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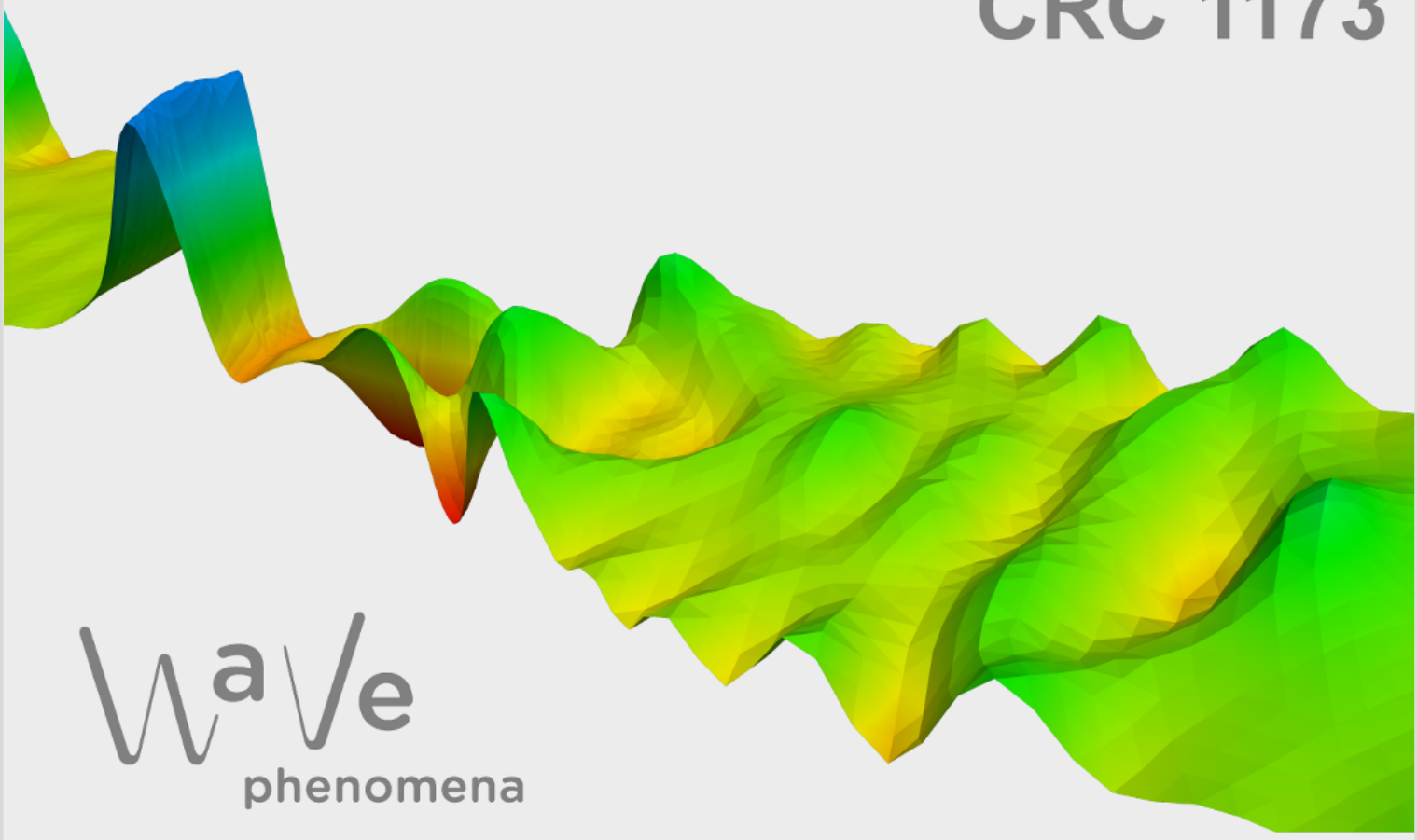

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MARTINGALE SOLUTIONS FOR THE STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION IN THE ENERGY SPACE

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ABSTRACT. We consider a stochastic nonlinear Schrödinger equation with multiplicative noise in an abstract framework that covers subcritical focusing and defocusing Stochastic NLSE in H^1 on compact manifolds and bounded domains. We construct a martingale solution using a modified Faedo-Galerkin-method based on the Littlewood-Paley-decomposition. For 2d manifolds with bounded geometry, we use Strichartz estimates to show pathwise uniqueness.

Keywords: Nonlinear Schrödinger Equation, Multiplicative Noise, Galerkin approximation, compactness method

1. INTRODUCTION

The article is concerned with the following nonlinear stochastic Schrödinger equation

$$\begin{cases} \mathrm{d}u(t) = (-\mathrm{i}Au(t) - \mathrm{i}F(u(t)))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in the energy space $E_A := \mathcal{D}(A^{\frac{1}{2}})$, where A is a selfadjoint, positive operator A with a compact resolvent in an L^2-space H, F a nonlinearity, W a multiplicative Stratonovich noise and B a linear bounded operator.
Three basic examples of A are

- the Laplace-Beltrami operator $-\Delta_g$ on a compact riemannian manifold (M, g) without boundary,
- the Laplacian on a bounded domain of $\mathbb{R}^d$ with Neumann or Dirichlet boundary conditions,
- fractional powers of the first two examples.

The model nonlinearities are the defocusing power nonlinearity $F_\alpha^+(u) := |u|^{\alpha-1}u$ with subcritical exponents in the sense that the embedding $E_A \hookrightarrow L^{\alpha+1}$ is compact and the focusing nonlinearity $F_\alpha^-(u) := -|u|^{\alpha-1}u$ with an additional restriction to the power α. The typical noise term has the form

$$-\mathrm{i}Bu(t) \circ \mathrm{d}W(t) = -\mathrm{i}\sum_{m=1}^{\infty} e_m u(t) \circ \mathrm{d}\beta_m(t) = -\frac{1}{2}\sum_{m=1}^{\infty} e_m^2 u(t) - \mathrm{i}\sum_{m=1}^{\infty} e_m u(t)\mathrm{d}\beta_m(t) \tag{1.2}$$

with a sequence of independant standard real Brownian motions $(\beta_m)_{m\in\mathbb{N}}$ and functions $(e_m)_{m\in\mathbb{N}}$ satisfying certain regularity and decay conditions that guarantee the convergence of the series on the RHS of (1.2) in E_A.

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The main aim of the paper is to construct a martingale solution of problem (1.1) by a modified Faedo-Galerkin approximation

$$\begin{cases} \mathrm{d}u_n(t) = (-\mathrm{i}Au_n(t) - \mathrm{i}P_nF\left(u_n(t)\right))\,\mathrm{d}t - \mathrm{i}S_nB(S_nu_n(t)) \circ \mathrm{d}W(t), \quad t > 0, \\ u_n(0) = P_nu_0, \end{cases} \tag{1.3}$$

in finite dimensional subspaces H_n of H spanned by some eigenvectors of A. Here, $P_n : H \to H_n$ are the standard orthogonal projections and $S_n : H \to H_n$ are selfadjoint operators derived from the Littlewood-Paley-decomposition associated to A. The reason for using the operators $(S_n)_{n\in\mathbb{N}}$ lies in the uniform estimate

$$\sup_{n\in\mathbb{N}} \|S_n\|_{L^p\to L^p} < \infty, \quad 1 < p < \infty,$$

which turns out to be necessary in the estimates of the noise and which is false if one replaces S_n by P_n.
On the other hand, the orthogonal projections P_n are used in the deterministic part, because they do not destroy the cancellation effects which lead to the mass and energy conservation

$$\|u\|_{L^2}^2 = \text{const}, \qquad \frac{1}{2}\|A^{\frac{1}{2}}u\|_{L^2}^2 + \hat{F}(u) = \text{const}$$

for solutions u of problem (1.1) in the deterministic setting, where $\hat{F}$ denotes the antiderivative of the nonlinearity F. Note that in the case $F_\alpha^\pm(u) = \pm|u|^{\alpha-1}u$, the antiderivative is given by $\hat{F}_\alpha^\pm = \pm\frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1}$.
In the stochastic case, the mass conservation $\|u_n\|_{L^2}^2 = \text{const}$ for solutions of (1.3) holds almost surely due to the Stratonovich form of the noise. Moreover, the conservation of the energy is carried over in the sense that a Gronwall type argument yields the uniform a priori estimates, for every $T > 0$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^2\Big] < \infty, \qquad \sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{L^{\alpha+1}(M)}^{\alpha+1}\Big] < \infty. \tag{1.4}$$

Combined with the Aldous condition $[A]$, see Definition 4.3, which is a stochastic version of the equicontinuity, the estimates (1.4) lead to the tightness of the sequence $(u_n)_{n\in\mathbb{N}}$ in the locally convex space

$$Z_T := C([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap C_w([0,T], E_A),$$

where $C_w([0,T], E_A)$ denotes the space of continuous functions with respect to the weak topology in E_A. The first two spaces in Z_T can be used to pass to the limit in the Galerkin equation (1.3), whereas the third space guarantees nice path properties of the limit.
The construction of a martingale solution is similar to [BM12] using Jakubowski's extension of the Skorohod Theorem to nonmetric spaces and the Martingale Representation Theorem from [DPZ14], chapter 8. Our main result is the following Theorem.

Theorem 1.1. *Let $T > 0$ and $u_0 \in E_A$. Under the assumptions 2.1, 2.4, 2.6, 2.7, there exists a martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ of equation (1.1) (see Definition 2.9), which satisfies*

$$u \in L^q(\tilde{\Omega}, L^\infty(0,T; E_A)) \tag{1.5}$$

for all $q \in [1,\infty)$ and

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)} \qquad \tilde{\mathbb{P}}\text{-a.s. for all } t \in [0,T].$$

As an application of Theorem 1.1, we get the following Corollary. Note that an analogous result holds in the case of a bounded domain, see Corollary 3.5.

Corollary 1.2. *Let (M, g) be a compact d-dimensional riemannian manifold without boundary. Under assumption 2.7 and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4}{(d-2)_+}\right)$,
ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4}{d}\right)$,

the equation

$$\begin{cases} \mathrm{d}u(t) = (\mathrm{i}\Delta_g u(t) - \mathrm{i}F(u(t))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t) \quad \text{in } H^1(M), \\ u(0) = u_0 \in H^1(M), \end{cases} \tag{1.6}$$

has a martingale solution with

$$u \in L^q(\tilde{\Omega}, L^\infty(0, T; H^1(M))), \tag{1.7}$$

for all $q \in [1, \infty)$ and

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)} \qquad \tilde{\mathbb{P}}\text{-a.s. for all } t \in [0, T].$$

Furthermore, we adress the question of uniqueness of the solution from Corollary 1.2 in two dimensions.

Corollary 1.3. *In the situation of Corollary 1.2 with $d = 2$, there exists a unique strong solution of (1.6) in $H^1(M)$ and the martingale solutions are unique in law.*

We obtain pathwise uniqueness by an improvement of the regularity of solutions based on the Strichartz estimates by Bernicot and Savoyeau from [BS14] and Brzeźniak and Millet from [BM14]. Ondreját showed in [Ond04] in a quite general setting, that this is sufficient to get a strong solution. In fact, our uniqueness result is more general than we have formulated in Corollary 1.3. On the one hand, we allow possibly non-compact of manifolds with bounded geometry. On the other hand, uniqueness holds in the strictly larger class $L^r(\Omega, L^\beta(0, T; H^s(M)))$ with $r > \alpha$, $\beta := \max\{2, \alpha\}$ and

$$s \in \begin{cases} (\frac{2\alpha-1}{2\alpha}, 1] & \text{for } \alpha \in (1, 3], \\ (\frac{\alpha(\alpha-1)-1}{\alpha(\alpha-1)}, 1] & \text{for } \alpha > 3. \end{cases}$$

For the details, we refer to Theorem 7.5.

The question of the existence and uniqueness for nonlinear Schrödinger equations with multiplicative noise was previously addressed in the $\mathbb{R}^d$-case by de Bouard and Debussche in [dBD99],[dBD03], Barbu, Röckner and Zhang in [BRZ14],[BRZ16] and Hornung in [Hor16] and in the case of compact 2d riemannian manifolds by Brzeźniak and Millet in [BM14]. They constructed mild solutions with a fixed point argument based on appropriate Strichartz estimates.

The present paper was motivated by the construction of a global solution of the cubic equation on compact 3d-manifolds M generalizing the existence part of Burq, Gérard, Tzvetkov in [BGT04], Theorem 3, to the stochastic setting. In three dimensions, the fixed point argument from [BM14] is restricted to higher regularity, because it needs Sobolev embeddings $H^{s,q} \hookrightarrow L^\infty$, which are more restrictive in 3D than in 2D. Hence, this approach only yields local solutions, which is the motivation for constructing a global solution in $H^1(M)$ with an approximation procedure based on the conservation laws of the NLSE without using the

dispersive properties of the Schrödinger group. We remark that in [BGT04], the authors also prove uniqueness for the deterministic NLSE in 3D. For the equation with noise this question will be adressed in a forthcoming paper.

Classical references for the construction of weak solutions of the deterministic NLSE by a Galerkin method are [Gaj78] and [Gaj79] for intervals and [Nas80] as well as [Vla87] for domains of arbitrary dimension. In particular, we give a new proof of these results. But we would like to remark that the deterministic case is significantly simpler since the spectral theoretic methods to construct the operators S_n are not needed.

Recently, stochastic NLSE on bounded 1D domains were studied by Keller and Lisei in [KL15] and [KL16]. In [KL15] the authors considered Itô noise with time dependent coefficients instead of a space dependent Stratonovich noise as in the present paper and used a transformation to a random equation (and in this sense this approach is similar to the one used in [BRZ16]) which is solved by a Galerkin approximation. In [KL16], the nonlinear term is different, namely $F(u) = \mathrm{i}|u|^{\alpha-1}u$ instead of $F(u) = \pm|u|^{\alpha-1}u$ in our notation and the authors use a direct Galerkin approximation for the stochastic equation. In this special 1d situation, the authors obtained strong unique solutions.

The paper is organized as follows. In the Sections 2 and 3, we fix the notation, formulate our Assumptions and present a number of typical examples of operators A, a model nonlinearity F and noise coefficients B covered by our framework. In Section 4, we are concerned with the compactness results that we will be using later on. In Section 5, we formulate the Galerkin approximation equations and proof the a priori estimates which are sufficient for compactness in view of Section 4. Section 6 is devoted to the proof of Theorem 1 and in Section 7, we focus on uniqueness in the case of 2d manifolds with bounded geometry.

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2. Notation and Assumptions

In this section, we want to fix the notations, explain the assumptions and formulate an abstract framework for the stochastic nonlinear Schrödinger equation.

Let (X, Σ, μ) be a σ-finite measure space with metric ρ satisfying the *doubling property*, i.e. $\mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$ and

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)). \tag{2.1}$$

Let $M \subset X$ be an open subset with finite measure and $L^q(M)$ for $p \in [1,\infty]$ the space of equivalence classes of $\mathbb{C}$-valued $q-$integrable functions. We further abbreviate $H := L^2(M)$. If functions $a, b \geq 0$ satisfy the inequality $a \leq C(A)b$ with a constant $C(A) > 0$ depending on the expression A, we write $a \lesssim_A b$. For two Banach spaces E, F, we denote by $\mathcal{L}(E,F)$ the space of linear bounded operators $B : E \to F$ and abbreviate $\mathcal{L}(E) := \mathcal{L}(E,E)$. Furthermore, we write $E \hookrightarrow F$, if E is continuously embedded in F; i.e. $E \subset F$ with natural embedding $j \in \mathcal{L}(E,F)$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the law of a random variable $X : \Omega \to E$ is denoted by $\mathbb{P}^X$.

Assumption and Notation 2.1. We assume the following:

i) Let A be a non-negative selfadjoint operator on H with domain $\mathcal{D}(A)$.

ii) There is a strictly positive selfadjoint operator S on H with compact resolvent commuting with A. Moreover, we assume that S has *generalized Gaussian* (p_0, p_0')*-bounds* for some $p_0 \in [1,2)$, i.e.

$$\|\mathbf{1}_{B(x,t^{\frac{1}{m}})} e^{-tS} \mathbf{1}_{B(y,t^{\frac{1}{m}})}\|_{\mathcal{L}(L^{p_0},L^{p_0'})} \leq C\mu(B(x,t^{\frac{1}{m}}))^{\frac{1}{p_0'}-\frac{1}{p_0}} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \tag{2.2}$$

for all $t>0$ and $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \geq 2$.

iii) The Hilbert space

$$(E_A, (\cdot,\cdot)_{E_A}) := (\mathcal{D}(A^{\frac{1}{2}}), (\cdot,\cdot)_{\mathcal{D}(A^{\frac{1}{2}})})$$

is called the *energy space* and $\|\cdot\|_{E_A}$ the *energy norm* associated to A. We denote the dual space of E_A by E_A^* and abbreviate the duality with $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{E_A^*,E_A}$, where the complex conjugation is taken over the second variable of the duality. Note that (E_A, H, E_A^*) is a Gelfand triple, i.e.

$$E_A \hookrightarrow H \cong H^* \hookrightarrow E_A^*.$$

iv) Let $\alpha \in (1, p_0'-1)$ be such that E_A is compactly embedded in $L^{\alpha+1}(M)$. We set

$$p_{\max} := \sup\{p \in (1,\infty] : E_A \hookrightarrow L^p(M) \quad \text{is continuous}\}$$

and note that $p_{\max} \in [\alpha+1, \infty]$. In the case $p_{\max} < \infty$, we assume that $E_A \hookrightarrow L^{p_{\max}}(M)$ is continuous, but not necessarily compact.

Remark 2.2. a) If $p_0 = 1$, then it is proved in [BK03] that (2.2) is equivalent to the usual upper Gaussian estimate, i.e. for all $t>0$ there is a measurable function $p(t,\cdot,\cdot) : M \times M \to \mathbb{R}$ with

$$T(t)f(x) = \int_M p(t,x,y)f(y)\mu(dy), \quad t>0, \quad \text{a.e. } x \in M$$

for all $f \in H$ and

$$|p(t,x,y)| \leq \frac{C}{\mu(B(x,t^{\frac{1}{m}}))} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \tag{2.3}$$

for all $t>0$ and almost all $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \geq 2$.

b) In our proof, the generalized Gaussian estimate (2.2) is only indirectly used to guarantee that A can be extrapolated from $L^2(M)$ to $L^p(M)$ for $p \in (p_0, p_0')$ with a Mihlin $\mathcal{M}^\beta$ functional calculus for some $\beta > 0$. For additional information about the Mihlin functional calculus, we refer to [Kri09] and [KW16].

We start with some conclusions which can be deduced from Assumption 2.1.

Lemma 2.3. a) *There is a positive selfadjoint operator $\hat{A}$ on E_A^* with $\mathcal{D}(\hat{A}) = E_A$ with $\hat{A} = A$ on H.*

b) *The embedding $E_A \hookrightarrow H$ is compact.*

c) *There is an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ and a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ and*

$$Sx = \sum_{n=1}^{\infty} \lambda_n \big(x, h_n\big)_H h_n, \quad x \in \mathcal{D}(S) = \left\{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |\big(x,h_n\big)_H|^2 < \infty\right\},$$

Proof. ad a). The operator $\hat{A}$ is defined by

$$\langle \hat{A}\varphi, \psi\rangle := \big(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\big)_H, \quad \varphi, \psi \in E_A.$$

The estimate

$$|\langle \hat{A}\varphi, \psi\rangle| \leq \|A^{\frac{1}{2}}\varphi\|_H \|A^{\frac{1}{2}}\psi\|_H \leq \|\varphi\|_{E_A}\|\psi\|_{E_A}$$

shows that $\hat{A}$ is well-defined and a bounded operator from E_A to E_A^* with $\|\hat{A}\| \leq 1$. Moreover, one can apply the Lax-Milgram-Theorem to see that $I + \hat{A}$ is a surjective isometry from E_A to E_A^*. If one equips E_A^* with the inner product

$$\big(f^*, g^*\big)_{E_A^*} := \big((I+\hat{A})^{-1}f^*, (I+\hat{A})^{-1}g^*\big)_{E_A}, \qquad f^*, g^* \in E_A^*,$$

one can show the symmetry of $\hat{A}$ as an unbounded operator in E_A^*. Hence, $\hat{A}$ is selfadjoint, because $-1 \in \rho(\hat{A})$.

ad b). The embedding $E_A \hookrightarrow L^{\alpha+1}(M)$ is compact by Assumption 2.1 iv) and $L^{\alpha+1}(M) \hookrightarrow H$ is continuous due to $\mu(M) < \infty$. Hence, $E_A \hookrightarrow H$ is compact.
ad c). Immediate consequence of the spectral theorem, since S has a compact resolvent. □

In most cases where this does not cause ambiguity or confusion, we also use the notations A for $\hat{A}$. We continue with the assumptions on the nonlinear part of our problem.

Assumption 2.4. Let $\alpha \in (1, p_0' - 1)$ be chosen as in Assumption 2.1. Then, we assume the following:

i) Let $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ be a function satisfying the following estimate

$$\|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \|u\|^{\alpha}_{L^{\alpha+1}(M)}, \quad u \in L^{\alpha+1}(M). \tag{2.4}$$

Note that this leads to $F : E_A \to E_A^*$ by Assumption 2.1 iv), because $E_A \hookrightarrow L^{\alpha+1}(M)$ implies $(L^{\alpha+1}(M))^* = L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$. We further assume and $F(0) = 0$ and

$$\mathrm{Re}\langle \mathrm{i}u, F(u)\rangle = 0, \quad u \in L^{\alpha+1}(M). \tag{2.5}$$

ii) The map $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ is continuously real Fréchet differentiable with

$$\|F'[u]\|_{L^{\alpha+1} \to L^{\frac{\alpha+1}{\alpha}}} \lesssim \|u\|^{\alpha-1}_{L^{\alpha+1}(M)}, \quad u \in L^{\alpha+1}(M). \tag{2.6}$$

iii) The map F has a real antiderivative $\hat{F}$, i.e. there exists a Fréchet-differentiable map $\hat{F} : L^{\alpha+1}(M) \to \mathbb{R}$ with

$$\hat{F}'[u]h = \mathrm{Re}\langle F(u), h\rangle, \quad u, h \in L^{\alpha+1}(M). \tag{2.7}$$

By Assumption 2.4 ii) and the mean value theorem for Fréchet differentiable maps, we get

$$\begin{aligned}\|F(x) - F(y)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} &\leq \sup_{t\in[0,1]} \|F'[tx + (1-t)y]\|\|x-y\|_{L^{\alpha+1}(M)} \\ &\lesssim \big(\|x\|_{L^{\alpha+1}(M)} + \|y\|_{L^{\alpha+1}(M)}\big)^{\alpha-1}\|x-y\|_{L^{\alpha+1}(M)}, \qquad x, y \in L^{\alpha+1}(M),\end{aligned} \tag{2.8}$$

which means that the nonlinearity is Lipschitz on bounded sets of $L^{\alpha+1}(M)$.

We will cover the following two standard types of nonlinearities.

Definition 2.5. Let F satisfy Assumption 2.4. Then, F is called *defocusing*, if $\hat{F}(u) \geq 0$ and *focusing*, if $\hat{F}(u) \leq 0$ for all $u \in L^{\alpha+1}(M)$.

