Mono- and polychromatic ground states for semilinear curl-curl wave equations

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

Dissertation

von

Andreas Hirsch

Referent: Prof. Dr. Wolfgang Reichel

Korreferent: Prof. Dr. Michael Plum

Datum der mündlichen Prüfung: 7. Juni 2017
Mai 2017

Acknowledgement

This thesis would not have been possible without the help of a lot of people. First of all, I would like to thank and express deepest gratitude to my supervisor Prof. Dr. Wolfgang Reichel for entrusting me with this research topic, spending endless time (day and night) with uncountably many helpful discussions, filtering positive aspects in half-baked ideas, always being patient and never losing optimism.

Moreover, I would like to thank Prof. Dr. Michael Plum for agreeing to be the second supervisor and various discussions on the estimation of convolution integrals and related topics which appeared during the process of finding polychromatic solutions.

I am grateful to my colleagues in the CRC 1173, especially in the Workgroup Nonlinear Partial Differential Equations and the Research Group Numerical Analysis who contributed a lot to a comfortable working atmosphere. In particular, I gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft through CRC 1173 where this research was partly funded by.

Special thanks goes to Martin Spitz, Peter Rupp and Carlos Hauser for numerous discussions on and off topic and lots of fun in and outside working hours. Thanks also to Martin for pointing out a new contribution concerning nondegeneracy. Unfortunately it didn't help at all or as he said: "Habe ich schon vermutet, dass es nicht hilft. Aber ich dachte es wäre fahrlässig es dir nicht zu schicken; es gibt ja auch Leute, die im Lotto gewinnen ...".

Last but not least I want to thank Anke as well as my family for invaluable support in all non-mathematical issues.

Contents

Introduction

In this thesis, we investigate the quasilinear curl-curl wave equation

$$\nabla \times \nabla \times E + \partial_t^2 (q(x)E + \tilde{f}(x, |E|^2)E) = 0 \tag{1}$$

and its semilinear variant

$$\nabla \times \nabla \times E + q_1(x)E + q_2(x)\partial_t^2 E + \tilde{f}(x, |E|^2)E = 0, \tag{2}$$

where $E\colon \mathbb{R}^4 \to \mathbb{C}^3, q, q_1, q_2\colon \mathbb{R}^3 \to \mathbb{R}$ and $\tilde{f}\colon \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$. We seek time-periodic solutions of (1) and (2). For a physical motivation of (1) and (2) and in particular a connection to the three-dimensional system of Maxwell equations without currents and charges

$$\begin{aligned} \nabla \times E + \partial_t B = 0, \quad \operatorname{div} D = 0, \\ \nabla \times H - \partial_t D = 0, \quad \operatorname{div} B = 0, \end{aligned}$$

together with a linear connection between B and H as well as a nonlinear relation between D and E, see Section 1.3 in [5].

Concerning the time-periodicity of solutions we pursue two different variants: a mono- and a polychromatic ansatz, which naturally result in the two parts of this thesis.

In the first part we make a *monochromatic* approach for E, i.e., $E(x, t) = U(x)e^{i\omega t}$ for $U\colon \mathbb{R}^3 \to \mathbb{R}^3$. When inserted in (1) or (2) it leads to an equation of the type

$$\nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U \text{ in } \mathbb{R}^3. \tag{3}$$

We assume $V \in L^\infty(\mathbb{R}^3)$ positive and $f\colon \mathbb{R}^3 \times [0, \infty) \to [0, \infty)$ being a non-negative superlinear Carathéodory function which grows at infinity in the second variable with a power at most $\frac{p-1}{2}$ for $p \in (1, 5)$. Notice that for the quasilinear system (1) only the case $V \leq 0$ in (3) is physically relevant.

In general, an ansatz of the form $E(x, t) = U(x)e^{i\omega t}$ is complex-valued and therefore not relevant from a physical point of view. If we slightly modify the nonlinearity $\tilde{f}(x, |E|^2)$ in (1) and (2) then we can generate real-valued solutions. Precisely, we have to replace $\tilde{f}(x, |E|^2)$ by $\hat{f}(x, \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} |E|^2 dt)$ and make an ansatz of the form $E(x, t) = U(x)\cos(\omega t)$, see for instance (1.8) in [67], (2.2) in [68] or Chapter 6 in [69]. Then due to $\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} |\cos(\omega t)|^2 dt = 1$ we conlcude that (1) and (2) with $\hat{f}$ instead of $\tilde{f}$ again reduce to (3).

In general, weak solutions of (3) arise as critical points of the functional

$$\mathcal{J}(U) = \int_{\mathbb{R}^3} \Big(|\nabla \times U|^2 + V(x)|U|^2 - F(x, |U|^2)\Big)\, dx$$

for $F(x,s) := \int_0^s f(x,\tau)d\tau$ and

$$U \in H := H(\nabla\times, \mathbb{R}^3) \cap L^{\frac{p+1}{2}}(\mathbb{R}^3).$$

Here, $H(\nabla\times, \mathbb{R}^3)$ is the space of functions $W \in L^2(\mathbb{R}^3)$ such that $\nabla \times W$ is defined in the sense of distributions and $\nabla \times W \in L^2(\mathbb{R}^3)$. Minimizing $\mathcal{J}$ on the whole of H is in general not possible since $\mathcal{J}$ is unbounded from below. Moreover, we have $\nabla \times \psi = 0$ for all smooth $\psi = \nabla\varphi$ with $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ which shows that the $\nabla\times$-operator has an infinite-dimensional kernel. Our strategy is to look for critical points of $\mathcal{J}$ on suitable subspaces of H. These subspaces can consist of functions with some prescribed symmetry. For instance, in the subspace of radial functions, i.e.,

$$U(x) = A^{-1}U(Ax) \text{ for all } A \in O(3), \tag{4}$$

the authors in [5] treat (3) with $f(x,|U|^2)U = \Gamma(x)|U|^{p-1}U, p > 1$ and characterize all radial distributional solutions if V, Γ radial and $0 \leq V\Gamma^{-1} \in L_{\text{loc}}^{\frac{p}{p-1}}(\mathbb{R}^3)$, see Theorem 1 in [5]. The symmetry assumption (4) is too restrictive since it was shown in Lemma 4 (a) in [5] that if $U \in L^1_{\text{loc}}(\mathbb{R}^3)$ satisfies (4) then $\nabla \times U = 0$ in distributional sense so that one ends up with an algebraic equation. This is also the fundamental problem of Theorem 2 in [9] since in case of $V \equiv 0$ no non-trivial radial solution can satisfy (3). Therefore, apart from the radial symmetry many researchers considered cylindrical symmetries which will play a major role in the first part of this thesis. The search for cylindrically symmetric solutions in semilinear equations of the type

$$\nabla \times \nabla \times U = W'(|U|^2)U \tag{5}$$

was initiated in the last decade by Azzollini, Benci, D'Aprile and Fortunato in [3]. Their ansatz is

$$U(x) = u(r,z)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \text{ for } x = (x_1,x_2,x_3) \in \mathbb{R}^3, r = \sqrt{x_1^2 + x_2^2}, z = x_3 \tag{6}$$

and $u\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$. Notice that U in (6) has vanishing divergence so that the $\nabla \times \nabla\times$ operator reduces to $-\Delta$.

Theorem 3 in [5] treats (3) with $f(x,|U|^2)U = \Gamma(x)|U|^{p-1}U, p \in (1,5)$ and periodic and cylindrically symmetric coefficients V and Γ. By Palais' principle of symmetric criticality [54] it can be shown that every critical point of $\mathcal{J}$ restricted to the subspace of functions having the form (6) is indeed a solution of (3). Zeng [74] also studies (3) in the cylindrical framework (6) together with a critical nonlinearity, i.e., $f(x,|U|^2)U = |U|^{p-1}U + |U|^4U, p \in (1,5)$.

Mederski [53] and Bartsch, Mederski [6] made progress with (3) for constant coefficients in a bounded domain and perfectly conducting boundary conditions $\nu \times U = 0$. For the bounded domain case see also the overview article [7] and the references therein for further results and open problems.

Although the cylindrical symmetry in (6) together with $\operatorname{div} U = 0$ plays a major role in many contributions to (3) there are results apart from that. For instance, in [22] D'Aprile and Siciliano study a different kind of cylindrical symmetry, namely an ansatz of the form

$$U(x) = u(r,z)\left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^T + \tilde{u}(r,z)\,(0,0,1)^T \text{ for } u,\tilde{u}\colon [0,\infty)\times\mathbb{R} \to \mathbb{R} \tag{7}$$

for (5), see Therorem 1.1 therein. Notice that functions of the form (7) are no longer divergence-free. Moreover, Mederski [52] gave an existence result for (3) without prescribing any additional symmetry for U under the assumption $V \leq 0$, smallness of V in $L^{\frac{3}{2}}(\mathbb{R}^3)$ and $f(x,|U|^2)U$ replaced by some powertype nonlinearity which behaves supercritical near zero and subcritical away from zero.

As mentioned above we also make use of the cylindrically symmetric ansatz (6). Plugging (6) into (3) we obtain the scalar equation

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right)-\frac{\partial^2 u}{\partial z^2}+V(r,z)u=f(r,z,r^2u^2)u \text{ in } [0,\infty)\times\mathbb{R}. \tag{8}$$

Here and throughout this thesis we often identify a point $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ with its cylindrical coordinates $x=(r,z)\in\Omega:=[0,\infty)\times\mathbb{R}$.

The first part of this thesis is subdivided in four chapters. In Chapter 1 we fix our notations and introduce the cylindrical Sobolev spaces in which we mainly work. In Chapter 2 we prove the following result.

Theorem 1. *Let f in* (8) *satisfy*

(i) $f:\Omega\times[0,\infty)\to\mathbb{R}$ *is a Carathéodory function with* $0\leq f(r,z,s)\leq c(1+s^{\frac{p-1}{2}})$ *for some* $c>0$ *and* $p\in(1,5)$,

(ii) $f(r,z,s)=o(1)$ *as* $s\to 0$ *uniformly in* $(r,z)\in\Omega$,

(iii) $f(r,z,s)$ *strictly increasing in* $s\in[0,\infty)$ *for all* $(r,z)\in\Omega$,

(iv) $\frac{F(r,z,s)}{s}\to\infty$ *as* $s\to\infty$ *uniformly in* $(r,z)\in\Omega$,

(v) for all $r\in[0,\infty)$, $s\geq 0$ *and* $\sigma>0$ *the function*

$$\varphi_\sigma(r,z,s):=f(r,z,(s+\sigma)^2)(s+\sigma)^2-f(r,z,s^2)s^2$$

is symmetrically nonincreasing in z.

Moreover, let $V\in L^\infty(\Omega)$ *be reversed Steiner-symmetric such that the map*

$$\|\cdot\|: H^1_{\mathrm{cyl}}(r^3drdz)\to\mathbb{R}; u\mapsto\left(\int_\Omega\left(|\nabla_{r,z}u|^2+V(r,z)u^2\right)r^3d(r,z)\right)^{\frac{1}{2}}$$

is an equivalent norm to $\|\cdot\|_{H^1_{\mathrm{cyl}}(r^3drdz)}$. *Then* (8) *has a ground state* $u\in H^1_{\mathrm{cyl}}(r^3drdz)$ *which is symmetric about* $\{z=0\}$.

Theorem 1 gives existence of ground states for nonlinearities which have not appeared in the literature before. For instance $f(r,z,s)=\Gamma(r,z)s^{\frac{p-1}{2}}$ where $\Gamma\in L^\infty(\Omega)$ is Steiner-symmetric, $\operatorname{ess\,inf}_\Omega\Gamma>0$ and $p\in(1,5)$ is a valid choice. Other examples (which have not occurred before) are given in Section 2.2. The existence of a ground state is established as a suitably constrained minimizer of a corresponding energy functional. Unfortunately, due to compactness issues we can not work in the whole Hilbert space. Symmetrization then allows us to work in a suitable cone of cylindrical functions which are Steiner symmetric with respect to z. Among other advantages this cone has the remarkable feature that compactness properties are available. The result of Chapter 2 is already accepted for puplication in *Zeitschrift für Analysis und ihre Anwendungen* in a joint paper with W. Reichel, see [40].

In Chapter 3 and 4 we specify the nonlinearity f and consider the vector-valued equation

$$\nabla \times \nabla \times U + V(x)U = \Gamma(x)\,|U|^{p-1}\,U, \quad 1 < p < 5 \tag{9}$$

where $V, \Gamma \in W^{1,\infty}(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} V > 0$, $\inf_{\mathbb{R}^3} \Gamma > 0$. Moreover, we assume that V and Γ are cylindrically symmetric and only depend on $r \in [0,\infty)$, i.e., the corresponding scalar equation reads

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} + V(r)u = \Gamma(r)r^{p-1}|u|^{p-1}u \text{ in } \Omega. \tag{10}$$

The existence result obtained in Chapter 2 then also applies for (10) but due to the special power-type nonlinearity we are able to deduce further properties of the corresponding ground states, or more general, arbitrary positive solutions of (10). For instance, we obtain regularity, symmetry and monotonicity of positive solutions of (10) in Chapter 3. Chapter 4 establishes a-priori bounds for positive solutions of (10) in

$$H_{\text{symm}} := \left\{v \in H^1_{\text{cyl}}(r^3 drdz) : v \text{ is symmetric about } \{z = 0\}\right\}$$

for $p \in (1,2)$. The result reads as follows.

Theorem 2. *Let $[p_\star, p^\star] \subset (1,2)$. Then there is a constant $C = C(p_\star, p^\star) > 0$ such that*

$$\|ru\|_{L^\infty([0,\infty)\times\mathbb{R})} \leq C$$

for every positive solution $u \in H_{\text{symm}}$ of (10) and every $p \in [p_\star, p^\star]$. Moreover, there is a constant $\tilde{C} = \tilde{C}(p_\star, p^\star) > 0$ such that $\|u\|_{H^1_{\text{cyl}}(r^3 drdz)} \leq \tilde{C}$ for all ground states $u \in H_{\text{symm}}$ of (10).

A-priori bounds are often a consequence of a Liouville theorem and indeed, one major ingredient in our proof is to provide a Liouville theorem for our cylindrical framework in the following sense:

Theorem 3. *Let $\bar{p} \in (1,2)$ and $c > 0$. Then there is no non-trivial, positive solution $u \in H^1_{\text{loc}}(r^3 drdz)$ of*

$$-\partial_r^2 u - \frac{3}{r}\partial_r u - \partial_z^2 u = c r^{\bar{p}-1} u^{\bar{p}} \text{ in } (0,\infty)\times\mathbb{R}.$$

Notice that due to $u \in H^1_{\text{loc}}(r^3 drdz)$ our test functions in Theorem 3 are not allowed to have support on $\{0\}\times\mathbb{R}$. Compared to Liouville theorems in the literature (see for instance Chapter I.8 in [59]) our range of exponents $\bar{p} \in (1,2)$ seems not optimal. For system $-\Delta U = |U|^{p-1}U$ in $\mathbb{R}^3$ we would expect a Liouville theorem for $\bar{p} \in (1,5)$ but in our case, do to the cylindrical ansatz (6), we can not make use of any positivity argument for the system so that the arguments by Gidas and Spruck ([37], [38]) are not directly applicable and we end up with a smaller range of admissible $\bar{p}$. The techniques in our proof do not allow for a larger range of exponents so that it is an open question whether Theorem 3 holds true for $\bar{p} > 2$. Apart from Theorem 3 the proof of Theorem 2 is based on scaling arguments which exploit (9), symmetries of (10) and arguments similar to the classical papers by Gidas and Spruck ([37], [38]), the starting point for the rich literature on Liouville theorems (see for instance

[62] for higher order differential operators or [24], [26], [58] for systems of elliptic equations to name only a few contributions).

Chapter 4 contains a further main result, this time for an equation on the bounded domain $\Omega_k := \{(r,z) \in \Omega : r^2 + z^2 < k^2\}$, $k > 0$. We consider

$$\begin{aligned} -\Delta_{5,\mathrm{cyl}} u + V(r)u &= \Gamma(r) r^{p-1} u^p \text{ in } \Omega_k, \\ u &= 0 \text{ on } \partial\Omega_k \setminus (\{0\} \times [-k,k]), \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \{0\} \times [-k,k], \end{aligned} \tag{11}$$

with $V, \Gamma \in W^{1,\infty}(\Omega_k)$, $\inf V, \inf \Gamma > 0$ and not depending on z, i.e., $V(r,z) = V(r), \Gamma(r,z) = \Gamma(r)$. The a-priori bounds are also valid in the bounded domain case which then leads to a uniqueness result as summarized in the following theorem.

Theorem 4. *Let $[p_\star, p^\star] \subset (1,2)$ and $k > 0$. Then there is a constant $C = C(p_\star, p^\star, k) > 0$ such that*

$$\|ru\|_{L^\infty(\Omega_k)} \le C$$

for every positive weak solution of (11) *and every $p \in [p_\star, p^\star]$. Moreover, there is $p_0 = p_0(k) > 1$ such that* (11) *has only one positive solution for $p \in (1, p_0)$.*

The second part of this thesis is devoted to a study of

$$\nabla \times \nabla \times U + V(x)\partial_t^2 U = \Gamma \, |U|^{p-1} \, U \text{ in } \mathbb{R}^3 \tag{12}$$

for $p \in (1, \frac{5}{3})$, constant $\Gamma > 0$ and a periodically distributed potential $V > 0$, see Chapter 5. In contrast to the elliptic equation (8) in the first part, in (12) we work in the hyperbolic regime. The goal is to find real-valued spatially localized time-periodic solutions (so called *breathers*) of (12). Here, we consider a polarized field of the form

$$U(x,t) = \begin{pmatrix} 0 \\ u(x_1,t) \\ 0 \end{pmatrix}, \quad x = (x_1, x_2, x_3)^T$$

so that (12) reduces to the $1+1$ dimensional nonlinear wave equation

$$-u_{xx} + V(x) u_{tt} = \Gamma |u|^{p-1} u \text{ in } \mathbb{R} \times \mathbb{R}. \tag{13}$$

In general, the phenomena of breathers in a $1+1$-dimensional nonlinear wave equation is quiet rare. For the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \text{ in } \mathbb{R} \times \mathbb{R} \tag{14}$$

an explicit family of breathers is given by

$$u_{m,\omega}(x,t) = 4 \arctan\left(\frac{m}{\omega} \frac{\sin(\omega t)}{\cosh(mx)} \right), m, \omega > 0, m^2 + \omega^2 = 1.$$

In most cases these breathers do not persist if the $\sin u$ nonlinearity in (14) is perturbed to $f(u)$ with $f(0) = 0, f'(0) > 0$, see [12] and [28].

The first existence results of breathers for a nonlinear wave equation apart from the Sine-Gordon equation was given by Blank, Chirilus-Bruckner, Lescarret, Schneider [13] with the help of bifurcation theory and center manifold theory. Precisely, they considered an equation of the type

$$s(x)u_{tt} = u_{xx} - q(x)u + u^3$$

with periodic $s, q\colon \mathbb{R} \to \mathbb{R}$ and guaranteed the existence of breathers for a very specific choice of s and q. Recently, Plum and Reichel [57] gave an existence result for breathers in the $3+1$-dimensional semilinear curl-curl wave equation

$$s(x)\partial_t^2 U + \nabla \times \nabla \times U + q(x)U \pm V(x)|U|^{p-1}U = 0, \quad p > 1,$$

for $V, q, s\colon \mathbb{R}^3 \to (0, \infty)$ radially symmetric, positive and satisfying further properties which we do not list here.

Our ansatz for (13) is a so-called *polychromatic* ansatz, i.e., we consider

$$u(x,t) = \sum_{k \in 2\mathbb{Z}+1} \hat{u}_k(x)e^{ik\omega t}, \quad \hat{u}_k(x) = \overline{\hat{u}_{-k}(x)}, \quad \omega > 0. \tag{15}$$

Polychromatic here refers to the fact that we do not consider only one frequency as done in the monochromatic approach but infinitely many frequencies. The reason why we only consider odd integers in (15) is that the nonlinearity as well as the differential operator in (13) respect the structure in (15). Moreover, the case $k = 0$ has to be excluded since otherwise zero is in the spectrum of the differential operator in (13). The advantage of the Fourier decomposition (15) is that u is real-valued and $\frac{2\pi}{\omega}$-periodic in time. In difference to the techniques in [13] we use variational tools to find a weak solution of (13). Our result reads as follows:

Theorem 5. *Let $p \in (1, \frac{5}{3})$ and let $V\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$V(x) = \alpha + \beta\delta_{\text{per}}(x),$$

where δ_{per} denotes a 2π- periodic δ-potential located in $(0, 2\pi)$, $\alpha > 0$ and $\beta = 16\alpha$. Then (13) possesses a non-trivial $8\pi\sqrt{\alpha}$-periodic weak solution in the sense explained in Section 5.8.

The search for polychromatic waves in nonlinear equations goes at least back to a paper of Tasgal, Malomed and Band, see [71]. Therein, the authors considered first and third harmonics of a so-called extended nonlinear coupled mode equation which approximates (13) for $p = 3$. The results were obtained by numerical means but they did not have a precise proof. Seven years later, Pelinovsky, Simpson and Weinstein [56] again considered the extended nonlinear coupled mode equation but this time with all discrete frequencies. We briefly sketch the idea how they proceeded in order to highlight the differences to our approach. They used a combination of numerical and analytical methods. In a first step they approximated the nonlinear coupled mode equation by an infinite system of coupled nonlinear Schrödinger equations. This system is embedded in a family of systems which contain a parameter ε and they are interested in solutions for $\varepsilon = 1$. Near $\varepsilon = 0$ which resembles a decoupled system they can rigorously proof the existence of solutions with the help of bifurcation theory. In order to achieve a result for $\varepsilon = 1$ they use numerical continuation methods. Indeed they *give convincing* *numerical evidence* (page 482 in [56]) that some branches continue to $\varepsilon = 1$ if they truncate their system to finitely many equations. This refers to truncating the series in (15) to a finite sum. But in considering the full problem of infinitely many equations, i.e., infinitely many discrete frequencies in (15) they are left with many open challenges. In particular, a purely variational approach seems out of reach in their setting. Even for the truncated version of their system there is no rigorous proof of existence and the authors in [56] take into account the non-existence of such solutions.

In contrast to these results Theorem 5 is obtained by purely analytical tools and the proof is rigorous. We use the Fourier-Floquet-Bloch transformation to diagonalize the wave operator and to obtain a suitable indefinite variational setting for (13). A key observation is that the family of operators acting on the Fourier coefficients $\hat{u}_k$ in (15) possesses a uniform spectral gap which contains zero and grows linearly in k. This will allow us to handle the nonlinearity and to prove the existence of a minimizer of the functional on the generalized Nehari manifold. Besides other difficulties one major problem is to ensure that the power-nonlinearity in (13) can be controlled in our functional analytic framework.

Part I.

Monochromatic waves

1. Preliminaries and notation

In this first chapter we want to fix our notations. In particular, we introduce several spaces which are used in part one of this thesis.
First we introduce cylindrical Lebesgue and Sobolev spaces on $\Omega := [0,\infty)\times\mathbb{R}$ in a canonical manner. Here we consider (r,z) as cylindrical coordinates $r = \sqrt{x_1^2 + \cdots + x_j^2}$, $z = x_{j+1}$ in $\mathbb{R}^{j+1}$.

Definition 1.1. *For $p \in [1,\infty)$, $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we set*

$$\begin{aligned} L^p_{\mathrm{cyl}}(r^j drdz) &:= \left\{ v\colon \Omega \to \mathbb{R} : \int_\Omega |v(r,z)|^p r^j d(r,z) < \infty \right\}, \\ W^{0,p}_{\mathrm{cyl}}(r^j drdz) &:= L^p_{\mathrm{cyl}}(r^j drdz), \\ W^{k,p}_{\mathrm{cyl}}(r^j drdz) &:= \left\{ v \in W^{k-1,p}_{\mathrm{cyl}}(r^j drdz) : \frac{\partial v^k}{\partial r^i \partial z^{k-i}} \in L^p_{\mathrm{cyl}}(r^j drdz) \, \text{for all } i \in \{0,\dots,k\} \right\}. \end{aligned}$$

In case of $p = 2$ we abbreviate $H^k_{\mathrm{cyl}}(r^j drdz) := W^{k,2}_{\mathrm{cyl}}(r^j drdz)$. Moreover, we introduce

$$L^\infty_{\mathrm{cyl}}(r^j drdz) := \left\{ v\colon \Omega \to \mathbb{R} : \operatorname{ess\,sup}_{(r,z)\in\Omega} |v(r,z)| < \infty \right\}$$

and in an analogous fashion $W^{k,\infty}_{\mathrm{cyl}}(r^j drdz)$ for $k \in \mathbb{N}$. For $u \in L^p_{\mathrm{cyl}}(r^j drdz)$ let

$$\|u\|_{L^p_{\mathrm{cyl}}(r^j drdz)} := \left(\int_\Omega |u(r,z)|^p \, r^j d(r,z) \right)^{1/p}$$

and for $u \in W^{k,p}_{\mathrm{cyl}}(r^j drdz)$ we set

$$\|u\|_{W^{k,p}_{\mathrm{cyl}}(r^j drdz)} := \begin{cases} \left(\sum_{0\le|\alpha|\le k} \|\partial^\alpha u\|^p_{L^p_{\mathrm{cyl}}(r^j drdz)} \right)^{1/p} & \text{for } p \in [1,\infty), \\ \max_{0\le|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty} & \text{for } p = \infty. \end{cases}$$

where we use the multi index notation, i.e., $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}_0^2$, $|\alpha| := \alpha_1 + \alpha_2$ and $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial r^{\alpha_1} \partial z^{\alpha_2}}$.

Then $L^p_{\mathrm{cyl}}(r^j drdz)$, $W^{k,p}_{\mathrm{cyl}}(r^j drdz)$ endowed with $\|\cdot\|_{L^p_{\mathrm{cyl}}(r^j drdz)}$ respectively $\|\cdot\|_{W^{k,p}_{\mathrm{cyl}}(r^j drdz)}$ are Banach spaces; for $p = 2$ even Hilbert spaces with scalar product

$$\begin{aligned} \langle u,v\rangle_{L^2_{\mathrm{cyl}}(r^j drdz)} &:= \int_\Omega u(r,z)v(r,z) r^j d(r,z) \text{ for } u,v \in L^2_{\mathrm{cyl}}(r^j drdz), \\ \langle u,v\rangle_{H^k_{\mathrm{cyl}}(r^j drdz)} &:= \sum_{0\le|\alpha|\le k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2_{\mathrm{cyl}}(r^j drdz)} \text{ for } u,v \in H^k_{\mathrm{cyl}}(r^j drdz). \end{aligned}$$

In the upcoming chapters, the cases $j = 1$ and $j = 3$ in Definition 1.1 will play a major role.
Next, we introduce the notion of cylindrical C^∞_c-functions.

Definition 1.2. *A function $u = u(r,z)$ belongs to $C_c^\infty([0,\infty)\times\mathbb{R})$ if and only if $u \in C^\infty([0,\infty)\times\mathbb{R})$,* supp *$u$ is compact in $[0,\infty)\times\mathbb{R}$ and $\frac{\partial^j u}{\partial r^j}(0,z) = 0$ for all odd integers $j \in 2\mathbb{N}-1$.*

Definition 1.2 implies equivalence between $u \in C_c^\infty([0,\infty)\times\mathbb{R})$ and $\tilde{u} \in C_c^\infty(\mathbb{R}^n)$ where $\tilde{u}(x) := u(|(x_1,\dots,x_{n-1})|, x_n)$. Thus, we conclude that $C_c^\infty([0,\infty)\times\mathbb{R})$ is dense in $H^1_{\mathrm{cyl}}(r^{n-2}drdz)$.
We now transfer these concepts to cylindrical functions on arbitrary subsets of Ω and point out several connections. For this purpose, let $\tilde{\Omega} \subseteq \Omega$ be relatively open in Ω and fix $n \in \mathbb{N}_{\geq 2}$. We denote a point in Ω by $(r, x_n) \in \Omega$. Define

$$\tilde{\Omega}_n := \{x = (x_1,\dots,x_n) \in \mathbb{R}^n : (|(x_1,\dots,x_{n-1})|, x_n) \in \tilde{\Omega}\},$$

which is nothing else than the n-dimensional counterpart of the two-dimensional set $\tilde{\Omega}$. Consider a function $\varphi\colon \tilde{\Omega} \to \mathbb{R}$. Then we can define its n-dimensional cylindrically symmetric counterpart on $\tilde{\Omega}_n$, i.e., the function

$$\varphi_n\colon \tilde{\Omega}_n \to \mathbb{R};\ \varphi_n(x_1,\dots,x_{n-1},x_n) := \varphi(|(x_1,\cdots,x_{n-1})|, x_n).$$

The relation between φ and φ_n is stated in the next lemma.

Lemma 1.3. *Let $k \in \mathbb{N}$ and φ, φ_n as above. Additionally, let $\Lambda := \left\{x_n \in \mathbb{R} : (0, x_n) \in \tilde{\Omega}\right\}$. Then the following assertions are equivalent.*

(a) $\varphi_n \in C^k(\tilde{\Omega}_n)$.

(b) $\varphi \in C^k(\tilde{\Omega})$ *and* $\frac{\partial^{2l+1}\varphi}{\partial r^{2l+1}} = 0$ *on* Λ *for all* $l \in \mathbb{N}_0$ *such that* $2l+1 \leq k$.

Proof. We only give a proof for the case $k = 1$. A repetition of the arguments below with appropriate calculations shows the claim also for $k > 1$.
(a)⇒(b): For $x_1 > 0$ we have by definition of φ_n the relation

$$\varphi_n(x_1, 0, \dots, 0, x_n) = \varphi(x_1, x_n).$$

Thus the regularity of φ_n on $\tilde{\Omega}_n$ transfers to the regularity of φ on $\tilde{\Omega}$. Moreover, let $x_n \in \Lambda$. By using $\varphi_n \in C^1(\tilde{\Omega}_n)$ and $\varphi_n(t, 0, \dots, 0, x_n) = \varphi_n(-t, 0, \dots, 0, x_n)$ we calculate

$$\begin{aligned}\frac{\partial\varphi}{\partial r}(0,x_n) &= \lim_{t\to 0^+} \frac{\varphi(t,x_n)-\varphi(0,x_n)}{t} = \lim_{t\to 0^+} \frac{\varphi_n(t,0,\dots,0,x_n)-\varphi_n(0,\dots,0,x_n)}{t} \\ &= \lim_{t\to 0^-} \frac{\varphi_n(t,0,\dots,0,x_n)-\varphi_n(0,\dots,0,x_n)}{t} = \lim_{t\to 0^-} -\frac{\varphi_n(-t,0,\dots,0,x_n)-\varphi_n(0,\dots,0,x_n)}{-t} \\ &= -\lim_{t\to 0^+} \frac{\varphi_n(t,0,\dots,0,x_n)-\varphi_n(0,\dots,0,x_n)}{t} = -\lim_{t\to 0^+} \frac{\varphi(t,x_n)-\varphi(0,x_n)}{t} = -\frac{\partial\varphi}{\partial r}(0,x_n),\end{aligned}$$

i.e., $\frac{\partial\varphi}{\partial r}(0, x_n) = 0$.
(b)⇒(a): For $x' \in \mathbb{R}^{n-1}$ define $r(x') := \sqrt{x_1^2 + \cdots + x_{n-1}^2}$. Since $r \in C^1(\mathbb{R}^{n-1}\setminus\{0\})$ we deduce that $\varphi_n(x_1,\dots,x_n) = \varphi(r(x'), x_n)$ satisfies $\varphi_n \in C^1(\tilde{\Omega}_n \setminus \Lambda)$. For $x_n \in \Lambda$ we now investigate differentiability of φ_n at the point $(0,\dots,0,x_n)$. Therefore, let $h' = (h_1,\dots,h_{n-1}) \in \mathbb{R}^{n-1}$. We deduce

$$\lim_{|h'|\to 0} \frac{\varphi_n(h_1,\dots,h_{n-1},x_n)-\varphi_n(0,\dots,0,x_n)}{|h'|} = \lim_{|h'|\to 0} \frac{\varphi(r(h'),x_n)-\varphi(0,x_n)}{|h'|} = 0,$$

since φ is differentiable at $(0, x_n)$ with $\frac{\partial\varphi}{\partial r}(0, x_n) = 0$. Hence,

$$\nabla\varphi_n(x) = \begin{cases} \left(\frac{\partial\varphi}{\partial r}(r(x'), x_n)\frac{x_1}{r(x')}, \ldots, \frac{\partial\varphi}{\partial r}(r(x'), x_n)\frac{x_{n-1}}{r(x')}, \frac{\partial\varphi}{\partial x_n}(r(x'), x_n)\right)^T & \text{, for } x' \neq 0, \\ \left(0, \ldots, 0, \frac{\partial\varphi}{\partial x_n}(0, x_n)\right)^T & \text{, for } x' = 0. \end{cases}$$

For $i = 1, \ldots, n-1$ this leads to

$$|\nabla\varphi_n(x', \tilde{x}_n)| = \left|\frac{\partial\varphi}{\partial r}(r(x'), \tilde{x}_n)\frac{x_i}{r(x')}\right| \leq \left|\frac{\partial\varphi}{\partial r}(r(x'), \tilde{x}_n)\right| \to 0 \text{ for } (x', \tilde{x}_n) \to (0', x_n),$$

which finally shows $\varphi_n \in C^1(\tilde{\Omega}_n)$. □

Lemma 1.3 clarifies the meaning of C^∞ functions and in analogy to Definition 1.2 we can now continue with the introduction of cylindrically symmetric functions with compact support.

Definition 1.4. *Let $\tilde{\Omega}$ be relatively open in Ω and $\varphi\colon \tilde{\Omega} \to \mathbb{R}$. Then we write $\varphi \in C_c^\infty(\tilde{\Omega})$ if and only if $\varphi \in C^\infty(\tilde{\Omega})$, $\operatorname{supp}\varphi := \overline{\left\{(r, x_n) \in \tilde{\Omega} : \varphi(r, x_n) \neq 0\right\}}$ is a compact subset of $\tilde{\Omega}$ and $\frac{\partial^k\varphi}{\partial r^k} = 0$ on Λ_{cpt} for all $k \in 2\mathbb{N} - 1$, where $\Lambda_{cpt} := \{x_n \in \mathbb{R} : (0, x_n) \in \operatorname{supp}\varphi\}$.*

With the definition above Lemma 1.3 implies $\varphi \in C_c^\infty(\tilde{\Omega})$ if and only if $\varphi_n \in C_c^\infty(\tilde{\Omega}_n)$.
We now need some additonal notation concerning cylindrical Sobolev spaces on arbitrary subsets of Ω. With the notation of Definition 1.4 we give the following definition.

Definition 1.5. *Let $\tilde{\Omega} \subseteq \Omega$ be relatively open in Ω. Then*

$$H^1_{0,\mathrm{cyl}}(\tilde{\Omega}, r^j drdz) := \overline{C_c^\infty(\tilde{\Omega})}^{\|\cdot\|_{H^1_{\mathrm{cyl}}(r^j drdz)}}.$$

Whenever there is no ambiguity we abbreviate $H^1_0(\tilde{\Omega}, r^j) := H^1_{0,\mathrm{cyl}}(\tilde{\Omega}, r^j drdz)$. This space so to say possesses Neumann boundary conditions on $\Lambda := \{z \in \mathbb{R} : (0, z) \in \tilde{\Omega}\}$ and Dirichlet boundary conditions on all other parts of $\partial\tilde{\Omega}$. Notice that $H^1_{0,\mathrm{cyl}}(\tilde{\Omega}, r^j drdz)$ agrees with the 'classical' definition of H^1_0-spaces in case of $\Lambda = \emptyset$. Moreover, we abbreviate $L^p(\tilde{\Omega}, r^j) := L^p_{\mathrm{cyl}}(\tilde{\Omega}, r^j drdz)$ and $H^{-1}(\tilde{\Omega}, r^j) := H^{-1}_{\mathrm{cyl}}(\tilde{\Omega}, r^j drdz)$.

Remark 1.6. *In particular, we have $u \in H^1_{0,\mathrm{cyl}}(\tilde{\Omega}, r^{n-2} drdz)$ if and only if $u_n \in H^1_0(\tilde{\Omega}_n)$.*

2. Existence of symmetric ground states for a general non-linearity

In this chapter we prove the existence of ground states to a class of partial differential equations with non-constant coefficients. The result of this chapter is published in [40].
As mentioned in the introduction, an ansatz of the type (6) for (3) leads to

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right)-\frac{\partial^2 u}{\partial z^2}+V(r,z)u=f(r,z,r^2u^2)u \text{ in } \Omega, \tag{2.1}$$

where we list the precise conditions on f in section 2.2.
Weak solutions of (2.1) arise as critical points of the functional

$$J(u)=\frac{1}{2}\int_\Omega\left(|\nabla_{r,z}u|^2+V(r,z)u^2\right)r^3d(r,z)-\int_\Omega\frac{1}{2r^2}F(r,z,r^2u^2)r^3d(r,z),\quad u\in H^1_{\mathrm{cyl}}(r^3drdz), \tag{2.2}$$

where $F(r,z,t):=\int_0^t f(r,z,s)\,ds$. A function $u\in H^1_{\mathrm{cyl}}(r^3drdz)$ which realises the least energy level of J among all non-trivial weak solutions of (2.1) is called *ground state* of (2.1). In other words,

$$J(u)=\inf_{v\in M}J(v),$$

where M denotes the Nehari-manifold

$$M:=\left\{v\in H^1_{\mathrm{cyl}}(r^3drdz)\setminus\{0\}:\int_\Omega\left(|\nabla_{r,z}v|^2+V(r,z)v^2\right)r^3d(r,z)=\int_\Omega f(r,z,r^2v^2)v^2r^3d(r,z)\right\}. \tag{2.3}$$

The outline of this chapter is as follows: In the next section we recall some inequalities from P.-L. Lions [48]. We then deduce some compactness properties in a cone of symmetric functions, see [47] and [48]. In Section 2.2 we formulate the precise conditions on f and state our result. The proof thereof uses an extension of the Nehari-manifold method due to Szulkin and Weth [70], the compactness results mentioned above and rearrangement inequalities for general nonlinearities, see [15].

2.1. Decay properties of symmetric functions

At first, we start with a well-known fact concerning radially symmetric functions and afterwards extend the result to cylindrically symmetric functions.
For $1\le\alpha,\beta\le\infty$ let

$$X^{\alpha,\beta}:=\left\{u\in L^\alpha(\mathbb{R}^n),\nabla u\in L^\beta(\mathbb{R}^n)\right\}\text{ and }X^{\alpha,\beta}_{\mathrm{rad}}:=\left\{u\in X^{\alpha,\beta}:u\text{ is radially symmetric}\right\}.$$

Lemma 2.1. *(see [48]) Let $n \geq 2, 1 \leq \alpha, \beta \leq \infty$. Then for every $u \in X^{\alpha,\beta}_{\text{rad}}$ it holds that*

$$|u(x)| \leq C \, \|\nabla u\|^{\theta}_{L^\beta} \, \|u\|^{1-\theta}_{L^\alpha} \, |x|^{-(n-1)\theta} \text{ for almost all } x \in \mathbb{R}^n,$$

where $\theta = \frac{\beta'}{\beta'+\alpha}, \beta' = \frac{\beta}{\beta-1}$ and C is independent of u.

Proof. We will only prove this Lemma for $1 < \alpha, \beta < \infty$, since we will not need the cases where α and/or β is equal to 1 and/or ∞. Since $C^\infty_{c,\text{rad}}(\mathbb{R}^n)$ is dense in $X^{\alpha,\beta}_{\text{rad}}$ it is sufficient to prove the estimate for $u \in C^\infty_{c,\text{rad}}(\mathbb{R}^n)$. Let $\gamma := \frac{1}{\theta} = \frac{\beta'+\alpha}{\beta'}$. Then, denoting by r the radial variable of $u \in C^\infty_{c\,\text{rad}}(\mathbb{R}^n)$ we get

$$\frac{d}{dr}\left(r^{n-1} \, |u|^\gamma\right) = (n-1) r^{n-2} \, |u|^\gamma + r^{n-1} \gamma \, |u|^{\gamma-2} \, u \frac{\partial u}{\partial r} \geq -\gamma \, |u|^{\gamma-1} \left|\frac{\partial u}{\partial r}\right| r^{n-1}.$$

Integrating from r to ∞ and expanding the domain of integration to all of $\mathbb{R}^n$ yields

$$r^{n-1} \, |u(r)|^\gamma \leq C \int_{\mathbb{R}^n} |u|^{\gamma-1} \, |\nabla u| \, dx \leq C \, \|\nabla u\|_{L^\beta} \, \|u\|^{\gamma-1}_{L^{(\gamma-1)\beta'}} \, ,$$

where we used Hölder's inequality in the last estimate. Finally, dividing by r^{n-1}, taking the γ-th root and using $\frac{1}{\gamma} = \theta$ gives the desired estimate

$$|u(r)| \leq C \, \|\nabla u\|^{\theta}_{L^\beta} \, \|u\|^{1-\theta}_{L^\alpha} \, r^{-(n-1)\theta}.$$

□

Now we give an extension of Lemma 2.1 for cylindrically symmetric functions which are Steiner-symmetric in the non-radial component. We will make use of the following notation: Let $t \in \mathbb{N}_{\geq 2}$ and $s \in \mathbb{N}$ such that $n = t + s$. We write points in $\mathbb{R}^n$ as (x, y) with $x \in \mathbb{R}^t$ and $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$. Furthermore, let

$$K^{\alpha,\beta}_{t,s} := \left\{ u \in X^{\alpha,\beta} \text{ s.t. } \begin{cases} u(\cdot, y) & \text{is a radially symmetric function for every } y \in \mathbb{R}^s \text{ and} \\ u(x, \cdot) & \text{is Steiner-symmetric w.r.t. } y_i, i = 1, \ldots, s, \text{ for every } x \in \mathbb{R}^t \end{cases} \right\}. \tag{2.4}$$

In particular, if $u \in K^{\alpha,\beta}_{t,s}$ then necessarily $u \geq 0$. In this setting we have the following decay estimate.

Lemma 2.2. *(see [48]) Let $t \in \mathbb{N}_{\geq 2}, s \in \mathbb{N}, n = t + s$ and $1 \leq \alpha, \beta \leq \infty$. Then for every $u \in K^{\alpha,\beta}_{t,s}$ we have*

$$|u(x, y)| \leq C \, \|\nabla_x u\|^{\theta}_{L^\beta(\mathbb{R}^n)} \, \|u\|^{1-\theta}_{L^\alpha(\mathbb{R}^n)} \, |x|^{-(t-1)\theta} \, |y_1 \cdots y_s|^{-\theta} \, ,$$

where $\theta = \frac{\beta'}{\beta'+\alpha}$, $\beta' = \frac{\beta}{\beta-1}$ and C is independent of u.

Proof. Again, we only give the proof in the case $1 < \alpha, \beta < \infty$. Let $u \in K^{\alpha,\beta}_{t,s}$. W.l.o.g. let $y_i > 0$ for all $i = 1, \ldots, s$. We define

$$v(x) := \int_0^{y_1} \cdots \int_0^{y_s} u(x, z) dz \text{ for } x \in \mathbb{R}^t.$$

By Jensen's inequality we obtain

$$\|v\|^{\alpha}_{L^\alpha(\mathbb{R}^t)} = \int_{\mathbb{R}^t} \left(\int_0^{y_1} \cdots \int_0^{y_s} u(x, z) dz \right)^{\alpha} dx \leq \int_{\mathbb{R}^t} \frac{1}{y_1 \cdots y_s} \left(\int_0^{y_1} \cdots \int_0^{y_s} (y_1 \cdots y_s)^\alpha u(x, z)^\alpha dz \right) dx$$

$$\le (y_1\cdots y_s)^{\alpha-1}\int_{\mathbb{R}^n}|u(x,z)|^\alpha\,d(x,z),\ \text{ so }\ \|v\|_{L^\alpha(\mathbb{R}^t)}\le (y_1\cdots y_s)^{\frac{1}{\alpha'}}\|u\|_{L^\alpha(\mathbb{R}^n)}\,. \tag{2.5}$$

In the same manner we receive

$$\begin{aligned}\|\nabla v\|^\beta_{L^\beta(\mathbb{R}^t)} &= \int_{\mathbb{R}^t}\left|\nabla_x\int_0^{y_1}\cdots\int_0^{y_s}u(x,z)dz\right|^\beta dx \le \int_{\mathbb{R}^t}\left(\int_0^{y_1}\cdots\int_0^{y_s}|\nabla_x u(x,z)|dz\right)^\beta dx\\ &\le (y_1\cdots y_s)^{\beta-1}\int_{\mathbb{R}^t}\int_0^{y_1}\cdots\int_0^{y_s}|\nabla_x u(x,z)|^\beta\,dzdx \le (y_1\cdots y_s)^{\beta-1}\int_{\mathbb{R}^n}|\nabla_x u(x,z)|^\beta\,d(x,z),\\ &\text{so }\ \|\nabla v\|_{L^\beta(\mathbb{R}^t)}\le (y_1\cdots y_s)^{\frac{1}{\beta'}}\|\nabla_x u\|_{L^\beta(\mathbb{R}^n)}\,. \end{aligned}\tag{2.6}$$

Using (2.5) and (2.6) we can apply Lemma 2.1 to the function v which is radially symmetric in $\mathbb{R}^t$ and deduce

$$\begin{aligned}|v(x)| &\le C\,\|\nabla v\|^\theta_{L^\beta(\mathbb{R}^t)}\|v\|^{1-\theta}_{L^\alpha(\mathbb{R}^t)}|x|^{-(t-1)\theta}\\ &\le C(y_1\cdots y_s)^{\frac{\theta}{\beta'}}\|\nabla_x u\|^\theta_{L^\beta(\mathbb{R}^n)}(y_1\cdots y_s)^{\frac{1-\theta}{\alpha'}}\|u\|^{1-\theta}_{L^\alpha(\mathbb{R}^n)}|x|^{-(t-1)\theta}\,.\end{aligned}\tag{2.7}$$

Moreover, since

$$\frac{\theta}{\beta'}+\frac{1-\theta}{\alpha'}=\frac{1}{\beta'+\alpha}+\frac{1-\frac{\beta'}{\beta'+\alpha}}{\alpha'}=\frac{1}{\beta'+\alpha}+\frac{\alpha}{\alpha'(\beta'+\alpha)}=\frac{\alpha}{\beta'+\alpha},$$

the exponent of the $(y_1\cdots y_s)$-term on the right hand side simplifies to $\frac{\alpha}{\beta'+\alpha}$. Due to the monotonicity-property in y-direction we also have $v(x)\ge y_1\cdots y_s u(x,y)$. Plugging both into (2.7) gives

$$|u(x,y)|\le C\,\|\nabla_x u\|^\theta_{L^\beta(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^\alpha(\mathbb{R}^n)}|x|^{-(t-1)\theta}(y_1\cdots y_s)^{\frac{\alpha}{\beta'+\alpha}-1}=C\,\|\nabla_x u\|^\theta_{L^\beta(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^\alpha(\mathbb{R}^n)}|x|^{-(t-1)\theta}(y_1\cdots y_s)^{-\theta}.$$

□

We now improve the estimates from above in case of having functions which are also nonincreasing in r-direction.

Lemma 2.3. *Let $p\in[1,\infty)$ and $v\in L^p_{\mathrm{rad}}(\mathbb{R}^n)$ be radially nonincreasing. Then*

$$v(r)\le C\,\|v\|_{L^p(\mathbb{R}^n)}\,r^{-\frac{n}{p}}\ \textit{for all } r>0. \tag{2.8}$$

Proof. Let $r>0$. From the monotonicity-assumption we receive

$$\left|\mathbb{S}^{n-1}\right|\frac{v^p(r)r^n}{n}\le\int_0^r\left|\mathbb{S}^{n-1}\right|v^p(\rho)\rho^{n-1}d\rho\le\|v\|^p_{L^p(\mathbb{R}^n)}$$

from which we conclude (2.8). □

Lemma 2.4. *Let $p\in[1,\infty)$ and $n\ge 2$. For a cylindrically symmetric function $v=v(r,z)\in L^p_{\mathrm{cyl}}(r^{n-2}drdz)$ which is nonincreasing in r as well as in z-direction, we have*

$$|v(r,z)|\le C\,\|v\|_{L^p_{\mathrm{cyl}}(r^{n-2}drdz)}\,r^{-\frac{n-1}{p}}\,|z|^{-\frac{1}{p}}\ \textit{for all } r>0, z\ne 0. \tag{2.9}$$

Proof. A function v which satisfies the assumptions is always non-negative. Without loss of generality let $z > 0$. We define $w(r) := \int_0^z v(r,s)ds$. Then w is radially symmetric in $n-1$ dimensions. Hölder's inequality yields

$$\|w\|^p_{L^p_{\mathrm{rad}}(r^{n-2}dr)} = \int_0^\infty \left(\int_0^z v(r,s)ds\right)^p r^{n-2}dr \le \int_0^\infty z^{p-1}\left(\int_0^z v(r,s)^p ds\right) r^{n-2}dr$$
$$\le z^{p-1}\int_\Omega v(r,z)^p r^{n-2} d(r,z) = z^{p-1}\|v\|^p_{L^p_{\mathrm{cyl}}(r^{n-2}drdz)}.$$

Lemma 2.3 and the monotonicity-property in z-direction gives

$$z\, v(r,z) \le w(r) \le C\|w\|_{L^p_{\mathrm{rad}}(r^{n-2}dr)}\, r^{-\frac{n-1}{p}} \le Cz^{\frac{p-1}{p}}\|v\|_{L^p_{\mathrm{cyl}}(r^{n-2}drdz)}\, r^{-\frac{n-1}{p}},$$

which finally proves (2.9). □

We prove an additional lemma which is used in the next section.

Lemma 2.5. *The set $K_{t,s} := K^{2,2}_{t,s}$ is a weakly closed cone in $H^1(\mathbb{R}^n)$.*

Proof. Take a sequence $(u_k)_{k\in\mathbb{N}} \subset K^{2,2}_{t,s}$ such that $u_k \rightharpoonup u \in H^1(\mathbb{R}^n)$ as $k \to \infty$. By the Sobolev embedding on bounded domains we deduce that a subsequence of u_k converges pointwise almost everywhere on $\mathbb{R}^n$ to u. Since every u_k enjoys the radial symmetry in the first component and the nonincreasing property in the second variable, the pointwise convergence implies that also u enjoys these properties, i.e., $u \in K^{2,2}_{t,s}$. □

2.2. Statement and proof of existence

We find ground states of (2.1) under additional assumptions on V and f. To state these assumptions we need the notion of Steiner-symmetrization, cf. Chapter 3 in [46]. The Steiner-symmetrization (also called symmetric-deacreasing rearrangement) of a cylindrical function $g = g(r,z)$ with respect to z is denoted by $g^\star$. We say that g is *Steiner-symmetric* if g coincides with its Steiner-symmetrization with respect to z, keeping the r-variable fixed. A function $h \in L^\infty(\Omega)$ is *reversed Steiner-symmetric* if $(\operatorname{ess\,sup} h - h)^\star = \operatorname{ess\,sup} h - h$ holds true. In other words h is even and symmetrically nondecreasing. Our assumptions on f are

(i) $f : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function with $0 \le f(r,z,s) \le c(1 + s^{\frac{p-1}{2}})$ for some $c > 0$ and $p \in (1,5)$,

(ii) $f(r,z,s) = o(1)$ as $s \to 0$ uniformly in $(r,z) \in \Omega$,

(iii) $f(r,z,s)$ strictly increasing in $s \in [0,\infty)$ for all $(r,z) \in \Omega$,

(iv) $\frac{F(r,z,s)}{s} \to \infty$ as $s \to \infty$ uniformly in $(r,z) \in \Omega$,

(v) for all $r \in [0,\infty)$, $s \ge 0$ and $\sigma > 0$ the function

$$\varphi_\sigma(r,z,s) := f(r,z,(s+\sigma)^2)(s+\sigma)^2 - f(r,z,s^2)s^2$$

is symmetrically nonincreasing in z.

Condition (i) refers to a subcritical growth of f. Conditions (ii)-(iv) are the ones by Szulkin and Weth (compare [70]) translated to our cylindrically symmetric setting. The last condition is needed later to prove that a Steiner symmetrized minimizing sequence is still a minimizing sequence of the functional J over M. This condition is due to Brock (see Theorem 5.1 in [15]).

The conditions on f are satisfied if for instance $f(r,z,s) = \Gamma(r,z)s^{\frac{p-1}{2}}$ where $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric, $\operatorname{ess\,inf}_\Omega \Gamma > 0$ and $p \in (1,5)$. This choice of f corresponds to the equation $\nabla\times\nabla\times U + V(r,z)U = \Gamma(r,z)\,|U|^{p-1}\,U$ in $\mathbb{R}^3$. Another possible choice is $f(r,z,s) = \Gamma(r,z)\log(1+s)$ where again $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric and $\operatorname{ess\,inf}_\Omega \Gamma > 0$. This nonlinearity appeared for instance in [49] and it does not satisfy the classical Ambrosetti-Rabinowitz condition. Another example is

$$f(r,z,s) = \begin{cases} s^{\frac{\tilde{p}-1}{2}}, & s \in [0,1], (r,z) \in \Omega, \\ s^{\frac{p(z)-1}{2}}, & s > 1, (r,z) \in \Omega, \end{cases}$$

where $\tilde{p} \in (1,5)$, p is Steiner symmetric and $1 < \inf p \le \sup p < 5$.

We aim to prove the following result.

Theorem 2.6. *Let $V \in L^\infty(\Omega)$ be reversed Steiner-symmetric such that the map*

$$\|\cdot\| : H^1_{\mathrm{cyl}}(r^3 drdz) \to \mathbb{R}; u \mapsto \left(\int_\Omega \left(|\nabla_{r,z} u|^2 + V(r,z)u^2 \right) r^3 d(r,z) \right)^{\frac{1}{2}} \tag{2.10}$$

is an equivalent norm to $\|\cdot\|_{H^1_{\mathrm{cyl}}(r^3drdz)}$. Additionally, let f satsify the assumptions (i)-(v). Then (2.1) *has a ground state $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ which is symmetric about $\{z = 0\}$.*

Remark 2.7. *(1) The assumption of norm-equivalence is for instance satisfied if $V \ge 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$, where $B_R^c := \{(r,z) \in \Omega : r^2 + z^2 > R^2\}$. Suppose not. Then there is a sequence $(u_k)_{k\in\mathbb{N}}$ such that $\|u_k\|_{L^2(r^3drdz)} = 1$ and $\int_\Omega \left(|\nabla_{r,z} u_k|^2 + V(r,z)u_k^2 \right) r^3 d(r,z) \to 0$ as $k \to \infty$. In particular,*

$$\int_\Omega |\nabla_{r,z} u_k|^2 r^3 d(r,z) \to 0 \text{ and } \int_{B_R^c} u_k^2 r^3 d(r,z) \to 0 \text{ as } k \to \infty. \tag{2.11}$$

Let χ denote a smooth cut-off function such that $\chi(r,z) = 1$ for $0 \le \sqrt{r^2+z^2} < R$ and $\chi(r,z) = 0$ for $\sqrt{r^2+z^2} \ge R+1$. Then $v_k := \chi u_k \in H^1_{0,\mathrm{cyl}}(B_{R+1}, r^3 drdz)$ and

$$|\nabla_{r,z} v_k|^2 = \chi^2 |\nabla_{r,z} u_k|^2 + |\nabla_{r,z}\chi|^2 u_k^2 + 2u_k\chi\nabla_{r,z}u_k \cdot \nabla_{r,z}\chi.$$

Hence, by (2.11)

$$\begin{aligned} \int_\Omega |\nabla_{r,z} v_k|^2 r^3 d(r,z) &\le 2\int_\Omega \chi^2 |\nabla_{r,z} u_k|^2 r^3 d(r,z) + 2\int_\Omega u_k^2 |\nabla_{r,z}\chi|^2 r^3 d(r,z) \\ &\le 2\int_\Omega |\nabla_{r,z} u_k|^2 r^3 d(r,z) + 2\|\nabla_{r,z}\chi\|_\infty^2 \int_{B_{R+1}\setminus B_R} u_k^2 r^3 d(r,z) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{2.12}$$

In particular, $\int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r,z) \to 0$ as $k \to \infty$. By Poincaré's inequality, $\|u_k\|_{L^2(r^3drdz)} = 1$ and (2.11) *we see*

$$C_P \int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r,z) \ge \int_{B_{R+1}} v_k^2 r^3 d(r,z) \ge \int_{B_R} u_k^2 r^3 d(r,z) = 1 - o(1),$$

contradicting (2.12). □

(2) Since Poincaré's inequality is applicable for domains bounded in one direction we can weaken $\inf_{B_R^c} V > 0$ *to* $\inf_{S^c} V > 0$ *for strips* $S = [0,\infty) \times [0,\rho]$ *with* $\rho > 0$ *or* $S = [r_0, r_1] \times [0,\infty)$ *with* $0 \le r_0 < r_1 < \infty$.

We now prove several statements which will finally lead to the proof of Theorem 2.6.

Lemma 2.8. *For* $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ *Hardy's inequality holds*

$$\int_\Omega \frac{u^2}{r^2} r^3 d(r,z) \le C_H \int_\Omega \left(\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right) r^3 d(r,z). \tag{2.13}$$

Moreover, if $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ *then* $ru \in H^1_{\mathrm{cyl}}(rdrdz)$ *and there is a constant* $C > 0$ *such that for* $2 \le q \le 6$

$$\|ru\|_{H^1_{\mathrm{cyl}}(rdrdz)}, \|ru\|_{L^q_{\mathrm{cyl}}(rdrdz)} \le C \|u\|_{H^1_{\mathrm{cyl}}(r^3 drdz)} \tag{2.14}$$

Proof. Hardy's inequality (2.13) is given in Lemma 9 (i) in [5]. For $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ we have ru, $\frac{\partial}{\partial z}(ru)$, $r\frac{\partial u}{\partial r} \in L^2_{\mathrm{cyl}}(rdrdz)$ and by (2.13) also $u \in L^2_{\mathrm{cyl}}(rdrdz)$. Since $\frac{\partial}{\partial r}(ru) = r\frac{\partial u}{\partial r} + u$ we conclude altogether $ru \in H^1_{\mathrm{cyl}}(rdrdz)$. By the Sobolev embedding in three dimensions this implies $ru \in L^q(rdrdz)$ for $q \in [2,6]$ and (2.13) yields

$$\begin{aligned} \|ru\|^2_{H^1_{\mathrm{cyl}}(rdrdz)} &= \int_\Omega \left(|\nabla_{r,z}(ru)|^2 + r^2u^2 \right) rd(r,z) \\ &\le 2 \int_\Omega \left(\left(r\frac{\partial u}{\partial z}\right)^2 + \left(r\frac{\partial u}{\partial r}\right)^2 + u^2 + r^2u^2 \right) rd(r,z) \le \tilde{C} \|u\|^2_{H^1_{\mathrm{cyl}}(r^3 drdz)}. \end{aligned} \tag{2.15}$$

□

Next we show that the functional J from (2.2) as well as the functional in the defintion of the Nehari-manifold are well-defined.

Lemma 2.9. *There is a constant* $C > 0$ *such that*

$$\int_\Omega f(r,z,r^2u^2)u^2r^3 d(r,z), \int_\Omega \frac{1}{2r^2} F(r,z,r^2u^2) r^3 d(r,z) \le C \left(\|u\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|u\|^{p+1}_{H^1_{\mathrm{cyl}}(r^3drdz)} \right) \tag{2.16}$$

for all $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$.

Proof. Clearly assumption (i) and (ii) show that for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$0 \le f(r,z,s) \le \epsilon + C_\epsilon s^{\frac{p-1}{2}}.$$

Hence

$$0 \le f(r,z,r^2u^2)u^2r^3 \le \left(\epsilon r^2u^2 + C_\epsilon |ru|^{p+1} \right) r, \tag{2.17}$$

$$0 \le \frac{1}{r^2} F(r,z,r^2u^2) r^3 \le \left(\epsilon r^2 u^2 + \tilde{C}_\epsilon |ru|^{p+1} \right) r. \tag{2.18}$$

Due to (2.14) this implies the claim. □

The following compactness result again due to Lions ([47] and [48]) is an important tool in the next lemma as well as the following chapters. We give a proof here since a proof is not included in the work by Lions (compare Théorème III.2 in [48], in particular the comment that the proof is exactly like the one of Théorème III.1 is not the case). Recall the notation $K_{t,s} := K^{2,2}_{t,s}$ from Lemma 2.5.

Theorem 2.10. *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence in $K_{4,1}$ such that $v_k \rightharpoonup v \in H^1_{\mathrm{cyl}}(r^3drdz)$ as $k \to \infty$. Then*

$$rv_k \to rv \text{ in } L^{p+1}(rdrdz) \text{ as } k \to \infty \text{ for } p \in (1,5). \tag{2.19}$$

Remark: In the proof we use twice the following principle: if $S \subset \mathbb{R}^m$ is a set of finite measure and $w_k : S \to \mathbb{R}$ a sequence of measurable functions such that $\|w_k\|_{L^r(S)} \leq C$ and $w_k \to w$ pointwise a.e. as $k \to \infty$ then $\|w_k - w\|_{L^q(S)} \to 0$ as $k \to \infty$ for $1 \leq q < r$. The proof is as follows: Egorov's theorem allows to choose $\Sigma \subset S$ such that $w_k \to w$ uniformly on Σ and $|S \setminus \Sigma| \leq \epsilon$ arbitrary small. By Hölder's inequality the remaining integral is estimated by $\int_{S\setminus\Sigma} |w_k - w|^q \, dx \leq \epsilon^{1-\frac{q}{r}} \|w_k - w\|^q_{L^r(S)}$.

Proof. Let $(v_k)_{k\in\mathbb{N}}$ be a sequence in $K_{4,1}$ such that $v_k \rightharpoonup v \in H^1_{\mathrm{cyl}}(r^3drdz)$ as $k \to \infty$. W.l.o.g. we choose a subsequence such that $v_k \to v$ pointwise almost everywhere as $k \to \infty$. By Lemma 2.5 one gets $v \in K_{4,1}$ and using Lemma 2.2 there exists a constant $C > 0$ such that

$$0 \leq v_k(r,z), v(r,z) \leq Cr^{-\frac{3}{2}}|z|^{-\frac{1}{2}} \text{ for all } k \in \mathbb{N} \text{ and almost all } (r,z) \in \Omega. \tag{2.20}$$

We prove (2.19) by splitting our domain Ω into four parts $\Omega_1, \ldots, \Omega_4$ and show (2.19) on each of these parts separately. The definitions of $\Omega_1, \ldots, \Omega_4$ are as follows: For $R > 0$ let

$$\begin{aligned}
\Omega_1 &:= \{(r,z) \in \Omega : r < R, |z| < R\}, \quad \Omega_2 := \{(r,z) \in \Omega : r \geq R, |z| \geq R\}, \\
\Omega_3 &:= \{(r,z) \in \Omega : r < R, |z| \geq R\}, \quad \Omega_4 := \{(r,z) \in \Omega : r \geq R, |z| < R\}.
\end{aligned}$$

Convergence on Ω_1: Follows from $rv_k \to rv$ in $L^q(K; r\,drdz)$ for every compact subset $K \subset [0,\infty)\times\mathbb{R}$ and every $q \in [1,6)$. This step works independently of the choice of $R > 0$.

Convergence on Ω_2: Let $\varepsilon > 0$. With the help of (2.20) we calculate

$$\begin{aligned}
\int_{\Omega_2} |rv_k - rv|^{p+1} \, rd(r,z) &\leq 2^{p+1} \int_{\Omega_2} r^{p+1} \left(|v_k|^{p+1} + |v|^{p+1}\right) rd(r,z) \\
&\leq 2^{p+1} C^{p-1} \int_{\Omega_2} r^{-\frac{p-1}{2}} \, |z|^{-\frac{p-1}{2}} \left(|v_k(r,z)|^2 + |v(r,z)|^2\right) r^3 d(r,z) \\
&\leq C_1 \left(\|v_k\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|v\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)}\right) R^{-(p-1)} \leq C_2 R^{-(p-1)}
\end{aligned}$$

which is less or equal ε if we choose $R > 0$ large enough.

Convergence on Ω_3: Due to symmetry in z-direction it is enough to focus on $\tilde{\Omega}_3 := \{(r,z) \in \Omega : r < R, z \geq R\}$. Let $\alpha > 0$ be arbitrary. Again by (2.20) we obtain

$$\{(r,z) \in \tilde{\Omega}_3 : v_k(r,z) > \alpha\} \subset \{(r,z) \in \tilde{\Omega}_3 : r\, z^{\frac{1}{3}} \leq C_\alpha\} =: S_\alpha,$$

where $C_\alpha = (C/\alpha)^{2/3}$ and C is the constant from (2.20). The set S_α has finite measure since

$$|S_\alpha| \leq \int_R^\infty \int_0^{C_\alpha z^{-1/3}} r^3 dr \, dz = \frac{C_\alpha^4}{4} \int_R^\infty z^{-\frac{4}{3}} dz = \frac{3}{4} C_\alpha^4 R^{-\frac{1}{3}} < \infty.$$

By the convergence principle from the remark above and since by (2.14) $\|rv_k\|_{L^6(rdrdz)} \leq \|v_k\|_{H^1_{\mathrm{cyl}}(r^3drdz)}$ is bounded we obtain $\int_{S_\alpha} r^{p-1}|v_k - v|^{p+1} r^3 d(r,z) \to 0$ as $k \to \infty$ for $1 \leq p < 5$. It remains to prove the convergence on $\tilde{\Omega}_3 \setminus S_\alpha$. For allmost all $(r,z) \in \tilde{\Omega}_3 \setminus S_\alpha$ we have that $v(r,z) = \lim_{k\to\infty} v_k(r,z) \leq \alpha$. Hence,

$$\int_{\tilde{\Omega}_3\setminus S_\alpha} r^{p-1}|v_k - v|^{p+1} r^3 d(r,z) \leq R^{p-1}(2\alpha)^{p-1} \int_\Omega |v_k - v|^2 r^3 d(r,z) \leq C\alpha^{p-1}.$$

In summary, since $\alpha > 0$ is arbitrary this shows (2.19) on Ω_3.
Convergence on Ω_4: Again it is enough to focus on $\tilde{\Omega}_4 := \{(r,z) \in \Omega : r \geq R, 0 \leq z < R\}$. Fix $z \in (0,R)$. Let us first show that

$$\int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0 \text{ as } k \to \infty. \tag{2.21}$$

Since $v_k(r,\cdot)$ is nonincreasing in its last component we deduce

$$\int_0^\infty r^q v_k^q(r,z) r\, dr \leq \frac{1}{z}\int_0^z \int_0^\infty r^q v_k^q(r,\zeta) r\, drd\zeta \leq \frac{1}{z}\int_\Omega r^q v_k^q(r,\zeta) r d(r,\zeta) \leq \frac{C}{z} \tag{2.22}$$

for all $q \in [2,6]$ by (2.14) . Thus for $q \in [2,6]$ the sequence $\|\cdot\, v_k(\cdot,z)\|_{L^q((0,\infty),rdr)}$ is uniformly bounded in $k \in \mathbb{N}$. Moreover, (2.20) implies $v_k(r,z) \leq C(z) r^{-\frac{3}{2}}$ uniformly in $k \in \mathbb{N}$. Hence for $\tilde{R} > R$

$$\begin{aligned}\int_{\tilde{R}}^\infty r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr &\leq (2C(z))^{p-1} \int_{\tilde{R}}^\infty r^{-\frac{p-1}{2}} |v_k(r,z) - v(r,z)|^2 r^3 dr \\ &\leq (2C(z))^{p-1} \tilde{R}^{\frac{1-p}{2}} \frac{C}{z}\end{aligned}$$

by (2.22). The last term can be made arbitrarily small provided $\tilde{R}$ is chosen big enough. To finish the proof of (2.21) it remains to prove $\int_R^{\tilde{R}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0$ as $k \to \infty$. Since for almost all $z \in (0,R)$ we have $v_k(\cdot,z) \to v(\cdot,z)$ pointwise almost everywhere on $(R,\tilde{R})$ as well as the boundedness of $\|\cdot\, v_k(\cdot,z)\|_{L^6((0,\infty),rdr)}$ by (2.22) we can apply the convergence principle from the remark above and deduce

$$\int_R^{\tilde{R}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0 \text{ as } k \to \infty.$$

Hence (2.21) is accomplished for almost all $z \in (0,R)$.
Defining $\varphi_k(z) := \int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr$ we have $\varphi_k \to 0$ as $k \to \infty$ pointwise almost everywhere in $[0,R)$. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $L^1([0,R),dz)$ since by (2.14)

$$\int_0^R \int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 drdz \leq C \int_\Omega r^{p-1}\left(|v_k|^{p+1} + |v|^{p+1}\right) r^3 d(r,z) \leq \tilde{C}.$$

Moreover, for $p \in (1,3]$, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}([0,R),dz)$ since

$$\left\|\frac{\partial\varphi_k}{\partial z}\right\|^2_{L^1([0,R],dz)} \leq \left(\int_0^R \int_R^\infty (p+1) r^{p-1}|v_k - v|^p \left|\frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z}\right| r^3 drdz\right)^2$$

$$
\begin{aligned}
&\leq \left(\int_\Omega (p+1) r^{p-1} |v_k - v|^p \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right| r^3 d(r,z) \right)^2 \\
&\leq C \int_\Omega r^{2p-2} |v_k - v|^{2p} r^3 d(r,z) \int_\Omega \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right|^2 r^3 d(r,z) \\
&= C \| r(v_k - v) \|^{2p}_{L^{2p}(rdrdz)} \int_\Omega \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right|^2 r^3 d(r,z) \leq C.
\end{aligned}
$$

Hence, by the compact embedding $W^{1,1}([0,R),dz) \hookrightarrow L^1([0,R),dz)$ we conclude that at least a subsequence of $(\varphi_k)_{k\in\mathbb{N}}$ is converging in $L^1([0,R),dz)$ to a limit function, which must be 0 since we have already asserted the pointwise a.e. convergence to 0 on $[0,R)$. This shows (2.19) on Ω_4 for $p \in (1,3]$. For $p \in (3,5)$ we make use of Hölder interpolation, namely,

$$
\|rv_k - rv\|^{p+1}_{L^{p+1}_{\mathrm{cyl}}(\Omega_4, rdrdz)} \leq \|rv_k - rv\|^{4\theta}_{L^4_{\mathrm{cyl}}(\Omega_4, rdrdz)} \|rv_k - rv\|^{6(1-\theta)}_{L^6_{\mathrm{cyl}}(\Omega_4, rdrdz)} \leq \tilde{C} \, \|rv_k - rv\|^{4\theta}_{L^4_{\mathrm{cyl}}(\Omega_4, rdrdz)} \to 0
$$

as $k \to \infty$, where $\theta \in (0,1)$ is chosen such that $p+1 = 4\theta + 6(1-\theta)$, i.e., $\theta = \frac{5-p}{2}$.
The combination of convergences on $\Omega_1, \ldots, \Omega_4$ finally proves (2.19). □

Lemma 2.11. *The functionals*

$$
I(v) = \int_\Omega \frac{1}{2r^2} F(r,z,r^2v^2) r^3 \, d(r,z), \qquad I'(v)[v] = \int_\Omega f(r,z,r^2v^2) v^2 r^3 \, d(r,z)
$$

are weakly sequentially continuous on the set $K_{4,1} \subset H^1_{\mathrm{cyl}}(r^3drdz)$.

