Tensorial Curvature Measures in Integral Geometry

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Abstract

The tensorial curvature measures are tensor-valued generalizations of the curvature measures of convex bodies. On convex polytopes, there exist further generalizations some of which also have continuous extensions to arbitrary convex bodies. The global tensorial curvature measures are the well-known Minkowski tensors, which are tensor-valued generalizations of the intrinsic volumes of convex bodies.

We prove two complete sets of integral geometric formulae, so called kinematic and Crofton formulae, for the (generalized) tensorial curvature measures. The kinematic formulae express the integral mean of the (generalized) tensorial curvature measures of the intersection of two given convex bodies (resp. polytopes), one of which is uniformly moved by a proper rigid motion, in terms of linear combinations of the (generalized) tensorial curvature measures of the given convex bodies (resp. polytopes). The Crofton formulae express the integral mean of the (generalized) tensorial curvature measures of the intersection of a given convex body (resp. polytope) with a uniformly moved affine subspace in terms of linear combinations of (generalized) tensorial curvature measures of the given convex body (resp. polytope). In the proof of the kinematic formulae, we proceed in a more direct way than in the classical proof of the principal kinematic formula for curvature measures, which uses the connection to the Crofton formula, to determine the involved constants explicitly. However, we apply this aforementioned connection to prove the Crofton formulae for the (generalized) tensorial curvature measures.

By globalization of these integral geometric formulae, we derive two complete sets of corresponding integral geometric formulae for the Minkowski tensors. Non-trivial adjustments of the above methods to the case of $\mathrm{SO}(n)$-covariant tensorial curvature measures (which are versions of the tensorial curvature measures with slightly different covariance properties and only occur in dimensions two and three), yield the corresponding integral formulae for them as well. From a different approach we finally obtain Crofton formulae for the tensorial curvature measures which are defined with respect to the uniformly moved affine subspace as the ambient space (and for their global versions).

Publications

Parts of this thesis are direct quotes from the following joint publications:

Daniel Hug and Jan A. Weis, *Crofton formulae for tensor-valued curvature measures*, Tensor Valuations and their Applications in Stochastic Geometry and Imaging (Markus Kiderlen and Eva B. Vedel Jensen, eds.), Lecture Notes in Mathematics, vol. 2177, Springer, 2017 (to appear), arXiv:1606.05131 (2016).

Daniel Hug and Jan A. Weis, *Kinematic formulae for tensorial curvature measures*, arXiv:1612.08427, 2016.

Daniel Hug and Jan A. Weis, *Crofton formulae for tensorial curvature measures: the general case*, Analytic aspects of convexity (Gabriele Bianchi, Andrea Colesanti, and Paolo Gronchi, eds.), Springer INdAM Series, Springer, 2017+ (to appear), arXiv:1612.08847 (2016).

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Contents

CHAPTER 1

Introduction

In the third of his famous twenty-three mathematical problems, which David Hilbert presented at the International Congress of Mathematicians in 1900 in Paris (see [40]), he raised the question if any polyhedron (in three-dimensional Euclidean space) can always be dissected into finitely many polyhedra which yield reassembled any given polyhedron of the same volume as the first. Contrarily to the well-studied equivalent for polygons (in two-dimensional Euclidean space), Hilbert conjectured the answer to be negative in general (referring to a correspondence between Gauß and Gerling; see [31, p. 240ff]). This was confirmed by Max Dehn in the same year, by introduction and application of the so called Dehn invariant (see [24]), which can be seen as the first important example of a real valuation (which is not a measure) on convex polytopes, and therefore, as the starting point of valuation theory, one facet of which form the tensorial curvature measures, the crucial objects of investigation in this thesis.

A mapping φ defined on the set $\mathcal{K}^n$ of convex bodies (non-empty, compact, convex sets) in Euclidean space $\mathbb{R}^n$ which takes values in an abelian group is called a *valuation* (or *additive*) if

$$\varphi(K) + \varphi(K') = \varphi(K \cup K') + \varphi(K \cap K')$$

whenever $K, K', K \cup K' \in \mathcal{K}^n$. This concept of additivity weakens the countable additivity, which basically defines a measure and is therefore essential in various fields all over mathematics. However, the theory of valuations is likewise remarkably fruitful, to a great extent due to the investigations of the algebraic structure of valuations in recent years, which led to deep new insights into integral geometry (for an overview thereof, see [4, 10, 30]).

Some of the most important classical valuations (and the first step towards introducing the tensorial curvature measures) are the *intrinsic volumes* $V_j : \mathcal{K}^n \to \mathbb{R}$, for $j \in \{0, \dots, n\}$, which occur as the coefficients of the monomials in the *Steiner formula*

$$\mathcal{H}^n(K + \varepsilon B^n) = \sum_{j=0}^{n} \kappa_{n-j} V_j(K) \varepsilon^{n-j}, \tag{1.1}$$

for a convex body $K \in \mathcal{K}^n$ and $\varepsilon \geq 0$. As usual in this context, $+$ denotes the Minkowski addition in $\mathbb{R}^n$, B^n is the Euclidean unit ball in $\mathbb{R}^n$ of n-dimensional volume κ_n, and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. The basic properties of the intrinsic volumes, isometry invariance, additivity and continuity (with respect to the Hausdorff metric), which are derived from corresponding properties of the volume functional, are crucial in characterizing them. More precisely, *Hadwiger's characterization theorem* states that $V_0, \dots, V_n$ form a basis of the vector space of continuous and isometry invariant real-valued valuations on $\mathcal{K}^n$ (see Hadwiger's original works [36, 37, 38] or Klain's new and shorter proof in [59], reproduced in [60, Theorem 9.1.1] and [83, Theorem 6.4.14]). This key result for intrinsic volumes is one of the fundamental tools in convex integral geometry, which will be explained later in this chapter in more detail.

A far reaching generalization of the intrinsic volumes is obtained by their localization as measures, associated with convex bodies, such that the intrinsic volumes are just the total measures. Specifically, this leads to the *support measures* which are weakly continuous, locally defined and motion equivariant valuations on convex bodies with values in the space of finite measures on Borel subsets of $\mathbb{R}^n \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the Euclidean unit sphere in $\mathbb{R}^n$. In the spirit of Hadwiger's characterization theorem, Glasauer characterized the support measures using these properties, where the additivity is not required (see [32]). The support measures are determined by a local version of the Steiner formula (1.1), and thus they provide a natural example of a localization of the intrinsic volumes. Their marginal measures on Borel subsets of $\mathbb{S}^{n-1}$ are called *area measures*, and the ones on Borel subsets of $\mathbb{R}^n$ are called *curvature measures*, both of which admit characterizations of Hadwiger type via suitable properties (including additivity) as well, proved by Schneider (see [79, 80]).

Already in the early seventies of the last century, Schneider, and Hadwiger and Schneider analyzed vector-valued versions of the intrinsic volumes, so called *quermassvectors* (*curvature centroids*), in terms of characterizations and integral geometry (see [39, 77, 78]). More recently, in 1997 McMullen extended this framework and initiated a study of tensor-valued generalizations of the (scalar) intrinsic volumes and the vector-valued quermassvectors (see [68]). This naturally raised the question for an analogue of Hadwiger's characterization theorem for basic additive, tensor-valued mappings on the space of convex bodies. As shown by Alesker in [2, 3], and further studied in [52], there exist natural tensor-valued functions, the *Minkowski tensors*, which generalize the intrinsic volumes and span the vector space of tensor-valued, continuous valuations on the space of convex bodies which are also isometry covariant. Although the basic Minkowski tensors span the corresponding vector space of tensor-valued valuations, they satisfy non-trivial linear relationships and hence are not a basis.

The next natural step is to combine local and tensor-valued extensions of the classical intrinsic volumes. This setting has recently been studied by Schneider (see [82]) and further analyzed by Hug and Schneider in their works on *tensorial support measures*, or *local Minkowski tensors* (see [47, 48, 49]). As their names suggest, these valuations can be seen as tensor-valued generalizations of the support measures. On the other hand, they can be considered as localizations of the (global) Minkowski tensors. Inspired by the characterization results obtained in [82, 47, 48, 49, 74], we consider tensor-valued curvature measures, the *tensorial curvature measures*, and their generalizations, some of which only occur for polytopes. Even though a general characterization of the tensorial curvature measures is still an open problem (see however [74] for results in the smooth setting), we prove their linear independence in this thesis, which is a crucial result in view of the upcoming integral formulae.

Minkowski tensors, tensorial curvature measures, and general local tensor valuations are useful morphological characteristics that allow to describe the geometry of complex spatial structure and are particularly well suited for developing structure-property relationships for tensor-valued or orientation-dependent physical properties; see [69, 86, 87] for surveys and Klatt's PhD thesis [61] for an in-depth analysis of various aspects (including random fields and percolation) of the interplay between physics and Minkowski tensors. These applications cover a wide spectrum ranging from nuclear physics [91], granular matter [57, 101, 76, 67], density functional theory [100], physics of complex plasmas [20], to physics of materials science [73]. Characterization and classification theorems for tensor valuations, uniqueness and reconstruction results [42, 64, 63, 62], which are accompanied by numerical algorithms [86, 87, 45, 21], stereological estimation procedures [65, 66], and integral geometric formulae, as considered in the present work, form the foundation for these and many other applications.

After the brief motivation of the tensorial curvature measures in the preceding paragraphs, we now turn to the integral geometric formulae, in which the former are applied in this thesis. The classical integral geometry goes back to a series of lectures held by Wilhelm Blaschke in Hamburg, in the 1930s (collected in [19]). He initiated investigations of problems in the field of convex and differential geometry, which arise from problems in classical geometric probability, but are nevertheless of geometric interest, independent of their stochastic applications. In particular, intersection formulae are key results in integral geometry, in which specific geometric quantities of the intersection of moving geometric objects are averaged with respect to invariant measures. For a classical approach to this topic see [75], for more recent developments see [85, Chap. 5] and [83, Chap. 4.4].

The two best known and most fundamental classical intersection formulae are the *principal kinematic formula* and the *classical Crofton formula*, which treat the intrinsic volume of the intersection of a convex body with another geometric object (in the principal kinematic formula this is a second convex body, in the Crofton formula this is an affine subspace) which is uniformly moved by a proper rigid motion. More precisely, the principal kinematic formula (see [83, (4.52)]) states, for two convex bodies $K, K' \in \mathcal{K}^n$ and $j \in \{0, \dots, n\}$, that

$$\int_{\mathrm{G}_n} V_j(K \cap gK')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \alpha_{njk}\, V_k(K)\, V_{n-k+j}(K'), \tag{1.2}$$

where G_n denotes the group of proper rigid motions of $\mathbb{R}^n$, μ is the motion invariant Haar measure on G_n, normalized in the usual way (see [85, p. 586]), and the constant

$$\alpha_{njk} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k+j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \tag{1.3}$$

is expressed in terms of specific values of the Gamma function. Furthermore, for a convex body $K \in \mathcal{K}^n$ and $k \in \{0, \dots, n\}$, $j \in \{0, \dots, k\}$, the classical Crofton formula (see [83, (4.59)]), the name of which originates from works of the Irish mathematician Morgan W. Crofton on integral geometry in $\mathbb{R}^2$ in the late 19th century (see [23]), states that

$$\int_{\mathrm{A}(n,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) = \alpha_{njk}\, V_{n-k+j}(K), \tag{1.4}$$

where $\mathrm{A}(n, k)$ is the affine Grassmannian of k-flats in $\mathbb{R}^n$, on which μ_k denotes the motion invariant Haar measure, normalized as in [85, p. 588], and α_{njk} is again defined as in (1.3). Both of these formulae can be proved by applying Hadwiger's characterization theorem, which yields a representation of the integrals in (1.2) and in (1.4) in terms of intrinsic volumes of the involved convex bodies. For the classical Crofton formula, the occurring coefficients of this representation are then determined by computation of the formula for suitable convex bodies. These further yield the coefficients in the principal kinematic formula, using the connection via Hadwiger's general integral geometric theorem (see [85, Theorem 5.1.2]).

It is natural to extend these integral formulae by applying the corresponding integrations to the functionals introduced above which generalize the intrinsic volumes. We briefly summarize various aspects of the progress concerning these extensions, which has been made during the past decades. In 1959, Federer (see [27, Theorem 6.11]) proved kinematic formulae for the curvature measures, even in the more general setting of sets with positive reach, which contain the classical kinematic formula as a very special case. More recently, kinematic formulae for support measures on convex bodies have been established by Glasauer in 1997 (see [32, Theorem 3.1]). These formulae are based on a special set operation on support elements of the involved convex bodies, which limits their usefulness for the present purpose, as explained in [33]. As mentioned above, Schneider and Hadwiger developed integral formulae for the vector-valued quermassvectors, early in the nineteen-seventies (see [39, 77, 78]). For a proof of a tensor-valued version of the integral formulae, an application of Alesker's characterization theorem for Minkowski tensors does not seem to be promising, due to the fact that the Minkowski tensors are not linearly independent and because of the inherent difficulty of evaluating them explicitly for sufficiently many examples. Nevertheless, major progress has been made in various works by different methods. Integral geometric Crofton formulae for general Minkowski tensors have been obtained in [51]. A specific case has been further studied and applied to problems in stereology in [65], for various extensions see [93]. A quite general study of various kinds of integral geometric formulae for translation invariant tensor valuations is carried out in [15], where also corresponding algebraic structures are explicitly determined. An approach to Crofton and thus kinematic formulae for translation invariant tensor valuations via integral geometric formulae for area measures (which are of independent interest) follows from [33] and [89]. Despite all these efforts and substantial progress, a complete set of kinematic and Crofton formulae for general Minkowski tensors has not been found so far. The current state of the art is described in several contributions of the lecture notes [58].

In the joint works with Daniel Hug [53, 55], we established two complete sets of kinematic and Crofton formulae for the tensorial curvature measures (and their generalizations), which had not been considered in the literature before. By globalization of these integral geometric formulae, we further derive two complete sets of the corresponding integral formulae for the Minkowski tensors. Following the approach of [48, 49], we introduce the $\mathrm{SO}(n)$-*covariant tensorial curvature measures*, which are versions of the tensorial curvature measures with slightly different covariance properties (and only occur in dimensions two and three). By non-trivial adjustments of the above methods to their case, we obtain integral geometric formulae for these tensorial valuations as well. A completely different approach, which is based on the techniques in [51], finally yields Crofton formulae for the tensorial curvature measures defined in the uniform affine subspace as the ambient space (and for their global versions), which is published in the joint work with Daniel Hug [54] (containing generalizations of [99]). The results listed in the present paragraph are the outcome of my research during the last two and a half years. Their placement in the structure of this thesis is provided in the last part of this chapter.

In the present thesis, we explore generalizations of integral geometric formulae to tensorial measure-valued valuations. Various other directions have been taken in extending the classical framework of integral geometry. Kinematic formulae for support functions have been studied by Weil in [96], by Goodey and Weil in [34], and by Schneider in [81]. Furthermore, there is recent related work on mean section bodies and Minkowski valuations by Schuster (see [88]), Goodey and Weil (see [35]), and Schuster and Wannerer (see [89]); a Crofton formula for Hessian measures of convex functions has been established and applied in [22]. Instead of changing the functionals involved in the integral geometric formulae, it is also natural and in fact required by applications in stochastic geometry to explore formulae where the integration is extended over subgroups of the motion group. The extremal cases are translative and rotational integral geometry, where the subgroup is $\mathbb{R}^n$ and $\mathrm{O}(n)$, respectively. The former is described in detail in [85, Chap. 6.4], recent progress for scalar- and measure-valued valuations and further references are provided in [97, 98, 46], applications to stochastic geometry are given in [43, 41, 44], where translative integral formulae for tensor-valued measures are established and applied. Rotational Crofton formulae for tensor valuations have recently been developed further by Auneau et al. in [8, 7] and Svane and Vedel Jensen in [93] (see also the literature cited there), applications to stereological estimation and bio-imaging are treated and discussed in [71, 103, 56]. Various other groups of isometries, also in Riemannian isotropic spaces, have been studied in recent years. Major progress has been made, for instance, in Hermitian integral geometry (in curved spaces), where the interplay between global and local results turned out to be crucial (see [13, 14, 28, 29, 95, 94, 92] and the survey [11]), but various other group actions have been studied successfully as well (see [5, 9, 10, 12, 16, 17, 18, 26]).

The thesis is structured as follows. In Chapter 2, we fix our notation and terminology, and provide a brief introduction to the basic concepts and definitions required in this thesis. Chapter 3 is intended to motivate and define different tensor-valued generalizations of the curvature measures, which is done in Section 3.1. Moreover, in Section 3.2, we recall the definition of the Minkowski tensors and the required background concerning these.

In Chapter 4, we establish a complete set of kinematic formulae for the generalized tensorial curvature measures on polytopes and for their non-vanishing extensions to convex bodies (see Section 4.1). The constants involved in these formulae are surprisingly simple (when compared to the previous results from the literature) and can be expressed as a concise product of Gamma functions. Although some information about tensorial kinematic formulae can be gained from abstract characterization results (as developed in [82, 47]), we believe that explicit results cannot be obtained by such an approach, at least not in a simple way. In contrast, in the proof (provided in Section 4.4), our argument starts as a tensor-valued version of the proof of the kinematic formula for curvature measures (see [83, Theorem 4.4.2]). But instead of first deriving Crofton formulae to obtain the coefficients of the appearing functionals, we have to proceed in a direct way. In fact, the explicit derivation of the constants in related Crofton formulae via the template method does not seem to be feasible. The main technical part of the present argument, which requires the calculation of rotational averages over Grassmannians and the rotation group (prepared in Section 4.2), is new even in the scalar setting. Therefore, we also provide the proof of the (scalar) principal kinematic formula for curvature measures, as an instructive preparation for the general (tensorial) proof in Section 4.3.

In Chapter 5, we provide a complete set of Crofton formulae for (generalized) tensorial curvature measures as a straightforward consequence of the kinematic formulae, and relate them to results in Chapter 8 (see Section 5.1). This complements the particular results for tensorial curvature measures in [93]. The current approach is basically an application of the kinematic formulae for (generalized) tensorial curvature measures derived in Chapter 4. The connection between local kinematic and local Crofton formulae is well-known for the (scalar) curvature measures. In that setting, it is used to determine the coefficients in the kinematic formula for curvature measures. In the tensorial framework however, we apply this relation reversely to derive the explicit Crofton formulae (see Section 5.2).

Since the tensorial curvature measures are local versions of the Minkowski tensors, it is rather straightforward to derive the corresponding sets of integral geometric formulae for Minkowski tensors as well. This is the subject of Chapter 6; see Section 6.1 for the kinematic formulae and Section 6.2 for the Crofton formulae. These results extend some of the integral formulae for translation invariant Minkowski tensors obtained by Bernig and Hug in [15] and significantly simplify the coefficients of the Crofton formulae proved in [51]. We prove the integral formulae by globalization of the corresponding ones for tensorial curvature measures (see Section 6.3). The total generalized tensorial curvature measures, which naturally occur in that approach, are the only challenging part of the proof. They can, however, be treated using a relation due to McMullen (see [68]).

Chapter 7 is devoted to the study of integral formulae for versions of the tensorial curvature measures with weakened covariance properties. More precisely, in dimension two and three there exist tensorial generalizations of the curvature measures which are covariant under proper rotations but not under orientation reversing rotations (see Section 3.1.4). We establish kinematic and Crofton formulae for these and for their global versions (in case they do not vanish); see Sections 7.1 and 7.2. The proofs are based on the ideas in the previous chapters. Nevertheless, at some point we have to deviate from the general approach, which is explained in detail in the proofs.

Finally, in Chapter 8, we derive Crofton formulae for tensorial curvature measures which are defined with respect to the uniformly moved affine subspace as the ambient space (intrinsic viewpoint); see Section 8.1. The approach of the proofs combines main ideas of the previous works [51] and [47] and also links it to [15] (see Section 8.2). From the general local results, we deduce further various special consequences for the total measures (obtained by globalization), the Minkowski tensors. For the latter, we restrict ourselves to the translation invariant case, which simplifies the involved constants. In a second step, we demonstrate how the arguments can be extended to the case where the curvature measures are considered in $\mathbb{R}^n$ (extrinsic viewpoint), even though these are basically special cases of the formulae in Chapter 5. In the case of the results for the extrinsic tensorial Crofton formulae, the connection to the approach in [15] via the methods of algebraic integral geometry is used and deepened (see Section 8.3).

As a matter of completeness, we recall several auxiliary integral geometric formulae known from the literature, which are required during this thesis, in Appendix A. Finally in Appendix B, we state and prove some explicit expressions for sums of Gamma functions and binomial coefficients. The proofs of these results are based on relations found with Zeilberger's algorithm (see [70]).

Chapter 2

Preliminaries

In this chapter, we set up the notation and terminology, and review several basic mathematical facts.

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, equipped with its usual topology generated by the standard scalar product $\langle \cdot\,, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. For a topological space X, we denote the Borel σ-algebra on X by $\mathcal{B}(X)$. We write $\mathcal{H}^j$, $j \in \{0, \ldots, n\}$, for the j-dimensional Hausdorff measure on $\mathcal{B}(\mathbb{R}^n)$. The unit ball in $\mathbb{R}^n$ centered at the origin is denoted by B^n, its boundary (the unit sphere) is denoted by $\mathbb{S}^{n-1}$, and the product space $\mathbb{R}^n \times \mathbb{S}^{n-1}$ is denoted by Σ^n. We further write κ_n for the volume of the unit ball and ω_n for its surface area, that is,

$$\kappa_n = \mathcal{H}^n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}, \qquad \omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

where Γ denotes the Gamma function, which is briefly introduced with some important properties in the end of this chapter.

For a set $A \subset \mathbb{R}^n$, we denote the affine, linear, and positive hull by $\mathrm{aff} A$, $\mathrm{lin} A$ and $\mathrm{pos} A$, respectively. The dimension of A is defined as the dimension of the affine hull of A, and denoted by $\dim A$. The interior, closure and boundary of A are respectively denoted by $\mathrm{int}\, A$, $\mathrm{cl}\, A$ and $\mathrm{bd}\, A$. We further write $\mathrm{relint}\, A$ (resp. $\mathrm{relbd}\, A$) for the relative interior (resp. relative boundary) of A, which is the interior (resp. boundary) of A relative to its affine hull as the ambient space. We define the Minkowski sum of A and a set $B \subset \mathbb{R}^n$ by $A + B := \{a + b : a \in A, b \in B\}$, and shortly write $A + x := A + \{x\}$, where $x \in \mathbb{R}^n$. We denote the indicator function of A by $\mathbb{1}_A : \mathbb{R}^n \to \mathbb{R}$, which is defined by $\mathbb{1}_A(x) := 1$ if

$x \in A$ and by $\mathbb{1}_A(x) := 0$ if $x \notin A$. Similarly, for an assertion P, we set $\mathbb{1}\{P\} = 1$ if P is true and $\mathbb{1}\{P\} = 0$ if P is false.

The rotation group (or orthogonal group) on $\mathbb{R}^n$, consisting of all distance preserving linear mappings from $\mathbb{R}^n$ onto itself, is denoted by $\mathrm{O}(n)$, and the proper rotation group on $\mathbb{R}^n$, which is the subgroup of $\mathrm{O}(n)$ consisting of orientation preserving rotations, is denoted by $\mathrm{SO}(n)$. We write ν for the Haar probability measure on both of these topological groups. By G_n, we denote the (rigid) motion group on $\mathbb{R}^n$, consisting of all distance and orientation preserving affine maps from $\mathbb{R}^n$ onto itself. Between G_n and $\mathbb{R}^n \times \mathrm{SO}(n)$, there is a bijection. In fact, we can write every rigid motion $g \in \mathrm{G}_n$ as the unique composition of a proper rotation $\vartheta \in \mathrm{SO}(n)$ and a translation $t_x : \mathbb{R}^n \to \mathbb{R}^n$ (which is defined by $t_x(y) := y + x$, for $y \in \mathbb{R}^n$), that is, $g = t_x \circ \vartheta$. We write μ for the Haar measure on G_n, normalized as in [85, p. 586], that is,

$$\mu(\cdot) = \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \mathbb{1}\{t_x \circ \vartheta \in \cdot\ \} \, \mathcal{H}^n(\mathrm{d}x) \, \nu(\mathrm{d}\vartheta).$$

By $\mathrm{G}(n,k)$ (resp. $\mathrm{A}(n,k)$), for $k \in \{0, \ldots, n\}$, we denote the Grassmannian (resp. affine Grassmannian) of k-dimensional linear (resp. affine) subspaces of $\mathbb{R}^n$. We write ν_k for the rotation invariant Haar probability measure on $\mathrm{G}(n,k)$, and μ_k for the motion invariant Haar measure on $A(n,k)$, normalized as in [85, p. 588], that is, for a fixed (but arbitrary) linear subspace $L \in \mathrm{G}(n,k)$,

$$\mu_k(\cdot) = \int_{\mathrm{SO}(n)} \int_{L^\perp} \mathbb{1}\{\vartheta(L+t) \in \cdot\} \, \mathcal{H}^{n-k}(\mathrm{d}t) \, \nu(\mathrm{d}\vartheta). \tag{2.1}$$

The directional space of an affine k-flat $E \in \mathrm{A}(n,k)$ is denoted by $E^0 \in \mathrm{G}(n,k)$, its orthogonal complement by $E^\perp \in \mathrm{G}(n, n-k)$, and the translate of E by a vector $t \in \mathbb{R}^n$ is denoted by $E_t := E + t$. For $k \in \{0, \ldots, n\}$ and $F \in \mathrm{G}(n,k)$, we denote the group of rotations of $\mathbb{R}^n$ mapping F (and hence also $F^\perp$) into itself by $\mathrm{SO}(F)$ (which is the same as $\mathrm{SO}(F^\perp)$) and write ν^F for the Haar probability measure on $\mathrm{SO}(F)$. For $l \in \{0, \ldots, n\}$, we set

$$\mathrm{G}(F,l) := \begin{cases} \{L \in \mathrm{G}(n,l) : L \subset F\}, & \text{if } l \le k, \\ \{L \in \mathrm{G}(n,l) : L \supset F\}, & \text{if } l > k. \end{cases}$$

Then $\mathrm{G}(F,l)$ is a homogeneous $\mathrm{SO}(F)$-space. Hence, there exists a unique Haar probability measure ν_l^F on $\mathrm{G}(F,l)$, which is $\mathrm{SO}(F)$ invariant. An introduction to invariant measures and group operations as needed here is provided in [85, Chap. 13], where, however, $\mathrm{SO}(F)$ is defined in a slightly different way.

The orthogonal projection of a vector $x \in \mathbb{R}^n$ to a linear subspace L of $\mathbb{R}^n$ is denoted by $p_L(x)$, and its direction is denoted by $\pi_L(x) := p_L(x)/\|p_L(x)\| \in \mathbb{S}^{n-1}$ for $x \notin L^\perp$. For two linear subspaces L, L' of $\mathbb{R}^n$, the subspace determinant $[L, L']$ is defined as follows (see

[85, Sect. 14.1]). If $\dim L + \dim L' \geq n$, one extends an orthonormal basis of $L \cap L'$ (the empty set if $L \cap L' = \{0\}$) to an orthonormal basis of L and to one of L'. Then $[L, L']$ is the volume of the parallelepiped spanned by all these vectors. If $\dim L + \dim L' < n$, we define $[L, L'] := [L^\perp, L'^\perp]$. Consequently, if $L = \{0\}$ or $L = \mathbb{R}^n$, then $[L, L'] = 1$, and $[L, L'] = 0$ if and only if L and L' are not in general position. For two sets $F, F' \subset \mathbb{R}^n$ (in this thesis F, F' will often be faces of polytopes), we define $[F, F'] := [F^0, (F')^0]$, where F^0 is the direction space of the affine hull of F.

In this paragraph we introduce the required notation from convex geometry, which can be found in detail in [83]. We call a non-empty, convex and compact subset of $\mathbb{R}^n$ a convex body, and denote the set of all convex bodies in $\mathbb{R}^n$ by $\mathcal{K}^n$. For an arbitrary set $A \subset \mathbb{R}^n$, the convex hull of A is denoted by $\mathrm{conv}A$. We call the convex hull of finitely many points a polytope, and denote the set of all non-empty polytopes in $\mathbb{R}^n$ by $\mathcal{P}^n$. For a convex body $K \in \mathcal{K}^n$, we call $K \cap E$ a support set of K, where $E \in \mathrm{A}(n, n-1)$ is a hyperplane (that is an $(n-1)$-dimensional subset of $\mathbb{R}^n$), whenever $(K \cap E) \subset \mathrm{relbd}\, K$. A j-dimensional support set, $j \in \{0, \dots, n-1\}$, of a polytope $P \in \mathcal{P}^n$ is called a j-face or simply a face (and the polytope itself is its only n-face, if it is n-dimensional). In particular, a $\dim P - 1$-face is referred to as facet. We further denote the set of j-faces of P by $\mathcal{F}_j(P)$. For a convex body $K \in \mathcal{K}^n$ and a point $x \in \mathbb{R}^n$, we denote the metric projection of x onto K by $p(K, x)$, which is the unique nearest point of x in K. We further set the unit vector $u(K, x) := (x - p(K, x))/\|x - p(K, x)\| \in \mathbb{S}^{n-1}$ to be the direction of the vector pointing from the metric projection $p(K, x)$ to $x \in \mathbb{R}^n \setminus K$, which we call an outer unit normal vector of K at $p(K, x)$. A pair $(x, u) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ is a support element of K if $x \in \mathrm{bd}\, K$ and u is an outer unit normal vector of K at x, and the set of all support elements of K is denoted by $\mathrm{Nor}\, K \subset \mathbb{R}^n \times \mathbb{S}^{n-1}$. The normal cone of a convex body $K \in \mathcal{K}^n$ at a point $x \in K$ is defined as

$$N(K, x) := \{u \in \mathbb{R}^n : x = p(K, x + u)\}.$$

We set the normal cone of a polytope $P \in \mathcal{P}^n$ at a face $F \in \mathcal{F}_j(P)$, $j \in \{0, \dots, n\}$, of P to be $N(P, F) := N(P, x)$, where $x \in \mathrm{relint}\, F$ can be chosen arbitrarily.

The algebra of symmetric tensors over $\mathbb{R}^n$ is denoted by $\mathbb{T}$ (the underlying $\mathbb{R}^n$ will be clear from the context), the vector space of symmetric tensors of rank $p \in \mathbb{N}_0$ is denoted by $\mathbb{T}^p$ (with $\mathbb{T}^0 = \mathbb{R}$). In $\mathbb{T}$ there are no non-zero zero divisors. Identifying $\mathbb{R}^n$ with its dual space via the given scalar product, we interpret a symmetric tensor of rank p as a symmetric p-linear map from $(\mathbb{R}^n)^p$ to $\mathbb{R}$. The symmetric tensor product of $k \in \mathbb{N}$ tensors $T_i \in \mathbb{T}^{p_i}$ over $\mathbb{R}^n$, where $p_i \in \mathbb{N}_0$, $i \in \{1, \dots, k\}$, is denoted by $T_1 \cdots T_k \in \mathbb{T}^{p_1 + \cdots + p_k}$, and (with $q_0 := 0$, $q_i := p_1 + \cdots + p_i$, $i \in \{1, \dots, k\}$) defined as

$$T_1 \cdots T_k(x_1, \dots, x_{q_k}) := \frac{1}{q_k!} \sum_{\sigma \in \mathcal{S}(q_k)} \prod_{i=0}^{k} T_i(x_{\sigma(q_{i-1}+1)}, \dots, x_{\sigma(q_i)}),$$

for $x_1, \ldots, x_{q_k} \in \mathbb{R}^n$, where $\mathcal{S}(q_k)$ is the symmetric group of permutations of the set $\{1, \ldots, q_k\}$. (For the sake of clarity, we deviate here from the standard notation $T_1 \odot \cdots \odot T_k$ for the symmetric tensor product.) For $q \in \mathbb{N}_0$ and a tensor $T \in \mathbb{T}$, we write T^q for the q-fold tensor product of T (with $T^0 := 1$). The symmetric tensor product is not only commutative, but the algebra of symmetric tensors satisfies a binomial theorem; that is,

$$(T + S)^k = \sum_{j=0}^{k} \binom{k}{j} T^j S^{k-j}, \qquad T, S \in \mathbb{T}^p, p \in \mathbb{N}_0.$$

A special tensor is the metric tensor $Q \in \mathbb{T}^2$, which is defined by

$$Q(x, y) := \langle x, y \rangle, \qquad x, y \in \mathbb{R}^n.$$

For an affine k-flat $E \subset \mathbb{R}^n$, $k \in \{0, \ldots, n\}$, the metric tensor $Q(E)$ associated with E is defined by $Q(E)(x, y) := \langle p_{E^0}(x), p_{E^0}(y) \rangle$, for $x, y \in \mathbb{R}^n$. Obviously, we observe that $Q = Q(E) + Q(E^\perp)$ and even more generally $Q(E + F) = Q(E) + Q(F)$, whenever $F \subset E^\perp$ is a second flat orthogonal to E. If $F \subset \mathbb{R}^n$ is a k-dimensional convex body (in this thesis F will often be the face of a polytope), then we again write $Q(F)$ for the metric tensor $Q(\mathrm{aff} F) = Q((\mathrm{aff} F)^0)$ associated with the affine subspace $\mathrm{aff} F$ spanned by F.

In the coefficients of the kinematic formula and in the proof of our main theorem, the classical Gamma function is involved. It can be defined via the Gaussian product formula

$$\Gamma(z) := \lim_{a \to \infty} \frac{a^z a!}{z(z+1) \cdots (z+a)}$$

for all $z \in \mathbb{C} \setminus \{0, -1, \ldots\}$ (see [6, (2.7)]). For $c \in \mathbb{R} \setminus \mathbb{Z}$ and $m \in \mathbb{N}_0$, this definition implies that

$$\frac{\Gamma(-c + m)}{\Gamma(-c)} = (-1)^m \frac{\Gamma(c + 1)}{\Gamma(c - m + 1)}. \tag{2.2}$$

The Gamma function has simple poles at the non-positive integers. The right side of relation (2.2) provides a continuation of the left side at $c \in \mathbb{N}_0$, where $\Gamma(c - m + 1)^{-1} = 0$ for $c < m$.

Another in this work repeatedly used relation concerning the Gamma function is *Legendre's duplication formula*, which states that

$$\Gamma(c)\,\Gamma(c + \tfrac{1}{2}) = 2^{1-2c} \sqrt{\pi}\,\Gamma(2c)$$

for $c > 0$ (see [6, (3.11)]).

Chapter 3

Tensorial Valuations on Convex Bodies and Polytopes

In this chapter, we introduce the geometric mappings which form the backbone of this work, the (generalized) tensorial curvature measures (see Section 3.1). As their names suggest, they extend the curvature measures of a convex body to tensor-valued measures (and contain the latter as scalar special cases). The curvature measures on convex bodies are additive, isometry invariant, locally defined and weakly continuous. They can even be characterized by these properties (see [80]). We show that the (generalized) tensorial curvature measures satisfy tensor-valued equivalents of these characteristic properties of the scalar curvature measures. However, there does not exist a tensor-valued counterpart of this characterization result so far (see nonetheless [74] for results in the smooth setting). Corresponding versions of these characteristics have been extensively studied by Schneider in [82], and Hug and Schneider in [47] for tensorial support measures, so called local Minkowski tensors. Weakening the isometry covariance property, leads to interesting new mappings in dimensions two and three (see Saienko's PhD thesis [74], and the works by Hug and Schneider [48, 49]). We transfer these mappings to the setting of tensorial curvature measures.

The tensorial curvature measures are local versions of the Minkowski tensors, for which we therefore also consider integral formulae in this thesis (mainly in Chapter 6 and in Chapter 8). In Section 3.2, we consequently recall the definitions of the latter. The Minkowski tensors have been studied first by McMullen (see [68]) and characterized shortly after by Alesker (see [2, 3]). We recall Alesker's characterization theorem and a relation, which is due to McMullen, between the Minkowski tensors and the (generalized) tensorial curvature measures. This relation is a usefull tool to derive integral formulae for Minkowski tensors from the corresponding ones for tensorial curvature measures.

3.1. The (Generalized) Tensorial Curvature Measures

The tensorial curvature measures are tensor-valued generalizations of the curvature measures. Therefore, in order to define them we start by introducing the support measures on convex bodies (which are measures on $\mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$), the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of which are the curvature measures.

For a convex body $K \in \mathcal{K}^n$, $\epsilon > 0$ and a Borel set $\eta \subset \Sigma^n$ (we recall that $\Sigma^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$),

$$M_\epsilon(K,\eta) := \Big\{x \in (K + \epsilon B^n) \setminus K\colon\ (p(K,x), u(K,x)) \in \eta\Big\}$$

is a local parallel set of K which satisfies the *local Steiner formula*

$$\mathcal{H}^n(M_\epsilon(K,\eta)) = \sum_{j=0}^{n-1} \kappa_{n-j}\Lambda_j(K,\eta)\epsilon^{n-j}, \qquad \epsilon \geq 0. \tag{3.1}$$

This relation determines the *support measures* $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ of K, which are finite Borel measures on $\mathcal{B}(\Sigma^n)$. Obviously, a comparison of (3.1) and the Steiner formula (1.1) yields

$$V_j(K) = \Lambda_j(K,\Sigma^n). \tag{3.2}$$

In other words, the intrinsic volumes are the total support measures. As mentioned before, the *curvature measures* of a convex body are the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of the support measures. Hence for $K \in \mathcal{K}^n$ and $j \in \{0,\dots,n-1\}$, they are defined as

$$\phi_j(K,\cdot) := \Lambda_j(K,\cdot \times \mathbb{S}^{n-1}). \tag{3.3}$$

Additionally, we define the nth curvature measure as $\phi_n(K,\cdot) := \mathcal{H}^n(K\cap\cdot)$. For further information on support measures, curvature measures and intrinsic volumes we refer to [83, Chap. 4.2].

3.1.1. The Tensorial Curvature Measures on Convex Bodies

The *tensorial curvature measures* are tensor-valued versions of the scalar curvature measures, obtained via generalization of relation (3.3). That is, for a convex body $K \in \mathcal{K}^n$, a Borel set $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $r,s \in \mathbb{N}_0$, they are given by

$$\phi_j^{r,s,0}(K,\beta) := c_{n,j}^{r,s,0} \int_{\beta\times\mathbb{S}^{n-1}} x^r u^s\, \Lambda_j(K,\mathrm{d}(x,u)), \tag{3.4}$$

for $j \in \{0, \dots, n-1\}$, where

$$c_{n,j}^{r,s,0} := \frac{1}{r!s!} \frac{\omega_{n-j}}{\omega_{n-j+s}}.$$

As in the scalar case, we extend this definition by further defining

$$\phi_n^{r,0,0}(K, \beta) := c_{n,n}^{r,0,0} \int_{K \cap \beta} x^r \, \mathcal{H}^n(\mathrm{d}x),$$

where $c_{n,n}^{r,0,0} := \frac{1}{r!}$. For the sake of convenience, we moreover set $\phi_j^{r,s,0} := 0$ for $j \notin \{0, \dots, n\}$ or $r \notin \mathbb{N}_0$ or $s \notin \mathbb{N}_0$ or $j = n$ and $s \neq 0$. Furthermore, we observe that, for $K \in \mathcal{K}^n$, $r = s = 0$, and $j \in \{0, \dots, n\}$, the scalar-valued tensorial curvature measures $\phi_j^{0,0,0}(K, \cdot)$ are simply the curvature measures $\phi_j(K, \cdot)$.

3.1.2. The Generalized Tensorial Curvature Measures on Polytopes

There exist generalizations of the tensorial curvature measures (some of which are exclusively defined on polytopes), which are as well of substantial interest in the context of tensor-valued curvature measures. Their significance is shown later in this chapter, as we prove that they are essentially linearly independent, which will be important for the representation of the integral formulae in the upcoming chapters. The most obvious way to introduce these generalizations uses a different, more intuitive interpretation of the support measures for polytopes.

The jth support measure $\Lambda_j(P, \cdot)$, $j \in \{0, \dots, n-1\}$, of a polytope $P \in \mathcal{P}^n$ is explicitly given by

$$\Lambda_j(P, \eta) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{N(P,F) \cap \mathbb{S}^{n-1}} \mathbb{1}_\eta(x, u) \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \mathcal{H}^j(\mathrm{d}x), \tag{3.5}$$

for $\eta \in \mathcal{B}(\Sigma^n)$.

Using representation (3.5), we can generalize the definition of the tensorial curvature measures in the following way. For a polytope $P \in \mathcal{P}^n$, we define the *generalized tensorial curvature measure*

$$\phi_j^{r,s,l}(P, \cdot), \qquad j \in \{0, \dots, n-1\}, \, r, s, l \in \mathbb{N}_0,$$

as the Borel measure on $\mathcal{B}(\mathbb{R}^n)$ which is given by

$$\phi_j^{r,s,l}(P, \beta) := c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} Q(F)^l \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u),$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$, where

$$c_{n,j}^{r,s,l} := \frac{1}{r!s!} \frac{\omega_{n-j}}{\omega_{n-j+s}} \frac{\omega_{j+2l}}{\omega_j} \text{ if } j \neq 0, \quad c_{n,0}^{r,s,0} := \frac{1}{r!s!} \frac{\omega_n}{\omega_{n+s}}, \quad \text{and } c_{n,0}^{r,s,l} := 1 \text{ for } l \geq 1.$$

As the coefficients $c_{n,j}^{r,s,0}$ of the tensorial curvature measures (defined in Section 3.1.1) match the just defined coefficients $c_{n,j}^{r,s,l}$ for $l = 0$, the tensorial curvature measures for polytopes are simply generalized tensorial curvature measures without the additional weight of powers of the metric tensor on the faces of the given polytope, meaning $l = 0$. We note that if $j = 0$ and $l \geq 1$, then we have $\phi_0^{r,s,l} \equiv 0$, as in that case $Q(F) = 0$ for all $F \in \mathcal{F}_0(P)$. In all other cases the factor $1/\omega_{n-j}$ in the definition of $\phi_j^{r,s,l}(P, \beta)$ and the factor ω_{n-j} involved in the constant $c_{n,j}^{r,s,l}$ cancel.

In a similar way, we extend the tensorial curvature measures $\phi_n^{r,0,0}$, which can even be done on $\mathcal{K}^n$. For a general convex body $K \in \mathcal{K}^n$, we define the *generalized tensorial curvature measure*

$$\phi_n^{r,0,l}(K, \cdot), \qquad r, l \in \mathbb{N}_0,$$

as the Borel measure on $\mathcal{B}(\mathbb{R}^n)$ which is given by

$$\phi_n^{r,0,l}(K, \beta) := c_{n,n}^{r,0,l}\, Q^l \int_{K \cap \beta} x^r \, \mathcal{H}^n(\mathrm{d}x),$$

for $\beta \in \mathcal{B}(\mathbb{R}^n)$, where $c_{n,n}^{r,0,l} := \frac{1}{r!} \frac{\omega_{n+2l}}{\omega_n}$. We observe that the generalized tensorial curvature measures $\phi_n^{r,0,l}$ are basically renormalized tensorial curvature measures $\phi_n^{r,0,0}$ multiplied with powers of the metric tensor, which therefore seem to be superfluous. However, they turn out to be quite useful in order to simplify the integral formulae in the upcoming chapters.

There exist further relations among some of the generalized tensorial curvature measures. In fact, we observe that the normal cone of a given polytope at each $(n-1)$-face is one-dimensional. Hence, the integrations on the normal cones in the definition of $\phi_{n-1}^{r,s,l}$, $r, s, l \in \mathbb{N}_0$, can be combined with the metric tensors on the corresponding faces. We state the resulting relations in the following lemma.

Lemma 3.1. *Let $r, s, l \in \mathbb{N}_0$. Then*

$$\phi_{n-1}^{r,s,l} = \sum_{m=0}^{l} (-1)^m \binom{l}{m} \frac{(s+2m)!}{s!} \frac{\omega_{s+2m+1}}{\omega_{s+1}} \frac{\omega_{n+2l-1}}{\omega_{n-1}} \, Q^{l-m} \phi_{n-1}^{r,s+2m,0}.$$

Lemma 3.1 is simply a "marginal version" of the analog result for generalized tensorial support measures by Hug and Schneider (see [47, Lemma 3.4]). The generalized tensorial curvature measures are the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of the latter, which we introduce in the following section. Therefore, we do not need to provide the proof of Lemma 3.1 here, as we basically obtain the assertion from [47, Lemma 3.4] by going over to the marginal measures (and renormalizing).

3.1.3. Properties of the (Generalized) Tensorial Curvature Measures

The systematic investigation of local tensorial valuations on convex bodies (and polytopes) has been initiated by Schneider in 2013 (see [82]) and further deepened by Hug and Schneider (see [47, 50]) in their works on generalized local Minkowski tensors. In this thesis we prefer to call them *generalized tensorial support measures*, as the original name is slightly imprecise. For a polytope (or in some cases, for a convex body), these are tensor-valued measures on $\mathcal{B}(\Sigma^n)$, the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of which are the generalized tensorial curvature measures. We recall the relevant definitions and results from the just mentioned works, in order to put the generalized tensorial curvature measures into their natural context and to emphasize some of their properties.

For $\eta \in \Sigma^n$, $t \in \mathbb{R}^n$ and $\vartheta \in \mathrm{SO}(n)$, we set $\eta + t := \{(x+t, u) : (x,u) \in \eta\}$ and $\vartheta\eta := \{(\vartheta x, \vartheta u) : (x,u) \in \eta\}$. For $p \in \mathbb{N}_0$, let $\tilde{T}_p(\mathcal{P}^n)$ denote the vector space of all mappings $\tilde{\Gamma} : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ such that

(SM1) $\tilde{\Gamma}(P, \cdot)$ is a $\mathbb{T}^p$-valued measure on $\mathcal{B}(\Sigma^n)$, for each $P \in \mathcal{P}^n$;

(SM2) $\tilde{\Gamma}$ is *isometry covariant*, that is, *translation covariant* of degree $r \in \mathbb{N}_0$ in the sense that, for all $P \in \mathcal{P}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, and $t \in \mathbb{R}^n$,

$$\tilde{\Gamma}(P+t, \eta+t) = \sum_{i=0}^{r} \tilde{\Gamma}_i(P, \eta) \frac{t^i}{i!},$$

with $\tilde{\Gamma}_i(P, \eta) \in \mathbb{T}^{p-i}$, and *rotation covariant* in the sense that, for all $P \in \mathcal{P}^n$, $\eta \in \mathcal{B}(\Sigma^n)$, and $\vartheta \in \mathrm{O}(n)$,

$$\tilde{\Gamma}(\vartheta P, \vartheta\eta) = \vartheta\tilde{\Gamma}(P, \eta);$$

(SM3) $\tilde{\Gamma}$ is *locally defined*, that is, for all $\eta \in \mathcal{B}(\Sigma^n)$ and $P, P' \in \mathcal{P}^n$ which satisfy $\eta \cap \operatorname{Nor} P = \eta \cap \operatorname{Nor} P'$, we have $\tilde{\Gamma}(P, \eta) = \tilde{\Gamma}(P', \eta)$.

For a polytope $P \in \mathcal{P}^n$, the *generalized tensorial support measure*

$$\tilde{\phi}_j^{r,s,l}(P, \cdot), \qquad j \in \{0, \ldots, n-1\},\ r, s, l \in \mathbb{N}_0,$$

is the Borel measure on $\mathcal{B}(\Sigma^n)$ which is defined by

$$\tilde{\phi}_j^{r,s,l}(P, \eta) := c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} Q(F)^l \int_F \int_{N(P,F) \cap \mathbb{S}^{n-1}} \mathbb{1}_\eta(x,u) x^r u^s \, \mathcal{H}^j(\mathrm{d}x) \, \mathcal{H}^{n-j-1}(\mathrm{d}u),$$

for $\eta \in \mathcal{B}(\Sigma^n)$. We note that the name of the generalized tensorial support measures is perfectly justified, as their definition is a tensor-valued generalization of representation (3.5).

It was shown in [82, 47] (where a different notation and normalization was used) that the mappings $Q^m \tilde{\phi}_j^{r,s,l}$, where $m, r, s, l \in \mathbb{N}_0$ satisfy $2m+r+s+2l = p$, where $j \in \{0, \ldots, n-1\}$, and where $l = 0$ if $j \in \{0, n-1\}$, form a basis of $\tilde{T}_p(\mathcal{P}^n)$. This fundamental characterization theorem highlights the importance of the generalized tensorial support measures. In particular, since the mappings $P \mapsto Q^m \tilde{\phi}_j^{r,s,l}(P, \cdot)$, $P \in \mathcal{P}^n$, are additive (as further shown in [47]), all mappings in $\tilde{T}_p(\mathcal{P}^n)$ are valuations.

Noting that

$$\phi_j^{r,s,l}(P, \beta) = \tilde{\phi}_j^{r,s,l}(P, \beta \times \mathbb{S}^{n-1}), \qquad j \in \{0, \ldots, n-1\},\ r, s, l \in \mathbb{N}_0,$$

for $P \in \mathcal{P}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$, it is clear that the mappings

$$\phi_j^{r,s,l} : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{T}^p, \qquad (P, \beta) \mapsto \phi_j^{r,s,l}(P, \beta),$$

where $p = r + s + 2l$, have similar properties as the generalized local Minkowski tensors. In order to show these, let $T_p(\mathcal{P}^n)$ denote the vector space of all mappings $\Gamma : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{T}^p$, $p \in \mathbb{N}_0$, such that

(CM1) $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure on $\mathcal{B}(\mathbb{R}^n)$, for each $P \in \mathcal{P}^n$;

(CM2) Γ is *isometry covariant*, that is, *translation covariant* of degree $r \in \mathbb{N}_0$ in the sense that, for all $P \in \mathcal{P}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $t \in \mathbb{R}^n$,

$$\Gamma(P + t, \beta + t) = \sum_{i=0}^{r} \Gamma_i(P, \beta) \frac{t^i}{i!},$$

where $\Gamma_i(P, \beta) \in \mathbb{T}^{p-i}$, and *rotation covariant* in the sense that

$$\Gamma(\vartheta P, \vartheta \beta) = \vartheta \Gamma(P, \beta),$$

for all $P \in \mathcal{P}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $\vartheta \in \mathrm{O}(n)$;

(CM3) Γ is *locally defined*, that is, if $\beta \subset \mathbb{R}^n$ is open and $P, P' \in \mathcal{P}^n$ are such that $P \cap \beta = P' \cap \beta$, then $\Gamma(P, \gamma) = \Gamma(P', \gamma)$ for all Borel sets $\gamma \subset \beta$;

(CM4) $P \mapsto \Gamma(P, \cdot)$, $P \in \mathcal{P}^n$, is *additive* (a *valuation*), that is, we have

$$\Gamma(P \cup P', \cdot) + \Gamma(P \cap P', \cdot) = \Gamma(P, \cdot) + \Gamma(P', \cdot),$$

for $P, P' \in \mathcal{P}(\mathbb{R}^n)$ with $P \cup P' \in \mathcal{P}(\mathbb{R}^n)$.

In (CM2), we moreover call Γ *translation invariant* if it is translation covariant of degree 0. Furthermore, we note that the notion of what we called "locally defined" in (CM3) is common in this context, though different from the definition in (SM3) for mappings $\tilde{\Gamma} : \mathcal{P}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ (see [83, Sect. 4.2]).

Since the generalized tensorial curvature measures are the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of the generalized tensorial support measures, we obtain the following theorem.

Theorem 3.2. *For $j \in \{0, \dots, n-1\}$, $r, s, l \in \mathbb{N}_0$, where $p = r + s + 2l$, we have*

$$\phi_j^{r,s,l} \in T_p(\mathcal{P}^n).$$

Moreover, the degree of translation covariance is r, with $(\phi_j^{r,s,l})_i = \phi_j^{r-i,s,l}$, $i \in \{0, \dots, r\}$.

Theorem 3.2 implies that the only translation invariant generalized tensorial curvature measures are $\phi_j^{0,s,l}$.

In this work, we sometimes refer to the (generalized) tensorial curvature measures a bit sloppily as valuations or as measures, even though they are maps which are valuations in the first component and measures in the second.

Proof. The properties (CM1), (CM2) and (CM4) immediately follow from the corresponding properties (SM1), (SM2) and the additivity of the generalized tensorial support measures, by going over to the marginal measure, that is, by setting $\eta = \beta \times \mathbb{S}^{n-1}$, where $\beta \in \mathcal{B}(\mathbb{R}^n)$.

As (CM3) differs slightly from (SM3), we prove directly that $\phi_j^{r,s,l}$ is locally defined (in the sense of (CM3)). Let $\beta \subset \mathbb{R}^n$ be open and $P, P' \in \mathcal{P}^n$ be such that $P \cap \beta = P' \cap \beta$. It is clear that $\dim P = \dim P' =: k$ if $P \cap \beta \neq \emptyset$ (and else we do not have to show anything). In fact, as β is open, we have $\dim P = \dim(P \cap \beta) = \dim(P' \cap \beta) = \dim P'$. Moreover, for every face $F \in \mathcal{F}_j(P)$ of P with $F \cap \beta \neq \emptyset$, there exists a unique face $F' \in \mathcal{F}_j(P')$ of P', such that $F' \cap \beta = F \cap \beta$ (and vice versa). We even have $N(P', F') = N(P, F)$. More precisely, as β is open, there exists a unique facet $G' \in \mathcal{F}_{k-1}(P')$ of P' with $F' \subset G'$ for every facet $G \in \mathcal{F}_{k-1}(P)$ of P with $F \subset G$, such that $G' \cap \beta = G \cap \beta \neq \emptyset$. Since $N(P, F)$ is determined by the outer normal vectors of all facets of P in which F lies (see [83, Lemma 2.4.9]), the normal cones of P at F and of P' at F' coincide. Therefore, we obtain

$$\begin{aligned} \phi_j^{r,s,l}(P, \gamma) = c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} & \underbrace{\sum_{F \in \mathcal{F}_j(P) \colon F \cap \beta \neq \emptyset}}_{= F' \in \mathcal{F}_j(P') \colon F' \cap \beta \neq \emptyset} \underbrace{Q(F)}_{= Q(F')}{}^l \int_{\underbrace{F \cap \gamma}_{= F' \cap \gamma}} x^r \, \mathcal{H}^j(\mathrm{d}x) \\ & \times \int_{\underbrace{N(P,F)}_{= N(P',F')} \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \\ = \phi_j^{r,s,l}(P', \gamma), & \end{aligned}$$

for $\gamma \in \mathcal{B}(\mathbb{R}^n)$ with $\gamma \subset \beta$. Hence, $\phi_j^{r,s,l}$ is locally defined. □

It has been shown by Hug and Schneider in [47] that the generalized tensorial support measure $\tilde{\phi}_j^{r,s,l}$ has a continuous extension to $\mathcal{K}^n$ which preserves all other properties if and only if $l \in \{0, 1\}$ (or if $j = n - 1$ and $l \in \mathbb{N}_0$, due to [47, Lemma 3.4], which is the analogon of Lemma 3.1 for generalized tensorial support measures). In fact, they even proved a characterization theorem for these extensions (see [47, Theorem 2.3]). These can be represented with suitable differential forms which are defined on the sphere bundle of $\mathbb{R}^n$ and evaluated on the normal cycle. This is the reason why they are called *smooth* (for more details, see for example [74] and [47]). A characterization theorem for smooth tensor-valued curvature measures has recently been found by Saienko [74].

