articles which inspired our approach: [Kovács and Lubich, 2016] provide an error analysis of parabolic equations with dynamic boundary conditions. The bulk-surface finite element method was introduced in [Elliott and Ranner, 2013] to discretize stationary coupled bulk-surface partial differential equations of elliptic type.

General assumption In this chapter, we only consider problems in $\Omega \subset \mathbb{R}^d$ where $d = 2$ or $d = 3$ and we assume that $\Gamma \in C^{k+1}$ for some $k \in \mathbb{N}$.

7.1 The bulk-surface finite element method

In this section, we introduce the bulk-surface finite element method from [Elliott and Ranner, 2013]. To keep the following exposition short, we only give the essential constructions and recapitulate the approximation results. For details and proofs, we refer to the mentioned article.

Computational domain Assume that $\Omega_\sharp \subset \mathbb{R}^d$ is a polygonal approximation of the smooth domain Ω and let $\mathcal{T}_h^\sharp$ be a triangulation of $\Omega_\sharp$ which consists of simplices, either triangles for $d = 2$ or tetrahedra for $d = 3$. In addition, we assume the following:

(i) The vertices of $\Gamma_\sharp := \partial\Omega_\sharp$ lie on Γ, so that $\Gamma_\sharp$ is an interpolation of Γ.

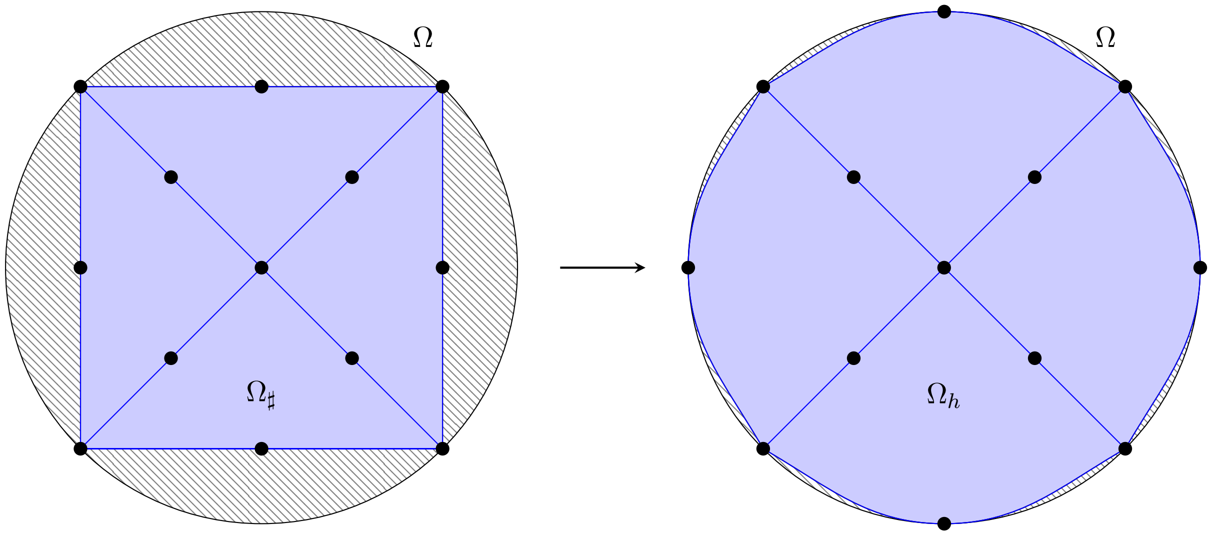


Figure 7.1: This sketch shows an example of an isoparametric triangulation of the disc Ω with $p = 2$. On the left we see the (very coarse) polygonal approximation $\Omega_\sharp$ of Ω with the triangulation $\mathcal{T}_h^\sharp$ and on the right the computational domain Ω_h with $\mathcal{T}_h$.

(ii) The maximal diameter of $\mathcal{T}_h^\sharp$

$$h := \max\left\{ \operatorname{diam}(K^\sharp) \mid K^\sharp \in \mathcal{T}_h^\sharp \right\}$$

is sufficiently small to guarantee that for every $x \in \Gamma_\sharp$ there is a unique normal projection $p(x) \in \Gamma$ s.t. $x - p(x)$ is orthogonal to the tangent plane of Γ at $p(x)$, cf. [Elliott and Ranner, 2013, Section 2.1].

(iii) The triangulation $\mathcal{T}_h^\sharp$ is quasi-uniform, i.e., there exists some constant $\rho > 0$ s.t.

$$\min\left\{ \operatorname{diam}(B_{K^\sharp}) \mid K^\sharp \in \mathcal{T}_h^\sharp \right\} \geq \rho h,$$

where $B_{K^\sharp}$ is the largest ball contained in $K^\sharp$.

(iv) Each $K^\sharp \in \mathcal{T}_h^\sharp$ has at most one face on $\Gamma_\sharp$.

Now let $\mathcal{T}_h^{\mathsf{e}}$ be the exact triangulation of Ω from [Elliott and Ranner, 2013, Sect. 4.1.1]. By construction, $\mathcal{T}_h^{\mathsf{e}}$ contains the internal elements of $\mathcal{T}_h^\sharp$ (all elements with at most one vertex on $\Gamma_\sharp$) and curved simplices at the boundary which exactly match Ω s.t.

$$\bigcup_{K^{\mathsf{e}} \in \mathcal{T}_h^{\mathsf{e}}} K^{\mathsf{e}} = \overline{\Omega}.$$

All elements $K^{\mathsf{e}} \in \mathcal{T}_h^{\mathsf{e}}$ can be expressed via smooth transformations $F_{K^{\mathsf{e}}}^{\mathsf{e}}$ of the unit simplex $\widehat{K}$

$$K^{\mathsf{e}} = F_{K^{\mathsf{e}}}^{\mathsf{e}}(\widehat{K}), \qquad F_{K^{\mathsf{e}}}^{\mathsf{e}} \colon \widehat{K} \to \mathbb{R}^d.$$

We proceed as in [Elliott and Ranner, 2013, Sect. 4.1.2]: Let $\phi_1, \ldots, \phi_{n_p}$ be the Lagrangian basis on $\widehat{K}$ of degree $p \geq 1$ corresponding to the nodal points $\widehat{\mathrm{x}}_1, \ldots, \widehat{\mathrm{x}}_{n_p}$. For each $K^{\mathsf{e}} \in \mathcal{T}_h^{\mathsf{e}}$ we consider the polynomial interpolation of $F_{K^{\mathsf{e}}}^{\mathsf{e}}$ of degree p given by

$$F_{K^{\mathsf{e}}}^{\mathsf{ip}}(\widehat{\mathrm{x}}) := \sum_{j=1}^{n_p} F_{K^{\mathsf{e}}}^{\mathsf{e}}(\widehat{\mathrm{x}}_j)\phi_j(\widehat{\mathrm{x}}), \qquad \widehat{\mathrm{x}} \in \widehat{K},$$

and define the corresponding polynomial simplex as $K := F^{\mathsf{ip}}_{K^{\mathsf{e}}}(\widehat{K}) \approx K^{\mathsf{e}}$. Then we call

$$\mathcal{T}_h := \mathcal{T}_h^{(p)} = \Big\{ K = F^{\mathsf{ip}}_{K^{\mathsf{e}}}(\widehat{K}) \mid K^{\mathsf{e}} \in \mathcal{T}_h^{\mathsf{e}} \Big\}$$

the mesh of isoparametric elements of degree p and

$$\Omega_h := \bigcup_{K \in \mathcal{T}_h} K \approx \Omega$$

the computational domain, cf. Figure 7.1. Consequently, we refer to $\Gamma_h := \partial\Omega_h$ as the computational surface and define the surface triangulation with isoparametric surface elements of degree p by

$$\mathcal{T}_h\big|_{\Gamma_h} := \big\{ F = K \cap \Gamma_h \mid K \in \mathcal{T}_h \text{ has one face on } \Gamma_h \big\}.$$

This construction admits quasi-uniform triangulations $\mathcal{T}_h$ and $\mathcal{T}_h\big|_{\Gamma_h}$ of Ω_h and Γ_h, respectively.

Notation We overload our notation and use $n\colon \Gamma_h \to \mathbb{R}^d$ for the unit outer normal on Γ_h and $\gamma\colon H^1(\Omega_h) \to L^2(\Gamma_h)$ for the trace operator on $H^1(\Omega_h)$. Moreover, we introduce the Hilbert space

$$\mathbb{H}_h^m := H^m(\Omega_h) \times H^m(\Gamma_h), \qquad m = 0, 1$$

and endow them with their canonical norm.

Finite element spaces Let $\mathcal{P}_p(\widehat{K})$ denote the space of polynomials of degree p on $\widehat{K}$, and let F_K be the transformation from $\widehat{K}$ to $K \in \mathcal{T}_h$. For the discretization of bulk-surface function spaces, we introduce finite element functions in the bulk Ω_h and on the surface Γ_h of degree $p \geq 1$

$$\begin{aligned} V_{h,p}^{\Omega} &:= \Big\{ v_h \in C(\Omega_h) \mid v_h|_K = \widehat{v}_h \circ (F_K)^{-1} \text{ with } \widehat{v}_h \in \mathcal{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \Big\}, \\ V_{h,p}^{\Gamma} &:= \Big\{ \vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h|_{\Gamma_h},\ v_h \in V_{h,p}^{\Omega} \Big\}, \end{aligned}$$

which is equivalent to the definition given in [Elliott and Ranner, 2013, Sect. 5.1]. An important part of this construction is the relation

$$\gamma(V_{h,p}^{\Omega}) = V_{h,p}^{\Gamma}. \tag{7.1a}$$

Lift operator Since in general $\Omega_h \neq \Omega$, the finite element solutions in Ω_h and on Γ_h need to be lifted to Ω and Γ, respectively. As proposed in [Elliott and Ranner, 2013, Sect. 4.2], we implement this lifting by means of the elementwise smooth diffeomorphism

$$G_h\colon \Omega_h \to \Omega, \qquad G_h|_K := F^{\mathsf{e}}_{K^{\mathsf{e}}} \circ F_K^{-1} \in C^{p+1}(K) \quad \text{for } p \leq k \text{ and } K \in \mathcal{T}_h.$$

For functions $v_h \in V_{h,p}^{\Omega}$ and $\vartheta_h \in V_{h,p}^{\Gamma}$, we define their lifts as

$$\begin{aligned} v_h^{\ell}(\mathrm{x}) &:= v_h\big(G_h^{-1}(\mathrm{x})\big), && \mathrm{x} \in \Omega, \\ \vartheta_h^{\ell}(\mathrm{x}) &:= \vartheta_h\big(G_h^{-1}(\mathrm{x})\big), && \mathrm{x} \in \Gamma. \end{aligned}$$

Note that this lift operation complies with the discrete spaces in the sense that

$$\gamma(v_h^{\ell}) = \gamma(v_h)^{\ell}, \qquad v_h \in V_{h,p}^{\Omega}. \tag{7.1b}$$

Stability of the lifts A crucial property of this lifting process is the norm equivalence stated in the following Lemma. It collects the results from [Elliott and Ranner, 2013, Prop. 4.9 and 4.13].

LEMMA 7.1.

