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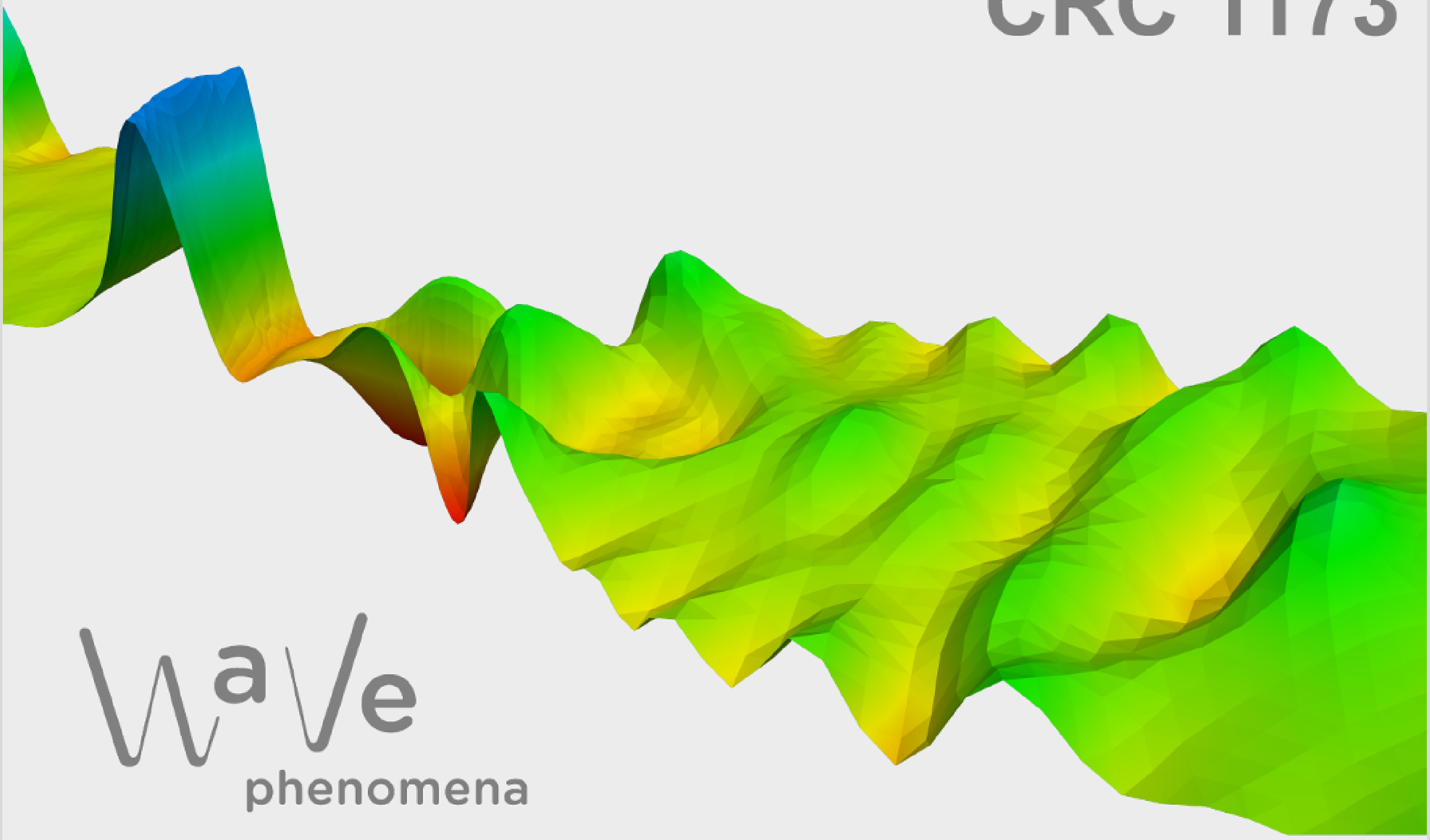

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A RADIATION CONDITION ARIZING FROM THE LIMITING ABSORPTION PRINCIPLE FOR A CLOSED FULL- OR HALF-WAVEGUIDE PROBLEM.

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Abstract. In this paper we consider the propagation of waves in a closed full- or half-waveguide where the index of refraction is periodic along the axis of the waveguide. Motivated by the limiting absorption principle, proven in the Appendix by a functional analytic perturbation theorem, we formulate a radiation condition which assures uniqueness of a solution and allows the existence of propagating modes. Our approach is quite different to the known one as, e.g., considered in [6] and allows an extension to open wave guides (see [10]). After application of the Floquet-Bloch transform we consider the Floquet-Bloch variable α as a parameter in the resulting quasi-periodic boundary value problem and study the behaviour of the solution when α tends to an exceptional value by a singular perturbation result which we have found in [4].

1. Introduction

The study of wave propagation in periodic structures has a long history. Most of the work concerns wave propagation in layered media (see, e.g., [12, 13] as classical references) but during the last two decades also more generel periodic structures have been investigated. We were motivated by the work of Fliss and Joly [5] who studied the asymptotic behaviour of time-harmonic waves in closed periodic wave guides. The challenge for these problems is to develop suitable radiation conditions which, from the mathematical point of view, assure uniqueness and existence of propagating modes; that is, existence of waves which do not decay along the direction of periodicity. A natural way is to characterize the proper waves by the limiting absorption principle; that is, these waves are limits (with respect to a certain topology) of H^1-solutions for complex wave numbers when the imaginary part tends to zero. Recently, in [10], we investigated the scattering of point sources by a periodic layer (periodicity along the axes of the layer) on a perfectly conducting line. We developed a rather elementary proof of the limiting absorption principle which directly provides the correct form of the propagating modes and leads to a natural radiation condition, extending the well known upwards propagation radiation condition for rough surfaces (in the sense of [2, 1]). This approach carries over without any difficulties, being even simpler, to the case of radiation problems in a closed waveguide. Because of this close analogy we transfer this approach for the closed waveguide into the Appendix, where we simplify the arguments by making use of a functional analytic perturbation argument based on a classical result bei Colton and Kress in [4]. We use the classical form of the singular perturbation result in [4] to give a direct proof of uniqueness and existence using only the radiation condition. We want to point out that our approach is rather different

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compared to the one in [6], and also compared to the independent study in [7] and leads to a form of the radiation condition which is equivalent to the well known formulations as, e.g., in [6] (see also [8]) but takes a slightly different form.

The radiation condition, motivated by the limiting absorption principle of the Appendix, demands a special decomposition of the solution into a field $u^{(1)}$ which decays along the axis of the layer and a finite combination of surface waves which can be understood as solutions of finite dimensional eigenvalue problem. The sign of the corresponding eigenvalues determine whether the surface wave travel to the right or left, respectively. We show uniqueness of the solution under this radiation condition. The proof of existence is based on the Floquet-Bloch transform and the singular perturbation result in [4].

The methods we apply are all well-known and in principle simple enough to extend our analysis to more involved scattering problems in linear elasticity or electromagnetics (which, however, has to be done). To reduce technical difficulties, in this paper we are however merely considering the simple Helmholtz equation in $\mathbb{R}^2$.

To briefly comment on this paper's structure, the following Section 2 discusses the closed waveguide problem with respect to uniqueness, existence, and well-posedness. Then we consider the half-waveguide problem in Section 3 where we followed an idea presented by Hoang in [7]. In the Appendix we prove a general functional analytic perturbation result and apply this to prove the limiting absorption principle for the closed wave guide problem.

2. The Closed Periodic Waveguide

Let $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0$ be the wave number, $W = \mathbb{R} \times (0,h)$ the waveguide, $n \in L^\infty(W)$ the index of refraction which is assumed to be $2\pi-$periodic with respect to x_1 and satisfies $n \geq n_0$ in W for some $n_0 > 0$. Furthermore, let $f \in L^2(W)$ be a source function. For f we will need some decay when $|x_1|$ tends to infinity. Therefore, for $r \geq 0$ we define the weighted space

$$L^2_r(W) \;=\; \big\{ f \in L^2(W) : x \mapsto (1+|x_1|^2)^{r/2} f(x) \text{ is in } L^2(W) \big\}$$

with canonical norm and assume that $f \in L^2_r(W)$ for some $r > 3/2$. We want to determine $u \in H^1_{loc}(W)$ such that

$$\Delta u + k^2 n\, u = -f \text{ in } W\,, \quad u = 0 \text{ on } \partial W\,. \tag{1}$$

Definition 2.1. *A function $u \in H^1_{loc}(W)$ with $u = 0$ on ∂W is called variational solution of (1) if*

$$\int_W \big[\nabla u \cdot \nabla \psi - k^2 n\, u\, \psi\big]\, dx \;=\; \int_W f\, \psi\, dx \tag{2}$$

for all $\psi \in H^1(W)$ with compact support.

We make the first assumption that k^2 does not belong to the point spectrum of the operator $-\frac{1}{n}\Delta$ (with the canonical domain of definition).

Assumption 2.2. *There is no non-trivial solution $u \in H^1_0(W)$ of $\Delta u + k^2 n\, u = 0$ (in the sense of (2)). Here, and in the following, $H^1_0(W) = \{u \in H^1(W) : u = 0$ on $\partial W\}$.*

This assumption is standard, see, e.g., [6], and is only necessary for real values of k. Indeed, we have:

Theorem 2.3. *For $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$ there exists a unique variational solution $u \in H^1(W)$ of problem* (2)*.*

The proof is a simple consequence of the theorem of Lax-Milgram.

For real values of k, however, there exist propagating modes as the simple example of constant index n shows. In [10] we have proven the limiting absorption principle for the - more complicated - half-open waveguide problem where the scattering medium is a periodic layer on a perfectly conducting plate. The analysis carries over to this case of a closed waveguides and motivates a radiation condition. We recall ans simplify the proof of the limiting absorption principle in the Appendix. The formulation of the radiation condition needs some preparation. From now on we assume that k is a fixed real and positive number.

We recall the (periodic) Floquet-Bloch transform $T_{per} : L^2(\mathbb{R}) \to L^2\big((0, 2\pi)\times(-1/2, 1/2)\big)$ which is defined by

$$(3) \qquad (T_{per}h)(t,\alpha) \;=\; \tilde{h}(t,\alpha) \;=\; \sum_{j\in\mathbb{Z}} h(t+2\pi j)\, e^{-i\alpha(t+2\pi j)}\,, \quad t \in \mathbb{R}\,,\ \alpha \in (-1/2, 1/2]\,.$$

The latter formula directly shows that for smooth functions h and fixed α the transformed function $t \mapsto T_{per}h(t,\alpha) = \tilde{h}(t,\alpha)$ is $2\pi-$periodic while for fixed t the function $\alpha \mapsto T_{per}h(t,\alpha) = \tilde{h}(t,\alpha)$ is $t-$quasi-periodic; that is, $\tilde{h}(t,\alpha+1) = e^{-it}\tilde{h}(t,\alpha)$. It is hence sufficient to consider $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ as image space of T_{per}. The inverse transform is given by

$$(T_{per}^{-1}g)(t) \;=\; \int_{-1/2}^{1/2} g(t,\alpha)\, e^{i\alpha t} d\alpha\,, \quad t \in \mathbb{R}\,,$$

where we extend $g(\cdot,\alpha)$ to a $2\pi-$periodic function in $\mathbb{R}$. In view of our waveguide problem, we apply the Floquet-Bloch transform to the variable x_1 and consider x_2 as a parameter. By an abuse of notation we denote this operator also by T_{per}. Setting $Q := (0, 2\pi)\times(0, h)$ one can then show that T_{per} is an isometry from $L^2(W)$ onto $L^2\big(Q\times(-1/2, 1/2)\big)$ which we identify with $L^2\big((-1/2, 1/2), L^2(Q)\big)$, such that

$$\|\tilde{h}\|^2_{L^2(Q\times(-1/2,1/2))} \;=\; \int\limits_{-1/2}^{1/2}\int\limits_{Q} |\tilde{h}(x,\alpha)|^2\, dx\, d\alpha \;=\; \int\limits_{W} |h(x)|^2\, dx \;=\; \|h\|^2_{L^2(W)}\,.$$

Further, the restriction of T_{per} to $H^1(W)$ is an isomorphism from $H^1(W)$ onto $L^2\big((-1/2,1/2), H^1_{per}(Q)\big)$ where $H^1_{per}(Q) = \{\tilde{h} \in H^1(Q) : \tilde{h} \text{ is } 2\pi\text{–periodic in } x_1\}$. For these properties of the Bloch transform we refer to [11, Section 6]. In the following we will mainly use the periodic Bloch transform T_{per} but sometimes later it will be convenient to consider also the quasi-periodic Bloch transform T_{qp}, defined by

$$(T_{qp}h)(x,\alpha) \;=\; e^{i\alpha x_1}(T_{per}h)(x,\alpha) \;=\; \sum_{j\in\mathbb{Z}} h(x_1{+}2\pi j, x_2)\, e^{-i\alpha 2\pi j}\,, \quad x \in Q\,,\ \alpha \in (-1/2, 1/2]\,.$$

We note that $T_{qp}h$ is one-periodic with respect to α. In [11] it has been shown that T_{qp} yields an isomorphism from the weighted space $L^2_r(W)$ onto $H^r_{per}\big((-1/2, 1/2), L^2(Q)\big)$.