For $l = 0$, the extension of the generalized tensorial support measures $\tilde{\phi}_j^{r,s,0}$, which are also called *tensorial support measures*, can be easily expressed via the jth support measure (analogously to the definition of the tensorial curvature measures; see (3.4)). For $l = 1$, Hug and Schneider found an explicit description of the extension of the generalized tensorial support measures $\tilde{\phi}_j^{r,s,1}$, $j \in \{0, \dots, n-2\}$, in [47], which instantly yields the extension of the corresponding generalized tensorial curvature measures, by globalization of the $\mathbb{S}^{n-1}$-coordinate. However, as the construction itself is neither instructive for nor required in this thesis we do not state it here, but refer to [47, Section 4] for the explicit description.

For the continuous extensions of the generalized tensorial curvature measures $\phi_j^{r,s,l}$, $l \in \{0, 1\}$, we deduce the properties (CM1), (CM2), (CM3), (CM4) (where $\mathcal{P}^n$ is replaced by $\mathcal{K}^n$) from the corresponding extensions of the generalized tensorial support measures. Furthermore, for $l \in \{0, 1\}$, we conclude that

(CM5) $\phi_j^{r,s,l}$ is *weakly continuous*, that is, for each sequence $(K_i)_{i\in\mathbb{N}}$ of convex bodies in $\mathcal{K}^n$ converging to a convex body $K \in \mathcal{K}^n$, the relation

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} f(x)\, \phi_j^{r,s,l}(K_i, \mathrm{d}x) = \int_{\mathbb{R}^n} f(x)\, \phi_j^{r,s,l}(K, \mathrm{d}x)$$

holds for all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.

We note that also the generalized tensorial curvature measures $\phi_n^{r,0,l}$ (defined on $\mathcal{K}^n$) satisfy all these properties.

It is an open problem whether the vector space $T_p(\mathcal{P}^n)$ is (analogously to $\tilde{T}_p(\mathcal{P}^n)$) spanned by the mappings $Q^m\phi_j^{r,s,l}$, where $m, r, s, l \in \mathbb{N}_0$ satisfy $2m + r + s + 2l = p$, where $j \in \{0, \dots, n-1\}$, and where $l = 0$ if $j \in \{0, n-1\}$, or where $j = n$ and $s = l = 0$. However, the linear independence of these mappings can be shown in a similar way as it is done for local Minkowski tensors in [47, Theorem 3.1]. We state this in the following theorem.

Theorem 3.3. *For $p \in \mathbb{N}_0$, the tensorial measure valued valuations*

$$Q^m\phi_j^{r,s,l} : \mathcal{P}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{T}^p$$

with $m, r, s, l \in \mathbb{N}_0$ and $j \in \{0, \dots, n\}$, where $2m + 2l + r + s = p$, but with $l = 0$ if $j \in \{0, n-1\}$ and with $s = l = 0$ if $j = n$, are linearly independent.

In Theorem 3.3, it is obvious that we have to require $l = 0$ if $j = 0$ (recall that $\phi_0^{r,s,l} \equiv 0$ for $l > 0$), and $l = 0$ if $j = n$ (recall that $\phi_n^{r,0,l}$ is basically a renormalization of the valuation $Q^l\phi_n^{r,0,0}$). Furthermore, Lemma 3.1 shows that the valuations $\phi_{n-1}^{r,s,l}$ are linearly dependent. Thus, we require $l = 0$ if $j = n-1$.

Proof. Suppose that

$$\sum_{\substack{j,m,r,s,l\\2m+2l+r+s=p}} a^{(0)}_{j,m,r,s,l} Q^m \phi_j^{r,s,l} = 0 \tag{3.6}$$

holds for some $a^{(0)}_{j,m,r,s,l} \in \mathbb{R}$, where $a^{(0)}_{0,m,r,s,l} = a^{(0)}_{n-1,m,r,s,l} = 0$ if $l \neq 0$ and $a^{(0)}_{n,m,r,s,l} = 0$ if $s \neq 0$ or $l \neq 0$. In the proof we will replace the constants $a^{(0)}_{j,m,r,s,l}$ by new constants $a^{(1)}_{j,m,r,s,l}$ without keeping track of the precise relations, since it will be sufficient to know that $a^{(0)}_{j,m,r,s,l} = 0$ if and only if $a^{(1)}_{j,m,r,s,l} = 0$.

For a fixed $j \in \{0, \dots, n\}$, let $P \in \mathcal{P}^n$ with $\operatorname{int} P \neq \emptyset$, $F \in \mathcal{F}_j(P)$, and $\beta \in \mathcal{B}(\operatorname{relint} F)$. Then, if $j < n$ we obtain for the generalized tensorial curvature measures

$$\begin{aligned}\phi_j^{r,s,l}(P,\beta) &= c_{n,j,r,s,l} \sum_{G \in \mathcal{F}_j(P)} Q(G)^l \int_{G \cap \beta} x^r \,\mathcal{H}^j(\mathrm{d}x) \int_{N(P,G)\cap \mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u)\\ &= c_{n,j,r,s,l} Q(F)^l \int_{\beta} x^r \,\mathcal{H}^j(\mathrm{d}x) \int_{N(P,F)\cap \mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u),\end{aligned}$$

where $c_{n,j,r,s,l} > 0$ is a constant, and $\phi_k^{r,s,l}(P,\beta) = 0$ for $k \neq j$. Moreover, we have

$$\phi_n^{r,0,0}(P,\beta) = \frac{1}{r!} \int_{\beta} x^r \,\mathcal{H}^n(\mathrm{d}x).$$

Hence, from (3.6) it follows that

$$\sum_{\substack{m,r,s,l\\2m+2l+r+s=p}} a^{(1)}_{j,m,r,s,l} Q^m Q(F)^l \int_{\beta} x^r \,\mathcal{H}^j(\mathrm{d}x) \int_{N(P,F)\cap \mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u) = 0,$$

where for $j = n$ the spherical integral is omitted (also in the following).

We may assume that $\int_\beta x^r \,\mathcal{H}^j(\mathrm{d}x) \neq 0$ (otherwise, we consider a translate of P and β). If we repeat the above calculations with multiples of P and β, a comparison of the degrees of homogeneity yields, for every $r \in \mathbb{N}_0$, that

$$\sum_{\substack{m,s,l\\2m+2l+s=p-r}} a^{(1)}_{j,m,r,s,l} Q^m Q(F)^l \int_{\beta} x^r \,\mathcal{H}^j(\mathrm{d}x) \int_{N(P,F)\cap \mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u) = 0.$$

Hence, due to the lack of zero divisors in the tensor algebra $\mathbb{T}$, we obtain

$$\sum_{\substack{m,s,l\\2m+2l+s=p-r}} a^{(1)}_{j,m,r,s,l} Q^m Q(F)^l \int_{N(P,F)\cap \mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u) = 0. \tag{3.7}$$

This shows that, in the case of $j = n$ (where the spherical integral with respect to u is omitted), we have $a^{(1)}_{n,m,r,s,l} = 0$ also for $s = l = 0$. Hence, in the following we may assume that $j < n$.

Let $L \in \mathrm{G}(n,j)$, $j < n$, and $u_0 \in L^\perp \cap \mathbb{S}^{n-1}$. For $j \le n-2$, let $u_0, u_1, \ldots, u_{n-j-1}$ be an orthonormal basis of $L^\perp$. In this case, we define the (pointed) polyhedral cone $C(u_0,\tau) := \mathrm{pos}\{u_0 \pm \tau\, u_1, \ldots, u_0 \pm \tau\, u_{n-j-1}\} \subset L^\perp$ for $\tau \in (0,1)$. Then, for any $v \in C(u_0,\tau) \cap \mathbb{S}^{n-1}$, we have $\langle v, u_0\rangle \ge 1/\sqrt{1+\tau^2}$, and therefore $\|u_0 - v\| \le \sqrt{2}\tau$. In fact, any $v \in C(u_0,\tau)\cap\mathbb{S}^{n-1}$ can be written as $v = \frac{x}{\|x\|}$, where $x \in \mathrm{conv}\{v_1^\pm, \ldots, v_{n-j-1}^\pm\}$ with $v_i^\pm = \frac{u_0 \pm \tau u_i}{\|u_0 \pm \tau u_i\|} = \frac{u_0 \pm \tau u_i}{\sqrt{1+\tau^2}} \in \mathbb{S}^{n-1}$, $i \in \{1,\ldots,n-j-1\}$. Thus we have $x = \sum \lambda_i^\epsilon v_i^\epsilon$, where we sum over all $i \in \{1,\ldots,n-j-1\}$ and all $\epsilon \in \{+,-\}$, with $\sum \lambda_i^\epsilon = 1$ and $\lambda_i^\epsilon \ge 0$, $i \in \{1,\ldots,n-j-1\}$, $\epsilon \in \{+,-\}$. This yields

$$\langle v, u_0\rangle = \frac{1}{\|x\|}\langle \sum \lambda_i^\epsilon v_i^\epsilon, u_0\rangle = \frac{1}{\sqrt{1+\tau^2}\|x\|}\sum \lambda_i^\epsilon \ge \frac{1}{\sqrt{1+\tau^2}},$$

as $\|x\| \le \sum \lambda_i^\epsilon \|v_i^\epsilon\| = 1$. This proves the assertion. For $j = n-1$ we simply put $C(u_0,\tau) := \mathrm{pos}\{u_0\}$.

Let $C(u_0,\tau)^\circ$ denote the polar cone of $C(u_0,\tau)$. Then $P := C(u_0,\tau)^\circ \cap [-1,1]^n \in \mathcal{P}^n$ and $F := L \cap [-1,1]^n \in \mathcal{F}_j(P)$ satisfy $N(P,F) = N(P,0) = C(u_0,\tau)$. With these choices, (3.7) turns into

$$\sum_{\substack{m,s,l\\ 2m+2l+s=p-r}} a^{(1)}_{j,m,r,s,l} Q^m Q(L)^l \int_{C(u_0,\tau)\cap\mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u) = 0. \tag{3.8}$$

Dividing (3.8) by $\mathcal{H}^{n-j-1}(C(u_0,\tau)\cap\mathbb{S}^{n-1})$ and passing to the limit as $\tau \to 0$, we get

$$\sum_{\substack{m,s,l\\ 2m+2l+s=p-r}} a^{(1)}_{j,m,r,s,l} Q^m Q(L)^l u_0^s = 0$$

for any $u_0 \in L^\perp \cap \mathbb{S}^{n-1}$. Here we use that

$$\begin{aligned}&\left|\mathcal{H}^{n-j-1}(C(u_0,\tau)\cap\mathbb{S}^{n-1})^{-1}\int_{C(u_0,\tau)\cap\mathbb{S}^{n-1}} u^s\,\mathcal{H}^{n-j-1}(\mathrm{d}u) - u_0^s\right|\\ &\qquad \le \max\{|u^s - u_0^s| : u \in C(u_0,\tau)\cap\mathbb{S}^{n-1}\} \to 0\end{aligned}$$

as $\tau \to 0$.

The rest of the proof follows similarly as in the proof of [47, Theorem 3.1]. □

Finally in this section, we confirm the measurability of certain maps concerning the (generalized) tensorial curvature measures, which we need for the integral formulae in the next chapters. Obviously, the measurability of the map $K \mapsto \phi_j^{r,s,0}(K,\cdot)$, $K \in \mathcal{K}^n$, is clear from the definition (3.4) and the measurability of $K \mapsto \Lambda_j(K,\cdot)$. In [47], it is shown that the map $K \mapsto \tilde{\phi}_j^{r,s,1}(K,\eta)$, $K \in \mathcal{K}^n$, is measurable for all $\eta \in \mathcal{B}(\Sigma^n)$. As the generalized tensorial curvature measures are the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of the generalized tensorial support measures, this immediately yields that the map $K \mapsto \phi_j^{r,s,1}(K,\beta)$, $K \in \mathcal{K}^n$, is measurable for all $\beta \in \mathcal{B}(\mathbb{R}^n)$. The measurability of the map $\phi_j^{r,s,l}(\cdot,\beta)$ on $\mathcal{P}^n$ follows from the next (more general) lemma for generalized tensorial support measures.

Lemma 3.4. *For $j \in \{0,\dots,n-1\}$, $r,s,l \in \mathbb{N}_0$, and $\eta \in \mathcal{B}(\Sigma^n)$, the mapping*

$$\mathcal{P}^n \ni P \mapsto \tilde{\phi}_j^{r,s,l}(P,\eta)$$

is measurable.

Lemma 3.4 already implies that $P \mapsto \phi_j^{r,s,l}(P,\beta)$, $P \in \mathcal{P}^n$, is measurable for all $\beta \in \mathcal{B}(\mathbb{R}^n)$ (by setting $\eta = \beta \times \mathbb{S}^{n-1}$).

Proof. For the proof, it is sufficient to consider the case where $r = s = 0$ and $\eta = \beta \times \omega$ with Borel sets $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$. In fact, a basic monotone class argument (see [25, Theorem 4.4.2]) then yields the assertion. For a locally compact Hausdorff space E with a countable base, let $\mathcal{F}(E)$ denote the system of closed subsets of E. With the Fell topology, $\mathcal{F}(E)$ becomes a compact Hausdorff space with a countable base and $\mathcal{F}'(E) := \mathcal{F}(E) \setminus \{\emptyset\}$ is a locally compact subspace. Then $\mathcal{K}^n$ and $\mathcal{P}^n$ are measurable subsets of $\mathcal{F}(\mathbb{R}^n)$ and the subspace topology on these subsets coincides with the topology induced by the Hausdorff metric (see [85, Theorem 12.3.4]). Further, let $\mathsf{N}(E)$ denote the set of counting measures on $\mathcal{B}(E)$. On $\mathsf{N}(E)$ we write $\mathcal{N}(E)$ for the σ-algebra generated by the evaluation maps $\eta \mapsto \eta(A)$, where $A \in \mathcal{B}(E)$. We refer to [85, Chapter 3.1 and Chapter 12.2] for details on these topics.

In the proof of [85, Lemma 10.1.2] it is shown that the map $\mathcal{P}^n \to \mathcal{F}(\mathcal{F}'(\mathbb{R}^n))$, $P \mapsto \mathcal{F}_k(P)$, is measurable. By [85, Lemma 3.1.4] it follows then that the map $\mathcal{P}^n \to \mathcal{P}^n \times \mathsf{N}(\mathcal{F}'(\mathbb{R}^n))$, $P \mapsto (P, \eta_{\mathcal{F}_k(P)})$ is also measurable, where $\eta_{\mathcal{F}_k(P)}$ is the simple counting measure with support $\mathcal{F}_k(P)$. Further, if $g : \mathcal{P}^n \times \mathcal{F}'(\mathbb{R}^n) \to [0,\infty]$ is measurable, then the map

$$\mathcal{P}^n \times \mathsf{N}(\mathcal{F}'(\mathbb{R}^n)) \to [0,\infty], \qquad (P,\eta) \mapsto \int g(P,F)\,\eta(\mathrm{d}F),$$

is measurable. Thus, to prove the assertion of the lemma, it is sufficient to show that, for all $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$, the map g defined by

$$g(P,F) := \mathbb{1}\{F \in \mathcal{F}_k(P)\}(Q(F)^l)_{i_1\dots i_{2l}}\,\mathcal{H}^k(F \cap \beta)\,\mathcal{H}^{n-1-k}(N(P,F) \cap \omega),$$

for $(P,F) \in \mathcal{P}^n \times \mathcal{F}'(\mathbb{R}^n)$, is measurable, where the definition is to be understood in the sense that $g(P,F) := 0$ if $F \notin \mathcal{F}_k(P)$ and where $(Q(F)^l)_{i_1\dots i_{2l}}$ is the coordinate of the tensor $Q(F)^l$ with respect to some basis of $\mathbb{T}^{2l}$. We note that $(\operatorname{aff} F)^0 = \bigcup\{\lambda(F - s(F)) : \lambda \in \mathbb{N}\}$, where $s(K) \in \operatorname{relint}(K)$ is the Steiner point of a convex body $K \in \mathcal{K}^n$ (see [83, p. 50]). Since the maps $\mathcal{P}^n \to \mathcal{F}(\mathcal{F}'(\mathbb{R}^n))$, $P \mapsto \mathcal{F}_k(P)$, and $s : \mathcal{K}^n \to \mathbb{R}^n$ are measurable, the measurability of the mapping $(P,F) \mapsto \mathbb{1}\{F \in \mathcal{F}_k(P)\}(Q(F)^l)_{i_1\dots i_{2l}}$ is implied by Theorems 12.2.3, 12.2.7 and 12.3.1 in [85]. Moreover, it follows that $M_k := \{(P,F) \in \mathcal{P}^n \times \mathcal{F}'(\mathbb{R}^n) : F \in \mathcal{F}_k(P)\}$ is measurable.

Next we show that the map $M_k \to \mathcal{F}'(\mathbb{R}^n)$, $(P,F) \mapsto N(P,F) \cap \mathbb{S}^{n-1}$, is measurable. For this, we observe that $s(F) \in \operatorname{relint}(F)$ implies that $N(P,F) = N(P,s(F))$. Since $M := \{(P,x) \in \mathcal{P}^n \times \mathbb{R}^n : x \in P\}$ is a measurable subset of $\mathcal{P}^n \times \mathbb{R}^n$, and $s : \mathcal{K}^n \to \mathbb{R}^n$ is measurable, it is sufficient to show that the map $T : M \to \mathcal{F}'(\mathbb{R}^n)$, $(P,x) \mapsto N(P,x) \cap \mathbb{S}^{n-1}$, is measurable. To see this, let $C \subset \mathbb{R}^n$ be compact. It is sufficient to prove that the set $M_C := \{(P,x) \in M : T(P,x) \cap C = \emptyset\}$ is open in M. Aiming at a contradiction, we assume that there are $(P_i,x_i) \in M \setminus M_C$, for $i \in \mathbb{N}$, with $(P_i,x_i) \to (P,x) \in M_C$ as $i \to \infty$. Then there are $u_i \in N(P_i,x_i) \cap \mathbb{S}^{n-1} \cap C$ for $i \in \mathbb{N}$. By compactness, there is a subsequence u_{i_j}, $j \in \mathbb{N}$, which converges to $u \in \mathbb{S}^{n-1} \cap C$. For a convex body $K \in \mathcal{K}^n$, a point $x \in K$, and $v \in \mathbb{R}^n$ we have $v \in N(K,x)$ if and only if $\langle v, x\rangle = h(K,v)$, where $h(K,v)$ is the support function $h(K,\cdot)$ of K evaluated at v (for details see [83, Section 1.7.1]). By assumption, we have $\langle u_{i_j}, x_{i_j}\rangle = h(P_{i_j}, u_{i_j})$ for $j \in \mathbb{N}$. Since the support function depends continuously on (K,v), it follows that $\langle u, x\rangle = h(P,u)$, and thus $u \in N(P,x)$. This yields $u \in N(P,x) \cap \mathbb{S}^{n-1} \cap C \neq \emptyset$, that is, $(P,x) \notin M_C$, a contradiction.

The measurability of the map g now follows by applying twice [102, Corollary 2.1.4], since the indicator function ensures that both of the Hausdorff measures $\mathcal{H}^k(F \cap \cdot)$ and $\mathcal{H}^{n-k-1}(N(P,F) \cap \mathbb{S}^{n-1} \cap \cdot)$ are locally finite. □

3.1.4. The SO(n)-Covariant Tensorial Curvature Measures

The characterization of the (generalized) tensorial support measures on polytopes and on convex bodies via their properties (SM1), (SM2), (SM3) in [82, 47] raises the interesting question for further classification results (potentially involving new mappings) with other properties. A natural idea is to consider *proper* rotation covariance instead of rotation covariance in (SM2), that is, replacing the rotation group $\mathrm{O}(n)$ by its subgroup, the proper rotation group $\mathrm{SO}(n)$. Subsequent to their previous works on local Minkowski tensors, Hug and Schneider proved characterization theorems for polytopes [48] and for convex bodies [49] in this slightly varied setting. Interestingly, these classifications do not require any new mappings in dimensions $n > 3$. Only for $n = 2, 3$, there appear further mappings which are covariant under $\mathrm{SO}(n)$, but not under $\mathrm{O}(n)$. The same discovery was already made by Saienko under different continuity and smoothness assumptions (see [74]). In this section, we introduce the marginal measures on $\mathcal{B}(\mathbb{R}^n)$ of these additional mappings.

We start in the two-dimensional Euclidean space $\mathbb{R}^2$. For a convex body $K \in \mathcal{K}^2$, we define the SO(2)-*covariant tensorial curvature measure*

$$\breve{\phi}_j^{r,s}(K,\cdot), \qquad j \in \{0,1\},\ r,s \in \mathbb{N}_0,$$

as the Borel measure on $\mathcal{B}(\mathbb{R}^2)$ which is given by

$$\breve{\phi}_j^{r,s}(K,\beta) := \omega_{2-j} \int_{\beta \times \mathbb{S}^1} x^r \overline{u} u^s \, \Lambda_j(K, \mathrm{d}(x,u))$$

for $\beta \in \mathcal{B}(\mathbb{R}^2)$. Here, $\overline{u} \in \mathbb{S}^1$ denotes the unique unit vector for which $(u, \overline{u})$ is a positively oriented orthonormal basis of $\mathbb{R}^2$, for $u \in \mathbb{S}^1$; that is, $\overline{u} = \breve{\rho} u$ for the rotation

$$\breve{\rho} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SO}(2).$$

For a polytope $P \in \mathcal{P}^2$, we can once more use the alternative representation of the support measures for polytopes (3.5), to obtain

$$\breve{\phi}_j^{r,s}(P, \beta) = \sum_{F \in \mathcal{F}_j(P)} \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u),$$

for $\beta \in \mathcal{B}(\mathbb{R}^2)$ and $j \in \{0, 1\}$, $r, s \in \mathbb{N}_0$.

Since the $\breve{\phi}_j^{r,s}$ occur in dimension two, they admit other, more intuitive representations for polytopes. If the polytope $P \in \mathcal{P}^2$ is of full dimension (that is $\dim P = 2$), then there exists a unique outer unit normal vector $u_F \in \mathbb{S}^1$, for every facet $F \in \mathcal{F}_1(P)$, meaning that $N(P, F) \cap \mathbb{S}^1 = \{u_F\}$. Hence, we have

$$\breve{\phi}_1^{r,s}(P, \beta) = \sum_{F \in \mathcal{F}_1(P)} \overline{u_F} u_F^s \int_{F \cap \beta} x^r \, \mathcal{H}^1(\mathrm{d}x),$$

for $r, s \in \mathbb{N}_0$ and $\beta \in \mathcal{B}(\mathbb{R}^2)$. On the other hand, if $\dim P = 1$ (in that case $P \in \mathcal{K}^2$ is even a convex body), then $\mathcal{F}_1(P) = \{P\}$, and there exists a unit vector $u_P \in \mathbb{S}^1$ such that $N(P, P) \cap \mathbb{S}^1 = \{\pm u_P\}$. Thus, we can rewrite

$$\begin{aligned} \breve{\phi}_1^{r,s}(P, \beta) &= \left(\overline{u_P} u_P^s - \overline{u_P}(-u_P)^s\right) \int_{P \cap \beta} x^r \, \mathcal{H}^1(\mathrm{d}x) \\ &= \mathbb{1}\{s \text{ odd}\} \; 2 \, \overline{u_P} u_P^s \int_{P \cap \beta} x^r \, \mathcal{H}^1(\mathrm{d}x), \end{aligned}$$

for $r, s \in \mathbb{N}_0$ and $\beta \in \mathcal{B}(\mathbb{R}^2)$. Lastly, there exists a unique vector $x_F \in \mathbb{R}^2$, for every 0-face $F \in \mathcal{F}_0(P)$ of a polytope $P \in \mathcal{P}^2$, such that $F = \{x_F\}$. Therefore, we have

$$\breve{\phi}_0^{r,s}(P, \beta) = \sum_{F \in \mathcal{F}_0(P)} x_F^r \int_{N(P,F) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u),$$

for $r, s \in \mathbb{N}_0$ and $\beta \in \mathcal{B}(\mathbb{R}^2)$.

In $\mathbb{R}^3$, we at first introduce tensor-valued SO(3)-covariant (yet not O(3)-covariant) generalizations of the curvature measures operating on the 1-faces of polytopes, some of which can then be continuously extended to the convex bodies. Therefore, let $P \in \mathcal{P}^3$ be a polytope. We can choose a unit vector $v_F \in F^0 \cap \mathbb{S}^2$ in the directional space of every one-dimensional face $F \in \mathcal{F}_1(P)$. Then, for $\beta \in \mathcal{B}(\mathbb{R}^3)$ and $r, s, l \in \mathbb{N}_0$, the SO(3)-*covariant* *tensorial curvature measures* are given by

$$\breve{\phi}^{r,s,l}(P,\beta) := \sum_{F\in\mathcal{F}_1(P)} v_F\, Q(F)^l \int_{F\cap\beta} x^r\,\mathcal{H}^1(\mathrm{d}x) \int_{N(P,F)\cap\mathbb{S}^2} (v_F\times u)u^s\,\mathcal{H}^1(\mathrm{d}u),$$

where $(a\times b)\in\mathbb{R}^3$ denotes the vector product of the two vectors $a,b\in\mathbb{R}^3$. Here in particular, we have that $(v_F\times u)\in\mathbb{S}^2$ extends the orthogonal unit vectors $v_F,u\in\mathbb{S}^2$ to a positively oriented orthonormal basis $(v_F,u,v_F\times u)$. Hence, the definition of $\breve{\phi}^{r,s,l}$ is independent of the choice of $v_F\in F^0\cap\mathbb{S}^2$. In fact, if we choose $-v_F$ instead, then $(-v_F)\times u = -(v_F\times u)$, and therefore, the definition stays unchanged, as the two newly appearing negative signs cancel each other.

Since the metric tensor on a one-dimensional face $F\in\mathcal{F}_1(P)$ of a polytope $P\in\mathcal{P}^3$ can be written as $Q(F)=v_F^2$, with the unit vector $v_F\in F^0\cap\mathbb{S}^2$, we obtain the alternative representation

$$\breve{\phi}^{r,s,l}(P,\beta) = \sum_{F\in\mathcal{F}_1(P)} v_F^{2l+1} \int_{F\cap\beta} x^r\,\mathcal{H}^1(\mathrm{d}x) \int_{N(P,F)\cap\mathbb{S}^2} (v_F\times u)u^s\,\mathcal{H}^1(\mathrm{d}u),$$

for $r,s,l\in\mathbb{N}_0$ and $\beta\in\mathcal{B}(\mathbb{R}^3)$.

The SO(3)-covariant tensorial curvature measures $\breve{\phi}^{r,s,l}(P,\cdot)$, $P\in\mathcal{P}^3$, are the marginal measures on $\mathcal{B}(\mathbb{R}^3)$ of the mappings introduced by Hug and Schneider in [82]. It has been shown by the same authors that only the ones with $l=0$ admit a continuous extension to the convex bodies (for details see [47, Section 3]). This extension can be transferred by globalization of the $\mathbb{S}^2$-coordinate. Therefore, the SO(3)-covariant tensorial curvature measures $\breve{\phi}^{r,s,0}$, $r,s\in\mathbb{N}_0$, can be continuously extended to the convex bodies.

As the SO(n)-covariant (that is, SO(2)- and SO(3)-covariant) tensorial curvature measures are the marginal measures of the corresponding mappings introduced and characterized by Hug and Schneider, we can deduce some of their properties as we did in Section 3.1.3 for the (generalized) tensorial curvature measures. That is, they satisfy (CM1), (CM2) (in which O(n) has to be replaced by SO(n)), (CM3), (CM4) and (CM5) (for the ones defined on $\mathcal{K}^n$). Remarkably, even though the SO(n)-covariant tensorial curvature measures are not O(n)-covariant, they satisfy some kind of covariance under orientation reversing rotations. That is, for $\vartheta_2\in\mathrm{O}(2)\setminus\mathrm{SO}(2)$ (resp. $\vartheta_3\in\mathrm{O}(3)\setminus\mathrm{SO}(3)$), we deduce from the O($n$)-invariance of the support measures and $\overline{\vartheta_2 u} = -\vartheta_2\overline{u}$ (resp. $(\vartheta_3 v_F)\times(\vartheta_3 u) = -\vartheta_3(v_F\times u)$) that

$$\breve{\phi}_j^{r,s}(\vartheta_2 K,\vartheta_2\beta_2) = -\vartheta_2\,\breve{\phi}_j^{r,s}(K,\beta_2), \qquad \text{resp.} \qquad \breve{\phi}^{r,s,l}(\vartheta_3 P,\vartheta_3\beta_3) = -\vartheta_3\,\breve{\phi}^{r,s,l}(P,\beta_3),$$

for $K\in\mathcal{K}^2$, $\beta_2\in\mathcal{B}(\mathbb{R}^2)$ (resp. $P\in\mathcal{P}^3$, $\beta_3\in\mathcal{B}(\mathbb{R}^3)$).

3.1.5. The Intrinsic (Generalized) Tensorial Curvature Measures

For a lower dimensional convex body which is contained in an affine subspace of $\mathbb{R}^n$, we can consider (generalized) tensorial curvature measures defined inside of the surrounding affine subspace. In this section, we define those valuations and call them intrinsic (generalized) tensorial curvature measures. Throughout this work, we sometimes refer to the original (generalized) tensorial curvature measures as *extrinsic (generalized) tensorial curvature measures*, when we want to clearly distinguish between them and their intrinsic versions.

Let $j, k \in \mathbb{N}_0$ with $j < k \leq n$, and $\mathcal{K}^n \ni K \subset E \in \mathrm{A}(n,k)$ be a convex body which is lying in an affine subspace E of $\mathbb{R}^n$. Then we denote the jth support measure of K defined with respect to E as the ambient space by $\Lambda_j^{(E)}(K,\cdot)$, which is a Borel measure on $\mathcal{B}(\mathbb{R}^n \times (E^0 \cap \mathbb{S}^{n-1}))$, concentrated on $\Sigma_E^n := E \times (E^0 \cap \mathbb{S}^{n-1})$. Then, for $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $r, s \in \mathbb{N}_0$, the *intrinsic tensorial curvature measures* are given by

$$\phi_{j,E}^{r,s,0}(K,\beta) := c_{k,j}^{r,s,0} \int_{\beta \times (E^0 \cap \mathbb{S}^{n-1})} x^r u^s \, \Lambda_j^{(E)}(K, \mathrm{d}(x,u)).$$

For $j = k$, we extend this as in the extrinsic case (but here directly for the generalized tensorial curvature measures) by defining

$$\phi_{k,E}^{r,0,l}(K,\beta) := c_{k,k}^{r,0,l} Q^l \int_{K \cap \beta} x^r \, \mathcal{H}^k(\mathrm{d}x),$$

for $l \in \mathbb{N}_0$.

Next, let $j, k \in \mathbb{N}_0$ with $j < k \leq n$, and $\mathcal{P}^n \ni P \subset E \in \mathrm{A}(n,k)$ be a polytope contained in an affine subspace of $\mathbb{R}^n$. Then, for $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $r, s, l \in \mathbb{N}_0$, the *intrinsic generalized tensorial curvature measures* are given by

$$\phi_{j,E}^{r,s,l}(P,\beta) := c_{k,j}^{r,s,l} \frac{1}{\omega_{k-j}} \sum_{F \in \mathcal{F}_j(P)} Q(F)^l \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{N_E(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{k-j-1}(\mathrm{d}u),$$

where $N_E(P,F) := N(P,F) \cap E^0$ denotes the normal cone of P at the face F, taken with respect to the linear subspace E^0. Of course, the intrinsic generalized tensorial curvature measures $\phi_{j,E}^{r,s,l}$ can again be continuously extended to $\mathcal{K}^n$, for $l = 0, 1$.

The intrinsic and extrinsic (generalized) tensorial curvature measures satisfy relations among each other. That is, we can express the extrinsic (generalized) tensorial curvature measures as a linear combination of intrinsic (generalized) tensorial curvature measures (and vice versa). These relations have been proved by McMullen (see [68, Theorem 5.1]) for the total tensorial curvature measures, the Minkowski tensors (introduced and further studied in Section 3.2). A generalization of these results to tensorial curvature measures by Schuster can be found in [90, Korollar 2.2.2]. In the following lemma, we state these relations extended to generalized tensorial curvature measures without proof, as one proceeds analogously to the proofs of the just mentioned results.

Lemma 3.5. *Let $j, k \in \mathbb{N}_0$ with $j < k \leq n$, and $\mathcal{P}^n \ni P \subset E \in \mathrm{A}(n,k)$. Then we have, for $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $r, s, l \in \mathbb{N}_0$,*

$$\phi_j^{r,s,l}(P,\beta) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \frac{1}{(4\pi)^m m!} \, Q(E^\perp)^m \phi_{j,E}^{r,s-2m,l}(P,\beta).$$

The same holds if P is replaced by a convex body $K \in \mathcal{K}^n$, for $l = 0, 1$.

3.2. The Minkowski Tensors

In the same way as the tensorial curvature measures generalize the (scalar) curvature measures, the Minkowski tensors are tensor-valued generalizations of the real-valued intrinsic volumes. They were the first tensor-valued valuations which were considered systematically. In 1997, McMullen introduced and further analyzed the Minkowski tensors (see [68]). This lead to a broad investigation in the newly established field of tensorial valuation and integration theory (part of which are the local tensorial valuations of Section 3.1 and the integral formulae in the following chapters). In this section, we recall their definitions and most important properties.

3.2.1. The Extrinsic Minkowski Tensors

As seen in equation (3.2), the intrinsic volumes are the total support measures. A tensor-valued generalization of this relation yields the *Minkowski tensors*. For $K \in \mathcal{K}^n$ and $j, r, s \in \mathbb{N}_0$ with $j < n$, they are defined by

$$\Phi_j^{r,s}(K) := c_{n,j}^{r,s,0} \int_{\Sigma^n} x^r u^s \, \Lambda_j(K, \mathrm{d}(x,u)),$$

and

$$\Phi_n^{r,0}(K) := c_{n,n}^{r,0,0} \int_K x^r \, \mathcal{H}^n(\mathrm{d}x).$$

As a matter of convenience, we again set $\Phi_j^{r,s} := 0$ for $j \notin \{0, \ldots, n\}$ or $r \notin \mathbb{N}_0$ or $s \notin \mathbb{N}_0$ or $j = n$ and $s \neq 0$.

Analogously to relations (3.2) and (3.3), we observe that the Minkowski tensors are the total tensorial curvature measures (due to their similar normalization). In other words, we have

$$\Phi_j^{r,s} = \phi_j^{r,s,0}(\cdot, \mathbb{R}^n),$$

for $j, r, s \in \mathbb{N}_0$ with $j \leq n$.

Obviously, the Minkowski tensors inherit some of the properties of the tensorial curvature measures (and therefore of the tensorial support measures). More precisely, for $j, r, s \in \mathbb{N}_0$ with $j \leq n$, we have that

(MT1) $\Phi_j^{r,s}$ is *isometry covariant*, that is, *translation covariant* of degree r in the sense that

$$\Phi_j^{r,s}(K+t) = \sum_{i=0}^{r} \Phi_j^{r-i,s}(K)\frac{t^i}{i!},$$

for all $K \in \mathcal{K}^n$, and $t \in \mathbb{R}^n$, and *rotation covariant* in the sense that

$$\Phi_j^{r,s}(\vartheta K) = \vartheta \Phi_j^{r,s}(K),$$

for all $K \in \mathcal{K}^n$, and $\vartheta \in \mathrm{O}(n)$;

(MT2) $\Phi_j^{r,s}$ is *continuous* with respect to the Hausdorff metric;

(MT3) $\Phi_j^{r,s}$ is *additive* (a *valuation*).

As pointed out before, McMullen initiated the investigation of the Minkowski tensors in 1997. In particular, he raised the question if the Minkowski tensors span the vector space of isometry covariant and continuous valuations on $\mathcal{K}^n$. This was positively answered by Alesker, who proved a tensor-valued characterization theorem for Minkowski tensors (see [2, 3]), as had been done by Hadwiger for intrinsic volumes.

Moreover, Alesker showed in [3, Theorem 4.1] that weakening the rotation covariance in (MT1) to *proper rotation covariance* (i.e. replacing the orthogonal group $\mathrm{O}(n)$ by its subgroup, the special orthogonal group $\mathrm{SO}(n)$) in the characterization does not yield more tensorial valuations than the Minkowski tensors in dimensions $n \geq 3$. But in dimension $n = 2$ there occur further $\mathrm{SO}(2)$-*covariant Minkowski tensors*, which are not $\mathrm{O}(2)$-covariant. For a convex body $K \in \mathcal{K}^2$, $r, s \in \mathbb{N}_0$ and $j \in \{0, 1\}$, these are defined by

$$\breve{\Phi}_j^{r,s}(K) := \omega_{2-j} \int_{\Sigma^2} x^r \overline{u} u^s \, \Lambda_j(K, \mathrm{d}(x,u)).$$

Obviously, these SO(2)-covariant Minkowski tensors are the total SO(2)-covariant tensorial curvature measures. The SO(3)-covariant tensorial curvature measures, in contrast, do not have a non-vanishing globalized counterpart. This follows either by Alesker's characterization theorem (see [3, Theorem 4.1]) or can be calculated in a direct way, as was done by Hug and Schneider in [49, Proposition 2].

The SO(2)-covariant Minkowski tensors satisfy several relations, which have been stated and proved by Hug and Schneider (see [49, Theorem 6]). As we apply some of them in the integral formulae in Chapter 7, we recall them here and refer to [49] for the proof. They read

$$\begin{aligned} \breve{\Phi}_1^{r,0} = 0, \qquad \breve{\Phi}_0^{0,s} &= 0, \qquad \text{for } r, s \in \mathbb{N}_0, \\ r\breve{\Phi}_1^{r-1,s} + s\breve{\Phi}_0^{r,s-1} &= 0, \qquad \text{for } r, s \in \mathbb{N}_0. \end{aligned} \tag{3.9}$$

3.2.2. The Intrinsic Minkowski Tensors

Apparently, there exist intrinsic versions of the Minkowski tensors. That is, for $j, k \in \mathbb{N}_0$ with $j < k \leq n$, and a convex body $\mathcal{K}^n \ni K \subset E \in \mathrm{A}(n,k)$ contained in an affine subspace of $\mathbb{R}^n$, the *intrinsic Minkowski tensors* are given by

$$\Phi_{j,E}^{r,s}(K) := c_{k,j}^{r,s,0} \int_{\Sigma_E^n} x^r u^s \, \Lambda_j(K, \mathrm{d}(x,u))$$

and

$$\Phi_{n,E}^{r,0}(K) := c_{k,k}^{r,0,0} \int_K x^r \, \mathcal{H}^k(\mathrm{d}x);$$

in all other cases, $\Phi_{j,E}^{r,s}$ is defined as the zero function. Of course, the intrinsic Minkowski tensors are the total intrinsic tensorial curvature measures.

Globalizing the results in Lemma 3.5 (which is basically the original statement by McMullen [68, Theorem 5.1]) yields a representation of the extrinsic Minkowski tensors in terms of in intrinsic Minkowski tensors.

3.2.3. McMullen's Lemma

As pointed out in Section 3.1, there exist more tensor-valued generalizations of the curvature measures for convex bodies than the "obviously appearing" tensorial curvature measures. The generalized tensorial curvature measures $\phi_j^{r,s,1}$ also admit a continuous extension to the convex bodies. In the upcoming (local) integral formulae, we will observe that these are not only a theoretical construct, but they appear naturally in the representation of kinematic and Crofton integrals of tensorial curvature measures for convex bodies. Since we aim to obtain integral formulae for Minkowski tensors by globalization of the corresponding local formulae, the valuations $\phi_j^{r,s,1}(\cdot, \mathbb{R}^n)$ likewise occur. Although the generalized tensorial curvature measures have no global counterpart, in the same way as the Minkowski tensors are essentially the total tensorial curvature measures, Alesker's characterization theorem shows that these are representable in terms of Minkowski tensors. The exact form of this representation has already been proved by McMullen (see [68, p. 269]). As it is an important tool in the upcoming proofs, we recall it in this section.

In order to represent the argument more clearly, we define (in the sense of McMullen) for a polytope $P \in \mathcal{P}^n$ with k-dimensional face $F \in \mathcal{F}_k(P)$, where $k \in \{0, \dots, n\}$,

$$\Upsilon_r(F) := \frac{1}{r!} \int_F x^r \, \mathcal{H}^k(\mathrm{d}x), \qquad r \in \mathbb{N}_0,$$

$$\Theta_s(P,F) := \frac{1}{s!} \frac{1}{\omega_{n-k+s}} \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-k-1}(\mathrm{d}u), \qquad s \in \mathbb{N}_0.$$

If $r, s \notin \mathbb{N}_0$, then we set $\Upsilon_r(F) := 0$ and $\Theta_s(P,F) := 0$.

We confirm the following relation

$$\Phi_k^{r,s}(P) = \sum_{F\in\mathcal{F}_k(P)} \Upsilon_r(F)\Theta_s(P,F),$$

using the just defined mappings. This is a tensor-valued generalization of a well-known representation for intrinsic volumes (see [83, eq. (4.23)]). Furthermore, we obtain

$$\phi_k^{r,s-2,1}(P,\mathbb{R}^n) = \tfrac{2\pi}{k} \sum_{F\in\mathcal{F}_k(P)} Q(F)\Upsilon_r(F)\Theta_{s-2}(P,F).$$

Next, we state McMullen's lemma.

Lemma 3.6 (McMullen)**.** *Let $P \in \mathcal{P}^n$ be a polytope, $r,s \in \mathbb{N}_0$ and $k \in \{0,\dots,n\}$. Then*

$$\begin{aligned}2\pi s\, \Phi_k^{r,s}(P) = &\sum_{F\in\mathcal{F}_k(P)} Q(F^\perp)\Upsilon_r(F)\Theta_{s-2}(P,F)\\ &+ \sum_{G\in\mathcal{F}_{k+1}(P)} Q(G)\Upsilon_{r-1}(G)\Theta_{s-1}(P,G).\end{aligned}$$

For $r = 0$, the second sum on the right side of the formula in Lemma 3.6 vanishes. If furthermore $s = 1$, the lemma simply states that

$$\Phi_k^{0,1} \equiv 0,$$

for $k \in \{0,\dots,n\}$. We note that Lemma 3.6 is essentially a global result which is derived by applying a version of the divergence theorem (for more details see [68, p. 269]).

Moreover, we can deduce a representation of the total generalized tensorial curvature measures $\phi_k^{r,s-2,1}(P,\mathbb{R}^n)$ in terms of Minkowski tensors from Lemma 3.6.

Lemma 3.7. *Let $K \in \mathcal{K}^n$ be a convex body, $r,s \in \mathbb{N}_0$ and $k \in \{0,\dots,n\}$. Then*

$$\phi_k^{r,s-2,1}(K,\mathbb{R}^n) = \frac{2\pi}{k}\sum_{p=0}^{r}\Big(Q\Phi_{k+p}^{r-p,s+p-2}(K) - 2\pi(s+p)\,\Phi_{k+p}^{r-p,s+p}(K)\Big). \tag{3.10}$$

In fact, the summation with respect to p on the right-hand side of the representation in Lemma 3.7 goes up to $\min\{r, n-k\}$.

Proof. We start the proof for a polytope $P \in \mathcal{P}^n$. Then we conclude from Lemma 3.6

$$\begin{aligned}\tfrac{k}{2\pi}\phi_k^{r,s-2,1}(P,\mathbb{R}^n) &= \sum_{F\in\mathcal{F}_k(P)} Q(F)\Upsilon_r(F)\Theta_{s-2}(P,F)\\ &= Q\Phi_k^{r,s-2}(P) - 2\pi s\,\Phi_k^{r,s}(P) + \sum_{G\in\mathcal{F}_{k+1}(P)} Q(G)\Upsilon_{r-1}(G)\Theta_{s-1}(P,G)\\ &= Q\Phi_k^{r,s-2}(P) - 2\pi s\,\Phi_k^{r,s}(P) + \phi_{k+1}^{r-1,s-1,1}(P,\mathbb{R}^n),\end{aligned}$$

where we have $\phi_{k+1}^{r-1,s-1,1}(\cdot,\mathbb{R}^n)\equiv 0$ if $r=0$ (or if $k=n$). Now a recursive application of Lemma 3.6 to the newly arising total generalized tensorial curvature measures of gradually decreasing degree of translation covariance yields

$$\begin{aligned}\tfrac{k}{2\pi}\phi_k^{r,s-2,1}(P,\mathbb{R}^n) &= Q\Phi_k^{r,s-2}(P)+Q\Phi_{k+1}^{r-1,s-1}(P)-2\pi s\,\Phi_k^{r,s}(P)-2\pi(s+1)\,\Phi_{k+1}^{r-1,s+1}(P)\\ &\quad+\sum_{G\in\mathcal{F}_{k+1}(P)}Q(G)\Upsilon_{r-2}(G)\Theta_s(P,G)\\ &=\sum_{p=0}^{r}\Big(Q\Phi_{k+p}^{r-p,s+p-2}(P)-2\pi(s+p)\,\Phi_{k+p}^{r-p,s+p}(P)\Big).\end{aligned}$$

Since the valuations on both sides of this relation are continuous, we obtain the assertion for a general convex body $K\in\mathcal{K}^n$ by approximation. □

Another interesting consequence from Lemma 3.6 is the following lemma, which has been proved by McMullen (see [68, Theorem 5.3]).

Lemma 3.8. *Let $k,r\in\mathbb{N}_0$. Then*

$$2\pi\sum_{s\in\mathbb{N}_0}s\,\Phi_{k-r+s}^{r-s,s}=Q\sum_{s\in\mathbb{N}_0}\Phi_{k-r+s}^{r-s,s-2}.$$

In fact, the summation with respect to s on the left-hand side of the relation in Lemma 3.8 starts at $\max\{r-k,0\}$ and goes up to $\min\{r,n-k+r\}$; the one on the right starts at $\max\{r-k,2\}$ and goes up to $\min\{r,n-k+r\}$. For the proof of Lemma 3.8, we sum the relation obtained in Lemma 3.6 over s, while keeping $r+s$ and $k+r$ constant. Then, we combine the metric tensors $Q(F^{\perp})+Q(F)=Q$ and obtain the assertion for polytopes. The rest follows by approximation.

Even though the Minkowski tensors span the vector space of isometry covariant and continuous valuations on $\mathcal{K}^n$, the relations in Lemma 3.8 show, that they do not form a basis thereof. However, Hug, Schneider and Schuster proved that these are essentially all linear dependences among the Minkowski tensors [52].

Chapter 4

Kinematic Formulae

In this chapter, we establish a complete set of explicit kinematic formulae for the generalized tensorial curvature measures $\phi_j^{r,s,l}$ of polytopes. In other words, for $P, P' \in \mathcal{P}^n$ and $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$, we express the integral mean value

$$\int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g) \tag{4.1}$$

in terms of the generalized tensorial curvature measures of P and P', evaluated at β and β', respectively (see Section 4.1). In fact, the precise result shows that only a selection of these measures is needed. Furthermore, for $l = 0, 1$, the tensorial measures $\phi_j^{r,s,l}$ can be continuously extended to mappings defined on $\mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n)$, and therefore in these two cases we also consider integral means of the form

$$\int_{\mathrm{G}_n} \phi_j^{r,s,l}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g), \tag{4.2}$$

for $K, K' \in \mathcal{K}^n$ and $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$. Although the formulae for convex bodies are a straightforward consequence of the ones for polytopes, it came as a surprise that the general formulae simplify for $l \in \{0, 1\}$ so that only tensorial curvature measures are involved which admit a continuous extension.

We note that, since the generalized tensorial curvature measures depend additively on the underlying convex body (resp. polytope), all integral formulae in this chapter remain true if the occurring convex bodies (resp. polytopes) are replaced by finite unions of convex bodies (resp. polytopes).

Our proof of these kinematic formulae proceeds more directly than the classical proof of the kinematic formula for curvature measures (see [83, Theorem 4.4.2]). In the latter one first expresses the kinematic integral in terms of curvature measures with some coefficients, which are then determined by application of the Crofton formula for curvature measures to specifically chosen convex bodies. Here we start as in the classical proof by first treating the translative part of the kinematic integral, but then find a direct way to compute its rotational part. As the scalar version of this proof gives a new direct proof for the scalar kinematic formula, we first provide this in Section 4.3 as an instructive introduction of the general tensorial proof in Section 4.4. The integral geometric machinery which we require for those proofs is introduced in Section 4.2 (or recalled from the literature in Appendix A).

Remark. The results in this chapter have already been submitted. To a great extent the present chapter contains direct quotes from the publication *Kinematic Formulae for Tensorial Curvature Measures*, a joint work with Daniel Hug, submitted in 2016 (see [53]).

4.1. The Results of Chapter 4

In this section, we state the formulae for the kinematic integrals for generalized tensorial curvature measures on polytopes (4.1) and on convex bodies (4.2).

4.1.1. Generalized Tensorial Curvature Measures on Polytopes

At first, we give the intersectional formula concerning the integrals (4.1).

Theorem 4.1. *For $P, P' \in \mathcal{P}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$, $j, l, r, s \in \mathbb{N}_0$ with $j \leq n$, and $l = 0$ if $j = 0$,*

$$\int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,k}^{s,l,i,m}\, Q^{m-i} \phi_k^{r,s-2m,l+i}(P, \beta) \phi_{n-k+j}(P', \beta'), \tag{4.3}$$

where

$$c_{n,j,k}^{s,l,i,m} := \frac{(-1)^i}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \times \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{k-j}{2} + m)}{\Gamma(\frac{k-j}{2})}.$$

Several remarkable facts concerning the coefficients $c_{n,j,k}^{s,l,i,m}$ should be observed. First, the ratio $(i + l - 2)!/(l - 2)!$ has to be interpreted in terms of Gamma functions and relation (2.2), for $l \in \{0, 1\}$, as further described in the two theorems in Section 4.1.2. Second, the coefficients are indeed independent of the tensorial parameter (and degree of translation covariance) r and depend only on l through the ratio $(i+l-2)!/(l-2)!$. Moreover, only tensors $\phi_k^{r,s-2m,p}(P,\beta)$ with $p \geq l$ show up on the right side of the kinematic formula. Using Legendre's duplication formula, we could shorten the given expressions for the coefficients $c_{n,j,k}^{s,l,i,m}$ even further. However, the present form has the advantage of exhibiting that the factors in the second line cancel each other if $s=0$ (and hence also $m=i=0$). Furthermore, in contrast to the classical kinematic formula, the coefficients are signed. We shall see below that for $l \in \{0,1\}$ all coefficients are non-negative.

In Theorem 4.1, we can simplify the coefficient $c_{n,j,k}^{s,l,i,m}$ for $k \in \{j,n\}$ and $j \leq n-1$ such that only one generalized tensorial curvature measure remains. From (2.2) it follows that

$$c_{n,j,j}^{s,l,i,m} = \mathbb{1}\{i=m=0\}.$$

Furthermore, since $\phi_n^{r,s,l}$ vanishes for $s \neq 0$ and the measures $Q^{\frac{s}{2}-i}\phi_n^{r,0,l+i}$, $i \in \{0,\dots,\frac{s}{2}\}$, can be combined, we can redefine

$$c_{n,j,n}^{s,l,i,m} := \mathbb{1}\{s \text{ even}, m=i=\tfrac{s}{2}\} \frac{1}{(2\pi)^s (\frac{s}{2})!} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})};$$

for more details see (4.6) and (4.9) in the proof of Theorem 4.1.

It should also be observed that the generalized tensorial curvature measures $\phi_{n-1}^{r,s-2m,l+i}$ can be expressed in terms of the tensorial curvature measures (multiplied with suitable powers of the metric tensor) $Q^{m'}\phi_{n-1}^{r,s',0}$, where $m', s' \in \mathbb{N}_0$ and $2m'+s' = s+2l$. We do not pursue this here, since the resulting coefficients do not simplify nicely; see, however, Chapter 5, where this is done for some specific Crofton formulae.

Theorem 4.1 states an equality for measures, hence the case $r=0$ of the theorem immediately implies the general case. In fact, algebraic induction and the inversion invariance of μ yield the following extension of Theorem 4.1.

Corollary 4.2. *Let $P, P' \in \mathcal{P}^n$, $j,l,r,s \in \mathbb{N}_0$ with $j \leq n$, and $l=0$ if $j=0$. Let f,h be tensor-valued continuous functions on $\mathbb{R}^n$. Then*

$$\begin{aligned}
&\int_{\mathrm{G}_n} \int_{\mathbb{R}^n} f(x)h(gx)\,\phi_j^{r,s,l}(P \cap g^{-1}P', \mathrm{d}x)\,\mu(\mathrm{d}g) \\
&\quad = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,k}^{s,l,i,m}\, Q^{m-i} \int_{\mathbb{R}^n} f(x)\,\phi_k^{r,s-2m,l+i}(P,\mathrm{d}x) \int_{\mathbb{R}^n} h(y)\,\phi_{n-k+j}(P',\mathrm{d}y).
\end{aligned}$$

In particular, we could choose $h(y) = y^{\bar{r}}$, $y \in \mathbb{R}^n$, for $\bar{r} \in \mathbb{N}_0$. We state and prove Theorem 4.1 in the present form, since this does not change the argument and globalization yields corresponding results for general Minkowski tensors (see Chapter 6).