(i) *There exist constants* $C_{\Omega,\Omega_h} > c_{\Omega,\Omega_h} > 0$ *s.t. for all* $v_h \in V_{h,p}^{\Omega}$

$$c_{\Omega,\Omega_h} \|v_h^\ell\|_{L^2(\Omega)} \le \|v_h\|_{L^2(\Omega_h)} \le C_{\Omega,\Omega_h} \|v_h^\ell\|_{L^2(\Omega)} \tag{7.2a}$$

$$c_{\Omega,\Omega_h} \|\nabla v_h^\ell\|_{L^2(\Omega)} \le \|\nabla v_h\|_{L^2(\Omega_h)} \le C_{\Omega,\Omega_h} \|\nabla v_h^\ell\|_{L^2(\Omega)}. \tag{7.2b}$$

(ii) *There exist constants* $C_{\Gamma,\Gamma_h} > c_{\Gamma,\Gamma_h} > 0$ *s.t. for all* $\vartheta_h \in V_{h,p}^{\Gamma}$

$$c_{\Gamma,\Gamma_h} \|\vartheta_h^\ell\|_{L^2(\Gamma)} \le \|\vartheta_h\|_{L^2(\Gamma_h)} \le C_{\Gamma,\Gamma_h} \|\vartheta_h^\ell\|_{L^2(\Gamma)} \tag{7.2c}$$

$$c_{\Gamma,\Gamma_h} \|\nabla_\Gamma \vartheta_h^\ell\|_{L^2(\Gamma)} \le \|\nabla_{\Gamma_h} \vartheta_h\|_{L^2(\Gamma_h)} \le C_{\Gamma,\Gamma_h} \|\nabla_\Gamma \vartheta_h^\ell\|_{L^2(\Gamma)}. \tag{7.2d}$$

Interpolation error bounds For the error analysis, we will use the approximation properties of the interpolation operators from the following Lemma.

LEMMA 7.2.

(i) *There exists an interpolation operator* $I_h^\Omega \colon H^2(\Omega) \to V_{h,p}^\Omega$ *s.t. for* $1 \le r \le p$

$$\|v - \left(I_h^\Omega v\right)^\ell\|_{L^2(\Omega)} + h\|v - \left(I_h^\Omega v\right)^\ell\|_{H^1(\Omega)} \le Ch^{r+1}\|v\|_{H^{r+1}(\Omega)}, \quad v \in H^{r+1}(\Omega).$$

(ii) *There exists an interpolation operator* $I_h^\Gamma \colon H^2(\Gamma) \to V_{h,p}^\Gamma$ *s.t. for* $1 \le r \le \min\{p,k\}$

$$\|\vartheta - \left(I_h^\Gamma \vartheta\right)^\ell\|_{L^2(\Gamma)} + h\|\vartheta - \left(I_h^\Gamma \vartheta\right)^\ell\|_{H^1(\Gamma)} \le Ch^{r+1}\|\vartheta\|_{H^{r+1}(\Gamma)}, \quad \vartheta \in H^{r+1}(\Gamma),$$

and $I_h^\Gamma \gamma(v) = \gamma(I_h^\Omega v)$ *for* $v \in H^2(\Omega;\Gamma)$.

Proof. The error bounds are shown in [Elliott and Ranner, 2013, Prop. 5.4] for nodal interpolation operators. Hence we obtain $I_h^\Gamma \gamma(v) = \gamma(I_h^\Omega v)$ as a consequence of the compliant nodes of $\mathcal{T}_h$ and $\mathcal{T}_h\big|_{\Gamma_h}$ and (7.1a). □

Domain error bounds To bound the errors of the discrete forms, it will be necessary to estimate the difference between integrals over the exact domain Ω and the computational domain Ω_h. The following Lemma is a collection of such geometric error estimates for different types of integrals. It is a straightforward generalization of [Elliott and Ranner, 2013, Lem. 6.2].

LEMMA 7.3. *Let* $v_h, \varphi_h \in V_{h,p}^\Omega$ *and* $\vartheta_h, \psi_h \in V_{h,p}^\Gamma$.

(i) *Suppose* $\omega_\Omega \in L^\infty(\Omega)$ *and* $\omega_{\Omega_h} \in L^\infty(\Omega_h)$, *and define* $\omega_\Omega^\Delta := \omega_\Omega - \omega_{\Omega_h}^\ell$. *Then we have*

$$|\int_\Omega \omega_\Omega v_h^\ell \varphi_h^\ell \,\mathrm{dx} - \int_{\Omega_h} \omega_{\Omega_h} v_h \varphi_h \,\mathrm{dx}| \le \left(Ch^p + \|\omega_\Omega^\Delta\|_{L^\infty(\Omega)}\right) \|u_h\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)} \tag{7.3a}$$

and

$$|\int_\Omega \omega_\Omega \nabla v_h^\ell \cdot \nabla \varphi_h^\ell \,\mathrm{dx} - \int_{\Omega_h} \omega_{\Omega_h} \nabla v_h \cdot \nabla \varphi_h \,\mathrm{dx}|$$
$$\le \left(Ch^p + \|\omega_\Omega^\Delta\|_{L^\infty(\Omega)}\right) \|\nabla v_h\|_{L^2(\Omega_h)} \|\nabla \varphi_h\|_{L^2(\Omega_h)}. \tag{7.3b}$$

(ii) *Suppose* $\omega_\Gamma \in L^\infty(\Gamma)$ *and* $\omega_{\Gamma_h} \in L^\infty(\Gamma_h)$, *and define* $\omega_\Gamma^\Delta := \omega_\Gamma - \omega_{\Gamma_h}^\ell$. *Then we have*

$$
\begin{aligned}
|\int_\Gamma \omega_\Gamma \vartheta_h^\ell \, \psi_h^\ell \, \mathrm{dx} - \int_{\Gamma_h} \omega_{\Gamma_h} \vartheta_h \, \psi_h \, \mathrm{dx}| \\
\leq \Big(Ch^{\min\{p,k\}+1} + \|\omega_\Gamma^\Delta\|_{L^\infty(\Gamma)} \Big) \|\vartheta_h\|_{L^2(\Gamma_h)} \|\psi_h\|_{L^2(\Gamma_h)} \quad (7.3c)
\end{aligned}
$$

and

$$
\begin{aligned}
|\int_\Gamma \omega_\Gamma \nabla_\Gamma \vartheta_h^\ell \cdot \nabla_\Gamma \psi_h^\ell \, \mathrm{dx} - \int_{\Gamma_h} \omega_{\Gamma_h} \nabla_{\Gamma_h} \vartheta_h \cdot \nabla_{\Gamma_h} \psi_h \, \mathrm{dx}| \\
\leq \Big(Ch^{\min\{p,k\}+1} + \|\omega_\Gamma^\Delta\|_{L^\infty(\Omega)} \Big) \|\nabla_{\Gamma_h} \vartheta_h\|_{L^2(\Gamma_h)} \|\nabla_{\Gamma_h} \psi_h\|_{L^2(\Omega_h)}. \quad (7.3d)
\end{aligned}
$$

(iii) *Suppose* $\vec{\omega}_\Omega \in L^\infty(\Omega)^d$ *and* $\vec{\omega}_{\Omega_h} \in L^\infty(\Omega)^d$, *and define* $\vec{\omega}_\Omega^\Delta := \vec{\omega}_\Omega - \vec{\omega}_{\Omega_h}^\ell$. *Then we have*

$$
\begin{aligned}
|\int_\Omega (\vec{\omega}_\Omega \cdot \nabla v_h^\ell) \varphi_h^\ell \, \mathrm{dx} - \int_{\Omega_h} (\vec{\omega}_{\Omega_h} \cdot \nabla v_h) \varphi_h \, \mathrm{dx}| \\
\leq \Big(Ch^p + \|\vec{\omega}_\Omega^\Delta\|_{L^\infty(\Omega)} \Big) \|\nabla v_h\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)}, \quad (7.3e)
\end{aligned}
$$

where we define the lift of $\vec{\omega}_{\Omega_h} = (w_i)_{i=1}^d$ *componentwise by* $\vec{\omega}_{\Omega_h}^\ell := \big(w_i^\ell\big)_{i=1}^d$.

Proof. The estimates in (i) and (ii) can be shown as in [Elliott and Ranner, 2013, Lem. 6.2]. To handle the additional error term due to weight functions $\omega_\Omega \neq \omega_{\Omega_h}^\ell$, proceed as in the proof of (iii).

(iii) Let DG_h denote the (elementwise defined) Jacobian of G_h. In a first step, we use integration by substitution to rewrite the integral over Ω_h as an integral over Ω. Then we split the error into three parts, each of whom is estimated separately

$$
\begin{aligned}
&\int_\Omega (\vec{\omega}_\Omega \cdot \nabla v_h^\ell) \varphi_h^\ell \, \mathrm{dx} - \int_{\Omega_h} (\vec{\omega}_{\Omega_h} \cdot \nabla v_h) \varphi_h \, \mathrm{dx} \\
&\qquad = \int_\Omega (\vec{\omega}_\Omega \cdot \nabla v_h^\ell) \varphi_h^\ell - \Big(\vec{\omega}_{\Omega_h}^\ell \cdot \big(\nabla v_h\big)^\ell \Big) \varphi_h^\ell |\det DG_h^{-1}| \, \mathrm{dx} \\
&\qquad \leq |\int_\Omega \Big((\vec{\omega}_\Omega - \vec{\omega}_{\Omega_h}^\ell) \cdot \nabla v_h^\ell \Big) \varphi_h^\ell \, \mathrm{d}x| + |\int_B \Big(\vec{\omega}_{\Omega_h}^\ell \cdot \big(\nabla v_h^\ell - \big(\nabla v_h\big)^\ell \big) \Big) \varphi_h^\ell \, \mathrm{dx}| \\
&\qquad\quad + |\int_\Omega \Big(\vec{\omega}_{\Omega_h}^\ell \cdot \big(\nabla v_h\big)^\ell \Big) \varphi_h^\ell \big(1 - |\det DG_h^{-1}|\big) \, \mathrm{dx}| \\
&\qquad \leq \Bigg(\|\vec{\omega}_\Omega^\Delta\|_{L^\infty(\Omega)} + \Big\|\vec{\omega}_{\Omega_h}^\ell\Big\|_{L^\infty(\Omega)} \|\big(DG_h^\intercal\big)^\ell - \mathsf{I}\|_{L^\infty(\Omega)} \\
&\qquad\quad + \Big\| |\vec{\omega}_{\Omega_h}^\ell| \Big\|_{L^\infty(\Omega)} \big\|1 - |\det DG_h^{-1}|\big\|_{L^\infty(\Omega)} \Bigg) \|\nabla v_h^\ell\|_{L^2(\Omega)} \|\varphi_h^\ell\|_{L^2(\Omega)} \\
&\qquad \leq C(\vec{\omega}_{\Omega_h}) \Big(\|\vec{\omega}_\Omega^\Delta\|_{L^\infty(\Omega)} + \|\big(DG_h^\intercal\big)^\ell - \mathsf{I}\|_{L^\infty(\Omega)} + \big\|1 - |\det DG_h^{-1}|\big\|_{L^\infty(\Omega)} \Big) \\
&\qquad\quad \cdot c_{\Omega,\Omega_h}^{-2} \|\nabla v_h\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)}.
\end{aligned}
$$

Here we used $\vec{\omega}_\Omega^\Delta = \vec{\omega}_\Omega - \vec{\omega}_{\Omega_h}^\ell$, $\big(DG_h^\intercal\big)^\ell \nabla \varphi_h^\ell = \big(\nabla \varphi_h\big)^\ell$ (which follows by the chain rule) and Lemma 7.1. To obtain the desired estimate, it remains to analyze

$$
\|\big(DG_h^\intercal\big)^\ell - \mathsf{I}\|_{L^\infty(\Omega)} \quad \text{and} \quad \big\|1 - |\det DG_h^{-1}|\big\|_{L^\infty(\Omega)}.
$$

Let B_h be the union of all elements $K \in \mathcal{T}_h$ with more than one vertex on the boundary and define $B_h^\ell := G_h(B_h)$. By construction of G_h, we have $DG_h = \mathsf{I}$ in $\Omega_h \setminus B_h$ and therefore

$$
\big(DG_h^\intercal\big)^\ell = \mathsf{I} \quad \text{and} \quad |\det DG_h^{-1}| = 1 \quad \text{in } \Omega \setminus B_h^\ell.
$$

Hence it follows from the approximation estimates in [Elliott and Ranner, 2013, Prop. 4.7]

$$\|(DG_h^\mathsf{T})^\ell - \mathsf{I}\|_{L^\infty(\Omega)} = \|(DG_h^\mathsf{T})^\ell - \mathsf{I}\|_{L^\infty(B_h^\ell)} \le \|DG_h^\mathsf{T} - \mathsf{I}\|_{L^\infty(B_h)} \le Ch^p,$$

$$\big\|1 - |\det DG_h^{-1}|\big\|_{L^\infty(\Omega)} = \big\|1 - |\det DG_h^{-1}|\big\|_{L^\infty(B_h^\ell)} \le C\frac{\||\det DG_h| - 1\|_{L^\infty(B_h)}}{\|(|\det DG_h|)^\ell\|_{L^\infty(B_h^\ell)}} \le Ch^p,$$

where we used $|\det DG_h^{-1}| = 1/(|\det DG_h|)^\ell$. This finishes the proof. □

7.2 A priori error bounds for the wave equation with kinetic boundary conditions

In this section, we discuss the numerical approximation of solutions of (6.14) with $\beta_\Gamma = 0$ using isoparametric elements of degree p in space and the Crank–Nicolson method in time.