Assumption 2.6. We assume either i) or i'):

i) Let F be defocusing and satisfy

$$\|u\|^{\alpha+1}_{L^{\alpha+1}(M)} \lesssim \hat{F}(u), \quad u \in L^{\alpha+1}(M). \tag{2.9}$$

i') Let F be focusing and satisfy

$$-\hat{F}(u) \lesssim \|u\|^{\alpha+1}_{L^{\alpha+1}(M)}, \quad u \in L^{\alpha+1}(M). \tag{2.10}$$

and there is $\theta \in (0, \frac{2}{\alpha+1})$ with

$$(H, E_A)_{\theta,1} \hookrightarrow L^{\alpha+1}(M). \tag{2.11}$$

Here $(\cdot,\cdot)_{\theta,1}$ denotes the real interpolation space and we remark that by [Tri95], Lemma 1.10.1, (2.11) is equivalent to

$$\|u\|^{\alpha+1}_{L^{\alpha+1}(M)} \lesssim \|u\|^{\beta_1}_H \|u\|^{\beta_2}_{E_A}, \quad u \in E_A. \tag{2.12}$$

for some $\beta_1 > 0$ and $\beta_2 \in (0,2)$ with $\alpha + 1 = \beta_1 + \beta_2$. Let us continue with the definitions and assumptions for the stochastic part.

Assumption 2.7. We assume the following:

i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, Y a separable real Hilbert space with ONB $(f_m)_{m\in\mathbb{N}}$ and W a Y-canonical cylindrical Wiener process adapted to a filtration $\mathbb{F}$ satisfying the usual conditions.

ii) Let $B : H \to \mathrm{HS}(Y, H)$ be a linear operator and set $B_m u := B(u) f_m$ for $u \in H$ and $m \in \mathbb{N}$. Additionally, we assume that B_m, $m \in \mathbb{N}$, are bounded selfadjoint operators on H with

$$\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(H)} < \infty \tag{2.13}$$

and that B_m are also bounded from E_A to E_A and from $L^{\alpha+1}(M)$ to $L^{\alpha+1}(M)$ and

$$\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)} < \infty, \qquad \sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} < \infty. \tag{2.14}$$

For the special case, when the B_m are pointwise multiplication operators, see section 3.5 below.

Remark 2.8. The estimates (2.13) and (2.14) imply

$$B \in \mathcal{L}(H, \mathrm{HS}(Y,H)), \quad B \in \mathcal{L}(E_A, \mathrm{HS}(Y, E_A)), \quad B \in \mathcal{L}(L^{\alpha+1}(M), \gamma(Y, L^{\alpha+1}(M))),$$

where $\gamma(Y, L^{\alpha+1}(M))$ denotes the spaces of γ-radonifying operators from Y to $L^{\alpha+1}(M)$.

Finally, we have sufficient background to formulate the problem which we want to solve. We investigate the following stochastic evolution equation in the Stratonovich form

$$\begin{cases} \mathrm{d}u(t) = (-\mathrm{i}Au(t) - \mathrm{i}F(u(t))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.15}$$

where the stochastic differential is defined by

$$-\mathrm{i}Bu(t)\circ\mathrm{d}W(t)=-\mathrm{i}Bu(t)\mathrm{d}W(t)+\frac{1}{2}\operatorname{tr}_Y\left(\mathcal{M}(u(t))\right)\mathrm{d}t, \tag{2.16}$$

with the bilinear form $\mathcal{M}(u)$ on $Y\times Y$ defined by

$$\mathcal{M}(u)(y_1,y_2):=-\mathrm{i}B'[u](-\mathrm{i}B(u)y_1)y_2,\quad u\in H,\quad y_1,y_2\in Y.$$

For the purpose of giving a rigorous definition of a solution to problem (2.15), it is useful to rewrite the equation in the Itô form. Therefore, we first compute

$$\begin{aligned}\operatorname{tr}_Y\left(\mathcal{M}(u)\right)&=\sum_{m=1}^{\infty}-\mathrm{i}B'[u](-\mathrm{i}B(u)f_m)f_m=-\sum_{m=1}^{\infty}B\left(B(u)f_m\right)f_m\\&=-\sum_{m=1}^{\infty}B\left(B_mu\right)f_m=-\sum_{m=1}^{\infty}B_m^2u.\end{aligned}$$

Hence, equation (2.15) will be understood in the following Itô form

$$\begin{cases}\mathrm{d}u(t)=(-\mathrm{i}Au(t)-\mathrm{i}F(u(t)+\mu\left(u(t)\right))\mathrm{d}t-\mathrm{i}Bu(t)\mathrm{d}W(t),\quad t\in(0,T),\\ u(0)=u_0,\end{cases} \tag{2.17}$$

where the linear operator μ defined by

$$\mu(u):=-\frac{1}{2}\sum_{m=1}^{\infty}B_m^2u,\qquad u\in H,$$

is the Stratonovich correction term.

Most of our paper will be concerned with the construction of a martingale solution.

Definition 2.9. Let $T>0$ and $u_0\in E_A$. A *martingale solution* of the equation (1.1) is a system $\left(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde W,\tilde{\mathbb{F}},u\right)$ such that

- a probability space $\left(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)$;
- a Y-valued cylindrical Wiener $\tilde W$ process on $\tilde\Omega$;
- a filtration $\tilde{\mathbb{F}}=\left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ with the usual conditions;
- a continuous, $\tilde{\mathbb{F}}$-adapted, E_A^*-valued process such that $u\in L^2(\Omega\times[0,T],E_A^*)$ and almost all paths are in $C_w([0,T],E_A)$,

such that the equation

$$u(t)=u_0+\int_0^t\left[-\mathrm{i}Au(s)-\mathrm{i}F(u(s))+\mu(u(s))\right]\mathrm{d}s-\mathrm{i}\int_0^tBu(s)\mathrm{d}\tilde W(s) \tag{2.18}$$

holds almost surely in E_A^* for all $t\in[0,T]$.

3. Examples

In this section, we consider concrete situations and verify that they are covered by the general framework presented in the last section.

3.1. **The Model Nonlinearities.** The class of the general nonlinearities from the Assumptions 2.4 and 2.6 covers the standard focusing and defocusing power nonlinearity.

Proposition 3.1. *Let $\alpha \in (1,\infty)$ be chosen as in Assumption 2.1. Define the following function*

$$F_\alpha^\pm(u) := \pm|u|^{\alpha-1}u, \qquad \hat{F}_\alpha^\pm(u) := \pm\frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}(M)}^{\alpha+1}, \qquad u \in L^{\alpha+1}(M).$$

Then, $F_\alpha^\pm$ satisfies Assumption 2.4 with antiderivative $\hat{F}_\alpha^\pm$.

Proof. Obviously, $F_\alpha^\pm : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ due to

$$\|F_\alpha^\pm(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} = \|u\|_{L^{\alpha+1}(M)}^\alpha, \qquad u \in L^{\alpha+1}(M).$$

Furthermore,

$$\mathrm{Re}\langle \mathrm{i}v, F_\alpha^\pm(v)\rangle = \pm\,\mathrm{Re}\int_M \mathrm{i}v|v|^{\alpha-1}\overline{v}\mathrm{d}\mu = \pm\,\mathrm{Re}\left[\mathrm{i}\|v\|_{L^{\alpha+1}(M)}^{\alpha+1}\right] = 0.$$

We can apply the following Lemma 3.2 with $p = \alpha+1$ and

$$\Phi(a,b) = \left(a^2+b^2\right)^{\frac{\alpha-1}{2}}\begin{pmatrix} a \\ b\end{pmatrix}, \qquad a,b \in \mathbb{R},$$

to obtain part ii) and iii) of Assumption 2.4. □

The next Lemma contains the differentiablity properties of the nonlinearity.

Lemma 3.2. *Let $(S,\mathcal{A},\mu)$ be a measure space and $\alpha > 1$.*

a) *Let $p > 1$. The map $G_1 : L^p(S) \to \mathbb{R}$ defined by $G_1(u) := \|u\|_{L^p(S)}^p$ is continuously Fréchet differentiable with*

$$G_1'[u]h = \mathrm{Re}\int_S |u|^{\alpha-1}u\overline{h}\mathrm{d}\mu$$

for all $u,h \in L^p(S)$.

b) *Let $p > \alpha$ and $\Phi : \mathbb{C} \to \mathbb{C}$ be continuously real differentiable. Assume that there is $C > 0$ with*

$$|\Phi(z)| \le C|z|^\alpha, \qquad |\Phi'(z)| \le C|z|^{\alpha-1}, \qquad z \in \mathbb{C}.$$

Then, the map

$$G : L^p(S) \to L^{\frac{p}{\alpha}}(S), \qquad G(u) := \Phi_1(u) + \mathrm{i}\Phi_2(u)$$

is continuously Fréchet differentiable with

$$G'[u]h = \Phi'(u)h, \qquad u,h \in L^p(S),$$

and

$$\|G'[u]\|_{L^p \to L^{\frac{p}{\alpha}}} \le C\|u\|_{L^p}^{\alpha-1}, \qquad u \in L^p(S).$$

3.2. **The Laplace-Beltrami Operator on compact manifolds.** In this subsection, we deduce Corollary 1.2 from Theorem 1.1. Let (M, g) be a compact d-dimensional riemannian manifold without boundary equipped with its canonical measure and $A := -\Delta_g$ be the Laplace-Beltrami operator on M. We choose $S := I - \Delta_g$. Then, S is selfadjoint, strictly positive and commutes with A. The manifold M has the doubling property and S has upper Gaussian bounds by [DOS02], Section 7.2., and in particular, S has generalized Gaussian bounds with $p_0 = 1$.

We have the following relation between the scale of Sobolev spaces from Appendix B and the fractional domains of the Laplace-Beltrami operator. By Proposition $B.2$ a), the scale of Sobolev spaces on M is given by

$$H^s(M) = R\left(S^{-\frac{s}{2}}\right) = \mathcal{D}\left(S^{\frac{s}{2}}\right) = \mathcal{D}\left((-\Delta_g)^{\frac{s}{2}}\right), \quad s > 0,$$

where the last identity can be deduced from the spectral theorem and $(1+\lambda)^s \eqsim_s 1 + \lambda^s$. In particular, we have $E_A = H^1(M)$.

Let $1 < \alpha < 1 + \frac{4}{(d-2)_+}$. Then, by Proposition $B.2$ c) and Lemma 2.3, the embeddings

$$E_A = H^1(M) \hookrightarrow H^{-1}(M) = E_A^*, \qquad E_A = H^1(M) \hookrightarrow L^{\alpha+1}(M)$$

are compact. Hence, Assumption (2.1) holds with our choice of A and S. The range of admissible powers in the focusing case is the content of the following Lemma.

Lemma 3.3. *F_α^+ satisfies Assumption 2.6 i) and F_α^- satisfies i') under the restriction $\alpha \in \left(1, 1 + \frac{4}{d}\right)$.*

Proof. Obviously, the assertion for F_α^+ is true. We consider F_α^-.
Case 1. Let $d \geq 3$. Then, $p_{\max} := \frac{2d}{d-2}$ is the maximal exponent with $H^1(M) \hookrightarrow L^{p_{\max}}(M)$. Since $\alpha \in (1, p_{\max} - 1)$, we can interpolate $L^{\alpha+1}(M)$ between H and $L^{p_{\max}}(M)$ and get

$$\|u\|_{L^{\alpha+1}(M)} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{L^{p_{\max}}(M)}^{\theta} \lesssim \|u\|_{L^2}^{1-\theta} \|u\|_{H^1(M)}^{\theta}.$$

with $\theta = \frac{d(\alpha-1)}{2(\alpha+1)} \in (0,1)$. The restriction $\beta_2 := \theta(\alpha+1) < 2$ from Assumption 2.6 i') leads to $\alpha < 1 + \frac{4}{d}$.

Case 2. In the case $d = 2$, Assumption i') is guaranteed for $\alpha \in (1,3)$. To see this, take $p > \frac{4}{3-\alpha}$ which is equivalent to $\theta(\alpha+1) < 2$ when $\theta \in (0,1)$ is chosen as

$$\theta = \frac{(\alpha-1)p}{(\alpha+1)(p-2)}.$$

We have $H^1(M) \hookrightarrow L^p(M)$ and as above, interpolation between H and $L^p(M)$ yields

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1} \lesssim \|u\|_{L^2}^{(\alpha+1)(1-\theta)} \|u\|_{E_A}^{(\alpha+1)\theta}.$$

Case 3. Let $d = 1$ and fix $\varepsilon \in (0, \frac{1}{2})$. Proposition $B.2$ yields

$$H^{\frac{1}{2}+\varepsilon}(M) \hookrightarrow L^\infty(M), \qquad H^{\frac{1}{2}+\varepsilon}(M) = \left[L^2(M), H^1(M)\right]_{\frac{1}{2}+\varepsilon}.$$

Hence,

$$\|v\|_{L^{\alpha+1}}^{\alpha+1} \leq \|v\|_{L^2}^2 \|v\|_{L^\infty}^{\alpha-1} \lesssim \|v\|_{L^2}^2 \|v\|_{H^{\frac{1}{2}+\varepsilon}}^{\alpha-1} \lesssim \|v\|_{L^2}^{2+(\frac{1}{2}-\varepsilon)(\alpha-1)} \|v\|_{H^1}^{(\frac{1}{2}+\varepsilon)(\alpha-1)}$$

The condition $(\frac{1}{2}+\varepsilon)(\alpha-1) < 2$ is equivalent to $\alpha < 1 + \frac{4}{1+2\varepsilon}$. Choosing ε small enough, we see that Assumption 2.6 i') is true for $\alpha \in (1,5)$. □

Remark 3.4. Note, that the 3-dimensional case with a cubic defocusing nonlinearity, i.e.

$$d = \alpha = 3, \quad F(u) = F_3^+(u) = |u|^2 u$$

is admissible in our framework. In the deterministic setting, i.e. $B = 0$, a global unique weak solution to this problem in $H^1(M)$ was constructed in [BGT04], Theorem 3. Uniqueness in the stochastic case will be proved in a forthcoming paper. In [BM14], the authors considered the stochastic problem, but only obtained global solutions in the 2-dimensional case.

3.3. **Laplacians on bounded domains.** We can apply Theorem 1.1 to the stochastic NLSE on bounded domains.

Corollary 3.5. *Let $M \subset \mathbb{R}^d$ be a bounded domain and Δ be the Laplacian with Dirichlet or Neumann boundary conditions. In the Neumann case, we assume that ∂M is Lipschitz. Under assumption 2.7 and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4}{(d-2)_+}\right)$,
ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4}{d}\right)$,

the equation

$$\begin{cases} \mathrm{d}u(t) = (\mathrm{i}\Delta u(t) - \mathrm{i}F(u(t))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t) \quad \textit{in } H^1(M), \\ u(0) = u_0 \in H^1(M), \end{cases} \tag{3.1}$$

has a martingale solution which satisfies

$$u \in L^q(\tilde{\Omega}, L^\infty(0, T; H^1(M)))$$

for all $q \in [1, \infty)$.

We remark, that one could consider uniformly elliptic operators and more general boundary conditions, but for the sake of simplicity, we concentrate on the present two examples.

Proof. We consider the Dirichlet form $a_V : V \times V \to \mathbb{C}$,

$$a_V(u, v) = \int_M \nabla u \cdot \nabla v \mathrm{d}x, \quad u, v \in V,$$

with associated operator $(A_v, \mathcal{D}(A_V))$ in the following two situations:

i) $V = H_0^1(M)$
ii) $V = H^1(M)$ and M has Lipschitz-boundary.

The operator $A_{H_0^1(M)} = \Delta_D$ is the Dirichlet Laplacian and $A_{H^1(M)} = \Delta_N$ is the Neumann Laplacian. In both cases, $V = E_{A_V}$ by the square root property (see [Ouh09], Theorem 8.1) and the embedding $E_{A_V} \hookrightarrow L^{\alpha+1}(M)$ is compact iff $1 < \alpha < p_{\max} - 1$ with $p_{\max} := 2 + \frac{4}{(d-2)_+}$. Hence, we obtain the same range of admissible powers α for the focusing and the defocusing nonlinearity as in the case of the Riemannian manifold without boundary.

In the Dirichlet case, we choose $S := A = -\Delta_D$, which is a strictly positive operator and [Ouh09], Theorem 6.10, yields the Gaussian estimate for the associated semigroup. Hence, we can directly apply Theorem 1.1 to construct a martingale solution of problem (3.1).

In the Neumann case, we have $0 \in \sigma(\Delta_N)$ and the kernel of the semigroup $\left(e^{-t\Delta_N}\right)_{t\geq 0}$ only satisfies the estimate

$$|p(t,x,y)| \leq \frac{C_\varepsilon}{\mu(B(x,t^{\frac{1}{m}}))} e^{\varepsilon t} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}$$

for all $t > 0$ and almost all $(x,y) \in M \times M$ with an arbitrary $\varepsilon > 0$, see [Ouh09], Theorem 6.10. In order to get a strictly positive operator with the Gaussian bound from Remark 2.2, we fix $\varepsilon > 0$ and choose $S := \varepsilon I - \Delta_N$. □

3.4. **The fractional NLSE.** In this subsection, we show how the range of admissible nonlinearities change when the Laplacians in the previous examples are replaced by their fractional powers $(-\Delta)^\beta$ for $\beta > 0$. Exemplary, we treat the case of a compact riemannian manifold without boundary. Similar results are also true for the Dirichlet and the Neumann Laplacian on a bounded domain.
In the setting of Section 3.2, we look at the fractional Laplace-Beltrami operator given by $A := (-\Delta_g)^\beta$ for $\beta > 0$, which is also a selfadjoint positive operator by the functional calculus and once again, we choose $S := I - \Delta_g$. We apply Theorem 1.1 with

$$E_A = \mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}\left((I-\Delta_g)^{\frac{\beta}{2}}\right) = H^\beta(M),$$

see Proposition $B.2$ a). Therefore the range of admissible pairs (α,β) in the defocusing case is given by

$$\beta > \frac{d}{2} - \frac{d}{\alpha+1} \quad \Leftrightarrow \quad \alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right),$$

since this is exactly the range of α and β with a compact embedding $E_A \hookrightarrow L^{\alpha+1}(M)$ (see Proposition $B.2$ c)). In the focusing case, analogous calculations as in the proof of Lemma 3.3 (with the distinction of $\beta > \frac{d}{2}$, $\beta = \frac{d}{2}$ and $\beta < \frac{d}{2}$) imply, that the range of exponents reduces to

$$\alpha \in \left(1, 1 + \frac{4\beta}{d}\right).$$

Hence, we get the following Corollary.

Corollary 3.6. *Let (M,g) be a compact d-dimensional riemannian manifold without boundary, $\beta > 0$ and*
$u_0 \in H^\beta(M)$. Under assumption 2.7 and either i) or ii)

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right)$,
ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{d}\right)$,

the equation

$$\begin{cases} \mathrm{d}u(t) = \left(-\mathrm{i}\,(-\Delta_g)^\beta u(t) - \mathrm{i}F(u(t))\right)dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t), \quad t > 0, \\ u(0) = u_0 \in H^\beta(M), \end{cases} \tag{3.2}$$

has a martingale solution $\left(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde W, \tilde{\mathbb{F}}, u\right)$ in $H^\beta(M)$ with

$$u \in L^q(\tilde\Omega, L^\infty(0,T; H^\beta(M))) \tag{3.3}$$

for all $q \in [1, \infty)$.

3.5. **The Model Noise.** In Corollaries 1.2 and 3.5, we considered the general linear noise from Assumption 2.7. If M is either a compact riemannian manifold or a bounded domain, let us consider the following example. Let $(B_m)_{m\in\mathbb{N}}$ the multiplication operators given by

$$B_m u := e_m u$$

for $u \in H$ with real valued functions e_m, $m \in \mathbb{N}$, that satisfy

$$e_m \in F := \begin{cases} H^{1,d}(M) \cap L^\infty(M), & d \geq 3, \\ H^{1,q}(M), & d = 2, \\ H^1(M), & d = 1, \end{cases} \tag{3.4}$$

for some $q > 2$ in the case $d = 2$. Moreover, we assume

$$\sum_{m=1}^{\infty} \|e_m\|_F^2 < \infty,$$

We get

$$\|e_m u\|_{L^p} \leq \|e_m\|_{L^\infty(M)} \|u\|_{L^p}, \qquad u \in L^p(M),$$

for $p \in [1, \infty]$. First, let $d \geq 3$. The Sobolev embedding $H^1(M) \hookrightarrow L^{p_{\max}}(M)$ for $p_{\max} = \frac{2d}{d-2}$ and the Hölder inequality with $\frac{1}{2} = \frac{1}{d} + \frac{1}{p_{\max}}$ yield

$$\begin{aligned} \|\nabla (e_m u)\|_{L^2} \leq & \|u \nabla e_m\|_{L^2} + \|e_m \nabla u\|_{L^2} \leq \|\nabla e_m\|_{L^d} \|u\|_{L^{p_{\max}}} + \|e_m\|_{L^\infty(M)} \|\nabla u\|_{L^2} \\ & \lesssim \left(\|\nabla e_m\|_{L^d} + \|e_m\|_{L^\infty(M)} \right) \|u\|_{H^1}, \qquad u \in H^1(M). \end{aligned}$$

Now, let $d = 2$ and $q > 2$ as in (3.4). Then, we have $F \hookrightarrow L^\infty(M)$. Furthermore, we choose $p > 2$ according to $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$ and observe $H^1(M) \hookrightarrow L^p(M)$. As above, we obtain

$$\|\nabla (e_m u)\|_{L^2} \lesssim \left(\|\nabla e_m\|_{L^q} + \|e_m\|_{L^\infty(M)} \right) \|u\|_{H^1} \lesssim \|e_m\|_{H^{1,q}} \|u\|_{H^1}, \qquad u \in H^1(M).$$

Hence, we conclude in both cases

$$\|e_m u\|_{H^1} \lesssim \|e_m\|_F \|u\|_{H^1}, \qquad m \in \mathbb{N}, \quad u \in H^1(M).$$

For $d = 1$, this inequality directly follows from the embedding $H^1(M) \hookrightarrow L^\infty(M)$. Therefore, we obtain

$$\sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(E_A)}^2 < \infty.$$

for arbitrary dimension d. The properties of B_m as operator in $\mathcal{L}(L^{\alpha+1}(M))$ and in $\mathcal{L}(L^2(M))$ can be deduced from the embedding $F \hookrightarrow L^\infty(M)$.

We close this section by remarks on natural generalizations of the linear, conservative noise considered in this paper which will be worked out in the second author's dissertation.

Remark 3.7. As in [BM14], Section 8, it is possible to replace the linear Stratonovich noise in Theorem 1.1, see also Assumption 2.7, by a nonlinear one of the form

$$B_m(u) := -\mathrm{i} B_m \left(g(|u|^2) u \right), \qquad \mu(u) := -\frac{1}{2} \sum_{m=1}^{\infty} B_m^2 \left(g(|u|^2)^2 u \right),$$

where we assume the Lipschitz and linear growth conditions

$$\|g(|u|^2)^j u\|_{E_A} \lesssim \|u\|_{E_A}, \qquad \|g(|u|^2)^j u\|_{L^p} \lesssim \|u\|_{L^p}, \qquad \|g(|u|^2)^j u - g(|v|^2)^j v\|_{L^p} \lesssim \|u-v\|_{L^p}$$

for $j \in \{1,2\}$ and $p \in \{\alpha+1, 2\}$. In the case of H^1-based energy spaces, i.e. the $A = -\Delta$ on a bounded domain or $A = -\Delta_g$ on a riemannian manifold, one can take $g \in C^2([0,\infty),\mathbb{R})$ which satisfies the following conditions:

$$\sup_{r>0} |g(r)| < \infty, \qquad \sup_{r>0} (1+r)|g'(r)| < \infty, \qquad \sup_{r>0} (1+r^{\frac{3}{2}})|g''(r)| < \infty. \tag{3.5}$$

This kind of nonlinearity is often called saturated and typical examples are given by

$$g_1(r) = \frac{r}{1+\sigma r}, \qquad g_2(r) = \frac{r(2+\sigma r)}{(1+\sigma r)^2}, \qquad g_3(r) = \frac{\log(1+\sigma r)}{1+\log(1+\sigma r)}, \qquad r \in [0,\infty),$$

for a constant $\sigma > 0$. For the Galerkin equation, we then take

$$\begin{cases} \mathrm{d}u_n = \left(-\mathrm{i}Au_n - \mathrm{i}P_n F(u_n) - \frac{1}{2}\sum_{m=1}^{\infty} S_n B_m^2 \left(g(|u_n|^2)^2 u_n \right) \right) \mathrm{d}t - \mathrm{i}\sum_{m=1}^{\infty} S_n B_m (g(|u_n|^2)u_n) \mathrm{d}\beta_m, \\ u_n(0) = P_n u_0. \end{cases}$$

Unfortunately, this approximation does not respect mass conservation, but one still has

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^2 \right] \lesssim 1, \tag{3.6}$$

which is enough for our purpose.