For fixed α the transformed $2\pi-$periodic field $\tilde{u}(\cdot,\alpha) = (T_{per}u)(\cdot,\alpha)$ is a variational solution to

$$(4) \qquad \begin{cases} \Delta \tilde{u}(\cdot,\alpha) \;+\; 2i\alpha \dfrac{\partial \tilde{u}(\cdot,\alpha)}{\partial x_1} \;+\; (k^2 n - \alpha^2)\,\tilde{u}(\cdot,\alpha) \;=\; -\tilde{f}(\cdot,\alpha) \quad \text{in } W\,, \\ \tilde{u}(\cdot,\alpha) = 0 \quad \text{on } \partial W\,. \end{cases}$$

To tackle the last problem variationally, we define $H^1_{0,per}(Q)$ to be the subspace of functions in $H^1_{per}(Q)$ that vanish on $\partial W \cap \overline{Q}$.

Then we seek $\tilde{u}_\alpha = \tilde{u}(\cdot,\alpha) \in H^1_{0,per}(Q)$ as a solution to the variational equation

$$(5) \qquad \int_Q \left[\nabla \tilde{u}_\alpha \cdot \nabla \overline{\psi} - 2i\alpha\,\overline{\psi}\,\frac{\partial \tilde{u}_\alpha}{\partial x_1} + (\alpha^2 - k^2 n)\,\tilde{u}_\alpha\,\overline{\psi} \right] dx \;=\; \int_Q \tilde{f}(\cdot,\alpha)\,\overline{\psi}\,dx$$

for all $\psi \in H^1_{0,per}(Q)$. We equip $H^1_{0,per}(Q)$ with the inner product $(u,v)_* = \int_Q \nabla u \cdot \nabla \overline{v}\,dx$ which leads to the norm $\|\cdot\|_*$ which is equivalent to the ordinary H^1-norm on $H^1_{0,per}(Q)$ (Poincare-Friedrich's inequality). Then we can rewrite the variational equation (5) as

$$(6) \qquad (\tilde{u}_\alpha,\psi)_* \;-\; a_\alpha(\tilde{u}_\alpha,\psi) \;=\; \int_Q \tilde{f}(\cdot,\alpha)\,\overline{\psi}\,dx \quad \text{for all } \psi \in H^1_{0,per}(Q)\,,$$

where

$$a_\alpha(v,\psi) \;:=\; \int_Q \left[2i\alpha\,\overline{\psi}\,\frac{\partial v}{\partial x_1} + (k^2 n - \alpha^2)\,v\,\overline{\psi} \right] dx\,, \quad v,\psi \in H^1_{0,per}(Q)\,.$$

By the representation of Riesz in the Hilbert space $H^1_{0,per}(Q)$ and the compact imbedding of $H^1_{0,per}(Q)$ in $L^2(Q)$ for every $\alpha \in (-1/2, 1/2]$ there exists a compact operator K_α such that $a_\alpha(u,\psi) = (K_\alpha u,\psi)_*$ for all $u,\psi \in H^1_{0,per}(Q)$; that is,

$$(7) \qquad (K_\alpha u,\psi)_* \;=\; \int_Q \left[2i\alpha\,\overline{\psi}\,\frac{\partial u}{\partial x_1} + (k^2 n - \alpha^2)\,u\,\overline{\psi} \right] dx \quad \text{for all } u,\psi \in H^1_{0,per}(Q)\,.$$

Furthermore, there exists $\tilde{f}_\alpha \in H^1_{0,per}(Q)$ with $\int_Q \tilde{f}(\cdot,\alpha)\,\overline{\psi}\,dx = (\tilde{f}_\alpha,\psi)_*$ for all $\psi \in H^1_{0,per}(Q)$. Then we can rewrite the variational equation (6) as an operator equation,

$$(8) \qquad \tilde{u}_\alpha \;-\; K_\alpha \tilde{u}_\alpha \;=\; \tilde{f}_\alpha \quad \text{in } H^1_{0,per}(Q)\,.$$

The values of $\alpha \in (-1/2, 1/2]$ for which $I - K_\alpha$ fails to be injective are called *exceptional values.*

Theorem 2.4. *Under Assumption 2.2 there exist only finitely many (possibly no) exceptional values which we collect in the set $\{\hat{\alpha}_j : j \in J\} \subset (-1/2, 1/2]$. Therefore, for every $j \in J$ the space*

$$X_j \;=\; \left\{ \phi \in H^2_{loc}(W) : \Delta\phi + k^2 n \phi = 0 \text{ in } W\,,\ \phi = 0 \text{ on } \partial W\,,\ \phi \text{ is } \hat{\alpha}_j\text{−quasi-periodic} \right\}$$

is finite dimensional. Set $m_j = \dim X_j$.

Proof: Only the finiteness of the set of exceptional values has to be shown. The operator $I - K_\alpha$ depends quadratically on α and has the form $I - K_\alpha = I - K_0 + \alpha B + \alpha^2 C$ where

K_0, B, C are also selfadjoint and C ist positive. Therefore, we can write $(I - K_\alpha)\tilde{u} = 0$ as a linear eigenvalue problem in the form

$$\begin{pmatrix} I - K_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \alpha \begin{pmatrix} B & C^{1/2} \\ -C^{1/2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $u_1 = \tilde{u}$ and $u_2 = \alpha\, C^{1/2}\tilde{u}$. This has the form $Mu = \alpha Du$ where M is Fredholm and D is compact. If there exists $\hat{\alpha} \in \mathbb{C}$ such that $M - \hat{\alpha}D$ is invertible then $Mu = \alpha Du$ is equivalent to the ordinary eigenvalue problem $\frac{1}{\alpha-\hat{\alpha}}\, u = [M - \hat{\alpha}D]^{-1}Du$ for the compact operator $[M - \hat{\alpha}D]^{-1}D$. Since its eigenvalues can accumulate only at zero the eigenvalues α can accumulate only at infinity which shows that there are at most finitely many in $[-1/2, 1/2]$. Therefore, we have to show the existence of at least one $\hat{\alpha} \in (-1/2, 1/2]$ such that $I - K_{\hat{\alpha}}$ is invertible. If this were not the case $\lambda = 1$ would be an eigenvalue of the selfadjoint operators K_α for every $\alpha \in (-1/2, 1/2]$. Let us fix an arbitrary $\hat{\alpha} \in (-1/2, 1/2]$. Then $\lambda = 1$ is an eigenvalue of $K_{\hat{\alpha}}$. Furthermore K_α depends analytically on α. By a general theorem on selfadjoint holomorphic families of operators (see [9], Chapter 7, Section 3) there exist $\lambda(\alpha)$ and non-trivial $\tilde{u}(\alpha) \in H^1_{per}(Q)$ which depend holomorphically on α in a neighborhood I of $\hat{\alpha}$ such that $\lambda(\hat{\alpha}) = 1$ and $K_\alpha\tilde{u}(\alpha) = \lambda(\alpha)\tilde{u}(\alpha)$ for all $\alpha \in I$. Because 1 is an isolated eigenvalue of K_α for every $\alpha \in (-1/2, 1/2]$ we conclude that $\lambda(\alpha) = 1$ for all $\alpha \in I$. Then $u(x) = \int_I \tilde{u}(x,\alpha)\, \exp(i\alpha x_1)\, d\alpha$ is in $H^1_0(W)$ and satisfies $\Delta u + k^2 n u = 0$ in W which contradicts Assumption 2.2. □

We note that this space X_j is related to the nullspace $\mathcal{N}(I - K_{\hat{\alpha}_j}) \subset H^2_{0,per}(Q)$ of $I - K_{\hat{\alpha}_j}$ in the sense that $\phi \in X_j$ if, and only if $\tilde{\phi} \in \mathcal{N}(I - K_{\hat{\alpha}_j})$ where $\tilde{\phi}(x) = \phi(x)e^{-i\hat{\alpha}_j x_1}$.

We choose a basis $\{\phi_{\ell,j} \in X_j : \ell = 1, \ldots, m_j\}$ of X_j for $j \in J$ as eigenfunctions of the following self adjoint eigenvalue problem

$$(9) \qquad -i \int_Q \frac{\partial \phi_{\ell,j}}{\partial x_1}\, \overline{\psi}\, dx \;=\; \lambda_{\ell,j}\, k \int_Q n\, \phi_{\ell,j}\, \overline{\psi}\, dx \quad \text{for all } \psi \in X_j$$

with eigenvalues $\lambda_{\ell,j} \in \mathbb{R}$, $\ell = 1, \ldots, m_j$, $j \in J$. We normalize the eigenfunctions as

$$k \int_Q n\, \phi_{\ell,j}\, \overline{\phi_{\ell',j}}\, dx \;=\; \delta_{\ell,\ell'} \quad \text{for all } \ell, \ell' = 1, \ldots, m_j\,.$$

If we transform (9) to the $2\pi-$periodic functions we have for $\tilde{\phi}_{j,\ell}(x) = \phi_{j,\ell}(x)e^{-i\hat{\alpha}_j x_1} \in \mathcal{N}(I - K_{\hat{\alpha}_j})$ that

$$-i \int_Q \left[\frac{\partial \tilde{\phi}_{\ell,j}}{\partial x_1} + i\hat{\alpha}_j\, \tilde{\phi}_{\ell,j} \right] \overline{\tilde{\psi}}\, dx \;=\; \lambda_{\ell,j}\, k \int_Q n\, \tilde{\phi}_{\ell,j}\, \overline{\tilde{\psi}}\, dx \quad \text{for all } \tilde{\psi} \in \mathcal{N}(I - K_{\hat{\alpha}_j})\,.$$

Using the definition (7) of the operator $K_{k,\alpha}$ (where we indicated also the dependence on k); that is,

$$(K_{k,\alpha}v, \psi)_* \;=\; \int_Q \left[2i\alpha\, \overline{\psi}\, \frac{\partial v}{\partial x_1} + (k^2 n - \alpha^2)\, v\, \overline{\psi} \right] dx\,, \quad v, \psi \in H^1_{0,per}(Q)\,,$$

we note that we can rewrite the eigenvalue problem (9) in the form

$$(10) \qquad -P_j \frac{\partial}{\partial \alpha} K_{\hat{\alpha}_j}\, \tilde{\phi}_{\ell,j} \;=\; \lambda_{j,\ell}\, P_j \frac{\partial}{\partial k} K_{\hat{\alpha}_j}\, \tilde{\phi}_{\ell,j}\,, \quad \ell = 1, \ldots, m_j\,,\ j \in J\,,$$

where P_j is the orthogonal projection onto $\mathcal{N}(I - K_{\hat{\alpha}_j})$.