Proof. Let us take a weakly convergent sequence $(v_k)_{k\in\mathbb{N}}$ in $K_{4,1}$ such that $v_k \rightharpoonup v$ in $H^1_{\mathrm{cyl}}(r^3drdz)$ and $v_k \to v$ pointwise a.e. in Ω. Our goal is now to show at least for a subsequence

$$
\int_\Omega \frac{1}{r^2} F(r,z,r^2v_k^2) r^3 d(r,z) \to \int_\Omega \frac{1}{r^2} F(r,z,r^2v^2) r^3 d(r,z) \text{ as } k \to \infty \tag{2.23}
$$

and

$$
\int_\Omega f(r,z,r^2v_k^2) v_k^2 r^3 d(r,z) \to \int_\Omega f(r,z,r^2v^2) v^2 r^3 d(r,z) \text{ as } k \to \infty. \tag{2.24}
$$

By (2.18) we find

$$
\frac{1}{r^2} \left| F(r,z,r^2v_k^2) - F(r,z,r^2v^2) \right| r^3 \leq \epsilon r^2 (v_k^2 + v^2) r + C_\epsilon \left(|rv_k|^{p+1} + |rv|^{p+1} \right) r
$$

and hence

$$
\left(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2 (v_k^2 + v^2) \right)^+ r \leq C_\epsilon \left(|rv_k|^{p+1} + |rv|^{p+1} \right) r. \tag{2.25}
$$

Theorem 2.10 implies

$$
rv_k \to rv \text{ in } L^{p+1}(rdrdz) \text{ as } k \to \infty \tag{2.26}
$$

so that we obtain a majorant $|rv_k|, |rv| \leq w \in L^{p+1}(r\,drdz)$ (cf. Lemma A.1 in [73]). Together with (2.25) this majorant allows to apply Lebesgue's dominated convergence theorem and yields

$$
\lim_{k\to\infty} \int_\Omega \left(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2 (v_k^2 + v^2) \right)^+ r\,drdz = 2\epsilon \|v\|^2_{L^2(r^3drdz)}. \tag{2.27}
$$

If we set

$$a_k := \int_\Omega |F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| r\,drdz$$

and

$$b_k := \epsilon \|r^2(v_k^2 + v^2)\|_{L^1(rdrdz)} = \epsilon(\|v_k\|^2_{L^2(r^3drdz)} + \|v\|^2_{L^2(r^3drdz)}) \le C\epsilon$$

then

$$\begin{aligned}
\limsup_{k\in\mathbb{N}} a_k &\le \limsup_{k\in\mathbb{N}} b_k + \limsup_{k\in\mathbb{N}} (a_k - b_k)^+ \\
&\le C\epsilon + \limsup_{k\in\mathbb{N}} \left(\int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big) rdrdz\right)^+ \\
&\le C\epsilon + \limsup_{k\in\mathbb{N}} \int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big)^+ rdrdz \\
&\le \epsilon(C + 2\|v\|^2_{L^2(r^3drdz)}) \text{ by (2.27).}
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this shows that $\lim_{k\to\infty} a_k = 0$ and therefore (2.23) holds. The proof of (2.24) is similar since $\Big(f(r,z,r^2v_k^2)r^2v_k^2 - f(r,z,r^2v^2)r^2v^2 - \epsilon r^2(v_k^2+v^2)\Big)^+ r$ satisfies an estimate just like (2.25) if we use (2.17) instead of (2.18). □

Here is our last lemma before we can give the proof of Theorem 2.6.

Lemma 2.12. *For $u \in H^1_{\text{cyl}}(r^3drdz)$, $u \ge 0$ we have $\|u^\star\| \le \|u\|$ where $\star$ denotes Steiner-symmetrization with respect to z and $\|\cdot\|$ is the equivalent norm from Theorem 2.6. Moreover*

$$I(u) \le I(u^\star) \quad \text{and} \quad I'(u)[u] \le I'(u^\star)[u^\star].$$

Proof. We begin by recalling several classical rearrangement inequalities from [45], [46]. Recall first the Pólya-Szegö inequality

$$\int_{\mathbb{R}^n} |\nabla f^\circledast|^2 dx \le \int_{\mathbb{R}^n} |\nabla f|^2 dx \tag{2.28}$$

for $f \in H^1(\mathbb{R}^n)$ and $^\circledast$ denoting Schwarz-symmetrization (also called symmetrically decreasing rearrangement). Furthermore we have for $0 \le f, g \in L^2(\mathbb{R}^n)$ the classical rearrangement inequality

$$\int_{\mathbb{R}} fg dx \le \int_{\mathbb{R}} f^\circledast g^\circledast dx \tag{2.29}$$

and the nonexpansivity of rearrangement

$$\int_{\mathbb{R}^n} |f^\circledast - g^\circledast|^2 dx \le \int_{\mathbb{R}^n} |f-g|^2\, dx. \tag{2.30}$$

From (2.28) we immediately receive for $u \in H^1_{\text{cyl}}(r^3drdz)$ that

$$\int_{\mathbb{R}} |\nabla_z u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_z u|^2 dz. \tag{2.31}$$

Next we want to establish a similar inequality for $\nabla_r u$. We do this first for $u \in C_c^\infty([0,\infty)\times\mathbb{R})$. With the help of (2.30) we find that

$$\int_{\mathbb{R}} \left|\frac{u^\star(r+t,z)-u^\star(r,z)}{t}\right|^2 dz \le \int_{\mathbb{R}} \left|\frac{u(r+t,z)-u(r,z)}{t}\right|^2 dz$$

for almost all $r,t \in [0,\infty)$. Sending $t \to 0$ and using Fatou's lemma on the left side of the inequality yields

$$\int_{\mathbb{R}} |\nabla_r u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_r u|^2 dz \tag{2.32}$$

for $u \in C_c^\infty([0,\infty)\times\mathbb{R})$ and almost all $r \in [0,\infty)$. Since Steiner Symmetrization is continuous in H^1 (see Theorem 1 in [16]) we obtain by approximation that (2.32) is indeed valid for all $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$. Together with (2.31) we obtain $\int_{\mathbb{R}} |\nabla_{r,z} u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_{r,z} u|^2 dz$ for almost all $r \ge 0$ and integration leads to

$$\int_{\mathbb{R}}\int_0^\infty |\nabla_{r,z} u^\star|^2 r^3 drdz \le \int_{\mathbb{R}}\int_0^\infty |\nabla_{r,z} u|^2 r^3 drdz. \tag{2.33}$$

Fixing $r \in [0,\infty)$ and applying (2.29) to $f(\cdot) = \operatorname{ess\,sup} V - V(r,\cdot)$ and $g(\cdot) = u^2(r,\cdot)$ gives

$$\begin{aligned}\int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, u^2(r,\cdot) dz &\le \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))^\star\, (u^2)^\star(r,\cdot) dz \\ &= \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, (u^\star)^2\,(r,\cdot) dz.\end{aligned}$$

Using $\|u(r,\cdot)\|_{L^2(\mathbb{R})} = \|u^\star(r,\cdot)\|_{L^2(\mathbb{R})}$ this results in

$$\int_{\mathbb{R}}\int_0^\infty V(r,z)\,(u^\star)^2\, r^3 drdz \le \int_{\mathbb{R}}\int_0^\infty V(r,z) u^2 r^3 drdz. \tag{2.34}$$

The combination of (2.33) and (2.34) yields the claimed inequality $\|u^\star\|^2 \le \|u\|^2$.
Assumption (v) on f allows to apply Theorem 5.1 in [15] and to deduce

$$I'(u)[u] = \int_\Omega f(r,z,r^2u^2)u^2 r^3 d(r,z) \le \int_\Omega f(r,z,r^2u^{\star 2})u^{\star 2} r^3 d(r,z) = I'(u^\star)[u^\star]. \tag{2.35}$$

Moroever, using (v) with $s = 0$ shows that for all $r \in [0,\infty)$, $\sigma > 0$ the function $z \mapsto f(r,z,\sigma^2)$ is symmetrically nonincreasing in z and hence

$$\Phi_\sigma(r,z,s) := F(r,z,r^2(s+\sigma)^2) - F(r,z,r^2s^2) = \int_{r^2s^2}^{r^2(s+\sigma)^2} f(r,z,t)\, dt$$

is symmetrically nonincreasing in z. Applying once more Theorem 5.1 in [15] yields

$$I(u) = \int_\Omega \frac{1}{2r^2} F(r,z,r^2u^2) r^3 d(r,z) \le \int_\Omega \frac{1}{2r^2} F(r,z,r^2u^{\star 2}) r^3 d(r,z) = I(u^\star).$$

This finishes the proof of the lemma. □

Finally, we are ready to give the proof of Theorem 2.6.

Proof. Recall from Lemma 2.11 the definition $I(u) := \int_\Omega \frac{1}{2r^2} F(r,z,r^2u^2) r^3 d(r,z)$ for $u \in H^1_{\text{cyl}}(r^3 drdz)$. We show that the assumptions (i)-(iii) of Theorem 12 in [70] are satisfied. Let $\varepsilon > 0$. The growth assumptions (i) and (ii) on f imply that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that the global estimate $0 \le f(r,z,s) \le \epsilon + C_\epsilon |s|^{\frac{p-1}{2}}$ holds. Together with (2.14) we obtain

$$\begin{aligned}
|I'(u)[v]| &= \left| \int_\Omega f(r,z,r^2u^2) uvr^3 d(r,z) \right| \\
&\le \varepsilon \int_\Omega |ru||rv| r d(r,z) + C_\epsilon \int_\Omega |ru|^p |rv| r d(r,z) \\
&\le \varepsilon C \|u\|_{H^1_{\text{cyl}}(r^3 drdz)} \|v\|_{H^1_{\text{cyl}}(r^3 drdz)} + \tilde{C}_\epsilon \|u\|^p_{H^1_{\text{cyl}}(r^3 drdz)} \|v\|_{H^1_{\text{cyl}}(r^3 drdz)}
\end{aligned}$$

Taking the supremum over all $v \in H^1_{\text{cyl}}(r^3 drdz)$ with $\|v\|_{H^1_{\text{cyl}}(r^3 drdz)} = 1$ we see that

$$I'(u) = o(\|u\|) \text{ as } u \to 0. \tag{2.36}$$

Moreover, due to assumption (iii) on f the map

$$s \mapsto \frac{I'(su)[u]}{s} = \int_\Omega f(r,z,s^2r^2u^2) u^2 r^3 d(r,z) \text{ is strictly increasing for all } u \ne 0 \text{ and } s > 0. \tag{2.37}$$

Next we claim that

$$\frac{I(su)}{s^2} \to \infty \text{ as } s \to \infty \text{ uniformly for } u \text{ on weakly compact subsets } W \text{ of } H^1_{\text{cyl}}(r^3 drdz) \setminus \{0\}. \tag{2.38}$$

Suppose not. Then there are $(u_k)_{k\in\mathbb{N}} \subset W$ and $s_k \to \infty$ as $k \to \infty$ such that $\frac{I(s_k u_k)}{s_k^2}$ is bounded as $k \to \infty$. But along a subsequence we have $u_k \rightharpoonup u \ne 0$ and $u_k(x) \to u(x)$ pointwise almost everywhere. Let $\Omega^\sharp := \{(r,z) \in \Omega : u(r,z) \ne 0\}$. Then $|\Omega^\sharp| > 0$ and on $\Omega^\sharp$ we have $|s_k u_k(r,z)| \to \infty$ as $k \to \infty$. Fatou's lemma and assumption (iv) on F imply

$$\frac{I(s_k u_k)}{s_k^2} = \int_\Omega \frac{F(r,z,s_k^2 r^2 u_k^2)}{2 s_k^2 r^2} r^3 d(r,z) \ge \int_{\Omega^\sharp} \frac{F(r,z,s_k^2 r^2 u_k^2)}{2 s_k^2 r^2 u_k^2} u_k^2 r^3 d(r,z) \to \infty \text{ as } k \to \infty,$$

a contradiction. In summary, (2.36), (2.37), (2.38) imply that (i)-(iii) of Theorem 12 in [70] are satisfied.

Now we take a sequence $(u_k)_{k\in\mathbb{N}} \subset M$ such that $J(u_k) \to \inf_M J$ as $k \to \infty$. Since $\|\nabla_{r,z} |u_k|\,\|_{L^2} = \|\nabla_{r,z} u_k\|_{L^2}$ we can assume that $u_k \ge 0$ for all $k \in \mathbb{N}$. Then Theorem 12 in [70] guarantees that for every k there is a unique $t_k > 0$ such that $v_k := t_k u_k^\star \in M$. We show next that $t_k \le 1$ for all $k \in \mathbb{N}$. Assume $t_k > 1$. Then

$$\begin{aligned}
\int_\Omega f(r,z,r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) &< \int_\Omega f(r,z,t_k^2 r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) \quad \text{by assumption (iii)} \\
&= \|u_k^\star\|^2 \quad \text{since } t_k u_k^\star \in M \\
&\le \|u_k\|^2 \quad \text{by Lemma 2.12} \\
&= \int_\Omega f(r,z,r^2 u_k^2) u_k^2 r^3 d(r,z) \quad \text{since } u_k \in M.
\end{aligned}$$

This contradicts the inequality $I'(u_k)[u_k] \leq I'(u_k^\star)[u_k^\star]$ from Lemma 2.12 and thus $t_k \leq 1$ for all $k \in \mathbb{N}$. Next notice that for fixed $(r,z,s) \in [0,\infty) \times \mathbb{R} \times [0,\infty)$ and $t \in (0,1]$ one has

$$\frac{d}{dt}\left(t^2 f(r,z,s^2)s^2 - F(r,z,t^2s^2)\right) = 2ts^2\left(f(r,z,s^2) - f(r,z,t^2s^2)\right) > 0$$

since f is strictly increasing in its last variable by assumption (iii). This shows that the map $t \mapsto t^2 f(r,z,s^2)s^2 - F(r,z,t^2s^2)$ is strictly increasing for $t \in [0,1]$. From this monotonicity and the inequality $I(t_k u_k) \leq I(t_k u_k^\star)$ from Lemma 2.12 we conclude

$$\begin{aligned} 2J(v_k) &= \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k^\star|^2 + V(r,z) t_k^2 u_k^{\star 2} - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^{\star 2})\right) r^3 d(r,z) \\ &\leq \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k|^2 + V(r,z) t_k^2 u_k^2 - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^2)\right) r^3 d(r,z) \\ &= \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2u_k^2) t_k^2 r^2 u_k^2 - F(r,z,r^2 t_k^2 u_k^2)\right) r^3 d(r,z) \\ &\leq \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2u_k^2) r^2 u_k^2 - F(r,z,r^2 u_k^2)\right) r^3 d(r,z) \\ &= 2J(u_k). \end{aligned} \tag{2.39}$$

So $(v_k)_{k\in\mathbb{N}} \subset M$ is also a minimizing sequence for J which belongs to $K_{4,1}$. The boundedness of $(v_k)_{k\in\mathbb{N}}$ is established in Proposition 14 in [70]. Hence, we find $v_\infty \in H^1_{\mathrm{cyl}}(r^3 drdz)$ such that $v_k \rightharpoonup v_\infty$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ along a subsequence as $k \to \infty$. In addition, $v_\infty \in K_{4,1}$ due to Lemma 2.5 and $v_\infty \neq 0$ by Proposition 14 in [70] where instead of the weak sequential continuity of I on all of $H^1_{\mathrm{cyl}}(r^3 drdz)$ we use it only on $K_{4,1}$ as stated in Lemma 2.11.

Let us show that $v_\infty \in M$. Since $v_\infty \neq 0$ we can choose $t_\infty > 0$ such that $t_\infty v_\infty \in M$. Arguing in the same manner as before for the sequence t_k we know that $t_\infty \leq 1$. Assume $t_\infty < 1$. Then as in (2.39) and using the weak sequential continuity on $K_{4,1}$ as shown in Lemma 2.11 we find

$$\begin{aligned} 2J(t_\infty v_\infty) &< \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2 v_\infty^2) r^2 v_\infty^2 - F(r,z,r^2 v_\infty^2)\right) r^3 d(r,z) \\ &= \lim_{k\to\infty} \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2 v_k^2) r^2 v_k^2 - F(r,z,r^2 v_k^2)\right) r^3 d(r,z) \\ &= 2\inf_M J \leq 2J(t_\infty v_\infty) \end{aligned}$$

which is a contradiction. So $t_\infty = 1$ and thus $v_\infty \in M$. Then by the weak lower semi-continuity of $\|\cdot\|$ and once again the weak sequential continuity of I we conclude

$$J(v_\infty) \leq \liminf_{k\to\infty} J(v_k) = \inf_M J \leq J(v_\infty).$$

Hence, $v_\infty \in K_{4,1}$ is a minimizer of J on M, i.e., a ground state of (2.1) which is Steiner symmetric in z with respect to $\{z = 0\}$. □

3. Further properties in the case of a power-nonlinearity

In this chapter we turn away from a general nonlinearity and focus on an odd subcritical power nonlinearity. This refers to $f(r,z,r^2u^2)u = \Gamma(r,z)r^{p-1}|u|^{p-1}u$ for $\Gamma\colon \Omega \to \mathbb{R}$ in the previous Chapter 2. Moreover, we also restrict the class of coefficients to those which are independent of $z \in \mathbb{R}$. Hence, (2.1) simplifies to

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} + V(r)u = \Gamma(r)r^{p-1}|u|^{p-1}u \text{ in } \Omega. \tag{3.1}$$

We assume that the coefficients V and Γ satisfy

(i) $V, \Gamma \in W^{1,\infty}([0,\infty))$,

(ii) $\inf V, \inf \Gamma > 0$.

In particular, the assumptions of Theorem 2.6 are satisfied and we conclude that there is a ground state $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ of (3.1) which is symmetric about $\{z = 0\}$. Moreover, due to $u \in K_{4,1}$ we have $u \geq 0$, cf. (2.4).

Since $W^{1,\infty}([0,\infty))$ corresponds to Lipschitz-continuous functions assumption (i) above can equivalently be replaced by $V, \Gamma \in W^{1,\infty}(\mathbb{R}^3)$ where here V and Γ are not considered in cylindrical coordinates but as functions of the variable $x \in \mathbb{R}^3$.

By Lemma A.5 weak solutions of (3.1) in $H^1_{\mathrm{cyl}}(r^3 drdz)$ correspond to weak solutions $U \in H^1(\mathbb{R}^3)$ of

$$\nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \text{ in } \mathbb{R}^3$$

where

$$U(x) = u(r,z)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, r = \sqrt{x_1^2 + x_2^2} \text{ and } z = x_3. \tag{3.2}$$

The overall goal of this chapter is to establish several properties of weak solutions of (3.1). We ensure regularity and exponential decay of weak solutions of (3.1) in Section 3.1. Section 3.2 shows that the linearization around a ground state u is a Fredholm operator. Some general considerations of cylindrical eigenfunctions are given in Section 3.3. The fact that the linearization around a ground state u possesses exactly one negative eigenvalue will be provided in Section 3.4. Section 3.5 ensures useful monotonicity and symmetry properties of weak solutions of (3.1).

3.1. Regularity and exponential decay

In this section we guarantee exponential decay of weak solutions of (3.1) and exclude a possible sign-change of ground states of (3.1). First, we introduce some notaion. Let

$$\tilde{L} := -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) - \frac{\partial^2}{\partial z^2} + V(r),$$

where $D(\tilde{L}) := H^2_{\text{cyl}}(r^3 drdz) \subset L^2_{\text{cyl}}(r^3 drdz)$. Since $\tilde{L}$ corresponds to a five-dimensional Schrödinger operator with cylindrical symmetry we abbreviate

$$-\Delta_{5,\text{cyl}} := -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) - \frac{\partial^2}{\partial z^2}.$$

By Lemma 11 in [5], the operator $\tilde{L}\colon H^2_{\text{cyl}}(r^3 drdz) \subset L^2_{\text{cyl}}(r^3 drdz) \to L^2_{\text{cyl}}(r^3 drdz)$ is self-adjoint. The energy functional corresponding to (3.1) reads

$$J\colon H^1_{\text{cyl}}(r^3 drdz) \to \mathbb{R}; u \mapsto \frac{1}{2}\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r)\,|u|^2\right) r^3 d(r,z) - \frac{1}{p+1}\int_\Omega \Gamma(r) r^{p-1}\,|u|^{p+1}\, r^3 d(r,z). \tag{3.3}$$

Like already done in (2.10) we equip $H^1_{\text{cyl}}(r^3 drdz)$ with the norm

$$\|u\|^2 := \int_\Omega \left(|\nabla_{r,z}u|^2 + V(r)\,|u|^2\right) r^3 d(r,z),$$

which is due to $V \in L^\infty([0,\infty))$ and $\inf V > 0$ equivalent to the norm given in Definition 1.1. Moreover, we have

$$\langle u, v\rangle := \int_\Omega (\nabla_{r,z}u \cdot \nabla_{r,z}v + V(r)uv)\, r^3 d(r,z).$$

Our first goal is to improve the regularity of vector-valued weak solutions U of the form (3.2) of

$$\nabla\times\nabla\times U + V(r)U = -\Delta U + V(r)U = \Gamma(r)\,|U|^{p-1}\,U \text{ in } \mathbb{R}^3 \tag{3.4}$$

and afterwards transfer this regularity via (3.2) to the regularity of scalar solutions u of (3.1). Precisely, Theorem A.6 tells us that every component of U is a $C^{2,\alpha}(\mathbb{R}^3)$ function with arbitrary $\alpha \in (0,1)$. Moreover, Lemma A.7 ensures $u \in C^2([0,\infty)\times\mathbb{R})$. Having established regularity of u we are able to deduce the following result.

Corollary 3.1. *Let u be a ground state solution of* (3.1). *Then either $u > 0$ or $u < 0$.*

Proof. Let u be a ground state solution of (3.1) and assume that u has non-vanishing positive and negative part, i.e., $u = u^+ - u^-$ with $u^+ \not\equiv 0, u^- \not\equiv 0$. Then due to the continuity of u we conclude that $|u|$ has to possess zeros, i.e., there is a point $(r^\star, z^\star) \in \Omega$ such that $|u(r^\star, z^\star)| = 0$. Moreover, since u is a ground state of (3.1) we infer that $|u|$ is also a ground state (recall Theorem 7.8 in [46]). Then Harnack's inequality (Theorem 8.20 in [39] and recall $u \in L^\infty(\Omega)$ due to Lemma A.7) gives $\sup_K |u| \leq C(K)\inf_K |u| = 0$, where K denotes an arbitrary compact subset of Ω containing the point $(r^\star, z^\star)$ i.e., $|u| \equiv 0$ in Ω, a contradiction. □

Due to Corollary 3.1 we restrict throughout the first part of this thesis to positive ground state solutions of (3.1).

Our next step is to ensure exponential decay for weak solutions of (3.4). Therefore, we need the notion of Kato classes which we repeat in the following definition.

Definition 3.2. *(see [65]) Let* $w_n(x,y) := |x-y|^{2-n}$ *for* $n \geq 3$ *and* $w_2(x,y) := -\log|x-y|$. *A measurable function* $v\colon \mathbb{R}^n \to \mathbb{R}$ *belongs to the Kato class* K_n, *if*

$$\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^n}\int_{\{|x-y|\leq\rho\}} w_n(x,y)\,|v(y)|\,dy = 0 \text{ in case } n\geq 2,$$
$$\sup_{x\in\mathbb{R}}\int_{\{|x-y|\leq 1\}} |v(y)|\,dy < \infty \text{ in case } n = 1.$$

If $O \subseteq \mathbb{R}^n$ *is open we denote by* $K_n(O)$ *the set of measurable functions* $v\colon \mathbb{R}^n \to \mathbb{R}$ *such that* $v\mathbb{1}_O$ *lies in the Kato class* K_n. *We denote* $v \in K_{n,\mathrm{loc}}(O)$ *if and only if* $v\mathbb{1}_K \in K_n(O)$ *for all compact* $K \subset O$.

We are now in a position to prove the exponential decay result.

Lemma 3.3. *Let* U *be a weak solution of* (3.4). *Then for every* $\mu \in (0, \sqrt{\operatorname{ess\,inf} V})$, *there is a constant* $C_\mu > 0$ *such that* $|U(x)| \leq C_\mu e^{-\mu|x|}$ *for* $|x|$ *sufficiently large.*

Proof. Let $W := V(r) - \Gamma(r)\,|U|^{p-1}$ on $\mathbb{R}^3$. For applying Proposition 5.2 in [51] (with $R = 0$) for every component of the $\mathbb{R}^3$-valued function U we have to check $W^- \in K_3(\mathbb{R}^3 \setminus \{0\})$ and $W^+ \in K_{3,\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, where $W^\pm$ denotes the positive/negative part of W.

The claim $W^+ \leq V \in K_{3,\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ is true, since

$$\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^3}\int_{\{|x-y|\leq\rho\}} \frac{V(x)}{|x-y|}dy \leq C\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^3}\int_{\{|z|\leq\rho\}}\frac{1}{|z|}dz = 4\pi C\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^3}\int_0^\rho r\,dr = 2\pi C\lim_{\rho\to 0}\sup_{x\in\mathbb{R}^3}\rho^2 = 0.$$

From Theorem A.6 we observe $U \in L^\infty(\mathbb{R}^3)$ and hence $W^- \leq C\,|U|^{p-1} \in L^\infty(\mathbb{R}^3)$. Then $W^- \in K_3(\mathbb{R}^3 \setminus \{0\})$ follows by replacing V by $\Gamma\,|U|^{p-1}$ in the calculation above.

To apply Proposition 5.2 in [51] (and then obtain exponential decay of weak solutions of (3.4)) we also have to get some information about $\sigma_{\mathrm{ess}}(-\Delta + V(r) - \Gamma(r)\,|U|^{p-1})$. Theorem 8.3.1 in [55] yields

$$\sigma_{\mathrm{ess}}(-\Delta + V(r) - \Gamma(r)\,|U|^{p-1}) = \sigma_{\mathrm{ess}}(-\Delta + V(r)), \tag{3.5}$$

since we know by Theorem A.6 that $\Gamma(r)\,|U|^{p-1} \in L^\infty(\mathbb{R}^3) \subseteq L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ and $\Gamma(r)\,|U(x)|^{p-1} \to 0$ as $|x| \to \infty$. Due to $\sigma_{\mathrm{ess}}(-\Delta + V(r)) \subseteq [\operatorname{ess\,inf} V, \infty)$ and $\operatorname{ess\,inf} V > 0$ all assumptions of Proposition 5.2 are verified and consequently every component of U has exponential decay at infinity, i.e., also $|U|$ and the claim follows. □

In particular, due to the exponential decay of U in (3.2) we deduce exponential decay of u.

3.2. Fredholm-property of second derivative

In this short section we ensure that the linear operator $J''(u)\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^1_{\mathrm{cyl}}(r^3drdz)$ where u is a positive solution of (3.1) is a Fredholm operator with index 0.

The first and second Fréchet derivatives of J in (3.3) are calculated to be

$$dJ(u)[v] = \int_\Omega \left(\nabla_{r,z}u \cdot \nabla_{r,z}v + V(r)uv - \Gamma(r)r^{p-1}|u|^{p-1}uv\right) r^3 d(r,z) \text{ for } v \in H^1_{\mathrm{cyl}}(r^3 drdz),$$
$$d^2J(u)[v,w] = \int_\Omega \left(\nabla_{r,z}w \cdot \nabla_{r,z}v + V(r)wv - p\Gamma(r)r^{p-1}|u|^{p-1}wv\right) r^3 d(r,z) \text{ for } v,w \in H^1_{\mathrm{cyl}}(r^3 drdz).$$

Via the Riesz representation theorem there are $J'(u) \in H^1_{\mathrm{cyl}}(r^3drdz)$ and a linear operator $J''(u) \in \mathcal{L}\left(H^1_{\mathrm{cyl}}(r^3drdz), H^1_{\mathrm{cyl}}(r^3drdz)\right)$ such that

$$\langle J'(u), v\rangle = dJ(u)[v] \text{ for all } v \in H^1_{\mathrm{cyl}}(r^3drdz),$$
$$\langle J''(u)v, w\rangle = d^2J(u)[v,w] \text{ for all } v,w \in H^1_{\mathrm{cyl}}(r^3drdz). \tag{3.6}$$

We will also write $\nabla J(u)$ instead of $J'(u)$ and $\nabla^2 J(u)$ instead of $J''(u)$.

Theorem 3.4. *The linear operator $J''(u)\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^1_{\mathrm{cyl}}(r^3drdz)$ defined by* (3.6) *is a Fredholm operator with index* 0 *provided u is a positive solution of* (3.1).

Proof. Let u be a positive solution of (3.1). We prove that $J''(u) = \mathrm{Id}_{\mathcal{H}} - \tilde{K}$, where $\tilde{K}\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^1_{\mathrm{cyl}}(r^3drdz)$ is a compact operator. Besides, let $\Phi\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^1_{\mathrm{cyl}}(r^3drdz)'$ denote the isometric Riesz isomorphism. For $v \in H^1_{\mathrm{cyl}}(r^3drdz)$ we define the mappings

$$\mathrm{Id}_v\colon H^1_{\mathrm{cyl}}(r^3drdz) \to \mathbb{R}; w \mapsto \int_\Omega (\nabla_{r,z}w \cdot \nabla_{r,z}v + V(r)wv)\, r^3 d(r,z) = \langle w, v\rangle \text{ and}$$
$$F_v\colon H^1_{\mathrm{cyl}}(r^3drdz) \to \mathbb{R}; w \mapsto \int_\Omega \Gamma(r)r^{p-1}u^{p-1}wvr^3 d(r,z).$$

Hence, by (3.6) we get

$$J''(u)v = \Phi^{-1}(d^2J(u)[v,\cdot]) = \Phi^{-1}(\mathrm{Id}_v) - p\Phi^{-1}(F_v) = v - p\Phi^{-1}(F_v).$$

We now show that $K\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^{-1}_{\mathrm{cyl}}(r^3drdz), v \mapsto F_v$ is a compact operator. For this purpose, let $(v_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^1_{\mathrm{cyl}}(r^3drdz)$. Hence there is $\hat{v} \in H^1_{\mathrm{cyl}}(r^3drdz)$ such that $v_k \rightharpoonup \hat{v}$ as $k \to \infty$ along a subsequence. Let $\varepsilon > 0$. We choose a bounded rectangle $Q = [0,\tilde{R}] \times [-\tilde{z},\tilde{z}] \subset \Omega$ such that

$$\left\|\Gamma(r)r^{p-1}u^{p-1}\right\|_{L^\infty(\Omega\setminus Q)} < \frac{\varepsilon}{2(\|\hat{v}\| + \tilde{C})},$$

where $\tilde{C} := \max_{k\in\mathbb{N}} \|v_k\|$. Using the compactness of the embedding $H^1_{\mathrm{cyl}}(Q, r^3drdz) \hookrightarrow L^2_{\mathrm{cyl}}(Q, r^3drdz)$, we receive for a further subsequence (again denoted by $(v_k)_{k\in\mathbb{N}}$):

$$\|\hat{v} - v_k\|_{L^2_{\mathrm{cyl}}(Q,r^3drdz)} < \frac{\varepsilon}{2\left\|\Gamma(r)r^{p-1}u^{p-1}\right\|_{L^\infty(\Omega)}}$$

for all k large enough. Altogether, we infer

$$\|K\hat{v} - Kv_k\|_{H^{-1}_{\mathrm{cyl}}(r^3drdz)} = \sup_{\|w\|_{H^1_{\mathrm{cyl}}(r^3drdz)} \le 1} \left| \int_\Omega \Gamma(r)r^{p-1}u^{p-1}w(\hat{v} - v_k)r^3 d(r,z)\right|$$

$$
\begin{aligned}
&\leq \sup_{\|w\|_{H^1_{\text{cyl}}(r^3drdz)}\leq 1} \int_Q \Gamma(r) r^{p-1} |u|^{p-1} |w| |\hat{v} - v_k| r^3 d(r,z) \\
&+ \sup_{\|w\|_{H^1_{\text{cyl}}(r^3drdz)}\leq 1} \int_{\Omega\setminus Q} \Gamma(r) r^{p-1} |u|^{p-1} |w| |\hat{v} - v_k| r^3 d(r,z) \\
&\leq \sup_{\|w\|_{H^1_{\text{cyl}}(r^3drdz)}\leq 1} \left\|\Gamma(r) r^{p-1} u^{p-1}\right\|_{L^\infty(\Omega)} \|w\|_{L^2_{\text{cyl}}(Q,r^3drdz)} \|\hat{v} - v_k\|_{L^2_{\text{cyl}}(Q,r^3drdz)} \\
&+ \sup_{\|w\|_{H^1_{\text{cyl}}(r^3drdz)}\leq 1} \left\|\Gamma(r) r^{p-1} u^{p-1}\right\|_{L^\infty(\Omega\setminus Q)} \|w\|_{L^2_{\text{cyl}}(\Omega\setminus Q,r^3drdz)} \|\hat{v} - v_k\|_{L^2_{\text{cyl}}(\Omega\setminus Q,r^3drdz)} \\
&\leq \left\|\Gamma(r) r^{p-1} u^{p-1}\right\|_{L^\infty(\Omega)} \|\hat{v} - v_k\|_{L^2_{\text{cyl}}(Q,r^3drdz)} + \left\|\Gamma(r) r^{p-1} u^{p-1}\right\|_{L^\infty(\Omega\setminus Q)} \left(\|\hat{v}\|_{\mathcal{H}} + \tilde{C}\right) < \varepsilon
\end{aligned}
$$

for $k \in \mathbb{N}$ large enough. Since ε was arbitrarily chosen we conclude $\|K\hat{v} - Kv_k\|_{H^{-1}_{\text{cyl}}(r^3drdz)} \to 0$ as $k \to \infty$, so K is a compact operator.

Because the set of compact operators is a two-sided ideal also $\tilde{K} := \Phi^{-1}K$ is compact. The fact that compact perturbations of the identity are Fredholm operators with index 0 is well-known (theorem of Riesz-Schauder, see [72] Satz VI.2.1) and this finishes the proof since $J''(u) = \mathrm{Id}_{H^1_{\text{cyl}}(r^3drdz)} - \tilde{K}$. □

3.3. Cylindrical eigenfunctions

We now state and prove some basic properties of first (and higher) eigenvalues of

$$L := -\Delta_{5,\text{cyl}} + V(r) - p\Gamma(r) r^{p-1} u^{p-1} \tag{3.7}$$

with $D(L) = H^2_{\text{cyl}}(r^3drdz)$ where u denotes a positive ground state solution of (3.1). L is a self-adjoint operator, see Corollary A.2. We will make use of a variational characterization of eigenvalues. For this purpose we cite (in our notation) the 'max-min principle' (Theorem XIII.1 in [61]): For $n \in \mathbb{N}$ define

$$\lambda_n(\Omega) := \sup_{\varphi_1,\ldots,\varphi_{n-1} \in H^1(r^3)} F(\varphi_1, \ldots, \varphi_{n-1}) \tag{3.8}$$

where

$$F(\varphi_1, \ldots, \varphi_m) := \inf_{\substack{\psi \in H^1(r^3): \|\psi\|_{L^2(r^3)}=1, \\ \psi \in [\varphi_1 \ldots \varphi_m]^{\perp_{L^2(r^3)}}}} b(\psi, \psi) \tag{3.9}$$

where b denotes the bilinear form associated to L, i.e.,

$$b(\varphi, \psi) := \int_\Omega \left(\nabla_{r,z}\varphi \cdot \nabla_{r,z}\psi + V(r)\varphi\psi - p\Gamma(r) r^{p-1} u^{p-1} \varphi\psi\right) r^3 d(r,z) \tag{3.10}$$

for all $\varphi, \psi \in H^1_{\text{cyl}}(r^3drdz)$. Note that $\varphi_1, \ldots, \varphi_m$ are not necessarily linearly independent. Then, for each fixed $n \in \mathbb{N}$, *either*

(a) there are n eigenvalues (counting degenerate eigenvalues according to their multiplicity) below the bottom of the essential spectrum, and $\lambda_n(\Omega)$ is the n-th eigenvalue counting multiplicity, *or*

(b) $\lambda_n(\Omega)$ is the bottom of the essential spectrum, i.e., $\lambda_n(\Omega) = \inf\{\lambda : \lambda \in \sigma_{\text{ess}}(L)\}$ and in that case $\lambda_n = \lambda_{n+1} = \lambda_{n+2} = \ldots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below $\lambda_n(\Omega)$.

We call $\lambda_1 := \lambda_1(\Omega)$ from (3.8) the *principal eigenvalue* of L, i.e.,

$$\lambda_1 = \inf_{\psi \in H^1_{\text{cyl}}(r^3 drdz)} \frac{b(\psi,\psi)}{\|\psi\|^2_{L^2(r^3)}} = \inf_{\substack{\psi \in H^1_{\text{cyl}}(r^3 drdz) \\ \|\psi\|_{L^2(r^3)}=1}} b(\psi,\psi) \tag{3.11}$$

although it may be that λ_1 is the bottom of the essential spectrum.
In the same manner, we write $\lambda_1(\tilde{\Omega})$ for $\tilde{\Omega} \subseteq \Omega$ if we consider L in (3.7) on the domain $D(L) = H^2_{\text{cyl}}(\tilde{\Omega}, r^3 drdz) \cap H^1_0(\tilde{\Omega}, r^3 drdz)$ where we always assume that the boundary of $\tilde{\Omega}$ is at least Lipschitz. Of course, then the infimum in (3.11) is taken over functions $\psi \in H^1_{0,\text{cyl}}(\tilde{\Omega}, r^3 drdz)$ such that $\|\psi\|_{L^2(\tilde{\Omega},r^3)} = 1$.
Now we are ready to give some general properties of first eigenfunctions which are well-known. The proof of the following lemma is inspired by chapter 6.5 in [34].

Lemma 3.5. *Let $\tilde{\Omega} \subseteq \Omega$ be a Lipschitz-domain and suppose that $\lambda_1(\tilde{\Omega})$ is attained by a first eigenfunction $v_1 \in H^1_{0,\text{cyl}}(\tilde{\Omega}, r^3 drdz)$. Then v_1*

(a) is continuously differentiable on $\overline{\tilde{\Omega}}$.

(b) vanishes at no point of $\tilde{\Omega}$.

Moreover, $\lambda_1(\tilde{\Omega})$ is simple.

Proof. (a) Assume $\tilde{\lambda} \in \mathbb{R}$ and $v \in H^1_{0,\text{cyl}}(\tilde{\Omega}, r^3 drdz)$ is a weak solution of

$$-\Delta_{5,\text{cyl}} v + V(r)v = \tilde{\lambda} v + \Gamma(r) p r^{p-1} u^{p-1} v.$$

Denote by $\tilde{\Omega}_5 \subseteq \mathbb{R}^5$ the 5-dimensional representation of $\tilde{\Omega}$ (see Section 1) and consider v as a cylindrical function in $\tilde{\Omega}_5$. We conclude from the equation above and Theorem 8.8 in [39] (,i.e., a local version of Lemma A.10 for $q = 2$) that $v \in H^2_{\text{loc}}(\tilde{\Omega}_5)$. By Sobolev embedding we conclude $v \in L^{10}_{\text{loc}}(\tilde{\Omega}_5)$, i.e. $v \in W^{2,10}_{\text{loc}}(\tilde{\Omega}_5)$. Since $W^{2,10}_{\text{loc}}(\tilde{\Omega}_5)$ is embedded in $C^1_{\text{loc}}(\overline{\tilde{\Omega}_5})$ (see Theorem 4.12 in [1]) Lemma 1.3 then yields the desired claim.
(b) Assume $v_1(r,z) = 0$ at some point $(r,z) \in \tilde{\Omega}$. Our first claim is that $\tilde{\Omega}^+ := \{(r,z) \in \tilde{\Omega} : v_1(r,z) > 0\}$ as well as $\tilde{\Omega}^- := \{(r,z) \in \tilde{\Omega} : v_1(r,z) < 0\}$ are both nonempty. If $\tilde{\Omega}^+ = \emptyset$ or $\tilde{\Omega}^- = \emptyset$ then $v_1 \le 0$ or $v_1 \ge 0$ and by the strong maximum principle we receive strict negativity/positivity of v_1 contradicting our initial assumption. So $\tilde{\Omega}^- \neq \emptyset$ and $\tilde{\Omega}^+ \neq \emptyset$. Define

$$v^+(r,z) := \begin{cases} v_1(r,z) & ,(r,z) \in \tilde{\Omega}^+, \\ 0 & ,(r,z) \in \tilde{\Omega}^- \end{cases}$$

and $v^- := v^+ - v_1$. Hence, $\nabla_{r,z} v^+ = \nabla_{r,z} v_1$ in $\tilde{\Omega}^+$, $\nabla_{r,z} v^+ = 0$ in $\tilde{\Omega}^-$ and similar equations hold true for v^-. Our goal is now to show that

$$b(v^+, v^+) = \lambda_1 \|v^+\|^2_{L^2(\tilde{\Omega},r^3)} \text{ and } b(v^-, v^-) = \lambda_1 \|v^-\|^2_{L^2(\tilde{\Omega},r^3)}. \tag{3.12}$$

Plugging $v_1 = v^+ - v^-$ into the bilinear form b and exploiting the disjoint support of v^+ and v^-, which leads to a vanishing of the mixed terms $b(v^+, v^-)$ and $b(v^-, v^+)$, we get

$$\lambda_1 = b(v_1, v_1) = b(v^+, v^+) + b(v^-, v^-) \geq \lambda_1 \|v^+\|_{L^2(\tilde{\Omega}, r^3)} + \lambda_1 \|v^-\|_{L^2(\tilde{\Omega}, r^3)} = \lambda_1, \tag{3.13}$$

where the definition of λ_1 as the infimum of the Rayleigh-quotient has been used. So (3.13) has to be an equality which proves (3.12). Hence, v^+ and v^- are minimizers of the Rayleigh-quotient $b(\cdot, \cdot)$ and hence both are weak solutions of $-\Delta_{5,\text{cyl}}\varphi + W(r,z)\varphi = \lambda_1\varphi$ where $W(r,z) := V(r) - p\Gamma(r)r^{p-1}u^{p-1}$ with Dirichlet boundary conditions on $\partial\tilde{\Omega} \setminus \{r = 0\}$ and Neumann boundary conditions on $\{r = 0\}$ if part of $\partial\tilde{\Omega}$. The maximum principle now implies $v^+ > 0$ or $v^+ \equiv 0$ in $\tilde{\Omega}$ and likewise $v^- > 0$ or $v^- \equiv 0$ in $\tilde{\Omega}$. Hence, $\tilde{\Omega}^+ = \emptyset$ or $\tilde{\Omega}^- = \emptyset$, contradicting our assumption above which finishes the proof of part (b).

Finally, assume that v and $\tilde{v}$ are two eigenfunctions corresponding to $\lambda_1(\tilde{\Omega})$. Then

$$\hat{v} := \tilde{v} - \frac{\int_{\tilde{\Omega}} v\tilde{v}r^3 d(r,z)}{\int_{\tilde{\Omega}} v^2 r^3 d(r,z)} v$$

is also an eigenfunction corresponding to $\lambda_1(\tilde{\Omega})$, i.e., does not change sign in $\tilde{\Omega}$. But $\hat{v}$ is $L^2_{\text{cyl}}(r^3 drdz)$-orthogonal to v which is contradicting the fact that the product $v\hat{v}$ is sign-preserving and non-zero in $\tilde{\Omega}$. This proves the last part. □

3.4. Spectral analysis

In this section, we turn away from regularity questions but instead focus on an investigation of the point spectrum of the linearization and its corresponding eigenfunctions. The overall goal of this section is to prove that the linearized operator

$$L := -\Delta_{5,\text{cyl}} + V(r) - p\Gamma(r)r^{p-1}u^{p-1}$$

with $D(L) = H^2_{\text{cyl}}(r^3 drdz)$ has exactly one negative eigenvalue, where u is a positive ground state solution of (3.1).

We first prove that L admits at least one negative eigenvalue. For this purpose, the ground state property of u is not needed. The proof is based on a comparison argument, see also Appendix B of [63].

Theorem 3.6. *Let u be a positive solution of* (3.1). *Then the operator L in* (3.7) *has at least one negative eigenvalue.*

Proof. We introduce

$$L_0 := -\Delta_{5,\text{cyl}} + V(r) - \Gamma(r)r^{p-1}u^{p-1}$$

with $D(L_0) = D(L)$. L_0 is a self-adjoint operator. We have $L_0 u = 0$ so that $0 \in \sigma(L_0)$. Since $u > 0$ we even conclude $\lambda_1(L_0) = 0$ and $\sigma(L_0) \subset [0, \infty)$, see Section 3.3. Notice that

$$L = L_0 - (p-1)\Gamma(r)r^{p-1}u^{p-1}.$$

Due to the exponential decay of positive solutions we infer that $(p-1)\Gamma(r)r^{p-1}u^{p-1}$ is a relatively compact perturbation so that $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(L_0) \subset [0,\infty)$ (see Section XIII.4 in [61]). Due to

$$b(u,u) = -\int_\Omega (p-1)\Gamma(r)r^{p-1}u^{p+1}r^3 d(r,z) < 0$$

we have $\lambda_1(L) < 0$ and $\lambda_1(L)$ can not belong to $\sigma_{\text{ess}}(L) \subset [0,\infty)$. Thus $\lambda_1(L)$ has to be an eigenvalue of L and the proof of Theorem 3.6 is done. □

To prove that L has at most one negative eigenvalue we have to restrict to ground states. The Nehari manifold from (2.3) reads in our setting

$$\begin{aligned} M &= \left\{v \in H^1_{\text{cyl}}(r^3 drdz) \setminus \{0\} : \int_\Omega \left(|\nabla_{r,z}v|^2 + V(r)v^2\right) r^3 d(r,z) = \int_\Omega \Gamma(r)r^{p-1}\,|v|^{p+1}\, r^3 d(r,z)\right\} \\ &= \{v \in H^1_{\text{cyl}}(r^3 drdz) \setminus \{0\} : J'(v)v = 0\}. \end{aligned}$$

We now prove a basic lemma concerning minimization of J subject to another constraint. Therefore, we also introduce

$$M_1 := \{w \in H^1_{\text{cyl}}(r^3 drdz) : \int_\Omega \Gamma(r)r^{p-1}\,|w|^{p+1}\, r^3 d(r,z) = 1\}.$$

Lemma 3.7. *For $w \in H^1_{\text{cyl}}(r^3 drdz) \setminus \{0\}$ define*

$$\hat{J}(w) := \frac{\int_\Omega \left(|\nabla_{r,z}w|^2 + V(r)w^2\right) r^3 d(r,z)}{\left(\int_\Omega \Gamma(r)r^{p-1}\,|w|^{p+1}\, r^3 d(r,z)\right)^{\frac{2}{p+1}}} =: \frac{J_1(w)}{J_2(w)}.$$

Then there is a one-to-one relation between minimizers $u \in M$ of J and minimizers $\tilde{w} \in M_1$ of J_1, namely

$$\tilde{w} = \frac{u}{\left(\int_\Omega \Gamma(r)r^{p-1}\,|u|^{p+1}\, r^3 d(r,z)\right)^{\frac{1}{p+1}}}. \tag{3.14}$$

Proof. We minimize

$$J(v) = \int_\Omega \left(\frac{1}{2}\left(|\nabla_{r,z}v|^2 + V(r)v^2\right) - \frac{1}{p+1}\Gamma(r)r^{p-1}\,|v|^{p+1}\right) r^3 d(r,z)$$

for v in M. In this case we have $J(v) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \left(|\nabla_{r,z}v|^2 + V(r)v^2\right) r^3 d(r,z)$. Since u is a ground state solution, we have $J(u) = \inf_{v\in M} J(v) =: c$. We shorten $\kappa := \frac{1}{2} - \frac{1}{p+1}$ and $d := \inf_{w\in M_1} J_1(w)$. Let $(u_k)_{k\in\mathbb{N}} \subset M$ be a minimizing sequence for J and set $w_k := \frac{u_k}{\left(\int_\Omega \Gamma(r)r^{p-1}|u_k|^{p+1}r^3 d(r,z)\right)^{\frac{1}{p+1}}} \in M_1$. Then:

$$\begin{aligned} d \le J_1(w_k) &= \frac{\int_\Omega \left(|\nabla_{r,z}u_k|^2 + V(r)u_k^2\right) r^3 d(r,z)}{\left(\int_\Omega \Gamma(r)r^{p-1}\,|u_k|^{p+1}\, r^3 d(r,z)\right)^{\frac{2}{p+1}}} = \left(\int_\Omega \left(|\nabla_{r,z}u_k|^2 + V(r)u_k^2\right) r^3 d(r,z)\right)^{\frac{p-1}{p+1}} \\ &= \left(\frac{J(u_k)}{\kappa}\right)^{\frac{p-1}{p+1}} \to \left(\frac{c}{\kappa}\right)^{\frac{p-1}{p+1}} \text{ as } k \to \infty. \end{aligned}$$

On the other hand, let $(w_k)_{k\in\mathbb{N}} \subset M_1$ be a minimizing sequence for J_1 and choose $t_k \in \mathbb{R}$ such that $u_k := t_k w_k \in M$, i.e., $t_k^2 \int_\Omega \left(|\nabla_{r,z} w_k|^2 + V(r) w_k^2\right) r^3 d(r,z) = t_k^{p+1} \int_\Omega \Gamma(r) r^{p-1} |w_k|^{p+1} r^3 d(r,z) = t_k^{p+1}$, so $t_k = \left(\int_\Omega \left(|\nabla_{r,z} w_k|^2 + V(r) w_k^2\right) r^3 d(r,z)\right)^{\frac{1}{p-1}}$. Hence,

$$c \le J(u_k) = \kappa \left(\int_\Omega \left(|\nabla_{r,z} w_k|^2 + V(r) w_k^2\right) r^3 d(r,z) \right)^{\frac{p+1}{p-1}} \to \kappa d^{\frac{p+1}{p-1}} \text{ as } k \to \infty.$$

These two inequalities result in $d = \left(\frac{c}{\kappa}\right)^{\frac{p-1}{p+1}}$. In particular, u is a minimizer of J on M if and only if $\tilde{w}$ given by (3.14) is a minimizer of J_1 on M_1. □

With the notation of Lemma 3.7 and the bilinear form b from (3.10) we prove another auxiliary statement.

Lemma 3.8. *The following statements hold true:*
(a) We have

$$\left(J_1''(\tilde{w}) - \hat{J}(\tilde{w}) J_2''(\tilde{w})\right)[\varphi,\psi] = 2b(\varphi,\psi) - \langle A\varphi, \psi\rangle_{L^2(r^3)} \tag{3.15}$$

for all $\varphi,\psi \in H^1_{\mathrm{cyl}}(r^3 drdz)$, where $A\colon L^2_{\mathrm{cyl}}(r^3 drdz) \to L^2_{\mathrm{cyl}}(r^3 drdz)$ is given by

$$A\varphi = c\left(\int_\Omega \Gamma(r) r^{p-1} |\tilde{w}|^{p-1} \tilde{w}\varphi r^3 d(r,z)\right) \Gamma(r) r^{p-1} |\tilde{w}|^{p-1} \tilde{w} \tag{3.16}$$

and the constant c is defined by $c := \hat{J}(\tilde{w}) 2(1-p) \left(\int_\Omega \Gamma(r) r^{p-1} |\tilde{w}|^{p+1} r^3 d(r,z)\right)^{-\frac{2p}{p+1}}$.
(b) We have

$$\left(J_1''(\tilde{w}) - \hat{J}(\tilde{w}) J_2''(\tilde{w})\right)[\varphi,\varphi] \ge 0 \textit{ for all } \varphi \in H^1_{\mathrm{cyl}}(r^3 drdz). \tag{3.17}$$

Proof. (a) Let $w,\varphi,\psi \in H^1_{\mathrm{cyl}}(r^3 drdz)$. For J_2 we calculate

$$J_2'(w)[\varphi] = 2\left(\int_\Omega \Gamma(r) r^{p-1} |w|^{p+1} r^3 d(r,z)\right)^{\frac{1-p}{p+1}} \int_\Omega \Gamma(r) r^{p-1} |w|^{p-1} w\varphi r^3 d(r,z) \text{ and}$$

$$J_2''(w)[\varphi,\psi] = 2(1-p)\left(\int_\Omega \Gamma(r) r^{p-1} |w|^{p+1} r^3 d(r,z)\right)^{-\frac{2p}{p+1}} \left(\int_\Omega \Gamma(r) r^{p-1} |w|^{p-1} w\varphi r^3 d(r,z)\right) \cdot$$
$$\left(\int_\Omega \Gamma(r) r^{p-1} |w|^{p-1} w\psi r^3 d(r,z)\right) + 2p\left(\int_\Omega \Gamma(r) r^{p-1} |w|^{p+1} r^3 d(r,z)\right)^{\frac{1-p}{p+1}} \int_\Omega \Gamma(r) r^{p-1} |w|^{p-1} \psi\varphi r^3 d(r,z).$$

This results in

$$\left(J_1''(\tilde{w}) - \hat{J}(\tilde{w}) J_2''(\tilde{w})\right)[\varphi,\psi] = 2\int_\Omega \left(\nabla_{r,z}\varphi \cdot \nabla_{r,z}\psi + V(r)\varphi\psi\right) r^3 d(r,z)$$
$$- \hat{J}(\tilde{w}) 2p \left(\int_\Omega \Gamma(r) r^{p-1} |\tilde{w}|^{p+1} r^3 d(r,z)\right)^{\frac{1-p}{p+1}} \int_\Omega \Gamma(r) r^{p-1} |\tilde{w}|^{p-1} \varphi\psi r^3 d(r,z) + \hat{J}(\tilde{w}) 2(p-1) \cdot$$

$$\left(\int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p+1}\,r^3 d(r,z)\right)^{-\frac{2p}{p+1}} \int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p-1}\,\tilde{w}\varphi r^3 d(r,z) \int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p-1}\,\tilde{w}\psi r^3 d(r,z). \quad (3.18)$$

So with $\tilde{w}$ given by (3.14) and $\hat{J}(u) = \hat{J}(\tilde{w})$ we further get

$$\begin{aligned} &\hat{J}(\tilde{w})\left(\int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p+1}\,r^3 d(r,z)\right)^{\frac{1-p}{1+p}} \int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p-1}\,\varphi\psi r^3 d(r,z) \\ &= \hat{J}(u)\frac{\int_\Omega \Gamma(r)r^{p-1}\,|u|^{p-1}\,\varphi\psi r^3 d(r,z)}{\left(\int_\Omega \Gamma(r)r^{p-1}\,|u|^{p+1}\,r^3 d(r,z)\right)^{\frac{p-1}{p+1}}} = \frac{\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r)u^2\right) r^3 d(r,z)}{\int_\Omega \Gamma(r)r^{p-1}\,|u|^{p+1}\,r^3 d(r,z)} \int_\Omega \Gamma(r)r^{p-1}\,|u|^{p-1}\,\varphi\psi r^3 d(r,z) \\ &= \int_\Omega \Gamma(r)r^{p-1}\,|u|^{p-1}\,\varphi\psi r^3 d(r,z), \end{aligned} \quad (3.19)$$

where the last equality is due to the fact that u solves $-\Delta_{5,\mathrm{cyl}}u + V(r)u = \Gamma(r)r^{p-1}\,|u|^{p+1}$. Plugging (3.19) into (3.18) we end up with (3.15).

(b) Recall $\hat{J}(w) = \frac{J_1(w)}{J_2(w)}$ for $w \in H^1_{\mathrm{cyl}}(r^3 drdz)$. Hence the chain rule gives

$$\begin{aligned} \hat{J}'(w)[\varphi] &= \frac{J_1'(w)[\varphi]J_2(w) - J_1(w)J_2'(w)[\varphi]}{J_2^2(w)} \text{ for } \varphi \in H^1_{\mathrm{cyl}}(r^3 drdz) \text{ and} \\ \hat{J}''(w)[\varphi,\psi] &= \frac{J_1''(w)[\varphi,\psi]J_2(w) - J_1(w)J_2''(w)[\varphi,\psi] + J_1'(w)[\varphi]J_2'(w)[\psi] - J_1'(w)[\psi]J_2'(w)[\varphi]}{J_2^2(w)} \\ &\quad - 2\frac{\hat{J}'(w)[\varphi]}{J_2(w)}J_2'(w)[\psi] \text{ for } (\varphi,\psi) \in H^1_{\mathrm{cyl}}(r^3 drdz) \times H^1_{\mathrm{cyl}}(r^3 drdz). \end{aligned}$$

At $\tilde{w}$, using $\hat{J}'(\tilde{w}) = 0$, i.e., $J_1'(\tilde{w})[\varphi]J_2(\tilde{w}) = J_1(\tilde{w})J_2'(\tilde{w})[\varphi]$ for all $\varphi \in H^1_{\mathrm{cyl}}(r^3 drdz)$ and setting $\psi = \varphi$, the equation above simplifies to

$$\hat{J}''(\tilde{w})[\varphi,\varphi] = \frac{J_1''(\tilde{w})[\varphi,\varphi]J_2(\tilde{w}) - J_1(\tilde{w})J_2''(\tilde{w})[\varphi,\varphi]}{J_2^2(\tilde{w})} = \frac{J_1''(\tilde{w})[\varphi,\varphi] - \hat{J}(\tilde{w})J_2''(\tilde{w})[\varphi,\varphi]}{J_2(\tilde{w})}. \quad (3.20)$$

Since $\tilde{w}$ is a minimizer of $\hat{J}$ we have $\hat{J}''(\tilde{w})[\varphi,\varphi] \geq 0$ for all $\varphi \in H^1_{\mathrm{cyl}}(r^3 drdz)$ and so (3.17) follows from (3.20) taking into account $J_2(\tilde{w}) > 0$. □

We are ready to prove the counterpart of Theorem 3.6.

Theorem 3.9. *Let u be a positive ground state of* (3.1). *Then the operator L in* (3.7) *has at most one negative eigenvalue.*

Proof. We have a look at the eigenvalues of A and its corresponding eigenfunctions: Let $\mu \in \mathbb{C}$ and $\varphi \in L^2_{\mathrm{cyl}}(r^3 drdz)$ be given with $A\varphi = \mu\varphi$, i.e.,

$$c\left(\int_\Omega \Gamma(r)r^{p-1}\,|\tilde{w}|^{p-1}\,\tilde{w}\varphi r^3 d(r,z)\right)\Gamma(r)r^{p-1}\,|\tilde{w}|^{p-1}\,\tilde{w} = \mu\varphi.$$

Then either $\mu = 0$ with corresponding eigenspace $\{\varphi \in L^2_{\text{cyl}}(r^3drdz) : \varphi \perp_{L^2(r^3)} \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}\}$ or $\mu = c\left\|\Gamma(r)r^{p-1}\tilde{w}^p\right\|^2_{L^2(r^3)}$ with eigenspace $\{\varphi \in L^2_{\text{cyl}}(r^3drdz) : \varphi = \hat{c}\Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$ with $\hat{c} \in \mathbb{R}\}$. We reformulate (3.15) as

$$2b(\varphi,\psi) = J^\star(\varphi,\psi) + \langle A\varphi,\psi\rangle_{L^2(r^3)}, \tag{3.21}$$

where $J^\star(\varphi,\psi) := \left(J_1''(\tilde{w}) - \hat{J}(\tilde{w})J_2''(\tilde{w})\right)[\varphi,\psi]$. We show that (3.21) allows at most one negative eigenvalue of L.

Assume by contradiction that L has two negative eigenvalues μ_1,μ_2, i.e., $L\varphi_i = \mu_i\varphi_i$ for $i = 1,2$ with $\varphi_1 \perp_{L^2(r^3)} \varphi_2$. Hence, $\langle L\varphi_i,\varphi_i\rangle_{L^2(r^3)} < 0$ for $i = 1,2$ and $\langle L\varphi_1,\varphi_2\rangle_{L^2(r^3)} = \langle L\varphi_2,\varphi_1\rangle_{L^2(r^3)} = 0$. Using these two statements, (3.21) and $J^\star(\varphi,\varphi) \geq 0$ for all $\varphi \in H^1_{\text{cyl}}(r^3drdz)$ we get

$$\begin{aligned}0 &> \alpha^2\langle L\varphi_1,\varphi_1\rangle_{L^2(r^3)} + \beta^2\langle L\varphi_2,\varphi_2\rangle_{L^2(r^3)} = \langle L(\alpha\varphi_1+\beta\varphi_2),\alpha\varphi_1+\beta\varphi_2\rangle_{L^2(r^3)}\\ &\geq \frac{1}{2}\langle A(\alpha\varphi_1+\beta\varphi_2),\alpha\varphi_1+\beta\varphi_2\rangle_{L^2(r^3)} \text{ for all } (\alpha,\beta) \in \mathbb{R}^2\setminus\{(0,0)\}.\end{aligned} \tag{3.22}$$

If $\varphi_1,\varphi_2 \perp_{L^2(r^3)} \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$ so is $\alpha\varphi_1+\beta\varphi_2 \perp_{L^2(r^3)} \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$. Thus

$$\langle A(\alpha\varphi_1+\beta\varphi_2),\alpha\varphi_1+\beta\varphi_2\rangle_{L^2(r^3)} = 0$$

holds true by definition of A. But this is a contradiction to (3.22).

If φ_1 is not $L^2_{\text{cyl}}(r^3drdz)$-perpendicular to $\Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$ we choose

$$\beta := 1 \text{ and } \alpha := -\frac{\int_\Omega \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}\varphi_2 r^3 d(r,z)}{\int_\Omega \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}\varphi_1 r^3 d(r,z)}$$

to obtain $\alpha\varphi_1+\beta\varphi_2 \perp_{L^2(r^3)} \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$. As before we receive $\langle A(\alpha\varphi_1+\beta\varphi_2),\alpha\varphi_1+\beta\varphi_2\rangle_{L^2(r^3)} = 0$ contradicting (3.22). If φ_2 is not $L^2_{\text{cyl}}(r^3drdz)$-perpendicular to $\Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}$ we choose

$$\alpha := 1 \text{ and } \beta := -\frac{\int_\Omega \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}\varphi_1 r^3 d(r,z)}{\int_\Omega \Gamma(r)r^{p-1}|\tilde{w}|^{p-1}\tilde{w}\varphi_2 r^3 d(r,z)}$$

which leads to the same contradiction as before and finally finishes the proof. □

In summary, we have shown that for every positive ground state u the operator L possesses exactly one simple negative eigenvalue. To close this section we show a symmetry property of the first eigenfunction.

Lemma 3.10. *The eigenfunction v_1 associated with the only and simple negative eigenvalue λ_1 of L is symmetric about $\{z=0\}$, i.e., an element of*

$$H_{\text{symm}} := \left\{v \in H^1_{\text{cyl}}(r^3drdz) : v \textit{ is symmetric about } \{z=0\}\right\}.$$

Proof. We know that $v_1 \in H^1_{\text{cyl}}(r^3drdz)$ satisfies

$$-\Delta_{5,\text{cyl}}v_1 + V(r)v_1 - p\Gamma(r)r^{p-1}|u|^{p-1}v_1 = \lambda_1 v_1 \text{ in } \Omega.$$

By Lemma 3.5 we may assume $v_1 > 0$ in Ω. Define $v_2(r,z) := v_1(r,-z)$ on Ω. Then due to $u(r,z) = u(r,-z)$ on Ω we obtain that v_2 is also an eigenfunction of L for the eigenvalue λ_1. Lemma 3.5 and $\|v_1\|_{L^2(r^3)} = 1 = \|v_2\|_{L^2(r^3)}$ imply $v_1 \equiv v_2$ which finishes the proof. □

3.5. Symmetry and monotonicity of positive solutions of (3.1)

In this section we prove that for every non-negative weak solution u of (3.1) there exists a $\theta \in \mathbb{R}$ such that u is symmetric about $\{z = \theta\}$. In addition, we prove that these solutions are (strictly) decaying in z-direction. We prove monotonicity and symmetry with the help of the moving plane method, see the paper of Li [44] or the classical paper of Gidas, Ni and Nirenberg [36] for similar results.

We first recall a maximum principle which we state now:

Theorem 3.11. *(Maximum principle)*
Let $G \subset \mathbb{R}^n$ be a domain, $u \in C^2(\bar{G}) \cap H_0^1(G)$, $u \not\equiv 0$ and $c \in L^\infty(G)$ such that $-\Delta u + cu \geq 0$ in G and $u \geq 0$ in G. Then $u > 0$ in G.

Moreover: Is $x_0 \in \partial G$ with $u(x_0) = 0$ and G satisfies an inner sphere condition at x_0 (that is: there exists an open ball $B \subset G$ such that $\bar{B} \cap \bar{G} = \{x_0\}$), then $\frac{\partial u}{\partial \nu}(x_0) < 0$, where ν denotes the outer unit normal vector to G.

Proof. We have

$$-\Delta u + c^+ u \geq -\Delta u + cu \geq 0 \text{ in } G$$

so that the statement follows from the strong maximum principle and the Hopf boundary lemma applied to $-\Delta u + c^+ u \geq 0$ in G, see Section 2.3 in [35]. □

Here is our main result for this section.

Theorem 3.12. *Every non-negative, non-trivial weak solution u of*

$$-\Delta_{5,\mathrm{cyl}} u(r,z) + V(r)u(r,z) = \Gamma(r) r^{p-1} u(r,z)^p \text{ in } \Omega \tag{3.23}$$

is strictly positive on $[0,\infty) \times \mathbb{R}$, i.e., on $\mathbb{R}^5$ and symmetric about $\{z = \theta\}$ for some $\theta \in \mathbb{R}$. Moreover, $\frac{\partial u}{\partial z}(r,z) < 0$ for all $z > \theta$ and arbitrary $r \geq 0$.

Proof. Our proof is nearly the same as Theorem 1.1 in [44], but for completeness we repeat it here adapted to our case.

First, we prove $u > 0$ in $\mathbb{R}^5$. Since $W := V(r) - \Gamma(r) r^{p-1} u^{p-1} \in L^\infty(\mathbb{R}^5), 0 \not\equiv u \geq 0$ in $\mathbb{R}^5$ and $-\Delta u + Wu = 0$ we conclude $u > 0$ in $\mathbb{R}^5$ by Theorem 3.11.