4.1.2. (Generalized) Tensorial Curvature Measures on Convex Bodies

The cases $l = 0, 1$ in Theorem 4.1 are of special interest, since we can formulate the kinematic formulae for general convex bodies in these cases.

Theorem 4.3. *For $K, K' \in \mathcal{K}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ and $j, r, s \in \mathbb{N}_0$ with $1 \leq j \leq n$,*

$$\int_{\mathrm{G}_n} \phi_j^{r,s,1}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,k}^{s,1,0,m}\, Q^m \phi_k^{r,s-2m,1}(K, \beta) \phi_{n-k+j}(K', \beta'),$$

where

$$c_{n,j,k}^{s,1,0,m} = \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})}.$$

Proof. We apply (2.2) to obtain

$$\frac{(i-1)!}{(-1)!} = \frac{\Gamma(i)}{\Gamma(0)} = \mathbb{1}\{i = 0\}.$$

Then, Theorem 4.1 yields the assertion in the polytopal case. For a convex body, we conclude the formula by approximating it by polytopes, since the valuations $\phi_k^{r,s-2m,1}$ have weakly continuous extensions to $\mathcal{K}^n$ (and the same is true for the curvature measures). □

Theorem 4.4. *For $K, K' \in \mathcal{K}^n$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ and $j, r, s \in \mathbb{N}_0$ with $j \leq n$,*

$$\int_{\mathrm{G}_n} \phi_j^{r,s,0}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{1} c_{n,j,k}^{s,0,i,m}\, Q^{m-i} \phi_k^{r,s-2m,i}(K, \beta) \phi_{n-k+j}(K', \beta'),$$

where

$$c_{n,j,k}^{s,0,i,m} = \frac{1}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})}.$$

Proof. We apply (2.2) to obtain

$$\frac{(i-2)!}{(-2)!} = \frac{\Gamma(i-1)}{\Gamma(-1)} = (-1)^i \frac{1}{\Gamma(2-i)} = \mathbb{1}\{i = 0\} - \mathbb{1}\{i = 1\}.$$

Then, Theorem 4.1 yields the assertion in the polytopal case. For a convex body, we conclude the formula by approximating it by polytopes, since for $i \in \{0, 1\}$ the valuations $\phi_k^{r,s-2m,i}$ have weakly continuous extensions to $\mathcal{K}^n$. Finally, we note that $c_{n,j,k}^{s,0,1,0} = 0$. □

It is crucial that the right sides of the formulae in Theorem 4.3 and Theorem 4.4 only involve the tensorial curvature measures $\phi_k^{r,s,0}$ and $\phi_k^{r,s,1}$, which are the ones with weakly continuous extensions to $\mathcal{K}^n$, and not $\phi_k^{r,s,i}$ with $i > 1$.

4.2. Some Auxiliary Results

Before we start with the proof of the kinematic formulae, we establish several auxiliary integral geometric results in this section. Some of these require facts from the literature which we, for the sake of completeness, provide in Appendix A. As a rule, these results hold for $n \geq 1$. If not stated otherwise, the case $n = 1$ (or even $n = 0$) can be checked directly.

From Lemma A.4 we deduce the first integral formula, which will be applied in the proofs of Lemma 4.6 and Proposition 4.9.

Lemma 4.5. *Let $i, j, k \in \mathbb{N}_0$ with $0 \leq k \leq n$. Then*

$$\int_{\mathrm{G}(n,k)} Q(L)^i Q(L^\perp)^j \, \nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+i)\Gamma(\frac{n-k}{2}+j)}{\Gamma(\frac{n}{2}+i+j)\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \, Q^{i+j}.$$

Proof. The cases where $k \in \{0, n\}$ can be checked easily by distinguishing whether $i, j = 0$ or not. Hence we can assume that $1 \leq k \leq n-1$. Let I denote the integral we are interested in. By expansion of $Q(L^\perp)^j = (Q - Q(L))^j$ and Lemma A.4 we obtain

$$\begin{aligned} I &= \sum_{l=0}^{j} (-1)^l \binom{j}{l} Q^{j-l} \int_{\mathrm{G}(n,k)} Q(L)^{i+l} \, \nu_k(\mathrm{d}L) \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{\Gamma(\frac{k}{2}+i+l)}{\Gamma(\frac{n}{2}+i+l)} \, Q^{i+j}. \end{aligned}$$

Then relation (B.1′) yields the assertion. □

The next lemma will be used in the proof of Theorem 4.1, in one of the boundary cases which are approached in Section 4.4.2.

Lemma 4.6. *Let $j, l, s \in \mathbb{N}_0$ with $j < n$, $L \in \mathrm{G}(n, j)$ and $u \in L^\perp \cap \mathbb{S}^{n-1}$. Then*

$$\int_{\mathrm{SO}(n)} Q(\vartheta L)^l (\vartheta u)^s \, \nu(\mathrm{d}\vartheta) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{j}{2}+l)\Gamma(\frac{s+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n+s}{2}+l)\Gamma(\frac{j}{2})} \, Q^{l+\frac{s}{2}},$$

if s is even. The same relation holds if the integration is extended over $\mathrm{O}(n)$*. If s is odd and $n \geq 2$ (or $n = 1$ and the integration is extended over* $\mathrm{O}(1)$*), then the integral vanishes.*

Proof. The case $n = 1$, $j = 0$ is checked directly by distinguishing $l = 0$ or $l \neq 0$. Hence let $n \geq 2$. Let I denote the integral we are interested in. Due to symmetry, $I = 0$ if s is odd. Therefore, let s be even. Let $\rho \in \mathrm{SO}(L^\perp)$. Then, by the right invariance of ν, it follows that

$$\begin{aligned} I &= \int_{\mathrm{SO}(n)} Q(\vartheta\rho L)^l (\vartheta\rho u)^s \,\nu(\mathrm{d}\vartheta) \\ &= \int_{\mathrm{SO}(n)} Q(\vartheta L)^l (\vartheta\rho u)^s \,\nu(\mathrm{d}\vartheta). \end{aligned}$$

Now we integrate over all such rotations $\rho \in \mathrm{SO}(L^\perp)$ and then apply Fubini's theorem in order to obtain

$$\begin{aligned} I &= \int_{\mathrm{SO}(L^\perp)} \int_{\mathrm{SO}(n)} Q(\vartheta L)^l (\vartheta\rho u)^s \,\nu(\mathrm{d}\vartheta)\, \nu^{L^\perp}(\mathrm{d}\rho) \\ &= \int_{\mathrm{SO}(n)} Q(\vartheta L)^l \,\vartheta \int_{\mathrm{SO}(L^\perp)} (\rho u)^s \,\nu^{L^\perp}(\mathrm{d}\rho)\, \nu(\mathrm{d}\vartheta). \end{aligned}$$

Lemma A.3, applied in $L^\perp$ with $\dim(L^\perp) \geq 1$, yields

$$\int_{\mathrm{SO}(L^\perp)} (\rho u)^s \,\nu^{L^\perp}(\mathrm{d}\rho) = \frac{1}{\omega_{n-j}} \int_{\mathbb{S}^{n-1}\cap L^\perp} v^s \,\mathcal{H}^{n-j-1}(\mathrm{d}v) = 2\frac{\omega_{n-j+s}}{\omega_{s+1}\omega_{n-j}}\, Q(L^\perp)^{\frac{s}{2}},$$

and hence we get

$$\begin{aligned} I &= 2\frac{\omega_{n-j+s}}{\omega_{s+1}\omega_{n-j}} \int_{\mathrm{SO}(n)} Q(\vartheta L)^l Q(\vartheta L^\perp)^{\frac{s}{2}} \,\nu(\mathrm{d}\vartheta) \\ &= 2\frac{\omega_{n-j+s}}{\omega_{s+1}\omega_{n-j}} \int_{\mathrm{G}(n,j)} Q(U)^l Q(U^\perp)^{\frac{s}{2}} \,\nu_j(\mathrm{d}U). \end{aligned}$$

From Lemma 4.5 we conclude that

$$I = 2\frac{\omega_{n-j+s}}{\omega_{s+1}\omega_{n-j}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{j}{2}+l)\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n+s}{2}+l)\Gamma(\frac{j}{2})\Gamma(\frac{n-j}{2})}\, Q^{l+\frac{s}{2}},$$

and thus we obtain the assertion. □

The following lemma is one of the tools which are required to treat the rotational part of the kinematic integral which is discussed in Section 4.4.3.

Lemma 4.7. *Let $u, v \in \mathbb{S}^{n-1}$, $i, t \in \mathbb{N}_0$ and $n \geq 1$. Then*

$$\int_{\mathrm{SO}(n)} (\rho v)^i \langle u, \rho v\rangle^t \,\nu(\mathrm{d}\rho) = \frac{\Gamma(\frac{n}{2})\Gamma(t+1)}{2^t\sqrt{\pi}\,\Gamma(\frac{n+i+t}{2})} \sum_{x=(\frac{i-t}{2})^+}^{\lfloor \frac{i}{2}\rfloor} \binom{i}{2x} \frac{\Gamma(x+\frac{1}{2})}{\Gamma(\frac{t-i}{2}+x+1)} u^{i-2x} Q^x,$$

if $i+t$ is even. The same relation holds if the integration is extended over $\mathrm{O}(n)$. If $i+t$ is odd and $n \geq 2$ (or $n = 1$ and the integration is extended over $\mathrm{O}(1)$), then the integral on the left side vanishes.

Proof. First, we assume that $n \geq 2$. Let I denote the integral we are interested in. By symmetry, it follows $I = 0$ if $i+t$ is odd. Thus, in the following we assume that $i+t$ is even. Applying the transformation

$$f : [-1, 1] \times (\mathbb{S}^{n-1} \cap u^{\perp}) \to \mathbb{S}^{n-1},\ (z, w) \mapsto zu + \sqrt{1 - z^2}w,$$

with Jacobian $\mathcal{J}f(z, w) = \sqrt{1 - z^2}^{n-3}$ to the integral I, we get

$$\begin{aligned} I &= \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} v^i \langle u, v\rangle^t \, \mathcal{H}^{n-1}(\mathrm{d}v) \\ &= \frac{1}{\omega_n} \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap u^{\perp}} \left(1 - z^2\right)^{\frac{n-3}{2}} \left(zu + \sqrt{1 - z^2}w\right)^i \left\langle u, zu + \sqrt{1 - z^2}w\right\rangle^t \mathcal{H}^{n-2}(\mathrm{d}w) \, \mathrm{d}z. \end{aligned}$$

Binomial expansion of $(zu + \sqrt{1 - z^2}w)^i$ yields

$$I = \frac{1}{\omega_n} \sum_{m=0}^{i} \binom{i}{m} u^{i-m} \underbrace{\int_{-1}^{1} z^{t+i-m} \left(1 - z^2\right)^{\frac{n+m-3}{2}} \mathrm{d}z}_{=\mathbb{1}\{m \text{ even}\} B(\frac{t+i-m+1}{2}, \frac{n+m-1}{2})} \underbrace{\int_{\mathbb{S}^{n-1}\cap u^{\perp}} w^m \, \mathcal{H}^{n-2}(\mathrm{d}w)}_{=:I'},$$

where $B(\cdot, \cdot)$ denotes the Beta function. However, we rewrite it by applying its connection to the Gamma function (see equation (2.13) in [6]). From Lemma A.3, we obtain

$$I' = \mathbb{1}\{m \text{ even}\} 2 \frac{\omega_{n+m-1}}{\omega_{m+1}} Q(u^{\perp})^{\frac{m}{2}},$$

and thus

$$\begin{aligned} I &= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i}{2m} \frac{\Gamma(\frac{t+i+1}{2} - m)\Gamma(\frac{n-1}{2} + m)}{\Gamma(\frac{n+i+t}{2})} \frac{2\omega_{n+2m-1}}{\omega_n \omega_{2m+1}} u^{i-2m} Q(u^{\perp})^m \\ &= \frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n+i+t}{2})} \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i}{2m} \Gamma(m + \tfrac{1}{2})\Gamma(\tfrac{t+i+1}{2} - m) u^{i-2m} Q(u^{\perp})^m. \end{aligned}$$

Since $Q(u^{\perp}) = Q - u^2$, binomial expansion yields

$$Q(u^{\perp})^m = \sum_{x=0}^{m} (-1)^{m+x} \binom{m}{x} u^{2m-2x} Q^x.$$

The fourfold application of Legendre's duplication formula to the occurring binomial coefficients and Gamma functions gives

$$\begin{aligned} \binom{i}{2m}\binom{m}{x}\Gamma(m + \tfrac{1}{2}) &= \binom{i}{2x}\Gamma(x + \tfrac{1}{2}) \frac{1}{(m - x)!} \frac{\Gamma(\frac{i+1}{2} - x)\Gamma(\frac{i}{2} - x + 1)}{\Gamma(\frac{i+1}{2} - m)\Gamma(\frac{i}{2} - m + 1)} \\ &= \binom{i}{2x}\Gamma(x + \tfrac{1}{2}) \binom{\lfloor \frac{i}{2} \rfloor - x}{m - x} \frac{\Gamma(\lfloor \frac{i+1}{2} \rfloor - x + \frac{1}{2})}{\Gamma(\lfloor \frac{i+1}{2} \rfloor - m + \frac{1}{2})}, \end{aligned}$$

and thus we obtain by a change of the order of summation

$$I = \frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n+i+t}{2})} \sum_{x=0}^{\lfloor\frac{i}{2}\rfloor} \binom{i}{2x} \Gamma(x+\tfrac{1}{2})\Gamma(\lfloor\tfrac{i+1}{2}\rfloor - x + \tfrac{1}{2}) u^{i-2x} Q^x \\ \times \sum_{m=x}^{\lfloor\frac{i}{2}\rfloor} (-1)^{m+x} \binom{\lfloor\frac{i}{2}\rfloor - x}{m-x} \frac{\Gamma(\frac{t+i+1}{2} - m)}{\Gamma(\lfloor\frac{i+1}{2}\rfloor - m + \frac{1}{2})}.$$

We denote the sum with respect to m by S_1. An index shift by x, applied to S_1, yields

$$S_1 = \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor - x} (-1)^m \binom{\lfloor\frac{i}{2}\rfloor - x}{m} \frac{\Gamma(\frac{t+i+1}{2} - x - m)}{\Gamma(\lfloor\frac{i+1}{2}\rfloor - x - m + \frac{1}{2})}.$$

Now we conclude from relation (B.1′) that

$$\begin{aligned} S_1 &= (-1)^{\lfloor\frac{i}{2}\rfloor - x} \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor - x} (-1)^m \binom{\lfloor\frac{i}{2}\rfloor - x}{m} \frac{\Gamma(\frac{t+i+1}{2} - \lfloor\frac{i}{2}\rfloor + m)}{\Gamma(\lfloor\frac{i+1}{2}\rfloor - \lfloor\frac{i}{2}\rfloor + m + \frac{1}{2})} \\ &= (-1)^{\lfloor\frac{i}{2}\rfloor - x} \frac{\Gamma(\frac{t+i+1}{2} - \lfloor\frac{i}{2}\rfloor)\Gamma(\overbrace{\lfloor\frac{i+1}{2}\rfloor + \lfloor\frac{i}{2}\rfloor}^{=i} - \frac{t+i+1}{2} - x + \frac{1}{2})}{\Gamma(\lfloor\frac{i+1}{2}\rfloor - x + \frac{1}{2})\Gamma(\lfloor\frac{i+1}{2}\rfloor - \frac{t+i+1}{2} + \frac{1}{2})} \\ &= \overbrace{(-1)^{i + \lfloor\frac{i+1}{2}\rfloor + \lfloor\frac{i}{2}\rfloor}}^{=(-1)^{2i}=1} \frac{\overbrace{\Gamma(\frac{t+i+1}{2} - \lfloor\frac{i}{2}\rfloor)\Gamma(\frac{t+i+1}{2} - \lfloor\frac{i+1}{2}\rfloor + \frac{1}{2})}^{=\Gamma(\frac{t+1}{2})\Gamma(\frac{t}{2}+1)}}{\Gamma(\lfloor\frac{i+1}{2}\rfloor - x + \frac{1}{2})\Gamma(\frac{t-i}{2} + x + 1)}, \end{aligned}$$

where we used (2.2) with $c = \frac{t+i+1}{2} - \lfloor\frac{i+1}{2}\rfloor - \frac{1}{2} \in \mathbb{N}_0$ and $m = i - \lfloor\frac{i+1}{2}\rfloor - x \in \mathbb{N}_0$. We notice that $S_1 = 0$ if $x < \frac{i-t}{2}$. Thus we obtain the assertion by another application of Legendre's duplication formula.

It remains to confirm the assertion if $n = 1$ and $i + t$ is even (all other assertions are easy to check). In this case, $u = \pm v$ and therefore the left-hand side of the asserted equation equals $(\pm 1)^t v^i$. Using first Legendre's duplication formula repeatedly, then relation (B.1), and finally again Legendre's duplication formula, we see that the right-hand side equals

$$\begin{aligned} &\frac{\Gamma(t+1)}{2^t\Gamma(\frac{1+i+t}{2})} \sum_{x=0}^{\lfloor\frac{i}{2}\rfloor} \binom{i}{2x} \frac{\Gamma(x+\frac{1}{2})}{\Gamma(\frac{t-i}{2} + x + 1)} (\pm 1)^i v^i \\ &= \frac{\sqrt{\pi}\Gamma(t+1)}{2^t\Gamma(\frac{1+i+t}{2})} \Gamma(\lfloor\tfrac{i+1}{2}\rfloor + \tfrac{1}{2}) \sum_{x=0}^{\lfloor\frac{i}{2}\rfloor} \binom{\lfloor\frac{i}{2}\rfloor}{x} \frac{1}{\Gamma(\frac{t-i}{2} + 1 + x)\Gamma(\lfloor\frac{i+1}{2}\rfloor + \frac{1}{2} - x)} (\pm 1)^i v^i \\ &= \frac{\sqrt{\pi}\Gamma(t+1)}{2^t\Gamma(\frac{1+i+t}{2})} \frac{\Gamma(\lfloor\frac{i+1}{2}\rfloor + \frac{1}{2} + \frac{t-i}{2} + 1 + \lfloor\frac{i}{2}\rfloor - 1)}{\Gamma(\frac{t-i}{2} + 1 + \lfloor\frac{i}{2}\rfloor)\Gamma(\frac{t-i}{2} + \lfloor\frac{i+1}{2}\rfloor + \frac{1}{2})} (\pm 1)^i v^i \\ &= (\pm 1)^i v^i, \end{aligned}$$

which confirms the assertion. □

The next lemma is an integral transformation formula which we require in the proof of Proposition 4.9.

Lemma 4.8. *Let $j, k, n \in \mathbb{N}_0$ with $j + k \leq n$, $n \geq 1$, and $U \in \mathrm{G}(n, j)$. Then, for any integrable function $f : \mathrm{G}(U, j+k) \to \mathbb{R}$,*

$$\int_{\mathrm{G}(U^\perp,k)} f(U+L)\, \nu_k^{U^\perp}(\mathrm{d}L) = \int_{\mathrm{G}(U,j+k)} f(L)\, \nu_{j+k}^{U}(\mathrm{d}L).$$

Proof. We consider the map $H : \mathrm{G}(U^\perp, k) \to \mathrm{G}(U, j+k)$, $L \mapsto U + L$, which is in fact well-defined, since $\dim(U \cap L) = 0$ and hence $\dim(L + U) = j + k$ for all $L \in \mathrm{G}(U^\perp, k)$. It is sufficient to show that $H(\nu_k^{U^\perp}) = \nu_{j+k}^U$, where $H(\nu_k^{U^\perp})$ denotes the image measure of $\nu_k^{U^\perp}$ under H.

Since $H(\nu_k^{U^\perp})$ and ν_{j+k}^U are probability measures, and ν_{j+k}^U is $\mathrm{SO}(U^\perp)$ invariant by definition, we only need to show that $H(\nu_k^{U^\perp})$ is $\mathrm{SO}(U^\perp)$ invariant. To verify this, we choose $A \in \mathcal{B}(\mathrm{G}(U, j+k))$ and $\vartheta \in \mathrm{SO}(U^\perp)$. Then we obtain

$$\begin{aligned} H(\nu_k^{U^\perp})(\vartheta A) &= \nu_k^{U^\perp}\Big(\{L \in \mathrm{G}(U^\perp, k) : U + L \in \vartheta A\}\Big) \\ &= \nu_k^{U^\perp}(\{L \in \mathrm{G}(U^\perp, k) : \vartheta^{-1}U + \vartheta^{-1}L \in A\}). \end{aligned}$$

The $\mathrm{SO}(U^\perp)$ invariance of $\nu_k^{U^\perp}$ yields

$$H(\nu_k^{U^\perp})(\vartheta A) = \nu_k^{U^\perp}(\{L \in \mathrm{G}(U^\perp, k) : U + L \in A\}) = H(\nu_k^{U^\perp})(A),$$

which completes the argument. □

The following proposition, which is a generalization of Lemma A.5 in the case $a = 2$, is one of the most important ingredients in the calculation of the rotational part of the kinematic integral which is stated in Section 4.4.3. Its proof uses several of the lemmas provided above.

Proposition 4.9. *Let $F \in \mathrm{G}(n, k)$ with $0 \leq j \leq k \leq n$ and $m, l \in \mathbb{N}_0$. Then*

$$\begin{aligned} &\int_{\mathrm{G}(n,n-k+j)} [F, L]^2 Q(L)^m Q(F \cap L)^l \, \nu_{n-k+j}(\mathrm{d}L) \\ &\quad = \frac{(n-k+j)!k!}{n!j!} \frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{j}{2}+l)\Gamma(\frac{k}{2})}{\Gamma(\frac{n}{2}+m+1)\Gamma(\frac{j}{2})\Gamma(\frac{k-j}{2})\Gamma(\frac{n-k+j}{2}+1)} \\ &\qquad \times \sum_{i=0}^{m} \binom{m}{i} \frac{(l+i-2)!}{(l-2)!} \frac{\Gamma(\frac{k-j}{2}+i)\Gamma(\frac{n-k+j}{2}+m-i+1)}{\Gamma(\frac{k}{2}+l+i)} Q^{m-i} Q(F)^{l+i}. \end{aligned}$$

For $l \leq 1$, we interpret the factor $\frac{(l+i-2)!}{(l-2)!}$ in Proposition 4.9 as stated in (2.2) and discussed in Section 4.1. Moreover, the factor $\Gamma(l + j/2)/\Gamma(j/2)$ vanishes if $j = 0$, $l \neq 0$ and cancels, that is, equals one if $j = l = 0$.

Proof. Let I denote the integral in which we are interested. If $j = k$, all summands on the right side of the asserted equation are zero except for $i = 0$. Thus it is easy to confirm the assertion. Now assume that $0 \le j < k \le n$, hence $n \ge 1$. If $j = l = 0$, then the assertion follows as a special case of Lemma A.5. If $j = 0, l \ne 0$ then both sides of the asserted equation vanish.

In the following, we consider the remaining cases where $0 < j < k$. Then Lemma A.1 yields

$$I = d_{n,j,k} \int_{\mathrm{G}(F,j)} \int_{\mathrm{G}(U,n-k+j)} [F,L]^{j+2} Q(L)^m Q(F \cap L)^l \, \nu^U_{n-k+j}(\mathrm{d}L) \, \nu^F_j(\mathrm{d}U).$$

For fixed $U \in \mathrm{G}(F,j)$, we have $\dim(F \cap L) = j = \dim U$ for ν^U_{n-k+j}-almost all $L \in \mathrm{G}(U, n-k+j)$ and $U \subset F \cap L$, hence $U = F \cap L$, and therefore

$$I = d_{n,j,k} \int_{\mathrm{G}(F,j)} Q(U)^l \int_{\mathrm{G}(U,n-k+j)} [F,L]^{j+2} Q(L)^m \, \nu^U_{n-k+j}(\mathrm{d}L) \, \nu^F_j(\mathrm{d}U).$$

An application of Lemma 4.8 shows that

$$I = d_{n,j,k} \int_{\mathrm{G}(F,j)} Q(U)^l \int_{\mathrm{G}(U^\perp,n-k)} [F, U+L]^{j+2} Q(U+L)^m \, \nu^{U^\perp}_{n-k}(\mathrm{d}L) \, \nu^F_j(\mathrm{d}U).$$

As $U \subset F$ and $L \subset U^\perp$, we have

$$[F, U+L] = [F \cap U^\perp, L]^{(U^\perp)}$$

and

$$Q(U+L)^m = (Q(U) + Q(L))^m = \sum_{\alpha=0}^{m} \binom{m}{\alpha} Q(L)^\alpha Q(U)^{m-\alpha}.$$

Thus we obtain

$$\begin{aligned} I = d_{n,j,k} \sum_{\alpha=0}^{m} \binom{m}{\alpha} \int_{\mathrm{G}(F,j)} Q(U)^{l+m-\alpha} \\ \times \int_{\mathrm{G}(U^\perp,n-k)} \left([F \cap U^\perp, L]^{(U^\perp)} \right)^{j+2} Q(L)^\alpha \, \nu^{U^\perp}_{n-k}(\mathrm{d}L) \, \nu^F_j(\mathrm{d}U). \end{aligned}$$

We observe that $\dim(U^\perp) = n - j > n - k \ge 0$, hence $\dim(U^\perp) \ge 1$. Therefore Lemma A.5 can be used to see that the integral with respect to L can be expressed as

$$\begin{aligned} e_{n-j,n-k,k-j,j+2} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)\Gamma(\frac{n}{2}+1+\alpha)} \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \\ \times \frac{\Gamma(\frac{n-k+j}{2}+1+\alpha-\beta)\Gamma(\frac{k-j}{2}+\beta)\Gamma(\frac{j}{2}+2)\Gamma(\frac{k-j}{2})}{\Gamma(\frac{k-j}{2})\Gamma(\frac{j}{2}+2-\beta)\Gamma(\frac{k-j}{2}+\beta)} Q(U^\perp)^{\alpha-\beta} Q(F \cap U^\perp)^\beta, \end{aligned}$$

and thus

$$\begin{aligned} I = d_{n,j,k}\, e_{n-j,n-k,k-j,j+2} \, \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-k+j}{2}+1)} \\ \times \sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}(-1)^{\beta}\binom{m}{\alpha}\binom{\alpha}{\beta}\frac{\Gamma(\frac{n-k+j}{2}+1+\alpha-\beta)\Gamma(\frac{j}{2}+2)}{\Gamma(\frac{n}{2}+1+\alpha)\Gamma(\frac{j}{2}+2-\beta)} \\ \times \int_{\mathrm{G}(F,j)} Q(U)^{l+m-\alpha}Q(U^{\perp})^{\alpha-\beta}Q(F\cap U^{\perp})^{\beta}\,\nu_j^F(\mathrm{d}U). \end{aligned}$$

Observing cancellations and using Legendre's duplication formula several times, we get

$$d_{n,j,k}\, e_{n-j,n-k,k-j,j+2} = \frac{(n-k+j)!k!}{n!j!}.$$

Expanding $Q(U^{\perp})^{\alpha-\beta} = (Q-Q(U))^{\alpha-\beta}$, we obtain

$$\begin{aligned} I = \frac{(n-k+j)!k!}{n!j!}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{j}{2}+2)}{\Gamma(\frac{n-k+j}{2}+1)}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{i=0}^{\alpha-\beta}(-1)^{\alpha+i}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{\alpha-\beta}{i} \\ \times \frac{\Gamma(\frac{n-k+j}{2}+1+\alpha-\beta)}{\Gamma(\frac{n}{2}+1+\alpha)\Gamma(\frac{j}{2}+2-\beta)}Q^i\int_{\mathrm{G}(F,j)} Q(U)^{l+m-\beta-i}Q(F\cap U^{\perp})^{\beta}\,\nu_j^F(\mathrm{d}U). \end{aligned}$$

Lemma 4.5, applied in F, yields

$$\begin{aligned} I = \frac{(n-k+j)!k!}{n!j!}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2})\Gamma(\frac{j}{2}+2)}{\Gamma(\frac{n-k+j}{2}+1)\Gamma(\frac{j}{2})\Gamma(\frac{k-j}{2})}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{\alpha}\sum_{i=0}^{\alpha-\beta}(-1)^{\alpha+i}\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{\alpha-\beta}{i} \\ \times \frac{\Gamma(\frac{n-k+j}{2}+1+\alpha-\beta)}{\Gamma(\frac{n}{2}+1+\alpha)}\frac{\Gamma(\frac{j}{2}+l+m-\beta-i)\Gamma(\frac{k-j}{2}+\beta)}{\Gamma(\frac{k}{2}+l+m-i)\Gamma(\frac{j}{2}+2-\beta)}Q^iQ(F)^{l+m-i}. \end{aligned}$$

Using the relation

$$\binom{m}{\alpha}\binom{\alpha}{\beta}\binom{\alpha-\beta}{i} = \binom{m}{i}\binom{m-i}{\beta}\binom{m-i-\beta}{\alpha-i-\beta}$$

and by a change of the order of summation, we conclude that

$$\begin{aligned} I = \frac{(n-k+j)!k!}{n!j!}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2})\Gamma(\frac{j}{2}+2)}{\Gamma(\frac{n-k+j}{2}+1)\Gamma(\frac{j}{2})\Gamma(\frac{k-j}{2})}\sum_{i=0}^{m}\binom{m}{i}Q^iQ(F)^{l+m-i} \\ \times \frac{1}{\Gamma(\frac{k}{2}+l+m-i)}\sum_{\beta=0}^{m-i}\binom{m-i}{\beta}\frac{\Gamma(\frac{j}{2}+l+m-\beta-i)\Gamma(\frac{k-j}{2}+\beta)}{\Gamma(\frac{j}{2}+2-\beta)} \\ \times \sum_{\alpha=i+\beta}^{m}(-1)^{\alpha+i}\binom{m-i-\beta}{\alpha-i-\beta}\frac{\Gamma(\frac{n-k+j}{2}+1+\alpha-\beta)}{\Gamma(\frac{n}{2}+1+\alpha)}. \end{aligned}$$

For the sum with respect to α, we obtain from relation (B.1′)

$$\sum_{\alpha=0}^{m-i-\beta}(-1)^{\alpha+\beta}\binom{m-i-\beta}{\alpha}\frac{\Gamma(\frac{n-k+j}{2}+i+1+\alpha)}{\Gamma(\frac{n}{2}+i+\beta+1+\alpha)}$$
$$=(-1)^{\beta}\frac{\Gamma(\frac{n-k+j}{2}+i+1)\Gamma(\frac{k-j}{2}+m-i)}{\Gamma(\frac{n}{2}+m+1)\Gamma(\frac{k-j}{2}+\beta)}.$$

Next, for the resulting sum with respect to β, we obtain again from relation (B.1′)

$$\sum_{\beta=0}^{m-i}(-1)^{m+i+\beta}\binom{m-i}{\beta}\frac{\Gamma(\frac{j}{2}+l+\beta)}{\Gamma(\frac{j}{2}+2-m+i+\beta)}=(-1)^{m+i}\frac{\Gamma(\frac{j}{2}+l)\Gamma(2-l)}{\Gamma(\frac{j}{2}+2)\Gamma(2-l-m+i)}$$
$$=\frac{\Gamma(\frac{j}{2}+l)\Gamma(l+m-i-1)}{\Gamma(\frac{j}{2}+2)\Gamma(l-1)},$$

where we used $j>0$ in the first step and (2.2) in the second (and distinguished the cases $l=0$, $l=1$, $l\geq 2$). Thus we get

$$I=\frac{(n-k+j)!k!}{n!j!}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{j}{2}+l)\Gamma(\frac{k}{2})}{\Gamma(\frac{n}{2}+m+1)\Gamma(\frac{j}{2})\Gamma(\frac{k-j}{2})\Gamma(\frac{n-k+j}{2}+1)}$$
$$\times\sum_{i=0}^{m}\binom{m}{i}\frac{(l+i-2)!}{(l-2)!}\frac{\Gamma(\frac{k-j}{2}+i)\Gamma(\frac{n-k+j}{2}+m-i+1)}{\Gamma(\frac{k}{2}+l+i)}Q^{m-i}Q(F)^{l+i},$$

where we reversed the order of summation. □

4.3. A Direct Proof of the Classical Kinematic Formula

In this section, we provide a proof of the classical kinematic formula for curvature measures. It is a scalar-valued version of the proof of Theorem 4.1, which is given in Section 4.4. This course of action is chosen for two reasons. On the one hand, it is a new and the first direct proof of this famous classical result, and therefore, an interesting result by itself. On the other hand, it is a special case of the general proof. Hence, providing it first, already gives an impression of the integral geometric ideas which are used again in a more involved setting in the following section.

First, we recall the classical kinematic formula for support measures (see [83, Theorem 4.4.2]) in the following theorem.

Theorem 4.10. *For $K,K'\in\mathcal{K}^n$, $\beta,\beta'\in\mathcal{B}(\mathbb{R}^n)$ and $j\in\{0,\dots,n\}$,*

$$\int_{G_n}\phi_j(K\cap gK',\beta\cap g\beta')\,\mu(\mathrm{d}g)=\sum_{k=j}^{n}\alpha_{njk}\,\phi_k(K,\beta)\,\phi_{n-k+j}(K',\beta'),$$

where the coefficient α_{njk} is defined as in equation (1.3).

The classical proof of Theorem 4.10 starts by expressing the kinematic integral as a linear combination of curvature measures of the given convex bodies, with undetermined coefficients though. These coefficients are then obtained by making use of the connection to the classical Crofton formula, which is proved by calculating the Crofton integral for specific choices of the given convex body. Here, in contrast, we determine the coefficients of the appearing curvature measures directly, by transforming the rotational part of the kinematic integral and then using an integral geometric formula known from the tensorial calculations by Hug, Schneider and Schuster (see [51, Lemma 4.4]).

We start as in the classical proof, and only prove the assertion for polytopes $P, P' \in \mathcal{P}^n$. The general case can then be obtained by approximation (see for example the classical proof in [83, p. 243f]). We denote the kinematic integral by I and decompose the integration with respect to g to get

$$I = \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \phi_j(P \cap (\vartheta P' + t), \beta \cap (\vartheta \beta' + t)) \, \mathcal{H}^n(\mathrm{d}t) \, \nu(\mathrm{d}\vartheta).$$

Now we apply Theorem 4.4.3 in [83] to determine the translative integral

$$\begin{aligned} I = \frac{1}{\omega_{n-j}} \sum_{k=j+1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} & \mathcal{H}^{n-k+j}(F' \cap \beta') \mathcal{H}^k(F \cap \beta) \\ & \times \int_{\mathrm{SO}(n)} [F, \vartheta F'] \, \mathcal{H}^{n-j-1} \left((N(P,F) + \vartheta N(P',F')) \cap \mathbb{S}^{n-1} \right) \, \nu(\mathrm{d}\vartheta) \\ + \phi_j(P, \beta) \phi_n(P', \beta') + \phi_n(P, \beta) \phi_j(P', \beta'), & \end{aligned} \tag{4.4}$$

where we see that the boundary cases $k = j, n$ are already treated. In the proof of the tensor-valued case, this is done in two steps, the examination of the translative integration in Section 4.4.1 and the determination of the boundary cases in Section 4.4.2.

So far, we have proceeded as in the classical proof. But now we deviate from it and calculate the value of the rotational part of the kinematic integration directly. In the general case this is done in Section 4.4.3.

In the following, we denote by $C(\omega) := \{\lambda x \in \mathbb{R}^n : x \in \omega, \lambda > 0\}$ the cone spanned by a set $\omega \subset \mathbb{S}^{n-1}$. For $j \geq n-1$, we note that the summation with respect to k is empty and thus in those two cases the assertion is proved. Hence, we assume $j < n-1$. In order to determine the remaining rotational integration in (4.4), we define the map

$$J : \mathcal{B}(F^\perp \cap \mathbb{S}^{n-1}) \times \mathcal{B}(F'^\perp \cap \mathbb{S}^{n-1}) \to \mathbb{R}$$

by

$$J(\omega, \omega') := \int_{\mathrm{SO}(n)} [F, \vartheta F'] \, \mathcal{H}^{n-j-1} \left((C(\omega) + \vartheta C(\omega')) \cap \mathbb{S}^{n-1} \right) \, \nu(\mathrm{d}\vartheta).$$

The Hausdorff measure can be rewritten as integration which we transform and obtain

$$J(\omega,\omega') = \frac{2}{\Gamma(\frac{n-j}{2})} \int_{\mathrm{SO}(n)} [F,\vartheta F'] \int_{C(\omega)+\vartheta C(\omega')} e^{-\|x\|^2}\, \mathcal{H}^{n-j}(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta).$$

Then we apply the transformation

$$T : \omega \times \omega' \times (0,\infty)^2 \to C(\omega) + \vartheta C(\omega'), \quad (u,v,t_1,t_2) \mapsto t_1 u + t_2 \vartheta v,$$

which is bijective for almost all $\vartheta \in \mathrm{SO}(n)$ (more precisely, for all $\vartheta \in \mathrm{SO}(n)$ such that $F^\perp$ and $\vartheta F'^\perp$ are linearly independent subspaces), and has Jacobian

$$\mathcal{J}T(u,v,t_1,t_2) = t_1^{n-k-1} t_2^{k-j-1} [F^\perp, \vartheta F'^\perp] = t_1^{n-k-1} t_2^{k-j-1} [F,\vartheta F'],$$

to the integration with respect to x in $J(\omega,\omega')$. This gives

$$\begin{aligned} J(\omega,\omega') &= \frac{2}{\Gamma(\frac{n-j}{2})} \int_{\mathrm{SO}(n)} [F,\vartheta F']^2 \int_\omega \int_{\omega'} \int_{(0,\infty)^2} t_1^{n-k-1} t_2^{k-j-1} e^{-\|t_1 u + t_2 \vartheta v\|^2} \\ &\qquad \times \mathcal{H}^2(\mathrm{d}(t_1,t_2))\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta) \\ &= \frac{2}{\Gamma(\frac{n-j}{2})} \int_{\mathrm{SO}(n)} [F,\vartheta F']^2 \int_\omega \int_{\omega'} \int_{(0,\infty)^2} t_1^{n-k-1} t_2^{k-j-1} e^{-(t_1^2+t_2^2+2t_1t_2\langle u,\vartheta v\rangle)} \\ &\qquad \times \mathcal{H}^2(\mathrm{d}(t_1,t_2))\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta). \end{aligned}$$

Because of the left and right invariance of ν, we obtain for arbitrary but fixed rotations $\sigma \in \mathrm{SO}(F^\perp)$ and $\rho \in \mathrm{SO}(F'^\perp)$

$$\begin{aligned} J(\omega,\omega') &= \frac{2}{\Gamma(\frac{n-j}{2})} \int_{\mathrm{SO}(n)} \overbrace{[F,\sigma^{-1}\vartheta\rho F']}^{=[F,\vartheta F']}{}^2 \int_\omega \int_{\omega'} \int_{(0,\infty)^2} t_1^{n-k-1} t_2^{k-j-1} e^{-(t_1^2+t_2^2+2t_1t_2\langle u,\sigma^{-1}\vartheta\rho v\rangle)} \\ &\qquad \times \mathcal{H}^2(\mathrm{d}(t_1,t_2))\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta) \\ &= \frac{2}{\Gamma(\frac{n-j}{2})} \int_{\mathrm{SO}(n)} [F,\vartheta F']^2 \int_\omega \int_{\omega'} \int_{(0,\infty)^2} t_1^{n-k-1} t_2^{k-j-1} \\ &\qquad \times \int_{\mathrm{SO}(F^\perp)} \int_{\mathrm{SO}(F'^\perp)} e^{-(t_1^2+t_2^2+2t_1t_2\langle \sigma u,\vartheta\rho v\rangle)}\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu^{F^\perp}(\mathrm{d}\sigma) \\ &\qquad \times \mathcal{H}^2(\mathrm{d}(t_1,t_2))\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta), \end{aligned}$$

where we integrated over all such rotations $\sigma \in \mathrm{SO}(F^\perp)$ and $\rho \in \mathrm{SO}(F'^\perp)$ in the second step and applied Fubini's theorem. The value of these two inner integrations is now independent of the specific choice of the unit vectors $u \in F^\perp \cap \mathbb{S}^{n-1}$ and $v \in F'^\perp \cap \mathbb{S}^{n-1}$. Therefore, we obtain for arbitrary but fixed unit vectors $u_0 \in F^\perp \cap \mathbb{S}^{n-1}$ and $v_0 \in F'^\perp \cap \mathbb{S}^{n-1}$

$$\begin{aligned} J(\omega,\omega') &= \frac{2}{\Gamma(\frac{n-j}{2})} \mathcal{H}^{n-k-1}(\omega)\, \mathcal{H}^{k-j-1}(\omega') \int_{\mathrm{SO}(n)} [F,\vartheta F']^2 \int_{(0,\infty)^2} \int_{\mathrm{SO}(F^\perp)} \int_{\mathrm{SO}(F'^\perp)} \\ &\qquad \times t_1^{n-k-1} t_2^{k-j-1} e^{-(t_1^2+t_2^2+2t_1t_2\langle \sigma u_0,\vartheta\rho v_0\rangle)}\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu^{F^\perp}(\mathrm{d}\sigma)\, \mathcal{H}^2(\mathrm{d}(t_1,t_2))\, \nu(\mathrm{d}\vartheta), \end{aligned}$$

Rewriting the integrations on $\mathrm{SO}(F^\perp)$ and $\mathrm{SO}(F'^\perp)$ as integrations on $F^\perp \cap \mathbb{S}^{n-1}$ and $F^\perp \cap \mathbb{S}^{n-1}$, and again applying the transformation T (with $\omega = F^\perp$ and $\omega' = F'^\perp$), we get

$$J(\omega,\omega') = \frac{\mathcal{H}^{n-k-1}(\omega)\,\mathcal{H}^{k-j-1}(\omega')}{\omega_{n-k}\omega_{k-j}} \int_{\mathrm{SO}(n)} [F,\vartheta F']\,\mathcal{H}^{n-j-1}\left((F^\perp + \vartheta F'^\perp)\cap \mathbb{S}^{n-1}\right)\,\nu(\mathrm{d}\vartheta).$$

We have $\dim(F^\perp + \vartheta F'^\perp) = n-j$ and thus $\mathcal{H}^{n-j-1}((F^\perp+\vartheta F'^\perp)\cap\mathbb{S}^{n-1}) = \omega_{n-j}$ for almost all $\vartheta \in \mathrm{SO}(n)$. Hence we can apply Lemma A.5 to determine the well-known remaining rotational integral and obtain

$$J(\omega,\omega') = \frac{\omega_{n-j}}{\omega_{n-k}\omega_{k-j}} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \mathcal{H}^{n-k-1}(\omega)\mathcal{H}^{k-j-1}(\omega').$$

Plugging $J(\omega,\omega')$ into (4.4), yields

$$\begin{aligned} I = \sum_{k=j+1}^{n-1} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{1}{\omega_{n-k}} \sum_{F\in\mathcal{F}_k(P)} \mathcal{H}^k(F\cap\beta)\mathcal{H}^{n-k-1}(N(P,F)\cap\mathbb{S}^{n-1}) \\ \times \frac{1}{\omega_{k-j}} \sum_{F'\in\mathcal{F}_{n-k+j}(P')} \mathcal{H}^{n-k+j}(F'\cap\beta')\mathcal{H}^{k-j-1}(N(P',F')\cap\mathbb{S}^{n-1}) \\ + \phi_j(P,\beta)\phi_n(P',\beta') + \phi_n(P,\beta)\phi_j(P',\beta'). \end{aligned}$$

The definitions of the curvature measures for polytopes and of the coefficient α_{njk} (where $\alpha_{njj} = \alpha_{njn} = 1$) yield the assertion.

4.4. The Proof of Theorem 4.1

In this section, we provide the proof of Theorem 4.1, which we, for the sake of clarity, divide into several steps. First, we treat the translative part of the kinematic integral. This can be done similarly to the proof of the translative integral formula for curvature measures. Then we consider two “boundary cases” separately. This is also done in the classical (scalar) proof. However, in the tensorial case it requires more involved techniques. In the third and main step, we approach the explicit calculation of the rotational part of the kinematic integral. Once this is accomplished, the proof is basically finished, as we obtain an explicit representation of the kinematic integral in terms of generalized tensorial curvature measures of the corresponding polytopes. Nevertheless, at this point the coefficients in the formula are still given in terms of several iterated sums of products of binomial coefficients and Gamma functions. Therefore, in the final step, we simplify these coefficients to attain the form provided in the statement of the theorem.

4.4.1. The Translative Part

The case $j = n$ is checked easily (since then $s = 0$). Hence we may assume that $j \le n-1$ in the following. Let I_1 denote the integral in which we are interested. We start by decomposing the measure μ and by substituting the definition of $\phi_j^{r,s,l}$ for polytopes to get

$$\begin{aligned}
I_1 &= \int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g) \\
&= \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \phi_j^{r,s,l}(P \cap (\vartheta P' + t), \beta \cap (\vartheta \beta' + t))\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta) \\
&= c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \sum_{G \in \mathcal{F}_j(P \cap (\vartheta P' + t))} Q(G)^l \int_{G \cap \beta \cap (\vartheta \beta' + t)} x^r\, \mathcal{H}^j(\mathrm{d}x) \\
&\qquad \times \int_{N(P \cap (\vartheta P' + t), G) \cap \mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u)\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta).
\end{aligned}$$

Let $\vartheta \in \mathrm{SO}(n)$ be fixed for the moment. Neglecting a set of translations $t \in \mathbb{R}^n$ of measure zero, we can assume that the following is true (see [83, p. 241]). For every j-face $G \in \mathcal{F}_j(P \cap (\vartheta P' + t))$ there are a unique $k \in \{j, \dots, n\}$, a unique $F \in \mathcal{F}_k(P)$ and a unique $F' \in \mathcal{F}_{n-k+j}(P')$ such that $G = F \cap (\vartheta F' + t)$. Conversely, for every $k \in \{j, \dots, n\}$, every $F \in \mathcal{F}_k(P)$ and every $F' \in \mathcal{F}_{n-k+j}(P')$, we have $F \cap (\vartheta F' + t) \in \mathcal{F}_j(P \cap (\vartheta P' + t))$ for almost all $t \in \mathbb{R}^n$ such that $F \cap (\vartheta F' + t) \neq \emptyset$. Hence, we get

$$\begin{aligned}
I_1 = c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \int_{\mathrm{SO}(n)} &\sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} Q(F^0 \cap (\vartheta F')^0)^l \\
&\times \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u) \\
&\times \int_{\mathbb{R}^n} \int_{F \cap (\vartheta F' + t) \cap \beta \cap (\vartheta \beta' + t)} x^r\, \mathcal{H}^j(\mathrm{d}x)\, \mathcal{H}^n(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta), \qquad (4.5)
\end{aligned}$$

where we use that the integral with respect to u is independent of the choice of a vector $t \in \mathbb{R}^n$ such that $\operatorname{relint} F \cap \operatorname{relint} (\vartheta F' + t) \neq \emptyset$.

As a next step, we calculate the integral with respect to t, which we denote by I_2. The argument is essentially the same as in [83, p. 241-2]. We include it for the the sake of completeness. For this, we can again assume that F and $\vartheta F'$ are in general position, that is, $[F, \vartheta F'] \neq 0$. We set $\alpha := F \cap \beta$ and $\alpha' := \vartheta F' \cap \vartheta \beta'$. We can assume that $\alpha \neq \emptyset$ and $\alpha' \neq \emptyset$, since otherwise both sides of the equation to be derived are zero. Let $s_0, t_0 \in \mathbb{R}^n$ be such that $s_0 \in \alpha \cap (\alpha' + t_0) \neq \emptyset$. Then we define $L_1 := F^0 \cap (\vartheta F')^0$, $L_2 := F^0 \cap L_1^\perp$, $L_3 := (\vartheta F')^0 \cap L_1^\perp$. Hence, for every $t \in \mathbb{R}^n$, there are uniquely determined vectors $t_i \in L_i$, $i = 1, 2, 3$, such that $t = t_0 + t_1 + t_2 + t_3$. The transformation formula for integrals then yields that

$$I_2 = [F, \vartheta F'] \int_{L_3} \int_{L_2} \int_{L_1} \int_{\alpha \cap (\alpha' + t_0 + t_1 + t_2 + t_3)} x^r\, \mathcal{H}^j(\mathrm{d}x)\, \mathcal{H}^j(\mathrm{d}t_1)\, \mathcal{H}^{k-j}(\mathrm{d}t_2)\, \mathcal{H}^{n-k}(\mathrm{d}t_3).$$

From

$$(\alpha - s_0 - t_2) \cap (\alpha' + t_0 - s_0 + t_1 + t_3) \subset F^0 \cap (\vartheta F')^0 = L_1,$$

we conclude

$$\alpha \cap (\alpha' + t_0 + t_1 + t_2 + t_3) \subset s_0 + L_1 + t_2.$$

Hence, for the two inner integrations with respect to x and t_1, an application of Fubini's theorem yields

$$\begin{aligned}
&\int_{L_1} \int_{\alpha\cap(\alpha'+t_0+t_1+t_2+t_3)\cap(s_0+L_1+t_2)} x^r \,\mathcal{H}^j(\mathrm{d}x)\,\mathcal{H}^j(\mathrm{d}t_1) \\
&= \int_{L_1} \int_{L_1\cap(\alpha-t_2-s_0)\cap(\alpha'+t_0-s_0+t_1+t_3)} (x+s_0+t_2)^r \;\mathcal{H}^j(\mathrm{d}x)\,\mathcal{H}^j(\mathrm{d}t_1) \\
&= \int \mathbb{1}\{x \in (\alpha - t_2 - s_0) \cap L_1\}\,(x+s_0+t_2)^r \\
&\qquad \times \int \mathbb{1}\{t_1 \in L_1, x \in \alpha' + t_0 - s_0 + t_1 + t_3\}\,\mathcal{H}^j(\mathrm{d}t_1)\,\mathcal{H}^j(\mathrm{d}x) \\
&= \int \mathbb{1}\{x \in (\alpha - t_2 - s_0) \cap L_1\}\,(x+s_0+t_2)^r \\
&\qquad \times \mathcal{H}^j([(\alpha' + t_0 - s_0) + t_3] \cap L_1)\,\mathcal{H}^j(\mathrm{d}x) \\
&= \mathcal{H}^j([(\alpha' + t_0 - s_0) + t_3] \cap L_1) \int_{(\alpha-s_0-t_2)\cap L_1} (x+s_0+t_2)^r \;\mathcal{H}^j(\mathrm{d}x) \\
&= \mathcal{H}^j((\alpha' + t_0 - s_0) \cap (L_1 + t_3)) \int_{(\alpha-s_0)\cap(L_1+t_2)} (x+s_0)^r \;\mathcal{H}^j(\mathrm{d}x).
\end{aligned}$$

This gives

$$\begin{aligned}
I_2 &= [F, \vartheta F'] \int_{L_3} \mathcal{H}^j\left((\alpha' + t_0 - s_0) \cap (L_1 + t_3)\right)\,\mathcal{H}^{n-k}(\mathrm{d}t_3) \\
&\qquad \times \int_{L_2} \int_{(\alpha-s_0)\cap(L_1+t_2)} (x+s_0)^r\,\mathcal{H}^j(\mathrm{d}x)\,\mathcal{H}^{k-j}(\mathrm{d}t_2) \\
&= [F, \vartheta F']\mathcal{H}^{n-k+j}(\alpha' + t_0 - s_0) \int_{\alpha-s_0} (x+s_0)^r\,\mathcal{H}^k(\mathrm{d}x) \\
&= [F, \vartheta F']\mathcal{H}^{n-k+j}(F' \cap \beta') \int_{F\cap\beta} x^r\,\mathcal{H}^k(\mathrm{d}x).
\end{aligned}$$

Thus, (4.5) can be rewritten as

$$\begin{aligned}
I_1 = c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{k=j}^{n} \sum_{F\in\mathcal{F}_k(P)} \sum_{F'\in\mathcal{F}_{n-k+j}(P')} \mathcal{H}^{n-k+j}(F' \cap \beta') \int_{F\cap\beta} x^r\,\mathcal{H}^k(\mathrm{d}x) \int_{\mathrm{SO}(n)} [F, \vartheta F'] \\
\times Q\left(F^0 \cap (\vartheta F')^0\right)^l \int_{N(P\cap(\vartheta P'+t), F\cap(\vartheta F'+t))\cap\mathbb{S}^{n-1}} u^s\,\mathcal{H}^{n-j-1}(\mathrm{d}u)\,\nu(\mathrm{d}\vartheta),
\end{aligned}$$

where $t \in \mathbb{R}^n$ is chosen as described after (4.5).