Remark 3.8. Another possible generalization of the noise is to drop the assumption that B_m, $m \in \mathbb{N}$, is selfadjoint. Then, the correction term μ has the form

$$\mu(u) := -\frac{1}{2}\sum_{m=1}^{\infty} B_m^* B_m u.$$

This kind of noise is called non-conservative and was considered in [BRZ16] and [Hor16]. The existence result is then based on the approximation

$$\begin{cases} \mathrm{d}u_n = \left(-\mathrm{i}Au_n - \mathrm{i}P_n F(u_n) - \frac{1}{2}\sum_{m=1}^{\infty} S_n B_m^* B_m u_n \right) \mathrm{d}t - \mathrm{i}\sum_{m=1}^{\infty} S_n B_m u_n \mathrm{d}\beta_m, \\ u_n(0) = P_n u_0, \end{cases}$$

and the a priori estimates as well as the convergence results can be proved analogously. We only have to replace mass conservation by the estimate (3.6). The uniqueness result in section 7, however, only holds for selfadjoint B_m, since this is the crucial assumption in Lemma 7.4.

4. Compactness and Tightness Criteria

This section is devoted to the compactness results which will be used to get a martingale solution of (1.1) by the Faedo-Galerkin method.

Let A and $\alpha > 1$ be chosen according to Assumption 2.1. We recall that the energy space E_A is defined by $E_A := \mathcal{D}(A^{\frac{1}{2}})$. We start with a criterion for convergence of a sequence in $C([0,T],\mathbb{B}^r_{E_A})$, where the ball $\mathbb{B}^r_{E_A})$ is equipped with the weak topology.

Lemma 4.1. *Let $r > 0$ and $(u_n)_{n\in\mathbb{N}} \subset L^\infty(0,T;E_A)$ be a sequence with the properties*

a) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;E_A)} \leq r$,
b) $u_n \to u$ *in* $C([0,T],E_A^*)$ *for* $n\to\infty$.

Then $u_n, u \in C([0,T],\mathbb{B}^r_{E_A})$ *for all* $n\in\mathbb{N}$ *and* $u_n \to u$ *in* $C([0,T],\mathbb{B}^r_{E_A})$ *for* $n\to\infty$.

Proof. The Strauss-Lemma $A.3$ and the assumptions guarantee that

$$u_n \in C([0,T],E_A^*)\cap L^\infty(0,T;E_A)\subset C_w([0,T],E_A)$$

for all $n\in\mathbb{N}$ and $\sup_{t\in[0,T]}\|u(t)\|_{E_A}\leq r$. Hence, we infer that $u_n\in C([0,T],\mathbb{B}^r_{E_A})$ for all $n\in\mathbb{N}$. For $h\in E_A$

$$\sup_{s\in[0,T]} |\langle u_n(s)-u(s),h\rangle| \leq \|u_n-u\|_{C([0,T],E_A^*)}\|h\|_{E_A}\to 0, \qquad n\to\infty.$$

By a) and Banach-Alaoglu, we get a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $v\in L^\infty(0,T;E_A)$ with $u_{n_k}\rightharpoonup^* v$ in $L^\infty(0,T;E_A)$ and by the uniqueness of the weak star limit in $L^\infty(0,T;E_A^*)$, we conclude $u=v\in L^\infty(0,T;E_A)$ with $\|u\|_{L^\infty(0,T;E_A)}\leq r$.
Let $\varepsilon>0$ and $h\in E_A^*$. By the density of E_A in E_A^*, we choose $h_\varepsilon\in E_A$ with $\|h-h_\varepsilon\|_{E_A^*}\leq\frac{\varepsilon}{4r}$ and obtain for large $n\in\mathbb{N}$

$$\begin{aligned}|\langle u_n(s)-u(s),h\rangle| &\leq |\langle u_n(s)-u(s),h-h_\varepsilon\rangle| + |\langle u_n(s)-u(s),h_\varepsilon\rangle|\\ &\leq \|u_n(s)-u(s)\|_{E_A}\|h-h_\varepsilon\|_{E_A^*} + |\langle u_n(s)-u(s),h_\varepsilon\rangle|\\ &\leq 2r\frac{\varepsilon}{4r}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

independant of $s\in[0,T]$. This implies $\sup_{s\in[0,T]}|\langle u_n(s)-u(s),h\rangle|\to 0$ for $n\to\infty$ and all $h\in E_A^*$, i.e. $u_n\to u$ in $C_w([0,T],E_A)$. By Lemma A.2, we obtain the assertion. □

We define a Banach space $\tilde{Z}_T$ by

$$\tilde{Z}_T := C([0,T],E_A^*)\cap L^{\alpha+1}(0,T;L^{\alpha+1}(M))$$

and a locally convex space Z_T by

$$Z_T := \tilde{Z}_T\cap C_w([0,T],E_A).$$

The latter is equipped with the Borel σ-algebra, i.e. the σ-algebra generated by the open sets in the locally convex topology of Z_T. In the next Proposition, we give a criterion for compactness in Z_T.

Proposition 4.2. *Let* K *be a subset of* Z_T *and* $r>0$ *such that*

a) $\sup_{u\in K}\|u\|_{L^\infty(0,T;E_A)}\leq r$;
b) K *is equicontinuous in* $C([0,T],E_A^*)$, *i.e.*

$$\lim_{\delta\to 0}\sup_{u\in K}\sup_{|t-s|\leq\delta}\|u(t)-u(s)\|_{E_A^*}=0.$$

Then, K *is relatively compact in* Z_T.

Proof. Let K be a subset of Z_T such that the assumptions $a)$ and $b)$ are fullfilled and $(z_n)_{n\in\mathbb{N}}\subset K$. We want to construct a subsequence converging in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$, $C([0,T],E_A^*)$ and $C_w([0,T],E_A)$.

Step 1: By $a)$, we can choose a constant $C>0$ and for each $n\in\mathbb{N}$ a null set I_n with $\|z_n(t)\|_{E_A}\leq C$ for all $t\in[0,T]\setminus I_n$. The set $I:=\bigcup_{n\in\mathbb{N}} I_n$ is also a nullset and for each $t\in[0,T]\setminus I$, the sequence $(z_n(t))_{n\in\mathbb{N}}$ is bounded in E_A.

Let $(t_j)_{j\in\mathbb{N}} \subset [0,T]\setminus I$ be a sequence, which is dense in $[0,T]$. By Lemma 2.3, the embedding $E_A \hookrightarrow H$ is compact, which yields that $E_A \hookrightarrow E_A^*$ is also compact. Therefore, we can choose for each $j\in\mathbb{N}$ a Cauchy subsequence in E_A^* again denoted by $(z_n(t_j))_{n\in\mathbb{N}}$. By a diagonalisation argument, one obtains a common Cauchy subsequence $(z_n(t_j))_{k\in\mathbb{N}}$.
Let $\varepsilon>0$. Assumption $b)$ yields $\delta>0$ with

$$\sup_{u\in K}\sup_{|t-s|\le\delta}\|u(t)-u(s)\|_{E_A^*}\le\frac{\varepsilon}{3}.$$

Now, we fix $t\in[0,T]$ and take $j\in\mathbb{N}$ with $|t_j-t|\le\delta$ and n,m large enough to have

$$\|z_n(t_j)-z_m(t_j)\|_{E_A^*}\le\frac{\varepsilon}{3}.$$

We conclude

$$\|z_n(t)-z_m(t)\|_{E_A^*}\le\|z_n(t)-z_n(t_j)\|_{E_A^*}+\|z_n(t_j)-z_m(t_j)\|_{E_A^*}+\|z_m(t_j)-z_m(t)\|_{E_A^*}\le\varepsilon,$$

which means that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T],E_A^*)$.

Step 2: The first step yields $z\in C([0,T],E_A^*)$ with $z_n\to z$ in $C([0,T],E_A^*)$ for $n\to\infty$ and assumption a) implies, that there is $r>0$ with $\sup_{n\in\mathbb{N}}\|z_n\|_{L^\infty(0,T;E_A)}\le r$.
Therefore, we obtain $z\in C([0,T],\mathbb{B}^r_{E_A})$ and $z_n\to z$ in $C([0,T],\mathbb{B}^r_{E_A})$ for $n\to\infty$ by Lemma 4.1. Hence, $z_n\to z$ in $C_w([0,T],E_A)$.

Step 3: We fix again $\varepsilon>0$. By the Lions Lemma $A.4$ with $X_0=E_A$, $X=L^{\alpha+1}(M)$, $X_1=E_A^*$, $p=\alpha+1$ and $\varepsilon_0=\frac{\varepsilon}{2T(2C)^{\alpha+1}}$ we get

$$\|v\|^{\alpha+1}_{L^{\alpha+1}(M)}\le\varepsilon_0\|v\|^{\alpha+1}_{E_A}+C_{\varepsilon_0}\|v\|^{\alpha+1}_{E_A^*} \tag{4.1}$$

for all $v\in E_A$. The first step allows us to choose $n,m\in\mathbb{N}$ large enough that

$$\|z_n-z_m\|^{\alpha+1}_{C([0,T],E_A^*)}\le\frac{\varepsilon}{2C_{\varepsilon_0}T}$$

The special choice $v=z_n(t)-z_m(t)$ for $t\in[0,T]$ in (4.1) and integration with respect to time yields

$$\begin{aligned}\|z_n-z_m\|^{\alpha+1}_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}&\le\varepsilon_0\|z_n-z_m\|^{\alpha+1}_{L^{\alpha+1}(0,T;E_A)}+C_{\varepsilon_0}\|z_n-z_m\|^{\alpha+1}_{L^{\alpha+1}(0,T;E_A^*)}\\&\le\varepsilon_0T\|z_n-z_m\|^{\alpha+1}_{L^\infty(0,T;E_A)}+C_{\varepsilon_0}T\|z_n-z_m\|^{\alpha+1}_{C([0,T],E_A^*)}\\&\le\varepsilon_0T\,(2C)^{\alpha+1}+C_{\varepsilon_0}T\|z_n-z_m\|^{\alpha+1}_{C([0,T],E_A^*)}\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence, the sequence $(z_n)_{n\in\mathbb{N}}$ is also Cauchy in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$. □

In the following, we want to obtain a criterion for tightness in Z_T. Therefore, we introduce the Aldous condition.

Definition 4.3. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E. Assume that for every $\varepsilon>0$ and $\eta>0$ there is $\delta>0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times one has

$$\sup_{n\in\mathbb{N}}\sup_{0<\theta\le\delta}\mathbb{P}\left\{\|X_n((\tau_n+\theta)\wedge T)-X_n(\tau_n)\|_E\ge\eta\right\}\le\varepsilon.$$

In this case, we say that $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$.

The following Lemma (see [Mot12], Lemma A.7) gives us a useful consequence of the Aldous condition $[A]$.

Lemma 4.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E, which satisfies the Aldous condition $[A]$. Then, for every $\varepsilon > 0$ there exists a measurable subset $A_\varepsilon \subset C([0,T],E)$ such that*

$$\mathbb{P}^{X_n}(A_\varepsilon) \geq 1-\varepsilon, \qquad \lim_{\delta\to 0}\sup_{u\in A_\varepsilon}\sup_{|t-s|\leq\delta}\|u(t)-u(s)\|_E = 0.$$

The deterministic compactness result in Proposition 4.2 and the last Lemma can be used to get the following criterion for tightness in Z_T.

Proposition 4.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;E_A)}\right] < \infty.$$

Then the sequence $\left(\mathbb{P}^{X_n}\right)_{n\in\mathbb{N}}$ is tight in Z_T, i.e. for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset Z_T$ with

$$\mathbb{P}^{X_n}(K_\varepsilon) \geq 1-\varepsilon$$

for all $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$. With $R_1 := \left(\frac{2}{\varepsilon}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;E_A)}\right]\right)^{\frac{1}{2}}$, we obtain

$$\mathbb{P}\left\{\|X_n\|_{L^\infty(0,T;E_A)} > R_1\right\} \leq \frac{1}{R_1^2}\mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;E_A)}\right] \leq \frac{\varepsilon}{2}.$$

By Lemma 4.4, one can use the Aldous condition $[A]$ to get a Borel subset A of $C([0,T],E_A^*)$ with

$$\mathbb{P}^{X_n}(A) \geq 1-\frac{\varepsilon}{2}, \quad n\in\mathbb{N}, \qquad \lim_{\delta\to 0}\sup_{u\in A}\sup_{|t-s|\leq\delta}\|u(t)-u(s)\|_{E_A^*} = 0.$$

We define $K := \overline{A\cap B}$ where $B := \left\{u\in Z_T : \|u\|_{L^\infty(0,T;E_A)} \leq R_1\right\}$. This set K is compact by Proposition 4.2 and we can estimate

$$\mathbb{P}^{X_n}(K) \geq \mathbb{P}^{X_n}(A\cap B) \geq \mathbb{P}^{X_n}(A) - \mathbb{P}^{X_n}(B^c) \geq 1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} = 1-\varepsilon$$

for all $n \in \mathbb{N}$. □

In metric spaces, one can apply Prokhorov Theorem (see [Par67], Theorem II.6.7) and Skorohod Theorem (see [Bil99], Theorem 6.7.) to obtain convergence from tightness. Since the space Z_T is a locally convex space, we use the following generalization to nonmetric spaces.

Proposition 4.6 (Skorohod-Jakubowski)**.** *Let $\mathcal{X}$ be a topological space such that there is a sequence of continuous functions $f_m : \mathcal{X} \to \mathbb{C}$ that separates points of $\mathcal{X}$. Let $\mathcal{A}$ be the σ-algebra generated by $(f_m)_m$. Then, we have the following assertions:*

a) *Every compact set $K \subset \mathcal{X}$ is metrizable.*
b) *Let $(\mu_n)_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $(\mathcal{X},\mathcal{A})$. Then, there are a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, random variables X_k, X for $k\in\mathbb{N}$ on a common probability space $(\tilde\Omega,\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ with $\tilde{\mathbb{P}}^{X_k} = \mu_{n_k}$ for $k\in\mathbb{N}$, and $X_k \to X$ $\tilde{\mathbb{P}}$-almost surely for $k\to\infty$.*

We stated Proposition 4.6 in the form of [BO11] (see also [Jak98]) where it was first used to construct martingale solutions for stochastic evolution equations. We apply this result to the concrete situation and obtain the final result of this section.

Corollary 4.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;E_A)}\right] < \infty.$$

Then, there are a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ and random variables $\tilde{X}_k$, $\tilde{X}$ for $k\in\mathbb{N}$ on a second probability space $(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ with $\tilde{\mathbb{P}}^{\tilde{X}_k}=\mathbb{P}^{X_{n_k}}$ for $k\in\mathbb{N}$, and $\tilde{X}_k\to\tilde{X}$ $\tilde{\mathbb{P}}$-almost surely in Z_T for $k\to\infty$.

Proof. We recall that $Z_T = C([0,T],E_A^*)\cap L^{\alpha+1}(0,T;L^{\alpha+1}(M))\cap C_w([0,T],E_A)$ is a locally convex space. Therefore, the assertion follows by an application of the Propositions 4.5 and 4.6 if for each of the spaces in the definition of Z_T we find a sequence $f_m : Z_T\to\mathbb{R}$ of continuous functions separating points which generates the Borel σ-algebra. The separable Banach spaces $C([0,T],E_A^*)$ and $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ have this property.
Let $\{h_m : m\in\mathbb{N}\}$ be a dense subset of E_A^*. Then, we define the countable set
$F := \{f_{m,t} : m\in\mathbb{N}, t\in[0,T]\cap\mathbb{Q}\}$ of functionals on $C_w([0,T],E_A)$ by

$$f_{m,t}(u) := \langle u(t), h_m\rangle$$

for $m\in\mathbb{N}$, $t\in[0,T]\cap\mathbb{Q}$ and $u\in C_w([0,T],E_A)$.
The set F separates points, since for $u,v\in C_w([0,T],E_A)$ with $f_{m,t}(u)=f_{m,t}(v)$ for all $m\in N$ and $t\in[0,T]\cap\mathbb{Q}$, we get $\langle u,h_m\rangle=\langle v,h_m\rangle$ on $[0,T]$ for all $m\in\mathbb{N}$ by continuous continuation and therefore $u=v$ on $[0,T]$.
Furthermore, the density of $\{h_m : m\in\mathbb{N}\}$ and the definition of the locally convex topology yield that $(f_{m,t})_{m\in\mathbb{N},t\in[0,T]\cap\mathbb{Q}}$ generate the Borel σ-algebra on $C_w([0,T],E_A)$. □

5. The Galerkin Approximation

In this section, we introduce the Galerkin approximation, which will be used for the proof of the existence of a solution to (1.1). We prove the well-posedness of the approximated equation and uniform estimates for the solutions that are sufficient to apply Corollary 4.7.

Recall from Lemma 2.3, that S has the representation

$$Sx=\sum_{m=1}^{\infty}\lambda_m\big(x,h_m\big)_H h_m,\quad x\in\mathcal{D}(S)=\left\{x\in H:\sum_{m=1}^{\infty}\lambda_m^2|\big(x,h_m\big)_H|^2<\infty\right\},$$

with an orthonormal basis $(h_m)_{m\in\mathbb{N}}$, $\lambda_m>0$ and $\lambda_m\to\infty$ as $m\to\infty$. For $n\in\mathbb{N}_0$, we set

$$H_n := \operatorname{span}\left\{h_m : m\in\mathbb{N},\lambda_m<2^{n+1}\right\}$$

and denote the orthogonal projection from H to H_n by P_n, i.e.

$$P_n x=\sum_{\lambda_m<2^{n+1}}\big(x,h_m\big)_H h_m,\qquad x\in H.$$

Note that $P_n=\mathbf{1}_{(0,2^{n+1})}(S)$ and hence, P_n and $A^{\frac12}$ commute, since S and A commute by Assumption (2.1). We obtain

$$\|P_nx\|^2_{E_A}=\|P_nx\|^2_H+\|A^{\frac12}P_nx\|^2_H=\|P_nx\|^2_H+\|P_nA^{\frac12}x\|^2_H\le\|x\|^2_{E_A},\qquad x\in E_A. \tag{5.1}$$

and

$$\|P_n v\|_{E_A^*} = \sup_{\|x\|_{E_A}\leq 1} |\big(P_n v, x\big)_H| \leq \|v\|_{E_A^*} \sup_{\|x\|_{E_A}\leq 1} \|P_n x\|_{E_A} \leq \|v\|_{E_A^*}$$

By density, we can extend P_n to an operator $P_n : E_A^* \to H_n$ with $\|P_n\|_{E_A^*\to E_A^*} \leq 1$ with

$$\langle v, P_n v\rangle \in \mathbb{R}, \qquad \langle v, P_n w\rangle = \big(P_n v, w\big)_H, \qquad v \in E_A^*, \quad w \in E_A. \tag{5.2}$$

Despite their nice behaviour as orthogonal projections, it turns out that the operators P_n, $n \in \mathbb{N}$, lack the crucial property needed in the proof of the a priori estimates of the stochastic terms. In general, they are not uniformly bounded from $L^{\alpha+1}(M)$ to $L^{\alpha+1}(M)$. To overcome this deficit, we construct another sequence $(S_n)_{n\in\mathbb{N}}$ of operators $S_n : H \to H_n$ using functional calculus techniques and the general Littlewood-Paley decomposition from [KW16].

We take a function $\dot\rho \in C_c^\infty(0,\infty)$ with $\operatorname{supp}\dot\rho \subset [\frac{1}{2}, 2]$ and $\sum_{m\in\mathbb{Z}} \dot\rho(2^{-m}t) = 1$ for all $t > 0$. We define $\rho_m = \dot\rho(2^{-m}\cdot)$ for $m \in \mathbb{N}$ and $\rho_0 := \sum_{m=-\infty}^0 \dot\rho(2^{-m}\cdot)$, so that we have $\sum_{m=0}^\infty \rho_m(t) = 1$ for all $t > 0$. The sequence $(\rho_m)_{m\in\mathbb{N}_0}$ is called *dyadic partition of unity*.

Lemma 5.1. *We have the norm equivalence*

$$\|x\|_{L^{\alpha+1}(M)} \eqsim \sup_{\|a\|_{l^\infty(\mathbb{N}_0)}\leq 1} \left\| \sum_{m=0}^\infty a_m \rho_m(S) x \right\|_{L^{\alpha+1}(M)}, \tag{5.3}$$

where the operators ρ_m, $m \in \mathbb{N}$, are defined by the functional calculus for selfadjoint operators.

Proof. By Assumption 2.1 ii), we obtain that the restriction of $(T(t))_{t\geq 0}$ to $L^{\alpha+1}(M)$ defines a c_0-semigroup on $L^{\alpha+1}(M)$, see Theorem 7.1. in [Ouh09]. We denote the corresponding generator by $S_{\alpha+1}$. Lemma 6.1. in [KW16] implies that the operator $S_{\alpha+1}$ is 0-sectorial and has a Mihlin M^β-calculus for some $\beta > 0$. For a definition of these properties, we refer to [KW16], Section 2. The estimate (5.3) follows from Theorem 4.1 in [KW16].

□

In the next Proposition, we use the estimate from Lemma 5.1 to construct the sequence $(S_n)_{n\in\mathbb{N}}$ which we will employ in our Galerkin approximation of the problem (1.1).

Proposition 5.2. *There exists a sequence $(S_n)_{n\in\mathbb{N}_0}$ of selfadjoint operators $S_n : H \to H_n$ for $n \in \mathbb{N}_0$ with $S_n\psi \to \psi$ in E_A for $n \to \infty$ and $\psi \in E_A$ and the uniform norm estimates*

$$\sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(H)} \leq 1, \quad \sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(E_A)} \leq 1, \quad \sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(L^{\alpha+1})} < \infty. \tag{5.4}$$

Proof. We fix $n \in \mathbb{N}$ and define the operators $S_n : H \to H$ for $n \in \mathbb{N}_0$ by $S_n := \sum_{m=0}^n \rho_m(S)$ via the functional calculus for selfadjoint operators. The operator $\rho_m(S)$ is selfadjoint for each m, since ρ_m is real-valued. Hence, S_n is selfadjoint. By the convergence property of the functional calculus, we get $S_n\varphi \to \varphi$ in E_A for all $\varphi \in E_A$. A straightforward calculation using the properties of the dyadic partition of unity leads to

$$S_n x = \sum_{\lambda_m < 2^n} \big(x, h_m\big)_H h_m + \sum_{\lambda_m \in [2^n, 2^{n+1})} \rho_n(\lambda_m)\big(x, h_m\big)_H h_m, \quad u \in H.$$

Therefore, S_n maps H to H_n and we have $\sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(H)} \leq 1$. The second estimate in (5.4) can be derived as in (5.1), since S_n and $A^{\frac{1}{2}}$ commute. To prove the third estimate, we

employ Lemma 5.2 with $(a_m)_{m\in\mathbb{N}_0}$ as $a_m = 1$ for $m \le n$ and $a_m = 0$ for $m > n$ and obtain for $x \in L^{\alpha+1}(M)$

$$\|S_n x\|_{L^{\alpha+1}(M)} = \left\| \sum_{m=0}^{\infty} a_m \rho_m(S) x \right\|_{L^{\alpha+1}(M)} \le \sup_{\|a\|_{l^\infty(\mathbb{N}_0)}\le 1} \left\| \sum_{m=0}^{\infty} a_m \rho_m(S) x \right\|_{L^{\alpha+1}(M)} \lesssim \|x\|_{L^{\alpha+1}(M)}.$$

□

Using the operators P_n and S_n, $n \in \mathbb{N}$, we approximate our original problem (1.1) by the stochastic differential equation in H_n given by

$$\begin{cases} \mathrm{d}u_n(t) = (-\mathrm{i}Au_n(t) - \mathrm{i}P_nF(u_n(t)))\,\mathrm{d}t - \mathrm{i}S_nB(S_nu_n(t)) \circ \mathrm{d}W(t), \\ u_n(0) = P_nu_0. \end{cases}$$

With the Stratonovich correction term

$$\mu_n := -\frac{1}{2}\sum_{m=1}^{\infty} (S_nB_mS_n)^2,$$

the approximated problem can also be written in the Itô form

$$\begin{cases} \mathrm{d}u_n(t) = (-\mathrm{i}Au_n(t) - iP_nF(u_n(t)) + \mu_n(u_n(t)))\,\mathrm{d}t - \mathrm{i}S_nB(S_nu_n(t))\mathrm{d}W(t), \\ u_n(0) = P_nu_0. \end{cases} \tag{5.5}$$

By the well known theory for finite dimensional stochastic differential equations with locally Lipschitz coefficients, we get a local wellposedness result for (5.5).