The next step is to show symmetry and monotonicity concering the z-direction. Therefore, consider the domain $\Sigma(\eta) := \{x \in \mathbb{R}^5 : x_5 < \eta\}$ for $\eta < 0$. First, assume $\eta \leq -K$ where $K > 0$ is chosen such that

$$\tau_K := \max\{p\Gamma(r) r^{p-1} u(x)^{p-1} : x \in \Sigma(K)\} < \inf V.$$

Notice, that the exponential decay of u (see Lemma 3.3 and the conclusion thereafter) guarantees the existence of a constant K with the desired properties. Define $u_\eta(x) := u(x_1, x_2, x_3, x_4, 2\eta - x_5)$ and set

$$w_\eta(x) := u(x_1, x_2, x_3, x_4, 2\eta - x_5) - u(x_1, \dots, x_5),$$

so $w_\eta = u_\eta - u$ and $w_\eta(x_1, \dots, x_4, \eta) = 0$ for all $(x_1, \dots, x_4) \in \mathbb{R}^4$. Both u and u_η are solutions of (3.23) and therefore, by using the mean value theorem the difference w_η satisfies

$$-\Delta w_\eta + V(r) w_\eta = \Gamma(r) r^{p-1} \left(u_\eta^p - u^p\right) = g'(r,\xi)(u_\eta - u)(x) \tag{3.24}$$

where $g(r,t) := \Gamma(r)r^{p-1}t^p, (t > 0)$ and ξ is between $u_\eta(x)$ and $u(x)$.
Since we want to prove $w_\eta \geq 0$ in $\Sigma(\eta)$ we assume that there exists a point $x \in \Sigma(\eta)$ such that $w_\eta(x) < 0$ and want to derive a contradiction. From $w_\eta(y) \geq -u(y) \to 0$ for $|y| \to \infty$ we know that w_η has to take its negative minimum at some point $\bar{x} \in \Sigma(\eta)$. Clearly, $\nabla w_\eta(\bar{x}) = 0$ and

$$\Delta_5 w_\eta(\bar{x}) \geq 0. \tag{3.25}$$

At $\bar{x} \in \Sigma(\eta)$ we deduce by using (3.24) and $\bar{r} := \sqrt{\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 + \bar{x}_4^2}$

$$0 > \Gamma(\bar{r})\bar{r}^{p-1}\left(u_\eta^p - u^p\right)(\bar{x}) = g'(\bar{r},\xi)(u_\eta - u)(\bar{x}),$$

with $0 < u_\eta(\bar{x}) \leq \xi \leq u(\bar{x})$. Notice $g'(\bar{r},\xi) \leq g'(\bar{r},u(\bar{x})) \leq \tau_K$, i.e., $g'(\bar{r},\xi) - V(\bar{r}) < 0$. At the minimum point $\bar{x}$, equation (3.24) reads

$$\Delta w_\eta(\bar{x}) + \left(g'(r,\xi) - V(r)\right) w_\eta(\bar{x}) = 0. \tag{3.26}$$

But this is a contradiction since $\Delta w_\eta(\bar{x}) \geq 0$ by (3.25) and $g'(r,\xi) - V(r) < 0$. Hence, $w_\eta \geq 0$ in $\Sigma(\eta)$ for all $\eta \leq -K$.
By continuity we conclude $w_\eta \geq 0$ in $\Sigma(\eta)$ for η in a maximal interval $(-\infty, \bar{\eta}]$. This maximal $\bar{\eta}$ has to be finite since otherwise $0 \leq w_\eta(x) = u(x_1,\ldots,x_4, 2\eta - x_5) - u(x)$ in $\Sigma(\eta)$ for all $\eta \in \mathbb{R}$ would lead to $0 \leq -u(x)$ in $\Sigma(\eta)$ by sending $\eta \to \infty$ and keeping x fixed contradicting the strict positivity of the ground state.
Now we are going to prove $w_{\bar{\eta}} \equiv 0$ in $\Sigma(\bar{\eta})$. Suppose $w_{\bar{\eta}} \not\equiv 0$ in $\Sigma(\bar{\eta})$. Having in mind

$$-\Delta w_{\bar{\eta}} + \left(V(r) - g'(r,\xi)\right) w_{\bar{\eta}} = 0, w_{\bar{\eta}} \geq 0, w_{\bar{\eta}} \in C^2(\Sigma(\bar{\eta}))$$

and $V(r) - g'(r,\cdot) \in L^\infty(\mathbb{R}^5)$ (due to the exponential decay of $u_{\bar{\eta}}$ and u at infinity) the maximum principle directly yields $w_{\bar{\eta}} > 0$ in $\Sigma(\bar{\eta})$. By the maximality of $\bar{\eta}$ we find a sequence $(\eta_k)_{k\in\mathbb{N}}$ with $\eta_k \searrow \bar{\eta}$ as $k \to \infty$ and corresponding points $y_k \in \Sigma(\eta_k)$ such that $w_{\eta_k}(y_k) < 0$. By decay at infinity we can again assume that y_k is chosen such that

$$w_{\eta_k}(y_k) = \min_{x\in\Sigma(\eta_k)} w_{\eta_k}(x),\ \nabla w_{\eta_k}(y_k) = 0. \tag{3.27}$$

If $|y_k| \to \infty$ as $k \to \infty$ we conclude $u(y_k) \to 0$ as $k \to \infty$. As before we would receive $g'(r,\xi_k) < V(r)$ as $k \to \infty$ where $0 < u_{\eta_k}(x) \leq \xi_k \leq u(x)$ and we end up with the same contradiction as above in (3.26). For this reason, the sequence $(y_k)_{k\in\mathbb{N}}$ must be bounded. Thus we find a subsequence of $(y_k)_{k\in\mathbb{N}}$ (again denoted by $(y_k)_{k\in\mathbb{N}}$) and a $\bar{x} \in \overline{\Sigma(\bar{\eta})}$ such that $y_k \to \bar{x}$ as $k \to \infty$. Since the function $w_{\bar{\eta}}$ is strictly positive in $\Sigma(\bar{\eta})$ we conclude that $\bar{x} \in \partial\overline{\Sigma(\bar{\eta})}$. Moreover, since $w_{\bar{\eta}}(x_1,\ldots,x_4,\bar{\eta}) = 0$ the Hopf boundary lemma (the addendum of the maximum principle above) is applicable and implies $\frac{\partial w_{\bar{\eta}}}{\partial \nu}(\bar{x}) < 0$, where ν is the outer unit normal to $\partial\Sigma(\bar{\eta})$, so $\frac{\partial w_{\bar{\eta}}}{\partial x_5}(\bar{x}) < 0$. But we derive $\nabla w_{\bar{\eta}}(\bar{x}) = 0$ from sending $k \to \infty$ in (3.27), a contradiction. Hence, $w_{\bar{\eta}} \equiv 0$ in $\Sigma(\bar{\eta})$, i.e., u is symmetric about $\{x_5 = \bar{\eta}\}$.
The value $\bar{\eta}$ is unique. Indeed, assume $w_\eta \equiv 0$ for some $\eta < \bar{\eta}$. Then u is symmetric about $\{x_5 = \eta\}$ and $\{x_5 = \bar{\eta}\}$, so u has to be periodic in x_5-direction. But this is a contradiction to the exponential decay of u at infinity and so we get $w_\eta \not\equiv 0$ for all $\eta < \bar{\eta}$.
Again the maximum principle implies $w_\eta > 0$ in $\Sigma(\eta)$ for $\eta < \bar{\eta}$ and Hopf boundary lemma yields $\frac{\partial w_\eta}{\partial x_5}(x) < 0$ for y lying on the hyperplane $\{x \in \mathbb{R}^5 : x_5 = \eta\}$, so $\frac{\partial u}{\partial x_5}(x) > 0$ for $x \in \Sigma(\bar{\eta})$. This proves our desired results concerning the x_5-direction. □

4. A Liouville theorem and a-priori bounds

In this chapter we again assume $V, \Gamma \in W^{1,\infty}([0,\infty))$, $\inf V, \inf \Gamma > 0$. The goal of this chapter is to provide a Liouville theorem which then leads to a-priori bounds for positive solutions $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ of

$$\left(-\partial_r^2 - \frac{3}{r}\partial_r - \partial_z^2 + V(r)\right) u(r,z) = \Gamma(r) r^{p-1} u^p(r,z) \text{ in } \Omega, \tag{4.1}$$

and positive solutions $u \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ where $\Omega_k := \{(r,z) \in \Omega : r^2 + z^2 < k^2\}$ and $k > 0$ of

$$\begin{aligned} \left(-\partial_r^2 - \frac{3}{r}\partial_r - \partial_z^2 + V(r)\right) u(r,z) &= \Gamma(r) r^{p-1} u^p(r,z) \text{ in } \Omega_k, \\ u &= 0 \text{ on } \partial\Omega_k \setminus (\{0\} \times [-k,k]), \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \{0\} \times [-k,k]. \end{aligned} \tag{4.2}$$

In (4.2) we assume that V, Γ satisfy $V, \Gamma \in W^{1,\infty}(\Omega_k)$, $\inf V, \inf \Gamma > 0$ and V, Γ are not depending on z. The mixed Dirichlet-Neumann boundary conditions in (4.2) are first written down formally. The right boundary conditions for the differential equation in (4.2) are included in the space $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ and explained in Definition 1.5 with $\tilde{\Omega} = \Omega_k$. In other words, (4.2) is valid for $u \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ if and only if for all $v \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$

$$\int_{\Omega_k} (\nabla_{r,z} u \cdot \nabla_{r,z} v + V(r) uv)\, r^3 d(r,z) = \int_{\Omega_k} \Gamma(r) r^{p-1} u^p v r^3 d(r,z)$$

holds true. Due to the cylindrical symmetry of functions in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ the Neumann-boundary conditions at $\{r = 0\} \times [-k,k]$ are incorporated in a natural way. In Lemma 4.14 we will see that weak solutions $u \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ of the differential equation in (4.2) are classically differentiable in r and z up to the boundary so that the boundary conditions in (4.2) can be understood in a pointwise sense.

We prove the following Liouville theorem:

Theorem 4.1. *Let $\bar{p} \in (1,2)$. Then there is no non-trivial, positive solution $u \in H^1_{\mathrm{loc}}(r^3 drdz)$ of*

$$-\partial_r^2 u - \frac{3}{r}\partial_r u - \partial_z^2 u = \Gamma(0) r^{\bar{p}-1} u^{\bar{p}} \text{ in } (0,\infty) \times \mathbb{R}. \tag{4.3}$$

Note that the test functions in (4.3) *are not allowed to have support on* $\{0\} \times \mathbb{R}$.

Remarks 4.2. *(a) It is an open question whether Theorem 4.1 holds true for a larger range of exponents than only for $\bar{p} \in (1,2)$. The techniques we use only work for $\bar{p} \in (1,2)$, but a natural candidate*

for the validity of a Liouville Theorem would be $\bar{p} \in (1, \frac{n+2}{n-2})$*, cf. Theorem 8.2. in [59]. For the system* $-\Delta U = |U|^{p-1}U$ *we have* $n = 3$*, i.e.,* $\bar{p} \in (1,5)$ *could be expected, see also the discussion in the introduction.*
(b) It is clear that the statement remains valid if $\Gamma(0)$ *in* (4.3) *is replaced by an arbitrary constant* $c > 0$*. Nevertheless, in the proof of Theorem 4.1 we always write* $\Gamma(0)$ *since this is the connection to* (4.1) *and* (4.2).

The resulting a-priori bounds for positive solutions read as follows.

Theorem 4.3. *Let* $[p_\star, p^\star] \subset (1,2)$*. Then there is a constant* $C = C(p_\star, p^\star) > 0$ *such that*

$$\|ru\|_{L^\infty([0,\infty)\times\mathbb{R})} \leq C \tag{4.4}$$

for every positive weak solution $u \in H_{symm}$ *of* (4.1) *and every* $p \in [p_\star, p^\star]$.

The proof of Theorem 4.3 is done by contradiction. Hence, assume that there are sequences $(p_j)_{j\in\mathbb{N}}$ and positive solutions $(u_j)_{j\in\mathbb{N}}$ in H_{symm} of (4.1) with exponents $p_j \in [p_\star, p^\star] \subset (1,2)$ such that $p_j \to \bar{p} \in (1,2)$ as $j \to \infty$ and $\|ru_j\|_{L^\infty(\Omega)} \to \infty$ as $j \to \infty$. Thus, there is a sequence $(r_j, z_j)_{j\in\mathbb{N}}$ in Ω such that

$$M_j := r_j u_j\left(r_j, z_j\right) := \sup_{(r,z)\in\Omega} ru_j(r,z) \to \infty. \tag{4.5}$$

Recall that by Theorem 3.12 we know that $z_j = 0$ for all $j \in \mathbb{N}$. In order to derive a contradiction we distinguish three cases (convergence is understood up to subsequences):

1) $r_j M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$

2) $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$

3) $r_j M_j^{\frac{p_j-1}{2}} \to c \in (0,\infty)$ as $j \to \infty$

Those three cases are investigated in the following Sections 4.2-4.4. One important ingredient on the way to Theorem 4.3 is that we can scale both, the scalar equation (4.1) as well as the $\mathbb{R}^3$-valued equation

$$-\Delta U + V(x)U = \Gamma(x)|U|^{p-1}U \text{ in } \mathbb{R}^3. \tag{4.6}$$

This chapter is structured as follows: We present both scaling procedures in Section 4.1 and highlight an important connection between these two variants which will later allow us to use both procedures next to each other. In Section 4.2 we lead the first case to a contradiction by investigating the scaling procedure for (4.6). Once case 1) is ruled out we can use the scaling procedure for (4.1) to deduce a non-zero, non-negative solution of a limit equation. The strategy is to prove that this limit equation only admits the trivial solution among the set of non-negative solutions. This is done in Section 4.3 for case 2). The most challenging case is the third one. In this case the limit equation is (4.3) and the proofs of Theorem 4.1 and Theorem 4.3 are treated in Section 4.4. Afterwards, in Section 4.5 we use the a-priori estimates in Theorem 4.3 to deduce a uniform $H^1_{\text{cyl}}(r^3 drdz)$-bound for ground states. In Section 4.6 we investigate (4.2) and obtain similar results as for (4.1). Due to the bounded domain, we do not have to restrict to ground states and therefore the following uniqueness result can be established.

Theorem 4.4. *Let $k > 0$. Then the following statements hold true:*
(a) There is $p_0 = p_0(k) > 1$ such that (4.2) *has only one positive solution for $p \in (1, p_0)$.*
(b) Let $p \in (1, 2)$ be arbitrary. Then the number of non-degenerate positive solutions of (4.2) *in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ is less or equal to one.*

Finally, in Section 4.7 we return to (4.1) and establish the finiteness of the number of ground states in H_{symm} under the assumption of non-degeneracy.

4.1. Two ways of scaling and their commonalities

We continue to use the notation in (4.5). The first scaling which is done on the level of the scalar equation (4.1). We set

$$v_j \colon [-r_j M_j^{\frac{p_j-1}{2}}, \infty) \times \mathbb{R} \to \mathbb{R};\ v_j(r,z) := \frac{r_j + rM_j^{\frac{1-p_j}{2}}}{M_j} u_j\left(r_j + rM_j^{\frac{1-p_j}{2}}, zM_j^{\frac{1-p_j}{2}}\right). \tag{4.7}$$

Hence, with $\Omega_j := [-r_j M_j^{\frac{p_j-1}{2}}, \infty) \times \mathbb{R}$ we have $v_j(0,0) = 1$ and $\|v_j\|_{L^\infty(\Omega_j)} = 1$ for all $j \in \mathbb{N}$. Furthermore, we introduce

$$\tilde{r} := r_j + rM_j^{\frac{1-p_j}{2}},\ \tilde{z} := zM_j^{\frac{1-p_j}{2}},$$

i.e., we have $u_j(\tilde{r}, \tilde{z}) = \frac{M_j}{\tilde{r}} v_j(M_j^{(p_j-1)/2}(\tilde{r} - r_j), M_j^{(p_j-1)/2}\tilde{z})$ in $[0,\infty) \times \mathbb{R}$. Formally, we compute

$$\begin{aligned}
\partial_{\tilde{r}} u_j &= -\frac{M_j}{\tilde{r}^2} v_j + \frac{M_j}{\tilde{r}} M_j^{(p_j-1)/2} \partial_r v_j, \qquad \partial^2_{\tilde{z}} u_j = \frac{M_j^{p_j}}{\tilde{r}} \partial^2_z v_j,\\
\partial^2_{\tilde{r}} u_j &= 2\frac{M_j}{\tilde{r}^3} v_j - 2\frac{M_j}{\tilde{r}^2} M_j^{(p_j-1)/2} \partial_r v_j + \frac{M_j^{p_j}}{\tilde{r}} \partial^2_r v_j.
\end{aligned}$$

In combination with (4.1) this yields

$$V(\tilde{r})u_j - \Gamma(\tilde{r})\tilde{r}^{p_j-1} u_j^{p_j} = \partial^2_{\tilde{r}} u_j + \partial^2_{\tilde{z}} u_j + \frac{3}{\tilde{r}} \partial_{\tilde{r}} u_j = \frac{M_j}{\tilde{r}^2} M_j^{(p_j-1)/2} \partial_r v_j - \frac{M_j}{\tilde{r}^3} v_j + \frac{M_j^{p_j}}{\tilde{r}} \left(\partial^2_r v_j + \partial^2_z v_j\right).$$

Finally, multiplication by $\tilde{r} M_j^{-p_j}$ implies

$$\begin{aligned}
&-\partial^2_r v_j(r,z) - \frac{1}{r_j M_j^{(p_j-1)/2} + r} \partial_r v_j(r,z) - \partial^2_z v_j(r,z)\\
&= \Gamma(r_j + rM_j^{\frac{1-p_j}{2}}) v_j^{p_j}(r,z) - M_j^{1-p_j} V(r_j + rM_j^{\frac{1-p_j}{2}}) v_j(r,z) - \frac{1}{\left(r_j M_j^{(p_j-1)/2} + r\right)^2} v_j(r,z).
\end{aligned} \tag{4.8}$$

The second scaling is done for (4.6). Notice that we have

$$\sup_{(r,z)\in\Omega} r u_j(r,z) = \sup_{x\in\mathbb{R}^3} |U_j(x)|,$$

where $U_j(x) = u_j(r,z)(-x_2, x_1, 0)^T$ and U_j satisfies (4.6) for $p = p_j$. Let $y_j = (y_{j,1}, y_{j,2}, y_{j,3})^T \in \mathbb{R}^3$ denote the point where $|U_j|$ attains its maximum, i.e., $|U_j(y_j)| = \sup_{x\in\mathbb{R}^3} |U_j(x)|$. Precisely, we choose $y_j = (r_j, 0, y_{j,3})^T$ which is possible due to the radial symmetry in the first two components. With this notation, we introduce

$$\tilde{U}_j(x) := \frac{1}{M_j} U_j(y_j + xM_j^{\frac{1-p_j}{2}}) \text{ for } x \in \mathbb{R}^3. \tag{4.9}$$

Again due to Theorem 3.12 we have $y_{j,3} = 0$ for all $j \in \mathbb{N}$. By definition of M_j we have $|\tilde{U}_j(0)| = 1$ as well as $\|\tilde{U}_j\|_{L^\infty(\mathbb{R}^3)} = 1$ for all $j \in \mathbb{N}$. We know that U_j satisfies

$$-\Delta U_j + V(x)U_j = \Gamma(x)|U_j|^{p_j-1}U_j, \tag{4.10}$$

with $\operatorname{div} U_j = 0$ for all $j \in \mathbb{N}$. Moreover, by elliptic regularity $U_j \in C^{2,\alpha}(\mathbb{R}^3)$ for all $\alpha \in (0,1)$ and $|\partial^\beta U_j(x)| \to 0$ as $|x| \to \infty$ for all multi-indices $\beta \in \mathbb{N}_0^3$ with $|\beta| \le 2$, see Theorem A.6. We introduce $\tilde{x} = y_j + xM_j^{\frac{1-p_j}{2}}$, i.e., $x = M_j^{\frac{p_j-1}{2}}(\tilde{x} - y_j)$. Then

$$U_j(\tilde{x}) = M_j\tilde{U}_j(M_j^{\frac{p_j-1}{2}}(\tilde{x} - y_j))$$

and (4.10) transfers to

$$-M_j^{p_j}\Delta\tilde{U}_j(x) + V(y_j + xM_j^{\frac{1-p_j}{2}})M_j\tilde{U}_j(x) = \Gamma(y_j + xM_j^{\frac{1-p_j}{2}})M_j^{p_j}|\tilde{U}_j(x)|^{p_j-1}\tilde{U}_j(x).$$

A division by $M_j^{p_j}$ leads to

$$-\Delta\tilde{U}_j(x) + M_j^{1-p_j}V(y_j + xM_j^{\frac{1-p_j}{2}})\tilde{U}_j(x) = \Gamma(y_j + xM_j^{\frac{1-p_j}{2}})|\tilde{U}_j(x)|^{p_j-1}\tilde{U}_j(x) \text{ in } \mathbb{R}^3. \tag{4.11}$$

Finally, we highlight an obvious but important connection between the two scalings, namely we have

$$r_j \to 0 \text{ as } j \to \infty \text{ if and only if } y_j \to 0 \text{ as } j \to \infty \tag{4.12}$$

due to $r_j = \sqrt{y_{j,1}^2 + y_{j,2}^2}$ and $y_{j,3} = 0$ for all $j \in \mathbb{N}$.

4.2. The first case

In this section we exclude that the maximum points y_j accumulate at zero very fast. Precisely, we show the following result.

Lemma 4.5. *The case $r_jM_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$ can not occur.*

Notice that the arguments given in the proof of Lemma 4.5 work for arbitrary $\bar{p} \in (1,\infty)$ and not only for $\bar{p} \in (1,2)$.

Proof of Lemma 4.5. Assume by contradiction that $r_jM_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$. Hence, $r_j \to 0$ as $j \to \infty$ since $M_j \to \infty$ and $p_j \to \bar{p} \in (1,2)$ as $j \to \infty$. In particular, also $y_j \to 0$ as $j \to \infty$ by (4.12). We now pass to the limit in (4.11) in case of $y_j \to 0$ as $j \to \infty$. Therefore, let $K \subset \mathbb{R}^3$ be compact and

$K' \subset\subset K$. We have $\Gamma(y_j + \cdot M_j^{\frac{1-p_j}{2}})|\tilde{U}_j|^{p_j-1}\tilde{U}_j - M_j^{1-p_j}V(y_j + \cdot M_j^{\frac{1-p_j}{2}}) \in L^\infty(K)$ and hence also in $L^q(K)$ for all $q \geq 2$. Thus, Theorem 9.11 in [39] implies

$$\begin{aligned}\|\tilde{U}_j\|_{W^{2,q}(K')} &\leq C_1(K',K)(\|\tilde{U}_j\|_{L^q(K)} + \|\Gamma(y_j + \cdot M_j^{\frac{1-p_j}{2}})|\tilde{U}_j|^{p_j-1}\tilde{U}_j - M_j^{1-p_j}V(y_j + \cdot M_j^{\frac{1-p_j}{2}})\|_{L^q(K)}) \\ &\leq C_2(K',K)\sqrt[q]{|K|}\left(1 + \|\Gamma\|_{L^\infty(\mathbb{R}^3)} + \|V\|_{L^\infty(\mathbb{R}^3)}\right) \leq C(K',K)\end{aligned} \tag{4.13}$$

for constants $C_1(K',K), C_2(K',K), C(K',K) > 0$. Herewith, $\|\tilde{U}_j\|_{W^{2,q}(K')}$ is uniformly bounded in $j \in \mathbb{N}$. Since $K' \subset\subset K$ and $K \subset \mathbb{R}^3$ was arbitrary this allows us to choose a subsequence which converges weakly in $W^{2,q}_{\text{loc}}(\mathbb{R}^3)$ and, since $q \geq 2$ was arbitrary, strongly in $C^1_{\text{loc}}(\mathbb{R}^3)$ to $\tilde{U} \in C^1_{\text{loc}}(\mathbb{R}^3) \cap W^{2,q}_{\text{loc}}(\mathbb{R}^3)$ for all $q \geq 2$. From (4.9) we infer $|\tilde{U}_j(0)| = 1$ for all $j \in \mathbb{N}$. In particular, we conclude $|\tilde{U}(0)| = 1$ due to $\tilde{U}_j \to \tilde{U}$ in $C^1_{\text{loc}}(\mathbb{R}^3)$ as $j \to \infty$.

In the following, we denote the components of $\tilde{U}$ and $\tilde{U}_j$ by $\tilde{U}_1, \tilde{U}_2$ and $\tilde{U}_3$ respectively $\tilde{U}_{j,1}, \tilde{U}_{j,2}$ and $\tilde{U}_{j,3}$, i.e., $\tilde{U}_{j,i}\colon \mathbb{R}^3 \to \mathbb{R}$ for all $i \in \{1,2,3\}$ and $j \in \mathbb{N}$. Hence, $\tilde{U}_{j,3} \equiv 0$ for all $j \in \mathbb{N}$ entails $\tilde{U}_3 \equiv 0$. We now prove that $\tilde{U}_1(0) = \tilde{U}_2(0) = 0$ which then contradicts the fact that $|\tilde{U}(0)| = 1$.

First claim: $\tilde{U}_1$ is odd in x_2.

Due to the special choice $y_j = (r_j, 0, 0)$ and the structure $U_j(x) = u_j(r,z)(-x_2, x_1, 0)^T$ for all $j \in \mathbb{N}$ we infer by (4.9) that $\tilde{U}_{j,1}$ is odd in x_2 for all $j \in \mathbb{N}$. The convergence $\tilde{U}_{j,1} \to \tilde{U}_1$ in $C^1_{\text{loc}}(\mathbb{R}^3)$ as $j \to \infty$ then implies that also $\tilde{U}_1$ is odd in x_2.

Second claim: $\tilde{U}_2(x) \geq 0$ for $x \in \mathbb{R}^3$ with $x_1 \geq 0$ and $\tilde{U}_2(x) \leq 0$ for $x \in \mathbb{R}^3$ with $x_1 < 0$.

Due to (4.9) we have

$$\begin{aligned}\tilde{U}_{j,2}(x_1,x_2,x_3) &= \frac{1}{M_j}u_j\left(\sqrt{(y_{j,1} + x_1 M_j^{\frac{1-p_j}{2}})^2 + x_2^2 M_j^{1-p_j}}, x_3 M_j^{\frac{1-p_j}{2}}\right)(y_{j,1} + x_1 M_j^{\frac{1-p_j}{2}}) \\ &= \underbrace{\frac{1}{M_j^{1+\frac{p_j-1}{2}}}u_j\left(\sqrt{(y_{j,1} + x_1 M_j^{\frac{1-p_j}{2}})^2 + x_2^2 M_j^{1-p_j}}, x_3 M_j^{\frac{1-p_j}{2}}\right)}_{=:g(x)}\underbrace{(y_{j,1}M_j^{\frac{p_j-1}{2}} + x_1)}_{=:h(x)}.\end{aligned}$$

We have $g(x) \geq 0$ for all $x \in \mathbb{R}^3$. Let $x \in \mathbb{R}^3$ with $x_1 \geq 0$. Then $h(x) \geq 0$ due to $y_{j,1} \geq 0$ and thus $\tilde{U}_{j,2}(x_1,x_2,x_3) \geq 0$. On the other hand, let $x \in \mathbb{R}^3$ with $x_1 < 0$ be given. Then by assumption of the lemma, there is $j_0 \in \mathbb{N}$ such that $r_j M_j^{\frac{p_j-1}{2}} = y_{j,1}M_j^{\frac{p_j-1}{2}} < |x_1|$ for all $j \geq j_0$. Hence, $h(x) < 0$ for all $j \geq j_0$ which entails $\tilde{U}_2(x) \leq 0$ and proves the second claim.

Due to the continuity of $\tilde{U}_2$ we conclude $\tilde{U}_2(0) = 0$. The first claim implies $\tilde{U}_1(0) = 0$. This violates $|\tilde{U}(0)| = 1$ and the proof is done. □

4.3. The second case

In the previous section we have seen that $r_j M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$ is impossible. Therefore, it remains to consider the two cases $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$ or there is $c \in (0,\infty)$ such that $r_j M_j^{\frac{p_j-1}{2}} \to c$ as $j \to \infty$. In both cases we make use of (4.8) with v_j from (4.7). The case $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$ is excluded by the next lemma.

Lemma 4.6. *The sequence $(r_j M_j^{\frac{p_j-1}{2}})_{j\in\mathbb{N}}$ is bounded.*

Notice that since we end up with an equation in $\mathbb{R}^2$, Theorem 8.4 in [59] works for arbitrary $\bar{p} \in (1,\infty)$ and therefore Lemma 4.6 is valid for all $\bar{p} \in (1,\infty)$ and not only for $\bar{p} \in (1,2)$.

Proof of Lemma 4.6. Assume by contradiction $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$. Let $K \subset \mathbb{R}^2$ be compact and $K' \subset\subset K$. Since $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$ there is $j_0 \in \mathbb{N}$ such that $K \subset \Omega_j$ for all $j \geq j_0$. From (4.8) we deduce that the coefficients in front of $\partial_r v_j$, $V(r_j + rM_j^{(1-p_j)/2})v_j$ and v_j converge to zero uniformly on K as $j \to \infty$. Additionally, the right hand side is an element of $L^\infty(K)$ and hence of $L^q(K)$ for all $q \geq 2$. Hence, by Theorem 9.11 in [39] respectively Lemma A.11 we conclude similar to (4.13)

$$\|v_j\|_{W^{2,q}(K')} \leq C_1(K',K)\sqrt[q]{|K|}\,(1 + \|\Gamma\|_{L^\infty} + \|V\|_{L^\infty}) \leq C(K',K),$$

for constants $C_1(K',K), C(K',K) > 0$ where $|K|$ denotes the two-dimensional measure of K. Thus, $\|v_j\|_{W^{2,q}(K')}$ is uniformly bounded in $j \in \mathbb{N}$. Since $K \subset \mathbb{R}^2$ was an arbitrary compact subset and $K' \subset\subset K$, this allows us to choose a subsequence which converges weakly in $W^{2,q}_{\text{loc}}(\mathbb{R}^2)$ and, since $q \geq 2$ was arbitrary, strongly in $C^1_{\text{loc}}(\mathbb{R}^2)$ to $v \in C^1_{\text{loc}}(\mathbb{R}^2) \cap W^{2,q}_{\text{loc}}(\mathbb{R}^2)$ for all $q \geq 2$. Since $\Gamma(r_j + rM_j^{(1-p_j)/2})v_j^{p_j} \geq \inf\Gamma\, v_j^{p_j}$ the limit inequality for v reads

$$-\partial_r^2 v - \partial_z^2 v \geq \inf\Gamma\, v^{\bar{p}} \text{ in } \mathbb{R}^2.$$

Moreover, $v \in L^\infty(\mathbb{R}^2)$ and $v > 0$ due to the maximum principle for superharmonic functions. Theorem 8.4 in [59] implies $v \equiv 0$, a contradiction to $1 = v_j(0,0) \to v(0,0)$ as $j \to \infty$. □

4.4. The third case

It remains to lead $r_j M_j^{\frac{p_j-1}{2}} \to c \in (0,\infty)$ as $j \to \infty$ to a contradiction. As mentioned at the beginning of this chapter, this case is the most difficult one and it is here that we need the restriction $\overline{p} \in (1,2)$. We first derive the limit equation for (4.8). Due to $r_j M_j^{(p_j-1)/2} \to c \in (0,\infty)$ as $j \to \infty$ the limit domain is $[-c,\infty)\times\mathbb{R}$. Moreover, $r_j \to 0$ as $j \to \infty$, so also $\tilde{r} = r_j + rM_j^{(1-p_j)/2} \to 0$ as $j \to \infty$. Let $K \subset (-c,\infty)\times\mathbb{R}$ be compact. We rearrange (4.8), namely,

$$\begin{aligned}
&-\partial_r^2 v_j(r,z) - \frac{1}{r_j M_j^{(p_j-1)/2} + r}\partial_r v_j(r,z) - \partial_z^2 v_j(r,z) + \frac{1}{\left(r_j M_j^{(p_j-1)/2} + r\right)^2} v_j(r,z) \\
&\quad = \Gamma(r_j + rM_j^{\frac{1-p_j}{2}})v_j^{p_j}(r,z) - M_j^{1-p_j} V(r_j + rM_j^{\frac{1-p_j}{2}})v_j(r,z),
\end{aligned}$$

Now the right hand side is bounded in $L^q_{\text{loc}}((-c,\infty)\times\mathbb{R}, rdrdz)$ so that Lemma A.11 implies that $(v_j)_{j\in\mathbb{N}}$ is bounded in $W^{2,q}_{\text{loc}}((-c,\infty)\times\mathbb{R}, rdrdz)$ for all $q \geq 2$. Notice that for compact $K \subset (-c,\infty)\times\mathbb{R}$ the denominators on the left hand side are bounded away from zero. We conclude similar to the other two cases that $(v_j)_{j\in\mathbb{N}}$ converges weakly in $W^{2,q}_{\text{loc}}((-c,\infty)\times\mathbb{R}, rdrdz)$ for all $q \geq 2$ and strongly in C^1_{loc} to $v \in W^{2,q}_{\text{loc}}((-c,\infty)\times\mathbb{R}, rdrdz) \cap C^1_{\text{loc}}((-c,\infty)\times\mathbb{R}) \cap L^\infty((-c,\infty)\times\mathbb{R})$. Hence, due to $c > 0$ and the Lipschitz-continuity of Γ the limit equation for v then reads

$$-\partial_r^2 v - \frac{1}{c+r}\partial_r v - \partial_z^2 v + \frac{1}{(c+r)^2} v = \Gamma(0)v^{\bar{p}} \text{ in } (-c,\infty)\times\mathbb{R}$$

Let $v = (c+r)u$, then u satisfies

$$-\partial_r^2 u - \frac{3}{c+r}\partial_r u - \partial_z^2 u = \Gamma(0)(c+r)^{\bar{p}-1}u^{\bar{p}} \text{ in } (-c,\infty)\times\mathbb{R},$$

where positivity of v is passed to positivity of u on $(-c,\infty)\times\mathbb{R}$. Notice that due to $c > 0$ we receive $1 = v(0,0) = (c+0)u(0,0)$, i.e., $u(0,0) = \frac{1}{c} \neq 0$. After a translation we receive

$$-\partial_r^2 u - \frac{3}{r}\partial_r u - \partial_z^2 u = \Gamma(0)r^{\bar{p}-1}u^{\bar{p}} \text{ in } (0,\infty)\times\mathbb{R}$$

and $u(c,0) = \frac{1}{c} \neq 0$, which is precisly (4.3). Notice that the left hand side of (4.3) can be seen as a five-dimensional Laplacian. Since we cannot conclude that (4.3) is valid for $r = 0$ the limit equation only makes sense in $\mathbb{R}^5 \setminus \{x \in \mathbb{R}^5 : x_1 = x_2 = x_3 = x_4 = 0\}$.

We assume that Theorem 4.1 holds true and finish the proof of Theorem 4.3. Assume by contradiction that (4.4) is violated. Then with the notation from the beginning of this chapter, we consider the sequence $\left(r_j M_j^{\frac{p_j-1}{2}}\right)_{j\in\mathbb{N}}$. Lemma 4.5 and Lemma 4.6 imply that $r_j M_j^{\frac{p_j-1}{2}} \to c \in (0,\infty)$ as $j \to \infty$. This then leads to a non-trivial, positive solution of (4.3). This contradicts Theorem 4.1 and finishes the proof. □

It remains to prove Theorem 4.1 which is done in the following. We first fix some additional notation. Let e_n denote the n-th unit vector and $\mathring{S}^n_\varepsilon := \{x \in S^n : |x \pm e_n| > \varepsilon\}$ for $\varepsilon \in (0,1)$ and S^n denotes the $n-1$ dimensional sphere in $\mathbb{R}^n$. Moreover, for $x \in \mathbb{R}^n$ we introduce spherical coordinates, i.e., $(\rho,\theta) := (|x|, \frac{x}{|x|}) \in [0,\infty)\times S^{n-1}$. The first eigenvalue of the negative Laplace-Beltrami operator $-\Delta_\theta$ on $\mathring{S}^n_\varepsilon$ is denoted by $\lambda_{1,D}(\mathring{S}^n_\varepsilon)$. We recall the following auxiliary result:

Lemma 4.7. *$\lambda_{1,D}(\mathring{S}^n_\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.*

Proof. This follows from the even more general results in [21] (formulae (1) and (2) and the references there), see also [19]. □

Consider $\mathcal{K}_{R,\varepsilon} := \{x \in \mathbb{R}^n : \frac{x}{|x|} \in \mathring{S}^n_\varepsilon, |x| > R\}$. In particular, the limit equation (4.3) makes sense in $\mathcal{K}_{R,\varepsilon}$ for all $\varepsilon > 0$ and all $R > 0$. Let $\varphi_{1,\varepsilon}$ denote the first Dirichlet eigenfunction of $-\Delta$ on $\mathring{S}^n_\varepsilon$ such that $\|\varphi_1\|_{L^\infty(\mathring{S}^n_\varepsilon)} = 1$. We set

$$v_\varepsilon(\rho,\theta) := \rho^\alpha \varphi_{1,\varepsilon}(\theta) \text{ in } \mathcal{K}_{R,\varepsilon}, \tag{4.14}$$

where $\alpha = \alpha(\varepsilon) \in \mathbb{R}$ is chosen such that $\Delta v_\varepsilon = 0$. Indeed, due to the Laplacian in spherical coordinates,

$$\Delta = \partial_\rho^2 + \frac{n-1}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_\theta,$$

and $-\Delta_\theta \varphi_{1,\varepsilon} = \lambda_{1,\varepsilon}\varphi_{1,\varepsilon}$ we obtain

$$-\Delta v_\varepsilon(\rho,\theta) = \left(-\alpha(\alpha-1) - (n-1)\alpha + \lambda_{1,\varepsilon}\right)\rho^{\alpha-2}\varphi_{1,\varepsilon}(\theta) = \left(-\alpha(\alpha+n-2) + \lambda_{1,\varepsilon}\right)\rho^{\alpha-2}\varphi_{1,\varepsilon}(\theta).$$

Therefore, $\alpha_\varepsilon = \alpha(\varepsilon)$ has to satisfy $-\alpha_\varepsilon^2 + \alpha_\varepsilon(2-n) + \lambda_{1,\varepsilon} = 0$, i.e.,

$$\alpha_\varepsilon = \frac{-(2-n) \pm \sqrt{(2-n)^2 + 4\lambda_{1,\varepsilon}}}{-2} = \frac{2-n}{2} \mp \sqrt{\left(\frac{2-n}{2}\right)^2 + \lambda_{1,\varepsilon}}.$$

We now choose the minus-sign, so

$$\alpha_\varepsilon = \frac{2-n}{2} - \sqrt{\left(\frac{2-n}{2}\right)^2 + \lambda_{1,\varepsilon}} = 2 - n - O(\varepsilon) \text{ as } \varepsilon \to 0, \tag{4.15}$$

cf. Lemma 4.7.
Here is another estimate of auxiliary character.

Lemma 4.8. *Let $\varepsilon \in (0, 2)$. Then there is a constant $C_1 = C_1(\varepsilon) > 0$ such that $r \geq C_1\rho$ in $\mathcal{K}_{R,\varepsilon}$.*

Proof. Let $x \in \mathcal{K}_{R,\varepsilon}$, i.e., $|\frac{x}{|x|} \pm e_n| \geq \varepsilon$. Therefore,

$$\varepsilon^2 \leq |\frac{x}{|x|} \pm e_n|^2 = 2(1 \pm \frac{x_n}{|x|}). \tag{4.16}$$

Solving (4.16) for $z = x_n$ we obtain $\mp x_n \leq (1 - \frac{\varepsilon^2}{2})|x|$, i.e., $|z| \leq (1 - \frac{\varepsilon^2}{2})\rho$. Finally, we infer

$$r^2 = \rho^2 - z^2 \geq \rho^2 - \left(1 - \frac{\varepsilon^2}{2}\right)^2 \rho^2 = \rho^2 \left(1 - \left(1 - \frac{\varepsilon^2}{2}\right)^2\right) = \rho^2\varepsilon^2 \left(1 - \frac{\varepsilon^2}{4}\right)$$

and the choice $C_1 := \varepsilon\sqrt{1 - \frac{\varepsilon^2}{4}}$ is possible. □

We now explicitly work in five dimensions and we start with an estimate of the nonlinearity in (4.3). Therefore, for compact subsets $\mathring{S}^5_{\varepsilon,c} \subset\subset \mathring{S}^5_\varepsilon$ let $\mathcal{K}^c_{R,\varepsilon} := \{x \in \mathbb{R}^5 : \frac{x}{|x|} \in \mathring{S}^5_{\varepsilon,c}, |x| > R\}$.

Lemma 4.9. *Let $K > 0, \bar{p} \in (1, 2)$ be given and u be a non-trivial, positive solution of (4.3). Then there are $R_0 = R_0(K) > 0, \varepsilon_0 \in (0, 1)$ and a compact subset $\mathring{S}^5_{\varepsilon_0,c} \subset\subset \mathring{S}^5_{\varepsilon_0}$ with $\operatorname{int} \mathring{S}^5_{\varepsilon_0,c} \neq \emptyset$ in $\mathring{S}^5_{\varepsilon_0}$ such that*

$$-\Delta_5 u \geq K\rho^{-2}u \textit{ for } x \in \mathcal{K}^c_{R_0,\varepsilon_0}. \tag{4.17}$$

Proof. We choose $\varepsilon_0 > 0$ so small that

$$\bar{p} < 1 - \frac{2}{1 + \alpha_{\varepsilon_0}} \tag{4.18}$$

holds true, where α_ε is chosen by (4.15). This choice of $\varepsilon_0 > 0$ is possible due to Lemma 4.7 and (4.15). The value ε_0 is fixed for the rest of the proof. We first show that there is $\delta_{\varepsilon_0} = \delta(\varepsilon_0) > 0$ such that

$$u \geq \delta_{\varepsilon_0}\rho^{\alpha_{\varepsilon_0}}\varphi_{1,\varepsilon_0}(\theta) \text{ in } \mathcal{K}_{1,\varepsilon_0}. \tag{4.19}$$

We establish (4.19) by a maximum principle on unbounded domains, see for instance Lemma 2.1 in [10] or (MP) on page 2295 in [14]. We apply the maximum principle to the function

$$w_\delta(\rho, \theta) := u - \delta v_{\varepsilon_0}(\rho, \theta) \text{ in } \mathcal{K}_{1,\varepsilon_0},$$

where $\delta > 0$ is determined now. Due to $u > 0$ in $\mathcal{K}_{1,\varepsilon_0}$ and the Dirichlet boundary conditions of v_{ε_0} on $\partial\mathcal{K}_{1,\varepsilon_0} \setminus \{x \in \mathbb{R}^5 : |x| = 1\}$ we have $w_\delta > 0$ on $\partial\mathcal{K}_{1,\varepsilon_0} \setminus \{x \in \mathbb{R}^5 : |x| = 1\}$ for arbitrary $\delta > 0$. Due to

$u > 0$ on the compact set $\partial\mathcal{K}_{1,\varepsilon_0} \cap \{x \in \mathbb{R}^5 : |x| = 1\}$ we conclude that $w_\delta \geq 0$ on $\{x \in \mathbb{R}^n : |x| = 1\}$ provided $\delta > 0$ is chosen sufficient small, say smaller than $\delta_{\varepsilon_0} > 0$. Moreover, due to $\|\varphi_{1,\varepsilon_0}\|_{L^\infty} = 1$ and $\alpha_{\varepsilon_0} < 0$ we have $w_{\delta_{\varepsilon_0}} \geq -\delta_{\varepsilon_0}$. Finally, we compute

$$\Delta w_\delta = -\Gamma(0) r^{\bar{p}-1} u^{\bar{p}} \leq 0 \text{ in } \mathcal{K}_{1,\varepsilon_0}.$$

Hence, (4.19) follows from the above cited maximum principle on unbounded domains.

Combining the estimate in (4.19) with Lemma 4.8 guarantees

$$r^{\bar{p}-1}u^{\bar{p}} \geq C_1^{\bar{p}-1}\rho^{\bar{p}-1}u^{\bar{p}-1}u \geq C_1^{\bar{p}-1}\rho^{\bar{p}-1}\delta_{\varepsilon_0}^{\bar{p}-1}\rho^{\alpha_{\varepsilon_0}(\bar{p}-1)}\varphi_{1,\varepsilon_0}^{\bar{p}-1}(\theta)u = C_2\rho^{(\alpha_{\varepsilon_0}+1)(\bar{p}-1)}\varphi_1^{\bar{p}-1}(\theta)u \text{ in } \mathcal{K}_{1,\varepsilon_0}$$

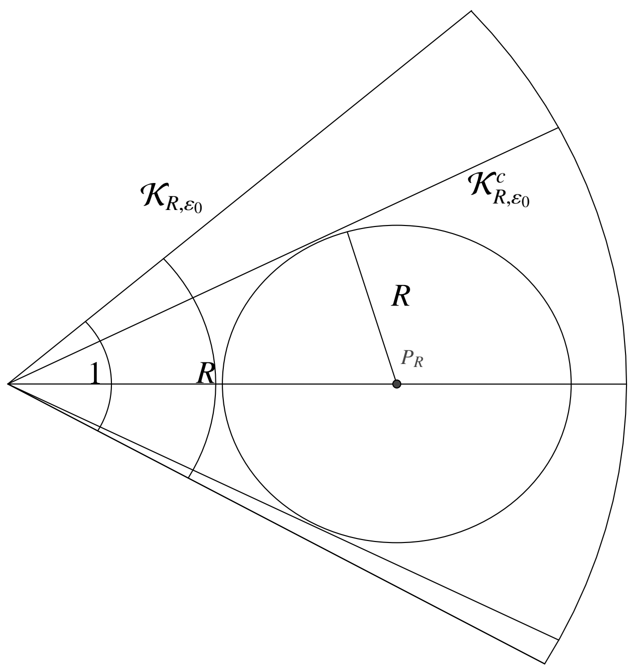
where $C_2 = C_2(\varepsilon_0) := C_1^{\bar{p}-1}\delta_{\varepsilon_0}^{\bar{p}-1} > 0$. We now choose $\emptyset \neq \mathring{S}^5_{\varepsilon_0,c} \subset\subset \mathring{S}^5_{\varepsilon_0}$. Thus there is $C_3 > 0$ such that $\varphi_{1,\varepsilon_0} \geq C_3 > 0$ on $\mathring{S}^5_{\varepsilon_0,c}$ and therefore

$$C_2\rho^{(\alpha_\varepsilon+1)(\bar{p}-1)}\varphi_1^{\bar{p}-1}(\theta)u \geq C_2C_3^{\bar{p}-1}\rho^{(\alpha_{\varepsilon_0}+1)(\bar{p}-1)}u = C_4\rho^{(\alpha_{\varepsilon_0}+1)(\bar{p}-1)}u \text{ in } \mathcal{K}^c_{1,\varepsilon_0} \tag{4.20}$$

with $C_4 = C_4(\varepsilon_0) := C_2C_3^{\bar{p}-1}$. By considering $\mathcal{K}^c_{R,\varepsilon_0} \subset \mathcal{K}^c_{1,\varepsilon_0}$ for all $R > 1$ we now establish (4.17) for sufficiently large R. This is possible if the exponents in (4.20) satisfy $(\alpha_{\varepsilon_0} + 1)(\bar{p} - 1) > -2$ which is already guaranteed by (4.18). Hence, due to the these exponents (4.17) is valid if $R > 0$ is chosen large enough, i.e., $R > R_0 = R_0(K)$. □

We continue to use the notation in front of Lemma 4.9.

Remark 4.10. *Let $R > 0$ be given. Since the interior of $\mathring{S}^5_{\varepsilon_0,c}$ w.r.t. $\mathring{S}^5_{\varepsilon_0}$ is non-empty there is a point $P_R \in \mathbb{R}^5$ such that $B_R(P_R) \subset \mathcal{K}^c_{R,\varepsilon_0}$. Moreover, we can choose P_R in such a way that there is $\tilde{C} > 0$ with $R \leq |x| \leq \tilde{C}R$ in $B_R(P_R)$ for all $R > 0$, see the sketch below. The estimate from below is trivial, since we can w.l.o.g. assume that one of the components of P_R has an absolute value larger than $2R$. The bound from above follows since by trigonometric identities P_R depends linerarly on R.*



The following result allows us to produce a contradiction to the statement of Lemma 4.9.

Lemma 4.11. *There is a constant $\hat{C} > 0$ such that*

$$\inf_{\psi \in H_0^1(B_R(P_R))} \frac{\int_{B_R(P_R)} |\nabla \psi|^2 dx}{\int_{B_R(P_R)} \frac{\psi^2}{\rho^2} dx} \leq \hat{C} \tag{4.21}$$

holds true uniformly in $R > 0$.

Proof. In order to show the statement it suffices to indicate for fixed $R > 0$ a function φ_R which satisfies (4.21) with a constant C not depending on $R > 0$. Let $R > 0$. Then we choose P_R from Remark 4.10. We choose φ_R to be the first Dirichlet eigenfunction of $-\Delta$ in $B_R(P_R)$. Notice that in $B_R(P_R)$ we have $R \leq |x| \leq \tilde{C}R$ for a constant $\tilde{C} > 0$, again by Remark 4.10. In particular, we have

$$\frac{1}{\tilde{C}^2 R^2} \int_{B_R(P_R)} |\varphi_R|^2 dx \leq \int_{B_R(P_R)} \frac{|\varphi_R|}{|x|^2} dx.$$

Due to the variational characterization and translation invariance of the first eigenvalue we have $\lambda_1(B_R(P_R)) = \frac{1}{R^2}\lambda_1(B_1(0))$. Hence, we conclude

$$\frac{\int_{B_R(P_R)} |\nabla \varphi_R|^2 dx}{\int_{B_R(P_R)} \frac{\varphi_R^2}{|x|^2} dx} \leq \tilde{C}^2 R^2 \frac{\int_{B_R(P_R)} |\nabla \varphi_R|^2 dx}{\int_{B_R(P_R)} \varphi_R^2 dx} = \tilde{C}^2 R^2 \lambda_1(B_R(P_R)) = \tilde{C}^2 \lambda_1(B_1(0))$$

and the choice $\hat{C} := \tilde{C}^2 \lambda_1(B_1(0))$ finishes the proof. □

Finally, we are ready to give the proof of Theorem 4.1. The idea is to combine the results of Lemma 4.9 and Lemma 4.11 to deduce a contradiction with the help of the so-called Agmon principle, see for instance Theorem 1.5.12 in [25].

Proof of Theorem 4.1: Let $\bar{p} \in (1, 2)$ and assume we have a non-trivial, positive solution of (4.3). We choose $K > \hat{C}$, where $\hat{C}$ is from Lemma 4.11. Then by Lemma 4.9 there are $R_0 > 0, \varepsilon_0 \in (0, 1)$ and compact $\mathring{S}^5_{\varepsilon_0,c} \subset\subset \mathring{S}^5_{\varepsilon_0}$ such that (4.17) holds true. Let $\varphi \in C_c^\infty(B_{R_0}(P_{R_0}))$ be arbitrary. We now apply the principle of Agmon. Therefore, we multiply (4.17) with $\frac{\varphi^2}{u}$ and integrate over $B_{R_0}(P_{R_0})$ which yield

$$\int_{B_{R_0}(P_{R_0})} \left(-\Delta u \frac{\varphi^2}{u} - K \frac{\varphi^2}{\rho^2} \right) dx \geq 0$$

An integration by parts (recall $\varphi \in C_c^\infty(B_{R_0}(P_{R_0}))$ gives

$$\int_{B_{R_0}(P_{R_0})} |\nabla \varphi|^2 dx \geq \int_{B_{R_0}(P_{R_0})} \left(|\nabla \varphi|^2 - |\nabla u \frac{\varphi}{u} - \nabla \varphi|^2 \right) dx = \int_{B_{R_0}(P_{R_0})} \nabla u \nabla \frac{\varphi^2}{u} dx = \int_{B_{R_0}(P_{R_0})} -\Delta u \frac{\varphi^2}{u} dx.$$

In summary,

$$\int_{B_{R_0}(P_{R_0})} \left(|\nabla \varphi|^2 - K \frac{\varphi^2}{\rho^2} \right) dx \geq 0 \text{ for all } \varphi \in C_c^\infty(B_{R_0}(P_{R_0})). \tag{4.22}$$

By density, (4.22) holds true for all $\varphi \in H_0^1(B_{R_0}(P_{R_0}))$ which contradicts Lemma 4.11. □

4.5. A-priori bounds for ground states

Theorem 4.3 already guarantees some a-priori bounds for positive solutions of (4.1). This section is devoted to a corollary which arises from Theorem 4.3 since inequality (4.4) allows for good uniform estimates on sequences of ground states of (4.1).

Corollary 4.12. *Let* $[p_\star, p^\star] \subset (1,2)$. *Then there is a constant* $C = C(p_\star, p^\star) > 0$ *such that*

$$\|u\|_{H^1_{\mathrm{cyl}}(r^3 drdz)} \leq C$$

for every positive ground state solution $u \in H_{\mathrm{symm}}$ *of* (4.1) *and every* $p \in [p_\star, p^\star]$.

Proof. We add a parameter to our notation of Nehari manifolds, i.e., for $p \in (1,2)$ set

$$M_p := \left\{ u \in H_{\mathrm{symm}} \setminus \{0\} : \int_\Omega \left(|\nabla_{r,z} u|^2 + V(r)u^2 \right) r^3 d(r,z) = \int_\Omega \Gamma(r) r^{p-1} \, |u|^{p+1} \, r^3 d(r,z) \right\}.$$

In addition, we set $N_p := \min_{M_p} J_p$, where also the energy functional now possesses the additonal information about the exponent of the nonlinearity. Recall that for $u \in M_p$ we have

$$J_p(u) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega \Gamma(r) r^{p-1} \, |u|^{p+1} \, r^3 d(r,z) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u\|^2$$

with $\|\cdot\|$ from (2.10). Furthermore, let $u_{2-\varepsilon}$ be a ground state solution of (4.1) for $p = 2-\varepsilon$. We scale $u_{2-\varepsilon}$ by a scalar $t_{p_j} \in \mathbb{R}$ such that $t_{p_j} u_{2-\varepsilon} \in M_{p_j}$. This condition forces

$$t_{p_j} = \left(\frac{\|u_{2-\varepsilon}\|^2}{\int_\Omega \Gamma(r) r^{p_j-1} |u_{2-\varepsilon}|^{p_j+1} r^3 d(r,z)} \right)^{\frac{1}{p_j-1}}.$$

Notice that due to $p_j \leq 2-\varepsilon$ for all $j \in \mathbb{N}$ and the uniform bound (4.4) we ensure

$$\int_\Omega \Gamma(r) r^{2-\varepsilon-1} |u_{2-\varepsilon}|^{2-\varepsilon+1} r^3 d(r,z) \leq C^{2-\varepsilon-p_j} \int_\Omega \Gamma(r) r^{p_j-1} |u_{2-\varepsilon}|^{p_j+1} r^3 d(r,z)$$

for a constant $C > 1$. Since $u_{2-\varepsilon} \in M_{2-\varepsilon}$ this implies

$$t_{p_j} \leq C^{\frac{2-\varepsilon-p_j}{p_j-1}} \left(\frac{\|u_{2-\varepsilon}\|^2}{\int_\Omega \Gamma(r) r^{2-\varepsilon-1} |u_{2-\varepsilon}|^{2-\varepsilon+1} r^3 d(r,z)} \right)^{\frac{1}{p_j-1}} = C^{\frac{2-\varepsilon-p_j}{p_j-1}} \leq C^{\frac{1-2\varepsilon}{\varepsilon}}. \tag{4.23}$$

Hence, making use of the minimization property of ground states, we conclude

$$\begin{aligned} 0 \leq N_{p_j} \leq J_{p_j}\left(t_{p_j} u_{2-\varepsilon} \right) &= \left(\frac{1}{2} - \frac{1}{p_j+1} \right) |t_{p_j}|^2 \|u_{2-\varepsilon}\|^2 \\ &= \frac{\frac{1}{2} - \frac{1}{p_j+1}}{\frac{1}{2} - \frac{1}{2-\varepsilon+1}} |t_{p_j}|^2 N_{2-\varepsilon} \leq \frac{3}{2} C^{2\frac{1-2\varepsilon}{\varepsilon}} N_{2-\varepsilon}, \end{aligned}$$

where in the last inequality we used (4.23) as well as

$$\frac{\frac{1}{2} - \frac{1}{p_j+1}}{\frac{1}{2} - \frac{1}{2-\varepsilon+1}} = \frac{(p_j-1)(2-\varepsilon+1)}{(1-\varepsilon)(p_j+1)} \leq \frac{2-\varepsilon+1}{p_j+1} \leq \frac{3}{2}.$$

This shows $\|u_{p_j}\| \leq C_\varepsilon$ uniformly in $j \in \mathbb{N}$ by definition of N_{p_j} and the statement follows from the equivalence of $\|\cdot\|$ and $\|\cdot\|_{H^1(r^3 drdz)}$. □

4.6. A problem on a bounded domain

We modify our original problem to obtain a related problem on a bounded domain. For this purpose, let $k > 0$ and $\Omega_k := \{(r,z) \in \Omega : r^2 + z^2 < k^2\}$. In $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ we consider the problem

$$\begin{aligned} -\Delta_{5,\mathrm{cyl}} u + V(r)u &= \Gamma(r) r^{p-1} u^p \text{ in } \Omega_k, \\ u &= 0 \text{ on } \partial\Omega_k \setminus (\{0\} \times [-k,k]), \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \{0\} \times [-k,k], \end{aligned} \tag{4.24}$$

with $V, \Gamma \in W^{1,\infty}(\Omega_k)$ not depending on z and $\inf V, \inf \Gamma > 0$. We recall a maximum principle in *small volume domains*, cf. [11]. We consider $-\Delta + c(x)$ with $c \in L^\infty(\Omega_{k,5})$, where $\Omega_{k,5} := \{x = (x_1, \ldots, x_5) \in \mathbb{R}^5 : (|(x_1, \ldots, x_4)|, x_5) \in \Omega_k\}$, see Section 1. Then the maximum principle in small volume domains for $-\Delta + c(x)$ reads as follows and notice that there is no assumption on the sign of $c(x)$.

Theorem 4.13. *(Proposition 1.1 in [11]) Assume $\hat{\Omega} \subset \Omega_{k,5}$ with $\operatorname{diam} \hat{\Omega} \le d$. Then there is $\delta > 0$ depending only on d and $\|c\|_\infty$ such that*

$$(-\Delta + c(x))w \ge 0 \text{ in } \hat{\Omega}, \ \limsup_{x \to \partial\hat{\Omega}} w(x) \le 0 \text{ and } |\hat{\Omega}| < \delta$$

imply $w \le 0$ in $\hat{\Omega}$.

We now state and prove symmetries of positive solutions of (4.24) on Ω_k which should not be surprising if we compare this result with Theorem 3.12 which is the analogue on Ω.

Lemma 4.14. *Let $p \in (1,5)$ and $V, \Gamma \in W^{1,\infty}(\Omega_k), \inf V, \inf \Gamma > 0$, $V(r,z) = V(r), \Gamma(r,z) = \Gamma(r)$. If u_k is a positive solution of (4.24) then u_k is twice differentiable in r and z up to the boundary, symmetric about $\{z = 0\}$ and decreasing in z-direction away from $\{z = 0\}$.*

Proof. Let u_k be a positive solution of (4.24). We identify u_k with a function in the five-dimensional ball $\Omega_{k,5}$ with Dirichlet boundary conditions. The regularity result can be obtained by a bootstrapping procedure like already done for the unbounded domain case in Theorem A.6 and Lemma A.7. Roughly speaking, we first prove $ru \in L^\infty(\Omega_k)$ by rewriting (4.24) as a system (Lemma A.5). With $ru \in L^\infty(\Omega_k)$ we get $W^{2,q}_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$-bounds for the right hand side of (4.24) and arbitrary $q \ge 2$ since the left hand side of (4.24) is nothing but a five-dimensional Schrödinger operator with cylindrical symmetry and $V \in W^{1,\infty}(\Omega_k)$ so that we can use classical regularity theory. Then Morrey's embedding and Schauder theory finishes the proof (Lemma A.7).

We now turn to the monotonicity and symmetry property. Therefore, we first show $\partial_{x_5} u_k > 0$ if $x_5 < 0$ by means of the moving plane method, compare Theorem 3.12. In the following, we drop the index k in u_k and simply write u. Let $\eta \in (-k, 0)$. Denote $\Sigma(\eta) := \{x \in \Omega_{k,5} : x_5 < \eta\}$. In particular, $2\eta - x_5 \in \Omega_{k,5}$ for $x \in \Sigma(\eta)$ and we can introduce

$$u_\eta(x) := u(x_1, \cdots, x_4, 2\eta - x_5) \text{ for } x \in \Sigma(\eta) \text{ and } w_\eta := u_\eta - u \text{ in } \Sigma(\eta).$$

Our goal is to show $w_\eta > 0$ for all $\eta \in (-k, 0)$. By the mean value theorem w_η satisfies

$$-\Delta w_\eta + V(r) w_\eta = \Gamma(r) r^{p-1} \left(u_\eta^p - u^p\right) = c(x,\eta) w_\eta \text{ in } \Sigma(\eta),$$

for $c(x,\eta)$ between $p\Gamma(r)r^{p-1}u_\eta^{p-1}(x)$ and $p\Gamma(r)r^{p-1}u^{p-1}(x)$, i.e., c is bounded. Moreover $w_\eta \gneqq 0$ on $\partial\Sigma(\eta)$ since $u = 0$ on $\partial\Omega_{k,5}$. We now show

$$w_\eta > 0 \text{ in } \Sigma(\eta) \text{ for all } \eta \in (-k, 0). \tag{4.25}$$

If $0 < \eta + k$ is small then Theorem 4.13 implies $w_\eta > 0$ in $\Sigma(\eta)$. Hence, there is a maximal $\bar{\mu} \le 0$ such that $w_\eta > 0$ in $\Sigma(\eta)$ for all $\eta \in (-k, \bar{\mu})$. In order to prove (4.25) we have to show

$$\bar{\mu} = 0. \tag{4.26}$$

Assume by contradiction $\bar{\mu} < 0$. Then by continuity we conclude $w_{\bar{\mu}} \ge 0$ in $\Sigma(\bar{\mu})$. Since $w_{\bar{\mu}} \not\equiv 0$ on $\partial\Sigma(\bar{\mu})$ we infer $w_{\bar{\mu}} > 0$ by the maximum principle. We show $w_{\bar{\mu}+\varepsilon} > 0$ in $\Sigma(\bar{\mu} + \varepsilon)$ for all $\varepsilon > 0$ sufficiently small. By Theorem 4.13 there is $\delta > 0$ such that the maximum principle holds for $L_{\bar{\mu}} := -\Delta + V(r) - c(x,\eta)$ in subsets of $\Sigma(\bar{\mu})$ with measure smaller than δ. Let K be a closed set in $\Sigma(\bar{\mu})$ such that $|\Sigma(\bar{\mu}) \setminus K| \le \frac{\delta}{2}$. In particular, $w_{\bar{\mu}} > 0$ in K. By continuity there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$|\Sigma(\bar{\mu} + \varepsilon) \setminus K| \le \delta \text{ and } w_{\bar{\mu}+\varepsilon} > 0 \text{ in } K. \tag{4.27}$$

By the mean value theorem we again deduce

$$-\Delta w_{\bar{\mu}+\varepsilon} + V(r)w_{\bar{\mu}+\varepsilon} = c(x, \bar{\mu} + \varepsilon)w_{\bar{\mu}+\varepsilon} \text{ in } \Sigma(\bar{\mu} + \varepsilon) \setminus K$$

and $w_{\bar{\mu}+\varepsilon} \not\equiv 0$ in $\Sigma(\bar{\mu} + \varepsilon) \setminus K$ due to $w_{\bar{\mu}+\varepsilon} > 0$ on ∂K. The maximum principle in narrow domains implies $w_{\bar{\mu}+\varepsilon} > 0$ in $\Sigma(\bar{\mu}+\varepsilon) \setminus K$. Hence, by (4.27) we get $w_{\bar{\mu}+\varepsilon} > 0$ in $\Sigma(\bar{\mu}+\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$ which contradicts the maximality of $\bar{\mu}$ and herewith proves (4.26), i.e., also (4.25).

In the next step we show $\partial_{x_5} u > 0$ if $x_5 < 0$ and prove the symmetry about $\{z = 0\}$. Since $w_\eta > 0$ in $\Sigma(\eta)$ and $w(x_1, \cdots, x_4, \eta) = 0$ Hopf's lemma applied on $\{x \in \Omega_{k,5} : x_5 = \eta\}$ implies

$$0 > \partial_{x_5} w_\eta(x_1, \cdots, x_4, \eta) = -2\partial_{x_5} u_\eta(x_1, \cdots, x_4, \eta) \text{ for all } \eta < 0 \text{ and all } (x_1, \dots, x_4, \eta) \in \Omega_{k,5}.$$

Moreover, $w_\eta > 0$ for all $\eta \in (-k, 0)$ implies $w_0 \ge 0$, i.e.,

$$u(x_1, \dots, x_4, x_5) \le u(x_1, \dots, x_4, -x_5) \text{ for } x_5 < 0. \tag{4.28}$$

We can now repeat all the arguments above for $\eta \in (0, k)$ and again end up with $w_\eta > 0$ for all $\eta \in (0, k)$. This entails $u(x_1, \dots, x_4, x_5) \le u(x_1, \dots, x_4, -x_5)$ for $x_5 > 0$ and herewith $u(x_1, \dots, x_4, x_5) = u(x_1, \dots, x_4, -x_5)$ by (4.28). This altogether finishes the proof for the z-direction. □

4.6.1. A-priori bounds for positive solutions on bounded domains

Before we give the a-priori bounds we again give two ways of scaling, similar to Section 4.1.

a) Scaling of (4.24): Let $(u_j)_{j \in \mathbb{N}}$ be a sequence of positive solutions of (4.24) for $p = p_j$ and $p_j \to \bar{p} \in [1, 2)$ as $j \to \infty$. Set

$$M_j := \max_{(r,z) \in \Omega_k} r u_j(r, z).$$

From Lemma 4.14 we infer that $M_j = r_j u_j(r_j, 0)$ for suitable $r_j \in [0, k)$. Thus, for a subsequence there is $r_\infty := \lim_{j\to\infty} r_j \in [0, k]$. We introduce

$$\begin{aligned} v_j(r,z) &:= \frac{r_j + rM_j^{\frac{1-p_j}{2}}}{M_j} u_j\left(r_j + rM_j^{\frac{1-p_j}{2}}, zM_j^{\frac{1-p_j}{2}}\right) \text{ for } j \in \mathbb{N} \text{ and } (r,z) \in \mathbb{R}^2 \\ &\text{such that } \left(r_j M_j^{\frac{p_j-1}{2}} + r\right)^2 + z^2 < k^2 M_j^{p_j-1}, r_j + rM_j^{\frac{1-p_j}{2}} \geq 0. \end{aligned} \tag{4.29}$$

In particular, $v_j(0,0) = 1$ for all $j \in \mathbb{N}$. Hence, using the same calculations as already carried out in front of (4.8) we deduce that v_j satisfies (4.8) with the domain of definition from (4.29). The conditions for the variable r in (4.29) can be expressed as

$$-r_j M_j^{\frac{p_j-1}{2}} \leq r < \left(\sqrt{k^2 - z^2 M_j^{1-p_j}} - r_j\right) M_j^{\frac{p_j-1}{2}} = \frac{(k-r_j)M_j^{\frac{p_j-1}{2}}(k+r_j)M_j^{\frac{p_j-1}{2}} - z^2}{\sqrt{k^2 M_j^{p_j-1} - z^2} + r_j M_j^{\frac{p_j-1}{2}}}. \tag{4.30}$$

b) Scaling of the vector-valued equation: As already done in Section 4.1 we can consider (4.24) as

$$\begin{aligned} -\Delta U + V(x)U &= \Gamma(x)|U|^{p-1}U \text{ in } B_k(0), \\ |U| &= 0 \text{ on } \partial B_k(0), \end{aligned} \tag{4.31}$$

where the two variants are connected via $U(x) = u(r,z)(-x_2, x_1, 0)^T$. Let $(U_j)_{j\in\mathbb{N}}$ be a sequence of solutions of (4.31) with exponent $p = p_j$. Again, let $y_j = (y_{j,1}, y_{j,2}, y_{j,3})^T \in B_k(0)$ denote the point where $|U_j|$ attains its maximum, i.e., $|U_j(y_j)| = \sup_{x\in B_k(0)} |U_j(x)|$. Once more, we can choose $y_j = (r_j, 0, 0)^T$. We introduce

$$\tilde{U}_j(x) := \frac{1}{M_j} U_j(y_j + xM_j^{\frac{1-p_j}{2}}) \text{ for } |y_j + xM_j^{\frac{1-p_j}{2}}| < k. \tag{4.32}$$

Then similar to (4.11) the function $\tilde{U}_j$ satisifies

$$-\Delta\tilde{U}_j(x) + M_j^{1-p_j} V(y_j + xM_j^{\frac{1-p_j}{2}})\tilde{U}_j(x) = \Gamma(y_j + xM_j^{\frac{1-p_j}{2}})|\tilde{U}_j(x)|^{p_j-1}\tilde{U}_j(x) \tag{4.33}$$

in $\{x \in \mathbb{R}^3 : |y_j + xM_j^{\frac{1-p_j}{2}}| < k\}$.
We are now able to prove the following result.

Theorem 4.15. *Let $p_j \to 1$ or $p_j \to \bar{p} \in (1,2)$ as $j \to \infty$. Then $(M_j)_{j\in\mathbb{N}}^{\frac{p_j-1}{2}}$ is bounded.*

Remark 4.16. *Theorem 4.15 entails the following a-priori bounds which is the analogue of Theorem 4.3: Let $[p_\star, p^\star] \subset (1,2)$ and $k > 0$. Then there is a constant $C = C(p_\star, p^\star, k) > 0$ such that*

$$\|ru\|_{L^\infty(\Omega_k)} \leq C$$

for every positive weak solution of (4.24) *and every $p \in [p_\star, p^\star]$.*

Proof of Theorem 4.15: Assume by contradiction that $M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$. We distinguish three cases with several subcases.

1) $r_\infty \in (0,k)$.

2) $r_\infty = k$ with subcases
2a) $(k-r_j)M_j^{\frac{p_j-1}{2}} \to \infty$, 2b) $(k-r_j)M_j^{\frac{p_j-1}{2}} \to 0$, 2c) $(k-r_j)M_j^{\frac{p_j-1}{2}} \to c \in (0,\infty)$ as $j \to \infty$.

3) $r_\infty = 0$ with subcases
3a) $r_jM_j^{\frac{p_j-1}{2}} \to \infty$, 3b) $r_jM_j^{\frac{p_j-1}{2}} \to 0$, 3c) $r_jM_j^{\frac{p_j-1}{2}} \to c \in (0,\infty)$ as $j \to \infty$.

The different treatment is needed since limit domain and limit equation which will arise from (4.8) and (4.29) depend on the quantity $\lim_{j\to\infty}(k-r_j)M_j^{\frac{p_j-1}{2}}$ respectively $\lim_{j\to\infty} r_jM_j^{\frac{p_j-1}{2}}$. Moreover, we distinguish $p_j \to 1$ from $p_j \to \bar{p} \in (1,2)$ since $p_j \to 1$ leads to a linear limit problem, whereas in case of $p_j \to \bar{p} \in (1,2)$ the limit problem stays non-linear.

The vector-valued scaling (4.31) and (4.33) will help us to rule out the cases 2b) and 3b), this is done in the postponed Lemma 4.17 and Lemma 4.18. In the following we investigate the other cases by passing to a limit equation. After this is done, we summarize all appearing cases and derive a contradiction in each of them.