Assumption 2.5. *We assume that $\lambda_{\ell,j} \neq 0$ for all $\ell = 1, \ldots, m_j$, $j \in J$.*

We will comment on this assumption and compare it to the corresponding assumption in [6] in the Appendix. We define the sets $L_j^\pm$ by

$$L_j^\pm \;:=\; \bigl\{\ell \in \{1,\ldots,m_j\} : \lambda_{\ell,j} \gtrless 0\bigr\} \;=\; \left\{\ell \in \{1,\ldots,m_j\} : \operatorname{Im} \int_Q \frac{\partial \phi_{\ell,j}}{\partial x_1}\, \overline{\phi_{\ell,j}}\, dx \gtrless 0\right\}.$$

Now we are able to formulate the radiation condition.

Definition 2.6. *(Radiation Condition)*
Let $\{\hat\alpha_j : j \in J\}$ be the (possibly empty) set of exceptional values for wave number $k > 0$ and let Assumptions 2.2 and 2.5 hold. Then the field u has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)} \in H_0^1(W)$ and $u^{(2)}$ has the form

$$u^{(2)}(x) \;=\; \psi^+(x_1) \sum_{j\in J} \sum_{\ell \in L_j^+} a_{\ell,j}^+\, \phi_{\ell,j}(x) \;+\; \psi^-(x_1) \sum_{j\in J} \sum_{\ell \in L_j^-} a_{\ell,j}^-\, \phi_{\ell,j}(x)\,, \quad x \in W\,, \tag{11}$$

for some $a_{\ell,j}^\pm \in \mathbb{C}$ where $\psi^\pm$ are given by

$$\psi^\pm(x_1) \;=\; \frac{1}{2}\left[1 \;\pm \frac{2}{\pi} \int_0^{x_1/2} \frac{\sin t}{t}\, dt\right], \quad x_1 \in \mathbb{R}\,. \tag{12}$$

We note that $\psi^+(x_1) + \psi^-(x_1) = 1$ and $\psi^+(x_1) \to 1$ as $x_1 \to \infty$ and $\psi^+(x_1) \to 0$ as $x_1 \to -\infty$.

The following lemma provides the essential tool for proving uniqueness.

Lemma 2.7. *Let $u^\pm = \sum_{j\in J} \sum_{\ell \in L_j^\pm} a_{\ell,j}^\pm\, \phi_{\ell,j}$ for some $a_{\ell,j}^\pm \in \mathbb{C}$. Set $\gamma_r = \{r\} \times (0,h)$ for $r \in \mathbb{R}$. Then*

$$\operatorname{Im} \int_{\gamma_r} \overline{u^\pm}\, \frac{\partial u^\pm}{\partial x_1}\, ds \;=\; \sum_{j\in J} \sum_{\ell \in L_j^\pm} \lambda_{\ell,j} |a_{\ell,j}^\pm|^2 \;\gtrless\; 0 \quad \textit{if} \quad u^\pm \neq 0\,.$$

Proof: Set $u_j^\pm = \sum_{\ell \in L_j^\pm} a_{\ell,j}^\pm\, \phi_{\ell,j}$ for $j \in J$. Then, for $j, j' \in J$,

$$\begin{aligned}
0 \;&=\; \int_{r+\partial Q} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial \nu} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial \nu}\right) ds \\
&=\; -\int_{\gamma_r} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) ds \;+\; \int_{\gamma_{r+2\pi}} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) ds \\
&=\; \left(e^{i(\hat\alpha_{j'} - \hat\alpha_j)2\pi} - 1\right) \int_{\gamma_r} \left(\overline{u_j^\pm}\, \frac{\partial u_{j'}^\pm}{\partial x_1} - u_{j'}^\pm\, \frac{\partial \overline{u_j^\pm}}{\partial x_1}\right) ds\,.
\end{aligned}$$

Therefore, the last integral vanishes for $j \neq j'$. Thus we have

$$
\begin{aligned}
& 2\,i\,\mathrm{Im} \int_{\gamma_r} \overline{u^\pm}\,\frac{\partial u^\pm}{\partial x_1}\,ds \\
&= \int_{\gamma_r} \left[\overline{u^\pm}\,\frac{\partial u^\pm}{\partial x_1} - u^\pm\,\frac{\partial \overline{u^\pm}}{\partial x_1} \right] ds \;=\; \sum_{j\in J} \int_{\gamma_r} \left[\overline{u_j^\pm}\,\frac{\partial u_j^\pm}{\partial x_1} - u_j^\pm\,\frac{\partial \overline{u_j^\pm}}{\partial x_1} \right] ds \\
&= 2\,i \sum_{j\in J} \mathrm{Im} \int_{\gamma_r} \overline{u_j^\pm}\,\frac{\partial u_j^\pm}{\partial x_1}\,ds\,.
\end{aligned}
$$

Now we fix $j \in J$ and consider only u_j^+. Setting $v(x) = (x_1 - r)u_j^+(x)$ yields $\frac{\partial v(x)}{\partial x_1} = u_j^+(x) + (x_1 - r)\,\frac{\partial u_j^+(x)}{\partial x_1}$ and $\Delta v + \hat{k}^2 n v = 2\frac{\partial u_j^+}{\partial x_1}$. Therefore,

$$
\begin{aligned}
2 \int\limits_Q \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,dx &= 2 \int\limits_{r+Q} \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,dx \;=\; \int\limits_{r+Q} \overline{u_j^+}\,\big(\Delta v + \hat{k}^2 n v\big)\,dx \\
&= \int\limits_{r+Q} v\,\big(\Delta \overline{u_j^+} + \hat{k}^2 n \overline{u_j^+}\big)\,dx \;+\; \int\limits_{r+\partial Q} \left(\overline{u_j^+}\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial \overline{u_j^+}}{\partial \nu} \right) ds \\
&= -\int\limits_{\gamma_r} |u_j^+|^2 ds \;+\; \int\limits_{\gamma_{r+2\pi}} \left[\overline{u_j^+} \left(u_j^+ + 2\pi\,\frac{\partial u_j^+}{\partial x_1} \right) - 2\pi u_j^+\,\frac{\partial \overline{u_j^+}}{\partial x_1} \right] ds \\
&= 2\pi \int\limits_{\gamma_r} \left(\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1} - u_j^+\,\frac{\partial \overline{u_j^+}}{\partial x_1} \right) ds \;=\; 4\pi\,i\;\mathrm{Im} \int\limits_{\gamma_r} \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds\,.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int\limits_Q \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,dx &= \sum_{\ell,\ell'\in L_j^+} \overline{a_{\ell,j}^+}\,a_{\ell',j}^+ \int\limits_Q \overline{\phi_{\ell,j}}\,\frac{\partial \phi_{\ell',j}}{\partial x_1}\,dx \\
&= ik \sum_{\ell,\ell'\in L_j^+} \overline{a_{\ell,j}^+}\,a_{\ell',j}^+\,\lambda_{\ell',j} \int\limits_Q n\,\overline{\phi_{\ell,j}}\,\phi_{\ell',j}\,dx \;=\; i \sum_{\ell\in L_j^+} \lambda_{\ell,j} |a_{\ell,j}^+|^2
\end{aligned}
$$

by the orthonormalization of $\phi_{\ell,j}$. Taking the imaginary part and summing over j yields the assertion. □

The following theorem is an essential ingredient for the proof of existence. It is taken from [3] (Theorem 1.32) where a slightly more general situation is considered.

Theorem 2.8. *Assume that X is a Hilbert space, I an open interval, and the families $y(\alpha) \in X$ and the linear and compact operators $K(\alpha) : X \to X$ are differentiable with respect to $\alpha \in I$. Let $L(\alpha) := I - K(\alpha)$ be bijective for $\alpha \neq \hat{\alpha}$ for some $\hat{\alpha} \in I$ but $\mathcal{N} := \mathcal{N}\big(I - K(\hat{\alpha})\big) \neq \{0\}$. Let $P : X \to \mathcal{N} = \mathcal{N}\big(I - K(\hat{\alpha})\big)$ be the orthogonal projection operator onto the null space $\mathcal{N}$ of $L(\hat{\alpha}) = I - K(\hat{\alpha})$. Assume, furthermore, that the Riesz number of $L(\hat{\alpha})$ is one; that is, the algebraic and geometric multiplicities of the eigenvalue 1 of $K(\hat{\alpha})$ coincide. Let $\hat{y}' := \frac{\partial}{\partial \alpha} y(\hat{\alpha}) \in \mathcal{N}$ and assume that the projection*

$C = P\frac{\partial}{\partial\alpha}K(\hat\alpha)\big|_{\mathcal N} : \mathcal N \to \mathcal N$ *of the derivative restricted to the nullspace* $\mathcal N$ *is one-to-one. Then the unique solution* $u(\alpha) \in X$ *of* $L(\alpha)u(\alpha) = y(\alpha)$ *for* $\alpha \neq \hat\alpha$ *converges in* X *to a solution* $u(\hat\alpha)$ *of* $L(\hat\alpha)u(\hat\alpha) = y(\hat\alpha)$ *as* α *tends to* $\hat\alpha$.
The solution $u(\hat\alpha)$ *is given by* $u(\hat\alpha) = \big(L(\hat\alpha) - P\big)^{-1} y(\hat\alpha) + \chi$ *where* $\chi \in \mathcal N$ *is the unique solution of* $C\chi = \hat y' - C\big(L(\hat\alpha) - P\big)^{-1} y(\hat\alpha)$.

Remarks 2.9. *(a) From the last characterization of* $u(\hat\alpha)$ *we observe that there exists* $c > 0$ *and* $\delta > 0$ *which are independent of* $y(\alpha)$ *such that* $\|u(\alpha)\| \leq c\big[\|y(\hat\alpha)\| + \|\partial y(\hat\alpha)/\partial\alpha\|\big]$ *for all* $\alpha \in I$ *with* $|\alpha - \hat\alpha| \leq \delta$.
(b) If $I \subset \mathbb C$ *is an open set in* $\mathbb C$ *and* $y(\alpha)$ *and* $K(\alpha)$ *depend holomorphicly on* $\alpha \in I$ *then, under the remaining assumptions of the theorem,* $u(\alpha)$ *depends holomorphicly on* $\alpha \in I$ *as well. Indeed, this follows from the facts that* u *is holomorphic in* $I \setminus \{\hat\alpha\}$ *and continuous in* $\hat\alpha$. *(Application of Riemann's theorem on removable singularities.)*

Before we prove the main result of this section we show how to determine the coefficients $a^\pm_{\ell,j}$ in the decomposition $u = u^{(1)} + u^{(2)}$.

Theorem 2.10. *Let Assumtions 2.2 and 2.5 hold. Define the functions* $F_{\ell,j}$ *by*

$$F_{\ell,j}(x,\alpha) \;=\; \frac{1}{2\pi}\left[\phi_{\ell,j}(x)\, i\,(\alpha - \hat\alpha_j) \;+\; 2\,\frac{\partial \phi_{\ell,j}(x)}{\partial x_1}\right] e^{i(\alpha-\hat\alpha_j)x_1} \tag{13}$$

for $x \in Q$, $\alpha \in (-1/2, 1/2]$, *and* $\ell = 1, \ldots, m_j$, $j \in J$. *Assume that the coefficients* $a^\pm_{\ell,j}$ *solve the linear quadratic system*

$$\begin{aligned} &\sum_{j\in J}\sum_{\ell\in L_j^+} a^+_{\ell,j}\big(F_{\ell,j}(\cdot,\hat\alpha_{j_0}),\phi_{\ell_0,j_0}\big)_{L^2(Q)} \;-\; \sum_{j\in J}\sum_{\ell\in L_j^-} a^-_{\ell,j}\big(F_{\ell,j}(\cdot,\hat\alpha_{j_0}),\phi_{\ell_0,j_0}\big)_{L^2(Q)} \\ &\qquad = \; -\big(\hat f(\cdot,\hat\alpha_{j_0}),\phi_{\ell_0,j_0}\big)_{L^2(Q)} \quad \text{for all } \ell_0 = 1,\ldots,m_{j_0}\,,\ j_0 \in J \end{aligned} \tag{14}$$

where $\hat f = T_{qp} f$ *is the quasi-periodic Floquet-Bloch transform of* f. *Then there exists* $u^{(1)} \in H^1_0(W) \cap H^2_{loc}(W)$ *such that* $u = u^{(1)} + u^{(2)}$ *with the representation (11) of* $u^{(2)}$ *solves (1).*
For every $R > 0$ *there exists* $c = c_R > 0$ *which is independent of* $f \in L^2_r(W)$ *such that* $\|u\|_{H^2(W_R)} \leq c\|f\|_{L^2_r(W)}$ *where* $W_R = (-R,R) \times (0,h)$.