Proposition 5.3. *For each $n \in \mathbb{N}$, there is a unique local solution u_n of* (5.5) *with continuous paths in H_n and maximal existence time τ_n, which is a blow-up time in the sense that we have $\limsup_{t\nearrow\tau_n(\omega)} \|u_n(t,\omega)\|_{H_n} = \infty$ for almost all $\omega \in \Omega$ with $\tau_n(\omega) < \infty$.*

The global existence for equation (5.5) is based on the conservation of the L^2-norm of solutions.

Proposition 5.4. *For each $n \in \mathbb{N}$, there is a unique global solution u_n of* (5.5) *with continuous paths in H_n and we have the estimate*

$$\|u_n(t)\|_{H_n} = \|u_n(t)\|_H = \|P_nu_0\|_H \le \|u_0\|_H \tag{5.6}$$

almost surely for all $t \ge 0$.

Proof. Step 1: We fix $n \in \mathbb{N}$ and take the unique maximal solution (u_n, τ_n) from Proposition 5.3. We show that the estimate (5.6) holds almost surely on $\{t \le \tau_n\}$. The function $\Phi : H_n \to \mathbb{R}$ defined by $\Phi(v) := \|v\|_H^2$ for $v \in H_n$ is twice continuously Fréchet-differentiable with

$$\Phi'[v]h_1 = 2\,\mathrm{Re}\,\big(v, h_1\big)_H, \qquad \Phi''[v]\,[h_1, h_2] = 2\,\mathrm{Re}\,\big(h_1, h_2\big)_H$$

for $v, h_1, h_2 \in H_n$. For the sequence $(\tau_{n,k})_{k\in\mathbb{N}}$ of stopping times

$$\tau_{n,k} := \inf\{t \in [0, \tau_n] : \|u_n(t)\|_{H_n} \ge k\} \wedge \tau_n, \qquad k \in \mathbb{N},$$

we have $\tau_{n,k} \nearrow \tau_n$ almost surely and the Itô process u_n has the representation

$$u_n(t) = P_nu_0 + \int_0^t [-\mathrm{i}Au_n(s) - \mathrm{i}P_nF(u_n(s)) + \mu_n(u_n(s))]\,\mathrm{d}s - \mathrm{i}\int_0^t S_nB(S_nu_n(s))\mathrm{d}W(s)$$

almost surely on $\{t \leq \tau_{n,k}\}$ for all $k \in \mathbb{N}$. We fix $k \in \mathbb{N}$. Since we have

$$\begin{aligned}
&\operatorname{tr}\Big(\Phi''[u_n(s)]\left(-\mathrm{i}S_nB\left(S_nu_n(s)\right), -\mathrm{i}S_nB\left(S_nu_n(s)\right)\right)\Big)\\
&\qquad=\sum_{m=1}^{\infty} 2\operatorname{Re}\big(-\mathrm{i}S_nB\left(S_nu_n(s)\right)f_m, -\mathrm{i}S_nB\left(S_nu_n(s)\right)f_m\big)_H\\
&\qquad=2\sum_{m=1}^{\infty}\|S_nB_mS_nu_n(s)\|_H^2
\end{aligned}$$

for $s \in \{t \leq \tau_{n,k}\}$, the Itô lemma yields

$$\begin{aligned}
\|u_n(t)\|_H^2 =&\|P_nu_0\|_H^2 + 2\int_0^t \operatorname{Re}\big(u_n(s), -\mathrm{i}Au_n(s) - \mathrm{i}P_nF\left(u_n(s)\right) + \mu_n(u_n(s))\big)_H\mathrm{d}s\\
&+ 2\int_0^t \operatorname{Re}\big(u_n(s), -\mathrm{i}S_nB(S_nu_n(s))\mathrm{d}W(s)\big)_H + \sum_{m=1}^{\infty}\int_0^t \|S_nB_mS_nu_n(s)\|_H^2\mathrm{d}s
\end{aligned}$$

almost surely in $\{t \leq \tau_{n,k}\}$. We fix $v \in H_n$ and $m \in \mathbb{N}$ and calculate

$$\begin{aligned}
\operatorname{Re}\big(v, -\mathrm{i}Av\big)_H &= \operatorname{Re}\Big[\mathrm{i}\|A^{\frac{1}{2}}v\|_H^2\Big] = 0,\\
\operatorname{Re}\big(v, -\mathrm{i}P_nF\left(v\right)\big)_H &= \operatorname{Re}\langle \mathrm{i}v, F\left(v\right)\rangle = 0,\\
2\operatorname{Re}\big(v, \mu_n(v)\big)_H &= -\sum_{m=1}^{\infty}\operatorname{Re}\big(v, \left(S_nB_mS_n\right)^2 v\big)_H = -\sum_{m=1}^{\infty}\|S_nB_mS_nv\|_H^2,
\end{aligned}$$

where we used (5.2) and Assumption 2.4 i) for the second term and the fact, that the operator $S_nB_mS_n$ is selfadjoint for the third term. Analogously, we get

$$\operatorname{Re}\big(v, -\mathrm{i}S_nB(S_nv)f_m\big)_H = \operatorname{Re}\big(v, -\mathrm{i}S_nB_mS_nv\big)_H = \operatorname{Re}\big[\mathrm{i}\big(v, S_nB_mS_nv\big)_H\big] = 0.$$

Thus, we obtain $\|u_n(t)\|_H^2 = \|P_nu_0\|_H^2 \leq \|u_0\|_H^2$ almost surely in $\{t \leq \tau_{n,k}\}$.

Step 2. To show $\tau_n = \infty$ almost surely, we assume the contrary. Therefore, there is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) > 0$ such that $\tau_n(\omega) < \infty$ and $\tau_{n,k}(\omega) \nearrow \tau_n(\omega)$ for all $\omega \in \Omega_0$. Hence, $\tau_{n,k} < \infty$ on Ω_0 and by the continuity of the paths of u_n and the definition of $\tau_{n,k}$, we get $\|u_n(\tau_{n,k}(\omega), \omega)\|_{H_n} = k$ for all $\omega \in \Omega_0$ and $k \in \mathbb{N}$. This is a contradiction to Step 1, where we obtained $\|u_n(t)\|_H \leq \|u_0\|_H$ almost surely in $\{t \leq \tau_{n,k}\}$. Therefore, u_n is a global solution and we have

$$\|u_n(t)\|_{H_n} = \|u_n(t)\|_H = \|P_nu_0\|_H \leq \|u_0\|_H$$

almost surely for all $t \geq 0$. □

The next goal is to find uniform energy estimates for the global solutions of the equation (5.5). Recall that by Assumption 2.4, the nonlinearity F has a real antiderivative denoted by $\hat{F}$.

Definition 5.5. We define the energy $\mathcal{E}(u)$ of $u \in E_A$ by

$$\mathcal{E}(u) := \frac{1}{2}\|A^{\frac{1}{2}}u\|_H^2 + \hat{F}(u), \qquad u \in E_A.$$

Note that $\mathcal{E}(u)$ is welldefined by the embedding $E_A \hookrightarrow L^{\alpha+1}(M)$. In contrast to the uniform L^2-estimate in $[0,\infty)$, we cannot exclude the growth of the energy in an infinity time interval. So, we fix $T>0$ from now on. As a preparation, we formulate a Lemma, which simplifies the arguments, when the Burkholder-Davis-Gundy inequality is used.

Lemma 5.6. *Let $r\in[1,\infty)$, $\varepsilon>0$, $T>0$ and $X\in L^q(\Omega,L^\infty(0,T))$. Then,*

$$\|X\|_{L^r(\Omega,L^2(0,t))}\le \varepsilon\|X\|_{L^r(\Omega,L^\infty(0,t))}+\frac{1}{4\varepsilon}\int_0^t \|X\|_{L^r(\Omega,L^\infty(0,s))}\mathrm{d}s,\qquad t\in[0,T].$$

Proof. By interpolation of $L^2(0,t)$ between $L^\infty(0,t)$ and $L^1(0,t)$ and the elementary inequality $\sqrt{ab}\le \varepsilon a+\frac{1}{4\varepsilon}b$ for $a,b\ge 0$ and $\varepsilon>0$, we obtain

$$\|X\|_{L^2(0,t)}\le \|X\|^{\frac12}_{L^\infty(0,t)}\|X\|^{\frac12}_{L^1(0,t)}\le \varepsilon\|X\|_{L^\infty(0,t)}+\frac{1}{4\varepsilon}\|X\|_{L^1(0,t)}.$$

Now, we take the $L^r(\Omega)$-norm and apply Minkowski's inequality to get

$$\begin{aligned}\|X\|_{L^r(\Omega,L^2(0,t))}&\le \varepsilon\|X\|_{L^r(\Omega,L^\infty(0,t))}+\frac{1}{4\varepsilon}\int_0^t\|X(s)\|_{L^r(\Omega)}\mathrm{d}s\\&\le \varepsilon\|X\|_{L^r(\Omega,L^\infty(0,t))}+\frac{1}{4\varepsilon}\int_0^t\|X\|_{L^r(\Omega,L^\infty(0,s))}\mathrm{d}s.\end{aligned}$$

□

The next Proposition is the key step to show that we can apply Corollary 4.7 to the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the equation (5.5) in the defocusing case.

Proposition 5.7. *Under Assumption 2.6 i), the following assertions hold.*

a) *For all $q\in[1,\infty)$ there is a constant $C=C(q,\|u_0\|_{E_A},\alpha,F,(B_m)_{m\in\mathbb{N}},T)>0$ with*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,T]}\big[\|u_n(t)\|_H^2+\mathcal{E}(u_n(t))\big]^q\Big]\le C$$

In particular, for all $r\in[1,\infty)$ there is $C_1=C_1(r,\|u_0\|_{E_A},\alpha,F,(B_m)_{m\in\mathbb{N}},T)>0$

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_n(t)\|_{E_A}^r\Big]\le C_1.$$

b) *The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$ in E_A^*.*

Proof. ad a): By Assumption 2.4 ii) and iii), the restriction of the energy $\mathcal{E}:H_n\to\mathbb{R}$ is twice continuously Fréchet-differentiable with

$$\begin{aligned}\mathcal{E}'[v]h_1&=\mathrm{Re}\langle Av+F(v),h_1\rangle;\\ \mathcal{E}''[v]\,[h_1,h_2]&=\mathrm{Re}\left(A^{\frac12}h_1,A^{\frac12}h_2\right)_H+\mathrm{Re}\langle F'[v]h_2,h_1\rangle\end{aligned}$$

for $v,h_1,h_2\in H_n$. We compute

$$\begin{aligned}&\mathrm{tr}\left(\mathcal{E}''[u_n(s)]\left(-\mathrm{i}S_nB\left(S_nu_n(s)\right),-\mathrm{i}S_nB\left(S_nu_n(s)\right)\right)\right)\\&\quad=\sum_{m=1}^\infty\mathcal{E}''[u_n(s)]\left(-\mathrm{i}S_nB_mS_nu_n(s),-\mathrm{i}S_nB_mS_nu_n(s)\right)\\&\quad=\sum_{m=1}^\infty\|A^{\frac12}S_nB_mS_nu_n(s)\|_H^2\mathrm{d}s+\sum_{m=1}^\infty\mathrm{Re}\langle F'[u_n(s)]\left(S_nB_mS_nu_n(s)\right),S_nB_mS_nu_n(s)\rangle\end{aligned}$$

and therefore, Itô's formula and Proposition 5.4 lead to the identity

$$\begin{aligned}\|u_n(t)\|_H^2 + \mathcal{E}\left(u_n(t)\right) =& \|P_n u_0\|_H^2 + \mathcal{E}\left(P_n u_0\right) \\ &+ \int_0^t \mathrm{Re}\langle Au_n(s) + F(u_n(s)), -\mathrm{i}Au_n(s) - \mathrm{i}P_n F(u_n(s))\rangle \mathrm{d}s \\ &+ \int_0^t \mathrm{Re}\langle Au_n(s) + F(u_n(s)), \mu_n(u_n(s))\rangle \mathrm{d}s \\ &+ \int_0^t \mathrm{Re}\langle Au_n(s) + F(u_n(s)), -\mathrm{i}S_n B\left(S_n u_n(s)\right) \mathrm{d}W(s)\rangle \\ &+ \frac{1}{2}\sum_{m=1}^{\infty} \int_0^t \|A^{\frac{1}{2}} S_n B_m S_n u_n(s)\|_H^2 \mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(s)]\left(S_n B_m S_n u_n(s)\right), S_n B_m S_n u_n(s)\rangle \mathrm{d}s \end{aligned} \tag{5.7}$$

almost surely for all $t \in [0,T]$. We can use (5.2) for

$$\mathrm{Re}\langle F(v), -\mathrm{i}P_n F(v)\rangle = \mathrm{Re}\left[\mathrm{i}\langle F(v), P_n F(v)\rangle\right] = 0;$$

$$\mathrm{Re}\left[\langle Av, -\mathrm{i}P_n F(v)\rangle + \langle F(v), -\mathrm{i}Av\rangle\right] = \mathrm{Re}\left[-\langle Av, \mathrm{i}F(v)\rangle + \overline{\langle Av, \mathrm{i}F(v)\rangle}\right] = 0;$$

$$\mathrm{Re}\left(Av, -\mathrm{i}Av\right)_H = \mathrm{Re}\left[\mathrm{i}\|Av\|_H^2\right] = 0$$

for all $v \in H_n$ to simplify (5.7) and get

$$\begin{aligned}\|u_n(t)\|_H^2 + \mathcal{E}\left(u_n(t)\right) =& \|P_n u_0\|_H^2 + \mathcal{E}\left(P_n u_0\right) + \int_0^t \mathrm{Re}\langle Au_n(s) + F(u_n(s)), \mu_n(u_n(s))\rangle \mathrm{d}s \\ &+ \int_0^t \mathrm{Re}\langle Au_n(s) + F(u_n(s)), -\mathrm{i}S_n B\left(S_n u_n(s)\right) \mathrm{d}W(s)\rangle \\ &+ \frac{1}{2}\sum_{m=1}^{\infty} \int_0^t \|A^{\frac{1}{2}} S_n B_m S_n u_n(s)\|_H^2 \mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(s)]\left(S_n B_m S_n u_n(s)\right), S_n B_m S_n u_n(s)\rangle \mathrm{d}s \end{aligned} \tag{5.8}$$

almost surely for all $t \in [0,T]$. Next, we fix $\delta > 0$, $q > 1$ and apply the Itô formula to the process on the LHS of (5.8) and the function $\Phi : (-\frac{\delta}{2}, \infty) \to \mathbb{R}$ defined by $\Phi(x) := (x+\delta)^q$. The derivatives are given by

$$\Phi'(x) = q\left(x+\delta\right)^{q-1}, \qquad \Phi''(x) = q(q-1)\left(x+\delta\right)^{q-2}, \quad x \in \left(-\frac{\delta}{2}, \infty\right).$$

With the short notation

$$Y(s) := \delta + \|u_n(s)\|_H^2 + \mathcal{E}\left(u_n(s)\right), \qquad s \in [0,T],$$

we obtain

$$Y(t)^q = \left[\delta + \|P_n u_0\|_H^2 + \mathcal{E}\left(P_n u_0\right)\right]^q + q\int_0^t Y(s)^{q-1}\, \mathrm{Re}\langle Au_n(s) + F(u_n(s)), \mu_n(u_n(s))\rangle \mathrm{d}s$$

$$
\begin{aligned}
&+ q \int_0^t Y(s)^{q-1} \operatorname{Re}\langle Au_n(s) + F(u_n(s)), -\mathrm{i}S_n B\left(S_n u_n(s)\right) \mathrm{d}W(s)\rangle \\
&+ \frac{q}{2} \sum_{m=1}^{\infty} \int_0^t Y(s)^{q-1} \|A^{\frac{1}{2}} S_n B_m S_n u_n(s)\|_H^2 \mathrm{d}s \\
&+ \frac{q}{2} \sum_{m=1}^{\infty} \int_0^t Y(s)^{q-1} \operatorname{Re}\langle F'[u_n(s)]\left(S_n B_m S_n u_n(s)\right), S_n B_m S_n u_n(s)\rangle \mathrm{d}s \\
&+ \frac{q}{2}(q-1) \sum_{m=1}^{\infty} \int_0^t Y(s)^{q-2} \left[\operatorname{Re}\langle Au_n(s) + F(u_n(s)), -\mathrm{i}S_n B_m S_n u_n(s)\rangle\right]^2 \mathrm{d}s
\end{aligned} \tag{5.9}
$$

almost surely for all $t \in [0,T]$. In order to treat the stochastic integral, we use Propositions 5.2 and 5.4 to estimate for fixed $s \in [0,T]$

$$
\begin{aligned}
|\left(Au_n(s), -\mathrm{i}S_n B_m S_n u_n(s)\right)_H| &\le \|A^{\frac{1}{2}} u_n(s)\|_H \|A^{\frac{1}{2}} S_n B_m S_n u_n(s)\|_H \\
&\le \|A^{\frac{1}{2}} u_n(s)\|_H \|S_n B_m S_n u_n(s)\|_{E_A} \\
&\le \|A^{\frac{1}{2}} u_n(s)\|_H \|S_n\|_{\mathcal{L}(E_A)}^2 \|B_m\|_{\mathcal{L}(E_A)} \|u_n(s)\|_{E_A} \\
&\le \left(\|u_n(s)\|_H^2 + \|A^{\frac{1}{2}} u_n(s)\|_H^2\right) \|B_m\|_{\mathcal{L}(E_A)} \\
&\lesssim Y(s) \|B_m\|_{\mathcal{L}(E_A)}
\end{aligned} \tag{5.10}
$$

and (2.4), (2.9) and Proposition 5.2 to estimate

$$
\begin{aligned}
|\langle F(u_n(s)), -\mathrm{i}S_n B_m S_n u_n(s)\rangle| &\le \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \|S_n B_m S_n u_n(s)\|_{L^{\alpha+1}(M)} \\
&\le \|u_n(s)\|_{L^{\alpha+1}(M)}^{\alpha+1} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2 \|B_m\|_{\mathcal{L}(L^{\alpha+1})} \\
&\lesssim \hat{F}(u_n(s)) \|B_m\|_{\mathcal{L}(L^{\alpha+1})} \\
&\lesssim Y(s) \|B_m\|_{\mathcal{L}(L^{\alpha+1})}.
\end{aligned} \tag{5.11}
$$

The Burkholder-Gundy-Davis inequality, the estimates (5.10) and (5.11), Assumption 2.7 and Lemma 5.6 applied to the process $X = Y^q$ with $r = 1$ yield for any $\varepsilon > 0$

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]} \left| \int_0^s Y(r)^{q-1} \operatorname{Re}\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_n B\left(S_n u_n(r)\right) \mathrm{d}W(r)\rangle \right| \Big] \\
&\lesssim \mathbb{E}\Big[\left(\int_0^t \sum_{m=1}^{\infty} \left| Y(r)^{q-1} \langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_n B_m S_n u_n(r)\rangle \right|^2 \mathrm{d}r \right)^{\frac{1}{2}} \Big] \\
&\lesssim \mathbb{E}\left[\left(\int_0^t Y(r)^{2q} \mathrm{d}r \right)^{\frac{1}{2}} \right] \le \varepsilon \mathbb{E}\Big[\sup_{s\in[0,t]} Y(s)^q \Big] + \frac{1}{4\varepsilon} \int_0^t \mathbb{E}\Big[\sup_{r\in[0,s]} Y(r)^q \Big] \mathrm{d}s
\end{aligned} \tag{5.12}
$$

The integrands of the deterministic integrals can be estimated by using the bounds (5.4), Proposition 5.4 for the linear and (2.4) as well as (2.9) for the nonlinear part. We fix $s \in [0,T]$ and get

$$
\begin{aligned}
\operatorname{Re}\left(Au_n(s), \left(S_n B_m S_n\right)^2 u_n(s)\right)_H &\le \|A^{\frac{1}{2}} u_n(s)\|_H \|A^{\frac{1}{2}} \left(S_n B_m S_n\right)^2 u_n(s)\|_H \\
&\le \|A^{\frac{1}{2}} u_n(s)\|_H \| \left(S_n B_m S_n\right)^2 u_n(s)\|_{E_A}
\end{aligned}
$$

$$\begin{aligned}
&\leq \|A^{\frac{1}{2}}u_n(s)\|_H \|S_n\|^4_{\mathcal{L}(E_A)} \|B_m\|^2_{\mathcal{L}(E_A)} \|u_n(s)\|_{E_A} \\
&\leq \left(\|u_n(s)\|^2_H + \|A^{\frac{1}{2}}u_n(s)\|^2_H\right) \|B_m\|^2_{\mathcal{L}(E_A)} \\
&\lesssim Y(s)\|B_m\|^2_{\mathcal{L}(E_A)} \qquad (5.13)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\langle F(u_n(s)), (S_nB_mS_n)^2 u_n(s)\rangle &\leq \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \|(S_nB_mS_n)^2 u_n(s)\|_{L^{\alpha+1}(M)} \\
&\lesssim \|u_n(s)\|^{\alpha+1}_{L^{\alpha+1}(M)} \|S_n\|^4_{\mathcal{L}(L^{\alpha+1})} \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} \\
&\lesssim \hat{F}(u_n(s))\|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} \lesssim Y(s)\|B_m\|^2_{\mathcal{L}(L^{\alpha+1})}; \qquad (5.14)
\end{aligned}$$

$$\begin{aligned}
\|A^{\frac{1}{2}}S_nB_mS_nu_n(s)\|^2_H &\leq \|S_nB_mS_nu_n(s)\|^2_{E_A} \leq \|S_n\|^4_{\mathcal{L}(E_A)} \|B_m\|^2_{\mathcal{L}(E_A)} \|u_n(s)\|^2_{E_A} \\
&\leq \|B_m\|^2_{\mathcal{L}(E_A)} \left(\|u_n(s)\|^2_H + \|A^{\frac{1}{2}}u_n(s)\|^2_H\right) \\
&\lesssim \|B_m\|^2_{\mathcal{L}(E_A)} Y(s) \qquad (5.15)
\end{aligned}$$

for $m \in \mathbb{N}$ and $s \in [0,T]$. By the bounds (5.4) of S_n and the Assumptions (2.6) and (2.9) on the nonlinearity

$$\begin{aligned}
\mathrm{Re}\langle F'[u_n(s)]\,(S_nB_mS_nu_n(s))\,, S_nB_mS_nu_n(s)\rangle &\lesssim \|F'[u_n(s)]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}} \|S_nB_mS_nu_n(s)\|^2_{L^{\alpha+1}(M)} \\
&\leq \|u_n(s)\|^{\alpha+1}_{L^{\alpha+1}(M)} \|S_n\|^4_{\mathcal{L}(L^{\alpha+1})} \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} \\
&\lesssim \hat{F}(u_n(s))\|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} \lesssim Y(s)\|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} \qquad (5.16)
\end{aligned}$$