Case 1): $r_\infty \in (0,k)$: Then $r_jM_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$. From (4.30) we deduce that the limit domain is $\mathbb{R}^2$. Hence, for a compact set $K \subset \mathbb{R}^2$ we can guarantee that there is $j_0 \in \mathbb{N}$ such that K is a subset of the domain of definition of v_j for all $j \geq j_0$. In this sense, due to $V, \Gamma \in L^\infty(\Omega_k)$ and $\|v_j\|_{L^\infty} = 1$ we infer that the right hand side of (4.8) is bounded in $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q \in [2,\infty)$. By elliptic estimates (see Theorem 9.11 in [39]) we obtain for an arbitrary compact subset $K' \subset\subset K$ and $q \in [2,\infty)$ chosen arbitrarily

$$\begin{aligned}\|v_j\|_{W^{2,q}(K')} \leq C_1(K',K)\Big(&\|v_j\|_{L^q(K)} + \|\Gamma(r_j + rM_j^{\frac{1-p_j}{2}})v_j^{p_j} \\ &-M_j^{1-p_j}V(r_j + rM_j^{\frac{1-p_j}{2}})v_j - \frac{1}{(r_jM_j^{(p_j-1)/2}+r)^2}v_j\|_{L^q(K)}\Big) \\ \leq{}& C_2(K',K)\sqrt[q]{|K|}\,(1 + \|\Gamma\|_{L^\infty} + \|V\|_{L^\infty}) \leq C_3(K',K)\end{aligned} \tag{4.34}$$

where $|K|$ denotes the two-dimensional volume of K. The estimate in (4.34) is done independently of $j \in \mathbb{N}$. Thus we conclude that $(v_j)_{j\in\mathbb{N}}$ converges weakly in $W^{2,q}_{\text{loc}}(\mathbb{R}^2)$ for all $q \geq 2$ and strongly in $C^1_{\text{loc}}(\mathbb{R}^2)$ to $v \in W^{2,q}_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Hence the limit equation reads

$$-\partial_r^2 v - \partial_z^2 v = \Gamma(r_\infty)v \text{ in } \mathbb{R}^2 \tag{4.35}$$

in case of $p_j \to 1$ as $j \to \infty$ and

$$-\partial_r^2 v - \partial_z^2 v = \Gamma(r_\infty)v^{\bar{p}} \text{ in } \mathbb{R}^2. \tag{4.36}$$

if $p_j \to \bar{p} \in (1,2)$ as $j \to \infty$. Notice that we have $v > 0$ in (4.36). But in case of (4.36) the non-existence result for classical non-negative solutions (see Theorem 8.4 in [59]) applies, a contradiction. Thus, (4.36) is ruled out. We will rule out (4.35) later.

Case 2): $r_\infty = k$: In particular, $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$. We investigate the subcases 2a) and 2c) mentioned before.

2a) $(k - r_j)M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$: By (4.30) we again obtain the limit domain $\mathbb{R}^2$ and similar to case 1) we conclude that the limit equation is (4.35) respectively (4.36).

2c) $(k - r_j)M_j^{\frac{p_j-1}{2}} \to c \in (0, \infty)$ as $j \to \infty$: From (4.30) we infer

$$
\begin{aligned}
r &< (k - r_j)M_j^{\frac{p_j-1}{2}} \frac{(k + r_j)M_j^{\frac{p_j-1}{2}}}{\sqrt{k^2 M_j^{p_j-1} - z^2} + r_j M_j^{\frac{p_j-1}{2}}} - \frac{z^2}{\sqrt{k^2 M_j^{p_j-1} - z^2} + r_j M_j^{\frac{p_j-1}{2}}} \\
&= (k - r_j)M_j^{\frac{p_j-1}{2}} \left(\sqrt{\frac{k^2}{(k + r_j)^2} - \frac{z^2}{(k + r_j)^2 M_j^{p_j-1}}} + \frac{r_j}{k + r_j} \right)^{-1} - \frac{z^2}{\sqrt{k^2 M_j^{p_j-1} - z^2} + r_j M_j^{\frac{p_j-1}{2}}} \to c
\end{aligned}
$$

as $j \to \infty$. Hence, the limit domain is $(-\infty, c) \times \mathbb{R}$. Since $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$ we can repeat the estimates in (4.34) in order to conclude that $(v_j)_{j\in\mathbb{N}}$ converges weakly in $W^{2,q}_{\text{loc}}((-\infty, -c) \times \mathbb{R})$ and strongly in $C^1((-\infty, -c) \times \mathbb{R})$ to $v \in W^{2,q}_{\text{loc}}((-\infty, -c) \times \mathbb{R}) \cap L^\infty((-\infty, -c) \times \mathbb{R})$. The limit equation is again (4.35) respectively (4.36) but this time in $(-\infty, c) \times \mathbb{R}$. The Dirichlet condition for u_j on $\partial\Omega_k \setminus (\{0\} \times [-k, k])$ carries over to a Dirichlet condition for v on the half-line $\{c\} \times \mathbb{R}$. This is done with the help of regularity theory up to the boundary, i.e., we first have to transform our expanding domain to a fixed domain which makes it possible to use appropriate results. Since this is a lengthy and routine calculation we skip it here. Notice that similar calculations are carried out in detail in the proof of Lemma 4.18.

The case (4.36) in $(-\infty, c) \times \mathbb{R}$ is ruled out by the classical non-existence result for classical non-negative solutions in a half-space (see Theorem 1.3 in [38]).

Case 3): $r_\infty = 0$: Again, we treat the two subcases 3a) and 3c).

3a) $r_j M_j^{\frac{p_j-1}{2}} \to \infty$ as $j \to \infty$: Once more, the limit domain is $\mathbb{R}^2$ and the limit equation is (4.35) respectively (4.36).

3c) $r_j M_j^{\frac{p_j-1}{2}} \to c \in (0, \infty)$ as $j \to \infty$: Here the limit domain is $(-c, \infty) \times \mathbb{R}$. Shifting the $1/r^2$-term in (4.8) to the left hand side, we can use Lemma A.11 to obtain uniform $W^{2,q}_{\text{loc}}((-c, \infty) \times \mathbb{R}, rdrdz)$ bounds of $(v_j)_{j\in\mathbb{N}}$ for all $q \in [2, \infty)$. In particular, $(v_j)_{j\in\mathbb{N}}$ converges weakly in $W^{2,q}_{\text{loc}}((-c, \infty) \times \mathbb{R}, rdrdz)$ and strongly in $C^1_{\text{loc}}((-c, \infty) \times \mathbb{R}, rdrdz)$ to $v \in C^1_{\text{loc}}((-c, \infty) \times \mathbb{R}, rdrdz) \cap L^\infty((-c, \infty) \times \mathbb{R})$. In particular, $v \not\equiv 0$ since $v_j(0, 0) = 1$ for all $j \in \mathbb{N}$. Herewith, we end up with

$$
-\partial_r^2 v - \partial_z^2 v - \frac{1}{c + r}\partial_r v + \frac{1}{(c + r)^2} v = \Gamma(0)v \text{ in } (-c, \infty) \times \mathbb{R} \tag{4.37}
$$

if $p_j \to 1$ as $j \to \infty$ and

$$
-\partial_r^2 v - \partial_z^2 v - \frac{1}{c + r}\partial_r v + \frac{1}{(c + r)^2} v = \Gamma(0)v^{\bar{p}} \text{ in } (-c, \infty) \times \mathbb{R} \tag{4.38}
$$

in case of $p_j \to \bar{p} \in (1,2)$ for a function $v \in C^1_{\text{loc}}((-c,\infty)\times\mathbb{R}) \cap W^{2,q}_{\text{loc}}((-c,\infty)\times\mathbb{R}, rdrdz)$ for all $q \in [2,\infty)$. Like already carried out in Section 4.4, $v = (c+r)u$ and a translation then transforms (4.38) in a non-trivial solution of (4.3). But this is ruled out by Theorem 4.1.

Summing up the previous cases, we end up with *either*

$$-\partial_r^2 v - \partial_z^2 v = \Gamma(r_\infty)v \text{ in } \mathbb{R}^2 \text{ or } (-\infty,c)\times\mathbb{R} \tag{4.39}$$

for fixed $r_\infty \in [0,k]$ and $c > 0$ (cases 1), 2a), 2c) and 3a)) *or* (4.37) with $c > 0$ (case 3c)).

We derive a contradiction in each of these cases which will then show that $(M_j^{\frac{p_j-1}{2}})_{j\in\mathbb{N}}$ is bounded. We first turn to (4.39) and afterwards deal with the remaining case (4.37) in 3c).

So from now on consider (4.39) and abbreviate $\Omega_\infty := (-\infty,c)\times\mathbb{R}$ respectively $\Omega_\infty = \mathbb{R}^2$. Since all v_j are known to be strictly positive the limit function v is non-negative. Due to the minimum priniciple for superharmonic functions (again compare Theorem 2.13 in [35]) we conclude that v has to be strictly positive. Let $R > 0$ and choose $P_R \in \Omega_\infty$ such that $B_R(P_R) \subset \Omega_\infty$. Moreover, let $(\lambda_1(B_R(P_R)), \varphi_{1,R})$ denote the pair of first eigenvalue and first eigenfunction of $-\Delta$ on the two-dimensional set $B_R(P_R)$ with Dirichlet boundary conditions. Since first eigenfunctions are positive we have $\int_{B_R(P_R)} v\varphi_{1,R}dx > 0$. By the variational characterization of the first eigenvalue (compare Lemma A.12) we obtain

$$\lambda_1(B_R(P_R)) = \frac{\lambda_1(B_1)}{R^2}.$$

Thus, we can choose $R > 0$ so large that $\lambda_1(B_R(P_R)) < \Gamma(r_\infty)$. Thus,

$$(\Gamma(r_\infty) - \lambda_1(B_R(P_R)))\int_{B_R(P_R)} v\varphi_{1,R}dx > 0.$$

Additionally, $(-\partial_r^2 - \partial_z^2)\varphi_{1,R} = \lambda_1(B_R(P_R))\varphi_{1,R}$ in $B_R(P_R)$ holds true, i.e., by (4.39) we conclude

$$\begin{aligned} 0 < (\Gamma(r_\infty) - \lambda_1(B_R(P_R)))\int_{B_R(P_R)} v\varphi_{1,R}dx &= \int_{B_R(P_R)} \left(\nabla v\cdot\nabla\varphi_{1,R} + v(\partial_r^2+\partial_z^2)\varphi_{1,R}\right)dx \\ &= \int_{\partial B_R(P_R)} v\frac{\partial\varphi_{1,R}}{\partial\nu}d\sigma \end{aligned} \tag{4.40}$$

where we have applied Green's formulae to obtain the last equality. But Hopf's lemma implies $\frac{\partial\varphi_{1,R}}{\partial\nu} < 0$ on $\partial B_R(P_R)$ which is a contradiction to (4.40). Hence, the sequence $(M_j^{\frac{p_j-1}{2}})_{j\in\mathbb{N}}$ is bounded in this case.

We now investigate case 3c), i.e., we consider (4.37) in $(-c,\infty)\times\mathbb{R}$ with $c > 0$. Again $v > 0$ on $(-c,\infty)\times\mathbb{R}$ by the strong minimum principle and the results of Lemma A.7. By a further translation we have

$$-\partial_r^2 v - \partial_z^2 v - \frac{1}{r}\partial_r v + \frac{1}{r^2}v = \Gamma(0)v \text{ in } (0,\infty)\times\mathbb{R}.$$

By Lemma A.12 we can choose $R > 0$ so large that $\lambda_1(A_{R,2R}) < \Gamma(0)$ holds true, where $A_{R,2R}$ denotes the annulus with inner radius R and outer radius $2R$. Similar to the case before $(\lambda_1(A_{R,2R}), \varphi_{1,R})$ denotes the pair of first eigenvalue and eigenfunction of $-\Delta_3 + \frac{1}{r^2}$ on $A_{R,2R}$. In analogue to (4.40) we derive

$$0 < (\Gamma(0) - \lambda_1(A_{R,R}))\int_{A_{R,2R}} v\varphi_{1,R}rd(r,z) = \int_{A_{R,2R}} \left(\nabla_{r,z}v\cdot\nabla_{r,z}\varphi_{1,R} + v\Delta_3\varphi_{1,R} + \frac{1}{r^2}v\varphi_{1,R}\right)rd(r,z)$$

$$= \int_{\partial A_{R,2R}} v \frac{\partial \varphi_{1,R}}{\partial \nu} d\sigma \leq 0,$$

a contradiction. Thus, again $(M_j^{\frac{p_j-1}{2}})_{j\in\mathbb{N}}$ is bounded which finishes the proof. □

We now exclude the cases 3b) and 2b) from above. Our arguments are independent of the limit of the sequence $(p_j)_{j\in\mathbb{N}}$. In both cases we make use of (4.32) and (4.33).

Lemma 4.17. *The case 3b) from above, i.e., $r_\infty = 0$ with $r_j M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$ can not occur.*

Proof. Assume by contradiction that $r_\infty = 0$ with $r_j M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$ holds true. Recall that $\tilde{U}_j$ is defined for all $x \in \mathbb{R}^3$ such that $|y_j M_j^{\frac{p_j-1}{2}} + x| < k M_j^{\frac{p_j-1}{2}}$, i.e., the limit domain is the entire space $\mathbb{R}^3$. The proof is finished by the same arguments as given in the proof of Lemma 4.5. □

We finally exclude case 2b). Here, the usual way of passing to a limit equation does not work since the domain of definition of $\tilde{U}_j$ in (4.32) is given by all $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that

$$\frac{|x|^2}{(k + r_j) M_j^{\frac{p_j-1}{2}}} + \frac{2r_j}{k + r_j} x_1 < (k - r_j) M_j^{\frac{p_j-1}{2}},$$

i.e., the limit domain is $(-\infty, 0) \times \mathbb{R}^2$ and the point 0 (where $|\tilde{U}(0)| = 1$) lies on the boundary of this set. Therefore, the local C^1_{loc} convergence on compact subsets is not sufficent to deduce a non-trivial solution of a limit equation. Instead, the argument we give to exclude this case is similar to an argument in the classical paper of Gidas and Spruck, see case 2) ($P \in \partial\Omega$) in Section 2 in [38] or the proof of Theorem 1 in [62]. We give the details here.

Lemma 4.18. *The case 2b) from above, i.e., $r_\infty = k$ with $(k - r_j) M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$ can not occur.*

Proof. Assume $r_\infty = k$ with $(k - r_j) M_j^{\frac{p_j-1}{2}} \to 0$ as $j \to \infty$. Our investigation is based on equation (4.31) for U_j expressed in 3-d Euclidian coordinates. In a first step we perform a transformation by flattening the boundary near the point $(k, 0, 0)^T \in \partial\Omega_k$ which allows us to switch to half-balls. The boundary of Ω_k near $(k, 0, 0)^T$ is parametrized via $x_1 = \psi(x_2, x_3) := \sqrt{k^2 - x_2^2 - x_3^2}$. Therefore, the transformation is as follows:

$$x_1' = \psi(x_2, x_3) - x_1, x_2' = x_2, x_3' = x_3. \tag{4.41}$$

Thus, $x \in \partial\Omega_k$ refers to $x_1' = 0$, whereas $x_1' > 0$ corresponds to points $x \in \Omega_k$. We rewrite (4.31) for U_j near $(k, 0, 0)^T$ in the coordinates $x' = (x_1', x_2', x_3')^T$. Due to (4.41) it holds

$$\partial_{x_1} = -\partial_{x_1'}, \;\; \partial_{x_2} = \partial_{x_2'} - \frac{x_2}{\psi(x_2, x_3)} \partial_{x_1'}, \;\; \partial_{x_3} = \partial_{x_3'} - \frac{x_3}{\psi(x_2, x_3)} \partial_{x_1'}.$$

Consequently,

$$\partial^2_{x_1} = \partial^2_{x_1'},$$

$$
\begin{aligned}
\partial^2_{x_2} &= \partial_{x_2}\left(\partial_{x_2'} - \frac{x_2}{\psi(x_2,x_3)}\partial_{x_1'}\right)\\
&= \left(\partial_{x_2'} - \frac{x_2}{\psi(x_2,x_3)}\partial_{x_1'}\right)\partial_{x_2'} - \frac{\psi(x_2,x_3)+\frac{x_2^2}{\psi(x_2,x_3)}}{\psi(x_2,x_3)^2}\partial_{x_1'} - \frac{x_2}{\psi(x_2,x_3)}\left(\partial_{x_2'} - \frac{x_2}{\psi(x_2,x_3)}\partial_{x_1'}\right)\partial_{x_1'}\\
&= \partial^2_{x_2'} + \frac{x_2^2}{\psi(x_2,x_3)^2}\partial^2_{x_1'} - \frac{2x_2}{\psi(x_2,x_3)}\partial^2_{x_1'x_2'} - \frac{\psi(x_2,x_3)+\frac{x_2^2}{\psi(x_2,x_3)}}{\psi(x_2,x_3)^2}\partial_{x_1'},\\
\partial^2_{x_3} &= \partial^2_{x_3'} + \frac{x_3^2}{\psi(x_2,x_3)^2}\partial^2_{x_1'} - \frac{2x_3}{\psi(x_2,x_3)}\partial^2_{x_1'x_3'} - \frac{\psi(x_2,x_3)+\frac{x_3^2}{\psi(x_2,x_3)}}{\psi(x_2,x_3)^2}\partial_{x_1'}.
\end{aligned}
$$

In summary,

$$
\Delta_x = \Delta_{x'} + \frac{x_2^2+x_3^2}{\psi(x_2,x_3)^2}\partial^2_{x_1'} - \frac{2x_2}{\psi(x_2,x_3)}\partial^2_{x_1'x_2'} - \frac{2x_3}{\psi(x_2,x_3)}\partial^2_{x_1'x_3'} - \frac{2\psi(x_2,x_3)+\frac{x_2^2+x_3^2}{\psi(x_2,x_3)}}{\psi(x_2,x_3)^2}\partial_{x_1'}.
$$

The maximum in $y_j = (r_j,0,0)^T$ of $|U_j|$ transforms to a maximum in $y_j' = (k-r_j,0,0)^T$ of $|W_j|$ where

$$
W_j(x') := U_j((\psi(x_2',x_3') - x_1', x_2', x_3')^T).
$$

Therefore, (4.31) near $(k,0,0)^T$ for U_j and $p = p_j$ turns into

$$
\begin{gathered}
\Big(-\Delta_{x'} - \frac{x_2^2+x_3^2}{\psi(x_2,x_3)^2}\partial^2_{x_1'} + \frac{2x_2}{\psi(x_2,x_3)}\partial^2_{x_1'x_2'} + \frac{2x_3}{\psi(x_2,x_3)}\partial^2_{x_1'x_3'} + \frac{2\psi(x_2,x_3)+\frac{x_2^2+x_3^2}{\psi(x_2,x_3)}}{\psi(x_2,x_3)^2}\partial_{x_1'}\Big)W_j \\
+ V(x')W_j = \Gamma(x')|W_j|^{p_j-1}W_j
\end{gathered}
\tag{4.42}
$$

in $\{|x'| < \tilde{k}\} \cap \{0 < x_1' < \delta\}$ for $0 < \tilde{k} < k$ and $\delta > 0$ small. We now perform a further scaling via

$$
\tilde{W}_j(x') = \frac{1}{M_j} W_j(M_j^{\frac{1-p_j}{2}} x' + y_j').
$$

In particular,

$$
|\tilde{W}_j(0)| = \frac{1}{M_j}|W_j(y_j')| = \frac{1}{M_j}|U_j(y_j)| = 1 \tag{4.43}
$$

and due to $y_{j,2}' = y_{j,3}' = 0$ we deduce Dirichlet-boundary conditions for $\tilde{W}_j$, namely

$$
\tilde{W}_j(-M_j^{\frac{p_j-1}{2}} y_{j,1}', x_2', x_3') = \frac{1}{M_j} W_j(0, x_2', x_3') = \frac{1}{M_j} U_j(\sqrt{k^2 - x_2^2 - x_3^2}, x_2, x_3) = 0. \tag{4.44}
$$

We now calculate the equation which is satisfied by $\tilde{W}_j$. Therefore, let $\tilde{x} := M_j^{\frac{1-p_j}{2}} x' + y_j'$. Hence, $W_j(\tilde{x}) = M_j \tilde{W}_j((\tilde{x} - y_j')M_j^{\frac{p_j-1}{2}})$ and $\partial_{\tilde{x}_i} W_j(\tilde{x}_j) = M_j^{\frac{p_j+1}{2}} \partial_{x_i'} \tilde{W}_j((\tilde{x} - y_j')M_j^{\frac{p_j-1}{2}})$ for $i = 1,2,3$. Thus,

$$
V(\tilde{x})W_j(\tilde{x}) - \Gamma(\tilde{x})|W_j(\tilde{x})|^{p_j-1}W_j(\tilde{x}) = M_j^{p_j}\Delta_{x'}\tilde{W}_j(x') + M_j^{p_j}\frac{\tilde{x}_2^2+\tilde{x}_3^2}{\psi(\tilde{x}_2,\tilde{x}_3)^2}\partial^2_{x_1'}\tilde{W}_j(x')
$$

$$-2M_j^{p_j}\frac{\tilde{x}_2}{\psi(\tilde{x}_2,\tilde{x}_3)}\partial^2_{x_1'x_2'}\tilde{W}_j(x') - 2M_j^{p_j}\frac{\tilde{x}_3}{\psi(\tilde{x}_2,\tilde{x}_3)}\partial^2_{x_1'x_3'}\tilde{W}_j(x') - \frac{2\psi(\tilde{x}_2,\tilde{x}_3)+\frac{\tilde{x}_2^2+\tilde{x}_3^2}{\psi(\tilde{x}_2,\tilde{x}_3)}}{\psi(\tilde{x}_2,\tilde{x}_3)^2}M_j^{\frac{p_j+1}{2}}\partial_{x_1'}\tilde{W}_j(x').$$

A multiplication with $M_j^{-p_j}$ gives

$$\begin{aligned} V(\tilde{x})\frac{1}{M_j^{p_j-1}}\tilde{W}_j(x') - \Gamma(\tilde{x})\tilde{W}_j(x')^{p_j} &= \Delta_{x'}\tilde{W}_j(x') + \frac{M_j^{1-p_j}(x_2'^2+x_3'^2)}{k^2-M_j^{1-p_j}(x_2'^2+x_3'^2)}\partial^2_{x_1'}\tilde{W}_j(x') \\ &-2\frac{M_j^{\frac{1-p_j}{2}}x_2'}{\sqrt{k^2-M_j^{1-p_j}((x_2'^2+x_3'^2))}}\partial^2_{x_1'x_2'}\tilde{W}_j(x') - 2\frac{M_j^{\frac{1-p_j}{2}}x_3'}{\sqrt{k^2-M_j^{1-p_j}(x_2'^2+x_3'^2)}}\partial^2_{x_1'x_3'}\tilde{W}_j(x') \\ &-\frac{2\sqrt{k^2-M_j^{1-p_j}(x_2'^2+x_3'^2)}+\frac{M_j^{1-p_j}(x_2'^2+x_3'^2)}{\sqrt{k^2-M_j^{1-p_j}(x_2'^2+x_3'^2)}}}{k^2-M_j^{1-p_j}(x_2'^2+x_3'^2)}M_j^{\frac{1-p_j}{2}}\partial_{x_1'}\tilde{W}_j(x') \end{aligned} \tag{4.45}$$

on $\left\{x'\in\mathbb{R}^3 : \sqrt{(x_1'+M_j^{(p_j-1)/2}(k-r_j))^2+x_2'^2+x_3'^2} < kM_j^{\frac{p_j-1}{2}}\right\}\cap\left\{x_1' > -(k-r_j)M_j^{\frac{p_j-1}{2}}\right\}$. Notice that the coefficients in front of $\partial^2_{x_1'}, \partial^2_{x_1'x_2'}, \partial^2_{x_1'x_3'}$ and $\partial_{x_1'}$ are converging to zero in L^∞ due to $M_j^{\frac{p_j-1}{2}} \to \infty$ as $j\to\infty$. The left hand side in (4.45) is also bounded in L^∞ since V, Γ and $\tilde{W}_j$ are bounded. From (4.43), (4.44) and the mean-value theorem we infer

$$\begin{aligned} 1 = |\tilde{W}_j(0)-\tilde{W}_j(-M_j^{\frac{p_j-1}{2}}y_j')| &= \sqrt{\sum_{i=1}^3 |\tilde{W}_{j,i}(0)-\tilde{W}_{j,i}(-M_j^{\frac{p_j-1}{2}}y_j')|^2} \\ &\le \sqrt{\sum_{i=1}^3 \sup_{\xi_i\in\{-tM_j^{(p_j-1)/2}y_j' : t\in[0,1]\}} |\nabla W_{j,i}(\xi_i)|^2}(k-r_j)M_j^{\frac{p_j-1}{2}} \le \|\nabla\tilde{W}_j\|_{L^\infty(B_{\rho_j}(0))}(k-r_j)M_j^{\frac{p_j-1}{2}}, \end{aligned} \tag{4.46}$$

where $\rho_j := M_j^{\frac{p_j-1}{2}}(k-r_j)$. Since $(k-r_j)M_j^{\frac{p_j-1}{2}} \to 0$ as $j\to\infty$ the estimate in (4.46) produces a contradiction if $(\|\nabla\tilde{W}_j\|_{L^\infty(B_{\rho_j}(0))})_{j\in\mathbb{N}}$ is uniformly bounded in $j\in\mathbb{N}$. Therefore, it remains to prove that $(\|\nabla\tilde{W}_j\|_{L^\infty(B_{\rho_j}(0))})_{j\in\mathbb{N}}$ is uniformly bounded in $j\in\mathbb{N}$ which is now done with the help of Corollary 6 in [62]. Obviously, we have $B_{\rho_j}(0)\subset B_1(0)\cap\{x_1' > -\rho_j\}$ for $j\in\mathbb{N}$ sufficiently large, i.e., for $j\in\mathbb{N}$ large enough we conclude

$$\|\nabla\tilde{W}_j\|_{L^\infty(B_{\rho_j}(0))} \le \|\nabla\tilde{W}_j\|_{L^\infty(B_1(0)\cap\{x_1'>-\rho_j\})}. \tag{4.47}$$

We perfrom a last translation in x_1' direction, i.e., $\hat{W}_j(x') := \tilde{W}_j(x'-M_j^{\frac{p_j-1}{2}}y_j')$ so that

$$\|\nabla\tilde{W}_j\|_{L^\infty(B_1(0)\cap\{x_1'>-\rho_j\})} \le \|\nabla\hat{W}_j\|_{L^\infty(B_2(0)\cap\{x_1'>0\})} \tag{4.48}$$

for j large enough. Notice that $B_1(0)\cap\{x_1' > -\rho_j\})$ is a subset of the domain of definition of $\tilde{W}_j$ and the Dirichlet-boundary conditions are satisfied due to (4.44). Moreover, the coefficients in front of the first and second order derivatives in the equation which is satisfied by $\hat{W}_j$ does not differ from the coefficients in the equation for $\tilde{W}_j$ since the coefficients of the first and second order derivatives do not depend on x_1', see (4.42). We are now in a position to apply Corollary 6 in [62]. Therefore, let $\hat{F}_j$ denote the right hand side in the equation for $\hat{W}_j$ and notice that F_j is uniformly bounded in L^∞. With this notation we infer for $j \in \mathbb{N}$ sufficiently large and $p \in (2, \infty)$ large enough

$$\begin{aligned}\|\nabla \hat{W}_j\|_{L^\infty(B_2(0)\cap\{x_1'>0\})} &\le C_{\mathrm{Sob}}\|\hat{W}_j\|_{W^{2,p}(B_2(0)\cap\{x_1'>0\})}\\ &\le C_{\mathrm{Sob}} C\left(\|\hat{F}_j\|_{L^p(B_2(0)\cap\{x_1'>0\})} + \|\hat{W}_j\|_{L^p(B_2(0)\cap\{x_1'>0\})}\right) \le C_{\mathrm{Sob}}\hat{C}\sqrt[p]{|B_2(0)|}.\end{aligned} \tag{4.49}$$

The combination of (4.47), (4.48) and (4.49) then yields the uniform bound of $(\|\tilde{W}_j\|_{L^\infty(B_{\rho_j}(0))})_{j\in\mathbb{N}}$ and finishes the proof. ◻

4.6.2. Uniqueness near $p = 1$

The next theorem ensures uniqueness of symmetric positive solutions of (4.24) near the exponent $p = 1$. We follow the approach by Damascelli, Grossi and Pacella in [23]. Our result can be seen as an extension which also works in the cylindrical setting for non-constant coefficients $V = V(r), \Gamma = \Gamma(r)$.

Theorem 4.19. *Fix $k > 0$. Then there exists $p_0 = p_0(k) > 1$ such that* (4.24) *has only one positive solution in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ for all $p \in (1, p_0)$.*

Proof. Let u_1 and u_2 be two distinct positive solutions of (4.24) in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$. Hence,

$$\begin{aligned}0 &= \int_{\Omega_k} \left(\nabla_{r,z}u_1 \cdot \nabla_{r,z}u_2 + V(r)u_1u_2 - \nabla_{r,z}u_1 \cdot \nabla_{r,z}u_2 - V(r)u_1u_2\right) r^3 d(r,z)\\ &= \int_{\Omega_k} \Gamma(r)r^{p-1}\left(u_1u_2^p - u_2u_1^p\right) r^3 d(r,z) = \int_{\Omega_k} \Gamma(r)r^{p-1}u_1u_2\left(u_2^{p-1} - u_1^{p-1}\right) r^3 d(r,z).\end{aligned}$$

Herewith, the possibilites $u_2 \ge u_1$ on Ω_k or $u_1 \ge u_2$ on Ω_k are ruled out. Thus, $w := u_1 - u_2$ has to change its sign on Ω_k.

Assume that the statement of the theorem is false, i.e., there are two sequences of positive solutions $(u_{1,j})_{j\in\mathbb{N}}$ and $(u_{2,j})_{j\in\mathbb{N}}$ of (4.24) for $p = p_j \to 1$ as $j \to \infty$ and $u_{1,j} \not\equiv u_{2,j}$ for all $j \in \mathbb{N}$. We set $\bar{u}_{1,j} := \frac{u_{1,j}}{M_{1,j}}$ and $\bar{u}_{2,j} := \frac{u_{2,j}}{M_{2,j}}$ on Ω_k where $M_{1,j} := \|ru_{1,j}\|_{L^\infty(\Omega_k)}$ and $M_{2,j} := \|ru_{2,j}\|_{L^\infty(\Omega_k)}$. In the following, we only give the arguments for $M_{1,j}$, the ones for $M_{2,j}$ are exactly the same. By Theorem 4.15 we know that $(M_{1,j})^{\frac{p_j-1}{2}}_{j\in\mathbb{N}}$ is bounded. Therefore, we have $M_{1,j}^{p_j-1} \to \mu^2 \in [0,\infty)$ along a subsequence as $j \to \infty$. The function $\bar{u}_{1,j}$ satisfies $\|r\bar{u}_{1,j}\|_{L^\infty(\Omega_k)} = 1$ and

$$\begin{aligned}-\Delta_{5,\mathrm{cyl}}\bar{u}_{1,j} + V(r)\bar{u}_{1,j} &= \Gamma(r)r^{p_j-1}M_{1,j}^{p_j-1}\bar{u}_{1,j}^{p_j} \text{ on } \Omega_k,\\ \bar{u}_{1,j} &= 0 \text{ on } \partial\Omega_k \setminus (\{0\}\times[-k,k]),\\ \frac{\partial \bar{u}_{1,j}}{\partial \nu} &= 0 \text{ on } \{0\}\times[-k,k].\end{aligned} \tag{4.50}$$

Recall that there is $C_1 > 0$ such that $M_{1,j}^{p_j-1} \le C_1$ for all $j \in \mathbb{N}$. By testing (4.50) with $\bar{u}_{1,j}$ we deduce

$$\begin{aligned}\int_{\Omega_k}\left(|\nabla_{r,z}\bar{u}_{1,j}^2 + V(r)\bar{u}_{1,j}^2\right) r^3 d(r,z) &= \int_{\Omega_k}\Gamma(r)r^{p_j-1}M_{1,j}^{p_j-1}\bar{u}_{1,j}^{p_j+1}r^3 d(r,z)\\ &\le \|\Gamma\|_\infty C_1 \int_{\Omega_k} rd(r,z) \le C_2\end{aligned} \tag{4.51}$$

uniformly in $j \in \mathbb{N}$. Therefore, $\|\bar{u}_{1,j}\|_{H^1(\Omega_k,r^3)}$ is uniformly bounded in $j \in \mathbb{N}$. In particular, the right hand side in (4.50) is uniformly bounded in $L^q(\Omega_k, r^3 drdz)$ for all $q \in (1, \frac{10}{3}]$ (recall that Ω_k corresponds to a five-dimensional ball). Due to global regularity theory (see for instance Lemma 9.17 in [39] or Theorem 5 in [62]) we obtain global bounds for $\|\bar{u}_{1,j}\|_{W^{2,\frac{10}{3}}(\Omega_k,r^3)}$. Another application of Sobolev's embedding quarantees global bounds for $(\bar{u}_{1,j})_{j\in\mathbb{N}}$ in $L^q(\Omega_k, r^3 drdz)$ for all $q \in (1,\infty]$. Therefore, $(\bar{u}_{1,j})_{j\in\mathbb{N}}$ is bounded in $W^{2,q}_{\mathrm{cyl}}(\Omega_k, r^3 drdz)$ for all $q \in (1,\infty)$ which finally leads to global bounds in $C^1(\overline{\Omega_k}, r^3 drdz)$. Therefore, $\bar{u}_{1,j} \to \bar{u} \in C^1(\overline{\Omega_k})$ and $\bar{u}$ satisfies

$$
\begin{aligned}
-\Delta_{5,\mathrm{cyl}}\bar{u} + V(r)\bar{u} &= \mu^2\Gamma(r)\bar{u} \text{ in } \Omega_k,\\
\bar{u} &= 0 \text{ on } \partial\Omega_k \setminus (\{0\}\times[-k,k]),\\
\frac{\partial\bar{u}}{\partial\nu} &= 0 \text{ on } \{0\}\times[-k,k].
\end{aligned}
\tag{4.52}
$$

The minimum-principle yields $\bar{u} > 0$ in Ω_k. Assume $\mu = 0$. Then $\bar{u}$ is an eigenfunction of $-\Delta_{5,\mathrm{cyl}}+V(r)$ to the eigenvalue 0, a contradiction since $\sigma(-\Delta_{5,\mathrm{cyl}} + V(r)) \subseteq [\operatorname{ess\,inf} V, \infty)$ and $\operatorname{ess\,inf} V > 0$. Hence $\mu > 0$. We now investigate the weighted eigenvalue-problem

$$
-\Delta_{5,\mathrm{cyl}}\varphi + V(r)\varphi = \lambda\Gamma(r)\varphi \text{ in } \Omega_k \text{ for } \lambda > 0. \tag{4.53}
$$

Minimizers of the Rayleigh-quotient

$$
B(\varphi,\varphi) = \frac{\int_{\Omega_k}\left(|\nabla_{r,z}\varphi|^2 + V(r)\varphi^2\right) r^3 d(r,z)}{\int_{\Omega_k}\Gamma(r)\varphi^2 r^3 d(r,z)}, \varphi \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)
$$

are weak solutions of (4.53) with $\lambda_1 := \min_{\varphi\in H^1(\Omega_k,r^3)} B(\varphi,\varphi)$. We calculate

$$
B(\varphi,\varphi) \geq \frac{\inf V}{\sup\Gamma} > 0
$$

due to our assumptions on V and Γ. Following the proof of Lemma 3.5 we conclude that minimizers do not change sign and that the eigenvalue λ_1 is simple. We denote the simple eigenfunction corresponding to λ_1 by φ_1. Due to $\bar{u} > 0$ in Ω_k and $\mu > 0$ we obtain

$$
\bar{u} = t\varphi_1 \text{ for } t > 0 \text{ and } \mu^2 = \lambda_1.
$$

The convergence discussed yields $\bar{u}_{1,j} \to t\varphi_1$ in $C^1(\overline{\Omega_k})$ as $j \to \infty$, i.e., we get $\bar{u}_{1,j}^{p_j-1} \to 1$ in compact subsets of Ω_k as $j \to \infty$. Herewith

$$
u_{1,j}^{p_j-1} = M_{1,j}^{p_j-1}\bar{u}_{1,j}^{p_j-1} \to \mu^2 = \lambda_1 \text{ as } j \to \infty
$$

uniformly in any compact subset of Ω_k. In the same manner, we conclude $\bar{u}_{2,j} \to \varphi_1$ in $C^1(\overline{\Omega_k})$ and $u_{2,j}^{p_j-1} \to \lambda_1$ in compact subsets of Ω_k as $j \to \infty$. For $j \in \mathbb{N}$, we introduce $w_j := \frac{u_{1,j}-u_{2,j}}{\|u_{1,j}-u_{2,j}\|_{L^\infty(\Omega_k)}}$. Hence, w_j satisfies

$$
\begin{aligned}
-\Delta_{5,\mathrm{cyl}}w_j + V(r)w_j &= \Gamma(r)r^{p_j-1}g_j(r,z)w_j \text{ in } \Omega_k,\\
w_j &= 0 \text{ on } \partial\Omega_k \setminus (\{0\}\times[-k,k]),\\
\frac{\partial w_j}{\partial\nu} &= 0 \text{ on } \{0\}\times[-k,k],
\end{aligned}
\tag{4.54}
$$

where $g_j(r,z) := \frac{u_{1,j}^{p_j}(r,z)-u_{2,j}^{p_j}(r,z)}{u_{1,j}(r,z)-u_{2,j}(r,z)}$ on the set $\tilde{\Omega}_j := \{(r,z)\in\Omega_k : u_{1,j}(r,z)\neq u_{2,j}(r,z)\}$ and $g_j(r,z)=0$ on the set $\Omega_k\setminus\tilde{\Omega}_j$. On arbitrary compact subsets $K\subset\Omega_k$ we conclude

$$g_j = \frac{u_{1,j}^{p_j}-u_{2,j}^{p_j}}{u_{1,j}-u_{2,j}} = \frac{M_{1,j}^{p_j}\bar{u}_{1,j}^{p_j}-M_{2,j}^{p_j}\bar{u}_{2,j}^{p_j}}{M_{1,j}\bar{u}_{1,j}-M_{2,j}\bar{u}_{2,j}} = p_j\xi_j(r,z)^{p_j-1},$$

where we applied the mean value theorem in the last step to obtain $\xi_j(r,z)$ between $M_{1,j}\bar{u}_{1,j}(r,z)$ and $M_{2,j}\bar{u}_{2,j}(r,z)$. Herewith there is a constant $C_1>0$ such that $\|g_j\|_{L^\infty(\Omega_k)}\leq C_1$ for all $j\in\mathbb{N}$ and $\xi_j^{p_j-1}\to\lambda_1$ in compact subsets of Ω_k as $j\to\infty$. Similar to (4.51) we deduce

$$\int_{\Omega_k}\left(|\nabla_{r,z}w_j|^2+V(r)w_j^2\right)r^3d(r,z) = \int_{\Omega_k}\Gamma(r)r^{p_j-1}g_j(r,z)w_j^2r^3d(r,z)\leq\|\Gamma\|_\infty C_1\int_{\Omega_k}r^{p_j+2}d(r,z)\leq C_2.$$

Therefore $w_j\rightharpoonup\tilde{w}\in H^1_{0,\mathrm{cyl}}(\Omega_k,r^3drdz)$ as $j\to\infty$. Again global regularity theory establishes uniform $W^{2,q}_{\mathrm{cyl}}(\Omega_k,r^3drdz)$-bounds for $(w_j)_{j\in\mathbb{N}}$ and all $q\in(1,\infty)$ which also guarantees $w_j\to\tilde{w}$ in $C^1(\overline{\Omega_k})$ as $j\to\infty$. Thus, due to the local convergence of $g_j\to\lambda_1$ as $j\to\infty$ the limit function $\tilde{w}$ satisfies

$$\begin{aligned}-\Delta_{5,\mathrm{cyl}}\tilde{w}+V(r)\tilde{w} &= \lambda_1\Gamma(r)\tilde{w} \text{ in } \Omega_k,\\ \tilde{w} &= 0 \text{ on } \partial\Omega_k\setminus(\{0\}\times[-k,k]),\\ \frac{\partial\tilde{w}}{\partial\nu} &= 0 \text{ on } \{0\}\times[-k,k].\end{aligned}$$

i.e., again there is $t>0$ such that $w_j\to t\varphi_1$ in $C^1(\overline{\Omega_k})$ as $j\to\infty$. This yields the desired contradiction since φ_1 does not change sign in Ω_k but at the very beginning of this proof we have seen that w_j has to change sign for every $j\in\mathbb{N}$. Hence, (4.24) has only one positive solution in $H^1_{0,\mathrm{cyl}}(\Omega_k,r^3drdz)$ near $p=1$ which finishes the proof. □

4.6.3. Uniqueness in the class of non-degenerate solutions

Theorem 4.19 shows uniqueness of positive solutions of (4.24) for p near one. We now restrict to the class of non-degenerate solutions and obtain with the help of the a-priori bounds in Theorem 4.15 the same result for the range of all $p\in(1,2)$. Here, a positive solution u of (4.24) is called *non-degenerate* if the linearized operator

$$-\Delta_{5,\mathrm{cyl}}+V(r)-p\Gamma(r)r^{p-1}u^{p-1}\colon H^1_{0,cyl}(\Omega_k,r^3drdz)\to H^{-1}_{0,\mathrm{cyl}}(\Omega_k,r^3drdz)$$

is invertible.

Theorem 4.20. *Let $p\in(1,2)$ and assume that every positive solution of* (4.24) *is non-degenerate. Then* (4.24) *has only one positive solution in $H^1_{0,\mathrm{cyl}}(\Omega_k,r^3drdz)$. In particular, the number of nondegenerate positive solutions $u\in H^1_{0,\mathrm{cyl}}(\Omega_k,r^3drdz)$ of* (4.24) *is less or equal to one.*

Proof. Let $\bar{p}>1$ be the maximal number such that the uniqueness of non-degenerate solutions in $H^1_{0,\mathrm{cyl}}(r^3drdz)$ of (4.24) is valid for $p\in(1,\bar{p})$. If $\bar{p}\geq 2$ there is nothing left to show, so let w.l.o.g. $\bar{p}<2$. We first show that the uniqueness then also holds true for $p=\bar{p}$.

Assume by contradiction that (4.24) has two distinct positive non-degenerate solutions for $p = \bar{p}$ denoted by $\bar{u}_1$ and $\bar{u}_2$. Hence, for

$$H\colon H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz) \times (1,2) \to H^{-1}_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz); \quad H(u,p) := -\Delta_{5,\mathrm{cyl}} u + V(r)u - \Gamma(r) r^{p-1}|u|^{p-1}u$$

we have $H(\bar{u}_1, \bar{p}) = 0 = H(\bar{u}_2, \bar{p})$ and $\frac{\partial H}{\partial u}(\bar{u}_1, \bar{p})$, $\frac{\partial H}{\partial u}(\bar{u}_2, \bar{p})$ are invertible. Notice that H is differentiable with respect to $p \in (1,2)$. Indeed, we have

$$\frac{\partial H(u,p)}{\partial p} = -p\Gamma(r) r^{p-1}|u|^{p-1} - (p-1)\Gamma(r) r^{p-2}|u|^{p-1}u \tag{4.55}$$

and we now prove that the right hand side of (4.55) is an element of $H^{-1}_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$. Due to $p \in (1,2)$ we compute with the help of (2.14) (notice $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz) \subset H^1_{\mathrm{cyl}}(r^3 drdz)$ by zero-extension)

$$\begin{aligned}
&\sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \left| \int_{\Omega_k} (p-1)\Gamma(r) r^{p-2}|u|^{p-1} u v r^3 d(r,z) \right| \\
&\le (p-1)\|\Gamma\|_{L^\infty} \sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \int_{\Omega_k} |ru|^p |v| r d(r,z) \\
&\le (p-1)\|\Gamma\|_{L^\infty} \left(\int_{\Omega_k} |ru|^{2p} r d(r,z) \right)^{\frac{1}{2}} \sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \left(\int_{\Omega_k} v^2 r d(r,z) \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Moreover, Hölder's inequality gives

$$\begin{aligned}
&\sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \left| \int_{\Omega_k} p\Gamma(r) r^{p-1}|u|^{p-1} v r^3 d(r,z) \right| \\
&\le p\|\Gamma\|_{L^\infty} \sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \int_{\Omega_k} r^{p-1}|u|^{p-1}|v| r^3 d(r,z) \\
&\le p\|\Gamma\|_{L^\infty} \|ru\|^{p-1}_{L^{p+1}_{\mathrm{cyl}}(\Omega_k, r^3 drdz)} \left(\int_{\Omega_k} r^3 d(r,z) \right)^{\frac{3-p}{2(p+1)}} \sup_{\|v\|_{H^1(\Omega_k, r^3 drdz)} \le 1} \|v\|_{L^2_{\mathrm{cyl}}(\Omega_k, r^3 drdz)} < \infty.
\end{aligned}$$

Thus, the implicit function theorem yields neighborhoods P_1 and P_2 of $\bar{p}$ and continuous functions $h_1\colon P_1 \to H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$, $h_2\colon P_2 \to H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ such that $h_1(\bar{p}) = \bar{u}_1 \neq \bar{u}_2 = h_2(\bar{p})$ and

$$H(h_1(p), p) = 0 \text{ for all } p \in P_1, \quad H(h_2(p), p) = 0 \text{ for all } p \in P_2.$$

By continuity of h_1 and h_2 and the fact that $\bar{u}_1 \not\equiv \bar{u}_2$ we infer that there are at least two non-degenerate solutions for p close to and smaller than $\bar{p}$, a contradiction to the choice of $\bar{p}$. Hence, we also have uniqueness for $p = \bar{p}$.

By definition of $\bar{p} < 2$ we find a sequence $(p_j)_{j\in\mathbb{N}}$ which decreases to $\bar{p}$ as $j \to \infty$ and two different non-degenerate positive solutions of (4.24) for $p = p_j$ denoted by $u_{1,j}$ and $u_{2,j}$ ($j \in \mathbb{N}$). Let $\bar{u}$ be the unique non-degenerate positive solution of (4.24) for $p = \bar{p}$. We show that

$$u_{1,j} \rightharpoonup \bar{u}, u_{2,j} \rightharpoonup \bar{u} \text{ in } H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz) \text{ as } j \to \infty, \tag{4.56}$$

where we only prove $u_{1,j} \rightharpoonup \bar{u}$ in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$ as $j \to \infty$ since the arguments for the sequence $(u_{2,j})_{j\in\mathbb{N}}$ are exactly the same. Due to Theorem 4.15 we have $\|ru_{1,j}\|_{L^\infty(\Omega_k)} \le C$ uniformly in $j \in \mathbb{N}$. We test (4.24) by $u_{1,j}$ and deduce

$$\int_{\Omega_k} \left(|\nabla_{r,z}u_{1,j}|^2 + V(r)u_{1,j}^2\right) r^3 d(r,z) = \int_{\Omega_k} \Gamma(r)r^{p-1}u^{p+1}r^3 d(r,z) \le C^{p+1}\,\|\Gamma\|_{L^\infty} \int_{\Omega_k} rd(r,z) \le \tilde{C},$$

i.e., we find a uniform $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$-bound. Thus, there is $u \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$ such that $u_{1,j} \rightharpoonup u$ in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$ as $j \to \infty$. Therefore, u is a weak solution of (4.24) for $p = \bar{p}$. Moreover, u is non-degenerate by assumption, i.e., by uniqueness, $u = \bar{u}$ which proves (4.56).

For $j \in \mathbb{N}$ we set $w_j := u_{1,j} - u_{2,j}$ and $\bar{w}_j := w_j\|w_j\|^{-1}_{H^1_0(\Omega_k,r^3)}$. Hence,

$$-\Delta_{5,\mathrm{cyl}}w_j + V(r)w_j = \Gamma(r)r^{p_j-1}\left(u_{1,j}^{p_j} - u_{2,j}^{p_j}\right) = \Gamma(r)p_j\xi_j(r,z)^{p_j-1}w_j \text{ in } \Omega_k,$$

where $\xi_j(r,z)$ is between $ru_{1,j}(r,z)$ and $ru_{2,j}(r,z)$ for $(r,z) \in \Omega_k$. A division by $\|u_{1,j}-u_{2,j}\|_{H^1_0(\Omega_k,r^3)}$ yields

$$-\Delta_{5,\mathrm{cyl}}\bar{w}_j + V(r)\bar{w}_j = \Gamma(r)p_j\xi_j(r,z)^{p_j-1}\bar{w}_j \text{ in } \Omega_k. \tag{4.57}$$

Since $\|\bar{w}_j\|_{H^1_0(\Omega_k,r^3)} = 1$ for all $j \in \mathbb{N}$ we deduce the existence of $\bar{w} \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$ such that $\bar{w}_j \rightharpoonup \bar{w}$ in $H^1_{0,\mathrm{cyl}}(\Omega_k, r^3drdz)$ along a subsequence as $j \to \infty$. Our goal is to identify $\bar{w}$ as a non-trivial solution of the linearization around a non-degenerate solution of (4.24). This then contradicts the non-degeneracy and finishes the proof. By testing (4.57) with $\bar{w}_j$ we obtain

$$1 = \|\bar{w}_j\|^2_{H^1_0(\Omega_k,r^3)} = p_j \int_{\Omega_k} \Gamma(r)\xi_j(r,z)^{p_j-1}\bar{w}_j^2 r^3 d(r,z). \tag{4.58}$$

We now show with the help of Lebesgue's dominated convergence theorem that we can pass to the limit in (4.58). Due to the compact embedding $H^1_{0,\mathrm{cyl}}(\Omega_k, rdrdz) \hookrightarrow L^4_{\mathrm{cyl}}(\Omega_k, rdrdz)$ we have $ru_{1,j}, ru_{2,j} \to r\bar{u}$ in $L^4_{\mathrm{cyl}}(\Omega_k, rdrdz)$ as $j \to \infty$ and $r\bar{w}_j \to r\bar{w}$ in $L^4_{\mathrm{cyl}}(\Omega_k, rdrdz)$ as $j \to \infty$. Thus, we find $g \in L^4(\Omega_k, r^3drdz)$ and a subsequence (again denoted by $(ru_{1,j})_{j\in\mathbb{N}}, (ru_{2,j})_{j\in\mathbb{N}}$) such that $|ru_{1,j}(r,z)|, |ru_{2,j}(r,z)| \le g(r,z)$ for almost all $(r,z) \in \Omega_k$ and $h \in L^4(\Omega_k, r^3drdz)$ such that $|\bar{w}_j| \le h(r,z)$ for almost all $(r,z) \in \Omega_k$ (see Lemma A.1 in [73]). Due to $p_j \to \bar{p} \in (1,2)$ we may assume $p_j < 2$ for all $j \in \mathbb{N}$. Then we estimate

$$\begin{aligned}|\xi_j(r,z)|^{p_j-1}\bar{w}_j(r,z)^2 &\le 2\max\{|ru_{1,j}(r,z)|, |ru_{2,j}(r,z)|\}^{p_j-1}h(r,z)^2\\ &\le 2g(r,z)^{p_j-1}h(r,z)^2 \le \begin{cases}2g(r,z)h(r,z)^2, & g(r,z) \ge 1,\\ 2h(r,z)^2, & g(r,z) < 1.\end{cases}\end{aligned}$$

Thus, $m(r,z) := 2\|\Gamma\|_{L^\infty}(1 + g(r,z))h(r,z)^2$ for $(r,z) \in \Omega_k$ is an L^1-majorant for $\Gamma(r)\xi_j(r,z)^{p_j-1}\bar{w}_j^2$ since

$$\begin{aligned}\int_{\Omega_k} m(r,z)r^3 d(r,z) &\le 2\|\Gamma\|_{L^\infty}\|h\|^2_{L^4_{\mathrm{cyl}}(\Omega_k,r)}\left(\int_{\Omega_k} r^4 rd(d,z)\right)^{\frac{1}{2}}\\ &\quad + 2\|\Gamma\|_{L^\infty}\|h\|^2_{L^4_{\mathrm{cyl}}(\Omega_k,r)}\|g\|_{L^4_{\mathrm{cyl}}(\Omega_k,r)}\left(\int_{\Omega_k} r^8 rd(r,z)\right)^{\frac{1}{4}} < \infty.\end{aligned}$$

Hence, Lebesgue's dominated convergence theorem and (4.58) imply

$$1 = p_j \int_{\Omega_k} \Gamma(r)\xi_j(r,z)^{p_j-1}\bar{w}_j^2 r^3 d(r,z) \to \bar{p} \int_{\Omega_k} \Gamma(r)r^{\bar{p}-1}\bar{u}^{\bar{p}-1}\bar{w}^2 r^3 d(r,z) \text{ as } j \to \infty.$$

Thus $\bar{w} \not\equiv 0$. In a similar manner, from (4.57) and the estimates obtained for the right hand side of (4.57) we conclude that $\bar{w} \in H^1_{0,\mathrm{cyl}}(\Omega_k, r^3 drdz)$ is a weak solution of the limit equation

$$-\Delta_{5,\mathrm{cyl}}\bar{w} + V(r)\bar{w} = \Gamma(r)\bar{p}r^{\bar{p}-1}\bar{u}^{\bar{p}-1}\bar{w} \text{ in } \Omega_k.$$

Hence, $\bar{w}$ is a non-trivial solution of the linearization of (4.24) around $\bar{u}$. This contradiction finishes the proof. ◻

4.7. Consequences of non-degeneracy in the unbounded domain case

In the previous section we have seen that non-degeneracy of all positive solutions in the bounded domain Ω_k leads to uniqueness of positive solutions. We now return to problem (4.1) on the full space and prove that a non-degeneracy assumption leads to finiteness of ground states.

Again we assume that the coefficients V, Γ satisfy $V, \Gamma \in W^{1,\infty}([0,\infty))$ and $\inf V, \inf \Gamma > 0$. Moreover, for this section we assume throughout the following assumption:

$$\begin{aligned} &0 \text{ is not an eivenvalue of the operator } L_{\mathrm{symm}} = -\Delta_{5,\mathrm{cyl}} + V(r) - p\Gamma(r)r^{p-1}u^{p-1} \\ &\text{with } D(L_{\mathrm{symm}}) = \{v \in H^2_{\mathrm{cyl}}(r^3 drdz) : v \text{ is symmetric about } \{z=0\}\}, \end{aligned} \tag{4.59}$$

where u denotes an arbitrary positive ground state solution of (4.1) with $p \in (1,2)$.

Remark 4.21. *We restrict to $p \in (1,2)$ in* (4.59) *since this is the range of p where we can use the a-priori bounds for ground states of* (4.1) *established in Section 4.5. These bounds get important in Section 4.7.2. The results in Section 4.7.1 are valid for all $p \in (1,5)$ under the assumption* (4.59).

We first give a reformulation of assumption (4.59).

Lemma 4.22. *The following statements are equivalent:*

(a) Assumption (4.59) *holds true.*

(b) If the second eigenvalue of the operator L_{symm} in (4.59) *exists, then it is positive for all ground state solutions of* (4.1).

Proof. Let u denote a ground state solution of (4.1). We deduce the equivalence of (a) and (b) by the following consideration. Since by Corollary A.2 we know that $\sigma_{\mathrm{ess}}(L_{\mathrm{symm}}) \subseteq [\operatorname{ess\,inf} V, \infty)$ and $\operatorname{ess\,inf} V > 0$, we conclude that zero does not belong to the essential spectrum of L_{symm}. From Section 3.4 we know that L_{symm} has exactly one negative eigenvalue λ_1. Since the associated eigenfunction φ_1 is symmetric with respect to $\{z=0\}$ (Lemma 3.10) and λ_1 is simple (Lemma 3.5) we infer that the second eigenvalue of L_{symm} is positive if and only if 0 is not an eigenvalue of L_{symm}. ◻

In the following we use the space

$$H_{\text{symm}} = \left\{ u \in H^1_{\text{cyl}}(r^3 drdz) \text{ such that } u \text{ is symmetric about } \{z = 0\} \right\}.$$

We give some consequences of assumption (4.59). The next lemma provides the connection between L_{symm} and $J''(u)$ as defined in (3.6).

Lemma 4.23. *Assume* (4.59). *Then the linear operator $J''(u) : H_{\text{symm}} \to H_{\text{symm}}$ is invertible.*

Proof. By Theorem 3.4 we only have to prove the injectivity of $J''(u)$. Assume that there is a $v \in H_{\text{symm}}$ such that $J''(u)v = 0$. Via the identification of the Riesz representation theorem we get

$$0 = \langle J''(u)v, w\rangle_{\mathcal{H}} = \int_\Omega \left(\nabla_{r,z} v \cdot \nabla_{r,z} w + V(r)vw - p\Gamma(r) r^{p-1} u^{p-1} vw \right) r^3 d(r, z) \text{ for all } w \in H_{\text{symm}}.$$

In other words, v is a weak solution in H_{symm} of $L_{\text{symm}} v = -\Delta_{5,\text{cyl}} v + V(r)v - p\Gamma(r) r^{p-1} u^{p-1} v = 0$. Thus $v \in H^2_{\text{cyl}}(r^3 drdz)$. Since $0 \notin \sigma(L_{\text{symm}})$ we conclude $v = 0$, i.e., $J''(u)\colon H_{\text{symm}} \to H_{\text{symm}}$ is one-to-one. □

We are now able to give a first application of Lemma 4.23 in terms of perturbation theory, i.e., we look at the perturbed equation

$$-\Delta_{5,\text{cyl}} u + V(r)u = (\Gamma(r) + \varepsilon h(r, z)) r^{p-1} |u|^{p-1} u, \tag{4.60}$$

where $\varepsilon > 0$ and $h \in L^\infty(\Omega)$ is symmetric about $\{z = 0\}$. We show the existence of solutions of (4.60) by the implicit function theorem. The associated perturbed energy functional on H_{symm} reads

$$J_\varepsilon(u) = \int_\Omega \left(\frac{|\nabla_{r,z} u|^2}{2} + \frac{V(r)}{2} u^2 - (\Gamma(r) + \varepsilon h(r, z)) \frac{r^{p-1}}{p+1} |u|^{p+1} \right) r^3 d(r, z).$$

This can be rewritten as

$$J_\varepsilon(u) = J_0(u) + \varepsilon \mathcal{G}(u), \tag{4.61}$$

where J_0 denotes the unperturbed functional (3.3) restricted to H_{symm} and

$$\mathcal{G}\colon H_{\text{symm}} \to \mathbb{R}; u \mapsto -\int_\Omega h(r, z) \frac{r^{p-1}}{p+1} |u|^{p+1} r^3 d(r, z).$$

We now find critical points of the perturbed functional J_ε, i.e., weak solutions in H_{symm} of (4.60).

Theorem 4.24. *Assume* (4.59) *and let $h \in L^\infty(\Omega)$ be symmetric about* $\{z = 0\}$ *and $u \in H_{\text{symm}}$ denote a ground state solution of the unperturbed problem* (4.1). *Then there is $\varepsilon_0 = \varepsilon_0(h) > 0$ such that* (4.60) *has a weak solution $u_\varepsilon \in H_{\text{symm}}$ for all ε with $|\varepsilon| < \varepsilon_0$. Moreover, $u_\varepsilon \to u$ in H_{symm} as $\varepsilon \to 0$.*

Proof. We have to find critical points of J_ε for ε sufficiently small. Let $u \in H_{\text{symm}}$ denote a ground state of the unperturbed equation, i.e., $J_0'(u) = 0$. In the sequel, we guarantee the existence of $w = w(\varepsilon) \in H_{\text{symm}}$ such that $J_\varepsilon'(u + w(\varepsilon)) = 0$ for ε sufficiently small. From (4.61) we immediately deduce $J_\varepsilon'(v) = J_0'(v) + \varepsilon \mathcal{G}'(v)$ for all $v \in H_{\text{symm}}$. Hence, we define

$$F\colon \mathbb{R} \times H_{\text{symm}} \to H_{\text{symm}}; (\varepsilon, w) \mapsto J_0'(u + w) + \varepsilon \mathcal{G}'(u + w).$$

Since u is a critical point of J_0 we infer that $F(0, 0) = 0$. We have $F \in C^1(\mathbb{R} \times H_{\text{symm}})$ and $D_w F(0, 0) = J_0''(u)$, i.e., $D_w F(0, 0)\colon H_{\text{symm}} \to H_{\text{symm}}$ is invertible by Lemma 4.23. Hence, the implicit function theorem is applicable and proves the existence of $\varepsilon_0 > 0$ as stated in our claim such that $J_\varepsilon(u + w(\varepsilon)) = 0$ for all ε such that $|\varepsilon| < \varepsilon_0$. Herewith, our weak solution is $u_\varepsilon = u + w(\varepsilon)$. In particular, the implicit function theorem yields $w(\cdot) \in C^1((-\varepsilon_0, \varepsilon_0))$ and $w(0) = 0$ which proves the additional claim $u_\varepsilon \to u$ in H_{symm} as $\varepsilon \to 0$. □

4.7.1. Extension of non-degeneracy to $H^1_{\mathrm{cyl}}(r^3drdz)$

Under assumption (4.59) we can drop the symmetry about $\{z = 0\}$ and deduce non-degeneracy of ground states in a larger space in the following sense.

Theorem 4.25. *Assume* (4.59) *and let* $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ *be a positive ground state of* (4.1). *Then for* $J''(u)\colon H^1_{\mathrm{cyl}}(r^3drdz) \to H^1_{\mathrm{cyl}}(r^3drdz)$ *we have* $\ker J''(u) = [\partial_z u]$.

Proof. Let $v \in H^1_{\mathrm{cyl}}(r^3drdz)$ be a solution of

$$-\Delta_{5,\mathrm{cyl}} v + V(r)v = p\Gamma(r)r^{p-1}u^{p-1}v \text{ in } \Omega.$$

By Theorem 3.12 there is $\theta \in \mathbb{R}$ such that u is symmetric about $\{z = \theta\}$. Since

$$v(r,z) = \frac{v(r,z) + v(r,2\theta - z)}{2} + \frac{v(r,z) - v(r,2\theta - z)}{2} =: v_1(r,z) + v_2(r,z) \text{ for } (r,z) \in \Omega$$

and v_1 is symmetric with respect to $\{z = \theta\}$ whereas v_2 is antisymmetric w.r.t. $\{z = \theta\}$ we have a splitting

$$H^1_{\mathrm{cyl}}(r^3drdz) = H_{\mathrm{symm},\theta} \oplus H_{\mathrm{antisymm},\theta},$$

where

$$H_{\mathrm{symm},\theta} := \{v \in H^1_{\mathrm{cyl}}(r^3drdz) : v(r,z) = v(r,2\theta - z) \text{ for almost all } (r,z) \in \Omega\},$$
$$H_{\mathrm{antisymm},\theta} := \{v \in H^1_{\mathrm{cyl}}(r^3drdz) : v(r,z) = -v(r,2\theta - z) \text{ for almost all } (r,z) \in \Omega\}.$$

With $\tilde{v}_1(r,z) := v_1(r,z-\theta)$ we have $\tilde{v}_1 \in H_{\mathrm{symm}}$, i.e., Lemma 4.23 yields $\tilde{v}_1 \equiv 0$. Thus also $v_1 \equiv 0$ holds true. We have to show $v_2 \in \left[\frac{\partial u}{\partial z}\right]$. Therefore, we again make a spectral analysis of L. Considered in $H^1_{\mathrm{cyl}}(r^3drdz)$ we know by Theorem 3.6 and Theorem 3.9 that L has exactly one negative eigenvalue with corresponding eigenfunction φ_1 and $\sigma_{\mathrm{ess}}(L) \subseteq [\operatorname{ess\,inf} V, \infty)$. Hence, considered in $H_{\mathrm{antisymm},\theta}$ we have $\sigma_{\mathrm{ess}}(L) \subseteq [\operatorname{ess\,inf} V, \infty)$ and no negative eigenvalues. Indeed, if $L|_{H_{\mathrm{antisymm},\theta}}$ would have another negative eigenvalue, then L would have either two negative eigenvalues or one negative eigenvalue with multiplicity two, a contradiction to Theorem 3.9. Since $L\, \partial_z u = 0$ we infer that 0 is the first eigenvalue of L considered in $H_{\mathrm{antisymm},\theta}$.

In general, every $w \in H_{\mathrm{antisymm},\theta}$ is by antisymmetry uniquely determined by its values in $\Omega^+ := (0,\infty)\times(\theta,\infty)$. We can repeat the steps in the proof of Lemma 3.5 that Dirichlet eigenfunctions corresponding to the first Dirichlet eigenvalue 0 do not change sign in Ω^+. As a consequence, the Dirichlet eigenvalue 0 is simple and since we already know that $\frac{\partial u}{\partial z} < 0$ in Ω^+ is a Dirichlet eigenfunction associated with the eigenvalue 0 we end up with $v_2 \in [\partial_z u]$. This finishes the proof. □

4.7.2. Finiteness of the number of ground states

The overall goal of this section is to use the non-degeneracy assumption (4.59) to prove that

$$-\Delta_{5,\mathrm{cyl}} u + V(r)u = \Gamma(r)r^{p-1}u^p \text{ in } \Omega \tag{4.62}$$

has only a finite number of ground states in H_{symm} for $p \in (1,2)$.

First of all, we introduce some notation. We work with the dual space of $H^1_{\text{cyl}}(r^3drdz)$ in the following. Therefore, we denote the dual of $H^1_{\text{cyl}}(r^3drdz)$ by $H^{-1}_{\text{cyl}}(r^3drdz)$ equipped with the norm

$$\|\varphi\|_{H^{-1}_{\text{cyl}}(r^3drdz)} := \sup_{\|v\|_{H^1(r^3)}=1} |\varphi(v)| \text{ for } \varphi \in H^{-1}_{\text{cyl}}(r^3drdz),$$

where we abbreviated $H^1(r^3) := H^1_{\text{cyl}}(r^3drdz)$. Recall the notion $K_{4,1} := K^{2,2}_{4,1} \subset H^1(r^3drdz)$ and its definition from (2.4). Next, we give an auxiliary result.

Lemma 4.26. *The operator* $-\Delta_{5,\text{cyl}} + V(r) \colon H^1_{\text{cyl}}(r^3drdz) \to H^{-1}_{\text{cyl}}(r^3drdz)$ *is an invertible isometry.*

Proof. We first prove that the weak formulation of $-\Delta + V(r)$ in $\mathbb{R}^5$ preserves cylindrical symmetry. Let $\left(a_{ij}\right)_{i,j=1}^4 = A \in O(4)$. Hence, $\delta_{ik} = \sum_{j=1}^4 a_{ij}a_{kj}$ for $i,k = 1,\dots,4$. We dentote a point in $\mathbb{R}^5$ by $(\tilde{x}, x_5)$, where $\tilde{x} = (x_1,\dots,x_4)$. For sufficiently smooth functions u and φ we formally compute

$$\begin{aligned}\sum_{j=1}^4 \frac{\partial}{\partial x_j}(u(A\tilde{x},x_5)) \cdot \frac{\partial}{\partial x_j}(\varphi(A\tilde{x},x_5)) &= \sum_{i,j,k=1}^4 \left(\frac{\partial u}{\partial x_i}\right)(A\tilde{x},x_5)a_{ij}\left(\frac{\partial \varphi}{\partial x_k}\right)(A\tilde{x},x_5)a_{kj}\\ &= (\nabla_{\tilde{x}}\, u \cdot \nabla_{\tilde{x}}\, \varphi)(A\tilde{x},x_5).\end{aligned}$$

For cylindrically symmetric functions $u,\varphi \in H^1(\mathbb{R}^5)$ we have $u(A\tilde{x},x_5) = u(\tilde{x},x_5)$ and $\varphi(A\tilde{x},x_n) = \varphi(\tilde{x},x_n)$. Hence, we receive

$$\int_{\mathbb{R}^5} \nabla(u(A\tilde{x},x_5)) \cdot \nabla(\varphi(A\tilde{x},x_5))\,dx = \int_{\mathbb{R}^5} (\nabla u \cdot \nabla\varphi)(\tilde{x},x_5)dx.$$

We next show that $-\Delta_{5,\text{cyl}} + V(r)$ is an isometry from $H^1_{\text{cyl}}(r^3drdz)$ to $H^{-1}_{\text{cyl}}(r^3drdz)$. For this purpose, take $u \in H^1_{\text{cyl}}(r^3drdz)$. Then

$$\begin{aligned}&\left\|V(r)u - \Delta_{5,\text{cyl}}u\right\|_{H^{-1}(r^3)} = \sup_{\{v:\|v\|_{H^1(r^3)}=1\}} \left|(V(r)u - \Delta_{5,\text{cyl}}u)(v)\right|\\ &= \sup_{\{v:\|v\|_{H^1(r^3)}=1\}} \left|\int_\Omega V(r)uvr^3d(r,z) + \int_\Omega \nabla_{r,z}u \cdot \nabla_{r,z}vr^3d(r,z)\right| = \sup_{\{v:\|v\|_{H^1(r^3)}=1\}} \left|\langle u,v\rangle_{H^1(r^3)}\right| = \|u\|_{H^1(r^3)},\end{aligned}$$

where we made use of the Hahn-Banach theorem in the last equality. Moreover, applying the Lemma of Lax-Milgram to the bounded and coercive bilinear form $a(v,w) := \int_\Omega (\nabla_{r,z}v \cdot \nabla_{r,z}w + V(r)vw)\,dx$ on $H^1(\mathbb{R}^5)$, we see that $-\Delta + V(r) \colon H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$ is bijective. Hence, $-\Delta_{5,\text{cyl}} + V(r) \colon H^1_{\text{cyl}}(r^3drdz) \to H^{-1}_{\text{cyl}}(r^3drdz)$ is invertible by the open mapping theorem. □

In the following, we say that a solution u of $G(u) = 0$ is *isolated* if there is $\delta > 0$ and a neighbourhood $B_\delta(u)$ such that $G^{-1}(0) \cap B_\delta(u) = \{u\}$.