The finite element approximation The approximate problem is to find a function $u_h\colon [0,T] \to V_{h,p}^\Omega$ which satisfies (5.1) where $f_h\colon [0,T] \to V_{h,p}^\Omega$ is a given function and the bilinear forms are defined by

$$m_h(v_h, \varphi_h) \coloneqq \int_{\Omega_h} v_h\varphi_h \,\mathrm{dx} + \int_{\Gamma_h} \mu_h v_h \varphi_h \,\mathrm{d}s, \tag{7.4a}$$

$$b_h(v_h, \varphi_h) \coloneqq \int_{\Omega_h} (\alpha_{\Omega_h} v_h + \beta_{\Omega_h}\cdot\nabla v_h)\varphi_h \,\mathrm{dx} + \int_{\Gamma_h} \alpha_{\Gamma_h} v_h\varphi_h \,\mathrm{d}s, \tag{7.4b}$$

$$a_h(v_h, \varphi_h) \coloneqq \int_{\Omega_h} a_{\Omega_h} v_h\varphi_h + c_{\Omega_h}\nabla v_h\cdot\nabla\varphi_h \,\mathrm{dx} + \int_{\Gamma_h} a_{\Gamma_h} v_h\varphi_h + c_{\Gamma_h}\nabla_{\Gamma_h} v_h\cdot\nabla_{\Gamma_h}\varphi_h \,\mathrm{d}s. \tag{7.4c}$$

We assume that the coefficients share the properties of their continuous counterparts:

(i) $\mu_h \in L^\infty(\Gamma_h)$ and $c_{\Omega_h} \in L^\infty(\Omega_h)$ are uniformly positive.

(ii) $c_{\Gamma_h} = c_\Gamma$ and $a_{\Gamma_h} = a_\Gamma$ are constants.

(iii) $\alpha_{\Gamma_h} \in L^\infty(\Gamma_h)$, $a_{\Omega_h}, \alpha_{\Omega_h} \in L^\infty(\Omega_h)$ are non-negative.

(iv) $\beta_{\Omega_h} \in L^\infty(\Omega_h)^d$ with $\operatorname{div}\beta_{\Omega_h} \in L^\infty(\Omega_h)$ satisfies $\operatorname{div}\beta_{\Omega_h} \le 0$ in Ω_h and $n\cdot\beta_{\Omega_h} \ge 0$ on Γ_h.

Convergence result of the finite element approximation We are now in the position to give a completely new convergence result for finite element discretizations of the wave equations with kinetic boundary conditions (6.14). Having the error bound for general non-conforming space discretizations of second-order wave-type problems from Theorem 5.5, it only remains to check its applicability and prove estimates for the errors in the data, the errors of the bilinear forms and the errors due to interpolation.

THEOREM 7.4. *Let $\Gamma \in C^{k+1}$ and let u be the solution of (6.14) with $\beta_\Gamma = 0$ from Corollary 6.7 (i). Assume that $u \in C^1([0,T]; H^{p+1}(\Omega;\Gamma)) \cap C^2([0,T]; H^p(\Omega;\Gamma))$ for $1 \le p \le k$ and that there exist constants $C_d, C_c > 0$ s.t.*

$$\|u_h^0 - I_h^\Omega u^0\|_{H^1(\Omega_h;\Gamma_h)} + \|v_h^0 - I_h^\Omega v^0\|_{\mathbb{H}_h^0} + \|f_h - Q_h^{H*}[f_\Omega, \mu^{-1}f_\Gamma]^\mathsf{T}\|_{L^\infty(\mathbb{H}_h^0)} \le C_d h^p,$$

and

$$\|g_\Gamma - g_{\Gamma_h}^\ell\|_{L^\infty(\Gamma)} \le C_c h^p, \qquad [g_\Gamma, g_{\Gamma_h}]^\mathsf{T} \in \Big\{[\mu,\mu_h]^\mathsf{T}, [\alpha_\Gamma, \alpha_{\Gamma_h}]^\mathsf{T}\Big\},$$
$$\|g_\Omega - g_{\Omega_h}^\ell\|_{L^\infty(\Omega)} \le C_c h^p, \qquad [g_\Omega, g_{\Omega_h}]^\mathsf{T} \in \Big\{[\alpha_\Omega, \alpha_{\Omega_h}]^\mathsf{T}, [\beta_\Omega, \beta_{\Omega_h}]^\mathsf{T}, [a_\Omega, a_{\Omega_h}]^\mathsf{T}, [c_\Omega, c_{\Omega_h}]^\mathsf{T}\Big\}.$$

Moreover, let u_h be the finite element solution in $V_{h,p}^\Omega$ with $0 < h \le 1$ as given above.

(i) *Then the lifted semi-discrete solution u_h^ℓ satisfies*

$$\|u_h^\ell(t) - u(t)\|_{H^1(\Omega;\Gamma)} + \|\big(u_h'\big)^\ell(t) - u'(t)\|_{\mathbb{H}^0} \le C\big(1 + te^{t/2}\big)h^p$$

for $t \in [0,T]$ and C independent of h and t.

(ii) *Furthermore, assume that $u \in C^4([0,T];\mathbb{H}^0) \cap C^4([0,T];H^1(\Omega;\Gamma))$ and let u_h^n and v_h^n be given by the Crank–Nicolson scheme* (5.9) *with $\tau \le 4$. Then $Q_h^V u_h^n$ and $Q_h^V v_h^n$ satisfy*

$$\|\big(u_h^n\big)^\ell - u(t_n)\|_{H^1(\Omega;\Gamma)} + \|\big(v_h^n\big)^\ell - u'(t_n)\|_{\mathbb{H}^0} \le C\big(1 + t_n e^{t_n/2}\big)\big(\tau^2 + h^p\big)$$

for $t_n \in [0,T]$, where C is independent of h and t_n.

Proof. Recall that, by Corollary 6.7, the second-order wave-type problem corresponding to (6.14) is well-posed in $V = H^1(\Omega;\Gamma)$ and $H = \mathbb{H}^0$, where V is embedded into H via $v \mapsto \big[v,\, \gamma(v)\big]^\mathsf{T}$.

(i) To apply Theorem 5.5, we formulate the approximate problem as a non-conforming space discretization of a second-order wave-type problem in $V_h := V_{h,p}^\Omega$. The bilinear forms were already defined in (7.4) and we choose

$$Q_h^V \varphi_h := \varphi_h^\ell$$

for the lift operator Q_h^V. To verify $Q_h^V(V_h) \subset V$, let $v_h \in V_{h,p}^\Omega$. Then we have $v_h^\ell \in H^1(\Omega)$ by Lemma 7.1 (i) and (7.1a) further yields $\gamma(v_h) \in V_{h,p}^\Gamma$. Therefore we find $\gamma(v_h^\ell) = \gamma(v_h)^\ell \in H^1(\Gamma)$ with (7.1b) and Lemma 7.1 (ii). Thus $Q_h^V(V_{h,p}^\Omega) \subset H^1(\Omega;\Gamma)$ as required. Now we show that Assumptions 5.1 is fulfilled under the stated assumptions: The assumptions on the discrete bilinear forms m_h, b_h, and a_h follow analogously to the considerations in Section 6.1 with $\widetilde{c}_G = 1$ and $\widetilde{\beta}_{\mathsf{qm}} = 0$. To see that the lift is stable in the sense of Assumption 5.1 (v), set $\mu^- := \min_\Gamma \mu$, $\mu_h^- := \min_{\Gamma_h} \mu_h$, $\mu^+ := \max_\Gamma \mu$, $\mu_h^+ := \max_{\Gamma_h} \mu_h$. Then we find for $\vartheta_h \in V_{h,p}^\Gamma$ by (7.2c)

$$\|\sqrt{\mu}\vartheta_h^\ell\|_{L^2(\Gamma)} \le \frac{\mu^+}{c_{\Gamma,\Gamma_h}}\|\vartheta_h\|_{L^2(\Gamma_h)} \le \frac{\mu^+}{c_{\Gamma,\Gamma_h}\mu_h^-}\|\sqrt{\mu}_h\vartheta_h\|_{L^2(\Gamma_h)},$$
$$\|\sqrt{\mu}_h\vartheta_h\|_{L^2(\Gamma_h)} \le \mu_h^+ C_{\Gamma,\Gamma_h}\|\vartheta_h^\ell\|_{L^2(\Gamma)} \le \frac{\mu_h^+ C_{\Gamma,\Gamma_h}}{\mu^-}\|\sqrt{\mu}\vartheta_h^\ell\|_{L^2(\Gamma)}.$$

Together with (7.2a), it is then easy to see that Assumption 5.1 (v) holds. Since Assumption 5.1 (vi) can be shown analogously, Assumptions 5.1 holds. Therefore Theorem 5.5 yields the general error bound with $\widehat{c}_{\mathsf{qm}} = 1/2$. It remains to show $E_i \le Ch^p$, $i = 1,2,3,4$. For that purpose, we choose the interpolation operator $I_h := I_h^\Omega$ and $Z^V = H^2(\Omega;\Gamma)$.

