Extremal and Ramsey Type Questions for Graphs and Ordered Graphs

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Abstract

In this thesis we study graphs and ordered graphs from an extremal point of view. Our main concern are the following three questions.

First, we study the chromatic number of ordered graphs, where an ordered graph is an ordinary graph equipped with a linear ordering of its vertex set. We find infinite families of χ-avoidable ordered forests H, where an ordered graph H is χ-avoidable if there are ordered graphs of arbitrarily large chromatic number not containing H as an ordered subgraph. This is in contrast to the unordered setting where for each unordered forest H on k vertices it is known that any graph which contains no subgraph isomorphic to H has chromatic number at most $k-1$.

Second, we study Ramsey equivalence of graphs. Two graphs are Ramsey equivalent if they have exactly the same sets of Ramsey graphs. Fox *et al.* [67] ask whether there are non-isomorphic connected graphs that are Ramsey equivalent. While it is known that the only connected graph that is isomorphic to a complete graph is this graph itself, not much is known in general. We prove that graphs of different chromatic number are not Ramsey equivalent provided some additional clique splitting property holds. Under stronger assumptions, we give more pairs of graphs that are not Ramsey equivalent. Our results show that the question above has a negative answer for all pairs of connected graphs involving a graph on at most five vertices.

Finally, we initiate the study of minimal ordered Ramsey graphs. We give a full characterization of pairs of ordered graphs having a forest as a Ramsey graph. Further, we study the question which pairs of ordered graphs have infinitely many minimal ordered Ramsey graphs. The answer to this question is known in the unordered setting except for certain pairs involving 2-connected graphs. We prove that, like in the unordered setting, any ordered graph containing a cycle is Ramsey infinite, while a pair of a monotone matching and any other ordered graph is Ramsey finite. Here, a monotone matching is an ordered matching where for each pair of edges the endpoints of one edge precede both endpoints of the other edge. The existence of χ-avoidable ordered forests breaks the application of a central argument from the unordered setting for pairs of ordered forests. We reduce the question for χ-unavoidable ordered trees to pairs of ordered stars and so-called almost increasing caterpillars. Our results show that there are Ramsey finite pairs of ordered stars and ordered trees of arbitrarily large diameter, which is in stark contrast to unordered graphs where any Ramsey finite pair of forests involves a matching or is a pair of star forests.

Acknowledgements

This thesis is based on great support by many people over many years. First and foremost, I want to thank all of them, while I shall mention some particular people below.

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I would like to thank my wife Rafaela very much for never-ending encouragement and being my home at any times. Having you by my side makes me happily looking out to everything what comes along. Finally I want to thank my parents for supporting my way since the very beginnings.

1

Introduction

1.1 Outline of the Thesis

This thesis contributes to the chromatic theory and to Ramsey theory for graphs and ordered graphs.

Chromatic Theory A large part of research in graph theory is devoted to many different variants of colorings of graphs, where the vertices, the edges, or larger subgraphs are colored according to some rules. On the one hand colorings provide an easy way to express various properties of a graph in an accessible form. For example decompositions of the vertices into independent sets (sets which do not induce any edges) correspond to proper colorings of the vertices, that is, colorings where adjacent vertices receive distinct colors. This particular kind of coloring alone has many applications where the edges represent conflicts and the vertices shall be arranged into groups avoiding conflicts within the groups. Also many other applications naturally come with a graph coloring or labeling. For an overview on this field we refer to the book of Jensen and Toft [88]. Often, it is easy to achieve a desired type of coloring using a large amount of colors and, naturally, one is interested in the smallest number of colors that is needed. In this case it is particularly interesting to ask for which classes of graphs this number is bounded by an absolute constant. A frequently studied class of graphs in this area is formed by the t-degenerate graphs, for some given positive integer t. This class consists of all graphs that admit an ordering of their vertices such that each vertex is adjacent to at most t of its predecessors in the ordering. In Chapter 2 we study generalizations of this type of graph class by means of restricted vertex orderings with respect to the proper colorings mentioned above (see Sections 1.2 and 1.3 for an introduction).

Ramsey Theory Ramsey type results guarantee that a given (small and quite restricted) structure necessarily appears as a substructure of any other structure, provided this other structure is just sufficiently large. In terms of colorings this usually means that any large colored structure of some type needs to contain monochromatic substructures of certain kinds. Also Ramsey type results come in many different variants and with various applications. We refer to the book of Graham, Rothschild, and Spencer [75] and to a survey by Rosta [130]. While the fundamental results in this area establish the pure existence of some threshold for the "largeness", it is an interesting task to explore the boundary between "sufficiently large" and "still too small" in more detail. In this thesis we follow this line of research in two ways, both concerned with (small) monochromatic subgraphs appearing in edge-colorings of (sufficiently large) graphs. Chapter 3 is concerned with pairs of (small) graphs that have exactly the same set of graphs which are "sufficiently large" (see Section 1.4 for an introduction). We provide several results supporting our intuition that such a behavior is quite rare. In Chapter 4 we consider graphs that carry a linear ordering of their vertices and study the question for which (small) ordered graphs there are only finitely many smallest ordered graphs that are "sufficiently large" (see Section 1.5 for an introduction).

Results from Articles and Preprints Large parts of Chapter 2 and Chapter 3 appear as joined work with Maria Axenovich and Torsten Ueckerdt [8, 9, 7]. In particular all the main results of these two chapters are shared with these articles.

Outline of the Remaining Introduction In Section 1.2 we introduce our results from Chapter 2 on the chromatic number of ordered graphs under local constraints, together with an introduction to ordered graphs in general. A summary of similar results for (unordered) graphs is given in Section 1.3. Section 1.4 gives a brief overview of Ramsey theory, focusing on structural results. This includes the notion of Ramsey equivalence and our contribution to this field, which is studied in Chapter 3 in detail. In Section 1.5 we consider Ramsey theory for ordered graphs and present the results of Chapter 4 on minimal ordered Ramsey graphs. Finally Section 1.6 contains basic graph theoretic definitions, notations, and facts.

1.2 Ordered Graphs: Chromatic Number and Local Constraints

A large part of this thesis is concerned with ordered graphs. Such graphs appear at many places in graph theory and computer science either implicitly or explicitly. Implicitly, many inductions and algorithms on graphs work along an arbitrary or carefully chosen ordering of the vertices. Moreover many concepts like (generalized) coloring numbers [92, 145], bandwidth [37], or graceful labelings [15] are defined in terms of vertex orderings, but a systematic study seems not at hand. Some overview is given in [55] from a computer science point of view and in [21, 72] with only some part devoted to labelings in the sense of linear order. Ordered graphs also appear naturally in many applications, for instance in biology [52] and archeology [82, 91]. Here, we define an *ordered graph* as a graph equipped with a linear order of its vertex set. See Figure 1.1 for an illustration of ordered graphs.

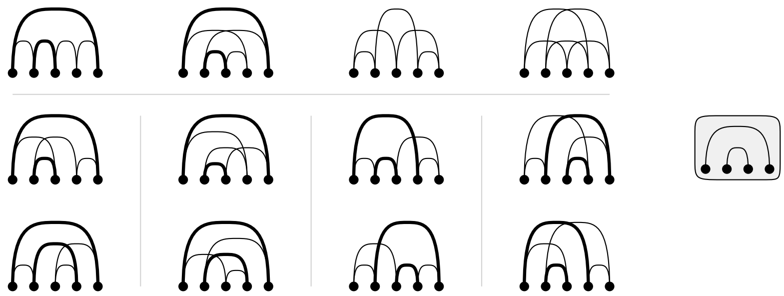


Figure 1.1: All pairwise non-isomorphic ordered 5-cycles. The ordered matching on the right is an ordered subgraph (bold edges) of all but two of these ordered cycles.

Recently, extremal properties of ordered graphs were studied with respect to their chromatic number [8, 7, 57], extremal number [57, 95, 121], and Ramsey number [11, 46]. The results of these articles show similarities but also significant differences between graphs and ordered graphs. In the following paragraph we introduce the results of our work from [8] and [7] which are discussed in Chapter 2 in detail. The results on extremal numbers are briefly outlined afterwards and the results of [11, 46] are presented in Section 1.5. Some other research on ordered graphs includes growth rates of hereditary properties of ordered graphs [12, 18], characterizations of classes of graphs by forbidden ordered subgraphs [51, 74], and the study of perfectly ordered graphs [41].

Our Contribution In Chapter 2 we study the chromatic number of ordered graphs from an extremal point of view. Here the chromatic number of an ordered graph G is the smallest number of color among colorings of the vertices of G where adjacent vertices receive different colors. Specifically we investigate for which ordered graphs H any ordered graph not containing H as a subgraph has bounded chromatic number. In Chapter 4 we initiate the study of minimal ordered Ramsey graphs. In particular we identify several classes of Ramsey finite as well as several classes of Ramsey infinite ordered graphs. While we give a brief outline of the results from Chapter 2 next, we refer to Section 1.5 for an introduction to the latter results.

Given an (unordered) forest H on k vertices it is well known that the chromatic number of any graph not containing H as a subgraph is at most $k-1$. Similar results also hold for hypergraphs [101] and directed graphs [25]. A summary of results of this kind for (unordered) graphs is given in Section 1.3. Surprisingly, the situation is different for ordered graphs. Call an ordered graph H *χ-avoidable* if for each integer k there is an ordered graph of chromatic number at least k which does not contain H as an ordered subgraph. It is easy to see that any ordered graph containing a cycle is χ-avoidable. However, we also describe infinite families of χ-avoidable ordered forests, called bonnets and tangled paths. We refer to Figure 1.2 for some examples of such ordered graphs and to Section 2.1 for proper definitions. Specifically we prove the following theorem.

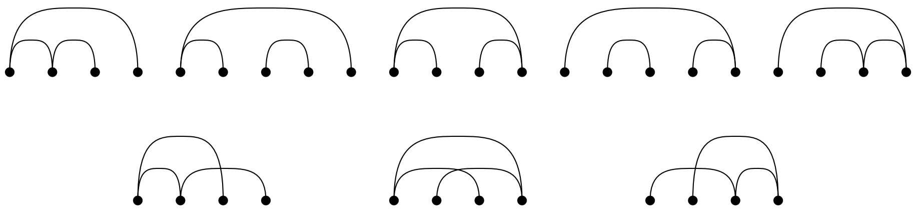


Figure 1.2: Some χ-avoidable ordered forests. The first row shows all bonnets, the second row shows some tangled paths.

Theorem 2.1. *If an ordered graph H contains a cycle, a bonnet, or a tangled path, then H is χ-avoidable.*

On top of that, these ordered forests are fairly simple to describe, the smallest ones on four or five vertices with only three edges. This clearly opens the search for a characterization of χ-avoidable ordered forests.

We characterize all so-called non-crossing ordered graphs that are χ-avoidable. The minimal χ-avoidable non-crossing ordered graphs are exactly the five bonnets (shown in the first row of Figure 1.2) and an ordered complete graph on three vertices. Note that any ordered supergraph of some χ-avoidable ordered graph is χ-avoidable itself. For crossing and connected ordered graphs we reduce the problem of characterizing such χ-avoidable ordered graphs to a well behaved class of trees, called monotonically alternating. Despite many insights, a full characterization of χ-avoidable ordered forests remains open.

For an ordered forest H that is not χ-avoidable we are also interested in the largest chromatic number of ordered graphs not containing H as an ordered subgraphs. We give several results in this direction. For non-crossing and connected such ordered forests H on k vertices this number is at most linear in k, while for non-crossing disconnected such H it is at most exponential. We do not have any upper bound for arbitrary such H and no good lower bounds in general.

Our results also have implications for other areas. In Section 1.5 we show some consequences for Ramsey theory of ordered graphs.

Extremal Numbers Some of the results mentioned in the previous paragraph rely on methods or results for extremal numbers of ordered graphs. The *ordered extremal number* $\mathrm{ex}_<(n, H)$, for an ordered graph H and a positive integer n, is the largest number of edges in an ordered graph on n vertices that does not contain a copy of H. Pach and Tardos [121] study the general behavior of ordered extremal numbers in detail. An *interval* of an ordered graph G is a set I of consecutive vertices of G, that is, for any $u, v \in I$, $z \in V(G)$ with $u \leqslant z \leqslant v$ we have $z \in I$. The *interval chromatic number* $\chi_<(G)$ is the smallest number of intervals, each inducing an independent set, needed to partition the vertices of an ordered graph G. The following statement for the extremal number $\mathrm{ex}_<(n, G)$ of ordered graphs is analogous to the Erdős-Stone-

Simonovits theorem [65], where the chromatic number is replaced by the interval chromatic number.

Theorem 1.1 ([121]). *Let G be an ordered graph. Then*

$$\mathrm{ex}_<(n,G) = (1 - \tfrac{1}{\chi_<(G)-1})\binom{n}{2} + o(n^2).$$

For ordered graphs with interval chromatic number 2, Pach and Tardos find a tight relation between the ordered extremal number and pattern avoiding matrices. For an ordered graph H with $\chi_<(H) = 2$ let $A(H)$ denote the 0-1-matrix where the rows correspond to the vertices in the first color and the columns to the vertices in the second color of a proper interval coloring of H in 2 colors and let $A(H)_{u,v} = 1$ if and only if uv is an edge in H. A 0-1-matrix B *avoids* another 0-1-matrix A if there is no submatrix in B which becomes equal to A after replacing some ones with zeros. For a 0-1-matrix A let $\mathrm{ex}(n,A)$ denote the largest number of ones in an $n \times n$ matrix avoiding A. In [121] it is shown that for each ordered graph H with $\chi_<(H) = 2$ there is a constant c such that $\mathrm{ex}(\lfloor \frac{n}{2} \rfloor, A(H)) \leqslant \mathrm{ex}_<(n,H) \leqslant c\,\mathrm{ex}(n, A(H)) \log n$.

Some of the extensive research on forbidden binary matrices and extremal functions for ordered graphs can be found in [22, 57, 71, 94, 95, 104].

Algorithmic Observations In order to decide whether two ordered graphs are isomorphic it is sufficient to check for each pair of positions in the vertex order whether the corresponding vertices form an edge or a non-edge in both ordered graphs. This clearly leads to an algorithm running in polynomial time (actually linear time in the number of edges).

Observation 1.1. *Whether two ordered graphs are isomorphic can be decided in polynomial time. In particular, for each fixed ordered graph H there is an algorithm that checks whether an ordered graph G contains a copy of H in time polynomial in the order of G.*

It is still not known whether for (unordered) graphs the graph isomorphism problem is in P or not. Recently, Babai [10] announced an algorithm that checks isomorphism of two graphs on n vertices in time $\exp(\log(n)^c)$ for some constant c (called quasi polynomial time). A common approach in this area is the computation of so-called canonical labelings (that is, a canonical vertex ordering), such that isomorphic graphs have the same canonical labeling [106]. This is the crucial part of such algorithms, as Observation 1.1 shows that checking whether two canonical labelings are equal is clearly polynomial.

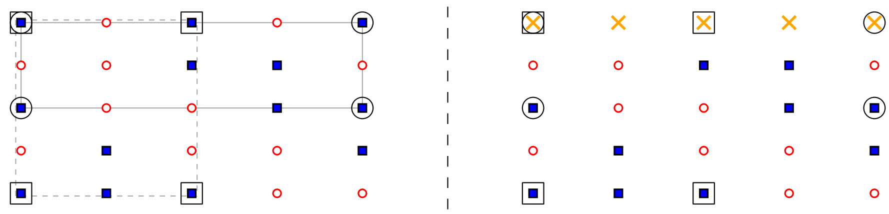


Figure 1.3: A 4-uniform hypergraph where hyperedges are formed by the four corners of any axis-aligned rectangle. One can see that for any 2-coloring (small disks and boxes) of the points there is a hyperedge with all its vertices of the same color (left), while the 3-coloring (small disks, boxes, and crosses) on the right is proper. So the chromatic number of this hypergraph is 3.

1.3 Chromatic Number of Graphs and Local Constraints: A Summary

In Chapter 2 below we consider the chromatic number of ordered graphs under local constraints. Since this type of question is studied extensively for (unordered) graphs we summarize some of the results from this area in this section. Recall that the *chromatic number* $\chi(H)$ of a (hyper)graph H is the smallest number of colors needed to color the vertices of H such that each edge of H contains at least two vertices of distinct colors. See Figure 1.3 for an illustration of this so-called proper colorings. Further, a *local constraint* means some property that is determined by the structure of subgraphs of constant size. The general question is whether some given constraints imply any upper bound on the chromatic number of the graph. Of course, the answer heavily depends on the constraints.

Forbidden Subgraphs First, we study graphs that do not contain subgraphs of some given kind. To begin with, a universal upper bound on the degrees of the vertices is strong enough to give an upper bound on the chromatic number. Indeed, it is easy to see that any graph of maximum degree d has chromatic number at most $d+1$. On the other hand consider some integer g and graphs of *girth* more than g, that is, graphs that do not contain any cycle of length at most g. Each set of g vertices of such a graph induces not more than a tree, which is easily colored with two colors in a proper way. A prominent result of Erdős [61] shows that there are such graphs of arbitrarily large chromatic number, that is, this constraint does not give any upper bound on the chromatic number.

For a set of graphs $\mathcal{H}$ let Forb($\mathcal{H}$) denote the set of graphs that do not contain any member of $\mathcal{H}$ as a subgraph. Now the question is whether there is a constant k that depends on $\mathcal{H}$ only such that $\chi(G) \leqslant k$ for each $G \in \text{Forb}(\mathcal{H})$. If such a number exists then call $\mathcal{H}$ *χ-unavoidable* and let $\kappa(\mathcal{H}) = \max\{\chi(G) \mid G \in \text{Forb}(\mathcal{H})\}$, otherwise call $\mathcal{H}$ *χ-avoidable* and let $\kappa(\mathcal{H}) = \infty$. In case $\mathcal{H} = \{H\}$ we shall write Forb(H), $\kappa(H)$, and call H itself χ-avoidable or χ-unavoidable respectively. The results above state that $\kappa(K_{1,d+1}) \leqslant d+1$ and $\kappa(\mathcal{H}) = \infty$ for any family $\mathcal{H}$ of graphs that contain a cycle each. More generally it is known that for each positive integer k we have $\kappa(H) = k$ if and only H is a forest on $k+1$ vertices. In particular we have the following observation.

Observation 1.2. *A graph H is χ-avoidable if and only if H contains a cycle.*

This statement generalizes to uniform hypergraphs due to results of [64, 101], see also [79]. A similar statement also holds for directed graphs H, with a similarly defined function $\kappa_{\text{dir}}(H)$ being finite if and only if the underlying (undirected) graph of H is acyclic. A result of Addalirio-Berry *et al.* [2], see also [25], implies that $\kappa_{\text{dir}}(H) \leqslant k^2/2 - k/2 - 1$ whenever H is a directed k-vertex graph whose underlying graph is acyclic. Dujmović and Wood [57] study a similar question for certain classes of χ-unavoidable ordered matchings. In Section 2.1 we study which ordered graphs are χ-avoidable. As mentioned above we shall present χ-avoidable ordered forests.

Forbidden Induced Subgraphs Another well studied constraint are forbidden induced subgraphs. Consider the family of all graphs not containing a given non-complete graph H as an induced subgraph. This family contains all complete graphs and hence has unbounded chromatic number. To overcome this problem one is interested how much the chromatic number of a graph G in such a family deviates from the order of the largest clique in G, denoted by $\omega(G)$. If for some family $\mathcal{G}$ of graphs there is a function $f_{\mathcal{G}}$ such that for each $G \in \mathcal{G}$ we have $\chi(G) \leqslant f_{\mathcal{G}}(\omega(G))$, then $\mathcal{G}$ is called *χ-bounded*. That is, a family $\mathcal{G}$ is χ-bounded if there is an upper bound on the chromatic number for each $G \in \mathcal{G}$ that depends only on $\mathcal{G}$ and $\omega(G)$. For example, the famous strong perfect graph theorem [38] states that if a graph G contains neither an induced odd cycle nor an induced copy of a complement of an odd cycle, then $\chi(G) \leqslant \omega(G)$. So the family of such graphs is χ-bounded. On the other hand, the family of graphs with no induced cycle of length g is not χ-bounded, as it contains all graphs of girth at least $g+1$. A famous conjecture of Gyárfás and Sumner states that for any forest H the family of graphs with no induced copies of H is χ-bounded. Recently Chudnovsky, Scott, and Seymour [39] made some progress towards this and many related conjectures. Observe that a family $\text{Forb}(\mathcal{H})$ is χ-bounded if and only if $\mathcal{H}$ is χ-unavoidable, as $\text{Forb}(\mathcal{H})$ does not contain large cliques.

Other Types of Constraints There are many other types of constraints known that give bounds on the chromatic number. We mention two such constraints here which are based on vertex orderings. We consider an ordered vertex set laid out along a horizontal line from *left* to *right* such that a vertex u is to the left of a vertex v if and only if $u < v$. The *coloring number* $\text{col}(G)$ of a graph G is the smallest integer t such there is an ordering of the vertices of G where each vertex

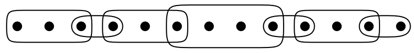


Figure 1.4: A 7-chain.

has at most $t-1$ neighbors to the right. It is easy to see that $\chi(G) \leqslant \text{col}(G)$. Zhu [145] gives tight relations between many other coloring parameters (specifically low tree-depth colorings) and generalized coloring numbers of graphs.

A *k-chain* is a (not necessarily uniform) hypergraph P with vertices $u_1 < \cdots < u_n$ and edges $E_1, \ldots, E_k$ such that each edge forms an interval in the ordering of P and $|E_i \cap E_{i+1}| = 1$, $i \in [k-1]$. See Figure 1.4 for an example. Similar to the Gallai-Hasse-Roy-Vitaver Theorem, Pluhár [124] shows that for each $k \geqslant 2$ an (unordered, not necessarily uniform) hypergraph admits a proper k-coloring of its vertices if and only if it admits an ordering of its vertices without any copy of a k-chain. We include a proof here for completeness. If a hypergraph $\mathcal{H}$ is k-colorable consider a partition $V(\mathcal{H}) = V_1 \dot\cup \cdots \dot\cup V_k$ such that no V_i contains an edge of $\mathcal{H}$, $i \in [k]$. Then each ordering of $V(\mathcal{H})$ where the vertices in V_i precede the vertices in V_{i+1}, $i \in [k-1]$, does not yield any copy of a k-chain. The other way round assume that the vertices of $\mathcal{H}$ are ordered without any copy of a k-chain. Color each vertex $u \in V(\mathcal{H})$ that is rightmost in some edge of $\mathcal{H}$ with the largest integer c such that u is rightmost in some copy of a c-chain in $\mathcal{H}$. Color all other vertices in color 0. This coloring uses at most k colors since there is no copy of a k-chain. Moreover it is proper, since for each edge in $\mathcal{H}$ the colors of the leftmost and the rightmost vertex are distinct.

Observe that this proof gives a proper k-coloring for any ordered hypergraph (that is, a hypergraph with a fixed linear ordering of its vertices) which does not contain any copy of a k-chain. This is exactly the kind of constraint which we consider in Section 2.1, that is, a forbidden ordered subgraph. Besides excluding some ordered subgraphs, other local constraints for ordered graphs exist that are based on the ordering. For example we might require that there are no short edges or any six vertices induce either two edges ab and cd with $a < c < b < d$ or two edges uv and xy with $u < x < y < v$, but not both. We consider a general framework to model many different constraints in Section 2.2.

Other Types Of Colorings Recently several groups of authors build on a uniform way to model different kinds of vertex colorings of hypergraphs [23, 36, 58]. The general idea is to prescribe for each edge of the hypergraph a set of admissible partitions into color classes. For instance, for the usual chromatic number any partition with at least two non-empty parts is admissible. It turns out that for many kinds of colorings there are hypergraphs which do not admit any coloring of this kind in any number of colors. Among many interesting results and questions in this area, Bujtás and Tuza [24] show that there are such kinds of colorings where it is NP-complete to decide whether a given hypergraph is colorable, even for interval

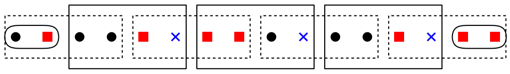


Figure 1.5: This interval hypergraph has no vertex-coloring where each edge of size two contains two vertices of different colors, the dashed edges contain at most two different colors, and each of the remaining edges contains at least three vertices of distinct colors.

hypergraphs (that is, there is an ordering of the vertices such that each edge forms an interval). See Figure 1.5 for an example of an uncolorable interval hypergraph. Note that "being an interval hypergraph" is clearly a local constraint in the sense as considered in Chapter 2, provided that the vertex ordering is fixed. Note further that many kinds of colorings in this area require edges of size more than 2, since they become trivial or do not exist for graphs.

1.4 Ramsey Theory for Graphs: Ramsey Equivalence

The origins of Ramsey theory date back to two theorems of Ramsey [126] from 1930, although some earlier results are nowadays called to be of "Ramsey type" as well. Theorem A from [126] states that no matter how the r-subsets of an infinite set A are colored using k colors, there will be an infinite subset A' of A such that all the r-subsets of A' are of the same color. Theorem B from [126] is a finite version of this statement. We give a formulation of Theorem B in terms of hypergraphs here.

Theorem 1.2 ([126])**.** *For any positive integers k, n, and r there is an integer N such that for each k-coloring of the edges of a complete r-uniform hypergraph K on N vertices, there is a copy of a complete r-uniform hypergraph on n vertices in K with all edges of the same color.*

While Ramsey was motivated by logic (he studied the question whether any sufficiently large logical expression of certain kind can be verified by a small so-called canonical model), this phenomenon quickly attracted lots of attention on its own. In particular when a few years later Erdős and Szekeres [60] reproved Ramsey's theorem along with Ramsey type results for integer sequences and points in the plane. For a broad introduction to the field we refer to the books of Graham, Rothschild, and Spencer [75] and Prömel [125], and to a recent survey of Conlon, Fox, and Sudakov [48]. See Figure 1.6 for an example of a Ramsey type result for points in the Euclidean plane and an example of a Ramsey type result for graphs. In this thesis we shall not consider such geometric type of questions but focus on graphs instead.

Naturally one might ask for the smallest N that satisfies Theorem 1.2 for given k, n, and r. Determining this number, called the *Ramsey number*, is a famous problem

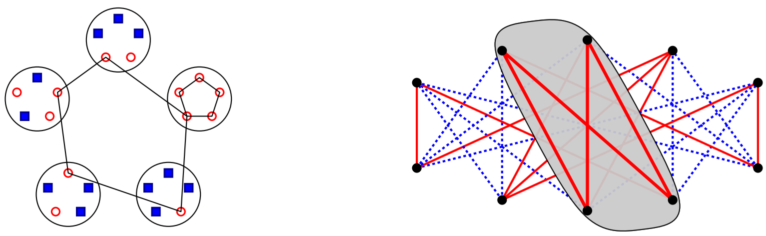


Figure 1.6: In any two coloring of the points on the left side there is a convex 5-gon with all corners of the same color. In any two coloring of the edges of the complete bipartite on the right side there is a 4-cycle with all edges of the same color. In both cases we actually find several such monochromatic objects.

in (graph) Ramsey theory. Here we focus on graphs and two colors, that is the case $k = r = 2$, and denote the Ramsey number by $r(K_n)$. In this case the Ramsey number is known exactly only for $n \leqslant 4$, and for larger n standard bounds of the form $2^{n/2} \leqslant r(K_n) \leqslant 2^{2n}$ were improved only slightly so far [45, 138]. Over the years Ramsey numbers and related concepts were studied for various classes of graphs, like degenerate graphs [100], bipartite graphs [44], or geometric graphs [63, 89], just to name a few.

Ramsey Graphs Ramsey's theorem implies that for any graph H there is a graph F such that for any coloring of the edges of F in two colors there is a monochromatic copy of H, that is, one with all edges of the same color. In this case we call the graph F a *Ramsey graph* of H and write $F \to H$ to indicate this fact[1].

Question 1.1. *How does $R(H) = \{F \mid F \to H\}$ look like for a given graph H?*

A full characterization of all Ramsey graphs is known for few classes of graphs like small matchings or stars [20, 31]. For example $R(2K_2) = \{F \mid 3K_2 \subseteq F \text{ or } C_5 \subseteq F\}$ and $R(K_{1,2}) = \{F \mid \Delta(F) \geqslant 3 \text{ or } \chi(F) \geqslant 3\}$ [31]. Therefore particular properties of the set $R(H)$ and its members are studied. This line of research was initiated by fundamental work of Nešetřil and Rödl [113] and Burr, Erdős, and Lovász [31]. Observe that the Ramsey number itself is the smallest order of graphs in $R(H)$. Apart from the order we may study any other graph parameter for graphs in $R(H)$. We shall summarize some of the known results for different parameters later in this section. Before that we outline our contribution to this area.

Ramsey Equivalence Only recently, Szabó *et al.* [139] observe that there are graphs having exactly the same set of Ramsey graphs and coined the term Ramsey equivalence for this phenomenon. So two graphs G and H are *Ramsey equivalent* if $R(G) = R(H)$. We write $G \stackrel{R}{\sim} H$ if G is Ramsey equivalent to H, and write $G \stackrel{R}{\not\sim} H$ otherwise. A series of papers [16, 67, 139] is concerned with graphs that

[1]Some authors use this notation with the arrow pointing from H to F

are Ramsey equivalent to complete graphs. They prove that any graph G which is Ramsey equivalent to a complete graph K_t, for some $t \geqslant 3$, is a vertex disjoint union of K_t and some graph of smaller clique number. While it is quite easy to find non-isomorphic Ramsey equivalent graphs in general (adding some isolated vertices to a graph usually yields a Ramsey equivalent graph, see Observation 3.1), it is not clear at all whether there is a Ramsey equivalent pair of non-isomorphic connected graphs. The result above shows that there is no such pair involving a complete graph. We give a more detailed discussion of these results in the beginning of Chapter 3.

Question 1.2 ([67]). *Are there two non-isomorphic connected graphs G and H with $G \overset{R}{\sim} H$?*

Our Contribution In Chapter 3 we identify many pairs of graphs that are not Ramsey equivalent. To this end we find graph parameters ρ such that $\rho(G) \neq \rho(H)$ implies that G and H are not Ramsey equivalent. Here, the difference in ρ helps to find a Ramsey graph of one of G or H that is not a Ramsey graph of the other. For example it is known that for each graph G there is $F \in R(G)$ with the same clique number as G [116]. Clearly, such F is not a Ramsey graph graph for any graph H of larger clique number than G and hence such H is not Ramsey equivalent to G. The only other structural parameter with this property that we know is the odd girth [115].

We prove that graphs of different chromatic number are not Ramsey equivalent, provided some additional clique splitting property holds. A graph is called *clique-splittable* if its vertex set can be partitioned into two subsets, each inducing a subgraph of smaller clique number.

Theorem 3.3. *If G and H are graphs, G is clique-splittable, and $\chi(G) < \chi(H)$, then $G \overset{R}{\not\sim} H$.*

Additionally we also handle graphs of the same chromatic number, although we require stronger assumptions on the graphs in this case. Our results show that for each connected graph G on at most five vertices there is no connected graph that is Ramsey equivalent to G and not isomorphic to G itself. So our results provide some evidence for a negative answer to Question 1.2, while a full answer remains open.

We also relate the notion of Ramsey equivalence to other Ramsey type results, like multicolor Ramsey numbers. If H is a subgraph of G and there is a graph F which is a Ramsey graph in k colors for H but not for G, for some $k \geqslant 2$, then G and H are not Ramsey equivalent (in two colors). Whether this also holds for graphs that are not in subgraph relation remains open. We discuss this and further open questions in the conclusions in Section 3.5.

Finally we include a result on Ramsey numbers of cycles with pendant or independent edges (which is not part of [9]).

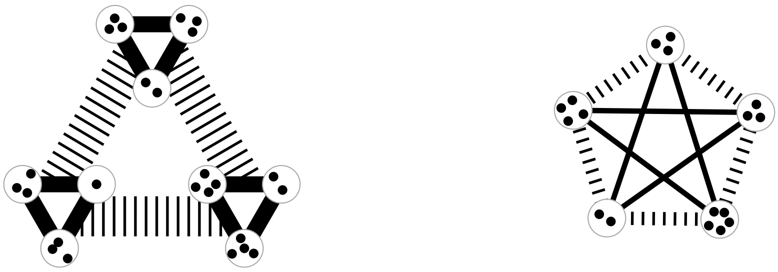


Figure 1.7: A 2-coloring (in bold and dashed) of the edges of a 9-chromatic graph where each color induces a subgraph of chromatic number at most 3 (left), and a 2-coloring of the edges of a 5-chromatic graph without monochromatic triangles (right). In both graphs encircled vertices form independent sets.

Properties of Ramsey Graphs In the remaining paragraphs of this section we summarize several structural results on the set of Ramsey graphs which are related to our results in Chapter 3 or Chapter 4. For a given graph parameter ρ that maps graphs to real numbers let $r_\rho = \min\{\rho(F) \mid F \in R(H)\}$, if this minimum exists. We shall summarize some of the known results for different parameters ρ next.

The *size Ramsey number* [62] is $r_e(H) = \min\{|E(F)| \mid F \in R(H)\}$. An argument contributed to Chvátal shows that $r_e(K_t) = \binom{r(K_t)}{2}$ while it is known that the size Ramsey number of any tree and any cycle is linear in the number of vertices [81]. Rödl and Szemerédi [129] show that for larger degrees any upper bound on $r_e(H)$ needs to depend on the order of H. Indeed, they provide for each n a construction of a graph H on at least n vertices of maximum degree 3 with $r_e(H) \geqslant cn(\log n)^\alpha$, for some absolute constants c, $\alpha > 0$. Kohayakawa *et al.* [97] announce a proof of a conjecture from [129], namely that for each d there are constants c, $\epsilon > 0$ such that for each n-vertex graph H of maximum degree d we have $r_e(H) \leqslant cn^{2-\epsilon}$. Regarding the maximum degree Δ the authors of [93] ask whether $r_\Delta(H)$ is bounded from above by a function depending on the maximum degree $\Delta(H)$ only. They prove that if H is a tree, then $2\Delta(H) - 1 \leqslant r_\Delta(H) \leqslant 4(\Delta(H) - 1)$. We observe that for a non-bipartite graph H, $r_\Delta(H) \geqslant 2\Delta(H)$, see Lemma 3.2.16. Burr *et al.* [31] prove that for any graph H with $\chi(H) = k$ we have $(k-1)^2 + 1 \leqslant r_\chi(H) \leqslant r(K_k)$. See Figure 1.7 for an illustration of the lower bound (left) and the tightness of the upper bound for complete graphs (right). While the upper bound is tight for any perfect graph H (it contains K_k), they conjecture that for each $k \geqslant 2$ there is a graph H with $\chi(H) = k$ and $r_\chi(H) = (k-1)^2 + 1$. Results of Zhu [146] and Paul and Tardif [123] show that this conjecture holds. Rödl and Ruciński [128] (see Theorem 1.4 below) give a threshold t_H for any graph H that is not a forest of stars or paths on three edges, such that almost all graphs with density larger than t_H are Ramsey graphs for H and almost all with smaller density are not, where the *density* of a graph G is $m(G) = \max\{|E(G')|/|V(G')| \mid G' \subseteq G\}$. Remarkable results of Nešetřil and Rödl show that for each graph H we have $r_\omega(H) = \omega(H)$ [116]

and $r_{-\mathrm{girth}_o}(H) = \mathrm{girth}_o(H)$ [115], provided $\mathrm{girth}_o(H) \neq \infty$ (here we consider the negative odd girth, denoted $-\mathrm{girth}_o$, since we are interested in the largest odd girth among graphs in $R(H)$). For example one can see that $F \in R(K_3)$ with $\omega(F) = \omega(K_3)$ where F is a graph obtained from $K_{3,5}$ by adding a triangle in the part of size 3 and a 5-cycle in the part of size 5. Similarly it is known that if H does not contain a copy of $K_{a,b}$, $a, b \geqslant 3$ or $1 = a \leqslant b \leqslant 2$, then there is $F \in R(H)$ such that F does not contain a copy of $K_{a,b}$ [118]. It is an open question whether this also holds for $r_{-\mathrm{girth}}$, that is, whether for each graph H there is $F \in R(H)$ with $\mathrm{girth}(H) = \mathrm{girth}(F)$ [111].