Lemma 4.27. *Assume* (4.59) *and let* $p \in (1,2)$. *Then every ground state in* H_{symm} *of* (4.62) *is isolated.*

Proof. Let $p \in (1,2)$ and consider the map

$$G \colon H_{\text{symm}} \to H_{\text{symm}};\ G(u) = u - \left(-\Delta_{5,\text{cyl}} + V(r)\right)^{-1}\left(\Gamma(r)r^{p-1}|u|^{p-1}u\right),$$

see Lemma 4.26 for the invertibility of $-\Delta_{5,\text{cyl}} + V(r)$. We have $G(u) = 0$ for every ground state $u \in H_{\text{symm}}$ of (4.62). By assumption (4.59) we know that u is non-degenerate. Hence, we infer by means of the implicit function theorem that u is an isolated solution, i.e., there is a neighbourhood $B_\delta(u)$ of u in H_{symm} such that $G^{-1}(0) \cap B_\delta(u) = \{u\}$, see Proposition 1.3 in [27]. □

We now give the result of the finiteness of ground states.

Lemma 4.28. *Assume* (4.59) *and let* $p \in (1,2)$. *Then the number of ground states in* H_{symm} *of* (4.62) *is finite.*

Proof. Assume by contradiction that there are infinitely many ground states in H_{symm}. Then we find a sequence of ground states $(u_n)_{n\in\mathbb{N}} \subset H_{\mathrm{symm}}$ such that $G(u_n) = 0$ for all $n \in \mathbb{N}$. In particular, from Corollary 4.12 we know that $\|u_n\|_{H^1_{\mathrm{cyl}}(r^3)} \leq C$ uniformly in $n \in \mathbb{N}$. Hence, there is $\bar{u} \in H_{\mathrm{symm}}$ such that $u_n \rightharpoonup \bar{u}$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ as $n \to \infty$. Again, $\bar{u} \in K_{4,1}$ by Lemma 2.5 and $\bar{u}$ is a weak solution of (4.62). Additionally, Theorem 2.10 implies $ru_n \to r\bar{u}$ in $L^{p+1}_{\mathrm{cyl}}(rdrdz)$ as $n \to \infty$. We conclude by the mean value theorem

$$\left|\int_\Omega \Gamma(r) r^{p-1}(|u_n|^{p+1} - |\bar{u}|^{p+1}) r^3 d(r,z)\right| \leq \|\Gamma\|_{L^\infty} (p+1) \int_\Omega |\xi_n^p(r,z)||ru_n(r,z) - r\bar{u}(r,z)| r d(r,z) \to 0$$

as $n \to \infty$ where $\xi_n(r,z)$ is between $ru_n(r,z)$ and $r\bar{u}(r,z)$ and ξ_n is uniformly bounded in $L^{p+1}_{\mathrm{cyl}}(rdrdz)$. Hence,

$$J(u_n) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega \Gamma(r) r^{p-1} |u_n|^{p+1} r^3 d(r,z) \to \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega \Gamma(r) r^{p-1} |\bar{u}|^{p+1} r^3 d(r,z) \text{ as } n \to \infty.$$

Since $\bar{u}$ is a weak solution of (4.62) we deduce

$$\int_\Omega \Gamma(r) r^{p-1} |\bar{u}|^{p+1} r^3 d(r,z) = \int_\Omega \left(|\nabla_{r,z}\bar{u}|^2 + V(r)\bar{u}^2\right) r^3 d(r,z).$$

Therefore also

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega \left(|\nabla_{r,z} u_n|^2 + V(r) u_n^2\right) r^3 d(r,z) \to \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega \left(|\nabla_{r,z}\bar{u}|^2 + V(r)\bar{u}^2\right) r^3 d(r,z)$$

as $n \to \infty$, i.e., $u_n \to \bar{u}$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ as $n \to \infty$. In summary, due to $J(u_n) = J(u_m)$ for all $n, m \in \mathbb{N}$ we obtain that $J(u_n) = J(\bar{u})$ holds true. So $\bar{u}$ is also a ground state to (4.62). Thus $\bar{u}$ is an accumulation point of ground states which contradicts Lemma 4.27. Consequently, the number of ground states is finite. $\square$

A. Appendix to part I

The content of the appendix of part one is split in two parts. We first investigate basic aspects of the cylindrical Laplacian and determine the spectrum. The second part is devoted to regularity questions in cylindrical spaces and related issues.

A.1. The cylindrical Laplacian

We start with a statement which clarifies the selfadjointness and the esssential spectrum of the cylindrical Laplacian. We then extend this to certain differential operators appearing throughout this thesis.

Theorem A.1. *Let*

$$-\Delta_{5,\mathrm{cyl}}\colon D(-\Delta_{5,\mathrm{cyl}}) \subset L^2_{\mathrm{cyl}}(r^3drdz) \to L^2_{\mathrm{cyl}}(r^3drdz);\; -\Delta_{5,\mathrm{cyl}}u = -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2}.$$

Then $-\Delta_{5,\mathrm{cyl}}$ is selfadjoint and $\sigma(-\Delta_{5,\mathrm{cyl}}) = \sigma_{\mathrm{ess}}(-\Delta_{5,\mathrm{cyl}}) = [0,\infty)$ in both of the following cases:

(a) $D(-\Delta_{5,\mathrm{cyl}}) = H^2_{\mathrm{cyl}}(r^3drdz)$,

(b) $D(-\Delta_{5,\mathrm{cyl}}) = \left\{u \in H^2_{\mathrm{cyl}}(r^3drdz) : u \text{ is symmetric about } \{z=0\}\right\}$*, where the symmetry about* $\{z = 0\}$ *is also incorporated in* $L^2_{\mathrm{cyl}}(r^3drdz)$.

Proof. The proof is done in two steps. The first one is to show the selfadjointness of $-\Delta_{5,\mathrm{cyl}}$ in both cases. The selfadjointness in case (a) is provided by Lemma 11 in [5].

In case (b) we also deduce selfadjointness of $-\Delta_{5,\mathrm{cyl}}$, since the operator respects the additional symmetry about $\{z = 0\}$, i.e., $-\Delta_{5,\mathrm{cyl}}u \in L^2_{\mathrm{cyl}}(r^3drdz)$ and $-\Delta_{5,\mathrm{cyl}}u$ is symmetric about $\{z = 0\}$ in case of $u \in D(-\Delta_{5,\mathrm{cyl}})$.

We now turn to the claim concerning the spectrum of $-\Delta_{5,\mathrm{cyl}}$. The idea is to show that for the radial Laplacian $-\Delta_{5,\mathrm{rad}} := -\frac{1}{\rho^4}\frac{\partial}{\partial\rho}(\rho^4\frac{\partial}{\partial\rho})$ where $\rho := |x| \in [0,\infty)$, we have $\sigma(-\Delta_{5,\mathrm{rad}}) = \sigma_{\mathrm{ess}}(-\Delta_{5,\mathrm{rad}}) = [0,\infty)$. Therefore, we make use of the fact that the Laplacian $-\Delta_5$ without any further symmetries satisfies $\sigma(-\Delta_5) = \sigma_{\mathrm{ess}}(-\Delta_5) = [0,\infty)$, see Theorem 7.6 in [41]. Notice that $-\Delta_{5,\mathrm{rad}}$ as well as $-\Delta_{5,\mathrm{cyl}}$ do not possess eigenvalues since $-\Delta_5$ only has essential spectrum. The claim of Theorem A.1 then follows since $D(-\Delta_{5,\mathrm{rad}}) \subseteq D(-\Delta_{5,\mathrm{cyl}}) \subseteq D(-\Delta_5)$.

It remains to show $\sigma(-\Delta_{\mathrm{rad}}) = [0,\infty)$. Since $-\Delta_{5,\mathrm{rad}}$ is a positive operator we conclude $\sigma(-\Delta_{5,\mathrm{rad}}) \subseteq [0,\infty)$. We now show $[0,\infty) \subseteq \sigma(-\Delta_{5,\mathrm{rad}})$. We take Weyl sequences for the Laplacian in five dimensions and transfer them to Weyl sequences for the radial Laplacian in five dimensions which then shows $\sigma(-\Delta_{5,\mathrm{rad}}) \subseteq [0,\infty)$. Let $\lambda > 0, u \in C_c^\infty(\mathbb{R}^5)$ and $f := -\Delta u - \lambda u$. We set

$$\bar{u}(\rho) := \frac{1}{\omega_5\rho^4}\int_{S_\rho(0)} u(x)d\sigma_x = \frac{1}{\omega_5}\int_{S_1(0)} u(\rho x)d\sigma_x,$$

where $\omega_n = |\mathbb{S}^{n-1}|$. In particular,

$$\bar{u}'(\rho) = \frac{1}{\omega_5}\int_{S_1(0)} \nabla u(\rho x)\cdot x d\sigma_x = \frac{1}{\omega_5 r^5}\int_{S_\rho(0)} \nabla u(x)\cdot x d\sigma_x = \frac{1}{\omega_5 r^4}\int_{B_\rho(0)} \Delta u(x)dx. \tag{A.1}$$

Multiplying f with $\frac{1}{\omega_5\rho^4}$ and integrating over S_ρ we obtain

$$\frac{1}{\omega_5\rho^4}\int_{S_\rho} (-\Delta u(x) - \lambda u(x))\, d\sigma_x = \frac{1}{\omega_5\rho^4}\int_{S_\rho} f(x)d\sigma_x.$$

From (A.1) we deduce

$$\bar{u}''(\rho) = -\frac{4}{\omega_5\rho^5}\int_{B_\rho(0)} \Delta u(x)dx + \frac{1}{\omega_5\rho^4}\int_{S_\rho(0)} \Delta u(x)dx,$$

i.e.,

$$-\frac{1}{\omega^5\rho^4}\int_{S_\rho(0)} \Delta u(x)dx = -\bar{u}''(\rho) - \frac{4}{\omega_5\rho^5}\int_{B_\rho(0)} \Delta u(x)dx = -\bar{u}''(\rho) - \frac{4}{\rho}\bar{u}'(\rho) = -\frac{1}{\rho^4}\left(\rho^4\bar{u}'(\rho)\right)'. \tag{A.2}$$

Hence, $\bar{u}$ satisfies

$$-\frac{1}{\rho^4}\left(\rho^4\bar{u}'(\rho)\right)' - \lambda\bar{u}(\rho) = \bar{f}, \tag{A.3}$$

where $\bar{f}(\rho) := \frac{1}{\omega_5\rho^4}\int_{S_\rho(0)} f(x)d\sigma_x$. We now consider Weyl sequences for $-\Delta_5$ with $\lambda \geq 0$. A Weyl sequence $(u_n)_{n\in\mathbb{N}}$ for $-\Delta_5$ satisfies $f_n := -\Delta u_n - \lambda u_n \to 0$ in $L^2(\mathbb{R}^5)$. We already proved that $\bar{u}_n$ satisfies (A.3) with $\bar{u}$ replaced by $\bar{u}_n$ and $\bar{f}$ replaced by $\bar{f}_n$. We now show that $\bar{f}_n \to 0$ in $L^2([0,\infty))$. Indeed, due to (A.3) and (A.2) we compute

$$\begin{aligned}
\|\bar{f}_n\|^2_{L^2([0,\infty))} &= \omega_5\int_0^\infty \bar{f}_n(\rho)^2\rho^4 d\rho = \omega_5\int_0^\infty \left(-\frac{1}{\rho^4}\left(\rho^4\bar{u}'(\rho)\right)' - \lambda\bar{u}(\rho)\right)^2\rho^4 d\rho\\
&= \frac{1}{\omega_5}\int_0^\infty \frac{1}{\rho^4}\left(\int_{S_\rho(0)} (-\Delta u_n(x) - \lambda u_n(x))d\sigma_x\right)^2 d\rho\\
&\leq \frac{1}{\omega_5}\int_0^\infty \frac{1}{\rho^4}|S_\rho(0)|\int_{S_\rho(0)} (-\Delta u_n(x) - \lambda u_n(x))^2 d\sigma_x d\rho\\
&= \int_0^\infty\int_{S_\rho(0)} (-\Delta u_n(x) - \lambda u_n(x))^2\, d\sigma_x d\rho = \|-\Delta u_n - \lambda u_n\|^2_{L^2(\mathbb{R}^5)} \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Since $\lambda \geq 0$ was arbitrary this calculation shows $\sigma(-\Delta_{5,\mathrm{rad}}) \subseteq [0,\infty)$ and the proof is done. □

Corollary A.2. *The operator $L := -\Delta_{5,\mathrm{cyl}} + V(r) - p\Gamma(r)r^{p-1}u^{p-1}$ is selfadjoint where u is a positive ground state solution of* (3.1)*, V and Γ satisfy the assumptions at the beginning of Chapter 3 and $D(L)$ is one of the two options from Theorem A.1. Furthermore, we have $\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(-\Delta_{5,\mathrm{cyl}} + V(r))$ and $\sigma(-\Delta_{5,\mathrm{cyl}} + V(r)) \subseteq [\operatorname{ess\,inf} V, \infty)$.*

Proof. The selfadjointness of L in both cases follows directly from Theorem 8.10 in [32] since L differs from $-\Delta_{5,\mathrm{cyl}}$ only by a multiplication operator which is bounded and symmetric with respect to the $L^2(r^3drdz)$- scalar product.

The conclusion $\sigma_{\mathrm{ess}}(-\Delta_{5,\mathrm{cyl}} + V(r)) = \sigma_{\mathrm{ess}}(-\Delta_{5,\mathrm{cyl}} + V(r) - p\Gamma(r)r^{p-1}u^{p-1})$ holds true since in both cases the term $-p\Gamma(r)r^{p-1}u^{p-1}$ is a compact perturbation (remember the exponential decay of positive solutions of (3.1)) and the essential spectrum is stable under such perturbations (see Section XIII.4 in [61]). The last statement follows due to Therorem A.1 and the boundedness of V. □

A.2. Regularity in a cylindrical framework and the operator $-\Delta_{3,\mathrm{cyl}} + \frac{1}{r^2}$

In this section we collect several basic statements which allow us to switch between different settings, namely vector-valued equations in $\mathbb{R}^3$ and a scalar equation in cylindrical coordinates. Once this is done, we recall aspects in L^p-regularity theory and transfer this to our cylindrical framework. We close this section by investigating one of the appearing operators in the cylindrical setting, precisely the operator $-\Delta_{3,\mathrm{cyl}} + \frac{1}{r^2}$. For this purpose, let

$$\mathcal{H}^1_{\mathrm{cyl}}(rdrdz) := \Big\{ v \in H^1_{\mathrm{cyl}}(rdrdz) : \int_\Omega \frac{v^2}{r^2} rd(r,z) < \infty \Big\}.$$

The first lemma highlights connections between three different spaces.

Lemma A.3. *Let $U\colon \mathbb{R}^3 \to \mathbb{R}^3, u\colon \Omega \to \mathbb{R}, \tilde{u}\colon \Omega \to \mathbb{R}$ be related as follows:*

$$U(x) = u(r,z)(-x_2, x_1, 0)^T = \frac{\tilde{u}(r,z)}{r}(-x_2, x_1, 0)^T \text{ for } x = (x_1, x_2, x_3), r = \sqrt{x_1^2 + x_2^2}, z = x_3. \tag{A.4}$$

Then the following statements are equivalent:

(a) $U \in H^1(\mathbb{R}^3)$,

(b) $u \in H^1_{\mathrm{cyl}}(r^3drdz)$,

(c) $\tilde{u} \in \mathcal{H}^1_{\mathrm{cyl}}(rdrdz)$.

Moreover, we have

$$\begin{aligned} \|U\|_{L^2(\mathbb{R}^3)} &= \|ru\|_{L^2_{cyl}(r^3drdz)} = \|\tilde{u}\|_{L^2_{\mathrm{cyl}}(rdrdz)} \text{ and} \\ \|U\|^2_{H^1(\mathbb{R}^3)} &= 2\pi \int_\Omega \left(|\nabla_{r,z}u|^2 + |u|^2 \right) r^3 d(r,z) = 2\pi \int_\Omega \left(|\nabla_{r,z}\tilde{u}|^2 + \frac{\tilde{u}^2}{r^2} + \tilde{u}^2 \right) rd(r,z). \end{aligned} \tag{A.5}$$

Proof. (a)⇔(b): This is included in Lemma 10 in [5].

(b)⇔(c): Obviously we have $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ if and only if $\frac{\tilde{u}}{r} \in H^1_{\mathrm{cyl}}(r^3drdz)$, i.e., if and only if

$$\partial_r \left(\frac{\tilde{u}}{r} \right) = \frac{\tilde{u}_r}{r} - \frac{\tilde{u}}{r^2}, \frac{\tilde{u}_z}{r}, \frac{\tilde{u}}{r} \in L^2_{\mathrm{cyl}}(r^3drdz).$$

Trivially, $\frac{\tilde{u}}{r} \in L^2_{\mathrm{cyl}}(r^3drdz)$ iff $\tilde{u} \in L^2_{\mathrm{cyl}}(rdrdz)$ and $\frac{\tilde{u}_z}{r} \in L^2_{\mathrm{cyl}}(r^3drdz)$ iff $\tilde{u}_z \in L^2_{\mathrm{cyl}}(rdrdz)$. Hence, it remains to show

$$\tilde{u}_r - \frac{\tilde{u}}{r} \in L^2_{\mathrm{cyl}}(rdrdz) \text{ if and only if } \tilde{u}_r, \frac{\tilde{u}}{r} \in L^2_{\mathrm{cyl}}(rdrdz).$$

The direction from right to left is clear. Vice versa, let $\tilde{u}_r - \frac{\tilde{u}}{r} \in L^2_{\mathrm{cyl}}(rdrdz)$. We use Hardy's inequality (2.13) to obtain

$$\int_\Omega \frac{\tilde{u}^2}{r^2} rd(r,z) = \int_\Omega \frac{u^2}{r^2} r^3 d(r,z) \le C \int_\Omega |\nabla_{r,z}u|^2 r^3 d(r,z) = C \int_\Omega \left(\left(\tilde{u}_r - \frac{\tilde{u}}{r} \right)^2 + \tilde{u}_z^2 \right) rd(r,z) < \infty.$$

Moreover,

$$\int_\Omega \tilde{u}_r^2 r d(r,z) = \int_\Omega \left(\tilde{u}_r - \frac{\tilde{u}}{r} + \frac{\tilde{u}}{r}\right)^2 r d(r,z) \leq 2\int_\Omega \left(\left(\tilde{u}_r - \frac{\tilde{u}}{r}\right)^2 + \left(\frac{\tilde{u}}{r}\right)^2\right) r d(r,z) < \infty$$

and the proof of the three equivalences is done.
The statement for the L^2-norms in (A.5) is clear. On the level of H^1-norms we formally calculate

$$\begin{aligned} \nabla U_1(x) &= \left(-\frac{x_1 x_2}{r^2}\tilde{u}_r + \frac{x_1 x_2}{r^3}\tilde{u}, -\frac{x_2^2}{r^2}\tilde{u}_r - \frac{r - x_2^2/r}{r^2}\tilde{u}, -\frac{x_2}{r}\tilde{u}_z\right)^T, \\ \nabla U_2(x) &= \left(\frac{r - x_1^2/r}{r^2}\tilde{u} + \frac{x_1^2}{r^2}\tilde{u}_r, \frac{x_1 x_2}{r^2}\tilde{u}_r - \frac{x_1 x_2}{r^3}\tilde{u}, \frac{x_1}{r}\tilde{u}_z\right)^T. \end{aligned} \tag{A.6}$$

Hence,

$$|\nabla U_1|^2 + |\nabla U_2|^2 = \frac{x_1^2}{r^4}\tilde{u}^2 + \frac{x_2^2}{r^2}\tilde{u}_r^2 + \frac{x_2^2}{r^2}\tilde{u}_z^2 + \frac{x_2^2}{r^4}\tilde{u}^2 + \frac{x_1^2}{r^2}\tilde{u}_r^2 + \frac{x_1^2}{r^2}\tilde{u}_z^2 = \tilde{u}_r^2 + \tilde{u}_z^2 + \frac{\tilde{u}^2}{r^2}. \tag{A.7}$$

which proves

$$\|\nabla U\|_{L^2(\mathbb{R})^3}^2 = 2\pi \int_\Omega \left(|\nabla_{r,z}\tilde{u}|^2 + \frac{\tilde{u}^2}{r^2}\right) r d(r,z).$$

In the same spirit

$$\begin{aligned} \nabla U_1(x) &= \left(-\frac{x_1 x_2}{r}u_r, -u - \frac{x_2^2}{r}u_r, -x_2 u_z\right)^T, \\ \nabla U_2(x) &= \left(\frac{x_1^2}{r}u_r + u, \frac{x_1 x_2}{r}u_r, x_1 u_z\right)^T. \end{aligned}$$

Thus,

$$|\nabla U_1(x)|^2 + |\nabla U_2(x)|^2 = r^2 u_r^2 + r^2 u_z^2 + 2(u^2 + r u u_r).$$

Notice that

$$\int_\Omega 2(u^2 + r u u_r) r d(r,z) = \int_\Omega \frac{d}{dr}\left(r^2 u^2\right) d(r,z) = 0$$

and therefore

$$\int_{\mathbb{R}^3} |\nabla U|^2 dx = 2\pi \int_\Omega (r^2 u_r^2 + r^2 u_z^2) r d(r,z) = 2\pi \int_\Omega |\nabla_{r,z} u|^2 r^3 d(r,z)$$

which finally establishes (A.5). □

Lemma A.3 entails the following result.

Corollary A.4. *$\mathcal{H}^1_{\mathrm{cyl}}(rdrdz)$ is a Hilbert space with respect to*

$$\langle f, g\rangle_{\mathcal{H}^1_{\mathrm{cyl}}} := \int_\Omega \left(\nabla_{r,z} f \cdot \nabla_{r,z} g + fg + \frac{fg}{r^2}\right) r d(r,z) \text{ for } f, g \in \mathcal{H}^1_{\mathrm{cyl}}(rdrdz).$$

Proof. Let $(\tilde{u}_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}^1_{\mathrm{cyl}}(rdrdz)$. We have to show that there is $\tilde{u} \in \mathcal{H}^1_{\mathrm{cyl}}(rdrdz)$ such that $\left\langle \tilde{u}_j - \tilde{u}, \tilde{u}_j - \tilde{u} \right\rangle_{\mathcal{H}^1_{\mathrm{cyl}}} \to 0$ as $j \to \infty$. By Lemma A.3, in particular (A.5) we infer that $\left(\frac{\tilde{u}_j}{r}(-x_2, x_1, 0)^T\right)_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\mathbb{R}^3)$. Hence, by pointwise almost everywhere identification along a subsequence there is $\tilde{u}$ such that $\frac{\tilde{u}_j}{r}(-x_2, x_1, 0)^T \to \frac{\tilde{u}}{r}(-x_2, x_1, 0)^T$ in $H^1(\mathbb{R}^3)$ as $j \to \infty$. Thus, Lemma A.3 yields $\tilde{u} \in \mathcal{H}^1_{\mathrm{cyl}}(rdrdz)$ and (A.5) gives $\left\langle \tilde{u}_j - \tilde{u}, \tilde{u}_j - \tilde{u} \right\rangle_{\mathcal{H}^1_{\mathrm{cyl}}} \to 0$ as $j \to \infty$. □

Here is a crucial result which together with Lemma A.3 ensures that the concept of a weak solution can be transferred between an $\mathbb{R}^3$-valued equation and its cylindrical counterpart.

Lemma A.5. *Let $\tilde{\Omega} \subseteq \Omega := [0, \infty) \times \mathbb{R}$ and*

$$\tilde{\Omega}_3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (|(x_1, x_2)|, x_3) \in \tilde{\Omega}\}. \tag{A.8}$$

Again, let $U\colon \tilde{\Omega}_3 \to \mathbb{R}^3, u\colon \tilde{\Omega} \to \mathbb{R}, \tilde{u}\colon \tilde{\Omega} \to \mathbb{R}$ be related as in (A.4). *Moreover, let V, Γ be cylindrically symmetric, i.e., $V(x) = V(r, z), \Gamma(x) = \Gamma(r, z)$. Then the following statements are equivalent:*

(a) $U \in H^1_0(\tilde{\Omega}_3)$ is a weak solution of

$$-\Delta U + V(x)U = \Gamma(x)\,|U|^{p-1}\,U \text{ in } \tilde{\Omega}_3, \tag{A.9}$$

(b) $u \in H^1_0(\tilde{\Omega}, r^3drdz)$ is a weak solution of

$$-\partial_r^2 u - \frac{3}{r}\partial_r u - \partial_z^2 u + V(r, z)u = \Gamma(r, z)r^{p-1}\,|u|^{p-1}\,u \text{ in } \tilde{\Omega}, \tag{A.10}$$

(c) $\tilde{u} \in H^1_0(\tilde{\Omega}, rdrdz)$ is a weak solution of

$$-\partial_r^2 \tilde{u} - \frac{1}{r}\partial_r \tilde{u} - \partial_z^2 \tilde{u} + \frac{1}{r^2}\tilde{u} + V(r, z)\tilde{u} = \Gamma(r, z)\,|\tilde{u}|^{p-1}\,\tilde{u} \text{ in } \tilde{\Omega}. \tag{A.11}$$

Proof. (a)⇔(b): Let $U(x) = u(r, z)\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}$ and $\varphi \in C_c^\infty(\tilde{\Omega})$ with $\Phi(x) := \varphi(r, z)(-x_2, x_1, 0)^T$ for $x \in \tilde{\Omega}_3$. By making use of the compact support of φ we obtain

$$\int_{\tilde{\Omega}_3} (2u\varphi + ru\varphi_r + ru_r\varphi)\,dx = 2\pi \int_{\tilde{\Omega}} (2u\varphi + ru\varphi_r + ru_r\varphi)\,rd(r, z) = 2\pi \int_{\tilde{\Omega}} \frac{d}{dr}(r^2 u\varphi)d(r, z) = 0.$$

From (A.6) we infer that

$$\begin{aligned}\int_{\tilde{\Omega}_3} (\nabla U)\cdot(\nabla\Phi)\,dx &= \int_{\tilde{\Omega}_3} \left(r^2 u_r\varphi_r + r^2 u_z\varphi_z + 2u\varphi + ru\varphi_r + ru_r\varphi\right)dx\\ &= 2\pi \int_{\Omega} \nabla_{r,z}u\cdot\nabla_{r,z}\varphi r^3 d(r, z).\end{aligned}$$

Moreover,

$$\int_{\tilde{\Omega}_3} \Gamma(x)\,|U|^{p-1}\,U\cdot\Phi dx = 2\pi \int_{\tilde{\Omega}} \Gamma(r)r^{p-1}\,|u|^{p-1}\,u\varphi r^3 d(r, z)$$

and in the same manner $\int_{\tilde{\Omega}_3} V(x)U \cdot \Phi dx = 2\pi \int_{\tilde{\Omega}} V(r)u\varphi r^3 d(r,z)$ holds. In summary

$$\begin{aligned}\int_{\tilde{\Omega}_3} &\left((\nabla U)\cdot(\nabla\Phi) + VU\cdot\Phi - \Gamma\,|U|^{p-1}\,U\cdot\Phi\right)dx\\ &= 2\pi\int_{\tilde{\Omega}}\left(\nabla_{r,z}u\cdot\nabla_{r,z}\varphi + Vu\varphi - \Gamma r^{p-1}\,|u|^{p-1}\,u\varphi\right)r^3 d(r,z).\end{aligned}$$

(b)⇔(c): Again let $\varphi \in C_c^\infty(\tilde{\Omega})$. For $u = \frac{\tilde{u}}{r}$ and $\tilde{\varphi} := r\varphi \in C_c^\infty(\tilde{\Omega})$ we obtain

$$\begin{aligned}\int_{\tilde{\Omega}}\nabla_{r,z}u\cdot\nabla_{r,z}\varphi r^3 d(r,z) &= \int_{\tilde{\Omega}}\left(\left(\frac{\tilde{u}_r}{r}-\frac{\tilde{u}}{r^2}\right)\left(\frac{\tilde{\varphi}_r}{r}-\frac{\tilde{\varphi}}{r^2}\right)+\frac{\tilde{u}_z\tilde{\varphi}_z}{r^2}\right)r^3 d(r,z)\\ &= \int_{\tilde{\Omega}}\left(\tilde{u}_r\tilde{\varphi}_r + \frac{1}{r^2}\tilde{u}\tilde{\varphi} + \tilde{u}_z\tilde{\varphi}_z\right)rd(r,z) - \int_{\tilde{\Omega}}\left(\tilde{u}_r\tilde{\varphi} + \tilde{\varphi}_r\tilde{u}\right)d(r,z)\\ &= \int_{\tilde{\Omega}}\left(\nabla_{r,z}\tilde{u}\cdot\nabla_{r,z}\tilde{\varphi} + \frac{1}{r^2}\tilde{u}\tilde{\varphi}\right)rd(r,z) - \int_{\tilde{\Omega}}\frac{d}{dr}\left(\tilde{u}\tilde{\varphi}\right)d(r,z) = \int_{\tilde{\Omega}}\left(\nabla_{r,z}\tilde{u}\cdot\nabla_{r,z}\tilde{\varphi} + \frac{1}{r^2}\tilde{u}\tilde{\varphi}\right)rd(r,z),\end{aligned}$$

where we made use of the compact support of $\tilde{\varphi}$ and $\tilde{\varphi}(0,z) = 0$ for $(0,z) \in \tilde{\Omega}$. Trivially,

$$\int_{\tilde{\Omega}} V(r)u\varphi r^3 d(r,z) = \int_{\tilde{\Omega}} V(r)\tilde{u}\tilde{\varphi} rd(r,z),$$
$$\int_{\tilde{\Omega}} \Gamma(r)r^{p-1}\,|u|^{p-1}\,u\varphi r^3 d(r,z) = \int_{\tilde{\Omega}} \Gamma(r)\,|\tilde{u}|^{p-1}\,\tilde{u}\tilde{\varphi} rd(r,z).$$

In summary,

$$\begin{aligned}\int_{\tilde{\Omega}}&\left(\nabla_{r,z}u\cdot\nabla_{r,z}\varphi + V(r)u\varphi - \Gamma(r)r^{p-1}\,|u|^{p-1}\,u\varphi\right)r^3 d(r,z)\\ &= \int_{\tilde{\Omega}}\left(\nabla_{r,z}\tilde{u}\cdot\nabla_{r,z}\tilde{\varphi} + \frac{1}{r^2}\tilde{u}\tilde{\varphi} + V(r)\tilde{u}\tilde{\varphi} - \Gamma(r)\,|\tilde{u}|^{p-1}\,\tilde{u}\tilde{\varphi}\right)rd(r,z)\end{aligned}$$

holds true. □

Theorem A.6. *Let $p \in (1,5)$ and $U \in H^1(\mathbb{R}^3)$ be a weak solution of* (A.9). *Then $U_i \in C^{2,\alpha}(\mathbb{R}^3)$ for all $\alpha \in (0,1)$, $i \in \{1,2,3\}$ and $|\partial^\beta U(x)| \to 0$ as $|x| \to \infty$ for each multi-index $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}_0^3$ with $|\beta| := |\beta_1| + |\beta_2| + |\beta_3| \leq 2$.*

Proof. The idea is to apply Theorem 8.1.1. in [18] to every component of (A.9). Nevertheless we repeat the details suitably adapted to our case.
Rewrite (A.9) as

$$-\Delta U_i + U_i = \Gamma(r)\,|U|^{p-1}\,U_i + (1 - V(r))U_i \quad \text{for } i = 1,2,3.$$

Hence,

$$\mathcal{F}^{-1}\left(\left(1 + 4\pi^2\,|\xi|^2\right)\mathcal{F}U_i\right) = \Gamma(r)\,|U|^{p-1}\,U_i + (1 - V(r))U_i \tag{A.12}$$

in the space of tempered distributions with $\mathcal{F}$ being the Fourier transform and $\mathcal{F}^{-1}$ its inverse.

By our assumption we have $U \in H^1(\mathbb{R}^3)$, in particular $U \in L^s(\mathbb{R}^3)$ for $2 \le s \le 6$ by the Sobolev embedding. Let $2 \le p < q < \infty$. If $U_i \in L^q(\mathbb{R}^3)$ for all $i \in \{1,2,3\}$ we get $|U|^{p-1} U_i \in L^{q/p}(\mathbb{R}^3)$ and by (A.12) we conclude $U_i \in \mathcal{H}^{2,\frac{q}{p}}(\mathbb{R}^3)$, the Bessel potential space. Since $\mathcal{H}^{2,\frac{q}{p}}(\mathbb{R}^3) = W^{2,\frac{q}{p}}(\mathbb{R}^3)$ (cf. Chapter V, Theorem 3 in [66]) we then receive $U_i \in W^{2,\frac{q}{p}}(\mathbb{R}^3)$. Recall the Sobolev embedding

$$W^{2,\frac{q}{p}}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3), \text{ if } r \ge \frac{q}{p} \text{ and } \frac{1}{r} \ge \frac{p}{q} - \frac{2}{3}. \tag{A.13}$$

Now let $(r_j)_{j\in\mathbb{N}_0}$ be defined by

$$\frac{1}{r_j} := p^j \left(\frac{1}{p+1} - \frac{2}{3(p-1)} + \frac{2}{3(p-1)p^j} \right).$$

It holds that $\frac{p-1}{p+1} < \frac{2}{3}$ due to $p < 5$, i.e., $\delta := \frac{2}{3} - \frac{p-1}{p+1} > 0$. This leads to

$$\begin{aligned} \frac{1}{r_{j+1}} - \frac{1}{r_j} &= p^{j+1} \left(\frac{1}{p+1} - \frac{2}{3(p-1)} \right) - p^j \left(\frac{1}{p+1} - \frac{2}{3(p-1)} \right) \\ &= (p-1)p^j \left(\frac{1}{p+1} - \frac{2}{3(p-1)} \right) = -p^j \delta \le -\delta \text{ for } j \in \mathbb{N}_0. \end{aligned}$$

So $\left(\frac{1}{r_j}\right)_{j\in\mathbb{N}_0}$ is a strictly decreasing sequence with $\frac{1}{r_j} \to -\infty$ for $j \to \infty$. For $j = 0$ we get $\frac{1}{r_0} = \frac{1}{p+1}$, so $r_0 > 0$. Herewith there is $k \in \mathbb{N}_0$ with $\frac{1}{r_l} > 0$ for all $0 \le l \le k$ and $\frac{1}{r_{k+1}} \le 0$.

Let $i \in \{1,2,3\}$. Our next step is to show $U_i \in L^{r_k}(\mathbb{R}^3)$. We already know $U_i \in L^{r_0}(\mathbb{R}^3)$ so we are done if we can show $U_i \in L^{r_{l+1}}(\mathbb{R}^3)$ provided $U_i \in L^{r_l}(\mathbb{R}^3)$ for $l \le k-1$. Assume $U_i \in L^{r_l}(\mathbb{R}^3)$ for some $l \le k-1$. We have

$$\frac{p}{r_l} - \frac{2}{3} = pp^j \left(\frac{1}{p+1} - \frac{2}{3(p-1)} + \frac{2}{3(p-1)p^j} \right) - \frac{2}{3} = \frac{1}{r_{l+1}} + \frac{2p}{3(p-1)} - \frac{2}{3(p-1)} - \frac{2}{3} = \frac{1}{r_{l+1}},$$

so by (A.13) we get $U_i \in L^r(\mathbb{R}^3)$ for all $r \ge \frac{r_l}{p}$ with $\frac{1}{r} \ge \frac{p}{r_l} - \frac{2}{3} = \frac{1}{r_{l+1}}$. In particular $U_i \in L^{r_{l+1}}(\mathbb{R}^3)$ since

$$\frac{r_{l+1}}{r_l} = \frac{1}{p}\left(1 + \frac{2}{3} r_{l+1}\right) \ge \frac{1}{p} \text{ due to } l \le k-1 \text{ and } r_{l+1} > 0.$$

Altogether $U_i \in L^{r_k}(\mathbb{R}^3)$. Applying the Sobolev embedding once more yields $U_i \in L^r(\mathbb{R}^3)$ for all $r \ge \frac{r_k}{p}$ with $\frac{1}{r} \ge \frac{p}{r_k} - \frac{2}{3} = \frac{1}{r_{k+1}} \le 0$. In particular, we may choose $r = \infty$ to get $U_i \in L^\infty(\mathbb{R}^3)$ for arbitrary $i \in \{1,2,3\}$ and so $U \in L^\infty(\mathbb{R}^3)$ and $|U|^{p-1} U_i \in L^q(\mathbb{R}^3)$ for all $2 \le q \le \infty$ and $i \in \{1,2,3\}$.

So far we have shown that the right hand side of (A.12) lies in $L^q(\mathbb{R}^3)$ for all $2 \le q \le \infty$. Hence, we deduce $U_i \in \mathcal{H}^{2,q}(\mathbb{R}^3) = W^{2,q}(\mathbb{R}^3)$ for all $2 \le q \le \infty$. This results in $\Gamma(r)\,|U|^{p-1} U_i + (1 - V(r))U_i \in W^{1,q}(\mathbb{R}^3)$ for all $2 \le q < \infty$ since

$$\|\Gamma(r)\,|U|^{p-1} U_i + (1-V(r))U_i\|_{L^q} \le \|\Gamma\|_{L^\infty} \|U\|_{L^\infty}^{p-1} \|U_i\|_{L^q} + (1 + \|V\|_{L^\infty})\|U_i\|_{L^\infty} < \infty$$

and

$$\|\nabla\Big(\Gamma(r)\,|U|^{p-1} U_i + (1-V(r))U_i\Big)\|_{L^q}$$

$$
\begin{aligned}
&= \| (\nabla\Gamma(r)) \, |U|^{p-1} \, U_i + \Gamma(r)\nabla \left(|U|^{p-1} \, U_i\right) + (1 - V(r))\nabla U_i - (\nabla V(r)) \, U_i\|_{L^q} \\
&\leq \|\partial_r\Gamma\|_\infty \, \|U\|_\infty^{p-1} \, \|U_i\|_q + \|\Gamma\|_\infty \left(\| \, |U|^{p-1} \, \nabla U_i + (p-1) \, |U|^{p-3} \, ((\nabla U)^T \cdot U)U_i\|_q\right) \\
&+ (1 + \|V\|_\infty)\|\nabla U_i\|_q + \|\partial_r V\|_\infty \|U_i\|_q < \infty,
\end{aligned}
$$

since

$$
\| \, |U|^{p-1} \, \nabla U_i + (p-1) \, |U|^{p-3} \, ((\nabla U)^T \cdot U)U_i\|_q \leq \|U\|_\infty^{p-1} \, \|\nabla U_i\|_q + (p-1) \, \|U\|_\infty^{p-1} \, \|\nabla U\|_q < \infty,
$$

where $\|\nabla U\|_{L^q}$ denotes the Frobenius L^q-norm of the matrix ∇U.
In summary, $(-\Delta + \mathrm{Id})U_i \in W^{1,q}(\mathbb{R}^3)$ for $2 \leq q < \infty$, so $(-\Delta + \mathrm{Id})D_j U_i \in L^q(\mathbb{R}^3)$ for $j \in \{1, 2, 3\}$ and $2 \leq q < \infty$, i.e., $\mathcal{F}^{-1}\left((1 + 4\pi^2 \, |\xi|^2)\mathcal{F} D_j U_i\right) \in L^q(\mathbb{R}^3)$ for all $2 \leq q < \infty$. As above we conclude $D_j U_i \in W^{2,q}(\mathbb{R}^3)$ for all $2 \leq q < \infty$. Since $j \in \{1, 2, 3\}$ was arbitrary, we arrive at $U_i \in W^{3,q}(\mathbb{R}^3)$ for all $2 \leq q < \infty$. Due to the Sobolev-Morrey embedding (see for example Theorem 6 (ii) in Chapter 5.6 of [34]), $U_i \in C^{2,\alpha}(\mathbb{R}^3)$ for all $\alpha \in (0, 1)$ and $\partial^\beta U_i(x) \to 0$ as $|x| \to \infty$ for all $\beta \in \mathbb{N}_0^3$ with $|\beta| \leq 2$. Since $i \in \{1, 2, 3\}$ was arbitrary, the proof is finished. □

We transfer the regularity statement from Theorem A.6 to the cylindrical framework.

Lemma A.7. *If $U(x) = u(r, z) \, (-x_2, x_1, 0)^T$ is a weak solution of* (A.9)*, then $u \in C^2([0, \infty) \times \mathbb{R})$, and $\partial^\beta u(r, z) \to 0$ as $r^2 + z^2 \to \infty$ for each multi-index $\beta \in \mathbb{N}_0^2$ with $|\beta| \leq 2$.*

Proof. By Lemma A.5, a weak solution U of (A.9) gives rise to a weak solution $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ of (A.10). On page 14 in [5] the equality of sets

$$
H^1_{\mathrm{cyl}}(r^3 drdz) = \left\{u : (0, \infty) \times \mathbb{R} \to \mathbb{R} : u \circ \Psi \in H^1(\mathbb{R}^5)\right\}, \tag{A.14}
$$

where

$$
\Psi \colon \mathbb{R}^5 \to \mathbb{R}^2, \ (y_1, \ldots, y_5) \mapsto \left(\sqrt{y_1^2 + \cdots + y_4^2}, y_5\right),
$$

is proved. Recall that $\|U\|_{L^\infty(\mathbb{R}^3)} = \|ru\|_{L^\infty([0,\infty)\times\mathbb{R})}$, i.e.,

$$
\|ru\|_{L^\infty([0,\infty)\times\mathbb{R})} < \infty \tag{A.15}
$$

by Theorem A.6. Hence, by the embedding $H^1_{\mathrm{cyl}}(r^3 drdz) \hookrightarrow L^q_{\mathrm{cyl}}(r^3 drdz) \subset L^q(\mathbb{R}^5)$ for all $2 \leq q \leq \frac{10}{3}$, which is a consequence of (A.14), we conclude with the help of (A.15) that the right hand side of (A.10) is an element of $L^q(\mathbb{R}^5)$ for all $2 \leq q \leq \frac{10}{3}$. Consequently, the bootstapping argument already applied in the proof of Theorem A.6 suitably adapted to the scalar case implies $u \in L^\infty(\mathbb{R}^5)$, i.e., $u \in L^\infty([0, \infty) \times \mathbb{R})$. Hence, $u \in W^{2,q}(\mathbb{R}^5)$ for all $q \geq 2$. We now show that the right hand side of (A.10) is in $W^{1,q}(\mathbb{R}^5)$ for all $q \geq 2$. We have

$$
\begin{aligned}
&\|\Gamma(r) r^{p-1} u^p\|_{L^q(\mathbb{R}^5)} \leq C \, \|\Gamma\|_\infty \, \|u\|_{L^q(\mathbb{R}^5)} < \infty, \\
&\|\nabla \left(\Gamma(r) r^{p-1} u^p\right) \|_{L^q(\mathbb{R}^5)} \leq \|\Gamma\|_\infty \, \|\nabla \left(r^{p-1} u^p\right) \|_q + \|\partial_r \Gamma\|_\infty \|r^{p-1} u^p\|_q \\
&= \|\Gamma\|_\infty \, \|p r^{p-1} u^{p-1} \nabla u + (p-1) u^p r^{p-3} \, (y_1, \ y_2, \ y_3, \ y_4, \ 0)^T \, \|_q + \|\partial_r \Gamma\|_\infty \|r^{p-1} u^p\|_q \\
&\leq p\|\Gamma\|_\infty \|r^{p-1} u^{p-1} \nabla u\|_q + (p-1)\|\Gamma\|_\infty \|r^{p-2} u^p\|_q + \|\partial_r \Gamma\|_\infty \|r^{p-1} u^p\|_q.
\end{aligned}
$$

If $p \geq 2$ these summands are finite due to $ru, u \in L^\infty(\mathbb{R}^5)$ and $u, |\nabla u| \in L^q(\mathbb{R}^5)$ for all $q \geq 2$. In case of $p \in (1,2)$ we split the second term into

$$\|r^{p-2}u^p\|_{L^q(\mathbb{R}^5)} \leq \|r^{p-2}u^p\|_{L^q(B_1(0))} + \|r^{p-2}u^p\|_{L^q(\mathbb{R}^5\setminus B_1(0))}.$$

The integral over the unbounded domain is finite due to the exponential decay of u (see Section 3.1). By exploiting (A.15) the integral over the unit ball in $\mathbb{R}^5$ can be estimated via

$$\|r^{p-2}u^p\|^q_{L^q(B_1(0))} \leq C \int_{-1}^{1}\int_0^1 \frac{1}{r^2} r^3 drdz < \infty.$$

In summary, we conclude $u \in W^{3,q}(\mathbb{R}^5)$ for all $q \geq 2$. Using Morrey's embedding theorem we end up with $u \in C^{2,\alpha}(\mathbb{R}^5)$ for all $\alpha \in (0,1)$, in particular, $u \in C^2(\mathbb{R}^5)$.
Exploiting the cylindrically symmetric profile of u, denoted by u_{cyl}, we receive

$$u(y_1,\ldots,y_5) = u_{\text{cyl}}(r,y_5) \text{ for all } (y_1,\ldots,y_5) \in \mathbb{R}^5 \text{ such that } r = \sqrt{y_1^2 + \cdots y_4^2}.$$

In particular, $u(y_1,0,0,0,y_5) = u_{\text{cyl}}(y_1,y_5)$ for $(y_1,y_5) \in [0,\infty)\times\mathbb{R}$, i.e., $u_{\text{cyl}} \in C^2([0,\infty)\times\mathbb{R})$. □

For $A \subseteq \mathbb{R}^3$ and $q \in [1,\infty)$ recall the definitions

$$\|U\|^q_{L^q(A)} := \int_A |U|^q\,dx, \quad \|U\|^q_{W^{1,q}(A)} := \int_A \left(\sum_{i=1}^3 |\nabla U_i(x)|^2 + |U|^2\right)^{q/2} dx,$$
$$\|U\|^q_{W^{2,q}(A)} := \int_A \left(\sum_{i,j,k=1}^3 \left|\frac{D^2 U_i}{\partial x_j \partial x_k}\right|^2 + \sum_{i=1}^3 |\nabla U_i(x)|^2 + |U|^2\right)^{q/2} dx.$$

We now establish a link between the norms of U and $\tilde{u}$.

Lemma A.8. *Let $A_3 \subseteq \mathbb{R}^3$ and $A \subseteq \Omega$ be related as in* (A.8) *and $U(x) = \frac{\tilde{u}(r,z)}{r}(-x_2,x_1,0)^T \in W_0^{k,q}(A_3)$. Then $\tilde{u} \in W^{k,q}(A, rdrdz)$ and*

$$\|\tilde{u}\|_{W^{k,q}(A,rdrdz)} \leq c\,\|U\|_{W^{k,q}(A_3)} \tag{A.16}$$

for all $k \in \{0,1,2\}$ and all $q \in (1,\infty)$ for a constant $c = c(q,k)$ independent of $U \in W_0^{k,q}(A_3)$.

Proof. For the sake of readability we suppress the domain of integration A respectively A_3.
We immediately obtain $\|U\|_{L^q} = \|\tilde{u}\|_{L^q(rdrdz)}$ which proves (A.16) for $k = 0$ with $c(q,0) = 1$ for all $q \in (1,\infty)$. In order to prove (A.16) for $k = 1$ we use (A.7). Altogether, we have

$$\begin{aligned}\|\tilde{u}\|^q_{W^{1,q}(rdrdz)} &\leq C\int \left(|\tilde{u}_r|^2 + |\tilde{u}_z|^2 + \tilde{u}^2\right)^{q/2} rd(r,z)\\ &\leq C\int \left(|\tilde{u}_r|^2 + |\tilde{u}_z|^2 + \frac{1}{r^2}\tilde{u}^2 + \tilde{u}^2\right)^{q/2} rd(r,z) \leq C\,\|U\|^q_{W^{1,q}},\end{aligned}$$

Hence, (A.16) for $k = 1$ is also proved and only the case $k = 2$ is left to show. Basically this is a routine calculation which we carry out next.

In the following, for $i \in \{1,2\}$ and $j \in \{1,2,3\}$ let ∇U_{ij} denote the j-th component of ∇U_i. Since the proof for $k \in \{0,1\}$ is already done we only have to verify

$$\int \left(|\tilde{u}_{rr}|^2 + |\tilde{u}_{rz}|^2 + |\tilde{u}_{zz}|^2\right)^{q/2} rd(r,z) \leq c(q,2) \int \left(\sum_{i,j,k=1}^{3} \left| \frac{D^2 U_i}{\partial x_j \partial x_k} \right|^2 \right)^{q/2} dx. \tag{A.17}$$

Therefore, with the help of (A.6) we calculate $\frac{d}{dx_k}\left(\nabla U_{ij}\right)$ for $i \in \{1,2\}$, $j,k \in \{1,2,3\}$. The result reads as follows:

$$\begin{aligned}
\frac{d}{dx_1}(\nabla U_{11}) &= -\frac{x_1^2 x_2}{r^3}\tilde{u}_{rr} - \frac{x_2 r^2 - 3x_1^2 x_2}{r^4}\tilde{u}_r + \frac{x_2 r^3 - 3x_1^2 x_2 r}{r^6}\tilde{u},\\
\frac{d}{dx_2}(\nabla U_{11}) &= \frac{d}{dx_1}(\nabla U_{12}) = -\frac{x_1 x_2^2}{r^3}\tilde{u}_{rr} + \frac{3x_1 x_2^2 - x_1 r^2}{r^4}\tilde{u}_r + \frac{x_1 r^3 - 3x_1 x_2^2 r}{r^6}\tilde{u},\\
\frac{d}{dx_2}(\nabla U_{12}) &= -\frac{x_2^3}{r^3}\tilde{u}_{rr} + 3\frac{x_2^3 - x_2 r^2}{r^4}\tilde{u}_r + 3\frac{x_2 r^3 - x_2^3 r}{r^6}\tilde{u},\\
\frac{d}{dx_3}(\nabla U_{11}) &= -\frac{d}{dx_3}(\nabla U_{22}) = -\frac{x_1 x_2}{r^2}\tilde{u}_{rz} + \frac{x_1 x_2}{r^3}\tilde{u}_z,\\
\frac{d}{dx_3}(\nabla U_{12}) &= -\frac{x_2^2}{r^2}\tilde{u}_{rz} - \frac{r^2 - x_2^2}{r^3}\tilde{u}_z,\\
\frac{d}{dx_3}(\nabla U_{21}) &= \frac{x_1^2}{r^2}\tilde{u}_{rz} + \frac{r^2 - x_1^2}{r^3}\tilde{u}_z,\\
\frac{d}{dx_1}(\nabla U_{21}) &= \frac{x_1^3}{r^3}\tilde{u}_{rr} + 3\frac{x_1 r^2 - x_1^3}{r^4}\tilde{u}_r + 3\frac{x_1^3 r - x_1 r^3}{r^6}\tilde{u},\\
\frac{d}{dx_2}(\nabla U_{21}) &= \frac{d}{dx_1}(\nabla U_{22}) = \frac{x_1^2 x_2}{r^3}\tilde{u}_{rr} - \frac{3x_1^2 x_2 - x_2 r^2}{r^4}\tilde{u}_r - \frac{x_2 r^3 - 3x_1^2 x_2 r}{r^6}\tilde{u},\\
\frac{d}{dx_2}(\nabla U_{22}) &= \frac{x_1 x_2^2}{r^3}\tilde{u}_{rr} + \frac{x_1 r^2 - 3x_1 x_2^2}{r^4}\tilde{u}_r - \frac{x_1 r^3 - 3x_1 x_2^2 r}{r^6}\tilde{u},\\
\frac{d}{dx_3}(\nabla U_{13}) &= -\frac{x_2}{r}\tilde{u}_{zz},\\
\frac{d}{dx_3}(\nabla U_{23}) &= \frac{x_1}{r}\tilde{u}_{zz}.
\end{aligned}$$

The terms $\frac{d}{dx_1}(\nabla U_{13})$, $\frac{d}{dx_2}(\nabla U_{13})$, $\frac{d}{dx_1}(\nabla U_{23})$ and $\frac{d}{dx_2}(\nabla U_{23})$ are not needed and they do not matter since they only enlarge the right hand side in (A.17). Using the expressions above we infer

$$\left|\frac{d}{dx_3}(\nabla U_{13})\right|^2 + \left|\frac{d}{dx_3}(\nabla U_{23})\right|^2 = \tilde{u}_{zz}.$$

In the same manner,

$$\begin{aligned}
&\left|\frac{d}{dx_3}(\nabla U_{11})\right|^2 + \left|\frac{d}{dx_3}(\nabla U_{12})\right|^2 + \left|\frac{d}{dx_3}(\nabla U_{21})\right|^2 + \left|\frac{d}{dx_3}(\nabla U_{22})\right|^2\\
&\qquad \geq \frac{x_1^4 + 2x_1^2 x_2^2 + x_2^4}{r^4}\tilde{u}_{rz}^2 + \tilde{u}_{rz}\tilde{u}_z \frac{2}{r^4}\left(-\frac{2x_1^2 x_2^2}{r} + r^3 - \frac{x_2^4 + x_1^4}{r}\right) = \tilde{u}_{rz}^2
\end{aligned}$$

where we neglected the positive coefficients for $\tilde{u}_{zz}^2$ in the first estimate. We now guarantee that

$$\sum_{i,j,k=1}^{2} \left| \frac{d}{dx_k} \left(\nabla U_{ij} \right) \right|^2 \geq \tilde{u}_{rr}^2 \tag{A.18}$$

which will then finish our proof. To verify (A.18) we rearrange the eight summands and merge for terms which contain $\tilde{u}_{rr}^2, \tilde{u}_r^2, \tilde{u}^2, \tilde{u}_{rr}\tilde{u}_r, \tilde{u}_{rr}\tilde{u}$ or $\tilde{u}_r\tilde{u}$.
First of all, we look at the coefficient in front of $\tilde{u}_{rr}^2$ which is

$$r^{-6} \left(x_1^4 x_2^2 + 2x_1^2 x_2^4 + x_2^6 + x_1^6 + 2x_1^4 x_2^2 + x_1^2 x_2^4 \right) = r^{-6}(x_1^2 + x_2^2)^3 = 1, \tag{A.19}$$

so this one is fine. We show that the coefficients in front of $\tilde{u}_{rr}\tilde{u}_r$ as well as $\tilde{u}_{rr}\tilde{u}$ vanish. Gathering the terms with $\tilde{u}_{rr}\tilde{u}_r$ we get

$$-6r^{-7}(3x_1^4x_2^2 + 3x_1^2x_2^4 + x_2^6 + x_1^6) + 6r^{-5}(2x_1^2x_2^2 + x_2^4 + x_1^4) = -\frac{6}{r} + \frac{6}{r} = 0.$$

Likewise, for the coefficient corresponding to $\tilde{u}_{rr}\tilde{u}$ we obtain

$$6r^{-8}(3x_1^4x_2^2 + 3x_1^2x_2^4 + x_2^6 + x_1^6) - 6r^{-6}(2x_1^2x_2^2 + x_2^4 + x_1^4) = \frac{6}{r^2} - \frac{6}{r^2} = 0.$$

Next, we calculate the remaining three coefficients. The coefficient for the mixed term $\tilde{u}_r\tilde{u}$ is

$$\begin{aligned}
&2r^{-10}\Big(- r(3x_1^2x_2 - x_2r^2)^2 - 2r(3x_1x_2^2 - x_1r^2)^2 - 9r(x_2r^2 - x_2^3)^2 \\
&- 9r(x_1r^2 - x_1^3)^2 - 2r(x_2r^2 - 3x_1^2x_2)^2 - r(x_1r^2 - 3x_1x_2^2)^2\Big) \\
&= 2r^{-9} \left(-3x_2^2(2x_1^2 - x_2^2)^2 - 9(x_1^2x_2^4 + x_1^4x_2^2) - 3x_1^2(2x_2^2 - x_1^2)^2 \right) \\
&= -6r^{-9}(x_1^2 + x_2^2)^3 = -6r^{-3}.
\end{aligned} \tag{A.20}$$

For the coefficient in front of $\tilde{u}^2$ we derive

$$\begin{aligned}
&3r^{-10} \left(x_2^2(x_2^2 - 2x_1^2)^2 + x_1^2(x_1^2 - 2x_2^2)^2 + 3x_2^2(x_2^2 - r^2)^2 + 3x_1^2(x_1^2 - r^2)^2 \right) \\
&= 3r^{-10} \left(x_1^2 + x_2^2 \right)^3 = 3r^{-4}.
\end{aligned} \tag{A.21}$$

Finally, the coefficent attached to $\tilde{u}_r^2$ simplifies to

$$\begin{aligned}
&3r^{-8} \left(x_2^2(2x_1^2 - x_2^2)^2 + x_1^2(x_1^2 - 2x_2^2)^2 + 3(x_2^2x_1^4 + x_1^2x_2^4) \right) \\
&= 3r^{-8} \left(x_1^2 + x_2^2 \right)^3 = 3r^{-2}.
\end{aligned} \tag{A.22}$$

Uniting the previous calculations in (A.19), (A.20), (A.21) and (A.22) we infer

$$\begin{aligned}
\sum_{i,j,k=1}^{2} \left| \frac{d}{dx_j} (\nabla U_{ik}) \right|^2 &\geq \tilde{u}_{rr}^2 + 3 \left(\frac{1}{r^2}\tilde{u}_r^2 - \frac{2}{r^3}\tilde{u}_r\tilde{u} + \frac{1}{r^4}\tilde{u}^2 \right) \\
&= \tilde{u}_{rr}^2 + \frac{3}{r^2} \left(\tilde{u}_r - \frac{1}{r}\tilde{u} \right)^2 \geq \tilde{u}_{rr}^2
\end{aligned}$$

which finally proves (A.18). □

We now turn to the promised regularity theory for our cylindrical setting. Therefore, recall the classical L^2, respectively L^p-regularity theory, see for instance Theorem 9.11 in [39]. There, an operator of the type

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u + \sum_{i=1}^{n} b^i(x)D_iu + c(x)u$$

is studied and one of the results reads as follows.

Theorem A.9. *Let Ξ be an open set in $\mathbb{R}^n$ and $u \in W^{2,p}_{\text{loc}}(\Xi) \cap L^p(\Xi)$, $p \in (1,\infty)$ be a strong solution of the equation $Lu = f$ in Ξ with $f \in L^p(\Xi)$. Moreover, let there be $\lambda, \Lambda > 0$ such that the coefficients of L satisfy*

$$\begin{aligned} a^{ij} \in C^0(\Xi), b^i, c \in L^\infty(\Xi) \text{ for } i, j = 1, \dots, n, \\ \sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \\ |a^{ij}|, |b^i|, |c| \leq \Lambda \text{ for } i, j = 1, \dots, n. \end{aligned}$$

Then for any domain $\Xi' \subset\subset \Xi$ we have

$$\|u\|_{W^{2,p}(\Xi')} \leq C\left(\|u\|_{L^p(\Xi)} + \|f\|_{L^p(\Xi)}\right).$$

where $C = C(p, \lambda, \Lambda, \Xi', \Xi, \omega(a^{ij}))$ and $\omega(a^{ij})$ denotes the modulus of continuity of the coefficient $a^{ij}, i, j = 1, \dots, n$ on Ξ'.

Next, we give an additional lemma which tells us how to boost weak solutions to $W^{2,q}_{\text{loc}}$-solutions in a general framework.

Lemma A.10. *Let $q \in [2,\infty)$, $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ be a weak solution of*

$$-\Delta u + V(x)u = f.$$

Then $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $\eta \in C^\infty_c(\mathbb{R}^n)$, $0 \leq \eta \leq 1$ denote a cut-off function. Formally, we deduce

$$\Delta(\eta u) = u\Delta\eta + \eta\Delta u + 2\nabla u \cdot \nabla\eta.$$

Thus,

$$-\Delta(\eta u) = u\Delta\eta + f\eta + 2\nabla u \cdot \nabla\eta - V(x)\eta u \in L^2_{\text{loc}}(\mathbb{R}^n). \tag{A.23}$$

Hence, for $A \subset\subset \mathbb{R}^n$ Theorem 8.8 in [39] implies $\eta u \in W^{2,2}(A)$. For $A_0 \subset\subset A$ with $\eta \equiv 1$ on A_0 we then conclude $\nabla u \in W^{1,2}(A_0) \hookrightarrow L^{\frac{2n}{n-2}}(A_0)$. Hence, the right hand side of (A.23) is in $L^{\frac{2n}{n-2}}_{\text{loc}}$ so that Theorem 9.15 in [39] implies $\eta u \in W^{2,\frac{2n}{n-2}}(A_0)$. Hence, for $A_1 \subset\subset A_0$ we have $\nabla u \in W^{1,\frac{2n}{n-2}}(A_1) \hookrightarrow L^{\frac{2n}{n-4}}(A_1)$. Thus, we can bound the right hand side of (A.23) in $L^{\frac{2n}{n-4}}(A_1)$, i.e., $\eta u \in W^{2,\frac{2n}{n-4}}(A_1)$. Repeating these steps finitely many times and using $W^{1,\frac{2n}{n-2r}}(A_r) \hookrightarrow L^{\frac{2n}{n-(2r+2)}}(A_r)$ for $r \in \mathbb{N}$ we obtain an embedding in $L^\infty(A_r)$ once $2r + 2 > n$. In particular, we have $\eta u \in L^q_{\text{loc}}(\mathbb{R}^n)$ so that $\eta u \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$ which finally entails $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$. This finishes the proof. $\square$

Analogously to Theorem A.9 we derive the following result:

Lemma A.11. *Let $q \in [1,\infty)$, $\tilde f \in L^q_{\mathrm{cyl,loc}}(rdrdz)$. Consider*

$$-\partial_r^2\tilde u - \frac{1}{r}\partial_r\tilde u - \partial_z^2\tilde u + \frac{1}{r^2}\tilde u = \tilde f \text{ in } \Omega \tag{A.24}$$

and let $\tilde u \in H^1_{\mathrm{cyl}}(rdrdz)$ be a weak solution of (A.24)*. Then $\tilde u \in W^{2,q}_{\mathrm{cyl,loc}}(rdrdz)$. Moreover, for compact sets $K' \subset\subset K \subset\subset \Omega$ there is $C = C(K', K, q) > 0$ such that*

$$\|\tilde u\|_{W^{2,q}(K',rdrdz)} \le C(K', K, q)\left(\|\tilde u\|_{L^q(K,rdrdz)} + \|\tilde f\|_{L^q(K,rdrdz)}\right). \tag{A.25}$$

Proof. By Lemma A.5 we set $U(x) = \frac{\tilde u}{r}(-x_2, x_1, 0)^T$, $F = \frac{\tilde f}{r}(-x_2, x_1, 0)^T$ and deduce $U \in H^1(\mathbb{R}^3)$, $F \in L^q(\mathbb{R}^3)$ as well as

$$-\Delta U = \nabla \times \nabla \times U = F \text{ in } \mathbb{R}^3$$

weakly. By Theorem A.9 (applied to each component of U) we obtain $U_i \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for $i \in \{1,2,3\}$ and

$$\|U_i\|_{W^{2,q}(K')} \le C(K', K, q)\left(\|U_i\|_{L^q(K)} + \|F_i\|_{L^q(K)}\right) \tag{A.26}$$

for compact sets $K' \subset\subset K \subset\subset \mathbb{R}^3$, all $i \in \{1,2,3\}$ and $C = C(K', K, q) > 0$. Let $\tilde K', \tilde K \subset \Omega$ denote the cylindrical counterpart of $K', K \subset \mathbb{R}^3$. Then the combination of (A.26) and Lemma A.8 implies

$$\begin{aligned}\|\tilde u\|_{W^{2,q}(\tilde K',rdrdz)} &\le C(q,2)\|U_i\|_{W^{2,q}(K')} \le C(q,2)C(K',K,q)\left(\|U_i\|_{L^q(K)} + \|F_i\|_{L^q(K)}\right)\\ &= C(q,2)C(K',K,q)\left(\|\tilde u\|_{L^q(\tilde K,rdrdz)} + \|\tilde f\|_{L^q(\tilde K,rdrdz)}\right)\end{aligned}$$

which verifies (A.25). □

Finally, let $R > 0$ and set $A_{R,2R} := \{(r,z) \in \Omega : R < \sqrt{r^2+z^2} < 2R\}$. We investigate the operator

$$-\Delta_{3,\mathrm{cyl}} + \frac{1}{r^2} : \mathcal{H}^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz) \subset \mathcal{H}^{-1}_{\mathrm{cyl}}(A_{R,2R}, rdrdz) \to \mathcal{H}^{-1}_{\mathrm{cyl}}(A_{R,2R}, rdrdz), \tag{A.27}$$

where

$$\mathcal{H}^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz) := \left\{v \in H^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz) : \int_{A_{R,2R}} \frac{v^2}{r^2} r d(r,z) < \infty\right\}$$

and $\mathcal{H}^{-1}_{\mathrm{cyl}}(A_{R,2R}, rdrdz) := \mathcal{H}^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz)'$ denotes the dual space of $\mathcal{H}^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz)$, recall Definition 1.5.

Lemma A.12. *The operator in* (A.27) *has positive discrete spectrum $(\lambda_i(A_{R,2R}))_{i\in\mathbb{N}}$ and $\lambda_i(A_{R,2R}) = \frac{1}{R^2}\lambda_i(A_{1,2})$ for all $i \in \mathbb{N}$.*

Proof. For $u, v \in \mathcal{H}^1_{0,\mathrm{cyl}}(A_{R,2R}, rdrdz)$ we consider

$$B(u,v) := \int_{A_{R,2R}} \left(\nabla_{r,z}u \cdot \nabla_{r,z}v + \frac{uv}{r^2}\right) rd(r,z).$$

Moreover, for $f \in L^2_{\text{cyl}}(A_{R,2R}, rdrdz)$ and $v \in \mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz)$ set

$$F_f(v) := \int_{A_{R,2R}} fvrd(r,z).$$

Then due to the Poincaré inequality on the bounded domain $A_{R,2R}$ and the Lemma of Lax-Milgram we deduce that $B(u,v) = F_f(v)$ has a unique solution $u \in \mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz)$. Therefore, we can define the operator $\tilde{K}\colon L^2_{\text{cyl}}(A_{R,2R}, rdrdz) \to \mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz), f \mapsto u$. Due to the compact embedding $\mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz) \hookrightarrow L^2_{\text{cyl}}(A_{R,2R}, rdrdz)$ we conclude that $K\colon L^2_{\text{cyl}}(A_{R,2R}, rdrdz) \to L^2_{\text{cyl}}(A_{R,2R}, rdrdz)$, $f \mapsto \tilde{K}f$ is compact. Moreover, K is symmetric w.r.t. $\langle \cdot, \cdot \rangle_{L^2(A_{R,2R}, rdrdz)}$ so that K possesses discrete spectrum $(\mu_i)_{i\in\mathbb{N}}$ with zero being the only accumulation point of eigenvalues. Note that zero is no eigenvalue of K. Otherwise, by definition of K there would be $\varphi \not\equiv 0$ such that

$$0 = \int_{A_{R,2R}} \varphi vrd(r,z) \text{ for all } v \in \mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz),$$

a contradiction. We denote the eigenfunctions of K by $(\varphi_i)_{i\in\mathbb{N}}$. Notice that $\varphi_i \in \mathcal{H}^1_{0,\text{cyl}}(A_{R,2R}, rdrdz)$ due to $\tilde{K}\varphi_i = K\varphi_i = \mu_i\varphi_i$. Then again by definition of K we deduce

$$\left(-\Delta_{3,\text{cyl}} + \frac{1}{r^2}\right)\varphi_i = \frac{1}{\mu_i}\varphi_i \text{ on } A_{R,2R} \tag{A.28}$$

so that the eigenvalues of $-\Delta_{3,\text{cyl}} + \frac{1}{r^2}$ are $\lambda_i(A_{R,2R}) := \frac{1}{\mu_i}$. By testing (A.28) with φ_i we conclude that $\lambda_i > 0$. For the last statement we consider

$$\left(-\Delta_{3,\text{cyl}} + \frac{1}{r^2}\right)\varphi_i = \lambda_i(A_{1,2})\varphi_i \text{ in } A_{1,2}$$

together with Dirichlet boundary conditions on $\partial A_{R,2R}$ for $i \in \mathbb{N}$ and define $w_i(r,z) = \varphi_i(\frac{r}{R}, \frac{z}{R})$ on $A_{R,2R}$. Then

$$\begin{aligned}
&\left(-\partial_r^2 - \frac{1}{r}\partial_r - \partial_z^2 + \frac{1}{r^2}\right) w_i(r,z) \\
&= \frac{1}{R^2}\left(-\partial_r^2\varphi_i\left(\frac{r}{R}, \frac{z}{R}\right) - \frac{R}{r}\partial_r\varphi_i\left(\frac{r}{R}, \frac{z}{R}\right) - \partial_z^2\varphi_i\left(\frac{r}{R}, \frac{z}{R}\right) + \frac{R^2}{r^2}\varphi_i\left(\frac{r}{R}, \frac{z}{R}\right)\right) \\
&= \frac{\lambda_i(A_{1,2})}{R^2}\varphi_i\left(\frac{r}{R}, \frac{z}{R}\right) = \frac{\lambda_i(A_{R,2R})}{R^2} w(r,z),
\end{aligned}$$

i.e., $\lambda_i(A_{R,2R}) = \frac{\lambda_i(A_{1,2})}{R^2}$ for all $i \in \mathbb{N}$. $\square$

Part II.

Polychromatic waves

5. Existence of polychromatic ground states in one dimension

In this chapter we investigate

$$\nabla \times \nabla \times E + V(x)\partial_t^2 E = \Gamma |E|^{p-1} E \text{ in } \mathbb{R}^3 \tag{5.1}$$

for $p \in (1, \frac{5}{3})$, a constant $\Gamma > 0$ and a potential $V\colon \mathbb{R}^3 \to \mathbb{R}$. In the previous chapters our approach consisted in a monochromatic ansatz $E(x,t) = U(x)e^{i\omega t}$ for $U\colon \mathbb{R}^3 \to \mathbb{R}^3$ (compare Chapter 2). In this chapter, we first reduce (5.1) to a scalar nonlinear wave equation by using polarized fields. Then we make a polychromatic ansatz via a Fourier expansion in time. The polarized fields have the form

$$E(x,t) = (0, u(x_1, t), 0)^T \tag{5.2}$$

for $x = (x_1, x_2, x_3)$ and $u\colon \mathbb{R}^2 \to \mathbb{R}$. Plugging (5.2) into (5.1) and abbreviating $x \coloneqq x_1$ we deduce

$$-u_{xx} + V(x)u_{tt} = \Gamma |u|^{p-1} u \text{ for } (x,t) \in \mathbb{R} \times \mathbb{R} \tag{5.3}$$

where also the potential V is assumed to only depend on the one-dimensional parameter x_1. We are looking for weak time-periodic solutions of (5.3) in a sense which will be clarified later in Definition 5.1.