4.4.2. The Boundary Cases

In the summation with respect to k, we have to consider the summands for $k = j$ and $k = n$ separately, since the application of some of the auxiliary results requires that $k - j \geq 1$ and $k \leq n - 1$. Starting with $k = j$ and denoting the corresponding summand by S_j, we get

$$\begin{aligned}
S_j &= c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \sum_{F' \in \mathcal{F}_n(P')} \mathcal{H}^n(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{\mathrm{SO}(n)} [F, \vartheta F'] \\
&\quad \times Q \left(F^0 \cap (\vartheta F')^0 \right)^l \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \underbrace{\mathcal{H}^n(P' \cap \beta')}_{= \phi_n(P', \beta')} \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{\mathrm{SO}(n)} \underbrace{[F, \vartheta P']}_{= [F, \mathbb{R}^n] = 1} \\
&\quad \times Q(F^0 \cap \underbrace{(\vartheta P')^0}_{= \mathbb{R}^n})^l \int_{\underbrace{N(P \cap (\vartheta P' + t), F \cap (\vartheta P' + t))}_{= N(P, F)} \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= \phi_j^{r,s,l}(P, \beta) \phi_n(P', \beta').
\end{aligned}$$

For $k = n$, we denote the corresponding summand by S_n and conclude from Fubini's theorem

$$\begin{aligned}
S_n &= c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_n(P)} \sum_{F' \in \mathcal{F}_j(P')} \mathcal{H}^j(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^n(\mathrm{d}x) \int_{\mathrm{SO}(n)} \underbrace{[F, \vartheta F']}_{= [\mathbb{R}^n, \vartheta F'] = 1} \\
&\quad \times Q(\underbrace{F^0}_{= \mathbb{R}^n} \cap (\vartheta F')^0)^l \int_{\underbrace{N(P \cap (\vartheta P' + t), P \cap (\vartheta F' + t))}_{= \vartheta N(P', F')} \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \int_{P \cap \beta} x^r \, \mathcal{H}^n(\mathrm{d}x) \sum_{F' \in \mathcal{F}_j(P')} \mathcal{H}^j(F' \cap \beta') \\
&\quad \times \int_{N(P', F') \cap \mathbb{S}^{n-1}} \int_{\mathrm{SO}(n)} Q\left(\vartheta F'\right)^l (\vartheta u)^s \, \nu(\mathrm{d}\vartheta) \, \mathcal{H}^{n-j-1}(\mathrm{d}u).
\end{aligned}$$

For this, we obtain from Lemma 4.6

$$\begin{aligned}
S_n &= \mathbb{1}\{s \text{ even}\} \, c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{j}{2} + l) \Gamma(\frac{s+1}{2})}{\sqrt{\pi} \Gamma(\frac{n+s}{2} + l) \Gamma(\frac{j}{2})} Q^{l + \frac{s}{2}} \int_{P \cap \beta} x^r \, \mathcal{H}^n(\mathrm{d}x) \\
&\quad \times \sum_{F' \in \mathcal{F}_j(P')} \mathcal{H}^j(F' \cap \beta') \int_{N(P', F') \cap \mathbb{S}^{n-1}} \mathcal{H}^{n-j-1}(\mathrm{d}u) \\
&= c_{n,j}^{s} \, \phi_n^{r, 0, \frac{s}{2} + l}(P, \beta) \phi_j(P', \beta'),
\end{aligned}$$

where

$$c_{n,j}^{s} := \mathbb{1}\{s \text{ even}\} \frac{2\omega_{n-j}}{s! \, \omega_{s+1} \omega_{n-j+s}} = \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \left(\frac{s}{2}\right)!} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})}. \tag{4.6}$$

Hence, we get

$$
\begin{aligned}
I_1 = c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{k=j+1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} & \mathcal{H}^{n-k+j}(F' \cap \beta') \\
& \times \int_{F \cap \beta} x^r \, \mathcal{H}^k(\mathrm{d}x) \int_{\mathrm{SO}(n)} [F, \vartheta F'] Q \left(F^0 \cap (\vartheta F')^0\right)^l \\
& \times \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
+ \phi_j^{r,s,l}(P, \beta)\phi_n(P', \beta') + c_{n,j}^s \, \phi_n^{r,0,\frac{s}{2}+l}(P, \beta)\phi_j(P', \beta').
\end{aligned}
$$

Furthermore, for any $t \in \mathbb{R}^n$ such that $\operatorname{relint} F \cap \operatorname{relint} (\vartheta F' + t) \neq \emptyset$ we obtain from [83, Theorem 2.2.1]

$$
N\left(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)\right) = N(P, F) + \vartheta N(P', F'),
$$

and thus

$$
\begin{aligned}
I_1 = c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{k=j+1}^{n-1} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} & \mathcal{H}^{n-k+j}(F' \cap \beta') \\
& \times \int_{F \cap \beta} x^r \, \mathcal{H}^k(\mathrm{d}x) \int_{\mathrm{SO}(n)} [F, \vartheta F'] Q \left(F^0 \cap (\vartheta F')^0\right)^l \\
& \times \int_{(N(P,F) + \vartheta N(P', F')) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
+ \phi_j^{r,s,l}(P, \beta)\phi_n(P', \beta') + c_{n,j}^s \, \phi_n^{r,0,\frac{s}{2}+l}(P, \beta)\phi_j(P', \beta'). \qquad (4.7)
\end{aligned}
$$

In the following, we denote by $C(\omega) := \{\lambda x \in \mathbb{R}^n : x \in \omega, \lambda > 0\}$ the cone spanned by a set $\omega \subset \mathbb{S}^{n-1}$. Moreover, if F is a face of P, we write $F^\perp$ for the linear subspace orthogonal to F^0. For the next and main step, we may assume that $j \leq n-2$ (since $j < k \leq n-1$). We define the mapping

$$
J : \mathcal{B}(F^\perp \cap \mathbb{S}^{n-1}) \times \mathcal{B}(F'^\perp \cap \mathbb{S}^{n-1}) \to \mathbb{T}^{2l+s}
$$

by

$$
\begin{aligned}
J(\omega, \omega') := \int_{\mathrm{SO}(n)} & [F, \vartheta F'] Q \left(F^0 \cap (\vartheta F')^0\right)^l \\
& \times \int_{(C(\omega) + \vartheta C(\omega')) \cap \mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-j-1}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta)
\end{aligned}
$$

for $\omega \in \mathcal{B}(F^\perp \cap \mathbb{S}^{n-1})$ and $\omega' \in \mathcal{B}(F'^\perp \cap \mathbb{S}^{n-1})$. Then J is a finite signed measure on $\mathcal{B}(F^\perp \cap \mathbb{S}^{n-1})$ in the first variable and a finite signed measure on $\mathcal{B}(F'^\perp \cap \mathbb{S}^{n-1})$ in the second variable, but this will not be needed in the following. In fact, we could specialize to the case $\omega = N(P, F) \cap \mathbb{S}^{n-1}$ and $\omega' = N(P', F') \cap \mathbb{S}^{n-1}$ throughout the proof.

Since $[F, \vartheta F']Q\left(F^0 \cap (\vartheta F')^0\right)^l$ depends only on the linear subspaces F^0 and $(\vartheta F')^0$, we can assume that $F \in \mathrm{G}(n,k)$ and $F' \in \mathrm{G}(n, n-k+j)$ for determining $J(\omega, \omega')$. Moreover, for ν-almost all $\vartheta \in \mathrm{SO}(n)$, the linear subspaces $F^\perp$ and $\vartheta(F'^\perp)$ are linearly independent. This will be tacitly used in the following.

4.4.3. The Rotational Part

In this section, ω, ω' are fixed and as described above. Due to the right invariance of ν, we obtain for $\rho \in \mathrm{SO}(F'^\perp)$

$$\begin{aligned}
J(\omega,\omega') &= \int_{\mathrm{SO}(n)} [F, \vartheta\rho F']Q(F \cap \vartheta\rho F')^l \int_{(C(\omega)+\vartheta\rho C(\omega'))\cap\mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u)\,\nu(\mathrm{d}\vartheta) \\
&= \int_{\mathrm{SO}(n)} [F, \vartheta F']Q(F \cap \vartheta F')^l \\
&\qquad \times \int_{\mathrm{SO}(F'^\perp)} \int_{(C(\omega)+\vartheta\rho C(\omega'))\cap\mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u)\,\nu^{F'^\perp}(\mathrm{d}\rho)\,\nu(\mathrm{d}\vartheta),
\end{aligned}$$

where we averaged over all such rotations $\rho \in \mathrm{SO}(F'^\perp)$ and applied Fubini's theorem in the second step. Next, we introduce a multiple J_1 of the inner integral of $J(\omega, \omega')$ and rewrite it by means of a polar coordinate transformation, that is,

$$\begin{aligned}
J_1 :=& \tfrac{1}{2}\Gamma\left(\tfrac{n-j+s}{2}\right) \int_{(C(\omega)+\vartheta\rho C(\omega'))\cap\mathbb{S}^{n-1}} u^s \,\mathcal{H}^{n-j-1}(\mathrm{d}u) \\
=& \int_0^\infty \int_{(C(\omega)+\vartheta\rho C(\omega'))\cap\mathbb{S}^{n-1}} (ru)^s e^{-\|ru\|^2} r^{n-j-1}\,\mathcal{H}^{n-j-1}(\mathrm{d}u)\,\mathrm{d}r \\
=& \int_{C(\omega)+\vartheta\rho C(\omega')} x^s e^{-\|x\|^2}\,\mathcal{H}^{n-j}(\mathrm{d}x).
\end{aligned}$$

The bijective transformation (here we assume that $\vartheta \in \mathrm{SO}(n)$ is such that $F^\perp$ and $\vartheta(F'^\perp)$ are linearly independent subspaces)

$$T : \omega \times \omega' \times (0,\infty)^2 \to C(\omega) + \vartheta\rho C(\omega'), \quad (u, v, t_1, t_2) \mapsto t_1 u + t_2 \vartheta\rho v,$$

has the Jacobian

$$\mathcal{J}T(u, v, t_1, t_2) = t_1^{n-k-1} t_2^{k-j-1} [F^\perp, \vartheta F'^\perp] = t_1^{n-k-1} t_2^{k-j-1} [F, \vartheta F'].$$

Hence, we obtain

$$\begin{aligned}
J_1 = \int_\omega \int_{\omega'} \int_{(0,\infty)^2} & t_1^{n-k-1} t_2^{k-j-1} [F, \vartheta F']\,(t_1 u + t_2\vartheta\rho v)^s\, e^{-\|t_1 u + t_2 \vartheta\rho v\|^2} \\
& \times \mathcal{H}^2(\mathrm{d}(t_1,t_2))\,\mathcal{H}^{k-j-1}(\mathrm{d}v)\,\mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}$$

Applying a polar coordinate transformation to the inner integral yields

$$J_1 = [F, \vartheta F'] \int_{\omega} \int_{\omega'} \int_0^{\frac{\pi}{2}} \int_0^{\infty} (r\cos(\alpha))^{n-k-1} (r\sin(\alpha))^{k-j-1} e^{-\|r\cos(\alpha)u + r\sin(\alpha)\vartheta\rho v\|^2} \\ \times (r\cos(\alpha)u + r\sin(\alpha)\vartheta\rho v)^s \, r \, \mathrm{d}r \, \mathrm{d}\alpha \, \mathcal{H}^{k-j-1}(\mathrm{d}v) \, \mathcal{H}^{n-k-1}(\mathrm{d}u).$$

Using binomial expansion, we get

$$J_1 = [F, \vartheta F'] \int_{\omega} \int_{\omega'} \sum_{i=0}^{s} \binom{s}{i} u^{s-i} (\vartheta\rho v)^i \int_0^{\frac{\pi}{2}} \cos(\alpha)^{n-k+s-i-1} \sin(\alpha)^{k-j+i-1} \\ \times \int_0^{\infty} r^{n-j+s-1} e^{-r^2(1+2\cos(\alpha)\sin(\alpha)\langle u, \vartheta\rho v\rangle)} \, \mathrm{d}r \, \mathrm{d}\alpha \, \mathcal{H}^{k-j-1}(\mathrm{d}v) \, \mathcal{H}^{n-k-1}(\mathrm{d}u).$$

We factor the occurring exponential function and expand the second part of it as a power series, to obtain

$$J_1 = [F, \vartheta F'] \int_{\omega} \int_{\omega'} \sum_{i=0}^{s} \binom{s}{i} u^{s-i} (\vartheta\rho v)^i \int_0^{\frac{\pi}{2}} \cos(\alpha)^{n-k+s-i-1} \sin(\alpha)^{k-j+i-1} \int_0^{\infty} r^{n-j+s-1} \\ \times e^{-r^2} \sum_{t=0}^{\infty} \frac{(-2r^2\cos(\alpha)\sin(\alpha)\langle u, \vartheta\rho v\rangle)^t}{t!} \, \mathrm{d}r \, \mathrm{d}\alpha \, \mathcal{H}^{k-j-1}(\mathrm{d}v) \, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\ = [F, \vartheta F'] \int_{\omega} \int_{\omega'} \sum_{i=0}^{s} \binom{s}{i} u^{s-i} (\vartheta\rho v)^i \sum_{t=0}^{\infty} \frac{(-2\langle u, \vartheta\rho v\rangle)^t}{t!} \int_0^{\infty} r^{n-j+s+2t-1} e^{-r^2} \, \mathrm{d}r \\ \times \int_0^{\frac{\pi}{2}} \cos(\alpha)^{n-k+s-i+t-1} \sin(\alpha)^{k-j+i+t-1} \, \mathrm{d}\alpha \, \mathcal{H}^{k-j-1}(\mathrm{d}v) \, \mathcal{H}^{n-k-1}(\mathrm{d}u),$$

where we interchanged the integrations with respect to r and to α with the series with respect to t by dominated convergence which can be applied for almost all (ϑ, u). In fact, for $\nu \otimes \mathcal{H}^{n-k-1}$-almost all pairs $(\vartheta, u) \in \mathrm{SO}(n) \times F^{\perp}$ we have $\vartheta^{-1}u \notin F'^{\perp}$ (and hence $\langle u, \vartheta\rho v\rangle < 1$, for all $\rho \in \mathrm{SO}(F'^{\perp})$ and $v \in F'^{\perp}$) such that the series converges absolutely and uniformly and yields an integrable upper bound. The integrations with respect to r and to α can now be simplified, applying the definitions of the Gamma function and of the Beta function and the relationship between them. This gives

$$J_1 = \frac{1}{4}[F, \vartheta F'] \int_{\omega} \int_{\omega'} \sum_{i=0}^{s} \binom{s}{i} u^{s-i} (\vartheta\rho v)^i \sum_{t=0}^{\infty} \frac{(-2\langle u, \vartheta\rho v\rangle)^t}{t!} \\ \times \Gamma\left(\tfrac{n-k+s-i+t}{2}\right) \Gamma\left(\tfrac{k-j+i+t}{2}\right) \mathcal{H}^{k-j-1}(\mathrm{d}v) \, \mathcal{H}^{n-k-1}(\mathrm{d}u).$$

The series with respect to t again converges absolutely for almost all (ϑ, u); in fact,

$$\sum_{t=0}^{\infty} \left| \frac{(-2\langle u, \vartheta\rho v\rangle)^t}{t!} \Gamma\left(\tfrac{n-k+s-i+t}{2}\right) \Gamma\left(\tfrac{k-j+i+t}{2}\right) \right| \\ \leq \sum_{t=0}^{\infty} \frac{(2|\langle u, \vartheta\rho v\rangle|)^t}{t!} \Gamma\left(\tfrac{n-k+s-i+t}{2}\right) \Gamma\left(\tfrac{k-j+i+t}{2}\right) < \infty,$$

which can be seen applying the ratio test, using as above that for $\nu \otimes \mathcal{H}^{n-k-1}$-almost all pairs $(\vartheta, u) \in \mathrm{SO}(n) \times F^\perp$ we have $\vartheta^{-1}u \notin F'^\perp$ (and hence $\langle u, \vartheta\rho v\rangle < 1$, for all $\rho \in \mathrm{SO}(F'^\perp)$ and $v \in F'^\perp$). Furthermore,

$$\begin{aligned}
&\int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(F'^\perp)} \int_\omega \int_{\omega'} \Bigg\| [F, \vartheta F']^2\, Q(F \cap \vartheta F')^l \sum_{i=0}^{s} \binom{s}{i} u^{s-i} (\vartheta\rho v)^i \sum_{t=0}^{\infty} \frac{(-2\langle u, \vartheta\rho v\rangle)^t}{t!} \\
&\qquad \times \Gamma\big(\tfrac{n-k+s-i+t}{2}\big)\Gamma\big(\tfrac{k-j+i+t}{2}\big)\Bigg\|\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu(\mathrm{d}\vartheta) \\
\le& \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(F'^\perp)} \int_\omega \int_{\omega'} \int_{(0,\infty)^2} t_1^{n-k-1} t_2^{k-j-1} [F, \vartheta F'] \| t_1 u + t_2 \vartheta\rho v\|^s e^{-\|t_1 u + t_2\vartheta\rho v\|^2} \\
&\qquad \times \mathcal{H}^2(\mathrm{d}(t_1, t_2))\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \mathcal{H}^{n-k-1}(\mathrm{d}u)\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu(\mathrm{d}\vartheta) \\
\le& \frac{1}{2\Gamma(\frac{n-j+s}{2})} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(F'^\perp)} \int_{(C(\omega)+\vartheta\rho C(\omega'))\cap \mathbb{S}^{n-1}} \|u\|^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u)\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu(\mathrm{d}\vartheta)
\end{aligned}$$

is finite. Therefore, Fubini's theorem yields

$$\begin{aligned}
J(\omega, \omega') = &\frac{1}{2\Gamma(\frac{n-j+s}{2})} \int_\omega \int_{\mathrm{SO}(n)} [F, \vartheta F']^2\, Q(F \cap \vartheta F')^l \\
&\times \sum_{i=0}^{s} \binom{s}{i} u^{s-i} \sum_{t=0}^{\infty} \frac{(-2)^t}{t!} \Gamma\big(\tfrac{n-k+s-i+t}{2}\big)\Gamma\big(\tfrac{k-j+i+t}{2}\big) \\
&\times \int_{\omega'} \int_{\mathrm{SO}(F'^\perp)} (\vartheta\rho v)^i \langle u, \vartheta\rho v\rangle^t\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \mathcal{H}^{k-j-1}(\mathrm{d}v)\, \nu(\mathrm{d}\vartheta)\, \mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}$$

Due to the right invariance of the measure $\nu^{F'^\perp}$, the integrand is now independent of the specific choice of $v \in F'^\perp \cap \mathbb{S}^{n-1}$. Thus, we obtain for an arbitrary but fixed unit vector $v_0 \in F'^\perp \cap \mathbb{S}^{n-1}$

$$\begin{aligned}
J(\omega, \omega') = &\frac{1}{2\Gamma(\frac{n-j+s}{2})}\, \mathcal{H}^{k-j-1}(\omega') \int_\omega \int_{\mathrm{SO}(n)} [F, \vartheta F']^2\, Q(F \cap \vartheta F')^l \\
&\times \sum_{i=0}^{s} \binom{s}{i} u^{s-i} \sum_{t=0}^{\infty} \frac{(-2)^t}{t!} \Gamma\big(\tfrac{n-k+s-i+t}{2}\big)\Gamma\big(\tfrac{k-j+i+t}{2}\big) \\
&\times \int_{\mathrm{SO}(F'^\perp)} (\vartheta\rho v_0)^i \langle u, \vartheta\rho v_0\rangle^t\, \nu^{F'^\perp}(\mathrm{d}\rho)\, \nu(\mathrm{d}\vartheta)\, \mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}$$

We denote the integral with respect to ρ by J_2 and obtain

$$\begin{aligned}
J_2 &= \vartheta \int_{\mathrm{SO}(F'^\perp)} (\rho v_0)^i \left\langle p_{F'^\perp}(\vartheta^{-1}u) + p_{F'}(\vartheta^{-1}u), \rho v_0 \right\rangle^t \nu^{F'^\perp}(\mathrm{d}\rho) \\
&= \|p_{\vartheta F'^\perp}(u)\|^t\, \vartheta \int_{\mathrm{SO}(F'^\perp)} (\rho v_0)^i \left\langle \pi_{F'^\perp}(\vartheta^{-1}u), \rho v_0 \right\rangle^t \nu^{F'^\perp}(\mathrm{d}\rho)
\end{aligned}$$

if $\vartheta^{-1}u \notin F'$ (which holds by an analogous argument as above for almost all pairs (ϑ, u)). We note that the integration over $\mathrm{SO}(F'^\perp)$ yields the same value as an integration over all $\vartheta \in \mathrm{O}(n)$ which fix F'^0 pointwise, since $\dim(F'^\perp) \in \{1, \ldots, n-1\}$ and $n \ge 2$. Hence, an application of Lemma 4.7 in $F'^{\perp}$ yields

$$J_2 = \mathbb{1}\{i+t \text{ even}\}\frac{\Gamma(\frac{k-j}{2})}{\sqrt{\pi}}\frac{\Gamma(t+1)}{2^t\Gamma(\frac{k-j+i+t}{2})}\|p_{\vartheta F'^{\perp}}(u)\|^t \times \sum_{x=(\frac{i-t}{2})^+}^{\lfloor\frac{i}{2}\rfloor}\binom{i}{2x}\frac{\Gamma(x+\frac{1}{2})}{\Gamma(\frac{t-i}{2}+x+1)}\pi_{\vartheta F'^{\perp}}(u)^{i-2x}Q(\vartheta F'^{\perp})^x.$$

Thus we conclude

$$\begin{aligned} J(\omega,\omega') = {} & \frac{\Gamma(\frac{k-j}{2})}{2\sqrt{\pi}\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\int_\omega\int_{\mathrm{SO}(n)}[F,\vartheta F']^2\, Q(F\cap\vartheta F')^l \\ & \times\sum_{i=0}^{s}(-1)^i\binom{s}{i}u^{s-i}\sum_{t=0}^{\infty}\mathbb{1}\{i+t \text{ even}\}\Gamma\big(\tfrac{n-k+s-i+t}{2}\big)\|p_{\vartheta F'^{\perp}}(u)\|^t \\ & \times\sum_{x=(\frac{i-t}{2})^+}^{\lfloor\frac{i}{2}\rfloor}\binom{i}{2x}\frac{\Gamma(x+\frac{1}{2})}{\Gamma(\frac{t-i}{2}+x+1)}\pi_{\vartheta F'^{\perp}}(u)^{i-2x}Q(\vartheta F'^{\perp})^x\,\nu(\mathrm{d}\vartheta)\,\mathcal{H}^{n-k-1}(\mathrm{d}u), \end{aligned}$$

where we used that $(-1)^t = (-1)^i$ provided that $i+t$ is even.

As the series with respect to t converges absolutely for almost all (ϑ,u) (using again that we have $\vartheta^{-1}u \notin F' \cup F'^{\perp}$, for $\nu\otimes\mathcal{H}^{n-k-1}$-almost all pairs $(\vartheta,u)\in \mathrm{SO}(n)\times F^{\perp}$), we can rearrange the order of the summations to get

$$\begin{aligned} J(\omega,\omega') = {} & \frac{\Gamma(\frac{k-j}{2})}{2\sqrt{\pi}\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\int_\omega\int_{\mathrm{SO}(n)}[F,\vartheta F']^2 Q(F\cap\vartheta F')^l\sum_{i=0}^{s}\sum_{x=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^i \\ & \times\binom{s}{i}\binom{i}{2x}\Gamma(x+\tfrac{1}{2})u^{s-i}\pi_{\vartheta F'^{\perp}}(u)^{i-2x}Q(\vartheta F'^{\perp})^x \\ & \times\sum_{t=i-2x}^{\infty}\mathbb{1}\{i+t \text{ even}\}\frac{\Gamma(\frac{n-k+s-i+t}{2})}{\Gamma(\frac{t-i}{2}+x+1)}\|p_{\vartheta F'^{\perp}}(u)\|^t\,\nu(\mathrm{d}\vartheta)\,\mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

We denote the series with respect to t by S_t. Then, for $\vartheta^{-1}u\notin F'^{\perp}$, we obtain (after an index shift)

$$\begin{aligned} S_t &= \sum_{t=0}^{\infty}\underbrace{\mathbb{1}\{2i-2x+t \text{ even}\}}_{=\mathbb{1}\{t \text{ even}\}}\frac{\Gamma(\frac{n-k+s+t-2x}{2})}{\Gamma(\frac{t}{2}+1)}\|p_{\vartheta F'^{\perp}}(u)\|^{i-2x+t} \\ &= \|p_{\vartheta F'^{\perp}}(u)\|^{i-2x}\sum_{t=0}^{\infty}\frac{\Gamma(\frac{n-k+s}{2}+t-x)}{\Gamma(t+1)}\|p_{\vartheta F'^{\perp}}(u)\|^{2t} \\ &= \Gamma(\tfrac{n-k+s}{2}-x)\|p_{\vartheta F'^{\perp}}(u)\|^{i-2x}\sum_{t=0}^{\infty}\binom{-\frac{n-k+s}{2}+x}{t}(-\|p_{\vartheta F'^{\perp}}(u)\|^2)^t, \end{aligned}$$

where the remaining series is just a binomial series. Hence, we get

$$\begin{aligned} S_t &= \Gamma(\tfrac{n-k+s}{2} - x)\|p_{\vartheta F'^\perp}(u)\|^{i-2x}(1 - \|p_{\vartheta F'^\perp}(u)\|^2)^{-\frac{n-k+s}{2}+x} \\ &= \Gamma(\tfrac{n-k+s}{2} - x)\|p_{\vartheta F'^\perp}(u)\|^{i-2x}\|p_{\vartheta F'}(u)\|^{-n+k-s+2x}. \end{aligned}$$

Expanding $Q(\vartheta F'^\perp)^x = (Q - Q(\vartheta F'))^x$ in $J(\omega, \omega')$, we obtain

$$\begin{aligned} J(\omega, \omega') = & \frac{\Gamma(\frac{k-j}{2})}{2\sqrt{\pi}\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\int_\omega\int_{\mathrm{SO}(n)}\sum_{i=0}^{s}\sum_{x=0}^{\lfloor\frac{i}{2}\rfloor}\sum_{y=0}^{x}(-1)^{i+y}\binom{s}{i}\binom{i}{2x}\binom{x}{y}\Gamma(x+\tfrac{1}{2}) \\ & \times \Gamma(\tfrac{n-k+s}{2} - x)u^{s-i}Q^{x-y}[F, \vartheta F']^2\|p_{\vartheta F'}(u)\|^{-n+k-s+2x} \\ & \times p_{\vartheta F'^\perp}(u)^{i-2x}Q(\vartheta F')^y Q(F\cap\vartheta F')^l\,\nu(\mathrm{d}\vartheta)\,\mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

Changing the order of the summation under the integral gives

$$\begin{aligned} J(\omega, \omega') = & \frac{\Gamma(\frac{k-j}{2})}{2\sqrt{\pi}\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\int_\omega\int_{\mathrm{SO}(n)}\sum_{x=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{y=0}^{x}\sum_{i=2x}^{s}(-1)^{i+y}\binom{s}{i}\binom{i}{2x}\binom{x}{y}\Gamma(x+\tfrac{1}{2}) \\ & \times \Gamma(\tfrac{n-k+s}{2} - x)u^{s-i}Q^{x-y}[F, \vartheta F']^2\|p_{\vartheta F'}(u)\|^{-n+k-s+2x}p_{\vartheta F'^\perp}(u)^{i-2x} \\ & \times Q(\vartheta F')^y Q(F\cap\vartheta F')^l\,\nu(\mathrm{d}\vartheta)\,\mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

We denote the integral with respect to ϑ in $J(\omega, \omega')$ by J_3 and transform it, to obtain

$$\begin{aligned} J_3 = & \int_{G(n,n-k+j)}\sum_{x=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{y=0}^{x}\sum_{i=2x}^{s}(-1)^{i+y}\binom{s}{i}\binom{i}{2x}\binom{x}{y}\Gamma(x+\tfrac{1}{2})\Gamma(\tfrac{n-k+s}{2} - x) \\ & \times u^{s-i}Q^{x-y}[F, G]^2\|p_G(u)\|^{-n+k-s+2x}p_{G^\perp}(u)^{i-2x}Q(G)^y Q(F\cap G)^l\,\nu_{n-k+j}(\mathrm{d}G). \end{aligned}$$

Since $n \geq 2$ and $1 \leq n-k+j \leq n-1$, Lemma A.2 yields

$$\begin{aligned} J_3 = & \frac{\omega_{n-k+j}}{2\omega_n}\int_{G(u^\perp,n-k+j-1)}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}\sum_{x=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{y=0}^{x}\sum_{i=2x}^{s}(-1)^{i+y}\binom{s}{i}\binom{i}{2x}\binom{x}{y}\Gamma(x+\tfrac{1}{2}) \\ & \times \Gamma(\tfrac{n-k+s}{2} - x)u^{s-i}Q^{x-y}|z|^{n-k+j-1}\left(1-z^2\right)^{\frac{k-j-2}{2}}[F, \mathrm{lin}\{U, zu+\sqrt{1-z^2}w\}]^2 \\ & \times \|p_{\mathrm{lin}\{U,zu+\sqrt{1-z^2}w\}}(u)\|^{-n+k-s+2x}Q(\mathrm{lin}\{U, zu+\sqrt{1-z^2}w\})^y \\ & \times Q(F\cap\mathrm{lin}\{U, zu+\sqrt{1-z^2}w\})^l p_{\mathrm{lin}\{U,zu+\sqrt{1-z^2}w\}^\perp}(u)^{i-2x} \\ & \times \mathcal{H}^{k-j-1}(\mathrm{d}w)\,\mathrm{d}z\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U). \end{aligned}$$

The required integrability will be clear from (4.8) below. Since $u, w \in U^\perp$, we obtain

$$\begin{aligned} p_{\mathrm{lin}\{U,zu+\sqrt{1-z^2}w\}^\perp}(u) &= u - p_{zu+\sqrt{1-z^2}w}(u) = u - z(zu+\sqrt{1-z^2}w) \\ &= \sqrt{1-z^2}\cdot(\sqrt{1-z^2}u - |z|\mathrm{sign}(z)w) \end{aligned}$$

and $\|p_{\mathrm{lin}\{U,zu+\sqrt{1-z^2}w\}}(u)\| = |z|$. Furthermore, since also $F \subset u^\perp$, we have

$$[F, \mathrm{lin}\{U, zu + \sqrt{1-z^2}w\}] = [F, U]^{(u^\perp)}|z|,$$
$$Q(\mathrm{lin}\{U, zu + \sqrt{1-z^2}w\}) = Q(U) + (|z|u + \sqrt{1-z^2}\mathrm{sign}(z)w)^2,$$

and, for all $z \in [-1,1] \setminus \{0\}$ and $w \in U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}$,

$$Q(F \cap \mathrm{lin}\{U, zu + \sqrt{1-z^2}w\}) = Q(F \cap U),$$

as $F \subset u^\perp$ and $U = \mathrm{lin}\{U, zu + \sqrt{1-z^2}w\} \cap u^\perp$. Using the fact that the integration with respect to w is invariant under reflection in the origin, we obtain

$$\begin{aligned} J_3 = \frac{\omega_{n-k+j}}{2\omega_n} &\int_{\mathrm{G}(u^\perp, n-k+j-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} \sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{i=2x}^{s} (-1)^{i+y} \binom{s}{i} \binom{i}{2x} \binom{x}{y} \Gamma(x + \tfrac{1}{2}) \\ &\times \Gamma(\tfrac{n-k+s}{2} - x) u^{s-i} Q^{x-y} |z|^{j-s+2x+1} \left(1 - z^2\right)^{\frac{k-j+i-2x-2}{2}} \left(\sqrt{1-z^2}u - |z|w\right)^{i-2x} \\ &\times \left([F, U]^{(u^\perp)}\right)^2 \left(Q(U) + (|z|u + \sqrt{1-z^2}w)^2\right)^y Q(F \cap U)^l \\ &\times \mathcal{H}^{k-j-1}(\mathrm{d}w)\, \mathrm{d}z\, \nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U). \end{aligned}$$

Binomial expansion yields

$$\left(\sqrt{1-z^2}u - |z|w\right)^{i-2x} = \sum_{\alpha=0}^{i-2x} (-1)^\alpha \binom{i-2x}{\alpha} (\sqrt{1-z^2}u)^{i-2x-\alpha} (|z|w)^\alpha.$$

A change of the order of summation gives

$$\begin{aligned} J_3 = \frac{\omega_{n-k+j}}{2\omega_n} &\int_{\mathrm{G}(u^\perp, n-k+j-1)} \left([F, U]^{(u^\perp)}\right)^2 \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} \sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{\alpha=0}^{s-2x} (-1)^y \binom{x}{y} \\ &\times w^\alpha \sum_{i=2x+\alpha}^{s} (-1)^{i+\alpha} \binom{s}{i} \binom{i}{2x} \binom{i-2x}{\alpha} \left(1 - z^2\right)^i \Gamma(\tfrac{n-k+s}{2} - x) \\ &\times \Gamma(x + \tfrac{1}{2}) u^{s-2x-\alpha} Q^{x-y} |z|^{j-s+2x+\alpha+1} \left(1 - z^2\right)^{\frac{k-j-4x-\alpha-2}{2}} Q(F \cap U)^l \\ &\times \left(Q(U) + (|z|u + \sqrt{1-z^2}w)^2\right)^y \mathcal{H}^{k-j-1}(\mathrm{d}w)\, \mathrm{d}z\, \nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U). \end{aligned}$$

With Lemma B.2 we conclude

$$\begin{aligned} J_3 = \frac{\omega_{n-k+j}}{2\omega_n} &\sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{\alpha=0}^{s-2x} (-1)^y \binom{x}{y} \binom{s}{2x} \binom{s-2x}{\alpha} \Gamma(x + \tfrac{1}{2}) \Gamma(\tfrac{n-k+s}{2} - x) u^{s-2x-\alpha} Q^{x-y} \\ &\times \int_{\mathrm{G}(u^\perp, n-k+j-1)} \left([F, U]^{(u^\perp)}\right)^2 \int_{-1}^{1} |z|^{j+s-2x-\alpha+1} \left(1 - z^2\right)^{\frac{k-j+\alpha-2}{2}} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^\alpha \\ &\times Q(F \cap U)^l \left(Q(U) + (|z|u + \sqrt{1-z^2}w)^2\right)^y \mathcal{H}^{k-j-1}(\mathrm{d}w)\, \mathrm{d}z\, \nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U). \quad (4.8) \end{aligned}$$

At this point we easily see that the integrals in J_3 are finite, since $j+s-2x-\alpha+1\geq 0$ and $k-j+\alpha-2\geq -1$. In fact, the absolute values of the integrands have finite integral, which also justifies the application of Lemma A.2 above. Therefore, we can change the order of summation and integration from now on. We write J_4 for the integral with respect to U multiplied by the factor $\omega_{n-k+j}/(2\omega_n)$.

By (twofold) binomial expansion of $(Q(U)+(|z|u+\sqrt{1-z^2}w\})^2)^y$ we obtain

$$J_4=\frac{\omega_{n-k+j}}{2\omega_n}\sum_{\beta=0}^{y}\binom{y}{\beta}\int_{\mathrm{G}(u^\perp,n-k+j-1)}\left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta}Q(F\cap U)^l$$
$$\times\int_{-1}^{1}|z|^{j+s-2x-\alpha+1}\left(1-z^2\right)^{\frac{k-j+\alpha-2}{2}}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}w^\alpha(|z|u+\sqrt{1-z^2}w)^{2\beta}$$
$$\times\,\mathcal{H}^{k-j-1}(\mathrm{d}w)\,\mathrm{d}z\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U),$$

and in the second step

$$J_4=\frac{\omega_{n-k+j}}{2\omega_n}\sum_{\beta=0}^{y}\sum_{\gamma=0}^{2\beta}\binom{y}{\beta}\binom{2\beta}{\gamma}u^{2\beta-\gamma}\int_{\mathrm{G}(u^\perp,n-k+j-1)}\left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta}$$
$$\times\,Q(F\cap U)^l\int_{-1}^{1}|z|^{j+s-2x-\alpha+2\beta-\gamma+1}\left(1-z^2\right)^{\frac{k-j+\alpha+\gamma-2}{2}}\,\mathrm{d}z$$
$$\times\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}}w^{\alpha+\gamma}\,\mathcal{H}^{k-j-1}(\mathrm{d}w)\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U).$$

Using Lemma A.3 and expressing the involved spherical volumes in terms of Gamma functions, we get

$$J_4=\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-k+j}{2})}\sum_{\beta=0}^{y}\sum_{\gamma=0}^{2\beta}\mathbb{1}\{\alpha+\gamma\text{ even}\}\binom{y}{\beta}\binom{2\beta}{\gamma}$$
$$\times\frac{\Gamma(\frac{j+s-\alpha-\gamma}{2}-x+\beta+1)\Gamma(\frac{\alpha+\gamma+1}{2})}{\Gamma(\frac{k+s}{2}-x+\beta+1)}u^{2\beta-\gamma}\int_{\mathrm{G}(u^\perp,n-k+j-1)}$$
$$\times\left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta}Q(F\cap U)^lQ(U^\perp\cap u^\perp)^{\frac{\alpha+\gamma}{2}}\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U).$$

With an index shift in the summation with respect to γ we obtain

$$J_4=\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-k+j}{2})}\sum_{\beta=0}^{y}\sum_{\gamma=\alpha}^{\alpha+2\beta}\mathbb{1}\{\gamma\text{ even}\}\binom{y}{\beta}\binom{2\beta}{\gamma-\alpha}$$
$$\times\frac{\Gamma(\frac{j+s-\gamma}{2}-x+\beta+1)\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{k+s}{2}-x+\beta+1)}u^{\alpha+2\beta-\gamma}\int_{\mathrm{G}(u^\perp,n-k+j-1)}$$
$$\times\left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta}Q(F\cap U)^lQ(U^\perp\cap u^\perp)^{\frac{\gamma}{2}}\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U).$$

We plug J_4 into J_3 and change the order of summation to get

$$\begin{aligned}J_3 = {} & \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-k+j}{2})} \sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{\beta=0}^{y} \sum_{\gamma=0}^{s-2x+2\beta} (-1)^y \mathbb{1}\{\gamma \text{ even}\} \binom{s}{2x}\binom{x}{y}\binom{y}{\beta} \\ & \times \Gamma(x+\tfrac{1}{2})\Gamma(\tfrac{n-k+s}{2}-x) \sum_{\alpha=(\gamma-2\beta)^+}^{\min\{s-2x,\gamma\}} \binom{s-2x}{\alpha}\binom{2\beta}{\gamma-\alpha} \\ & \times \frac{\Gamma(\frac{j+s-\gamma}{2}-x+\beta+1)\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{k+s}{2}-x+\beta+1)} u^{s-2x+2\beta-\gamma} Q^{x-y} \int_{G(u^\perp, n-k+j-1)} \\ & \times \left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta} Q(F\cap U)^l Q(U^\perp \cap u^\perp)^{\frac{\gamma}{2}} \, \nu_{n-k+j-1}^{u^\perp}(\mathrm{d}U).\end{aligned}$$

From Vandermonde's identity we conclude that

$$\sum_{\alpha=(\gamma-2\beta)^+}^{\min\{s-2x,\gamma\}} \binom{s-2x}{\alpha}\binom{2\beta}{\gamma-\alpha} = \binom{s-2x+2\beta}{\gamma},$$

and thus

$$\begin{aligned}J_3 = {} & \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-k+j}{2})} \sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{\beta=0}^{y} \sum_{\gamma=0}^{\lfloor \frac{s}{2} \rfloor - x+\beta} (-1)^y \binom{s}{2x}\binom{x}{y}\binom{y}{\beta}\binom{s-2x+2\beta}{2\gamma} \Gamma(x+\tfrac{1}{2}) \\ & \times \Gamma(\tfrac{n-k+s}{2}-x) \frac{\Gamma(\frac{j+s}{2}-x+\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-x+\beta+1)} u^{s-2x+2\beta-2\gamma} Q^{x-y} \\ & \times \int_{G(u^\perp, n-k+j-1)} \left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta} Q(F\cap U)^l Q(U^\perp \cap u^\perp)^{\gamma} \, \nu_{n-k+j-1}^{u^\perp}(\mathrm{d}U).\end{aligned}$$

Furthermore, the term $Q(U^\perp \cap u^\perp)^\gamma = (Q(u^\perp) - Q(U))^\gamma$ can be expanded so that we obtain

$$\begin{aligned}J(\omega,\omega') = {} & \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k-j}{2})}{2\pi\Gamma(\frac{n-k+j}{2})\Gamma(\frac{n-j+s}{2})} \mathcal{H}^{k-j-1}(\omega') \sum_{x=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{y=0}^{x} \sum_{\beta=0}^{y} \sum_{\gamma=0}^{\lfloor \frac{s}{2} \rfloor - x+\beta} \sum_{\delta=0}^{\gamma} (-1)^{y+\delta} \\ & \times \binom{s}{2x}\binom{x}{y}\binom{y}{\beta}\binom{s-2x+2\beta}{2\gamma}\binom{\gamma}{\delta} \Gamma(x+\tfrac{1}{2})\Gamma(\tfrac{n-k+s}{2}-x) \\ & \times \frac{\Gamma(\frac{j+s}{2}-x+\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-x+\beta+1)} Q^{x-y} \int_{\omega} u^{s-2x+2\beta-2\gamma} Q(u^\perp)^{\gamma-\delta} \\ & \times \int_{G(u^\perp, n-k+j-1)} \left([F,U]^{(u^\perp)}\right)^2 Q(U)^{y-\beta+\delta} Q(F\cap U)^l \, \nu_{n-k+j-1}^{u^\perp}(\mathrm{d}U) \\ & \times \mathcal{H}^{n-k-1}(\mathrm{d}u).\end{aligned}$$

Reversing the order of summation, first with respect to β, and then with respect to y, we get

$$
\begin{aligned}
J(\omega,\omega') = {} & \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k-j}{2})}{2\pi\Gamma(\frac{n-k+j}{2})\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\sum_{x=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{y=0}^{x}\sum_{\beta=0}^{x-y}\sum_{\gamma=0}^{\lfloor\frac{s}{2}\rfloor-y-\beta}\sum_{\delta=0}^{\gamma}(-1)^{x+y+\delta} \\
& \times\binom{s}{2x}\binom{x}{y}\binom{x-y}{\beta}\binom{s-2y-2\beta}{2\gamma}\binom{\gamma}{\delta}\Gamma(x+\tfrac{1}{2})\Gamma(\tfrac{n-k+s}{2}-x) \\
& \times\frac{\Gamma(\frac{j+s}{2}-y-\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-y-\beta+1)}Q^y\int_\omega u^{s-2y-2\beta-2\gamma}Q(u^\perp)^{\gamma-\delta} \\
& \times\int_{\mathrm{G}(u^\perp,n-k+j-1)}\left([F,U]^{(u^\perp)}\right)^2Q(U)^{\beta+\delta}Q(F\cap U)^l\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U) \\
& \times\mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}
$$

A change of the order of summation yields

$$
\begin{aligned}
J(\omega,\omega') = {} & \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k-j}{2})}{2\pi\Gamma(\frac{n-k+j}{2})\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\sum_{y=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{\beta=0}^{\lfloor\frac{s}{2}\rfloor-y}\sum_{\gamma=0}^{\lfloor\frac{s}{2}\rfloor-y-\beta}\sum_{\delta=0}^{\gamma}(-1)^{y+\delta} \\
& \times\sum_{x=y+\beta}^{\lfloor\frac{s}{2}\rfloor}(-1)^x\binom{s}{2x}\binom{x}{y}\binom{x-y}{\beta}\Gamma(x+\tfrac{1}{2})\Gamma(\tfrac{n-k+s}{2}-x) \\
& \times\binom{s-2y-2\beta}{2\gamma}\binom{\gamma}{\delta}\frac{\Gamma(\frac{j+s}{2}-y-\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-y-\beta+1)} \\
& \times Q^y\int_\omega u^{s-2y-2\beta-2\gamma}Q(u^\perp)^{\gamma-\delta}\int_{\mathrm{G}(u^\perp,n-k+j-1)}\left([F,U]^{(u^\perp)}\right)^2 \\
& \times Q(U)^{\beta+\delta}Q(F\cap U)^l\,\nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U)\,\mathcal{H}^{n-k-1}(\mathrm{d}u).
\end{aligned}
$$

By Legendre's duplication formula, applied several times, we obtain

$$
\begin{aligned}
& \binom{s}{2x}\binom{x}{y}\binom{x-y}{\beta}\Gamma(x+\tfrac{1}{2}) \\
& \quad = \binom{s}{2y+2\beta}\binom{y+\beta}{y}\Gamma(y+\beta+\tfrac{1}{2})\frac{\Gamma(\frac{s+1}{2}-y-\beta)\Gamma(\frac{s}{2}-y-\beta+1)}{\Gamma(\frac{s+1}{2}-x)\Gamma(\frac{s}{2}-x+1)(x-y-\beta)!} \\
& \quad = \binom{s}{2y+2\beta}\binom{y+\beta}{y}\Gamma(y+\beta+\tfrac{1}{2})\binom{\lfloor\frac{s}{2}\rfloor-y-\beta}{x-y-\beta}\frac{\Gamma(\lfloor\frac{s+1}{2}\rfloor-y-\beta+\frac{1}{2})}{\Gamma(\lfloor\frac{s+1}{2}\rfloor-x+\frac{1}{2})}.
\end{aligned}
$$

We denote the resulting sum with respect to x by S_x. An index shift and a change of the order of summation imply that

$$
S_x = \sum_{x=0}^{\lfloor\frac{s}{2}\rfloor-y-\beta}(-1)^{\lfloor\frac{s}{2}\rfloor+x}\binom{\lfloor\frac{s}{2}\rfloor-y-\beta}{x}\frac{\Gamma(\frac{n-k+s}{2}-\lfloor\frac{s}{2}\rfloor+x)}{\Gamma(\lfloor\frac{s+1}{2}\rfloor-\lfloor\frac{s}{2}\rfloor+x+\frac{1}{2})}.
$$

Hence, an application of relation (B.1′) and then of relation (2.2) with $c=\frac{n-k+s}{2}-\lfloor\frac{s+1}{2}\rfloor-\frac{1}{2}$

and $m = \lfloor \frac{s}{2} \rfloor - y - \beta \in \mathbb{N}_0$ yield

$$\begin{aligned} S_x &= (-1)^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma(\frac{n-k+s}{2} - \lfloor \frac{s}{2} \rfloor)\Gamma(\overbrace{\lfloor \frac{s+1}{2} \rfloor + \lfloor \frac{s}{2} \rfloor}^{=s} - \frac{n-k+s}{2} - y - \beta + \frac{1}{2})}{\Gamma(\lfloor \frac{s+1}{2} \rfloor - y - \beta + \frac{1}{2})\Gamma(\lfloor \frac{s+1}{2} \rfloor - \frac{n-k+s}{2} + \frac{1}{2})} \\ &= \overbrace{(-1)^{s + \lfloor \frac{s}{2} \rfloor + \lfloor \frac{s+1}{2} \rfloor + y + \beta}}^{=(-1)^{y+\beta}} \frac{\overbrace{\Gamma(\frac{n-k+s}{2} - \lfloor \frac{s}{2} \rfloor)\Gamma(\frac{n-k+s}{2} - \lfloor \frac{s+1}{2} \rfloor + \frac{1}{2})}^{=\Gamma(\frac{n-k}{2})\Gamma(\frac{n-k+1}{2})}}{\Gamma(\lfloor \frac{s+1}{2} \rfloor - y - \beta + \frac{1}{2})\Gamma(\frac{n-k+1}{2} + y + \beta - \frac{s}{2})}, \end{aligned}$$

where we used that $c \geq 0$, except for $k = n-1$ and odd s when $c = -1/2$. We note that $S_x = 0$ if $n-k+s$ is odd and $n-k+1 \leq s - 2y - 2\beta$. Thus, we obtain

$$\begin{aligned} J(\omega, \omega') &= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{n-k+1}{2})\Gamma(\frac{k-j}{2})}{2\pi\Gamma(\frac{n-k+j}{2})\Gamma(\frac{n-j+s}{2})} \mathcal{H}^{k-j-1}(\omega') \sum_{y=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{s}{2} \rfloor - y} \sum_{\gamma=0}^{\lfloor \frac{s}{2} \rfloor - y - \beta} \sum_{\delta=0}^{\gamma} (-1)^{\beta+\delta} \\ &\quad \times \binom{s}{2y+2\beta} \binom{y+\beta}{y} \binom{s-2y-2\beta}{2\gamma} \binom{\gamma}{\delta} \frac{\Gamma(\frac{j+s}{2} - y - \beta - \gamma + 1)\Gamma(\gamma + \frac{1}{2})}{\Gamma(\frac{k+s}{2} - y - \beta + 1)} \\ &\quad \times \frac{\Gamma(y + \beta + \frac{1}{2})}{\Gamma(\frac{n-k+1}{2} + y + \beta - \frac{s}{2})} Q^y \int_\omega u^{s-2y-2\beta-2\gamma} Q(u^\perp)^{\gamma-\delta} \int_{\mathrm{G}(u^\perp, n-k+j-1)} \\ &\quad \times \left([F,U]^{(u^\perp)}\right)^2 Q(U)^{\beta+\delta} Q(F \cap U)^l \, \nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U) \, \mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

We conclude from Proposition 4.9 that

$$\begin{aligned} &\int_{\mathrm{G}(u^\perp, n-k+j-1)} \left([F,U]^{(u^\perp)}\right)^2 Q(U)^{\beta+\delta} Q(F \cap U)^l \, \nu^{u^\perp}_{n-k+j-1}(\mathrm{d}U) \\ &\qquad = \frac{(n-k+j-1)!k!}{(n-1)!j!} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{j}{2} + l)\Gamma(\frac{k}{2})}{\Gamma(\frac{n+1}{2} + \beta + \delta)\Gamma(\frac{j}{2})\Gamma(\frac{k-j}{2})\Gamma(\frac{n-k+j+1}{2})} \sum_{i=0}^{\beta+\delta} \binom{\beta+\delta}{i} \\ &\qquad\quad \times \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{k-j}{2} + i)\Gamma(\frac{n-k+j+1}{2} + \beta + \delta - i)}{\Gamma(\frac{k}{2} + l + i)} Q(u^\perp)^{\beta+\delta-i} Q(F)^{l+i}, \end{aligned}$$

and hence we get

$$\begin{aligned} J(\omega, \omega') &= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})(n-k+j-1)!}{(n-1)!\Gamma(\frac{n-k+j}{2})\Gamma(\frac{n-k+j+1}{2})} \frac{k!\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{n-k+1}{2})\Gamma(\frac{j}{2} + l)}{2\pi j!\Gamma(\frac{j}{2})\Gamma(\frac{n-j+s}{2})} \mathcal{H}^{k-j-1}(\omega') \\ &\quad \times \sum_{y=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{s}{2} \rfloor - y} \sum_{\gamma=0}^{\lfloor \frac{s}{2} \rfloor - y - \beta} \sum_{\delta=0}^{\gamma} \sum_{i=0}^{\beta+\delta} (-1)^{\beta+\delta} \binom{s}{2y+2\beta} \binom{y+\beta}{y} \binom{s-2y-2\beta}{2\gamma} \binom{\gamma}{\delta} \\ &\quad \times \binom{\beta+\delta}{i} \frac{\Gamma(\frac{j+s}{2} - y - \beta - \gamma + 1)\Gamma(\gamma + \frac{1}{2})}{\Gamma(\frac{k+s}{2} - y - \beta + 1)} \frac{\Gamma(y + \beta + \frac{1}{2})}{\Gamma(\frac{n-k+1}{2} + y + \beta - \frac{s}{2})} \\ &\quad \times \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{k-j}{2} + i)}{\Gamma(\frac{k}{2} + l + i)} \frac{\Gamma(\frac{n-k+j+1}{2} + \beta + \delta - i)}{\Gamma(\frac{n+1}{2} + \beta + \delta)} Q^y Q(F)^{l+i} \\ &\quad \times \int_\omega u^{s-2y-2\beta-2\gamma} Q(u^\perp)^{\beta+\gamma-i} \, \mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

To simplify the right-hand side we apply Legendre's duplication formula three times. Then binomial expansion of $Q(u^\perp)^{\beta+\gamma-i} = (Q-u^2)^{\beta+\gamma-i}$ and an index shift in the resulting sum yield

$$\begin{aligned} J(\omega,\omega') = {} & \frac{k!(n-k-1)!\Gamma(\frac{k}{2})\Gamma(\frac{j}{2}+l)}{2^{n-j}\sqrt{\pi}j!\Gamma(\frac{j}{2})\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\sum_{y=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{\beta=0}^{\lfloor\frac{s}{2}\rfloor-y}\sum_{\gamma=0}^{\lfloor\frac{s}{2}\rfloor-y-\beta}\sum_{\delta=0}^{\gamma}\sum_{i=0}^{\beta+\delta}\sum_{m=y+i}^{y+\beta+\gamma} \\ & \times (-1)^{m+y+\gamma+\delta}\binom{s}{2y+2\beta}\binom{y+\beta}{y}\binom{s-2y-2\beta}{2\gamma}\binom{\gamma}{\delta}\binom{\beta+\delta}{i}\binom{\beta+\gamma-i}{m-y-i} \\ & \times \frac{(i+l-2)!}{(l-2)!}\frac{\Gamma(y+\beta+\frac{1}{2})}{\Gamma(\frac{n-k+1}{2}+y+\beta-\frac{s}{2})}\frac{\Gamma(\frac{j+s}{2}-y-\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-y-\beta+1)} \\ & \times \frac{\Gamma(\frac{k-j}{2}+i)}{\Gamma(\frac{k}{2}+l+i)}\frac{\Gamma(\frac{n-k+j+1}{2}+\beta+\delta-i)}{\Gamma(\frac{n+1}{2}+\beta+\delta)}Q^{m-i}Q(F)^{l+i}\int_\omega u^{s-2m}\,\mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

An index shift in the summation with respect to β implies that

$$\begin{aligned} J(\omega,\omega') = {} & \frac{k!(n-k-1)!\Gamma(\frac{k}{2})\Gamma(\frac{j}{2}+l)}{2^{n-j}\sqrt{\pi}j!\Gamma(\frac{j}{2})\Gamma(\frac{n-j+s}{2})}\mathcal{H}^{k-j-1}(\omega')\sum_{y=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{\beta=y}^{\lfloor\frac{s}{2}\rfloor}\sum_{\gamma=0}^{\lfloor\frac{s}{2}\rfloor-\beta}\sum_{\delta=0}^{\gamma}\sum_{i=0}^{\beta+\delta-y}\sum_{m=y+i}^{\beta+\gamma} \\ & \times (-1)^{m+y+\gamma+\delta}\binom{s}{2\beta}\binom{\beta}{y}\binom{s-2\beta}{2\gamma}\binom{\gamma}{\delta}\binom{\beta+\delta-y}{i}\binom{\beta+\gamma-y-i}{m-y-i} \\ & \times \frac{(i+l-2)!}{(l-2)!}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})}\frac{\Gamma(\frac{j+s}{2}-\beta-\gamma+1)\Gamma(\gamma+\frac{1}{2})}{\Gamma(\frac{k+s}{2}-\beta+1)}\frac{\Gamma(\frac{k-j}{2}+i)}{\Gamma(\frac{k}{2}+l+i)} \\ & \times \frac{\Gamma(\frac{n-k+j+1}{2}+\beta+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\beta+\delta-y)}Q^{m-i}Q(F)^{l+i}\int_\omega u^{s-2m}\,\mathcal{H}^{n-k-1}(\mathrm{d}u). \end{aligned}$$

By a change of the order of summation we finally obtain

$$J(\omega,\omega') = \mathcal{H}^{k-j-1}(\omega')\sum_{m=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{i=0}^{m} b_{n,j,k}^{s,l,i}\,\hat{a}_{n,j,k}^{s,i,m}\,Q^{m-i}Q(F)^{l+i}\int_\omega u^{s-2m}\,\mathcal{H}^{n-k-1}(\mathrm{d}u),$$

where

$$\begin{aligned} b_{n,j,k}^{s,l,i} &:= \frac{\Gamma(\frac{k}{2})}{2^{n-j}\sqrt{\pi}\Gamma(\frac{j}{2})\Gamma(\frac{n-j+s}{2})}\frac{k!(n-k-1)!}{j!}\frac{(i+l-2)!}{(l-2)!}\frac{\Gamma(\frac{j}{2}+l)\Gamma(\frac{k-j}{2}+i)}{\Gamma(\frac{k}{2}+l+i)}, \\ \hat{a}_{n,j,k}^{s,i,m} &:= \sum_{y=0}^{m-i}\sum_{\beta=y}^{\lfloor\frac{s}{2}\rfloor}\sum_{\gamma=(m-\beta)^+}^{\lfloor\frac{s}{2}\rfloor-\beta}\sum_{\delta=(i-\beta+y)^+}^{\gamma}(-1)^{m+y+\gamma+\delta}\binom{s}{2\beta}\binom{\beta}{y}\binom{s-2\beta}{2\gamma}\binom{\gamma}{\delta} \\ & \quad\times\binom{\beta+\delta-y}{i}\binom{\beta+\gamma-y-i}{m-y-i}\Gamma(\beta+\tfrac{1}{2})\Gamma(\gamma+\tfrac{1}{2}) \\ & \quad\times\frac{\Gamma(\frac{j+s}{2}-\beta-\gamma+1)}{\Gamma(\frac{k+s}{2}-\beta+1)\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})}\frac{\Gamma(\frac{n-k+j+1}{2}+\beta+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\beta+\delta-y)}. \end{aligned}$$

4.4.4. The Simplification of the Coefficients

In this section, we simplify the coefficients $\hat{a}^{s,i,m}_{n,j,k}$ by a repeated change of the order of summation and by repeated application of relations (B.1) and (B.1′).