These results, among others, lead to the notion of Ramsey classes. We describe this concept only for the very special case of edge colorings of graphs here, although it leads to far reaching generalizations and deep results, see [86, 110]. A class $\mathcal{F}$ of graphs is a *Ramsey class* if for any of its members $G \in \mathcal{F}$ there is a Ramsey graph of G in $\mathcal{F}$, that is $R(G) \cap \mathcal{F} \neq \emptyset$. The results above show that for any integer k, $k \geqslant 3$, the families $\{G \mid \omega(G) < k\}$ and $\{G \mid \mathrm{girth}_o(G) \geqslant k\}$ are Ramsey classes. Many results on Ramsey classes deal with induced subgraphs or ordered graphs [112, 117].

Minimal Ramsey Graphs Observe that any supergraph of a Ramsey graph is a Ramsey graph itself. Therefore, in order to describe $R(H)$ it is sufficient to describe the *minimal Ramsey graphs* of H, that is, the graphs in $R(H)$ where each proper subgraph is not in $R(H)$ anymore. A fundamental question asks whether for a given graph H there are infinitely many minimal Ramsey graphs. A series of several results establishes the following theorem.

Theorem 1.3 ([27, 66, 114, 127, 128])**.** *A graph H has only finitely many minimal Ramsey graphs if and only if H is a vertex disjoint union of a star with an odd number of edges, a matching, and isolated vertices.*

Besides the number of minimal Ramsey graphs we may again study any graph parameter for these graphs. Observe that all the parameters ρ considered in the previous paragraph are subgraph monotone in the sense that $\rho(H') \leqslant \rho(H)$ if H' is a subgraph of H. Moreover, if $r_\rho(H)$ exists for some graph H and some subgraph monotone parameter ρ, then there is a minimal Ramsey graph F of H with $\rho(F) = r_\rho(H)$. Furthermore observe that for parameters ρ that are not subgraph monotone, r_ρ might be trivial. For example it is easy to see that every graph H has a Ramsey graph of minimum degree 0 by adding an isolated vertex to some graph in $R(H)$. To overcome this issue we modify the definition of r_ρ and, slightly abusing notation, let $r_\rho = \min\{\rho(F) \mid F \in R(H), F \text{ minimal}\}$. As argued above this does not make any difference for the parameters considered in the previous paragraph. A number of results are known for the minimum degree δ. We have $r_\delta(K_t) = (t-1)^2$, [31, 69], $r_\delta(K_{a,b}) = 2\min\{a,b\} - 1$, [69], $r_\delta(H) = 1$ if H is a tree, [139], or when H is $K_{t,t}$ plus a pendant edge, [67], and $r_\delta(C_n) = 3$ for even $n \geqslant 4$, [139]. Further results deal with minimum degrees of minimal Ramsey graph for more colors [68] and of minimal Ramsey hypergraphs [43]. Besides the results already stated, Burr *et al.* [31] also consider the smallest maximum degree of Ramsey graphs, as well as the largest maximum degree and the smallest connectivity of minimal Ramsey graphs. Finally let us state an observation on the largest order of minimal Ramsey graphs.

Observation 1.3. *A graph H has infinitely many minimal Ramsey graphs if and only if the order of minimal Ramsey graphs of H is unbounded, that is, for each integer n there is a minimal Ramsey graph F of H with $|V(F)| \geqslant n$.*

Ramsey Finite Graphs A graph is called *Ramsey finite* if it has only finitely many non-isomorphic minimal Ramsey graphs, otherwise it is *Ramsey infinite*. Here we summarize known results on Ramsey finite pairs of graphs. We study Ramsey finite pairs of ordered graphs in Chapter 4 (see Section 1.5 for an introduction). For some of the results below we find similar results in the ordered setting, while others do not have a corresponding result for ordered graphs.

The following results are due to to Rödl and Ruciński [127, 128], see also Nenadov and Steger [109] for an alternative proof. The 2-*density* of a graph G is $m_2(G) = \max\left\{\frac{|E(H)|-1}{|V(H)|-2} \mid H \subseteq G, |V(H)| \geqslant 3\right\}$. Let $G(n,p)$ denote a random graph on n vertices that contains each edge with probability p independently from the other edges.

Theorem 1.4 ([128])**.** *Let H be a graph that is not a forest of stars or paths on three edges. Then there are positive constants $c = c(H)$ and $C = C(H)$ such that*

$$\lim_{n\to\infty} \operatorname{Prob}(G(n,p) \to H) = \begin{cases} 0, & \text{if } p \leqslant c\, n^{-1/m_2(H)}, \\ 1, & \text{if } p \geqslant C\, n^{-1/m_2(H)}. \end{cases}$$

Lemma 1.4.1 ([127])**.** *Let F and H be graphs with $m_2(H) > 1$. If $\max_{F' \subseteq F} \left(\frac{|E(F')|}{|V(F')|}\right) \leqslant m_2(H)$, then $F \not\to H$.*

Note that $m_2(H) > 1$ if and only if H contains a cycle. Combining these two results one can show that for each graph H containing a cycle and each integer t there is a constant c' such that with high probability (probability tending to 1 as $n \to \infty$) a random graph $G = G(n, c'n^{-1/m_2(H)})$ is a Ramsey graph for H while each subgraph of G on t vertices is not Ramsey (Corollary 4(a) [128]). Observation 1.3 yields the following corollary (see [102]).

Corollary 1.5 ([127, 128])**.** *If a graph G contains a cycle, then G is Ramsey infinite.*

An analog of Theorem 1.4 in the asymmetric case is not known in general. Kohayakawa and Kreuter [96] conjecture that Theorem 1.4 holds for any pair of graphs (H_1, H_2) with $m_2(H_1) \leqslant m_2(H_2)$ using an asymmetric version of the 2-density given by $m_2(H_1, H_2) = \max\left\{\frac{|E(H')|}{|V(H')|-2+1/m_2(H_1)} \mid H' \subseteq H_2, |E(H')| \geqslant 1\right\}$.

H, H' cyclic	$H = H'$ $\Rightarrow$ infinite	$H \neq H'$ partial results
H cyclic H' forest	H' matching $\Rightarrow$ finite	otherwise infinite
H, H' forests		otherwise finite $\Leftrightarrow$ H, H' special star forests

Table 1.1: Summary of results on Ramsey finiteness for graphs H and H'.

The 1-statement of this conjecture, that is, the case $p \geqslant C\, n^{-1/m_2(H_1,H_2)}$, holds for (H_1, H_2) under certain conditions on H_1 and H_2 [80, 98, 103], while the 0-statement, that is, the case $p \leqslant c\, n^{-1/m_2(H_1,H_2)}$, is wide open [78, 103].

Nešetřil and Rödl [114] prove that (H, H') is Ramsey infinite if both H and H' are 3-connected or both are of chromatic number at least 3. Results in [32] show that it is sufficient to consider 2-connected graphs. Bollobás *et al.* [19] prove that (H, C_k) is Ramsey infinite for each cycle C_k if H is 2-connected and contains no induced cycles of length at least ℓ, provided $k \geqslant \ell \geqslant 4$.

Each pair of a forest and a graph containing a cycle is handled by one of the following two theorems.

Theorem 1.6 ([29])**.** *If H is a matching, then (H, H') is Ramsey finite for each graph H'.*

Theorem 1.7 ([102])**.** *If H is a forest that is not a matching and H' is a graph that contains a cycle, then (H, H') is Ramsey infinite.*

Based on several previous results [27, 28, 114], Faudree [66] gives a characterization of all Ramsey finite pairs of forests, up to the value of some parameter n_0. Note that a pair (G, G') of graphs is Ramsey finite if and only if $(G + K_1, G')$ is Ramsey finite. Hence it is sufficient to consider graphs without isolated vertices. A summary of all the results stated here is given in Table 1.1.

Theorem 1.8 ([66])**.** *Let H, H' be forests without isolated vertices. Then there is n_0 such that (H, H') is Ramsey finite if and only if one of the following statements holds.*

(a) At least one of H or H' is a matching.

(b) Both H and H' are vertex disjoint unions of a matching and a star with an odd number of edges.

(c) One of H or H' is a vertex disjoint union of a matching and at least two stars with m_1 respectively m_2 edges, while the other is a vertex disjoint union of a matching on n edges and a star with n_1 edges. Moreover m_1, n_1 are odd, $m_1 \geqslant n_1 + m_2 - 1$, and $n \geqslant n_0$.

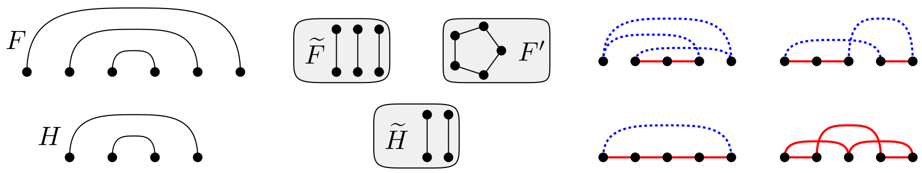


Figure 1.8: An ordered matching H and an ordered Ramsey graph F of H (left). An (unordered) 5-cycle F' is a Ramsey graph of the underlying graph $\widetilde{H}$ of H, but no ordering of the vertices of F' yields an ordered Ramsey graph of H (right). Here we show just a few possible orderings of F'. In Section 4.3 we show that any ordered Ramsey graph of H contains a copy of F (which is not possible for orderings of F').

1.5 Ramsey Theory for Ordered Graphs: Minimal Ordered Ramsey Graphs

To some extent orderings are part of Ramsey theory from the very beginnings where results on colorings of the integers were discovered, most prominent by Schur and Van der Waerden [135, 141]. Also many proofs rely on choosing the vertices in a suitable order, including the original proof of Ramsey [126] and the results by Erdős and Szekeres [60]. Later on, explicit results for ordered graphs, and more general ordered structures, were obtained which we will summarize later in this section.

First, we shall observe some basic connections between ordered and unordered Ramsey graphs. Then we summarize our contribution to this field and some of the results known so far. Let $R_<(H)$ denote the set of ordered Ramsey graphs of an ordered graph H. Note that the existence of ordered Ramsey graphs follows immediately from the existence of Ramsey graphs for complete graphs.

Observation 1.4. *Let H be an ordered graph and let $\widetilde{H}$ be its underlying graph. If $F \in R_<(H)$, then the underlying graph of F is in $R(\widetilde{H})$. If H is a complete graph, then also the reverse statement holds, otherwise it may fail.*

See Figure 1.8 for an example of an ordered graph H and a Ramsey graph F' of $\widetilde{H}$ which does not form an ordered Ramsey graph of H in any ordering.

Our Contribution In Chapter 4 we initiate the study of (minimal) ordered Ramsey graphs. First we characterize all pairs of ordered graphs that have an ordered Ramsey graph that is a forest. In the unordered setting it is easy to see that this holds if and only if both graphs are forests, one of which is a star forest. For ordered graphs there are even stronger restrictions and for many pairs of ordered star forests each Ramsey graph contains a cycle.

Our main concern is the question which ordered graphs have infinitely many minimal ordered Ramsey graphs. Here, like in the unordered setting, an ordered graph F is a *minimal ordered Ramsey graph* of some ordered graph G if F is an ordered Ramsey graph of G and each proper ordered subgraph of F is not an ordered

Ramsey graph of G. Similarly, an ordered graph G is *Ramsey finite* if there are only finitely many minimal ordered Ramsey graphs of G and *Ramsey infinite* otherwise. An introduction to Ramsey finite and infinite (unordered) graphs is given at the end of the previous section (see Table 1.1).

We identify wide classes of Ramsey finite respectively Ramsey infinite pairs of ordered graphs. For ordered graphs containing cycles we obtain results similar to those known for (unordered) graphs. Indeed, similar to Corollary 1.5 we prove that every ordered graph that contains a cycle is Ramsey infinite. Our proof of this result follows a recent approach due to Nenadov and Steger [109] using the hypergraph container method.

Theorem 4.2. *Each ordered graph that contains a cycle is Ramsey infinite.*

Moreover, similar to Theorem 1.6 we prove that any pair of some so-called monotone matching and an arbitrary ordered graph is Ramsey finite. An ordered matching with vertices $u_1 < \cdots < u_{2k}$ is *monotone* if its edges are of the form $u_{2i-1}u_{2i}$, $1 \leqslant i \leqslant k$.

Corollary 4.4. *If H' is a monotone matching, then (H, H') is Ramsey finite for each ordered graph H.*

We conjecture that also an analog of Theorem 1.7 holds for ordered graphs, that is, each pair of an ordered forest that is not a monotone matching and an ordered graph that contains a cycle is Ramsey infinite.

Contrary to the previous results, we show results for pairs of ordered forests that differ significantly from the unordered setting. Recall from Theorem 1.8 that any Ramsey finite pair of (unordered) graphs involves a matching or is a pair of star forests (of special kind). In contrast to this, we find Ramsey finite pairs of ordered stars and ordered trees of arbitrarily large diameter.

Any pair of (unordered) forests that are both not star forests is Ramsey infinite due to a result by Nešetřil and Rödl [114]. As their result is based on the fact that each (unordered) forest is not χ-avoidable, this approach does not work for χ-avoidable ordered forests. We give some more details of the approach from [114] in Chapter 4. So far we have only few results for χ-avoidable or disconnected ordered forests. On the other hand we can easily adopt the approach from [114] to prove Ramsey infiniteness for any pair of χ-unavoidable ordered forests which does not have forests as Ramsey graphs. Extending this result, we show that any Ramsey finite pair of χ-unavoidable connected ordered forests is a pair of an ordered star and a so-called almost increasing caterpillar.

Ordered Ramsey Numbers Many results in Ramsey theory for ordered graphs deal with ordered Ramsey numbers, denoted $r_<$. We summarize known bounds on ordered Ramsey numbers next. Again we start with some basic relations between ordered and unordered Ramsey numbers. Let H be an ordered graph and $\widetilde{H}$ its underlying graph. Due to Observation 1.4 we have $r(\widetilde{H}) \leqslant r_<(H)$ and equality holds if H is a complete graph. There are other cases where ordered and unordered Ramsey numbers coincide. For example $r_<(P_s^{\mathrm{mon}}, K_t) = r(P_s, K_t) = (s-1)(t-1)+1$ [46], where P_s^{mon} is an ordered path $v_1 \cdots v_s$ with $v_1 < \cdots < v_s$ and P_s is an (unordered) path on s vertices.

Due to several applications, mostly geometric Erdős-Szekeres type results, Ramsey numbers of monotone (hyper)paths received particular attention [108]. For given positive integers ℓ and r a *monotone r-uniform ℓ-hyperpath* is an ordered r-uniform hypergraph with edges $E_1, \ldots, E_t$, where each edge forms an interval in the vertex ordering and $E_i \cap E_{i+1}$ consists of the ℓ rightmost vertices in E_i and the ℓ leftmost vertices in E_{i+1}, $i \in [t-1]$. Building on previous results of Moshkovitz and Shapira [107], Cox and Stolee [50] prove that the ordered Ramsey number of such paths P grows like a tower of height $\Delta(P) - 2$ as a function in the number of edges of P (note that the maximum degrees of all sufficiently large monotone r-uniform ℓ-hyperpaths coincide for fixed ℓ and r). In contrast to this, the Ramsey numbers of unordered hyperpaths, and more general of any hypergraph of bounded maximum degree, are linear in the size. Indeed for any uniformity r and any positive integer d there is a constant $c(r,d)$ such that for each (unordered) r-uniform hypergraph H on n vertices and of maximum degree at most d we have $r(H) \leqslant c(r,d)\,n$ [40, 47]. In an even more striking contrast to this result, Conlon *et al.* [46] and independently Balko *et al.* [11] prove the existence of ordered matchings with superpolynomial Ramsey numbers.

Theorem 1.9 ([11, 46])**.** *There exists a constant $c > 0$ such that for each even $n \geqslant 2$ there is an ordered matching M on n vertices with $r_<(M) \geqslant n^{c\frac{\log(n)}{\log\log(n)}}$.*

In fact, Conlon *et al.* [46] prove that this lower bound holds for almost all ordered matchings on n vertices. Both papers present several results and open problems concerned with Ramsey numbers of sparse ordered graphs. We just present the following upper bound.

Theorem 1.10 ([46])**.** *There exists a constant $c > 0$ such that for any d-degenerate ordered graph H on n vertices we have $r_<(H) \leqslant 2^{cd\log^2(2n/d)}$.*

The authors note that this bound is almost tight for very small or very large d. This shows that for dense graphs the ordered Ramsey numbers behave similar to the unordered Ramsey numbers. Another line of research is concerned with generalizations of Ramsey numbers for graphs with partially ordered vertex sets [49].

Ramsey Theory for Induced Ordered Graphs Finally we state the following result on induced ordered Ramsey graphs, due to Nešetřil and Rödl [117] and independently Abramson and Harrington [1].

Theorem 1.11 ([1, 117])**.** *Let k denote a positive integer and let A and H be ordered graphs. Then there is an ordered graph F such that for any k-coloring of the induced copies of A in F there is an induced copy of H in F with all its induced copies of A of the same color.*

This results is in stark contrast to the unordered setting were an analogous statement holds if and only if A is a complete or an empty graph [112]. Dellamonica and Rödl [53] present a strengthening of Theorem 1.11 to distance preserving copies of ordered graphs. See also [125] for an introduction to this type of results.

1.6 Definitions, Notation, and Basic Facts

Here we give a self contained summary of the concepts and notations used in this thesis. Readers familiar with basic knowledge of graph theory may easily skip this section as any less common notion will be introduced at appropriate places in the text. References to the definitions can be found in the indices. For a general introduction to graph theory we refer to the books of Diestel [56] and West [143].

General

For finite sets U and V let 2^V denote the set of all subsets of V, let $U\dot{\cup}V$ denote the disjoint union of U and V, and let $U \times V = \{(u, v) \mid u \in U, v \in V\}$. For a positive integer n let $[n] = \{1, \dots, n\}$, let $\mathcal{S}_n$ denote the set of permutations of $[n]$, and let $\mathbb{Z}^n = \{(z_1, \dots, z_s)^\top \mid z_i \in \mathbb{Z}, i \in [n]\}$. For a positive integer n and two vectors x, $y \in \mathbb{Z}^n$, $x + y$ denotes the componentwise addition, and $x \leqslant y$ and $x \geqslant y$ denote the standard componentwise comparability of x and y. The vector $(p, \dots, p)^\top \in \mathbb{Z}^n$ is denoted by $\boldsymbol{p}$.

Graphs and Hypergraphs

A *hypergraph* is pair $G = (V, \mathcal{E})$ where V is the set of *vertices* and $\mathcal{E} \subseteq 2^V$ is the set of *edges* of G, where $|E| \geqslant 2$ for each $E \in \mathcal{E}$. Two hypergraphs G and G' are *isomorphic* if there is a bijection $f : V(G) \to V(G')$ such that $E \subset V(G)$ is an edge in G if and only if $f(E)$ is an edge in G'. For a hypergraph G let $V(G)$ denote its vertex set and let $E(G)$ denote its edge set. The *order* of a hypergraph G is $|V(G)|$ and its *size* is $|E(G)|$. All (hyper)graphs considered here are finite (that is, V is finite) and do not contain loops (edges of size 1) or parallel edges (as $\mathcal{E}$ is not a multiset). Two vertices that are contained in the same edge are called *adjacent* and *neighbors* of each other.

A *subhypergraph*[2] of a hypergraph G is a hypergraph G' with $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G')$. A subhypergraph G' of G is *induced* if $E(G') = E(G) \cap$

[2] Sometimes it is only required that each edge of G' is contained in some edge of G. Here we shall use this more restrictive notion.

$2^{V(G')}$. A *proper subhypergraph* is a subhypergraph G' of G such that $V(G') \neq V(G)$ or $E(G') \neq E(G)$. A *copy* of a some hypergraph G' in a hypergraph G is a subhypergraph of G isomorphic to G'. If $U \subseteq V(G)$, $F \subseteq E(G)$ let $G[U]$, $G - U$, and $G - F$ denote the hypergraphs $(U, E(G) \cap 2^U)$, $(V(G) \setminus U, E(G) \cap 2^{V(G)\setminus U})$, and $(V(G), E(G) \setminus F)$, respectively. In particular if u, $v \in V(G)$ then $G - \{u, v\}$ is the hypergraph obtained by removing u and v from G, not the edge uv only. If $u \in V(G)$, $e \in E(G)$, then let $G - u = G - \{u\}$ and let $G - e = G - \{e\}$.

A *vertex disjoint union* $G + H$ of hypergraphs G and H is a hypergraph F such that there is a partition $V(F) = V_1 \dot\cup V_2$ where $F[V_1]$ is isomorphic to G, $F[V_2]$ is isomorphic to H, and each edge of F is contained in V_1 or in V_2. For a hypergraph G and a positive integer n let nG denote a vertex disjoint union of n copies of G.

A hypergraph is *r-uniform* if all its edges are of size r. A *graph* is a 2-uniform hypergraph, and the terms *subgraph* and *copy of a graph* are defined accordingly. We write $G' \subseteq G$ if G' is a subgraph of G' and call G a *supergraph* of G'. For convenience we write $uv = vu = \{u, v\}$ for the edges of a graph. The *Cartesian product* $G \times H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and edge set $\{\{(u, v), (x, y)\} \mid u = x, vy \in E(H) \text{ or } v = y, ux \in E(G)\}$.

The *degree* $d(u) = d_G(u)$ of a vertex u in a hypergraph G is the total number of edges in G that contain u and we usually omit the subscript. A vertex of degree 1 is called a *leaf*. A vertex u of a hypergraph is *isolated* if it is not contained in any edge and an edge uv is *isolated* if all its vertices are leafs. A *pendant edge* is an edge incident to a leaf and some vertex of degree at least 2 . The *minimum degree* $\delta(G)$ is the smallest and the *maximum degree* $\Delta(G)$ is the largest degree of vertices in G. A hypergraph is *r-regular* if all its vertices are of degree r and it is *t-degenerate* if each subhypergraph of G has a vertex of degree at most t.

Observation 1.5. *A hypergraph G is t-degenerate if and only if there is an order of $V(G)$ such that each vertex is adjacent to at most t of its predecessors in the order.*

A hypergraph G is *k-partite* if there is a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_k$ such that $|E \cap V_i| \leqslant 1$ for each $E \in E(G)$ and $i \in [k]$. A 2-partite hypergraph is also called *bipartite*. A vertex coloring of a hypergraph is *proper* if each edge contains at least two vertices of distinct colors. The *chromatic number* $\chi(G)$ of a hypergraph G is the smallest number of colors among all proper of colorings of G.

Observation 1.6. *If a hypergraph G is k-partite then $\chi(G) \leqslant k$. The reverse statement holds for graphs, but may fail for hypergraphs with larger edges.*

The *independence number* $\alpha(G)$ of a hypergraph G is the size of a largest set of vertices not containing any edge of G completely. The *clique number* $\omega(G)$ of an r-uniform hypergraph G is the size of a largest set X of vertices such that each subset of X of size r is an edge in G. Observe that $\omega(G)$, $\alpha(G) \geqslant \min\{|V(G)|, r-1\}$ for any r-uniform hypergraph G.

A *path of length* n, $n \geqslant 0$, is a graph with $n+1$ vertices and n edges $e_1, \dots, e_n$ such that $e_i \cap e_j \neq \emptyset$ if and only if $|i-j| = 1$. For a path P with vertices $v_0, \dots, v_n$ and edges $v_{i-1}v_i$, $i \in [n]$, we write $P = v_0 \cdots v_n$. For vertices u and v in some graph G, a *u-v-path* P is a path in G starting with u and ending with v, i.e., a path $v_1 \cdots v_n$ with $u = v_1$, $v = v_n$. Given a path $P = v_1 \cdots v_n$ and some $i \in [n]$ let $v_iP = v_i \dots v_n$ and $Pv_i = v_1 \cdots v_i$. Similarly for a neighbor $v \notin V(P)$ of v_1 in G let $vP = vv_1 \cdots v_n$. A graph G is *connected* if there is a u-v-path for any two distinct vertices u and v in G. The *distance* of two vertices u, v of a connected graph G is the length of a shortest u-v-path in G. The *diameter* of a connected graph G is the largest distance among pairs of vertices from G.

A *cycle of length* n, $n \geqslant 2$, is a hypergraph that consists of n distinct hyperedges $E_0, \dots, E_{n-1}$, such that there are n distinct vertices $v_0, \dots, v_{n-1}$ with $\{v_i, v_{i+1}\} \subseteq E_i$, $0 \leqslant i \leqslant n-1$ (indices taken modulo n). The *girth*, denoted $\operatorname{girth}(G)$, of a hypergraph is the shortest length of a cycle in G. In particular if two edges of a hypergraph share two vertices then the girth is 2. If G contains no cycles then G is called *acyclic* or *forest* and we write $\operatorname{girth}(G) = \infty$. A *tree* is a connected forest. Similarly the *odd girth*, denoted $\operatorname{girth}_{\mathrm{o}}(G)$, of a hypergraph is the shortest length of a cycle of odd length in G. If no such cycle exists then we write $\operatorname{girth}_{\mathrm{o}}(G) = \infty$.

Observation 1.7. *Let G be a hypergraph. We have* $\operatorname{girth}(G) = \infty$ *if and only if G is a forest, and* $\operatorname{girth}_{\mathrm{o}}(G) = \infty$ *if and only if G is bipartite.*

Let K_n denote a complete graph on n vertices, let $K_{n,m}$ denote a complete bipartite graph with parts of size n and m, let M_n denote a (perfect) matching with n edges, and let P_n and C_n denote a path respectively a cycle on n vertices. Let $H_{t,d}$ denote a graph on $t+1$ vertices such that one vertex has degree d and the other vertices induce a copy of K_t.

Ordered Graphs

An *ordered graph* is a graph equipped with a linear ordering of its vertex set. For vertices u and v of an ordered graph G where u precedes v in the ordering of G we write $u \leqslant v$ and, if $u \neq v$, we write $u < v$. We consider the vertices of an ordered graph laid out along a horizontal line from *left* to *right* such that a vertex u is *to the left* of a vertex v if $u < v$, and *to the right* if $v < u$. We refer to the (unordered) graph $\widetilde{G} = (V(G), E(G))$ as the *underlying graph* of an ordered graph G. An *(ordered) subgraph* of an ordered graph G is a subgraph of $\widetilde{G}$ that inherits the ordering of vertices from G. Two ordered graphs G and G' with $V(G) = (u_1, \dots, u_n)$ and $V(G') = (v_1, \dots, v_m)$ are *isomorphic* if $n = m$ and u_iu_j is an edge in G if and only if v_iv_j is an edge in G' for all i, $j \in [n]$. A *copy* of an ordered graph H in some ordered graph G is an ordered subgraph of G that is isomorphic to H. For two sets $U, U' \subseteq V(G)$ we write $U \preceq U'$ ($U \prec U'$) if $u \leqslant u'$ ($u < u'$) for all $u \in U$ and $u' \in U'$.

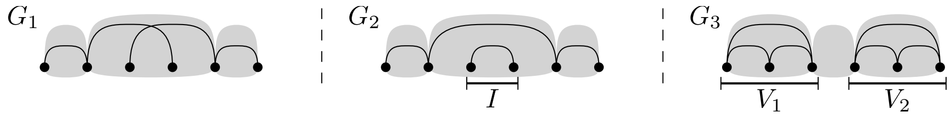


Figure 1.9: Three ordered graphs with exactly three segments each (gray bubbles), where G_1 is intervally and segmentally connected, G_2 is segmentally but not intervally connected, and G_3 is neither intervally nor segmentally connected. Here G_2 is not intervally connected since there is no edge between I and the remaining vertices, and G_3 is not segmentally connected since there is no edge between V_1 and V_2.

For two subgraphs of G', G'' of G we write $G' \preceq G''$ ($G' \prec G''$) if $V(G') \preceq V(G'')$ ($V(G') \prec V(G'')$). A vertex v is *between* vertices u and w if $u \leqslant v \leqslant w$.

An *interval* of an ordered graph G is a set I of consecutive vertices of G, i.e., for any u, $v \in I$, $z \in V(G)$ with $u \leqslant z \leqslant v$ we have $z \in I$. A *segment* of an ordered graph G is a maximal induced subgraph G' of G such that $V(G')$ forms an interval in $V(G)$ and for each $z \in V(G')$ that is neither leftmost nor rightmost in G' there is an edge uv in G' with $u < z < u$. An *inner cut vertex* of an ordered graph G is a vertex that is contained in precisely two segments. See Figure 1.9.

Observation 1.8. *An ordered graph G is the union of its segments where two segments are either vertex disjoint or share exactly one inner cut vertex of G. In particular, the number of inner cut vertices of G is exactly one less than the number of its segments.*

We say that an inner cut vertex v of an ordered graph G *splits* G into ordered graphs G_1 and G_2, where G_1 is induced by all vertices u with $u \leqslant v$ in G and G_2 is induced by all vertices u with $v \leqslant u$.

An ordered graph G with at least two vertices is *segmentally connected* if for any partition $V_1 \dot\cup V_2 = V(G)$ of the vertices of G into two disjoint intervals V_1 and V_2 there is an edge with one endpoint in V_1 and the other endpoint in V_2. An ordered graph G is *intervally connected* if for each nonempty interval I of vertices of G, that does not contain all vertices of G, there is an edge in G with one endpoint in I and one endpoint not in I. See Figure 1.9.

Observation 1.9. *Let G be an ordered graph on at least two vertices. Then the following properties hold.*

(a) If G is connected, then G is intervally connected.

(b) If G is intervally connected, then G is segmentally connected.

(c) G is segmentally connected if and only if each segment of G contains at least one edge.

Two edges uv and $u'v'$ of an ordered graph *cross* if $u < u' < v < v'$. An ordered graph G is called *non-crossing* if it contains no crossing edges and *crossing*

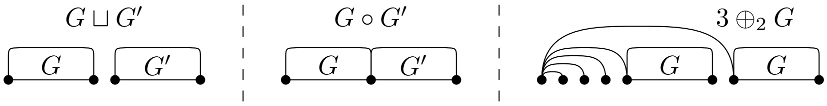


Figure 1.10: Three different types of unions of ordered graphs.

otherwise. Two distinct ordered subgraphs G_1 and G_2 of the same ordered graph G *cross each other* if there is an edge in G_1 crossing an edge in G_2. A vertex in an ordered graph G is called *reducible*, if it is of degree 1 in G, is leftmost or rightmost in G, and has a common neighbor with the vertex next to it.

Any ordinary graph parameter like chromatic number, clique number, or girth is determined by the underlying graph of an ordered graph. The *interval chromatic number* $\chi_<(G)$ of an ordered graph G is the smallest size of a partition of $V(G)$ into intervals that are independent sets in G. For an ordered graph H let $\mathrm{Forb}_<(H)$ denote the set of ordered graphs that do not contain H as a subgraph. For a positive integer n and an ordered graph H, let $\mathrm{ex}_<(n, H)$ denote the *ordered extremal number*, i.e., the largest number of edges in an ordered graph on n vertices in $\mathrm{Forb}_<(H)$. An ordered graph is called χ*-avoidable* if for each integer k there is $G \in \mathrm{Forb}_<(H)$ with $\chi(G) \geqslant k$, and χ*-unavoidable* otherwise.

We shall frequently use the following unions of ordered graphs G and G'. The *intervally disjoint union* $G \sqcup G'$ of ordered graphs G and G' is a vertex disjoint union of G and G' where all vertices of G are to left of all vertices of G'. The *concatenation* $G \circ G'$ is obtained from $G \sqcup G'$ by identifying the rightmost vertex in the copy of G with the leftmost vertex in the copy of G'. For an integer $b > 0$ we shall write $\sqcup_b G$ and $\circ_b G$ for an intervally disjoint union respectively a concatenation of b copies of G. Moreover $a \oplus_b G$ denotes the ordered graph obtained from $\sqcup_{a+1} K_1 \sqcup (\sqcup_b G)$ by connecting the leftmost vertex of this union with the first a vertices next to it and the leftmost vertex of each of the b copies of G. See Figure 1.10 for an illustration. The *reverse* $\overline{G}$ of an ordered graph G is the ordered graph obtained by reversing the ordering of the vertices in G.

The *length* of an edge xy, $x < y$, of an ordered graph G is the number of vertices $v \in V(G)$ such that $x \leqslant v < y$. A shortest edge among all the edges incident to a vertex x is referred to as a *shortest edge incident to* x. Note that there is either one or two shortest edges incident to a given vertex in a connected ordered graph on at least two vertices. Let U be a set of vertices of an ordered tree T, such that each vertex in U has exactly one shortest edge incident to it. For such a set U, let $S(U)$ be the set of edges e_u such that e_u is a shortest edge incident to u, $u \in U$. We call an ordered tree T *monotonically alternating* if there is a partition $V(T) = L \dot{\cup} R$, with $L \prec R$, such that L and R are independent sets in T, $E = S(L) \cup S(R)$, and neither $S(L)$ nor $S(R)$ contains a pair of crossing edges. See Figure 1.11.

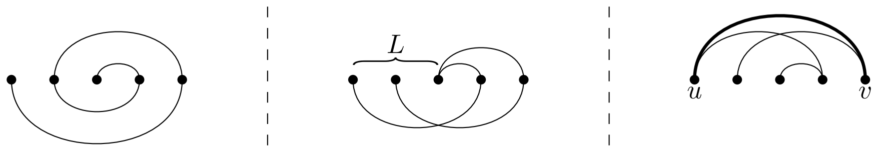


Figure 1.11: The ordered path on the left is monotonically alternating. The path in the middle contains crossing edges in $S(L)$ and the edge uv on the right is neither shortest incident to u nor to v. So both paths are not monotonically alternating.