The following calculations are only done on a formal level, we rigorously justify them later. Our ansatz for $u\colon \mathbb{R}^2 \to \mathbb{R}$ reads as follows

$$u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t} \text{ with } \omega > 0,\ \hat{u}_k\colon \mathbb{R} \to \mathbb{C} \text{ and } \overline{\hat{u}_k} = \hat{u}_{-k} \text{ on } \mathbb{R} \text{ for all } k \in \mathbb{Z}_{\text{odd}}. \tag{5.4}$$

Here and throughout this chapter we use the following notation: $\mathbb{Z}_{\text{odd}} \coloneqq 2\mathbb{Z}+1, \mathbb{Z}_{\text{even}} \coloneqq 2\mathbb{Z}$ and $\mathbb{N}_{\text{odd}} \coloneqq 2\mathbb{N}_0 + 1$. For $k_0 \in \mathbb{N}_{\text{odd}}$ we abbreviate $\mathbb{Z}_{\text{odd},k_0} \coloneqq \{k \in \mathbb{Z}_{\text{odd}} \text{ such that } |k| \leq k_0\}$.

Formally,

$$\overline{u(x,t)} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \overline{\hat{u}_k(x)}e^{-ik\omega t} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_{-k}(x)e^{-ik\omega t} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t} = u(x,t),$$

i.e., u is real-valued. Additionally, u from (5.4) is indeed $T \coloneqq \frac{2\pi}{\omega}$-periodic in time since

$$u(x, t+\frac{2\pi}{\omega}) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega(t+\frac{2\pi}{\omega})} = \sum_{k\in\mathbb{Z}_{odd}} \hat{u}_k(x)e^{ik\omega t}e^{2\pi ik} = u(x,t).$$

For $k \in \mathbb{Z}_{\text{odd}}$ we can write $k = 2m+1$ with $m \in \mathbb{Z}_{\text{even}}$. Thus, u is $\frac{T}{2} = \frac{\pi}{\omega}$-antiperiodic in time due to

$$u(x, t+\frac{\pi}{\omega}) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega(t+\frac{\pi}{\omega})} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t}e^{2\pi mi}e^{i\pi} = -u(x,t).$$

Concerning the potential we assume that $V\colon \mathbb{R} \to \mathbb{R}$ is periodic and has the special form

$$V(x) = \alpha + \beta\delta_{\mathrm{per}}(x), \tag{5.5}$$

where δ_{per} denotes a 2π- periodically distributed delta potential. We assume that the delta distribution is not located at the end points 0 and 2π (or integer multiples of 2π) but somewhere in between. We abbreviate the set of the location of delta potentials

$$I_\delta := \{x \in \mathbb{R} : x = \varsigma + 2\pi n : n \in \mathbb{Z}\}.$$

The parameter $\beta \in \mathbb{R}$ refers to the strength of the delta interaction whereas $\alpha \in \mathbb{R}$ is a shift.
We set $D := \mathbb{R} \times [0, T)$. Here is our concept of weak solutions.

Definition 5.1. *We call $u \in L^2(D)$ of the form* (5.4) *a weak T-periodic solution of* (5.3) *if u is T-periodic in the second component and*

$$\sum_{k\in\mathbb{Z}_{\mathrm{odd}}} b_k(\hat{u}_k, \hat{v}_k) = \frac{\Gamma}{T}\int_D |u|^{p-1}u\bar{v}\,d(x,t) \tag{5.6}$$

holds true for all v which have a representation $v(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd},k_0}} \hat{v}_k(x)e^{ik\omega t}$ with $k_0 \in \mathbb{N}_{\mathrm{odd}}$ such that $\hat{v}_k \in H^1(\mathbb{R})$ and $\overline{\hat{v}_k} = \hat{v}_{-k}$ for all $k \in \mathbb{Z}_{\mathrm{odd},k_0}$ where

$$b_k(\hat{u}_k, \hat{v}_k) := \int_{\mathbb{R}} \left(\hat{u}_k'(x)\overline{\hat{v}_k'(x)} - \frac{k^2}{16}\hat{u}_k(x)\overline{\hat{v}_k(x)}\right)dx - k^2\sum_{n\in\mathbb{Z}} \hat{u}_k(\varsigma + 2\pi n)\overline{\hat{v}_k(\varsigma + 2\pi n)}.$$

Now we can state the main result of this chapter.

Theorem 5.2. *Let $p \in (1, \frac{5}{3})$, $\Gamma > 0$ constant and $V\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic δ-potential given by* (5.5) *with $\alpha > 0$ and $\beta = 16\alpha$. Then* (5.3) *possesses a non-trivial weak $8\pi\sqrt{\alpha}$-periodic solution in the sense of Definition 5.1.*

In the following we write $f(x_+) := \lim_{y\searrow x} f(y)$ and $f(x_-) := \lim_{y\nearrow x} f(y)$ for a piecewise continuous function f and $x \in \mathbb{R}$.

Remark 5.3. *A slighlty different way of introducing the concept of a weak solution reads as follows: We call $u \in L^2(D)$ a weak T-periodic solution of* (5.3) *if u is T-periodic in the second component and*

$$\int_D u(-\overline{v_{xx}} + V(x)\overline{v_{tt}})d(x,t) = \Gamma\int_D |u|^{p-1}u\bar{v}\,d(x,t) \tag{5.7}$$

holds true for all v which have a representation $v(x,t) = \sum_{k\in\mathbb{Z}_{\mathrm{odd},k_0}} \hat{v}_k(x)e^{ik\omega t}$ with $\overline{\hat{v}_k} = \hat{v}_{-k}$ for all $k \in \mathbb{Z}_{\mathrm{odd},k_0}$ and

$$\begin{aligned} v_k \in N_k :=& \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R}), f \text{ abs. cont. on } \mathbb{R}, f' \text{ abs. cont. on } \mathbb{R}\setminus I_\delta,\\ & f'(x_+) - f'(x_-) = -k^2 f(x) \text{ for all } x \in I_\delta\} \text{ for all } k \in \mathbb{Z}_{\mathrm{odd}}. \end{aligned}$$

As N_k turns out to be the domain of a self-adjoint operator in $L^2(\mathbb{R})$, the set N_k is dense in $L^2(\mathbb{R})$ for all $k \in \mathbb{Z}_{\mathrm{odd}}$, see Section 5.1. Moreover, we will see that b_k is the associated bilinear form of this operator and that (5.7) *is satisfied if* (5.6) *holds true.*

This chapter is structured as follows: In the next section we briefly recall parts of the general theory of delta point interactions (c.f. [2]) and how domain and spectrum of self-adjoint Schrödinger operators involving delta point potentials can be characterized. In Section 5.2 we specify our operator (in the sense that we choose the parameters from Theorem 5.2) and study the spectrum of this operator. It turns out that 0 is in a spectral gap. More precisely, for every $k \in \mathbb{Z}_{\mathrm{odd}}$ we define a suitable operator L_k which corresponds in a sense to the frequency $ik\omega$ in (5.4) and we guarantee that 0 is in a spectral gap of all these operators $(L_k)_{k \in \mathbb{Z}_{\mathrm{odd}}}$. In Section 5.3 we first perform formal calculations with the polychromatic ansatz which then lead to a Hilbert space in which we seek for appropriate solutions of a Floquet-Bloch transformed variant of (5.3). After having established a functional analytic framework we study the consequences of the *uniform* spectral gap in Section 5.4. The following Section 5.5 is devoted to regularity issues which will allow us to incorporate the nonlinearity in a variational setting. This enables us in Section 5.6 to find solutions of the variant of (5.3) by minimizing a suitable functional on the so-called generalized Nehari manifold. Finally, Section 5.7 ensures that indeed (5.6) is valid which proves Theorem 5.2. In order to keep the presentation comprehensible some technical points are shifted to Appendix B.

5.1. The delta point interaction in one dimension

We consider the one-dimensional differential expression

$$Lu := -u'' + (\tilde{\alpha} + \tilde{\beta}\delta_{\mathrm{per}}(x))u \text{ on } \mathbb{R}, \tag{5.8}$$

where $\tilde{\alpha} \in \mathbb{R}$ and $\tilde{\beta} \in \mathbb{R} \setminus \{0\}$ corresponds to the strength of the delta potential. We always assume that the point interaction is located at $\varsigma \in (0, 2\pi)$ but not on the boundary.

One way to rigorously define (5.8) is to incorporate the action of the delta potential in the domain of a differential expression L on a densely defined subspace of $L^2(\mathbb{R})$. With the notation introduced previously we set

$$\begin{aligned} D(L) := \Big\{ & u \in L^2(\mathbb{R}) : u \text{ abs. cont. on } \mathbb{R}, u' \text{ abs. cont. on } \mathbb{R} \setminus I_\delta, \\ & u'(x_+) - u'(x_-) = \tilde{\beta} u(x) \text{ for all } x \in I_\delta \text{ and } -u'' + \tilde{\alpha} u \in L^2(\mathbb{R}) \Big\}. \end{aligned} \tag{5.9}$$

For $u \in D(L)$ from (5.9) the operator L in (5.8) is self-adjoint by Theorem 1 in [20]. In (5.9) the functions are interpreted in a classical sense. We rewrite the domain of definition in (5.9) by making use of weak derivatives. Here, u'' is not a function anymore but a distribution. Thus,

$$\begin{aligned} D(L) = \{u \in L^2(\mathbb{R}) : Lu \in L^2(\mathbb{R})\} = \Big\{ & u \in H^1(\mathbb{R}), u|_{\varsigma+2\pi n, \varsigma+2\pi(n+1))} \in H^2(\varsigma + 2\pi n, \varsigma + 2\pi(n+1)) \\ & \text{for all } n \in \mathbb{Z}, \sum_{n \in \mathbb{Z}} \|u''\|^2_{L^2(\varsigma+2\pi n, \varsigma+2\pi(n+1))} < \infty, u'(x_+) - u'(x_-) = \tilde{\beta} u(x) \text{ for all } x \in I_\delta \Big\}, \end{aligned}$$

see (3.16) in the bachelor thesis of Martin Belica [8] for the last equality sign. We now introduce the concept of a weak solution of $Lu = f$. For $f \in L^2(\mathbb{R})$ we say that $u \in H^1(\mathbb{R})$ is a weak solution of $Lu = f$ if

$$\int_{\mathbb{R}} (u'(x)\varphi'(x) + \tilde{\alpha} u(x)\varphi(x))\, dx + \tilde{\beta} \sum_{n \in \mathbb{Z}} u(\varsigma + 2\pi n)\varsigma(\varsigma + 2\pi(n+1)) = \int_{\mathbb{R}} f(x)\varphi(x) dx$$

holds true for all $\varphi \in C_c^\infty(\mathbb{R})$. Therefore, for $u, v \in H^1(\mathbb{R})$ we consider the bilinear form b associated to L given by

$$b(u,v) = \int_{-\infty}^{\infty} \left(u'(x)\overline{v'(x)} + \tilde{\alpha} u(x)\overline{v(x)} \right) dx + \tilde{\beta} \sum_{n\in\mathbb{Z}} u(\varsigma + 2\pi n)\overline{v(\varsigma + 2\pi n)}. \tag{5.10}$$

Lemma 5.4. *The bilinear form b in* (5.10) *is well-defined on $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.*

Proof. We only have to treat the term $\sum_{n\in\mathbb{Z}} u(\varsigma + 2\pi n)v(\varsigma + 2\pi n)$. Recall the one dimensional Sobolev embedding $H^1(I) \hookrightarrow C(I)$ for a bounded intervall $I \subset \mathbb{R}$. For $n \in \mathbb{Z}$ and $0 < \varepsilon < \pi$ we infer

$$|u(\varsigma + 2\pi n)| \leq \|u\|_{L^\infty(\varsigma+2\pi n-\varepsilon, \varsigma+2\pi n+\varepsilon)} \leq C_{\text{Sob}} \|u\|_{H^1(\varsigma+2\pi n-\varepsilon,\varsigma+2\pi n+\varepsilon))}$$

and the same for v. Hence, we obtain

$$\begin{aligned} \left| \sum_{n\in\mathbb{Z}} u(\varsigma + 2\pi n)\overline{v(\varsigma + 2\pi n)} \right| &\leq C_{\text{Sob}}^2 \sum_{n\in\mathbb{Z}} \|u\|_{H^1(\varsigma+2\pi n-\varepsilon,\varsigma+2\pi n+\varepsilon))} \|v\|_{H^1(\varsigma+2\pi n-\varepsilon,\varsigma+2\pi n+\varepsilon))} \\ &\leq C_{\text{Sob}}^2 \|u\|_{H^1(\mathbb{R})} \|v\|_{H^1(\mathbb{R})}. \end{aligned}$$

□

It can be shown that bilinear form b and operator L are related via

$$b(u,v) = \langle Lu, v\rangle_{L^2(\mathbb{R})} \text{ for all } u \in D(L), v \in H^1(\mathbb{R}),$$

see Theorem VIII.15 in [60].

In the following, we recall the definition of the discriminant $D(\cdot)$ (compare Chapter 1 and § 2.1 in [31]) and present its precise form when associated to (5.8). The discriminant allows us to gain a sufficient control of $\sigma(L)$.

Definition 5.5. *Consider for $u \in D(L)$ the expression*

$$-u'' + (\tilde{V}(x) - \lambda)u \quad \text{on } \mathbb{R} \tag{5.11}$$

with 2π-periodic potential $\tilde{V}: \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Let $\Lambda_1(\cdot,\lambda), \Lambda_2(\cdot,\lambda)$ be a foundamental system of (5.11) *on* $[0, 2\pi]$ *with* $\Lambda_1(0,\lambda) = 1, \Lambda_1'(0,\lambda) = 0, \Lambda_2(0,\lambda) = 0, \Lambda_2'(0,\lambda) = 1$. *Then*

$$D: \mathbb{R} \to \mathbb{R}, \; D(\lambda) := \Lambda_1(2\pi,\lambda) + \Lambda_2'(2\pi,\lambda)$$

is called the discriminant associated to (5.11).

Lemma 5.6. *The dicriminant $D(\cdot)$ associated to* (5.8) *reads*

$$D(\lambda) = \begin{cases} \frac{\tilde{\beta}}{\sqrt{\lambda-\tilde{\alpha}}} \sin(2\pi\sqrt{\lambda-\tilde{\alpha}}) + 2\cos(2\pi\sqrt{\lambda-\tilde{\alpha}}) & \text{for } \lambda - \tilde{\alpha} > 0, \\ 2 + 2\pi\tilde{\beta} & \text{for } \lambda - \tilde{\alpha} = 0, \\ \frac{\tilde{\beta}}{\sqrt{-(\lambda-\tilde{\alpha})}} \sinh(2\pi\sqrt{-(\lambda-\tilde{\alpha})}) + 2\cosh(2\pi\sqrt{-(\lambda-\tilde{\alpha})}) & \text{for } \lambda - \tilde{\alpha} < 0. \end{cases} \tag{5.12}$$

The proof of Lemma 5.6 and further explanations can be found in Appendix B.1. The relevance of the function D becomes clear by the following characterization of $\sigma(L)$.

Theorem 5.7. *We have $\sigma(L) = \{\lambda \in \mathbb{R} : |D(\lambda)| \leq 2\}$.*

Proof. In [17] it is shown that the classical Sturm-Liouville theory can be generalized to include delta-point interactions, see also the appendix of [20]. Herewith, the result follows for instance from Chapter 2 in [31], precisely Theorem 2.3.1 and the discussion thereafter. □

5.2. The spectrum of the operator family $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$

Plugging ansatz (5.4) in the left-hand side of (5.3) we formally compute

$$L_{x,t}u := -u_{xx} + V(x)u_{tt} = \sum_{k\in\mathbb{Z}_{\text{odd}}} (-\hat{u}_k'' - \omega^2 k^2 V(x)\hat{u}_k)e^{ik\omega t}.$$

For $k \in \mathbb{Z}_{\text{odd}}$ we abbreviate

$$L_k := -\frac{d^2}{dx^2} - k^2\omega^2 V(x) = -\frac{d^2}{dx^2} - \alpha\omega^2 k^2 - \beta\omega^2 k^2 \delta_{\text{per}}(x). \tag{5.13}$$

Note that L_k has the form (5.8). Due to Lemma 5.4 and Theorem VIII.15 in [60] for $f, g \in H^1(\mathbb{R})$ the corresponding bilinear form reads $b_k\colon H^1(\mathbb{R}) \to \mathbb{C}$,

$$b_k(f,g) = \int_{\mathbb{R}} \Big(f'(x)\overline{g'(x)} - \alpha\omega^2 k^2 f(x)\overline{g(x)}\Big)\,dx - \beta\omega^2 k^2 \sum_{n\in\mathbb{Z}} f(\varsigma + 2\pi n)\overline{g(\varsigma + 2\pi n)}. \tag{5.14}$$

5.2.1. Spectral properties of L_k

In this section we compute the spectrum of L_k depending on $k \in \mathbb{Z}_{\text{odd}}$ by making use of Theorem 5.7. Since k appears in L_k only as k^2 we restrict to $k \in \mathbb{N}_{\text{odd}}$ in the rest of this section. We give conditions on the parameters $(\omega, \alpha, \beta) \in \mathbb{R}^3_+$ such that zero lies uniformly in a spectral gap of L_k for all $k \in \mathbb{N}_{\text{odd}}$ in the sense that there is a constant $c > 0$ independent of $k \in \mathbb{N}_{\text{odd}}$ such that

$$(-c|k|, c|k|) \subset \rho(L_k) \text{ for all } k \in \mathbb{N}_{\text{odd}}. \tag{5.15}$$

The last part of this section reveals that the spectral gap can not grow superlinearly in $k \in \mathbb{N}_{\text{odd}}$ which means that (5.15) is optimal up to constants.

By Lemma 5.6 the discriminant D_k associated to L_k reads

$$D_k(\lambda) = \begin{cases} -\frac{\beta\omega^2k^2}{\sqrt{\lambda+\alpha\omega^2k^2}} \sin(2\pi\sqrt{\lambda + \alpha\omega^2k^2}) + 2\cos(2\pi\sqrt{\lambda + \alpha\omega^2k^2}) & \text{for } \lambda > -\alpha\omega^2k^2, \\ 2 - 2\pi\beta\omega^2k^2 & \text{for } \lambda = -\alpha\omega^2k^2, \\ -\frac{\beta\omega^2k^2}{\sqrt{-\lambda-\alpha\omega^2k^2}} \sinh(2\pi\sqrt{-\lambda - \alpha\omega^2k^2}) + 2\cosh(2\pi\sqrt{-\lambda - \alpha\omega^2k^2}) & \text{for } \lambda < -\alpha\omega^2k^2. \end{cases} \tag{5.16}$$

Before we turn to our main result we give an auxiliary estimate.

Lemma 5.8. *For $m \in \mathbb{N}_0$ we have*

$$m + \frac{1}{6} < \sqrt{m^2 + m + \frac{1}{4} - \frac{2m+1}{25}} < \sqrt{m^2 + m + \frac{1}{4} + \frac{2m+1}{25}} < m + \frac{5}{6}.$$

Proof. Obviously we have $m + \frac{1}{4} - \frac{2m+1}{25} = \frac{23}{25}m + \frac{21}{100} > \frac{m}{3} + \frac{1}{36}$. Hence,

$$\sqrt{m^2 + m + \frac{1}{4} - \frac{2m+1}{25}} > \sqrt{m^2 + \frac{m}{3} + \frac{1}{36}} = m + \frac{1}{6}$$

which establishes the first of the desired inequalities. The second inequality is clear. As before we compute

$$m^2 + m + \frac{1}{4} + \frac{2m+1}{25} = m^2 + \frac{27}{25}m + \frac{29}{100} < m^2 + \frac{5}{3}m + \frac{25}{36} = \left(m + \frac{5}{6}\right)^2$$

which finishes the proof. ◻

We choose our parameters $(\omega, \alpha, \beta) \in \mathbb{R}^3_+$ for the rest of this chapter to be related as follows:

$$\alpha > 0, \omega = \frac{1}{4\sqrt{\alpha}} \text{ and } \beta = 16\alpha. \tag{5.17}$$

This is precisely the assumption of Theorem 5.2 since (5.17) leads to $T = \frac{2\pi}{\omega} = 8\pi\sqrt{\alpha}$. With this choice we formulate a result which treats the case $k \geq 3$.

Lemma 5.9. *Suppose* (5.17). *Then*

$$(-\frac{k}{100}, \frac{k}{100}) \subset \rho(L_k) \textit{ for all } k \in 2\mathbb{N}+1.$$

Proof. By Theorem 5.7 we have to show $|D_k(\lambda)| > 2$ for all $\lambda \in (-\frac{k}{100}, \frac{k}{100})$ and all $k \in 2\mathbb{N}+1$. Since $-\frac{k}{100} > -\alpha\omega^2 k^2 = -\frac{k^2}{16}$ for all $k \in \mathbb{N}$ we only have to deal with the first case of the case distinction in (5.16). Due to (5.17) we have to guarantee that

$$\left|2\cos\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right) - \frac{k^2}{\sqrt{\lambda + \frac{k^2}{16}}}\sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| > 2 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{5.18}$$

The idea is to simplify (5.18) in order to find sufficient conditions which imply the validity of (5.18). Since $\left|2\cos\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| \leq 2$ it is sufficient for (5.18) to prove

$$\frac{k^2}{\sqrt{\lambda + \frac{k^2}{16}}}\left|\sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| > 4 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{5.19}$$

Note the inequality

$$\frac{\sqrt{29}}{20}k = \sqrt{\frac{k^2}{100} + \frac{k^2}{16}} > \sqrt{\lambda + \frac{k^2}{16}} \text{ for } |\lambda| < \frac{k^2}{100}$$

which is in particular valid for $|\lambda| < \frac{k}{100}$, i.e., the range of λ in (5.19). Hence, a sufficient condition for the validity of (5.19) and therefore also of (5.18) is to verify

$$\left|\sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\right| > \frac{\sqrt{29}}{5k} \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1. \tag{5.20}$$

To establish estimate (5.20) we take a closer look at the argument of the sine-function in (5.20). Since $k \in 2\mathbb{N}+1$ we can write $k = 2m+1$ with $m \in \mathbb{N}$ and therefore

$$2\sqrt{\lambda + \frac{k^2}{16}} = \sqrt{4\lambda + m^2 + m + \frac{1}{4}} \in \left(\sqrt{m^2 + m + \frac{1}{4} - \frac{2m+1}{25}}, \sqrt{m^2 + m + \frac{1}{4} + \frac{2m+1}{25}}\right)$$

for $|\lambda| < \frac{k}{100}$. Hence, Lemma 5.8 implies

$$2\sqrt{\lambda + \frac{k^2}{16}} \in \left(m + \frac{1}{6}, m + \frac{5}{6}\right) \text{ for } |\lambda| < \frac{k}{100}. \tag{5.21}$$

The periodicity of the sine-function ensures

$$\left|\sin\left(\pi(m+\frac{1}{6})\right)\right| = \left|\sin\left(\frac{\pi}{6}\right)\right| = \frac{1}{2} = \left|\sin\left(-\frac{\pi}{6}\right)\right| = \left|\sin\left(\pi(m+\frac{5}{6})\right)\right|. \tag{5.22}$$

The monotonicity of the sine-function and (5.21), (5.22) then gives

$$\left|\sin\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| \geq \left|\sin\left(\frac{\pi}{6}\right)\right| = \frac{1}{2} \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1.$$

In summary,

$$\left|\sin\left(2\pi\sqrt{\lambda+\frac{k^2}{16}}\right)\right| - \frac{\sqrt{29}}{5k} \geq \frac{1}{2} - \frac{\sqrt{29}}{5k} \geq \frac{1}{2} - \frac{\sqrt{29}}{15} > 0 \text{ for } |\lambda| < \frac{k}{100} \text{ and all } k \in 2\mathbb{N}+1 \tag{5.23}$$

which verifies (5.20) and finishes the proof. □

The estimate (5.23) in the preceeding proof is the only reason why we focus on $k \geq 3$ in Lemma 5.9. Thus, we now have to deal with the case $k = 1$.

Lemma 5.10. *Suppose* (5.17). *Then* $0 \in \rho(L_1)$.

Proof. Using (5.16) we obtain by direct computation

$$D_1(0) = -4\sin\left(\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right) = -4 < -2.$$

Hence, $0 \in \rho(L_1)$ by the characterization in Theorem 5.7. □

Summarizing Lemma 5.9 and Lemma 5.10 we obtain the following result.

Lemma 5.11. *Suppose* (5.17). *Then there is a constant* $c > 0$ *such that*

$$(-c|k|, c|k|) \subset \rho(L_k) \textit{ for all } k \in \mathbb{N}_{\text{odd}}. \tag{5.24}$$

Proof. By Lemma 5.10 and since the resolvent set is open there is a constant $\tilde{c} > 0$ such that $(-\tilde{c}, \tilde{c}) \subset \rho(L_1)$. With the help of Lemma 5.9 we obtain (5.24) with $c := \min\left\{\tilde{c}, \frac{1}{100}\right\}$. □

We next show that for each $k \in \mathbb{N}_{\text{odd}}$ the operator L_k has spectrum to the left of $-\frac{k}{100}$ as well as to the right of $\frac{k}{100}$ which justifies the notion of a spectral *gap*.

Lemma 5.12. *Suppose* (5.17). *Then for each* $k \in \mathbb{Z}_{\text{odd}}$ *we have* $\sigma(L_k) \cap (-\infty, -\frac{1}{100}) \neq \emptyset$ *as well as* $\sigma(L_k) \cap (\frac{1}{100}, \infty) \neq \emptyset$.

Proof. We first show that $-\frac{k^4}{4} - \frac{k^2}{16} \in \sigma(L_k)$ which implies $\sigma(L_k) \cap (-\infty, -\frac{1}{100}) \neq \emptyset$. Since $-\frac{k^4}{4} - \frac{k^2}{16} < -\frac{k^2}{16} < -\frac{1}{100}$ and $\cosh(x) - \sinh(x) = e^{-x}$ a direct computation implies

$$\left|D_k\left(-\frac{k^4}{4} - \frac{k^2}{16}\right)\right| = 2\left|\cosh(\pi k^2) - \sinh(\pi k^2)\right| = 2e^{-\pi k^2} < 2e^{-\pi} < 2,$$

i.e., $-\frac{k^4}{4} - \frac{k^2}{16} \in \sigma(L_k)$. On the other hand, since $\frac{k^4}{4} - \frac{k^2}{16} > \frac{1}{100}$ we have

$$\left|D_k\left(\frac{k^4}{4} - \frac{k^2}{16}\right)\right| = \left|-2\sin(\pi k^2) + 2\cos(\pi k^2)\right| = 2\left|\cos(\pi k^2)\right| = 2.$$

Thus, $\frac{k^4}{4} - \frac{k^2}{16} \in \sigma(L_k)$ which gives $\sigma(L_k) \cap (\frac{1}{100}, \infty) \neq \emptyset$. □

Finally, we give a result which in a sense complements the statement of Lemma 5.11. It guarantees that the spectral gap containing zero can not grow superlinear in k, i.e., the growth rate in Lemma 5.11 is optimal up to a constant factor.

Lemma 5.13. *Suppose* (5.17) *and let* $f\colon \mathbb{N}_{\mathrm{odd}} \to [0,\infty)$ *be a function with* $f(k) \to \infty$ *as* $k \to \infty$. *Then there is no constant* $c > 0$ *such that*

$$(-c|k|f(|k|), c|k|f(|k|)) \subset \rho(L_k) \text{ for all } k \in \mathbb{Z}_{\mathrm{odd}}. \tag{5.25}$$

Proof. Again it suffices to restrict to $k \in \mathbb{N}_{\mathrm{odd}}$. Suppose that the growth rate in (5.25) holds true for a constant $c > 0$. W.l.o.g. we may assume that $c < \frac{1}{16}$ and $f(k) < k$ for all $k \in \mathbb{N}_{\mathrm{odd}}$ so that $-c|k|f(k) > -\frac{k^2}{16} = -\alpha\omega^2 k^2$. Therefore, only the first case in (5.16) plays a role. In particular, by the characterization in Theorem 5.7 we have

$$\Big| -\frac{k^2}{\sqrt{\lambda + \frac{k^2}{16}}} \sin\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right) + 2\cos\left(2\pi\sqrt{\lambda + \frac{k^2}{16}}\right)\Big| > 2 \text{ for } |\lambda| < ckf(k) \text{ and all } k \in \mathbb{N}_{\mathrm{odd}}. \tag{5.26}$$

We show that for $k \in \mathbb{N}_{\mathrm{odd}}$ sufficiently large there is $\lambda^\star \in (-ckf(k), ckf(k))$ such that $2\pi\sqrt{\lambda^\star + \frac{k^2}{16}}$ equals an integer multiple of π which then entails $|D_k(\lambda^\star)| = 2$, i.e., a contradiction to (5.26). For this purpose, we again write $k = 2m+1$ for $m \in \mathbb{N}_0$ and investigate the range of

$$\Lambda_k : [-ckf(k), ckf(k)] \to \mathbb{R}, \lambda \mapsto 4\lambda + \frac{k^2}{4}.$$

We claim

$$\Lambda_k(-ckf(k)) < m^2 < \Lambda_k(ckf(k)) \tag{5.27}$$

for k sufficiently large. Since $\frac{k^2}{4} = m^2 + m + \frac{1}{4}$ and $4ckf(k) > 0$ the second inequality in (5.27) holds true for all $k \in \mathbb{N}_{\mathrm{odd}}$. The first inequality holds true if and only if

$$-4c(2m+1)f(2m+1) + m + \frac{1}{4} < 0$$

which is true for m sufficiently large due to $f(k) \to \infty$ as $k \to \infty$. Therefore, (5.27) is verified provided k is sufficiently large. Hence the mean value theorem guarantees the existence of $\lambda^\star \in (ckf(k), ckf(k))$ such that $\Lambda_k(\lambda^\star) = m^2$. As already mentioned above this contradicts (5.26) since $|D_k(\lambda^\star)| = 2$. □

5.3. The functional analytic framework

In this section we first use the Floquet-Bloch decomposition in order to derive a suitable functional analytic framework for our problem. This leads to a Hilbert space in which we work for the rest of this chapter.

5.3.1. Calculations via Floquet-Bloch decomposition

In this short section we introduce some notation which will later help us to treat the indefinite quadratic part of the energy functional arising from the family of operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$. For basics on the Floquet-transformation $\mathcal{T}$ see Section B.2. Let $\mathcal{P} := [0, 2\pi)$ denote the interval of periodicity and $\mathcal{B} := [-\frac{1}{2}, \frac{1}{2})$ the Brillouin zone. The sequence of Bloch waves for the operator L_k is denoted by $(\psi_{j,k})_{j\in\mathbb{N}_0}$, where $\psi_{j,k}\colon \mathcal{P}\times\mathcal{B} \to \mathbb{C}$ for all $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}$. For $s \in \mathcal{B}$ they satisfy

$$L_k\psi_{j,k}(\cdot, s) = \lambda_{j,k}(s)\psi_{j,k}(\cdot, s) \text{ in } \mathcal{P} \tag{5.28}$$

together with the quasiperiodicity condition

$$\psi_{j,k}(x + 2\pi, s) = e^{2\pi is}\psi_{j,k}(x, s) \text{ for all } (x, s, j, k) \in \mathcal{P}\times\mathcal{B}\times\mathbb{N}_0\times\mathbb{Z}_{\text{odd}}. \tag{5.29}$$

For fixed $s \in \mathcal{B}, k \in \mathbb{Z}_{\text{odd}}$ due to $V_k \in H^{-1}(\mathbb{R})$ we have $\psi_{j,k}(\cdot, s) \in H^1_{\text{loc}}(\mathbb{R})$ for $j \in \mathbb{N}_0$ and $(\psi_{j,k})_{j\in\mathbb{N}_0}$ are a $\langle\cdot,\cdot\rangle_{L^2(\mathcal{P})}$-orthonormal and complete system of eigenfunctions in $L^2(\mathcal{P})$ and

$$\lambda_{1,k}(s) \le \lambda_{2,k}(s) \le \cdots \le \lambda_{j,k}(s) \to \infty \text{ as } j \to \infty,$$

see also Section 3.4 in [30]. Recall $\sigma(L_k) = \bigcup_{j\in\mathbb{N}_0, s\in\mathcal{B}} \lambda_{j,k}(s)$ for all $k \in \mathbb{Z}_{\text{odd}}$, see for instance § 2.3, § 2.4 and Theorem 5.3.2 in [31] or Section 3.6 in [30]. Using the completeness of the Bloch waves (see Theorem B.3), for $\hat{u}_k \in L^2(\mathbb{R})$ we obtain

$$\hat{u}_k(x) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \left\langle \mathcal{T}\hat{u}_k(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{\mathcal{P}} \psi_{j,k}(x, s)ds \text{ in } L^2(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}}. \tag{5.30}$$

In particular, due to (5.28) we formally have

$$L_k\hat{u}_k(x) = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \left\langle \mathcal{T}\hat{u}_k(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{\mathcal{P}} \lambda_{j,k}(s)\psi_{j,k}(x, s)ds \text{ for all } k \in \mathbb{Z}_{\text{odd}}. \tag{5.31}$$

We justify (5.31) later on in Corollary B.6. In the following, for a function $\hat{u}_k\colon \mathbb{R} \to \mathbb{C}$ with $\hat{u}_k \in L^2(\mathbb{R})$ and an index $k \in \mathbb{Z}_{\text{odd}}$ we abbreviate

$$\begin{aligned} \tilde{\hat{u}}_k &:= \mathcal{T}\hat{u}_k, \\ \tilde{\hat{u}}_{j,k}(s) &:= \left\langle \tilde{\hat{u}}_k(\cdot, s), \psi_{j,k}(\cdot, s)\right\rangle_{\mathcal{P}}, \end{aligned} \tag{5.32}$$

where $\langle f, g\rangle_{\mathcal{P}} := \int_{\mathcal{P}} f(x)\overline{g(x)}dx$ for $f, g \in L^2(\mathcal{P})$. The indefinite quadratic functional (see (5.33)) gets now motivated by the calculations in the following lemma where we continue to use the previous notation. After the minimization procedure in Section 5.6 we do the back-transformation on a rigorous level in Section 5.7.

Lemma 5.14. *Fix $k \in \mathbb{Z}_{\text{odd}}$ and let $\hat{u}_k \in D(L_k), \hat{v}_k \in L^2(\mathbb{R})$. With the notation of* (5.32) *we have*

$$\int_{\mathbb{R}} L_k\hat{u}_k\overline{\hat{v}_k}dx = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)}ds. \tag{5.33}$$

Proof. We proceed in two steps. First we show that

$$\langle \hat{u}_k, \hat{v}_k \rangle_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)} ds \text{ for all } \hat{u}_k, \hat{v}_k \in L^2(\mathbb{R}). \tag{5.34}$$

Since $\mathcal{T}: L^2(\mathbb{R}) \to L^2(\mathcal{P} \times \mathcal{B})$ is unitary we have

$$\langle \hat{u}_k, \hat{v}_k \rangle_{L^2(\mathbb{R})} = \langle \mathcal{T}\hat{u}_k, \mathcal{T}\hat{v}_k \rangle_{L^2(\mathcal{P}\times\mathcal{B})} = \int_{\mathcal{B}} \langle (\mathcal{T}\hat{u}_k)(\cdot, s), (\mathcal{T}\hat{v}_k)(\cdot, s) \rangle_{L^2(\mathcal{P})} \, ds. \tag{5.35}$$

Recall that the family $(\psi_{j,k}(\cdot, s))_{j\in\mathbb{N}_0}$ is an orthonormal basis in $L^2(\mathcal{P})$ for all $s \in \mathcal{B}$. Therefore, we receive

$$\begin{aligned} \langle (\mathcal{T}\hat{u}_k)(\cdot, s), (\mathcal{T}\hat{v}_k)(\cdot, s) \rangle_{L^2(\mathcal{P})} &= \sum_{j\in\mathbb{N}_0} \left\langle (\mathcal{T}\hat{u}_k)(\cdot, s), \psi_{j,k}(\cdot, s) \right\rangle_{L^2(\mathcal{P})} \overline{\left\langle (\mathcal{T}\hat{v}_k)(\cdot, s), \psi_{j,k}(\cdot, s) \right\rangle_{L^2(\mathcal{P})}} \\ &= \sum_{j\in\mathbb{N}_0} \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)}. \end{aligned} \tag{5.36}$$

Matching (5.35) and (5.36) we derive

$$\langle \hat{u}_k, \hat{v}_k \rangle_{L^2(\mathbb{R})} = \int_{\mathcal{B}} \sum_{j\in\mathbb{N}_0} \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)} ds = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)} ds \tag{5.37}$$

and it remains to verify the permutation of summation and integration in (5.37) to affirm (5.34). Indeed, by the inequality of Cauchy-Schwarz we have

$$\left| \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)} ds \right| \leq \sum_{j\in\mathbb{N}_0} \|\tilde{\hat{u}}_{j,k}\|_{L^2(\mathcal{B})} \|\tilde{\hat{v}}_{j,k}\|_{L^2(\mathcal{B})} \leq \left(\sum_{j\in\mathbb{N}_0} \|\tilde{\hat{u}}_{j,k}\|^2_{L^2(\mathcal{B})} \right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{N}_0} \|\tilde{\hat{v}}_{j,k}\|^2_{L^2(\mathcal{B})} \right)^{\frac{1}{2}}.$$

Moreover, Bessel's inquality gives

$$\begin{aligned} \sum_{j\in\mathbb{N}_0} \|\tilde{\hat{u}}_{j,k}\|^2_{L^2(\mathcal{B})} &= \lim_{n\to\infty} \sum_{j=0}^{n} \int_{\mathcal{B}} | \left\langle (\mathcal{T}\hat{u}_k)(\cdot, s), \psi_{j,k}(\cdot, s) \right\rangle_{L^2(\mathcal{P})} |^2 ds \\ &= \lim_{n\to\infty} \int_{\mathcal{B}} \sum_{j=0}^{n} | \left\langle (\mathcal{T}\hat{u}_k)(\cdot, s), \psi_{j,k}(\cdot, s) \right\rangle_{L^2(\mathcal{P})} |^2 ds \\ &\leq \int_{\mathcal{B}} \|\mathcal{T}\hat{u}_k(\cdot, s)\|^2 ds = \|\hat{u}_k\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

The same calculation holds true with $\tilde{\hat{u}}_{j,k}, \hat{u}_k$ replaced by $\tilde{\hat{v}}_{j,k}, \hat{v}_k$. This justifies the last step in (5.37) and proves (5.34).

In a second step, we exploit (5.34) for $\hat{u}_k \in D(L_k)$, $\hat{v}_k \in L^2(\mathbb{R})$ and Corollary B.6 in order to establish (5.33). Precisely,

$$\langle L_k\hat{u}_k, \hat{v}_k \rangle_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \left\langle (\mathcal{T}L_k\hat{u}_k)(\cdot, s), \psi_{j,k}(\cdot, s) \right\rangle_{L^2(\mathcal{P})} \overline{\tilde{\hat{v}}_{j,k}(s)} ds = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s) \tilde{\hat{u}}_{j,k}(s)\overline{\tilde{\hat{v}}_{j,k}(s)} ds$$

and the proof is done. □

In a next step, we slightly generalize the statement of Lemma 5.14.

Corollary 5.15. *For $k \in \mathbb{Z}_{\text{odd}}$ we have*

$$b_k(\hat{u}_k, \hat{v}_k) = \sum_{j \in \mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s) \tilde{\hat{u}}_{j,k}(s) \overline{\tilde{\hat{v}}_{j,k}(s)} ds$$

with b_k given by (5.14) *and all $\hat{u}_k, \hat{v}_k \in D(b_k) = H^1(\mathbb{R})$.*

Proof. Recall that

$$b_k(\hat{u}_k, \hat{v}_k) = \langle L_k \hat{u}_k, \hat{v}_k \rangle \text{ for all } \hat{u}_k \in D(L_k) \text{ and all } \hat{v}_k \in D(b_k).$$

The statement now follows from Lemma 5.14 and the fact that $D(L_k)$ is dense in $D(b_k)$ for all $k \in \mathbb{Z}_{\text{odd}}$, see for instance Chapter IV, Theorem 2.4 (v) in [32]. There it is proved that the domain of a lower semi-bounded operator $L \geq -C$ with $C \geq 0$ is dense in the domain of the associated bilinear form b with respect to the norm induced by $\|\cdot\| := \sqrt{b(\cdot,\cdot) + C\|\cdot\|^2}$ whenever the operator is constructed from the bilinear form by a so-called Friedrichs-extension (which in particular applies for our case L_k and b_k). The statement $D(b_k) = H^1(\mathbb{R})$ follows from (5.14) and Theorem VIII.15 in [60]. □

5.3.2. The right Hilbert space

Finally, we are ready to introduce a Hilbert space in which we seek solutions. Due to Lemma 5.14 set

$$\begin{aligned} \mathcal{H} := \Big\{ \tilde{u} = (\tilde{\hat{u}}_{j,k})_{j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd}}} : & \tilde{\hat{u}}_{j,k} : \mathcal{B} \to \mathbb{C} \text{ measurable for all } (j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}, \\ & \overline{\tilde{\hat{u}}_{j,k}(s)} = \tilde{\hat{u}}_{j,-k}(-s) \text{ for all } (j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B} \\ & \text{and} \sum_{j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)| |\tilde{\hat{u}}_{j,k}(s)|^2 ds < \infty \Big\}, \end{aligned}$$

where we consider the space over the field $\mathbb{R}$ and not $\mathbb{C}$. We equip $\mathcal{H}$ with the canonical inner product and norm

$$\langle \tilde{u}, \tilde{v} \rangle_{\mathcal{H}} := \sum_{j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)| \tilde{\hat{u}}_{j,k}(s) \overline{\tilde{\hat{v}}_{j,k}(s)} ds \quad \text{and} \quad \|\tilde{u}\|_{\mathcal{H}} := \sqrt{\langle \tilde{u}, \tilde{u} \rangle} \text{ for } \tilde{u}, \tilde{v} \in \mathcal{H}.$$

The following lemma justifies the condition $\overline{\tilde{\hat{u}}_{j,k}(-s)} = \tilde{\hat{u}}_{j,-k}(s)$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ incorporated in $\mathcal{H}$.

Lemma 5.16. *Let $\tilde{u} \in \mathcal{H}$ and $\hat{u}_k$ be given by*

$$\hat{u}_k(x) = \sum_{j \in \mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s) \psi_{j,k}(x,s) ds. \tag{5.38}$$

Then $\overline{\hat{u}_k} = \hat{u}_{-k}$ for all $k \in \mathbb{Z}_{\text{odd}}$.

Proof. Since $\psi_{j,k}$ satisfies (5.28) and (5.29) we conclude

$$\psi_{j,k} = \psi_{j,-k} \text{ on } \mathcal{P} \times \mathcal{B} \text{ for all } (j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}}. \tag{5.39}$$

Taking the complex conjugates of (5.28) and (5.29) leads to

$$\begin{aligned} -\overline{\psi''_{j,k}(x,s)} - k^2 V(x)\overline{\psi_{j,k}(x,s)} &= \lambda_{j,k}(s)\overline{\psi_{j,k}(x,s)}, \\ \overline{\psi_{j,k}(x+2\pi,s)} &= \overline{\psi_{j,k}(x,s)}e^{-2\pi i s}. \end{aligned}$$

This reveals that $\psi_{j,k}(\cdot,s)$ can be chosen such that

$$\overline{\psi_{j,k}(\cdot,s)} = \psi_{j,k}(\cdot,-s) \text{ for all } (j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}. \tag{5.40}$$

In order to finish the proof it suffices to ensure $\int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\overline{\psi_{j,k}(x,s)}ds = \int_{\mathcal{B}} \tilde{\hat{u}}_{j,-k}(s)\overline{\psi_{j,-k}(x,s)}ds$ for all $j \in \mathbb{N}_0$ since the claim then follows from (5.38). Note that since $\mathcal{B}$ is symmetric about $\{s = 0\}$ the condition $\overline{\tilde{\hat{u}}_{j,k}(s)} = \tilde{\hat{u}}_{j,-k}(-s)$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ is equivalent to $\overline{\tilde{\hat{u}}_{j,k}(-s)} = \tilde{\hat{u}}_{j,-k}(s)$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$. Therefore, in the following calculation we first use (5.40), then profit from the fact that $\mathcal{B}$ is symmetric about $\{s = 0\}$ and finally exploit (5.39). Hence, for $j \in \mathbb{N}_0$ we deduce

$$\begin{aligned} \int_{\mathcal{B}} \overline{\tilde{\hat{u}}_{j,k}(s)\psi_{j,k}(x,s)}ds &= \int_{\mathcal{B}} \overline{\tilde{\hat{u}}_{j,k}(s)}\psi_{j,k}(x,-s)ds = \int_{\mathcal{B}} \overline{\tilde{\hat{u}}_{j,k}(-s)}\psi_{j,k}(x,s)ds \\ &= \int_{\mathcal{B}} \overline{\tilde{\hat{u}}_{j,k}(-s)}\psi_{j,-k}(s)ds = \int_{\mathcal{B}} \tilde{\hat{u}}_{j,-k}(s)\psi_{j,-k}(s)ds \end{aligned}$$

which finishes the proof. □

Next, we introduce some further notation which we use later to deal with the indefinite character of the problem. We introduce the projections $\mathcal{P}^+$ and $\mathcal{P}^-$ by

$$\begin{aligned} \mathcal{H}^+ &:= \mathcal{P}^+\mathcal{H} := \{\tilde{u} \in \mathcal{H} : \tilde{u}_{j,k} \equiv 0 \text{ whenever } \lambda_{j,k}(s) < 0 \text{ for all } s \in \mathcal{B}\}, \\ \mathcal{H}^- &:= \mathcal{P}^-\mathcal{H} := \{\tilde{u} \in \mathcal{H} : \tilde{u}_{j,k} \equiv 0 \text{ whenever } \lambda_{j,k}(s) > 0 \text{ for all } s \in \mathcal{B}\}. \end{aligned}$$

Moreover, we consider the bilinear form $B\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$,

$$B(\tilde{u},\tilde{v}) = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{u}_{j,k}(s)\overline{\tilde{v}_{j,k}(s)}ds \text{ for } \tilde{u},\tilde{v} \in \mathcal{H}.$$

Then we have a splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (recall that by Lemma 5.11 there is no triple $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ such that $\lambda_{j,k}(s) = 0$) with

$$B(\tilde{u},\tilde{u}) = \|\mathcal{P}^+\tilde{u}\|^2_{\mathcal{H}} - \|\mathcal{P}^-\tilde{u}\|^2_{\mathcal{H}} \text{ for all } \tilde{u} \in \mathcal{H}. \tag{5.41}$$

Therefore, we abbreviate $\tilde{u}^+ := \mathcal{P}^+\tilde{u}$ and $\tilde{u}^- := \mathcal{P}^-\tilde{u}$. Additionally,

$$\|\tilde{u}\|^2_{\mathcal{H}} = \|\tilde{u}^+\|^2_{\mathcal{H}} + \|\tilde{u}^-\|^2_{\mathcal{H}},$$

i.e., $\|\tilde{u}^+\|_{\mathcal{H}}, \|\tilde{u}^-\|_{\mathcal{H}} \le \|\tilde{u}\|_{\mathcal{H}}$ for all $\tilde{u} \in \mathcal{H}$.

In the rest of this section we deal with regularity questions. Roughly speaking, we show that elements of $\mathcal{H}$ already lead to $H^1(\mathbb{R})$ regularity of the functions $\hat{u}_k$ given by (5.30).

Lemma 5.17. *Fix $k \in \mathbb{Z}_{\text{odd}}$. Then*

$$D(L_k) = \{u = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\psi_{j,k}(x,s)ds : \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \lambda^2_{j,k}(s)|\tilde{\hat{u}}_{j,k}(s)|^2 ds < \infty\}$$

$$D(b_k) = \{u = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\hat{u}}_{j,k}(s)\psi_{j,k}(x,s)ds : \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{\hat{u}}_{j,k}(s)|^2 ds < \infty\}.$$

Proof. We know that $D(L_k) = \{\hat{u}_k : L_k\hat{u}_k \in L^2(\mathbb{R})\}$. In particular, by Lemma 5.14 and $\lambda_{j,k}(s) \in \mathbb{R}$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd}} \times \mathcal{B}$ we deduce

$$\|L_k\hat{u}_k\|^2_{L^2(\mathbb{R})} = \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)|^2|\tilde{\hat{u}}_{j,k}(s)|^2 ds$$

which proves the claim concerning $D(L_k)$. The second part concerning $D(b_k)$ then follows from Corollary 5.15 and the second representation theorem (Theorem 2.8 and Section IV.4 in [32], or see Section 10.2 in [64]). □

We give an application of Lemma 5.17.

Corollary 5.18. *Let $(\tilde{\hat{u}}_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \in \mathcal{H}$. Then $\hat{u}_k$ from* (5.30) *satisfies $\hat{u}_k \in H^1(\mathbb{R})$ for all $k \in \mathbb{Z}_{\text{odd}}$.*

Proof. Recall that by definition $(\tilde{\hat{u}}_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \in \mathcal{H}$ if and only if

$$\sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{\hat{u}}_{j,k}(s)|^2 ds < \infty, \tag{5.42}$$

where $\tilde{\hat{u}}_{j,k}(s) := \left\langle \mathcal{T}\hat{u}_k(\cdot,s), \psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}$. In particular we deduce from (5.42) that

$$\sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{\hat{u}}_{j,k}(s)|^2 ds < \infty$$

for all $k \in \mathbb{Z}_{\text{odd}}$. Therefore $\hat{u}_k \in D(b_k) = H^1(\mathbb{R})$ by Lemma 5.17 and (5.14). □

5.4. Fine tuning of prefactors and resulting optimal estimates

We now give further estimates which incorporate the k-dependance. For $k \in \mathbb{Z}_{\text{odd}}$ we abbreviate $V_k(x) := -\frac{k^2}{16} - k^2\delta_{\text{per}}(x)$. We first introduce some notation. Recall

$$\langle L_k u, \varphi\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \lambda d\,\langle P_\lambda u, \varphi\rangle \text{ for } u \in D(L_k), \varphi \in L^2(\mathbb{R}),$$

where $(P_\lambda)_{\lambda\in\mathbb{R}}$ denotes the projection-valued measure (see for instance [61] or Definition VII.1.9 and Theorem VII.3.2 in [72]). We next introduce for $v \in H^1(\mathbb{R})$ the splitting $v = P^+v + P^-v$ with

$$P^+v := v^+ := \int_0^\infty \lambda d\,\langle P_\lambda v, \cdot\rangle\,, \quad P^-v := v^- := \int_{-\infty}^0 \lambda d\,\langle P_\lambda v, \cdot\rangle\,.$$

Then $D(b_k) = D(b_k^+) \oplus D(b_k^-)$ with $D(b_k^\pm) := P^\pm D(b_k)$. Notice that due to the representation in Corollary 5.15 $D(b_k)^+$ refers to $\tilde{v}_{j,k}(s) = 0$ whenever $\lambda_{j,k}(s) < 0$. Vive versa, $D(b_k)^-$ transfers to $\tilde{v}_{j,k}(s) = 0$ if $\lambda_{j,k}(s) > 0$. Subsection 5.2.1 entails the following corollary.

Corollary 5.19. *There is $c > 0$ such that*

$$b_k(v^+, v^+) - b_k(v^-, v^-) \geq c\,|k|\,\|v\|^2_{L^2(\mathbb{R})} \text{ for all } v \in H^1(\mathbb{R}) \text{ and all } k \in \mathbb{Z}_{\mathrm{odd}}. \tag{5.43}$$

Moreover, $c\,|k|$ in (5.43) is optimal in the sense that (5.43) does not hold true for any $\tilde{c} > 0$ and any coercive $f\colon \mathbb{N}_{\mathrm{odd}} \to [0, \infty)$ with $c\,|k|$ replaced by $\tilde{c}\,|k|\,f(|k|)$.

Proof. Recall that for a self-adjoint lower semi-bounded operator $A\colon D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ we have

$$\inf_{f \in D(A)} \frac{\langle Af, f\rangle_{L^2(\mathbb{R})}}{\|f\|^2_{L^2(\mathbb{R})}} = \inf \sigma(A). \tag{5.44}$$

The idea is now to split the indefinite operator L_k into a positive definite and a negative definite operator $L_k^\pm$, apply (5.44) and then use the density of $D(L_k)$ in $H^1(\mathbb{R})$ (see Corollary 5.15). We introduce a further splitting, namely for $u \in L^2(\mathbb{R})$ we split $u = P_2^+ u + P_2^- u$ with

$$P_2^+ u := \int_0^\infty 1 d\,\langle P_\lambda u, \cdot\rangle\,, \quad P_2^- u := \int_{-\infty}^0 1 d\,\langle P_\lambda u, \cdot\rangle\,.$$

Assume for a moment that

$$L_k^\pm\colon P_2^\pm D(L_k) \subset P_2^\pm L^2(\mathbb{R}) \to P_2^\pm L^2(\mathbb{R}), L_k^\pm u := L_k u \tag{5.45}$$

are self-adjoint operators. Then L_k^+ is positive definite and L_k^- is negative definite. Thus we conclude from (5.44) and Lemma 5.11 that

$$\inf_{u \in P_2^+ D(L_k)} \frac{\left\langle L_k^+ u, u\right\rangle_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}} \geq \tilde{c}|k|, \inf_{u \in P_2^- D(L_k)} -\frac{\left\langle L_k^- u, u\right\rangle_{L^2(\mathbb{R})}}{\|u\|^2_{L^2(\mathbb{R})}} \geq \tilde{c}|k| \tag{5.46}$$

for $\tilde{c} > 0$. The combination of (5.46) and Corollary 5.15 then shows

$$\langle L_k^+ P_2^+ u, P_2^+ u\rangle_{L^2(\mathbb{R})} - \langle L_k^- P_2^- u, P_2^- u\rangle \geq \tilde{c}|k| \left(\|P_2^+ u\|^2_{L^2(\mathbb{R})} + \|P_2^- u\|^2_{L^2(\mathbb{R})}\right) \geq \frac{\tilde{c}}{2}|k|\|u\|^2_{L^2(\mathbb{R})}$$

and (5.43) then follows from the above mentioned density statement. It remains to verify (5.45). We show (5.45) for L_k^+, the statement for L_k^- follows in the same manner. Due to

$$\langle L_k^+ u, \varphi\rangle = \int_0^\infty \lambda d\,\langle P_\lambda u, \varphi\rangle = \int_0^\infty \lambda d\,\langle u, P_\lambda \varphi\rangle = \langle u, L_k^+ \varphi\rangle$$

we have that L_k^+ is symmetric. Moreover, let $u \in \mathcal{P}_2^+ L^2(\mathbb{R})$. Since L_k and the projection-valued measure P_λ commute we also know that L_k and P_2^+ commute. Hence, $L_k u = L_k P_2^+ u = P_2^+ L_k u \in P_2^+ L^2(\mathbb{R})$, i.e., also the mapping property of L_k^+ in (5.45) is proved. Since $\sigma(L_k^+) = \sigma(L_k) \cap (0, \infty)$ we obtain from Theorem VIII.3 in [60] that L_k^+ is self-adjoint and the proof is done.

The second part of the claim then follows from Lemma 5.13. □

The benefit of an estimate like (5.43) lies in the k-dependance since later we want to sum over $k \in \mathbb{Z}_{\mathrm{odd}}$. We want to construct a similar lower bound with $\|v'\|^2_{L^2(\mathbb{R})}$ instead of $\|v\|^2_{L^2(\mathbb{R})}$ in the right hand side of (5.43), i.e., we want to prove the following result.

Theorem 5.20. *There is a constant $c > 0$ such that*

$$b_k(v^+, v^+) - b_k(v^-, v^-) \geq \frac{c}{|k|^3}\|v'\|^2_{L^2(\mathbb{R})} \text{ for all } v \in H^1(\mathbb{R}) \text{ and all } k \in \mathbb{Z}_{\text{odd}}. \tag{5.47}$$

Proof. We prove (5.47) by several case distinctions. Let $\lambda \in (0, 1)$ be fixed for the whole proof.
1): Let $v \in D(b_k)^+$. We distinguish two cases.
a): $\int_{\mathbb{R}} \left(v'^2 + \frac{V_k}{1-\lambda}v^2\right) dx \geq 0$: Then a mulitplication by $1 - \lambda > 0$ directly implies

$$\int_{\mathbb{R}} \left(v'^2 + V_k v^2\right) dx \geq \lambda \int_{\mathbb{R}} v'^2 dx. \tag{5.48}$$

b): $-\int_{\mathbb{R}} \left(v'^2 + \frac{V_k}{1-\lambda}v^2\right) dx \geq 0$: Recall by (B.10) that

$$\beta\omega^2 k^2 \sum_{n\in\mathbb{Z}} v(\varsigma + 2\pi n)^2 \leq \beta\omega k^2 \left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right)\|v\|^2_{L^2(\mathbb{R})} + \beta\omega^2 k^2 \frac{\varepsilon}{2}\|v'\|^2_{L^2(\mathbb{R})}. \tag{5.49}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} v'^2 dx \leq -\int_{\mathbb{R}} \frac{V_k}{1-\lambda}v^2 dx &= \frac{\alpha\omega^2 k^2}{1-\lambda}\|v\|^2_{L^2(\mathbb{R})} + \frac{\beta\omega^2 k^2}{1-\lambda}\sum_{n\in\mathbb{Z}} v^2(\varsigma + 2\pi n)\\ &\leq \frac{\omega^2 k^2}{1-\lambda}\left(\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right)\right)\|v\|^2_{L^2(\mathbb{R})} + \frac{\beta\omega^2 k^2}{1-\lambda}\frac{\varepsilon}{2}\|v'\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

In particular, for $\varepsilon = \varepsilon_k := \frac{1-\lambda}{\beta\omega^2 k^2}$ we have $\frac{\beta\omega^2 k^2}{1-\lambda}\frac{\varepsilon_k}{2} < 1$ and thus

$$\|v'\|^2_{L^2(\mathbb{R})} \leq \frac{2\omega^2 k^2}{1-\lambda}\left(\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)\right)\|v\|^2_{L^2(\mathbb{R})}.$$

In summary, we conclude

$$\frac{\int_{\mathbb{R}}\left(v'^2 + V_k(x)v^2\right) dx}{\int_{\mathbb{R}} v'^2 dx} = \frac{\int_{\mathbb{R}}\left(v'^2 + V_k(x)v^2\right) dx}{\|v\|^2_{L^2(\mathbb{R})}}\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}} \geq c|k|\frac{1-\lambda}{2\omega^2 k^2}\frac{1}{\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)}. \tag{5.50}$$

Since ε_k is of order $\frac{1}{k^2}$ we infer that

$$c|k|\frac{1-\lambda}{2\omega^2 k^2}\frac{1}{\alpha + \beta\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)} = O\left(\frac{1}{|k|^3}\right).$$

Therefore, merging (5.48) and (5.50) we deduce $\int_{\mathbb{R}}\left(v'^2 + V_k(x)v^2\right) dx \geq \frac{c}{|k|^3}\int_{\mathbb{R}} v'^2 dx$ for all $v \in D(b_k)^+$ for a constant $c > 0$.
2) Let $v \in D(b_k)^-$, i.e., $\int_{\mathbb{R}}\left(v'^2 + V_k v^2\right) dx \leq -c|k|\int_{\mathbb{R}} v^2 dx$. Hence, by (5.49) with $\varepsilon = \varepsilon_k = \frac{1}{\beta\omega^2 k^2}$ we deduce

$$\int_{\mathbb{R}} v'^2 dx \leq \left(\alpha\omega^2 k^2 - c|k|\right)\|v\|^2_{L^2(\mathbb{R})} + \beta\omega^2 k^2\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)\|v\|^2_{L^2(\mathbb{R})} + \frac{\beta\omega^2 k^2\varepsilon_k}{2}\|v'\|^2_{L^2(\mathbb{R})}$$

which entails

$$\|v'\|^2_{L^2(\mathbb{R})} \le 2\left(\alpha\omega^2k^2 - c|k| + \beta\omega^2k^2\left(\frac{1}{2\pi} + \frac{1}{2\varepsilon_k}\right)\right)\|v\|^2_{L^2(\mathbb{R})}. \tag{5.51}$$

In analogy to the first case we now conclude

$$\frac{-\int_{\mathbb{R}}\left(v'^2 + V_k v^2\right)dx}{\|v'\|^2_{L^2(\mathbb{R})}} = \frac{-\int_{\mathbb{R}}\left(v'^2 + V_k v^2\right)dx}{\|v\|^2_{L^2(\mathbb{R})}}\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}} \ge c|k|\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}$$

and due to (5.51) the fraction $\frac{\|v\|^2_{L^2(\mathbb{R})}}{\|v'\|^2_{L^2(\mathbb{R})}}$ is of order $\frac{1}{|k|^4}$ which together with the factor $c|k|$ establishes our claim also in the case $v \in D(b_k)^-$.
Finally, merging the two estimates for $D(b_k)^+$ and $D(b_k)^-$ and exploiting $a^2 + b^2 \ge \frac{1}{2}(a+b)^2$ we end up with

$$b_k(v^+, v^+) - b_k(v^-, v^-) \ge \frac{\tilde{c}}{|k|^3}\int_{\mathbb{R}}\left(\left(v^{+'}\right)^2 + \left(v^{-'}\right)^2\right)dx \ge \frac{\tilde{c}}{2|k|^3}\int_{\mathbb{R}} v'^2 dx$$

for a constant $\tilde{c} > 0$ and the proof is done. □

5.5. Further regularity results in space and time

In Corollary 5.18 we were able to deduce $H^1(\mathbb{R})$-regularity in space of the sequence $(\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}}$. The goal of this section is to establish an embedding which transfers regularity of the sequence $(\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}}$ to regularity of the composite function in space and time $u = u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t}$. The main result is the following theorem.

Theorem 5.21. *The linear operator $\mathcal{S}\colon \mathcal{H} \to L^q(D)$,*

$$(\mathcal{S}\tilde{u})(x,t) := \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\tilde{u}}_{j,k}(s)\psi_{j,k}(x,s)ds\; e^{ik\omega t}$$

is bounded for all $q \in [2, \frac{8}{3}]$ where $D = \mathbb{R} \times [0,T)$.

We split the proof of Theorem 5.21 in several steps. First, we give two auxiliary lemmata which are needed later.

Lemma 5.22. *Let $v = (v_1, v_2)^T \in \mathbb{R}^2$. Then there is a constant $c_1 > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{1 - \cos(v\cdot x)}{|x|^{\frac{5}{2}}}dx = c_1\sqrt[4]{v_1^2 + v_2^2}.$$

Proof. The proof of this statement is done in arbitrary dimensions within the proof of Proposition 4.1 in [29]. The first step is to show that w.l.o.g. it suffices to prove the statement for $v = (|v|, 0)$ since

$$\int_{\mathbb{R}^2} \frac{1 - \cos(v\cdot x)}{|x|^{\frac{5}{2}}}dx = \int_{\mathbb{R}^2} \frac{1 - \cos(|v|x_1)}{|x|^{\frac{5}{2}}}dx,$$

see (4.8) in [29]. Afterwards the integral can be computed explicitly by the substitution $y = |v|x$. □

The proof of the next lemma is again given in Appendix B.1.

Lemma 5.23. *For $R > 0$ we have*

$$\int_0^\infty \int_0^R \frac{x^2}{(x^2+y^2)^{\frac{5}{4}}} dxdy \leq 4R(1+R).$$

In order to obtain sufficient regularity of the composite function $u = u(x,t)$ we make use of several intermediate spaces. These auxiliary spaces are introduced now. Let

$$\hat{H} := \Big\{ (\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}} : \hat{u}_k \in H^1(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ and } \sum_{k\in\mathbb{Z}_{\text{odd}}} \left(|k|\, \|\hat{u}_k\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3} \|\hat{u}_k'\|^2_{L^2(\mathbb{R})} \right) < \infty \Big\}$$

with

$$\|(\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}}\|_{\hat{H}} := \sqrt{\sum_{k\in\mathbb{Z}_{\text{odd}}} \left(|k|\, \|\hat{u}_k\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3} \|\hat{u}_k'\|^2_{L^2(\mathbb{R})} \right)}.$$

Moreover, for $r > 0$ and $D = \mathbb{R} \times [0, T)$ let

$$\tilde{H}^r(D) := \Big\{ u \colon D \to \mathbb{R}; u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x) e^{ik\omega t} \text{ s.t. } \overline{\hat{u}_k(x)} = \hat{u}_{-k}(x) \text{ for all } k \in \mathbb{Z}_{\text{odd}}$$
$$\text{and } \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1 + \xi^2 + k^2)^r |\mathcal{F}\hat{u}_k(\xi)|^2 d\xi < \infty \Big\},$$

where $\mathcal{F}$ denotes the Fourier transform with respect to the space-variable $x \in \mathbb{R}$. We equip $\tilde{H}^r(D)$ with

$$\|u\|_{\tilde{H}^r(D)} := \sqrt{\sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1 + \xi^2 + k^2)^r |\mathcal{F}\hat{u}_k(\xi)|^2 d\xi}.$$

Notice that $u \in \tilde{H}^r(D)$ is T-periodic in the second component. Additionally, for $r \in (0,1)$ and $\Omega \subseteq \mathbb{R}^2$ open recall the fractional Sobolev space (see [29])

$$H^r(\Omega) := \Big\{ u \in L^2(\Omega) : \frac{|u(x,s) - u(y,t)|}{|(x,s) - (y,t)|^{1+r}} \in L^2(\Omega \times \Omega) \Big\}$$

with

$$\|u\|_{H^r(\Omega)} := \sqrt{\int_\Omega |u(x,t)|^2 d(x,t) + \int_\Omega \int_\Omega \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(x,t) d(y,s)}.$$

Finally, we also introduce a periodic fractional Sobolev space. Therefore, let $D_n := \mathbb{R} \times (-nT, nT)$ for $n \in \mathbb{N}$. Then

$$H^r_{\text{per}}(\mathbb{R}^2) := \{ u \colon \mathbb{R}^2 \to \mathbb{R} : u \in H^r(D_n) \text{ for all } n \in \mathbb{N} \text{ and } u \text{ is } T-\text{perodic in the second component} \}$$

with $\|u\|_{H^r_{\text{per}}(\mathbb{R}^2)} := \|u\|_{H^r(D_1)}$.

Here are another two lemmata of auxiliary character.

Lemma 5.24. *Let $n \in \mathbb{N}$ and $r \in (0,1)$. Then there is a constant $c = c(n,r) > 0$ such that*

$$\|u\|_{H^r(D_n)} \leq c(n,r)\|u\|_{H^r(D_1)} \tag{5.52}$$

for all $u \in H^r_{\mathrm{per}}(\mathbb{R}^2)$.

Proof. We only show (5.52) for $n = 2$. The case $n > 2$ can be established by the same techniques. We have $\|u\|^2_{L^2(D_2)} = 2\|u\|^2_{L^2(D_1)}$, i.e., it remains to bound the expression

$$\int_{D_2}\int_{D_2} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(x,t)d(y,s)$$

by a constant multiple of $\|u\|^2_{H^r(D_1)}$. The idea is to split the domain of integration $D_2 \times D_2$ in several parts. Due to symmetry of the integrand in the variables t and s it is enough to consider the three cases

$$1)\ t,s \in (-T,T) \quad 2)\ t \in [T,2T), s \in (0,2T) \quad 3)\ t \in [T,2T), s \in (-2T,0)$$

which are treated one after another.

1) $t,s \in (-T,T)$: We directly obtain $\int_{\mathbb{R}\times(-T,T)}\int_{\mathbb{R}\times(-T,T)} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{2(1+r)}} d(x,t)d(y,s) \leq \|u\|^2_{H^r(D_1)}$.