Substituting the inequalities (5.12) to (5.16), into the identity (5.9), we get for each $t \in [0,T]$

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]} Y(s)^q\Big] &\lesssim_q \left[\delta + \|P_nu_0\|^2_H + \mathcal{E}(P_nu_0)\right]^q + \mathbb{E}\int_0^t \sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(E_A)} Y(s)^q \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t \sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} Y(s)^q \mathrm{d}s \\
&\quad + \varepsilon\mathbb{E}\Big[\sup_{r\in[0,t]} Y(s)^q\Big] + \frac{1}{4\varepsilon}\int_0^t \mathbb{E}\Big[\sup_{s\in[0,r]} Y(s)^q\Big]\mathrm{d}r \\
&\quad + \mathbb{E}\sum_{m=1}^\infty \int_0^t \|B_m\|^2_{\mathcal{L}(E_A)} Y(s)^q \mathrm{d}s + \mathbb{E}\int_0^t \sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} Y(s)^q \mathrm{d}s \\
&\quad + \mathbb{E}\int^t Y(s)^q \sum_{m=1}^\infty \max\{\|B_m\|^2_{\mathcal{L}(E_A)}, \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})}\}\mathrm{d}s \\
&\lesssim \left[\delta + \|u_0\|^2_H + \mathcal{E}(P_nu_0)\right]^q + \mathbb{E}\int_0^t Y(s)^q \mathrm{d}s \\
&\quad + \varepsilon\mathbb{E}\Big[\sup_{r\in[0,t]} Y(s)^q\Big] + \frac{1}{4\varepsilon}\int_0^t \mathbb{E}\Big[\sup_{s\in[0,r]} Y(s)^q\Big]\mathrm{d}r \\
&\lesssim_T \left[\delta + \|u_0\|^2_H + \mathcal{E}(P_nu_0)\right]^q + \varepsilon\mathbb{E}\Big[\sup_{r\in[0,t]} Y(s)^q\Big] + \int_0^t \mathbb{E}\Big[\sup_{s\in[0,r]} Y(s)^q\Big]\mathrm{d}r. \qquad (5.17)
\end{aligned}$$

Choosing $\varepsilon > 0$ small enough in inequality (5.17), the Gronwall lemma yields

$$\mathbb{E}\big[\sup_{s\in[0,t]} Y(s)^q\big] \leq C\left[\delta + \|u_0\|_H^2 + \mathcal{E}(P_n u_0)\right]^q e^{Ct}, \qquad t \in [0,T],$$

with a constant $C > 0$, which is uniform in $n \in \mathbb{N}$. Because of

$$\mathcal{E}(P_n u_0) \lesssim \|A^{\frac{1}{2}} P_n u_0\|_H^2 + \|P_n u_0\|_{L^{\alpha+1}(M)}^{\alpha+1} \lesssim \|P_n u_0\|_{E_A}^2 + \|P_n u_0\|_{E_A}^{\alpha+1} \lesssim 1,$$

we obtain the assertion of Proposition 5.7, part a).

ad b): Now, we continue with the proof of the Aldous condition. We have

$$\begin{aligned} u_n(t) - P_n u_0 &= -\,\mathrm{i}\int_0^t A u_n(s)\mathrm{d}s - \mathrm{i}\int_0^t P_n F(u_n(s))\mathrm{d}s + \int_0^t \mu_n(u_n(s))\mathrm{d}s \\ &\quad - \mathrm{i}\int_0^t S_n B(S_n u_n(s))\mathrm{d}W(s) \\ &=: J_1(t) + J_2(t) + J_3(t) + J_4(t) \end{aligned}$$

in H_n almost surely for all $t \in [0,T]$ and therefore

$$\|u_n((\tau_n+\theta)\wedge T) - u_n(\tau_n)\|_{E_A^*} \leq \sum_{k=1}^4 \|J_k((\tau_n+\theta)\wedge T) - J_k(\tau_n)\|_{E_A^*}$$

for each sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times and $\theta > 0$. Hence, we get

$$\mathbb{P}\left\{\|u_n((\tau_n+\theta)\wedge T) - u_n(\tau_n)\|_{E_A^*} \geq \eta\right\} \leq \sum_{k=1}^4 \mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T) - J_k(\tau_n)\|_{E_A^*} \geq \frac{\eta}{4}\right\} \quad (5.18)$$

for a fixed $\eta > 0$. We aim to apply Tschebyscheff's inequality and estimate the expected value of each term in the sum. We use part a) for

$$\begin{aligned} \mathbb{E}\|J_1((\tau_n+\theta)\wedge T) - J_1(\tau_n)\|_{E_A^*} &\leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|Au_n(s)\|_{E_A^*}\mathrm{d}s \leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|A^{\frac{1}{2}}u_n(s)\|_H \mathrm{d}s \\ &\lesssim \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}\big] \leq \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}^2\big]^{\frac{1}{2}} \leq \theta C_1; \end{aligned}$$

the embedding $L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$ and the nonlinear estimates (2.4) and (2.9) for

$$\begin{aligned} \mathbb{E}\|J_2((\tau_n+\theta)\wedge T) - J_2(\tau_n)\|_{E_A^*} &\leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|P_n F(u_n(s))\|_{E_A^*}\mathrm{d}s \\ &\leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{E_A^*}\mathrm{d}s \lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)}\mathrm{d}s \\ &\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|u_n(s)\|_{L^{\alpha+1}(M)}^{\alpha}\mathrm{d}s \lesssim \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}^{\alpha}\big] \leq \theta C_2 \end{aligned}$$

Propositions 5.2 and 5.4 for

$$\mathbb{E}\|J_3((\tau_n+\theta)\wedge T) - J_3(\tau_n)\|_{E_A^*} = \frac{1}{2}\mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\left(S_n B_m S_n\right)^2 u_n(s)\mathrm{d}s\right\|_{E_A^*}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\|\left(S_nB_mS_n\right)^2u_n(s)\|_{E_A^*}\mathrm{d}s\\
&\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\|\left(S_nB_mS_n\right)^2u_n(s)\|_{H}\mathrm{d}s\\
&\leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\|B_m\|^2_{\mathcal{L}(H)}\|u_n(s)\|_{H}\mathrm{d}s\\
&\lesssim \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_H\big]=C_3\theta
\end{aligned}
$$

Finally, we use the Itô isometry and again the Propositions 5.2 and 5.4 for

$$
\begin{aligned}
\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|^2_{E_A^*}&\leq \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}S_nB\left(S_nu_n(s)\right)\mathrm{d}W(s)\right\|_H^2\\
&=\mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|S_nB\left(S_nu_n(s)\right)\|^2_{\mathrm{HS}(Y,H)}\mathrm{d}s\right]\\
&=\mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\|S_nB_mS_nu_n(s)\|^2_H\mathrm{d}s\right]\\
&\leq\mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\|B_m\|^2_{\mathcal{L}(H)}\|u_n(s)\|^2_H\mathrm{d}s\right]\\
&\lesssim\theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|^2_H\big]=\theta C_4
\end{aligned}
$$

By the Tschebyscheff inequality, we obtain for a given $\eta>0$

$$
\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{4}{\eta}\mathbb{E}\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\leq\frac{4C_k\theta}{\eta}\tag{5.19}
$$

for $k\in\{1,2,3\}$ and

$$
\mathbb{P}\left\{\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{16}{\eta^2}\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|^2_{E_A^*}\leq\frac{16C_4\theta}{\eta^2}.\tag{5.20}
$$

Let us fix $\varepsilon>0$. Due to estimates (5.19) and (5.20) we can choose $\delta_1,\dots,\delta_4>0$ such that

$$
\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{\varepsilon}{4}
$$

for $0<\theta\leq\delta_k$ and $k=1,\dots,4$. With $\delta:=\min\{\delta_1,\dots,\delta_4\}$, using (5.18) we get

$$
\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\geq\eta\right\}\leq\varepsilon
$$

for all $n\in\mathbb{N}$ and $0<\theta\leq\delta$ and therefore, the Aldous condition $[A]$ holds in E_A^*. □

We continue with the a priori estimate for solutions of (5.5) with a focusing nonlinearity. Note that this case is harder since the expression

$$
\|v\|_H^2+\mathcal{E}(v):=\|v\|^2_{E_A}+\hat{F}(v),\qquad v\in H_n,
$$

does not dominate $\|v\|^2_{E_A}$, because $\hat{F}$ is negative.

Proposition 5.8. *Under Assumption 2.6 i'), the following assertions hold:*

a) *For all $r \in [1,\infty)$, there is a constant $C = C(r, \|u_0\|_{E_A}, \alpha, F, (B_m)_{m\in\mathbb{N}}, T) > 0$ with*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^r\Big] \le C$$

b) *The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$ in E_A^*.*

Proof. Let $\varepsilon > 0$. Assumption 2.6 i') and Young's inequality imply that there are $\gamma > 0$ and $C_\varepsilon > 0$ such that

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1} \lesssim \varepsilon\|u\|_{E_A}^2 + C_\varepsilon\|u\|_H^\gamma, \qquad u \in E_A, \tag{5.21}$$

and therefore by Proposition 5.4, we infer that

$$\begin{aligned} -\hat{F}(u_n(t)) &\lesssim \|u_n(t)\|_{L^{\alpha+1}(M)}^{\alpha+1} \lesssim \varepsilon\|u_n(t)\|_{E_A}^2 + C_\varepsilon\|u_n(t)\|_H^\gamma \\ &\lesssim \varepsilon\|A^{\frac12}u_n(t)\|_H^2 + \varepsilon\|u_0\|_H^2 + C_\varepsilon\|u_0\|_H^\gamma, \qquad t\in[0,T]. \end{aligned} \tag{5.22}$$

By the same calculations as in the proof of Proposition 5.7 we get

$$\begin{aligned} \frac12\|A^{\frac12}u_n(s)\|_H^2 =& \mathcal{E}(u_n(s)) - \hat{F}(u_n(s)) \\ =& -\hat{F}(u_n(s)) + \mathcal{E}(P_nu_0) + \int_0^s \mathrm{Re}\langle Au_n(r) + F(u_n(r)), \mu_n(u_n(r))\rangle \mathrm{d}r \\ &+ \int_0^s \mathrm{Re}\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_nB(S_nu_n(r))\,\mathrm{d}W(r)\rangle \\ &+ \frac12\sum_{m=1}^\infty \int_0^s \|A^{\frac12}S_nB_mS_nu_n(r)\|_H^2 \mathrm{d}r \\ &+ \frac12\int_0^s \sum_{m=1}^\infty \mathrm{Re}\langle F'[u_n(r)](S_nB_mS_nu_n(r)), S_nB_mS_nu_n(r)\rangle \mathrm{d}r \end{aligned} \tag{5.23}$$

almost surely for all $s\in[0,T]$. In the following, we fix $q\in[1,\infty)$ and $t\in(0,T]$ and want to apply the $L^q(\Omega, L^\infty(0,t))$-norm to the identity (5.23). We will use the notation

$$X(s) := \Big[\|u_0\|_H^2 + \|A^{\frac12}u_n(s)\|_H^2 + \|u_n(s)\|_{L^{\alpha+1}(M)}^{\alpha+1}\Big], \qquad s\in[0,T], \tag{5.24}$$

and estimate the stochastic integral by the Burkholder-Gundy-Davis inequality and the estimates (5.10) and (5.11) as well as Lemma 5.6

$$\begin{aligned} &\Big\|\int_0^\cdot \mathrm{Re}\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_nB(S_nu_n(r))\,\mathrm{d}W(r)\rangle\Big\|_{L^q(\Omega,L^\infty(0,t))} \\ &\lesssim \Big\|\Big(\sum_{m=1}^\infty |\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_nB_mS_nu_n(r)\rangle|^2\Big)^{\frac12}\Big\|_{L^q(\Omega,L^2([0,t]))} \\ &\lesssim \|X\|_{L^q(\Omega,L^2([0,t]))} \\ &\le \varepsilon\|X\|_{L^q(\Omega,L^\infty(0,t))} + \frac{1}{4\varepsilon}\int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s \end{aligned} \tag{5.25}$$

By (5.22), we get

$$\|-\hat{F}(u_n)\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \varepsilon \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + \varepsilon\|u_0\|_H^2 + C_\varepsilon\|u_0\|_H^\gamma \tag{5.26}$$

For the following estimates, we will use (5.13)-(5.16) and the Minkowski inequality and obtain

$$\left\| \int_0^\cdot \mathrm{Re}\langle Au_n(s) + F(u_n(s)), \mu_n(u_n(s))\rangle \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \left\| \int_0^t X(s)\mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)}\mathrm{d}s; \tag{5.27}$$

$$\left\| \sum_{m=1}^\infty \int_0^\cdot \|A^{\frac{1}{2}}S_nB_mS_nu_n(s)\|_H^2 \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \left\| \int_0^t X(s)\mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)}\mathrm{d}s; \tag{5.28}$$

$$\begin{aligned} &\left\| \int_0^\cdot \sum_{m=1}^\infty \mathrm{Re}\langle F'[u_n(s)]\left(S_nB_mS_nu_n(s)\right), S_nB_mS_nu_n(s)\rangle \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} \\ &\qquad \lesssim \left\| \int_0^t X(s)\mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)}\mathrm{d}s. \end{aligned} \tag{5.29}$$

By (5.23) and the estimates (5.25)-(5.29), we get

$$\begin{aligned} \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim{}& \left\| \|A^{\frac{1}{2}}u_n(t)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \varepsilon + \varepsilon\|u_0\|_H^2 + C_\varepsilon\|u_0\|_H^\gamma + \|u_0\|_{E_A} \\ &+ \int_0^t \|X(s)\|_{L^q(\Omega)}\mathrm{d}s + \varepsilon\|X\|_{L^q(\Omega,L^\infty(0,t))} + \frac{1}{4\varepsilon}\int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s \\ &+ \int_0^t \|X(s)\|_{L^q(\Omega)}\mathrm{d}s \end{aligned} \tag{5.30}$$

In order to estimate the terms with X by the LHS of (5.30), we exploit (5.21) to get

$$\begin{aligned} \|X\|_{L^q(\Omega,L^\infty(0,t))} &\le \|u_0\|_H^2 + \mathbb{E}\big[\sup_{s\in[0,t]} \|A^{\frac{1}{2}}u_n(s)\|_H^{2q}\big]^{\frac{1}{q}} + \mathbb{E}\big[\sup_{s\in[0,t]} \|u_n(s)\|_{L^{\alpha+1}(M)}^{(\alpha+1)q}\big]^{\frac{1}{q}} \\ &\lesssim \|u_0\|_H^2 + \mathbb{E}\big[\sup_{s\in[0,t]} \|A^{\frac{1}{2}}u_n(s)\|_H^{2q}\big]^{\frac{1}{q}} \\ &\quad + \varepsilon\mathbb{E}\big[\sup_{s\in[0,t]} \|A^{\frac{1}{2}}u_n(s)\|_H^{2q}\big]^{\frac{1}{q}} + \varepsilon\|u_0\|_H^2 + C_\varepsilon\|u_0\|_H^\gamma \\ &\lesssim \left\| \sup_{s\in[0,t]} \|A^{\frac{1}{2}}u_n(s)\|_H^2 \right\|_{L^q(\Omega)} + \|u_0\|_H^2 + \|u_0\|_H^\gamma. \end{aligned}$$

Hence, by (5.24), we obtain

$$\begin{aligned} \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim{}& \varepsilon \left\| \|A^{\frac{1}{2}}u_n(t)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + \varepsilon\|u_0\|_H^2 + C_\varepsilon\|u_0\|_H^\gamma + \|u_0\|_{E_A} \\ &+ \int_0^t \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s + t\|u_0\|_H^2 + t\|u_0\|_H^\gamma \\ &+ \varepsilon \left\| \|A^{\frac{1}{2}}u_n(s)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + \varepsilon\|u_0\|_H^2 + \varepsilon\|u_0\|_H^\gamma. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we get

$$\left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \leq C_1(\|u_0\|_{E_A}, T, q) + \int_0^t C_2(q) \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,s))} \mathrm{d}s, \qquad t \in [0,T],$$

and thus, the Gronwall Lemma yields

$$\left\| \|A^{\frac{1}{2}} u_n(s)\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \leq C_1(\|u_0\|_{E_A}, T, q) e^{C_2(q)t}, \qquad t \in [0,T].$$

This implies that there is $C > 0$ with

$$\sup_{n \in \mathbb{N}} \mathbb{E}\Big[\sup_{t \in [0,T]} \|u_n(t)\|_{E_A}^{2q} \Big] \leq C.$$

since the H-norm is conserved by Proposition 5.4. Therefore, we obtain the assertion for $r \geq 2$. Finally, the case $r \in [1,2)$ is an application of Hölder's inequality.

ad b): Analogous to the proof of Proposition 5.7 b). □

6. Construction of a martingale solution

The aim of this section is the construction of a solution of equation (1.1) by a suitable limiting process in the Galerkin equation (5.5) using the results from the previous sections. Let us recall that

$$Z_T := C([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap C_w([0,T], E_A).$$

Proposition 6.1. *Let $(u_n)_{n \in \mathbb{N}}$ be the sequence of solutions to the Galerkin equation* (5.5).

a) *There are a subsequence $(u_{n_k})_{k \in \mathbb{N}}$, a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ and random variables $v_k, v : \tilde{\Omega} \to Z_T$ with $\tilde{\mathbb{P}}^{v_k} = \mathbb{P}^{u_{n_k}}$ such that $v_k \to v$ $\tilde{\mathbb{P}}$-a.s. in Z_T for $k \to \infty$.*

b) *We have $v_k \in C([0,T], H_k)$ $\tilde{\mathbb{P}}$-a.s. and for all $r \in [1,\infty)$, there is $C > 0$ with*

$$\sup_{k \in \mathbb{N}} \tilde{\mathbb{E}}\left[\|v_k\|_{L^\infty(0,T;E_A)}^r \right] \leq C$$

c) *For all $r \in [1,\infty)$, we have*

$$\tilde{\mathbb{E}}\left[\|v\|_{L^\infty(0,T;E_A)}^r \right] \leq C$$

with the same constant $C > 0$ as in b).

For the precise dependence of the constants, we refer to the Propositions 5.7 and 5.8.

Proof. ad a): The estimates to apply Corollary 4.7 are provided by Propositions 5.7 and 5.8.

ad b): Since we have $u_{n_k} \in C([0,T], H_k)$ $\mathbb{P}$-a.s. and $C([0,T], H_k)$ is closed in $C([0,T], E_A^*)$ and therefore a Borel set , we conclude $v_k \in C([0,T], H_k)$ $\tilde{\mathbb{P}}$-a.s. by the identity of the laws. Furthermore, the map $C([0,T], H_k) \ni u \mapsto \|u\|_{L^\infty(0,T;E_A)}^2 \in [0,\infty)$ is continuous and therefore measurable, so that we can conclude that

$$\begin{aligned} \tilde{\mathbb{E}}\left[\|v_k\|_{L^\infty(0,T;E_A)}^2 \right] &= \int_{C([0,T],H_k)} \|u\|_{L^\infty(0,T;E_A)}^r \mathrm{d}\tilde{\mathbb{P}}^{v_k}(u) = \int_{C([0,T],H_k)} \|u\|_{L^\infty(0,T;E_A)}^r \mathrm{d}\mathbb{P}^{u_{n_k}}(u) \\ &= \mathbb{E}\left[\|u_{n_k}\|_{L^\infty(0,T;E_A)}^r \right]. \end{aligned}$$

Use the Propositions 5.7 in the defocusing respectively 5.8 in the focusing case to get the assertion.

ad c): We have $v_n \to v$ almost surely in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ by part a). From part b) and the embedding $L^\infty(0,T;E_A) \hookrightarrow L^{\alpha+1}(0,T;L^{\alpha+1}(M))$, we obtain that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^{\alpha+1}(\tilde{\Omega}\times[0,T]\times M)$. By Vitali's Theorem (see [Els07], Theorem VI, 5.6), we conclude

$$v_n \to v \quad \text{in} \quad L^2(\tilde{\Omega}, L^{\alpha+1}(0,T;L^{\alpha+1}(M)))$$

for $n\to\infty$. On the other hand, part b) yields $\tilde{v}\in L^r(\tilde{\Omega},L^\infty(0,T;E_A))$ for all $r\in[1,\infty)$ with norm less than the constant $C=C(\|u_0\|_{E_A},T,r)>0$ and a subsequence $(v_{n_k})_{k\in\mathbb{N}}$, such that $v_{n_k}\rightharpoonup^* \tilde{v}$ for $k\to\infty$. Especially, $v_{n_k}\rightharpoonup^*\tilde{v}$ for $k\to\infty$ in $L^2(\tilde{\Omega},L^{\alpha+1}(0,T;L^{\alpha+1}(M)))$ and hence,

$$v=\tilde{v}\in L^r(\tilde{\Omega},L^\infty(0,T;E_A))$$

□

The next Lemma shows, how convergence in Z_T can be used for the convergence of the terms appearing in the Galerkin equation.