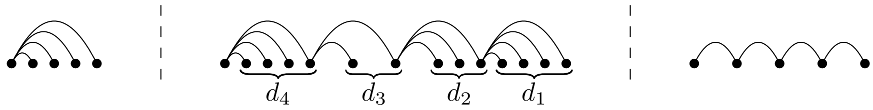


Figure 1.12: A right star (left), a right caterpillar with defining sequence $d_1 = 4$, $d_2 = 3$, $d_3 = 2$, $d_4 = 4$ (middle), and a monotone path (right).

A *bonnet* is an ordered graph on four or five vertices $u_1 < u_2 \leqslant u_3 < u_4 \leqslant u_5$ with edge set $\{u_1u_2, u_1u_5, u_3u_4\}$, or on vertices $u_1 \leqslant u_2 < u_3 \leqslant u_4 < u_5$ with edge set $\{u_1u_5, u_4u_5, u_2u_3\}$. See Figure 1.2 (first row). An ordered path $P = u_1 \cdots u_n$ is a *monotone path* if $u_1 < \cdots < u_n$. An ordered path $P = u_1 \cdots u_n$ is a *tangled path* if for a vertex u_i, $1 < i < n$, that is either leftmost or rightmost in P there is an edge in the subpath $u_1, \ldots, u_i$ that crosses an edge in the subpath $u_i \ldots u_n$. See Figure 1.2 (second row). A *monotone k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_{2i-1}u_{2i}$, $1 \leqslant i \leqslant k$. An *all crossing k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges u_iu_{i+k}, $1 \leqslant i \leqslant k$, for some $k \geqslant 2$. A *nested k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges u_iu_{2k+1-i}, $1 \leqslant i \leqslant k$, for some $k \geqslant 2$. A *right star* $\vec{S}_k$ is an ordered star with k leafs to the right of the center. A right or a left star with two edges is also called a *bend*. A *right caterpillar* is an ordered tree consisting of segments $S_i \preceq \cdots \preceq S_1$, for some $i \geqslant 1$, where each segment is a right star with at least one edge. The *defining sequence* of a right caterpillar with segments $S_i \preceq \cdots \preceq S_1$ is $|E(S_1)|, \ldots, |E(S_i)|$. Similarly we define *left caterpillars*. See Figure 1.12 for an illustration.

Ramsey Theory

Given graphs $H_1, \ldots, H_r$ a graph F is a *Ramsey graph*[3] of $(H_1, \ldots, H_r)$ if for any r-coloring of the edges of F there is for some $i \in [r]$ a *monochromatic copy* of H_i in color i, that is, one with all edges of color i. In this case we write $F \xrightarrow{r} (H_1, \ldots, H_r)$ to indicate this fact. Then let $R(H_1, \ldots, H_r) = \{F \mid F \xrightarrow{r} (H_1, \ldots, H_r)\}$. The *Ramsey number* of $(H_1, \ldots, H_r)$ is $r(H_1, \ldots, H_r) = \min\{|V(F)| \mid F \in R(H_1, \ldots, H_r)\}$. In case $r = 2$ we write $F \to (H_1, H_2)$ and if $H_1 = H_2 = H$ we write $F \to H$, $R(H) = R(H, H)$, $r(H) = r(H, H)$. For $t \geqslant 2$ we write $r(t) = r(K_t)$. For graphs F,

[3] In some articles a Ramsey graph is graph with no large cliques and independent sets, corresponding to a 2-coloring of K_n with no monochromatic copies of small cliques.

G and for $\epsilon > 0$ we write $F \xrightarrow{\epsilon} G$ if for any set $S \subseteq V(F)$ with $|S| \geqslant \epsilon|V(F)|$, we have $F[S] \to G$. A graph $F \in R(H_1, \ldots, H_r)$ is a *minimal Ramsey graph* if $F' \notin R(H_1, \ldots, H_r)$ for each proper subgraph F' of F.

We state the same definitions for ordered graphs and introduce separate notations to avoid confusion. Given ordered graphs $H_1, \ldots, H_r$ an ordered graph F is an *ordered Ramsey graph* of $(H_1, \ldots, H_r)$ if for any r-coloring of the edges of F there is for some $i \in [r]$ a monochromatic copy of H_i in color i, that is, one with all edges of color i. In this case we write $F \xrightarrow{r} (H_1, \ldots, H_r)$ to indicate this fact. Then let $R_<(H_1, \ldots, H_r) = \{F \mid F \xrightarrow{r} (H_1, \ldots, H_r)\}$. The *ordered Ramsey number* of $(H_1, \ldots, H_r)$ is $r_<(H_1, \ldots, H_r) = \min\{|V(F)| \mid F \in R_<(H_1, \ldots, H_r)\}$. In case $r = 2$ we write $F \to (H_1, H_2)$ and if $H_1 = H_2 = H$ we write $F \to H$, $R_<(H) = R_<(H, H)$, $r_<(H) = r_<(H, H)$. A graph $F \in R_<(H_1, \ldots, H_r)$ is a *minimal ordered Ramsey graph* if $F' \notin R_<(H_1, \ldots, H_r)$ for each proper ordered subgraph F' of F.

2

Chromatic Number of Ordered Graphs

In this chapter we study the chromatic number of ordered graphs under local constraints. First we consider forbidden ordered subgraphs in Section 2.1. As already mentioned in the introduction the behavior is very different compared to (unordered) graphs. We start with a discussion why the the main tool in the aforementioned results for graphs as well as for directed graphs, the greedy embedding method, does not work for ordered graphs. Then we present our main results concerned with χ-avoidable and χ-unavoidable ordered graphs.

In Section 2.2 we present a general framework to model local constraints for ordered graphs. First we show how to model different constraints, including forbidden ordered subgraphs, using this framework. Then we show how to use the abstract, algebraic setting of this framework to deduce bounds on the chromatic number of ordered graphs. For more specific results of this kind we refer to [7].

Section 2.3 contains some structural lemmas and provides several reductions that are used in the proofs of the main results and that might be of independent interest. Then we give the proofs of Theorems 2.1–2.4 and 2.6 in Section 2.4. A summary of our results for forbidden ordered forests with at most three edges is presented in Section 2.5. Finally, Section 2.6 contains conclusions and open questions.

2.1 Forbidden Ordered Subgraphs

In this section we consider the behavior of the chromatic number of ordered graphs with forbidden ordered subgraphs. For an ordered graph H on at least two vertices[1] $\mathrm{Forb}_<(H)$ is the set of all ordered graphs that do not contain a copy of H. We consider the function $\kappa_<(H) = \sup\{\chi(G) \mid G \in \mathrm{Forb}_<(H)\}$ and call H *χ-unavoidable* if $\kappa_< \neq \infty$ and *χ-avoidable* otherwise.

Recall Observation 1.2 in Section 1.3 which states that an (unordered) graph H is χ-avoidable if and only if H contains a cycle. The case when H contains a cycle was settled by the existence of graphs of arbitrarily large chromatic number and girth [61]. Any ordering of such a graph yields an ordered graph of arbitrarily

[1] If H has only one vertex, then $\mathrm{Forb}_<(H)$ consists only of the graph with empty vertex set and one can think of $\kappa_<(H)$ as being equal to 0. However, we will avoid this pathological case throughout.

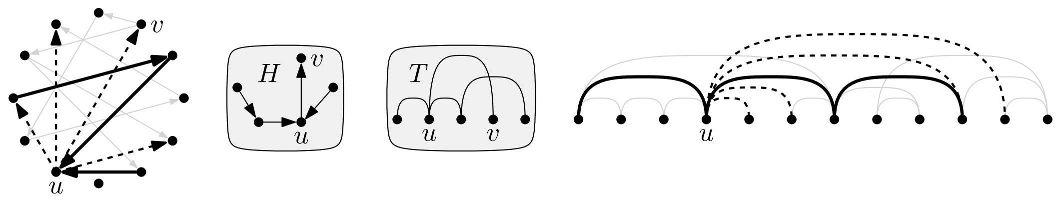


Figure 2.1: Left: A copy of $H - v$ (bold edges) in a directed graph where (the copy of) u has many outgoing edges. We see that we can embed v and find a copy of H. Right: A copy of an ordered tree $T - v$ (bold edges) in some ordered graph where (the copy of) u has many neighbors to the right but there is no possibility to embed v in order to obtain a copy of T.

large chromatic number and girth. Hence any ordered graph containing a cycle is still χ-avoidable, see Theorem 2.1 below. Here we show that it is no longer true that an ordered graph H is χ-unavoidable if H is acyclic. Specifically we present χ-avoidable ordered forests in Theorem 2.1. When H is connected, we reduce the problem of determining whether H is χ-avoidable to a well behaved class of trees, which we call monotonically alternating trees. We completely classify non-crossing χ-avoidable ordered graphs H. In case of non-crossing χ-avoidable H, we also provide specific upper bounds on $\kappa_<(H)$ in terms of the number of vertices in H. Note that $\kappa_<(H) \geqslant |V(H)| - 1$ for any ordered graph H, since a complete graph on $|V(H)| - 1$ vertices is in $\text{Forb}_<(H)$.

2.1.1 The Greedy Embedding Fails

Next we shall have a closer look at a proof of χ-unavoidability for directed graphs whose underlying graph is acyclic [25], and see why this approach fails for ordered graphs. Let H be such a directed graph on k vertices and let G be a directed graph of chromatic number at least k^2. We prove that G contains a copy of H by induction on k. If $k \leqslant 2$ this is easy to see. So suppose that $k \geqslant 3$ and let uv be an edge in H where v is a leaf in (the underlying forest of) H. Say uv is oriented from u to v. Let O denote the set of vertices of G with at most $k - 1$ outgoing edges. Then the subgraph of G induced by O is $2(k-1)$-degenerate and hence $(2k-1)$-colorable. Thus the vertices in $V(G) \setminus O$ induce a graph of chromatic number at least $k^2 - (2k - 1) = (k - 1)^2$. Inductively, this graph contains a copy H' of $H - v$. Consider the vertex corresponding to u in H'. Since it has k outgoing edges in G and $|V(H')| = k-1$, we can extend H' to a copy of H in G. See Figure 2.1 (left). An even simpler approach can be used in the undirected, unordered setting. A graph G of chromatic number at least k contains a subgraph G' of minimum degree $k - 1$. Now one can embed any forest on k vertices in G' greedily. We refer to this approach as the *greedy embedding*.

Now, when mimicking this approach for an ordered tree T we fail in the last step (the greedy embedding). Indeed, consider an ordered tree T on k vertices and an

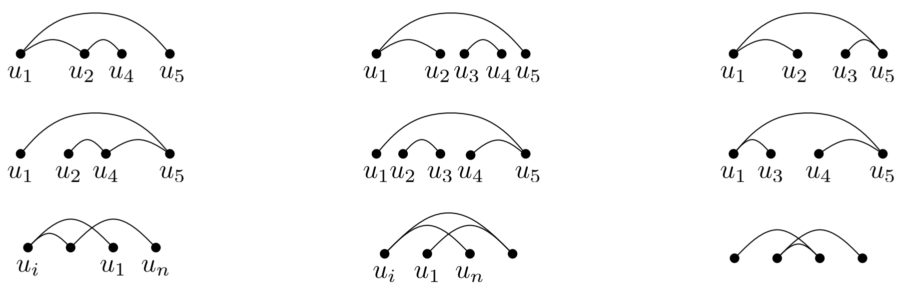


Figure 2.2: All bonnets (first two rows), two tangled paths (last row, left and middle), and a crossing path that is not tangled (last row, right).

edge uv in T where v is a leaf in T and to the right of u. Say we found a copy of $T - v$ where the copy of u has k neighbors to the right. Then there still might be no copy of T, as shown in Figure 2.1 (right).

Observation 2.1. *The greedy embedding fails for ordered forests.*

2.1.2 The Main Results

In order to state our main result we recall some definitions. Two edges uv and $u'v'$ of an ordered graph *cross* if $u < u' < v < v'$ and an ordered graph H is called *non-crossing* if it contains no crossing edges. The *bonnets* are exactly those five ordered graphs given in the first two rows of Figure 2.2. An ordered path $P = u_1 \cdots u_n$ is *tangled* if for a vertex u_i, $1 < i < n$, that is either leftmost or rightmost in P there is an edge in the subpath $u_1 \cdots u_i$ that crosses an edge in the subpath $u_i \cdots u_n$. See Figure 2.2 (last row, left and middle). Note that there are crossing paths which are not tangled, see for example Figure 2.2 (last row, right).

Theorem 2.1. *If an ordered graph H contains a cycle, a bonnet, or a tangled path, then H is χ-avoidable.*

Next we describe the structure of ordered trees that neither contain a bonnet nor a tangled path. Recall that a *segment* of an ordered graph G is a maximal induced subgraph G' of G such that $V(G')$ forms an interval in $V(G)$ and for each $z \in V(G')$ that is neither leftmost nor rightmost in G' there is an edge uv in G' with $u < z < u$. See Figure 2.3. Note that the segments of an ordered graph G are uniquely determined and each segment has at least two vertices, provided $|V(G)| \geqslant 2$. So, G is the union (more precisely the concatenation) of its segments where two segments are either vertex disjoint or share exactly one vertex which is an inner cut vertex of G. For a set of vertices U the set $S(U)$ denotes the set of edges e_u such that e_u is a shortest edge incident to u, $u \in U$. Recall that an ordered tree T is *monotonically alternating* if there is a partition $V(T) = L \dot\cup R$, with $L \prec R$, such that L and R are independent sets in T, $E = S(L) \cup S(R)$, and neither $S(L)$ nor $S(R)$ contains a pair of crossing edges. See Figure 2.4.

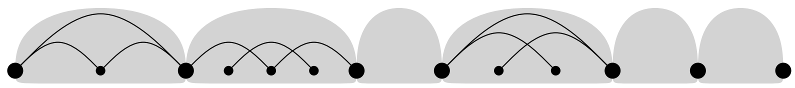


Figure 2.3: Segments of an ordered graph. The bold vertices are either inner cut-vertices, leftmost, or rightmost.

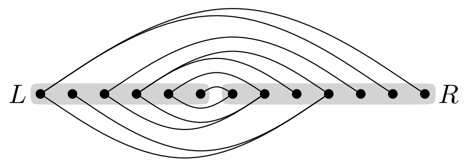


Figure 2.4: A monotonically alternating tree. Each edge on top is the shortest edge incident to a vertex in R and each edge at the bottom is the shortest edge incident to a vertex in L.

Theorem 2.2. *An ordered tree T contains neither a bonnet nor a tangled path if and only if each segment of T is monotonically alternating. In particular if H is a connected χ-unavoidable ordered graph, then each segment in H is a monotonically alternating tree.*

Note that a non-crossing graph does not contain tangled paths. We characterize all non-crossing χ-avoidable ordered graphs in the following theorem.

Theorem 2.3. *Let T be a non-crossing ordered graph on k vertices, $k \geqslant 2$. Then T is χ-unavoidable if and only if T is a forest that does not contain a bonnet.*

Moreover, if T is χ-unavoidable then $k-1 \leqslant \kappa_<(T) \leqslant 2^k$. If, in addition T is connected, then $\kappa_<(T) \leqslant 2k-3$. Finally, for each $k \geqslant 4$ there is an χ-unavoidable non-crossing ordered tree T with $\kappa_<(T) \geqslant k$, while for $k = 2, 3$ we have $\kappa_<(T) = k-1$.

For certain classes of ordered forests we prove better upper bounds on $\kappa_<$. Theorem 2.4 below summarizes several results on ordered forests which are either not covered by Theorem 2.3 or improve the upper bound from Theorem 2.3 significantly.

A *monotone k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_{2i-1}u_{2i}$, $1 \leqslant i \leqslant k$, a *nested k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_i u_{2k+1-i}$, $1 \leqslant i \leqslant k$, and an *all crossing k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_i u_{i+k}$, $1 \leqslant i \leqslant k$. See Figure 2.5 We may omit the parameter k if it is not important. A *generalized star* is a union of a star and isolated vertices. Finally we introduce a special family of star forests called tuple matchings. For positive integers m and t and a permutation π of $[t]$, an *m-tuple t-matching* $M = M(t, m, \pi)$ is an ordered graph with vertices $v_1 < \cdots < v_{t(m+1)}$, where each edge is of the form $v_i v_{t+j+m(\pi(i)-1)}$ for $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant m$. So an m-tuple t-matching is a vertex disjoint union of t stars on m edges each, where $v_1, \ldots, v_t$ are the centers of the stars that are to the left of all leaves and the leaves of each star form an interval in M, so that these intervals are

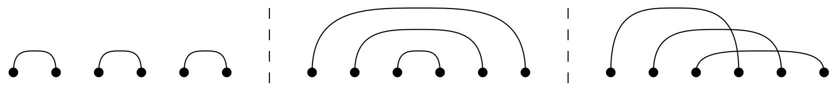


Figure 2.5: A monotone matching (left), a nested matching (middle), and an all crossing matching (right).

ordered according to the permutation π. The third item in the following theorem is an immediate corollary of a result by Weidert [142] who provides a linear upper bound on the extremal function for M. The other results are based on linear upper bounds for the extremal functions of nestings due to Dujmović and Wood [57], on the extremal function of crossings due to Capoyleas and Pach [35] and lower bounds for ordered Ramsey numbers due to Conlon *et al.* [46] and independently Balko *et al.* [11].

Theorem 2.4. *Let T be an ordered forest on k vertices, $k \geqslant 2$.*

(a) If each segment of T is either a generalized star, a nested 2-matching, or an all crossing 2-matching, then $\kappa_<(T) = k - 1$.

(b) If each segment of T is either a nested matching, an all crossing matching, a generalized star, or a non-crossing tree without bonnets, then $k-1 \leqslant \kappa_<(T) \leqslant 2k-3$.

(c) If T is an m-tuple t-matching for some positive integers m and t, then $k-1 \leqslant \kappa_<(T) \leqslant 2^{10k\log(k)}$.

(d) There is a positive constant c such that for each even integer $k \geqslant 4$ there is a matching M on k vertices with $\kappa_<(M) \geqslant 2^{c\frac{\log(k)^2}{\log\log(k)}}$.

2.1.3 Connections to Other Parameters

Connection to Extremal Numbers In order to prove that some ordered forest is χ-unavoidable we can use the following connection between the extremal number $\mathrm{ex}_<(n, H)$ and the function $\kappa_<(H)$ for ordered graphs. If there is a constant c such that $\mathrm{ex}_<(n, H) < c\,n$ for every n, then

$$\kappa_<(H) \leqslant 2c. \tag{2.1}$$

In particular $\kappa_<(H)$ is finite. Indeed, if $\mathrm{ex}_<(n, H) < c\,n$ then any $G \in \mathrm{Forb}_<(H)$ has less than $c\,|V(G)|$ edges, and hence has a vertex of degree less than $2c$. Moreover, if $G \in \mathrm{Forb}_<(H)$, then each subgraph of G is in $\mathrm{Forb}_<(H)$. Thus each subgraph has a vertex of degree less than $2c$ and so G is $(2c-1)$-degenerate. Therefore $\chi(G) \leqslant 2c$. Recall that $\mathrm{ex}_<(n, H)$ can be linear in n only if $\chi_<(H) \leqslant 2$, due to Theorem 1.1. Recall also that when $\mathrm{ex}(n, A(H))$ is linear in n, one can guarantee that $\mathrm{ex}_<(n, H) \in O(n \log n)$, but this is not enough to claim that $\kappa_<(H) \neq \infty$.

For some ordered graphs H with interval chromatic number 2, one can show that $\mathrm{ex}_<(n, H)$ is indeed linear [104, 121, 142]. This in turn, implies that $\kappa_<(H)$ is finite as argued above. On the other hand, we see that there is no equivalence between $\kappa_<(H)$ being finite and $\mathrm{ex}_<(n, H)$ being linear in n because there are dense ordered graphs avoiding H for some ordered graphs H with small $\kappa_<(H)$. A specific example for such a graph H is an ordered path $u_1u_2u_3$, with $u_1 < u_2 < u_3$. One can see from Theorem 2.4 that $\kappa_<(H) = 2$ but $\mathrm{ex}_<(n, H) = n^2/4 + o(n^2)$ by Theorem 1.1. Here a complete bipartite ordered graph G on n vertices with all vertices of one bipartition class to the left of all other vertices does not contain H and has $n^2/4$ edges.

Connection to Ordered Ramsey Numbers There are also connections between the Ramsey numbers $r_<(H)$ for ordered graphs and the function $\kappa_<(H)$. If the edges of K_n, $n = r(H) - 1$, are colored in two colors without monochromatic copies of H, then both color classes form ordered graphs G_1 and G_2 not containing H as an ordered subgraph. Then G_1 or G_2 has chromatic number at least $\sqrt{n}$, since a product of proper colorings of G_1 and G_2 yields a proper coloring of K_n. Therefore $\kappa_<(H) \geqslant \sqrt{r_<(H) - 1}$. This leads to the lower bound in Theorem 2.4(d) for certain ordered matchings. An overview of ordered Ramsey numbers is given in Section 1.5.

2.2 Local Constraints: A General Framework

In this section we present a general framework to model local constraints for ordered graphs. In order to formulate the framework we need to introduce a slightly different notion of ordered graphs. An *integer graph* is a graph (V, E) with $V \subseteq \mathbb{Z}$. Note that an integer graph has an ordered set of vertices and hence is an ordered graph in the sense of the previous section. On the other hand, any ordered graph is clearly isomorphic to infinitely many integer graphs (as ordered graphs). Two integer graphs G and G' are isomorphic if $V(G) = V(G')$ and they are isomorphic as ordered graphs. A discussion on the differences between ordered and integer graphs is given in the conclusions in Section 2.6. There we show that the notion of integer graphs is mainly for technical reasons.

We shall define local constraints on integer graphs based on so-called conflicts of edges that can be expressed as linear inequalities in the coordinates of the endpoints of the edges. To make this precise we need to introduce some notation. For a fixed integer graph $G = (V, E)$, each edge $e \in E$ is associated with the (ordered) tuple (u, v) where $e = uv$ and $u < v$. Recall that for a positive integer s and two vectors x, $y \in \mathbb{Z}^s$, $x + y$ denotes the componentwise addition, and $x \leqslant y$ and $x \geqslant y$ denote the standard componentwise comparability of x and y. Further $\boldsymbol{p}$ denotes the vector $(p, \dots, p)^\top \in \mathbb{Z}^n$. For s, $t \in \mathbb{Z}$, s, $t \geqslant 1$, a matrix $M \in \mathbb{Z}^{s \times 2t}$, and a permutation $\pi \in \mathcal{S}_t$ of $[t]$, let $\pi(M)$ denote the matrix where for each $i \in [t]$ the $(2i-1)^{\text{st}}$ and

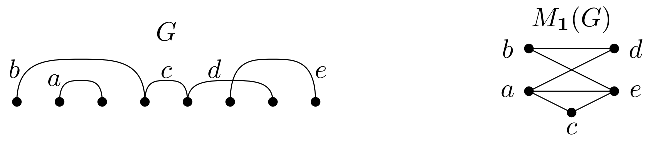


Figure 2.6: An integer graph G and its conflict graph $M_{\mathbf{1}}(G)$ for $M = (+\ 0\ 0\ -)$.

the $(2i)^{\text{th}}$ column of M are permuted to positions $2\pi(i) - 1$ respectively $2\pi(i)$. For example if $M = (1, 2, 3, 4, 5, 6)$ and $\pi = 321$, then $\pi(M) = (5, 6, 3, 4, 1, 2)$.

For integers s and t with $s \geqslant 1$, $t \geqslant 2$, a matrix $M \in \mathbb{Z}^{s \times 2t}$, and a vector $p \in \mathbb{Z}^{2t}$ the *conflict hypergraph* $M_p(G)$ of an integer graph G with respect to M and p is the t-uniform hypergraph with

$$\begin{aligned} V(M_p(G)) &= E(G), \\ E(M_p(G)) &= \{E \subseteq E(G) \mid\ |E| = t, E = \{(u_1, v_1), \ldots, (u_t, v_t)\}, \\ &\qquad\qquad \exists \pi \in \mathcal{S}_t : \pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top \geqslant p\}. \end{aligned}$$

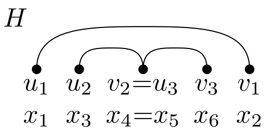
We say that $E \subseteq E(G)$ is *conflicting* if $E \in E(M_p(G))$. Observe that the labeling of the edges in E does not affect the definition of a conflict hypergraph, since $M_p(G) = \pi(M)_p(G)$ for each $\pi \in \mathcal{S}_t$. Similarly, permuting the rows of M and p (with the same permutation) does not affect conflicts. Now we shall give constraints on an integer graph by restricting the structure of the conflict hypergraph. When writing matrices we shall write "+" instead of $+1$ and "−" instead of -1 for convenience.

For example consider the matrix $(+\ 0\ 0\ -)$. Then two edges (u_1, v_1) and (u_2, v_2) are in conflict if and only if $u_1 - v_2 \geqslant 1$ or $u_2 - v_1 \geqslant 1$. That is, one of the edges has both endpoints to the right of the endpoints of the other edge. See Figure 2.6 for an example.

Forbidden Subgraph Given an ordered graph H (without isolated vertices[2]) we can model the fact that an integer graph is in $\text{Forb}_<(H)$ as follows. Let $t = |E(H)|$ and let $E(H) = \{(a_1, b_1), \ldots, (a_t, b_t)\}$. For distinct i, $j \in [2t]$ let $r(i, j)$ denote the vector in $\mathbb{Z}^{2t}$ that equals -1 in coordinate i, equals 1 in coordinate j, and is 0 otherwise. Observe that for any vector $x = (x_1, \ldots, x_{2t})^\top \in \mathbb{Z}^{2t}$ we have $r(i, j)^\top x = x_j - x_i \geqslant 1$ if and only if $x_i < x_j$, and $r(i, j)^\top x$, $r(j, i)^\top x \geqslant 0$ if and only if $x_i = x_j$. Let $(x_1, \ldots, x_{2t}) = (a_1, b_1, \ldots, a_t, b_t)$, let $A = \{(i, j) \mid i, j \in [2t], i < j, x_i = x_j\}$, and let $B = \{(i, j) \mid i, j \in [2t], i < j, (i, j) \notin A\}$. Further let $s = 2|A| + |B|$. We define a matrix $M \in \mathbb{Z}^{s \times 2t}$ and a vector $p \in \mathbb{Z}^s$ as follows[3]. Let $M^{(1)}, \ldots, M^{(s)}$ denote the rows of M and let $p = (p_1, \ldots, p_s)^\top$. Then for any pair $(i, j) \in A$ there are σ, $\sigma' \in [s]$ such that $M^{(\sigma)} = r(i, j)$, $M^{(\sigma')} = r(j, i)$, and $p_\sigma = p_{\sigma'} = 0$. Further for any pair $(i, j) \in B$ there is $\sigma \in [s]$ with $M^{(\sigma)} = r(i, j)$ and $p_\sigma = 1$. Moreover there

[2]The model is based on conflicting edges, so isolated vertices cannot be handled.

[3]For convenience we include several rows in the definition of M which are unnecessary by transitive relations between vertices.



$$M = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline - & 0 & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - \\ 0 & 0 & - & + & 0 & 0 \\ 0 & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & + & - & 0 \\ 0 & 0 & 0 & 0 & - & + \end{array}$$

$$p = (+ + + 0 0 +)^\top$$

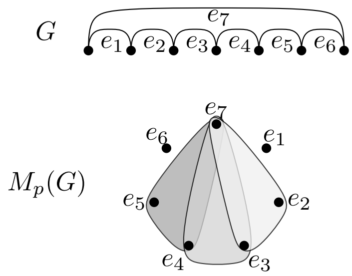


Figure 2.7: An ordered graph H, a matrix M, and a vector p such that $M_p(G)$ is empty if and only if G does not contain a copy of H. Here we do not state rows of M which are unnecessary due to transitive relations between vertices.

are no other rows in M. See Figure 2.7 for an example. We claim that an integer graph G contains a copy of H if and only if $M_p(G)$ is not empty. Indeed if $\bar{H}$ is a copy of H in G then label its edges $(\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_t, \bar{b}_t)$ such that the edge $(\bar{a}_i, \bar{b}_i)$ in $\bar{H}$ corresponds to the edge (a_i, b_i) in H, $i \in [t]$. Then $M(\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_t, \bar{b}_t)^\top \geqslant p$ and thus $M_p(G)$ is not empty. The other way round assume that $M_p(G)$ is not empty. Then there is a set of edges $\bar{E} = \{(\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_t, \bar{b}_t)\}$ in G such that $M(\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_t, \bar{b}_t)^\top \geqslant p$. We see that the subgraph formed by $\bar{E}$ is isomorphic to H (as an ordered graph).

Some ordered matchings can be modeled in another way. Recall the definitions of monotone, nested, and all crossing matchings from Figure 2.5. Let

$$M^{\text{mon}} = (+ 0 0 -), \quad M^{\text{nest}} = \begin{pmatrix} + & 0 & - & 0 \\ 0 & - & 0 & + \end{pmatrix}, \quad M^{\text{cross}} = \begin{pmatrix} - & 0 & + & 0 \\ 0 & + & - & 0 \\ 0 & - & 0 & + \end{pmatrix}.$$

Note that for each of these three matrices M the conflict hypergraph is a graph. One can easily check that an integer graph G does not contain a monotone, a nested, respectively an all crossing matching on $w+1$ edges if and only if $\omega(M_{\mathbf{1}}(G)) \leqslant w$, where $M = M^{\text{mon}}$, M^{nest}, M^{cross} respectively. Ordered graphs not having a large such matching appear in the literature under the names arch, queue, respectively stack layouts [57] (the latter is also called book or page embedding). We remark that the edges of an ordered graph that contains no monotone (respectively nested) matching on $w+1$ edges can be decomposed into w sets, each not containing a monotone (respectively nested) pair of edges, while for an all crossing matching $\theta(w \log(w))$ such sets are required in the worst case [57]. For a matrix M, a vector p, and an integer $w \geqslant t-1$ let $\kappa_\omega(M, p, w) = \max\{\chi(G) \mid G \text{ integer graph}, \omega(M_p(G)) \leqslant w\}$ if this maximum exists and $\kappa_\omega(M, p, w) = \infty$ otherwise.

Theorem 2.5. *For each integer $w \geqslant 1$*

(a) $\kappa_\omega(M^{\text{mon}}, \mathbf{1}, w) = 2w + 1$ *(Theorem 2.4),*

(b) $2w + 1 \leqslant \kappa_\omega(M^{\text{nest}}, \mathbf{1}, w) \leqslant 4w$ *[57],*

(c) $2w + 1 \leqslant \kappa_\omega(M^{\text{cross}}, \mathbf{1}, w) \leqslant 4w$ *[35].*

Interestingly, nested and all crossing matchings behave very similar from this point of view. Some insight is given by Ozsvárt [119] with enumerations of $\mathrm{Forb}_<(H)$ for all ordered graphs H on two edges without isolated vertices. In particular there is an order (meaning the number of vertices) and size preserving bijection between $\mathrm{Forb}_<(H_1)$ and $\mathrm{Forb}_<(H_2)$ where H_1 is a nested and H_2 is an all crossing 2-matching.

Other constraints In [7] we consider conflicting pairs of edges, that is, matrices $M \in \mathbb{Z}^{s\times 4}$ and conflict graphs. Here we generalize some of our results to conflict hypergraphs. For a matrix $M \in \mathbb{Z}^{s\times 2t}$, a vector $p \in \mathbb{Z}^s$, and an integer $a \geqslant t-1$ let $\kappa_\alpha(M,p,a) = \max\{\chi(G) \mid G \text{ integer graph}, \alpha(M_p(G)) \leqslant a\}$ if this maximum exists and $\kappa_\alpha(M,p,a) = \infty$ otherwise. Note that $M_p(G)$ is a t-uniform hypergraph and hence has independence number and clique number at least $\min\{|E(G)|, t-1\}$. Further note that $\kappa_\omega(M,p,w) = X_{\mathbf{cli}}(M,p,w)$ and $\kappa_\alpha(M,p,a) = X_{\mathbf{ind}}(M,p,a)$ in the notation of [7]. For $x > 0$ let[4] $b(x)$ be the largest integer k with $\binom{k}{2} \leqslant x$. Since $\alpha(M_p(K)), \omega(M_p(K)) \leqslant \binom{k}{2}$ for any complete integer graph K on k vertices we have for all matrices M, vectors p, and integers a, w, with a, $w \geqslant t-1$, that

$$\kappa_\alpha(M,p,a) \geqslant b(a), \quad \kappa_\omega(M,p,w) \geqslant b(w). \tag{2.2}$$

We call a matrix M *translation invariant* if $M\mathbf{1} = 0$. Observe that a matrix $M \in \mathbb{Z}^{s\times 2t}$ is translation invariant if and only if for each $\tau \in \mathbb{Z}$ and all vectors $x \in \mathbb{Z}^{2t}$, $p \in \mathbb{Z}^s$ we have $Mx \leqslant p$ if and only if $M(x+\boldsymbol{\tau}) \leqslant p$. That is, whether or not a set of edges is conflicting does not depend on the absolute coordinates of the edges but on their relative positions to each other. The first part of Theorem 2.6 below shows that for many matrices M that are not translation invariant $\kappa_\omega(M,p,w) = \kappa_\alpha(M,p,a) = \infty$ for all values of a, p, and w.

Theorem 2.6. *Let a, s, t, $w \in \mathbb{Z}$, with $s \geqslant 1$, $t \geqslant 2$, a, $w \geqslant t-1$, let $p \in \mathbb{Z}^s$, and let $M \in \mathbb{Z}^{s\times 2t}$. If M is not translation invariant, then $\kappa_\omega(M,p,w) = \infty$. Moreover, if $M\mathbf{1} > \mathbf{0}$ or $M\mathbf{1} < \mathbf{0}$, then $\kappa_\alpha(M,p,a) = \infty$.*
If $s = 1$ and $M = (m_1, \dots, m_{2t})$ is translation invariant, then the following holds.

(a) If $\sum_{i=1}^t m_{2i} \geqslant \max\{p, 0\}$, then $\kappa_\alpha(M,p,a) = \infty$ and $\kappa_\omega(M,p,w) = b(w)$.

(b) If $\sum_{i=1}^t m_{2i} < p$ and for each $i \in [t]$ we have $m_{2i-1} + m_{2i} = 0$, $m_{2i} \leqslant 0$, then $\kappa_\alpha(M,p,a) = b(a)$ and $\kappa_\omega(M,p,w) = \infty$.