2) $t \in [T,2T), s \in (0,2T)$: With the substitution $(\tilde{t},\tilde{s}) = (t-T, s-T)$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}\times[T,2T)}\int_{\mathbb{R}\times[0,2T)} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2(1+r)}} d(y,s)d(x,t) \\
&= \int_{\mathbb{R}\times[0,T)}\int_{\mathbb{R}\times[-T,T)} \frac{|u(x,\tilde{t}\,) - u(y,\tilde{s})|^2}{|(x,\tilde{t}\,) - (y,\tilde{s})|^{2(1+r)}} d(y,\tilde{s})d(x,\tilde{t}\,) \leq \|u\|^2_{H^r(D_1)}.
\end{aligned}$$

3) $t \in [T,2T), s \in (-2T,0)$: In this case we estimate

$$\begin{aligned}
&\int_{\mathbb{R}\times[T,2T)}\int_{\mathbb{R}\times(-2T,0)} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2+2r}} d(y,s)d(x,t) \\
&\leq 2\int_{\mathbb{R}\times[T,2T)} |u(x,t)|^2 \int_{\mathbb{R}\times(-2T,0)} \frac{1}{|(x,t) - (y,s)|^{2+2r}} d(y,s)d(x,t) \\
&\quad + 2\int_{\mathbb{R}\times(-2T,0)} |u(y,s)|^2 \int_{\mathbb{R}\times[T,2T)} \frac{1}{|(x,t) - (y,s)|^{2+2r}} d(x,t)d(y,s)
\end{aligned} \tag{5.53}$$

For shorter notation we set

$$\begin{aligned}
I_{x,t} &:= \int_{\mathbb{R}\times(-2T,0)} \frac{1}{|(x,t) - (y,s)|^{2+2r}} d(y,s) \text{ for } (x,t) \in \mathbb{R}\times[T,2T), \\
I_{y,s} &:= \int_{\mathbb{R}\times[T,2T)} \frac{1}{|(x,t) - (y,s)|^{2+2r}} d(x,t) \text{ for } (y,s) \in \mathbb{R}\times(-2T,0).
\end{aligned}$$

Then with $(z,\delta) := (y-x, s-t), t \in [T,2T)$ and polar coordinates $\tilde{r} := \sqrt{z^2+\delta^2}$ we infer

$$I_{x,t} = \int_{\mathbb{R}\times(-2T-t,-t)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta) \leq \int_{\mathbb{R}\times(-4T,-T)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta) \leq \int_{\mathbb{R}^2\setminus B_T(0)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta)$$

$$= 2\pi \int_T^\infty \frac{1}{\tilde{r}^{2+2r}} \tilde{r} d\tilde{r} = \frac{\pi}{rT^{2r}}.$$

In the same manner, with $(z, \delta) := (x - y, t - s)$, $s \in (-2T, 0)$ we deduce

$$I_{y,s} = \int_{\mathbb{R}\times[T-s,2T-s)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta) \le \int_{\mathbb{R}\times[T,4T)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta) \le \int_{\mathbb{R}^2\setminus B_T(0)} \frac{1}{|(z,\delta)|^{2+2r}} d(z,\delta) = \frac{\pi}{rT^{2r}}.$$

In summary, (5.53) implies

$$\begin{aligned}
&\int_{\mathbb{R}\times[T,2T)} \int_{\mathbb{R}\times(-2T,0)} \frac{|u(x,t) - u(y,s)|^2}{|(x,t) - (y,s)|^{2+2r}} d(y,s) d(x,t) \\
&\le \frac{2\pi}{rT^{2r}} \left(\int_{\mathbb{R}\times[T,2T)} |u(x,t)|^2 d(x,t) + \int_{\mathbb{R}\times(-2T,0)} |u(y,s)|^2 d(y,s) \right) \\
&\le \frac{4\pi}{rT^{2r}} \|u\|^2_{L^2(D_1)} \le \frac{4\pi}{rT^{2r}} \|u\|^2_{H^r(D_1)}
\end{aligned}$$

and the proof is done. □

Lemma 5.25. *For $(z, \delta) \in \mathbb{R}^2$ and $u \in L^2(D)$ we have*

$$\int_D |u(x,s) - u(x+z, s+\delta)|^2 d(x,s) = 2T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1 - \cos(k\omega\delta + \xi z)) |(\mathcal{F}\hat{u}_k)(\xi)|^2 d\xi.$$

Proof. By using that u is real-valued we directly calculate

$$\begin{aligned}
&\int_D |u(x,s) - u(x+z, s+\delta)|^2 d(x,s) \qquad (5.54)\\
&= \int_{\mathbb{R}} \int_0^T \sum_{k\in\mathbb{Z}_{\text{odd}}} (\hat{u}_k(x+z) e^{ik\omega(s+\delta)} - \hat{u}_k(x) e^{ik\omega s}) \sum_{k\in\mathbb{Z}_{\text{odd}}} (\hat{u}_k(x+z) e^{ik\omega(s+\delta)} - \hat{u}_k(x) e^{ik\omega s}) ds dx \\
&= \int_{\mathbb{R}} \int_0^T \sum_{k\in\mathbb{Z}_{\text{odd}}, l\in\mathbb{Z}_{\text{even}}} \Big((\hat{u}_k(x+z) e^{ik\omega(s+\delta)} - \hat{u}_k(x) e^{ik\omega s}) (\hat{u}_{l-k}(x+z) e^{i(l-k)\omega(s+\delta)} - \hat{u}_{l-k}(x) e^{i(l-k)\omega s}) \Big) ds dx \\
&= \int_{\mathbb{R}} \int_0^T \sum_{k\in\mathbb{Z}_{\text{odd}}, l\in\mathbb{Z}_{\text{even}}} \Big(\hat{u}_k(x+z) \hat{u}_{l-k}(x+z) e^{il\omega(s+\delta)} - \hat{u}_k(x+z) \hat{u}_{l-k}(x) e^{i\omega(k\delta + ls)} \\
&\qquad\qquad - \hat{u}_k(x) \hat{u}_{l-k}(x+z) e^{i\omega(ls + (l-k)\delta)} + \hat{u}_k(x) \hat{u}_{l-k}(x) e^{il\omega s} \Big) ds dx \\
&= T \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}_{\text{odd}}} \Big(\hat{u}_k(x+z) \hat{u}_{-k}(x+z) - \hat{u}_k(x+z) \hat{u}_{-k}(x) e^{i\omega k\delta} - \hat{u}_k(x) \hat{u}_{-k}(x+z) e^{-i\omega k\delta} + \hat{u}_k(x) \hat{u}_{-k}(x) \Big) dx,
\end{aligned}$$

where in the last equation we interchanged the order of summation and integration over s which is possible due to Fubini. Notice that we have

$$|e^{ix} - 1|^2 = 2 - (e^{ix} + e^{-ix}) = 2(1 - \cos(x)). \qquad (5.55)$$

Thus, with the help of Plancharel's Theorem, basic calculation rules for the Fourier transform and (5.55) we can continue the chain of equalities in (5.54) by

$$\int_D |u(x,s) - u(x+z, s+\delta)|^2 d(x,s)$$

$$= T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \left(|\hat{u}_k(x+z)|^2 + |\hat{u}_k(x)|^2 - e^{ik\omega\delta}\hat{u}_k(x+z)\overline{\hat{u}_k(x)} - e^{-ik\omega\delta}\hat{u}_k(x)\overline{\hat{u}_k(x+z)}\right) dx$$
$$= T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} |\hat{u}_k(x+z)e^{ik\omega\delta} - \hat{u}_k(x)|^2 dx = T \sum_{k\in\mathbb{Z}_{\text{odd}}} \|\hat{u}_k(\cdot+z)e^{ik\omega\delta} - \hat{u}_k(\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= T \sum_{k\in\mathbb{Z}_{\text{odd}}} \|\mathcal{F}\left(\hat{u}_k(\cdot+z)e^{ik\omega\delta} - \hat{u}_k(\cdot)\right)\|^2_{L^2(\mathbb{R})} = T \sum_{k\in\mathbb{Z}_{\text{odd}}} \|(e^{i(k\omega\delta+z\cdot)} - 1)(\mathcal{F}\hat{u}_k)(\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= 2T \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1-\cos(k\omega\delta+\xi z))|(\mathcal{F}\hat{u}_k)(\xi)|^2 d\xi. \qquad \square$$

In the end, we have all ingredients to deduce several embeddings from $\mathcal{H}$ into the spaces introduced previously. The first result summarizes the outcome of Section 5.4 and demonstrates a connection between $\mathcal{H}$ and the spaces $\tilde{H}^{\frac{1}{4}}(D), H^{\frac{1}{4}}(D)$ as well as $H^{\frac{1}{4}}_{\text{per}}(\mathbb{R}^2)$.

Theorem 5.26. *The following linear operators are bounded:*

$$\mathcal{S}_1\colon \mathcal{H} \to \hat{H}, (\mathcal{S}_1\tilde{u})_k(x) := \hat{u}_k(x) := \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{\tilde{u}}_{j,k}(s)\psi_{j,k}(x,s)ds \textit{ for } k\in\mathbb{Z}_{\text{odd}},$$
$$\mathcal{S}_2\colon \hat{H} \to \tilde{H}^{\frac{1}{4}}(D), \left(\mathcal{S}_2(\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}}\right)(x,t) := \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t},$$
$$\mathcal{S}_3\colon \tilde{H}^{\frac{1}{4}}(D) \to H^{\frac{1}{4}}_{\text{per}}(\mathbb{R}^2), \mathcal{S}_3 u(x,t) := u(x,s), \textit{ where } s = t \mod T,$$
$$\mathcal{S}_4\colon \tilde{H}^{\frac{1}{4}}(D) \to H^{\frac{1}{4}}(D), \mathcal{S}_4 u(x,t) := u(x,s), \textit{ where } s = t \mod T.$$

Proof. We investigate the four operators separately.
1) Boundedness of $\mathcal{S}_1$: Due to $b_k(v^+,v^+) - b_k(v^-,v^-) = \sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{v}_{j,k}(s)|^2 ds$, Corollary 5.19 and Theorem 5.20 we know that there is $C>0$ such that

$$|k|\,\|v\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3}\|v'\|^2_{L^2(\mathbb{R})} \le C \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{v}_{j,k}(s)|^2 ds \tag{5.56}$$

for all $v\in H^1(\mathbb{R})$. Setting $v = \hat{u}_k$ in (5.56) and summing over $k\in\mathbb{Z}_{\text{odd}}$ gives

$$\|\left((\mathcal{S}_1\tilde{u})_k\right)_{k\in\mathbb{Z}_{\text{odd}}}\|^2_{\hat{H}} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \left(|k|\,\|\hat{u}_k\|^2_{L^2(\mathbb{R})} + \frac{1}{|k|^3}\|\hat{u}_k'\|^2_{L^2(\mathbb{R})}\right) \le C \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{\tilde{u}}_{j,k}(s)|^2 ds = C\|\tilde{u}\|^2_{\mathcal{H}},$$

which proves the boundedness of $\mathcal{S}_1$.
2) Boundedness of $\mathcal{S}_2$: By Plancharel's identity we obtain

$$\|\hat{u}_k'\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}(\hat{u}_k')\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \xi^2|(\mathcal{F}\hat{u}_k)(\xi)|^2 d\xi.$$

Recall Young's inequality $ab \le \frac{a^4}{4} + \frac{3b^{\frac{4}{3}}}{4}$ for $a,b\ge 0$. Thus, we infer

$$\|\mathcal{S}_2(\hat{u}_k)_{k\in\mathbb{Z}_{\text{odd}}}\|_{\tilde{H}^{\frac{1}{4}}(D)} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} (1+\xi^2+k^2)^{\frac{1}{4}}|\mathcal{F}\hat{u}_k(\xi)|^2 d\xi = \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \left(\frac{1+\xi^2+k^2}{|k|^3}\right)^{\frac{1}{4}} |k|^{\frac{3}{4}}\,|\mathcal{F}\hat{u}_k(\xi)|^2 d\xi$$

$$
\begin{aligned}
&\le \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} \left(\frac{1}{4} \frac{1+\xi^2+k^2}{|k|^3} + \frac{3}{4}|k| \right) |\mathcal{F}\hat{u}_k(\xi)|^2 d\xi \\
&\le \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} \left(\frac{1}{4} \frac{\xi^2}{|k|^3} + \frac{5}{4}|k| \right) |\mathcal{F}\hat{u}_k(\xi)|^2 d\xi = \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \frac{1}{4\,|k|^3} \|\hat{u}_k'\|^2_{L^2(\mathbb{R})} + \frac{5}{4} \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} |k| \|\hat{u}_k\|^2_{L^2(\mathbb{R})} \\
&\le \frac{5}{4} \| (\hat{u}_k)_{k\in\mathbb{Z}_{\mathrm{odd}}} \|^2_{\hat{H}}
\end{aligned}
$$

which shows the boundedness of $\mathcal{S}_2$.

3) Boundedness of $\mathcal{S}_3$: Fix $n \in \mathbb{N}$. Then due to periodicity

$$
\begin{aligned}
\|u\|^2_{L^2(D_n)} &= \int_{D_n} |u(x,t)|^2 d(x,t) = 2n \int_D |u(x,t)|^2 d(x,t) \\
&= 2nT \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} |\hat{u}_k(x)|^2 dx \le 2nT \|u\|^2_{\tilde{H}^{\frac{1}{4}}(D)}.
\end{aligned}
\tag{5.57}
$$

Moreover, with the help of the substitution $(z,\delta) := (y-x, s-t)$, Fubini and the periodicity of u in the second component we obtain

$$
\begin{aligned}
&\int_{D_n} \int_{D_n} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t) d(y,s) \\
&= \int_{D_n} \int_{\mathbb{R}\times(-nT-t, nT-t)} \frac{|u(x,t)-u(x+z,t+\delta)|^2}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) d(x,t) \\
&\le \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} \int_{D_n} |u(x,t)-u(x+z,t+\delta)|^2 d(x,t) d(z,\delta) \\
&= \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} \|u(z+\cdot,\delta+\cdot)-u(\cdot,\cdot)\|^2_{L^2(D_n)} d(z,\delta) \\
&= 2n \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} \|u(z+\cdot,\delta+\cdot)-u(\cdot,\cdot)\|^2_{L^2(D)} d(z,\delta).
\end{aligned}
\tag{5.58}
$$

Due to Lemma 5.25 and Lemma 5.22 we conclude

$$
\begin{aligned}
&2n \int_{\mathbb{R}^2} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} \|u(z+\cdot,\delta+\cdot)-u(\cdot,\cdot)\|^2_{L^2(D)} d(z,\delta) \\
&= 4nT \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{1-\cos(k\omega\delta+\xi z)}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) |\mathcal{F}\hat{u}_k(\xi)|^2 d\xi \\
&= 4nTc_1 \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathbb{R}} \sqrt[4]{\omega^2k^2+\xi^2} |(\mathcal{F}\hat{u}_k)(\xi)|^2 d\xi \le \tilde{c}(n) T \|u\|^2_{\tilde{H}^{\frac{1}{4}}(D)}
\end{aligned}
\tag{5.59}
$$

for a constant $\tilde{c}(n) > 0$. The combination of (5.57), (5.58) and (5.59) implies

$$
\|\mathcal{S}_3 u\|_{H^{\frac{1}{4}}_{\mathrm{per}}(\mathbb{R}^2)} \le (2+\tilde{c}(1)) T \|u\|^2_{\tilde{H}^{\frac{1}{4}}(D)},
$$

i.e., the boundedness of $\mathcal{S}_3$. The boundedness of $\mathcal{S}_4$ follows in the same spirit. □

The next lemma contains a crucial step in our regularity considerations.

Lemma 5.27. *Let*

$$\varphi\colon \mathbb{R} \to \mathbb{R}, \varphi(t) := \begin{cases} 1 & , \text{ if } t \in [-T, T], \\ 2 - \frac{1}{T}t & , \text{ if } t \in (T, 2T), \\ 2 + \frac{1}{T}t & , \text{ if } t \in (-2T, -T), \\ 0 & , \text{ if } t \in (-\infty, -2T] \cup [2T, \infty). \end{cases}$$

Then $\varphi u \in H^{\frac{1}{4}}(\mathbb{R}^2)$ and the multiplication operator $\mathcal{S}_5\colon H^{\frac{1}{4}}_{\text{per}}(\mathbb{R}^2) \to H^{\frac{1}{4}}(\mathbb{R}^2), u \mapsto \varphi u$ is bounded.

Proof. Let $u \in H^{\frac{1}{4}}_{\text{per}}(\mathbb{R}^2)$. Notice that φ is Lipschitz-continuous with Lipschitz constant $\frac{1}{T}$. By definition of φ and the periodicity of u in the second component we have

$$\|\varphi u\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}\times(-2T,2T)} |\varphi(t)u(x,t)|^2 d(x,t) \leq 2\|u\|^2_{L^2(D_1)} \leq 2\|u\|^2_{H^{\frac{1}{4}}_{\text{per}}(\mathbb{R}^2)}.$$

It remains to bound the expression

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(t)u(x,t) - \varphi(s)u(y,s)|^2}{|(x,t) - (y,s)|^{\frac{5}{2}}} d(x,t)d(y,s)$$

by constant multiples of $\|\cdot\|_{L^2(D_1)}$ and $\|\cdot\|_{H^{\frac{1}{4}}(D_1)}$. Therefore, we split the domain of integration into nine subdomains, namely,

$$\begin{aligned}
\Omega_1 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t, s \in (-2T, 2T)\}, \\
\Omega_2 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t, s \in [2T, \infty)\}, \\
\Omega_3 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t, s \in (-\infty, -2T]\}, \\
\Omega_4 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-2T, 2T), s \in [2T, \infty)\}, \\
\Omega_5 &:= \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-2T, 2T), t \in [2T, \infty)\}, \\
\Omega_6 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-2T, 2T), s \in (-\infty, -2T]\}, \\
\Omega_7 &:= \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-2T, 2T), t \in (-\infty, -2T]\}, \\
\Omega_8 &:= \{(x,t,y,s) \in \mathbb{R}^4 : t \in (-\infty, -2T], s \in [2T, \infty)\}, \\
\Omega_9 &:= \{(x,t,y,s) \in \mathbb{R}^4 : s \in (-\infty, -2T], t \in [2T, \infty)\}.
\end{aligned}$$

With $I_r := \int_{\Omega_r} \frac{|\varphi(t)u(x,t) - \varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t,y,s)$ for $r \in \{1, 2, \ldots, 9\}$ and Fubini we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(t)u(x,t) - \varphi(s)u(y,s)|^2}{|(x,t) - (y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) = \sum_{r=1}^{9} I_r.$$

Due to symmetry in the variables (x,t) and (y,s) we infer that $I_4 = I_5 = I_6 = I_7$. Since $\varphi \equiv 0$ on $(-\infty, -2T] \cup [2T, \infty)$ we have $I_2 = I_3 = I_8 = I_9 = 0$. Therefore, it is sufficient to estimate I_1 and I_4 which will be done in the following.

Estimation of I_1: We have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \frac{|\varphi(t)u(x,t) - \varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \\ &\leq 2 \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \left(\frac{|\varphi(t)(u(x,t)-u(y,s))|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} + \frac{|(\varphi(t)-\varphi(s))u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} \right) d(x,t)d(y,s) \end{aligned}$$

and both summands will be treated separately. With the help of $\varphi \leq 1$ and Lemma 5.24 for $n = 2$ we infer

$$\begin{aligned} &\int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \frac{|\varphi(t)(u(x,t)-u(y,s))|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \\ &\leq \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \frac{|u(x,t)-u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \leq \|u\|^2_{H^{\frac{1}{4}}(D_2)} \leq c(2,\frac{1}{4})\|u\|^2_{H^{\frac{1}{4}}(D_1)} \end{aligned}$$

with the constant $c(2, \frac{1}{4})$ from Lemma 5.24.
For the second summand we use the Lipschitz-continuity of φ and the substitution $(z,\delta) = (x-y, t-s)$ in order to estimate

$$\begin{aligned} &\int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \frac{|(\varphi(t)-\varphi(s))u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \\ &\leq \frac{1}{T^2} \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times(-2T,2T)} \frac{|t-s|^2|u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \\ &= \frac{1}{T^2} \int_{\mathbb{R}\times(-2T,2T)} |u(y,s)|^2 \int_{\mathbb{R}\times(-2T-s,2T-s)} \frac{\delta^2}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta)d(y,s) \\ &\leq \frac{1}{T^2} \int_{\mathbb{R}\times(-2T,2T)} |u(y,s)|^2 d(y,s) \int_{\mathbb{R}\times(-4T,4T)} \frac{\delta^2}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) \\ &= \frac{8}{T^2} \int_{\mathbb{R}\times(-T,T)} |u(y,s)|^2 d(y,s) \int_{(0,\infty)\times(0,4T)} \frac{\delta^2}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) \leq \frac{128}{T}(1+4T)\,\|u\|^2_{L^2(D_1)} \end{aligned}$$

due to the periodicity of u in the second component and Lemma 5.23.
Estimation of I_4: First of all, notice that for $T > 0$ and $t < 2T$ by polar coordinates

$$\int_{\mathbb{R}\times[2T-t,\infty)} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) \leq \int_{\mathbb{R}^2\setminus B_{2T-t}(0)} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta) = 2\pi \int_{2T-t}^{\infty} \frac{1}{r^{\frac{5}{2}}} r\,dr = \frac{4\pi}{\sqrt{2T-t}}.$$

Thus, the substitution $(z,\delta) := (y-x, s-t)$ and the Lipschitz-continuity of φ imply

$$\begin{aligned} I_4 &= \int_{\mathbb{R}\times(-2T,2T)} \int_{\mathbb{R}\times[2T,\infty)} \frac{|\varphi(t)u(x,t)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(y,s)d(x,t) \\ &= \int_{\mathbb{R}\times(-2T,2T)} |\varphi(t)|^2|u(x,t)|^2 \int_{\mathbb{R}\times[2T-t,\infty)} \frac{1}{|(z,\delta)|^{\frac{5}{2}}} d(z,\delta)d(x,t) \\ &\leq \int_{\mathbb{R}\times(-2T,2T)} |\varphi(2T)-\varphi(t)|^2|u(x,t)|^2 \frac{4\pi}{\sqrt{2T-t}} d(x,t) \end{aligned}$$

$$\leq \frac{4\pi}{T^2} \int_{\mathbb{R}\times(-2T,2T)} (2T-t)^{\frac{3}{2}} |u(x,t)|^2 d(x,t)$$
$$\leq \frac{32\pi}{\sqrt{T}} \int_{\mathbb{R}\times(-2T,2T)} |u(x,t)|^2 d(x,t) = \frac{64\pi}{\sqrt{T}} \|u\|^2_{L^2(D_1)}.$$

The combination of the estimates of I_1 and I_4 together with the symmetry considerations yield

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\varphi(t)u(x,t)-\varphi(s)u(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \leq c(2,\frac{1}{4})\|u\|^2_{H^{\frac{1}{4}}(D_1)} + \left(\frac{64\pi}{\sqrt{T}} + \frac{128}{T}(1+4T)\right)\|u\|^2_{L^2(D_1)}$$

where again $c(2,\frac{1}{4})$ denotes the constant from Lemma 5.24. This finishes the proof. □

We now give the last chain of embeddings.

Corollary 5.28. *For any $u \in H^{\frac{1}{4}}_{\mathrm{per}}(\mathbb{R}^2)$ and any $q \in [2, \frac{8}{3}]$ we have*

$$\|u\|_{L^q(D)} \leq \|\varphi u\|_{L^q(\mathbb{R}^2)} \leq c(q)\|\varphi u\|_{H^{\frac{1}{4}}(\mathbb{R}^2)} \leq c(q)\sqrt{c}\|u\|_{H^{\frac{1}{4}}_{\mathrm{per}}(\mathbb{R}^2)}$$

with φ and $c > 0$ from Lemma 5.27 and a constant $c(q) > 0$ not depending on u.

Proof. The last inequality is precisely Lemma 5.27 whereas the first inequality is trivial due to $\varphi u \equiv u$ on D. The second inequality follows from an embedding theorem for fractional Sobolev spaces, precisely, $H^{\frac{1}{4}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \in [2, \frac{8}{3}]$, see for instance Theorem 6.5 in [29] or Theorem 1.66 in [4]. □

After all these calculations we are finally ready to give the proof of Theorem 5.21.
Proof of Theorem 5.21: With the linear operators from Theorem 5.26 we have $\mathcal{S}\tilde{u} = (\mathcal{S}_3 \circ \mathcal{S}_2 \circ \mathcal{S}_1)\tilde{u}$ in $H^{\frac{1}{4}}_{\mathrm{per}}(\mathbb{R}^2)$. Due to the boundedness of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_5$ and Corollary 5.28 we conclude

$$\|\mathcal{S}\tilde{u}\|_{L^q(D)} \leq \|\mathcal{S}_5(\mathcal{S}\tilde{u})\|_{L^q(\mathbb{R}^2)} \leq c(q)\|\mathcal{S}_5(\mathcal{S}\tilde{u})\|_{H^{\frac{1}{4}}(\mathbb{R}^2)} \leq c(q)C\|\mathcal{S}\tilde{u}\|_{H^{\frac{1}{4}}_{\mathrm{per}}(\mathbb{R}^2)} \leq c(q)\tilde{C}\|\tilde{u}\|_{\mathcal{H}},$$

where $C := \|\mathcal{S}_5\| > 0$ and $\tilde{C} := C\|\mathcal{S}_3\|\|\mathcal{S}_2\|\|\mathcal{S}_1\| > 0$. □

5.5.1. Compatibility of nonlinearity and Hilbert space

In summary, Theorem 5.21 guarantees that $\mathcal{S}$ transforms elements of $\mathcal{H}$ into $L^q(D)$-functions for all $q \in [2, \frac{8}{3}]$. In this section, we use this result to control the nonlinearity in (5.3).
By standard calculations (compare Proposition 1.12 in [73]) we infer that the functional

$$\mathcal{J}_1 \colon L^{p+1}(D) \to \mathbb{R}; \mathcal{J}_1(u) = \int_D |u(x,t)|^{p+1} d(x,t), \, p \in [1, \frac{5}{3}]$$

is continuously Fréchet-differentiable with

$$\mathcal{J}_1'(u)[v] = (p+1)\int_D |u(x,t)|^{p-1}\, u(x,t)v(x,t)d(x,t) \text{ for all } u, v \in L^{p+1}(D) \text{ real-valued.}$$

Recall that the linear transformation

$$\mathcal{S}\colon \mathcal{H} \to L^{p+1}(D); \tilde{u} \mapsto (\mathcal{S}\tilde{u})(x,t) := \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\tilde{u}}_{j,k}(s)\psi_{j,k}(x,s)ds e^{ik\omega t}$$

is bounded by Theorem 5.21. Moreover, let

$$J_1\colon \mathcal{H} \to \mathbb{R}, J_1 := \mathcal{J}_1 \circ \mathcal{S}. \tag{5.60}$$

Then $J_1 \in C^1(\mathcal{H};\mathbb{R})$ and by chain rule we have

$$J_1'(\tilde{u})[\tilde{v}] = \mathcal{J}_1'(\mathcal{S}\tilde{u})[\mathcal{S}\tilde{v}] \text{ for all } \tilde{u}, \tilde{v} \in \mathcal{H}. \tag{5.61}$$

5.6. Minimization on the generalized Nehari manifold

In this section we minimize a functional J on a suitable set, the so-called generalized Nehari manifold which is introduced later. Due to Section 5.5.1 we are able to define $J\colon \mathcal{H} \to \mathbb{R}$ by

$$J(\tilde{u}) := \frac{1}{2}J_0(\tilde{u}) - \frac{\Gamma}{T(p+1)}J_1(\tilde{u})$$

with

$$\begin{aligned} J_0(\tilde{u}) &:= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)|\tilde{\tilde{u}}_{j,k}(s)|^2 ds = \|\tilde{u}^+\|_{\mathcal{H}}^2 - \|\tilde{u}^-\|_{\mathcal{H}}^2, \\ J_1(\tilde{u}) &:= \int_D |\mathcal{S}\tilde{u}|^{p+1} d(x,t). \end{aligned}$$

By standard calculations (compare again Proposition 1.12 in [73]) we obtain the following result.

Lemma 5.29. *We have $J \in C^1(\mathcal{H})$ with*

$$J'(\tilde{u})[\tilde{v}] = \operatorname{Re} \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\tilde{u}}_{j,k}(s)\overline{\tilde{\tilde{v}}_{j,k}(s)}ds - \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{v} d(x,t).$$

In particular we have $J'(\tilde{u}) = 0$ for $\tilde{u} \in \mathcal{H}$ if and only if

$$\operatorname{Re} \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\tilde{u}}_{j,k}(s)\overline{\tilde{\tilde{v}}_{j,k}(s)}ds = \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{v} d(x,t) \text{ for all } \tilde{v} \in \mathcal{H}. \tag{5.62}$$

5.6.1. A variant of a lemma of P.L.Lions

Next, we modify Lions Lemma (see for instance Lemma 1.21 in [73]). Therefore, we need to work with sequences in $\mathcal{H}$ which we denote by $(\tilde{u}_n)_{n\in\mathbb{N}}$, i.e., for each $n \in \mathbb{N}$ we have $\tilde{u}_n = \left((\tilde{\tilde{u}}_{j,k})_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}}\right)_n$. Recall that we can interpret a function $\mathcal{S}\tilde{u}$ with $\tilde{u} \in \mathcal{H}$ as a function on D which is periodically continued in the second component. This is needed since in Lemma 5.31 we consider $\mathcal{S}\tilde{u}$ on balls $B_r(y)$ which can exceed the set D.

Lemma 5.30. *Let $r > 0$ and $T > 0$. Then there is a sequence $(y_l)_{l\in\mathbb{N}}$ in $\mathbb{R} \times [0, T)$ such that*

(a) $D \subset \bigcup_{l\in\mathbb{N}} B_r(y_l)$,

(b) Each point $y \in D$ is contained in at most four balls $B_r(y_l)$.

Proof. We choose $(y_l)_{l\in\mathbb{N}}$ to be an enumeration of the lattice $r\mathbb{Z}^2 \cap D$ where we assume w.l.o.g. that $r < T$ (otherwise $r\mathbb{Z}^2 \cap D = \emptyset$ and the statement of Lemma 5.30 is obvious). The statement then follows. □

Here is our variant of Lions' Lemma.

Lemma 5.31. *Let $q \in [2, \frac{8}{3})$ and $r > 0$ be given. Moreover, let $(\tilde{u}_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ and*

$$\sup_{z\in D} \int_{B_r(z)} |\mathcal{S}\tilde{u}_n|^q d(x,t) \to 0 \text{ as } n \to \infty. \tag{5.63}$$

Then $\mathcal{S}\tilde{u}_n \to 0$ in $L^{\tilde{q}}(D)$ as $n \to \infty$ for all $\tilde{q} \in (2, \frac{8}{3})$.

Proof. Fix $\tilde{u} \in \mathcal{H}$ and $y \in D$. Then by Hölder interpolation for $s \in (q, \frac{8}{3})$ there is $\lambda = \frac{s-q}{\frac{8}{3}-q}\frac{8}{3s}$ such that

$$\|\mathcal{S}\tilde{u}\|_{L^s(B_r(y))} \leq \|\mathcal{S}\tilde{u}\|^{1-\lambda}_{L^q(B_r(y))} \|\mathcal{S}\tilde{u}\|^{\lambda}_{L^{\frac{8}{3}}(B_r(y))}.$$

For $s = 2 + \frac{q}{4}$ we have $\lambda = \frac{2}{s}$ and in particular

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(B_r(y))} \leq \|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(y))} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y))} \leq \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y))} \sup_{z\in D} \|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))}. \tag{5.64}$$

We now choose the sequence $(y_l)_{l\in\mathbb{N}}$ from Lemma 5.30, then use (5.64) for $y = y_l$ and perform a summation over $l \in \mathbb{N}$. Due to Lemma 5.30 we obtain

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(D)} \leq \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^s_{L^s(B_r(y_l))} \leq \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y_l))} \sup_{z\in D} \|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))}.$$

The following Lemma 5.32 guarantees the existence of $C > 0$ such that

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y_l))} \leq C\|\tilde{u}\|^2_{\mathcal{H}}.$$

In summary,

$$\|\mathcal{S}\tilde{u}\|^s_{L^s(D)} \leq C\|\tilde{u}\|^2_{\mathcal{H}} \sup_{z\in D} \|\mathcal{S}\tilde{u}\|^{(1-\lambda)s}_{L^q(B_r(z))} \tag{5.65}$$

for any $\tilde{u} \in \mathcal{H}$. Plugging the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ into (5.65), assumption (5.63) entails $\|\mathcal{S}\tilde{u}_n\|_{L^s(D)} \to 0$ as $n \to \infty$. For $\tilde{q} \in (2, \frac{8}{3})$ again Hölder interpolation yields

$$\|\mathcal{S}\tilde{u}_n\|_{L^{\tilde{q}}(D)} \leq \begin{cases} \|\mathcal{S}\tilde{u}_n\|^{1-\lambda}_{L^2(D)} \|\mathcal{S}\tilde{u}_n\|^{\lambda}_{L^s(D)} & \text{, for } \lambda = \frac{s(\tilde{q}-2)}{\tilde{q}(s-2)} \text{ if } \tilde{q} \in (2, s), \\ \|\mathcal{S}\tilde{u}_n\|^{1-\lambda}_{L^{\frac{8}{3}}(D)} \|\mathcal{S}\tilde{u}_n\|^{\lambda}_{L^s(D)} & \text{, for } \lambda = \frac{s(\frac{8}{3}-\tilde{q})}{\tilde{q}(\frac{8}{3}-s)} \text{ if } \tilde{q} \in (s, \frac{8}{3}), \end{cases}$$

which finally yields the desired result $\|\mathcal{S}\tilde{u}_n\|_{L^{\tilde{q}}(D)}$ as $n \to \infty$ for all $\tilde{q} \in (2, \frac{8}{3})$. □

Lemma 5.32. *With the notation of Lemma 5.30 there is a constant $C > 0$ such that*

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y_l))} \leq C\|\tilde{u}\|^2_{\mathcal{H}} \textit{ for all } \tilde{u} \in \mathcal{H}.$$

Proof. Recall the compact embedding $H^{\frac{1}{4}}(B_r(y_l)) \hookrightarrow L^{\frac{8}{3}}(B_r(y_l))$ (Corollary 7.2 in [29]). Note that due to Lemma 5.30 we can divide the balls $B_r(y_l), l \in \mathbb{N}$ in two classes N_1 and N_2, where the set N_1 contains all balls which are completely in D and N_2 contains all the others which protrude beyond D. For $B_r(y_{l_1}), B_r(y_{l_2}) \in N_1$ the Sobolev constant in the compact embedding above is the same since (as in the classical case) they are invariant under translations. The Sobolev constant for the class N_2 may differ from the one for N_1 but again the constant stays invariant for all $B_r(y_l) \in N_2$ since by Lemma 5.30 we can choose the balls in such a way that they always protrude beyond D in the same way. Thus there is $\tilde{c} > 0$ such that

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y_l))} \leq \tilde{c} \sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{H^{\frac{1}{4}}(B_r(y_l))} \quad \text{for all } l \in \mathbb{N}. \tag{5.66}$$

We abbreviate $\tilde{D}_r := \bigcup_{l\in\mathbb{N}} B_r(y_l)$. Due to the overlapping property in Lemma 5.30 we calculate

$$\begin{aligned}
\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{H^{\frac{1}{4}}(B_r(y_l))} &= \sum_{l\in\mathbb{N}} \left(\int_{B_r(y_l)} |\mathcal{S}\tilde{u}|^2 d(x,t) + \int_{B_r(y_l)} \int_{B_r(y_l)} \frac{|(\mathcal{S}\tilde{u})(x,t) - (\mathcal{S}\tilde{u})(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \right) \\
&\leq 4 \int_{\tilde{D}_r} |(\mathcal{S}\tilde{u})(x,t)|^2 d(x,t) + 4 \int_{\tilde{D}_r} \int_{\tilde{D}_r} \frac{|(\mathcal{S}\tilde{u})(x,t) - (\mathcal{S}\tilde{u})(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s).
\end{aligned} \tag{5.67}$$

Due to $r < T$ (recall the proof of Lemma 5.30) and Lemma 5.24 we conclude

$$\begin{aligned}
&\int_{\tilde{D}_r} |(\mathcal{S}\tilde{u})(x,t)|^2 d(x,t) + \int_{\tilde{D}_r} \int_{\tilde{D}_r} \frac{|(\mathcal{S}\tilde{u})(x,t) - (\mathcal{S}\tilde{u})(y,s)|^2}{|(x,t)-(y,s)|^{\frac{5}{2}}} d(x,t)d(y,s) \\
&\leq \|\mathcal{S}\tilde{u}\|^2_{H^{\frac{1}{4}}(\mathbb{R}\times[-T,2T])} \leq \hat{c}\|\mathcal{S}\tilde{u}\|^2_{H^{\frac{1}{4}}(D)}
\end{aligned} \tag{5.68}$$

Finally, Theorem 5.26 (recall $\mathcal{S}\tilde{u} = (\mathcal{S}_4 \circ \mathcal{S}_2 \circ \mathcal{S}_1)\tilde{u}$ in $H^{\frac{1}{4}}(D)$) and the combination of (5.66), (5.67), (5.68) gives

$$\sum_{l\in\mathbb{N}} \|\mathcal{S}\tilde{u}\|^2_{L^{\frac{8}{3}}(B_r(y_l))} \leq 4\tilde{c}\hat{c}\|\mathcal{S}\tilde{u}\|^2_{H^{\frac{1}{4}}(D)} \leq C\|\tilde{u}\|^2_{\mathcal{H}}$$

and the proof is done. □

5.6.2. The minimization process

The exposition in this section is closely related to the one in [70]. We first verify the assumption (B_1) at the beginning of Chapter 4 in [70].

Lemma 5.33. *The following statements hold true:*

(a) J_1 is weakly lower semicontinuous,

$$J_1(0) = 0 \quad \textit{and} \quad \frac{\Gamma}{2T(p+1)} J_1'(\tilde{u})[\tilde{u}] > \frac{\Gamma}{T(p+1)} J_1(\tilde{u}) > 0 \textit{ for } \tilde{u} \neq 0. \tag{5.69}$$

(b) $\lim_{\tilde{u}\to 0} \frac{J_1'(\tilde{u})}{\|\tilde{u}\|_{\mathcal{H}}} = 0$ *and* $\lim_{\tilde{u}\to 0} \frac{J_1(\tilde{u})}{\|\tilde{u}\|_{\mathcal{H}}^2} = 0$.

(c) For a weakly compact set $U \subset \mathcal{H} \setminus \{0\}$ *we have*

$$\lim_{s\to\infty} \frac{J_1(s\tilde{u})}{s^2} = \infty$$

uniformly with respect to $\tilde{u} \in U$.

Proof. (a) Since J_1 is continuous and convex (recall $\mathcal{S}$ is linear) it is in particular weakly continuous. Due to $p > 1$ we obtain

$$\frac{\Gamma}{2T(p+1)} J_1'(\tilde{u})[\tilde{u}] = \frac{\Gamma}{2T} J_1(\tilde{u}) > \frac{\Gamma}{T(p+1)} J_1(\tilde{u}).$$

We now also verify the last inequality in (5.69). It suffices to prove that $\mathcal{S}\colon \mathcal{H} \to L^{p+1}(D)$ is one-to-one. Therefore, let $\tilde{u} \in \mathcal{H}$ be given with $\mathcal{S}\tilde{u} = 0$. In particular, $\mathcal{S}\tilde{u} \in L^2(D)$ and

$$0 = \|\mathcal{S}\tilde{u}\|_{L^2(D)}^2 = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\mathrm{odd}}} \int_{\mathcal{B}} |\tilde{\hat{u}}_{j,k}(s)|^2 ds,$$

i.e., $\tilde{u} = 0$ and (5.69) is verified.

(b) By embeddings we have

$$J_1'(\tilde{u})[\tilde{v}] = \frac{\Gamma}{T} \int_D |\mathcal{S}\tilde{u}|^{p-1} \mathcal{S}\tilde{u} \mathcal{S}\tilde{v} d(x,t) \le C \|\tilde{u}\|_{\mathcal{H}}^p \|\tilde{v}\|_{\mathcal{H}}.$$

In particular, we conclude

$$\left\| \frac{J_1'(\tilde{u})}{\|\tilde{u}\|} \right\|_{\mathcal{H}} \le C \|\tilde{u}\|_{\mathcal{H}}^{p-1} \to 0 \text{ as } \tilde{u} \to 0 \text{ in } \mathcal{H}.$$

Moreover, $J_1(\tilde{u}) \le C\|\tilde{u}\|_{\mathcal{H}}^{p+1}$ and according to this $\lim_{\tilde{u}\to 0} \frac{J_1(\tilde{u})}{\|\tilde{u}\|_{\mathcal{H}}^2} = 0$ since $p > 1$.

(c) Let $U \subset \mathcal{H} \setminus \{0\}$ be weakly compact and $\delta := \inf_{\tilde{u}\in U} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}$. We show that $\delta > 0$. There is a sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ in U with $\|\mathcal{S}\tilde{u}_n\|_{L^{p+1}(D)} \to \delta$ as $n \to \infty$. Since U is weakly compact there is $\tilde{u} \in U$ and a subsequence such that $\tilde{u}_{n_m} \rightharpoonup \tilde{u}$ in $\mathcal{H}$ as $m \to \infty$. In particular, $\mathcal{S}\tilde{u}_{n_m} \to \mathcal{S}\tilde{u}$ in $L^2(D_{\mathrm{loc}})$ as $m \to \infty$ and therefore by a further diagonal argument we can assume w.l.o.g. that $\mathcal{S}\tilde{u}_{n_m} \to \mathcal{S}\tilde{u}$ pointwise almost everywhere in D. In particular, Fatou's lemma gives

$$\delta = \liminf_{m\to\infty} \|\mathcal{S}\tilde{u}_{n_m}\|_{L^{p+1}(D)}^{p+1} \ge \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^{p+1} > 0$$

due to $0 \notin U$. Thus, for an arbitrary sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to \infty$ as $n \to \infty$ we infer

$$\inf_{\tilde{u}\in U} \frac{J_1(s_n\tilde{u})}{s_n^2} = \inf_{\tilde{u}\in U} s_n^{p-1} J_1(\tilde{u}) = s_n^{p-1} \inf_{\tilde{u}\in U} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^{p+1} = s_n^{p-1}\delta^{p+1} \to \infty \text{ as } n \to \infty$$

and the last equality sign shows that this statement holds true uniformly in U. $\square$

We next introduce some additional notation. Let

$$\mathcal{M} := \{\tilde{u} \in \mathcal{H} \setminus \mathcal{H}^- : J'(\tilde{u})[\tilde{u}] = 0 \text{ and } J'(\tilde{u})[\tilde{v}] = 0 \text{ for all } \tilde{v} \in \mathcal{H}^-\}$$

denote the so-called generalized Nehari manifold. Moreover, for $\tilde{u} \in \mathcal{H}$ we set

$$\mathcal{H}(\tilde{u}) := \mathbb{R}^+\tilde{u} \oplus \mathcal{H}^- = \mathbb{R}^+\tilde{u}^+ \oplus \mathcal{H}^-,$$

where $\mathbb{R}^+ = [0, \infty)$. Finally, let S denote the unit ball in $\mathcal{H}$ and define $S^+ := S \cap \mathcal{H}^+$.
The next two statements guarantee (B_2) and (B_3) of Chapter 4 in [70].

Lemma 5.34. *The following statements hold true:*

(a) For each $\tilde{w} \in \mathcal{H} \setminus \mathcal{H}^-$ there exists a unique nontrivial critical point $m_1(\tilde{w})$ of $J|_{\mathcal{H}(\tilde{w})}$. Moreover, $m_1(\tilde{w})$ is the unique global maximum of $J|_{\mathcal{H}(\tilde{w})}$ as well as $J(m_1(\tilde{w})) > 0$.

(b) There exists $\delta > 0$ such that $\|m_1(\tilde{w})^+\|_{\mathcal{H}} \geq \delta$ for all $\tilde{w} \in \mathcal{H} \setminus \mathcal{H}^-$.

(c) For each compact subset $K \subset \mathcal{H} \setminus \mathcal{H}^-$ there exists a constant $C = C(K)$ such that $\|m_1(\tilde{w})\|_{\mathcal{H}} \leq C(K)$ for all $\tilde{w} \in K$.

Proof. (a) Obviously, we have $\mathcal{H}(\tilde{w}) = \mathcal{H}\left(\frac{\tilde{w}^+}{\|\tilde{w}^+\|_{\mathcal{H}}}\right)$, so w.l.o.g. let $\tilde{w} \in S^+$. We divide the statement of part (a) in three steps which automatically give the desired result.
First claim: There is $R > 0$ such that $J(\tilde{u}) \leq 0$ for all $\tilde{u} \in \mathcal{H}(\tilde{w})$ with $\|\tilde{u}\|_{\mathcal{H}} \geq R$.
Suppose not. Then there is a sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ in $\mathcal{H}(\tilde{w})$ with $\|\tilde{u}_n\|_{\mathcal{H}} \geq n$ and $J(\tilde{u}_n) > 0$ for all $n \in \mathbb{N}$. Set $\tilde{v}_n := \frac{\tilde{u}_n}{\|\tilde{u}_n\|_{\mathcal{H}}}$, so there is $\tilde{v} \in \mathcal{H}(\tilde{w})$ such that $\tilde{v}_{n_m} \rightharpoonup \tilde{v}$ as $m \to \infty$. Due to

$$0 < \frac{J(\tilde{u}_{n_m})}{\|\tilde{u}_{n_m}\|^2_{\mathcal{H}}} = \frac{1}{2}\left(\|\tilde{v}^+_{n_m}\|^2_{\mathcal{H}} - \|\tilde{v}^-_{n_m}\|^2_{\mathcal{H}}\right) - \frac{\Gamma}{T(p+1)} \frac{J_1(\|\tilde{u}_{n_m}\|_{\mathcal{H}}\tilde{v}_{n_m})}{\|\tilde{u}_{n_m}\|^2_{\mathcal{H}}} \tag{5.70}$$

Lemma 5.33 (c) entails $\tilde{v} = 0$ since otherwise (5.70) can not hold true as $m \to \infty$. On the other hand, $J_1 \geq 0$ implies $\|\tilde{v}^+_{n_m}\|_{\mathcal{H}} > \|\tilde{v}^-_{n_m}\|_{\mathcal{H}}$ so that $\|\tilde{v}^+_{n_m}\|_{\mathcal{H}} \geq \delta$ for a further subsequence and $\delta \in (0, 1]$. Due to $\tilde{v}_{n_m} \in \mathcal{H}(\tilde{w})$ and $\tilde{w} \in S^+$ we have $\tilde{v}^+_{n_m} = r_m\tilde{w}$ for $r_m \geq \delta$. This implies $\delta \leq r_m = \|\tilde{v}^+_{n_m}\|_{\mathcal{H}} \leq \|\tilde{v}_{n_m}\|_{\mathcal{H}} = 1$. Thus there is $r \in [\delta, 1]$ such that $r_m \to r$ as $m \to \infty$ which entails $\tilde{v}^+_{n_m} \to r\tilde{w} \neq 0$ as $m \to \infty$. This contradicts $\tilde{v} = 0$.
Second claim: $J|_{\mathcal{H}(\tilde{w})}$ has a maximizer $\tilde{u}_1 \in \mathcal{M} \cap \mathcal{H}(\tilde{w})$.
Let $r > 0$. By the structure of J and $\tilde{w} \in S^+$ we have

$$\frac{J(r\tilde{w})}{r^2} = \frac{1}{2} - r^{p-1}\frac{\Gamma}{T(p+1)}J_1(\tilde{w})$$

and therefore $J(r\tilde{w}) > 0$ provided r is chosen sufficiently small. Hence, $\sup J|_{\mathcal{H}(\tilde{w})} > 0$. The first claim implies that a maximizing sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ has to be bounded, so there is $\tilde{u}_1 \in \mathcal{H}$ such that $\tilde{u}_{n_m} \rightharpoonup \tilde{u}_1$ as $m \to \infty$. Similar as already done in the first claim after (5.70) we infer that $\tilde{u}_1^+ = r\tilde{w}$ for $r > 0$, i.e., $\tilde{u}_1 \in \mathcal{H}(\tilde{w})$. Lemma 5.33 (a) implies that J is weakly upper semicontinuous and so $J(\tilde{u}_1) = \max J|_{\mathcal{H}(\tilde{w})} \in (0, \infty)$. The variational principle of Ekeland (see Theorem 2.2 in [33]) implies $J'(\tilde{u}_1)|_{\mathcal{H}(\tilde{w})} \equiv 0$, i.e., $\tilde{u}_1 \in \mathcal{M} \cap \mathcal{H}(\tilde{w})$.

Third claim: If $\tilde{u}_1 \in \mathcal{M} \cap \mathcal{H}(\tilde{w})$ and $\tilde{u}_2 \in \mathcal{H}(\tilde{w})$ with $\tilde{u}_1 \neq \tilde{u}_2$ then $J(\tilde{u}_2) < J(\tilde{u}_1)$.
We write $\tilde{u}_2 = (1+r)\tilde{u}_1 + \tilde{v}$ for $\tilde{v} \in \mathcal{H}^-$ and $r \geq -1$. Due to $\tilde{u}_1 \in \mathcal{M}$ we further have $J'(\tilde{u}_1)[\tilde{u}_2] = 0$. We have

$$
\begin{aligned}
&\frac{1}{2}\left(B((1+r)\tilde{u}_1 + \tilde{v}, (1+r)\tilde{u}_1 + \tilde{v}) - B(\tilde{u}_1, \tilde{u}_1)\right) \\
&= \frac{1}{2}\Big(((1+r)^2 - 1)B(\tilde{u}_1, \tilde{u}_1) + 2(1+r)B(\tilde{u}_1, \tilde{v}) + B(\tilde{v}, \tilde{v})\Big) \\
&= B(\tilde{u}_1, r(\frac{r}{2} + 1)\tilde{u}_1 + (1+r)\tilde{v}) - \frac{\|\tilde{v}^-\|_{\mathcal{H}}^2}{2}.
\end{aligned}
$$

Together with $z := r(\frac{r}{2}+1)\tilde{u}_1 + (1+r)\tilde{v}$ and

$$
0 = J'(\tilde{u}_1)[\tilde{z}] = B(\tilde{u}_1, \tilde{z}) - \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}_1|^{p-1}\mathcal{S}\tilde{u}_1\mathcal{S}\tilde{z}d(x,t)
$$

this leads to the expression

$$
\begin{aligned}
J(\tilde{u}_2) - J(\tilde{u}_1) &= \frac{1}{2}\left(B((1+r)\tilde{u}_1 + \tilde{v}, (1+r)\tilde{u}_1 + \tilde{v}) - B(\tilde{u}_1, \tilde{u}_1)\right) + \frac{\Gamma}{T(p+1)}\left(J_1(\tilde{u}_1) - J_1((1+r)\tilde{u}_1 + \tilde{v})\right) \\
&= -\frac{\|\tilde{v}^-\|_{\mathcal{H}}^2}{2} + \frac{\Gamma}{T(p+1)}\left(J_1(\tilde{u}_1) - J_1((1+r)\tilde{u}_1 + \tilde{v})\right) + \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}_1|^{p-1}\mathcal{S}\tilde{u}_1\mathcal{S}\tilde{z}d(x,t) < 0
\end{aligned}
$$

due to Lemma 38 in [70].
(b) First, consider $\tilde{v} \in \mathcal{H}^+$. Then we have

$$
\lim_{\tilde{v}\to 0}\frac{J(\tilde{v})}{\|\tilde{v}\|_{\mathcal{H}}^2} = \lim_{\tilde{v}\to 0}\left(\frac{1}{2} - \frac{\Gamma}{T(p+1)}\frac{J_1(\tilde{v})}{\|\tilde{v}\|_{\mathcal{H}}^2}\right) = \frac{1}{2}
$$

due to Lemma 5.33 (b). Thus there is $\rho_0 > 0$ such that $J(\tilde{v}) \geq \frac{1}{4}\|\tilde{v}\|_{\mathcal{H}}^2$ for all $\tilde{v} \in \mathcal{H}^+$ with $\|\tilde{v}\|_{\mathcal{H}} \leq \rho_0$. Hence for $\rho \in (0, \rho_0)$ we find $\eta = \frac{\rho^2}{4}$ with $J(\tilde{v}) \geq \eta$ for all $\tilde{v} \in \mathcal{H}^+$ with $\|\tilde{v}\|_{\mathcal{H}} = \rho$. Now, let $\tilde{w} \in \mathcal{H}\backslash\mathcal{H}^-$. Due to the structure of J we infer that

$$
\frac{\|m_1(\tilde{w})^+\|_{\mathcal{H}}^2}{2} \geq J(m_1(\tilde{w})). \tag{5.71}
$$

Since $m_1(\tilde{w})$ is the maximizer of $J|_{\mathcal{H}(\tilde{w})}$ we conclude

$$
J(m_1(\tilde{w})) \geq J\left(\rho\frac{\tilde{w}^+}{\|\tilde{w}^+\|_{\mathcal{H}}}\right) \geq \eta. \tag{5.72}
$$

and the combination of (5.71) and (5.72) finishes the proof of part (b).
(c) Since $m_1(\tilde{w}) = m_1\left(\frac{\tilde{w}^+}{\|\tilde{w}^+\|_{\mathcal{H}}}\right)$ it again suffices to consider a compact set $K \subset S^+$. Suppose the statement is violated. Then there is a sequence $(\tilde{w}_n)_{n\in\mathbb{N}}$ in K with $\|m_1(\tilde{w}_n)\|_{\mathcal{H}} \geq n$ for all $n \in \mathbb{N}$. W.l.o.g. we can assume that there is $\tilde{w} \in K$ such that $\tilde{w}_n \to \tilde{w}$ as $n \to \infty$. Due to the representation $m_1(\tilde{w}_n) = r_n\tilde{w}_n + \tilde{v}_n$ with $r_n \geq 0$ and $\tilde{v}_n \in \mathcal{H}^-$ we obtain $\|m_1(\tilde{w}_n)\|_{\mathcal{H}}^2 = r_n^2 + \|\tilde{v}_n\|_{\mathcal{H}}^2$. Since

$$
0 < J(m_1(\tilde{w}_n)) \leq \frac{1}{2}\left(\|m_1(\tilde{w}_n)^+\|_{\mathcal{H}}^2 - \|m_1(\tilde{w}_n)^-\|_{\mathcal{H}}^2\right) = \frac{1}{2}\left(r_n^2 - \|\tilde{v}_n\|_{\mathcal{H}}^2\right)
$$

we deduce $\|\tilde{v}_n\|_{\mathcal{H}} < r_n$ for all $n \in \mathbb{N}$ and $r_n \to \infty$ as $n \to \infty$ because of $\|m_1(\tilde{w}_n)\|_{\mathcal{H}} \to \infty$ as $n \to \infty$. Hence, the sequence $\left(\frac{\tilde{v}_n}{r_n}\right)_{n\in\mathbb{N}}$ is bounded and therefore we find $\tilde{v} \in \mathcal{H}^-$ such that $\frac{\tilde{v}_{n_m}}{r_{n_m}} \rightharpoonup \tilde{v}$ as $m \to \infty$. Next, we set

$$\tilde{u}_n := \frac{m_1(\tilde{w}_n)}{r_n} = \tilde{w}_n + \frac{\tilde{v}_n}{r_n}$$

so that

$$\tilde{u}_{n_m} \rightharpoonup \tilde{w} + \tilde{v} =: \tilde{u} \text{ as } m \to \infty. \tag{5.73}$$

The set $U := \{\tilde{u}\} \cup \{\tilde{u}_{n_m} : m \in \mathbb{N}\}$ is weakly compact. Due to (5.73) and $\tilde{w} \in S^+ \neq 0$ we conclude $\tilde{u} \neq 0$, i.e., $0 \notin U$. Finally,

$$\begin{aligned} 0 < J(m_1(\tilde{w}_{n_m})) &= r_{n_m}^2 \left(\frac{1}{2}\|\tilde{u}_{n_m}^+\|_{\mathcal{H}}^2 - \frac{1}{2}\|\tilde{u}_{n_m}^-\|_{\mathcal{H}}^2 - \frac{\Gamma}{T(p+1)} \frac{J_1(r_{n_m}\tilde{u}_{n_m})}{r_{n_m}^2} \right) \\ &\le r_{n_m}^2 \left(\frac{C}{2} - \frac{\Gamma}{T(p+1)} \frac{J_1(r_{n_m}\tilde{u}_{n_m})}{r_{n_m}^2} \right) \to -\infty \end{aligned}$$

by Lemma 5.33 (c) which gives the desired contradiction. □

Lemma 5.34 enables us to consider the two maps

$$m_1 \colon \mathcal{H} \setminus \mathcal{H}^- \to \mathcal{M} \text{ and } m_2 := m_1|_{S^+} \colon S^+ \to \mathcal{M}.$$

By Proposition 31 in [70] m_1 is continuous whereas m_2 is a homeomorphism. We introduce

$$\Psi \colon S^+ \to \mathbb{R}; \tilde{w} \mapsto J(m_2(\tilde{w})).$$

The next result is proved in Proposition 32 and Corollary 33 in [70].

Lemma 5.35. *(a)* $\Psi \in C^1(S^+, \mathbb{R})$ *with*

$$D_T\Psi(\tilde{w})[\tilde{z}] = \|m_2(\tilde{w})^+\|_{\mathcal{H}} J'(m_2(\tilde{w}))[\tilde{z}] \quad \text{for } \tilde{w} \in S^+, \tilde{z} \in T_{\tilde{u}}S^+, \tag{5.74}$$

where D_T stands for the derivative in tangential direction of the sphere and $T_{\tilde{u}}S^+$ denotes the tangent space of S^+ at the point $\tilde{u}$.

(b) If $(\tilde{w}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence for Ψ then $(\tilde{u}_n)_{n\in\mathbb{N}} := (m_2(\tilde{w}_n))_{n\in\mathbb{N}}$ is a Palais-Smale sequence for J.

Finally, we can turn to our overall goal of this section and verify the following statement.

Theorem 5.36. *The functional J admits a ground state $\tilde{u}$, i.e. $\tilde{u} \in \mathcal{M}$ satisfies* (5.62) *and* $J(\tilde{u}) = \inf_{\tilde{v}\in\mathcal{M}} J(\tilde{v})$.

Proof. We take a minimizing sequence $(\tilde{w}_n)_{n\in\mathbb{N}}$ in S^+ for Ψ and set $\tilde{u}_n := m_2(\tilde{w}_n)$. In particular, as a consequence of Ekeland's variational principle the minimizing sequence $(\tilde{w}_n)_{n\in\mathbb{N}}$ can be chosen in such a way that $\|D_T\Psi(\tilde{w}_n)\| \to 0$ as $n \to \infty$. Due to (5.74) we deduce

$$D_T\Psi(\tilde{w}_n) = \|\tilde{u}_n^+\|_{\mathcal{H}} J'(\tilde{u}_n)|_{T_{\tilde{w}_n}S^+} \to 0 \text{ as } n \to \infty$$

and Lemma 5.34 (b) entails $J'(\tilde{u}_n)|_{T_{\tilde{w}_n}S^+} \to 0$ as $n \to \infty$. Since $\tilde{u}_n \in \mathcal{M}$ the derivatives of J at $\tilde{u}_n$ in normal direction vanish, i.e., in summary we have $J'(\tilde{u}_n) \to 0$ as $n \to \infty$.

Lemma 5.37 which is suffixed to this proof guarantees that $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded. Thus, there is $\tilde{u} \in \mathcal{H}$ such that $\tilde{u}_{n_m} \rightharpoonup \tilde{u}$ as $m \to \infty$. We now proceed in three steps:

First claim: $J'(\tilde{u}) = 0$.

We choose a dense subset $M \subset \mathcal{H}$ such that elements $\tilde{v} \in M$ enjoy the property that $\mathcal{S}\tilde{v}$ has compact support in D. By Theorem B.9 we know that such a set M exists. Moreover, the compact support in D allows to apply compact Sobolev embeddings. Therefore, for $\tilde{v} \in M$ we conclude

$$\begin{aligned} J'(\tilde{u})[\tilde{v}] &= \langle \tilde{u}^+, \tilde{v}^+ \rangle - \langle \tilde{u}^-, \tilde{v}^- \rangle - \frac{\Gamma}{T} \int_D |\mathcal{S}\tilde{u}|^{p-1} \mathcal{S}\tilde{u}\mathcal{S}\tilde{v} d(x,t) \\ &= \lim_{m\to\infty} J'(\tilde{u}_{n_m})[\tilde{v}] = 0. \end{aligned}$$

Since M is dense in $\mathcal{H}$ and $J'(\tilde{u})$ is continuous we deduce $J'(\tilde{u}) = 0$.

Second claim: W.l.o.g. we may choose $\tilde{u} \in \mathcal{H}$ which satisfies $J'(\tilde{u}) = 0$ as well as $\tilde{u}^+ \neq 0$.

We first show that

$$\liminf_{n\to\infty} \sup_{z\in D} \int_{B_1(z)} |\mathcal{S}\tilde{u}_n|^2 d(x,t) > 0. \tag{5.75}$$

Suppose (5.75) is violated. Then Lemma 5.31 implies $\|\mathcal{S}\tilde{u}_n\|_{L^{p+1}(D)} \to 0$ as $n \to \infty$ along a subsequence which we again denote by $(\tilde{u}_n)_{n\in\mathbb{N}}$. Therefore, we conclude

$$\int_D |\mathcal{S}\tilde{u}_n|^{p-1} \mathcal{S}\tilde{u}_n \mathcal{S}\tilde{u}_n^+ d(x,t) \to 0 \text{ as } n \to \infty.$$

Due to $J'(\tilde{u}_n)\tilde{u}_n^+ \to 0$ as $n \to \infty$ and

$$J'(\tilde{u}_n)\tilde{u}_n^+ = \|\tilde{u}_n^+\|_{\mathcal{H}}^2 - \frac{\Gamma}{T} \int_D |\mathcal{S}\tilde{u}_n|^{p-1} \mathcal{S}\tilde{u}_n \mathcal{S}\tilde{u}_n^+ d(x,t)$$

we obtain $\|\tilde{u}_n^+\|_{\mathcal{H}}^2 \to 0$ as $n \to \infty$, a contradiction to Lemma 5.34 (b). Therefore, (5.75) is valid and we find $\delta > 0$, a sequence $(y_n)_{n\in\mathbb{N}}$ in D and a subsequence of $(\tilde{u}_n)_{n\in\mathbb{N}}$ (again denoted by $(\tilde{u}_n)_{n\in\mathbb{N}}$) such that

$$\int_{B_1(y_n)} |\mathcal{S}\tilde{u}_n|^2 d(x,t) \geq \delta > 0 \text{ for all } n \in \mathbb{N}. \tag{5.76}$$

The idea is to shift $\mathcal{S}\tilde{u}_n$ in such a way that we can make use of compact embeddings for the shifted sequence. For this purpose, notice that for $y_n = (x_n, t_n)^T$ with $x_n \in \mathbb{R}, t_n \in [0, T)$ we have $x_n = 2\pi m_n + r_n$ with $m_n \in \mathbb{Z}, r_n \in [0, 2\pi)$ for all $n \in \mathbb{N}$. Let $y_n' := (r_n, t_n)^T \in [0, 2\pi) \times [0, T)$. Using this notation we define $\tilde{v}_n = \left(\tilde{\tilde{v}}_{n_{j,k}}\right)_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}}$ for $n \in \mathbb{N}$ via

$$\tilde{\tilde{v}}_{n_{j,k}}(s) := e^{2\pi i m_n s} \tilde{\tilde{u}}_{n_{j,k}}(s).$$

Notice that $\tilde{v}_n \in \mathcal{H}$ for all $n \in \mathbb{N}$ due to

$$\sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)| |\tilde{\tilde{v}}_{n_{j,k}}(s)|^2 ds = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} |\lambda_{j,k}(s)| |\tilde{\tilde{u}}_{n_{j,k}}(s)|^2 ds < \infty$$

and

$$\overline{\tilde{\hat{v}}_{n_{j,k}}(s)} = e^{-2\pi i m_n s}\overline{\tilde{\hat{u}}_{n_{j,k}}(s)} = e^{-2\pi i m_n s}\tilde{\hat{u}}_{n_{j,-k}}(-s) = \tilde{\hat{v}}_{n_{j,-k}}(-s)$$

where we used that $\tilde{u}_n \in \mathcal{H}$ for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} \mathcal{S}\tilde{u}_n(y_n) &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\hat{u}}_{n_{j,k}}(s)\psi_{j,k}(x_n, s)ds e^{ik\omega t_n} \\ &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\hat{u}}_{n_{j,k}}(s)e^{2\pi i s m_n}\psi_{j,k}(r_n, s)ds e^{ik\omega t_n} \\ &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\hat{v}}_{n_{j,k}}(s)\psi_{j,k}(r_n, s)ds e^{ik\omega t_n} = \mathcal{S}\tilde{v}_n(y_n'). \end{aligned} \tag{5.77}$$

The calculation in (5.77) together with (5.76) and $y_n' \in [0, 2\pi) \times [0, T)$ for all $n \in \mathbb{N}$ show

$$\int_{\tilde{B}} |\mathcal{S}\tilde{v}_n|^2 d(x,t) \geq \int_{B_1(y_n')} |\mathcal{S}\tilde{v}_n|^2 d(x,t) \geq \delta \text{ for all } n \in \mathbb{N}$$

where $\tilde{B} := [-1, 2\pi + 1] \times [-1, T + 1]$. Hence, the compact embedding (see for instance Corollary 7.2 in [29]) on the compact set $\tilde{B}$ yields $\tilde{v} \in \mathcal{H}$ with $\|\mathcal{S}\tilde{v}^+\|_{L^{p+1}(D)} \neq 0$. We now prove some additional properties of $(\tilde{v}_n)_{n\in\mathbb{N}}$ which ensure that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is also a Palais-Smale sequence. We have $\|\tilde{v}_n\|^2_{\mathcal{H}} = \|\tilde{u}_n\|^2_{\mathcal{H}}$ as well as $B(\tilde{u}_n, \tilde{u}_n) = B(\tilde{v}_n, \tilde{v}_n)$ with B from (5.41). This entails

$$\|\tilde{u}_n^+\|_{\mathcal{H}} = \|\tilde{v}_n^+\|_{\mathcal{H}} \text{ and } \|\tilde{u}_n^-\|_{\mathcal{H}} = \|\tilde{v}_n^-\|_{\mathcal{H}} \text{ for all } n \in \mathbb{N}. \tag{5.78}$$

In particular $\|\tilde{v}_n^+\|_{\mathcal{H}} \neq 0$ for all $n \in \mathbb{N}$. From (5.77) we infer that $\int_D |\mathcal{S}\tilde{u}_n|^{p+1} d(x,t) = \int_D |\mathcal{S}\tilde{v}_n|^{p+1} d(x,t)$. This and (5.78) entails $J(\tilde{u}_n) = J(\tilde{v}_n)$. In order to prove that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Palais-smale sequence it remains to show that

$$\|J'(\tilde{v}_n)\| \to 0 \text{ as } n \to \infty. \tag{5.79}$$

Therefore, we treat J_0' and J_1' separately. For $\tilde{w} \in \mathcal{H}$ we calculate

$$\begin{aligned} J_0'(\tilde{u}_n)[\tilde{w}] &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\hat{u}}_{n_{j,k}}(s)\overline{\tilde{w}_{j,k}(s)}ds = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\hat{v}}_{n_{j,k}}(s)e^{-2\pi i m_n s}\overline{\tilde{w}_{j,k}(s)}ds \\ &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s)\tilde{\hat{v}}_{n_{j,k}}(s)\overline{\tilde{w}_{j,k}(s)e^{2\pi i m_n s}} = J_0'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}]. \end{aligned} \tag{5.80}$$

Similarly, we obtain

$$\begin{aligned} \mathcal{S}\tilde{u}_n(x,t) = \mathcal{S}(\tilde{v}_n e^{-2\pi i m_n s})(x,t) &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} e^{-2\pi i m_n s}\tilde{v}_{n_{j,k}}(s)\psi_{j,k}(x,s)ds e^{ik\omega t} \\ &= \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \tilde{v}_{n_{j,k}}(s)\psi_{j,k}(x - 2\pi m_n, s)ds e^{ik\omega t} = \mathcal{S}\tilde{v}_n(x - 2\pi m_n, t). \end{aligned}$$

Thus,

$$
\begin{aligned}
J_1'(\tilde{u}_n)[\tilde{w}] &= \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}_n|^{p-1}\mathcal{S}\tilde{u}_n\mathcal{S}\tilde{w}\,d(x,t) \\
&= \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{v}_n(\cdot-2\pi m_n,\cdot)|^{p-1}\mathcal{S}\tilde{v}_n(\cdot-2\pi m_n,\cdot)\mathcal{S}\tilde{w}\,d(x,t) \\
&= \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{v}_n|^{p-1}\mathcal{S}\tilde{v}_n\mathcal{S}\tilde{w}(\cdot+2\pi m_n,\cdot)\,d(x,t) = J_1'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}].
\end{aligned} \tag{5.81}
$$

From (5.80) and (5.81) we conclude that

$$
J'(\tilde{u}_n)[\tilde{w}] = J'(\tilde{v}_n)[\tilde{w}e^{2\pi i m_n s}]. \tag{5.82}
$$

Moreover, for $m \in \mathbb{Z}$ notice that $\Lambda_m : \mathcal{H} \to \mathcal{H}, \tilde{w} \mapsto \tilde{w}e^{-2\pi i m s}$ is a bijection with inverse $\Lambda_m^{-1} : \mathcal{H} \to \mathcal{H}, \tilde{w} \mapsto \tilde{w}e^{2\pi i m s}$. Thus, by (5.82) we conclude that $\|J'(\tilde{u}_n)\| = \|J'(\tilde{v}_n)\|$, i.e., $\|J'(\tilde{v}_n)\| \to 0$ as $n \to \infty$. In summary, we have shown that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence so that $J'(\tilde{v}) = 0$ follows as in the first claim.

Since we have already established $\tilde{v} \neq 0$ with $J'(\tilde{v}) = 0$ the last part of the claim, namely $\tilde{v}^+ \neq 0$ can be deduced from the following consideration. Assume by contradiction that $\tilde{v}^+ = 0$, i.e., $\tilde{v} = \tilde{v}^-$. Then testing $J'(\tilde{v}) = 0$ with $\tilde{v}$ we infer

$$
-\frac{1}{2}\|\tilde{v}^-\|_{\mathcal{H}}^2 = \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{v}|^{p+1}\,d(x,t),
$$

a contradiction since the left hand side is negative whereas the right hand side is positive. Hence, $\tilde{v} \in \mathcal{M}$.

Third claim: $\tilde{u}$ minimizes J on $\mathcal{M}$.