Proof: Let $a^\pm_{\ell,j}$ solve the system (14) and define $u^{(2)}$ by (11). In order that $u = u^{(1)} + u^{(2)}$ solves (1), the field $u^{(1)} \in H^1_0(W)$ has to solve

$$\begin{aligned} \Delta u^{(1)}(x) + k^2 n(x)\, u^{(1)}(x) \;&=\; -f(x) \;-\; \big[\Delta u^{(2)}(x) + k^2 n(x)\, u^{(2)}(x)\big] \\ &=\; -f(x) \;-\; \sum_{j\in J}\left[u_j^+(x)\,\frac{d^2\psi^+(x_1)}{dx_1^2} \;+\; 2\,\frac{d\psi^+(x_1)}{dx_1}\,\frac{\partial u_j^+(x)}{\partial x_1}\right] \\ &\qquad -\; \sum_{j\in J}\left[u_j^-(x)\,\frac{d^2\psi^-(x_1)}{dx_1^2} \;+\; 2\,\frac{d\psi^-(x_1)}{dx_1}\,\frac{\partial u_j^-(x)}{\partial x_1}\right] \end{aligned} \tag{15}$$

where $u_j^\pm = \sum_{\ell\in L_j^\pm} a_{\ell,j}^\pm \phi_{\ell,j}$. We set $\varphi^\pm(x_1) = \frac{d\psi^\pm(x_1)}{dx_1} = \pm\frac{1}{2\pi}\frac{\sin(x_1/2)}{(x_1/2)}$ and note that the right-hand side of (15) is in $L^2(W)$. We take the quasi-periodic Floquet-Bloch transform

$$(T_{qp}v)(x,\alpha) \;=\; \hat v(x,\alpha) \;=\; \sum_{m\in\mathbb{Z}} v(x+2\pi m e^{(1)})\, e^{-2\pi i\, m\alpha}$$

to both sides and note that, for any $\varphi\in L^2(\mathbb{R})$,

$$T_{qp}(u_j^\pm\,\varphi)(x,\alpha) \;=\; u_j^\pm(x)\sum_{m\in\mathbb{Z}}\varphi(x_1+2\pi m)\,e^{2\pi m(\hat\alpha_j-\alpha)i} \;=\; u_j^\pm(x)\,\hat\varphi(x_1,\alpha-\hat\alpha_j)\,.$$

For $x\in Q$ and $\alpha\in(-1/2,1/2]$, this yields

$$\begin{aligned}
&\Delta\hat u^{(1)}(x,\alpha)+k^2n(x)\hat u^{(1)}(x,\alpha) \;=\\
&\quad -\hat f(x,\alpha) \;-\; \sum_{j\in J}\left[u_j^+(x)\,\frac{\partial\hat\varphi^+(x_1,\alpha-\hat\alpha_j)}{\partial x_1} \;+\; 2\,\hat\varphi^+(x_1,\alpha-\hat\alpha_j)\,\frac{\partial u_j^+(x)}{\partial x_1}\right]\\
&\qquad -\;\sum_{j\in J}\left[u_j^-(x)\,\frac{\partial\hat\varphi^-(x_1,\alpha-\hat\alpha_j)}{\partial x_1} \;+\; 2\,\hat\varphi^-(x_1,\alpha-\hat\alpha_j)\,\frac{\partial u_j^-(x)}{\partial x_1}\right].
\end{aligned}$$

From the inversion formula for the quasi-periodic Floquet-Bloch transform we directly compute the Floquet-Bloch transform of $\varphi^\pm$,

$$\frac{1}{2\pi}\int_{-1/2}^{1/2} e^{i\alpha x_1}d\alpha \;=\; \frac{1}{2\pi}\,\frac{\sin(x_1/2)}{x_1/2} \;=\; \pm\varphi^\pm(x_1)\,,$$

such that $\hat\varphi^\pm(x_1,\alpha)=\pm\,\exp(i\alpha x_1)/(2\pi)$. Therefore,

$$\begin{aligned}
&\Delta\hat u^{(1)}(x,\alpha)+k^2n(x)\hat u^{(1)}(x,\alpha)\\
&\quad=\; -\hat f(x,\alpha) \;-\; \frac{1}{2\pi}\sum_{j\in J}\left[u_j^+(x)\,i\,(\alpha-\hat\alpha_j) \;+\; 2\,\frac{\partial u_j^+(x)}{\partial x_1}\right]e^{i(\alpha-\hat\alpha_j)x_1}\\
&\qquad +\;\frac{1}{2\pi}\sum_{j\in J}\left[u_j^-(x)\,i\,(\alpha-\hat\alpha_j) \;+\; 2\,\frac{\partial u_j^-(x)}{\partial x_1}\right]e^{i(\alpha-\hat\alpha_j)x_1}\\
&\quad=\; -\hat f(x,\alpha) \;-\; \frac{1}{2\pi}\sum_{j\in J}\sum_{\ell\in L_j^+} a_{\ell,j}^+\left[\phi_{\ell,j}(x)\,i\,(\alpha-\hat\alpha_j) \;+\; 2\,\frac{\partial\phi_{\ell,j}(x)}{\partial x_1}\right]e^{i(\alpha-\hat\alpha_j)x_1}\\
&\qquad +\;\frac{1}{2\pi}\sum_{j\in J}\sum_{\ell\in L_j^-} a_{\ell,j}^-\left[\phi_{\ell,j}(x)\,i\,(\alpha-\hat\alpha_j) \;+\; 2\,\frac{\partial\phi_{\ell,j}(x)}{\partial x_1}\right]e^{i(\alpha-\hat\alpha_j)x_1}\\
&\quad=\; -\hat f(x,\alpha) \;-\; \sum_{j\in J}\sum_{\ell\in L_j^+} a_{\ell,j}^+F_{\ell,j}(x,\alpha) \;+\; \sum_{j\in J}\sum_{\ell\in L_j^-} a_{\ell,j}^-F_{\ell,j}(x,\alpha) \;=:\; h(x,\alpha)
\end{aligned}\tag{16}$$

for $x\in Q$ and $\alpha\in(-1/2,1/2]$. We note that for $\alpha\notin\{\hat\alpha_j : j\in J\}$ this equation (16) has a unique solution. For fixed $\hat\alpha_{j_0}$ with $j_0\in J$ it is well known that (16) is solvable if, and only if, the right hand side is orthogonal (in $L^2(Q)$) to the space $\tilde X_{j_0}$ of solutions of the homogeneous problem for $\alpha=\hat\alpha_{j_0}$. This condition is satisfied because the coefficients satisfy the system (14).

We show that $\lim_{\alpha\to\hat\alpha_j}\hat u^{(1)}(\cdot,\alpha)$ exists in $H_0^1(Q)$; that is, $\alpha\mapsto\hat u^{(1)}(\cdot,\alpha)$ is continuous.

This would in particular imply that $\hat{u}^{(1)} \in L^2\big((-1/2,1/2), H_0^1(Q)\big)$ and would provide the solution $u = u^{(1)} + u^{(2)}$ to the waveguide problem by taking the inverse Floquet-Bloch transform. To show continuity it is the aim to apply Theorem 2.8. First, we denote by $h(\alpha, x)$ the right hand side of (16) and transform equation (16) in its $2\pi-$periodic version; that is,

$$\Delta\tilde{u}(x,\alpha) \;-\; 2i\alpha\frac{\partial\tilde{u}(x,\alpha)}{\partial x_1} \;+\; (k^2 n(x) - \alpha^2)\,\tilde{u}(x,\alpha) \;=\; e^{-i\alpha x_1} h(x,\alpha) \quad \text{in } Q\times(-1/2,1/2]$$

for the $2\pi-$periodic field $\tilde{u}(x,\alpha) = e^{-i\alpha x_1}\hat{u}^{(1)}(x,\alpha)$. As in (8) we can write this as

$$(I - K_\alpha)\tilde{u}(\cdot,\alpha) \;=\; y_\alpha \quad \text{in } H^1_{0,per}(Q) \tag{17}$$

where $y_\alpha \in H^1_{0,per}(Q)$ is the Riesz representation of the functional $\psi \mapsto -\int_W e^{-i\alpha x_1} h(\alpha,x)\overline{\psi(x)}\,dx$. We note from our assumption on f and the mapping property of T_{qp} that $\hat{f} = T_{qp} f \in H^r_{per}\big((-1/2,1/2), L^2(Q)\big)$. Since $r > 3/2$ Sobolev's imbedding theorem guarantees that $\alpha \mapsto \hat{f}(\cdot,\alpha)$ is differentiable which implies that also $\alpha \mapsto h(\cdot,\alpha)$ and thus $\alpha \mapsto y_\alpha$ is differentiable at every $\hat{\alpha}_j$ (the latter as a mapping into $H^1_{0,per}(Q)$). In order to apply Theorem 2.8 it remains to show that the projection of the derivative $P\frac{\partial}{\partial\alpha}K_\alpha$ is one-to-one for $\alpha = \hat{\alpha}_j$ on the nullspace of $I - K_{\hat{\alpha}_j}$. From the definition of K_α we have that

$$\left(\frac{\partial}{\partial\alpha}K_\alpha v, \psi\right)_* \;=\; \frac{\partial}{\partial\alpha}a_\alpha(v,\psi) \;=\; 2i\int_Q \overline{\psi}\left[\frac{\partial v}{\partial x_1} + i\alpha\, v\right]dx\,, \quad v,\psi \in H^1_{0,per}(Q)\,.$$

Therefore, $P\frac{\partial}{\partial\alpha}K_\alpha v = 0$ at $\alpha = \hat{\alpha}_j$ for $v \in \mathcal{N}(I - K_{\hat{\alpha}_j})$ reads as $\int_Q \overline{\psi}\left[\frac{\partial v}{\partial x_1} + i\hat{\alpha}_j\, v\right]dx = 0$ for all $\psi \in \mathcal{N}(I - K_{\hat{\alpha}_j})$ which transforms into the quasi-periodic equation

$$\int_Q \overline{\varphi}\,\frac{\partial\tilde{v}}{\partial x_1}\,dx = 0 \quad \text{for all } \varphi \in \tilde{X}_j\,,$$

where $\tilde{v}(x) = e^{i\hat{\alpha}_j x_1} v(x)$. Assumption 2.5 yields that $\tilde{v}$ vanishes and thus also v. Therefore, all of the assumptions of Theorem 2.8 are satisfied which shows that $u = u^{(1)} + u^{(2)}$ solves (1).