First, an index shift by β, applied to the summation with respect to γ, gives

$$\hat{a}^{s,i,m}_{n,j,k} = \sum_{y=0}^{m-i} \sum_{\beta=y}^{\lfloor \frac{s}{2} \rfloor} \sum_{\gamma=\max\{\beta,m\}}^{\lfloor \frac{s}{2} \rfloor} \sum_{\delta=(i-\beta+y)^+}^{\gamma-\beta} (-1)^{m+y+\gamma+\beta+\delta} \binom{s}{2\beta} \binom{\beta}{y} \binom{s-2\beta}{2\gamma-2\beta} \\ \times \binom{\gamma-\beta}{\delta} \binom{\beta+\delta-y}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(\beta+\tfrac{1}{2})\Gamma(\gamma-\beta+\tfrac{1}{2}) \\ \times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-\beta+1)\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\beta+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\beta+\delta-y)}.$$

A change of the order of summation yields

$$\hat{a}^{s,i,m}_{n,j,k} = \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\beta=y}^{\gamma} \sum_{\delta=(i-\beta+y)^+}^{\gamma-\beta} (-1)^{m+y+\gamma+\beta+\delta} \binom{s}{2\beta} \binom{\beta}{y} \binom{s-2\beta}{2\gamma-2\beta} \binom{\gamma-\beta}{\delta} \\ \times \binom{\beta+\delta-y}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(\beta+\tfrac{1}{2})\Gamma(\gamma-\beta+\tfrac{1}{2}) \\ \times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-\beta+1)\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\beta+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\beta+\delta-y)}.$$

Shifting the index of the summation with respect to δ by β, we obtain

$$\hat{a}^{s,i,m}_{n,j,k} = \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\beta=y}^{\gamma} \sum_{\delta=\max\{\beta,i+y\}}^{\gamma} (-1)^{m+y+\gamma+\delta} \binom{s}{2\beta} \binom{\beta}{y} \binom{s-2\beta}{2\gamma-2\beta} \\ \times \binom{\gamma-\beta}{\delta-\beta} \binom{\delta-y}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(\beta+\tfrac{1}{2})\Gamma(\gamma-\beta+\tfrac{1}{2}) \\ \times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-\beta+1)\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\delta-y)}.$$

A change of the order of summation gives

$$\hat{a}^{s,i,m}_{n,j,k} = \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\delta=i+y}^{\gamma} \sum_{\beta=y}^{\delta} (-1)^{m+y+\gamma+\delta} \binom{s}{2\beta} \binom{\beta}{y} \binom{s-2\beta}{2\gamma-2\beta} \\ \times \binom{\gamma-\beta}{\delta-\beta} \binom{\delta-y}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(\beta+\tfrac{1}{2})\Gamma(\gamma-\beta+\tfrac{1}{2}) \\ \times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-\beta+1)\Gamma(\frac{n-k+1}{2}+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\delta-y)}.$$

We conclude from an index shift by y, applied to the summation with respect to β,

$$\hat{a}_{n,j,k}^{s,i,m} = \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\delta=i+y}^{\gamma} \sum_{\beta=0}^{\delta-y} (-1)^{m+y+\gamma+\delta} \binom{s}{2y+2\beta} \binom{y+\beta}{y} \binom{s-2y-2\beta}{2\gamma-2y-2\beta}$$
$$\times \binom{\gamma-y-\beta}{\delta-y-\beta} \binom{\delta-y}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(y+\beta+\tfrac{1}{2}) \Gamma(\gamma-y-\beta+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-y-\beta+1)\Gamma(\frac{n-k+1}{2}+y+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\delta-y-i)}{\Gamma(\frac{n+1}{2}+\delta-y)},$$

and by $-i-y$, applied to the summation with respect to δ,

$$\hat{a}_{n,j,k}^{s,i,m} = \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\delta=0}^{\gamma-y-i} \sum_{\beta=0}^{i+\delta} (-1)^{i+m+\gamma+\delta} \binom{s}{2y+2\beta} \binom{y+\beta}{y} \binom{s-2y-2\beta}{2\gamma-2y-2\beta}$$
$$\times \binom{\gamma-y-\beta}{i+\delta-\beta} \binom{i+\delta}{i} \binom{\gamma-y-i}{m-y-i} \Gamma(y+\beta+\tfrac{1}{2}) \Gamma(\gamma-y-\beta+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-y-\beta+1)\Gamma(\frac{n-k+1}{2}+y+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\delta)}{\Gamma(\frac{n+1}{2}+i+\delta)}.$$

With Legendre's duplication formula (applied three times) we obtain

$$\binom{s}{2y+2\beta} \binom{y+\beta}{y} \binom{s-2y-2\beta}{2\gamma-2y-2\beta} \binom{\gamma-y-\beta}{i+\delta-\beta} \binom{i+\delta}{i} \binom{\gamma-y-i}{m-y-i}$$
$$\times \Gamma(y+\beta+\tfrac{1}{2}) \Gamma(\gamma-y-\beta+\tfrac{1}{2})$$
$$= \binom{s}{2i} \binom{m-i}{y} \binom{\gamma-i}{m-i} \binom{s-2i}{2\gamma-2i} \binom{\gamma-y-i}{\delta} \binom{i+\delta}{\beta} \Gamma(i+\tfrac{1}{2}) \Gamma(\gamma-i+\tfrac{1}{2}),$$

and hence

$$\hat{a}_{n,j,k}^{s,i,m} = \Gamma(i+\tfrac{1}{2}) \binom{s}{2i} \sum_{y=0}^{m-i} \sum_{\gamma=m}^{\lfloor \frac{s}{2} \rfloor} \sum_{\delta=0}^{\gamma-y-i} \sum_{\beta=0}^{i+\delta} (-1)^{i+m+\gamma+\delta}$$
$$\times \binom{m-i}{y} \binom{\gamma-i}{m-i} \binom{s-2i}{2\gamma-2i} \binom{\gamma-y-i}{\delta} \binom{i+\delta}{\beta} \Gamma(\gamma-i+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2}-\gamma+1)}{\Gamma(\frac{k+s}{2}-y-\beta+1)\Gamma(\frac{n-k+1}{2}+y+\beta-\frac{s}{2})} \frac{\Gamma(\frac{n-k+j+1}{2}+\delta)}{\Gamma(\frac{n+1}{2}+i+\delta)}.$$

Now we define $a_{n,j,k}^{s,i,m} := (\Gamma(i+\frac{1}{2})\binom{s}{2i})^{-1} \hat{a}_{n,j,k}^{s,i,m}$. We first use relation (B.1) and then apply relation (B.1′) twice. Thus we obtain

$$\sum_{\beta=0}^{i+\delta}\binom{i+\delta}{\beta}\frac{1}{\Gamma(\frac{k+s}{2}-y-\beta+1)\Gamma(\frac{n-k-s+1}{2}+y+\beta)}$$
$$=\frac{\Gamma(\frac{n+1}{2}+i+\delta)}{\Gamma(\frac{k+s}{2}-y+1)\Gamma(\frac{n-k-s+1}{2}+i+y+\delta)\Gamma(\frac{n+1}{2})},$$

$$\sum_{\delta=0}^{\gamma-y-i}(-1)^{\delta}\binom{\gamma-y-i}{\delta}\frac{\Gamma(\frac{n-k+j+1}{2}+\delta)}{\Gamma(\frac{n-k-s+1}{2}+i+y+\delta)}$$
$$=\frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(-\frac{j+s}{2}+\gamma)}{\Gamma(\frac{n-k-s+1}{2}+\gamma)\Gamma(-\frac{j+s}{2}+i+y)}$$
$$=(-1)^{i+\gamma+y}\frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{j+s}{2}-i-y+1)}{\Gamma(\frac{n-k-s+1}{2}+\gamma)\Gamma(\frac{j+s}{2}-\gamma+1)},$$

where we used (2.2) with $c=\frac{j+s}{2}-i-y\geq 0$ and $m=\gamma-i-y\in\mathbb{N}_0$ in the second step, and

$$\sum_{y=0}^{m-i}(-1)^{m+y}\binom{m}{y}\frac{\Gamma(\frac{j+s}{2}-i-y+1)}{\Gamma(\frac{k+s}{2}-y+1)}=\sum_{y=0}^{m-i}(-1)^{i+y}\binom{m}{y}\frac{\Gamma(\frac{j+s}{2}-m+y+1)}{\Gamma(\frac{k+s}{2}+i-m+y+1)}$$
$$=(-1)^{i}\frac{\Gamma(\frac{j+s}{2}-m+1)\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k+s}{2}+1)\Gamma(\frac{k-j}{2}+i)}.$$

This gives

$$a_{n,j,k}^{s,i,m}=(-1)^{i}\frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{j+s}{2}-m+1)\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{k+s}{2}+1)\Gamma(\frac{k-j}{2}+i)}$$
$$\times\sum_{\gamma=m}^{\lfloor\frac{s}{2}\rfloor}\binom{\gamma-i}{m-i}\binom{s-2i}{2\gamma-2i}\frac{\Gamma(\gamma-i+\frac{1}{2})}{\Gamma(\frac{n-k-s+1}{2}+\gamma)}.$$

We deduce from Legendre's duplication formula that

$$\binom{\gamma-i}{m-i}\binom{s-2i}{2\gamma-2i}\Gamma(\gamma-i+\tfrac{1}{2})=\frac{\sqrt{\pi}}{(m-i)!}\frac{\Gamma(\frac{s+1}{2}-i)\Gamma(\frac{s}{2}-i+1)}{(\gamma-m)!\Gamma(\frac{s+1}{2}-\gamma)\Gamma(\frac{s}{2}-\gamma+1)}$$
$$=\sqrt{\pi}\binom{\lfloor\frac{s}{2}\rfloor-i}{m-i}\binom{\lfloor\frac{s}{2}\rfloor-m}{\gamma-m}\frac{\Gamma(\lfloor\frac{s+1}{2}\rfloor-i+\frac{1}{2})}{\Gamma(\lfloor\frac{s+1}{2}\rfloor-\gamma+\frac{1}{2})}.$$

Denoting the remaining sum in $a_{n,j,k}^{s,i,m}$ with respect to γ by S_4, we obtain

$$S_4=\sum_{\gamma=0}^{\lfloor\frac{s}{2}\rfloor-m}\binom{\lfloor\frac{s}{2}\rfloor-m}{\gamma}\frac{1}{\Gamma(\lfloor\frac{s+1}{2}\rfloor-m-\gamma+\frac{1}{2})\Gamma(\frac{n-k-s+1}{2}+m+\gamma)},$$

for which relation (B.1) yields

$$\begin{aligned}S_4 &= \frac{\Gamma(\frac{n-k-s}{2}+\lfloor\frac{s+1}{2}\rfloor+\lfloor\frac{s}{2}\rfloor-m)}{\Gamma(\lfloor\frac{s+1}{2}\rfloor-m+\frac{1}{2})\Gamma(\frac{n-k+1}{2}+\lfloor\frac{s}{2}\rfloor-\frac{s}{2})\Gamma(\frac{n-k}{2}+\lfloor\frac{s+1}{2}\rfloor-\frac{s}{2})}.\\ &= \frac{\Gamma(\frac{n-k+s}{2}-m)}{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\lfloor\frac{s+1}{2}\rfloor-m+\frac{1}{2})}.\end{aligned}$$

We obtain from Legendre's duplication formula

$$\begin{aligned}&\sqrt{\pi}\binom{\lfloor\frac{s}{2}\rfloor-i}{m-i}\Gamma(\lfloor\tfrac{s+1}{2}\rfloor-i+\tfrac{1}{2})S_4\\ &\quad= \frac{\Gamma(\frac{n-k+s}{2}-m)}{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})}\frac{\sqrt{\pi}}{(m-i)!}\frac{\Gamma(\frac{s}{2}-i+1)\Gamma(\frac{s+1}{2}-i)}{\Gamma(\frac{s}{2}-m+1)\Gamma(\frac{s+1}{2}-m)}\\ &\quad= \frac{\Gamma(\frac{n-k+s}{2}-m)}{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})}\binom{s-2i}{2m-2i}\Gamma(m-i+\tfrac{1}{2}).\end{aligned}$$

This gives

$$\begin{aligned}a_{n,j,k}^{s,i,m} &= (-1)^i\binom{s-2i}{2m-2i}\Gamma(m-i+\tfrac{1}{2})\frac{\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{k+s}{2}+1)}\\ &\quad\times\frac{\Gamma(\frac{n-k+s}{2}-m)\Gamma(\frac{j+s}{2}-m+1)\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2}+i)}.\end{aligned}$$

Next, using

$$\binom{s}{2i}\binom{s-2i}{2m-2i}\Gamma(i+\tfrac{1}{2})\Gamma(m-i+\tfrac{1}{2}) = \frac{s!}{(s-2m)!}\frac{\pi}{2^{2m}i!(m-i)!},$$

$$\frac{(n-k-1)!k!}{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})j!} = \frac{2^{n-j-1}}{\sqrt{\pi}}\frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k+1}{2})}{\Gamma(\frac{j}{2}+1)\Gamma(\frac{j+1}{2})},$$

and

$$\frac{\Gamma(\frac{j}{2}+l)\Gamma(\frac{n-k+s}{2}-m)\Gamma(\frac{k}{2})}{\Gamma(\frac{n-j+s}{2})\Gamma(\frac{j}{2})\Gamma(\frac{k}{2}+l+i)} = \sqrt{\pi}^{j-k-2i-2m}\frac{\omega_{n-j+s}\omega_j\omega_{k+2l+2i}}{\omega_{j+2l}\omega_{n-k+s-2m}\omega_k},$$

we get

$$\begin{aligned}c_{n,j,k}^{s,l,i,m} &:= \frac{\omega_{n-k}\omega_{k-j}}{\omega_{n-j}}\frac{c_{n,j}^{r,s,l}}{c_{n,k}^{r,s-2m,l+i}}\binom{s}{2i}\Gamma(i+\tfrac{1}{2})b_{n,j,k}^{s,l,i}a_{n,j,k}^{s,i,m}\\ &= (-1)^i\binom{s}{2i}\binom{s-2i}{2m-2i}\Gamma(i+\tfrac{1}{2})\Gamma(m-i+\tfrac{1}{2})\frac{(n-k-1)!k!}{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n-k}{2})j!}\\ &\qquad\times\frac{(i+l-2)!}{(l-2)!}\frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{j+s}{2}-m+1)\Gamma(\frac{k-j}{2}+m)}{2^{n-j}\sqrt{\pi}\Gamma(\frac{n+1}{2})\Gamma(\frac{k+s}{2}+1)}\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\Gamma(\frac{j}{2}+l)\Gamma(\frac{n-k+s}{2}-m)\Gamma(\frac{k}{2})}{\Gamma(\frac{n-j+s}{2})\Gamma(\frac{j}{2})\Gamma(\frac{k}{2}+l+i)} \frac{2\sqrt{\pi}^{k-j}}{\Gamma(\frac{k-j}{2})} \frac{\omega_{n-k}}{\omega_{n-j}} \frac{c_{n,j}^{r,s,l}}{c_{n,k}^{r,s-2m,l+i}} \\
= (-1)^i &\frac{1}{4^m i!(m-i)!} \frac{1}{\pi^{i+m}} \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \\
&\times \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})},
\end{aligned}
$$

which yields the assertion.

Finally, returning to (4.7) and using the definition of the tensorial curvature measures, we get

$$
\begin{aligned}
I_1 = &\sum_{k=j+1}^{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,k}^{s,l,i,m}\, Q^{m-i} \frac{1}{\omega_{n-k}} c_{n,k}^{r,s-2m,l+i} \\
&\times \sum_{F \in \mathcal{F}_k(P)} Q(F)^{l+i} \int_{F\cap\beta} x^r\, \mathcal{H}^k(\mathrm{d}x) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s-2m}\, \mathcal{H}^{n-k-1}(\mathrm{d}u) \\
&\times \frac{1}{\omega_{k-j}} \sum_{F' \in \mathcal{F}_{n-k+j}(P')} \mathcal{H}^{n-k+j}(F'\cap\beta')\mathcal{H}^{k-j-1}\left(N(P',F')\cap\mathbb{S}^{n-1}\right) \\
&+ \phi_j^{r,s,l}(P,\beta)\phi_n(P',\beta') + c_{n,j}^{s}\, \phi_n^{r,0,\frac{s}{2}+l}(P,\beta)\phi_j(P',\beta') \\
= &\sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,k}^{s,l,i,m}\, Q^{m-i} \phi_k^{r,s-2m,l+i}(P,\beta)\phi_{n-k+j}(P',\beta').
\end{aligned}
$$

In the last step, we use that for $k = j$ we have $c_{n,j,j}^{s,l,i,m} = \mathbb{1}\{i = m = 0\}$. Moreover, in the case $k = n$ we use that $\phi_n^{r,s-2m,l+i}$ vanishes for $m \neq \frac{s}{2}$. Hence, for even s we have to simplify the sum

$$
\begin{aligned}
&\sum_{i=0}^{\frac{s}{2}} c_{n,j,n}^{s,l,i,\frac{s}{2}}\, Q^{\frac{s}{2}-i} \phi_n^{r,0,l+i}(P,\beta)\phi_j(P',\beta') \\
&\qquad = \sum_{i=0}^{\frac{s}{2}} c_{n,j,n}^{s,l,i,\frac{s}{2}} \frac{\omega_{n+2l+2i}}{\omega_{n+s+2l}} \phi_n^{r,0,\frac{s}{2}+l}(P,\beta)\phi_j(P',\beta').
\end{aligned}
$$

For this, an application of relation (B.1′) yields

$$
\begin{aligned}
&\frac{1}{(2\pi)^s\,(\frac{s}{2})!} \frac{\Gamma(\frac{n+s}{2}+l)}{\Gamma(l-1)} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+s}{2}+1)} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})} \sum_{i=0}^{\frac{s}{2}} (-1)^i \binom{\frac{s}{2}}{i} \frac{\Gamma(i+l-1)}{\Gamma(\frac{n}{2}+l+i)} \\
&\qquad \times \phi_n^{r,0,\frac{s}{2}+l}(P,\beta)\phi_j(P',\beta') \\
&= c_{n,j}^{s}\, \phi_n^{r,0,\frac{s}{2}+l}(P,\beta)\phi_j(P',\beta'),
\end{aligned}
\tag{4.9}
$$

as required. This completes the proof.

CHAPTER 5

Crofton Formulae

In this chapter, we establish a complete set of Crofton formulae for the tensorial curvature measures of polytopes. That is, for $P \in \mathcal{P}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$, we explicitly express integrals of the form

$$\int_{\mathrm{A}(n,k)} \phi_j^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$

in terms of generalized tensorial curvature measures of P, evaluated at β. Furthermore, since the tensorial measures $\phi_j^{r,s,l}$ can be continuously extended to mappings defined on $\mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n)$, for $l = 0, 1$, we also consider the Crofton integrals

$$\int_{\mathrm{A}(n,k)} \phi_j^{r,s,l}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$

for $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, $l = 0, 1$. Moreover, we point out several special cases of these formulae which can be simplified even further.

Since the generalized tensorial curvature measures depend additively on the underlying convex body (resp. polytope), all integral formulae in this chapter remain true if the occurring convex bodies (resp. polytopes) are replaced by finite unions of convex bodies (resp. polytopes).

In the proof of the Crofton formula for generalized tensorial curvature measures on polytopes, we make use of a well-known connection to the corresponding kinematic formula, which is already applied in the proof of the classical Crofton formula for curvature measures (see [83, Theorem 4.4.5]). More intuitively, we choose the polytope in the kinematic integral that is uniformly moved by the rigid motion to be of a suitable dimension and "sufficiently large" in a sense that the integration goes over to the Crofton integral in which the affine hull of the polytope is moved uniformly.

Remark. The results in this chapter have already been submitted. To a great extent the present chapter contains direct quotes from the publication *Crofton Formulae for Tensorial Curvature Measures: The General Case*, a joint work with Daniel Hug, submitted in 2016 (see [55]).

5.1. The Results of Chapter 5

We present the Crofton formulae in two steps. We start with results for the generalized tensorial curvature measures on polytopes. Then we state the formulae for the ones with existing extension to convex bodies. For the latter we deduce some special cases and point out the connection to the extrinsic results from Chapter 8.

5.1.1. Generalized Tensorial Curvature Measures on Polytopes

As explained before, we start with the Crofton formulae for the generalized tensorial curvature measures on polytopes. First, we separately state a formula for $j = k$.

Theorem 5.1. *Let $P \in \mathcal{P}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $k, r, s, l \in \mathbb{N}_0$ with $k \leq n$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_k^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \left(\frac{s}{2}\right)!} \frac{\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)} \phi_n^{r,0,\frac{s}{2}+l}(P, \beta).$$

Theorem 5.1 generalizes Theorem 2.1 in [51]. In fact, setting $l = 0$ and $\beta = \mathbb{R}^n$ one obtains the known result for Minkowski tensors. If $l \in \{0, 1\}$, one can even formulate Theorem 5.1 for a convex body, as in both of these cases all appearing valuations are defined on $\mathcal{K}^n$. For $k = n$, the integral on the left-hand side of the formula in Theorem 5.1 is trivial. However, we note that on the right-hand side the quotient of the Gamma functions has to be interpreted as $\mathbb{1}\{s = 0\}$, according to (2.2).

Next, we state the formulae for general $j < k$.

Theorem 5.2. *Let $P \in \mathcal{P}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $j, k, r, s, l \in \mathbb{N}_0$ with $j < k \leq n$, and with $l = 0$ if $j = 0$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_j^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} d_{n,j,k}^{s,l,i,m}\, Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta),$$

where

$$d_{n,j,k}^{s,l,i,m} := \frac{(-1)^i}{(4\pi)^m m!}\frac{\binom{m}{i}}{\pi^i}\frac{(i+l-2)!}{(l-2)!}\frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})}$$
$$\times \frac{\Gamma(\frac{n-k+j}{2}+1)}{\Gamma(\frac{n-k+j+s}{2}+1)}\frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{j}{2}+1)}\frac{\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n-k}{2})}.$$

For $k = n$ the coefficient in Theorem 5.2 has to be interpreted as

$$d_{n,j,n}^{s,l,i,m} = \mathbb{1}\{i = m = 0\},$$

according to (2.2), so that the result is a tautology in this case.

The coefficients $d_{n,j,k}^{s,l,i,m}$ in Theorem 5.2 are well-known from the kinematic formulae in Theorem 4.1. In fact, they satisfy the relation

$$d_{n,j,k}^{s,l,i,m} = c_{n,j,n-k+j}^{s,l,i,m}. \tag{5.1}$$

However, we restate them here for the sake of clarity.

Several remarkable facts concerning the coefficients $d_{n,j,k}^{s,l,i,m}$ should be recalled from Chapter 4. First, the ratio $(i+l-2)!/(l-2)!$ has to be interpreted in terms of Gamma functions and relation (2.2) if $l \in \{0,1\}$. The corresponding special cases will be considered separately in the following two theorems and the subsequent corollaries. Second, due to our normalization of the generalized tensorial curvature measures, the coefficients are independent of the tensorial parameter r and depend only on l through the ratio $(i+l-2)!/(l-2)!$. Third, only tensors $\phi_{n-k+j}^{r,s-2m,p}(P,\beta)$ with $p \geq l$ show up on the right side of the kinematic formula. Using Legendre's duplication formula, we could shorten the given expressions for the coefficients $d_{n,j,k}^{s,l,i,m}$ even further. However, the present form has the advantage of exhibiting that the factors in the second line cancel each other if $s = 0$ (and hence also $m = i = 0$). Furthermore, in general the coefficients are signed in contrast to the classical kinematic formula. We shall see below that for $l \in \{0,1\}$ all coefficients are non-negative.

5.1.2. (Generalized) Tensorial Curvature Measures on Convex Bodies

For $l \in \{0,1\}$, the generalized tensorial curvature measures $\phi_j^{r,s,l}$ can be continuously extended to all convex bodies. In these two cases, Theorem 5.1 (in which $j = k$) holds for general convex bodies as well. For this reason, we restrict our attention to the cases where $j < k$ in the following. The next theorems are stated without a proof, as they basically follow from Theorem 5.2 and approximation of the given convex body by polytopes (using the weak continuity of the curvature measures and the usual arguments needed to take care of exceptional positions).

We start with the formula for $l = 1$.

Theorem 5.3. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $j,k,r,s \in \mathbb{N}_0$ with $0 < j < k \le n$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_j^{r,s,1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} d_{n,j,k}^{s,1,0,m}\, Q^m \phi_{n-k+j}^{r,s-2m,1}(K,\beta),$$

where

$$d_{n,j,k}^{s,1,0,m} = \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \times \frac{\Gamma(\frac{n-k+j}{2}+1)}{\Gamma(\frac{n-k+j+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n-k}{2})}.$$

Next, we state the formula for $l = 0$.

Theorem 5.4. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $j,k,r,s \in \mathbb{N}_0$ with $j < k \le n$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_j^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{1} d_{n,j,k}^{s,0,i,m}\, Q^{m-i} \phi_{n-k+j}^{r,s-2m,i}(K,\beta),$$

where

$$d_{n,j,k}^{s,0,i,m} = \frac{1}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \times \frac{\Gamma(\frac{n-k+j}{2}+1)}{\Gamma(\frac{n-k+j+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m+1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n-k}{2})}.$$

In Theorem 5.4, we have $d_{n,j,k}^{s,0,1,0} = 0$ so that, in fact, the undefined tensor Q^{-1} does not appear.

For the special case $j = k-1$, we deduce two more Crofton formulae. The first concerns the generalized tensorial curvature measures $\phi_{k-1}^{r,s,1}$.

Corollary 5.5. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $k,r,s \in \mathbb{N}_0$ with $0 < k < n$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_{k-1}^{r,s,1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m,1}(K,\beta),$$

where

$$\iota_{n,k}^{s,m} := \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}.$$

Due to the easily verified relation

$$\phi_{n-1}^{r,s-2m,1} = \frac{2\pi}{n-1}\left(Q\phi_{n-1}^{r,s-2m,0} - 2\pi(s-2m+2)\phi_{n-1}^{r,s-2m+2,0}\right), \tag{5.2}$$

Corollary 5.5 can be transformed in such a way that only the tensorial curvature measures $\phi_{n-1}^{r,s-2m,0}$ are involved on the right-hand side of the preceding formula. This is presented in the following corollary.

Corollary 5.6. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then,*

$$\int_{\mathrm{A}(n,k)} \phi_{k-1}^{r,s,1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor + 1} \lambda_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta),$$

where

$$\begin{aligned} \lambda_{n,k}^{s,m} := &\frac{\pi}{(n-1)(4\pi)^{m-1} m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - m)\Gamma(\frac{n-k}{2} + m - 1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\ &\times \Big(2m(\tfrac{k+s+1}{2} - m) - (s - 2m + 2)(\tfrac{n-k}{2} + m - 1)\Big), \end{aligned}$$

for $m \in \{1, \dots, \lfloor \frac{s}{2} \rfloor\}$, and

$$\begin{aligned} \lambda_{n,k}^{s,0} &:= -\frac{4\pi^2 (s+2)}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})}, \\ \lambda_{n,k}^{s,\lfloor \frac{s}{2} \rfloor + 1} &:= \frac{2\pi}{(n-1)(4\pi)^{\lfloor \frac{s}{2} \rfloor} (\lfloor \frac{s}{2} \rfloor)!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - \lfloor \frac{s}{2} \rfloor)\Gamma(\frac{n-k}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}. \end{aligned}$$

The second special case concerns the tensorial curvature measures $\phi_{k-1}^{r,s,0}$. Although this result is also derived in a different way from the intrinsic Crofton formulae in Chapter 8 (see Theorem 8.11 and its proof in Section 8.3), we state it and derive it here as a special case of the present more general approach.

Corollary 5.7. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta),$$

where

$$\kappa_{n,k}^{s,m} := \frac{k-1}{n-1} \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2} - m)\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}$$

if $m \neq \frac{s-1}{2}$, and

$$\kappa_{n,k}^{s,\frac{s-1}{2}} := \frac{k(n+s-2)}{2(n-1)} \frac{1}{(4\pi)^{\frac{s-1}{2}} \frac{s-1}{2}!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{n-k}{2})}.$$

Finally, we state the remaining case where $k = 1$ (see also [54, Theorem 4.13]).

Corollary 5.8. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and $r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{A}(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) = \frac{\Gamma(\frac{s}{2} - \lfloor \frac{s}{2} \rfloor + 1)}{\sqrt{\pi}(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor !} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+s+1}{2})} Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta).$$

Comparing Corollary 5.7 and Corollary 5.8 to the corresponding results in Chapter 8, it should be observed that the normalization of the tensorial measures in Chapter 8 is different from the current normalization (although the measures are denoted in the same way).

5.2. The Proofs of the Crofton Formulae

In this section, we prove the Crofton formulae which have been stated in Section 5.1. The proof uses the connection to the corresponding (more general) kinematic formulae. For the classical scalar-valued curvature measures this connection is well-known (see for example [83, Theorem 4.4.5]).

We start by proving both, the Crofton formulae in Theorem 5.1 and Theorem 5.2, at once using Theorem 4.1.

Proof of Theorem 5.1 and Theorem 5.2. Let $P \in \mathcal{P}^n$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$. First, we prove the identity

$$J := \int_{\mathrm{A}(n,k)} \phi_j^{r,s,l}(P \cap E, \beta)\, \mu_k(\mathrm{d}E) = \int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gE_k, \beta \cap g\alpha)\, \mu(\mathrm{d}g) \tag{5.3}$$

for an arbitrary (but fixed) $E_k \in \mathrm{G}(n,k)$ and $\alpha \in \mathcal{B}(E_k)$ with $\mathcal{H}^k(\alpha) = 1$, where we define J to be the Crofton integral in which we are interested. This is shown as follows. Using (2.1), we obtain

$$J = \int_{\mathrm{SO}(n)} \int_{E_k^\perp} \int_{\mathbb{R}^n} \mathbb{1}_\beta(x)\, \phi_j^{r,s,l}(P \cap \rho(E_k + t_1), \mathrm{d}x)\, \mathcal{H}^{n-k}(\mathrm{d}t_1)\, \nu(\mathrm{d}\rho).$$

For $t_1 \in E_k^\perp$ and $x \in \rho(E_k + t_1)$ we have

$$x \in \rho(\alpha + t_1 + t_2) \Leftrightarrow t_2 \in -\alpha + \rho^{-1}x - t_1,$$

for all $t_2 \in E_k$. Moreover, $-\alpha + \rho^{-1}x - t_1 \subset E_k$, since $\alpha \subset E_k$ and $x \in \rho(E_k + t_1)$ yields $\rho^{-1}x - t_1 \in E_k$. Thus, we get

$$\mathcal{H}^k\left(\{t_2 \in E_k : x \in \rho(\alpha + t_1 + t_2)\}\right) = \mathcal{H}^k(-\alpha + \rho^{-1}x - t_1) = \mathcal{H}^k(\alpha) = 1,$$

and hence we have

$$\begin{aligned} J &= \int_{\mathrm{SO}(n)} \int_{E_k^\perp} \int_{\mathbb{R}^n} \mathbb{1}_\beta(x) \int_{E_k} \mathbb{1}\{x \in \rho(\alpha + t_1 + t_2)\} \, \mathcal{H}^k(\mathrm{d}t_2) \\ &\qquad \times \phi_j^{r,s,l}(P \cap \rho(E_k + t_1), \mathrm{d}x) \, \mathcal{H}^{n-k}(\mathrm{d}t_1) \, \nu(\mathrm{d}\rho) \\ &= \int_{\mathrm{SO}(n)} \int_{E_k^\perp} \int_{E_k} \int_{\mathbb{R}^n} \mathbb{1}_{\beta \cap \rho(\alpha + t_1 + t_2)}(x) \, \phi_j^{r,s,l}(P \cap \rho(E_k + t_1 + t_2), \mathrm{d}x) \\ &\qquad \times \mathcal{H}^k(\mathrm{d}t_2) \, \mathcal{H}^{n-k}(\mathrm{d}t_1) \, \nu(\mathrm{d}\rho). \end{aligned}$$

Finally, Fubini's theorem yields

$$\begin{aligned} J &= \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \phi_j^{r,s,l}(P \cap \rho(E_k + t), \beta \cap \rho(\alpha + t)) \, \mathcal{H}^n(\mathrm{d}t) \, \nu(\mathrm{d}\rho) \\ &= \int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gE_k, \beta \cap g\alpha) \, \mu(\mathrm{d}g), \end{aligned}$$

which concludes the proof of (5.3).

Let $\alpha \in \mathcal{B}(\mathbb{R}^n)$ be compact with $\alpha \subset E_k$ and $\mathcal{H}^k(\alpha) = 1$. Then choose $P' \in \mathcal{P}^n$ with $P' \subset E_k$ and $\alpha \subset \operatorname{relint} P'$, such that the following holds, for all $g \in \mathrm{G}_n$: If $g^{-1}P \cap \alpha \neq \emptyset$, then $g^{-1}P \cap E_k \subset P'$. Hence, if $P \cap g\alpha \neq \emptyset$, then $P \cap gE_k = P \cap gP'$. Thus we obtain

$$J = \int_{\mathrm{G}_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\alpha) \, \mu(\mathrm{d}g),$$

and therefore, by Theorem 4.1

$$J = \sum_{p=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} c_{n,j,p}^{s,l,i,m} \, Q^{m-i} \phi_p^{r,s-2m,l+i}(P, \beta) \phi_{n-p+j}(P', \alpha).$$

Hence, if $k = j$ we get

$$\begin{aligned} J &= \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \, (\frac{s}{2})!} \frac{\Gamma(\frac{n-k+s}{2})}{\Gamma(\frac{n-k}{2})} \, \phi_n^{r,0,\frac{s}{2}+l}(P, \beta) \, \underbrace{\phi_k(P', \alpha)}_{= \mathcal{H}^k(\alpha) = 1} \\ &= \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \frac{s}{2}!} \frac{\Gamma(\frac{n-k+s}{2})}{\Gamma(\frac{n-k}{2})} \, \phi_n^{r,0,\frac{s}{2}+l}(P, \beta), \end{aligned}$$

and for $j < k$ we get

$$\begin{aligned} J &= \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} \underbrace{c_{n,j,n-k+j}^{s,l,i,m}}_{=: d_{n,j,k}^{s,l,i,m}} \, Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta) \, \underbrace{\phi_k(P', \alpha)}_{= \mathcal{H}^k(\alpha) = 1} \\ &= \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^{m} d_{n,j,k}^{s,l,i,m} \, Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta), \end{aligned}$$

since $\phi_q(P', \alpha) = 0$ for $q \neq k$. □

Next, we prove Corollary 5.5 and Corollary 5.6, which are derived from Theorem 5.3. The first follows immediately, whereas the second subsequently is obtained by an application of relation (5.2).

Proof of Corollary 5.5 and Corollary 5.6. In both cases, we denote the integral we are interested in by I. First, we consider Corollary 5.5 and hence $l = 1$ in the general formulae. In this case, Theorem 5.3 yields

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m,1}(K, \beta),$$

where

$$\iota_{n,k}^{s,m} := d_{n,k-1,k}^{s,1,0,m} = \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - m)\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})},$$

which already proves Corollary 5.5.

Next, we turn to the proof of Corollary 5.6. From Corollary 5.5 and (5.2), we conclude that

$$\begin{aligned}
I &= \frac{2\pi}{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m} Q^{m+1} \phi_{n-1}^{r,s-2m,0}(K, \beta) - 2\pi(s - 2m + 2)\iota_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\
&= \frac{2\pi}{n-1} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor + 1} \iota_{n,k}^{s,m-1}\, Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\
&\quad - \frac{2\pi}{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} 2\pi(s - 2m + 2)\iota_{n,k}^{s,m}\, Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\
&= \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \frac{2\pi}{n-1} \left(\iota_{n,k}^{s,m-1} - 2\pi(s - 2m + 2)\iota_{n,k}^{s,m} \right) Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\
&\quad + \frac{2\pi}{n-1} \iota_{n,k}^{s,\lfloor \frac{s}{2} \rfloor}\, Q^{\lfloor \frac{s}{2} \rfloor + 1} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta) - \frac{4\pi^2(s+2)}{n-1} \iota_{n,k}^{s,0}\, \phi_{n-1}^{r,s+2,0}(K, \beta).
\end{aligned}$$

Denoting the coefficients by $\lambda_{n,k}^{s,m}$, we obtain for $m \in \{1, \ldots, \lfloor \frac{s}{2} \rfloor\}$

$$\begin{aligned}
\lambda_{n,k}^{s,m} = &\frac{\pi}{(n-1)(4\pi)^{m-1} m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - m)\Gamma(\frac{n-k}{2} + m - 1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
&\times \left(2m(\tfrac{k+s+1}{2} - m) - (s - 2m + 2)(\tfrac{n-k}{2} + m - 1) \right),
\end{aligned}$$

and

$$\lambda_{n,k}^{s,0} = -\frac{4\pi^2(s+2)}{n-1}\iota_{n,k}^{s,0} = -\frac{4\pi^2(s+2)}{n-1}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})},$$

$$\begin{aligned}\lambda_{n,k}^{s,\lfloor\frac{s}{2}\rfloor+1} &= \frac{2\pi}{n-1}\iota_{n,k}^{s,\lfloor\frac{s}{2}\rfloor} \\ &= \frac{2\pi}{(n-1)(4\pi)^{\lfloor\frac{s}{2}\rfloor}(\lfloor\frac{s}{2}\rfloor)!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-\lfloor\frac{s}{2}\rfloor)\Gamma(\frac{n-k}{2}+\lfloor\frac{s}{2}\rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})},\end{aligned}$$

where $\lambda_{n,k}^{s,0}$ is defined according to the general definition, but $\lambda_{n,k}^{s,\lfloor\frac{s}{2}\rfloor+1}$ differs slightly for odd s. □

Finally, we prove Corollary 5.7 and Corollary 5.8, which are immediate consequences of Theorem 5.4.

Proof of Corollary 5.7 and Corollary 5.8. We denote the integral we are interested in by I and establish both corollaries simultaneously. Theorem 5.4 yields

$$I = \sum_{m=0}^{\lfloor\frac{s}{2}\rfloor} d_{n,k-1,k}^{s,0,0,m}\, Q^m \phi_{n-1}^{r,s-2m,0}(K,\beta) + \sum_{m=1}^{\lfloor\frac{s}{2}\rfloor} d_{n,k-1,k}^{s,0,1,m}\, Q^{m-1}\phi_{n-1}^{r,s-2m,1}(K,\beta),$$

where

$$d_{n,k-1,k}^{s,0,i,m} := \frac{1}{4^m(m-i)!}\frac{1}{\pi^{i+m}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}\Gamma(\tfrac{k+s+1}{2}-m)\Gamma(\tfrac{n-k}{2}+m).$$

From (5.2) we obtain

$$\begin{aligned}I = \sum_{m=0}^{\lfloor\frac{s}{2}\rfloor} d_{n,k-1,k}^{s,0,0,m}\, Q^m \phi_{n-1}^{r,s-2m,0}(K,\beta) &+ \frac{2\pi}{n-1}\sum_{m=1}^{\lfloor\frac{s}{2}\rfloor} d_{n,k-1,k}^{s,0,1,m}\, Q^m \phi_{n-1}^{r,s-2m,0}(K,\beta) \\ - \frac{4\pi^2}{n-1}\sum_{m=1}^{\lfloor\frac{s}{2}\rfloor} d_{n,k-1,k}^{s,0,1,m}&(s-2m+2)\, Q^{m-1}\phi_{n-1}^{r,s-2m+2,0}(K,\beta),\end{aligned}$$

where we used that $d_{n,k-1,k}^{s,0,1,0} = 0$. This can be rewritten in the form

$$\begin{aligned}I &= \sum_{m=0}^{\lfloor\frac{s}{2}\rfloor}\left(d_{n,k-1,k}^{s,0,0,m} + \frac{2\pi}{n-1}d_{n,k-1,k}^{s,0,1,m}\right) Q^m\phi_{n-1}^{r,s-2m,0}(K,\beta) \\ &\quad - \frac{4\pi^2}{n-1}\sum_{m=0}^{\lfloor\frac{s}{2}\rfloor-1} d_{n,k-1,k}^{s,0,1,m+1}(s-2m)\, Q^m\phi_{n-1}^{r,s-2m,0}(K,\beta) \\ &= \sum_{m=0}^{\lfloor\frac{s}{2}\rfloor-1}\left(d_{n,k-1,k}^{s,0,0,m} + \frac{2\pi}{n-1}d_{n,k-1,k}^{s,0,1,m} - \frac{4\pi^2(s-2m)}{n-1}d_{n,k-1,k}^{s,0,1,m+1}\right) Q^m\phi_{n-1}^{r,s-2m,0}(K,\beta) \\ &\quad + \left(d_{n,k-1,k}^{s,0,0,\lfloor\frac{s}{2}\rfloor} + \frac{2\pi}{n-1}d_{n,k-1,k}^{s,0,1,\lfloor\frac{s}{2}\rfloor}\right) Q^{\lfloor\frac{s}{2}\rfloor}\phi_{n-1}^{r,s-2\lfloor\frac{s}{2}\rfloor,0}(K,\beta).\end{aligned}$$

Denoting the corresponding coefficients of the summand $Q^m\phi_{n-1}^{r,s-2m,0}(K,\beta)$ by $\kappa_{n,k}^{s,m}$, we obtain

$$\begin{aligned}
\kappa_{n,k}^{s,m} &= \left(1+\frac{2m}{n-1}\right)\frac{1}{(4\pi)^m m!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
&\quad -\frac{s-2m}{n-1}\frac{1}{(4\pi)^m m!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m+1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
&= \frac{1}{(4\pi)^m m!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
&\qquad \times \underbrace{\left(\tfrac{n+2m-1}{n-1}(\tfrac{k+s-1}{2}-m)-\tfrac{s-2m}{n-1}(\tfrac{n-k}{2}+m)\right)}_{=\frac{k-1}{n-1}\frac{n+s-1}{2}} \\
&= \frac{k-1}{n-1}\frac{1}{(4\pi)^m m!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})},
\end{aligned} \tag{5.4}$$

for $m\in\{0,\dots,\lfloor\frac{s}{2}\rfloor-1\}$. For $k=1$, we immediately get $\kappa_{n,1}^{s,m}=0$ in these cases. Furthermore, we have

$$\kappa_{n,k}^{s,\lfloor\frac{s}{2}\rfloor} = \left(1+\frac{2\lfloor\frac{s}{2}\rfloor}{n-1}\right)\frac{1}{(4\pi)^{\lfloor\frac{s}{2}\rfloor}\lfloor\frac{s}{2}\rfloor!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-\lfloor\frac{s}{2}\rfloor)\Gamma(\frac{n-k}{2}+\lfloor\frac{s}{2}\rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}.$$

If s is even and $k>1$, this coincides with (5.4) for $m=\frac{s}{2}$. If s is odd, we have

$$\kappa_{n,k}^{s,\frac{s-1}{2}} = \frac{k(n+s-2)}{2(n-1)}\frac{1}{(4\pi)^{\frac{s-1}{2}}\frac{s-1}{2}!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{n-k}{2})}$$

and thus the assertion of Corollary 5.7. For $k=1$, we obtain

$$\kappa_{n,1}^{s,\lfloor\frac{s}{2}\rfloor} = \frac{\Gamma(\frac{s}{2}-\lfloor\frac{s}{2}\rfloor+1)}{\sqrt{\pi}(4\pi)^{\lfloor\frac{s}{2}\rfloor}\lfloor\frac{s}{2}\rfloor!}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2}+\lfloor\frac{s}{2}\rfloor)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+s+1}{2})}$$

and thus the assertion of Corollary 5.8. $\square$

CHAPTER 6

Integral Formulae for Minkowski Tensors

Since the Minkowski tensors are the total tensorial curvature measures, it is a natural consequent step to globalize the kinematic and Crofton formulae for tensorial curvature measures (see Chapter 4 and Chapter 5), in order to obtain the corresponding formulae for Minkowski tensors. Even though these integral formulae are well-studied in the translation invariant case, using methods from algebraic integral geometry (see [15]), and there exist Crofton formulae (which can be applied to further obtain kinematic formulae) for general Minkowski tensors (see [51]), the aim of this chapter is to establish two complete sets of these integral geometric formulae, which not only generalize the translation invariant results in [15], but also allow a considerably less technical representation than in the formulae which were derived in [51].

Recalling the integral formulae for the tensorial curvature measures on convex bodies (see Theorem 4.4 and Theorem 5.4), we observe that the representations of the integrals involve the generalized tensorial curvature measures $\phi_j^{r,s,1}$, which do not have a direct global counterpart. However, McMullen's Lemma 3.6 (or rather the consequence thereof, Lemma 3.7) yields a representation of the globalization of these measures in terms of Minkowski tensors. This will be the crucial ingredient of the upcoming proofs. As this representation is most simple in the translation invariant case, we state the integral formulae separately for translation invariant and for general Minkowski tensors.

We note that, since the Minkowski tensors depend additively on the underlying convex body, all integral formulae in this chapter remain true if the occurring convex bodies are replaced by finite unions thereof.

6.1. Kinematic Formulae

In this section, we provide the complete set of kinematic formulae for the Minkowski tensors of convex bodies. In other words, for $K, K' \in \mathcal{K}^n$ we express the integral mean value

$$\int_{G_n} \Phi_j^{r,s}(K \cap gK')\,\mu(\mathrm{d}g)$$

in terms of the Minkowski tensors of K and K'. In fact, one does so using only a selection of them (in particular, only scalar Minkowski tensors, that is, intrinsic volumes of K').

We proceed in two steps, first we state the formulae for the translation invariant Minkowski tensors, which are then followed by the formulae for general Minkowski tensors. The proof is basically an application of the kinematic formulae for the tensorial curvature measures (obtained in Chapter 4), combined with Lemma 3.7.

6.1.1. Translation Invariant Minkowski Tensors

As explained before, we start by stating the kinematic formula for translation invariant Minkowski tensors $\Phi_j^{0,s}$, $j, s \in \mathbb{N}_0$ with $j \leq n$, where $s = 0$ if $j = n$. Here, the proof and the representation of the kinematic integrals are more simple as in the general case, as one can combine several coefficients of the occurring Minkowski tensors.

Theorem 6.1. *For $K, K' \in \mathcal{K}^n$ and $j, s \in \mathbb{N}_0$ with $j \leq n$, where $s = 0$ if $j = n$,*

$$\int_{G_n} \Phi_j^{0,s}(K \cap gK')\,\mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e_{n,j,k}^{s,m,0}\, Q^m \Phi_k^{0,s-2m}(K)\, V_{n-k+j}(K'),$$

where

$$e_{n,j,k}^{s,m,0} := \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+s}{2})} \frac{\Gamma(\frac{j+s}{2} - m)}{\Gamma(\frac{j}{2})} \frac{\Gamma(\frac{k-j}{2} + m)}{\Gamma(\frac{k-j}{2})},$$

for $m = 0, \ldots, \lfloor \frac{s}{2} \rfloor - 1$, and

$$e_{n,j,k}^{s,\lfloor \frac{s}{2} \rfloor,0} := \frac{\frac{k}{2} + \lfloor \frac{s}{2} \rfloor}{(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+s}{2} + 1)} \frac{\Gamma(\frac{j+s}{2} - \lfloor \frac{s}{2} \rfloor + 1)}{\Gamma(\frac{j}{2} + 1)} \frac{\Gamma(\frac{k-j}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{k-j}{2})}.$$

In Theorem 6.1 one could, in fact, define the coefficient $e_{n,j,k}^{s,\lfloor \frac{s}{2} \rfloor,0}$ as for general m, since the definitions coincide for even s, and their difference for odd s is irrelevant, as $\Phi_k^{0,1} \equiv 0$. However, for later use we already define them here in the correct way.

For $k = j$, we note that the coefficient in Theorem 6.1 is given by

$$e_{n,j,j}^{s,m,0} = \mathbb{1}\{m = 0\}.$$

For $k = n$, we note that the Minkowski tensors $\Phi_n^{0,s-2m}$ vanish if $m \neq \frac{s}{2}$. In the case of $m = \frac{s}{2}$ (and hence s even), the corresponding coefficient is given by

$$e_{n,j,n}^{s,\frac{s}{2},0} = \frac{1}{(2\sqrt{\pi})^s \frac{s}{2}!} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+s}{2})} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})}.$$

If $j = 0$, then the kinematic integral equals zero, for odd s, and further, for even s, the only non-vanishing coefficients on the right side of the kinematic formula are $e_{n,0,k}^{s,\frac{s}{2},0}$, $k \in \{1, \dots, n\}$, as in that case the quotient $\Gamma(\frac{j+s}{2} - m)/\Gamma(\frac{j}{2})$ is read as $\mathbb{1}\{m = \frac{s}{2}\}$, due to the continuation of the Gamma function (2.2).

6.1.2. General Minkowski Tensors

In this section, we state the kinematic formula for general Minkowski tensors. The representation of the kinematic integral will be more involved compared to the translation invariant case treated in Theorem 6.1, as Lemma 3.6 adds Minkowski tensors to the formula which did not appear before, and can thus not be combined with the rest. However, the representation of the geometric integral is still remarkably more simple than the ones obtained for the Crofton formulae in [51] (which can be applied to obtain kinematic formulae).

Theorem 6.2. *For $K, K' \in \mathcal{K}^n$ and $j, r, s \in \mathbb{N}_0$ with $j \leq n$, where $s = 0$ if $j = n$,*

$$\int_{G_n} \Phi_j^{r,s}(K \cap gK')\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \sum_{p=0}^{r} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e_{n,j,k}^{s,m,p}\, Q^m \Phi_{k+p}^{r-p,s-2m+p}(K)\, V_{n-k+j}(K'),$$

where the coefficients $e_{n,j,k}^{s,m,0}$ are defined as in Theorem 6.1. For all further $p = 1, \dots, r$ the coefficients are

$$e_{n,j,k}^{s,m,p} := \frac{m\frac{k-p}{k} - \frac{s+p}{2}\frac{k-j}{k}}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2} - m)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})},$$

for $m = 0, \dots, \lfloor \frac{s}{2} \rfloor - 1$, and

$$e_{n,j,k}^{s,\lfloor \frac{s}{2} \rfloor,p} := \frac{1}{(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} - 1 \rfloor!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2} - \lfloor \frac{s}{2} \rfloor + 1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{k-j}{2})}.$$

In Theorem 6.2, we note that for $p > 0$ and $k = j$, we have $e_{n,j,j}^{s,m,p} = 0$. If $p > 0$ and $k = n$, then the Minkowski tensors $\Phi_{k+p}^{r-p,s-2m+p}$ vanish. For further simplifications of the coefficients, see the remark after Theorem 6.1.

6.2. Crofton Formulae

In this section, we state the complete set of Crofton formulae for Minkowski tensors, which can be derived from the corresponding formulae for tensorial curvature measures in Chapter 5. That is, for $K \in \mathcal{K}^n$, we explicitly express integrals of the form

$$\int_{\mathrm{A}(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

as a linear combination of Minkowski tensors of K (multiplied with suitable powers of the metric tensor). Similar to the kinematic formulae, we only need a selection of these tensorial valuations.

At first, we consider the case $j = k$. For the matter of completeness, we mention this formula here, even though it is a well-known result, derived in [51] using a completely different approach.

Theorem 6.3. *Let $K \in \mathcal{K}^n$ and $k, r, s, l \in \mathbb{N}_0$ with $k \leq n$, where $s = 0$ if $k = n$. Then,*

$$\int_{\mathrm{A}(n,k)} \Phi_k^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \mathbb{1}\{s \text{ even}\} \frac{1}{(4\pi)^{\frac{s}{2}} \frac{s}{2}!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k+s}{2})}{\Gamma(\frac{n+s}{2})\Gamma(\frac{n-k}{2})} Q^{\frac{s}{2}} \Phi_n^{r,0}(K).$$

It is not necessary to state a proof of Theorem 6.3. In fact, the formula is an immediate consequence of Theorem 5.1, which is derived by simply setting $l = 0$ and $\beta = \mathbb{R}^n$. If $k = n$, then the Minkowski tensor on the left side of the formula vanishes if $s \neq 0$, and so does the factor $\Gamma(\frac{n-k+s}{2})/\Gamma(\frac{n-k}{2})$ on the right side.