(E_2) The upper bound for E_2 follows from the approximation results for the interpolation operator I_h^Ω: Let $1 \le r \le \min\{p,k\}$, then

$$\|(\mathsf{I} - Q_h^V I_h)v\|_m + h\|(\mathsf{I} - Q_h^V I_h)v\|_{\widetilde{a}} \le Ch^{r+1}\|v\|_{H^{r+1}(\Omega;\Gamma)}, \quad v \in H^{r+1}(\Omega;\Gamma), \tag{7.5}$$

since by (7.1b) and Lemma 7.2

$$\begin{aligned}\|(\mathsf{I} - Q_h^V I_h)v\|_{\widetilde{a}} &\le C(a_\Omega, c_\Omega, a_\Gamma, c_\Gamma)\|v - \big(I_h^\Omega v\big)^\ell\|_{H^1(\Omega;\Gamma)} \\ &\le C\Big(\|v - \big(I_h^\Omega v\big)^\ell\|_{H^1(\Omega)} + \|\,\gamma(v - \big(I_h^\Omega v\big)^\ell)\|_{H^1(\Gamma)}\Big) \\ &\le Ch^r\|v\|_{H^{r+1}(\Omega)} + C\|\,\gamma(v) - \big(I_h^\Gamma\,\gamma(v)\big)^\ell\|_{H^1(\Gamma)} \\ &\le Ch^r\|v\|_{H^{r+1}(\Omega;\Gamma)},\end{aligned}$$

and, in a similar way,

$$\begin{aligned}\|(\mathsf{I} - Q_h^V I_h)v\|_m &\le C(\mu)\Big(\|v - \big(I_h^\Omega v\big)^\ell\|_{L^2(\Omega)} + \|\,\gamma(v - \big(I_h^\Omega v\big)^\ell)\|_{L^2(\Gamma)}\Big)\\ &\le Ch^{r+1}\|v\|_{H^{r+1}(\Omega;\Gamma)}.\end{aligned}$$

(E_3) To show the upper bound for E_3, we derive estimates for $\Delta\widetilde{a}$ and Δm. Let $v_h \in \widetilde{V}_h$. Then we obtain from (7.3a)–(7.3d) with the assumptions on the coefficients

$$\begin{aligned}|\Delta\widetilde{a}(v_h,\varphi_h)| &\le C\Big(h^p + \|a_\Omega - a_{\Omega_h}^\ell\|_{L^\infty(\Omega)} + \|c_\Omega - c_{\Omega_h}^\ell\|_{L^\infty(\Omega)}\Big)\|v_h\|_{H^1(\Omega_h)}\|\varphi_h\|_{H^1(\Omega_h)}\\ &\quad + C\Big(h^{\min\{p,k\}+1} + \|a_\Gamma - a_{\Gamma_h}^\ell\|_{L^\infty(\Gamma)} + \|c_\Gamma - c_{\Gamma_h}^\ell\|_{L^\infty(\Gamma)}\Big)\|v_h\|_{H^1(\Gamma_h)}\|\varphi_h\|_{H^1(\Gamma_h)}\\ &\le C\big(h^p + h^{\min\{p,k\}+1}\big)\|v_h\|_{\widetilde{a}_h}\|\varphi_h\|_{\widetilde{a}_h},\end{aligned}$$

where we used $\|\cdot\|_{\widetilde{a}} \sim \|\cdot\|_{H^1(\Omega_h;\Gamma_h)}$ in the last inequality. Now set $v_h = Q_h^{V*}v$ for some $v \in V$. By definition of the adjoint lift Q_h^{V*} and the Cauchy–Schwarz inequality, we find

$$\|Q_h^{V*}v\|_{\widetilde{a}_h}^2 = \widetilde{a}_h\big(Q_h^{V*}v, Q_h^{V*}v\big) = \widetilde{a}\big(v, Q_h^V Q_h^{V*}v\big) \le \|v\|_{\widetilde{a}}\|Q_h^V Q_h^{V*}v\|_{\widetilde{a}}.$$

Using (5.2c) and then dividing by $\|Q_h^{V*}v\|_{\widetilde{a}_h}$, we infer $\|Q_h^{V*}v\|_{\widetilde{a}_h} \le c_V^{-1}\|v\|_{\widetilde{a}}$. This yields

$$\|\Delta\widetilde{a}(Q_h^{V*}v)\|_{\widetilde{V}_h^*} = \max_{\|\varphi_h\|_{\widetilde{a}}=1}|\Delta a\big(Q_h^{V*}v,\varphi_h\big)| \le Ch^p\|v\|_{\widetilde{a}}. \tag{7.6}$$

The bound $\|\Delta m(Q_h^{H*}v)\|_{H_h^*} \le Ch^p\|v\|_m$, $v \in H$ follows analogously.

(E_1) By Remark 5.6 (i), it holds

$$\begin{aligned}\|u_h^0 - Q_h^{V*}u^0\|_{\widetilde{a}_h} &\le \|u_h^0 - I_h u^0\|_{\widetilde{a}_h} + 2C_V\|(\mathsf{I} - Q_h^V I_h)u^0\|_{\widetilde{a}} + C_V^2\|\Delta\widetilde{a}(Q_h^{V*}u^0)\|_{\widetilde{V}_h^*}\\ &\le Ch^p\Big(1 + \|u^0\|_{H^{p+1}(\Omega;\Gamma)}\Big),\end{aligned}$$

where we used the assumptions on the data, (7.5), and (7.6) for the second estimate. The remaining terms in E_1 are sufficiently small by assumption ($f = \big[f_\Omega, \mu^{-1}f_\Gamma\big]^\mathsf{T}$ for (6.14)) so that altogether $E_1 \le Ch^p$.

(E_4) It is easy to see that $\mathcal{B} \in \mathcal{L}(V,H)$ for (6.14). Therefore, we obtain from Remark 5.6 (ii) and (7.5)

$$E_4 \le C\Big(h^p\|u'\|_{H^{p+1}(\Omega;\Gamma)} + \|\Delta b(I_h u')\|_{L^\infty(0,T;H_h^*)}\Big).$$

For an upper bound of the last term, let $\psi_h \in V_{h,p}^\Omega$ with $\|\psi_h\|_{m_h} = 1$. By (7.3a), (7.3c), (7.3e), and the assumptions on the coefficients, we get

$$\begin{aligned}\|\Delta b(I_h u')\|_{H_h^*} &= \max_{\|\psi_h\|_{m_h}=1}|\Delta b\big(I_h u',\psi_h\big)|\\ &\le \max_{\|\psi_h\|_{m_h}=1} Ch^p\Big(\|I_h u'\|_{L^2(\Omega_h)} + \|\nabla I_h u'\|_{L^2(\Omega_h)} + \|I_h u'\|_{L^2(\Gamma_h)}\Big)\|\psi_h\|_{m_h}\\ &\le Ch^p\|I_h u'\|_{H^1(\Omega_h)}.\end{aligned}$$

This gives the final estimate, since (5.2b) and (7.5) imply

$$\begin{aligned}\|I_h u'\|_{H^1(\Omega_h)} &\le C_H\Big(\|u'\|_{H^1(\Omega)} + \|(\mathsf{I} - Q_h^V I_h)u'\|_{H^1(\Omega_h)}\Big)\\ &\le C_H\|u'\|_{H^1(\Omega)} + Ch^p\|u'\|_{H^{p+1}(\Omega;\Gamma)}.\end{aligned}$$

(ii) By assumption it is $u \in C^4([0,T];H) \cap C^3([0,T];\widetilde{V})$ and $\tau\widehat{c}_{\mathsf{qm}} < 2$, since $\widehat{c}_{\mathsf{qm}} = 1/2$. Thus Corollary 5.9 applies and the desired estimate is a consequence of $E_i \le Ch^p$, $i = 1,2,3,4$. □

7.3 A priori error bounds for the wave equation with acoustic boundary conditions

In this section, we discuss the numerical approximation of solutions of (6.17) using isoparametric bulk and surface elements of degree p in space and the Crank–Nicolson method in time.

The finite element approximation The approximate problem is to find functions $u_h \colon [0,T] \to V_{h,p}^{\Omega}$ and $\delta_h \colon [0,T] \to V_{h,p}^{\Gamma}$ s.t. $\vec{u}_h := \big[u_h, \delta_h\big]^\intercal$ satisfies (5.1). Here $f_h := \big[f_{\Omega_h}, f_{\Gamma_h}\big]^\intercal$ is given by the functions $f_{\Omega_h} \colon [0,T] \to V_{h,p}^{\Omega}$ and $f_{\Gamma_h} \colon [0,T] \to V_{h,p}^{\Gamma}$, and the bilinear forms are defined as

$$m_h\big(\vec{v}_h, \vec{\varphi}_h\big) := \int_{\Omega_h} v_h \varphi_h \,\mathrm{d}x + \int_{\Gamma_h} m_\Gamma \vartheta_h \psi_h \,\mathrm{d}s, \tag{7.7a}$$

$$b_h\big(\vec{v}_h, \vec{\varphi}_h\big) := c_\Omega \int_{\Gamma_h} v_h\, \psi_h - \vartheta_h\, \varphi_h \,\mathrm{d}s, \tag{7.7b}$$

$$a_h\big(\vec{v}_h, \vec{\varphi}_h\big) := \int_{\Omega_h} a_\Omega v_h \varphi_h + c_\Omega \nabla v_h \cdot \nabla \varphi_h \,\mathrm{d}x + \int_{\Gamma_h} k_\Gamma \vartheta_h\, \psi_h + c_\Gamma \nabla_{\Gamma_h} \vartheta_h \cdot \nabla_{\Gamma_h} \psi_h \,\mathrm{d}s \tag{7.7c}$$

for $\vec{v}_h = \big[v_h, \vartheta_h\big]^\intercal, \vec{\varphi}_h = \big[\varphi_h, \psi_h\big]^\intercal \in V_{h,p}^{\Omega} \times V_{h,p}^{\Gamma}$. Furthermore, assume that u_h and δ_h have initial values $u_h(0) = u_h^0$, $u_h'(0) = v_h^0$, $\delta_h(0) = \delta_h^0$, $\delta_h'(0) = \vartheta_h^0$. Note that the coefficients are exact, and recall that $c_\Gamma, c_\Omega, m_\Gamma > 0$ and $a_\Omega, k_\Gamma \geq 0$.

Convergence result of the finite element approximation For the numerical analysis of this space discretization, we proceed as in Theorem 7.4.

THEOREM 7.5. *Let $\Gamma \in C^{k+1}$ and let u and δ be the solutions of* (6.17) *from Corollary 6.9 (i). Assume that $\big[u, \delta\big]^\intercal \in C^1\big([0,T]; \mathbb{H}^{p+1}\big) \cap C^2\big([0,T]; \mathbb{H}^{p}\big)$ for $1 \leq p \leq k$ and let there exist some constant $C_d > 0$ s.t.*

$$\|u_h^0 - I_h^\Omega u^0\|_{H^1(\Omega_h)} + \|\delta_h^0 - I_h^\Gamma \delta^0\|_{H^1(\Gamma_h)} + \|v_h^0 - I_h^\Omega v^0\|_{L^2(\Omega_h)} + \|\vartheta_h^0 - I_h^\Gamma \vartheta^0\|_{L^2(\Gamma_h)} \leq C_d h^p$$

and

$$\|\big[f_{\Omega_h}, f_{\Gamma_h}\big]^\intercal - Q_h^{H*}\big[f_\Omega, m_\Gamma^{-1} f_\Gamma\big]^\intercal\|_{L^\infty(\mathbb{H}^0)} \leq C_d h^p.$$

Moreover, let u_h and δ_h be the finite element solutions in $V_{h,p}^{\Omega}$ and $V_{h,p}^{\Gamma}$, respectively, with $0 < h \leq 1$.

(i) *Then the lifted semi-discrete solutions u_h^ℓ and δ_h^ℓ satisfy*

$$\begin{aligned}&\|u_h^\ell(t) - u(t)\|_{H^1(\Omega)} + \|\big(u_h'\big)^\ell(t) - u'(t)\|_{L^2(\Omega)} \\ &\qquad + \|\delta_h^\ell(t) - \delta(t)\|_{H^1(\Gamma)} + \|\big(\delta_h'\big)^\ell(t) - \delta'(t)\|_{L^2(\Gamma)} \leq C e^{\widehat{c}_{\mathsf{qm}} t}\big(1+t\big) h^p\end{aligned}$$

for $t \in [0,T]$ and $\widehat{c}_{\mathsf{qm}} = (\min\{c_\Omega, c_\Gamma\})^{1/2}/2$, where C is independent of h and t.