(c) If $\sum_{i=1}^t m_{2i} > 0$ or ($\sum_{i=1}^t m_{2i} = 0$ and there are i, $j \in [t]$ with $m_{2i}, m_{2j-1} \neq 0$), then $\kappa_\alpha(M,p,a) = \infty$.

(d) If for each $i \in [t]$ we have $m_{2i} \leqslant 0$ and $|m_{2i}| \geqslant m_{2i-1}$ and for some $i \in [t]$ we have $m_{2i} < 0$, then $\kappa_\omega(M,p,w) = \infty$.

[4] For $x \geqslant 1$ we have $\binom{\sqrt{2x}}{2} < x < \binom{\sqrt{2x}+1}{2}$. Thus $b(x) = \lfloor\sqrt{2x}\rfloor$ if $x < \binom{\lfloor\sqrt{2x}\rfloor+1}{2}$ and $b(x) = \lfloor\sqrt{2x}\rfloor + 1$ otherwise.

In all cases above the vertices of any complete graph can be embedded into $\mathbb{Z}$ *such that the conflict hypergraph is either empty or complete* t*-uniform.*

In case M and p do not satisfy any of the requirements of Theorem 2.6, the exact behavior of $\kappa_\alpha(M,p,a)$ and $\kappa_\omega(M,p,w)$ can be non-trivial. In [7] we determine $\kappa_\alpha(M,p,a)$ and $\kappa_\omega(M,p,w)$ exactly for all $M \in \{-1,0,1\}^{1\times 4}$ and a, p, $w \in \mathbb{Z}$, a, $w \geqslant 1$. We also consider the matrix M^{nest} defined above and generalize the results of Dujmović and Wood [57] for this matrix to arbitrary p. We state the results from [7] in Table 2.1 without proof.

2.3 Structural Lemmas and Reductions

In this section we first analyze the structure of ordered trees without bonnets and tangled paths. This leads to a proof of Theorem 2.2 in Section 2.4. Afterwards we establish several cases when $\kappa_<(H)$ can be upper bounded in terms of $\kappa_<(H')$ for a subgraph H' of H. This allows us to reduce the problem of whether $\kappa_<(H) \neq \infty$ to the problem of whether $\kappa_<(H') \neq \infty$. These reductions are the crucial tools in the proof of Theorem 2.3 in Section 2.4.

Lemma 2.3.1. *Let T be an ordered tree that does not contain a tangled path and let $u < v < w$ be vertices in T. If uw is an edge in T, then all vertices of the path connecting u and v in T are between u and w.*

Proof. Let P be the path in T that starts with v and ends with the edge uw. Let ℓ denote the leftmost vertex in P. Assume for the sake of contradiction that $\ell < u$. Then the path $vP\ell$ contains neither u nor w and therefore crosses the edge uw. Hence the paths $P\ell$ and ℓP cross and P is tangled, a contradiction. Therefore $\ell = u$. Due to symmetric arguments w is the rightmost vertex in P. Hence all vertices in P are between u and w. □

Lemma 2.3.2. *Let T be an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment. Deleting any leaf from T yields an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment.*

Proof. Let uv be an edge in T incident to a leaf u and let $T' = T - u$. Then clearly T' is an ordered tree that contains neither a bonnet nor a tangled path. For the sake of contradiction assume that T' has at least two segments and let x be an inner cut vertex in T'. Then $x \neq u$,v and is between u and v in T, since x is not an inner cut vertex in T. By reversing T if necessary we may assume that $v < x < u$. Let P be the v-x-path in T'. All vertices in P are between v and u by Lemma 2.3.1 applied to u, v and x. In addition no vertex in P is to the right of x since x is an inner cut vertex in T'. So all vertices in P are between v and x. Let vw denote the first edge of P and let xy denote an edge in T' with $x < y$. Such an edge xy exists since the

M	p	$\kappa_\alpha(M,p,a)$	$\kappa_\omega(M,p,w)$
$M\mathbf{1} \neq 0$	$p \in \mathbb{Z}$	∞	∞
(0000)	$p \leqslant 0$	∞	$b(w)$
	$p > 0$	$b(a)$	∞
$(+0-0)$ $(-0+0)$ $(0+0-)$ $(0-0+)$	$p \leqslant 0$	∞	$b(w)$
	$p > 0$	$a+1$	$pw+1.$
$(-+00)$	$p \leqslant 1$	∞	$b(w)$
$(00-+)$	$p \geqslant 1$	∞	$b(w-1)+p-1$
$(+-00)$	$p \leqslant 0$	$b(a)-p$	∞
$(00+-)$	$p \geqslant 0$	$b(a)$	∞
$(-+-+)$	$p \leqslant 2$	∞	$b(w)$
	$p \geqslant 2$	∞	$b(w - p \bmod 2) + \lceil \frac{p-2}{2} \rceil$
$(+-+-)$	$p \leqslant -1$	$b(a-(1-p) \bmod 2) + \left\lceil \frac{-(p+1)}{2} \right\rceil$	∞
	$p \geqslant -1$	$b(a)$	∞
$(+--+)$	$p \leqslant 0$	∞	$b(w)$
$(-++-)$	$p > 0$	∞	$pw+1$
$(++--)$	$p \leqslant 0$	∞	$b(w)$
$(--++)$	$p > 0$	∞	$\lfloor \frac{pw+3}{2} \rfloor$
$(+00-)$	$p \leqslant 0$	$\lfloor \sqrt{4a+3} \rfloor - p$	$w+1$
$(0-+0)$	$p \geqslant 0$	$\lfloor \sqrt{4a+7} \rfloor - 1$	$(p+1)w+1$
$(-00+)$	$p \leqslant 1$	∞	$b(w)$
$(0+-0)$	$p \geqslant 2$	∞	$b(w)+p-2$
$\left(\begin{smallmatrix} + & 0 & - & 0 \\ 0 & + & 0 & - \end{smallmatrix}\right)$	$p \leqslant 0$	$2(1-p)a+1 \leqslant \cdot \leqslant 4(1-p)a$	$\lfloor \frac{w+3}{2} \rfloor$
	$p > 0$	$a+1$	$pw+1$
$\left(\begin{smallmatrix} + & 0 & - & 0 \\ 0 & - & 0 & + \end{smallmatrix}\right)$	$p \leqslant 0$	$(1-p)a+1$	$w+1$
$= M^{\text{nest}}$	$p > 0$	$\lfloor \frac{a+3}{2} \rfloor$	$2pw+1 \leqslant \cdot \leqslant 4pw$

Table 2.1: Summary of results from [7] on values of $\kappa_\alpha(M,p,a)$ and $\kappa_\omega(M,p,w)$ for a, p, $w \in \mathbb{Z}$, a, $w \geqslant 1$, and matrices M. The first row covers all non-translation invariant $M \in \mathbb{Z}^{1\times 4}$, rows 2 to 11 cover all translation invariant $M \in \{-1,0,1\}^{1\times 4}$, and the last two rows cover two $M \in \{-1,0,1\}^{2\times 4}$. Gray entries also from Theorem 2.6 here.

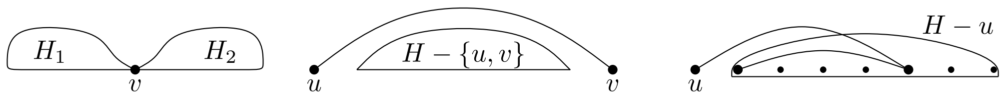


Figure 2.8: An inner cut vertex v splitting an ordered graph into ordered graphs H_1 and H_2 (left), an isolated edge uv in an ordered graph H (middle), and a reducible vertex u (right).

inner cut vertex x is not rightmost in T' and T' is connected. If $u < y$, then $uvPxy$ is a tangled path in T. If $y < u$, then u, v, w, x and y form a bonnet in T. In both cases we have a contradiction and hence T' has only one segment. □

Lemma 2.3.3. *If T is an ordered tree that contains neither a bonnet nor a tangled path and that has only one segment, then $\chi_<(T) \leqslant 2$.*

Proof. We prove the claim by induction on $k = |V(T)|$. If $k \leqslant 2$, then clearly $\chi_<(T) \leqslant 2$. So assume that $k \geqslant 3$. Let u denote a leaf in T, v its neighbor in T, and let $T' = T - u$. Then T' has only one segment and contains neither a bonnet nor a tangled path due to Lemma 2.3.2. Inductively $\chi_<(T') \leqslant 2$, i.e., there is a partition $L \dot\cup R = V(T')$, with $L \prec R$, such that all edges in T' are between L and R. By reversing T if necessary we assume that $v \in L$. For the sake of contradiction assume that $\chi_<(T) > 2$. Then $u < \ell$ for the rightmost vertex ℓ in L, possibly $\ell = v$. Let $w \in R$ denote one fixed neighbor of v in T'. Then all vertices of the path connecting ℓ and v in T' are between v and w due to Lemma 2.3.1. In particular ℓ is incident to an edge ℓx, $x \in R$, with $x \leqslant w$. Hence $u < v$, since otherwise there is a bonnet on vertices v, u, ℓ, x, and w in T. If there is a vertex y, $u < y < v$, then all vertices of the path connecting y and u in T are between u and v due to Lemma 2.3.1. But this is not possible since $y, v \in L$ and all the neighbors of y are in R. Hence u is immediately to the left of v in T. Note that u is not leftmost in T, since otherwise v is an inner cut vertex in T. Consider the path P connecting a vertex left of u to ℓ in T. This path contains distinct vertices $p, q \in L$, $r \in R$, such that pr and rq are edges in P and $p < u < v \leqslant q < r$. Hence there is a bonnet, a contradiction. This shows that $\chi_<(T) \leqslant 2$. □

We now present several reductions. Let us mention that some of the following arguments are similar to reductions used for extremal numbers of matrices [121, 140].

Recall, that an inner cut vertex v of an ordered graph H *splits* H into ordered graphs H_1 and H_2, where H_1 is induced by all vertices u with $u \leqslant v$ in H and H_2 is induced by all vertices u with $v \leqslant u$. See Figure 2.8 (left).

Reduction Lemma 2.1. *If an inner cut vertex v splits an ordered graph H into ordered graphs H_1 and H_2 with $\kappa_<(H_1), \kappa_<(H_2) \neq \infty$, then*

$$\kappa_<(H) \leqslant \kappa_<(H_1) + \kappa_<(H_2).$$

Proof. Consider an ordered graph $G \in \mathrm{Forb}_<(H)$. Let V_1 denote the set of vertices in G that are rightmost in some copy of H_1 in G. Further let $V_2 = V(G) \setminus V_1$. Then $G[V_2] \in \mathrm{Forb}_<(H_1)$ by the choice of V_1. Moreover $G[V_1] \in \mathrm{Forb}_<(H_2)$, since otherwise the leftmost vertex u in a copy of H_2 in $G[V_1]$ is also a rightmost vertex in a copy of H_1 and hence plays the role of v in a copy of H in G. Thus $\chi(G) \leqslant \chi(G[V_1]) + \chi(G[V_2]) \leqslant \kappa_<(H_2) + \kappa_<(H_1)$ and since $G \in \mathrm{Forb}_<(H)$ was arbitrary we have $\kappa_<(H) \leqslant \kappa_<(H_1) + \kappa_<(H_2)$. □

Reduction Lemma 2.2. *If v is an isolated vertex in an ordered graph H with $|V(H)| \geqslant 3$ and $\kappa_<(H - v) \neq \infty$, then $\kappa_<(H) \leqslant 2\,\kappa_<(H - v)$.*

Proof. Consider an ordered graph $G \in \mathrm{Forb}_<(H)$. If v is not leftmost or rightmost in H, then let V_1 be the set of vertices of G that are odd in the ordering of G and let $V_2 = V(G) \setminus V_1$. Then $G[V_1], G[V_2] \in \mathrm{Forb}_<(H - v)$, since for any two vertices $u < w$ in V_i there is a vertex $v \in V_{3-i}$ with $u < v < w$, $i = 1, 2$. Hence $\chi(G) \leqslant \chi(G[V_1]) + \chi(G[V_2]) \leqslant 2\kappa_<(H - v)$. If v is the leftmost or the rightmost in H, assume without loss of generality the former. Then clearly $G - u \in \mathrm{Forb}_<(H - v)$ for the leftmost vertex u of G. Thus $\chi(G) \leqslant 1 + \chi(G - u) \leqslant 1 + \kappa_<(H - v) \leqslant 2\kappa_<(H - v)$. Since $G \in \mathrm{Forb}_<(H)$ was arbitrary we have $\kappa_<(H) \leqslant 2\kappa_<(H - v)$ in both cases. □

Reduction Lemma 2.3. *Let u and v be the leftmost and rightmost vertices in an ordered graph H, $|V(H)| \geqslant 4$. If uv is an isolated edge in H and $\kappa_<(H - \{u, v\}) \neq \infty$, then*

$$\kappa_<(H) \leqslant 2\,\kappa_<(H - \{u, v\}) + 1.$$

Proof. See Figure 2.8 (middle). Let $H' = H - \{u, v\}$ and consider an ordered graph $G \in \mathrm{Forb}_<(H)$. If G does not contain a copy of H', then $\chi(G) \leqslant \kappa_<(H') \leqslant 2\kappa_<(H') + 1$. So, assume that G contains a copy of H'. Let $V_1 \dot\cup \cdots \dot\cup V_p$ denote a partition of $V(G)$ into disjoint intervals with $V_1 \prec \cdots \prec V_p$, v_i being the leftmost vertex in V_i, $1 \leqslant i \leqslant p$, such that $G[V_i] \in \mathrm{Forb}_<(H')$, $1 \leqslant i \leqslant p$, and $G[V_i \cup \{v_{i+1}\}]$ contains a copy of H', $1 \leqslant i < p$. Note that one can find such a partition greedily by iteratively choosing a largest interval from the left that does not induce any copy of H' in G. If $p \geqslant 3$, there are no edges xy with $x \in V_i$ and $v_{i+2} < y$, since otherwise xy together with a copy of H' in $G[V_{i+1} \cup \{v_{i+2}\}]$ forms a copy of H, $1 \leqslant i \leqslant p - 2$.

Choose a set Φ of $2\,\kappa_<(H') + 1$ distinct colors. Let $\Phi_1, \ldots, \Phi_p \subset \Phi$ denote subsets of colors such that $|\Phi_i| = \kappa_<(H')$, $1 \leqslant i \leqslant p$, $\Phi_i \cap \Phi_{i+1} = \emptyset$, $1 \leqslant i < p$, and, if $p \geqslant 3$, $\Phi_{i+2} \setminus (\Phi_i \cup \Phi_{i+1}) \neq \emptyset$, $1 \leqslant i \leqslant p - 2$. Note that such sets Φ_i can be chosen greedily from Φ. Since $G[V_i] \in \mathrm{Forb}_<(H')$ we can color $G[V_i]$ properly with colors from Φ_i, $1 \leqslant i \leqslant p$, such that, if $i \geqslant 3$, v_i is colored with a color in $\Phi_i \setminus (\Phi_{i-1} \cup \Phi_{i-2})$. This yields a proper coloring of G using colors from the set Φ only. Hence $\chi(G) \leqslant 2\,\kappa_<(H') + 1$. Since $G \in \mathrm{Forb}_<(H)$ was arbitrary we have $\kappa_<(H) \leqslant 2\,\kappa_<(H - \{u, v\}) + 1$. □

Recall, that a vertex in an ordered graph H is called *reducible*, if it is a leaf in H, is leftmost or rightmost in H and has a common neighbor with the vertex next to it. See Figure 2.8 (right).

Reduction Lemma 2.4. *Let H denote an ordered graph with $|V(H)| \geqslant 3$. If u is a reducible vertex in H and $\kappa_<(H-u) \neq \infty$, then*

$$\kappa_<(H) \leqslant 2\,\kappa_<(H-u).$$

Moreover, for each $G \in \mathrm{Forb}_<(H)$ there is $G' \subseteq G$ such that G' is 1-degenerate and deleting the edges of G' from G yields a graph from $\mathrm{Forb}_<(H-u)$.

Proof. By reversing H if necessary we may assume that the reducible vertex u is leftmost in H. Let $G \in \mathrm{Forb}_<(H)$. Let E denote the set of edges in G consisting for each vertex w in G of the longest edge to the left incident to w in G, if such an edge exists.

Assume that there is a copy H' of $H-u$ in $G-E$. Let v denote the vertex in H' corresponding to the vertex immediately to the right of u in H and let w denote the vertex in H' corresponding to the neighbor of u in H. Then v is leftmost in H' and there is an edge between v and w in H'. Thus, there is an edge xw in E incident to w in G with $x < v$. Hence H' extends to a copy of H in G with the edge xw, a contradiction. This shows that $G - E \in \mathrm{Forb}_<(H-u)$.

Finally observe that the graph G' with the edge-set E is 1-degenerate and hence 2-colorable. This shows that $\chi(G) \leqslant \chi(G')\chi(G-E) \leqslant 2\kappa_<(H-u)$ and since $G \in \mathrm{Forb}_<(H)$ was arbitrary we have $\kappa_<(H) \leqslant 2\kappa_<(H-u)$. □

Having Reduction Lemma 2.4 at hand, we are now ready to prove that every non-crossing monotonically alternating tree T satisfies $\kappa_<(T) \neq \infty$.

Lemma 2.3.4. *If T is a non-crossing monotonically alternating tree with at least two vertices, then*

$$\kappa_<(T) \leqslant 2|V(T)| - 3.$$

Proof. Let $k = |V(T)|$ and $G \in \mathrm{Forb}_<(T)$. We shall prove that G can be edge-decomposed into $(k-2)$ 1-degenerate graphs by induction on k.

If $k = 2$, then T consists of a single edge only. Hence G has an empty edge-set and there is nothing to prove.

So consider $k \geqslant 3$ and assume that the induction statement holds for all smaller values of k. Assume for the sake of contradiction that the leftmost vertex u and the rightmost w in T are of degree at least 2. Then the longest and the shortest edge incident to w do not coincide. Let e be the longest edge incident to w. Since in a monotonically alternating tree each edge is the shortest edge incident to its left or right endpoint, e is the shortest edge incident to its left endpoint. In particular, $e \neq uw$ because u is incident to another edge e', shorter than uw. Thus e and e'

cross since $\chi_<(T) \leqslant 2$, a contradiction. Hence the leftmost or the rightmost vertex is a leaf in T.

By reversing T if necessary we assume that u is of degree 1. We shall show that u is a reducible leaf. To do so, we need to show that the vertex x that is immediately to the right of u is adjacent to the neighbor v of u. Assume for the sake of contradiction that x is not adjacent to v. Note that v is adjacent to a leaf, so it is not a leaf itself. Let e'' be an edge incident to v, $e'' \neq uv$. Then an edge incident to x crosses either uv or e'' since $\chi_<(T) \leqslant 2$, a contradiction. Thus x is adjacent to v and u is a reducible leaf in T.

Therefore, by Reduction Lemma 2.4, there is a 1-degenerate subgraph G' of G such that removing the edges of G' from G yields a graph $G'' \in \mathrm{Forb}_<(T-u)$. Observe that the tree $T-u$ is non-crossing and monotonically alternating with $k > |V(T-u)| = k-1 \geqslant 2$. Hence G'' can be edge-decomposed into $(k-3)$ 1-degenerate graphs $G_1, \ldots, G_{k-3}$ by induction. Thus the graphs $G_1, \ldots, G_{k-3}, G'$ decompose G into $(k-2)$ 1-degenerate graphs, proving the induction step.

If $k = 2$, we know that G has no edges and $\chi(G) = 1 \leqslant 2|V(T)| - 3$. So assume that $k \geqslant 3$. Since G is a union of $(k-2)$ 1-degenerate graphs, each subgraph of G is a union of $(k-2)$ 1-degenerate graphs, so each subgraph G^* of G on at least one vertex that has at most $(k-2)(|V(G^*)|-1)$ edges, and thus has a vertex of degree at most $2(k-2)-1$. Therefore G is $(2(k-2)-1)$-degenerate, so $\chi(G) \leqslant 2(k-2) \leqslant 2|V(T)|-3$. Since $G \in \mathrm{Forb}_<(H)$ was arbitrary we have $\kappa_<(H) \leqslant 2|V(T)| - 3$. □

Reduction Lemma 2.5. *Let T denote an ordered matching with at least two edges. If uv is an edge in T, u and v are consecutive, and $\kappa_<(T - \{u,v\}) \neq \infty$, then*

$$\kappa_<(T) \leqslant 3\,\kappa_<(T - \{u,v\}).$$

Proof. Let $G \in \mathrm{Forb}_<(T)$ with vertices $v_1 < \cdots < v_n$. We shall prove that $\chi(G) \leqslant 3\,\kappa_<(T - \{u,v\})$ by induction on $n = |V(G)|$. If $n \leqslant 3\,\kappa_<(T - \{u,v\})$, then the claim holds trivially. So assume that $n > 3\,\kappa_<(T - \{u,v\}) \geqslant 3$. If there are two consecutive vertices x, y in G that are not adjacent, then let G' denote the graph obtained by identifying x and y. Then $G' \in \mathrm{Forb}_<(T)$ and $\chi(G) \leqslant \chi(G')$. Hence $\chi(G) \leqslant \chi(G') \leqslant 3\,\kappa_<(T - \{u,v\})$ by induction. If each pair of consecutive vertices in G forms an edge, then consider a partition $V(G) = V_0 \dot\cup V_1 \dot\cup V_2$ such that $V_i = \{v_j \in V(G) \mid j \equiv i \pmod 3\}$. Observe that for each pair of vertices $x, y \in V_i$ there are at least two adjacent vertices from $V(G) \setminus V_i$ between x and y. Hence $G[V_i] \in \mathrm{Forb}_<(T - \{u,v\})$, $i = 0,1,2$, since any copy of $T - \{u,v\}$ in $G[V_i]$ extends to a copy of T in G. Hence $\chi(G) \leqslant 3\,\kappa_<(T - \{u,v\})$ and since $G \in \mathrm{Forb}_<(T)$ was arbitrary we have $\kappa_<(T) \leqslant 3\,\kappa_<(T - \{u,v\})$. □

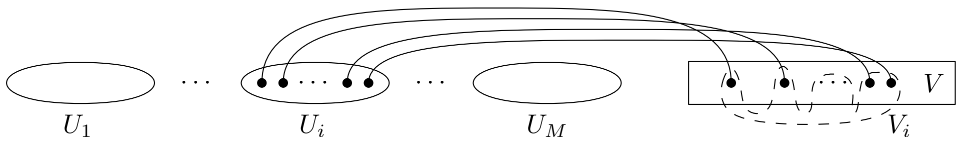


Figure 2.9: A graph G_k obtained by Tutte's construction from a graph G_{k-1}. Here $G_k[U_i] = G_{k-1}$, $1 \leqslant i \leqslant M$.

Remark. *We do not use Reduction Lemma 2.5 for proving our theorems. Instead, it is only used to derive upper bounds on $\kappa_<(T)$ for some small ordered forests T on three edges in Section 2.5.*

2.4 Proofs of Theorems

Proof of Theorem 2.1

We shall prove that if an ordered graph H contains a cycle, a tangled path or a bonnet then for each positive integer k there is an ordered graph $G \in \text{Forb}_<(H)$ with $\chi(G) \geqslant k$. In case of cycles we shall use graphs of high girth and large chromatic number, while for bonnets and tangled paths we shall provide ordered versions of classical constructions of graphs of large chromatic number. Specifically we use Tutte's construction in case of tangled paths and shift graphs in case of bonnets.

First assume that H contains a cycle of length ℓ. Fix a positive integer k and consider a graph G of girth at least $\ell + 1$ with $\chi(G) \geqslant k$ that exists by [61]. Then no ordering of the vertices of G gives an ordered subgraph isomorphic to H. This shows that for any positive integer k, $\kappa_<(H) \geqslant k$ and hence H is χ-avoidable.

A tangled path is minimal if it does not contain a proper subpath that is tangled. Next we shall show that for each minimal tangled path P and each $k \geqslant 1$ there is an ordered graph $G_k \in \text{Forb}_<(P)$ with $\chi(G_k) \geqslant k$.

By reversing P if necessary we assume that in P the paths Pu and uP cross for the rightmost vertex u in P. We will prove the claim by induction on k. If $k \leqslant 3$ let $G_k = K_k$ that has no crossing edges and thus no tangled paths. Consider $k \geqslant 4$ and let G_{k-1} denote an n-vertex graph of chromatic number at least $k-1$ that does not contain a copy of P. Such a graph exists by induction. The following construction is due to Tutte (alias Blanche Descartes) for unordered graphs [54]. Let $N = (k-1)(n-1)+1$ and $M = \binom{N}{n}$. Consider pairwise disjoint sets of vertices $U_1, \ldots, U_M, V$ such that $|U_i| = n$, $i = 1, \ldots, M$, $|V| = N$ and $U_1 \prec \cdots \prec U_M \prec V$. Let $V_1, \ldots, V_M$ be the n-element subsets of V. Let each U_i, $i = 1, \ldots, M$, induce a copy of G_{k-1}. Finally let there be a perfect matching between U_i and V_i such that the j^{th} vertex in U_i is matched to the j^{th} vertex in V_i, $i = 1, \ldots, M$. See Figure 2.9.

First we shall show that $\chi(G_k) \geqslant k$. If there are at most $k-1$ colors assigned to the vertices of G_k, then by Pigeonhole Principle there are n vertices of V of the same color, i.e., there is a set V_i with all vertices of the same color, say color 1. Since

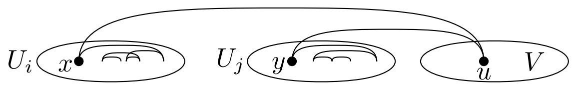


Figure 2.10: A path in G_k with rightmost vertex $u \in V$ is not tangled if Pu and uP are not tangled.

each vertex of U_i is adjacent to a vertex in V_i, no vertex in U_i is colored 1, so if the coloring is proper, then $G[U_i]$ uses at most $k-2$ colors. Hence the coloring is not proper, since $\chi(G[U_i]) = \chi(G_{k-1}) \geqslant k-1$. Therefore $\chi(G_k) \geqslant k$.

Next, we shall show that G_k does not contain a copy of P. Assume that there is such a copy P' of P in G_k with rightmost vertex u of P'. Let x and y be the neighbors of u in P', i.e., P' is a union of paths $P'yu$ and uxP'. Then $u \in V$ and $x, y \notin V$, since $G[U_i]$ does not contain a copy of P and there are no edges in $G_k[V]$. Let $x \in U_i$ and $y \in U_j$. Note that $i \neq j$ because the edges between U_i and V form a matching. The path uxP' is a proper subpath of P' and hence is not tangled. Recall that for each edge zw with $z \in U_i$, $w \in V$, and $w < u$, we have $z < x$ due to the construction of the matching between U_i and V_i. Hence the path uxP' does not contain any vertex $w \in V$ with $w < u$, since otherwise the path $uxP'w$ has a vertex left of x contradicting Lemma 2.3.1 applied to u, x and w. Hence $V(xP') \subseteq U_i$, because there are no edges between U_i's and u is rightmost in P'. See Figure 2.10. Similarly, all vertices of $P'y$ are contained in U_j. Thus $P'u$ and uP' do not cross. However, P' is a copy of P with respective subpaths crossing, a contradiction. Hence $G_k \in \mathrm{Forb}_<(P)$.

Now, if an ordered graph H contains a tangled path, then it contains a minimal tangled path. Thus H is χ-avoidable.

Next, let B be a bonnet. By reversing B if necessary, we assume that B has vertices $u < v \leqslant x, y \leqslant w$ and edges uv, uw, xy. A shift graph $S(n)$ is defined on vertices $\{(i,j) \mid 1 \leqslant i < j \leqslant n\}$ and edges $\{\{(i,j),(j,t)\} \mid 1 \leqslant i < j < t \leqslant n\}$. We will show that some ordering of $S(n)$ does not contain B. Let $G = S(n)$ be a shift graph with vertices ordered lexicographically, i.e., $(x_1, x_2) < (y_1, y_2)$ if and only if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. Assume that G contains vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $w = (w_1, w_2)$ that form a copy of B with $u < v \leqslant x, y \leqslant w$ and edges uv, uw, xy. Then $u_2 = v_1$, $u_2 = w_1$, $x_2 = y_1$. Thus $v_1 = w_1$. However, since $v \leqslant x, y \leqslant w$, we have that $v_1 \leqslant x_1, y_1 \leqslant w_1$, so $x_1 = y_1 = v_1 = w_1$. But $x_2 = y_1$, thus $x_2 = x_1$, a contradiction. Thus $G \in \mathrm{Forb}_<(B)$. We claim that $\chi(G) \geqslant \log(n) \geqslant c \log |V(G)|$. Indeed consider a proper coloring ϕ of G using $\chi(G)$ colors and sets of colors $\Phi_i = \{\phi(i,j) \mid i < j \leqslant n\}$, $1 \leqslant i \leqslant n$. Then $\phi(i,j) \notin \Phi_j$, since a vertex (i,j) is adjacent to all vertices (j,t), $j < t \leqslant n$. Therefore $\Phi_i \neq \Phi_j$ for all $j < i$. Hence all the sets of colors are distinct. This shows that $2^{\chi(G)} \geqslant n$, since there are at most $2^{\chi(G)}$ distinct subsets of colors. This proves that $\chi(G) \geqslant \log(n)$. Thus, for any k, there is an ordered graph of chromatic number at least k in $\text{Forb}_<(B)$. So, if an ordered graph H contains a bonnet, then H is χ-avoidable.. □

Proof of Theorem 2.2

Let T' be a segment of an ordered tree that does not contain a bonnet or a tangled path. We shall prove that T' is monotonically alternating by induction on $k = |V(T')|$. Every ordered tree on at most two vertices is monotonically alternating. So suppose $k \geqslant 3$. We have $\chi_<(T') = 2$ due to Lemma 2.3.3.

Claim. *The leftmost or the rightmost vertex in T' is of degree* 1.

Proof of Claim. As T' is a segment of an ordered tree, the subgraph induced by $V(T')$ is connected and thus both the leftmost vertex u and the rightmost vertex v in T' are of degree at least 1. For the sake of contradiction assume that both u and v are of degree at least 2. If u and v are adjacent then the edge uv, another edge incident to u and another edge incident v form a tangled path since $\chi_<(T') = 2$, a contradiction. If u and v are not adjacent let P denote the path in T' connecting u and v. It uses at most one of the edges incident to u. Then any other edge zu incident to u crosses the edge in P that is incident to v since $\chi_<(T') = 2$. Hence zP forms a tangled path, a contradiction. This shows that at least one of u or v is a leaf in T'. △

By reversing T' if necessary we assume that the leftmost vertex u is a leaf in T'. The ordered tree $T' - u$ is monotonically alternating by induction and Lemma 2.3.2. Consider the partition $V(T') = L \dot{\cup} R$, with $L \prec R$ and L and R being independent sets. Such a partition is unique since T' is connected. Let v be the neighbor of u in T'. Since $\chi_<(T') = 2$, $v \in R$. Since T' is connected, $k \geqslant 3$ and u is leftmost in T', the edge uv is not the shortest edge incident to v. Hence $uv \notin S(R)$ and therefore $S(R)$ has no crossing edges by induction. Clearly $uv \in S(L)$ since uv is the only edge incident to u and thus it is the shortest edge incident to u. If uv crosses some edge xy in T', $x < y$, then all vertices in the path connecting v and x are between x and y due to Lemma 2.3.1 applied to x, y and v. Therefore xy is not the shortest edge incident to x and hence $xy \notin S(L)$. This shows that $S(L)$ has no crossing edges and thus T' is monotonically alternating.

The other way round assume that each segment of an ordered tree T is monotonically alternating. We need to show that each segment contains neither a bonnet nor a tangled path. Let T' denote a segment of T, $V(T') = L \cup R$, $L \prec R$ and $E(T') = S(L) \cup S(R)$, so each edge is either a shortest edge incident to a vertex in R or a shortest edge incident to a vertex in L. Then $\chi_<(T') \leqslant 2$ and hence T' does not contain a bonnet. We will prove that T' does not contain a tangled path by induction on $k = |V(T')|$. If $k \leqslant 3$, then there are no crossing edges in T' and hence no tangled path. Suppose $k \geqslant 4$.

Assume that the leftmost vertex u and the rightmost vertex w in T' are of degree at least 2. If $uw \in E(T')$ then $uw \notin S(L)$ and $uw \notin S(R)$, a contradiction. So, $uw \notin E(T')$. Consider the longest edge xw incident to w. Then $x \neq u$ and since $xw \notin S(R)$, $xw \in S(L)$. Then the shortest edge incident to u crosses xw, a contradiction since $S(L)$ does not contain crossing edges. Hence the leftmost or the rightmost vertex is a leaf in T'.

By reversing T' if necessary we assume that the leftmost vertex u is a leaf. We see that $T' - u$ is monotonically alternating, thus by induction it does not contain a tangled path. Hence if T' has a tangled path P, then P contains an edge uv crossing some other edge in P, where v is the neighbor of u in T'. Then the rightmost vertex r in P is of degree 2 and to the right of v, since P is tangled and u is leftmost and of degree 1 in T'. Let x and y, $x < y$, be neighbors of r in P. Then xr is the shortest edge incident to x, since any shorter edge forms a tangled path with r and y in $T' - u$. This is a contradiction since uv and xr cross and T' is monotonically alternating. Thus T' has no tangled path.

Finally we prove the last statement of the theorem. If H is a connected χ-unavoidable ordered graph, then H is a tree that contains neither a bonnet nor a tangled path due to Theorem 2.1. Hence each segment of H is a monotonically alternating tree. □

Proof of Theorem 2.3

Let T be a non-crossing χ-unavoidable ordered graph. Then T is acyclic, contains no tangled path and no bonnet by Theorem 2.1. Hence T is a non-crossing ordered forest with no bonnet.

On the other hand let T be a non-crossing forest with no bonnet. Recall that $\kappa_<(H) \geqslant k-1$ for each ordered k-vertex graph H because $K_{k-1} \in \text{Forb}_<(H)$. We shall prove that T is χ-unavoidable. Let $k = |V(T)|$ and consider any ordered graph $G \in \text{Forb}_<(T)$. We will prove by induction on k that $\chi(G) \leqslant 2^k$ and $\chi(G) \leqslant 2k-3$ if T is a tree. If $k=2$, then clearly $\chi(G) = 1$. So consider $k \geqslant 3$.

If T is a tree, then each segment of T is a monotonically alternating tree, by Theorem 2.2. If there is only one segment in T, then $\kappa_<(T) \leqslant 2k-3$ by Lemma 2.3.4. If there is more than one segment in T, then there is an inner cut vertex splitting T into two trees T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 2.1 and induction we have $\kappa_<(T) \leqslant \kappa_<(T_1) + \kappa_<(T_2) \leqslant 2|V(T_1)| - 3 + 2|V(T_2)| - 3 = 2(|V(T)|+1) - 6 = 2k-4$.