It remains to show boundedness of the operator $f \mapsto u|_{W_R}$. Let again $h(\cdot,\alpha)$ be the right hand side of (16). We apply part (a) of Remark 2.9 to the equation (17) and note that the problem is well-posed for $\alpha \notin \{\hat{\alpha}_j : j \in J\}$. Setting $I = [-1/2, 1/2]$ this yields

$$\begin{aligned}
\max_{\alpha\in I}\|\hat{u}^{(1)}(\cdot,\alpha)\|_{H^1(Q)} &= \max_{\alpha\in I}\|\tilde{u}(\cdot,\alpha)\|_{H^1(Q)} \\
&\le c_1\big[\max_{\alpha\in I}\|y_\alpha\|_{H^1(Q)} + \max_{\alpha\in I}\|\partial y_\alpha/\partial\alpha\|_{H^1(Q)}\big] \\
&\le c_2\big[\max_{\alpha\in I}\|h(\cdot,\alpha)\|_{L^2(Q)} + \max_{\alpha\in I}\|\partial h(\cdot,\alpha)/\partial\alpha\|_{L^2(Q)}\big]
\end{aligned}$$

for some constants $c_1, c_2 > 0$ independent of h. Now we use the fact that $h(\cdot,\alpha)$ is explicitely given by $\hat{f}(\cdot,\alpha)$ and $\hat{f}(\cdot,\hat{\alpha}_j)$, $j \in J$, see (16) and (14), which yields the estimate

$$\begin{aligned}
\max_{\alpha\in I}\|\hat{u}^{(1)}(\cdot,\alpha)\|_{H^1(Q)} &\le c_3\left[\max_{\alpha\in I}\|\hat{f}(\cdot,\alpha)\|_{L^2(Q)} + \max_{\alpha\in I}\|\partial\hat{f}(\cdot,\alpha)/\partial\alpha\|_{L^2(Q)}\right] \\
&\le c_4\,\|\hat{f}\|_{H^r((-1/2,1/2),L^2(Q))}
\end{aligned}$$

where $r > 3/2$. The last estimate follows by Sobolev's imbeding theorem. Taking the inverse Floquet-Bloch transform yields $\|u^{(1)}\|_{H^1(W)} \le \tilde{c}\|f\|_{L^2_r(W)}$ which shows boundedness of $f \mapsto u^{(1)}$ from $L^2_r(W)$ into $H^1(W)$. Finally, by standard regularity results, the operator $f \mapsto u^{(1)}|_{W_R}$ is also bounded from $L^2_r(W)$ into $H^2(W_R)$ which ends the proof. $\square$

Now we can state and prove the main theorem.

Theorem 2.11. *Let Assumptions 2.2 and 2.2 hold. For every $f \in L^2_r(W)$ with $r > 3/2$ there exists a unique solution u of (1) which vanishes on ∂W and satisfies the radiation condition of Definition 2.6. Furthermore, the operator $T_R : f \mapsto u|_{W_R}$ is bounded from $L^2_r(W)$ into $H^2(W_R)$ for any $R > 0$ where $W_R = (-R, R) \times (0, h)$.*

Proof: First we show uniqueness. Let u be a solution of the problem for $f = 0$ which satisfies the radiation condition. Then $u(x) = u^{(1)}(x) + \sum_{j\in J}\big[\psi^+(x_1)u_j^+(x) + \psi^-(x_1)u_j^-(x)\big]$ where $u_j^\pm = \sum_{\ell\in L_j^\pm} a_{\ell,j}^\pm\, \phi_{\ell,j}$ and $u^{(1)} \in H^1_0(W)$. We apply Green's first theorem in the rectangle $W_R := (-R, R) \times (0, h)$:

$$
\begin{aligned}
0 &= -\int_{W_R} \overline{u}\,\big[\Delta u + k^2 n u\big]\,dx \\
&= \int_{W_R} \big[|\nabla u|^2 - k^2 n |u|^2\big]\,dx \;+\; \int_{\gamma_{-R}} \overline{u}\,\frac{\partial u}{\partial x_1}\,ds \;-\; \int_{\gamma_R} \overline{u}\,\frac{\partial u}{\partial x_1}\,ds
\end{aligned}
\tag{18}
$$

where again $\gamma_{\pm R} = \{\pm R\}\times(0, h)$. Now we consider the vertical boundary parts separately and substitute the form of u. This gives the sum of the following nine integrals.

$$
\begin{aligned}
I_r^{(1)} &= \int_{\gamma_r} \overline{u^{(1)}}\,\frac{\partial u^{(1)}}{\partial x_1}\,ds\,, \\
I_r^{(\pm 2)} &= \sum_{j\in J}\int_{\gamma_r} \overline{u^{(1)}}\,\frac{\partial}{\partial x_1}\big(\psi^\pm u_j^\pm\big)\,ds\,, \\
I_r^{(\pm 3)} &= \psi^\pm(r)\sum_{j\in J}\int_{\gamma_r} \overline{u_j^\pm}\,\frac{\partial u^{(1)}}{\partial x_1}\,ds\,, \\
I_r^{(4)} &= \psi^+(r)\sum_{j,\ell\in J}\int_{\gamma_r} \overline{u_j^+}\,\frac{\partial}{\partial x_1}\big(\psi^+ u_\ell^+\big)\,ds\,, \\
I_r^{(5)} &= \psi^+(r)\sum_{j,\ell\in J}\int_{\gamma_r} \overline{u_j^+}\,\frac{\partial}{\partial x_1}\big(\psi^- u_\ell^-\big)\,ds\,, \\
I_r^{(6)} &= \psi^-(r)\sum_{j,\ell\in J}\int_{\gamma_r} \overline{u_j^-}\,\frac{\partial}{\partial x_1}\big(\psi^+ u_\ell^+\big)\,ds\,, \\
I_r^{(7)} &= \psi^-(r)\sum_{j,\ell\in J}\int_{\gamma_r} \overline{u_j^-}\,\frac{\partial}{\partial x_1}\big(\psi^- u_\ell^-\big)\,ds
\end{aligned}
$$

for $r = \pm R$. First we set $r = R$ and let R tend to infinity. Then all integrals converge to zero except of $I_R^{(4)}$ which behaves as

$$
I_R^{(4)} = \sum_{j,\ell\in J}\int_{\gamma_R} \overline{u_j^+}\,\frac{\partial u_\ell^+}{\partial x_1}\,ds \;+\; o(1)\,, \quad R \to \infty\,,
$$

and thus by Lemma 2.7 (for any $R_0 > 0$)

$$\lim_{R\to\infty}\big(\mathrm{Im}\ I_R^{(4)}\big) \;=\; \mathrm{Im}\sum_{j\in J}\int_{\gamma_{R_0}} \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds \;\geq\; 0\,.$$

Analogously,

$$\lim_{R\to\infty}\big(\mathrm{Im}\ I_{-R}^{(7)}\big) \;=\; \mathrm{Im}\sum_{j\in J}\int_{\gamma_{-R_0}} \overline{u_j^-}\,\frac{\partial u_j^-}{\partial x_1}\,ds \;\leq\; 0\,,$$

and all other integrals tend to zero as $r = -R$ tends to $-\infty$. Therefore, taking the imaginary part in (18) yields

$$0 \;=\; \mathrm{Im}\ I_{-R}^{(7)} \;-\; \mathrm{Im}\ I_R^{(4)} \;+\; o(1)\,,\quad R\to\infty\,,$$

which yields that $\sum_{j\in J}\int_{\gamma_{R_0}} \overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds = \sum_{j\in J}\int_{\gamma_{R_0}} \overline{u_j^-}\,\frac{\partial u_j^-}{\partial x_1}\,ds = 0$ for all R_0 and thus $u_j^\pm = 0$ for all j by Lemma 2.7. This implies that also $a_{\ell,j}^\pm = 0$ because the functions $\big\{\phi_{\ell,j} : \ell = 1,\dots,m_j\big\}$ are linearly independent. Therefore, $u = u^{(1)} \in H_0^1(W)$ and thus u has to vanish by Assumption 2.2. This proves uniqueness.

To show existence we proceed differently as in Theorem 6.8 of [10] (where we proved existence by the limiting absorption principle) and make use of Theorem 2.10. We have to show that the quadratic linear system (14) admits a solution. It suffices to show uniqueness for this system. Therefore, let $\{a_{\ell,j}^+, a_{\ell,j}^-\}$ be a solution of (14) for vanishing right hand side. According to Theorem 2.10 there exists $u^{(1)} \in H_0^1(W)$ such that $u = u^{(1)} + u^{(2)}$ solves (1) for $f = 0$. The uniqueness part of this proof yields $u^{(1)} = u^{(2)} = 0$ and thus $a_{\ell,j}^+ = 0$ and $a_{\ell,j}^- = 0$ for all ℓ and j because the functions $\phi_{\ell,j}$ are linearly independent. □

Corollary 2.12. *Let f have compact support. Then $u^{(1)}$ decays exponentially as $|x_1|$ tends to infinity.*

Proof: The Floquet-Bloch transform $\tilde f$ is holomorphic because the series in (3) reduces to a finite sum. Therefore, also the right hand side $h(\cdot,\alpha)$ of (16) is holomorphic and thus y_α in (17). Therefore, by the remark (b) following Theorem 2.8 the solution $\hat u^{(1)}$ of (16) depends holomorphicly on α which in turn implies that $u^{(1)}$ decays exponentially. □

3. The Half-Waveguide Problem

We define $W = \mathbb{R}\times(0,h)$ as before and $W^\pm = \mathbb{R}_{\gtrless 0}\times(0,h)$ to be the positive and negative, respectively, half-waveguides. We set $\gamma_r = \{r\}\times(0,h)$ and $\Gamma^\pm = \partial W \cap \overline{W^\pm}$ to be the horizontal part of the boundary of $W^\pm$. Furthermore, let $H_0^{1/2}(\gamma_0)$ be defined as

$$H_0^{1/2}(\gamma_0) \;=\; \big\{u|_{\gamma_0} : u \in H_0^1(W)\big\}\,,$$

and assume that $\varphi \in H_0^{1/2}(\gamma_0)$ and $k > 0$ and $n \in L^\infty(W)$ be given such that n is bounded below by some constant $n_0 > 0$ and $2\pi-$periodic with respect to x_1. Analogously to the case of the full waveguide we make the following assumption.

Assumption 3.1. *The only solution $u^\pm \in H_0^1(W^\pm)$ of $\Delta u + k^2 n\,u = 0$ in $W^\pm$ is the trivial one. Here, $H_0^1(W^\pm) = \{u \in H^1(W^\pm) : u = 0$ on $\partial W^\pm\}$.*

It is our aim to determine $u \in H^1_{loc}(W^+)$ such that

$$\Delta u + k^2 n\, u = 0 \text{ in } W^+\,, \quad u = 0 \text{ on } \Gamma^+\,, \quad u = \varphi \text{ on } \gamma_0\,, \tag{19}$$

and u satisfies the following radiation condition:

Definition 3.2. *(Radiation Condition)*
Let $\{\hat{\alpha}_j : j \in J\}$ be the (possibly empty) set of exceptional values for the full wave guide (see definition after (8)) and let Assumptions 2.2, 2.5, and 3.1 hold. Then the field u has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)} \in H^1(W^+)$ with $u^{(1)} = 0$ on Γ^+ and $u^{(2)}$ has the form

$$u^{(2)}(x) \;=\; \sum_{j\in J} \sum_{\ell \in L_j^+} a^+_{\ell,j}\, \phi_{\ell,j}(x)\,, \quad x \in W^+\,, \tag{20}$$

for some $a^+_{\ell,j} \in \mathbb{C}$.

Uniqueness is proven in just the same way as in Theorem 2.11. To show existence we transform the problem to the full waveguide problem as in [7]. We study the following auxiliary problem.

For arbitrary fixed constant $\sigma > 0$ and $g \in H^{1/2}_0(\gamma_0)$ determine $v^\pm \in H^1(W^\pm)$ with

$$\Delta v^\pm - \sigma^2 v^\pm = 0 \text{ in } W^\pm\,, \quad v^\pm = 0 \text{ on } \Gamma^\pm\,, \quad v^\pm = g \text{ on } \gamma_0\,. \tag{21}$$

Lemma 3.3. *The problem (21) has a unique solution $v^\pm \in H^1(W^\pm)$. Furthermore, $v^\pm$ decays exponentially with rate σ; that is, the function $x \mapsto v^\pm(x) e^{\sigma|x_1|}$ is in $H^1(W^\pm)$. The solution operator*

$$T_1 \,:\, g \;\mapsto\; v := \begin{cases} v^+ & \text{in } W^+\,, \\ v^- & \text{in } W^-\,, \end{cases}$$

is bounded from $H^{1/2}_0(\gamma_0)$ into $L^2_r(W)$ for all $r > 0$.