(ii) *Furthermore, assume that $\vec{u} \in C^4([0,T]; \mathbb{H}^0) \cap C^4([0,T]; \mathbb{H}^1)$ and let $\vec{u}_h^n = \big[u_h^n, \delta_h^n\big]^\intercal$, $\vec{v}_h^n = \big[v_h^n, \vartheta_h^n\big]^\intercal$ be given by the Crank–Nicolson scheme* (5.9). *For $\tau \widehat{c}_{\mathsf{qm}} \leq 2$ with $\widehat{c}_{\mathsf{qm}} = (\min\{c_\Omega, c_\Gamma\})^{1/2}/2$, the approximations satisfy*

$$\begin{aligned}&\|\big(u_h^n\big)^\ell - u(t_n)\|_{H^1(\Omega)} + \|\big(v_h^n\big)^\ell - u'(t_n)\|_{L^2(\Omega)} \\ &\qquad + \|\big(\delta_h^n\big)^\ell - \delta(t_n)\|_{H^1(\Gamma)} + \|\big(\vartheta_h^n\big)^\ell(t) - \delta'(t_n)\|_{L^2(\Gamma)} \leq C e^{\widehat{c}_{\mathsf{qm}} t_n}\big(1+t_n\big)\big(\tau^2 + h^p\big)\end{aligned}$$

for $t \in [0,T]$ and C is independent of h and t.

Proof. Recall that, by Corollary 6.9, the second-order wave-type problem corresponding to (6.17) is well-posed in $V = \mathbb{H}^1$ and $H = \mathbb{H}^0$. In this proof, let $\vec{v} = \big[v, \vartheta\big]^\intercal$, $\vec{\varphi} = \big[\varphi, \psi\big]^\intercal$, $\vec{v}_h = \big[v_h, \vartheta_h\big]^\intercal$, $\vec{\varphi}_h = \big[\varphi_h, \psi_h\big]^\intercal$.

(i) To apply Theorem 5.5, we formulate the approximate problem as a non-conforming space discretization of a second-order wave-type problem in $V_h := V_{h,p}^\Omega \times V_{h,p}^\Gamma$. The bilinear forms were already defined in (7.7) and we choose the lift operator

$$Q_h^V \vec{\varphi}_h := \big[\varphi_h^\ell, \psi_h^\ell\big]^\intercal. \tag{7.8}$$

Since the coefficients in $\|\cdot\|_m$ and $\|\cdot\|_{\widetilde{a}}$ are constant, Lemma 7.1 directly implies that $Q_h^V : V_h \to V$ is stable in the sense of Assumptions 5.1 (v) and 5.1 (vi). Moreover, the discrete bilinear forms m_h, b_h, and a_h satisfy Assumptions 5.1 with $\widetilde{c}_G = \widetilde{\alpha} = \min\{c_\Omega, c_\Gamma\}$ and $\widetilde{\beta}_{\mathsf{qm}} = 0$. This can be shown exactly as in the continuous case. Therefore Theorem 5.5 yields the general error bound with $\widehat{c}_{\mathsf{qm}} = 1/2\widetilde{c}_G^{-1/2}$ and it remains to show $E_i \le Ch^p$, $i = 1,2,3,4$. For that purpose, we choose the interpolation operator $I_h := \big(I_h^\Omega, I_h^\Gamma\big) : Z^V \to V_h$, $Z^V = \mathbb{H}^2$.

(E_2) The upper bound for E_2 follows from the approximation results from Lemma 7.2: Let $1 \le r \le \min\{p,k\}$, then we have for $\vec{v} \in \mathbb{H}^{r+1}$

$$\begin{aligned}\|(\mathsf{I} - Q_h^V I_h)\vec{v}\|_m + h\|(\mathsf{I} - Q_h^V I_h)\vec{v}\|_{\widetilde{a}} &\le \|v - \big(I_h^\Omega v\big)^\ell\|_{L^2(\Omega)} + m_\Gamma\|\vartheta - \big(I_h^\Gamma \vartheta\big)^\ell\|_{L^2(\Gamma)} \\ &\quad + \max\Big\{\sqrt{a_\Omega + \widetilde{c}_G}, \sqrt{c_\Omega}\Big\} h\|v - \big(I_h^\Omega v\big)^\ell\|_{H^1(\Omega)} \\ &\quad + \max\Big\{\sqrt{k_\Gamma + m_\Gamma \widetilde{c}_G}, \sqrt{c_\Gamma}\Big\} h\|\vartheta - \big(I_h^\Gamma \vartheta\big)^\ell\|_{H^1(\Gamma)} \\ &\le Ch^{r+1}\|\vec{v}\|_{\mathbb{H}^{r+1}}.\end{aligned} \tag{7.9}$$

(E_3) We proceed as in the proof of Theorem 7.4 to show

$$\|\Delta m(Q_h^{H*}\vec{v})\|_{H_h^*} \le Ch^p\|v\|_{L^2(\Omega)} + Ch^{p+1}\|\vartheta\|_{L^2(\Gamma)}, \qquad \vec{v} \in \mathbb{H}^0 \tag{7.10a}$$

$$\|\Delta \widetilde{a}(Q_h^{V*}\vec{v})\|_{\widetilde{V}_h^*} \le Ch^p\|v\|_{H^1(\Omega)} + Ch^{p+1}\|\vartheta\|_{H^1(\Gamma)}, \qquad \vec{v} \in \mathbb{H}^1. \tag{7.10b}$$

Using the assumed regularity of the exact solution, this yields $E_3 \le Ch^p$.

(E_1) As in the proof of Theorem 7.4 (with $\big[u^0, \delta^0\big]^\intercal$ instead of u^0), it can be show that the approximation properties of the initial values are sufficient for

$$\|\big[u_h^0, \delta_h^0\big]^\intercal - Q_h^{V*}\big[u^0, \delta^0\big]^\intercal\|_{\widetilde{a}_h} \le Ch^p\Big(1 + \|u^0\|_{H^{p+1}(\Omega;\Gamma)}\Big).$$

Since $f = \big[f_\Omega, m_\Gamma^{-1} f_\Gamma\big]^\intercal$ for (6.17), the assumptions on the data also guarantee that the remaining terms in E_1 are bounded by a constant times h^p.

(E_4) It remains to study $E_4 = \max_{\|\vec{\varphi}_h\|_{m_h}=1}|b\big(\vec{u}', Q_h^V\vec{\varphi}_h\big) - b_h\big(I_h\vec{u}', \vec{\varphi}_h\big)|$. First, we rewrite b as

$$b\big(\vec{v}, \vec{\varphi}\big) = \frac{c_\Omega}{m_\Gamma}\Big(m\big(\big[0, \gamma(v)\big]^\intercal, \big[0, \psi\big]^\intercal\big) - m\big(\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi)\big]^\intercal\big)\Big). \tag{7.11}$$

Using (7.1b) and (7.8), we transform that the first term in E_4 to

$$\begin{aligned}\frac{m_\Gamma}{c_\Omega} b\big(\vec{v}, Q_h^V\vec{\varphi}_h\big) &= m\big(\big[0, \gamma(v)\big]^\intercal, \big[0, \psi_h^\ell\big]^\intercal\big) - m\big(\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi_h^\ell)\big]^\intercal\big) \\ &= m\big(\big[0, \gamma(v)\big]^\intercal, \big[0, \psi_h^\ell\big]^\intercal\big) - m\big(\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi_h)^\ell\big]^\intercal\big) \\ &= m\big(\big[0, \gamma(v)\big]^\intercal, Q_h^V\big[0, \psi_h\big]^\intercal\big) - m\big(\big[0, \vartheta\big]^\intercal, Q_h^V\big[0, \gamma(\varphi_h)\big]^\intercal\big) \\ &= m_h\big(Q_h^{H*}\big[0, \gamma(v)\big]^\intercal, \big[0, \psi_h\big]^\intercal\big) - m_h\big(Q_h^{H*}\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi_h)\big]^\intercal\big).\end{aligned}$$

Since b_h also admits a representation like (7.11), we have for $\vec{\varphi}_h$ with $\|\vec{\varphi}_h\|_{m_h} = 1$

$$
\begin{aligned}
&|b\big(\vec{v}, Q_h^V \vec{\varphi}_h\big) - b_h\big(Q_h^{H*}\vec{v}, \vec{\varphi}_h\big)| \\
&\quad = |\frac{c_\Omega}{m_\Gamma}\Big(m_h\big((Q_h^{H*} - I_h)\big[0, \gamma(v)\big]^\intercal, \big[0, \psi_h\big]^\intercal\big) - m_h\big((Q_h^{H*} - I_h)\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi_h)\big]^\intercal\big)\Big)| \\
&\quad \leq \frac{c_\Omega}{m_\Gamma}\Big(|m_h\big((Q_h^{H*} - I_h)\big[0, \gamma(v)\big]^\intercal, \big[0, \psi_h\big]^\intercal\big)| + |m_h\big((Q_h^{H*} - I_h)\big[0, \vartheta\big]^\intercal, \big[0, \gamma(\varphi_h)\big]^\intercal\big)|\Big) \\
&\quad \leq \frac{c_\Omega}{m_\Gamma}\Big(\|(Q_h^{H*} - I_h)\big[0, \gamma(v)\big]^\intercal\|_{m_h} + \|(Q_h^{H*} - I_h)\big[0, \vartheta\big]^\intercal\|_{m_h}\sqrt{m_\Gamma}\|\,\gamma(\varphi_h)\|_{L^2(\Gamma_h)}\Big),
\end{aligned}
$$

where we applied the Cauchy–Schwarz inequality for m_h in the last step. Using the continuity of the trace operator and the inverse inequality from [Brenner and Scott, 2008, Lem. 4.5.3], we find

$$
\|\,\gamma(\varphi_h)\|_{L^2(\Gamma_h)} \leq \|\gamma\|_{L^2(\Gamma_h)\leftarrow H^1(\Omega_h)}\|\varphi_h\|_{H^1(\Omega_h)} \leq Ch^{-1}\|\varphi_h\|_{L^2(\Omega_h)} \leq Ch^{-1}
$$

and therefore

$$
E_4 \leq C\Big(\|(Q_h^{H*} - I_h)\big[0, \gamma(u')\big]^\intercal\|_{m_h} + h^{-1}\|(Q_h^{H*} - I_h)\big[0, \delta'\big]^\intercal\|_{m_h}\Big). \tag{7.12}
$$

A bound for such terms follows from (5.2b), $P_h^H = Q_h^{H*}Q_h^V$, and (5.5b), which yield

$$
\begin{aligned}
\|(Q_h^{H*} - I_h)\vec{v}\|_{m_h} &\leq C_H\Big(\|(P_h^H - \mathsf{I})\vec{v}\|_m + \|(\mathsf{I} - Q_h^V I_h)\vec{v}\|_m\Big) \\
&\leq 2C_H\|(\mathsf{I} - Q_h^V I_h)\vec{v}\|_m + C_H^2\|\Delta m(Q_h^{H*}\vec{v})\|_{H_h^*}, \qquad \vec{v} \in H.
\end{aligned}
$$

Hence the first term in (7.12) is bounded by

$$
\|(Q_h^{H*} - I_h)\big[0, \gamma(u')\big]^\intercal\|_{m_h} \leq Ch^p\|\,\gamma(u')\|_{H^p(\Gamma)} + Ch^{p+1}\|\,\gamma(u')\|_{L^2(\Gamma)} \leq Ch^p\|u'\|_{H^{p+1}(\Omega)},
$$

where we used (7.9), (7.10a) and the continuity of the trace operator $\gamma\colon H^{k+1}(\Omega) \to H^k(\Gamma)$, $1 \leq k \leq p$, cf. [Han and Atkinson, 2009, Thm. 7.3.11]. For the second term in (7.12), we find with (7.9) and (7.10a)

$$
\|(Q_h^{H*} - I_h)\big[0, \delta'\big]^\intercal\|_{m_h} \leq Ch^{p+1}\|\delta'\|_{H^{p+1}(\Gamma)} + Ch^{p+1}\|\delta'\|_{L^2(\Gamma)} \leq Ch^{p+1}\|\delta'\|_{H^{p+1}(\Gamma)}.
$$

Summing both estimates, we arrive at

$$
E_4 \leq Ch^p\Big(\|u'\|_{H^{p+1}(\Omega)} + \|\delta'\|_{H^{p+1}(\Gamma)}\Big),
$$

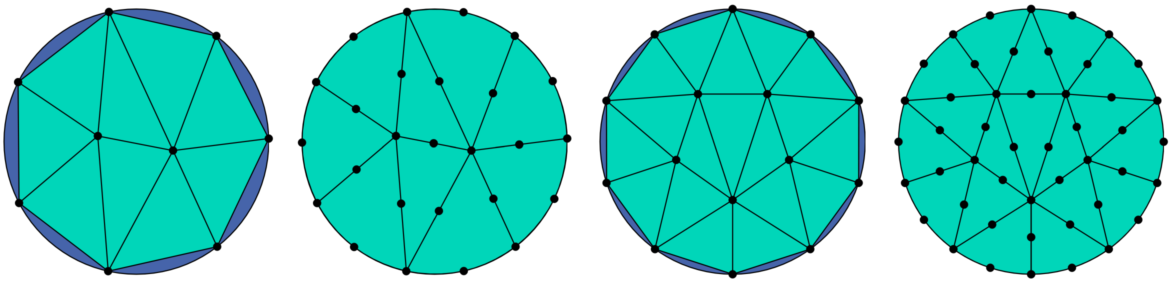
which finishes this proof.