If T is a forest we consider several cases. If T has more than one segment, then there is an inner cut vertex splitting T into two forests T_1 and T_2 that are clearly also non-crossing and contain no bonnet. Thus by Reduction Lemma 2.1 and induction we have $\kappa_<(T) \leqslant \kappa_<(T_1) + \kappa_<(T_2) \leqslant 2^{|V(T_1)|} + 2^{|V(T_2)|} = 2^t + 2^{k+1-t} \leqslant 2^k$ with $t = |V(T_1)| \geqslant 2$. If T has an isolated vertex u, then by Reduction Lemma 2.2

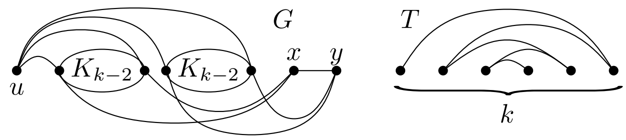


Figure 2.11: An ordered graph G with chromatic number k not containing a non-crossing ordered tree T on k vertices without bonnets on the right, $k = 6$.

and induction we have $\kappa_<(T) \leqslant 2\kappa_<(T-u) \leqslant 2 \cdot 2^{k-1} = 2^k$. Finally, if T has no isolated vertices and exactly one segment, then consider the leftmost and rightmost vertices u and v of T. Since u and v are not isolated in this case, and T is non-crossing with no inner cut vertices, uv is an edge. If uv is isolated, then $k \geqslant 4$ (since there is no isolated vertex) and by Reduction Lemma 2.3 and induction we have $\kappa_<(T) \leqslant 2 \cdot \kappa_<(T - \{u, v\}) + 1 \leqslant 2 \cdot 2^{k-2} + 1 \leqslant 2^k$. If uv is not isolated, then either u or v, say u, is a leaf of T, since T is non-crossing and does not contain a bonnet. Let xv denote the longest edge incident to v in $T - u$. Note that x exists since the edge uv is not isolated. Then there is no other vertex between u and x, since such a vertex would be isolated in the non-crossing forest T without bonnets. Thus, u is a reducible vertex, so by Reduction Lemma 2.4 and induction we have $\kappa_<(T) \leqslant 2\kappa_<(T-u) \leqslant 2 \cdot 2^{k-1} = 2^k$.

Next, we provide a k-vertex non-crossing χ-unavoidable ordered tree with no bonnet such that $\kappa_<(T) \geqslant k$. Let T be a monotonically alternating path on $k \geqslant 4$ vertices with leftmost vertex of degree 1, as in Figure 2.11 (right). Then T is non-crossing and contains no bonnet, and hence is χ-unavoidable by Theorem 2.3. Further let G denote a graph on vertices $u < x_1 < \cdots < x_{k-2} < y_1 < \cdots < y_{k-2} < x < y$ such that xy is an edge and $\{u, x_1, \ldots, x_{k-2}\}$, $\{u, y_1, \ldots, y_{k-2}\}$, $\{x, x_1, \ldots, x_{k-2}\}$, and $\{y, y_1, \ldots, y_{k-2}\}$ induce complete graphs on $k-1$ vertices each. See Figure 2.11 (left).

We shall show that $G \in \mathrm{Forb}_<(T)$ and $\chi(G) \geqslant k$. Consider a proper vertex coloring of G using colors $1, \ldots, k-1$. Without loss of generality u has color 1. Then all colors $2, \ldots, k-1$ are used on the vertices $x_1, \ldots, x_{k-2}$ as well as on $y_1, \ldots, y_{k-2}$. Hence both x and y are of color 1, a contradiction. Thus $\chi(G) \geqslant k$.

Assume that there is a copy P of T in G. Let v be the leftmost and w be the rightmost vertex in P. Note that vw is an edge and that there are k vertices between v and w (including v and w). Therefore vw is one of the edges uy_i, $1 \leqslant i \leqslant k-2$, x_jx, $1 \leqslant j \leqslant k-2$, or y_1y. In the first case $V(P) \subseteq \{u, y_1, \ldots, y_{k-2}\}$, in the second case $V(P) \subseteq \{x_1, \ldots, x_{k-2}, x\}$ and in the last case either $P = y_1, y, x$ or $V(P) \subseteq \{y, y_1, \ldots, y_{k-2}\}$. Since T has at least 4 vertices, $P \neq y_1, y, x$. So in any case P has at most $k-1$ vertices, a contradiction since T has k vertices. Hence $G \in \mathrm{Forb}_<(T)$.

Finally it is easy to see that $\kappa_<(T) = k - 1$ for any ordered tree T on at most 3 vertices using Reduction Lemmas 2.1 and 2.4. □

Proof of Theorem 2.4

(a) Let T be an ordered forest on k vertices where each segment is a generalized star, a nested 2-matching, or an all crossing 2-matching. Let $T_1, \dots, T_s$ denote the segments of T and $k_i = |V(T_i)|$, $1 \leqslant i \leqslant s$. Let T' be a segment of T. If T' is a generalized star on k' vertices, then the center of the star is leftmost (or rightmost) in T'. Let $G \in \mathrm{Forb}_<(T')$. Then each vertex in G has at most $k'-2$ neighbors to the right (or to the left). Thus each such graph can be greedily colored from right to left (or left to right) with at most $k'-1$ colors. This shows that $\kappa_<(T') \leqslant |V(T')|-1$. If T' is a nested 2-matching, then $\kappa_<(T') = 3 = |V(T')|-1$ due to [57] (Lemma 9). If T' is an all crossing 2-matching, then $\kappa_<(T') = 3 = |V(T')|-1$, since any graph not containing T' is outerplanar and outerplanar graphs have chromatic number at most 3. We apply Reduction Lemma 2.1 and the results above which yield $\kappa_<(T) \leqslant \sum_{i=1}^{s} \kappa_<(T_i) \leqslant \sum_{i=1}^{s}(k_i - 1) = k-1$.

(b) Let T be an ordered forest on k vertices where each segment is a generalized star, a non-crossing tree without bonnets, an all crossing or a nested matching. Let $T_1, \dots, T_s$ denote the segments of T and $k_i = |V(T_i)| \geqslant 2$. Let T' be a segment of T. If T' is a nested or all an all crossing k'-matching, $k' \geqslant 2$, then $\kappa_<(T') \leqslant 4(k'-1) \leqslant 2|V(T')|-3$ due to equation (2.1), since any graph $G \in \mathrm{Forb}_<(T')$ contains less than $2(k'-1)|V(G)|$ edges due to Dujmović and Wood [57] (for nestings), respectively Capoyleas and Pach [35] (for crossings), see also Theorem 2.5. Further $\kappa_<(T') \leqslant 2|V(T')|-3$ if T' is a non-crossing tree without bonnets due to Theorem 2.3. Hence Reduction Lemma 2.1 yields $\kappa_<(T) \leqslant \sum_{i=1}^{s} \kappa_<(T_i) \leqslant \sum_{i=1}^{s}(2k_i - 3) \leqslant 2k-3$.

(c) Let $T = M(t, m, \pi)$ for some positive integers m and t and a permutation π of $[t]$. If $t = 1$, then $\kappa_<(T) = m$ due to the results above, since $M(1, m, \pi)$ is a star on $m+1$ vertices. Weidert [142] proves that $\mathrm{ex}_<(n, M(t, 1, \pi)) \leqslant \mathrm{ex}_<(n, M(t, 2, \pi)) \leqslant 11t^4\binom{2t^2}{2t}n < t^4(2t^2)^{2t}n$ for any positive integer $t \geqslant 2$ and any permutation π of $[t]$. Moreover if $m \geqslant 2$, then

$$\mathrm{ex}_<(n, M(t, m, \pi)) \leqslant 2^{t(m-2)}\mathrm{ex}_<(n, M(t, 2, \pi))$$

due to a reduction by Tardos [140]. Therefore $\mathrm{ex}_<(n, M(t, m, \pi)) < 2^{tm}t^{4+4t}n$. Thus, using the fact that $|V(T)| = k = tm + t$ and equation (2.1) we have that $\kappa_<(M(t, m, \pi)) \leqslant 2^{tm+9t\log(t)} \leqslant 2^{10k\log k}$.

(d) Conlon *et al.* [46] and independently Balko *et al.* [11] prove that that there is a positive constant c such that for any sufficiently large positive integer k there is an ordered matching on $2k$ vertices with ordered Ramsey number at least $2^{c\frac{\log(k)^2}{\log\log(k)}}$. If, for some ordered graph H, the edges of a complete ordered graph G on $N = r_<(H)-1$ vertices are colored in two colors without monochromatic copies of H, then both color classes form ordered graphs G_1 and G_2 in $\mathrm{Forb}_<(H)$. Then one of the G_i's has chromatic number at least $\sqrt{N}$, since a product of proper colorings of G_1 and G_2 yields a proper coloring of G using $\chi(G_1)\chi(G_2) \geqslant \chi(G) = N$ colors. This shows that there is a positive constant c' such that for all positive integers k and ordered matchings H on k vertices with $\kappa_<(H) \geqslant 2^{c' \frac{\log(k)^2}{\log\log(k)}}$. □

Proof of Theorem 2.6

Let a, s, t, $w \in \mathbb{Z}$, with $s \geqslant 1$, $t \geqslant 2$, a, $w \geqslant t-1$, $p \in \mathbb{Z}^s$, and $M \in \mathbb{Z}^{s\times 2t}$.

First of all assume that M is not translation invariant. Consider some arbitrary integer k, $k \geqslant 2$, and let G be any integer graph with $\chi(G) \geqslant k$, $|E(G)| \geqslant t$. For an integer τ, let G_τ denote the ordered graph obtained from G by adding τ to every vertex (so G_τ contains an edge $(x+\tau, y+\tau)$ if and only if (x, y) is an edge in G). Clearly, we have $M(x+\boldsymbol{\tau}) = Mx+M\boldsymbol{\tau} = Mx+\tau(M\mathbf{1})$ for any vector $x \in \mathbb{Z}^{2t}$. Hence, since $M\mathbf{1} \neq \mathbf{0}$, there is a large or small enough $\tau = \tau(G, M, p)$ such that for some row $M^{(\sigma)}$ of M and any $u_1, v_1, \ldots, u_t, v_t \in V(G_\tau)$ we have $M^{(\sigma)}(u_1, v_1, \ldots, u_t, v_t)^\top < p$, i.e., the conflict graph $M_p(G_\tau)$ is empty and its clique number is $t-1 \leqslant w$. Since k was arbitrary, we have $\kappa_\omega(M, p, w) = \infty$.

If additionally $M\mathbf{1} > \mathbf{0}$ (respectively $M\mathbf{1} < \mathbf{0}$), then there is a large (respectively small) enough $\tau = \tau(G, M, p)$ such that $M_p(G_\tau)$ is a complete t-uniform hypergraph. Thus $\alpha(M_p(G_\tau)) = t-1 \leqslant a$, implying that $\kappa_\alpha(M, p, a) = \infty$.

Now assume that $s = 1$ and $M = (m_1, \ldots, m_{2t}) \in \mathbb{Z}^{1\times 2t}$ is translation invariant. Let $m_o = \sum_{i=1}^t m_{2i-1}$ and $m_e = \sum_{i=1}^t m_{2i}$. Then $m_e + m_o = 0$ since M is translation invariant and for each vector $(u_1, v_1, \ldots, u_t, v_t)^\top \in \mathbb{Z}^{2t}$ we have

$$\begin{aligned}
\sum_{\pi\in\mathcal{S}_t} \pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top &= \sum_{\pi\in\mathcal{S}_t}\sum_{i=1}^t \left(m_{2\pi(i)-1}u_i + m_{2\pi(i)}v_i\right) \\
&= \sum_{i=1}^t \left(u_i \sum_{\pi\in\mathcal{S}_t} m_{2\pi(i)-1} + v_i \sum_{\pi\in\mathcal{S}_t} m_{2\pi(i)}\right) \\
&= \sum_{i=1}^t \left(u_i(t-1)!\sum_{j=1}^t m_{2j-1} + v_i(t-1)!\sum_{j=1}^t m_{2j}\right) \\
&= (t-1)!\, m_o \sum_{i=1}^t u_i + (t-1)!\, m_e \sum_{i=1}^t v_i. \qquad (2.3)
\end{aligned}$$

Consider some arbitrary integer k, and let G be an arbitrary fixed integer graph with $\chi(G) \geqslant k$ and $|E(G)| \geqslant t$. Note that there is an edge between any two color classes in an optimal proper coloring of G and hence

$$|E(G)| \geqslant \tbinom{k}{2}. \qquad (2.4)$$

We shall prove that in each of the cases below the conflict hypergraph $M_p(G)$ is either empty or complete t-uniform.

(a) We assume that $m_e = \sum_{i=1}^{t} m_{2i} \geqslant \max\{p, 0\}$. Consider a set of t distinct edges $E = \{(u_1, v_1), \ldots, (u_t, v_t)\}$ in G. Then $\sum_{i=1}^{t} v_i \geqslant \sum_{i=1}^{t}(u_i + 1) = t + \sum_{i=1}^{t} u_i$ and

$$\begin{aligned}
\sum_{\pi \in \mathcal{S}_t} \pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top &\overset{(2.3)}{=} (t-1)!\, m_o \sum_{i=1}^{t} u_i + (t-1)!\, m_e \sum_{i=1}^{t} v_i \\
&\geqslant (t-1)!\, m_o \sum_{i=1}^{t} u_i + (t-1)!\, m_e \sum_{i=1}^{t} u_i + t!\, m_e \\
&= (t-1)!\, \underbrace{(m_e + m_o)}_{=0} \sum_{i=1}^{t} u_i + t!\, m_e \\
&\geqslant t!\, p.
\end{aligned}$$

Hence we have $\pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top \geqslant p$ for some $\pi \in \mathcal{S}_t$. Therefore E is conflicting and $M_p(G)$ is a complete t-uniform hypergraph on $|E(G)| \geqslant t-1$ vertices with $\alpha(M_p(G)) = t - 1 \leqslant a$. This implies that $\kappa_\alpha(M, p, a) = \infty$ since G was arbitrary.

Secondly, as $\chi(G) \geqslant k$, we have $\omega(M_p(G)) = |E(G)| \overset{(2.4)}{\geqslant} \binom{k}{2}$. Thus if $\omega(M_p(G)) \leqslant w$, then $k \leqslant b(w)$. This shows that $\kappa_\omega(M, p, w) \leqslant b(w)$. Thus $\kappa_\omega(M, p, w) = b(w)$, by Inequality (2.2).

(b) We assume that $m_e = \sum_{i=1}^{t} m_{2i} < p$ and for each $i \in [t]$ we have $m_{2i-1} + m_{2i} = 0$, $m_{2i} \leqslant 0$. Due to the last assumption we have for any edge (u, v) in G that $m_{2i}v \leqslant m_{2i}(u+1)$ for each $i \in [t]$. Hence for any set $E = \{(u_1, v_1), \ldots, (u_t, v_t)\}$ of t distinct edges in G we have

$$\begin{aligned}
M(u_1, v_1, \ldots, u_t, v_t)^\top &= \sum_{i=1}^{t} m_{2i-1} u_i + \sum_{i=1}^{t} m_{2i} v_i \\
&\leqslant \sum_{i=1}^{t} m_{2i-1} u_i + \sum_{i=1}^{t} m_{2i}(u_i + 1) \\
&= \sum_{i=1}^{t} \underbrace{(m_{2i-1} + m_{2i})}_{=0} u_i + \sum_{i=1}^{t} m_{2i} \\
&< p.
\end{aligned}$$

This shows that $\pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top < p$ for each $\pi \in \mathcal{S}_t$, since the labeling of E is arbitrary. Hence E is not conflicting and $M_p(G)$ is an empty hypergraph on $|E(G)|$ vertices. So $\alpha(M_p(G)) = |E(G)| \overset{(2.4)}{\geqslant} \binom{k}{2}$ and $\omega(M_p(G)) = t - 1 \leqslant w$. Similarly to the first item $\kappa_\alpha(M, p, a) = b(a)$ and $\kappa_\omega(M, p, w) = \infty$ since G was arbitrary.

(c) We assume that $m_e = \sum_{i=1}^{t} m_{2i} > 0$ or ($m_e = \sum_{i=1}^{t} m_{2i} = 0$ and $m_{2i-1} \neq 0$ for some $i \in [t]$). Fix some integer q with

$$q \geqslant \max\{2, |p| + \textstyle\sum_{i=1}^{2t} |m_i|\} \tag{2.5}$$

and let $V = \{q^a \mid a \in \mathbb{Z}, a \geqslant 2\}$. We claim that if $V(G) \subset V$, then $M_p(G)$ is a complete graph. Indeed, consider a set $E = \{(q^{a_1}, q^{b_1}), \ldots, (q^{a_t}, q^{b_t})\}$ of t distinct edges in G with $a_i < b_i$, $i \in [t]$. If $m_e > 0$, then $m_e \geqslant 1$ and

$$\begin{aligned}
\sum_{\pi \in \mathcal{S}_t} \pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top &\overset{(2.3)}{=} (t-1)!\, m_o \sum_{i=1}^{t} q^{a_i} + (t-1)!\, m_e \sum_{i=1}^{t} q^{b_i} \\
&\geqslant (t-1)!\, m_o \sum_{i=1}^{t} q^{a_i} + (t-1)!\, m_e \sum_{i=1}^{t} q^{a_i+1} \\
&\geqslant (t-1)!\, m_o \sum_{i=1}^{t} q^{a_i} + (t-1)!\, m_e \sum_{i=1}^{t} (q^{a_i} + q) \\
&= (t-1)!\, \underbrace{(m_o + m_e)}_{=0} \sum_{i=1}^{t} q^{a_i} + t!\, q\, m_e \\
&= t!\, q\, m_e \geqslant t!\, q \overset{(2.5)}{\geqslant} t!\, p.
\end{aligned}$$

Thus $\pi(M)(q^{a_1}, q^{b_1}, \ldots, q^{a_t}, q^{b_t})^\top \geqslant p$ for some $\pi \in \mathcal{S}_t$.

Now suppose that $m_e = \sum_{i=1}^{t} m_{2i} = 0$ and there are $i, j \in [t]$ with $m_{2i}, m_{2j-1} \neq 0$. Then $m_o = \sum_{i=1}^{t} m_{2i-1} = 0$ since M is translation invariant. We distinguish two further cases. First suppose that $b_i \neq b_j$ for some distinct $i, j \in [t]$. Without loss of generality assume that $b_1 \leqslant \cdots \leqslant b_t$ and $m_2 \leqslant m_4 \leqslant \cdots \leqslant m_{2t}$ (otherwise relabel the edges or permute the columns of M). Let j denote the smallest integer in $[t]$ such that $b_j = b_t$ (that is, there are $t - j + 1$ edges in E having q^{b_t} as right endpoint). Then $j > 1$ since not all the b_i are equal. Moreover $\sum_{i=j}^{t} m_{2i} \geqslant 1$ (since $m_{2i} \neq 0$ for some $i \in [t]$ and $m_e = 0$) and

$$\begin{aligned}
M(q^{a_1}, q^{b_1}, \ldots, q^{a_t}, q^{b_t})^\top &= \sum_{i=1}^{t} m_{2i-1} q^{a_i} + \sum_{i=1}^{t} m_{2i} q^{b_i} \\
&= \sum_{i=1}^{t} m_{2i-1} q^{a_i} + \sum_{i=1}^{j-1} m_{2i} q^{b_i} + \sum_{i=j}^{t} m_{2i} q^{b_t} \\
&\geqslant -\sum_{i=1}^{t} |m_{2i-1}| q^{b_t - 1} - \sum_{i=1}^{j-1} |m_{2i}| q^{b_t - 1} + q^{b_t} \\
&= q^{b_t - 1} \Big(q - \sum_{i=1}^{t} |m_{2i-1}| - \sum_{i=1}^{j-1} |m_{2i}| \Big) \\
&\overset{(2.5)}{\geqslant} q^{b_t - 1} |p| \geqslant p.
\end{aligned}$$

Finally suppose that $b_i = b_j$ for any i, $j \in [t]$. Then $\sum_{i=1}^t m_{2i}q^{b_i} = 0$ and all the a_i are distinct (as all edges have the same right endpoint). We assume without loss of generality that $a_1 \leqslant \cdots \leqslant a_t$ and $m_1 \leqslant m_3 \leqslant \cdots \leqslant m_{2t-1}$ (otherwise relabel the edges or permute the columns of M). Then $a_i \leqslant a_t - 1$ for each $i \in [t-1]$, $m_{2t-1} \geqslant 1$, and

$$\begin{aligned}
M(q^{a_1}, q^{b_1}, \ldots, q^{a_t}, q^{b_t})^\top &= \sum_{i=1}^t m_{2i-1}q^{a_i} + \underbrace{\sum_{i=1}^t m_{2i}q^{b_i}}_{=0} \\
&= \sum_{i=1}^{t-1} m_{2i-1}q^{a_i} + m_{2t-1}q^{a_t} \\
&\geqslant -\sum_{i=1}^{t-1} |m_{2i-1}|q^{a_i} + m_{2t-1}q^{a_t} \\
&\geqslant q^{a_t} - \sum_{i=1}^{t-1} |m_{2i-1}|q^{a_t-1} \\
&= q^{a_t-1}\Big(q - \sum_{i=1}^{t-1} |m_{2i-1}|\Big) \\
&\overset{(2.5)}{\geqslant} q^{a_t-1}|p| \geqslant p.
\end{aligned}$$

This shows that any set of t edges with endpoints in V is in conflict. Therefore $M_p(G)$ is a complete t-uniform hypergraph if $V(G) \subset V$. This shows that $\kappa_\alpha(M, p, a) = \infty$ since G was arbitrary..

(d) We assume that for each $i \in [t]$ we have $m_{2i} \leqslant 0$ and $|m_{2i}| \geqslant m_{2i-1}$ and for some $i \in [t]$ we have $m_{2i} < 0$. Fix some integer q with $q > \max\{2, |p|+m_1, \ldots, |p|+m_{2t-1}\}$ and let $V = \{q^a \mid a \in \mathbb{Z}, a \geqslant 1\}$. Then for each $i \in [t]$ we have

$$m_{2i-1} + m_{2i}q \leqslant 0. \tag{2.6}$$

Moreover by the choice of q we have for any $i \in [t]$ with $m_{2i} < 0$ that

$$m_{2i-1} + m_{2i}q \leqslant m_{2i-1} - q < -|p|. \tag{2.7}$$

We claim that if $V(G) \subset V$, then $M_p(G)$ is an empty graph. Indeed, consider a set $E = \{(q^{a_1}, q^{b_1}), \ldots, (q^{a_t}, q^{b_t})\}$ of t distinct edges in G with $a_i < b_i$, $i \in [t]$. Then

$$\begin{aligned}
M(q^{a_1}, q^{b_1}, \ldots, q^{a_t}, q^{b_t})^\top &= \sum_{i=1}^t m_{2i-1}q^{a_i} + \sum_{i=1}^t m_{2i}q^{b_i} \\
&\leqslant \sum_{i=1}^t m_{2i-1}q^{a_i} + \sum_{i=1}^t m_{2i}q^{a_i+1}
\end{aligned}$$

$$\overset{(2.6)}{\leqslant} \sum_{i\in[t],m_{2i}<0}^{t} (m_{2i-1} + m_{2i}q)$$

$$\overset{(2.7)}{<} - \sum_{i\in[t],m_{2i}<0}^{t} |p| \leqslant p.$$

This shows that $\pi(M)(u_1, v_1, \ldots, u_t, v_t)^\top < p$ for each $\pi \in \mathcal{S}_t$, since the labeling of E is arbitrary. Thus no set of t edges with vertices in V is in conflict. Therefore $M_p(G)$ is empty if $V(G) \subset V$. Similarly to above we have $\kappa_\omega(M, p, w) = \infty$. □

2.5 Summary for Small Forests

In this section we evaluate our results from Section 2.1 for the function $\kappa_<$ for all ordered forests on at most three edges and without isolated vertices. Recall that P_k denotes a path on k vertices, M_k a matching on k edges, and S_k a star with k leaves (note that $M_1 = S_1 = P_2$ and $P_3 = S_2$). Then the set of all (unordered) forests without isolated vertices and at most three edges is given by

$$\{P_2, S_2, M_2, S_3, P_4, S_2 + P_2, M_3\}.$$

Let G denote a graph on n vertices and a automorphisms. Then the number $\text{ord}(G)$ of non-isomorphic orderings of G equals $\text{ord}(G) = \frac{n!}{a}$. Hence

$$\text{ord}(P_2) = \tfrac{2!}{2} = 1,\ \text{ord}(S_2) = \tfrac{3!}{2} = 3,\ \text{ord}(M_2) = \tfrac{4!}{8} = 3,\ \text{ord}(S_3) = \tfrac{4!}{3!} = 4,$$
$$\text{ord}(P_4) = \tfrac{4!}{2} = 12,\ \text{ord}(S_2 + P_2) = \tfrac{5!}{2\cdot 2} = 30,\ \text{ord}(M_3) = \tfrac{6!}{6\cdot 4\cdot 2} = 15.$$

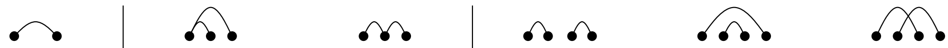
Recall that the *reverse* $\overline{T}$ of an ordered graph T is the ordered graph obtained by reversing the ordering of the vertices in T. Note that $\kappa_<(T) = \kappa_<(\overline{T})$ for any ordered graph T since $G \in \text{Forb}_<(T)$ if and only if $\overline{G} \in \text{Forb}_<(\overline{T})$. Table 2.2 shows all ordered forests T without isolated vertices and at most three edges and their $\kappa_<$ values, where only one of T and $\overline{T}$ is listed. When T and $\overline{T}$ are not isomorphic ordered graphs the entry in the table is marked with an $*$. For example there are only two instead of three entries for S_2 and similarly for the other graphs.

2.6 Conclusions

In this chapter we consider ordered graphs, that is, graphs equipped with a linear ordering of their vertices. We consider several local constraints and study whether they provide upper bounds on the chromatic number.

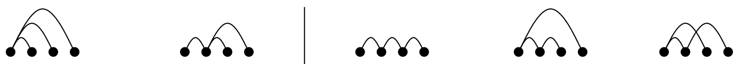
Forbidden Ordered Subgraphs The first part deals with forbidden ordered subgraphs. An ordered graph H (on at least two vertices) is χ-unavoidable if

T



$\kappa_<$: 1 (Thm. 2.4)	2 $*$ (Thm. 2.4)	2 (Thm. 2.4)	3 (Thm. 2.4)	3 (Thm. 2.4)	3 (Thm. 2.4)

T

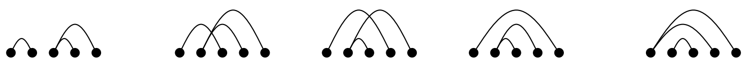


$\kappa_<$: 3 $*$ (Thm. 2.4)	3 $*$ (Thm. 2.4)	3 (Thm. 2.4)	∞ $*$ (bonnet)	∞ $*$ (tangled)

T

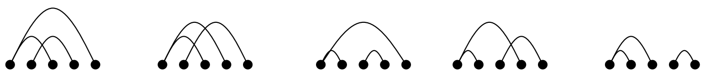
$\kappa_<$: 3 $*$ (Thm. 2.4)	∞ (bonnet)	∞ (tangled)	4 $*$ (Lem. 2.3.4, Fig. 2.11)	$\leqslant 4$ (Red. 2.4)

T



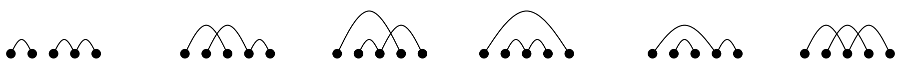
$\kappa_<$: 4 $*$ (Thm. 2.4)	$\leqslant 6$ $*$ (Red. 2.4)	? $*$	$\leqslant 6$ $*$ (Red. 2.3)	$\leqslant 6$ $*$ (Lem. 2.3.4)

T



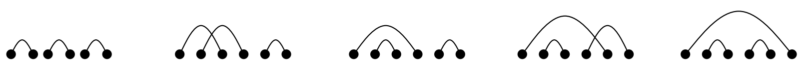
$\kappa_<$: ? $*$	$\neq \infty$ $*$ (Thm. 2.4)	∞ $*$ (bonnet)	? $*$	4 $*$ (Thm. 2.4)

T



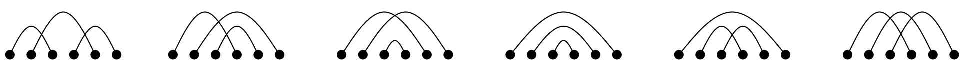
$\kappa_<$: 4 $*$ (Thm. 2.4)	4 $*$ (Thm. 2.4)	? $*$	$\leqslant 6$ (Red. 2.3)	4 $*$ (Thm. 2.4)	?

T



$\kappa_<$: 5 (Thm. 2.4)	5 $*$ (Thm. 2.4)	5 $*$ (Thm. 2.4)	$\leqslant 9$ $*$ (Red. 2.5)	$\leqslant 7$ (Red. 2.3)

T



$\kappa_<$: ?	$\neq \infty$ $*$ (Thm. 2.4)	$\leqslant 9$ (Red. 2.5)	$\leqslant 8$ (Thm. 2.4)	$\leqslant 7$ (Red. 2.3)	$\leqslant 8$ (Thm. 2.4)

Table 2.2: All ordered forests T on at most three edges without isolated vertices and their $\kappa_<$ value.

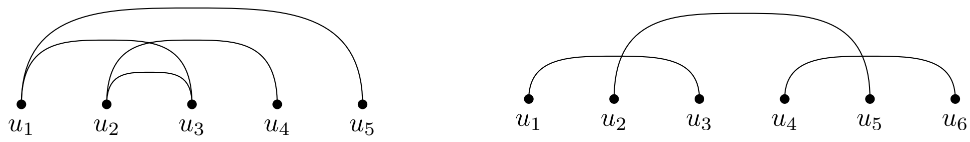


Figure 2.12: Ordered graphs for which we don't know whether they are χ-avoidable.

$\kappa_<(H) = \sup\{\chi(G) \mid G \in \text{Forb}_<(H)\}$ is finite, and χ-avoidable otherwise. We prove that there are χ-avoidable ordered forests, in contrast to unordered and directed graphs. To this end we explicitly describe several infinite classes of minimal χ-avoidable ordered forests, called bonnets and tangled paths. A full characterization of χ-avoidable ordered graphs remains open.

Question 2.1. *Which ordered forests are χ-avoidable?*

We completely answer Question 2.1 for non-crossing ordered graphs H. Theorem 2.3 states that a non-crossing ordered graph H is χ-avoidable if and only if it contains a cycle or a bonnet. For crossing connected ordered graphs, we reduce Question 2.1 to monotonically alternating trees.

Question 2.2. *Are there χ-avoidable monotonically alternating trees?*

We do not have an answer to Question 2.2 even for some monotonically alternating paths. A smallest unknown such path is $u_5u_1u_3u_2u_4$, where $u_1 < \cdots < u_5$. See Figure 2.12 (left). The situation becomes even more unclear for crossing disconnected graphs. From Theorem 2.4 we see that each ordered matching with interval chromatic number 2 is χ-unavoidable. For many other ordered matchings H we do not know whether they are χ-unavoidable. A smallest such matching has edges u_1u_3, u_2u_5, and u_4u_6 where $u_1 < \cdots < u_6$. See Figure 2.12 (right).

Given a χ-unavoidable ordered graph also the actual value of the function $\kappa_<$ is of interest. Recall that for any (unordered) forest H' we have $\kappa(H') = |V(H')| - 1$. Suppose that H is a non-crossing χ-unavoidable ordered graph on k vertices. We prove that, if H connected, then $k - 1 \leqslant \kappa_<(H) \leqslant 2k - 3$ and, if H is disconnected, then $k - 1 \leqslant \kappa_<(H) \leqslant 2^k$. Let

$$\kappa_<(k) = \max\{\kappa_<(H) \mid |V(H)| = k, \kappa_<(H) \neq \infty\}.$$

The value and the asymptotic behavior of this function remain open, even when restricted to non-crossing graphs. Our results show that for connected non-crossing ordered graphs $\kappa_<(H)$ is linear in $|V(H)|$ while for general non-crossing ordered graphs $\kappa_<(H)$ might be exponential. Specifically for each $k \geqslant 4$ we give ordered graphs H on k vertices with $\kappa_<(H) = k - 1$, as well as an χ-unavoidable ordered graph H on k vertices with $\kappa_<(H) \geqslant k$, see Theorems 2.3 and 2.4. The latter result provides a lower bound of $\kappa_<(k) \geqslant k$ for $k \geqslant 4$, slightly improving the trivial lower bound of $k - 1$. Again, this is in contrast to (unordered) graphs where $\max\{\kappa(H) \mid$

H forest, $|V(H)| = k\} = k - 1$ as we have seen before. Note that we do not know whether the matchings in the last statement of Theorem 2.4 are χ-unavoidable.

Question 2.3. *Is there a constant c such that $\kappa_<(k) \leqslant c\,k$ for each $k \in \mathbb{Z}$, $k \geqslant 2$?*

Note that Reduction Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5 apply to crossing ordered graph as well. We find a more precise version of Reduction Lemma 2.2 and other types of reductions, similar to reductions for matrices in [140], but none of these lead to significantly better upper bounds in Theorems 2.3 and 2.4 or a new class of χ-unavoidable ordered forests.

All the results above are concerned with one ordered graph that is forbidden as an ordered subgraph. We propose two variants of this problem. First one might study several simultaneously forbidden ordered subgraphs. Clearly, if H is some (unordered) graph and $\mathcal{H}$ is the set of all ordered graphs having H as their underlying graph, then $\mathrm{Forb}_<(\mathcal{H})$ consists of all ordered graphs with underlying graph in Forb(H). The problem becomes more interesting when several but not all orderings of some graph are forbidden. Recall that the reverse of an ordered graph H is the ordered graph obtained by reversing the order of vertices in H.

Question 2.4. *Let B be a bonnet and let $\overline{B}$ be the reverse of B. Is $\kappa_<(\{B, \overline{B}\}) \neq \infty$?*

Second, we are interested in forbidden induced ordered subgraphs. This leads to the concept of χ-bounded classes of ordered graphs. For an ordered graph H let $\mathrm{Forb}_<^{\mathrm{ind}}(H)$ denote the set of ordered graphs not containing an induced copy of H. Clearly $\mathrm{Forb}_<(H) \subseteq \mathrm{Forb}_<^{\mathrm{ind}}(H)$. Hence $\mathrm{Forb}_<^{\mathrm{ind}}(H)$ is not χ-bounded for any χ-avoidable ordered graphs. In particular we are interested in the following question. Note that the conjecture of Gyárfás and Sumner states that the corresponding question for (unordered) graphs has a negative answer.