6.2.1. Translation Invariant Minkowski Tensors

We proceed with the Crofton formulae in the case of $j < k$, and start with the translation invariant Minkowski tensors.

Theorem 6.4. *Let $K \in \mathcal{K}^n$ and $j, k, r, s \in \mathbb{N}_0$ with $j < k \leq n$, where $s = 0$ if $j = n$. Then,*

$$\int_{\mathrm{A}(n,k)} \Phi_j^{0,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e_{n,j,n-k+j}^{s,m,0}\, Q^m \Phi_{n-k+j}^{0,s-2m}(K),$$

where the coefficients $e_{n,j,n-k+j}^{s,m,0}$ are defined as in Theorem 6.1.

For $j = 0$, we have $\Gamma(\frac{j+s}{2} - m)/\Gamma(\frac{j}{2}) = \mathbb{1}\{m = \frac{s}{2}\}$. Thus in that case, the only remaining summand on the right-hand side of the Crofton formula in Theorem 6.4 is $e_{n,j,k}^{s,\frac{s}{2}} Q^{\frac{s}{2}} \Phi_{n-k+j}^{0,0}(K)$, if s is even (else the integral on the left-hand side vanishes). We note that Theorem 6.4 coincides with Theorem 3 in [15], which was derived by a completely different algebraic approach.

6.2.2. General Minkowski Tensors

Finally, we state the Crofton formula for general Minkowski tensors. Similar to the kinematic formulae, we conclude from Lemma 3.6 that the representation of the Crofton integrals involves more Minkowski tensors than in the translation invariant case.

Theorem 6.5. *Let $K \in \mathcal{K}^n$ and $j, k, r, s \in \mathbb{N}_0$ with $j < k \leq n$. Then,*

$$\int_{\mathrm{A}(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \sum_{p=0}^{r} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e_{n,j,n-k+j}^{s,m,p}\, Q^m \Phi_{n-k+j+p}^{r-p,s-2m+p}(K),$$

where the coefficients $e_{n,j,n-k+j}^{s,m,p}$ are defined as in Theorem 6.1 and 6.2.

In Theorem 6.5, for $j = k - 1$, the only summand remaining in the summation with respect to p is the one for $p = 0$, as the Minkowski tensors, which occur for $p > 0$, vanish.

6.3. The Proofs

6.3.1. The Proofs of the Kinematic Formulae

The proofs of the kinematic formulae are applications of the kinematic formulae for tensorial curvature measures (obtained in Chapter 4). Even though one can prove Theorem 6.1 and Theorem 6.2 together at once, as the latter implies the first, we split the proofs in order to emphasize the difference in the coefficients of the appearing translation invariant and the general Minkowski tensors.

We start with the translation invariant case.

Proof of Theorem 6.1. We only prove the assertion for polytopes $P, P' \in \mathcal{P}^n$. The rest follows by an approximation argument. We denote the integral under investigation by I. Then Theorem 4.4 with $\beta = \beta' = \mathbb{R}^n$ yields

$$I = \sum_{k=j+1}^{n-1} \Bigg(\sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,k}^{s,0,m}\, Q^m \Phi_k^{0,s-2m}(P) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,k}^{s,1,m}\, Q^{m-1} \phi_k^{0,s-2m,1}(P, \mathbb{R}^n) \Bigg) V_{n-k+j}(P')$$
$$+ \Phi_j^{0,s}(P)\, V_n(P') + c_{n,j}^{s}\, Q^{\frac{s}{2}} \Phi_n^{0,0}(P)\, V_j(P').$$

We conclude from Lemma 3.7

$$\phi_k^{0,s-2m,1}(K, \mathbb{R}^n) = \frac{2\pi}{k} \sum_{F \in \mathcal{F}_k(P)} Q(F) \Upsilon_0(F) \Theta_{s-2m}(P, F)$$
$$= \frac{2\pi}{k}\, Q \Phi_k^{0,s-2m}(P) - \frac{4\pi^2}{k} (s - 2m + 2)\, \Phi_k^{0,s-2m+2}(P).$$

Hence, we have

$$\begin{aligned} I = \sum_{k=j+1}^{n-1} \Bigg(& \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,k}^{s,0,m}\, Q^m \Phi_k^{0,s-2m}(P) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \frac{2\pi}{k} c_{n,j,k}^{s,1,m}\, Q^m \Phi_k^{0,s-2m}(P) \\ & - \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor -1} \frac{4\pi^2}{k} (s-2m) c_{n,j,k}^{s,1,m+1}\, Q^m \Phi_k^{0,s-2m}(P) \Bigg) V_{n-k+j}(P') \\ & + \Phi_j^{0,s}(P)\, V_n(P') + c_{n,j}^{s} \frac{\omega_{n+s}}{\omega_n} Q^{\frac{s}{2}}\, V_n(P)\, V_j(P'). \end{aligned}$$

Combining all the sums with respect to m gives

$$\begin{aligned} I = & \sum_{k=j+1}^{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \left(c_{n,j,k}^{s,0,m} + \frac{2\pi}{k} c_{n,j,k}^{s,1,m} - \frac{4\pi^2}{k}(s-2m) c_{n,j,k}^{s,1,m+1} \right) Q^m \Phi_k^{0,s-2m}(P)\, V_{n-k+j}(P') \\ & + \Phi_j^{0,s}(P)\, V_n(P') + c_{n,j}^{s} \frac{\omega_{n+s}}{\omega_n} Q^{\frac{s}{2}}\, V_n(P)\, V_j(P'), \end{aligned}$$

which holds as $c_{n,j,k}^{s,1,0} = 0$ and, for $m = \lfloor \frac{s}{2} \rfloor$, either $(s-2m) = 0$ (if s is even) or $\Phi_k^{0,s-2m} \equiv 0$ (if s is odd). Now we simplify the occurring coefficients

$$\begin{aligned} e_{n,j,k}^{s,m,0} &:= c_{n,j,k}^{s,0,m} + \frac{2\pi}{k} c_{n,j,k}^{s,1,m} - \frac{4\pi^2}{k}(s-2m) c_{n,j,k}^{s,1,m+1} \\ &= \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+s}{2}+1)} \\ &\quad \times \Bigg(\left(1 + \frac{2m}{k}\right) \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})} \\ &\qquad - \frac{s-2m}{k} \frac{\Gamma(\frac{j+s}{2} - m)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m+1)}{\Gamma(\frac{k-j}{2})} \Bigg) \\ &= \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2} - m)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})} \\ &\quad \times \underbrace{\left(\frac{k+2m}{k} (\frac{j+s}{2} - m) - \frac{s-2m}{k} (\frac{k-j}{2} + m) \right)}_{= \frac{j}{k} \frac{k+s}{2}} \\ &= \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+s}{2})} \frac{\Gamma(\frac{j+s}{2} - m)}{\Gamma(\frac{j}{2})} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})}, \end{aligned}$$

for $j < k < n$ and $m = 0, \ldots, \lfloor \frac{s}{2} \rfloor - 1$. Even though it is irrelevant here, as explained earlier, we define the coefficients for $m = \lfloor \frac{s}{2} \rfloor$ in a slightly different way by

$$\begin{aligned} e_{n,j,k}^{s,\lfloor \frac{s}{2} \rfloor,0} &:= c_{n,j,k}^{s,0,\lfloor \frac{s}{2} \rfloor} + \frac{2\pi}{k} c_{n,j,k}^{s,1,\lfloor \frac{s}{2} \rfloor} \\ &= \frac{\frac{k}{2} + \lfloor \frac{s}{2} \rfloor}{(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor !} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2} - \lfloor \frac{s}{2} \rfloor + 1)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{k-j}{2})}, \end{aligned}$$

which is necessary in Theorem 6.2. For even s this coincides with the general definition, whereas for odd s they differ.

As the continuation of the coefficients to $k = j, n$ is

$$e^{s,m,0}_{n,j,j} = \mathbb{1}\{m = 0\}$$

and

$$e^{s,m,0}_{n,j,n} = \mathbb{1}\{s \text{ even}\} \frac{1}{(2\sqrt{\pi})^s \frac{s}{2}!} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+s}{2})} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})} = c^s_{n,j},$$

we can briefly write

$$I = \sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e^{s,m,0}_{n,j,k}\, Q^m \Phi^{0,s-2m}_k(P)\, V_{n-k+j}(P'),$$

since $\Phi^{0,s-2m}_n$ vanishes for $m \neq \frac{s}{2}$. $\square$

In the proof of the general case, we observe that the coefficients of the translation invariant Minkowski tensors are the same as the ones which we derived in Theorem 6.1. However, the coefficients of the other Minkowski tensors have to be defined in a slightly different way.

Proof of Theorem 6.2. Again, we only prove the assertion for polytopes $P, P' \in \mathcal{P}^n$. The rest follows by an approximation argument. We denote the integral under investigation by I. Then Theorem 4.1 with $\beta = \beta' = \mathbb{R}^n$ yields, as in the proof of Theorem 6.1,

$$\begin{aligned} I = \sum_{k=j+1}^{n-1} \Bigg(\sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c^{s,0,m}_{n,j,k}\, Q^m \Phi^{r,s-2m}_k(P) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c^{s,1,m}_{n,j,k}\, Q^{m-1} \phi^{r,s-2m,1}_k(P, \mathbb{R}^n) \Bigg) V_{n-k+j}(P') \\ + \Phi^{r,s}_j(P)\, V_n(P') + c^s_{n,j} \frac{\omega_{n+s}}{\omega_n}\, Q^{\frac{s}{2}} \Phi^{r,0}_n(P)\, V_j(P'). \end{aligned}$$

We conclude from Lemma 3.7

$$\begin{aligned} I = &\sum_{k=j+1}^{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c^{s,0,m}_{n,j,k}\, Q^m \Phi^{r,s-2m}_k(P)\, V_{n-k+j}(P') \\ &+ \sum_{k=j+1}^{n-1} \frac{2\pi}{k} \sum_{p=0}^{r} \Bigg(\sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c^{s,1,m}_{n,j,k}\, Q^m \Phi^{r-p,s-2m+p}_{k+p}(P) \\ &\qquad - 2\pi \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} (s - 2m + p + 2) c^{s,1,m}_{n,j,k}\, Q^{m-1} \Phi^{r-p,s-2m+p+2}_{k+p}(P) \Bigg) V_{n-k+j}(P') \\ &+ \Phi^{r,s}_j(P)\, V_n(P') + c^s_{n,j} \frac{\omega_{n+s}}{\omega_n}\, Q^{\frac{s}{2}} \Phi^{r,0}_n(P)\, V_j(P'). \end{aligned}$$

For $p = 0$, we can combine all of the coefficients as in the proof of the translation invariant case in Theorem 6.1. For this purpose it is important that the definition of the coefficients for $m = \frac{s-1}{2}$ (if s is odd) differs from the general ones, as for $r > 0$, we have $\Phi_k^{r,1} \not\equiv 0$ in general. Thus, we obtain

$$
\begin{aligned}
I = &\sum_{k=j}^{n} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} e_{n,j,k}^{s,m,0}\, Q^m \Phi_k^{r,s-2m}(P)\, V_{n-k+j}(P') \\
&+ \sum_{k=j+1}^{n-1} \frac{2\pi}{k} \sum_{p=1}^{r} \Bigg(\sum_{m=0}^{\lfloor \frac{s}{2} \rfloor -1} (c_{n,j,k,s}^{s,1,m} - 2\pi(s-2m+p) c_{n,j,k}^{s,1,m+1})\, Q^m \Phi_{k+p}^{r-p,s-2m+p}(P) \\
&\qquad + c_{n,j,k}^{s,1,\lfloor \frac{s}{2} \rfloor}\, Q^{\lfloor \frac{s}{2} \rfloor} \Phi_{k+p}^{r-p,s-2\lfloor \frac{s}{2} \rfloor + p}(P) \Bigg) V_{n-k+j}(P'),
\end{aligned}
$$

as $c_{n,j,k}^{s,1,0} = 0$. Now we rename the remaining coefficients as

$$
e_{n,j,k}^{s,m,p} := \tfrac{2\pi}{k} \left(c_{n,j,k}^{s,1,m} - 2\pi(s-2m+p) c_{n,j,k}^{s,1,m+1} \right),
$$

which we simplify via

$$
\begin{aligned}
e_{n,j,k}^{s,m,p} = &\frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+s}{2}+1)} \frac{\Gamma(\frac{j+s}{2}-m)}{\Gamma(\frac{j}{2}+1)} \frac{\Gamma(\frac{k-j}{2}+m)}{\Gamma(\frac{k-j}{2})} \\
&\times \underbrace{\tfrac{2}{k} \left(m \left(\tfrac{j+s}{2} - m \right) - \tfrac{s-2m+p}{2} \left(\tfrac{k-j}{2} + m \right) \right)}_{= m \frac{k-p}{k} - \frac{s+p}{2} \frac{k-j}{k}},
\end{aligned}
$$

and denote $e_{n,j,k}^{s,\lfloor \frac{s}{2} \rfloor,p} := \frac{2\pi}{k} c_{n,j,k}^{s,1,\lfloor \frac{s}{2} \rfloor}$, for $p > 0$, which gives the assertion. □

6.3.2. The Proofs of the Crofton Formulae

There are several possible ways of proving the Crofton formulae for Minkowski tensors. One can use the connection of the Crofton formula and the kinematic formula to derive the results (for the scalar (local) case see [83, Theorem 4.4.5], for the tensorial (local) case see the proof of Theorem 5.1 and Theorem 5.2). However, here this is done by globalizing the Crofton formulae for the global curvature measures and then applying Lemma 3.6 to the results. Again, we split the proof into the translation invariant and the general case, and start with the first.

Proof of Theorem 6.4. We only prove the assertion for a polytope $P \in \mathcal{P}^n$. The rest follows by an approximation argument. We denote the integral under investigation by I. Then Theorem 5.4 with $\beta = \mathbb{R}^n$ yields

$$
I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,0,m}\, Q^m \Phi_{n-k+j}^{0,s-2m}(P) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,1,m}\, Q^{m-1} \phi_{n-k+j}^{0,s-2m,1}(P, \mathbb{R}^n),
$$

where we applied relation (5.1) to use the same notation as in the kinematic formulae. As in the preceding proofs, we obtain from Lemma 3.7

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,0,m} \, Q^m \Phi_{n-k+j}^{0,s-2m}(P) + \tfrac{2\pi}{n-k+j} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,1,m} \, Q^m \Phi_{n-k+j}^{0,s-2m}(P)$$
$$- \tfrac{4\pi^2}{n-k+j} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} (s-2m+2) c_{n,j,n-k+j}^{s,1,m} \, Q^{m-1} \Phi_{n-k+j}^{0,s-2m+2}(P).$$

Comparing this to the proof of Theorem 6.1, we observe, that we simply obtain the same coefficients in a different order. Thus, the proof is already complete. □

In the proof of the Crofton formulae for general Minkowski tensors, we observe that the coefficients of the translation invariant Minkowski tensors are the same as the ones which we derived in Theorem 6.1. However, the coefficients of the other Minkowski tensors have to be defined in a slightly different way.

Proof of Theorem 6.5. We only prove the assertion for a polytope $P \in \mathcal{P}^n$. The rest follows by an approximation argument. We denote the integral under investigation by I. Then Theorem 5.4 with $\beta = \mathbb{R}^n$ (and application of (5.1)) yields

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,0,m} \, Q^m \Phi_{n-k+j}^{r,s-2m}(P) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,1,m} \, Q^{m-1} \phi_{n-k+j}^{r,s-2m,1}(P, \mathbb{R}^n).$$

As before, we conclude from Lemma 3.7

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} c_{n,j,n-k+j}^{s,0,m} \, Q^m \Phi_{n-k+j}^{r,s-2m}(P) + \tfrac{2\pi}{n-k+j} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \sum_{p=0}^{r} c_{n,j,n-k+j}^{s,1,m} \, Q^m \Phi_{n-k+j+p}^{r-p,s-2m+p}(P)$$
$$- \tfrac{4\pi^2}{n-k+j} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \sum_{p=0}^{r} c_{n,j,n-k+j}^{s,1,m} (s-2m+p+2) \, Q^{m-1} \Phi_{n-k+j+p}^{r-p,s-2m+p+2}(P).$$

Similarly to the preceding proof we observe, that we obtain the same coefficients as in the proof of Theorem 6.2, which concludes the proof. □

CHAPTER 7

Integral Formulae for SO(n)-Covariant Valuations

In the preceding three chapters we derived several complete sets of kinematic and Crofton formulae for tensorial curvature measures and for Minkowski tensors. The aim of this chapter is to establish the corresponding integral geometric formulae for SO(n)-covariant tensorial curvature measures (introduced in Section 3.1.4) and for their total measures, the SO(n)-covariant Minkowski tensors (see Section 3.2.1). These valuations only occur in dimensions two and three (the total measures even vanish in dimension three).

Hence, for $K, K' \in \mathcal{K}^2$ and $\beta, \beta' \in \mathcal{B}(\mathbb{R}^2)$, our purpose is to express the integral mean values

$$\int_{\mathrm{G}_2} \breve{\phi}_j^{r,s}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g), \tag{7.1}$$

and

$$\int_{\mathrm{A}(2,1)} \breve{\phi}_j^{r,s}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E), \tag{7.2}$$

in terms of (SO(2)-covariant) tensorial curvature measures of K evaluated at β and, in the kinematic case (7.1), of K' evaluated at β'. Furthermore, we deduce the corresponding integral formulae for the SO(2)-covariant Minkowski tensors.

Moreover, for $P, P' \in \mathcal{P}^3$ and $\beta, \beta' \in \mathcal{B}(\mathbb{R}^3)$, we develop explicit formulae for the kinematic integrals

$$\int_{\mathrm{G}_3} \breve{\phi}^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g), \tag{7.3}$$

and for the Crofton integrals

$$\int_{\mathrm{A}(3,k)} \breve{\phi}^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E). \tag{7.4}$$

For the SO(3)-covariant tensorial curvature measures with continuous extensions to the convex bodies (that is, for $l = 0$), we provide analog formulae for $K, K' \in \mathcal{K}^3$. Since the total SO(3)-covariant tensorial curvature measures vanish, we do not consider global versions of the formulae in dimension three.

7.1. SO(2)-Covariant Tensorial Valuations

In this section, we start our investigations in dimension two. At first, we state the kinematic formulae (7.1) for the local and the global valuations. In the second step, we provide the Crofton formulae (7.2).

7.1.1. Kinematic Formulae

We begin with the kinematic formulae for SO(2)-covariant tensorial curvature measures in the following theorem.

Theorem 7.1. *Let $K, K' \in \mathcal{K}^2$, $\beta, \beta' \in \mathcal{B}(\mathbb{R}^2)$ and $r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{G}_2} \breve{\phi}_1^{r,s}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \breve{\phi}_1^{r,s}(K, \beta)\phi_2(K', \beta'),$$

and

$$\int_{\mathrm{G}_2} \breve{\phi}_0^{r,s}(K \cap gK', \beta \cap g\beta')\, \mu(\mathrm{d}g)$$
$$= \breve{\phi}_0^{r,s}(K, \beta)\phi_2(K', \beta') + \mathbb{1}\{s \text{ even}\} \frac{\Gamma(\frac{s+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{s+4}{2})} Q^{\frac{s}{2}} \breve{\phi}_1^{r,0}(K, \beta)\phi_1(K', \beta').$$

Remarkably, in Theorem 7.1, the representations of the kinematic integrals contain SO(2)-covariant tensorial curvature measures of the convex body K, and (scalar) curvature measures of the convex body K' (which are O(2)-invariant). Moreover, they only involve a selection of measures, which is even smaller than the corresponding formulae for the classical tensorial curvature measures in Theorem 4.4.

We easily conclude the corresponding kinematic formulae for the SO(2)-covariant Minkowski tensors.

Corollary 7.2. *Let $K, K' \in \mathcal{K}^2$, and $r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{G}_2} \breve{\Phi}_1^{r,s}(K \cap gK')\, \mu(\mathrm{d}g) = \breve{\Phi}_1^{r,s}(K)\, V_2(K'),$$

and

$$\int_{\mathrm{G}_2} \breve{\Phi}_0^{r,s}(K \cap gK')\, \mu(\mathrm{d}g) = \breve{\Phi}_0^{r,s}(K)\, V_2(K').$$

Corollary 7.2 is easily obtained by setting $\beta = \beta' = \mathbb{R}^n$ and applying the definition of the SO(2)-covariant Minkowski tensors. Moreover, we note that the kinematic integral for $\breve{\Phi}_0^{r,s}$ can be represented by only one summand, as the second summand (occurring in Corollary 7.2) vanishes, due to relation (3.9) which states that $\breve{\Phi}_1^{r,0} = 0$, $r \in \mathbb{N}_0$.

7.1.2. Crofton Formulae

We proceed with the Crofton formulae for SO(2)-covariant tensorial curvature measures. Here, the only interesting results arise for the intersectional integrals on the affine Grassmannian A(2, 1). In fact, for $E \in \mathrm{A}(2,0)$, we have $\breve{\phi}_0^{r,s}(K \cap E, \cdot) = 0$, which is an immediate consequence of Lemma 7.7 in Section 7.3.1. Therefore, the Crofton integral vanishes in this case. The following theorem treats the remaining case.

Theorem 7.3. *Let $K \in \mathcal{K}^2$, $\beta \in \mathcal{B}(\mathbb{R}^2)$ and $r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{A}(2,1)} \breve{\phi}_1^{r,s}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) = 0,$$

and

$$\int_{\mathrm{A}(2,1)} \breve{\phi}_0^{r,s}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) = \mathbb{1}\{s \text{ even}\} \frac{\Gamma(\frac{s+1}{2})}{\sqrt{\pi}\Gamma(\frac{s+4}{2})} Q^{\frac{s}{2}} \breve{\phi}_1^{r,0}(K, \beta).$$

Theorem 7.3 can be proved using the same method as in the proofs of Theorem 5.1 and Theorem 5.2. Therefore, we do not provide it here. Furthermore, there is a global version of Theorem 7.3, which can be obtained by setting $\beta = \mathbb{R}^n$ (in the same way as Corollary 7.2 was deduced from Theorem 7.1). For the matter of completeness, we state it here.

Corollary 7.4. *Let $K \in \mathcal{K}^2$, and $r, s \in \mathbb{N}_0$ and $j \in \{0, 1\}$. Then*

$$\int_{\mathrm{A}(2,1)} \breve{\Phi}_j^{r,s}(K \cap E)\, \mu_1(\mathrm{d}E) = 0.$$

Remarkably, the global Crofton formulae for both of the SO(2)-covariant tensorial curvature measures vanish. This fact again follows from relation (3.9) which already simplified the kinematic formulae.

7.2. SO(3)-Covariant Tensorial Valuations

In this section, we proceed with our investigations in dimension three. At first, we state the kinematic formulae (7.3) for the SO(3)-covariant tensorial curvature measures on polytopes and (if existing) for their continuous extensions on convex bodies. In the second step, we provide the Crofton formulae (7.4). As the total SO(3)-covariant tensorial curvature measures vanish, there are no global results.

7.2.1. Kinematic Formulae

We begin with the kinematic formulae for SO(3)-covariant tensorial curvature measures in the following theorem.

Theorem 7.5. *Let* $P, P' \in \mathcal{P}^3$*,* $\beta, \beta' \in \mathcal{B}(\mathbb{R}^3)$ *and* $r, s, l \in \mathbb{N}_0$*. Then*

$$\int_{\mathrm{G}_3} \breve{\phi}^{r,s,l}(P \cap gP', \beta \cap g\beta')\, \mu(\mathrm{d}g) = \breve{\phi}^{r,s,l}(P, \beta)\, \phi_3(P', \beta').$$

In Theorem 7.5, we can replace the polytopes by convex bodies $K, K' \in \mathcal{K}^3$ if $l = 0$, since in this case there exist continuous extensions of the SO(3)-covariant tensorial curvature measures to $\mathcal{K}^3$. We do not consider global versions of these kinematic formulae, as the global counterparts of the SO(3)-covariant tensorial curvature measures vanish.

As in Theorem 7.1 in $\mathbb{R}^2$, we note, that the representations of the kinematic integrals in Theorem 7.5 consist of SO(3)-covariant tensorial curvature measures of the polytope P, and (scalar) curvature measures of the polytope P' (which are O(3)-invariant). Moreover, the representations in Theorem 7.5 are more simple than the corresponding ones for the generalized tensorial curvature measures in Theorem 4.1.

7.2.2. Crofton Formulae

The Crofton formulae for SO(3)-covariant tensorial curvature measures are rather simple, as the only potentially interesting intersectional integrals vanish. This is stated in the following theorem.

Theorem 7.6. *Let* $P \in \mathcal{P}^3$*,* $\beta \in \mathcal{B}(\mathbb{R}^3)$ *and* $k, r, s, l \in \mathbb{N}_0$ *with* $0 < k < 3$*. Then*

$$\int_{\mathrm{A}(3,k)} \breve{\phi}^{r,s,l}(P \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = 0.$$

Similar to the two dimensional case, we do not provide the proof of Theorem 7.6, but refer to the proof of the Crofton formulae for generalized tensorial curvature measures (see Theorem 5.1 and Theorem 5.2), which is easily transferred to the current situation.

7.3. The Proofs

In this section, we provide the proofs of the main results of this chapter, Theorem 7.1 and Theorem 7.5. Before we provide these in Section 7.3.2, we state and prove an integral geometric lemma in Section 7.3.1, which is then applied several times in the upcoming proofs.

7.3.1. An Auxiliary Lemma

In the proofs of Theorem 7.1 and Theorem 7.5, we need the following lemma in dimensions two and three. Nevertheless, here we state a more general version.

Lemma 7.7. *Let $u_1, \ldots, u_n \in \mathbb{S}^{n-1}$ be an orthonormal basis of $\mathbb{R}^n$, and $s_1, \ldots, s_n \in \mathbb{N}_0$, where s_i is odd for some $i \in \{1, \ldots, n\}$. Then*

$$\int_{\mathrm{SO}(n)} \vartheta(u_1^{s_1} \cdots u_n^{s_n})\, \nu(\mathrm{d}\vartheta) = 0.$$

The special cases of Lemma 7.7 that we need are the following. For $u \in \mathbb{S}^2$, and $s \in \mathbb{N}_0$, we have

$$\int_{\mathrm{SO}(2)} \vartheta(\overline{u} u^s)\, \nu(\mathrm{d}\vartheta) = 0. \tag{7.5}$$

This is basically equation (43) in [49, Theorem 6] (resp. relation (3.9) in this thesis), which states that the SO(2)-covariant Minkowski tensors $\breve{\Phi}_0^{0,s}$, $s \in \mathbb{N}_0$, vanish. This is also proved there. However, as a matter of completeness, we still provide a different proof here.

For $u, v \in \mathbb{S}^3$ with $v \in u^\perp$, and $s_1, s_2 \in \mathbb{N}_0$, we have

$$\int_{\mathrm{SO}(3)} \vartheta((u \times v) u^{s_1} v^{s_2})\, \nu(\mathrm{d}\vartheta) = 0. \tag{7.6}$$

Further, if s_1 is odd and $w \in u^\perp$ then we obtain

$$\int_{\mathrm{SO}(w^\perp)} \vartheta((u \times v) u^{s_1} v^{s_2})\, \nu^{w^\perp}(\mathrm{d}\vartheta) = 0. \tag{7.7}$$

In fact, we can split the occurring vectors into their components in $w^\perp$ and in $\operatorname{lin} w$, and then apply equation (7.5) in $w^\perp$. That is, denoting the integral in (7.7) by I,

$$I = \int_{\mathrm{SO}(w^\perp)} \vartheta\Big((p_{w^\perp}(u \times v) + \underbrace{p_{\operatorname{lin} w}(u \times v)}_{=\langle u \times v, w\rangle w}) u^{s_1} (p_{w^\perp}(v) + \underbrace{p_{\operatorname{lin} w}(v)}_{=\langle v, w\rangle w})^{s_2}\Big)\, \nu^{w^\perp}(\mathrm{d}\vartheta),$$

where u remains unchanged as $w \in u^\perp$. Since we have $\langle p_{w^\perp}(v), u\rangle = \langle v, p_{w^\perp}(u)\rangle = 0$ and $\langle p_{w^\perp}(u \times v), u\rangle = \langle u \times v, p_{w^\perp}(u)\rangle = 0$, it follows that $p_{w^\perp}(v), p_{w^\perp}(u \times v) \in u^\perp \cap w^\perp$ and hence these two vectors are multiples of each other. That is, we can write $p_{w^\perp}(v) = c_1 x$ and

$p_{w^\perp}(u \times v) = c_2 x$, where $x \in \mathbb{S}^2 \cap w^\perp$ and $c_1, c_2 \in [-1,1]$ (which are chosen independently of ρ). Applying the binomial theorem to the summations yields

$$I = \sum_{i=0}^{s_2} \binom{s_2}{i} c_1^i (\langle v, w\rangle w)^{s_2-i} \Big(c_2 \int_{\mathrm{SO}(w^\perp)} \vartheta\Big(u^{s_1} x^{i+1}\Big)\, \nu^{w^\perp}(\mathrm{d}\vartheta) \\ + \langle u \times v, w\rangle w \int_{\mathrm{SO}(w^\perp)} \vartheta\Big(u^{s_1} x^{i}\Big)\, \nu^{w^\perp}(\mathrm{d}\vartheta)\Big),$$

where we used that $\vartheta w = w$ for $\vartheta \in \mathrm{SO}(w^\perp)$. Then it follows from equation (7.5) applied in $w^\perp$ that both of the remaining integrals vanish, as u, x form an orthonormal basis in $w^\perp$.

Now we provide the proof of Lemma 7.7.

Proof of Lemma 7.7. We prove the assertion by induction on $n \in \mathbb{N}$ with $n \geq 2$. We denote the integral under investigation by I_n.

Induction start: For $n = 2$, let $u_1, u_2 \in \mathbb{S}^1$ be an orthonormal basis of $\mathbb{R}^2$, and $s_1, s_2 \in \mathbb{N}_0$, where without loss of generality s_1 is odd. Due to the invariance of the Haar measure ν, the tensor $I_2 \in \mathbb{T}^{s_1+s_2}$ is rotation invariant. That is, for a rotation $\rho \in \mathrm{SO}(2)$, we have

$$\rho I_2 = \int_{\mathrm{SO}(2)} \rho\vartheta\big(u_1^{s_1} u_2^{s_2}\big)\, \nu(\mathrm{d}\vartheta) = \int_{\mathrm{SO}(2)} \rho\vartheta\big(u_1^{s_1} u_2^{s_2}\big)\, \nu(\mathrm{d}\vartheta) = I_2.$$

Therefore, if $s_1 + s_2$ is odd, then I_2 is of odd tensor rank, and thus $I_2 = 0$. So we assume $s_1 + s_2$ to be even, meaning that s_1 and s_2 are odd. As $Q^{\frac{s_1+s_2}{2}}$ is up to scalar multiples the only rotation invariant tensor of rank $s_1 + s_2$, there is a constant $c \in \mathbb{R}$ such that $I_2 = c\, Q^{\frac{s_1+s_2}{2}}$ holds. Finally, to determine c, let $v \in \mathbb{S}^1$. Then we have

$$c = \langle v^{s_1+s_2}, c\, Q^{\frac{s_1+s_2}{2}}\rangle = \int_{\mathrm{SO}(2)} \langle v, \vartheta u_1\rangle^{s_1} \langle v, \vartheta u_2\rangle^{s_2}\, \nu(\mathrm{d}\vartheta).$$

Without loss of generality, let $u_2 := \overline{u}_1$. Then, since $\mathrm{SO}(2)$ is commutative, we can rewrite

$$c = \tfrac{1}{\omega_2} \int_{\mathbb{S}^1} \langle v, u\rangle^{s_1} \langle v, \overline{u}\rangle^{s_2}\, \mathcal{H}^1(\mathrm{d}u).$$

A simple transformation of the integral yields

$$c = \tfrac{1}{\omega_2} \int_{-1}^{1} \int_{\underbrace{\mathbb{S}^1 \cap v^\perp}_{=\{\overline{v}, -\overline{v}\}}} \sqrt{1-z^2}^{\,-1} \underbrace{\langle v, zv + \sqrt{1-z^2} w\rangle}_{=z}{}^{s_1} \underbrace{\langle v, z\overline{v} + \sqrt{1-z^2}\,\overline{w}\rangle}_{=\sqrt{1-z^2}\langle v, \overline{w}\rangle}{}^{s_2}\, \mathcal{H}^0(\mathrm{d}w)\, \mathrm{d}z = 0$$

$$= \tfrac{1}{\omega_2} \int_{-1}^{1} \sqrt{1-z^2}^{\,s_2-1} z^{s_1}\, \mathrm{d}z \int_{\{\overline{v}, -\overline{v}\}} \langle v, \overline{w}\rangle^{s_2}\, \mathcal{H}^0(\mathrm{d}w).$$

Since both integrals vanish due to the parity of s_1 and s_2, we have $I_2 = 0$.

Induction step: Assume the assertion to be proved for all dimensions up to $n - 1 \geq 2$. Then, let $u_1, \ldots, u_n \in \mathbb{S}^{n-1}$ be an orthonormal basis of $\mathbb{R}^n$, and $s_1, \ldots, s_n \in \mathbb{N}_0$, where without loss of generality s_1 is odd. For $\rho \in \mathrm{SO}(u_n^\perp)$, the invariance of ν gives

$$I_n = \int_{\mathrm{SO}(n)} (\vartheta\rho u_1)^{s_1} \cdots (\vartheta\rho u_{n-1})^{s_{n-1}} (\vartheta \underbrace{\rho u_n}_{=u_n})^{s_n} \, \nu(\mathrm{d}\vartheta).$$

Hence, we obtain with an application of Fubini's theorem

$$I_n = \int_{\mathrm{SO}(n)} \vartheta\Big(u_n^{s_n} \int_{\mathrm{SO}(u_n^\perp)} \rho(u_1^{s_1} \cdots u_{n-1}^{s_{n-1}}) \, \nu^{u_n^\perp}(\mathrm{d}\rho) \Big) \, \nu(\mathrm{d}\vartheta).$$

The induction hypothesis applied in $u_n^\perp$ to the integral with respect to ρ yields the assertion. □

7.3.2. The Proofs of the Main Results

We start with the proof of Theorem 7.1. Interestingly, the method of proof of the kinematic formulae for generalized tensorial curvature measures does not simply carry over to the present case. In fact, we change the approach, when the integration over the intersection of normal cones needs to be evaluated (see equation (7.9) and the following). This turns out to be quite promising in such a low dimension (it is further transferred to the proof of Theorem 7.5 in dimension three, which we provide subsequently) and might even help to find an easier proof of Theorem 4.1.

Proof of Theorem 7.1. We prove both of the formulae at once and only for polytopes $P, P' \in \mathcal{P}^2$. The general case then follows by approximation. We denote the kinematic integrals by I_j, $j \in \{0, 1\}$. Then we start by decomposing the measure μ to get

$$\begin{aligned} I_j &= \int_{\mathrm{G}_2} \breve{\phi}_j^{r,s}(P \cap gP', \beta \cap g\beta') \, \mu(\mathrm{d}g) \\ &= \int_{\mathrm{SO}(2)} \int_{\mathbb{R}^2} \breve{\phi}_j^{r,s}(P \cap (\vartheta P' + t), \beta \cap (\vartheta\beta' + t)) \, \mathcal{H}^2(\mathrm{d}t) \, \nu(\mathrm{d}\vartheta) \\ &= \int_{\mathrm{SO}(2)} \int_{\mathbb{R}^2} \sum_{G \in \mathcal{F}_j(P \cap (\vartheta P' + t))} \int_{G \cap \beta \cap (\vartheta\beta' + t)} x^r \, \mathcal{H}^j(\mathrm{d}x) \\ &\qquad\qquad \times \int_{N(P \cap (\vartheta P' + t), G) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u) \, \mathcal{H}^2(\mathrm{d}t) \, \nu(\mathrm{d}\vartheta). \end{aligned}$$

Now we proceed as in the proof of the kinematic formulae for generalized tensorial curvature measures to obtain

$$\begin{aligned} I_j = \sum_{k=j}^{2} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{2-k+j}(P')} \mathcal{H}^{2-k+j}(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^k(\mathrm{d}x) \\ \times \int_{\mathrm{SO}(2)} [F, \vartheta F'] \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta), \end{aligned} \tag{7.8}$$

where the integral with respect to u is independent of the choice of a vector $t \in \mathbb{R}^2$ such that $\operatorname{relint} F \cap \operatorname{relint}(\vartheta F' + t) \neq \emptyset$. Next, we consider the two summands for $k = j$ and for $k = 2$ separately. In the first case, we get

$$\begin{aligned}
&\sum_{F \in \mathcal{F}_j(P)} \sum_{F' \in \mathcal{F}_2(P')} \mathcal{H}^2(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \\
&\qquad \times \int_{\mathrm{SO}(2)} [F, \vartheta F'] \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= \mathcal{H}^2(P' \cap \beta') \sum_{F \in \mathcal{F}_j(P)} \int_{F \cap \beta} x^r \, \mathcal{H}^j(\mathrm{d}x) \int_{\mathrm{SO}(2)} \int_{N(P,F) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= \breve{\phi}_j^{r,s}(P, \beta) \phi_2(P', \beta').
\end{aligned}$$

In the second case, Fubini's theorem yields

$$\begin{aligned}
&\sum_{F \in \mathcal{F}_2(P)} \sum_{F' \in \mathcal{F}_j(P')} \mathcal{H}^j(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^2(\mathrm{d}x) \\
&\qquad \times \int_{\mathrm{SO}(2)} [F, \vartheta F'] \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^{1-j}(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
&= \sum_{F' \in \mathcal{F}_j(P')} \mathcal{H}^j(F' \cap \beta') \int_{P \cap \beta} x^r \, \mathcal{H}^2(\mathrm{d}x) \int_{N(P',F') \cap \mathbb{S}^1} \int_{\mathrm{SO}(2)} \vartheta(\overline{u} u^s) \, \nu(\mathrm{d}\vartheta) \, \mathcal{H}^{1-j}(\mathrm{d}u) \\
&= 0,
\end{aligned}$$

which follows from equation (7.5), the special case of Lemma 7.7 in $\mathbb{R}^2$.

If $j = 1$, then these are the only two cases for k to be considered. If $j = 0$, then there is also the case $k = 1$. Since we obtain from [83, Theorem 2.2.1]

$$N\left(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)\right) = N(P, F) + \vartheta N(P', F'),$$

for any $t \in \mathbb{R}^2$ such that $\operatorname{relint} F \cap \operatorname{relint}(\vartheta F' + t) \neq \emptyset$, we get

$$\begin{aligned}
S_1 :=& \sum_{F \in \mathcal{F}_1(P)} \sum_{F' \in \mathcal{F}_1(P')} \mathcal{H}^1(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^1(\mathrm{d}x) \\
&\qquad \times \int_{\mathrm{SO}(2)} [F, \vartheta F'] \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta) \\
=& \sum_{F \in \mathcal{F}_1(P)} \sum_{F' \in \mathcal{F}_1(P')} \mathcal{H}^1(F' \cap \beta') \int_{F \cap \beta} x^r \, \mathcal{H}^1(\mathrm{d}x) \\
&\qquad \times \int_{\mathrm{SO}(2)} [F, \vartheta F'] \int_{(N(P,F) + \vartheta N(P',F')) \cap \mathbb{S}^1} \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta)
\end{aligned} \tag{7.9}$$

for the summand $k = 1$ in (7.8) if $j = 0$.

Up to now we proceeded accordingly to the proof of Theorem 4.1. But as it turns out, now we have to use a different approach. In fact, an evaluation of the remaining integral with respect to ϑ in (7.9) according to the original proof does not seem to be promising, as one soon reaches a dead end. However, since the normal cones of P at F and of P' at F' are one-dimensional, we can make use of the simple structure of the Minkowski sum thereof. This is done in the following.

Let u_F (resp. $u_{F'}$) denote the unique outer unit normal of F (resp. F'). That is, $N(P,F) = \{u_F\}$ (resp. $N(P,F') = \{u_{F'}\}$) and thus $N(P,F)+\vartheta N(P',F') = \operatorname{pos}\{u_F, \vartheta u_{F'}\}$. Therefore, we obtain for the integration with respect to ϑ in (7.9), denoted by J,

$$J = \int_{\mathrm{SO}(2)} \underbrace{[F, \vartheta F']}_{=[u_F, \vartheta u_{F'}]} \int_{\mathbb{S}^1} \mathbb{1}\{u \in \operatorname{pos}\{u_F, \vartheta u_{F'}\}\} \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta).$$

Next we observe that the equality $\mathbb{1}\{u \in \operatorname{pos}\{u_F, \vartheta u_{F'}\}\} = \mathbb{1}\{u \in \operatorname{int} \operatorname{pos}\{u_F, \vartheta u_{F'}\}\}$ holds for $\mathcal{H}^1$-almost all $u \in \mathbb{S}^1$. The occuring condition can be rewritten as follows

$$\begin{aligned}
u \in \operatorname{int} \operatorname{pos}\{u_F, \vartheta u_{F'}\} \quad &\Leftrightarrow \quad \exists \lambda, \mu > 0 : u = \lambda u_F + \mu \vartheta u_{F'} \\
&\Leftrightarrow \quad \exists \lambda, \mu > 0 : \vartheta u_{F'} = \frac{1}{\mu} u - \frac{\lambda}{\mu} u_F \\
&\Leftrightarrow \quad \exists \tilde{\lambda}, \tilde{\mu} > 0 : \vartheta u_{F'} = \tilde{\mu} u + \tilde{\lambda}(-u_F) \\
&\Leftrightarrow \quad \vartheta u_{F'} \in \operatorname{int} \operatorname{pos}\{u, -u_F\}.
\end{aligned}$$

Hence, we have $\mathbb{1}\{u \in \operatorname{pos}\{u_F, \vartheta u_{F'}\}\} = \mathbb{1}\{\vartheta u_{F'} \in \operatorname{pos}\{u, -u_F\}\}$ for $\mathcal{H}^1$-almost all $u \in \mathbb{S}^1$, and therefore

$$J = \int_{\mathrm{SO}(2)} [u_F, \vartheta u_{F'}] \int_{\mathbb{S}^1} \mathbb{1}\{\vartheta u_{F'} \in \operatorname{pos}\{u, -u_F\}\} \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u) \, \nu(\mathrm{d}\vartheta).$$

A transformation of the integration with respect to ϑ and Fubini's theorem yield

$$J = \frac{1}{\omega_2} \int_{\mathbb{S}^1} \overline{u} u^s \int_{\mathbb{S}^1} \mathbb{1}\{v \in \operatorname{pos}\{u, -u_F\}\} \, [u_F, v] \, \mathcal{H}^1(\mathrm{d}v) \, \mathcal{H}^1(\mathrm{d}u).$$

For the inner integration with respect to v, we get

$$\begin{aligned}
\int_{\mathbb{S}^1} \mathbb{1}\{v \in \operatorname{pos}\{u, -u_F\}\} \, [u_F, v] \, \mathcal{H}^1(\mathrm{d}v) &= \int_0^{\angle(u, -u_F)} \sin(\alpha) \, \mathrm{d}\alpha \\
&= 1 - \cos(\angle(u, -u_F)) \\
&= 1 + \langle u, u_F \rangle, \qquad (7.10)
\end{aligned}$$

where $\angle(u, -u_F) \in [0, \pi]$ denotes the angle between u and $-u_F$. This gives

$$\begin{aligned}
J &= \frac{1}{\omega_2} \int_{\mathbb{S}^1} (1 + \langle u, u_F \rangle) \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u) \\
&= \frac{1}{\omega_2} \int_{\mathbb{S}^1} \langle u, u_F \rangle \overline{u} u^s \, \mathcal{H}^1(\mathrm{d}u),
\end{aligned}$$

where we applied equation (7.5) (the special case of Lemma 7.7 in $\mathbb{R}^2$) again, in the second step. Transforming the remaining integral in J yields

$$\begin{aligned} J &= \frac{1}{\omega_2} \int_{-1}^{1} \int_{\underbrace{\mathbb{S}^1 \cap u_F^\perp}_{=\{\overline{u}_F, -\overline{u}_F\}}} \sqrt{1-z^2}^{-1} \left\langle z u_F + \sqrt{1-z^2} w, u_F \right\rangle \\ &\quad \times \overline{(z u_F + \sqrt{1-z^2} w)} \left(z u_F + \sqrt{1-z^2} w \right)^s \mathcal{H}^0(\mathrm{d}w)\, \mathrm{d}z \\ &= \frac{1}{\omega_2} \int_{-1}^{1} \int_{\{\overline{u}_F, -\overline{u}_F\}} z \sqrt{1-z^2}^{-1} \left(z \overline{u}_F + \sqrt{1-z^2} \overline{w} \right) \left(z u_F + \sqrt{1-z^2} w \right)^s \mathcal{H}^0(\mathrm{d}w)\, \mathrm{d}z. \end{aligned}$$

Then we conclude from the binomial theorem

$$\begin{aligned} J &= \frac{1}{\omega_2} \sum_{i=0}^{s} \binom{s}{i} \overline{u}_F u_F^{s-i} \int_{-1}^{1} z^{s-i+2} \sqrt{1-z^2}^{i-1} \, \mathrm{d}z \int_{\{\overline{u}_F, -\overline{u}_F\}} w^i \, \mathcal{H}^0(\mathrm{d}w) \\ &\quad + \frac{1}{\omega_2} \sum_{i=0}^{s} \binom{s}{i} u_F^{s-i} \int_{-1}^{1} z^{s-i+1} \sqrt{1-z^2}^{i} \, \mathrm{d}z \int_{\{\overline{u}_F, -\overline{u}_F\}} \overline{w} w^i \, \mathcal{H}^0(\mathrm{d}w). \end{aligned}$$

The definition of the beta function and

$$\int_{\{\overline{u}_F, -\overline{u}_F\}} \overline{w}^l w^i \, \mathcal{H}^0(\mathrm{d}w) = \mathbb{1}\{i + l \text{ even}\}\, 2(-u_F)^l \overline{u}_F^i, \qquad l \in \{0, 1\},$$

give

$$\begin{aligned} J &= \frac{2}{\omega_2} \sum_{i=0}^{s} \mathbb{1}\{s - i \text{ even}\} \mathbb{1}\{i \text{ even}\} \binom{s}{i} B(\tfrac{s-i+3}{2}, \tfrac{i+1}{2}) \overline{u}_F^{i+1} u_F^{s-i} \\ &\quad - \frac{2}{\omega_2} \sum_{i=0}^{s} \mathbb{1}\{s - i \text{ odd}\} \mathbb{1}\{i \text{ odd}\} \binom{s}{i} B(\tfrac{s-i+2}{2}, \tfrac{i+2}{2}) \overline{u}_F^{i} u_F^{s-i+1}. \end{aligned}$$

We combine the resulting summations with respect to i and get

$$\begin{aligned} J = \mathbb{1}\{s \text{ even}\} \frac{2}{\omega_2 \Gamma(\frac{s+4}{2})} \Bigg[\sum_{i=0}^{s} \mathbb{1}\{i \text{ even}\} \binom{s}{i} \Gamma(\tfrac{s-i+3}{2}) \Gamma(\tfrac{i+1}{2}) \overline{u}_F^{i+1} u_F^{s-i} \\ - \mathbb{1}\{i \text{ odd}\} \binom{s}{i} \Gamma(\tfrac{s-i+2}{2}) \Gamma(\tfrac{i+2}{2}) \overline{u}_F^{i} u_F^{s-i+1} \Bigg]. \end{aligned}$$

Furthermore, evaluating the indicator functions, we obtain

$$\begin{aligned} J = \mathbb{1}\{s \text{ even}\} \frac{2}{\omega_2 \Gamma(\frac{s+4}{2})} \Bigg[\sum_{i=0}^{\frac{s}{2}} \binom{s}{2i} \Gamma(\tfrac{s-2i+3}{2}) \Gamma(\tfrac{2i+1}{2}) \overline{u}_F^{2i+1} u_F^{s-2i} \\ - \binom{s}{2i+1} \Gamma(\tfrac{s-2i+1}{2}) \Gamma(\tfrac{2i+3}{2}) \overline{u}_F^{2i+1} u_F^{s-2i} \Bigg]. \end{aligned}$$

The two summands inside the summation with respect to i can be combined as follows

$$\begin{aligned} J = \mathbb{1}\{s \text{ even}\}\frac{2}{\omega_2\Gamma(\frac{s+4}{2})}\Big[\sum_{i=0}^{\frac{s}{2}}(\tfrac{s-2i+1}{2})\binom{s}{2i}\Gamma(\tfrac{s-2i+1}{2})\Gamma(\tfrac{2i+1}{2})\overline{u}_F^{2i+1}u_F^{s-2i} \\ - \tfrac{s-2i}{2}\binom{s}{2i}\Gamma(\tfrac{s-2i+1}{2})\Gamma(\tfrac{2i+1}{2})\overline{u}_F^{2i+1}u_F^{s-2i}\Big] \\ = \mathbb{1}\{s \text{ even}\}\frac{1}{\omega_2\Gamma(\frac{s+4}{2})}\sum_{i=0}^{\frac{s}{2}}\binom{s}{2i}\Gamma(\tfrac{s-2i+1}{2})\Gamma(\tfrac{2i+1}{2})\overline{u}_F^{2i+1}u_F^{s-2i}. \end{aligned}$$

Applying the binomial theorem to $\overline{u}_F^{2i} = (Q - u_F^2)^i$ yields

$$J = \mathbb{1}\{s \text{ even}\}\frac{1}{\omega_2\Gamma(\frac{s+4}{2})}\sum_{i=0}^{\frac{s}{2}}\sum_{m=0}^{i}(-1)^{i-m}\binom{s}{2i}\binom{i}{m}\Gamma(\tfrac{s-2i+1}{2})\Gamma(\tfrac{2i+1}{2})Q^m\overline{u}_F u_F^{s-2m}.$$

Then a change of the order of summation and an index shift give

$$\begin{aligned} J = \mathbb{1}\{s \text{ even}\}\frac{1}{\omega_2\Gamma(\frac{s+4}{2})}\sum_{m=0}^{\frac{s}{2}}\sum_{i=0}^{\frac{s}{2}-m}(-1)^{i}\binom{s}{2i+2m}\binom{i+m}{m}\Gamma(\tfrac{s-2m-2i+1}{2})\Gamma(\tfrac{2i+2m+1}{2}) \\ \times Q^m\overline{u}_F u_F^{s-2m}. \end{aligned}$$

Next we apply Legendre's duplication formula three times to the coefficients of the remaining tensors to obtain

$$\begin{aligned} \binom{s}{2i+2m}\binom{i+m}{m}\Gamma(\tfrac{s-2m-2i+1}{2})\Gamma(\tfrac{2i+2m+1}{2}) &= \frac{s!}{m!i!}\frac{(i+m)!\Gamma(\frac{2i+2m+1}{2})}{(2i+2m)!}\frac{\Gamma(\frac{s-2i-2m+1}{2})}{(s-2i-2m)!} \\ &= \sqrt{\pi}\Gamma(\tfrac{s+1}{2})\binom{\frac{s}{2}}{m}\binom{\frac{s}{2}-m}{i} \end{aligned}$$

and hence

$$\begin{aligned} J &= \mathbb{1}\{s \text{ even}\}\frac{\sqrt{\pi}\Gamma(\frac{s+1}{2})}{\omega_2\Gamma(\frac{s+4}{2})}\sum_{m=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{m}\underbrace{\sum_{i=0}^{\frac{s}{2}-m}(-1)^i\binom{\frac{s}{2}-m}{i}}_{=\mathbb{1}\{m=\frac{s}{2}\}}Q^m\overline{u}_F u_F^{s-2m} \\ &= \mathbb{1}\{s \text{ even}\}\frac{\sqrt{\pi}\Gamma(\frac{s+1}{2})}{\omega_2\Gamma(\frac{s+4}{2})}Q^{\frac{s}{2}}\overline{u}_F. \end{aligned}$$

Plugging this into (7.9) gives

$$S_1 = \mathbb{1}\{s \text{ even}\}\frac{\sqrt{\pi}\Gamma(\frac{s+1}{2})}{\omega_2\Gamma(\frac{s+4}{2})}Q^{\frac{s}{2}}\underbrace{\sum_{F\in\mathcal{F}_1(P)}\int_{F\cap\beta}x^r\,\mathcal{H}^1(\mathrm{d}x)\overline{u}_F}_{=\breve{\phi}_1^{r,0}(P,\beta)}\underbrace{\sum_{F'\in\mathcal{F}_1(P')}\mathcal{H}^1(F'\cap\beta')}_{=2\phi_1(P',\beta')}.$$

Combining this with the results for the summands where $k = 0, 2$, yields the assertion. □

Next, we prove Theorem 7.5. As in the proof of Theorem 7.1, we change the approach of the proof of Theorem 4.1 at a certain point (see equation (7.11)). Then we proceed as in two dimensions, which works similarly (with some minor difficulties) as the codimension of the involved faces again equals one.