(ii) Note that by assumption $u \in C^4([0,T];H) \cap C^3([0,T];V)$ and $\tau\widehat{c}_{\mathsf{qm}} < 2$. Thus Corollary 5.9 applies and the desired estimate is a consequence of $E_i \leq Ch^p$, $i = 1, 2, 3, 4$. □

7.4 Numerical experiments

In this section, we present the results of our numerical experiments.

Implementation We implemented the linear and quadratic isoparametric finite element method for the wave equation in dimension $d = 2$ in Matlab. Our assembly and visualization routines are based on the P2Q2Iso2D code provided in [Bartels et al., 2006]. For the triangulation of the domain we use the distmesh package from [Persson and Strang, 2004] and the corresponding quadratic nodes stem from a code written for [Kovács, 2016].



(a) $h \approx 0.6$, $p = 1$ (b) $h \approx 0.6$, $p = 2$ (c) $h \approx 0.45$, $p = 1$ (d) $h \approx 0.45$, $p = 2$

Figure 7.2: Triangulations of the unit disc Ω (in blue) with isoparametric elements of degree p. Note that we remesh the domain to obtain finer triangulations. The black dots indicate the nodes of the finite element basis.

Computational domain We numerically solve two examples of the scalar wave equation in the unit disc

$$\Omega = \{\mathrm{x} = [\mathrm{x}_1, \mathrm{x}_2]^\intercal \in \mathbb{R}^2 \mid |\mathrm{x}| < 1\}.$$

To guarantee a consistent mesh quality over all numerical tests, we remesh the domain to obtain finer triangulations, cf. Figure 7.2.

Example with kinetic boundary conditions The first example is the wave equation with kinetic boundary conditions

$$u_{tt}(t, \mathrm{x}) - \Delta u(t, \mathrm{x}) = f_\Omega(t, \mathrm{x}), \qquad \mathrm{x} \in \Omega,\ t \geq 0, \tag{7.13a}$$
$$u_{tt}(t, \mathrm{x}) - \Delta_\Gamma u(t, \mathrm{x}) = f_\Gamma(t, \mathrm{x}) - \partial_n u(t, \mathrm{x}), \qquad \mathrm{x} \in \Gamma,\ t \geq 0. \tag{7.13b}$$

Let $f_\Omega(t, \mathrm{x}) = -4\pi^2 \sin(2\pi t)\mathrm{x}_1\mathrm{x}_2$ and $f_\Gamma(t, \mathrm{x}) = (6 - 4\pi^2) \sin(2\pi t)\mathrm{x}_1\mathrm{x}_2$. Then

$$u(t, \mathrm{x}) = \sin(2\pi t)\mathrm{x}_1\mathrm{x}_2, \qquad \mathrm{x} \in \Omega,\ t \geq 0, \tag{7.14}$$

is a solution of (7.13), since $-\Delta_\Gamma \mathrm{x}_1\mathrm{x}_2 = 4\mathrm{x}_1\mathrm{x}_2$ for the unit sphere Γ. The finite element discretization of (7.13) is given in Section 7.2. We consider the case $p = 1, 2$ with exact coefficients and $f_{\Omega_h} = I_h^\Omega f_\Omega$, $f_{\Gamma_h} = I_h^\Gamma f_\Gamma$. For the initial values at $t = 0$ of the finite element approximations, we use $u_h^0 = I_h^\Omega u(0, \cdot)$ and $v_h^0 = I_h^\Omega u_t(0, \cdot)$.

Example with acoustic boundary conditions As a second example, we approximate the solution of a wave equation with acoustic boundary conditions

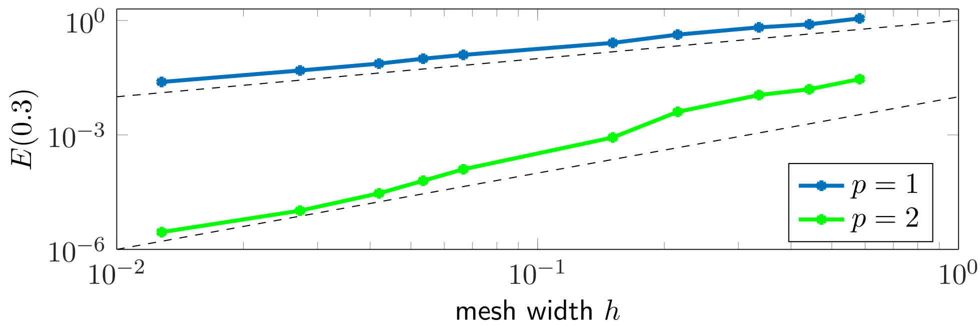
$$u_{tt}(t, \mathrm{x}) - \Delta u(t, \mathrm{x}) = f_\Omega(t, \mathrm{x}), \qquad \mathrm{x} \in \Omega,\ t \geq 0, \tag{7.15a}$$
$$\delta_{tt}(t, \mathrm{x}) - \Delta_\Gamma \delta(t, \mathrm{x}) + u_t(t, \mathrm{x}) = f_\Gamma(t, \mathrm{x}), \qquad \mathrm{x} \in \Gamma,\ t \geq 0, \tag{7.15b}$$
$$\delta_t(t, \mathrm{x}) = \partial_n u(t, \mathrm{x}), \qquad \mathrm{x} \in \Gamma,\ t \geq 0. \tag{7.15c}$$

Choosing $f_\Omega(t, \mathrm{x}) = -4\pi^3 \cos(2\pi t)\mathrm{x}_1\mathrm{x}_2$ and $f_\Gamma(t, \mathrm{x}) = (4 - 6\pi^2) \sin(2\pi t)\mathrm{x}_1\mathrm{x}_2$, this problem has the solution

$$u(t, \mathrm{x}) = \pi \cos(2\pi t)\mathrm{x}_1\mathrm{x}_2, \qquad \mathrm{x} \in \Omega,\ t \geq 0, \tag{7.16a}$$
$$\delta(t, \mathrm{x}) = \sin(2\pi t)\mathrm{x}_1\mathrm{x}_2, \qquad \mathrm{x} \in \Gamma,\ t \geq 0. \tag{7.16b}$$

We described the finite element method for (7.15) in Section 7.3. As in the first example, we choose the initial values and source terms as interpolations of the exact data.

(a) Space discretization errors of (7.13).



(b) Space discretization errors of (7.15).

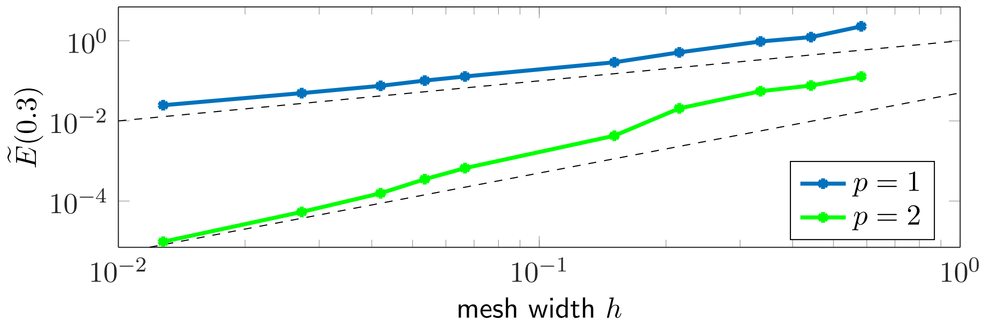


Figure 7.3: Error of the isoparametric finite element approximations of order $p = 1$ (blue line) and $p = 2$ (green line). The dashed lines indicate slopes 1 and 2.

Details of the experiments Let $\Omega_{h,2}$ be the approximation of Ω with quadratic isoparametric finite elements and let $I^{\Omega}_{h,2} : H^2(\Omega) \to V^{\Omega}_{h,2}$, $I^{\Gamma}_{h,2} : H^2(\Gamma) \to V^{\Gamma}_{h,2}$ be the corresponding interpolation operators. We investigate the convergence of $u_h(t) \in V^{\Omega}_{h,p}$, $p = 1, 2$ to (7.14) by considering the error

$$e_h(t) := u_h(t) - I^{\Omega}_{h,2} u(t)$$

in the energy norm

$$E(t) := \Big(\|e_h(t)\|^2_{H^1(\Omega_{h,2};\Gamma_{h,2})} + \|e'_h(t)\|^2_{L^2(\Omega_{h,2})\times L^2(\Gamma_{h,2})} \Big)^{1/2},$$

where $\Gamma_{h,2} := \partial\Omega_{h,2}$. To compare the linear finite element approximations in $V^{\Omega}_{h,1}$ and $V^{\Gamma}_{h,1}$ with the quadratic interpolations of the exact solutions, we lift them from $\Omega_{h,1}$ and $\Gamma_{h,1}$ to $\Omega_{h,2}$ and $\Gamma_{h,2}$, respectively. For the approximation of (7.16), we consider the error

$$\vec{e}_h(t) := \begin{bmatrix} u_h(t) - I^{\Omega}_{h,2} u(t) \\ \delta_h(t) - I^{\Gamma}_{h,2} \delta(t) \end{bmatrix}$$

in the energy norm

$$\widetilde{E}(t) := \Big(\|\vec{e}_h(t)\|^2_{H^1(\Omega_{h,2})\times H^1(\Gamma_{h,2})} + \|\vec{e_h}'(t)\|^2_{L^2(\Omega_{h,2})\times L^2(\Gamma_{h,2})} \Big)^{1/2}.$$

We will also write $E(t_n)$ and $\widetilde{E}(t_n)$ for the energy norms of the full discrete errors $e^n_h := u^n_h - I^{\Omega}_{h,2} u(t_n)$ and $\vec{e}^{\,n}_h := \vec{u}^{\,n}_h - \Big(I^{\Omega}_{h,2}, I^{\Gamma}_{h,2} \Big) \vec{u}(t_n)$, respectively.

Convergence of the space discretization The error plots in Figure 7.3 confirm that the space discretizations converge with $\mathcal{O}(h^p)$, which confirms the error bounds from Theorems 7.4 and 7.5. For the time integration of these examples, we used a Gauss Runge–Kutta method with $s = 3$ stages and time step size $\tau \approx h$.