Lemma 6.2. *Let $z_n\in C([0,T],H_n)$ for $n\in\mathbb{N}$ and $z\in Z_T$. Assume $z_n\to z$ for $n\to\infty$ in Z_T. Then, for $t\in[0,T]$ and $\psi\in E_A$ as $n\to\infty$*

$$\big(z_n(t),\psi\big)_H \to \langle z(t),\psi\rangle,$$

$$\int_0^t \big(Az_n(s),\psi\big)_H \mathrm{d}s \to \int_0^t \langle Az(s),\psi\rangle \mathrm{d}s,$$

$$\int_0^t \big(\mu_n\left(z_n(s)\right),\psi\big)_H \mathrm{d}s \to \int_0^t \langle \mu\left(z(s)\right),\psi\rangle \mathrm{d}s,$$

$$\int_0^t \big(P_nF(z_n(s)),\psi\big)_H \mathrm{d}s \to \int_0^t \langle F(z(s)),\psi\rangle \mathrm{d}s.$$

Proof. Step 1: We fix $\psi\in E_A$ and $t\in[0,T]$. Recall, that the assumption implies $z_n\to z$ for $n\to\infty$ in $C([0,T],E_A^*)$. This can be used to deduce

$$\big|\big(z_n(t),\psi\big)_H - \langle z(t),\psi\rangle\big| \le \|z_n-z\|_{C([0,T],E_A^*)}\|\psi\|_{E_A}\to 0.$$

By $z_n\to z$ in $C_w([0,T],E_A)$ we get $\sup_{s\in[0,T]}|\langle z_n(s)-z(s),\varphi\rangle|\to 0$ for $n\to\infty$ and all $\varphi\in E_A^*$. We plug in $\varphi=A\psi$ and use $\langle Az_n(s),\psi\rangle=\langle z_n(s),A\psi\rangle$ for $n\in\mathbb{N}$ and $s\in[0,t]$ to get

$$\begin{aligned}\int_0^t \big|\big(Az_n(s),\psi\big)_H - \langle z(s),A\psi\rangle\big|\,\mathrm{d}s &= \int_0^t |\langle z_n(s)-z(s),A\psi\rangle|\,\mathrm{d}s\\ &\le T\sup_{s\in[0,T]}|\langle z_n(s)-z(s),A\psi\rangle|\to 0,\quad n\to\infty.\end{aligned}$$

Step 2: First, we fix $m\in\mathbb{N}$. Using that the operators B_m and S_n are selfadjoint, we get

$$\begin{aligned}&\int_0^t \Big|\Big((S_nB_mS_n)^2z_n(s),\psi\Big)_H - \langle B_m^2z(s),\psi\rangle\Big|\,\mathrm{d}s\\ &\le \int_0^t \Big|\Big((S_n-I)B_mS_n^2B_mS_nz_n(s),\psi\Big)_H\Big|\,\mathrm{d}s + \int_0^t\Big|\Big(B_m(S_n^2-I)B_mS_nz_n(s),\psi\Big)_H\Big|\,\mathrm{d}s\\ &\quad+\int_0^t\Big|\Big(B_m^2(S_n-I)z_n(s),\psi\Big)_H\Big|\,\mathrm{d}s + \int_0^t\big|\langle B_m^2\left(z_n(s)-z(s)\right),\psi\rangle\big|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
\leq & T\|z_n\|_{C([0,T],E_A^*)}\|B_m\|^2_{\mathcal{L}(E_A)}\|S_n\|^3_{\mathcal{L}(E_A)}\|(S_n-I)\psi\|_{E_A} \\
& + T\|z_n\|_{C([0,T],E_A^*)}\|S_n\|_{\mathcal{L}(E_A)}\|B_m\|_{\mathcal{L}(E_A)}\|S_n+I\|_{\mathcal{L}(E_A)}\|(S_n-I)\left(B_m\psi\right)\|_{E_A} \\
& + T\|z_n\|_{C([0,T],E_A^*)}\|(S_n-I)\left(B_m^2\psi\right)\|_{E_A} \\
& + T\|z_n-z\|_{C([0,T],E_A^*)}\|B_m^2\|_{\mathcal{L}(E_A)}\|\psi\|_{E_A} \longrightarrow 0, \quad n\to\infty,
\end{aligned}
$$

since $S_n\varphi \to \varphi$ in E_A for $\varphi \in E_A$ by Proposition 5.2 and $z_n \to z$ in $C([0,T],E_A^*)$. By the estimate

$$
\begin{aligned}
& \int_0^t \left|\Big((S_nB_mS_n)^2 z_n(s),\psi\Big)_H - \langle B_m^2 z(s),\psi\rangle\right| \mathrm{d}s \\
& \quad \leq T\|\psi\|_{E_A}\left[\|(S_nB_mS_n)^2\|_{\mathcal{L}(E_A)}\|z_n\|_{C([0,T],E_A^*)} + \|B_m^2\|_{\mathcal{L}(E_A)}\|z\|_{C([0,T],E_A^*)}\right] \\
& \quad \lesssim_{T,\psi} \|B_m\|^2_{\mathcal{L}(E_A)} \in l^1(\mathbb{N})
\end{aligned}
$$

and Lebesgue's convergence Theorem, we obtain

$$
\sum_{m=1}^{\infty}\int_0^t \left|\Big((S_nB_mS_n)^2 z_n(s),\psi\Big)_H - \langle B_m^2 z(s),\psi\rangle\right| \mathrm{d}s \longrightarrow 0, \qquad n\to\infty,
$$

and therefore

$$
\int_0^t \left(\mu_n\left(z_n(s)\right),\psi\right)_H \mathrm{d}s \to \int_0^t \langle \mu\left(z(s)\right),\psi\rangle \mathrm{d}s, \qquad n\to\infty.
$$

Step 3. Before we prove the last assertion, we recall $z_n \to z$ in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ for $n\to\infty$. We estimate

$$
\begin{aligned}
& \int_0^t \left|\big(P_nF(z_n(s)),\psi\big)_H - \langle F(z(s)),\psi\rangle\right| \mathrm{d}s \\
& \quad \leq \int_0^t |\langle F(z_n(s)),(P_n-I)\psi\rangle| \,\mathrm{d}s + \int_0^t |\langle F(z_n(s))-F(z(s)),\psi\rangle| \,\mathrm{d}s
\end{aligned} \tag{6.1}
$$

where we used (5.2). For the first term in (6.1), we look at

$$
\begin{aligned}
\int_0^t |\langle F(z_n(s)),(P_n-I)\psi\rangle| \,\mathrm{d}s &\leq \|F(z_n)\|_{L^1(0,T;E_A^*)}\|(P_n-I)\psi\|_{E_A} \\
&\lesssim \|F(z_n)\|_{L^1(0,T;L^{\frac{\alpha+1}{\alpha}}(M))}\|(P_n-I)\psi\|_{E_A} \\
&\lesssim \|z_n\|^{\alpha}_{L^{\alpha}(0,T;L^{\alpha+1}(M))}\|(P_n-I)\psi\|_{E_A} \\
&\lesssim \|z_n\|^{\alpha}_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}\|(P_n-I)\psi\|_{E_A} \longrightarrow 0, \quad n\to\infty.
\end{aligned}
$$

By Assumption (2.4) (see (2.6)), we get

$$
\|F(z_n(s))-F(z(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \left(\|z_n(s)\|_{L^{\alpha+1}(M)} + \|z(s)\|_{L^{\alpha+1}(M)}\right)^{\alpha-1}\|z_n(s)-z(s)\|_{L^{\alpha+1}(M)}
$$

for $s\in[0,T]$. Now, we apply Hölder's inequality in time with $\frac{1}{\alpha+1}+\frac{1}{\alpha+1}+\frac{\alpha-1}{\alpha+1}=1$

$$
\begin{aligned}
\|F(z_n)-F(z)\|_{L^1(0,T;L^{\frac{\alpha+1}{\alpha}}(M))} &\leq T^{\frac{1}{\alpha+1}}\left(\|z_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))} + \|z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}\right)^{\alpha-1} \\
&\qquad \|z_n-z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))} \to 0, \qquad n\to\infty.
\end{aligned}
$$

This leads to the last claim. □

By the application of the Skorohod-Jakubowski Theorem, we have replaced the Galerkin solutions u_n by the processes v_n on $\tilde{\Omega}$. Now, we want to transfer the properties given by the Galerkin equation (5.5). Therefore, we define the process $N_n : \tilde{\Omega} \times [0,T] \to H_n$ by

$$N_n(t) = -v_n(t) + P_n u_0 + \int_0^t \left[-\mathrm{i}Av_n(s) - \mathrm{i}P_n F(v_n(s)) + \mu_n(v_n(s))\right] \mathrm{d}s$$

for $n \in \mathbb{N}$ and $t \in [0,T]$ and in the following lemma, we prove its martingale property. Note that in this section, we consider H as a real Hilbert space equipped with the real scalar product $\mathrm{Re}\left(u,v\right)_H$ for $u,v \in H$ in order to be consistent with the martingale theory from [DPZ14] we use.

Lemma 6.3. *For each $n \in \mathbb{N}$, the process N_n is an H-valued continuous square integrable martingale w.r.t the filtration $\tilde{\mathcal{F}}_{n,t} := \sigma\left(v_n(s) : s \le t\right)$. The quadratic variation of N_n is given by*

$$\langle\langle N_n \rangle\rangle_t \psi = -\sum_{m=1}^{\infty} \int_0^t S_n B_m S_n v_n(s) \,\mathrm{Re}\left(S_n B_m S_n v_n(s), \psi\right)_H \mathrm{d}s$$

for all $\psi \in H$.

Proof. Fix $n \in \mathbb{N}$. We define $M_n : \Omega \times [0,T] \to H_n$ by

$$M_n(t) := -u_n(t) + P_n u_0 + \int_0^t \left[-\mathrm{i}Au_n(s) - \mathrm{i}P_n F(u_n(s)) + \mu_n(u_n(s))\right] \mathrm{d}s$$

for $t \in [0,T]$. Since u_n is a solution of the Galerkin equation (5.5), we obtain the representation

$$M_n(t) = \mathrm{i} \int_0^t S_n B_m (S_n u_n(s)) \mathrm{d}W(s)$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. The estimate

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^{\infty} \int_0^T \|S_n B_m S_n u_n(s)\|_H^2 \mathrm{d}s\right] &\le \sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(H)}^2 \mathbb{E}\left[\int_0^T \|u_n(s)\|_H^2 \mathrm{d}s\right] \\ &\le T \sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(H)}^2 \|u_0\|_H^2 < \infty\end{aligned}$$

yields, that M_n is a square integrable continuous martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. From the definition of M_n we get, that for each $t \in [0,T]$, $M_n(t)$ is measurable w.r.t. the smaller σ-field $\mathcal{F}_{n,t} := \sigma\left(u_n(s) : s \le t\right)$.
The adjoint of the operator $\Phi_n(s) := \mathrm{i}S_n B(S_n u_n(s)) : Y \to H$ for $s \in [0,T]$ is given by $\Phi^*(s)\psi = \sum_{m=1}^{\infty} \mathrm{Re}\left(\mathrm{i}S_n B_m S_n u_n(s), \psi\right)_H f_m$ for $\psi \in H$. Therefore

$$\Phi(s)\Phi^*(s)\psi = \sum_{m=1}^{\infty} \mathrm{Re}\left(\mathrm{i}S_n B_m S_n u_n(s), \psi\right)_H \mathrm{i}S_n B_m S_n u_n(s)$$

for $\psi \in H$ and $s \in [0,T]$. Hence, M_n is a $(\mathcal{F}_{n,t})$-martingale with quadratic variation

$$\langle\langle M_n \rangle\rangle_t \psi = \sum_{m=1}^{\infty} \int_0^t \mathrm{i}S_n B_m S_n u_n(s) \,\mathrm{Re}\left(\mathrm{i}S_n B_m S_n u_n(s), \psi\right)_H \mathrm{d}s$$

for $\psi \in H$ (see [DPZ14], Theorem 4.27). This property can be rephrased as

$$\mathbb{E}\left[\mathrm{Re}\left(M_n(t) - M_n(s), \psi\right)_H h(u_n|_{[0,s]})\right] = 0$$

and

$$\mathbb{E}\Bigg[\Bigg(\mathrm{Re}\left(M_n(t), \psi\right)_H \mathrm{Re}\left(M_n(t), \varphi\right)_H - \mathrm{Re}\left(M_n(s), \psi\right)_H \mathrm{Re}\left(M_n(s), \varphi\right)_H \\ - \sum_{m=1}^{\infty} \int_0^t \mathrm{Re}\left(\mathrm{i}S_n B_m S_n u_n(s), \psi\right)_H \mathrm{Re}\left(\mathrm{i}S_n B_m S_n u_n(s), \varphi\right)_H \mathrm{d}s\Bigg) h(u_n|_{[0,s]})\Bigg] = 0$$

for all $\psi, \varphi \in H$ and bounded, continuous functions h on $C([0,T], H)$.

We use the identity of the laws of u_n and v_n on $C([0,T], H_n)$ to obtain

$$\tilde{\mathbb{E}}\left[\mathrm{Re}\left(N_n(t) - N_n(s), \psi\right)_H h(u_n|_{[0,s]})\right] = 0$$

and

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\mathrm{Re}\left(N_n(t), \psi\right)_H \mathrm{Re}\left(N_n(t), \varphi\right)_H - \mathrm{Re}\left(N_n(s), \psi\right)_H \mathrm{Re}\left(N_n(s), \varphi\right)_H \\ - \sum_{m=1}^{\infty} \int_0^t \mathrm{Re}\left(\mathrm{i}S_n B_m S_n v_n(s), \psi\right)_H \mathrm{Re}\left(\mathrm{i}S_n B_m S_n v_n(s), \varphi\right)_H \mathrm{d}s\Bigg) h(v_n|_{[0,s]})\Bigg] = 0$$

for all $\psi, \varphi \in H$ and bounded, continuous functions h on $C([0,T], H_n)$. Hence, N_n is a continuous square integrable martingale w.r.t $\tilde{\mathcal{F}}_{n,t} := \sigma\left(v_n(s) : s \le t\right)$ and the quadratic variation is given as claimed in the lemma. □

We define a process N on $\tilde{\Omega} \times [0,T]$ by

$$N(t) := -v(t) + u_0 + \int_0^t \left[-\mathrm{i}Av(s) - \mathrm{i}F(v(s)) + \mu(v(s))\right] \mathrm{d}s, \quad t \in [0,T].$$

By Proposition 6.1, we infer that $v \in C([0,T], E_A^*)$ almost surely and

$$\|F(v)\|_{L^\infty(0,T;E_A^*)} \lesssim \|F(v)\|_{L^\infty(0,T;L^{\frac{\alpha+1}{\alpha}}(M))} = \|v\|^\alpha_{L^\infty(0,T;L^{\alpha+1}(M))} < \infty \quad \text{a.s.}$$

$$\|Av\|_{L^\infty(0,T;E_A^*)} \le \|v\|_{L^\infty(0,T;E_A)} < \infty \quad \text{a.s.}$$

Because of $\mu \in \mathcal{L}(E_A^*)$, we infer that $\mu(v) \in C([0,T], E_A^*)$ almost surely. Hence, N has E_A^*-valued continous paths.

Let $\iota : E_A \hookrightarrow H$ be the usual embedding, $\iota^* : H \to E_A$ its Hilbert-space-adjoint, i.e. $\left(\iota u, v\right)_H = \left(u, \iota^* v\right)_{E_A}$ for $u \in E_A$ and $v \in H$. Further, we set $L := (\iota^*)' : E_A^* \to H$ as the dual operator of ι^* with respect to the Gelfand triple $E_A \hookrightarrow H \eqsim H^* \hookrightarrow E_A^*$.

In the next Lemma, we use the martingale property of N_n for $n \in \mathbb{N}$ and a limiting process based on Proposition 6.1 and Lemma 6.2. to conclude that LN is also an H-valued martingale.

Lemma 6.4. *The process LN is an H-valued continuous square integrable martingale with respect to the filtration $\tilde{\mathbb{F}} = \left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$, where $\tilde{\mathcal{F}}_t := \sigma\left(v(s) : s \le t\right)$. The quadratic variation is given by*

$$\langle\langle LN \rangle\rangle_t \zeta = \sum_{m=1}^{\infty} \int_0^t \mathrm{i}LB_m v(s) \, \mathrm{Re}\left(\mathrm{i}LB_m v(s), \zeta\right)_H \mathrm{d}s$$

for all $\zeta \in H$.

Proof. Step 1: Let $t \in [0,T]$. We will first show that $\tilde{\mathbb{E}}\left[\|N(t)\|^2_{E_A^*}\right] < \infty$. By Lemma 6.2, we have $N_n(t) \to N(t)$ almost surely in E_A^* for $n \to \infty$. By the Davis inequality for continuous martingales (see [Par76]), Lemma 6.3 and Proposition 6.1 , we conclude

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \|N_n(t)\|_H^{\alpha+1}\right] &\lesssim \tilde{\mathbb{E}}\left[\left(\sum_{m=1}^{\infty}\int_0^T \|S_nB_mS_nv_n(s)\|_H^2 \mathrm{d}s\right)^{\frac{\alpha+1}{2}}\right] \\
&\leq \left(\sum_{m=1}^{\infty}\|B_m\|^2_{\mathcal{L}(H)}\right)^{\frac{\alpha+1}{2}} \tilde{\mathbb{E}}\left[\left(\int_0^T \|v_n(s)\|_H^2 \mathrm{d}s\right)^{\frac{\alpha+1}{2}}\right] \\
&\lesssim \tilde{\mathbb{E}}\left[\int_0^T \|v_n(s)\|_H^{\alpha+1}\mathrm{d}s\right] \lesssim \tilde{\mathbb{E}}\left[\int_0^T \|v_n(s)\|^{\alpha+1}_{L^{\alpha+1}(M)}\mathrm{d}s\right] \\
&\leq T\sup_{n\in\mathbb{N}}\tilde{\mathbb{E}}\left[\|v_n\|^{\alpha+1}_{L^\infty(0,T;L^{\alpha+1}(M))}\right] \leq TC. \qquad (6.2)
\end{aligned}$$

Since $\alpha+1>2$, we deduce $N(t) \in L^2(\tilde{\Omega}, E_A^*)$ by the Vitali Theorem and $N_n(t) \to N(t)$ in $L^2(\tilde{\Omega}, E_A^*)$ for $n \to \infty$.

Step 2: Let $\psi, \varphi \in E_A$ and h be a bounded continuous function on $C([0,T], E_A^*)$.
For $0 \leq s \leq t \leq T$, we define the random variables

$$f_n(t,s) := \mathrm{Re}\left(N_n(t)-N_n(s),\psi\right)_H h(u_n|_{[0,s]}), \qquad f(t,s) := \mathrm{Re}\langle N(t)-N(s),\psi\rangle h(u|_{[0,s]}).$$

The $\tilde{\mathbb{P}}$-a.s.-convergence $v_n \to v$ in Z_T for $n\to\infty$ yields by Lemma 6.2 $f_n(t,s) \to f(t,s)$ $\tilde{\mathbb{P}}$-a.s. for all $0 \leq s \leq t \leq T$. We use $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $a,b \geq 0$ and $p \geq 1$ and the estimate (6.2) for

$$\begin{aligned}
\tilde{\mathbb{E}}|f_n(t,s)|^{\alpha+1} &\leq 2^\alpha \|h\|_\infty^{\alpha+1}\|\psi\|_H^{\alpha+1}\tilde{\mathbb{E}}\left[\|N_n(t)\|_H^{\alpha+1}+\|N_n(s)\|_H^{\alpha+1}\right] \\
&\leq 2^\alpha \|h\|_\infty^{\alpha+1}\|\psi\|_H^{\alpha+1} 2TC
\end{aligned}$$

In view of the Vitali Theorem, we get

$$0 = \lim_{n\to\infty}\tilde{\mathbb{E}} f_n(t,s) = \tilde{\mathbb{E}} f(t,s), \qquad 0 \leq s \leq t \leq T.$$

Step 3: For $0 \leq s \leq t \leq T$, we define

$$g_{1,n}(t,s) := \Big(\mathrm{Re}\left(N_n(t),\psi\right)_H \mathrm{Re}\left(N_n(t),\varphi\right)_H - \mathrm{Re}\left(N_n(s),\psi\right)_H \mathrm{Re}\left(N_n(s),\varphi\right)_H\Big) h(v_n|_{[0,s]})$$

and

$$g_1(t,s) := \Big(\mathrm{Re}\langle N(t),\psi\rangle \,\mathrm{Re}\langle N(t),\varphi\rangle - \mathrm{Re}\langle N(s),\psi\rangle \,\mathrm{Re}\langle N(s),\varphi\rangle\Big) h(v|_{[0,s]}).$$

By Lemma 6.2, we obtain $g_{1,n}(t,s) \to g_1(t,s)$ $\tilde{\mathbb{P}}$-a.s. for all $0 \leq s \leq t \leq T$. In order to get uniform integrability, we set $r := \frac{\alpha+1}{2} > 1$ and estimate

$$\begin{aligned}
\tilde{\mathbb{E}}|g_{1,n}(t,s)|^r \leq& 2^r\|h\|_\infty^r \tilde{\mathbb{E}}\left[|\,\mathrm{Re}\left(N_n(t),\psi\right)_H \mathrm{Re}\left(N_n(t),\varphi\right)_H|^r + |\,\mathrm{Re}\left(N_n(s),\psi\right)_H \mathrm{Re}\left(N_n(s),\varphi\right)_H|^r\right] \\
\leq& 2^r\|h\|_\infty^r\|\psi\|_H^r\|\varphi\|_H^r\tilde{\mathbb{E}}\left[\|N_n(t)\|_H^{\alpha+1}+\|N_n(s)\|_H^{\alpha+1}\right] \leq 2^r\|h\|_\infty^r\|\psi\|_H^r\|\varphi\|_H^r 2TC,
\end{aligned}$$

where we used (6.2) again. As above, Vitali's Theorem yields

$$0 = \lim_{n\to\infty} \tilde{\mathbb{E}} g_{1,n}(t,s) = \tilde{\mathbb{E}} g_1(t,s), \qquad 0 \le s \le t \le T$$

Step 4: For $0 \le s \le t \le T$, we define

$$g_{2,n}(t,s) := h(v_n|_{[0,s]}) \sum_{m=1}^{\infty} \int_s^t \operatorname{Re}\left(S_n B_m S_n v_n(s), \psi\right)_H \operatorname{Re}\left(S_n B_m S_n v_n(\tau), \varphi\right)_H \mathrm{d}\tau$$

$$g_2(t,s) := h(v|_{[0,s]}) \sum_{m=1}^{\infty} \int_s^t \operatorname{Re}\langle B_m v(\tau), \psi\rangle \operatorname{Re}\langle B_m v(\tau), \varphi\rangle \mathrm{d}\tau.$$

Because of $h(v_n|_{[0,s]}) \to h(v|_{[0,s]})$ $\tilde{\mathbb{P}}$-a.s. and the continuity of die inner product $L^2([s,t]\times\mathbb{N})$, the convergence

$$\operatorname{Re}\left(S_n B_m S_n v_n, \psi\right)_H \to \operatorname{Re}\langle B_m v, \psi\rangle$$

$\tilde{\mathbb{P}}$-a.s. in $L^2([s,t]\times\mathbb{N})$ already implies $g_{2,n}(t,s) \to g_2(t,s)$ $\tilde{\mathbb{P}}$-a.s. Therefore, we consider

$$\begin{aligned}
&\| \operatorname{Re}\left(S_n B_m S_n v_n, \psi\right)_H - \operatorname{Re}\langle B_m v, \psi\rangle\|_{L^2([s,t]\times\mathbb{N})} \\
&\quad \le \| \operatorname{Re}\left(B_m S_n v_n, (S_n - I)\,\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} + \| \operatorname{Re}\left(v_n, (S_n - I)\, B_m\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\
&\qquad + \| \operatorname{Re}\langle B_m\,(v_n - v)\,, \psi\rangle\|_{L^2([s,t]\times\mathbb{N})} \\
&\quad \le \|B_m S_n v_n\|_{L^2([s,t]\times\mathbb{N}, E_A^*)} \| (S_n - I)\,\psi\|_{E_A} + \| \operatorname{Re}\left(v_n, (S_n - I)\, B_m\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\
&\qquad + \|\psi\|_{E_A} \|B_m(v_n - v)\|_{L^2([s,t]\times\mathbb{N}, E_A^*)} \\
&\quad \le \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)}\right)^{\frac12} T^{\frac12} \|v_n\|_{C([0,T],E_A^*)} \| (P_n - I)\,\psi\|_{E_A} + \| \operatorname{Re}\left(v_n, (S_n - I)\, B_m\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\
&\qquad + \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)}\right)^{\frac12} T^{\frac12} \|v_n - v\|_{C([0,T],E_A^*)} \|\psi\|_{E_A}.
\end{aligned}$$

The first and the third term tend to 0 as $n \to \infty$ by Proposition 6.1 and for the second one, this follows by the estimate

$$|\operatorname{Re}\left(v_n(s), (S_n - I)\, B_m\psi\right)_H|^2 \le 4\|v_n(s)\|^2_{E_A^*} \|B_m\|^2_{\mathcal{L}(E_A)} \|\psi\|^2_{E_A} \in L^1([s,t]\times\mathbb{N})$$

and Lebesgue's convergence Theorem. Hence, we conclude

$$\| \operatorname{Re}\left(S_n B_m S_n v_n, \psi\right)_H - \operatorname{Re}\langle B_m v, \psi\rangle\|_{L^2([s,t]\times\mathbb{N})} \to 0$$

$\tilde{\mathbb{P}}$-a.s. as $n \to \infty$. Furthermore, we estimate

$$\sum_{m=1}^{\infty} \int_s^t |\operatorname{Re}\left(S_n B_m S_n v_n(\tau), \psi\right)_H|^2 \mathrm{d}\tau \le \int_0^T \|v_n(\tau)\|^2_{E_A^*} \mathrm{d}\tau \|\psi\|^2_{E_A} \sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)}$$

and continue with $r := \frac{\alpha+1}{2} > 1$ and

$$\begin{aligned}
\tilde{\mathbb{E}}|g_{2,n}(t,s)|^r &\le \tilde{\mathbb{E}}\Big[\| \operatorname{Re}\langle S_n B_m S_n v_n, \psi\rangle\|^r_{L^2([s,t]\times\mathbb{N})} \| \operatorname{Re}\langle S_n B_m S_n v_n, \varphi\rangle\|^r_{L^2([s,t]\times\mathbb{N})} |h(v_n|_{[0,s]})|^r\Big] \\
&\le \tilde{\mathbb{E}}\left[\left(\int_0^T \|v_n(\tau)\|^2_{E_A^*} \mathrm{d}\tau\right)^r\right] \|\psi\|^r_{E_A} \|\varphi\|^r_{E_A} \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)}\right)^r \|h\|^r_{\infty}
\end{aligned}$$

$$\lesssim \tilde{\mathbb{E}}\left[\int_0^T \|v_n(\tau)\|_{E_A^*}^{\alpha+1}\mathrm{d}\tau\right] \lesssim \sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\left[\|v_n\|^{\alpha+1}_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}\right] \leq CT.$$

Using Vitali's Theorem, we obtain

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[g_{2,n}(t,s)\right] = \tilde{\mathbb{E}}\left[g_2(t,s)\right], \qquad 0 \leq s \leq t \leq T.$$

Step 5: From step 2, we have

$$\tilde{\mathbb{E}}\left[\mathrm{Re}\langle N(t) - N(s), \psi\rangle h(u|_{[0,s]})\right] = 0 \tag{6.3}$$

and step 3, step 4 and Lemma 6.3 yield

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\mathrm{Re}\langle N(t),\psi\rangle \,\mathrm{Re}\langle N(t),\varphi\rangle - \mathrm{Re}\langle N(s),\psi\rangle \,\mathrm{Re}\langle N(s),\varphi\rangle \\ + \sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\langle B_m v(\tau),\psi\rangle \,\mathrm{Re}\langle B_m v(\tau),\varphi\rangle \mathrm{d}\tau\Bigg) h(v|_{[0,s]})\Bigg] = 0. \tag{6.4}$$

Now, let $\eta, \zeta \in H$. Then $\iota^*\eta, \iota^*\zeta \in E_A$ and for all $z \in E_A^*$, we have $\mathrm{Re}\left(Lz,\eta\right)_H = \mathrm{Re}\langle z, \iota^*\eta\rangle$. By the first step, LN is a continuous, sqare integrable process in H and the identities (6.3) and (6.4) imply

$$\tilde{\mathbb{E}}\left[\mathrm{Re}\left(LN(t) - LN(s),\eta\right)_H h(u|_{[0,s]})\right] = 0$$

and

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\mathrm{Re}\left(LN(t),\eta\right)_H \mathrm{Re}\left(LN(t),\zeta\right)_H - \mathrm{Re}\left(LN(s),\eta\right)_H \mathrm{Re}\left(LN(s),\zeta\right)_H \\ + \sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\left(LB_m v(\tau),\eta\right)_H \mathrm{Re}\left(LB_m v(\tau),\zeta\right)_H \mathrm{d}\tau\Bigg) h(v|_{[0,s]})\Bigg] = 0.$$

Hence, LN is a continuous, square integrable martingale in H with respect to the $\tilde{\mathcal{F}}_{n,t} := \sigma\left(v(s) : s \leq t\right)$ and quadratic variation

$$\langle\langle LN\rangle\rangle_t \zeta = \sum_{m=1}^{\infty}\int_0^t \mathrm{i}LB_m v(s)\,\mathrm{Re}\left(\mathrm{i}LB_m v(s),\zeta\right)_H \mathrm{d}s$$

for all $\zeta \in H$. □

Finally, we can prove our main result Theorem 1.1 using the Martingale Representation Theorem from [DPZ14], Theorem 8.2.