Proof of Theorem 7.5. We denote the kinematic integral by I. A decomposition of the measure μ yields

$$\begin{aligned} I &= \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \breve{\phi}^{r,s,l}(P \cap (\vartheta P' + t), \beta \cap (\vartheta \beta' + t))\, \mathcal{H}^3(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{\mathrm{SO}(3)} \int_{\mathbb{R}^3} \sum_{F \in \mathcal{F}_1(P \cap (\vartheta P' + t))} v_F^{2l+1} \int_{F \cap \beta \cap (\vartheta \beta' + t)} x^r\, \mathcal{H}^1(\mathrm{d}x) \int_{N(P \cap (\vartheta P' + t), F) \cap \mathbb{S}^2} \\ &\qquad \times (v_F \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^3(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta). \end{aligned}$$

In the same way as in the proof of Theorem 4.1, we obtain

$$\begin{aligned} I = \sum_{k=1}^{3} \sum_{F \in \mathcal{F}_k(P)} \sum_{F' \in \mathcal{F}_{4-k}(P')} \mathcal{H}^{4-k}(F' \cap \beta') \int_{F \cap \beta} x^r\, \mathcal{H}^k(\mathrm{d}x) \int_{\mathrm{SO}(3)} [F, \vartheta F']\, v_{F \cap (\vartheta F' + t)}^{2l+1} \\ \times \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^2} (v_{F \cap (\vartheta F' + t)} \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta), \end{aligned}$$

where the integral with respect to u is independent of the choice of a vector $t \in \mathbb{R}^3$ such that $\operatorname{relint} F \cap \operatorname{relint}(\vartheta F' + t) \neq \emptyset$. By S_k, $k = 1, 2, 3$, we denote the summands of the summation with respect to k, and calculate each of them separately. For S_1 we get

$$\begin{aligned} S_1 &= \sum_{F \in \mathcal{F}_1(P)} \sum_{F' \in \mathcal{F}_3(P')} \mathcal{H}^3(F' \cap \beta') \int_{F \cap \beta} x^r\, \mathcal{H}^1(\mathrm{d}x) \int_{\mathrm{SO}(3)} [F, \vartheta F']\, v_{F \cap (\vartheta F' + t)}^{2l+1} \\ &\qquad \times \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^2} (v_{F \cap (\vartheta F' + t)} \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta) \\ &= \mathcal{H}^3(P' \cap \beta') \sum_{F \in \mathcal{F}_1(P)} v_F^{2l+1} \int_{F \cap \beta} x^r\, \mathcal{H}^1(\mathrm{d}x) \int_{N(P, F) \cap \mathbb{S}^2} (v_F \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u) \\ &= \breve{\phi}^{r,s,l}(P, \beta)\, \phi_3(P', \beta'). \end{aligned}$$

Next, we show that both of the other summands vanish. In fact, for $k = 3$, we obtain

$$\begin{aligned} S_3 &= \sum_{F \in \mathcal{F}_3(P)} \sum_{F' \in \mathcal{F}_1(P')} \mathcal{H}^1(F' \cap \beta') \int_{F \cap \beta} x^r\, \mathcal{H}^3(\mathrm{d}x) \int_{\mathrm{SO}(3)} [F, \vartheta F']\, v_{F \cap (\vartheta F' + t)}^{2l+1} \\ &\qquad \times \int_{N(P \cap (\vartheta P' + t), F \cap (\vartheta F' + t)) \cap \mathbb{S}^2} (v_{F \cap (\vartheta F' + t)} \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{P \cap \beta} x^r\, \mathcal{H}^3(\mathrm{d}x) \sum_{F' \in \mathcal{F}_1(P')} \mathcal{H}^1(F' \cap \beta') \\ &\qquad \times \int_{\mathrm{SO}(3)} \int_{N(\vartheta P', \vartheta F') \cap \mathbb{S}^2} v_{\vartheta F'}^{2l+1} (v_{\vartheta F'} \times u)\, u^s\, \mathcal{H}^1(\mathrm{d}u)\, \nu(\mathrm{d}\vartheta). \end{aligned}$$

Then, Lemma 7.7 in $\mathbb{R}^3$ (more precisely, equation (7.6)) yields after an application of Fubini's theorem

$$\begin{aligned} S_3 &= \int_{P\cap\beta} x^r\,\mathcal{H}^3(\mathrm{d}x) \sum_{F'\in\mathcal{F}_1(P')} \mathcal{H}^1(F'\cap\beta') \\ &\quad\times \int_{N(P',F')\cap\mathbb{S}^{n-1}} \int_{\mathrm{SO}(3)} (\vartheta v_{F'})^{2l+1} (\vartheta v_{F'}\times\vartheta u)\,(\vartheta u)^s\,\nu(\mathrm{d}\vartheta)\,\mathcal{H}^1(\mathrm{d}u) \\ &= 0, \end{aligned}$$

as $v_{F'}, u, v_{F'}\times u$ form an orthonormal basis of $\mathbb{R}^3$.

For $k=2$, we have

$$\begin{aligned} S_2 = \sum_{F\in\mathcal{F}_2(P)} \sum_{F'\in\mathcal{F}_2(P')} \mathcal{H}^2(F'\cap\beta') \int_{F\cap\beta} x^r\,\mathcal{H}^2(\mathrm{d}x) \int_{\mathrm{SO}(3)} [F,\vartheta F']\, v_{F\cap(\vartheta F'+t)}^{2l+1} \\ \times \int_{N(P\cap(\vartheta P'+t),F\cap(\vartheta F'+t))\cap\mathbb{S}^2} (v_{F\cap(\vartheta F'+t)}\times u)\,u^s\,\mathcal{H}^1(\mathrm{d}u)\,\nu(\mathrm{d}\vartheta). \end{aligned} \tag{7.11}$$

We denote the integral with respect to ϑ by J_1 and apply once more [83, Theorem 2.2.1] to obtain

$$N\left(P\cap(\vartheta P'+t), F\cap(\vartheta F'+t)\right) = N(P,F)+\vartheta N(P',F').$$

Since $\dim F = \dim F' = 2$, there exist unique outer unit normal vectors u_F (resp. $u_{F'}$) of F (resp. F'), such that $N(P,F)=\{u_F\}$ (resp. $N(P',F')=\{u_{F'}\}$). Consequently, we have $N(P\cap(\vartheta P'+t), F\cap(\vartheta F'+t)) = \mathrm{pos}\{u_F,\vartheta u_{F'}\}$. It follows that

$$\begin{aligned} J_1 = \int_{\mathrm{SO}(3)} \int_{\mathrm{lin}\{u_F,\vartheta u_{F'}\}\cap\mathbb{S}^2} \mathbb{1}\left\{u\in\mathrm{pos}\{u_F,\vartheta u_{F'}\}\right\}[u_F,\vartheta u_{F'}] \\ \times v_{F\cap(\vartheta F'+t)}^{2l+1}\,(v_{F\cap(\vartheta F'+t)}\times u)\,u^s\,\mathcal{H}^1(\mathrm{d}u)\,\nu(\mathrm{d}\vartheta). \end{aligned}$$

In the same manner as in the proof of Theorem 7.1, we have

$$\mathbb{1}\left\{u\in\mathrm{pos}\{u_F,\vartheta u_{F'}\}\right\} = \mathbb{1}\left\{\vartheta u_{F'}\in\mathrm{pos}\{u,-u_F\}\right\},$$

for $\mathcal{H}^1$-almost all $u\in(\mathrm{lin}\{u_F,\vartheta u_{F'}\}\cap\mathbb{S}^2)$. We further set

$$v_{F\cap(\vartheta F'+t)} = \frac{u_F\times\vartheta u_{F'}}{\|u_F\times\vartheta u_{F'}\|} \in \left(\mathrm{lin}\{u_F,\vartheta u_{F'}\}^\perp\cap\mathbb{S}^2\right),$$

which is defined for ν-almost all $\vartheta\in\mathrm{SO}(3)$. We recall, that the calculations are independent of the choice of $v_{F\cap(\vartheta F'+t)}\in(\mathrm{lin}\{u_F,\vartheta u_{F'}\}^\perp\cap\mathbb{S}^2)$. Therefore, it follows that

$$\begin{aligned} J_1 = \int_{\mathrm{SO}(3)} \int_{\mathrm{lin}\{u_F,\vartheta u_{F'}\}\cap\mathbb{S}^2} \mathbb{1}\left\{\vartheta u_{F'}\in\mathrm{pos}\{u,-u_F\}\right\}[u_F,\vartheta u_{F'}] \\ \times \left(\tfrac{u_F\times\vartheta u_{F'}}{\|u_F\times\vartheta u_{F'}\|}\right)^{2l+1} \left(\tfrac{u_F\times\vartheta u_{F'}}{\|u_F\times\vartheta u_{F'}\|}\times u\right) u^s\,\mathcal{H}^1(\mathrm{d}u)\,\nu(\mathrm{d}\vartheta). \end{aligned}$$

A transformation of the integration with respect to ϑ yields

$$J_1 = \frac{1}{\omega_3} \int_{\mathbb{S}^2} \int_{\operatorname{lin}\{u_F, w\} \cap \mathbb{S}^2} \mathbb{1}\{w \in \operatorname{pos}\{u, -u_F\}\} [u_F, w] \\ \times \left(\tfrac{u_F \times w}{\|u_F \times w\|}\right)^{2l+1} \left(\tfrac{u_F \times w}{\|u_F \times w\|} \times u\right) u^s \, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^2(\mathrm{d}w).$$

From another transformation of the integral with respect to w, we conclude

$$J_1 = \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, v\} \cap \mathbb{S}^2} |\langle v, w\rangle| \int_{\operatorname{lin}\{u_F, w\} \cap \mathbb{S}^2} \mathbb{1}\{w \in \operatorname{pos}\{u, -u_F\}\} [u_F, w] \\ \times \left(\tfrac{u_F \times w}{\|u_F \times w\|}\right)^{2l+1} \left(\tfrac{u_F \times w}{\|u_F \times w\|} \times u\right) u^s \, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^1(\mathrm{d}w)\, \mathcal{H}^1(\mathrm{d}v),$$

where we can rewrite $|\langle v, w\rangle| = \sqrt{1 - |\langle u_F, w\rangle|^2} = [u_F, w]$. We further apply

$$\frac{u_F \times w}{\|u_F \times w\|} = \begin{cases} u_F \times v, & \text{if } \langle w, v\rangle > 0, \\ -u_F \times v, & \text{if } \langle w, v\rangle < 0, \end{cases}$$

to obtain

$$J_1 = \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, v\} \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, w\} \cap \mathbb{S}^2} \mathbb{1}\{w \in \operatorname{pos}\{u, -u_F\}\} [u_F, w]^2 \\ \times (u_F \times v)^{2l+1} ((u_F \times v) \times u) u^s \, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^1(\mathrm{d}w)\, \mathcal{H}^1(\mathrm{d}v).$$

As $v \in u_F^\perp$, it follows that $\operatorname{lin}\{u_F, w\} = \operatorname{lin}\{u_F, v\}$, for $\mathcal{H}^1$-almost all $w \in \operatorname{lin}\{u_F, v\}$. Then Fubini's theorem yields

$$J_1 = \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, v\} \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, v\} \cap \mathbb{S}^2} \mathbb{1}\{w \in \operatorname{pos}\{u, -u_F\}\} [u_F, w]^2 \, \mathcal{H}^1(\mathrm{d}w) \\ \times (u_F \times v)^{2l+1} ((u_F \times v) \times u) u^s \, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^1(\mathrm{d}v).$$

We denote the inner integration with respect to w by J_2 and get

$$\begin{aligned} J_2 &= \int_{\operatorname{pos}\{u, -u_F\} \cap \mathbb{S}^2} [u_F, w]^2 \, \mathcal{H}^1(\mathrm{d}w) \\ &= \int_0^{\angle(u, -u_F)} \sin(\alpha)^2 \, \mathrm{d}\alpha \\ &= \angle(u, -u_F) - \sin(\angle(u, -u_F)) \cos(\angle(u, -u_F)) \\ &=: c(\angle(u, -u_F)), \end{aligned}$$

where $\angle(u, -u_F) \in [0, \pi]$ denotes the angle between u and $-u_F$. Thus, we obtain

$$J_1 = \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\operatorname{lin}\{u_F, v\} \cap \mathbb{S}^2} c(\angle(u, -u_F)) (u_F \times v)^{2l+1} ((u_F \times v) \times u) u^s \, \mathcal{H}^1(\mathrm{d}u)\, \mathcal{H}^1(\mathrm{d}v).$$

For $\rho \in \mathrm{SO}(u_F^\perp)$, it follows from the rotation invariance of $\mathcal{H}^1$ that

$$\begin{aligned}
J_1 &= \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\underbrace{\mathrm{lin}\{u_F, \rho v\}}_{=\rho \mathrm{lin}\{u_F, v\}} \cap \mathbb{S}^2} c(\angle(u, -u_F)) \\
&\quad \times (\underbrace{u_F \times \rho v}_{=\rho(u_F \times v)})^{2l+1} ((\underbrace{u_F \times \rho v}_{=\rho(u_F \times v)}) \times u) u^s \, \mathcal{H}^1(\mathrm{d}u) \, \mathcal{H}^1(\mathrm{d}v) \\
&= \frac{1}{2\omega_3} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\mathrm{lin}\{u_F, v\} \cap \mathbb{S}^2} c(\underbrace{\angle(\rho u, -u_F)}_{=\angle(u, -u_F)}) \\
&\quad \times \big(\rho(u_F \times v)\big)^{2l+1} \rho\big((u_F \times v) \times u\big) (\rho u)^s \, \mathcal{H}^1(\mathrm{d}u) \, \mathcal{H}^1(\mathrm{d}v).
\end{aligned}$$

Then we can integrate over all such rotations $\rho \in \mathrm{SO}(u_F^\perp)$ and obtain

$$\begin{aligned}
J_1 = \frac{1}{2\omega_3} &\int_{\mathrm{SO}(u_F^\perp)} \int_{u_F^\perp \cap \mathbb{S}^2} \int_{\mathrm{lin}\{u_F, v\} \cap \mathbb{S}^2} c(\angle(u, -u_F)) \\
&\times \big(\rho(u_F \times v)\big)^{2l+1} \rho\big((u_F \times v) \times u\big) (\rho u)^s \, \mathcal{H}^1(\mathrm{d}u) \, \mathcal{H}^1(\mathrm{d}v) \, \nu^{u_F^\perp}(\mathrm{d}\rho).
\end{aligned}$$

A further application of Fubini's theorem yields

$$\begin{aligned}
J_1 = \frac{1}{2\omega_3} &\int_{u_F^\perp \cap \mathbb{S}^2} \int_{\mathrm{lin}\{u_F, v\} \cap \mathbb{S}^2} c(\angle(u, -u_F)) \\
&\times \int_{\mathrm{SO}(u_F^\perp)} \rho\Big((u_F \times v)^{2l+1} \big((u_F \times v) \times u\big) u^s\Big) \, \nu^{u_F^\perp}(\mathrm{d}\rho) \, \mathcal{H}^1(\mathrm{d}u) \, \mathcal{H}^1(\mathrm{d}v).
\end{aligned}$$

Now it follows from equation (7.7) (the consequence of Lemma 7.7) that the inner integral with respect to ρ vanishes, which finishes the proof. □

CHAPTER 8

Intrinsic Crofton Formulae

In the present chapter, we use a different normalization of the (intrinsic) generalized tensorial curvature measures and the (intrinsic) Minkowski tensors, defined in Chapter 3, in order to simplify the representations of the upcoming formulae. That is, we define

$$\hat{\phi}_j^{r,s,l} := \omega_{n-j} \left(c_{n,j}^{r,s,l}\right)^{-1} \phi_j^{r,s,l} \qquad \text{and} \qquad \hat{\phi}_n^{r,0,0} := \left(c_{n,n}^{r,0,0}\right)^{-1} \hat{\phi}_n^{r,0,0}$$

on $\mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n)$, for $j, r, s \in \mathbb{N}_0$ with $j < n$, $l \in \{0, 1\}$, and set the corresponding total measures as

$$\hat{\Phi}_j^{r,s} := \hat{\phi}_j^{r,s,0}(\cdot, \mathbb{R}^n) \qquad \text{and} \qquad \hat{\Phi}_n^{r,0} := \hat{\phi}_n^{r,0,0}(\cdot, \mathbb{R}^n).$$

Furthermore, for $E \in \mathrm{A}(n, k)$ with $k \in \{0, \ldots, n\}$, we define

$$\hat{\phi}_{j,E}^{r,s,l} := \omega_{k-j} \left(c_{k,j}^{r,s,l}\right)^{-1} \phi_{j,E}^{r,s,l} \qquad \text{and} \qquad \hat{\phi}_{k,E}^{r,0,0} := \left(c_{k,k}^{r,0,0}\right)^{-1} \hat{\phi}_{k,E}^{r,0,0}$$

on $(\mathcal{K}^n \cap \mathcal{B}(E)) \times \mathcal{B}(\mathbb{R}^n)$, where $j, r, s \in \mathbb{N}_0$ with $j < k$, $l \in \{0, 1\}$, and set the corresponding total measures as

$$\hat{\Phi}_{j,E}^{r,s} := \hat{\phi}_{j,E}^{r,s,0}(\cdot, \mathbb{R}^n) \qquad \text{and} \qquad \hat{\Phi}_{n,E}^{r,0} := \hat{\phi}_{n,E}^{r,0,0}(\cdot, \mathbb{R}^n)$$

However, we still refer to these valuations as (intrinsic) tensorial curvature measures, resp. (intrinsic) Minkowski tensors, since they are simply renormalized versions thereof.

The aim of this chapter is to state and prove a set of Crofton formulae for the intrinsic tensorial curvature measures on convex bodies. More precisely, for $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $j, k, i, r, s \in \mathbb{N}_0$ with $0 \le j \le k \le n$, we express the integral mean value

$$\int_{\mathrm{A}(n,k)} Q(E)^i \, \hat{\phi}^{r,s,0}_{j,E}(K \cap E, \beta \cap E) \, \mu_k(\mathrm{d}E) \tag{8.1}$$

in terms of (generalized) tensorial curvature measures of K, evaluated at β. In fact, we show that this expression requires only a selection of these valuations.

By globalization of (8.1), we then deduce explicit expressions for the Crofton integrals

$$\int_{\mathrm{A}(n,k)} Q(E)^i \, \hat{\Phi}^{0,s}_{j,E}(K \cap E) \, \mu_k(\mathrm{d}E) \tag{8.2}$$

in terms of Minkowski tensors of K. However, in the global case we restrict our investigations to the translation invariant intrinsic Minkowski tensors (setting $r = 0$). Crofton formulae for general (intrinsic) Minkowski tensors have already been established in [51]. The proofs in this chapter are based on the approach used to prove these. Nevertheless, the restriction to the translation invariant case allows substantial simplifications of the appearing coefficients.

Via the relation between extrinsic and intrinsic tensorial curvature measures depicted in Lemma 3.5, one can derive extrinsic Crofton formulae from the intrinsic results. This is what we explain in detail for $j = k - 1$, meaning we investigate Crofton integrals of the type

$$\int_{\mathrm{A}(n,k)} \hat{\phi}^{r,s,0}_{k-1}(K \cap E, \beta \cap E) \, \mu_k(\mathrm{d}E), \tag{8.3}$$

and we again obtain simplified coefficients in that case. These formulae are special cases of the Crofton formulae stated and proved in Chapter 5. Nevertheless, they are provided here as well, as the approach is completely different from the one in Chapters 4 and 5. Moreover, we introduce an alternative representation of the tensorial curvature measures (similar to the so called Ψ-basis of the Minkowski tensors, introduced in [15, Proposition 4.10]) and show quite simple Crofton formulae for the thus obtained valuations.

We note that, since the tensorial curvature measures (resp. Minkowski tensors) depend additively on the underlying convex body, all integral formulae in this chapter remain true if the occurring convex bodies are replaced by finite unions thereof.

Remark. The results in this chapter have already been published. To a great extent the present chapter contains direct quotes from the publication *Crofton formulae for tensor-valued curvature measures*, a joint work with Daniel Hug, appearing as Chapter 4 in the lecture notes *Tensor Valuations and their Applications in Stochastic Geometry and Imaging* edited by Kiderlen and Vedel Jensen (see [54] in [58]). Parts of this chapter (more precisely, Theorem 8.4 and the global results in Section 8.1.2 with the deduction thereof) are already included in my Master's thesis from 2014 (see [99]).

8.1. The Results of Chapter 8

In this section, we state the formulae for the Crofton integrals (8.1), (8.2) and (8.3). Along the way, we present some special cases which are either new results or have already been proved by other authors using different methods.

8.1.1. Intrinsic Tensorial Curvature Measures

At first, we give the formulae concerning the integrals (8.1) and (8.2), and start with the local versions, where we distinguish the cases $j = k$ and $j < k$. In the first theorem we consider the former.

Theorem 8.1. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i, k, r, s \in \mathbb{N}_0$ with $k < n$. Then*

$$\int_{\mathrm{A}(n,k)} Q(E)^i \, \hat{\phi}^{r,s,0}_{k,E}(K \cap E, \beta \cap E) \, \mu_k(\mathrm{d}E) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2} + i)}{\Gamma(\frac{n}{2} + i)\Gamma(\frac{k}{2})} \, Q^i \hat{\phi}^{r,0,0}_n(K, \beta)$$

if $s = 0$; for $s \neq 0$ the integral on the left is zero.

If $s = 0$ in Theorem 8.1, then relation (2.2) allows us to interpret the coefficient of the tensor on the right-hand side as 0, if $k = 0$ and $i \neq 0$, and as 1, if $k = i = 0$. A global version of Theorem 8.1 is obtained by simply setting $\beta = \mathbb{R}^n$.

Next, we proceed with the case $j < k$.

Theorem 8.2. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i, j, k, r, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$. Then*

$$\int_{\mathrm{A}(n,k)} Q(E)^i \, \hat{\phi}^{r,s,0}_{j,E}(K \cap E, \beta \cap E) \, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} Q^z \Big(\lambda^{(0)}_{n,k,j,s,i,z} \, \hat{\phi}^{r,s+2i-2z,0}_{n-k+j}(K, \beta) + \lambda^{(1)}_{n,k,j,s,i,z} \, \hat{\phi}^{r,s+2i-2z-2,1}_{n-k+j}(K, \beta) \Big),$$

where for $\varepsilon \in \{0, 1\}$ we set

$$\gamma_{n,k,j} := \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi},$$
$$\lambda^{(\varepsilon)}_{n,k,j,s,i,z} := \sum_{p=0}^{i} \sum_{q=(z-p+\varepsilon)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q-\varepsilon}{z} \Gamma(q + \tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)} \frac{\Gamma(\frac{k-1}{2} + p)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + p + q)} \vartheta^{(\varepsilon)}_{n,k,j,p,q},$$
$$\vartheta^{(0)}_{n,k,j,p,q} := (n-k+j)(\tfrac{k-1}{2} + p), \qquad \vartheta^{(1)}_{n,k,j,p,q} := p(n-k) - q(k-1).$$

In Theorem 8.2, if $j = k - 1$, then the tensorial curvature measures and the generalized tensorial curvature measures are linearly dependent. In this case, the right-hand side can be expressed as a linear combination of the valuations $Q^z \hat{\phi}_{n-1}^{r,s+2i-2z,0}(K,\cdot)$, whereas the measures $Q^z \hat{\phi}_{n-1}^{r,s+2i-2z,1}(K,\cdot)$ are not needed. An explicit description of this case is given in Corollary 8.9 for $i = 0$ and in (8.12) for $i \in \mathbb{N}_0$.

If the additional metric tensor is omitted as a weight function, that is in the case $i = 0\,(= p)$, then the coefficients $\lambda^{(\varepsilon)}_{n,k,j,s,0,z}$ in Theorem 8.2 simplify to a single sum.

Apparently, the coefficients in Theorem 8.2 are not well-defined in the (here excluded) case $k = 1$ and $j = 0$, as $\Gamma(0)$ is involved in the numerator of $\lambda^{(\varepsilon)}_{n,1,0,s,i,z}$. Although this issue can be resolved by the proper interpretation of the (otherwise ambiguous) expression $\Gamma(p) \cdot p = \Gamma(p+1)$ as 1 for $p = 0$, we prefer to state and derive this case separately. In fact, our analysis leads to substantial simplifications of the constants, as our next result shows.

Theorem 8.3. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i, r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{A}(n,1)} Q(E)^i \, \hat{\phi}_{0,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$
$$= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z}\, Q^{\frac{s}{2}+i-z} \hat{\phi}_{n-1}^{r,2z,0}(K,\beta)$$

for even s. If s is odd, then

$$\int_{\mathrm{A}(n,1)} Q(E)^i \, \hat{\phi}_{0,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2}+i)}\, Q^{\frac{s-1}{2}+i} \hat{\phi}_{n-1}^{r,1,0}(K,\beta).$$

We note that in Theorem 8.3 the Crofton integral is expressed only by tensorial curvature measures $\hat{\phi}_{n-1}^{r,z,0}$ (multiplied with suitable powers of the metric tensor), whereas generalized tensorial curvature measures are not needed. A global version of Theorem 8.3 is obtained by simply setting $\beta = \mathbb{R}^n$.

A translation invariant, global version of Theorem 8.2 allows us to combine several of the summands on the right-hand side of the formula with the help of McMullen's Lemma 3.6.

Theorem 8.4. *Let $K \in \mathcal{K}^n$ and $i, j, k, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$. Then*

$$\int_{\mathrm{A}(n,k)} Q(E)^i \, \hat{\Phi}_{j,E}^{0,s}(K \cap E)\, \mu_k(\mathrm{d}E) = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda^{(0)}_{n,k,j,s,i,z}\, Q^z \hat{\Phi}_{n-k+j}^{0,s+2i-2z}(K),$$

where $\gamma_{n,k,j}$ and $\lambda^{(0)}_{n,k,j,s,i,z}$ are defined as in Theorem 8.2, but

$$\vartheta^{(0)}_{n,k,j,s,i,z,p,q} := (n-k+j)\big(\tfrac{k-1}{2}+p\big) - \big(p(n-k) - q(k-1)\big)\big(1 + \tfrac{k-j-1}{s+2i-2z-1}\big(1 - \tfrac{z}{p+q}\big)\big)$$

replaces $\vartheta^{(0)}_{n,k,j,p,q}$, except if s is odd and $z = \lfloor \frac{s}{2} \rfloor + i$, where $\lambda^{(0)}_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i} := 0$.

In Theorem 8.4, if $p = q = 0$, then the definition of $\lambda^{(0)}_{n,k,j,s,i,z}$ implies that also $z = 0$ and thus, $\vartheta^{(0)}_{n,k,j,s,i,0,0,0}$ is well-defined with $\frac{z}{p+q} = 1$.

8.1.2. Some Special Cases

In the following, we state some special cases of the just given theorems. For that purpose, we restrict to the case of Crofton formulae for unweighted intrinsic Minkowski tensors or tensorial curvature measures, meaning $i = 0$.

Corollary 8.5. *Let $K \in \mathcal{K}^n$ and $k, j, s \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \hat{\Phi}^{0,s}_{j,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \delta_{n,k,j,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \eta_{n,k,j,s,z}\, Q^z \hat{\Phi}^{0,s-2z}_{n-k+j}(K),$$

where

$$\delta_{n,k,j,s} := \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{k+1}{2})}{\pi\Gamma(\frac{n-k+j+s}{2}+1)},$$

$$\eta_{n,k,j,s,z} := \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q}\binom{q}{z} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{j+s}{2}-q+1)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+q)} \times \left(\tfrac{n-k+j}{2} + q + \tfrac{(k-j-1)(q-z)}{s-2z-1}\right),$$

but $\eta_{n,k,j,s,\lfloor \frac{s}{2} \rfloor} := 0$ if s is odd.

Specific choices of s

Next we collect some special cases of Corollary 8.5, which are obtained for specific choices of $s \in \mathbb{N}_0$ by applications of Legendre's duplication formula and elementary calculations.

Corollary 8.6. *Let $K \in \mathcal{K}^n$ and $k, j \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \hat{\Phi}^{0,2}_{j,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+3}{2})\Gamma(\frac{j+1}{2})} \left(\tfrac{n-k}{4(n-k+j)}\, Q\hat{\Phi}^{0,0}_{n-k+j}(K) + \tfrac{n-k+nj+j}{2(n-k+j)}\, \hat{\Phi}^{0,2}_{n-k+j}(K) \right).$$

Corollary 8.7. *Let $K \in \mathcal{K}^n$ and $k, j \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \hat{\Phi}^{0,3}_{j,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \frac{j+1}{n-k+j+1} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j}{2})} \hat{\Phi}^{0,3}_{n-k+j}(K).$$

As $\Gamma(\frac{j}{2})^{-1} = 0$, for $j = 0$, the integral in Corollary 8.7 equals 0 in this case. However, as the integrand on the left-hand side is already 0, this is not surprising. The same is true for any odd number $s \in \mathbb{N}$ and $j = 0$.

Corollary 8.7 immediately leads to a result which was obtained and applied by Bernig and Hug in [15, Lemma 4.13].

Corollary 8.8. *Let $K \in \mathcal{K}^n$. Then*

$$\int_{\mathrm{A}(n,2)} \hat{\Phi}^{0,3}_{1,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \binom{n}{2}^{-1} \hat{\Phi}^{0,3}_{n-1}(K).$$

THE CHOICE $j = k - 1$

Furthermore, we obtain simple Crofton formulae for the specific choice $j = k - 1$ in the local and in the global case.

Corollary 8.9. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \hat{\phi}^{r,s,0}_{k-1,E}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \delta_{n,k,k-1,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \xi_{n,k,s,z}\, Q^z \hat{\phi}^{r,s-2z,0}_{n-1}(K, \beta),$$

where

$$\xi_{n,k,s,z} := \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma\left(q + \tfrac{1}{2}\right) \frac{\Gamma(\frac{k+s+1}{2} - q)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + q)}.$$

Corollary 8.9 will be derived from Theorem 8.2 in the same way as Theorem 8.4 is proved. More specifically, we apply the special case of Lemma 3.1 where $l = 1$, that is,

$$\hat{\phi}^{r,s,1}_{n-1} = Q \hat{\phi}^{r,s,0}_{n-1} - \hat{\phi}^{r,s+2,0}_{n-1}, \tag{8.4}$$

which can be considered as a local version of McMullen's Lemma 3.6 in the particular case where $j = n - 1$. Although $k = 1$ is excluded in Corollary 8.9, the result is formally consistent with Theorem 8.3 (for $i = 0$), which can be checked by simplifying the coefficients $\xi_{n,1,s,z}$ with the help of Zeilberger's algorithm.

A global version of Corollary 8.9 is obtained by setting $\beta = \mathbb{R}^n$.

Finally, Theorem 8.3 can be globalized to give a result, which was obtained in [65] by a completely different approach.

Corollary 8.10. *Let $K \in \mathcal{K}^n$ and $s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{A}(n,1)} \hat{\Phi}^{0,s}_{0,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \frac{2\omega_{n+s+1}}{\pi \omega_{s+1} \omega_n} \sum_{z=0}^{\frac{s}{2}} \frac{(-1)^z}{1 - 2z} \binom{\frac{s}{2}}{z} Q^{\frac{s}{2} - z} \hat{\Phi}^{0,2z}_{n-1}(K)$$

for even s. For odd s the integral on the left-hand side equals 0.

We note that if $s \in \mathbb{N}$ is odd, then the Crofton integral in Theorem 8.3 is a non-zero measure, as the tensorial curvature measures $\hat{\phi}^{r,1,0}_{n-1}(K, \cdot)$ are non-zero (if the underlying set K is at least $(n-1)$-dimensional), whereas $\hat{\Phi}^{0,1}_{n-1} \equiv 0$ in the global case considered in Corollary 8.10.

8.1.3. Extrinsic Tensorial Curvature Measures

In the following, we state Crofton formulae for tensorial curvature measures for $j = k - 1$. The method also applies to the cases where $j \leq k - 2$, but it remains to be explored to which extent the constants can be simplified then by the current approach. However, in Chapter 5 we have presented all the remaining cases, though obtained by a completely different approach.

As for the intrinsic versions, we have to distinguish between the cases $k > 1$ and $k = 1$. We start with the former, which is basically a (renormalized) special case of Theorem 5.4.

Theorem 8.11. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then*

$$\int_{\mathrm{A}(n,k)} \hat{\phi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z}\, Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K, \beta),$$

where

$$\kappa_{n,k,s,z} := \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2}+z)\Gamma(\frac{k+s-1}{2}-z)}{\Gamma(\frac{s}{2}-z+1)z!}$$

if $z \neq \frac{s-1}{2}$, and

$$\kappa_{n,k,s,\frac{s-1}{2}} := \pi^{\frac{n-k-1}{2}} \frac{2k(n+s-2)}{(n-1)(n-k+s-1)} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n+s+1}{2})}. \tag{8.5}$$

In Theorem 8.11, if s is odd the coefficient $\kappa_{n,k,s,(s-1)/2}$ has to be defined separately, as the proof shows. In fact, one easily checks that the difference amounts to a factor $k(n+s-2)[(k-1)(n+s-1)]^{-1}$. For even s, the constants involved in the proof of Theorem 8.11 can be simplified by a direct calculation to arrive at the asserted result. However, if s is odd, we need the connection to the work [15] to simplify the constants. Since this connection breaks down for $z = (s-1)/2$, s odd, a separate direct calculation is required for this case, and that finally yields the correct constant in (8.5). The result is also consistent with the special case $k = 1$ which is considered next. The more structural viewpoint in Chapter 5 provides another explanation for the case distinction required for the coefficients in the preceding Crofton formula (see Corollary 5.7, which is a renormalized version of Theorem 8.11).

For $k = 1$ the Crofton integrals can be represented with a single tensorial measure, as the following theorem shows.

Theorem 8.12. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $r, s \in \mathbb{N}_0$. Then*

$$\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) = \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\lfloor \frac{s+1}{2} \rfloor + \frac{1}{2})}{\Gamma(\frac{n}{2} + \lfloor \frac{s+1}{2} \rfloor)} Q^{\lfloor \frac{s}{2} \rfloor} \hat{\phi}_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta).$$

Theorem 8.12 is a renormalized version of Corollary 5.8. It can be easily checked that this formula for $k = 1$ can be obtained from Theorem 8.11 (which actually holds for $k > 1$) by a formal specialization and proper interpretation of expressions which a priori are not well-defined. For this to work, it is indeed crucial that for odd values of s and $z = (s-1)/2$ the definition in (8.5) applies.

In [15, Proposition 4.10], an alternative basis of the vector space of continuous, translation invariant and rotation covariant $\mathbb{T}^p$-valued valuations on $\mathcal{K}^n$ was introduced, based on the trace free part of the Minkowski tensors, which was called the Ψ-basis. In the same spirit (but locally and with the current normalization), we now define

$$\hat{\psi}_k^{r,s,0} := \hat{\phi}_k^{r,s,0} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j \hat{\phi}_k^{r,s-2j,0}$$

for $r, s \in \mathbb{N}_0$ and $k \in \{0, \ldots, n-1\}$. Interpreting this definition in the right way if $n = 2$ and $s = 0$ (where $\hat{\psi}_k^{r,0,0} = \hat{\phi}_k^{r,0,0}$), we can also write

$$\hat{\psi}_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j \hat{\phi}_k^{r,s-2j,0}. \tag{8.6}$$

In particular, $\hat{\psi}_k^{r,s,0} = \hat{\phi}_k^{r,s,0}$ for $s \in \{0, 1\}$. Conversely, we have

$$\hat{\phi}_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-2j)}{\Gamma(\frac{n}{2}+s-j)} Q^j \hat{\psi}_k^{r,s-2j,0}. \tag{8.7}$$

Although this will not be needed explicitly, it shows how we can switch between a $\hat{\phi}$-representation and a $\hat{\psi}$-representation of tensorial curvature measures.

The main advantage of the new local tensor valuations given in (8.6) is that the Crofton formula takes a particularly simple form.

Corollary 8.13. *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and let $k, r, s \in \mathbb{N}_0$ with $0 < k < n$.*

If $s \notin \{0, 1\}$, then

$$\begin{aligned} &\int_{A(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\ &\qquad = \pi^{\frac{n-k}{2}} \frac{k-1}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n-k+s+1}{2})} \hat{\psi}_{n-1}^{r,s,0}(K, \beta). \end{aligned}$$

If $s = 0$, then

$$\int_{A(n,k)} \hat{\psi}_{k-1}^{r,0,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \pi^{\frac{n-k+1}{2}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n+1}{2})} \hat{\psi}_{n-1}^{r,0,0}(K, \beta).$$

If $s = 1$, then

$$\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,1,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \pi^{\frac{n-k}{2}} \frac{k}{n} \frac{1}{\Gamma(\frac{n-k+2}{2})} \hat{\psi}_{n-1}^{r,1,0}(K, \beta).$$

For $r = 0$ and $\beta = \mathbb{R}^n$, Corollary 8.13 coincides with [15, Corollary 6.1] (in the case corresponding to $j = k - 1$). If $s \in \{0, 1\}$, then $\hat{\psi}_k^{r,s,0} = \hat{\phi}_k^{r,s,0}$ and Corollary 8.13 coincides with Theorem 8.11 (resp. Theorem 8.12, for $k = 1$). If $k = 1$, then the integral in Corollary 8.13 vanishes, except for $s \in \{0, 1\}$.

8.2. The Proofs for the Intrinsic Results

In this section, we prove the intrinsic Crofton formulae stated in Section 8.1.1. The approach applied here is heavily based on the proofs in [51]. Therefore, before we can start, we derive an integral formula which is required in the following proofs (similarly to the procedure in [51]).

8.2.1. Auxiliary Integral Formulae

With the preliminary integral formulae from [51], recalled in Appendix A, we are able to establish the following integral formula, which is a slightly modified version of [51, Proposition 4.7].

Proposition 8.14. *Let $i, j, k, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$, $F \in \mathrm{G}(n, n-k+j)$ and $u \in F^\perp \cap \mathbb{S}^{n-1}$. Then*

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \Big(\lambda_{n,k,j,s,i,z}^{(0)} u^2 + \lambda_{n,k,j,s,i,z}^{(1)} Q(F) \Big) Q^z u^{s+2i-2z-2},$$

where the coefficients are defined as in Theorem 8.2.

In Proposition 8.14 we exclude the case $k = 1$, the reason of which becomes clear in the following proof, which further leads to undefined coefficients in that case (see the explanation after Theorem 8.2). However, these coefficients can be properly interpreted, such that the assertion also holds for $k = 1$. Nevertheless, we derive this case separately in Proposition 8.15.

Proof. The integral geometric transformation formula in Lemma A.2 yields

$$\begin{aligned}
&\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L) \\
&\quad = \frac{\omega_k}{2\omega_n} \int_{\mathrm{G}(u^\perp,k-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1} (1-t^2)^{\frac{n-k-2}{2}} \pi_{\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}}(u)^s \\
&\qquad \times Q\big(\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}\big)^i \|p_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u)\|^{j-k} \\
&\qquad \times \big[F, \mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}\big]^2 \, \mathcal{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).
\end{aligned}$$

As

$$\begin{aligned}
Q\big(\mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}\big) &= Q(U) + \big(|t|u + \sqrt{1-t^2}\mathrm{sign}(t)w\big)^2, \\
\pi_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u) &= |t|u + \sqrt{1-t^2}\mathrm{sign}(t)w, \\
\|p_{\mathrm{lin}\{U,tu+\sqrt{1-t^2}w\}}(u)\| &= |t|, \\
[F, \mathrm{lin}\{U, tu+\sqrt{1-t^2}w\}] &= [F,U]^{(u^\perp)}|t|
\end{aligned}$$

hold for all $t \in [-1,1] \setminus \{0\}$, we obtain

$$\begin{aligned}
&\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L) \\
&\quad = \frac{\omega_k}{2\omega_n} \int_{\mathrm{G}(u^\perp,k-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{j+1} (1-t^2)^{\frac{n-k-2}{2}} \big([F,U]^{(u^\perp)}\big)^2 \big(|t|u + \sqrt{1-t^2}w\big)^s \\
&\qquad \times \big(Q(U) + (|t|u+\sqrt{1-t^2}w)^2\big)^i \, \mathcal{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu_{k-1}^{u^\perp}(\mathrm{d}U),
\end{aligned}$$

where we used the fact that the integration with respect to w is invariant under reflections in the origin. Then we apply the binomial theorem to the terms $(Q(U) + (|t|u+\sqrt{1-t^2}w)^2)^i$ and $(|t|u+\sqrt{1-t^2}w)^{s+2p}$ and get

$$\begin{aligned}
&\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L) \\
&\quad = \frac{\omega_k}{2\omega_n} \sum_{p=0}^{i} \sum_{q=0}^{s+2p} \binom{i}{p} \binom{s+2p}{q} \int_{\mathrm{G}(u^\perp,k-1)} \int_{-1}^{1} |t|^{j+s+2p-q+1} (1-t^2)^{\frac{n-k+q-2}{2}} \, \mathrm{d}t \\
&\qquad \times \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \, \mathcal{H}^{n-k-1}(\mathrm{d}w) \big([F,U]^{(u^\perp)}\big)^2 u^{s+2p-q} Q(U)^{i-p} \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).
\end{aligned}$$

Since Lemma A.3 yields

$$\int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \, \mathcal{H}^{n-k-1}(\mathrm{d}w) = \mathbb{1}\{q \text{ even}\} 2 \frac{\omega_{n-k+q}}{\omega_{q+1}} Q(U^\perp \cap u^\perp)^{\frac{q}{2}},$$

we can rewrite the integration with respect to t in terms of the Beta function and apply the relation of the latter to the Gamma function, to obtain

$$\begin{aligned}\int_{\mathrm{G}(n,k)} & Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\ &= \frac{\omega_k}{\omega_n} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \frac{\Gamma(\frac{j+s}{2}+p-q+1)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-k+j+s}{2}+p+1)} \frac{\omega_{n-k+2q}}{\omega_{2q+1}} \\ &\quad \times u^{s+2p-2q} \int_{\mathrm{G}(u^\perp,k-1)} Q(U^\perp \cap u^\perp)^q \big([F,U]^{(u^\perp)}\big)^2 Q(U)^{i-p} \,\nu_{k-1}^{u^\perp}(\mathrm{d}U).\end{aligned}$$

Applying the binomial theorem to $Q(U^\perp \cap u^\perp)^q = (Q(u^\perp) - Q(U))^q$ yields

$$\begin{aligned}\int_{\mathrm{G}(n,k)} & Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\ &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{k}{2})} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^{q} (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{j+s}{2}+p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+p+1)} \\ &\quad \times u^{s+2p-2q} Q(u^\perp)^{q-y} \int_{\mathrm{G}(u^\perp,k-1)} \big([F,U]^{(u^\perp)}\big)^2 Q(U)^{i-p+y} \,\nu_{k-1}^{u^\perp}(\mathrm{d}U). \qquad (8.8)\end{aligned}$$

We conclude from Lemma A.6, which is applied in $u^\perp$ to the remaining integral on the right-hand side of (8.8),

$$\begin{aligned}\int_{\mathrm{G}(n,k)} & Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\ &= \frac{(n-k+j)!(k-1)!}{\sqrt{\pi}(n-1)!j!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \Gamma(q+\tfrac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{j+s}{2}+p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+p+1)} u^{s+2p-2q} \sum_{y=0}^{q} (-1)^y \binom{q}{y} \frac{\Gamma(\frac{k-1}{2}+i-p+y)}{\Gamma(\frac{n+1}{2}+i-p+y)} \\ &\quad \times \Big((\tfrac{k-1}{2}+i-p+y) Q(u^\perp)^{i-p+q} + \tfrac{k-n}{n-k+j}(i-p+y) Q(u^\perp)^{i-p+q-1} Q(F) \Big).\end{aligned}$$

Relation (B.1′) applied twice to the summations with respect to y and Legendre's duplication formula applied three times to the Gamma functions involving n, k and $n-k$ yield together with the definitions of $\gamma_{n,k,j}$ and $\vartheta^{(\varepsilon)}_{n,k,j,p,q}$, $\varepsilon \in \{0,1\}$,

$$\begin{aligned}\int_{\mathrm{G}(n,k)} & Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\ &= \gamma_{n,k,j} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q+\tfrac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)} \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+p+q)} \\ &\quad \times u^{s+2i-2p-2q} \Big(\vartheta^{(0)}_{n,k,j,p,q} Q(u^\perp)^{p+q} - \vartheta^{(1)}_{n,k,j,p,q} Q(u^\perp)^{p+q-1} Q(F) \Big),\end{aligned}$$

where we changed the order of summation with respect to p. From the binomial theorem applied to $Q(u^\perp)^{p+q} = (Q - u^2)^{p+q}$ we obtain

$$\begin{aligned}
&\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\
&= \gamma_{n,k,j} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)} \\
&\quad \times \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+p+q)} \Bigg(\sum_{z=0}^{p+q} (-1)^{p+q-z} \binom{p+q}{z} \vartheta^{(0)}_{n,k,j,p,q} Q^z u^{s+2i-2z} \\
&\qquad + \sum_{z=0}^{p+q-1} (-1)^{p+q-z} \binom{p+q-1}{z} \vartheta^{(1)}_{n,k,j,p,q} Q^z u^{s+2i-2z-2} Q(F) \Bigg).
\end{aligned}$$

A change of the order of summation, such that we sum with respect to z first, gives

$$\begin{aligned}
&\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) \\
&= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} (\lambda^{(0)}_{n,k,j,s,i,z} u^2 + \lambda^{(1)}_{n,k,j,s,i,z} Q(F)) Q^z u^{s+2i-2z-2},
\end{aligned}$$

which concludes the proof. $\square$

Next we state and prove the extension of Proposition 8.14 to the case of $k = 1$.

Proposition 8.15. *Let $i, s \in \mathbb{N}_0$, $F \in \mathrm{G}(n, n-1)$ and $u \in F^\perp \cap \mathbb{S}^{n-1}$. Then*

$$\begin{aligned}
&\int_{\mathrm{G}(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \,\nu_1(\mathrm{d}L) \\
&= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z} u^{2z} Q^{\frac{s}{2}+i-z}
\end{aligned}$$

for even s. If s is odd, then

$$\int_{\mathrm{G}(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2}+i)} u Q^{\frac{s-1}{2}+i}.$$

Proof. The proof basically works as the proof of Proposition 8.14. But we do not need to apply Lemma A.6, as (8.8) simplifies to

$$\begin{aligned}
&\int_{\mathrm{G}(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \,\nu_k(\mathrm{d}L) \\
&= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^{q} (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+p-q+1)}{\Gamma(\frac{n+s+1}{2}+p)} \\
&\quad \times u^{s+2p-2q} Q(u^\perp)^{q-y} \int_{\mathrm{G}(u^\perp,0)} ([F,U]^{(u^\perp)})^2 Q(U)^{i-p+y} \,\nu^{u^\perp}_{k-1}(\mathrm{d}U).
\end{aligned}$$

Since the remaining integral on the right-hand side equals 1, if $p = i$ and $y = 0$, and in all the other cases it equals 0, we obtain

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \binom{s+2i}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+i-q+1)}{\Gamma(\frac{n+s+1}{2}+i)} u^{s+2i-2q} Q(u^\perp)^q.$$

Applying the binomial theorem to $Q(u^\perp)^q = (Q - u^2)^q$ yields

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \sum_{z=0}^{q} (-1)^{q-z} \binom{s+2i}{2q} \binom{q}{z} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+i-q+1)}{\Gamma(\frac{n+s+1}{2}+i)} u^{s+2i-2z} Q^z.$$

A change of the order of summation and Legendre's duplication formula applied to the Gamma functions involving q give

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{(s+2i)!\Gamma(\frac{n}{2})}{2^{s+2i}\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \frac{1}{z!} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2}+i-q)(q-z)!} u^{s+2i-2z} Q^z.$$

If s is even, we conclude from Lemma B.3 applied to the summation with respect to q and from another application of Legendre's duplication formula that

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^{\frac{s}{2}+i-z+1} \binom{\frac{s}{2}+i}{z} \frac{1}{s+2i-2z-1} u^{s+2i-2z} Q^z.$$

A change of the order of summation with respect to z then yields the assertion.

On the other hand, if s is odd, the binomial theorem gives, for $\lfloor \frac{s}{2} \rfloor + i \neq z$,

$$\begin{aligned} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2}+i-q)(q-z)!} &= \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - z} (-1)^q \binom{\lfloor \frac{s}{2} \rfloor + i - z}{q} \\ &= \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} (1-1)^{\lfloor \frac{s}{2} \rfloor + i - z} \\ &= 0. \end{aligned} \tag{8.9}$$

For $\lfloor \frac{s}{2} \rfloor + i = z$, the sum on the left-hand side of (8.9) equals 1. Hence, we finally obtain

$$\int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2}+i)} u Q^{\lfloor \frac{s}{2} \rfloor + i},$$

if s is odd. □

8.2.2. The Proofs

Now we possess all the required tools to provide the proofs of the main results of this chapter and start with the proof of Theorem 8.1.

Proof of Theorem 8.1. Let $L \in \mathrm{G}(n,k)$ and $t \in L^\perp$. Then we have

$$\hat{\phi}_{k,L_t}^{r,s,0}(K \cap L_t, \beta \cap L_t) = \mathbb{1}\{s = 0\} \int_{K \cap \beta \cap L_t} x^r \,\mathcal{H}^k(\mathrm{d}x)$$

and thus, for $s \neq 0$,

$$\begin{aligned} &\int_{\mathrm{A}(n,k)} Q(E)^i \hat{\phi}_{k,E}^{r,s,0}(K \cap E, \beta \cap E) \,\mu_k(\mathrm{d}E) \\ &\quad = \int_{\mathrm{G}(n,k)} \int_{L^\perp} Q(L_t)^i \hat{\phi}_{k,L_t}^{r,s,0}(K \cap L_t, \beta \cap L_t) \,\mathcal{H}^{n-k}(\mathrm{d}t) \,\nu_k(\mathrm{d}L) \\ &\quad = 0. \end{aligned}$$

Furthermore, for $s = 0$ Fubini's theorem yields

$$\begin{aligned} &\int_{\mathrm{A}(n,k)} Q(E)^i \hat{\phi}_{k,E}^{r,0,0}(K \cap E, \beta \cap E) \,\mu_k(\mathrm{d}E) \\ &\quad = \int_{\mathrm{G}(n,k)} Q(L)^i \int_{L^\perp} \int_{K \cap \beta \cap L_t} x^r \,\mathcal{H}^k(\mathrm{d}x) \,\mathcal{H}^{n-k}(\mathrm{d}t) \,\nu_k(\mathrm{d}L) \\ &\quad = \int_{\mathrm{G}(n,k)} Q(L)^i \,\nu_k(\mathrm{d}L) \int_{K \cap \beta} x^r \,\mathcal{H}^n(\mathrm{d}x). \end{aligned}$$

Then we conclude the proof with Lemma A.4 and the definition of $\hat{\phi}_n^{r,0,0}$. □

We turn to the proof of Theorem 8.2. The integral formula in Lemma A.7 allows us to rewrite the integration with respect to the intrinsic support measure, appearing in the Crofton integral of Theorem 8.2 (in the definition of the tensorial curvature measures). After this integral transformation we apply the other integral formulae to calculate the remaining integrals.

Proof of Theorem 8.2. First, we prove the formula for a polytope $P \in \mathcal{P}^n$. The general result then follows by an approximation argument.

As a matter of convenience, we name the integral of interest I. Then Lemma A.7 yields

$$\begin{aligned} I &= \omega_{k-j} \int_{\mathrm{G}(n,k)} Q(L)^i \int_{L^\perp} \int_{L_t \times (L \cap \mathbb{S}^{n-1})} \mathbb{1}_\beta(x) x^r u^s \\ &\qquad \times \Lambda_j^{(L_t)}(P \cap L_t, \mathrm{d}(x,u))\, \mathcal{H}^{n-k}(\mathrm{d}t)\, \nu_k(\mathrm{d}L) \\ &= \sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(\mathrm{d}x) \int_{\mathrm{G}(n,k)} Q(L)^i \\ &\qquad \times \int_{N(P,F) \cap \mathbb{S}^{n-1}} \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2\, \mathcal{H}^{k-j-1}(\mathrm{d}u)\, \nu_k(\mathrm{d}L). \end{aligned}$$

With Fubini's theorem we conclude

$$\begin{aligned} I = &\sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} \\ &\times \int_{\mathrm{G}(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2\, \nu_k(\mathrm{d}L)\, \mathcal{H}^{k-j-1}(\mathrm{d}u). \end{aligned} \tag{8.10}$$

Then we obtain from Proposition 8.14

$$\begin{aligned} I = \gamma_{n,k,j} &\sum_{F \in \mathcal{F}_{n-k+j}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(\mathrm{d}x) \\ &\times \Bigg(\sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z}\, \mathcal{H}^{k-j-1}(\mathrm{d}u) \\ &\quad + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z Q(F) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^{s+2i-2z-2}\, \mathcal{H}^{k-j-1}(\mathrm{d}u) \Bigg). \end{aligned}$$

With the definition of the tensorial curvature measures we get

$$\begin{aligned} I = \gamma_{n,k,j} &\sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \hat{\phi}_{n-k+j}^{r,s+2i-2z,0}(P,\beta) \\ &+ \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \hat{\phi}_{n-k+j}^{r,s+2i-2z-2,1}(P,\beta). \end{aligned}$$

Combining the two sums yields the assertion in the polytopal case.

As pointed out before, there exists a weakly continuous extension of the generalized tensorial curvature measures $\hat{\phi}_{n-k+j}^{r,s+2i-2z-2,1}$ from the set of all polytopes to $\mathcal{K}^n$. The same is true for the tensorial curvature measures $\hat{\phi}_{n-k+j}^{r,s+2i-2z,0}$. Hence, approximating a convex body $K \in \mathcal{K}^n$ by polytopes yields the assertion in the general case. □

Now we prove Theorem 8.3, which deals with the case $k = 1$ excluded in the statement of Theorem 8.2.

Proof of Theorem 8.3. The proof basically works as the one of Theorem 8.2. Again, we prove the formula for a polytope $P \in \mathcal{P}^n$. We call the integral of interest I and proceed as in the previous proof in order to obtain (8.10). Now we apply Proposition 8.15 and obtain

$$I = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z} Q^{\frac{s}{2}+i-z} \times \sum_{F \in \mathcal{F}_{n-1}(P)} \int_{F \cap \beta} x^r \mathcal{H}^{n-k+j}(\mathrm{d}x) \int_{N(P,F)\cap \mathbb{S}^{n-1}} u^{2z}\, \mathcal{H}^0(\mathrm{d}u),$$

if s is even. Hence, we conclude the assertion with the definition of $\hat{\phi}_{n-1}^{r,2z,0}$.

If s is odd, Proposition 8.15 yields

$$I = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2}+i)} Q^{\frac{s-1}{2}+i} \hat{\phi}_{n-1}^{r,1,0}(P,\beta).$$

As sketched in the proof of Theorem 8.2, the general result follows by an approximation argument. □

For the proof of Theorem 8.4, we first globalize Theorem 8.2 and then apply Lemma 3.7 (which is a direct consequence of McMullen's Lemma 3.6) to treat the appearing valuations $\hat{\phi}_{n-k+j}^{0,s+2i-2z-2,1}(\cdot,\mathbb{R}^n)$. For the sake of convenience, we add here the special case of Lemma 3.7 which we need (and with the renormalized valuations that we use in this chapter).