Since $\tilde{u} \in \mathcal{M}$ by the second claim we obviously have $J(\tilde{u}) \geq \inf_{\mathcal{M}} J$. To finish the proof we have to show the reverse inequality. For this purpose, recall $\inf_{\mathcal{M}} J = \inf_{S^+} \Psi$. In particular, $(\tilde{u}_n)_{n\in\mathbb{N}}$ is a minimizing sequence for $J|_{\mathcal{M}}$. Therefore, Fatou's lemma implies

$$
\begin{aligned}
\inf_{\mathcal{M}} J = \lim_{m\to\infty} J(\tilde{u}_{n_m}) &= \frac{\Gamma}{T}\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_D |\mathcal{S}\tilde{u}_{n_m}|^{p+1}\,d(x,t) \\
&\geq \frac{\Gamma}{T}\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_D |\mathcal{S}\tilde{u}|^{p+1}\,d(x,t) = J(\tilde{u}).
\end{aligned}
$$

□

Lemma 5.37. *Any Palais-Smale sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ of $J|_{\mathcal{M}}$ is bounded.*

Proof. We show that there is a constant $C > 0$ such that

$$
\|\tilde{u}\|_{\mathcal{H}} \leq CJ(\tilde{u})^{\frac{p}{p+1}} \text{ for all } \tilde{u} \in \mathcal{M}.
$$

Due to $\tilde{u} \in \mathcal{M}$ we have $J'(\tilde{u})[\tilde{u} - \tilde{u}^-] = 0$. Thus,

$$
\begin{aligned}
\|\tilde{u}^+\|_{\mathcal{H}}^2 &= J'(\tilde{u})[\tilde{u}^+] + \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{u}^+\,d(x,t) \\
&= J'(\tilde{u})[\tilde{u} - \tilde{u}^-] + \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{u}^+\,d(x,t) \leq \frac{\Gamma}{T}\|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^p\|\mathcal{S}\tilde{u}^+\|_{L^{p+1}(D)}.
\end{aligned} \tag{5.83}
$$

Recall that $\tilde{u} \in \mathcal{M}$ implies $J(\tilde{u}) = \frac{\Gamma(p-1)}{2T(p+1)} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^{p+1}$. Therefore, by Theorem 5.21 we derive

$$\frac{\Gamma}{T} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^{p} \|\mathcal{S}\tilde{u}^+\|_{L^{p+1}(D)} \le C(p,T,\Gamma) \left(\frac{\Gamma(p-1)}{2T(p+1)} \|\mathcal{S}\tilde{u}\|_{L^{p+1}(D)}^{p+1} \right)^{\frac{p}{p+1}} \|\tilde{u}^+\|_{\mathcal{H}} \tag{5.84}$$

with $C(p,T,\Gamma) \coloneqq C^{\frac{2(p+1)}{p-1}} \sqrt[p]{\frac{\Gamma(p-1)}{2T(p+1)}}$. In summary, (5.83) and (5.84) show

$$\|\tilde{u}^+\|_{\mathcal{H}} \le C(p,T,\Gamma) J(\tilde{u})^{\frac{p}{p+1}}. \tag{5.85}$$

Analogously, one shows

$$\|\tilde{u}^-\|_{\mathcal{H}} \le C J(\tilde{u})^{\frac{p}{p+1}}. \tag{5.86}$$

Both (5.85) and (5.86) finish the proof. □

5.7. The back-transformation to space and time

In this section we prove Theorem 5.2, i.e., we prove that the ground state $\tilde{u}$ of J obtained previously in Theorem 5.36 leads to a weak solution of (5.3) in the sense of Definition 5.1.

Proof of Theorem 5.2. Let $\tilde{u} \in \mathcal{H}$ denote the critical point of J from Theorem 5.36. In particular, $\tilde{u}$ satisifes

$$\sum_{j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \lambda_{j,k}(s) \tilde{\hat{u}}_{j,k}(s) \overline{\tilde{\hat{v}}_{j,k}(s)} ds = \frac{\Gamma}{T} \int_D |\mathcal{S}\tilde{u}|^{p-1} \mathcal{S}\tilde{u} \mathcal{S}\tilde{v} d(x,t) \text{ for all } \tilde{v} \in \mathcal{H}.$$

In the following we fix a test function v which satisfies the conditions prescribed in Definition 5.1, i.e., there is $k_0 \in \mathbb{N}_{\text{odd}}$ such that $v(x,t) = \sum_{k \in \mathbb{Z}_{\text{odd},k_0}} \hat{v}_k(x) e^{ik\omega t}$ and $\hat{v}_k \in H^1(\mathbb{R})$, $\overline{\hat{v}_k} = \hat{v}_{-k}$ for all $k \in \mathbb{Z}_{\text{odd},k_0}$. In particular, by Lemma 5.17 we conclude that $\tilde{v} = (\tilde{\hat{v}}_{j,k})_{j \in \mathbb{N}_0, k \in \mathbb{Z}_{\text{odd},k_0}} \in \mathcal{H}$. Recall that

$$b_k(\hat{u}_k, \hat{v}_k) = \sum_{j \in \mathbb{N}_0} \int_{\mathcal{B}} \lambda_{j,k}(s) \tilde{\hat{u}}_{j,k}(s) \overline{\tilde{\hat{v}}_{j,k}(s)} ds$$

for all $\hat{u}_k, \hat{v}_k \in H^1(\mathbb{R})$ by Corollary 5.15. Therefore,

$$\sum_{k \in \mathbb{Z}_{\text{odd}}} b_k(\hat{u}_k, \hat{v}_k) = \frac{\Gamma}{T} \int_D |\mathcal{S}\tilde{u}|^{p-1} \mathcal{S}\tilde{u} \mathcal{S}\tilde{v} d(x,t) \tag{5.87}$$

and (5.6) is satisfied. Thus, we see that $u = \mathcal{S}\tilde{u}$ is the desired weak solution of (5.3) which finishes the proof. □

5.8. Remarks on a further reaching solution concept

In this short section we briefly sketch an idea how to generalize the concept of a weak solution $\tilde{u} \in \mathcal{H}$ to our problem (5.3). The idea is to rewrite the quadratic expression $\sum_{k \in \mathbb{Z}_{\text{odd}}} b_k(\hat{u}_k, \hat{v}_k)$ in (5.87) as suitable duality pairings, see Definition 7.2.1.b in [34]. Therefore, let

$$M_1 \coloneqq \Big\{ \Psi \colon D \to \mathbb{R}, \Psi(x,t) = \sum_{k \in \mathbb{Z}_{\text{odd}}} \hat{\psi}_k(x) e^{ik\omega t} : \overline{\hat{\psi}_k} = \hat{\psi}_{-k} \text{ for all } k \in \mathbb{Z}_{\text{odd}},$$

$$\hat{\psi}_k \in L^2(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ and } \sum_{k\in\mathbb{Z}_{\text{odd}}} \frac{1}{|k|^3}\|\hat{\psi}_k\|^2_{L^2(\mathbb{R})} < \infty\Big\},$$

$$M_2 := \Big\{\Psi\colon D \to \mathbb{R}, \Psi(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{\psi}_k(x)e^{ik\omega t} : \overline{\hat{\psi}_k} = \hat{\psi}_{-k} \text{ for all } k \in \mathbb{Z}_{\text{odd}},$$

$$\hat{\psi}_k \in H^1(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ and } \sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} \frac{1}{|k|^3}|\hat{\psi}_k(\varsigma + 2\pi n)|^2 < \infty\Big\}$$

equipped with

$$\|\Psi\|_{M_1} := \sqrt{\sum_{k\in\mathbb{Z}_{\text{odd}}} \frac{1}{|k|^3}\|\hat{\psi}_k\|^2_{L^2(\mathbb{R})}} \quad \text{and} \quad \|\Psi\|_{M_2} := \sqrt{\sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} \frac{1}{|k|^3}|\hat{\psi}_k(\varsigma + 2\pi n)|^2}.$$

Thus, we have

$$M_1^\star = \Big\{\Phi\colon D \to \mathbb{R}, \Phi(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{\varphi}_k(x)e^{ik\omega t} : \overline{\hat{\varphi}_k} = \hat{\varphi}_{-k} \text{ for all } k \in \mathbb{Z}_{\text{odd}},$$

$$\hat{\varphi}_k \in L^2(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ and } \sum_{k\in\mathbb{Z}_{\text{odd}}} |k|^3\|\hat{\varphi}_k\|^2_{L^2(\mathbb{R})} < \infty\Big\},$$

$$M_2^\star = \Big\{\Phi\colon D \to \mathbb{R}, \Phi(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{\varphi}_k(x)e^{ik\omega t} : \overline{\hat{\varphi}_k} = \hat{\varphi}_{-k} \text{ for all } k \in \mathbb{Z}_{\text{odd}},$$

$$\hat{\varphi}_k \in H^1(\mathbb{R}) \text{ for all } k \in \mathbb{Z}_{\text{odd}} \text{ and } \sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} |k|^3|\hat{\varphi}_k(\varsigma + 2\pi n)|^2 < \infty\Big\},$$

where the duality pairings $\langle\cdot,\cdot\rangle_{M_1}, \langle\cdot,\cdot\rangle_{M_2}$ are given by

$$\langle\Psi,\Phi\rangle_{M_1\times M_1^\star} := \sum_{k\in\mathbb{Z}_{\text{odd}}} \int_{\mathbb{R}} \hat{\psi}_k\overline{\hat{\varphi}_k}dx,$$

$$\langle\Psi,\Phi\rangle_{M_2\times M_2^\star} := \sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} \hat{\psi}_k(\varsigma + 2\pi n)\overline{\hat{\varphi}_k(\varsigma + 2\pi n)}.$$

Recall that the ground state $\tilde{u} \in \mathcal{H}$ satisfies (5.87). For $\tilde{u} \in \mathcal{H}$ and

$$\mathcal{S}\tilde{u}(x,t) = \sum_{j\in\mathbb{N}_0, k\in\mathbb{Z}_{\text{odd}}} \int_{\mathcal{B}} \tilde{\tilde{u}}_{j,k}(s)\psi_{j,k}(x,s)ds e^{ik\omega t} = \sum_{k\in\mathbb{Z}_{\text{odd}}} \hat{u}_k(x)e^{ik\omega t}$$

we conclude $(\mathcal{S}\tilde{u})_x \in M_1, (\mathcal{S}\tilde{u})_t \in M_1$ since

$$\sum_{k\in\mathbb{Z}_{\text{odd}}} \frac{1}{|k|^3}\|\hat{u}_k'\|^2_{L^2(\mathbb{R})} < \infty, \sum_{k\in\mathbb{Z}_{\text{odd}}} |k|\|\hat{u}_k\|^2_{L^2(\mathbb{R})} < \infty,$$

where the derivatives of $\mathcal{S}\tilde{u}$ w.r.t. x and t have to be understood in distributional sense (recall that we only know $\mathcal{S}\tilde{u} \in H^{\frac{1}{4}}(D)$). Moreover, due to (B.10) with $\varepsilon = \frac{1}{|k|^2}$ we deduce

$$\sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} \frac{1}{|k|^3}|(ik\omega\hat{u}_k)(\varsigma + 2\pi n)|^2 = \sum_{k\in\mathbb{Z}_{\text{odd}}, n\in\mathbb{Z}} \frac{\omega^2}{|k|}|\hat{u}_k(\varsigma + 2\pi n)|^2$$

$$
\begin{aligned}
&\leq \sum_{k\in\mathbb{Z}_{\mathrm{odd}}} \frac{\omega^2}{|k|}\left(\left(\frac{1}{2\pi}+\frac{k^2}{2}\right)\|\hat{u}_k\|^2_{L^2(\mathbb{R})}+\frac{1}{2k^2}\|\hat{u}_k'\|^2_{L^2(\mathbb{R})}\right)\\
&\leq \omega^2 \sum_{k\in\mathbb{Z}_{\mathrm{odd}}}\left(|k|\|\hat{u}_k\|^2_{L^2(\mathbb{R})}+\frac{1}{2|k|^3}\|\hat{u}_k'\|^2_{L^2(\mathbb{R})}\right)\leq C\|\tilde{u}\|^2_{\mathcal{H}},
\end{aligned}
$$

i.e., $(\mathcal{S}\tilde{u})_t \in M_2$. Thus, we can rewrite (5.87) as

$$
\begin{aligned}
\langle(\mathcal{S}\tilde{u})_x,(\mathcal{S}\tilde{v})_x\rangle_{M_1\times M_1^\star} - \alpha\,\langle(\mathcal{S}\tilde{u})_t,(\mathcal{S}\tilde{v})_t\rangle_{M_1\times M_1^\star} - \beta\,\langle(\mathcal{S}\tilde{u})_t,(\mathcal{S}\tilde{v})_t\rangle_{M_2\times M_2^\star}\\
= \frac{\Gamma}{T}\int_D |\mathcal{S}\tilde{u}|^{p-1}\mathcal{S}\tilde{u}\mathcal{S}\tilde{v}d(x,t).
\end{aligned}
$$

In particular, for $k_0 \in \mathbb{N}_{\mathrm{odd}}$ the solution concept sketched above works for

$$
v(x,t) := \sum_{k\in\mathbb{Z}_{\mathrm{odd},k_0}} \hat{v}_k(x)e^{ik\omega t}
$$

with $\hat{v}_k \in H^1(\mathbb{R})$ and $\overline{\hat{v}_k} = \hat{v}_{-k}$ for all $k \in \mathbb{Z}_{\mathrm{odd},k_0}$ and therefore it is an extension of our concept of a weak solution in Definition 5.1.

B. Appendix to part II

This appendix is tripartite. We first give the proofs of Lemma 5.6 and Lemma 5.23. Afterwards, we recall some basic aspects of Floquet-Bloch theory. In the end, we give the details concerning an argument in the proof of Theorem 5.36.

B.1. The proofs of Lemma 5.6 and Lemma 5.23

We now give the proof of Lemma 5.6, which is meanwhile classic, see for instance formula (8) in [43] or the results of Lotoreichik and Simonov [50] who obtained related expressions for similar cases.
Proof of Lemma 5.6: We introduce $\mu := \lambda - \alpha$ and investigate

$$-f'' + \beta\delta_{\mathrm{per}}(x)f = \mu f \text{ on } [0, 2\pi). \tag{B.1}$$

We distinguish the cases $\mu > 0$ and $\mu < 0$.

1) $\mu > 0$: We introduce sectionally defined functions Φ_1, Φ_2, Φ_3 and Φ_4 where each of the pairs $\{\Phi_1, \Phi_3\}, \{\Phi_2, \Phi_4\}$ together with a matching condition at $x = \varsigma$ describes a fundamental solution of (B.1) on $[0, 2\pi)$.

A fundamental system for (B.1) on $[0, \varsigma)$ is given by

$$\Phi_1(x) = \cos(\sqrt{\mu}x),\ \Phi_2(x) = \sin(\sqrt{\mu}x),$$

whereas on $(\varsigma, 2\pi)$ we obtain

$$\begin{aligned} \Phi_3(x) &= \bar{c}_3 \sin(\sqrt{\mu}x) + \bar{c}_4 \cos(\sqrt{\mu}x), \\ \Phi_4(x) &= \tilde{c}_3 \sin(\sqrt{\mu}x) + \tilde{c}_4 \cos(\sqrt{\mu}x), \end{aligned} \tag{B.2}$$

where $\bar{c}_3, \bar{c}_4, \tilde{c}_3, \tilde{c}_4 \in \mathbb{R}$ are chosen such that the jump conditions at $x = \varsigma$ are satisfied, i.e.,

$$\begin{aligned} \Phi_3'(\varsigma) &= \Phi_1'(\varsigma) + \beta\Phi_1(\varsigma), \\ \Phi_4'(\varsigma) &= \Phi_2'(\varsigma) + \beta\Phi_2(\varsigma). \end{aligned} \tag{B.3}$$

Moreover, the continuity of fundamental solutions at $x = \varsigma$ forces

$$\Phi_1(\varsigma) = \Phi_3(\varsigma),\ \Phi_2(\varsigma) = \Phi_4(\varsigma). \tag{B.4}$$

The combination of (B.3) and (B.4) leads to the linear systems

$$\begin{pmatrix} \sin(\sqrt{\mu}\varsigma) & \cos(\sqrt{\mu}\varsigma) \\ \cos(\sqrt{\mu}\varsigma) & -\sin(\sqrt{\mu}\varsigma) \end{pmatrix} \begin{pmatrix} \bar{c}_3 \\ \bar{c}_4 \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\mu}\varsigma) \\ -\sin(\sqrt{\mu}\varsigma) + \frac{\beta}{\sqrt{\mu}} \cos(\sqrt{\mu}\varsigma) \end{pmatrix} \tag{B.5}$$

and

$$\begin{pmatrix} \sin(\sqrt{\mu}\varsigma) & \cos(\sqrt{\mu}\varsigma) \\ \cos(\sqrt{\mu}\varsigma) & -\sin(\sqrt{\mu}\varsigma) \end{pmatrix} \begin{pmatrix} \tilde{c}_3 \\ \tilde{c}_4 \end{pmatrix} = \begin{pmatrix} \sin(\sqrt{\mu}\varsigma) \\ \cos(\sqrt{\mu}\varsigma) + \frac{\beta}{\sqrt{\mu}} \sin(\sqrt{\mu}\varsigma) \end{pmatrix}. \tag{B.6}$$

The solutions of (B.5), (B.6) are given by

$$\begin{pmatrix} \bar{c}_3 \\ \bar{c}_4 \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\sqrt{\mu}} \cos^2(\sqrt{\mu}\varsigma) \\ 1 - \frac{\beta}{\sqrt{\mu}} \sin(\sqrt{\mu}\varsigma)\cos(\sqrt{\mu}\varsigma) \end{pmatrix}, \begin{pmatrix} \tilde{c}_3 \\ \tilde{c}_4 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\beta}{\sqrt{\mu}} \sin(\sqrt{\mu}\varsigma)\cos(\sqrt{\mu}\varsigma) \\ -\frac{\beta}{\sqrt{\mu}} \sin^2(\sqrt{\mu}\varsigma) \end{pmatrix}.$$

By plugging these constants into (B.2) and denoting the sectionally defined pairs $\{\Phi_1, \Phi_3\}$, $\{\Phi_2, \Phi_4\}$ by Ψ_1 and Ψ_2 we obtain a fundamental system

$$\Psi_1(x) = \begin{cases} \cos(\sqrt{\mu}x) & , x \in [0, \varsigma), \\ \frac{\beta}{\sqrt{\mu}} \cos(\sqrt{\mu}\varsigma)^2 \sin(\sqrt{\mu}x) + (1 - \frac{\beta}{\sqrt{\mu}} \sin(\sqrt{\mu}\varsigma)\cos(\sqrt{\mu}\varsigma))\cos(\sqrt{\mu}x) & , x \in [\varsigma, 2\pi), \end{cases}$$
$$\Psi_2(x) = \begin{cases} \sin(\sqrt{\mu}x) & , x \in [0, \varsigma), \\ (1 + \frac{\beta}{\sqrt{\mu}} \sin(\varsigma\sqrt{\mu})\cos(\varsigma\sqrt{\mu}))\sin(\sqrt{\mu}x) - \frac{\beta}{\sqrt{\mu}} \sin(\varsigma\sqrt{\mu})^2 \cos(\sqrt{\mu}x) & , x \in [\varsigma, 2\pi), \end{cases}$$

where the system $\{\Psi_1, \frac{1}{\sqrt{\mu}}\Psi_2\}$ satisifies $\Psi_1(0) = 1, \Psi_1'(0) = 0, \frac{1}{\sqrt{\mu}}\Psi_2(0) = 0$ and $\frac{1}{\sqrt{\mu}}\Psi_2'(0) = 1$. Hence,

$$\begin{aligned} D(\mu) &= \Psi_1(2\pi) + \frac{1}{\sqrt{\mu}}\Psi_2'(2\pi) \\ &= \frac{\beta}{\sqrt{\mu}}(\cos(\sqrt{\mu}\varsigma)^2 + \sin(\sqrt{\mu}\varsigma)^2)\sin(2\pi\sqrt{\mu}) + 2\cos(2\pi\sqrt{\mu}) \\ &= \frac{\beta}{\sqrt{\mu}} \sin(2\pi\sqrt{\mu}) + 2\cos(2\pi\sqrt{\mu}). \end{aligned}$$

The substitution $\mu = \lambda - \alpha$ then implies the first part of the statement.

2) $\mu < 0$: We keep the notation of case 1) but here the fundamental solutions read

$$\Phi_1(x) = \cosh(\sqrt{-\mu}x),\ \Phi_2(x) = \sinh(\sqrt{-\mu}x) \text{ on } [0, \varsigma)$$

and

$$\Phi_3(x) = \bar{c}_3 \sinh(\sqrt{-\mu}x) + \bar{c}_4 \cosh(\sqrt{-\mu}x), \Phi_4(x) = \tilde{c}_3 \sinh(\sqrt{-\mu}x) + \tilde{c}_4 \cosh(\sqrt{-\mu}x) \text{ on } (\varsigma, 2\pi).$$

The requirement of continuity and the jump condition for the derivatives translate to the linear systems

$$\begin{pmatrix} \sinh(\sqrt{-\mu}\varsigma) & \cosh(\sqrt{-\mu}\varsigma) \\ \cosh(\sqrt{-\mu}\varsigma) & \sinh(\sqrt{-\mu}\varsigma) \end{pmatrix} \begin{pmatrix} \bar{c}_3 \\ \bar{c}_4 \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{-\mu}\varsigma) \\ \sinh(\sqrt{-\mu}\varsigma) + \frac{\beta}{\sqrt{-\mu}} \cosh(\sqrt{-\mu}\varsigma) \end{pmatrix}$$

and

$$\begin{pmatrix} \sinh(\sqrt{-\mu}\varsigma) & \cosh(\sqrt{-\mu}\varsigma) \\ \cosh(\sqrt{-\mu}\varsigma) & \sinh(\sqrt{-\mu}\varsigma) \end{pmatrix} \begin{pmatrix} \tilde{c}_3 \\ \tilde{c}_4 \end{pmatrix} = \begin{pmatrix} \sinh(\sqrt{-\mu}\varsigma) \\ \cosh(\sqrt{-\mu}\varsigma) + \frac{\beta}{\sqrt{-\mu}} \sinh(\sqrt{-\mu}\varsigma) \end{pmatrix}.$$

The solutions hereof are given by

$$\begin{pmatrix}\bar{c}_3\\ \bar{c}_4\end{pmatrix} = \begin{pmatrix} \frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)^2 \\ 1-\frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)\sinh(\sqrt{-\mu}\varsigma)\end{pmatrix},\ \begin{pmatrix}\tilde{c}_3\\ \tilde{c}_4\end{pmatrix} = \begin{pmatrix} 1+\frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)\sinh(\sqrt{-\mu}\varsigma) \\ -\frac{\beta}{\sqrt{-\mu}}\sinh(\sqrt{-\mu}\varsigma)^2\end{pmatrix}.$$

In summary a fundamental system reads

$$\Psi_1(x) = \begin{cases} \cosh(\sqrt{-\mu}x), & x\in[0,\varsigma),\\ \frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)^2\sinh(\sqrt{-\mu}x) & \\ +(1-\frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)\sinh(\sqrt{-\mu}\varsigma))\cosh(\sqrt{-\mu}x), & x\in[\varsigma,2\pi),\end{cases}$$

$$\Psi_2(x) = \begin{cases} \sinh(\sqrt{-\mu}x), & x\in[0,\varsigma),\\ (1+\frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)\sinh(\sqrt{-\mu}\varsigma))\sinh(\sqrt{-\mu}x) & \\ -\frac{\beta}{\sqrt{-\mu}}\sinh(\sqrt{-\mu}\varsigma)^2\cosh(\sqrt{-\mu}x), & x\in[\varsigma,2\pi),\end{cases}$$

where the system $\{\Psi_1, \frac{1}{\sqrt{-\mu}}\Psi_2\}$ satisifies $\Psi_1(0) = 1, \Psi_1'(0) = 0, \frac{1}{\sqrt{-\mu}}\Psi_2(0) = 0$ and $\frac{1}{\sqrt{-\mu}}\Psi_2'(0) = 1$. Hence,

$$\begin{aligned} D(\mu) &= \Psi_1(2\pi) + \frac{1}{\sqrt{-\mu}}\Psi_2'(2\pi)\\ &= \frac{\beta}{\sqrt{-\mu}}\cosh(\sqrt{-\mu}\varsigma)^2\sinh(2\pi\sqrt{-\mu}) - \frac{\beta}{\sqrt{-\mu}}\sinh(\sqrt{-\mu}\varsigma)^2\sinh(2\pi\sqrt{-\mu}) + 2\cosh(2\pi\sqrt{-\mu})\\ &= \frac{\beta}{\sqrt{-\mu}}\sinh(2\pi\sqrt{-\mu}) + 2\cosh(2\pi\sqrt{-\mu}).\end{aligned}$$

Again the substitution $\mu = \lambda - \alpha$ yields the desired claim.

The value of $D(0)$ in (5.12) arises since $\mu \mapsto D(\mu)$ has to be continuous at $\mu = 0$ and

$$\lim_{\mu\to 0^+}\frac{\beta}{\sqrt{\mu}}\sin(2\pi\sqrt{\mu}) + 2\cos(2\pi\sqrt{\mu}) = 2+\beta T = \lim_{\mu\to 0^-}\frac{\beta}{\sqrt{-\mu}}\sinh(2\pi\sqrt{-\mu}) + 2\cosh(2\pi\sqrt{-\mu})$$

which can be seen by Taylor expansion. This finishes the proof. □

The section is closed by the estimation of the integral in Lemma 5.23.

Proof of Lemma 5.23: With the help of the substitution $z = \frac{x}{y}$ we calculate

$$\begin{aligned} \int_0^\infty\int_0^R \frac{x^2}{(x^2+y^2)^{\frac{5}{4}}}dxdy &= \int_0^\infty\int_0^R \frac{x^2}{y^{\frac{5}{2}}(\frac{x^2}{y^2}+1)^{\frac{5}{4}}}dxdy = \int_0^\infty\int_0^{\frac{R}{y}} \sqrt{y}\frac{z^2}{(1+z^2)^{\frac{5}{4}}}dzdy\\ &\le \int_0^\infty\int_0^{\frac{R}{y}} \frac{R}{\sqrt{y}}\frac{z}{(1+z^2)^{\frac{5}{4}}}dzdy = \int_0^\infty \frac{R}{\sqrt{y}}\left[-\frac{2}{\sqrt[4]{1+z^2}}\right]_0^{\frac{R}{y}}dy = \int_0^\infty \frac{2R}{\sqrt{y}}\left(1-\frac{1}{\sqrt[4]{1+\frac{R^2}{y^2}}}\right)dy.\end{aligned}$$

We split the last integral in two parts. We have

$$\int_0^1 \frac{2R}{\sqrt{y}}\left(1-\frac{1}{\sqrt[4]{1+\frac{R^2}{y^2}}}\right)dy \le \int_0^1\frac{2R}{\sqrt{y}}dy = 4R. \tag{B.7}$$

Notice that $1 + \frac{R^2}{y^2} \leq \left(1 + \frac{R}{y}\right)^4$ for all $y > 0$, i.e., $\sqrt[4]{1 + \frac{R^2}{y^2}} - 1 \leq \frac{R}{y}$ for all $y > 0$. Therefore, we conclude

$$\begin{aligned}\int_1^\infty \frac{2R}{\sqrt{y}}\left(1 - \frac{1}{\sqrt[4]{1 + \frac{R^2}{y^2}}}\right)dy &= \int_1^\infty \frac{2R}{\sqrt{y}} \frac{\sqrt[4]{1 + \frac{R^2}{y^2}} - 1}{\sqrt[4]{1 + \frac{R^2}{y^2}}} dy \leq 2R \int_1^\infty \frac{1}{\sqrt{y}}\left(\sqrt[4]{1 + \frac{R^2}{y^2}} - 1\right)dy \\ &\leq 2R^2 \int_1^\infty \frac{1}{y^{\frac{3}{2}}} dy = 4R^2.\end{aligned} \tag{B.8}$$

The combination of (B.7) and (B.8) then yields the desired estimate. □

B.2. Basics on Floquet transformation and Bloch waves

In this section we consider and recall the notion of Bloch waves and the Floquet transformation. In comparison to differential operators with constant coefficients plain waves $e^{-is\cdot x}$ are substituted by so called Bloch waves, whereas the Fourier transform is replaced by the Floquet transformation. The basic statements on the Floquet transformation listed here can be found in [30] and [42]. The bachelor thesis of Martin Belica [8] shows that the main results stay valid for our δ-potential type operators. Altough these concepts work in arbitrary finite dimension we only recall the definitions and statements for dimension one which allows us to keep additional notation at a minimum.

Let $\mathcal{P} := [0, 2\pi)$ denote the interval of periodicity and $\mathcal{B} := [-\frac{1}{2}, \frac{1}{2})$ denote the Brillouin zone. We now recall the Floquet transformation.

Definition B.1. *Let $f \in L^2(\mathbb{R}), x \in \mathcal{P}, s \in \mathcal{B}$. The Floquet transformation $\mathcal{T}$ is given by*

$$(\mathcal{T} f)(x, s) := \frac{1}{\sqrt{|\mathcal{B}|}} \sum_{n\in\mathbb{Z}} f(x - 2\pi n)e^{2\pi isn} = \sum_{n\in\mathbb{Z}} f(x - 2\pi n)e^{2\pi isn}.$$

Theorem B.2. *The Floquet transformation $\mathcal{T}\colon L^2(\mathbb{R}) \to L^2(\mathcal{P} \times \mathcal{B})$ is well-defined and an isometric isomorphism with inverse*

$$\left(\mathcal{T}^{-1} g\right)(x - 2\pi n) = \frac{1}{\sqrt{|\mathcal{B}|}} \int_{\mathcal{B}} g(x, l)e^{-2\pi iln} dl$$

for $g \in L^2(\mathcal{P} \times \mathcal{B}), x \in \mathcal{P}$ and $n \in \mathbb{Z}$.

For fixed $s \in \bar{\mathcal{B}}$ we consider the quasi-periodic eigenvalue problem

$$\begin{aligned}\left(-\frac{d^2}{dx^2} + V\right)\psi(\cdot, s) &= \lambda(s)\psi(\cdot, s) \text{ in } \mathcal{P}, \\ \psi(x + 2\pi, s) &= e^{2\pi is}\psi(x, s) \text{ for } x \in \mathcal{P},\end{aligned} \tag{B.9}$$

with $V\colon \mathbb{R} \to \mathbb{R}$ being 2π-periodic. Fix $s \in \mathcal{B}$. Then it can be shown that (B.9) has a complete, $L^2(\mathcal{P})$-orthonormal system $(\psi_j(\cdot, s))_{j\in\mathbb{N}_0}$ of eigenfunctions. The corresponding eigenvalues are denoted by $\lambda_j(s)$ and are ordered in increasing way (double eigenvalues are counted twice), i.e.,

$$\lambda_0(s) \leq \lambda_1(s) \leq \cdots \leq \lambda_j(s) \to \infty \text{ as } j \to \infty.$$

The eigenfunctions $\psi_j(\cdot, s)$ are called Bloch waves. By varying over $s \in \mathcal{B}$ we obtain that the Bloch waves are complete in $L^2(\mathbb{R})$ in the following sense:

Theorem B.3. *Let $f \in L^2(\mathbb{R})$ and define*

$$f_n(x) := \frac{1}{\sqrt{|B|}} \sum_{j=0}^{n} \int_B \Big\langle (\mathcal{T} f)(\cdot, s), \psi_j(\cdot, s) \Big\rangle_{L^2(\mathcal{P})} \psi_j(x, s) ds \text{ for } x \in \mathbb{R}.$$

Then $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$.

We now pass to the family of operators $(L_k)_{k \in \mathbb{Z}_{\text{odd}}}$ from Section 5.2. Therefore, notation gets modified and the eigenfunctions of L_k are denoted by $(\psi_{j,k})_{j \in \mathbb{N}_0}$, the corresponding eigenvalues by $(\lambda_{j,k}(s))_{j \in \mathbb{N}_0}$. Here is another technical lemmata which is used in the following.

Lemma B.4. *The following statements hold true.*

(a) *Let $f \in D(L_k)$. Then $f(2\pi n), f'(2\pi n) \to 0$ as $n \to \pm\infty$.*

(b) *Let $f \in H^1(\mathbb{R})$. Then for $\varepsilon > 0$ we have*

$$\sum_{n \in \mathbb{Z}} |f(\varsigma + 2\pi n)|^2 \leq \left(\frac{1}{2\pi} + \frac{1}{2\varepsilon}\right) \|f\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2} \|f'\|^2_{L^2(\mathbb{R})}. \tag{B.10}$$

Proof. (a) Recall $\varsigma \in (0, 2\pi)$. Therefore, $\varepsilon := \frac{\min\{\varsigma, 2\pi - \varsigma\}}{2} > 0$. We first show that

$$f(2\pi n)^2 \leq \frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right) \|f(2\pi n + \cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} + \frac{1}{2} \|f'(2\pi n + \cdot)\|^2_{L^2(-\varepsilon,\varepsilon)}. \tag{B.11}$$

Indeed, for $n \in \mathbb{Z}$ we set $u_n(x) := f(2\pi n + x)$. Then (B.11) is equivalent to

$$u_0(0)^2 \leq \frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right) \|u_n\|^2_{L^2(-\varepsilon,\varepsilon)} + \frac{1}{2} \|u_n'\|^2_{L^2(-\varepsilon,\varepsilon)}. \tag{B.12}$$

We compute

$$\begin{aligned} u_n(0)^2 &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \frac{d}{dt} \left[(t + \varepsilon) u_n(t)^2\right] dt = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 u_n(t)^2 dt + \frac{2}{\varepsilon} \int_{-\varepsilon}^0 (t + \varepsilon) u_n(t) u_n'(t) dt \\ &\leq \frac{1}{\varepsilon} \int_{-\varepsilon}^0 u_n(t)^2 dt + 2 \int_{-\varepsilon}^0 |u_n(t) u_n'(t)| dt. \end{aligned} \tag{B.13}$$

In the same manner

$$u_n(0)^2 = -\frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{d}{dt} \left[(\varepsilon - t) u_n(t)^2\right] dt \leq \frac{1}{\varepsilon} \int_0^{\varepsilon} u_n(t)^2 dt + 2 \int_0^{\varepsilon} |u_n(t) u_n'(t)| dt. \tag{B.14}$$

By adding (B.13) and (B.14) we conclude

$$u_n(0)^2 \leq \frac{1}{2\varepsilon} \|u_n\|^2_{L^2(-\varepsilon,\varepsilon)} + \|u_n\|_{L^2(-\varepsilon,\varepsilon)} \|u_n'\|_{L^2(-\varepsilon,\varepsilon)} \leq \frac{1}{2}\left(1 + \frac{1}{\varepsilon}\right) \|u_n\|^2_{L^2(-\varepsilon,\varepsilon)} + \frac{1}{2} \|u_n'\|^2_{L^2(-\varepsilon,\varepsilon)}$$

which establishes (B.12) and herewith also (B.11). Notice that

$$\sum_{n \in \mathbb{Z}} \|f(2\pi n + \cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} \leq \|f\|^2_{L^2(\mathbb{R})} < \infty \text{ and } \sum_{n \in \mathbb{Z}} \|f'(2\pi n + \cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} \leq \|f'\|^2_{L^2(\mathbb{R})} < \infty.$$

Thus,

$$\|f(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)}, \|f'(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} \to 0 \text{ as } n \to \pm\infty. \tag{B.15}$$

so that $f(2\pi n) \to 0$ as $n \to \pm\infty$ follows from (B.11) and (B.15).
We now turn to the proof of $f'(2\pi n) \to 0$ as $n \to \pm\infty$. Due to

$$\sum_{n\in\mathbb{Z}} \|f''(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} \le \sum_{n\in\mathbb{Z}} \|f''\|^2_{L^2(\varsigma+2\pi n,\varsigma+2\pi(n+1))} < \infty$$

we infer that

$$\|f''(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} \to 0 \text{ as } n \to \pm\infty. \tag{B.16}$$

Replacing f, f' by f', f'' in the calculations for (B.11) we can show that

$$f'(2\pi n)^2 \le \frac{1}{2}\left(1+\frac{1}{\varepsilon}\right)\|f'(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)} + \frac{1}{2}\|f''(2\pi n+\cdot)\|^2_{L^2(-\varepsilon,\varepsilon)}$$

and the proof is finished by (B.15) and (B.16).
(b) This estimate requires some changes to the strategy in (a) so we give the details. For $n \in \mathbb{Z}$ we set $u_n(x) := f(\varsigma + 2\pi n + x)$ and show

$$u_n(0)^2 \le \left(\frac{1}{2\pi}+\frac{1}{2\varepsilon}\right)\|u_n\|^2_{L^2(-\pi,\pi)} + \frac{\varepsilon}{2}\|u_n'\|^2_{L^2(-\pi,\pi)}. \tag{B.17}$$

Then (B.10) follows from (B.17) by summation over $n \in \mathbb{Z}$. We have

$$\begin{aligned} u_n(0)^2 &= \frac{1}{\pi}\int_{-\pi}^0 \frac{d}{dt}\left[(t+\pi)u_n(t)^2\right]dt = \frac{1}{\pi}\int_{-\pi}^0 u_n(0)^2 dt + \frac{2}{\pi}\int_{-\pi}^0 (t+\pi)u_n u_n' dt \\ &\le \frac{1}{\pi}\int_{-\pi}^0 u_n^2 dt + 2\int_{-\pi}^0 |u_n u_n'| dt. \end{aligned} \tag{B.18}$$

In the same spirit we establish

$$u_n(0)^2 \le \frac{1}{\pi}\int_0^\pi u_n^2 dt + 2\int_0^\pi |u_n u_n'| dt. \tag{B.19}$$

Adding (B.18) and (B.19) leads with the help of Young's inequality to

$$\begin{aligned} u_n(0)^2 &\le \frac{1}{2\pi}\|u_n\|^2_{L^2(-\pi,\pi)} + \|u_n\|^2_{L^2(-\pi,\pi)}\|u_n'\|^2_{L^2(-\pi,\pi)} \\ &\le \frac{1}{2\pi}\|u_n\|^2_{L^2(-\pi,\pi)} + \frac{1}{2\varepsilon}\|u_n\|^2_{L^2(-\pi,\pi)} + \frac{\varepsilon}{2}\|u_n'\|^2_{L^2(-\pi,\pi)} \end{aligned}$$

which verifies (B.17) and herewith also (B.10). □

We next consider a quasi-periodic problem on the interval $\mathcal{P}$ and derive several connections to the family of operators $(L_k)_{k\in\mathbb{Z}_{\text{odd}}}$. Precisely, for $s \in \mathcal{B}$ we set

$D(L_k^{\text{quasi}}) := \{f \in L^2(\mathcal{P}), f$ cont. on $[0,2\pi], f'$ cont. on $[0,\varsigma) \cup (\varsigma,2\pi], f'' \in L^2(0,\varsigma), f'' \in L^2(\varsigma,2\pi),$

$$f'(\varsigma^+) - f'(\varsigma^-) = -k^2 f(\varsigma), f(2\pi) = e^{2\pi i s} f(0), f'(2\pi) = e^{2\pi i s} f'(0)\}.$$

If $f \in D(L_k^{\text{quasi}})$ then f has a periodic extension on $\mathbb{R}$ and therefore $L_k^{\text{quasi}} f := (L_k f)|_{\mathcal{P}}$ makes sense. Vice versa, if $f \in D(L_k)$ then $f|_{\mathcal{P}} \in D(L_k^{\text{quasi}})$. Moreover, L_k^{quasi} has pure point spectrum, namely $\sigma(L_k^{\text{quasi}}) = \bigcup_{j \in \mathbb{N}_0} \lambda_{j,k}(s)$ due to the definition of L_k^{quasi}. We now highlight an important connection between L_k^{quasi} and L_k.

Lemma B.5. *Let $f \in D(L_k), g \in D(L_k^{\text{quasi}})$. Then*

$$\langle \mathcal{T} L_k f(\cdot, s), g \rangle_{\mathcal{P}} = \left\langle \mathcal{T} f(\cdot, s), L_k^{\text{quasi}} g \right\rangle_{\mathcal{P}}$$

holds true for all $s \in \mathcal{B}$.

Proof. Let $s \in \mathcal{B}$. We have

$$\langle \mathcal{T} L_k f(\cdot, s), g \rangle_{\mathcal{P}} = \int_0^{2\pi} \sum_{n \in \mathbb{Z}} (L_k f)(x - 2\pi n) e^{2\pi i s n} \overline{g(x)} dx.$$

We show that permutation of summation and integration is allowed. Therefore, for $m \in \mathbb{N}$ we set

$$h_m(x) := \sum_{n \in \mathbb{Z}, |n| \le m} (L_k f)(x - 2\pi n) e^{2\pi i s n} \overline{g(x)}.$$

Then

$$|h_m(x)| \le |\overline{g(x)}| \sum_{n \in \mathbb{Z}} |(L_k f)(x - 2\pi n)| \tag{B.20}$$

and the expression on the right hand side of (B.20) is in $L^1(\mathcal{P})$ since by monotone convergence we have

$$\begin{aligned} &\int_0^{2\pi} |\overline{g(x)}| \sum_{n \in \mathbb{Z}} |(L_k f)(x - 2\pi n)| dx = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} |\overline{g(x)}| |(L_k f)(x - 2\pi n)| dx \\ &\le \|g\|_{L^2(\mathcal{P})} \sum_{n \in \mathbb{Z}} \|L_k f(\cdot - 2\pi n)\|_{L^2(\mathcal{P})} = \|g\|_{L^2(\mathcal{P})} \|L_k f\|_{L^2(\mathbb{R})} < \infty. \end{aligned}$$

Thus, summation and integration can be interchanged and by partial integration we calculate

$$\begin{aligned} \langle \mathcal{T} L_k f(\cdot, s), g \rangle_{\mathcal{P}} &= \int_0^{2\pi} \sum_{n \in \mathbb{Z}} (L_k f)(x - 2\pi n) e^{2\pi i s n} \overline{g(x)} dx = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} (L_k f)(x - 2\pi n) e^{2\pi i s n} \overline{g(x)} dx \\ &= \sum_{n \in \mathbb{Z}} \left(\int_0^{2\pi} f(x - 2\pi n) e^{2\pi i s n} L_k \overline{g}(x) dx + \left[f'(x - 2\pi n) e^{2\pi i s n} \overline{g(x)} - f(x - 2\pi n) e^{2\pi i s n} \overline{g'(x)} \right]_0^{2\pi} \right). \end{aligned} \tag{B.21}$$

The proof is done if we can show that

$$\sum_{n \in \mathbb{Z}} \left[f'(x - 2\pi n) e^{2\pi i s n} \overline{g(x)} - f(x - 2\pi n) e^{2\pi i s n} \overline{g'(x)} \right]_0^{2\pi} = 0 \tag{B.22}$$

since from (B.21) we then infer the desired result by again interchanging summation and integration. It remains to verify (B.22). This is done now. By using the quasi-periodicity of g and rearranging terms we deduce

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\left[f'(x-2\pi n)e^{2\pi isn}\overline{g(x)}-f(x-2\pi n)e^{2\pi isn}\overline{g'(x)}\right]_0^{2\pi}\\&=\sum_{n\in\mathbb{Z}}e^{2\pi isn}\left(\left(f'(2\pi(1-n))e^{-2\pi is}-f'(-2\pi n)\right)\overline{g(0)}-\left(f(2\pi(1-n))-f(-2\pi n)\right)\overline{g'(0)}\right).\end{aligned}\tag{B.23}$$

Notice that (B.23) is a telescoping series and (B.22) then follows from the fact that

$$f'(2\pi(1-n)),f'(-2\pi n),f(2\pi(1-n)),f(-2\pi n)\to 0\text{ as }n\to\pm\infty,$$

see Lemma B.4. □

As already introduced in (5.32), for a function $f\in L^2(\mathbb{R})$ and $j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd}}$ we set

$$\tilde{f}_{j,k}(s):=\left\langle(\mathcal{T}f)(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{L^2(\mathcal{P})},$$

where $(\psi_{j,k}(\cdot,s))_{j\in\mathbb{N}_0}$ denotes the set of Bloch waves for the operator L_k. The next statement can be found in Theorem XIII.98 (c) in [61]. In the proof we profit from Lemma B.5.

Corollary B.6. *Let $\hat{u}_k\in D(L_k)$. Then for $s\in\mathcal{B}$ we have*

$$\left\langle\mathcal{T}L_k\hat{u}_k(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}=\lambda_{j,k}(s)\tilde{\hat{u}}_{j,k}(s)\tag{B.24}$$

and

$$L_k\hat{u}_k=\sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}}\lambda_{j,k}(s)\tilde{\hat{u}}_{j,k}(s)\psi_{j,k}(x,s)ds\text{ in }L^2(\mathbb{R}).\tag{B.25}$$

Proof. Lemma B.5 and the definition of $\psi_{j,k}$ entails

$$\left\langle\mathcal{T}L_k\hat{u}_k(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}=\left\langle\mathcal{T}\hat{u}_k(\cdot,s)),L_k\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}=\lambda_{j,k}(s)\left\langle\mathcal{T}\hat{u}_k(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}=\lambda_{j,k}(s)\tilde{\hat{u}}_{j,k}(s).$$

Thus, (B.24) is established. To show (B.25) notice that due to $\hat{u}_k\in D(L_k)$ we have $w:=L_k\hat{u}_k\in L^2(\mathbb{R})$ and therefore

$$w=\sum_{j\in\mathbb{N}_0}\int_{\mathcal{B}}\tilde{w}_{j,k}(s)\psi_{j,k}(x,s)ds\quad\text{in }L^2(\mathbb{R})$$

with $\tilde{w}_{j,k}(s)=\left\langle\mathcal{T}w(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}$. By (B.24) we conclude

$$\tilde{w}_{j,k}(s)=\left\langle\mathcal{T}L_k\hat{u}_k(\cdot,s),\psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}=\lambda_{j,k}(s)\tilde{\hat{u}}_{j,k}(s)$$

and the proof is done. □

Since the spectrum of L_k is bounded from below we know that there is $M_k > 0$ such that $\lambda_{j,k}(s) \geq -M_k$ for all $(j,s) \in \mathbb{N}_0 \times \mathcal{B}$. We introduce $\mu_{j,k}(s) := \lambda_{j,k}(s) + M_k + 1 \geq 1$ as well as $L_{M,k} := L_k + M_k + 1$. Then $L_{M,k}$ is a positive operator. The corresponding bilinear forms on $H^1(\mathbb{R})$ are denoted by b_k and $b_{M,k}$. Moreover, the study of one operator L_k for $k \in \mathbb{Z}_{\text{odd}}$ can be seen as a monochromatic aspect whereas the polychromatic functions only appear if those L_k operators are summed over $k \in \mathbb{Z}_{\text{odd}}$. Thus, we define

$$\mathcal{H}_{k,\text{mono}} := \{\tilde{u} = (\tilde{u}_{j,k})_{j\in\mathbb{N}_0} : \tilde{u}_{j,k} \colon \mathcal{B} \to \mathbb{C} \text{ measurable s.t.} \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \mu_{j,k}(s)|\tilde{u}_{j,k}(s)|^2 ds < \infty\}$$

with

$$\|\tilde{u}\|_{\mathcal{H}_{k,\text{mono}}} := \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \mu_{j,k}(s)|\tilde{u}_{j,k}(s)|^2 ds.$$

Here is a first result concerning the space $\mathcal{H}_{k,\text{mono}}$.

Lemma B.7. *Let $\tilde{u} \in \mathcal{H}_{k,\text{mono}}$. Then*

$$u_k(x) := \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds \in H^1(\mathbb{R}).$$

Proof. For $n \in \mathbb{N}_0$ set $u_k^n(x) := \sum_{j=0}^n \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_j(x,s)ds$. Then $u_k^n \to u_k$ in $L^2(\mathbb{R})$ as $n \to \infty$. Moreover, we have $u_k^n \in D(L_{M,k})$ for all $n \in \mathbb{N}_0$. Since $D(L_{M,k}) \subset D(b_{M,k}) = H^1(\mathbb{R})$ we obtain $u_k^n \in H^1(\mathbb{R})$ for all $n \in \mathbb{N}_0$. Parseval's identity (see (5.34)) entails for $m \geq n \in \mathbb{N}_0$

$$\langle L_{M,k}(u_k^n - u_k^m), u_k^n - u_k^m\rangle_{L^2(\mathbb{R})} = \sum_{j=n+1}^{m} \int_{\mathcal{B}} \mu_{j,k}(s)|\tilde{u}_{j,k}(s)|^2 ds \to 0 \text{ as } n \leq m \to \infty. \tag{B.26}$$

In particular, $(u_k^n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $H^1(\mathbb{R})$. Thus, $u_k^n \to u_k$ in $H^1(\mathbb{R})$ which proves our claim. $\square$

Remark B.8. *From* (B.26) *and* $\left\langle L_{M,k}(u_k^n - u_k^m), u_k^n - u_k^m\right\rangle_{L^2(\mathbb{R})} = b_{M,k}(u_k^n - u_k^m, u_k^n - u_k^m)$ *we deduce that the norms* $\|\cdot\|_{H^1(\mathbb{R})}$ *and* $\|\tilde{\cdot}\|_{\mathcal{H}_{k,\text{mono}}}$ *are equivalent.*

B.3. A technical point in the proof of Theorem 5.36

In this section we close the proof of the first claim of Theorem 5.36. Therefore, let

$$\mathcal{N} := \{M \subset \mathcal{H} : \overline{M}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H}, \operatorname{supp} \mathcal{S}\tilde{u} \cap \operatorname{int} D \text{ compact for all } \tilde{u} \in M\}.$$

Once more, we need some additional notation. We introduce $\mathcal{S}_{k,\text{mono}} \colon \mathcal{H}_{k,\text{mono}} \to H^1(\mathbb{R})$ by

$$\mathcal{S}_{k,\text{mono}}\tilde{u}(x) := u_k(x) := \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds.$$

Recall that the mapping property of $\mathcal{S}_{k,\text{mono}}$ is described by Lemma B.7. Moreover, for $k_0 \in \mathbb{N}_{\text{odd}}$ let $\mathbb{Z}_{\text{odd},k_0} := \{k \in \mathbb{Z}_{\text{odd}} : |k| \le k_0\}$ and set

$$\mathcal{H}_{k_0} := \Big\{\tilde{u} = (\tilde{u}_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd},k_0}} : \tilde{u}_{j,k} \colon \mathcal{B} \to \mathbb{C} \text{ measurable for all } (j,k) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd},k_0},$$
$$\overline{\tilde{u}_{j,k}(s)} = \tilde{u}_{j,-k}(-s) \text{ for all } (j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd},k_0} \times \mathcal{B} \text{ and } (\tilde{u}_{j,k})_{j\in\mathbb{N}_0} \in \mathcal{H}_{k,\text{mono}} \text{ for all } k \in \mathbb{Z}_{\text{odd},k_0}\Big\}$$

with

$$\|\tilde{u}\|_{\mathcal{H}_{k_0}} := \sqrt{\sum_{j\in\mathbb{N}_0,k\in\mathbb{Z}_{\text{odd},k_0}} \int_{\mathcal{B}} |\lambda_{j,k}(s)||\tilde{u}_{j,k}(s)|^2 ds}.$$

Finally, let $H_c := \{v \in H^1(\mathbb{R}) : \operatorname{supp} v \text{ compact}\}$,

$$H_{k_0} := \Big\{u \in H^1(D) : u(x,t) = \sum_{k\in\mathbb{Z}_{\text{odd},k_0}} u_k(x)e^{ik\omega t} : u_k \in H^1(\mathbb{R}) \text{ and } \overline{u_k} = u_{-k}\Big\}$$

and

$$H_{k_0,c} := \{u \in H_{k_0} : \operatorname{supp} u_k \text{ compact for all } k \in \mathbb{Z}_{\text{odd},k_0}\}.$$

The corresponding mapping reads $\mathcal{S}_{k_0} \colon \mathcal{H}_{k_0} \to H_{k_0}$ given by

$$\mathcal{S}_{k_0}\tilde{u}(x,t) := \sum_{k\in\mathbb{Z}_{\text{odd},k_0}} \sum_{j\in\mathbb{N}_0} \int_{\mathcal{B}} \tilde{u}_{j,k}(s)\psi_{j,k}(x,s)ds\, e^{ik\omega t} = \sum_{k\in\mathbb{Z}_{\text{odd},k_0}} \mathcal{S}_{k,\text{mono}}\tilde{u}_k(x)e^{ik\omega t}.$$

Notice that $\mathcal{S}_{k_0}\tilde{u}$ is real-valued due to the condition $\overline{\tilde{u}_{j,k}(s)} = \tilde{u}_{j,-k}(-s)$ for all $(j,k,s) \in \mathbb{N}_0 \times \mathbb{Z}_{\text{odd},k_0} \times \mathcal{B}$ incorporated in $\mathcal{H}_{k_0}$ and Lemma 5.16.
In the proof of Theorem 5.36 we chose an element of $\mathcal{N}$. This gets justified now.

Theorem B.9. $\mathcal{N} \neq \emptyset$.

Proof. With the notation introduced above we verify

$$\overline{\mathcal{S}_{k_0}^{-1}(H_{k_0,c})}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H}_{k_0} \text{ for all } k_0 \in \mathbb{N}_{\text{odd}} \tag{B.27}$$

and

$$\overline{\bigcup_{k_0\in\mathbb{N}_{\text{odd}}} \mathcal{H}_{k_0}}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H}. \tag{B.28}$$

Assume that (B.27) and (B.28) are valid. Then

$$\overline{\bigcup_{k_0\in\mathbb{N}_{\text{odd}}} \mathcal{S}_{k_0}^{-1}(H_{k_0,c})}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H} \tag{B.29}$$

and therefore $\bigcup_{k_0\in\mathbb{N}_{\text{odd}}} \mathcal{S}^{-1}(H_{k_0,c}) \in \mathcal{N}$. Indeed, take $\tilde{u} \in \mathcal{H}$ and $\varepsilon > 0$. Due to (B.28) we find $\tilde{u}^1 \in \bigcup_{k_0\in\mathbb{N}_{\text{odd}}} \mathcal{H}_{k_0}$ with $\|\tilde{u}^1 - \tilde{u}\|_{\mathcal{H}} \le \frac{\varepsilon}{2}$. In particular, there is $k_1 \in \mathbb{N}_{\text{odd}}$ such that $\tilde{u}^1 \in \mathcal{H}_{k_1}$. (B.27) guarantees $\tilde{u}^{1,0} \in \mathcal{S}_{k_1}^{-1}(H_{k_1,c})$ with $\|\tilde{u}^{1,0} - \tilde{u}^1\|_{\mathcal{H}} \leq \frac{\varepsilon}{2}$. Hence, $\mathcal{S}_{k_1}\tilde{u}^{1,0} \in H_{k_1,c}$ so that $\mathcal{S}_{k_1}\tilde{u}^{1,0}$ has compact support in D and $\|\tilde{u}^{1,0} - \tilde{u}\|_{\mathcal{H}} \leq \|\tilde{u}^{1,0} - \tilde{u}^1\|_{\mathcal{H}} + \|\tilde{u}^1 - \tilde{u}\|_{\mathcal{H}} \leq \varepsilon$, i.e., (B.29) holds true.

Thus it remains to prove (B.27) and (B.28) which is done in the following. The proof of (B.28) is immediate by the definition of $\mathcal{H}_{k_0}$ so only (B.27) needs to be proved.

For this purpose, in a first step we show that $\overline{\mathcal{S}_{k,\mathrm{mono}}^{-1}(H_c^1(\mathbb{R}))}^{\|\cdot\|_{\mathcal{H}_{k,\mathrm{mono}}}} = \mathcal{H}_{k,\mathrm{mono}}$ for $k \in \mathbb{Z}_{\mathrm{odd}}$. The mapping $\mathcal{S}_{k,\mathrm{mono}}$ is bijective and continuous: Indeed, mapping properties of $\mathcal{S}_{k,\mathrm{mono}}$, injectivity as well as the continuity follow from Lemma B.7 and the equivalence of norms mentioned in Remark B.8. Take $u \in H^1(\mathbb{R})$. Then $\tilde{u}_{j,k}(s) := \left\langle (\mathcal{T}u)(\cdot,s), \psi_{j,k}(\cdot,s)\right\rangle_{\mathcal{P}}$ satisfies $\mathcal{S}_{k,\mathrm{mono}}\tilde{u} = u$ which shows that $\mathcal{S}_{k,\mathrm{mono}}$ is also onto. Therefore, $\mathcal{S}_{k,\mathrm{mono}}$ has also a continuous inverse. Thus

$$\overline{\mathcal{S}_{k,\mathrm{mono}}^{-1}(H_c^1(\mathbb{R}))}^{\|\cdot\|_{\mathcal{H}_{k,\mathrm{mono}}}} = \mathcal{S}_{k,\mathrm{mono}}^{-1}\left(\overline{H_c^1(\mathbb{R})}^{\|\cdot\|_{H^1(\mathbb{R})}}\right) = \mathcal{S}_{k,\mathrm{mono}}^{-1}(H^1(\mathbb{R})) = \mathcal{H}_{k,\mathrm{mono}}.$$

We now use this density result to verify (B.27). Fix $k_0 \in \mathbb{N}_{\mathrm{odd}}$. The spaces $H_{k_0}, H_{k_0,c}$ and $\mathcal{H}_{k_0}$ are by definition isomorphic to $k_0 + 1$ copies of the "monochromatic" variants, i.e.,

$$H_{k_0} \simeq \underbrace{H^1(\mathbb{R}) \times \cdots \times H^1(\mathbb{R})}_{k_0+1 \text{ times}}, H_{k_0,c} \simeq \underbrace{H_c^1(\mathbb{R}) \times \cdots \times H_c^1(\mathbb{R})}_{k_0+1 \text{ times}} \text{ and } \mathcal{H}_{k_0} \simeq \underbrace{\mathcal{H}_{-k_0,\mathrm{mono}} \times \cdots \times \mathcal{H}_{k_0,\mathrm{mono}}}_{k_0+1 \text{ times}}.$$

Notice that due to the equivalence of $\|\cdot\|_{H^1(\mathbb{R})}$ and $\|\tilde{\cdot}\|_{\mathcal{H}_{k,\mathrm{mono}}}$ we infer that on the $k_0 + 1$ copies the norms $\|\tilde{u}\|_{\mathcal{H}_{k_0}}$ and $\sum_{k\in\mathbb{Z}_{\mathrm{odd},k_0}} \|\tilde{u}_k\|_{\mathcal{H}_{k,\mathrm{mono}}}$ are equivalent. The desired density result now follows from the density in the monochromatic case and the fact that we only consider a finite number of copies. Hence, also (B.27) is established and the proof is done. □

Bibliography

[1] Adams, R. A., Fournier, J. J.: Sobolev spaces (Vol. 140). Academic press, (2003).

[2] Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable models in quantum mechanics. Springer Science & Business Media, (2012).

[3] Azzollini, A., Benci, V., D'Aprile, T. and Fortunato, D.: Existence of static solutions of the semilinear Maxwell equations. Ricerche di matematica, 55(2), 123-137, (2006).

[4] Bahouri, H., Chemin, J. Y., and Danchin, R.: Fourier analysis and nonlinear partial differential equations (Vol. 343). Springer Science & Business Media, (2011).

[5] Bartsch, T., Dohnal, T., Plum, M. and Reichel, W.: Ground States of a Nonlinear Curl-Curl Problem in Cylindrically Symmetric Media. Nonlinear Differential Equations and Applications NoDEA, 23(5), 52, online first, August 2016.

[6] Bartsch, T. and Mederski, J.: Ground and bound state solutions of semilinear time-harmonic Maxwell equations in a bounded domain. Archive for Rational Mechanics and Analysis, 215(1), 283-306, (2015).

[7] Bartsch, T. and Mederski, J.: Nonlinear time-harmonic Maxwell equations in domains. arXiv preprint arXiv:1610.06338, (2016).

[8] Belica, Martin: On the spectra of the Schrödinger operator with periodic delta potential. Bachelor thesis, Faculty for Mathematics, Karlsruhe Institute of Technology,(2016).

[9] Benci, V. and Fortunato, D.: Towards a unified field theory for classical electrodynamics. Archive for rational mechanics and analysis, 173(3), 379-414, (2004).

[10] Berestycki, H., Caffarelli, L. A. and Nirenberg, L.: Monotonicity for elliptic equations in unbounded Lipschitz domains. Communications on pure and applied mathematics, 50(11), 1089-1111, (1997).

[11] Berestycki, H. and Nirenberg, L.: On the method of moving planes and the sliding method. Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society, 22(1), 1-37, (1991).

[12] Birnir, B., McKean, H. P. and Weinstein, A: The rigidity of sine-gordon breathers. Communications on Pure and Applied Mathematics, 47(8), 1043-1051,(1994).

[13] Blank, C., Chirilus-Bruckner, M., Lescarret, V., and Schneider, G.: Breather solutions in periodic media. Communications in mathematical physics, 302(3), 815-841, (2011).

[14] Bonfiglioli, A., and Lanconelli, E.: Maximum Principle on unbounded domains for sub-Laplacians: A Potential Theory approach. Proceedings of the American Mathematical Society, 130(8), 2295-2304, (2002).

[15] Brock, F.: Continuous rearrangement and symmetry of solutions of elliptic problems. Proceedings of the Indian Academy of Sciences-Mathematical Sciences. Vol. 110. No. 2. Springer India, (2000).

[16] Burchard, A.: Steiner symmetrization is continuous in $W^{1,p}$. Geometric & Functional Analysis GAFA 7.5: 823-860, (1997).

[17] Buschmann, D., Stolz, G. and Weidmann, J.: One-dimensional Schrödinger operators with local point interactions. J. reine angew. Math 467: 169-186, (1995).

[18] Cazenave, T.: Semilinear schrödinger equations. Vol. 10. American Mathematical Soc. (2003).

[19] Chavel, I., Feldman, E.: Spectra of manifolds less a small domain. Duke Math. J, 56(2), 399-414, (1988).

[20] Christ, C. S. and Stolz, G.: Spectral theory of one-dimensional Schrödinger operators with point interactions. Journal of mathematical analysis and applications, 184(3), 491-516, (1994).

[21] Courtois, G.: Spectrum of manifolds with holes. Journal of Functional Analysis, 134(1), 194-221, (1995).

[22] D'Aprile, T. and Siciliano, G.: Magnetostatic solutions for a semilinear perturbation of the Maxwell equations. Advances in Differential Equations, 16(5/6), 435-466,(2011).

[23] Damascelli, L., Grossi, M. and Pacella, F.: Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle. In Annales de l'IHP Analyse non linéaire (Vol. 16, No. 5, pp. 631-652), (1999).

[24] Dancer, E. N., Wei, J. and Weth, T: A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. In Annales de l'Institut Henri Poincare (C) Non Linear Analysis (Vol. 27, No. 3, pp. 953-969). Elsevier Masson, (2010).

[25] Davies, E. B.: Heat kernels and spectral theory (Vol. 92). Cambridge University Press, (1990).

[26] de Figueiredo, D. G. and Sirakov, B.: Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems. Mathematische Annalen, 333(2), 231-260, (2005).

[27] Deimling, K.: Nonlinear functional analysis. Springer-Verlag, Berlin, 105, (1985).

[28] Denzler, J.: Nonpersistence of breather families for the perturbed sine Gordon equation. Communications in Mathematical physics, 158(2), 397-430,(1993).

[29] Di Nezza, E., Palatucci, G. and Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bulletin des Sciences Mathématiques, 136(5), 521-573, (2012).

[30] Dörfler, W., Lechleiter, A., Plum, M., Schneider, G., Wieners, C.: Photonic crystals: Mathematical analysis and numerical approximation (Vol. 42). Springer Science & Business Media, (2011).

[31] Eastham, M. S. P.: The spectral theory of periodic differential equations. Scottish Academic Press, distributed by Chatto & Windus, London, (1973).

[32] Edmunds, D. E., Evans, W. D.: Spectral theory and differential operators, Oxford, (1987).

[33] Ekeland, I.: On the variational principle. Journal of Mathematical Analysis and Applications, 47(2), 324-353, (1974).

[34] Evans, L.: Partial differential equations, American Mathematical Society, second edition, (2010).

[35] Fraenkel, L. E.: An introduction to maximum principles and symmetry in elliptic problems. No. 128. Cambridge University Press, 2000.

[36] Gidas, B., Ni, W. M. and Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. Adv. Math. Suppl. Stud. A, 7, 369-402, (1981).

[37] Gidas, B. and Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. Communications on Pure and Applied Mathematics, 34(4), 525-598, (1981).

[38] Gidas, B., and Spruck, J.: A priori bounds for positive solutions of nonlinear elliptic equations. Communications in Partial Differential Equations 6.8, 883-901, (1981).

[39] Gilbarg, D. and Trudinger, N.: Elliptic partial differential equations of second order. Springer, (2001).

[40] Hirsch, A., Reichel, W.: Existence of cylindrically symmetric ground states to a nonlinear curl-curl equation with non-constant coefficients. arXiv preprint arXiv:1606.04415, (2016). To appear in ZAA, (2017).

[41] Hislop, P. D., Sigal, I. M.: Introduction to spectral theory: With applications to Schrödinger operators (Vol. 113). Springer Science and Business Media, (2012).

[42] Keller, J. B., Odeh, F.: Partial differential equations with periodic coefficients and Bloch waves in crystals. Journal of Mathematical Physics, 5(11), 1499-1504, (1964).

[43] Kurasov, P., and Larson, J.: Spectral asymptotics for Schrödinger operators with periodic point interactions., Journal of Mathematical Analysis and Applications 266.1, 127-148, (2002).

[44] Li, C.: Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains. Communications in partial differential equations, 16(4-5), 585-615, (1991).

[45] Lieb, E. H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Studies in Applied Mathematics 57: 93-105 (1977).

[46] Lieb, E. H., Loss, M.: Analysis, volume 14 of graduate studies in mathematics. American Mathematical Society, Providence, RI, 4, (2001).

[47] Lions, P.-L.: Minimization problems in $L^1(\mathbb{R}^3)$. Journal of Functional Analysis, 41(2), 236-275, (1981).

[48] Lions, P.-L.: Symétrie et compacité dans les espaces de Sobolev. Journal of Functional Analysis 49.3: 315-334, (1982).

[49] Liu, S.: On superlinear problems without the Ambrosetti and Rabinowitz condition. Nonlinear Analysis: Theory, Methods & Applications, 73(3), 788-795, (2010).

[50] Lotoreichik, V. and Simonov, S.: Spectral Analysis of the Half-Line Kronig–Penney Model with Wigner–Von Neumann Perturbations., Reports on Mathematical Physics 74.1, 45-72, (2014).

[51] Mandel, R. and Reichel, W.: Distributional Solutions of the Stationary Nonlinear Schrödinger Equation: Singularities, Regularity and Exponential Decay, ZAA, issue 1, pp. 55–82, (2013).

[52] Mederski, J.: Ground States of Time-Harmonic Semilinear Maxwell Equations in $\mathbb{R}^3$ with Vanishing Permittivity. Archive for Rational Mechanics and Analysis, 218(2), 825-861, (2015).

[53] Mederski, J.: The Brezis-Nirenberg problem for the curl-curl operator., arXiv preprint arXiv:1609.03989, (2016).

[54] Palais, R. S.: The principle of symmetric criticality. Communications in Mathematical Physics 69.1: 19-30, (1979).

[55] Pankov, A.: Lecture Notes on Schrödinger Equations, Nova Science Publisher, (2007).

[56] Pelinovsky, D. E., Simpson, G. and Weinstein, M. I.: Polychromatic solitary waves in a periodic and nonlinear Maxwell system. SIAM Journal on Applied Dynamical Systems 11.1: 478-506, (2012).

[57] Plum, M., Reichel, W.: A breather construction for a semilinear curl-curl wave equation with radially symmetric coefficients, Journal of Elliptic and Parabolic Equations, Vol 2, 371-387, (2016).

[58] Poláčik, P., Quittner, P., and Souplet, P.: Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems. Duke Mathematical Journal, 139(3), 555-579, (2007).

[59] Quittner, P. and Souplet, P.: Superlinear parabolic problems: blow-up, global existence and steady states. Springer Science & Business Media, (2007).

[60] Reed, M. and Simon, B.: Methods of modern mathematical physics I. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, Functional analysis, (1980).

[61] Reed, M. and Simon, B.: Analysis of Operators, Vol. IV of Methods of Modern Mathematical Physics, (1978).

[62] Reichel, W., Weth, T.: A priori bounds and a Liouville theorem on a half-space for higher-order elliptic Dirichlet problems. Mathematische Zeitschrift, 261(4), 805-827, (2009).

[63] Rose, H., Weinstein. M.: On the bound states of the nonlinear Schrödinger equation with a linear potential. Physica D: Nonlinear Phenomena 30.1: 207-218, (1988).

[64] Schmüdgen, K.: Unbounded self-adjoint operators on Hilbert space (Vol. 265). Springer Science & Business Media, (2012).

[65] Simon, B.: Schrödinger semigroups. Bull.Amer.Math.Soc. 7 , 447-526, (1982).

[66] Stein, E.: Singular integrals and differentiability properties of functions, Vol. 2, Princeton University Press, (1970).

[67] Stuart, C. A.: Self-trapping of an electromagnetic field and bifurcation from the essential spectrum. Archive for Rational Mechanics and Analysis, 113(1), 65-96, (1991).

[68] Stuart, C. A.: Guidance properties of nonlinear planar waveguides. Archive for rational mechanics and analysis, 125(2), 145-200, (1993).

[69] Sutherland, R. L.: Handbook of nonlinear optics, CRC press, (2003).

[70] Szulkin, A. and Weth, T.: The method of Nehari manifold. Handbook of nonconvex analysis and applications, 597-632, (2010).

[71] Tasgal, R.S., Band, Y. B. and Malomed, B. A.: Gap solitons in a medium with third-harmonic generation. Physical Review E 72.1: 016624, (2005).

[72] Werner, D.: Funktionalanalysis. Springer-Verlag, (2007).

[73] Willem, M.: Minimax theorems. No. 24. Springer Science & Business Media, (1996).

[74] Zeng, X.: Cylindrically symmetric ground state solutions for curl-curl equations with critical exponent., arXiv preprint arXiv:1609.09598 (2016).

Eidesstattliche Erklärung:

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und nur unter Zuhilfenahme der ausgewiesenen Hilfsmittel angefertigt habe. Sämtliche Stellen der Arbeit, die im Wortlaut oder dem Sinn nach anderen gedruckten oder im Internet veröffentlichten Werken entnommen sind, habe ich durch genaue Quellenangaben kenntlich gemacht.

Karlsruhe, den 10.05.2017