Proof: We take $\tilde{g} \in H^1_0(W)$ with $\tilde{g}|_{\gamma_0} = g$ and $\tilde{g} = 0$ for $|x_1| \geq 1$ such that $\|\tilde{g}\|_{H^1(W)} \leq c\|g\|_{H^{1/2}_0(\gamma_0)}$ where c is independent of g. We consider only the positive waveguide W^+ and search for a solution in the form $v^+(x) = \tilde{g}(x) + e^{-\sigma x_1} \tilde{v}^+(x)$ with $\tilde{v}^+ \in H^1_0(W^+)$. If $v^\pm$ solves (21) then $\tilde{v}^+$ solves

$$\Delta \tilde{v}^+ - 2\sigma \frac{\partial \tilde{v}^+}{\partial x_1} = -e^{\sigma x_1}(\Delta \tilde{g} - \sigma^2 \tilde{g}) \text{ in } W^+\,, \quad \tilde{v}^\pm = 0 \text{ on } \partial W^+\,;$$

that is in variational form

$$\int_{W^+} \left[\nabla \tilde{v}^+ \cdot \nabla \psi + \sigma \left(\psi \frac{\partial \tilde{v}^+}{\partial x_1} - \tilde{v}^+ \frac{\partial \psi}{\partial x_1} \right) \right] dx \;=\; -\int_{W_1^+} \left[\nabla \tilde{g} \cdot \nabla\big(e^{\sigma x_1}\psi\big) + \sigma^2 \tilde{g}\big(e^{\sigma x_1}\psi\big) \right]$$

for all $\psi \in H^1_0(W^\pm)$. Here, $W_1^+ = (0,1) \times (0,h)$. The bilinear form on the left hand side is coercive in $H^1_0(W^+)$ and the right hand side is bounded. The theorem of Lax-Milgram yields existence and uniqueness of a solution $\tilde{v}^+$ and, furthermore, boundedness of the mapping $\tilde{g} \mapsto \tilde{v}^+$ from $H^1(W_1^+)$ into $H^1_0(W^+)$. This ends the proof for W^+. For the negative waveguide the arguments have to be modified accordingly. $\square$

We define a second operator $T_2 : L^2_r(W) \to H^{1/2}_0(\gamma_0)$ by $T_2 h = w|_{\gamma_0}$ where $w \in H^1_{0,loc}(W)$ is the unique solution (by Theorem 2.11) of

$$\Delta w + k^2 n w = (\sigma^2 + k^2 n)\, h \text{ in } W\,, \quad w = 0 \text{ on } \partial W\,, \tag{22}$$

and w satisfies the radiation condition of Definition 2.6. From Theorem 2.11, the compact imbedding of $H^2(W_R)$ into $H^1(W_R)$ and the boundedness of the trace operator we observe that T_2 is compact. Since T_1 is bounded by Lemma 3.3 we conclude that $T = T_1 \circ T_2$ is compact from $H_0^{1/2}(\gamma_0)$ into itself. Assume that for some given $\varphi \in H_0^{1/2}(\gamma_0)$ the equation $Tg - g = \varphi$ has a solution $g \in H_0^{1/2}(\gamma_0)$. Then $u := w - v^+$ solves the half-waveguide problem (19) in W^+ for this choice of g where v^+ solves (21) in W^+ and w solves (22) for $h = T_1 v$. Indeed, $u|_{\gamma_0} = w|_{\gamma_0} - g = Tg - g = \varphi$ and

$$\Delta u + k^2 n u = (\sigma^2 + k^2 n)\, v - (\sigma^2 + k^2 n)\, v^+ = 0 \text{ in } W^+ , \quad u = 0 \text{ on } \Gamma .$$

Also, u satisfies the radiation condition. Indeed, using the decomposition of w as in the radiation condition of Definition 2.6 we have

$$\begin{aligned} u(x) \;&=\; w(x) \;-\; v^+(x) \\ &=\; w^{(1)}(x) - v^+(x) \;+\; \psi^+(x_1) \sum_{j\in J} \sum_{\ell \in L_j^+} a_{\ell,j}^+\, \phi_{\ell,j}(x) \;+\; \psi^-(x_1) \sum_{j\in J} \sum_{\ell \in L_j^-} a_{\ell,j}^-\, \phi_{\ell,j}(x) \\ &=\; w^{(1)}(x) - v^+(x) + \psi^-(x_1) \sum_{j\in J} \sum_{\ell \in L_j^-} a_{\ell,j}^-\, \phi_{\ell,j}(x) + [\psi^+(x_1) - 1] \sum_{j\in J} \sum_{\ell \in L_j^+} a_{\ell,j}^+\, \phi_{\ell,j}(x) \\ &\quad + \sum_{j\in J} \sum_{\ell \in L_j^+} a_{\ell,j}^+\, \phi_{\ell,j}(x) \end{aligned}$$

which provides the decomposition as in Definition 3.2 because the first four terms on the right hand side are in $H_0^1(W^+)$.

Therefore, we have almost shown the second main result of this paper.

Theorem 3.4. *For every $\varphi \in H_0^{1/2}(\gamma_0)$ there exists a unique solution u of (19) in W^+ which satisfies the radiation condition of Definition 3.2. Furthermore, the operator $S_R : \varphi \mapsto u|_{W_R^+}$ is bounded from $H_0^{1/2}(\gamma_0)$ into $H^1(W_R^+)$ for any $R > 0$ where $W_R^+ = (0, R) \times (0, h)$.*

Proof: As mentioned above, uniqueness for both, the positive waveguide W^+ and the negative waveguide W^- (with respect to the analogously defined radiation condition) can be shown as in the proof of Theorem 2.11 and is omitted. For existence we have to show that the equation $Tg - g = \varphi$ admits a solution. Since T is compact it suffices to show uniqueness. Therefore, let $Tg - g = 0$ and define v as a solution of (21) as in Lemma 3.3 and w as the solution of (22) for $h = T_1 v$. Then $(w - v^\pm)|_{\gamma_0} = w|_{\gamma_0} - g = Tg - g = 0$ and

$$\Delta(w - v^\pm) + k^2 n (w - v^\pm) = (\sigma^2 + k^2 n)\, v^\pm - (\sigma^2 + k^2 n)\, v^\pm = 0 \text{ in } W^\pm , \quad w - v^\pm = 0 \text{ on } \Gamma^\pm .$$

Therefore, $w - v^\pm$ solves the homogeneous problem in $W^\pm$ and also the radiation condition. The uniqueness property in both half-waveguides yields $w - v^\pm = 0$ in $W^\pm$. Therefore, $v = w \in H^1(W)$ satsifies $\Delta v - \sigma^2 v = 0$ in W and $v = 0$ on ∂W and thus has to vanish by Green's theorem. This proves that $g = 0$ and ends the proof. $\square$

Remark: From the proof and the fact that the limiting absorption principle holds for the full waveguide problem we observe that the limiting absorption principle holds also for the half-waveguide problem.

4. Appendix

In this appendix we want to sketch the arguments which lead to a limiting absorption principle. Basis is the following theorem which extends Theorem 1.32 of [3].

Theorem 4.1. *Let $U \subset \mathbb{C}$ be an open set containing 0 and $I = [-\alpha_0, \alpha_0] \subset \mathbb{R}$ be a closed interval containing 0. Let $K(k,\alpha) : H \to H$ be a family of compact operators from a (complex) Hilbert space H into itself and $f(k,\alpha) \in H$ such that $(k,\alpha) \mapsto K(k,\alpha)$ is twice continuously differentiable on $U \times \operatorname{int}(I)$ and $(k,\alpha) \mapsto f(k,\alpha)$ is Lipschitz continuous on $U \times I$. Set $L(k,\alpha) = I - K(k,\alpha)$ and assume the following:*

(a) *The null space $\mathcal{N} := \mathcal{N}\big(L(0,0)\big)$ is not trivial and the Riesz number of of $L(0,0)$ is one; that is, the algebraic and geometric multiplicities of the eigenvalue 1 of $K(0,0)$ coincide. Let $P : H \to \mathcal{N} \subset H$ be the projection operator onto $\mathcal{N}$ corresponding to the direct decomposition $H = \mathcal{N} \oplus \mathcal{R}\big(L(0,0)\big)$,*

(b) *$L(k,\alpha)$ is one-to-one; that is, also onto, for all $(k,\alpha) \in U \times I$, $(k,\alpha) \neq (0,0)$,*

(c) *$A := P\frac{\partial}{\partial k}K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and positive definite and $B := P\frac{\partial}{\partial \alpha}K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and one-to-one.*

Let $u(\varepsilon,\alpha) \in H$ be the unique solution of $L(i\varepsilon,\alpha)u(\varepsilon,\alpha) = f(i\varepsilon,\alpha)$ for all $(\varepsilon,\alpha) \in (0,\varepsilon_0) \times I$. Then there exists $\varepsilon_1 \in (0,\varepsilon_0)$ and $\alpha_1 \in (0,\alpha_0)$ such that u has the form

$$u(\varepsilon,\alpha) \;=\; \tilde{u}(\varepsilon,\alpha) \;-\; \sum_{\ell=1}^{m} \frac{f_\ell}{i\varepsilon - \lambda_\ell \alpha}\,\phi_\ell \quad \text{for } (\varepsilon,\alpha) \in (0,\varepsilon_1) \times (-\alpha_1,\alpha_1)\,.$$

Here, $\|\tilde{u}(\varepsilon,\alpha)\|_H$ is uniformly bounded with respect to (ε,α), and $\big\{\lambda_\ell, \phi_\ell : \ell = 1,\ldots,m\big\}$ is an orthonormal eigensystem of the following generalized eigenvalue problem in the $m-$dimensional space $\mathcal{N}$:

$$-B\phi_\ell \;=\; \lambda_\ell\, A\phi_\ell \quad \text{in } \mathcal{N} \quad \text{with normalization} \quad \big(A\phi_\ell, \phi_{\ell'}\big)_H \;=\; \delta_{\ell,\ell'} \tag{23}$$

for $\ell,\ell' = 1,\ldots,m$. Finally, $f_\ell = \big(Pf(0,0),\phi_\ell\big)_H$ are the expansion coefficients of $A^{-1}Pf(0,0)$ with respect to the inner product $(A\cdot,\cdot)_H$.