Lemma 8.16. *Let $P \in \mathcal{P}^n$ and $j, s \in \mathbb{N}_0$ with $j \le n-1$. Then*

$$\tfrac{n-j+s}{s+1} \hat{\Phi}_j^{0,s+2}(P) = \sum_{F \in \mathcal{F}_j(P)} Q(F^\perp)\mathcal{H}^j(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^s\, \mathcal{H}^{n-j-1}(\mathrm{d}u).$$

With the help of Lemma 8.16, now we can prove Theorem 8.4.

Proof of Theorem 8.4. We only prove the formula for a polytope $P \in \mathcal{P}^n$. As before, the general result follows by an approximation argument.

We briefly write I for the Crofton integral under investigation. Starting from the special case of Theorem 8.2 where $r = 0$ and $\beta = \mathbb{R}^n$, we obtain

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \hat{\Phi}_{n-k+j}^{0,s+2i-2z}(P) + \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i-1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \times \sum_{F \in \mathcal{F}_{n-k+j}(P)} Q(F)\mathcal{H}^{n-k+j}(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z-2}\, \mathcal{H}^{k-j-1}(\mathrm{d}u).$$

With $Q(F) = Q - Q(N(P,F))$ and Lemma 8.16 we get

$$\sum_{F \in \mathcal{F}_{n-k+j}(P)} Q(F)\mathcal{H}^{n-k+j}(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z-2}\, \mathcal{H}^{k-j-1}(\mathrm{d}u) = Q\hat{\Phi}_{n-k+j}^{0,s+2i-2z-2}(P) - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \hat{\Phi}_{n-k+j}^{0,s+2i-2z}(P)$$

and thus

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda^{(0)}_{n,k,j,s,i,z} Q^z \hat{\Phi}^{0,s+2i-2z}_{n-k+j}(P)$$
$$+ \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i - 1} \lambda^{(1)}_{n,k,j,s,i,z} Q^z \big(Q \hat{\Phi}^{0,s+2i-2z-2}_{n-k+j}(P) - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \hat{\Phi}^{0,s+2i-2z}_{n-k+j}(P)\big).$$

Combining these sums yields

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} (\lambda^{(0)}_{n,k,j,s,i,z} + \lambda^{(1)}_{n,k,j,s,i,z-1} - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \lambda^{(1)}_{n,k,j,s,i,z}) Q^z \hat{\Phi}^{0,s+2i-2z}_{n-k+j}(P).$$

In fact, we have $\lambda^{(1)}_{n,k,j,s,i,-1} = 0$ and, furthermore for even s, as the sum with respect to q is empty, $\lambda^{(1)}_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i}$ also vanishes. On the other hand, for odd s, as $\hat{\Phi}^{0,1}_{n-k+j} \equiv 0$, the last summand of the sum with respect to z actually vanishes and thus its coefficient does not have to be determined and is defined as zero.

Hence, we obtained a representation of the integral with the desired Minkowski tensors. It remains to determine the coefficients explicitly. First, we consider the case where ($k > 1$ and) $z \in \{1, \dots, \lfloor \frac{s}{2} \rfloor + i - 1\}$. We get

$$\begin{aligned} \lambda^{(0)}_{n,k,j,s,i,z} + \lambda^{(1)}_{n,k,j,s,i,z-1} &= \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1) \Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1) \Gamma(\frac{n+1}{2} + p + q)} \\ &\quad \times \Big((n-k+j)(\tfrac{k-1}{2} + p) - \tfrac{z}{p+q}\big(p(n-k) - q(k-1)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} \lambda^{(1)}_{n,k,j,s,i,z} &= \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\ &\quad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1) \Gamma(\frac{k-1}{2} + p) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1) \Gamma(\frac{n+1}{2} + p + q)} \\ &\quad \times \tfrac{p+q-z}{p+q}\big(p(n-k) - q(k-1)\big). \end{aligned} \tag{8.11}$$

Hence we conclude

$$\begin{aligned}
&\lambda^{(0)}_{n,k,j,s,i,z} + \lambda^{(1)}_{n,k,j,s,i,z-1} - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1}\lambda^{(1)}_{n,k,j,s,i,z} \\
&\quad = \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q+\tfrac{1}{2}) \\
&\qquad \times \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)\Gamma(\frac{n+1}{2}+p+q)} \\
&\qquad \times \Big((n-k+j)(\tfrac{k-1}{2}+p) - \tfrac{p(n-k)-q(k-1)}{p+q}\Big(p+q+\tfrac{(k-j-1)(p+q-z)}{s+2i-2z-1}\Big)\Big).
\end{aligned}$$

The case $z = \lfloor \frac{s}{2} \rfloor + i$, for even s, follows similarly. For $z = 0$, we have $\lambda^{(1)}_{n,k,j,s,i,-1} = 0$ and (8.11) still holds, if one cancels the remaining $\frac{p+q-z}{p+q} = 1$. □

Finally, we provide the argument for Corollary 8.9, which is the special case of Theorem 8.2 obtained for $i = 0$ and $j + 1 = k \geq 2$. The proof makes use of relation (8.4) to rewrite the generalized tensorial curvature measures $\hat{\phi}^{r,s-2z-2,1}_{n-1}$.

Proof of Corollary 8.9. With the specific choices of the indices, we obtain

$$\begin{aligned}
\lambda^{(\varepsilon)}_{n,k,k-1,s,0,z} = &\sum_{q=z+\varepsilon}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q-\varepsilon}{z} \Gamma(q+\tfrac{1}{2}) \\
&\qquad \times \frac{\Gamma(\frac{k+s+1}{2}-q)}{\Gamma(\frac{n+s+1}{2})} \frac{\Gamma(\frac{k-1}{2})\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+q)} \vartheta^{(\varepsilon)}_{n,k,k-1,0,q},
\end{aligned}$$

with

$$\vartheta^{(0)}_{n,k,k-1,0,q} = \tfrac{1}{2}(n-1)(k-1), \qquad \vartheta^{(1)}_{n,k,k-1,0,q} := -q(k-1),$$

and

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}.$$

Let us denote the Crofton integral by I. Then, (8.4) applied to the generalized tensorial curvature measures $\hat{\phi}^{r,s-2z-2,1}_{n-1}$ in Theorem 8.2 yields that

$$\begin{aligned}
I &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z (\lambda^{(0)}_{n,k,k-1,s,0,z} - \lambda^{(1)}_{n,k,k-1,s,0,z}) \hat{\phi}^{r,s-2z,0}_{n-1}(K,\beta) \\
&\quad + \gamma_{n,k,k-1} \sum_{z=1}^{\lfloor \frac{s}{2} \rfloor + 1} Q^z \lambda^{(1)}_{n,k,k-1,s,0,z-1} \hat{\phi}^{r,s-2z,0}_{n-1}(K,\beta) \\
&= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z \underbrace{(\lambda^{(0)}_{n,k,k-1,s,0,z} + \lambda^{(1)}_{n,k,k-1,s,0,z-1} - \lambda^{(1)}_{n,k,k-1,s,0,z})}_{=:\lambda} \hat{\phi}^{r,s-2z,0}_{n-1}(K,\beta),
\end{aligned}$$

where

$$\lambda = \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-q)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+q)}$$
$$\times \left[\binom{q}{z} \frac{1}{2}(n-1)(k-1) - \binom{q-1}{z-1}(-1)q(k-1) - \binom{q-1}{z}(-1)q(k-1) \right]$$
$$= \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-q)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+q)} \binom{q}{z} (k-1) \left(\tfrac{n-1}{2}+q\right)$$
$$= 2\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-q)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+q)},$$

from which the assertion follows. □

8.3. The Proofs for the Extrinsic Results

Our starting point is the relation, due to McMullen, between the intrinsic and the extrinsic Minkowski tensors (see [68, Theorem 5.1]). The localization of this result stated in Lemma 3.5 (which has to be renormalized to fit the setting of this chapter) can be combined with the binomial theorem applied to the relation $Q = Q(E) + Q(E^\perp)$, where $E \subset \mathbb{R}^n$ is any k-flat, which yields the following lemma.

Lemma 8.17. *Let $j, k, r, s \in \mathbb{N}_0$ with $j < k < n$, let $K \in \mathcal{K}^n$ with $K \subset E \in \mathrm{A}(n,k)$ and $\beta \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\hat{\phi}_j^{r,s,0}(K,\beta) = \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-j+s}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{k-j+s}{2}-m)}{4^m\, m!(s-2m)!} Q^l Q(E)^{m-l} \hat{\phi}_{j,E}^{r,s-2m,0}(K,\beta).$$

We start with the proof of Theorem 8.11, for which we use Theorem 8.2 after an application of Lemma 8.17.

Proof of Theorem 8.11. Lemma 8.17 for $j = k-1$ gives

$$\int_{\mathrm{A}(n,k)} \hat{\phi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} Q^l$$
$$\times \int_{\mathrm{A}(n,k)} Q(E)^{m-l} \hat{\phi}_{k-1,E}^{r,s-2m,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E).$$

For $j = k-1$ we can argue as in the proof of Corollary 8.9 to see that Theorem 8.2 implies that

$$\int_{\mathrm{A}(n,k)} Q(E)^i \hat{\phi}^{r,s,0}_{k-1,E}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,k-1,s,i,z}\, Q^z \hat{\phi}^{r,s+2i-2z,0}_{n-1}(K \cap E, \beta \cap E), \tag{8.12}$$

where

$$\lambda_{n,k,k-1,s,i,z} = (k-1) \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z}$$
$$\times \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} + i - p - q)}{\Gamma(\frac{n+s+1}{2} + i - p)} \frac{\Gamma(\frac{k-1}{2} + p)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + p + q)}.$$

(Of course, for $i = 0$ we recover Corollary 8.9.) Hence, we obtain

$$\int_{\mathrm{A}(n,k)} \hat{\phi}^{r,s,0}_{k-1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - l} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l}$$
$$\times \lambda_{n,k,k-1,s-2m,m-l,z}\, Q^{l+z} \hat{\phi}^{r,s-2l-2z,0}_{n-1}(K, \beta).$$

An index shift of the summation with respect to z yields

$$\int_{\mathrm{A}(n,k)} \hat{\phi}^{r,s,0}_{k-1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} \sum_{z=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l}$$
$$\times \lambda_{n,k,k-1,s-2m,m-l,z-l}\, Q^{z} \hat{\phi}^{r,s-2z,0}_{n-1}(K, \beta).$$

Changing the order of summation gives

$$\int_{\mathrm{A}(n,k)} \hat{\phi}^{r,s,0}_{k-1}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{z} \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l}$$
$$\times \lambda_{n,k,k-1,s-2m,m-l,z-l}\, Q^{z} \hat{\phi}^{r,s-2z,0}_{n-1}(K, \beta). \tag{8.13}$$

The coefficients of the tensorial curvature measures on the right-hand side of (8.13) do not depend on the choice of $r \in \mathbb{N}_0$ or $\beta \in \mathcal{B}(\mathbb{R}^n)$. Thus, we can set

$$\int_{\mathrm{A}(n,k)} \hat{\phi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z}\, Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K, \beta),$$

where the coefficient $\kappa_{n,k,s,z}$ is uniquely defined in the obvious way. By choosing $r = 0$ and $\beta = \mathbb{R}^n$, we can compare this to the Crofton formula for translation invariant Minkowski tensors in [15]. In fact, since the measures $Q^z \hat{\phi}_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n)$, $z \in \{0, \dots, \lfloor s/2 \rfloor\} \setminus \{(s-1)/2\}$, are linearly independent, we can conclude from the Crofton formula for the translation invariant Minkowski tensors in [15, Theorem 3] that

$$\kappa_{n,k,s,z} = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2}+z)\Gamma(\frac{k+s-1}{2}-z)}{\Gamma(\frac{s}{2}-z+1)z!}$$

for $z \neq (s-1)/2$. If $z = (s-1)/2$, then $\hat{\phi}_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n) = \hat{\Phi}_{n-1}^{0,1}(K) = 0$, and hence we do not get any information about the corresponding coefficient from the global theorem. Consequently, we have to calculate $\kappa_{n,k,s,(s-1)/2}$ directly, which is what we do later in the proof.

But first we demonstrate that the coefficients of the tensorial curvature measures in (8.13) can be determined also by a direct calculation if s is even. In fact, we obtain

$$\begin{aligned}
S &:= \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} \lambda_{n,k,k-1,s-2m,m-l,z-l} \\
&= (k-1) \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} \sum_{p=l}^{m} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^{m+l+p+q-z} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \\
&\quad \times \binom{m}{l} \binom{m-l}{p-l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}$$

Changing the order of summation gives

$$\begin{aligned}
S = (k-1) &\sum_{p=l}^{\lfloor \frac{s}{2} \rfloor} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^{l+q-z} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \\
&\times \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)} \\
&\times \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \binom{m}{l} \binom{m-l}{p-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!}.
\end{aligned}$$

We denote the sum with respect to m by T and conclude

$$
\begin{aligned}
T &= \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \binom{m}{l} \binom{m-l}{p-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \\
&= \frac{1}{l!(p-l)!} \sum_{m=p}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m+p} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,(m-p)!(s-2m)!}.
\end{aligned}
$$

An index shift yields

$$
T = \frac{1}{2^s l!(p-l)!} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^m \frac{2^{s-2p-2m}\Gamma(\frac{s+1}{2}-p-m)}{m!(s-2p-2m)!}.
$$

Legendre's duplication formula gives

$$
T = \frac{\sqrt{\pi}}{2^s l!(p-l)!} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-p-m+1)}.
$$

If s is even, the binomial theorem yields

$$
\begin{aligned}
T &= \frac{\sqrt{\pi}}{2^s l!(p-l)!(\frac{s}{2}-p)!} \sum_{m=0}^{\frac{s}{2}-p} (-1)^m \binom{\frac{s}{2}-p}{m} \\
&= \frac{\sqrt{\pi}}{2^s l!(p-l)!(\frac{s}{2}-p)!} (1-1)^{\frac{s}{2}-p} \\
&= \mathbb{1}\{p = \frac{s}{2}\} \frac{\sqrt{\pi}}{2^s l!(\frac{s}{2}-l)!}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
S &= \frac{(k-1)\sqrt{\pi}}{2^s l!(\frac{s}{2}-l)!} \sum_{q=(z-\frac{s}{2})^+}^{0} (-1)^{l+q-z} \binom{\frac{s}{2}+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \\
&\quad \times \frac{\Gamma(\frac{k+1}{2}-q)}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2}-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+s-1}{2}+q-l)} \\
&= (-1)^{l-z} \frac{(k-1)\sqrt{\pi}\Gamma(\frac{1}{2})}{2^s l!(\frac{s}{2}-l)!} \binom{\frac{s}{2}-l}{z-l} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2}-l)\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+s-1}{2}-l)} \\
&= (-1)^{l-z} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+1}{2})} \frac{(k-1)\pi}{2^s l!(\frac{s}{2}-l)!} \binom{\frac{s}{2}-l}{z-l} \frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}.
\end{aligned}
$$

Furthermore, Legendre's duplication formula yields

$$
s!S = (-1)^{l-z} \frac{(k-1)\sqrt{\pi}\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})} \underbrace{\binom{\frac{s}{2}}{l}\binom{\frac{s}{2}-l}{z-l}}_{=\binom{\frac{s}{2}}{z}\binom{z}{l}} \frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}.
$$

Thus, we obtain

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\hat{\phi}_{k-1}^{r,s,0}(K\cap E,\beta\cap E)\,\mu_k(\mathrm{d}E)\\ &= \gamma_{n,k,k-1}\frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})}\frac{(k-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})}\sum_{z=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{z}\\ &\quad\times\sum_{l=0}^{z}(-1)^{l-z}\binom{z}{l}\frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}\,Q^z\hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta).\end{aligned}$$

From relation (B.1′) we conclude

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\hat{\phi}_{k-1}^{r,s,0}(K\cap E,\beta\cap E)\,\mu_k(\mathrm{d}E)\\ &= \gamma_{n,k,k-1}\frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})}\frac{(k-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+s-1}{2})}\\ &\quad\times\sum_{z=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{z}\Gamma(\tfrac{k+s-1}{2}-z)\Gamma(\tfrac{n-k}{2}+z)Q^z\hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta).\end{aligned}$$

With

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1}\frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!}\frac{\Gamma(\frac{n-k+1}{2})}{2\pi}$$

we get

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\hat{\phi}_{k-1}^{r,s,0}(K\cap E,\beta\cap E)\,\mu_k(\mathrm{d}E)\\ &= \frac{(n-2)!}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!}\frac{\Gamma(\frac{k+1}{2})}{(k-2)!}\frac{\pi^{\frac{n-k-1}{2}}\Gamma(\frac{s+1}{2})}{2\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})}\\ &\quad\times\sum_{z=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{z}\Gamma(\tfrac{k+s-1}{2}-z)\Gamma(\tfrac{n-k}{2}+z)Q^z\hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta).\end{aligned}$$

Legendre's formula applied three times gives

$$\begin{aligned}\int_{\mathrm{A}(n,k)} &\hat{\phi}_{k-1}^{r,s,0}(K\cap E,\beta\cap E)\,\mu_k(\mathrm{d}E)\\ &= \frac{k-1}{n-1}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}\frac{\pi^{\frac{n-k}{2}}\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})}\\ &\quad\times\sum_{z=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{z}\Gamma(\tfrac{k+s-1}{2}-z)\Gamma(\tfrac{n-k}{2}+z)Q^z\hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta),\end{aligned}$$

which confirms the coefficients for even s.

On the other hand, if s is odd, then Lemma B.3 yields

$$\begin{aligned}
T &= \frac{\sqrt{\pi}}{2^s l!(p-l)!} \sum_{m=0}^{\frac{s-1}{2}-p} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-p-m+1)} \\
&= \frac{\sqrt{\pi}}{2^s l!(p-l)!} \left(\sum_{m=0}^{\frac{s+1}{2}-p} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-p-m+1)} - (-1)^{\frac{s+1}{2}-p} \frac{1}{(\frac{s+1}{2}-p)!\Gamma(\frac{1}{2})} \right) \\
&= \frac{\sqrt{\pi}}{2^s l!(p-l)!} \left((-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi}(-s+2p)(\frac{s+1}{2}-p)!} - (-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi}(\frac{s+1}{2}-p)!} \right),
\end{aligned}$$

which can be further simplified as

$$\begin{aligned}
T &= (-1)^{\frac{s-1}{2}-p} \frac{\sqrt{\pi}}{2^s l!(p-l)!} \frac{1}{\sqrt{\pi}(\frac{s+1}{2}-p)!} (\tfrac{1}{s-2p}+1) \\
&= (-1)^{\frac{s-1}{2}-p} \frac{1}{2^{s-1}(s-2p)(\frac{s-1}{2}-p)!l!(p-l)!} \\
&= (-1)^{\frac{s-1}{2}-p} \frac{2\Gamma(\frac{s}{2}+1)}{\sqrt{\pi}(s-2p)s!} \binom{\frac{s-1}{2}}{p} \binom{p}{l}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
s! \sum_{l=0}^{z} S = \frac{2(k-1)\Gamma(\frac{s}{2}+1)}{\sqrt{\pi}} &\sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \\
&\times \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \\
&\times \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\int_{A(n,k)} \hat{\phi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = 2(k-1)\gamma_{n,k,k-1} \frac{\pi^{\frac{n-k-1}{2}}\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\frac{s-1}{2}} Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta) \\
&\qquad \times \sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \\
&\qquad \times \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}$$

With

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}$$

we get

$$\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\phi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{(n-2)!}{(n-k-1)!(k-2)!} \frac{\pi^{\frac{n-k-3}{2}} \Gamma(\frac{n-k+1}{2}) \Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\frac{s-1}{2}} Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta) \\
&\qquad \times \sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \\
&\qquad \times \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}$$

We denote the threefold sum with respect to l, p and q by R. Hence, R multiplied with the factor in front of the sum with respect to z equals $\kappa_{n,k,s,z}$. A direct calculation for R still remains an open task. However, for the proof this is not required.

Finally, if s is odd we calculate the only so far unknown coefficient $\kappa_{n,k,s,(s-1)/2}$. For $z = (s-1)/2$ we see that the sum over q only contains one summand, namely $q = (s-1)/2 - p$. Hence, we obtain

$$\begin{aligned}
R &= \Gamma(\tfrac{k}{2}+1) \sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+l} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \Gamma(\tfrac{s}{2}-p) \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k+s-1}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)\Gamma(\frac{n+s}{2}-l-1)} \\
&= \Gamma(\tfrac{k}{2}+1) \sum_{p=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} \binom{\frac{s-1}{2}}{p} \Gamma(\tfrac{s}{2}-p) \frac{\Gamma(\frac{n-k+s-1}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \frac{\Gamma(\frac{k-1}{2}+p-l)}{\Gamma(\frac{n+s}{2}-l-1)}.
\end{aligned}$$

Then relation (B.1′) yields

$$\begin{aligned}
R &= \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+p} \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{s}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)} \\
&= \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^{p} \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{1}{2}+p)}{\Gamma(\frac{n}{2}+1+p)}.
\end{aligned}$$

Again, we apply relation (B.1′) and obtain

$$R = \sqrt{\pi} \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+s}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)\Gamma(\frac{n+s+1}{2})}.$$

Thus, we conclude

$$\begin{aligned}\kappa_{n,k,s,\frac{s-1}{2}} &= \frac{(n-2)!}{(n-k-1)!(k-2)!}\frac{\pi^{\frac{n-k-3}{2}}\Gamma(\frac{n-k+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})}R \\ &= \pi^{\frac{n-k-2}{2}}\frac{(n-2)!}{\Gamma(\frac{n+1}{2})}\frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})}{(k-2)!}\frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!}\frac{(n+s-2)\Gamma(\frac{s}{2}+1)}{(n-k+s-1)\Gamma(\frac{n+s+1}{2})}.\end{aligned}$$

Applying three times Legendre's formula gives

$$\kappa_{n,k,s,\frac{s-1}{2}} = \pi^{\frac{n-k-1}{2}}\frac{2k(n+s-2)}{(n-1)(n-k+s-1)}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})}\frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n+s+1}{2})},$$

which completes the argument. □

Next we prove Theorem 8.12. As in the previous proof, one can compare the Crofton integral to the global one obtained in [15, Theorem 3]. However, we deduce it directly from Theorem 8.3.

Proof of Theorem 8.12. Lemma 8.17 yields

$$\begin{aligned}\int_{A(n,1)} \hat\phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E) &= \frac{\pi^{\frac{n-1}{2}}s!}{\Gamma(\frac{n+s}{2})}\sum_{m=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,m!(s-2m)!}Q^l \\ &\quad\times\int_{A(n,1)} Q(E)^{m-l}\hat\phi_{0,E}^{r,s-2m,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E).\end{aligned}$$

If $s\in\mathbb{N}_0$ is even, we conclude from Theorem 8.3

$$\begin{aligned}&\int_{A(n,1)} \hat\phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E) \\ &= \frac{\pi^{\frac{n-1}{2}}s!}{\Gamma(\frac{n+s}{2})}\sum_{m=0}^{\frac{s}{2}}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,m!(s-2m)!} \\ &\quad\times\frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}-l)}{\pi\Gamma(\frac{n+s+1}{2}-l)}\sum_{z=0}^{\frac{s}{2}-l}(-1)^z\binom{\frac{s}{2}-l}{z}\frac{1}{1-2z}\,Q^{\frac{s}{2}-z}\hat\phi_{n-1}^{r,2z,0}(K,\beta).\end{aligned}$$

A change of the order of summation yields

$$\begin{aligned}&\int_{A(n,1)} \hat\phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E) \\ &= \frac{\pi^{\frac{n-1}{2}}s!}{\Gamma(\frac{n+s}{2})}\sum_{l=0}^{\frac{s}{2}}\sum_{m=l}^{\frac{s}{2}}(-1)^{m-l}\binom{m}{l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,m!(s-2m)!} \\ &\quad\times\frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}-l)}{\pi\Gamma(\frac{n+s+1}{2}-l)}\sum_{z=0}^{\frac{s}{2}-l}(-1)^z\binom{\frac{s}{2}-l}{z}\frac{1}{1-2z}\,Q^{\frac{s}{2}-z}\hat\phi_{n-1}^{r,2z,0}(K,\beta).\end{aligned}$$

Legendre's duplication formula gives for the sum with respect to m, which we denote by S,

$$\begin{aligned} S &= \frac{\sqrt{\pi}}{2^s} \sum_{m=l}^{\frac{s}{2}} (-1)^{m-l} \binom{m}{l} \frac{1}{m!\Gamma(\frac{s}{2}-m+1)} \\ &= \frac{\sqrt{\pi}}{2^s l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-l-m+1)}. \end{aligned}$$

As seen before, we conclude from the binomial theorem

$$\begin{aligned} S &= \frac{\sqrt{\pi}}{2^s(\frac{s}{2}-l)!l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \binom{\frac{s}{2}-l}{m} \\ &= \mathbb{1}\{l = \frac{s}{2}\} \frac{\Gamma(\frac{s+1}{2})}{s!}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\ &\qquad = \frac{\pi^{\frac{n-3}{2}} \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s}{2})} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} Q^{\frac{s}{2}} \hat{\phi}_{n-1}^{r,0,0}(K,\beta) \\ &\qquad = \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s}{2})\Gamma(\frac{n+1}{2})} Q^{\frac{s}{2}} \hat{\phi}_{n-1}^{r,0,0}(K,\beta). \end{aligned}$$

On the other hand, if $s \in \mathbb{N}$ is odd, we conclude from Theorem 8.3

$$\begin{aligned} &\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\ &= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\frac{s-1}{2}} \sum_{l=0}^{m} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} \frac{\Gamma(\frac{s}{2}-l+1)}{\Gamma(\frac{n+s+1}{2}-l)} Q^{\frac{s-1}{2}} \hat{\phi}_{n-1}^{r,1,0}(K,\beta). \end{aligned}$$

A change of the order of summation yields

$$\begin{aligned} &\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\ &= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} \sum_{m=l}^{\frac{s-1}{2}} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} \frac{\Gamma(\frac{s}{2}-l+1)}{\Gamma(\frac{n+s+1}{2}-l)} Q^{\frac{s-1}{2}} \hat{\phi}_{n-1}^{r,1,0}(K,\beta). \end{aligned}$$

Legendre's duplication formula gives for the sum with respect to m, which we denote by S,

$$S = \frac{\sqrt{\pi}}{2^s l!} \sum_{m=0}^{\frac{s-1}{2}-l} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-l-m+1)}.$$

Then Lemma B.3 yields

$$
\begin{aligned}
S &= \frac{\sqrt{\pi}}{2^s l!}\Bigg(\sum_{m=0}^{\frac{s+1}{2}-l} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2}-l-m+1)} - (-1)^{\frac{s+1}{2}-l} \frac{1}{(\frac{s+1}{2}-l)!\Gamma(\frac{1}{2})}\Bigg) \\
&= \frac{\sqrt{\pi}}{2^s l!}\Bigg((-1)^{\frac{s-1}{2}-l} \frac{1}{\sqrt{\pi}(s-2l)(\frac{s+1}{2}-l)!} - (-1)^{\frac{s+1}{2}-l} \frac{1}{\sqrt{\pi}(\frac{s+1}{2}-l)!}\Bigg) \\
&= (-1)^{\frac{s-1}{2}-l} \frac{1}{2^{s-1} l!(s-2l)(\frac{s-1}{2}-l)!}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\
&\quad = \frac{\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2})s!}{2^s\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}-l} \frac{1}{l!(\frac{s-1}{2}-l)!} \frac{\Gamma(\frac{s}{2}-l)}{\Gamma(\frac{n+s+1}{2}-l)}\, Q^{\frac{s-1}{2}} \hat{\phi}_{n-1}^{r,1,0}(K,\beta) \\
&\quad = \frac{\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2})s!}{2^s\Gamma(\frac{s+1}{2})\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^{l} \binom{\frac{s-1}{2}}{l} \frac{\Gamma(l+\frac{1}{2})}{\Gamma(\frac{n+2}{2}+l)}\, Q^{\frac{s-1}{2}} \hat{\phi}_{n-1}^{r,1,0}(K,\beta).
\end{aligned}
$$

Then relation (B.1′) gives

$$
\begin{aligned}
&\int_{\mathrm{A}(n,1)} \hat{\phi}_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\
&\quad = \frac{s!}{2^s\Gamma(\frac{s+1}{2})} \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{n+1}{2})}\, Q^{\frac{s-1}{2}} \hat{\phi}_{n-1}^{r,1,0}(K,\beta).
\end{aligned}
$$

Now the assertion follows from Legendre's duplication formula. □

Finally, we show that the Crofton formula has a very simple form in the ψ-representation of tensorial curvature measures, which is stated in Corollary 8.13.

Proof of Corollary 8.13. The cases $s \in \{0,1\}$ are checked directly, hence we can assume $s \geq 2$ in the following. Using (8.6) we get

$$
\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j \\
&\qquad\qquad \times \int_{\mathrm{A}(n,k)} \hat{\phi}_{k-1}^{r,s-2j,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E). \qquad (8.14)
\end{aligned}
$$

Then, for $k \neq 1$, Theorem 8.11 yields

$$\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} \\
&\qquad \times \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - j} \kappa_{n,k,s-2j,z}\, Q^{z+j} \hat{\phi}_{n-1}^{r,s-2j-2z,0}(K,\beta) \\
&\quad = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{z=j}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} \\
&\qquad \times \kappa_{n,k,s-2j,z-j}\, Q^{z} \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta),
\end{aligned}$$

where

$$\begin{aligned}
\kappa_{n,k,s-2j,z-j} = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2}-j)\Gamma(\frac{s}{2}-j+1)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)} \\
\times \frac{\Gamma(\frac{n-k}{2}+z-j)\Gamma(\frac{k+s-1}{2}-z)}{\Gamma(\frac{s}{2}-z+1)(z-j)!},
\end{aligned}$$

if $z \neq (s-1)/2$. On the other hand, if $z = (s-1)/2$, then the coefficient needs to be multiplied by the factor $\frac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)}$ (see the comment after Theorem 8.11).

Applying Legendre's duplication formula twice, we thus obtain

$$\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{s!}{2^s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma(\frac{k+s-1}{2}-z)}{z!\Gamma(\frac{n}{2}+s-1)\Gamma(\frac{s}{2}-z+1)} Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta) \\
&\qquad \times \sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\frac{n-k}{2}+z-j)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)} \\
&\qquad \times \left(1 - \mathbb{1}\{z = \tfrac{s-1}{2}\} \left(1 - \tfrac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)}\right)\right),
\end{aligned}$$

Denoting the sum with respect to j by S_z, an application of Lemma B.4 shows that

$$\begin{aligned}
S_z &= \sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\frac{n-k}{2}+z-j)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)} \\
&= (-1)^z \frac{\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})\Gamma(\frac{k+s-1}{2})\Gamma(\frac{n}{2}+s-z-1)}{\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})\Gamma(\frac{s+1}{2}-z)\Gamma(\frac{k+s-1}{2}-z)},
\end{aligned} \tag{8.15}$$

for $z \neq (s-1)/2$ and $k > 1$. On the other hand, for $z = (s-1)/2 =: t$, we obtain from

Lemma B.4 and Lemma B.5 (since $s > 1$ and thus $t > 0$)

$$\begin{aligned}
S_t &= \tfrac{k}{k-1} \sum_{j=0}^{t} (-1)^j \binom{t}{j} \left(1 - \tfrac{1}{n+2t-2j}\right) \frac{\Gamma(\frac{n}{2}+2t-j)\Gamma(\frac{n-k}{2}+t-j)}{\Gamma(\frac{n-k}{2}+t-j+1)\Gamma(\frac{n}{2}+t-j)} \\
&= \tfrac{k}{k-1} \Bigg(\sum_{j=0}^{t} (-1)^j \binom{t}{j} \frac{\Gamma(\frac{n}{2}+2t-j)\Gamma(\frac{n-k}{2}+t-j)}{\Gamma(\frac{n-k}{2}+t-j+1)\Gamma(\frac{n}{2}+t-j)} \\
&\qquad - \sum_{j=0}^{t} (-1)^j \binom{t}{j} \tfrac{1}{\frac{n-k}{2}+t-j} \frac{\Gamma(\frac{n}{2}+2t-j)}{\Gamma(\frac{n}{2}+t-j+1)} \Bigg) \\
&= (-1)^t \frac{\Gamma(\frac{n-k}{2})\Gamma(t+1)\Gamma(\frac{k}{2}+t)}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2}+t+1)},
\end{aligned}$$

which coincides with (8.15) for $z = (s-1)/2$.

Thus, we have

$$\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}} \Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \frac{s!\Gamma(\frac{s+1}{2})}{2^s} \\
&\qquad \times \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \frac{\Gamma(\frac{n}{2}+s-z-1)}{z!\Gamma(\frac{n}{2}+s-1)\Gamma(\frac{s}{2}-z+1)\Gamma(\frac{s+1}{2}-z)} Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta).
\end{aligned}$$

Applying Legendre's duplication formula twice, we get

$$\begin{aligned}
&\int_{\mathrm{A}(n,k)} \hat{\psi}_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \\
&\qquad \times \frac{1}{\sqrt{\pi}} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \binom{s}{2z} \frac{\Gamma(z+\frac{1}{2})\Gamma(\frac{n}{2}+s-z-1)}{\Gamma(\frac{n}{2}+s-1)} Q^z \hat{\phi}_{n-1}^{r,s-2z,0}(K,\beta).
\end{aligned}$$

With (8.6) we obtain the assertion for $k \neq 1$.

On the other hand, if $k = 1$, then Theorem 8.12 yields for (8.14) that

$$\begin{aligned}
&\int_{\mathrm{A}(n,1)} \hat{\psi}_{0}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{\pi^{\frac{n-3}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+s-1)} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \Gamma(j+\tfrac{1}{2}) \\
&\qquad \times \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\lfloor \frac{s+1}{2} \rfloor - j + \frac{1}{2})}{\Gamma(\frac{n}{2}+\lfloor \frac{s+1}{2} \rfloor - j)} Q^{\lfloor \frac{s}{2} \rfloor} \hat{\phi}_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K,\beta).
\end{aligned}$$

Denoting the sum with respect to j by S and applying Legendre's duplication formula three times, we conclude that

$$S = \sqrt{\pi}\Gamma(\lfloor \tfrac{s+1}{2} \rfloor + \tfrac{1}{2}) \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{\lfloor \frac{s}{2} \rfloor}{j} \frac{\Gamma(\frac{n}{2} + s - j - 1)}{\Gamma(\frac{n}{2} + \lfloor \frac{s+1}{2} \rfloor - j)}.$$

Since $s \geq 2$, relation (B.1′) yields $S = 0$ due to (2.2), and hence the assertion. □

APPENDIX A

Basic Integral Formulae

In this chapter, we recall various integral formulae, which are known for some time and have been applied several times in different works on integral geometry.

A basic tool in this work is the following integral geometric transformation formula, which is a special case of [85, Theorem 7.2.6].

Lemma A.1. *Let* $0 \leq j \leq k \leq n$ *be integers,* $F \in \mathrm{G}(n, k)$*, and let* $f : \mathrm{G}(n, n-k+j) \to \mathbb{R}$ *be integrable. Then*

$$\int_{\mathrm{G}(n,n-k+j)} f(L)\, \nu_{n-k+j}(\mathrm{d}L) = d_{n,j,k} \int_{\mathrm{G}(F,j)} \int_{\mathrm{G}(U,n-k+j)} [F, L]^j f(L)\, \nu^U_{n-k+j}(\mathrm{d}L)\, \nu^F_j(\mathrm{d}U)$$

with

$$d_{n,j,k} := \prod_{i=1}^{k-j} \frac{\Gamma(\frac{i}{2})\Gamma(\frac{n-k+j+i}{2})}{\Gamma(\frac{j+i}{2})\Gamma(\frac{n-k+i}{2})}.$$

The preceding lemma yields the next result, which is again an integral geometric transformation formula (which is required in Chapter 4 and Chapter 8). It can be used to carry out an integration over linear Grassmann spaces recursively. Here we (implicitly) require that $n \geq 2$.

Lemma A.2 ([51, Corollary 4.2])**.** *Let* $u \in \mathbb{S}^{n-1}$ *and let* $h : \mathrm{G}(n, k) \to \mathbb{T}$ *be an integrable function, where and* $0 < k < n$*. Then*

$$\int_{\mathrm{G}(n,k)} h(L)\, \nu_k(\mathrm{d}L) = \frac{\omega_k}{2\omega_n} \int_{\mathrm{G}(u^\perp,k-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}} \times h\big(\mathrm{lin}\{U, tu + \sqrt{1-t^2}w\}\big)\, \mathcal{H}^{n-k-1}(\mathrm{d}w)\, \mathrm{d}t\, \nu^{u^\perp}_{k-1}(\mathrm{d}U).$$

The following lemmas can be derived from Lemma A.2 (for proofs see [84, (24)], [51, Lemma 4.3, Proposition 4.5, and Corollary 4.6]).

Lemma A.3 ([84, (24)]). *Let $s \in \mathbb{N}_0$ and $n \geq 1$. Then*

$$\int_{\mathbb{S}^{n-1}} u^s \, \mathcal{H}^{n-1}(\mathrm{d}u) = \mathbb{1}\{s \text{ even}\} \, 2 \frac{\omega_{n+s}}{\omega_{s+1}} \, Q^{\frac{s}{2}}.$$

The next lemma is used in the proofs of Lemma A.5 below and Lemma 4.5 in Chapter 4.

Lemma A.4 ([51, Lemma 4.3]). *Let $i, k \in \mathbb{N}_0$ with $k \leq n$ and $n \geq 1$. Then*

$$\int_{\mathrm{G}(n,k)} Q(L)^i \, \nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i)\Gamma(\frac{k}{2})} \, Q^i.$$

In Lemma A.4, we interpret the coefficient of the tensor on the right-hand side of the equation as 0 if $k = 0$ and $i \neq 0$, and as 1 if $k = i = 0$. This follows from relation (2.2) and fits the value of the integral on the right.

The following lemma extends Lemma A.4 (but the latter is used in the proof of Lemma A.5). It will be needed in the proof of Proposition 4.9 in Chapter 4 (of which Lemma A.5 in the case $a = 2$ is a special case).

Lemma A.5 ([51, Proposition 4.5]). *Let $a, i \in \mathbb{N}_0$, $k, r \in \{0, \dots, n\}$ with $k + r \geq n \geq 1$, and let $F \in \mathrm{G}(n, r)$. Then*

$$\begin{aligned}\int_{\mathrm{G}(n,k)} [F, L]^a Q(L)^i \, \nu_k(\mathrm{d}L) = e_{n,k,r,a} \, &\frac{\Gamma(\frac{n+a}{2})}{\Gamma(\frac{n+a}{2}+i)\Gamma(\frac{k+a}{2})} \sum_{\beta=0}^{i} (-1)^\beta \binom{i}{\beta} \Gamma(\tfrac{k+a}{2}+i-\beta) \\ &\times \frac{\Gamma(\frac{n-k}{2}+\beta)\Gamma(\frac{a}{2}+1)\Gamma(\frac{r}{2})}{\Gamma(\frac{n-k}{2})\Gamma(\frac{a}{2}+1-\beta)\Gamma(\frac{r}{2}+\beta)} \, Q^{i-\beta} Q(F)^\beta\end{aligned}$$

with

$$e_{n,k,r,a} := \prod_{p=0}^{n-r-1} \frac{\Gamma(\frac{n-p}{2})\Gamma(\frac{k-p+a}{2})}{\Gamma(\frac{n-p+a}{2})\Gamma(\frac{k-p}{2})}.$$

Proof. Although this lemma is stated in [51, Proposition 4.5] only for $k, r \geq 1$, it is easy to check that it remains true for $k = 0$ (and $r = n$) and for $r = 0$ (and $k = n$) with $n \geq 1$ as well as for $n = k = r = 1$.

The only non-trivial case that has to be checked concerns the assertion for $k = 0$, $r = n$ and $i \geq 1$, where we have to show that the right side is the zero tensor. For this we can assume that $a > 0$, since the case $a = 0$ is covered by Lemma A.4. Up to irrelevant constants, the factor on the right side equals

$$\sum_{\beta=0}^{i}(-1)^{\beta}\binom{i}{\beta}\Gamma(\tfrac{a}{2}+i-\beta)\frac{\Gamma(\frac{n}{2}+\beta)}{\Gamma(\frac{a}{2}+1-\beta)\Gamma(\frac{n}{2}+\beta)}$$
$$=(-1)^{i}\sum_{\beta=0}^{i}(-1)^{\beta}\binom{i}{\beta}\frac{\Gamma(\frac{a}{2}+\beta)}{\Gamma(\frac{a}{2}+1-i+\beta)}=0,$$

which follows from relation (B.1′), since $\Gamma(1-i)^{-1}=0$ for $i\geq 1$. □

Interestingly, the sum representation of the integral in Lemma A.5 simplifies significantly for the special choice of $a=2$, as in that case the factor $\Gamma(\frac{a}{2}+1-\beta)^{-1}$ equals 0 for all $\beta>1$. Therefore, we state this result in a separate lemma, even though it is an immediate consequence of the preceding lemma.

Lemma A.6 ([51, Corollary 4.6])**.** *Let $a,i\in\mathbb{N}_0$, $k,r\in\{0,\dots,n\}$ with $k+r\geq n\geq 1$, and let $F\in \mathrm{G}(n,r)$. Then*

$$\int_{\mathrm{G}(n,k)}[F,L]^2Q(L)^i\,\nu_k(\mathrm{d}L)=\frac{r!k!}{n!(k+r-n)!}\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i+1)\Gamma(\frac{k}{2}+1)}$$
$$\times\left((\tfrac{k}{2}+i)Q^i+i\tfrac{k-n}{r}Q^{i-1}Q(F)\right).$$

In Lemma A.6, we interpret the second summand on the right-hand side of the equation as 0, if $i=0$, which is consistent with [51, Lemma 4.4]. If $r=0$, we also interpret the second summand as 0 and the integral on the left equals Q^i. If $k=0$, we interpret the right side as $\mathbb{1}\{s=0\}$, since we read $\Gamma(\frac{k}{2}+i)(\frac{k}{2}+i)$ for $i=0$.

We conclude this chapter with the following integral formula (see [51, p. 503]), which is a special case of [72, Theorem 3.1] and is required in the proof of the intrinsic Crofton formulae.

Lemma A.7. *Let $P\in\mathcal{P}^n$ be a polytope and $0\leq j<k<n$. Let further $L\in \mathrm{G}(n,k)$ and $g:\mathbb{R}^n\times(\mathbb{S}^{n-1}\cap L)\to\mathbb{T}$ be a measurable bounded function. Then*

$$\int_{L^\perp}\int_{L_t\times(L\cap\mathbb{S}^{n-1})}g(x,u)\,\Lambda_j^{(L_t)}(P\cap L_t,\mathrm{d}(x,u))\,\mathcal{H}^{n-k}(\mathrm{d}t)$$
$$=\frac{1}{\omega_{k-j}}\sum_{F\in\mathcal{F}_{n-k+j}(P)}\int_{F\times(N(P,F)\cap\mathbb{S}^{n-1})}g(x,\pi_L(u))\|p_L(u)\|^{j-k}[F,L]^2\,\mathcal{H}^{n-1}(\mathrm{d}(x,u)).$$

APPENDIX B

Explicit Sum Expressions

In this chapter, we establish closed form expressions for sums which are required in the preceding parts. All of them can be proved using special relations obtained by application of Zeilberger's algorithm (see [70]).

Remark. The results in this chapter have already been published or submitted in different works. Lemma B.1 and Lemma B.2 can be found in *Kinematic Formulae for Tensorial Curvature Measures*, a joint work with Daniel Hug, submitted in 2016 (see [53]). Lemma B.3, Lemma B.4 and Lemma B.5 can be found in *Crofton Formulae for Tensor-Valued Curvature Measures*, a joint work with Daniel Hug in the lecture notes *Tensor Valuations and their Applications in Stochastic Geometry and Imaging* edited by Kiderlen and Vedel Jensen (see [54], which is Chapter 4 in [58]).

Lemma B.1. *Let $q \in \mathbb{N}_0$, $b, c \in \mathbb{R}$. Then*

$$\sum_{y=0}^{q} \binom{q}{y} \frac{1}{\Gamma(b+y)\Gamma(c-y)} = \frac{\Gamma(b+c+q-1)}{\Gamma(c)\Gamma(b+q)\Gamma(b+c-1)}. \tag{B.1}$$

In this work, we often use a consequence of Lemma B.1: With (2.2) we obtain for $a > 0$ and $b \in \mathbb{R}$ the relation

$$\sum_{y=0}^{q} (-1)^y \binom{q}{y} \frac{\Gamma(a+y)}{\Gamma(b+y)} = \frac{\Gamma(a)\Gamma(b-a+q)}{\Gamma(b+q)\Gamma(b-a)}, \tag{B.1$'$}$$

which can be found as Lemma 15.6.4 in [1] under the additional assumption $a < b$ (and thus $b > 0$). Since this case is not sufficient for our purposes, we deduce the current more general version via Zeilberger's algorithm.

Proof. For the proof, we can assume that $b, c \notin \mathbb{Z}$. The general case then follows from a continuity argument. We set

$$F(q,y) := \binom{q}{y} \frac{1}{\Gamma(b+y)\Gamma(c-y)}, \qquad q, y \in \mathbb{N}_0.$$

Then we have $F(q,y) = 0$ if $y \notin \{0, \dots, q\}$. We set

$$f(q) := \sum_{y=0}^{q} F(q,y), \qquad q \in \mathbb{N}_0.$$

Furthermore, for $q, y \in \mathbb{N}_0$ we define

$$G(q,y) := \begin{cases} \frac{y(b+y-1)}{q-y+1} F(q,y), & \text{for } y \in \{0, \dots, q\}, \\ G(q,q) - (b+q)F(q+1,q) & \\ \qquad +(b+c+q-1)F(q,q), & \text{for } y = q+1, \\ 0, & \text{for } y \geq q+2. \end{cases}$$

A direct calculation yields

$$-(b+q-1)F(q,y) + (b+c+q-2)F(q-1,y) = G(q-1,y+1) - G(q-1,y)$$

for $y \in \mathbb{N}_0$ and $q \in \mathbb{N}$. Summing this relation for all $y \in \{0, \dots, q\}$ gives

$$-(b+q-1)f(q) + (b+c+q-2)f(q-1) = 0,$$

and thus recursively

$$\begin{aligned} f(q) &= \frac{(b+c+q-3)(b-a+q-2)}{(b+q-2)(b+q-1)} f(q-2) \\ &\vdots \\ &= \frac{(b+c-1)\cdots(b+c+q-2)}{b\cdots(b+q-1)} f(0) \\ &= \frac{\Gamma(b+c+q-1)\Gamma(b)}{\Gamma(b+q)\Gamma(b+c-1)} f(0). \end{aligned}$$

With $f(0) = \frac{1}{\Gamma(b)\Gamma(c)}$ we obtain the assertion. □

The next two lemmas can also be proved with the help of Zeilberger's algorithm. We, however, show the closed sum expressions in a shorter, more direct way.

Lemma B.2. *Let* $\alpha, \beta, \gamma \in \mathbb{N}$, $0 < j < n$. *Then*

$$\sum_{i=2x+\alpha}^{s} (-1)^{i+\alpha} \binom{s}{i} \binom{i}{2x} \binom{i-2x}{\alpha} \left(1-z^2\right)^i$$
$$= \binom{s}{2x} \binom{s-2x}{\alpha} z^{2s-4x-2\alpha} (1-z^2)^{2x+\alpha}.$$

Proof. We start with an index shift

$$\sum_{i=2x+\alpha}^{s} (-1)^{i+\alpha} \binom{s}{i} \binom{i}{2x} \binom{i-2x}{\alpha} \left(1-z^2\right)^i$$
$$= \left(1-z^2\right)^{2x+\alpha} \sum_{i=0}^{s-2x-\alpha} \binom{s}{i+2x+\alpha} \binom{i+2x+\alpha}{2x} \binom{i+\alpha}{\alpha} \left(z^2-1\right)^i.$$

It is easy to check that

$$\binom{s}{i+2x+\alpha} \binom{i+2x+\alpha}{2x} \binom{i+\alpha}{\alpha} = \binom{s}{2x} \binom{s-2x}{\alpha} \binom{s-2x-\alpha}{i}.$$

Then the binomial theorem yields

$$\sum_{i=0}^{s-2x-\alpha} \binom{s-2x-\alpha}{i} \left(z^2-1\right)^i = z^{2s-4x-2\alpha},$$

and thus the assertion. □

Lemma B.3. *Let* $a \in \mathbb{N}_0$. *Then*

$$\sum_{q=0}^{a} \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} = \frac{(-1)^a}{\sqrt{\pi}(1-2a)a!}.$$

Proof. For the sum S on the left-hand side of the asserted equation, we obtain

$$S = \sum_{q=0}^{a} \left(\frac{2q}{2a-1} \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} + \frac{2q+2}{2a-1} \frac{(-1)^q}{\Gamma(a-q-\frac{1}{2})(q+1)!} \right),$$

where we use that $(-\frac{1}{2})\Gamma(-\frac{1}{2}) = \sqrt{\pi}$. Due to cancellation in this telescoping sum, the assertion follows immediately. □

Lemma B.4. *Let* $a, b, c \in \mathbb{R}$ *and* $z \in \mathbb{N}_0$ *with* $a > z \geq 0$ *and* $b > 0$. *Then*

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)}$$
$$= (-1)^z \frac{\Gamma(a-z)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(c)} \frac{\Gamma(a-c+1)}{\Gamma(a-c+1-z)} \frac{\Gamma(c-b)}{\Gamma(c-b-z)}.$$

The factor $\Gamma(a-c+1)$ (resp. $\Gamma(c-b)$) in Lemma B.4 does not cause any problems for $c-a \in \mathbb{N}$ (resp. $b-c \in \mathbb{N}_0$), as the also appearing $\Gamma(a-c+1-z)$ (resp. $\Gamma(c-b-z)$) cancels out the singularity. On the other hand, in our applications of the lemma, we only need the cases where $a-c+1 > z$ and $c-b > z$.

Proof. We set

$$F(z,j) := (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)},$$

for $j \in \{0,\dots,z\}$, and $F(z,j) = 0$ in all other cases, and

$$f(z) := \sum_{j=0}^{z} F(z,j).$$

Furthermore, we define the function

$$G(z,j) := \begin{cases} -\frac{j(a-j)(b+z-j)}{z-j+1} F(z,j), & \text{for } j \in \{0,\dots,z\}, \\ G(z,z) + (a-z-1)F(z+1,z) & \\ \qquad +(c-b-z-1)(a-c-z)F(z,z), & \text{for } j = z+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$\begin{aligned} &(a-z)F(z,j) + (c-b-z)(a-c-z+1)F(z-1,j) \\ &\quad = G(z-1,j+1) - G(z-1,j) \end{aligned}$$

for $j \in \mathbb{N}_0$. Summing this relation over $j \in \{0,\dots,z\}$ gives

$$(a-z)f(z) + (c-b-z)(a-c-z+1)f(z-1) = 0$$

and thus

$$\begin{aligned} f(z) &= \frac{(c-b-z)(c-b-z+1)(a-c-z+1)(a-c-z+2)}{(a-z)(a-z+1)} f(z-2) \\ &\;\;\vdots \\ &= (-1)^z \frac{\Gamma(c-b)\Gamma(a-c+1)\Gamma(a-z)}{\Gamma(c-b-z)\Gamma(a-c+1-z)\Gamma(a)} f(0), \end{aligned}$$

where

$$\frac{\Gamma(c-b)}{\Gamma(c-b-z)} = (c-b-z)\cdots(c-b-1)$$

is well-defined, even for $b-c \in \mathbb{N}_0$, and a similar statement holds for $\frac{\Gamma(a-c+1)}{\Gamma(a-c+1-z)}$. With

$$f(0) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c+1)}$$

we obtain the assertion. □

Lemma B.5. *Let $a, b \in \mathbb{R}$ with $a, b > 0$ and $t \in \mathbb{N}$. Then*

$$\sum_{j=0}^{t} (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)} = \frac{\Gamma(a-b+t)\Gamma(b)\Gamma(t+1)}{\Gamma(a-b+1)\Gamma(b+t+1)}.$$

The factor $\Gamma(a-b+t)$ in Lemma B.5 does not cause any problems for $b-a-t \in \mathbb{N}_0$, as the also appearing $\Gamma(a-b+1)$ cancels out the singularity. In our application of the lemma, we will additionally know that $a > b$.

Proof. We set

$$F(t,j) := (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)},$$

for which we see that $F(t,j) = 0$ if $j \notin \{0, \ldots, t\}$, and

$$f(t) := \sum_{j=0}^{t} F(t,j).$$

Furthermore, we define the function

$$G(t,j) := \begin{cases} \frac{j(a+j)(a+2t+1)(t^2+t(a+1)-j+1)(b+j)}{t(t-j+1)(a+t)(a+t+1)} F(t,j), & \text{for } j \in \{0, \ldots, t\}, \\ G(t,t) - (b+t+1)F(t+1,t) & \\ \qquad +(t+1)(a-b+t)F(t,t), & \text{for } j = t+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$-(b+t)F(t,j) + t(a-b+t-1)F(t-1,j) = G(t-1,j+1) - G(t-1,j)$$

for $j \in \mathbb{N}_0$. Summing this relation over $j \in \{0, \ldots, t\}$ gives

$$-(b+t)f(t) + t(a-b+t-1)f(t-1) = 0$$

and thus

$$
\begin{aligned}
f(t) &= \frac{(t-1)t(a-b+t-2)(a-b+t-1)}{(b+t-1)(b+t)} f(t-2) \\
&\vdots \\
&= \frac{\Gamma(t+1)\Gamma(a-b+t)\Gamma(b+2)}{\Gamma(a-b+1)\Gamma(b+t+1)} f(1).
\end{aligned}
$$

With

$$f(1) = \frac{1}{b} - \frac{1}{b+1} = \frac{1}{b(b+1)}$$

we obtain the assertion. □

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