Convergence of the full discretization We show the full discretization errors for time integration with different schemes in Figure 7.4. In both experiments, we see that the Crank–Nicolson method and the Gauss Runge–Kutta method with $s = 1$ converge quadratically until the space discretization error dominates. These results confirm the convergence rates from Theorems 7.4 and 7.5. The Gauss Runge–Kutta method with $s = 2$ stages converges with $\mathcal{O}(\tau^4)$, although we expect $\mathcal{O}(\tau^{s+1})$, cf. [Pazur, 2013]. This full convergence rate with $\mathcal{O}(\tau^{2s})$ can be explained with [Brenner et al., 1982, Thm. 1], since the exact solutions of our examples are sufficiently smooth.

(a) Full discretization errors of (7.13).

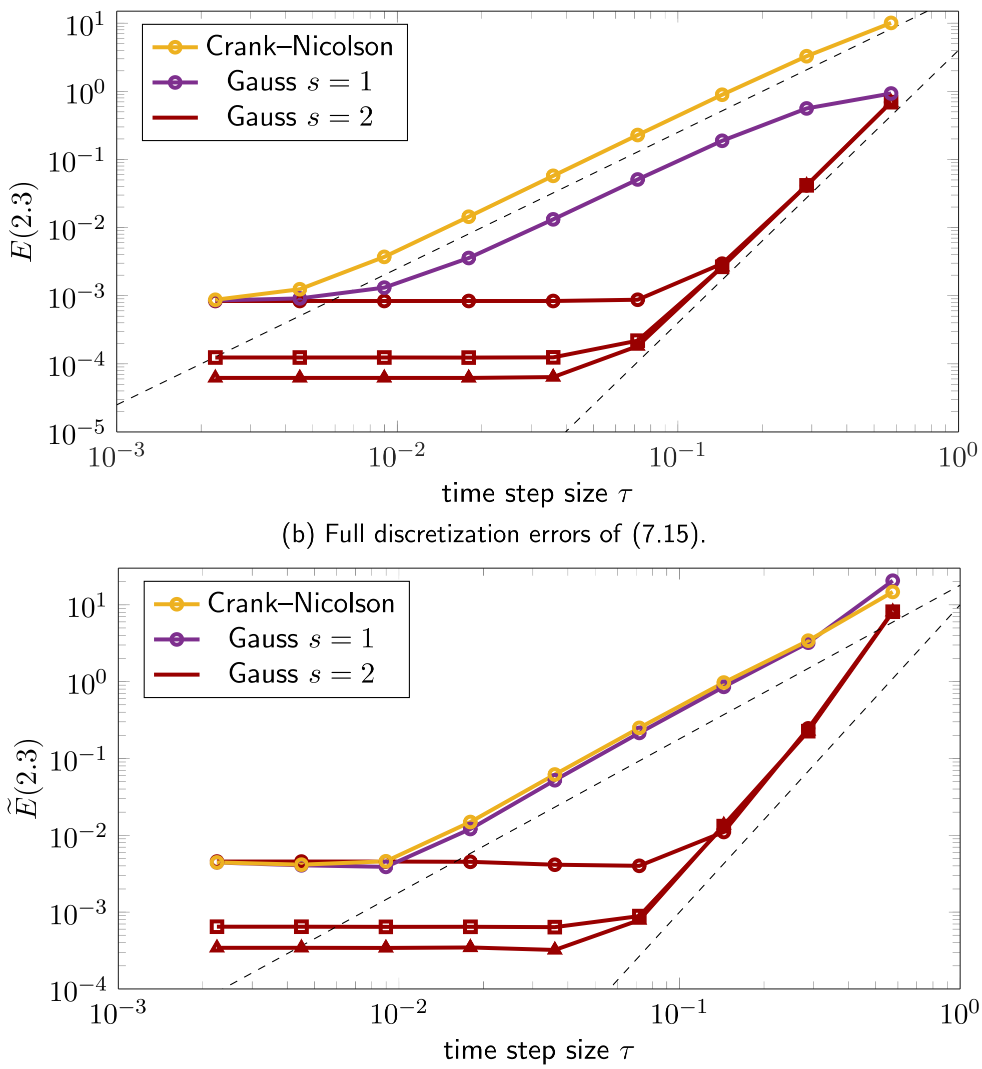


Figure 7.4: Error at time $t = 2.3$ of the full discretizations of (7.13) with isoparametric finite elements approximations of order $p = 2$ and mesh width $h = 0.1508$. The yellow line shows the error of the Crank–Nicolson method w.r.t. to the time step size τ. The violet and red lines correspond to Gauss Runge–Kutta methods with $s = 1$ and $s = 2$ stages, respectively. The dashed line indicates slope 2 and 4. For the Gauss methods with $s = 2$ we additionally plot the error on mesh sizes $h = 0.0666$ (square) and $h = 0.0535$ (triangle).

Bibliography

[Abdulle and Grote, 2011] Abdulle, A. and Grote, M. (2011). Finite Element Heterogeneous Multiscale Method for the Wave Equation. *Multiscale Modeling & Simulation*, 9(2):766–792.

[Amann, 1995] Amann, H. (1995). *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA.

[Antoine et al., 2008] Antoine, X., Arnold, A., Besse, C., Ehrhardt, M., and Schädle, A. (2008). A review of transparent and artificial boundary conditions techniques for linear and nonlinear Schrödinger equations. *Commun. Comput. Phys.*, 4(4):729–796.

[Arendt et al., 2011] Arendt, W., Batty, C. J. K., Hieber, M., and Neubrander, F. (2011). *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition.

[Baker, 1976] Baker, G. A. (1976). Error estimates for finite element methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 13(4):564–576.

[Baker and Bramble, 1979] Baker, G. A. and Bramble, J. H. (1979). Semidiscrete and single step fully discrete approximations for second order hyperbolic equations. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 13(2):75–100.

[Baker and Dougalis, 1976] Baker, G. A. and Dougalis, V. A. (1976). The effect of quadrature errors on finite element approximations for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 13(4):577–598.

[Bales and Lasiecka, 1995] Bales, L. and Lasiecka, I. (1995). Negative norm estimates for fully discrete finite element approximations to the wave equation with nonhomogeneous Dirichlet boundary data. *Mathematics of Computation*, 64(209):89–115.

[Banjai et al., 2015] Banjai, L., Lubich, C., and Sayas, F.-J. (2015). Stable numerical coupling of exterior and interior problems for the wave equation. *Numerische Mathematik*, 129(4):611–646.

[Banks, 2012] Banks, H. T. (2012). *A Functional Analysis Framework for Modeling, Estimation and Control in Science and Engineering*. CRC Press.

[Bartels et al., 2006] Bartels, S., Carstensen, C., and Hecht, A. (2006). Isoparametric FEM in Matlab. *Journal of Computational and Applied Mathematics*, 192(2):219–250.

[Barucq, Helene et al., 2012] Barucq, Helene, Diaz, Julien, and Duprat, Veronique (2012). Micro-differential boundary conditions modelling the absorption of acoustic waves by 2d arbitrarily-shaped convex surfaces. *Communications in Computational Physics*, 11:674–690.

[Beale, 1976] Beale, J. T. (1976). Spectral properties of an acoustic boundary condition. *Indiana University Mathematics Journal*, 25(9):895–917.

[Beale and Rosencrans, 1974] Beale, J. T. and Rosencrans, S. I. (1974). Acoustic boundary conditions. *Bulletin of the American Mathematical Society*, 80(6):1276–1278.

[Ben Belgacem et al., 1997] Ben Belgacem, F., Bernardi, C., Costabel, M., and Dauge, M. (1997). Un résultat de densité pour les équations de Maxwell. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 324(6):731–736.

[Benzoni-Gavage and Serre, 2007] Benzoni-Gavage, S. and Serre, D. (2007). *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford.

[Brenner et al., 1982] Brenner, P., Crouzeix, M., and Thomée, V. (1982). Single step methods for inhomogeneous linear differential equations in Banach space. *RAIRO-Analyse numérique*, 16(1):5–26.

[Brenner and Scott, 2008] Brenner, S. C. and Scott, R. (2008). *The mathematical theory of finite element methods*, volume 15. Springer Science & Business Media.

[Burazin and Erceg, 2016] Burazin, K. and Erceg, M. (2016). Non-Stationary Abstract Friedrichs Systems. *Mediterranean Journal of Mathematics*, 13(6):3777–3796.

[Burman et al., 2010] Burman, E., Ern, A., and Fernández, M. (2010). Explicit Runge–Kutta Schemes and Finite Elements with Symmetric Stabilization for First-Order Linear PDE Systems. *SIAM Journal on Numerical Analysis*, 48(6):2019–2042.

[Campos, 2007] Campos, L. M. (2007). On 36 Forms of the Acoustic Wave Equation in Potential Flows and Inhomogeneous Media. *Applied Mechanics Reviews*, 60(4):149–171.

[Ciarlet, 2002] Ciarlet, P. G. (2002). *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[Cockburn et al., 2014] Cockburn, B., Qiu, W., and Solano, M. (2014). A priori error analysis for HDG methods using extensions from subdomains to achieve boundary conformity. *Mathematics of Computation*, 83(286):665–699.

[Cohen and Pernet, 2016] Cohen, G. and Pernet, S. (2016). *Finite Element and Discontinuous Galerkin Methods for Transient Wave Equations*. Springer, New York, NY, 1st ed. 2017 edition.

[Cohen, 2002] Cohen, G. C. (2002). *Higher-order numerical methods for transient wave equations*. Scientific Computation. Springer-Verlag, Berlin.

[Di Pietro and Ern, 2012] Di Pietro, D. A. and Ern, A. (2012). *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg.

[Disser et al., 2015] Disser, K., Meyries, M., and Rehberg, J. (2015). A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces. *Journal of Mathematical Analysis and Applications*, 430(2):1102–1123.

[Dunca, 2017] Dunca, A. A. (2017). On an optimal finite element scheme for the advection equation. *Journal of Computational and Applied Mathematics*, 311:522–528.

[Dupont, 1973] Dupont, T. (1973). L^2-estimates for galerkin methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 10(5):880–889.

[Elliott and Ranner, 2013] Elliott, C. M. and Ranner, T. (2013). Finite element analysis for a coupled bulk–surface partial differential equation. *IMA Journal of Numerical Analysis*, 33(2):377–402.

[Emmrich and Thalhammer, 2010] Emmrich, E. and Thalhammer, M. (2010). Convergence of a Time Discretisation for Doubly Nonlinear Evolution Equations of Second Order. *Foundations of Computational Mathematics*, 10(2):171–190.

[Engel et al., 1999] Engel, K.-J., Nagel, R., Brendle, S., Campiti, M., Hahn, T., Metafune, G., Nickel, G., Pallara, D., Perazzoli, C., Rhandi, A., Romanelli, S., and Schnaubelt, R. (1999). *One-Parameter Semigroups for Linear Evolution Equations*. Springer, New York, 2000 edition.

[Fattorini, 1985] Fattorini, H. O. (1985). *Second order linear differential equations in Banach spaces*, volume 108 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam.

[Figotin and Reyes, 2015] Figotin, A. and Reyes, G. (2015). Lagrangian variational framework for boundary value problems. *Journal of Mathematical Physics*, 56(9):093506.

[Frota et al., 2011] Frota, C. L., Medeiros, L. A., and Vicente, A. (2011). Wave equation in domains with non-locally reacting boundary. *Differential Integral Equations*, 24(11-12):1001–1020.

[Fujita et al., 2001] Fujita, H., Saito, N., and Suzuki, T. (2001). *Operator theory and numerical methods*, volume 30 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam.

[Gal and Tebou, 2017] Gal, C. and Tebou, L. (2017). Carleman Inequalities for Wave Equations with Oscillatory Boundary Conditions and Application. *SIAM Journal on Control and Optimization*, 55(1):324–364.

[Gal et al., 2003] Gal, C. G., Goldstein, G. R., and Goldstein, J. A. (2003). Oscillatory boundary conditions for acoustic wave equations. volume 3, pages 623–635.