Proof: In the first part we follow very closely the proof of Theorem 1.32 in [3] (with essentially the same symbols). We define $L_0^+ := \big(L(0,0) - P\big)^{-1}$ and $M(\varepsilon,\alpha) := L_0^+\big(L(i\varepsilon,\alpha) - L(0,0)\big)$ and set $\psi(\varepsilon,\alpha) = \big(I - M(\varepsilon,\alpha)\big)^{-1} L_0^+ f(i\varepsilon,\alpha)$. We note that $\big(I - M(\varepsilon,\alpha)\big)^{-1}$ exists for sufficiently small $|(\varepsilon,\alpha)|$ because $M(\varepsilon,\alpha) \to 0$ as (ε,α) tends to zero. For the same reason $\psi(\varepsilon,\alpha)$ converges to $\psi(0,0)$ as $(\varepsilon,\alpha) \to (0,0)$ and thus is bounded in a neighborhood of $(0,0)$. Then we make an ansatz for $u(\varepsilon,\alpha)$ in the form

$$u(\varepsilon,\alpha) \;=\; \psi(\varepsilon,\alpha) \;+\; \big(I - M(\varepsilon,\alpha)\big)^{-1}\chi(\varepsilon,\alpha)\,.$$

In the proof of Theorem 1.32 in [3] it is shown that $u(\varepsilon,\alpha)$ solves $L(i\varepsilon,\alpha)u(\varepsilon,\alpha) = f(i\varepsilon,\alpha)$ if, and only if, $\chi(\varepsilon,\alpha) \in \mathcal{N}$ solves

$$P\big(L(i\varepsilon,\alpha) - L(0,0)\big)\big(I - M(\varepsilon,\alpha)\big)^{-1}\chi(\varepsilon,\alpha) \;=\; Pf(i\varepsilon,\alpha) - P\big(L(i\varepsilon,\alpha) - L(0,0)\big)\psi(\varepsilon,\alpha)\,.$$

We study this equation differently as in the proof of Theorem 1.32 in [3]. First we abbreviate this as

$$C(\varepsilon,\alpha)\chi(\varepsilon,\alpha) \;=\; g(\varepsilon,\alpha)$$

with obvious meanings of $C(\varepsilon,\alpha)$ and $g(\varepsilon,\alpha)$ and note that $g(\varepsilon,\alpha) \in \mathcal{N}$ and $C(\varepsilon,\alpha)$ maps the finite dimensional subspace $\mathcal{N}$ into itself. Furthermore, $\frac{\partial C(0,0)}{\partial \varepsilon} = -iA$ and

$\frac{\partial C(0,0)}{\partial\alpha} = -B$ where $A,B:\mathcal{N}\to\mathcal{N}$ are the derivatives of the assumption (c). Next, we look at the linearized problem

$$-\big[i\varepsilon A+\alpha B\big]\tilde\chi(\varepsilon,\alpha) \;=\; g(0,0)$$

Expanding $g(0,0)=Pf(0,0)$ in the form $g(0,0)=\sum_{\ell=1}^m\big(Pf(0,0),\phi_\ell\big)_H A\phi_\ell=\sum_{\ell=1}^m f_\ell A\phi_\ell$ we observe that the solution $\tilde\chi(\varepsilon,\alpha)$ is given by

$$\tilde\chi(\varepsilon,\alpha) \;=\; -\sum_{\ell=1}^m \frac{f_\ell}{i\varepsilon-\lambda_\ell\alpha}\phi_\ell\,.$$

Therefore, we decompose $u(\varepsilon,\alpha)$ in the form $u(\varepsilon,\alpha)=\tilde u(\varepsilon,\alpha)+\tilde\chi(\varepsilon,\alpha)$ with

$$\tilde u \;=\; \psi \;+\; \big(I-M(\varepsilon,\alpha)\big)^{-1}[\chi-\tilde\chi] \;+\; \Big[\big(I-M(\varepsilon,\alpha)\big)^{-1}-I\Big]\tilde\chi \tag{24}$$

where we dropped the argument (ε,α). We have to prove boundedness of $\|\tilde u(\varepsilon,\alpha)\|_H$. First we show the existence of $c>1$ such that

$$\frac{1}{c}\,\|v\|_H \;\le\; \sqrt{\varepsilon^2+\alpha^2}\,\big\|[i\varepsilon A+\alpha B]^{-1}v\big\|_H \;\le\; c\,\|v\|_H \tag{25}$$

for all $v\in\mathcal{N}$ and $(\varepsilon,\alpha)\in U\times I$. Indeed, let $u=[i\varepsilon A+\alpha B]^{-1}v$. Then, as before, $v=\sum_{\ell=1}^m\frac{u_\ell}{i\varepsilon-\lambda_\ell\alpha}\phi_\ell$ where u_ℓ are the coefficients in the expansion $A^{-1}u=\sum_{\ell=1}^m u_\ell\phi_\ell$. By the orthonormality of $\{\phi_\ell:\ell=1,\dots,m\}$ with respect to $(Au,v)_H$ we have $(Av,v)_H=\sum_{\ell=1}^m\frac{|u_\ell|^2}{\varepsilon^2+\lambda_\ell^2\alpha^2}$ and $(u,A^{-1}u)_H=\sum_{\ell=1}^m|u_\ell|^2$. Since $\lambda_\ell\neq0$ for all ℓ by assumption (c) there exist $c_-,c_+>0$ with $c_-^2\le\lambda_\ell^2\le c_+^2$ for all ℓ. Also, the norms $\sqrt{(Av,v)_H}$ and $\sqrt{(u,A^{-1}u)_H}$ are equivalent to $\|v\|_H$ and $\|u\|_H$, respectively, which proves (25).
Now we consider the difference $v=\tilde\chi-\chi$ and have

$$\begin{aligned}\big[i\varepsilon A+\alpha B\big]v(\varepsilon,\alpha) \;&=\; g(\varepsilon,\alpha)\;-\;g(0,0)\;-\;\big[C(\varepsilon,\alpha)+(i\varepsilon A+\alpha B)\big]\chi(\varepsilon,\alpha)\\ &=\; g(\varepsilon,\alpha)\;-\;g(0,0)\;+\;\big[C(\varepsilon,\alpha)+(i\varepsilon A+\alpha B)\big]v(\varepsilon,\alpha)\\ &\quad-\;\big[C(\varepsilon,\alpha)+(i\varepsilon A+\alpha B)\big]\tilde\chi(\varepsilon,\alpha)\,.\end{aligned}$$

From $\|(i\varepsilon A+\alpha B)v(\varepsilon,\alpha)\|_H\ge c\sqrt{\varepsilon^2+\alpha^2}\|v(\varepsilon,\alpha)\|_H$ and $\|\tilde\chi(\varepsilon,\alpha)\|_H\le\frac{c}{\sqrt{\varepsilon^2+\alpha^2}}$ and $\|g(0,0)-g(\varepsilon,\alpha)\|_H\le c\sqrt{\varepsilon^2+\alpha^2}$ and $\|C(\varepsilon,\alpha)-(i\varepsilon A+\alpha B)\|\le c\,(\varepsilon^2+\alpha^2)$ we conclude that

$$c_1\sqrt{\varepsilon^2+\alpha^2}\big[1-\sqrt{\varepsilon^2+\alpha^2}\big]\,\|v(\varepsilon,\alpha)\|_H \;\le\; c_2\sqrt{\varepsilon^2+\alpha^2}\;+\;c_3\sqrt{\varepsilon^2+\alpha^2}$$

which shows boundedness of $\|v(\varepsilon,\alpha)\|_H$ for sufficiently small $\sqrt{\varepsilon^2+\alpha^2}$. Also, $\big\|\big[\big(I-M(i\varepsilon,\alpha)\big)^{-1}-I\big]\big\|\le c\sqrt{\varepsilon^2+\alpha^2}$ for sufficiently small $\sqrt{\varepsilon^2+\alpha^2}$ which shows boundedness of $\|\tilde u(\varepsilon,\alpha)\|_H$ for sufficiently small $\sqrt{\varepsilon^2+\alpha^2}$ by (24). $\square$

A different proof of this theorem in the case that K depends holomorphicly on $(k,\alpha)\in\mathbb{C}^2$ is given in [10] by using the total projection and a similarity transform (see [9]).

We apply this theorem to the equation (8) but indicate the dependence on k; that is,

$$u_{k,\alpha}\;-\;K_{k,\alpha}u_{k,\alpha}\;=\;\tilde f_\alpha\quad\text{in } H^1_{0,per}(Q)\,, \tag{26}$$

where

$$(K_{k,\alpha}v,\psi)_* \;=\; \int_Q\left[2i\alpha\,\overline{\psi}\,\frac{\partial v}{\partial x_1}+(k^2n-\alpha^2)\,v\,\overline{\psi}\right]dx\,,\quad v,\psi\in H^1_{0,per}(W)\,,$$

and $(\tilde{f}_\alpha, \psi)_* = -\int_Q \tilde{f}(\cdot,\alpha)\,\overline{\psi}\,dx$ for all $\psi \in H^1_{0,per}(Q)$. We study this equation in a neighborhood of the point $(\hat{k}, \hat{\alpha}_j)$ where $\hat{\alpha}_j$ is an exceptional value of $\hat{k} > 0$. The point $(\hat{k}, \hat{\alpha}_j)$ replaces $(0,0)$ in Theorem 4.1. We have to check the assumptions of this theorem. The smoothness assumptions are satisfied. This is obvious for the operator $K_{k,\alpha}$ which depends even holomorphicly on k and α. The right hand side is continously differentiable with respect to α because $f \in L^2_r(W)$.

Furthermore, we note that $K_{\hat{k},\hat{\alpha}_j}$ is selfadjoint. Therefore, $L_{\hat{k},\hat{\alpha}_j} = I - K_{\hat{k},\hat{\alpha}_j}$ has Riesz number one. It remains to check assumption (c). Obviously,

$$\begin{aligned}
\frac{\partial}{\partial k}(K_{k,\alpha}v,\psi)_* &= 2k \int_Q n\,v\,\overline{\psi}\,dx\,,\\
\frac{\partial}{\partial \alpha}(K_{k,\alpha}v,\psi)_* &= 2i \int_Q \overline{\psi}\left[\frac{\partial v}{\partial x_1} + i\alpha\, v\right] dx \;=\; 2i \int_Q \overline{\psi(x)\,e^{i\alpha x_1}}\,\frac{\partial}{\partial x_1}\big(v(x)\,e^{i\alpha x_1}\big)\,dx
\end{aligned}$$

for $v,\psi \in H^1_{0,per}(W)$ which proves assumption (c) if Assumption 2.5 holds. Furthermore, these representations show that ϕ_ℓ is an eigenfunction of (23) corresponding to the eigenvalue λ_ℓ if, and only if, $\phi_\ell(x)\,e^{i\alpha x_1}$ is an eigenfunction of (9) corresponding to the eigenvalue λ_ℓ. Application of Theorem 4.1 yields the decomposition

$$u_{\hat{k}+i\varepsilon,\alpha} \;=\; \tilde{u}_{\hat{k}+i\varepsilon,\alpha} \;-\; \sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{i\varepsilon - \lambda_{\ell,j}(\alpha - \hat{\alpha}_j)}\,\phi_{\ell,j} \tag{27}$$

for $(\varepsilon,\alpha) \in (0,\varepsilon_0)\times(\hat{\alpha}_j - \delta, \hat{\alpha}_j + \delta)$ for some $\delta > 0$, $\varepsilon_0 > 0$. Here, $\|\tilde{u}_{\hat{k}+i\varepsilon,\alpha}\|_{H^1(Q)}$ is uniformly bounded with respect to ε and α and $\tilde{u}_{\hat{k}+i\varepsilon,\alpha} \to \tilde{u}_{\hat{k},\alpha}$ as $\varepsilon \to 0$ for all $\alpha \neq \hat{\alpha}_j$ because $u_{\hat{k}+i\varepsilon,\alpha}$ and the second term on the right hand side of (27) converge. Lebesgue's Theorem on dominated converges yields convergence of $\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} \tilde{u}_{\hat{k}+i\varepsilon,\alpha}\,e^{i\alpha x_1} d\alpha$ to $\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} \tilde{u}_{\hat{k},\alpha}\,e^{i\alpha x_1} d\alpha$ in $H^1(W)$ as $\varepsilon \to 0$. Now we consider the sum in (27). We compute

$$\begin{aligned}
&\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} \frac{1}{i\varepsilon - \lambda_{\ell,j}(\alpha-\hat{\alpha}_j)}\,e^{i\alpha x_1}d\alpha\\
&= e^{i\hat{\alpha}_j x_1}\int_{-\delta}^{\delta} \frac{1}{i\varepsilon - \lambda_{\ell,j}\alpha}\,e^{i\alpha x_1}d\alpha \;=\; -e^{i\hat{\alpha}_j x_1}\int_{-\delta}^{\delta} \frac{i\varepsilon + \lambda_{\ell,j}\alpha}{\varepsilon^2 + \lambda_{\ell,j}^2\alpha^2}\,e^{i\alpha x_1}d\alpha\\
&= -2i\,e^{i\hat{\alpha}_j x_1}\left[\frac{1}{|\lambda_{\ell,j}|}\int_0^{|\lambda_{\ell,j}|\delta/\varepsilon} \frac{\cos\big(\varepsilon x_1 t/|\lambda_{\ell,j}|\big)}{1+t^2}\,dt \;+\; \lambda_{\ell,j}\int_0^{\delta x_1} \frac{t\sin t}{x_1^2\varepsilon^2 + \lambda_{\ell,j}^2 t^2}\,dt\right]
\end{aligned}$$

which converges to

$$-\frac{i\pi}{|\lambda_{\ell,j}|}\,e^{i\hat{\alpha}_j x_1}\left[1 \;+\; \operatorname{sign}\lambda_{\ell,j}\,\frac{2}{\pi}\int_0^{\delta x_1} \frac{\sin t}{t}\,dt\right]$$

as $\varepsilon \to 0$ uniformly with respect to $|x_1| \le R$ for any $R > 0$. Now we combine all of the terms and have for the inverse Floquet-Bloch transform with $I = (-1/2,1/2)\setminus\bigcup_{j\in J}(\hat{\alpha}_j -$