Question 2.5. *Is there a χ-unavoidable ordered graph H such that $\mathrm{Forb}_<^{\mathrm{ind}}(H)$ is not χ-bounded?*

Question 2.6. *If G is a triangle-free ordered graph that contains no induced copy of a monotone path with three edges, then $\chi(G) \leqslant 3$?*

Besides the structural questions above, there are also interesting algorithmic questions for ordered graphs. First of all, recall from Observation 1.1 in Section 1.2 that the problem of deciding whether two ordered graphs are isomorphic is clearly in P. Also for each fixed χ-unavoidable ordered graph H we can check in time polynomial in n whether some ordered graph G on n vertices contains H as a subgraph. The following two problems seem more challenging. We are interested, first, in the computational complexity of deciding whether an ordered graph H is χ-unavoidable and, second, in computing proper colorings for graphs in $\mathrm{Forb}_<(H)$ with at most $\kappa_<(H)$ colors. Based on our structural observations for (so far known) χ-avoidable

Figure 2.13: A nested matching (left) and an all crossing matching (right) where left endpoints as well as the right endpoints are pairwise at distance at least 3 in $\mathbb{Z}$.

ordered graphs, we think that recognizing χ-unavoidable ordered graphs might be solvable in polynomial time. Of course, an answer to the following question heavily depends on how the class of χ-avoidable ordered graphs looks like.

Question 2.7. *Is there a polynomial time algorithm that decides for any given ordered graph H whether H is χ-avoidable or χ-unavoidable?*

Finding proper colorings for graphs in $\text{Forb}_<(H)$ is likely to be hard in general. For several χ-unavoidable ordered graphs H, though, we prove that the graphs in $\text{Forb}_<(H)$ are $(\kappa_<(H)-1)$-degenerate. Then a proper coloring of any $G \in \text{Forb}_<(H)$ with $\kappa_<(H)$ colors can be found in running time linear in the number of vertices of G [105]. Moreover, the reductions in Section 2.3 can be used to actually compute proper colorings. However, note that Reduction Lemma 2.1 is the only reduction that yields a tight upper bound on $\kappa_<$ for each ordered graphs satisfying its assumptions.

The General Framework In Section 2.2 we present a generalization of a framework from [7] to model local constraints for ordered graphs. This framework is based on conflicting sets of t edges, $t \geqslant 2$, which are given as linear inequalities on the coordinates of the endpoints of the edges. To this end we consider integer graphs, that is ordered graphs whose vertex set is a subset of the integers. We show how to describe conflicts in terms of pairs of a matrix $M \in \mathbb{Z}^{s\times 2t}$ and a vector $p \in \mathbb{Z}^s$ and introduce the (t-uniform) conflict hypergraph $M_p(G)$. Then restrictions on the structure of the conflict hypergraph yield local constrains on the integer graph. We consider two restrictions in detail, namely bounding either the independence number or the clique number of $M_p(G)$ from above. Note that this includes the possibility to exclude any conflicting edges, since $M_p(G)$ is empty if and only if $\omega(M_p(G)) \leqslant t-1$, or any non-conflicting edges, since $M_p(G)$ is complete t-uniform if and only if $\alpha(M_p(G)) \leqslant t-1$.

In this framework many known constraints can be modeled as well as generalized. For example we generalize results of Dujmović and Wood [57] on nested matchings in [7]. We show that for all integers p, $w > 0$ and some integer graph G we have $\chi(G) \leqslant 4pw$ provided that G does not contain a nested matching on $w+1$ edges where for any pair a, b of left endpoints as well as of right endpoints we have $|a-b| \geqslant p$. See Figure 2.13. So studying arbitrary translation invariant matrices

M yields a unified way to study new constraints. Moreover we show in Theorem 2.6 that some extremal questions can be answered for a given pair of a matrix and an integer vector based on the algebraic description only. However it seems that the framework cannot be used to model forbidden induced ordered subgraphs.

We determine the exact values of $\kappa_\alpha(M,p,a)$ and $\kappa_\omega(M,p,w)$ for all $M \in \{-1,0,1\}^{1\times 4}$, a, p, $w \in \mathbb{Z}$ with a, $w \geqslant 1$ in [7] (see Table 2.1) and for infinite classes of matrices $M \in \mathbb{Z}^{1\times 2t}$ in Theorem 2.6. It remains open to determine the values of $\kappa_\alpha(M,p,a)$ and $\kappa_\omega(M,p,w)$ for other matrices M and parameters a, p, w. Specifically it is an interesting challenge to determine the exact values for the matrices M^{nest} and M^{cross}, where Theorem 2.5 provides upper and lower bounds.

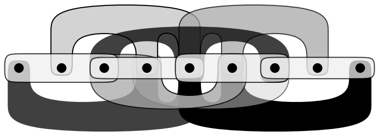
It is also not clear how operations on the matrix (like permutations of columns, multiplication by constants, or addition of two matrices) affect the conflict hypergraph. We give some partial results of this kind in [7].

Finally we see that for fixed M and p in all known cases $\kappa_\alpha(M,p,x)$ and $\kappa_\omega(M,p,x)$ are of order $\theta(x^{1/2})$, $\theta(x)$, or equal ∞, as functions in x (see Table 2.1).

Question 2.8. *Are there s, $t \in \mathbb{Z}$, a matrix $M \in \mathbb{Z}^{s\times 2t}$, and $p \in \mathbb{Z}^s$, such that $\kappa_\alpha(M,p,x)$ or $\kappa_\omega(M,p,x)$ is superlinear as a function of x but not equal to infinity?*

Ordered Graphs and Integer Graphs Recall that the framework uses integer graphs instead of general ordered graphs. As it turns out this is condition is basically for technical reasons only. At first glance, considering arbitrary matrices M leads to some dubious constraints that solely rely on the vertices to be integers and have no particular meaning for general ordered graphs. For example two edges are in conflict with respect to the matrix $(1\,0\,1\,0) \in \mathbb{Z}^{1\times 4}$ and some $p \in \mathbb{Z}$ if and only if the sum of their left endpoints is at least p. This constraint is easily achieved, as well as avoided, for any pair of edges in a given integer graph by shifting the vertices within $\mathbb{Z}$. The first result of Theorem 2.6 shows that our extremal questions become trivial for matrices that are not translation invariant. Therefore we only need to consider translation invariant matrices, that is, matrices which yield conflicts that are preserved by translations of the vertices.

Further note that adding isolated vertices to an integer graph has no affect on the conflict hypergraph. For an integer graph G let $\bar{G}$ denote the integer graph obtained by adding isolated vertices $z \in \mathbb{Z}$ to G whenever $z \notin V(G)$ and there are u, $v \in V(G)$ with $u < z < v$. Two integer graphs G and G' are *indistinguishable* whenever $\bar{G}$ and $\bar{G}'$ are isomorphic as ordered graphs. This establishes an equivalence relation on integer graphs. Let $[G]$ denote the set of integer graphs indistinguishable from G. Clearly $M_p(G')$ and $M_p(G'')$ are isomorphic whenever G', $G'' \in [G]$ and M is translation invariant. Note that each equivalence class $[G]$ contains a unique integer graph with vertex set $[n]$ for some $n \geqslant 1$. This establishes a 1-1-correspondence between ordered graphs and equivalence classes of integer graphs. Intuitively speaking, each integer graph corresponds to the ordered graph obtained by filling each

G  $M = \begin{pmatrix} + & 0 & 0 & - & 0 & 0 \\ 0 & + & 0 & 0 & - & 0 \\ 0 & 0 & + & 0 & 0 & - \end{pmatrix}$ $p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$E(G) = \{(123), (145), (245), (345), (346), (357), (467), (567), (568), (569), (789)\}$$

Figure 2.14: A 3-uniform shift chain that is not 2-colorable (due to Fulek, see [120]).

gap in its vertex set by an isolated vertex. Altogether we see that each translation invariant matrix and any vector yield a local constraint for general ordered graphs.

One might argue that isolated vertices in an ordered graph should not affect local constraints. To overcome this issue one might only consider connected ordered graphs. We think that this does not make much difference for the constraints given by our framework.

Question 2.9. *Are there s, $t \in \mathbb{Z}$, a matrix $M \in \mathbb{Z}^{s\times 2t}$, and $p \in \mathbb{Z}^s$, such that $\kappa_\alpha(M, p, x)$ or $\kappa_\omega(M, p, x)$ equals infinity for some $x \geqslant t - 1$, while the function restricted to connected integer graphs is finite?*

Generalizations Finally let us mention some possible generalizations of this framework. First, other parameters than the independence or clique number of the conflict hypergraph can be considered. For example it might be interesting to study the affect of bounding the degrees or the chromatic number. Second, the framework can be extended to (ordered) hypergraphs in a straightforward way by considering conflicting sets of hyperedges. For example Pálvölgyi [122] and Pach and Pálvölgyi [120] ask whether there is a sufficiently large r such that all r-uniform ordered hypergraphs of the following kind are properly 2-colorable (that is, have property B). A shift-chain G is an r-uniform hypergraph with linearly ordered vertex set such that for any two hyperedges $(u_1, \dots, u_r)$ and $(v_1, \dots, v_r)$ we have either $u_i \leqslant v_i$ for each $i \in [r]$ or $u_i \geqslant v_i$ for each $i \in [r]$. In our (generalized) framework this is equivalent to $\alpha(M_{\mathbf{0}}(G)) = 1$ (that is $M_{\mathbf{0}}(G)$ is a complete graph) for $M \in \mathbb{Z}^{r\times 2r}$ where for each $i \in [r]$ and the i^{th} row $M^{(i)}$ of M we have that $M^{(i)}_i = 1$, $M^{(i)}_{r+i} = -1$, and $M^{(i)}_j = 0$, $j \in [2r] \setminus \{i, r+i\}$. See Figure 2.14 for an example in case $r = 3$. Note that $M_{\mathbf{0}}(G)$ is a graph in this example, since conflicts are defined for pairs of hyperedges. Note further that in case $r = 2$ a graph is a shift chain if and only if it does not contain a nested 2-matching.

3

Ramsey Equivalence

3.1 Introduction

Two graphs G and H are *Ramsey equivalent* if $\mathcal{R}(G) = \mathcal{R}(H)$, that is, they have exactly the same set of Ramsey graphs. We write $G \overset{R}{\sim} H$ if G is Ramsey equivalent to H, and write $G \overset{R}{\nsim} H$ otherwise. Note that this indeed establishes an equivalence relation. Although many properties of Ramsey graphs are already studied for a long time, as we have seen in Section 1.4, the notion of Ramsey equivalence of graphs was raised only recently by Szabó *et al.* [139]. Of course each graph is Ramsey equivalent to itself. However, it is quite easy to obtain Ramsey equivalent pairs of non-isomorphic graphs.

Observation 3.1. *For any graph G we have $G \overset{R}{\sim} G + tK_1$ for any t with $0 \leqslant t \leqslant r(G) - |V(G)|$.*

So for any graph G with $r(G) > |V(G)|$ there is some Ramsey equivalent non-isomorphic graph H. Note that K_2 and $K_{1,2}$ are the only graphs G without isolated vertices and $r(G) = |V(G)|$, see Lemma 3.2.1. It is easy to see that each graph that is a union of one of these two graphs and some isolated vertices is Ramsey equivalent to itself only, see Lemma 3.2.2. We think that they are the only such graphs, so the following question has a positive answer.

Question 3.1. *Let G be a graph with $G \notin \{K_2 + tK_1 \mid t \geqslant 0\} \cup \{K_{1,2} + tK_1 \mid t \geqslant 0\}$. Is there a graph H that is not isomorphic to G such that $G \overset{R}{\sim} H$?*

Before summarizing our and previously known results we state two further simple observations.

Observation 3.2. *Let G and H be graphs with $G \subseteq H$. If $G \overset{R}{\sim} H$, then $G \overset{R}{\sim} H'$ for each graph H' with $G \subseteq H' \subseteq H$.*

Observation 3.3. *For any graph G we have $G \overset{R}{\nsim} G + G$.*

Observation 3.3 holds since for each graph G, each minimal Ramsey graph F of G, and some edge e in F there is a coloring of $F - e$ without no monochromatic copies of H. Then, no matter which color is assigned to e, all monochromatic copies of G in F contain e. So there is no monochromatic copy of $G + G$ and $F \not\rightarrow G + G$. See Figure 3.1 (right).

Figure 3.1: A coloring of K_6 without monochromatic copies of $K_3 + K_2$ (left) and a coloring of a minimal Ramsey graph F of G where all monochromatic copies of G contain a given edge e (right).

Outline In the following paragraphs we summarize known results on Ramsey equivalent pairs of graphs. Most of these are concerned with the question which graphs are Ramsey equivalent to complete graphs. In the last part of this section we present our contribution to this field.

In Section 3.2 we present preliminary lemmas and observations, used in the proofs of the main theorems in Section 3.3 or in Section 3.4 when we consider small so-called "distinguishing" graphs. Finally concluding remarks and open questions are stated in Section 3.5.

3.1.1 Previous Results

Szabó *et al.* [139], Fox *et al.* [67], and Bloom and Liebenau [16] give more results which pairs of graphs of the form $(G, G + H)$ are Ramsey equivalent. For given s and t let $p(t, s)$ denote the largest integer p such that $K_t \overset{R}{\sim} K_t + pK_s$. The first two items of the following theorem are due to Observations 3.1 and 3.3, and the coloring in Figure 3.1 (left), which shows that $K_6 \not\to K_3 + K_2$.

Theorem 3.1. *Let s and t be integers with $t \geqslant 2$, $1 \leqslant s < t$. Then*

(a) $p(t, 1) = r(K_t) - t$ *[67],*

(b) $p(t, t) = p(3, 2) = 0$ *[67],*

(c) $p(t, t-1) = 1$ *[16, 67],*

(d) if $s \leqslant t - 2$, *then* $p(t, s) \geqslant \frac{r(t,t-s+1)-2t}{2s}$ *[139],*

(e) if $s \geqslant 3$, *then* $p(t, s) \leqslant \frac{r(t,t-s+1)-1}{s}$ *[67],*

(f) $p(t, 2) \leqslant \frac{r(t,t)-t}{2}$ *[67].*

As it turns out, for all Ramsey equivalent pairs of non-isomorphic graphs that are known so far at least one the graphs is disconnected. In particular Fox *et al.* prove the following result.

Theorem 3.2 ([67])**.** *Let $t \geqslant 1$ and H be a graph. If $K_t \overset{R}{\sim} H$, then $H = K_t + H'$ for some graph H' with $\omega(H') < t$.*

In particular the only connected graph that is Ramsey equivalent to K_t is K_t itself.

3.1.2 The Main Question

Fox *et al.* [67] formulate the question whether there are is a Ramsey equivalent pair of connected graphs that are not isomorphic, which we already stated as Question 1.2 in Section 1.4. This is the main question for this chapter. Note that $G \overset{R}{\not\sim} H$ if and only if there exists a graph Γ such that $\Gamma \to H$ and $\Gamma \not\to G$ or $\Gamma \to G$ and $\Gamma \not\to H$. In this case we call Γ a graph, *distinguishing* G and H. So in order to prove that $G \overset{R}{\not\sim} H$, it is sufficient to explicitly construct a distinguishing graph. Similarly it is sufficient to show that $r_\rho(G) \neq r_\rho(H)$ for some graph parameter ρ. In particular two graphs of different Ramsey number are not Ramsey equivalent. Szabó *et al.* [139] and Fox *et al.* [67] follow this approach with ρ being the minimum degree. They prove $r_\delta(H_{t,1}) = t-1$, [67, 139], where $H_{t,d}$ is the graph with one vertex of degree d and t other vertices forming a copy of K_t, $d \leqslant t$. We have $r_\delta(K_t) = (t-1)^2$ [31, 69] and hence $K_t \overset{R}{\not\sim} H_{t,1}$ for $t \geqslant 3$. Thus $K_t \overset{R}{\not\sim} H$ for any graph H containing $H_{t,1}$ by Observation 3.2.

Another approach is to identify a graph parameter ρ, such that $\rho(G) \neq \rho(H)$ already implies that $G \overset{R}{\not\sim} H$. In this case, we say that ρ is a *Ramsey distinguishing parameter*. The only structural graph parameters that we know to be Ramsey distinguishing are the clique number and the odd girth. Results of Nešetřil and Rödl [115, 116] show that if $\omega(H) = \omega$ and $\text{girth}_\text{o}(H) = g \neq \infty$ then there are Ramsey graphs G, $G' \in R(H)$ such that $\omega(G) = \omega$ and $\text{girth}_\text{o}(G') = g$. Note that if Question 1.2 has a negative answer, then any graph parameter is a Ramsey distinguishing parameter for the class of connected graphs. Due to the result on the clique number, $K_t \overset{R}{\not\sim} H$ if H does not contain a copy of K_t. This result together with the result $K_t \overset{R}{\not\sim} H_{t,1}$ from above proves Theorem 3.2.

3.1.3 The Main Results

We provide a supporting evidence for a negative answer to Question 1.2 by the following theorems, focusing on another graph parameter, the chromatic number.

Observation 3.4. *If G and H are graphs, $\chi(G) = 2$, and $\chi(H) > 2$ then $G \overset{R}{\not\sim} H$.*

Indeed, a sufficiently large complete bipartite graph is a Ramsey graph for any fixed bipartite graph [14]. But it contains only bipartite graphs and thus is a Ramsey graph for such graphs only. Here, we prove that for several large classes of connected graphs, the chromatic number is a Ramsey distinguishing parameter. Recall that a graph is called *clique-splittable* if its vertex set can be partitioned into two subsets, each inducing a subgraph of smaller clique number. Note that any graph G with $\chi(G) \leqslant 2\omega(G) - 2$ is clique-splittable. In particular all cliques and all planar graphs containing a triangle are clique-splittable. The triangle-free clique-splittable graphs are precisely the bipartite graphs.

Theorem 3.3. *If G and H are graphs, G is clique-splittable, and $\chi(G) < \chi(H)$, then $G \overset{R}{\not\sim} H$.*

Corollary 3.4. *If G and H are graphs, $\chi(G) \leqslant 2\omega(G) - 2$, and $\chi(G) \neq \chi(H)$, then $G \overset{R}{\not\sim} H$.*

Theorem 3.3 distinguishes pairs of graphs of distinct chromatic number under some splittability condition. The following theorem requires stronger assumptions but also applies to graphs of the same chromatic number.

Theorem 3.5. *Let a connected graph G satisfy the following two properties:*

1) *There is an independent set $S \subset V(G)$ such that $\omega(G - S) < \omega(G)$.*

2) *There is a proper $\chi(G)$-vertex-coloring of G in which some two color classes induce a subgraph of a matching.*

Let H be a connected graph which is not isomorphic to G, such that either $H \subseteq G$ or $\chi(H) \geqslant \chi(G)$. Then $G \overset{R}{\not\sim} H$.

In Theorems 3.3 and 3.5 we distinguish pairs of graphs under certain properties. Call a graph G *Ramsey isolated* if $G \overset{R}{\not\sim} H$ for any connected graph H not isomorphic to G. Note that Question 1.2 asks whether every connected graph is Ramsey isolated or not. We apply the previous results to identify large families of Ramsey isolated graphs.

Theorem 3.6.

(a) If G is connected, $\chi(G) = \omega(G)$, and there is a proper $\chi(G)$-vertex-coloring of G in which some two color classes induce a subgraph of a matching in G, then G is Ramsey isolated.

(b) Each path and each star is Ramsey isolated.

(c) Each connected graph on at most five vertices is Ramsey isolated.

If F distinguishes G and H then F has at least $\min\{r(G), r(H)\}$ vertices. The distinguishing graphs used in the proof of Theorem 3.6 are rather large, except for stars. However in Section 3.4 we prove that for all but at most 16 pairs G, H of non-isomorphic connected graphs on at most five vertices there is a distinguishing graph on $\min\{r(G), r(H)\}$ vertices.

A tree T on k vertices is called *balanced* if deleting some edge splits T into components of order at most $\lceil \frac{k+1}{2} \rceil$ each. The Erdős-Sós-Conjecture states that $\text{ex}(n, T) \leqslant \frac{k-2}{2} n$ for any tree T on k vertices. We remark that recently, Ajtai, Komlós, Simonovits, and Szemerédi announce a proof of the conjecture for large k [3, 5, 4]. We state here a much weaker conjecture.

Conjecture 3.1. *There is a positive* ϵ *and an integer* n_ϵ *such that* $\mathrm{ex}(n,T) \leqslant \frac{k-1-\epsilon}{2}n$ *for any tree on* k *vertices and* $n > n_\epsilon$.

Theorem 3.7. *Let* T_k *and* T_ℓ *denote trees on* k *respectively* ℓ *vertices with* $k < \ell$. *Then* $T_k \overset{R}{\not\sim} T_\ell$ *if one of the following conditions holds.*

(a) Conjecture 3.1 is true.

(b) The tree T_k *is balanced.*

The next theorem makes use of multicolor Ramsey numbers. We write $G \overset{R}{\not\sim}_k H$ if G and H are not Ramsey equivalent in k colors, that is, there is a graph Γ such that for any k-coloring of its edges there is a monochromatic copy of G, and there is such a coloring avoiding monochromatic copies of H, or vice versa.

Theorem 3.8. *If* G *and* H *are graphs then* $G \overset{R}{\not\sim} H$ *if one of the following conditions holds.*

(a) There is a graph F *such that* $r(G,G,F) < r(H,H,F)$.

(b) $G \subseteq H$ *and there is* $k \geqslant 2$ *with* $G \overset{R}{\not\sim}_k H$.

Finally we start the investigation of cycles, denoted C_n. As a first step we establish the following result on Ramsey numbers of cycles, see also Lemma 3.2.9. Note that for large n one can add many chords to a cycle C_n without changing the Ramsey number [137].

Theorem 3.9.

(a) Let $n \geqslant 5$ *and let* G *be a graph obtained from* C_n *by adding a pendant edge. Then* $r(G) > r(C_n)$.

(b) For each $n \geqslant 2$ *we have* $r(C_{2n}) < r(C_{2n}+K_2)$ *and* $r(C_{2n-1}) = r(C_{2n-1}+kK_2)$ *if and only if* $0 \leqslant k \leqslant \frac{n}{3}$.

3.2 Preliminary Observations and Results

Graphs which are Ramsey Equivalent to Itself Only

Lemma 3.2.1. *If* G *has no isolated vertex, then* $r(G) = |V(G)|$ *if and only if* $G \in \{K_2, K_{1,2}\}$.

Proof. Consider a graph G on n vertices without isolated vertices. In particular $n \geqslant 2$. If G is disconnected or not isomorphic to $K_{1,n-1}$, then color a copy of $K_{1,n-1}$ in K_n red and all other edges blue. Then there is no monochromatic G in K_n and $r(G) > n$. So suppose G is isomorphic to $K_{1,n-1}$. Due to [34], $r(K_{1,n-1}) = 2n-2$ if n is even, and $r(K_{1,n-1}) = 2n-3$ if n is odd. Hence $r(G) = r(K_{1,n-1}) = n$ if and only if $n \in \{2,3\}$, that is, $G \in \{K_{1,1}, K_{1,2}\} = \{K_2, K_{1,2}\}$. □

Lemma 3.2.2. *If $G \in \{K_2 + tK_1 \mid t \geqslant 0\} \cup \{K_{1,2} + tK_1 \mid t \geqslant 0\}$ and $G \overset{R}{\sim} H$, then H is isomorphic to G.*

Proof. First assume that $G = K_2 + tK_1$ for some $t \geqslant 0$. Then each Ramsey graph of G contains an edge and t isolated vertices. Hence the only minimal Ramsey graph of G is G itself. It is easy to see that G is a minimal Ramsey graph for G only (either G is not Ramsey or G is not minimal). Hence G is not Ramsey equivalent to any other graph.

Next assume that $G = K_{1,2} + tK_1$ for some $t \geqslant 0$. Then $K_3 + tK_1$ has the same number of vertices as G and it is easy to see that $K_3 + tK_1 \to G$. Let G' denote a graph not isomorphic to G. If $G' \subseteq K_2 + t'K_1$ for some $t' \geqslant 0$, then $G \overset{R}{\nsim} G'$ by the first part of the lemma. If $G' = K_{1,2} + t'K_1$ for some $t' \neq t$, then $r(G) \neq r(G')$ and hence $G \overset{R}{\nsim} G'$. In any other case G' contains either two independent edges, a copy of $K_{1,3}$, or a copy of K_3. In each case we see that $K_3 + tK_1 \not\to G'$ and $G' \overset{R}{\nsim} G$. Altogether G is not Ramsey equivalent to any other graph. □

Lemmas for the Main Proofs

The following lemma is an easy generalization of the Focusing Lemma in [67].

Lemma 3.2.3 (Focusing Lemma, [67])**.** *Let $(A \cup B, E)$ be a bipartite graph with a 2-edge-coloring. Then there is a subset $B' \subseteq B$, $|B'| \geqslant |B|/2^{|A|}$, such that for each $a \in A$ all edges from a to B' are of the same color.*

Lemma 3.2.4 ([116])**.** *For any graph G there is a graph $F \in R(G)$ with $\omega(G) = \omega(F)$.*

Lemma 3.2.5 ([64])**.** *For any integers r, $g \geqslant 2$ and any $\epsilon > 0$ there is an integer n and an r-uniform hypergraph on n vertices with girth at least g and independence number less than ϵn.*

From this lemma one easily derives the following well-known result.

Lemma 3.2.6 ([64])**.** *For any integers r, g, $k \geqslant 2$ there is an r-uniform hypergraph with girth at least g and chromatic number at least k.*

Proof. Let $\epsilon < \frac{1}{k}$ and let $\mathcal{H}$ denote an r-uniform hypergraph with girth at least g and independence number at most $\epsilon|V(\mathcal{H})|$, which exists by Lemma 3.2.5. In any k-coloring of $V(\mathcal{H})$ there is a set of at least $\frac{1}{k}|V(\mathcal{H})| > \epsilon|V(\mathcal{H})|$ vertices of the same color. Hence this color class induces an edge of $\mathcal{H}$. Thus the coloring is not proper and $\chi(\mathcal{H}) > k$. □

Recall that for graphs F, G and for $\epsilon > 0$ we write $F \overset{\epsilon}{\to} G$ if for any set $S \subseteq V(F)$ with $|S| \geqslant \epsilon|V(F)|$, we have $F[S] \to G$.

Lemma 3.2.7 ([67])**.** *For any $\epsilon > 0$ and any graph H, there is a graph F with $\omega(F) = \omega(H)$ and $F \overset{\epsilon}{\to} H$.*

Proof. Let F' be a graph such that $F' \to H$ and $\omega(F') = \omega(H)$. Such a graph exists by Lemma 3.2.4. Further let $\mathcal{H}$ denote a $|V(F')|$-uniform hypergraph of girth at least 4 and no independent set of size $\epsilon|V(\mathcal{H})|$, which exists by Lemma 3.2.5. Construct a graph F by placing a copy of F' on the vertices of each hyperedge of $\mathcal{H}$. Then F is a graph on $|V(\mathcal{H})|$ vertices with $\omega(F) = \omega(F') = \omega(H)$. Each vertex set of size at least $\epsilon|V(\mathcal{H})|$ induces a hyperedge in $\mathcal{H}$ and thus a copy of F' in F which arrows H. □

We have the following corollary, since any graph which arrows H contains H.

Lemma 3.2.8. *For any $\epsilon > 0$ and any graph H, there is a graph F with $\omega(F) = \omega(H)$ and each set of $\epsilon|V(F)|$ vertices in F containing a copy of H.*

Lemma 3.2.9. *Let G and H be graphs and let n be a positive integer. If $\mathrm{ex}(n, G) < \mathrm{ex}(n, H)/2$ or if H is connected and $\mathrm{ex}(n, G) < \sqrt{n}\,\mathrm{ex}(\sqrt{n}, H)$, then $G \overset{R}{\not\sim} H$. In particular, if G is a forest and H contains a cycle, then $G \overset{R}{\not\sim} H$.*

Proof. Assume first that $\mathrm{ex}(n, G) < \mathrm{ex}(n, H)/2$. Let F be a graph on n vertices with $\mathrm{ex}(n, H) \geqslant 2\,\mathrm{ex}(n, G) + 1$ edges without a copy of H. In any 2-coloring of the edges of F one of the color classes contains at least $\mathrm{ex}(n, G) + 1$ edges, and thus a copy of G. Hence $F \to G$, but $F \not\to H$.

Assume now that H is connected and $\mathrm{ex}(n^2, G) < n\,\mathrm{ex}(n, H)$. Let F be a graph on n vertices and $\mathrm{ex}(n, H)$ edges not containing H. Let $F^* = F \times F$ be the Cartesian product of F with itself, that is, $V(F^*) = V(F) \times V(F)$ and $\{(u, v), (x, y)\} \in E(F^*)$ if and only if $u = x$ and $vy \in E(F)$ or $v = y$ and $ux \in E(F)$. Then F^* has n^2 vertices and $2n\,\mathrm{ex}(n, H)$ edges. In any 2-edge-coloring of F^* there is a color class with at least $n\,\mathrm{ex}(n, H)$ edges. This color class contains a copy of G, thus $F^* \to G$. On the other hand, we can color the edges of F^* without creating monochromatic copies of H by coloring an edge $\{(u, v), (x, y)\}$ red if $u = x$ and blue otherwise. Note that each color class is a vertex disjoint union of n copies of F and thus does not contain H, as H is connected. Thus $F^* \not\to H$.

For the second part of the statement let G be any forest and H be any graph with a cycle C. We have $\mathrm{ex}(n, G) \leqslant |V(G)|n$ and due to [99] we have $\mathrm{ex}(n, H) \geqslant \mathrm{ex}(n, C) \in \Omega(n^{1+\frac{1}{|V(C)|-1}})$. Therefore, for sufficiently large n we have $\mathrm{ex}(n, G) < \mathrm{ex}(n, C)/2$ and thus $G \overset{R}{\not\sim} H$ by the first part of the Lemma. □

Next we state a few technical lemmas used in proving Theorem 3.6(c). Let Z_1 and Z_4 denote the graphs obtained from C_4 by adding two, respectively three, pendant edges at some vertex, let Z_5 denote the graph obtained from $K_{2,3}$ by adding a pendant edge at a vertex of degree 3, see Figure 3.2.

Lemma 3.2.10. $r_\delta(Z_4) = 1$.

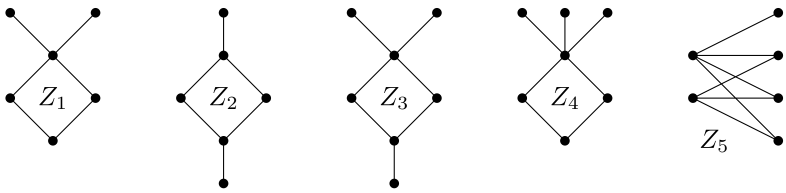


Figure 3.2: The graphs Z_1, Z_2, Z_3, Z_4 and Z_5.

Proof. Let Γ denote the graph obtained from K_9 by adding 9 new vertices and a matching between these and the vertices in K_9. We have $K_9 \not\to Z_4$, due to any 4-factorization, and we shall show $\Gamma \to Z_4$. Then each minimal Ramsey graph of Z_4 contained in Γ contains at least one of the vertices of degree 1 and thus $r_\delta(Z_4) = 1$.

Consider the copy K of K_9 in Γ and let c denote a 2-edge-coloring of K with no monochromatic copy of Z_4. We shall show that there is a red and a blue copy of Z_1 in K with the same vertex x of degree 4, see Claim 2. Then there is a monochromatic copy of Z_4 in Γ no matter which color is assigned to the edge pendant at x. Thus $\Gamma \to Z_4$.

Claim 1. *There is no vertex in K with 5 incident edges of the same color under c.*

Proof of Claim 1. For the sake of contradiction assume u in K has 5 incident red edges. Let N denote the 5 neighbors of u incident to these edges and x, y, z denote the vertices in K not incident to these edges. Then there is at most one red edge between N and each vertex in $\{x, y, z\}$, as otherwise there is a red copy of Z_4 in K. So there are two distinct vertices v, w in N such that there are only blue edges between $\{v, w\}$ and $\{x, y, z\}$. Since each vertex in $\{x, y, z\}$ is incident to 4 blue edges to N, each of the vertices in $\{x, y, z\}$ is the degree 4 vertex in a blue copy of Z_1 with another vertex from $\{x, y, z\}$ and four vertices from N. Thus there are only red edges between u and $\{x, y, z\}$ and only red edges within $\{x, y, z\}$, as otherwise there is a blue copy of Z_4. But then $\{u, x, y, z\}$ forms a red copy of C_4 with 3 red edges pendant at u (those to N), a monochromatic copy of Z_4, a contradiction. This proves Claim 1. △

Claim 2. *There is a vertex in K which is the vertex of degree 4 in a red and a blue copy of Z_1 under c.*

Proof of Claim 2. By Claim 1 the red and the blue subgraph of K under c are 4-regular. Consider a vertex u in K and let N_r and N_b denote the sets of neighbors in K adjacent to u via red respectively blue edges. If there are vertices $v \in N_b$ and $w \in N_r$ with two red edges between v and N_r and two blue edges between w and N_b, then u is the degree 4 vertex in a red and in a blue copy of Z_1 and we are done. So we assume that there is at most one blue edge between N_b and each vertex in N_r. Since $|N_r| = 4$ and the blue subgraph is 4-regular, each vertex in N_r sends at most 3 blue edges to the other vertices in N_r and at least one blue edge to N_b. Hence there is exactly one blue edge between N_b and each vertex in N_r and N_r forms a blue copy of K_4. If the blue edges between N_r and N_b form a matching, then N_b induces a blue copy of C_4 and two independant red edges. Then each vertex in N_b is the vertex of degree 4 in a blue and in a red copy of Z_1 and we are done. If the blue edges between N_r and N_b do not form a matching then there is vertex $v \in N_b$ with exactly two blue and two red edges between v and N_r (three or four blue edges is not possible). Hence v is contained in a blue copy of C_4 with three vertices from N_r and there are two blue edges pendant at v, one to u and one within N_b. Moreover v is contained in a red copy of C_4 with u and two vertices from N_r and there are two red edges pendant at v within N_b. This proves Claim 2. △

Altogether this proves the lemma. □

Lemma 3.2.11. *In any 2-edge-coloring of $K_{3,13} - e$ without a monochromatic copy of Z_5 the vertex of degree 2 is incident to exactly one red and one blue edge.*

Proof. Let x be the vertex of degree 2 in $K_{3,13} - e$, let B denote the vertices of degree 3 and $A = V(K_{3,13} - e) \setminus (B \cup \{x\})$. Then $|A| = 3$, $|B| = 12$ and there is a complete bipartite graph between A and B. Consider a 2-edge-coloring of $K_{3,13} - e$ without a monochromatic copy of Z_5. We consider the edges between A and B first. Since $|A| = 3$, each vertex in B is incident to at least 2 red or 2 blue edges by pigeonhole principle. Let $B_r \subseteq B$ denote the set of vertices in B incident to at least 2 red edges and $B_b = B \setminus B_r$ those incident to 2 blue edges. Without loss of generality assume $|B_r| \geqslant |B_b|$, thus $|B_r| \geqslant 6$. For $v \in B_r$ let A_v denote the vertices in A adjacent to v via red edges. Then $|A_v| \geqslant 2$. Assume there is a set $A' \subseteq A$ of size 2 and distinct vertices $v_1, v_2, v_3 \in B_r$ with $A' = A_{v_i}$, $1 \leqslant i \leqslant 3$. Then all edges between A' and $B \setminus \{v_1, v_2, v_3\}$ are blue, as otherwise there is a red copy of Z_5. But these edges form a blue copy of $K_{2,9}$ which contains a copy of Z_5, a contradiction. Hence for each set $A' \subseteq A$ of size 2 there are most two vertices v in B_r with $A' = A_v$. Since $|A| = 3$ and $|B_r| \geqslant 6$, there are exactly 2 vertices $v \in B_r$ with $A' = A_v$ for each such A' and thus $|B_r| = 6$. Hence $|B_b| = |B \setminus B_r| = 6$ and the same arguments applied to B_b and the blue edges show that for each $A' \in \binom{A}{2}$ there are exactly 2 vertices in B_b adjacent to A' with only blue edges too. Now consider the edges incident to x. If there are 2 red edges, then together with the neighbors of x and some 3 vertices from B_r there is a red copy of Z_5, a contradiction. The same argument holds for the blue edges and hence there is exactly one red and one blue edge incident to x. □

Lemma 3.2.12. $r_\delta(Z_5) = 1$.