[Gander, 2015] Gander, M. J. (2015). 50 Years of Time Parallel Time Integration. In *Multiple Shooting and Time Domain Decomposition Methods*. Springer, Cham.

[Gilbarg and Trudinger, 2001] Gilbarg, D. and Trudinger, N. S. (2001). *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the 1998 edition.

[Goldstein, 2006] Goldstein, G. R. (2006). Derivation and physical interpretation of general boundary conditions. *Advances in Differential Equations*, 11(4):457–480.

[Graber, 2012] Graber, P. J. (2012). *The Wave Equation with Generalized Nonlinear Acoustic Boundary Conditions*. Ph.D., University of Virginia.

[Graber and Lasiecka, 2014] Graber, P. J. and Lasiecka, I. (2014). Analyticity and Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions. *Semigroup Forum*, 88(2):333–365.

[Graber and Shomberg, 2016] Graber, P. J. and Shomberg, J. L. (2016). Attractors for strongly damped wave equations with nonlinear hyperbolic dynamic boundary conditions. *Nonlinearity*, 29(4):1171.

[Grisvard, 2011] Grisvard, P. (2011). *Elliptic Problems in Nonsmooth Domains*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.

[Grote and Sim, 2011] Grote, M. J. and Sim, I. (2011). Local nonreflecting boundary condition for time-dependent multiple scattering. *Journal of Computational Physics*, 230(8):3135–3154.

[Guaily and Epstein, 2013] Guaily, A. G. and Epstein, M. (2013). Boundary conditions for hyperbolic systems of partial differentials equations. *Journal of Advanced Research*, 4(4):321–329.

[Guidetti et al., 2004] Guidetti, D., Karasözen, B., and Piskarev, S. (2004). Approximation of Abstract Differential Equations. *Journal of Mathematical Sciences*, 122(2):3013–3054.

[Hagstrom and Lau, 2007] Hagstrom, T. and Lau, S. (2007). Radiation boundary conditions for Maxwell's equations: a review of accurate time-domain formulations. Technical Report 3.

[Hagstrom et al., 2008] Hagstrom, T., Mar-Or, A., and Givoli, D. (2008). High-order local absorbing conditions for the wave equation: Extensions and improvements. *Journal of Computational Physics*, 227(6):3322–3357.

[Hairer et al., 2010] Hairer, E., Lubich, C., and Wanner, G. (2010). *Geometric numerical integration*, volume 31 of *Springer Series in Computational Mathematics*. Springer, Heidelberg. Reprint of the second (2006) edition.

[Hairer and Wanner, 2010] Hairer, E. and Wanner, G. (2010). *Solving ordinary differential equations. II*, volume 14 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin. Second revised edition, paperback.

[Han and Atkinson, 2009] Han, W. and Atkinson, K. E. (2009). *Theoretical Numerical Analysis*, volume 39 of *Texts in Applied Mathematics*. Springer New York, New York, NY.

[Hochbruck et al., 2014] Hochbruck, M., Jahnke, T., and Schnaubelt, R. (2014). Convergence of an ADI splitting for Maxwell's equations. *Numerische Mathematik*, 129(3):535–561.

[Hochbruck and Pažur, 2015] Hochbruck, M. and Pažur, T. (2015). Implicit Runge–Kutta Methods and Discontinuous Galerkin Discretizations for Linear Maxwell's Equations. *SIAM Journal on Numerical Analysis*, 53(1):485–507.

[Hochbruck and Stohrer, 2016] Hochbruck, M. and Stohrer, C. (2016). Finite element heterogeneous multiscale method for time-dependent Maxwell's equations. CRC 1173 preprint 2016/25, Karlsruhe Institute of Technology.

[Hochbruck and Sturm, 2016] Hochbruck, M. and Sturm, A. (2016). Error Analysis of a Second-Order Locally Implicit Method for Linear Maxwell's Equations. *SIAM Journal on Numerical Analysis*, 54(5):3167–3191.

[Ito and Kappel, 2002] Ito, K. and Kappel, F. (2002). *Evolution equations and approximations*, volume 61 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co., Inc., River Edge, NJ.

[Joly, 2012] Joly, P. (2012). An elementary introduction to the construction and the analysis of perfectly matched layers for time domain wave propagation. *SeMA Journal*, 57(1):5–48.

[Kashiwabara et al., 2015] Kashiwabara, T., Colciago, C., Dedè, L., and Quarteroni, A. (2015). Well-Posedness, Regularity, and Convergence Analysis of the Finite Element Approximation of a Generalized Robin Boundary Value Problem. *SIAM Journal on Numerical Analysis*, 53(1):105–126.

[Kato, 1985] Kato, T. (1985). *Abstract Differential Equations and Nonlinear Mixed Problems*. Scuola normale superiore.

[Kirsch and Hettlich, 2014] Kirsch, A. and Hettlich, F. (2014). *The Mathematical Theory of Time-Harmonic Maxwell's Equations: Expansion-, Integral-, and Variational Methods*. Springer, New York, 2015 edition.

[Kovács and Lubich, 2016] Kovács, B. and Lubich, C. (2016). Numerical analysis of parabolic problems with dynamic boundary conditions. *IMA Journal of Numerical Analysis*, 37(1):1.

[Kovács, 2016] Kovács, B. (2016). High-order evolving surface finite element method for parabolic problems on evolving surfaces. *arXiv:1606.07234 [math]*.

[Layton, 1983] Layton, W. (1983). Stable Galerkin Methods for Hyperbolic Systems. *SIAM Journal on Numerical Analysis*, 20(2):221–233.

[Lescarret and Zuazua, 2015] Lescarret, V. and Zuazua, E. (2015). Numerical approximation schemes for multi-dimensional wave equations in asymmetric spaces. *Mathematics of Computation*, 84(291):119–152.

[Lions and Magenes, 1972] Lions, J.-L. and Magenes, E. (1972). *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York-Heidelberg. Die Grundlehren der mathematischen Wissenschaften, Band 181.

[Lubich and Mansour, 2015] Lubich, C. and Mansour, D. (2015). Variational discretization of wave equations on evolving surfaces. *Mathematics of Computation*, 84(292):513–542.

[Monk, 2003] Monk, P. (2003). *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York.

[Morse and Ingard, 1987] Morse, P. M. and Ingard, K. U. (1987). *Theoretical Acoustics*. Princeton University Press, Princeton, N.J.

[Mugnolo, 2006a] Mugnolo, D. (2006a). Abstract wave equations with acoustic boundary conditions. *Mathematische Nachrichten*, 279(3):299–318.

[Mugnolo, 2006b] Mugnolo, D. (2006b). Abstract wave equations with acoustic boundary conditions. *Math. Nachr.*, 279(3):299–318.

[Mugnolo, 2011] Mugnolo, D. (2011). Damped wave equations with dynamic boundary conditions. *J. Appl. Anal.*, 17(2):241–275.

[Mugnolo and Romanelli, 2006] Mugnolo, D. and Romanelli, S. (2006). Dirichlet forms for general Wentzell boundary conditions, analytic semigroups, and cosine operator functions. *Electronic Journal of Differential Equations*, 2006(118):1–20.

[Nicaise, 2017] Nicaise, S. (2017). Convergence and stability analyses of hierarchic models of dissipative second order evolution equations. *Collectanea Mathematica*, pages 1–30.

[Nickel, 2004] Nickel, G. (2004). A semigroup approach to dynamic boundary value problems. *Semigroup Forum*, 69(2):159–183.

[Nobile and Vergara, 2008] Nobile, F. and Vergara, C. (2008). An Effective Fluid-Structure Interaction Formulation for Vascular Dynamics by Generalized Robin Conditions. *SIAM Journal on Scientific Computing*, 30(2):731–763.

[Nédélec, 1986] Nédélec, J. C. (1986). A new family of mixed finite elements in r3. *Numerische Mathematik*, 50(1):57–81.

[Pazur, 2013] Pazur, T. (2013). *Error analysis of implicit and exponential time integration of linear Maxwell's equations*. Ph.D., Karlsruhe Institute of Technology.

[Pazy, 1992] Pazy, A. (1992). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York, 1st ed. 1983. corr. 2nd printing 1992 edition.

[Persson and Strang, 2004] Persson, P. and Strang, G. (2004). A Simple Mesh Generator in MATLAB. *SIAM Review*, 46(2):329–345.

[Picard et al., 2014] Picard, R., Trostorff, S., and Waurick, M. (2014). Well-posedness via Monotonicity. An Overview. *arXiv:1401.5294 [math-ph]*.

[Roubíček, 2013] Roubíček, T. (2013). *Nonlinear Partial Differential Equations with Applications*. Springer Basel, Basel.

[Sanz-Serna and Palencia, 1985] Sanz-Serna, J. M. and Palencia, C. (1985). A General Equivalence Theorem in the Theory of Discretization Methods. *Mathematics of Computation*, 45(171):143–152.

[Schnaubelt and Weiss, 2010] Schnaubelt, R. and Weiss, G. (2010). Two classes of passive time-varying well-posed linear systems. *Mathematics of Control, Signals, and Systems*, 21(4):265–301.

[Showalter, 1994] Showalter, R. E. (1994). *Hilbert space methods for partial differential equations*. Electronic Monographs in Differential Equations, San Marcos, TX. Electronic reprint of the 1977 original.

[Showalter, 2013] Showalter, R. E. (2013). *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. American Mathematical Soc.

[Sturm, 2017] Sturm, A. (2017). *Locally Implicit Time Integration for Linear Maxwell's Equations*. Ph.D., Karlsruhe Institute of Technology.

[ter Elst et al., 2012] ter Elst, A. F. M., Meyries, M., and Rehberg, J. (2012). Parabolic equations with dynamical boundary conditions and source terms on interfaces. *arXiv:1206.0600 [math]*.

[ter Elst et al., 2015] ter Elst, A. F. M., Sauter, M., and Vogt, H. (2015). A generalisation of the form method for accretive forms and operators. *Journal of Functional Analysis*, 269(3):705–744.

[Trostorff, 2014] Trostorff, S. (2014). A characterization of boundary conditions yielding maximal monotone operators. *Journal of Functional Analysis*, 267(8):2787–2822.

[Tucsnak and Weiss, 2009] Tucsnak, M. and Weiss, G. (2009). *Observation and control for operator semigroups*. Birkhäuser Verlag, Basel.

[Vedurmudi et al., 2016] Vedurmudi, A. P., Goulet, J., Christensen-Dalsgaard, J., Young, B. A., Williams, R., and van Hemmen, J. L. (2016). How Internally Coupled Ears Generate Temporal and Amplitude Cues for Sound Localization. *Physical Review Letters*, 116(2):028101.

[Vitillaro, 2013] Vitillaro, E. (2013). Strong solutions for the wave equation with a kinetic boundary condition. 594:295–307.

[Vitillaro, 2015] Vitillaro, E. (2015). On the the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and source. *arXiv:1506.00910 [math]*. arXiv: 1506.00910.

[Vitillaro, 2016] Vitillaro, E. (2016). On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources. *arXiv:1601.07075 [math]*.

[Xiao and Liang, 2004] Xiao, T.-J. and Liang, J. (2004). Complete second order differential equations in Banach spaces with dynamic boundary conditions. *Journal of Differential Equations*, 200(1):105–136.

[Zeidler, 1990a] Zeidler, E. (1990a). *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York.

[Zeidler, 1990b] Zeidler, E. (1990b). *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York.

[Zhang et al., 2004] Zhang, B., Boström, A., and Niklasson, A. J. (2004). Antiplane shear waves from a piezoelectric strip actuator: exact versus effective boundary condition solutions. *Smart Materials and Structures*, 13(1):161.

[Zhao, 2004] Zhao, J. (2004). Analysis of finite element approximation for time-dependent Maxwell problems. *Mathematics of Computation*, 73(247):1089–1105.