$\delta, \hat\alpha_j + \delta)$:

$$\begin{aligned}
\int_{-1/2}^{1/2} u_{\hat k + i\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;&=\; \int_I u_{\hat k + i\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;+\; \sum_{j\in J} \int_{\hat\alpha_j-\delta}^{\hat\alpha_j+\delta} u_{\hat k + i\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \\
&=\; \int_I u_{\hat k + i\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \;+\; \sum_{j\in J} \int_{\hat\alpha_j-\delta}^{\hat\alpha_j+\delta} \tilde u_{\hat k + i\varepsilon,\alpha}(x)\, e^{i\alpha x_1} d\alpha \\
&\qquad +\; \sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\, \phi_{\ell,j}(x) \int_{\hat\alpha_j-\delta}^{\hat\alpha_j+\delta} \frac{1}{i\varepsilon - \lambda_{\ell,j}(\alpha-\hat\alpha_j)}\, e^{i\alpha x_1} d\alpha
\end{aligned}$$

which converges to

$$\int_{1/2}^{1/2} v_\alpha(x)\, e^{i\alpha x_1} d\alpha \;-\; 2i\pi \sum_{j\in J}\sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \phi_{\ell,j}(x)\, e^{i\hat\alpha_j x_1} \frac{1}{2}\left[1 \;+\; \operatorname{sign}\lambda_{\ell,j}\, \frac{2}{\pi} \int_0^{\delta x_1} \frac{\sin t}{t}\, dt\right]$$

where

$$v_\alpha \;=\; \begin{cases} u_{\hat k,\alpha} & \text{in } I\,, \\ \tilde u_{\hat k,\alpha} & \text{in } (\hat\alpha_j - \delta, \hat\alpha_j + \delta)\,,\; j \in J\,. \end{cases}$$

Finally we observe that $\int_0^{\delta x_1} \frac{\sin t}{t}\, dt - \int_0^{x_1/2} \frac{\sin t}{t}\, dt$ decays as $1/|x_1|$. Therefore, we have shown the following limiting absorption principle.

Theorem 4.2. *Let Assumptions 2.2 and 2.5 hold. For every $f \in L^2_r(W)$ with $r > 3/2$ the solution u of (1) which vanishes on ∂W and satisfies the radiation condition of Definition 2.6.*

Now we will comment on Assumption 2.5. We go back to equation (5) and write it in the form (indicating the dependence also on k)

$$b_\alpha(\tilde u_{k,\alpha}, \psi) \;-\; k^2 d(\tilde u_{k,\alpha}, \psi) \;=\; \int_Q \tilde f(\cdot,\alpha)\, \overline{\psi}\, dx$$

for all $\psi \in H^1_{0,per}(Q)$ where

$$\begin{aligned}
b_\alpha(v,\psi) \;&=\; \int_Q \left[\nabla v \cdot \nabla\overline{\psi} - 2i\alpha\, \overline{\psi}\, \frac{\partial v}{\partial x_1} + \alpha^2\, v\, \overline{\psi}\right] dx \\
&=\; \int_Q \nabla\big(e^{i\alpha x_1} v(x)\big) \cdot \nabla\overline{\big(e^{i\alpha x_1}\psi(x)\big)}\, dx\,, \\
d(v,\psi) \;&=\; \int_Q n\, v\, \overline{\psi}\, dx\,, \qquad v,\psi \in H^1_{0,per}(Q)\,.
\end{aligned}$$

From the second form of b_α we observe that b_α is coercive. By the Theorems of Riesz and Lax-Milgram there exist selfadjoint operators $\mathcal{B}_\alpha$ and $\mathcal{D}$ from $H^1_{0,per}(Q)$ into itself such that $\mathcal{D}$ is compact and positive and $\mathcal{B}_\alpha$ is coercive and $b_\alpha(v,\psi) = (\mathcal{B}_\alpha v, \psi)_*$ and $d(v,\psi) = (\mathcal{D}v,\psi)_*$ for all $v,\psi \in H^1_{0,per}(Q)$. Then we can write (8) in the form

$$\big[I \;-\; K_{\hat k,\hat\alpha}\big]\tilde u_{k,\alpha} \;=\; \big[\mathcal{B}_\alpha \;-\; k^2\mathcal{D}\big]\tilde u_{k,\alpha} \;=\; \tilde f_\alpha \tag{28}$$

where $\tilde{f}_\alpha \in H^1_{0,per}(Q)$ is the Riesz representation of the right hand side of (5). Introducing an eigenvalue system $\{\mu_\ell(\alpha), \psi_\ell(\alpha) : \ell \in \mathbb{N}\}$ for $\alpha \in (-1/2, 1/2]$ of the generalized eigenvalue problem

$$\mathcal{D}\psi_\ell(\alpha) \;=\; \mu_\ell(\alpha)\mathcal{B}_\alpha\psi_\ell(\alpha)\,, \quad \ell \in \mathbb{N}\,, \tag{29}$$

for $\alpha \in (-1/2, 1/2]$ as done in [6], one is lead to the equation $\big[1 - k^2\mu_\ell(\alpha)\big]a_\ell(k,\alpha) = f_\ell(\alpha)$ where $a_\ell(k,\alpha)$ and $f_\ell(\alpha)$ are the coefficients of $\tilde{u}_{k,\alpha}$ and $\mathcal{B}^{-1}\tilde{f}_\alpha$, respectively, with respect to $\{\psi_\ell(\alpha) : \ell \in \mathbb{N}\}$[1]. For Im $k > 0$ and $\alpha \in (-1/2, 1/2]$ we have $1 - k^2\mu_\ell(\alpha) \neq 0$ for all ℓ because $\mu_\ell(\alpha)$ is real valued and strictly positive. For $k = \hat{k} \in \mathbb{R}_{>0}$, however, there might exist values of $\alpha = \hat{\alpha}$ for which $\hat{\ell}$ exists such that $1 - \hat{k}^2\mu_{\hat{\ell}}(\hat{\alpha}) = 0$. We set $L := \{\ell : \mu_\ell(\hat{\alpha}) = \mu_{\hat{\ell}}(\hat{\alpha})\}$ and note that span$\{\psi_\ell : \ell \in L\}$ is the solution space of the homogeneous form of equation (28); that is, also of (5). Therefore, $\hat{\alpha}$ is an exceptional value in the sense of above (just before Theorem 2.4) and span$\{\psi_\ell(\hat{\alpha}) : \ell \in L\} = \mathcal{N} = \mathcal{N}\big(I - K_{\hat{k},\hat{\alpha}}\big)$. We recall that P was the - now orthogonal - projection onto $\mathcal{N} = \mathcal{N}(I - K_{\hat{k},\hat{\alpha}}) = \mathcal{N}\big(\mathcal{B}_{\hat{\alpha}} - \hat{k}^2\mathcal{D}\big)$. As in [6] there exists a neighborhood $U \subset \mathbb{C}$ of $\hat{\alpha}$ and holomorphic extensions of $\mu_\ell(\alpha)$ and $\psi_\ell(\alpha)$ such that (29) holds for all $\alpha \in U$.

The wave number $\hat{k}$ is called *forbidden* (in the sense of Joly and Fliss, [6]) if there exists $\hat{\alpha} \in (-1/2, 1/2]$ and $\hat{\ell} \in \mathbb{N}$ such that $1 - \hat{k}^2\mu_{\hat{\ell}}(\hat{\alpha}) = 0$ and $\mu'_{\hat{\ell}}(\hat{\alpha}) = 0$.

We want to compare this condition with Assumption 2.5. Differentiating equation (29) with respect to α and setting $\alpha = \hat{\alpha}$ yields

$$\big(\mathcal{D} \;-\; \mu_\ell(\hat{\alpha})\mathcal{B}_{\hat{\alpha}}\big)\psi'_\ell(\hat{\alpha}) \;=\; \mu'_\ell(\hat{\alpha})\mathcal{B}_{\hat{\alpha}}\psi_\ell(\hat{\alpha}) \;+\; \mu_\ell(\hat{\alpha})\mathcal{B}'_{\hat{\alpha}}\psi_\ell(\hat{\alpha})\,. \tag{30}$$

For $\ell \in L$ we observe that the left hand side is in the range of $\mathcal{B}_{\hat{\alpha}} - \hat{k}^2\mathcal{D}$ and therefore $P\big(\mathcal{D} - \mu_\ell(\hat{\alpha})\mathcal{B}_{\hat{\alpha}}\big)\psi'_\ell(\hat{\alpha}) = 0$; that is,

$$P\frac{\partial}{\partial\alpha}K_{\hat{k},\hat{\alpha}}\,\psi_\ell(\hat{\alpha}) \;=\; -P\mathcal{B}'_{\hat{\alpha}}\psi_\ell(\hat{\alpha}) \;=\; \frac{\mu'_\ell(\hat{\alpha})}{\mu_\ell(\hat{\alpha})}\,P\mathcal{B}_{\hat{\alpha}}\psi_\ell(\hat{\alpha}) \quad \text{for all } \ell \in L\,.$$

Furthermore,

$$P\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}}\psi_\ell(\hat{\alpha}) \;=\; 2\hat{k}\,P\mathcal{D}\psi_\ell(\hat{\alpha}) \;=\; 2\hat{k}\,\mu_\ell(\hat{\alpha})\,P\mathcal{B}_{\hat{\alpha}}\psi_\ell(\hat{\alpha})$$

for all $\ell \in L$. Eliminating $P\mathcal{B}_{\hat{\alpha}}\psi_\ell(\hat{\alpha})$ from the previous equations yields

$$P\frac{\partial}{\partial\alpha}K_{\hat{k},\hat{\alpha}}\,\psi_\ell(\hat{\alpha}) \;=\; \frac{\mu'_\ell(\hat{\alpha})}{2\hat{k}\,\mu_\ell(\hat{\alpha})^2}P\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}}\,\psi_\ell(\hat{\alpha}) \;=\; \frac{1}{2}\,\hat{k}^3\mu'_\ell(\hat{\alpha})\,P\frac{\partial}{\partial k}K_{\hat{k},\hat{\alpha}}\,\psi_\ell(\hat{\alpha})$$

for all $\ell \in L$. This shows that $\psi_\ell(\hat{\alpha})$ is an eigenfunction with corresponding eigenvalue $\lambda_\ell = -\frac{1}{2}\,\hat{k}^3\mu'_\ell(\hat{\alpha})$ of the generalized eigenvalue problem (10) for $\hat{\alpha} = \hat{\alpha}_j$. Therefore, Assumption 2.5 coincides with the assumption that $\hat{k}$ is not a forbidden frequency.

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[1] Note that $\mu_\ell(\alpha)$ corresponds to $1/\lambda_\ell(\alpha)$ in [6]

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