Proof. Consider the graph Γ obtained from a complete bipartite graph on partite sets $X = \{x_1, x_2, x_3\}$ and $W = \{\ell_1, \ell_2, \ell_3, r_1, r_2, w\}$ by adding 3 new vertices and a matching between these and the vertices in X. See Figure 3.3 for an illustration. We construct a graph F as follows. Let C denote the complete bipartite graph on

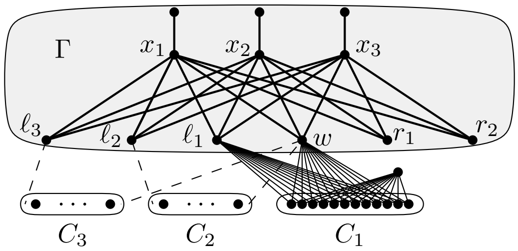


Figure 3.3: A graph F which is Ramsey for Z_5.

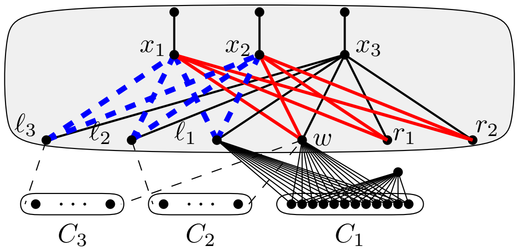


Figure 3.4: A red and a blue copy of $K_{2,3}$ in a 2-edge-coloring of F with the same degree 3 vertices.

parts A and B with $|A| = 3$ and $|B| = 12$. For each i, $1 \leqslant i \leqslant 3$, take a copy C_i of C and identify $\{w, \ell_i\}$ with two vertices in the smaller part of C_i. An illustration is given in Figure 3.3. From now on ℓ_i and w refer to the identified vertices. We prove $F \to Z_5$ next. Consider a 2-edge-coloring of F. Then each $x_j \in X$ together with each C_i, $1 \leqslant i \leqslant 3$, induces a copy of $K_{3,13} - e$. Hence there is a monochromatic copy of Z_5 by Lemma 3.2.11, if there are ℓ_i and x_j such that both edges between x_j and $\{w, \ell_i\}$ are of the same color, $1 \leqslant i, j \leqslant 3$. So assume that the edges between x_j and $\{w, \ell_i\}$ are of different colors, $1 \leqslant i, j \leqslant 3$. By pigeonhole principle we may assume that there are 2 vertices in X, say x_1, x_2, such that the edge between x_j and w is red, $j = 1, 2$. Then the edges $x_j\ell_1, x_j\ell_2, x_j\ell_3$ are blue for $j = 1, 2$. Thus there is a blue copy of $K_{2,3}$ between $\{x_1, x_2\}$ and $\{\ell_1, \ell_2, \ell_3\}$. Hence the edges x_jr_1, x_jr_2 are red, or otherwise there is a blue copy of Z_5. Thus there is a red copy of $K_{2,3}$ between $\{x_1, x_2\}$ and $\{w, r_1, r_2\}$, see Figure 3.4. Altogether there is a monochromatic copy of Z_5 no matter which color is assigned to the edge pendant at x_1. Thus $F \to Z_5$. Let F' denote the graph obtained from F by removing the vertices of degree 1. It remains to show that $F' \not\to Z_5$. Then every minimal Ramsey graph of Z_5 in F contains at least one of the degree 1 vertices and hence $r_\delta(Z_5) = 1$. Consider the following coloring of F'. Color all edges between $\{x_1, x_2\}$ and $\{w, r_1, r_2\}$ and all edges between x_3 and $\{\ell_1, \ell_2, \ell_3\}$ red. Color all other edges between X and W blue. Finally color the edges of each C_i, $1 \leqslant i \leqslant 3$, without a monochromatic copy of $K_{2,3}$. Such a coloring exists since $K_{3,12} \not\to K_{2,3}$ [69]. Next we show that there is no monochromatic copy of Z_5 under this coloring. Each copy of $K_{2,3}$ in F' which does not contain any vertex of X is contained in (exactly) one of the C_i and hence is not monochromatic. Moreover each copy of $K_{2,3}$ in F' which contains a vertex from X

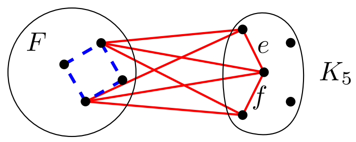


Figure 3.5: A red (solid lines) copy of W_4 in some 2-edge-coloring of Γ.

and a vertex from $V(C_i) \setminus \{w, \ell_i\}$ for some i, contains w and ℓ_i and hence a red and a blue edge. Thus the only monochromatic copies of $K_{2,3}$ in F' are contained in Γ. The part on 2 vertices of all monochromatic copies of $K_{2,3}$ in Γ is contained in X. Hence there is no monochromatic copy of Z_5 because each $x \in X$ is incident to exactly 3 red and 3 blue edges. $\square$

The wheel W_4 is the graph on 5 vertices obtained from a cycle C_4 of length 4 by adding a new vertex adjacent to all vertices of the cycle.

Lemma 3.2.13. *If H is connected and has at least 6 vertices, then $H \overset{R}{\not\to} W_4$.*

Proof. We assume $\omega(H) = 3 = \omega(W_4)$ due to Lemma 3.2.4. Then $\chi(H) \geqslant 3$. Let $\epsilon = 2^{-5}$ and let F be a graph with $F \overset{\epsilon}{\to} C_4$ and $\omega(F) = 2$, which exists by Lemma 3.2.7. We construct a graph Γ by taking the vertex disjoint union of F and a copy K of K_5 and placing a complete bipartite graph between F and K. We shall show that $\Gamma \to W_4$, but $\Gamma \not\to H$.

Color all edges within F and within K red and all other edges blue. Since $\omega(F) = 2 < \omega(H)$, $H \not\subseteq F$. Since $|V(H)| \geqslant 6$, $H \not\subseteq K$. Since H is connected there is no red copy of H. The blue subgraph is a complete bipartite graph and $\chi(H) \geqslant 3$. Thus there is no blue copy of H.

It remains to show that $\Gamma \to W_4$. Consider a 2-edge-coloring of Γ. By the Focusing Lemma (Lemma 3.2.3) there is a set V of $2^{-5}|V(F)| = \epsilon|V(F)|$ vertices in F such that between V and each vertex in K all edges are of the same color. Since $F \overset{\epsilon}{\to} C_4$ there is a monochromatic copy C of C_4 in $F[V]$. Assume without loss of generality that C is blue. If there is a vertex in K which sends a blue star to C then there is a blue copy of W_4 and we are done. So assume all vertices in K send red stars to C. If there is no blue copy of W_4 in K, then there are two adjacent red edges e and f in K (since the complement of W_4 in K_5 is a maximum matching). Then e, f and any two vertices from C form a red copy of W_4, with the vertex of degree 4 being the common vertex of e and f, see Figure 3.5. Hence $\Gamma \to W_4$. $\square$

Lemma 3.2.14. *If H is connected and has at least 6 vertices, then $H \overset{R}{\not\to} K_{2,3}$.*

Proof. We assume that H is bipartite and contains a cycle by Observation 3.4 and Lemma 3.2.9, respectively. Since $K_{3,13} \to K_{2,3}$ [69] we assume $K_{3,13} \to H$, since otherwise $H \overset{R}{\not\to} K_{2,3}$. Hence one of the partite sets of H contains at most 2 vertices, since otherwise coloring the edges of $K_{3,13}$ with a red copy of $K_{2,13}$ and an edge

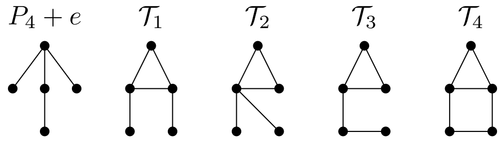


Figure 3.6: The graphs $P_4 + e$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$.

disjoint blue copy of $K_{1,13}$ does not yield a monochromatic copy of H. So H is $K_{2,b}$ with some edges pendant at the part of size 2 for some $b \in \mathbb{N}$ with $b \geqslant 2$.

Suppose H has at least 8 vertices. We claim $r(H) > 10$. Indeed, color the edges of a copy of $K_{5,5}$ in K_{10} red and all other edges blue. Then the blue edges form vertex disjoint copies of K_5 and do not contain copies of H since H is connected. The red subgraph contains no copies of H as one of the bipartition classes of H has at least 6 vertices (the other has only 2). Hence $H \overset{R}{\not\sim} K_{2,3}$ because $r(K_{2,3}) = 10$ [84].

Suppose $|V(H)| \in \{6, 7\}$. Then H is isomorphic to one of the graphs Z_i given in Figure 3.2 or a supergraph of Z_5. Note that Z_5 contains $K_{2,3}$. We have $r(Z_1) = 7$, $r(Z_2) = 8$ by [26] and $r(Z_3) = 9$, $r(Z_4) = 10$ by [83]. Thus $H \overset{R}{\not\sim} K_{2,3}$ if H is isomorphic to one of the graphs Z_i, $1 \leqslant i \leqslant 3$, because $r(K_{2,3}) = 10$. Moreover $H \overset{R}{\not\sim} K_{2,3}$ if H is isomorphic to Z_4 or Z_5 because $r_\delta(Z_4) = r_\delta(Z_5) = 1$ by Lemma 3.2.10 respectively Lemma 3.2.12, but $r_\delta(K_{2,3}) \geqslant \delta(K_{2,3}) = 2$. So assume H is a supergraph of Z_5. Let F denote a minimal Ramsey graph of Z_5 with $\delta(F) = 1$ and obtain a graph F' by removing a vertex of degree 1 from F. Then $F \to K_{2,3}$ because $K_{2,3} \subseteq Z_5$ and $F' \to K_{2,3}$ (since F is not minimal Ramsey for $K_{2,3}$). But $F' \not\to Z_5$, thus $F' \not\to H$, and hence $H \overset{R}{\not\sim} K_{2,3}$. □

The following lemmas give results for the graphs in Figure 3.6.

Lemma 3.2.15. $H_{5,2} \to P_4 + e$.

Proof. Let u denote the vertex of degree 2 in $H_{5,2}$ and let let $e = vw$ denote the edge incident to both neighbors of u. Let x, y, z denote the other vertices. Assume there is a 2-edge-coloring of $H_{5,2}$ without a monochromatic copy of $P_4 + e$. Without loss of generality assume the edge uv is blue.

Case 1: The edge e and two more edges vx, vy incident to v are red. Then either there is a red copy of $P_4 + e$ containing these edges or zx, zy, zw, and uw are blue. Then these form a blue copy of $P_4 + e$.

Case 2: The edge e and at most one other edge incident to v is red. Then assume vx, vy are blue. Then wx, wy, zx, zy, and uw are red and yield a copy of $P_4 + e$.

Case 3: The edges e and vx are blue. Then wy, wz, xy, and xz are red. Then the edges vy, vz, and uw are blue, so the blue subgraph contains a copy of $P_4 + e$.

Case 4: The edge e is blue but all the edges vx, vy, vz are red. Then wx, wy, wz are blue. Together with uv and vw there is a blue copy of $P_4 + e$. □

Lemmas for Section 3.4

Lemma 3.2.16. *If a graph H is not bipartite then $r_\delta(H) \geqslant 2\Delta(H)$. The lower bound is tight.*

Proof. Let $\Delta = \Delta(H)$ and suppose F is a graph with $\Delta(F) \leqslant 2\Delta - 1$. It is sufficient to prove that $F \not\rightarrow H$. Consider a partition $V_1 \dot\cup V_2$ of $V(F)$ with the maximum number of edges between V_1 and V_2. If there is a vertex $v \in V_1$ with at least Δ neighbors in V_1, then v has at most $\Delta - 1$ neighbors in V_2. Thus the partition $(V_1 \setminus \{v\}) \dot\cup (V_2 \cup \{v\})$ has at least one more edge between the parts than the original partition, a contradiction. Hence both $F[V_1]$ and $F[V_2]$ have maximum degree at most $\Delta - 1$. Color all edges between V_1 and V_2 red and all other edges blue. Then the red subgraph is bipartite and the blue subgraph has maximum degree at most $\Delta - 1$. Thus $F \not\rightarrow H$.

The lower bound is tight since $K_{2\Delta+1} \rightarrow H$, $\Delta \geqslant 3$, where H is the graph of maximum degree Δ obtained from $K_{1,\Delta}$ by adding an edge between two leaves. □

Lemma 3.2.17. $H_{5,3} \rightarrow P_5$.

Proof. Assume there is a 2-edge-coloring of $H_{5,3}$ without a monochromatic copy of P_5. Let u denote the vertex of degree 3 in $H_{5,3}$, let x, y, z denote its neighbors and v, w the remaining vertices. There are two edges of the same color incident to u, assume ux, uy are red. Since there is no monochromatic copy of $K_{2,3}$ (it contains a copy of P_5) there is at least one edge from $\{x, y\}$ to $\{v, w, z\}$ in red.

Case 1: There are $r, r' \in \{v, w, z\}$ such that the edges xr, yr' are red. Then $r = r'$ and all edges from $\{x, y\}$ to $\{v, w, z\} \setminus \{r\}$ are blue. But then the edge from r to $\{v, w, z\} \setminus \{r\}$ can be neither red nor blue.

Case 2: Without loss of generality, x has only blue edges to $\{v, w, z\}$. Then yp is red for some $p \in \{v, w, z\}$ and all edges from p to $\{v, w, z\} \setminus \{p\}$ are blue. This yields a blue copy of C_4 on $\{x, v, w, z\}$. Since all edges which are incident to this C_4 (but not contained) are red we can find a red copy of P_5. □

Lemma 3.2.18. $K_{5,5} \rightarrow C_4 + e$.

Proof. Consider a 2-edge-coloring of the edges of $K_{5,5}$ and a vertex v. First we shall prove that there is a monochromatic copy of C_4. Let V denote the partite set of $K_{5,5}$ containing v. Then v is incident to three edges vx, vy, vz of the same color, say red. From each of the four vertices in $V \setminus \{v\}$ at most one edge to $\{x, y, z\}$ is red, otherwise there is a red copy of $K_{2,2}$. But then there are two vertices in $\{x, y, z\}$ and two vertices in $V \setminus \{v\}$ forming a blue copy of $K_{2,2}$ by pigeonhole principle. This shows that there is a monochromatic copy K of C_4, say in red. There is either a red edge between K and the vertices not in K or all these edges are blue. In either case there is a monochromatic copy of $C_4 + e$. □

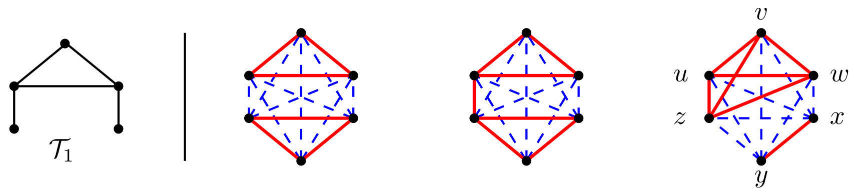


Figure 3.7: All possible 2-colorings of K_6 without a monochromatic copy of $\mathcal{T}_1$, up to isomorphism and swapping the colors.

Lemma 3.2.19. $H_{6,5} \to H_{3,1}$.

Proof. Consider a 2-edge-coloring of the edges of $H_{6,5}$. Since $H_{6,5}$ contains a copy of K_6 there is a monochromatic copy K of K_3, say in red. Let U denote the vertices of $H_{6,5}$ not in K. If there is a red edge between K and U, then there is a red copy of $H_{3,1}$. Otherwise all edges between K and U are blue. If there a blue edge induced by U, then there is a blue copy of $H_{3,1}$. Otherwise U induces a red copy of $H_{3,1}$. □

Lemma 3.2.20. *Figure 3.7 shows all 2-edge-coloring of K_6 without a monochromatic copy of $\mathcal{T}_1$, up to isomorphism and renaming the colors.*

Proof. Consider a 2-edge-coloring of K_6 on vertices u, v, w, x, y, z without monochromatic copies of $\mathcal{T}_1$. We may assume that $K = \{u, v, w\}$ forms a red copy of K_3 since $r(K_3) = 6$. Clearly there are no two independent red edges between K and $K^c = \{x, y, z\}$. If there is a vertex in K incident to at least two red edges to K^c, then all edges between the two other vertices in K and K^c are blue. Thus these blue edges form a blue copy of $K_{2,3}$, and hence no edge in K^c is blue. But if all edges in K^c are red, then there is a red copy of K_3 with two vertices in K^c and one vertex in K and a pendant red edge in K and a pendant edge in K^c. So we may assume that at most one vertex in K^c is adjacent to K in red, say z. Then $\{x, y\}$ and K form a blue copy of $K_{2,3}$. This shows that xy is red. We consider the cases how many edges between x, y and z are red.

Case 1: The edge xy is the only red edge. Then $\{x, y\}$ and $\{u, v, w, z\}$ induce a blue copy of $K_{2,4}$ and any additional blue edge within this copy of $K_{2,4}$ yields a blue copy of $\mathcal{T}_1$. Hence the red edges form a copy of K_4 plus disjoint copy of K_2, which corresponds to the rightmost coloring of Figure 3.7.

Case 2: There are at least two red edges. Then z is not part of any red copy of K_3 on $\{u, v, w, z\}$, since this K_3 would have a red pendant edge in K and another one in K^c. Hence there is at most one red edge from z to K. Thus there is a blue copy of $K_{3,3} - e$ between K and K^c. This shows that no edge in K^c is blue and the coloring corresponds to the left or the middle coloring in Figure 3.7. □

Let $\mathcal{K}$ and $\mathcal{K}'$ denote the graphs given in Figure 3.8. Also recall the graphs $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ given in Figure 3.6.

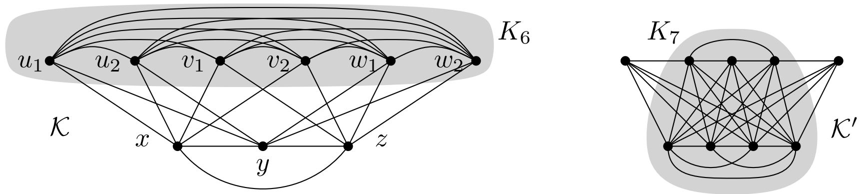


Figure 3.8: The graphs $\mathcal{K}$ (left) and $\mathcal{K}'$ (right).

Lemma 3.2.21. $\mathcal{K} \to \mathcal{T}_1$.

Proof. Assume c is a 2-coloring of the edges of $\mathcal{K}$, labeled like in Figure 3.8, without a monochromatic copy of $\mathcal{T}_1$. Let K denote the copy of K_6 in $\mathcal{K}$. Due to Lemma 3.2.20, c restricted to K is isomorphic to one of three colorings of K_6 given in Figure 3.7. We shall distinguish cases based on the coloring of K under c.

Case 1: The red subgraph of K under c consists of two disjoint K_3 and the blue edges in K form a copy of $K_{3,3}$. If one of these blue edges is contained in a blue copy of K_3 with a vertex from $\{x, y, z\}$, then there is a blue copy of $\mathcal{T}_1$. Note that each vertex from $\{x, y, z\}$ has four neighbors in K, these neighborhoods intersect pairwise in exactly two vertices, and no vertex from K is contained in all three neighborhoods. Therefore we can find three vertex disjoint copies of K_3 each with exactly one vertex from each of the red copy of K_3 in K and exactly one vertex from $\{x, y, z\}$. Since there is a red edge from K to $\{x, y, z\}$ in each of these, one of the red copies of K_3 in K has two independent pendant red edges. This gives a red copy of $\mathcal{T}_1$, a contradiction.

Case 2: The red subgraph of K under c consists of two disjoint K_3 connected by a single edge e. Then all edges from K to $\{x, y, z\}$ are blue if not adjacent to e. Then there are two vertices in K, not incident to e, each having two blue edges to $\{x, y, z\}$ but only one common neighbor in $\{x, y, z\}$. Since they are connected by a blue edge in K, this gives a blue copy of $\mathcal{T}_1$, a contradiction.

Case 3: The red subgraph of K_6 consists of a copy of K_4 and a vertex disjoint edge e. Then all edges from this copy of K_4 to $\{x, y, z\}$ are blue. The blue edges in K form a copy of $K_{2,4}$. Again, if one of these blue edges in K forms a blue triangle with a vertex from $\{x, y, z\}$, then there is a blue copy of $\mathcal{T}_1$. Thus all edges from e to $\{x, y, z\}$ are red. If $e \notin \{v_1v_2, u_1u_2, w_1w_2\}$, then e together with $\{x, y, z\}$ forms a red copy of $\mathcal{T}_1$. So assume $e = w_1w_2$. If the edge xy is blue, then $\{x, y, u_1, u_2, v_1\}$ gives a blue copy of $\mathcal{T}_1$. If it is red, then $\{x, y, z, w_1, w_2\}$ gives a red copy of $\mathcal{T}_1$, a contradiction.

Altogether we proved that there is no 2-edge-coloring of $\mathcal{K}$ without a monochromatic copy of $\mathcal{T}_1$. ∎

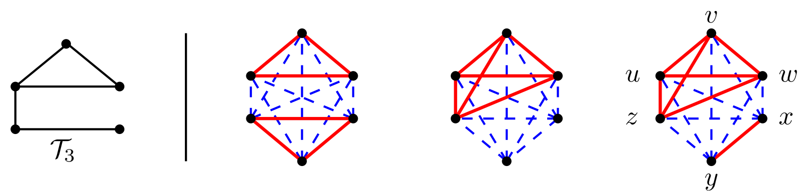


Figure 3.9: All possible 2-colorings of K_6 without a monochromatic copy of $\mathcal{T}_3$.

Lemma 3.2.22. *Figure 3.9 shows all 2-edge-coloring of K_6 without a monochromatic copy of $\mathcal{T}_3$, up to isomorphism and renaming the colors.*

Proof. Consider a 2-edge-coloring of K_6 on vertices u, v, w, x, y, z without monochromatic copies of $\mathcal{T}_3$. We may assume that $K = \{u, v, w\}$ forms a red copy of K_3.

If all edges from K to K^c are blue, then no edge among K^c is blue. Thus the coloring corresponds to the left one in Figure 3.9. So assume the edge uz is red. If $\{u, v, w, z\}$ forms a red copy of K_4, then all edges from this K_4 to $\{x, y\}$ are blue. No matter which color is assigned to xy, the coloring has no monochromatic copy of $\mathcal{T}_3$ and corresponds to the middle or right coloring in Figure 3.9.

So assume further that $\{u, v, w, z\}$ is not a red copy of K_4 (but uz is still red), without loss of generality wz is blue. Since uz is red, xz and yz are blue.

Case 1: The edge wx is blue. Then $\{w, x, z\}$ is a blue copy of K_3 with pendant blue edge yz. Thus uy, vy are red. But then wy needs to be blue (otherwise $\{v, w, y\}$ is red copy of K_3 with pendant red path vuz) and there is a blue copy of K_4. Thus the coloring corresponds to the middle or right coloring of Figure 3.9 with switched colors, as argued above.

Case 2: The edge wx is red. Then xy is blue and vx, vz are blue. Then vy needs to be red, since $\{v, x, y\}$ is a blue copy of K_3 with pendant blue path xzw otherwise. Then wy is blue, since otherwise $\{v, w, y\}$ is a red copy of K_3 with pendant red path vuz otherwise. But now $\{w, y, z\}$ is a blue copy of K_3 with pendant blue path zxv, a contradiction. □

Lemma 3.2.23. $\mathcal{K} \to \mathcal{T}_3$.

Proof. Assume c is a 2-coloring of the edges of $\mathcal{K}$, labeled like in Figure 3.8, without a monochromatic copy of $\mathcal{T}_3$. Let K denote the copy of K_6 in $\mathcal{K}$. Due to Lemma 3.2.22, c restricted to K is isomorphic to one of three colorings of K_6 given in Figure 3.9. We shall distinguish cases based on the coloring of K under c.

Case 1: The red subgraph of K_6 consists of two disjoint copies of K_3. Then the blue edges in K form a copy of $K_{3,3}$. If one of these blue edges is contained in a blue copy of K_3 with a vertex from $\{x, y, z\}$, then there is a blue copy of $\mathcal{T}_3$. On the other hand no vertex in $K^c = \{x, y, z\}$ sends a red edge to each of the red copies of K_3 in K. Since there are 4 edges from each vertex in $\{x, y, z\}$ to K, each is incident to at least one red edge and one blue edge. If one of the edges induced by $\{x, y, z\}$ is red, then there is a red copy of $\mathcal{T}_3$ with a red copy of K_3 from K and an edge between them. So $\{x, y, z\}$ induces a blue copy of K_3 which forms a blue copy of $\mathcal{T}_3$ with an edge to K and another contained in K.

Case 2: The red subgraph of K_6 consists of a red copy of K_4 only. Then no edge incident to this copy of K_4 is red. Let a, b denote the vertices in K not contained in the red copy of K_4. Then every blue edge between $\{x, y, z\}$ and the red copy of K_4 is part of a blue copy of $\mathcal{T}_3$ together with a, b, and another vertex in K.

Case 3: The red subgraph of K_6 consists of a red copy of K_4 and a disjoint red edge e. Again all edges incident to the red copy of K_4 are blue and no blue edge in K is contained in a blue copy of K_3 with a vertex from K^c. Thus all edges from e to K^c are red. Assume $e \notin \{v_1v_2, u_1u_2, w_1w_2\}$, say it is v_2w_2. If xy is blue then $\{x, y, u_1, v_1, z\}$ forms a blue copy of $\mathcal{T}_3$. If xy is red then $\{x, y, v_2, w_2, z\}$ gives a red copy of $\mathcal{T}_3$. So assume $e = w_1w_2$. If the edge xy is blue, then $\{x, y, u_1, v_1, z\}$ forms a blue copy of $\mathcal{T}_3$. If it is red, then $\{x, y, z, w_1, w_2\}$ gives a red copy of $\mathcal{T}_3$. □

Lemma 3.2.24. *For each $i \in \{1, 2, 3\}$ a 2-coloring of K_8 does not have a monochromatic copy of $\mathcal{T}_i$ if and only if one of the color classes induces two vertex disjoint copies of K_4.*

Proof. First of all note that a 2-edge-coloring of K_8 with one color class inducing two vertex disjoint copies of K_4's does not contain a monochromatic copy of $\mathcal{T}_i$ for each $i \in \{1, 2, 3\}$.

On the other hand, consider an arbitrary 2-edge-coloring of K_8 without no monochromatic copies of $\mathcal{T}_i$ for a fixed $i \in \{1, 2, 3\}$. There is a monochromatic copy K of $H_{3,1}$, say in red (i.e., a red copy of K_3 with a pendant edge), since $r(H_{3,1}) = 7$.

Suppose that there is no monochromatic copy of $\mathcal{T}_1$. Then none of the two vertices of degree 2 in K is incident to another red edge in K_8. Thus there is a blue copy of $K_{2,4}$. Then the part with four vertices in this $K_{2,4}$ contains no further blue edge and induces a red copy of K_4. But then no edge incident to this copy of K_4 is red, and there is a blue copy of $K_{4,4}$. Since no other edge might be blue then, there are two disjoint red copies of K_4.

Suppose there is no monochromatic copy of $\mathcal{T}_2$. The vertex of degree 3 in K has no other incident red edge. So it is the center of a blue copy of $K_{1,4}$. The degree 1 vertices in this copy of $K_{1,4}$ do not induce a blue edge, so they induce a red copy of K_4. But then no edge incident to this K_4 is red and there is a blue copy of $K_{4,4}$ between K and the other vertices. As argued above the red edges form two disjoint copies of K_4s and the blue edges form a copy of $K_{4,4}$.

Suppose there is no monochromatic copy of $\mathcal{T}_3$. Let K^c denote the set of vertices not in K. Then any edge connecting the vertex v of degree 1 in K to a vertex in K^c is blue. Assume there is a red edge from K to a vertex $u \in K^c$. Then any edge connecting u to a vertex in $K^c \setminus \{u\}$ is blue. Then each edge e within $K^c \setminus \{u\}$ or from $K^c \setminus \{u\}$ to $K \setminus \{v\}$ is red, since otherwise there is a blue copy of $\mathcal{T}_3$ spanned by u, v and e. But then there is red copy of $\mathcal{T}_3$, a contradiction.

So all edges between K and K^c are blue. Then there is no other blue edge and the red edges form two disjoint copies of K_4. □

Lemma 3.2.25. *$H_{8,5} \to \mathcal{T}_i$ for each $i \in \{1,2,3\}$.*

Proof. Assume there is a 2-edge-coloring of $H_{8,5}$ without a monochromatic copy of $\mathcal{T}_i$ for some $i \in \{1,2,3\}$. We may assume that within the copy of K_8 the red edges form two disjoint copies of K_4 with a blue copy of $K_{4,4}$ in-between by Lemma 3.2.24. Let v denote the vertex of degree 5. Then v has only blue incident edges since every neighbor of v is part of a red copy of K_4. But v has a neighbor in each of the red copies of K_4's. Thus v together with these two vertices forms a blue copy of K_3 which is contained in a blue copy of $\mathcal{T}_i$ for all $i \in \{1,2,3\}$, a contradiction. □

Lemma 3.2.26. *A 2-edge-coloring of K_7 does not have a monochromatic copy of $\mathcal{T}_1$ if and only if one of the color classes induces vertex disjoint copies of K_3 and K_4.*

Proof. The proof is very similar to the proof of Lemma 3.2.24. □

Lemma 3.2.27. *$\mathcal{K}' \to \mathcal{T}_1$.*

Proof. Assume there is a 2-edge-coloring of $\mathcal{K}'$ without a monochromatic copy of $\mathcal{T}_1$. Due to Lemma 3.2.26 we may assume that the copy of K_7 in $\mathcal{K}'$ is colored such that the blue edges induce a copy of $K_{3,4}$ and the red subgraph consists of two disjoint copies of K_4 and K_3. Let K denote the red copy of K_3 and u, v the two vertices of degree 5 in $\mathcal{K}'$. Then each edge from $\{u,v\}$ to the red copy of K_4 is blue and there are at least two such edges incident to each of u, v. Thus each edge from u or v to K is red, since there is a blue copy of $\mathcal{T}_1$ otherwise. Due to construction of $\mathcal{K}'$ there are two independent edges from K to $\{u,v\}$ and thus a red copy of $\mathcal{T}_1$, a contradiction. □

We checked the following lemma by a straightforward algorithm computing all solutions of an equivalent satisfiability problem. It would be nice to have a proof that avoids the use of computers as well as lengthy case analysis. Note that each of the colorings described in the following lemma does not contain a monochromatic copy of C_5. Since C_5 is a subgraph of $\mathcal{T}_4$ this shows that a coloring of K_8 has no monochromatic copy of C_5 if and only if there is no monochromatic copy of $\mathcal{T}_4$.

Lemma 3.2.28. *A 2-edge-coloring of K_8 does not have a monochromatic copy of $\mathcal{T}_4$ if and only if one of the color classes induces two disjoint copies of K_4 with at most one edge of the same color in-between (i.e., the other color spans a copy of $K_{4,4}$ or $K_{4,4} - e$).*

Lemma 3.2.29. $H_{8,6} \to C_5, \mathcal{T}_4$.

Proof. Since C_5 is a subgraph of $\mathcal{T}_4$ it is sufficient to prove $H_{8,6} \to \mathcal{T}_4$. Assume there is a 2-edge-coloring of $H_{8,6}$ without a monochromatic copy of $\mathcal{T}_4$. Due to Lemma 3.2.28 we assume that the coloring of the copy of K_8 in $H_{8,6}$ has two disjoint red copies of K_4 connected by at most one red edge. Let v denote the vertex of degree 6. It is incident to at most one red edge to each of the red copies of K_4. Thus there is a blue copy of K_3 with v and one vertex from each red copy of K_4. But this forms a blue copy of $\mathcal{T}_4$ together with some of the other blue edges, a contradiction. □

Lemma 3.2.30. *A 2-edge-coloring of K_9 does not have a monochromatic copy of $H_{3,2} = K_4 - e$ if and only if each color class is isomorphic to the Cartesian product $K_3 \times K_3$.*

Proof. First of all observe that $K_3 \times K_3$ does not contain a copy of $H_{3,2}$ since every edge is contained in exactly one copy of K_3. Moreover the complement of $K_3 \times K_3$ (as a subgraph of K_9) is isomorphic to $K_3 \times K_3$. Hence the edges of K_9 can be 2-colored without a monochromatic copy of $H_{3,2}$ using two edge disjoint copies of $K_3 \times K_3$.

On the other hand consider a 2-edge-coloring c of K_9 without a monochromatic copy of $H_{3,2} = K_4 - e$. We shall assign labels $v_{i,j}$, $1 \leqslant i, j \leqslant 3$, to the vertices of K_9 such that this labeling corresponds to an arrangement of the vertices in a 3×3 grid where the red subgraph spans all rows and columns and all other edges are blue.

There is a monochromatic copy of $\mathcal{T}_5$ under c, say in red, since $r(\mathcal{T}_5) = 9$, see Table 3.1. Let $K = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{3,1}\}$ denote the vertices of this copy of $\mathcal{T}_5$ such that $v_{1,1}$ is the vertex of degree 4 and the edges of this red copy of $\mathcal{T}_5$ span the first row and first column in the grid, see Figure 3.11. Observe that no edge spanned by K is red except for the edges in the red copy of $\mathcal{T}_5$. Indeed if another edge is red, then there is a red copy of $H_{3,2}$ in K. Let K^c denote the vertices not in K. We shall use the following claim.