Proof of Theorem 1.1. We choose $H = L^2(M)$, $Q = I$ and $\Phi(s) := \mathrm{i}LB\left(v(s)\right)$ for all $s \in [0,T]$. The adjoint $\Phi(s)^*$ is given by $\Phi(s)^*\zeta := \sum_{m=1}^{\infty}\mathrm{Re}\left(\mathrm{i}LB_m v(s),\zeta\right)_H f_m$ and hence,

$$\left(\Phi(s)Q^{\frac{1}{2}}\right)\left(\Phi(s)Q^{\frac{1}{2}}\right)^*\zeta = \Phi(s)\Phi(s)^*\zeta = \sum_{m=1}^{\infty}\mathrm{Re}\left(\mathrm{i}LB_m v(s),\zeta\right)_H \mathrm{i}LB_m v(s)$$

for $\zeta \in H$. Clearly, v is continuous in E_A^* and adapted to the filtration $\tilde{\mathbb{F}}$ given by $\tilde{\mathcal{F}}_t = \sigma\left(v(s) : 0 \leq s \leq t\right)$ for $s \in [0,T]$. Hence, Φ is continuous in H and adapted to $\tilde{\mathbb{F}}$ and therefore progressively measurable.

By an application of Theorem 8.2 in [DPZ14] to the process LN from Lemma 6.4, we obtain a cylindrical Wiener process $\tilde{W}$ on Y defined on a probability space

$$(\Omega', \mathcal{F}', \mathbb{P}') = \left(\tilde{\Omega} \times \tilde{\tilde{\Omega}}, \tilde{\mathcal{F}} \otimes \tilde{\tilde{\mathcal{F}}}, \tilde{\mathbb{P}} \otimes \tilde{\tilde{\mathbb{P}}}\right)$$

with

$$LN(t) = \int_0^t \Phi(s) \mathrm{d}\tilde{W}(s) = \int_0^t \mathrm{i}LB\left(v(s)\right) \mathrm{d}\tilde{W}(s)$$

for $t \in [0,T]$. The estimate

$$\begin{aligned}
\|Bv\|^2_{L^2([0,T]\times\Omega,\mathrm{HS}(Y,E_A^*))} =& \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|B_m v(s)\|^2_{E_A^*} \mathrm{d}s \lesssim \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|B_m v(s)\|^2_{E_A} \mathrm{d}s \\
\leq& \mathbb{E}\int_0^T \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)}\right) \|v(s)\|^2_{E_A} \mathrm{d}s \lesssim \mathbb{E}\int_0^T \|v(s)\|^2_{E_A} \mathrm{d}s \\
\leq& T\|v\|^2_{L^2(\Omega, L^\infty(0,T;E_A))} \leq TC
\end{aligned}$$

yields that the stochastic integral $\int_0^{\cdot} B\left(v(s)\right) \mathrm{d}\tilde{W}(s)$ is a continuous martingale in E_A^* and using the continuity of the operator L, we get

$$\int_0^t \mathrm{i}LB\left(v(s)\right) \mathrm{d}\tilde{W}(s) = L\left(\int_0^t \mathrm{i}B\left(v(s)\right) \mathrm{d}\tilde{W}(s)\right)$$

for all $t \in [0,T]$. The definition of N and the injectivity of L yield the equality

$$\int_0^t \mathrm{i}Bv(s) \mathrm{d}\tilde{W}(s) = -v(t) + u_0 + \int_0^t \left[-\mathrm{i}Av(s) - \mathrm{i}F(v(s)) + \mu(v(s))\right] \mathrm{d}s \tag{6.5}$$

in E_A^* for $t \in [0,T]$. The weak continuity of the paths of v in E_A and the estimates for property (1.5) have already been shown in Proposition 6.1. Hence, the system $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, v\right)$ is a martingale solution of equation (1.1). □

It remains to prove the mass conservation from Theorem 1.1. In Proposition 5.4, we proved a similar result for the approximating equation. Since this property is not invariant under the limiting procedure from above, we have to repeat the calculation in infinite dimensions and justify it by a regularization procedure.

Proposition 6.5. *Let* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ *be a martingale solution of* (1.1)*. Then, we have* $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ *almost surely for all* $t \in [0,T]$.

Proof. Step 1. Given $\lambda > 0$, we define $R_\lambda := \lambda\left(\lambda + A\right)^{-1}$. Using the series representation, one can verify

$$\begin{gathered}
R_\lambda f \to f \quad \text{in} \quad X, \quad \lambda \to \infty, \quad f \in X \\
\|R_\lambda\|_{\mathcal{L}(X)} \leq 1
\end{gathered} \tag{6.6}$$

for $X \in \{H, E_A, E_A^*\}$. Moreover, $R_\lambda(E_A^*) = E_A$ and hence, the equation

$$R_\lambda u(t) = R_\lambda u_0 + \int_0^t \left[-\mathrm{i}R_\lambda Au(s) - \mathrm{i}R_\lambda F(u(s)) + R_\lambda \mu(u(s))\right] \mathrm{d}s - \mathrm{i}\int_0^t R_\lambda Bu(s) \mathrm{d}\tilde{W}(s) \tag{6.7}$$

holds almost surely in E_A for all $t \in [0,T]$. The function $\mathcal{M} : H \to \mathbb{R}$ defined by $\mathcal{M}(v) := \|v\|_H^2$ is twice continuously Fréchet-differentiable with

$$\mathcal{M}'[v]h_1 = 2\,\mathrm{Re}\left(v, h_1\right)_H, \qquad \mathcal{M}''[v]\,[h_1,h_2] = 2\,\mathrm{Re}\left(h_1, h_2\right)_H$$

for $v, h_1, h_2 \in H$. Therefore, we get

$$\begin{aligned}\|R_\lambda u(t)\|_H^2 =& \|R_\lambda u_0\|_H^2 + 2\int_0^t \mathrm{Re}\left(R_\lambda u(s), -\mathrm{i}R_\lambda A u(s) - \mathrm{i}R_\lambda F(u(s)) + R_\lambda \mu(u(s))\right)_H \mathrm{d}s \\ &- 2\int_0^t \mathrm{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B u(s)\mathrm{d}\tilde{W}(s)\right)_H + \sum_{m=1}^\infty \int_0^t \|R_\lambda B_m u(s)\|_H^2 \mathrm{d}s \end{aligned} \tag{6.8}$$

almost surely for all $t \in [0,T]$.

Step 2. In the following, we deal with the behaviour of the terms in (6.8) for $\lambda \to \infty$. Since R_λ and A commute, we get

$$\mathrm{Re}\left(R_\lambda u(s), -\mathrm{i}R_\lambda A u(s)\right)_H = \mathrm{Re}\left(R_\lambda u(s), -\mathrm{i}A R_\lambda u(s)\right)_H = 0, \quad s \in [0,T], \quad \lambda > 0. \tag{6.9}$$

For $s \in [0,T]$, we have

$$\begin{aligned}&\mathrm{Re}\left(R_\lambda u(s), -\mathrm{i}R_\lambda F(u(s))\right)_H \to \mathrm{Re}\langle u(s), -\mathrm{i}F(u(s))\rangle = 0 \\ &\mathrm{Re}\left(R_\lambda u(s), R_\lambda \mu(u(s))\right)_H \to \mathrm{Re}\left(u(s), \mu(u(s))\right)_H, \qquad \lambda \to \infty. \end{aligned} \tag{6.10}$$

by (6.6). In order to apply the dominated convergence Theorem by Lebesgue, we estimate

$$\begin{aligned}&|\,\mathrm{Re}\big(R_\lambda u(s), -\mathrm{i}R_\lambda F(u(s)) + R_\lambda \mu(u(s))\big)_H| \\ &\qquad \le \|u(s)\|_{E_A}\, \|-\mathrm{i}F(u(s)) + \mu(u(s))\|_{E_A^*} \\ &\qquad \lesssim \|u(s)\|_{E_A} \left(\|F(u(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} + \sum_{m=1}^\infty \|B_m\|_{\mathcal{L}(H)}^2 \|u(s)\|_H \right) \\ &\qquad \lesssim \|u(s)\|_{E_A} \left(\|u(s)\|_{L^{\alpha+1}(M)}^\alpha + \|u(s)\|_H \right) \\ &\qquad \lesssim \|u(s)\|_{E_A}^{\alpha+1} + \|u(s)\|_{E_A}^2 \end{aligned}$$

using (6.6) and the Sobolev embeddings $L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$ and $E_A \hookrightarrow L^{\alpha+1}(M)$.
Since $u \in C_w([0,T], E_A)$ almost surely and $C_w([0,T],E_A) \subset L^\infty(0,T;E_A)$, we obtain

$$\int_0^t \mathrm{Re}\left(R_\lambda u(s), -\mathrm{i}R_\lambda F(u_1(s)) + R_\lambda \mu(u(s))\right)_H \mathrm{d}s \to \int_0^t \mathrm{Re}\left(u(s), \mu(u(s))\right)_H \mathrm{d}s, \qquad \lambda \to \infty,$$

almost surely for all $t \in [0,T]$. Moreover, the pointwise convergence

$$\|R_\lambda B_m u(s)\|_H \to \|B_m u(s)\|_H, \qquad m \in \mathbb{N}, \quad \text{f.a.a. } s \in [0,T]$$

and the estimate

$$\|R_\lambda B_m u(s)\|_H^2 \le \|B_m\|_{\mathcal{L}(H)}^2 \|u(s)\|_H^2 \in L^1([0,T]\times\mathbb{N})$$

lead to, by Lebesgue DCT,

$$\sum_{m=1}^\infty \int_0^t \|R_\lambda B_m u(s)\|_H^2 \mathrm{d}s \to \sum_{m=1}^\infty \int_0^t \|B_m u(s)\|_H^2 \mathrm{d}s, \quad \lambda \to \infty \tag{6.11}$$

almost surely for all $t \in [0,T]$. For the stochastic term, we fix $k \in \mathbb{N}$ and define a stopping time τ_K by

$$\tau_K := \inf\{t \in [0,T] : \|u(t)\|_H > K\}.$$

Then, we infer that

$$\operatorname{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B_m u(s)\right)_H \to \operatorname{Re}\left(u(s), \mathrm{i}Bu(s)\right)_H = 0 \quad \text{a.s.}, \quad m \in \mathbb{N}, s \in [0,T]$$

and

$$\begin{aligned}\mathbf{1}_{[0,\tau_K]}(s)|\operatorname{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B_m u(s)\right)_H|^2 &\le \mathbf{1}_{[0,\tau_K]}(s)\|u(s)\|_H^4\|B_m\|_{\mathcal{L}(H)}^2 \\ &\le K^4\|B_m\|_{\mathcal{L}(H)}^2 \in L^1(\tilde{\Omega}\times[0,T]\times\mathbb{N})\end{aligned}$$

to get

$$\tilde{\mathbb{E}}\sum_{m=1}^{\infty}\int_0^{\tau_K}\left[\operatorname{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B_m u(s)\right)_H\right]^2 \mathrm{d}s \to 0, \quad \lambda \to \infty,$$

by Lebesgue. The Itô isometry and the Doob inequality yield

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,\tau_K]}\left|\int_0^t \operatorname{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B u(s)\mathrm{d}W(s)\right)_H\right|^2\right] \to 0, \quad \lambda \to \infty,$$

After passing to a subsequence, we get

$$\int_0^t \operatorname{Re}\left(R_\lambda u(s), \mathrm{i}R_\lambda B u(s)\mathrm{d}W(s)\right)_H \to 0, \quad \lambda \to \infty, \tag{6.12}$$

almost surely in $\{t \le \tau_K\}$. By

$$\bigcup_{K\in\mathbb{N}}\{t \le \tau_K\} = [0,T] \qquad \text{a.s.},$$

we conclude that (6.12) holds almost surely on $[0,T]$.

Step 3. Using (6.9), (6.11) and (6.12) in (6.8), we obtain

$$\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \operatorname{Re}\left(u(s), \mu(u(s))\right)_H \mathrm{d}s + \sum_{m=1}^{\infty}\int_0^t \|B_m u(s)\|_H^2 \mathrm{d}s$$

almost surely for all $t \in [0,T]$. By the selfadjointness of B_m, $m \in \mathbb{N}$, we simplify

$$2\operatorname{Re}\left(u(s), \mu(u(s))\right)_H = -\sum_{m=1}^{\infty}\operatorname{Re}\left(u(s), B_m^2 u(s)\right)_H = -\sum_{m=1}^{\infty}\|B_m u(s)\|_H^2.$$

Therefore, we have $\|u(t)\|_H^2 = \|u_0\|_H^2$ almost surely for all $t \in [0,T]$. □

7. Regularity and Uniqueness of solutions on 2D manifolds

In this section, we want to study pathwise uniqueness of solutions to (1.1) and we consider the case of a 2-dimensional riemannian manifold without boundary M. We drop the assumption that M is compact and replace it by

$$M \text{ is complete, has a positive injectivity radius and a bounded geometry.} \tag{7.1}$$

We refer to [Tri92], chapter 7, for the definitions of the notions above and background references on differential geometry. We emphasize that (7.1) is satisfied by compact manifolds,

see [Aub13]. Let $A = -\Delta_g$ be the Laplace-Beltrami operator $F = F_\alpha^\pm$ be the model nonlinearity from section 3. The proof is based on an additional regularity of the solution, which we obtain by applying the deterministic and the stochastic Strichartz estimates from [BS14] and [BM14].

In two dimensions, the mapping properties of the nonlinearity improve, as we will see in the first Lemma.

Lemma 7.1. *Let $d = 2$, $\alpha > 1$ and $s \in (\frac{\alpha-1}{\alpha}, 1]$. Then, we have $F_\alpha^\pm : H^s(M) \to H^{\tilde{s}}(M)$ for all $\tilde{s} \in (0, 1 - \alpha + s\alpha)$ and*

$$\|F_\alpha^\pm(u)\|_{H^{\tilde{s}}} \lesssim \|u\|_{H^s}^\alpha, \qquad u \in H^s(M).$$

Proof. Step 1. We start with the proof of

$$\|F_\alpha^\pm(u)\|_{H^{s,r}(\mathbb{R}^2)} \lesssim \|u\|_{H^s(\mathbb{R}^2)}^\alpha \tag{7.2}$$

for $r \in (1, \frac{2}{(1-s)\alpha+s})$. For a general M, the estimate follows by the definition of fractional Sobolev spaces via charts, see Appendix B. By [CW91], Proposition 3.1, we obtain

$$\||\nabla|^s F_\alpha^\pm(u)\|_{L^r} \lesssim \|u\|_{L^q}^{\alpha-1} \||\nabla|^s u\|_{L^2}, \qquad \frac{1}{r} = \frac{1}{2} + \frac{\alpha-1}{q}.$$

Furthermore,

$$\|F_\alpha^\pm(u)\|_{L^r} = \|u\|_{L^{r\alpha}}^\alpha$$

and (7.2) follows from the Sobolev embeddings

$$H^s(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2), \qquad H^s(\mathbb{R}^2) \hookrightarrow L^{r\alpha}(\mathbb{R}^2).$$

for $r \in (1, \frac{2}{(1-s)\alpha+s}]$.

Step 2. The assertion follows from (7.2) and the Sobolev embedding $H^{s,r}(\mathbb{R}^2) \hookrightarrow H^{\tilde{s}}(\mathbb{R}^2)$. □

In the following Proposition, we reformulate problem (1.1) in a mild form and use this to show additional regularity properties of solutions of (1.1). Let us therefore recall the notation

$$\mu = -\frac{1}{2}\sum_{m=1}^\infty B_m^2.$$

Proposition 7.2. *Assume $d = 2$ and choose $2 < p, q < \infty$ with*

$$\frac{2}{p} + \frac{2}{q} = 1.$$

Let $s \in (\frac{\alpha-1}{\alpha}, 1]$, $\alpha > 1$, $r > 1$ and $\beta := \max\{\alpha, 2\}$. Let $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ be a solution to (1.1) with $F = F_\alpha^\pm$ and assume

$$u \in L^{r\alpha}(\tilde{\Omega}, L^\beta(0, T; H^s(M))). \tag{7.3}$$

Then, for each $\tilde{s} \in (0, 1 - \alpha + s\alpha)$ and $\varepsilon \in (0, 1)$, we have

$$u \in L^r(\tilde{\Omega}, C([0, T], H^{\tilde{s}}(M)) \cap L^q(0, T; H^{\tilde{s}-\frac{1+\varepsilon}{q}, p}(M))) \tag{7.4}$$

and almost surely in $H^{\tilde{s}}(M)$ for all $t \in [0,T]$

$$\mathrm{i}u(t) = \mathrm{i}e^{-\mathrm{i}tA}u_0 + \int_0^t e^{-\mathrm{i}(t-\tau)A}F_\alpha^\pm(u(\tau))\mathrm{d}\tau + \int_0^t e^{-\mathrm{i}(t-\tau)A}\mu(u(\tau))\mathrm{d}\tau + \int_0^t e^{-\mathrm{i}(t-\tau)A}B(u(\tau))\mathrm{d}W(\tau). \tag{7.5}$$

Remark 7.3. Of course, (7.4) also holds for $\varepsilon \geq 1$, but then $u \in L^r(\Omega, L^q(0,T; H^{s-\frac{1+\varepsilon}{q},p}(M)))$ would be trivial by the Sobolev embedding $H^{\tilde{s}}(M) \hookrightarrow H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M)$. Being able to choose $\varepsilon \in (0,1)$ means a gain of regularity which will be used below via $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ for an appropriate choice of the parameters.

Proof of Proposition 7.2. Step 1. First, we will show that it is possible to rewrite the equation (2.18) from the definition of solutions for (1.1) in the mild form (7.5).
We note that for each $\tilde{s} < 0$ the semigroup $\left(e^{-\mathrm{i}tA}\right)_{t\geq 0}$ on $L^2(M)$ extends to a semigroup $(T_{\tilde{s}}(t))_{t\geq 0}$ with the generator $A_{\tilde{s}}$ that extends A to $\mathcal{D}(A_{\tilde{s}}) = H^{\tilde{s}+2}(M)$. To keep the notation simple, we also call this semigroup $\left(e^{-\mathrm{i}tA}\right)_{t\geq 0}$.
We apply the Itô formula to $\Phi \in C^{1,2}([0,t] \times H^{s-2}(M), H^{s-4}(M))$ defined by

$$\Phi(\tau, x) := e^{-\mathrm{i}(t-\tau)A}x, \qquad \tau \in [0,t], \quad x \in H^{s-2}(M)$$

and obtain

$$\mathrm{i}u(t) = \mathrm{i}e^{-\mathrm{i}tA}u_0 + \int_0^t e^{-\mathrm{i}(t-\tau)A}F_\alpha^\pm(u(\tau))\mathrm{d}\tau + \int_0^t e^{-\mathrm{i}(t-\tau)A}\mu(u(\tau))\mathrm{d}\tau + \int_0^t e^{-\mathrm{i}(t-\tau)A}B(u(\tau))\mathrm{d}W(\tau)$$

almost surely in $H^{s-4}(M)$ for all $t \in [0,T]$.

Step 2. Using the Strichartz estimates from Lemma $B.4$ we deal with the free term and each convolution term on the right hand site to get (7.4) and the identity (7.5) in $H^{\tilde{s}}(M)$. For this purpose, we define

$$Y_T := L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M)) \cap L^\infty(0,T; H^{\tilde{s}}(M)).$$

By (B.7) we obtain

$$\|e^{-\mathrm{i}tA}u_0\|_{L^r(\tilde{\Omega},Y_T)} \lesssim \|u_0\|_{H^{\tilde{s}}} \lesssim \|u_0\|_{H^s} < \infty$$

and by (B.5) and Lemma 7.1, we get

$$\left\| \int_0^t e^{-\mathrm{i}(t-\tau)A}F_\alpha^\pm(u(\tau))\mathrm{d}\tau \right\|_{Y_T} \lesssim \|F_\alpha^\pm(u)\|_{L^1(0,T;H^{\tilde{s}})} \lesssim \|u\|^\alpha_{L^\alpha(0,T;H^s)}.$$

Integration over $\tilde{\Omega}$ and (7.3) yields

$$\left\| \int_0^t e^{-\mathrm{i}(t-\tau)A}F_\alpha^\pm(u(\tau))\mathrm{d}\tau \right\|_{L^r(\tilde{\Omega},Y_T)} \lesssim \|u\|^\alpha_{L^{r\alpha}(\tilde{\Omega},L^\alpha(0,T;H^1))} < \infty.$$

To estimate the other convolutions, we need that μ is bounded in $H^{\tilde{s}}(M)$ and B is bounded from $H^{\tilde{s}}(M)$ to $\mathrm{HS}(Y, H^{\tilde{s}}(M))$. This can be deduced from the following estimate, which follows from complex interpolation (see [Lun09], Theorem 2.1.6), Hölder's inequality and Assumption 2.7:

$$\sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(H^{\tilde{s}})} \leq \sum_{m=1}^\infty \|B_m\|^{2\tilde{s}}_{\mathcal{L}(H^1)} \|B_m\|^{2(1-\tilde{s})}_{\mathcal{L}(H)}$$

$$\leq \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(H^1)}\right)^{\tilde{s}} \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(H)}\right)^{1-\tilde{s}} < \infty. \tag{7.6}$$

Therefore, by (B.5), (7.6) and (7.3)

$$\begin{aligned}\left\| \int_0^t e^{-\mathrm{i}(t-\tau)A} \mu(u(\tau)) \mathrm{d}\tau \right\|_{L^r(\tilde{\Omega}, Y_T)} &\lesssim \|\mu(u)\|_{L^r(\tilde{\Omega}, L^1(0,T;H^{\tilde{s}}))} \lesssim \|u\|_{L^r(\tilde{\Omega}, L^1(0,T;H^{\tilde{s}}))} \\ &\lesssim \|u\|_{L^{r\alpha}(\tilde{\Omega}, L^\beta(0,T;H^s)} < \infty.\end{aligned}$$

The estimates (B.6), (7.6) and (7.3) imply

$$\begin{aligned}\left\| \int_0^t e^{-\mathrm{i}(t-\tau)A} B(u(\tau)) \mathrm{d}W(\tau) \right\|_{L^r(\tilde{\Omega}, Y_T)} &\lesssim \|B(u)\|_{L^r(\tilde{\Omega}, L^2(0,T;\mathrm{HS}(Y,H^{\tilde{s}}))} \lesssim \|u\|_{L^r(\tilde{\Omega}, L^2(0,T;H^{\tilde{s}}))} \\ &\lesssim \|u\|_{L^{r\alpha}(\tilde{\Omega}, L^\beta(0,T;H^s))} < \infty.\end{aligned}$$

Hence, the mild equation (7.5) holds almost surely in $H^{\tilde{s}}(M)$ for each $t \in [0,T]$ and thus, we get (7.4) by the pathwise continuity of deterministic and stochastic integrals. □

As a preparation for the proof of pathwise uniqueness, we show a formula for the L^2-norm of the difference of two solutions of (1.1).

Lemma 7.4. *Let* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u_j\right)$, $j = 1, 2$, *be solutions of* (1.1) *with* $F = F_\alpha^\pm$ *for* $\alpha > 1$. *Then,*

$$\|u_1(t) - u_2(t)\|^2_{L^2} = 2 \int_0^t \mathrm{Re}\left(u_1(\tau) - u_2(\tau), -\mathrm{i}F_\alpha^\pm(u_1(\tau)) + \mathrm{i}F_\alpha^\pm(u_2(\tau))\right)_{L^2} \mathrm{d}\tau \tag{7.7}$$

almost surely for all $t \in [0,T]$.

Proof. The proof is similar to Proposition 6.5. In fact, it is even simpler, since the regularity of $F_\alpha^\pm$ due to Lemma 7.1 simplifies the proof of the convergence for $\lambda \to \infty$. □

Finally, we are ready to prove the pathwise uniqueness of solutions to (1.1).

Theorem 7.5. *Let* $d = 2$ *and* $F(u) = F_\alpha^\pm(u) = \pm|u|^{\alpha-1}u$ *with* $\alpha \in (1, \infty)$. *Let* $r > \alpha$, $\beta \geq \max\{\alpha, 2\}$ *and*

$$s \in \begin{cases} (1 - \frac{1}{2\alpha}, 1] & \text{for } \alpha \in (1,3], \\ (1 - \frac{1}{\alpha(\alpha-1)}, 1] & \text{for } \alpha > 3. \end{cases}$$

Then, solutions of problem (1.1) *are pathwise unique in* $L^r(\tilde{\Omega}, L^\beta(0,T;H^s(M)))$, *i.e. given two solutions* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u_j\right)$ *with*

$$u_j \in L^r(\tilde{\Omega}, L^\beta(0,T;H^s(M))),$$

for $j = 1, 2$, *we have* $u_1(t) = u_2(t)$ *almost surely in* $L^2(M)$ *for all* $t \in [0,T]$.

Proof. Step 1. Take two solutions $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u_j\right)$ of (1.1) with $u_j \in L^r(\tilde{\Omega}, L^\infty(0,T;H^s(M)))$ for $j = 1, 2$, and define $w := u_1 - u_2$. From Lemma 7.4, we conclude

$$\|w(t)\|^2_{L^2} = 2 \int_0^t \mathrm{Re}\left(w(\tau), -\mathrm{i}F(u_1(\tau)) + \mathrm{i}F(u_2(\tau))\right)_{L^2} \mathrm{d}\tau$$

almost surely for all $t \in [0,T]$. The estimate

$$|F_\alpha^\pm(z_1) - F_\alpha^\pm(z_2)| \lesssim \left(|z_1|^{\alpha-1} + |z_2|^{\alpha-1}\right)|z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{C},$$

yields

$$\begin{aligned}\|w(t)\|_{L^2}^2 &\lesssim \int_0^t \int_M |w(\tau,x)|^2 \left[|u_1(\tau,x)|^{\alpha-1} + |u_2(\tau,x)|^{\alpha-1}\right] \mathrm{d}x\mathrm{d}\tau \\ &\leq \int_0^t \|w(\tau)\|_{L^2}^2 \left[\|u_1(\tau)\|_{L^\infty(M)}^{\alpha-1} + \|u_2(\tau)\|_{L^\infty(M)}^{\alpha-1}\right] \mathrm{d}\tau \end{aligned} \tag{7.8}$$

almost surely for all $t \in [0,T]$.

Step 2. First, we deal with the case $\alpha \in (1,3]$. By $s > 1 - \frac{1}{2\alpha}$, we can choose $q > 2$ and $\varepsilon \in (0,1)$ with

$$1 - \frac{1}{2\alpha} < 1 - \frac{1}{q\alpha} + \frac{\varepsilon}{q\alpha} < s.$$

Hence, there is $\tilde{s} \in (0, 1-\alpha+s\alpha)$ with $\tilde{s} > 1 - \frac{1}{q} + \frac{\varepsilon}{q}$. If we choose $p > 2$ according to $\frac{2}{p} + \frac{2}{q} = 1$, Proposition $B.2$ leads to $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ because of

$$\left(\tilde{s} - \frac{1+\varepsilon}{q}\right) - \frac{2}{p} = \tilde{s} - \frac{\varepsilon}{q} + \frac{1}{q} - 1 > 0.$$

Moreover, we have $u_j \in L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))$ almost surely for $j = 1,2$ by Proposition 7.2. Hence, the process b defined by

$$b(\tau) := \left[\|u_1(\tau)\|_{L^\infty}^{\alpha-1} + \|u_2(\tau)\|_{L^\infty}^{\alpha-1}\right], \qquad \tau \in [0,T], \tag{7.9}$$

satisfies

$$\|b\|_{L^1(0,T)} \lesssim \|u_1\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p})}^{\alpha-1} + \|u_2\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p})}^{\alpha-1} < \infty \quad \text{a.s.}, \tag{7.10}$$

where we used $q > 2 \geq \alpha - 1$ and the Hölder inequality in time. Because of (7.8), we can apply Gronwall's Lemma to get

$$u_1(t) = u_2(t) \quad \text{a.s. in } L^2(M) \text{ for all } t \in [0,T].$$

Step 3. Now, let $\alpha > 3$. Then, we set $q := \alpha - 1$ and choose $p > 2$ with $\frac{2}{p} + \frac{2}{q} = 1$. Using $s > 1 - \frac{1}{\alpha(\alpha-1)}$, we fix $\varepsilon \in (0,1)$ with

$$1 - \frac{1}{\alpha(\alpha-1)} < 1 - \frac{1}{q\alpha} + \frac{\varepsilon}{q\alpha} < s.$$

As above, we can choose $\tilde{s} \in (0, 1-\alpha+s\alpha)$ with $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ and $u_j \in L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))$ almost surely for $j = 1,2$. We obtain $b \in L^1(0,T)$ almost surely for b from (7.9) and Gronwall's Lemma implies

$$u_1(t) = u_2(t) \quad \text{a.s. in } L^2(M) \text{ for all } t \in [0,T].$$

□

Remark 7.6. In [BM14], Brzeźniak and Millet proved pathwise uniqueness of solutions in the space $L^q(\Omega, C([0,T], H^1(M)) \cap L^q([0,T], H^{1-\frac{1}{q},p}(M)))$ with $\frac{2}{q} + \frac{2}{p} = 1$ and $q > \alpha + 1$. Since they used the deterministic Strichartz estimates from [BGT04] instead of [BS14], their result is restricted to compact manifolds M. Comparing the two results, we see that the assumptions of Theorem 7.5 are weaker with respect to space and time. On the other hand, the assumptions on the required moments is slightly weaker in [BM14].

Remark 7.7. A similar Uniqueness-Theorem can also be proved on bounded domains in $\mathbb{R}^2$ using the Strichartz inequalities by Blair, Smith and Sogge from [BSS12]. We also want to mention the classical strategy by Vladimirov (see [Vla87], [Oga90], [OO91] and [Caz03]) to prove uniqueness of H^1-solutions using Trudinger type inequalities which can be seen as the limit case of Sobolev's embedding, see also [AF03], Theorem 8.27. Since this proof only relies on the formula (7.7) and the property of solutions to be in H^1, it can be directly transfered to the stochastic setting. This strategy does not use Strichartz estimates, but it suffers from a restriction to $\alpha \in (1,3]$ and it cannot be transfered to H^s for $s < 1$.

Now, we give the definition of the concepts of strong solutions and uniqueness in law used in Corollary 1.3.

Definition 7.8. a) Let $T > 0$ and $u_0 \in E_A$. Then, a *strong solution* of the equation (1.1) is a continuous, $\tilde{\mathbb{F}}$-adapted process with values in E_A^* such that $u \in L^2(\Omega \times [0,T], E_A^*)$ and almost all paths are in $C_w([0,T], E_A)$ with

$$u(t) = u_0 + \int_0^t [-\mathrm{i}Au(s) - \mathrm{i}F(u(s)) + \mu(u(s))]\,\mathrm{d}\tau - \mathrm{i}\int_0^t Bu(s)\mathrm{d}W(s)$$

almost surely in E_A^* for all $t \in [0,T]$.

b) The solutions of (1.6) are called *unique in law*, if for all martingale solutions $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j, W_j, \mathbb{F}_j, u_j)$ with $u_j(0) = u_0$, for $j = 1,2$, we have $\mathbb{P}_1^{u_1} = \mathbb{P}_2^{u_2}$ almost surely in $C([0,T], L^2(M))$.

We finish this section with the proof of Corollary 1.3.

Proof of Corollary 1.3. The existence of a martingale solution from Corollary 1.2 and the pathwise uniqueness from Theorem 7.5 yield the assertion by [Ond04], Theorem 2 and 12.1. □

APPENDIX A. AUXILARY RESULTS FROM FUNCTIONAL ANALYSIS

In this appendix, we collect some abstract notions and results needed in Section 4. For a Banach space X and $r > 0$, we denote

$$\mathbb{B}_X^r := \{u \in X : \|u\|_X \leq r\}.$$

The weak topology on $\mathbb{B}_X^r$ is metrizable if the dual X^* is separable and a metric is given by

$$q(x_1, x_2) = \sum_{k=1}^{\infty} 2^{-k} |\langle x_1 - x_2, x_k^* \rangle|, \qquad x_1, x_2 \in X,$$

for a dense sequence $(x_k^*)_{k\in\mathbb{N}} \in (B_{X^*}^1)^{\mathbb{N}}$, see [Bre10], Theorem 3.29. If X is also separable, then $C([0,T], \mathbb{B}_X^r)$ is a complete separable metric space with metric $\rho(u,v) := \sup_{t\in[0,T]} q(u(t), v(t))$ for $u, v \in C([0,T], \mathbb{B}_{E_A}^r)$.

Definition A.1. We define

$$C_w([0,T],X) := \{u : [0,T] \to X : [0,T] \ni t \to \langle u(t), x^* \rangle \in \mathbb{C} \text{ is cont. for all } x^* \in X^*\}$$

and equip $C_w([0,T],X)$ with the locally convex topology induced by the family P of seminorms given by

$$P := \{p_{x^*} : x^* \in X^*\}, \qquad p_{x^*}(u) := \sup_{t\in[0,T]} |\langle u(t), x^*\rangle|.$$

We continue with some auxiliary results.

Lemma A.2. *Let $r > 0$ and $u_n, u \in C_w([0,T],X)$ with $\sup_{t\in[0,T]} \|u_n(t)\|_X \le r$ and $u_n \to u$ in $C_w([0,T],X)$. Then, we have $u_n \to u$ in $C([0,T],\mathbb{B}^r_X)$.*

Proof. By Lebesgue's Convergence Theorem,

$$\rho(u_n,u) \le \sum_{k=1}^{\infty} 2^{-k} \sup_{t\in[0,T]} |\langle u_n(t) - u(t), x_k^*\rangle| \to 0, \qquad n \to \infty,$$

where we used the definition of convergence in $C_w([0,T],X)$ for fixed $k \in \mathbb{N}$ and

$$\sup_{t\in[0,T]} |\langle u_n(t) - u(t), x_k^*\rangle \le \left(\sup_{t\in[0,T]} \|u_n(t)\|_X + \sup_{t\in[0,T]} \|u(t)\|_X \right) \|x_k^*\|_{X^*} \le 2r.$$

□

Lemma A.3 (Strauss)**.** *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and $T > 0$. Then, we have the inclusion*

$$L^\infty(0,T;X) \cap C_w([0,T],Y) \subset C_w([0,T],X).$$

Proof. See [Tem77], Chapter 3, Lemma 1.4. □

Lemma A.4 (Lions)**.** *Let X, X_0, X_1 be Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$ where the first embedding is compact. Assume furthermore that X_0, X_1 are reflexive and $p \in [1,\infty)$. Then, for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ with*

$$\|x\|_X^p \le \varepsilon \|x\|_{X_0}^p + C_\varepsilon \|x\|_{X_1}^p, \quad x \in X_0.$$

Proof. See [Lio69], p. 58. □

Appendix B. Sobolev Spaces on manifolds and Strichartz estimates

In the Sections 3 and 7, we need some results about Sobolev spaces on manifolds and their connection with the fractional domains of the Laplace-Beltrami operator. In this appendix, we recall the basic definitions and Sobolev embeddings. Moreover, we state the deterministic and stochastic Strichartz estimates for the Schrödinger group $\left(e^{it\Delta_g}\right)_{t\in\mathbb{R}}$.

Let (M,g) be a d-dimensional riemannian manifold without boundary with

$$M \text{ is complete, has a positive injectivity radius and a bounded geometry.} \tag{B.1}$$

Definition B.1. a) Let $s \ge 0$, $p \in (1,\infty)$, $\mathcal{A} := (U_i,\kappa_i)_{i\in I}$ be an atlas of M and $(\Psi_i)_{i\in I}$ a partition of unity subordinate to $\mathcal{A}$. Then, we define the fractional Sobolev spaces $H^{s,p}(M)$ by

$$H^{s,p}(M) := \left\{ f \in L^p(M) : \|f\|_{H^{s,p}(M)} := \left(\sum_{i\in I} \|(\Psi_i f)\circ\kappa_i^{-1}\|^p_{H^{s,p}(\mathbb{R}^d)} \right)^{\frac{1}{p}} < \infty \right\},$$

where $H^{s,p}(\mathbb{R}^d)$ is the Sobolev space on $\mathbb{R}^d$. For $p = 2$, we write $H^s(M) := H^{s,2}(M)$.

b) For $p \in (1, \infty)$, we define $W^{1,p}(M)$ as the completion of $C^\infty(M)$ in the norm

$$\|f\|_{W^{1,p}(M)} := \|f\|_{L^p(M)} + \|\nabla f\|_{L^p(M)}, \qquad f \in C^\infty(M).$$

Note that in b), ∇f is an element of the tangential bundle of M. We refer to [Lab15] for further details. A useful characterization of fractional Sobolev spaces is in terms of the fractional powers of the Laplace-Beltrami operator. By Strichartz, [Str83] Theorem 3.5, the restriction of $\left(e^{t\Delta_g}\right)_{t\geq 0}$ to $L^2(M) \cap L^p(M)$ extends to a strongly continuous semigroup on $L^p(M)$. We fix $p \in (1, \infty)$ and $s > 0$. The generator $(\Delta_p, \mathcal{D}(\Delta_p))$ is called the Laplace-Beltrami operator on $L^p(M)$. The negative fractional powers of $I - \Delta_p$ are defined by

$$\mathcal{D}((I-\Delta_p)^{-\alpha}) := \left\{ f \in L^p(M) : \int_0^\infty t^{\alpha-1} e^{-t} e^{t\Delta_p} \mathrm{d}t \quad \text{exists} \right\}$$
$$(I-\Delta_p)^{-\alpha} f := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} e^{t\Delta_p} f \, \mathrm{d}t$$

for $\alpha > 0$. Note that in the case $p = 2$ this coincides with the definition via the functional calculus because of the identity $\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda t} f \mathrm{d}t = \lambda^{-\alpha}$ for $\lambda > 0$.
In the following Proposition, we list characterizations and embedding properties of the Sobolev spaces from Definition $B.1$.

Proposition B.2. *Let $s \geq 0$ and $p \in (1, \infty)$.*

a) *We have $H^{s,p}(M) = R((I-\Delta_p)^{-\frac{s}{2}})$ with $\|f\|_{H^{s,p}} \eqsim \|v\|_{L^p}$ for $f = (I-\Delta_p)^{-\frac{s}{2}} v$. Furthermore, we have $H^{1,p}(M) = W^{1,p}(M)$.*

b) *For $s > \frac{d}{p}$, we have $H^{s,p}(M) \hookrightarrow L^\infty(M)$.*

c) *Let $0 \leq s_2 < s_1 < \theta$ and $\frac{d}{d+\theta} < p_1, p_2 < \infty$ with*

$$s_1 - \frac{d}{p_1} \geq s_2 - \frac{d}{p_2} \tag{B.2}$$

Then, the embedding $H^{s_1,p_1}(M) \hookrightarrow H^{s_2,p_2}(M)$ holds. If the inequality (B.2) *is strict and M is compact, the embedding is compact.*

d) *For $s, s_0, s_1 \geq 0$ and $p, p_0, p_1 \in (1, \infty)$ and $\theta \in (0, 1)$ with*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

we have $[H^{s_0,p_0}(M), H^{s_1,p_1}(M)]_\theta = H^{s,p}(M)$.

Proof. ad a): See [Tri92], Theorem 7.4.5.
ad b): See [Bol15], Theorem III.1.2. d).
ad c): See [Yan02], Theorem 3.1, ii), for the strict inequality and [Yan03], Theorem 2 for the limit case. Note that the author considers the more general framework of Triebel-Lizorkin spaces $F^s_{p,q}(X)$ on spaces X of homogeneous type. The embedding in the case of fractional Sobolev spaces on compact manifolds is a consequence of the identity $H^{s,p}(M) = F^s_{p,2}(M)$ for $s \in \mathbb{R}$ and $1 < p < \infty$, see [Tri92], Section 7.4.5.
ad d): See [Tri92], section 7.4.5, Remark 2. □

In the next Lemma, we recall the deterministic homogeneous Strichartz estimate due to Bernicot and Savoyeau, see [BS14], Corollary 6.2.

Lemma B.3. *Let $\varepsilon > 0$, $T > 0$ and $2 < p < \infty$, $2 < q \leq \infty$ with*

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}, \qquad (q,p,d) \neq (2,\infty,2).$$

Then,

$$\|e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;L^q(M))} \lesssim_{T,\varepsilon} \|x\|_{H^{\frac{1+\varepsilon}{q}}(M)}, \qquad x \in H^{\frac{1+\varepsilon}{q}}(M). \tag{B.3}$$

We remark that in the special case of compact M, Burq, Gérard and Tzvetkov proved (B.3) even for $\varepsilon = 0$. But for our application in Section 7, this is not needed, such that we can prove uniqueness on non-compact manifolds with $d = 2$ and (B.1).

From Lemma $B.3$, one can deduce the following Strichartz estimates for the stochastic and deterministic convolutions in fractional Sobolev spaces. Note that we choose the probability space Ω and the Y-valued Wiener process W as in Assumption 2.7.

Lemma B.4. *In the situation of Lemma $B.3$, we take $s \in (\frac{1+\varepsilon}{q}, 1]$ and $r \in (1,\infty)$.*

a) *We have the homogeneous Strichartz estimate*

$$\|e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p}(M))} \lesssim_{T,\varepsilon} \|x\|_{H^s(M)} \tag{B.4}$$

for $x \in H^{\frac{1+\varepsilon}{q}}(M)$ and the inhomogeneous Strichartz estimate

$$\left\| \int_0^\cdot e^{\mathrm{i}(\cdot-\tau)\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p}(M))} \lesssim_{T,\varepsilon} \|f\|_{L^1(0,T;H^s(M))} \tag{B.5}$$

for $f \in L^1(0,T;H^s(M))$.

b) *We have the stochastic Strichartz estimate*

$$\left\| \int_0^\cdot e^{\mathrm{i}(\cdot-\tau)\Delta_g} B(\tau)\mathrm{d}W(\tau) \right\|_{L^r(\Omega,L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p}(M)))} \lesssim_{T,\varepsilon} \|B\|_{L^r(\Omega;L^2(0,T;\mathrm{HS}(Y,H^s(M))))} \tag{B.6}$$

for all adapted processes in $B \in L^r(\Omega;L^2(0,T;\mathrm{HS}(Y,H^s(M))))$.

Proof. Proposition $B.2$ a) and Lemma $B.3$ yield

$$\begin{aligned}
\|e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p}(M))} &\eqsim \|(1-\Delta_g)^{\frac{s}{2}-\frac{1+\varepsilon}{2q}} e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;L^p(M))} \\
&= \|e^{\mathrm{i}t\Delta_g}(1-\Delta_g)^{\frac{s}{2}-\frac{1+\varepsilon}{2q}} x\|_{L^q(0,T;L^p(M))} \\
&\lesssim_{T,\varepsilon} \|(1-\Delta_g)^{\frac{s}{2}-\frac{1+\varepsilon}{2q}} x\|_{H^{\frac{1+\varepsilon}{2q}}(M)} \eqsim \|x\|_{H^s(M)}.
\end{aligned} \tag{B.7}$$

From (B.7), we get

$$\left\| \int_0^\cdot e^{\mathrm{i}(\cdot-\tau)\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{L^q(0,T;H^{s-\frac{1+\varepsilon}{q},p}(M))} \lesssim_{T,\varepsilon} \left\| \int_0^\cdot e^{-\mathrm{i}\tau\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{H^s(M)} \lesssim \|f\|_{L^1(0,T;H^s(M))} \tag{B.8}$$

and Theorem 3.10 in [BM14] implies

$$\left\| \int_0^\cdot e^{\mathrm{i}(\cdot-\tau)\Delta_g} B(\tau)\mathrm{d}W(\tau) \right\|_{L^r(\Omega,L^q(0,T;L^p(M)))} \lesssim_{T,\varepsilon} \|B\|_{L^r(\Omega;L^2(0,T;\mathrm{HS}(Y,H^{\frac{1+\varepsilon}{q}}(M))))}. \tag{B.9}$$

With the same procedure as in (B.7), one can deduce the estimate (B.6). □

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