Claim 1. *If C is a red copy of K_3 and uv is a vertex disjoint blue edge, then there is a vertex x in C such that xu and xv are blue, and there are two independent red and two independent blue edges between $C - x$ and uv. See Figure 3.10 for an illustration.*

Proof of Claim 1. Indeed, there is at most one red edge between each vertex in $\{u, v\}$ and C and for at most one vertex in C both edges to uv are blue. Hence for

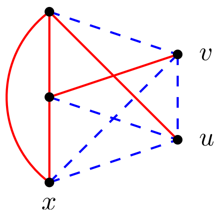


Figure 3.10: The unique 2-coloring of K_5 (up to isomorphism) without a monochromatic copy of $H_{3,2}$ provided that there is a red copy of K_3 with a disjoint blue edge.

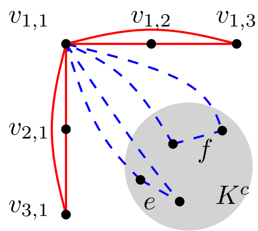


Figure 3.11: The partial labeling of vertices of K_9 under a 2-coloring without a monochromatic copy of $H_{3,2}$ in the proof of Lemma 3.2.30, with solid red and dashed blue edges.

exactly one vertex in C both edges to uv are blue and there are exactly two further independent blue edges between C and uv. This proves Claim 1. △

By assumption the four vertices in K^c do not induce a monochromatic copy of $H_{3,2}$ and hence there are at least two blue edges e, f. Let C_1, C_2 denote the red copies of K_3 in K. We shall apply Claim 1 to each of the pairs $\{e, C_1\}$, $\{e, C_2\}$, $\{f, C_1\}$, $\{f, C_2\}$. There is a vertex x_i in C_i, $i = 1, 2$, such that both edges between x_i and a blue edge in K^c are blue by Claim 1. Then $x_1 = x_2 = v_{1,1}$, since otherwise there is blue copy of $H_{3,2}$. Hence the blue edges in K^c are independent, since two adjacent blue edges together with $v_{1,1}$ form a blue copy of $H_{3,2}$. Thus e and f are the only blue edges in K^c. See Figure 3.11 for the partial labeling. Furthermore there are two independent red edges and two independent blue edges from each of the edges e and f to each $C_i - v_{1,1}$, $i = 1, 2$, by Claim 1. It remains to find labels for the vertices in e and f.

Claim 2. *For any two vertices $u \in \{v_{1,2}, v_{1,3}\}$, $v \in \{v_{2,1}, v_{3,1}\}$ there is exactly one vertex w in K^c such that uw and vw are red.*

Proof of Claim 2. Indeed, assume there are two such vertices w, w' in K^c for some pair u,v. Then the edge ww' is red by Claim 1 and there is a red copy of $H_{3,2}$. Thus there is at most one such vertex. Assume there is no such vertex in K^c for some pair. Then there is a red copy of $H_{3,2}$, since there are two independent red edges between each of e and f and each C_i, $i = 1, 2$, by Claim 1, a contradiction. This proves Claim 2. △

Let $v_{2,2}$ denote the vertex which is adjacent to $v_{1,2}$ and $v_{2,1}$ in red which exists by Claim 2. Without loss of generality assume $v_{2,2}$ is incident to e. Let $v_{3,3}$ denote the other vertex incident to e. Due to Claim 1 applied to e and C_1 and C_2, the edges $v_{3,3}v_{1,3}$ and $v_{3,3}v_{3,1}$ are red and the edges $v_{2,2}v_{1,3}$, $v_{2,2}v_{3,1}$, $v_{3,3}v_{1,2}$ and $v_{3,3}v_{2,1}$ are blue. With the same arguments we choose $f = v_{3,2}v_{2,3}$ accordingly. This shows that the red color class is isomorphic to $K_3 \times K_3$. □

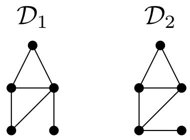


Figure 3.12: The graphs $\mathcal{D}_1$ and $\mathcal{D}_2$.

The following lemma give results for the graphs in Figure 3.12 used in Section 3.4.

Lemma 3.2.31. *$H_{9,6} \to H$ for each $H \in \{H_{3,2}, \mathcal{D}_1, \mathcal{D}_2\}$.*

Proof. Consider a 2-edge-coloring of $H_{9,6}$. Let v denote the vertex of degree 6. We shall show that it contains each of the graphs from $\{H_{3,2}, \mathcal{D}_1, \mathcal{D}_2\}$ as a monochromatic subgraph.

Either there is a monochromatic copy of $H_{3,2}$ in the copy of K_9 in $H_{9,6}$ or we may assume by Lemma 3.2.30 that the copy of K_9 in $H_{9,6}$ is an edge disjoint union of a red copy of $K_3 \times K_3$ and a blue copy of $K_3 \times K_3$. Each edge in the copy of K_9 belongs to a unique monochromatic triangle. If v sends two blue edges to vertices u, w, where uw is blue, then the blue triangle containing uw in K_9 together with v form a blue copy of $H_{3,2}$. Thus, we may assume that neighborhood of v via blue edges forms a red clique, and, similarly, its neighborhood via red edges forms a red clique. Since degree of v is 6, and the largest monochromatic clique in the copy of K_9 is a triangle, these cliques must be triangles. However, there are no two disjoint red and blue triangles in the copy of K_9, so we arrive at a contradiction. Thus, there is a monochromatic copy of $H_{3,2}$.

Assume that the monochromatic copy K of $H_{3,2}$ is red. First, we assume that K does not contain v. Let K^c denote the set of vertices from the copy of K_9 that are not in K. If there is a red edge between a vertex of degree 3 of K and K^c, we have a monochromatic copy of $\mathcal{D}_1$. If there is a red edge between a vertex of degree 2 of K and K^c, we have a monochromatic copy of $\mathcal{D}_2$. If all edges between degree 3 vertices of K and K^c are blue and there are two adjacent blue edges in K^c, then there is a blue copy of $\mathcal{D}_1$. If all edges between degree 3 vertices of K and K^c are blue and there are no two adjacent blue edges in K^c, then K^c forms a red copy of K_5 minus a matching, and thus contains a copy of $\mathcal{D}_1$. If all edges between degree 2 vertices of K and K^c are blue, then there is a blue copy of $\mathcal{D}_2$ or there is no blue edge induced by K^c, In the latter case K_c induces a red copy of K_5 that contains a red copy of $\mathcal{D}_2$.

Now, assume that any monochromatic copy of $H_{3,2}$ contains v, i.e., there is no monochromatic copy of $H_{3,2}$ in a copy of K_9 of $H_{9,6}$. Hence the coloring of this copy of K_9 is like it is described in Lemma 3.2.30. Then, it is easy to see that K and an appropriate edge of this copy of K_9 form a monochromatic copy of $\mathcal{D}_1$ and, similarly, a monochromatic copy of $\mathcal{D}_2$. □

Finally we prove a result on multicolor Ramsey numbers used in the conclusions to show that for some graphs of the same 2-color Ramsey number, the multicolor Ramsey numbers differ.

Lemma 3.2.32. *If $k \geqslant 3$ is odd and $\frac{k+1}{2}$ is even, then $r_k(P_4 + e) > 2k + 2$.*

Proof. We shall give a coloring of the edges of K_{2k+2} without monochromatic copy of $P_4 + e$. Let $t = \frac{k+1}{2}$ and let $V_1, V_2, \ldots, V_t$ denote a partition of the vertices of K_{2k+2} such that $|V_i| = 4$, $i \in [t]$. Color all edges induced by each set V_i in color 1, $i = 1, \ldots, t$. Then there is no copy of $P_4 + e$ in color 1. It remains to color the edges between the parts. The edges between two parts V_i and V_j form copies of $K_{4,4}$, $1 \leqslant i < j \leqslant t$. So these can be decomposed into two edge-disjoint copies $A_{i,j}$ and $B_{i,j}$ of an 8-cycle C_8. Consider a complete graph K with vertex set $[t]$. Since $t = \frac{k+1}{2}$ is even there is a decomposition of the edges of K into $t-1$ perfect matchings $M_1, \ldots, M_{t-1}$. For each $\ell \in [t-1]$ and each edge $ij \in M_\ell$ in K color the edges of $A_{i,j}$ in color 2ℓ and the edges of $B_{i,j}$ in color $2\ell + 1$. Then each color from $\{2, \ldots, 2(t-1)+1\}$ induces a vertex disjoint union of cycles. In particular there is no monochromatic copy of $P_4 + e$, since $\Delta(P_4 + e) = 3$. Moreover the number of colors used equals $2(t-1) + 1 = k$. This shows that $r_k(P_4 + e) > 2k + 2$ □

3.3 Proofs of Theorems

Proof of Theorem 3.3

If $\chi(G) = 2$ then G is not Ramsey equivalent to any graph H of higher chromatic number by Observation 3.4. So, assume that $\chi(G) \geqslant 3$.

We shall construct a graph Γ such that $\Gamma \to G$ and $\Gamma \not\to H$. Let $\omega = \omega(G)$, $k = \chi(G)$, $\chi(H) > k$. We assume $\omega(G) = \omega(H)$, otherwise $G \overset{R}{\not\sim} H$ by Lemma 3.2.4. Let $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that $G_1 = G[V_1]$ and $G_2 = G[V_2]$ each have clique number less than ω. Let $G^\star$ be a vertex disjoint union of G_1 and G_2, in particular $\omega(G^\star) < \omega$.

The building blocks of Γ are a hypergraph $\mathcal{H}$ and graphs F and F' such that:

- $\mathcal{H}$ is a 3-chromatic, k-uniform hypergraph of girth at least $|V(H)|+1$. It exists by Lemma 3.2.6.

- F is a graph such that $\omega(F) < \omega$ and every set of at least $\epsilon_1 |V(F)|$ vertices in F contains a copy of $G^\star$, where $\epsilon_1 = 2^{-|V(\mathcal{H})|}$. Such a graph exists by Lemma 3.2.8.

- F' is a graph such that $\omega(F') = \omega(F) < \omega$ and $F' \overset{\epsilon}{\to} F$ for $\epsilon = 2^{-|V(\mathcal{H})||V(F)|}$. Such a graph exists by Lemma 3.2.7.

Note that $|V(\mathcal{H})|$ depends on $|V(H)|$ and $\chi(G)$; $|V(F)|$ in turn depends on $|V(\mathcal{H})|$, $\omega(G)$, and G, so $|V(F)|$ depends only on H and G. So, ϵ and ϵ_1 are constants depending on H and G.

Construct a graph Γ by replacing the vertices $v_1, \dots, v_n$ of $\mathcal{H}$ with pairwise vertex disjoint copies of F' on vertex sets $V_1, \dots, V_n$ and placing a complete bipartite graph between two copies of F' if and only if the corresponding vertices belong to the same hyperedge of $\mathcal{H}$, see Figure 3.13.

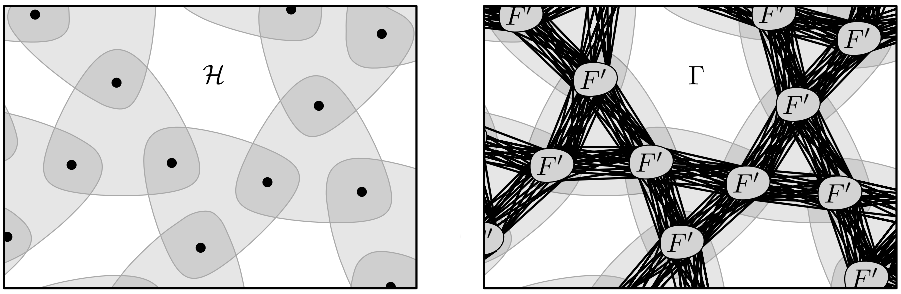


Figure 3.13: Left: The k-uniform hypergraph $\mathcal{H}$, $k = 3$. Right: The graph Γ.

To show that $\Gamma \not\to H$ color each edge with both endpoints in some V_i red, $i = 1, \dots, n$, and all other edges blue. The red subgraph is a vertex disjoint union of copies of F', its clique number is strictly less than ω, so it does not contain H, whose clique number is ω. The blue subgraph is a union of complete k-partite graphs induced by V_i, $i = 1, \dots, n$. To see that the blue subgraph does not contain a copy of H, consider any copy of H in Γ and consider sets $V_{i_1}, \dots, V_{i_\ell}$ intersecting the vertex set of this copy. Since $\mathcal{H}$ has girth at least $|V(H)| + 1$, $v_{i_1}, \dots, v_{i_\ell}$ do not form a cycle in $\mathcal{H}$, thus the blue graph induced by $V_{i_1}, \dots, V_{i_\ell}$ is k-partite. However, $\chi(H) > k$, so the blue subgraph does not contain a copy of H.

Next we shall show that $\Gamma \to G$. Consider a 2-edge-coloring of Γ. Recall that $n = |V(\mathcal{H})| = n(k, |V(H)|)$. We write $v_i \sim v_j$ if there is a hyperedge in $\mathcal{H}$ containing both v_i and v_j.

Claim 1. *For any m, $1 \leqslant m \leqslant n$, any i, $1 \leqslant i \leqslant m$, V_i contains a subset V_i' that is the vertex set of a monochromatic copy of F and such that for any $v \in V_i'$ and any j with $v_i \sim v_j$, $i < j \leqslant m$, all edges from v to V_j', are of the same color.*

Proof of Claim 1. We prove Claim 1 by induction on m using the Focusing Lemma (Lemma 3.2.3). When $m = 1$, we see that $\Gamma[V_1]$ is isomorphic to F' and $F' \stackrel{\epsilon}{\to} F$. So in particular $F' \to F$ and there is a monochromatic copy of F on some vertex set V_1'. Assume that $V_1', V_2', \dots, V_m'$ form vertex sets of monochromatic copies of F satisfying the conditions of Claim 1. Apply the Focusing Lemma to the bipartite graph with parts $U_m = V_1' \cup \dots \cup V_m'$ and V_{m+1}. It gives a subset $V_{m+1}^* \subseteq V_{m+1}$ such that for any $v \in U_m$, all edges between v and V_{m+1}^*, if any, are of the same color

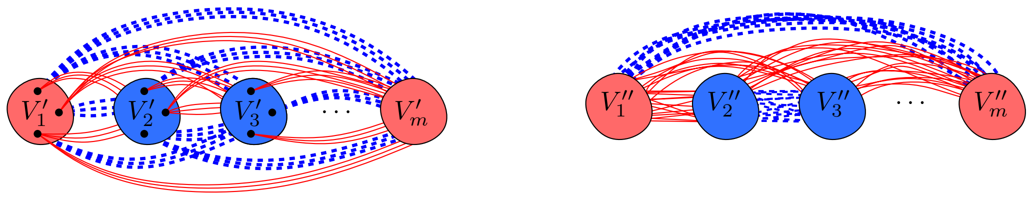


Figure 3.14: Illustrations of Claim 1 (left) and Claim 2 (right). Thin solid edges are red and thick dashed edges blue.

and such that $|V^*_{m+1}| \geqslant 2^{-|V_1' \cup \cdots \cup V_m'|}|V_{m+1}| = 2^{-m|V(F)|}|V_{m+1}| \geqslant \epsilon|V_{m+1}|$. Thus $\Gamma[V^*_{m+1}]$ contains a monochromatic copy of F, because $\Gamma[V_{m+1}]$ is isomorphic to F' and $F' \stackrel{\epsilon}{\to} F$. Call the vertex set of this copy V'_{m+1}. △

Claim 2. *For any m, $1 \leqslant m \leqslant n$, and i, $m \leqslant i \leqslant n$, each V_i contains a subset $V_i'' \subseteq V_i'$ that is the vertex set of a monochromatic copy of $G^\star$ and such that for each j with $v_i \sim v_j$, $i < j \leqslant n$, V_i'', V_j'' are partite sets of a monochromatic complete bipartite graph.*

Proof of Claim 2. We prove Claim 2 by induction on $n - m$ using the pigeonhole principle. When $m = n$, we see that V_n' forms the vertex set of a monochromatic copy of F, that in turn contains a monochromatic copy of $G^\star$. Denote the vertex set of this $G^\star$ as V_n''. Assume that $V_m'', V_{m+1}'', \ldots, V_n''$ form vertex sets of monochromatic copies of $G^\star$ satisfying the conditions of Claim 2. Consider V'_{m-1} and recall from Claim 1 that each vertex in V'_{m-1} sends only red or only blue edges to each V_i'' with $v_{m-1} \sim v_i$, $i = m, \ldots, n$. If $v_{m-1} \sim v_n$ then at least half of the vertices in V'_{m-1} send monochromatic stars of the same color to V_n''. If $v_{m-1} \sim v_{n-1}$ then at least half of those send monochromatic stars of the same color to V''_{n-1}, and so on. So at least $2^{-(n-m)}|V'_{m-1}|$ vertices of V'_{m-1} send monochromatic stars of the same color to each V_i'' with $v_{m-1} \sim v_i$ for $i = m, \ldots, n$. We denote the set of these vertices by V^*_{m-1}. Since V'_{m-1} forms the vertex set of a monochromatic copy of F, and $|V^*_{m-1}| \geqslant 2^{-(n-m)}|V'_{m-1}| \geqslant \epsilon_1|V'_{m-1}|$, the definition of F implies that $\Gamma[V^*_{m-1}]$ contains a monochromatic copy of $G^\star$. We denote the vertex set of this copy by V''_{m-1}. △

Applying Claim 2 with $m = 1$, we see that each vertex v_i of $\mathcal{H}$ corresponds to a monochromatic copy of $G^\star$ with vertex set V_i'', such that all edges between any two such copies from a common hyperedge have the same color. Assigning the color of this $G^\star$ to v_i gives a 2-coloring of $V(\mathcal{H})$. Since $\chi(\mathcal{H}) > 2$, there is a monochromatic hyperedge, without loss of generality with red vertices $v_1, \ldots, v_k$. Thus in Γ there are k red copies of $G^\star$ on vertex sets $V_1'', \ldots, V_k''$, such that V_i'', V_j'' are partite sets of monochromatic complete bipartite graphs, for all i, j, $1 \leqslant i < j \leqslant k$, see Figure 3.15. If at least one such bipartite graph is red, then there is a red copy of G obtained by taking a red copy of $G_1 \subseteq G^*$ from one part and a red copy of $G_2 \subseteq G^*$ from

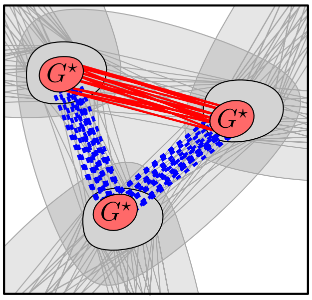


Figure 3.15: A set of k red copies of $G^\star$ corresponding to the k vertices of a hyperedge of $\mathcal{H}$. Here $k = 3$. The complete bipartite graph between any two of these k copies is also monochromatic.

the other part. So we can assume that all such bipartite graphs are blue, forming a complete k-partite graph with each part of size $|V(G)|$. Since $\chi(G) = k$, there is a blue copy of G. Thus $\Gamma \to G$. Since $\Gamma \not\to H$, we have that $G \overset{R}{\not\sim} H$. This concludes the proof of Theorem 3.3. □

Proof of Corollary 3.4. Let G and H be two graphs such that $\chi(G) \neq \chi(H)$ and $\chi(G) \leqslant 2\omega(G) - 2$. Consider an arbitrary proper $\chi(G)$-vertex-coloring of G. Let V_1 denote the union of $\lfloor \frac{\chi(G)}{2} \rfloor$ color classes and $V_2 = V(G) \setminus V_1$. Since $\omega(G) \geqslant \frac{\chi(G)}{2} + 1$, every maximum clique contains a vertex from both sets V_i, $i = 1, 2$. Thus, G is clique splittable. So, if $\chi(H) > \chi(G)$ then $G \overset{R}{\not\sim} H$ by Theorem 3.3. If $\chi(H) < \chi(G)$, then $\chi(H) < \chi(G) \leqslant 2\omega(H) - 2$ (where we assume $\omega(H) = \omega(G)$ by Lemma 3.2.4). Thus, H is clique-splittable with the same arguments as above. Hence $G \overset{R}{\not\sim} H$ by Theorem 3.3. □

Proof of Theorem 3.5

In the first part of the proof we shall construct a graph Γ with $\Gamma \to G$ and $\Gamma \not\to H$ if $H \not\subseteq G$ and $\chi(H) \geqslant \chi(G)$. Our construction is similar to the one from Lemma 3.9 in [77]. In the second part of the proof we suppose that $H \subseteq G$. In this case we can either apply this construction with roles of G and H switched or use Theorem 3.3.

Consider a connected graph H. We may assume $\omega(G) = \omega(H)$ by Lemma 3.2.4. Note that G is clique-splittable since $\omega(S) = 0$ and $\omega(G - S) < \omega(G)$. Further note that if G is bipartite, the conditions of the theorem imply that G is a union of a matching and a set of independent vertices. However, G is assumed to be connected, and thus it must be a single edge. Since a single edge is Ramsey isolated, we can assume that $\chi(G) \geqslant 3$.

In the first part of the proof, we assume that $H \not\subseteq G$ and $\chi(H) \geqslant \chi(G)$. Let $s = |S|$, $k = \chi(G) \leqslant \chi(H)$ and let m denote the size of a matching induced by two color classes of some proper k-vertex-coloring of G. Note that $m \geqslant 1$ since there is at least one edge between any two color classes. Further let $n = |V(G)|$, $\omega = \omega(G)$ and G_S be a vertex disjoint union of $G - S$ and s independent vertices, i.e., G_S is the graph obtained from G by deleting all edges incident to S. Then $\omega(G_S) < \omega$. Let G' be a vertex disjoint union of $m' = (k-2)(s-1) + m$ copies of G_S and G'_0 be a vertex disjoint union of m' copies of G. Let $\epsilon = 2^{-|V(G')|k} = 2^{-m'nk}$. Let F be a graph with $F \stackrel{\epsilon}{\to} G'$ and $\omega(F) = \omega(G') < \omega$, which exists by Lemma 3.2.7. We construct a graph Γ by taking the vertex disjoint union of a copy of G'_0 and $k-2$ copies of F denoted by $F_1, \dots, F_{k-2}$ and placing a complete bipartite graph between F_i and F_j, $1 \leqslant i < j \leqslant k-2$ and between F_i and G'_0, $i = 1, \dots, k-2$, see Figure 3.16.

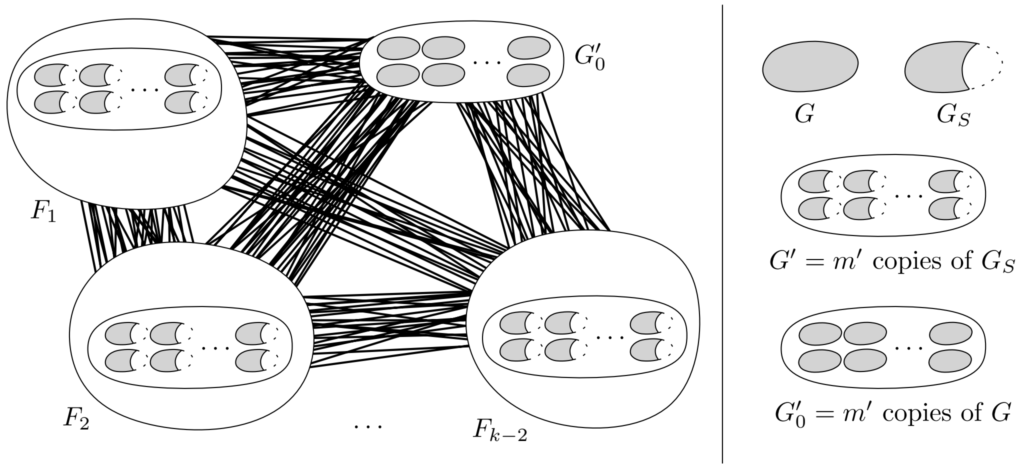


Figure 3.16: The graph Γ consisting of $k-2$ copies $F_1, \dots, F_{k-2}$ of F and one copy of G'_0 and all possible edges between distinct copies. We have $F \stackrel{\epsilon}{\to} G'$, G' consists of m' copies of G_S and G'_0 consists of m' copies of G.

We shall show that $\Gamma \to G$, but $\Gamma \not\to H$. Color all edges within each F_i and within G'_0 red and all other edges blue. Since $\omega(F) < \omega = \omega(H)$, $H \not\subseteq F$, and thus $H \not\subseteq F_i$, $i = 1, \dots, k-2$. Since $H \not\subseteq G$ and H is connected, we have that $H \not\subseteq G'_0$. Thus there is no red copy of H. On the other hand, the blue subgraph is a complete $(k-1)$-partite graph, but $\chi(H) \geqslant \chi(G) = k$. Thus there is no blue copy of H. It remains to show that $\Gamma \to G$. Consider a 2-edge-coloring of Γ. Assume for the sake of contradiction that there is no monochromatic copy of G. We prove the following claim, similar to Claim 1 in the proof of Theorem 3.3, by induction on p (up to renaming colors), see Figure 3.17 for an illustration.

Claim. *For each p, $1 \leqslant p \leqslant k-2$, and each i, $1 \leqslant i \leqslant p$, there is a red copy G'_i of G' in F_i. Moreover for each i, $0 \leqslant i < p$, each vertex v in G'_i and each j, $i < j \leqslant p$, all edges between v and G'_j are of the same color.*

Proof of Claim. There is a set V_1 of $2^{-m'n}|V(F_1)| \geqslant \epsilon|V(F)|$ vertices in F_1 such that for each vertex in G'_0 all edges to V_1 are of the same color by the Focusing Lemma (Lemma 3.2.3). Since $F \stackrel{\epsilon}{\to} G'$ there is a monochromatic copy G'_1 of G' in $F_1[V_1]$. Assume without loss of generality that G'_1 is red. This proves the Claim for $p = 1$, and for $k = 3$.

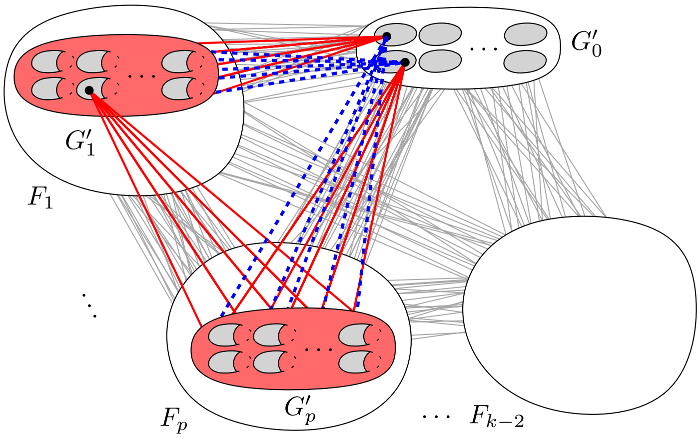


Figure 3.17: Illustrating the statement of the Claim.

Suppose $k-2 \geqslant p \geqslant 2$ and there are red subgraphs $G'_1, \dots, G'_{p-1}$, satisfying the conditions of the Claim. We apply the Focusing Lemma to the complete bipartite graph with one part $V(G'_0) \cup \cdots \cup V(G'_{p-1})$ and the other part $V(F_p)$. There is a set $V_p \subseteq V(F_p)$ of size $2^{-|V(G')|p}|V(F_p)| \geqslant \epsilon|V(F)|$, such that for each vertex v in $G'_0, \dots, G'_{p-1}$ all edges from v to V_p are of the same color. Since $F \overset{\epsilon}{\to} G'$ there is a monochromatic copy G'_p of G' in $F_p[V_p]$. It remains to prove that G'_p is red. Assume G'_p is blue. Consider the vertices of G'_1. All of them send monochromatic stars to G'_p. At most $s-1$ of these stars are blue, as otherwise these stars together with a blue subgraph of G'_p isomorphic to G_S form a blue copy of G. Since the number of vertex disjoint copies of G_S in G'_1 is $m' > s-1$, there is a red copy G^* of G_S in G'_1 whose vertices send only red stars to G'_p. Taking G^* and s vertices from G'_p gives a red copy of G, a contradiction. So we may assume that G'_p is red, which completes the proof of the Claim. $\triangle$

Consider the red G'_i, $1 \leqslant i \leqslant k-2$, given by the Claim for $p = k-2$. We say that a vertex in $V(G'_i)$, $i = 0, \dots, k-3$ is bad for G'_j if it sends a red star to G'_j, for some $j > i$. For each G'_j there are at most $s-1$ bad vertices, since otherwise there is a red copy of G. Hence, there are at most $(k-2)(s-1)$ bad vertices overall. Since G'_0 has $m' = (k-2)(s-1) + m$ vertex disjoint copies of G, there are at least $m \geqslant 1$ copies $G^0_1, \dots, G^0_m$ of G in G'_0 without bad vertices. Since each G'_i, $i = 1, \dots k-2$, has $m' = (k-2)(s-1) + m$ disjoint copies of G_S, there is at least one copy G''_i of G_S in G'_i without bad vertices, $i = 1, \dots, k-2$. Note that all G''_is are red, $i = 1, \dots, k-2$, all edges between them are blue, and all edges between a G''_i and G^0_j are blue, $i = 1, \dots, k-2$, $j = 1, \dots, m$, see Figure 3.18. By assumption each G^0_j, $j = 1, \dots, m$, has a blue edge, since otherwise there is a red copy of G. But then we can find a blue copy of G by identifying these blue edges with the matching of size m induced by the union of two color classes of G, picking the other vertices of these two color classes from G^0_1 and the vertices of the other $k-2$ color classes of G from G''_i, $i = 1, \dots, k-2$. Since $|V(G''_i)| = |V(G)|$, there is sufficient number of

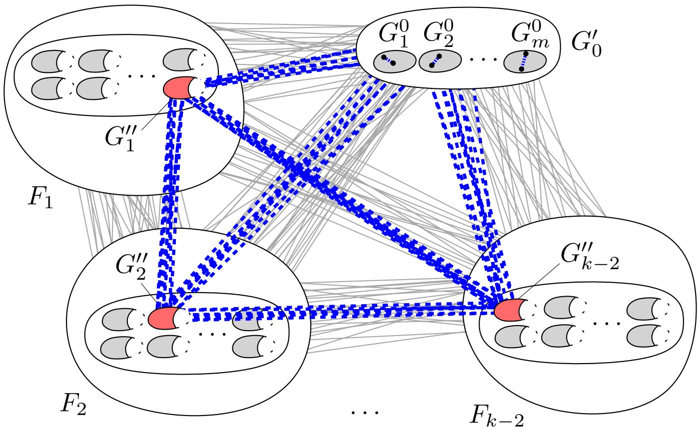


Figure 3.18: One red copy of G_S in each of $F_1, \dots, F_{k-2}$ and m copies $G_1^0, \dots, G_m^0$ of G in G_0' where all edges between distinct copies are blue. If each G_i^0, $i = 1, \dots, m$, has a blue edge we find a blue copy of G in here.

vertices for each color class. Altogether we have a contradiction to our assumption that there are no monochromatic copies of G. Hence $\Gamma \to G$. This concludes the proof in case when $H \not\subseteq G$ and $\chi(H) \geqslant \chi(G)$.

Now, in the second part of the proof, we assume that $H \subseteq G$. Then $\chi(H) \leqslant \chi(G)$. Since we assume that $\omega(G) = \omega(H)$, we have $\omega(H - S) < \omega(H)$. Thus, H is clique-splittable. Assume first that $\chi(H) < \chi(G)$. Then we have $G \overset{R}{\not\sim} H$ by Theorem 3.3, applied with roles of G and H switched. The last case to consider is when $\chi(H) = \chi(G)$ (and $H \subseteq G$). Now any proper $\chi(G)$-vertex-coloring of G with two color classes inducing a subgraph of a matching gives such a coloring of H, too. Thus, the first part of the proof applied with roles of G and H switched shows that $G \overset{R}{\not\sim} H$. □

Proof of Theorem 3.6

Proof of 3.6(a). Since $\chi(G) = \omega(G)$ and in some proper $\chi(G)$-vertex-coloring of G two color classes induce a subgraph of a matching, G satisfies the requirements of Theorem 3.5. Let H be an arbitrary graph. If $\omega(H) \neq \omega(G)$ then $H \overset{R}{\not\sim} G$ by Lemma 3.2.4. So, we can assume that $\omega(H) = \omega(G)$. If $H \subseteq G$ or $\chi(H) \geqslant \chi(G)$, then $G \overset{R}{\not\sim} H$ by Theorem 3.5. If $H \not\subseteq G$ and $\chi(H) < \chi(G)$, then $\omega(H) = \omega(G) = \chi(G) > \chi(H)$. Thus $\chi(H) < \omega(H)$, a contradiction. □

Proof of 3.6(b). To see that a star $S = K_{1,t}$ is not Ramsey equivalent to any other graph, observe that $K_{1,2t-1}$ is a minimal Ramsey graph for S, but $K_{1,2t-1}$ is minimal Ramsey for neither any connected subgraph of S nor any connected graph that is not a subgraph of S.

It remains to show that a path is not Ramsey equivalent to any other connected graph. Let $G = P_m$ be a path on m vertices, and let H be a connected graph not isomorphic to G. If H is a path of different length, then $G \overset{R}{\not\sim} H$ since $r(P_m) =$

H	$r(H)$	H	$r(H)$	H	$r(H)$	H	$r(H)$
K_2	2	P_5	6	C_5	9	$\mathcal{D}_4$	10
P_3	3	$K_{1,4}$	7	$K_{2,3}$	10	B_3	14
K_3	6	P_4+e	6	$\mathcal{T}_4$	9	$H_{4,1}$	18
$K_{1,3}$	6	C_4+e	6	$\mathcal{T}_5$	9	$H_{4,2}$	18
P_4	5	$\mathcal{T}_1$	9	$\mathcal{D}_1$	10	W_4	15
C_4	6	$\mathcal{T}_2$	9	$\mathcal{D}_2$	10	$H_{4,3}$	22
$H_{3,1}$	7	$\mathcal{T}_3$	9	$\mathcal{D}_3$	10	K_5	≥ 43
$H_{3,2}$	10						
K_4	18						

Table 3.1: The connected graphs on at most five vertices with their Ramsey numbers [42, 84]. The different vertex symbols indicate proper colorings for the graphs which satisfy the conditions of Theorem 3.6(a).

$m + \lfloor \frac{m}{2} \rfloor - 1$ [73] and hence $r(G) \neq r(H)$. So assume H is not a path. If H is not a tree, then by Lemma 3.2.9 we have $G \overset{R}{\not\to} H$. Otherwise, H is a tree and $\Delta(H) \geqslant 3$. Then $r_\Delta(H) \geqslant 2\Delta(H) - 1 \geqslant 5$ [85], while an easy argument due to Alon *et al.* [6] shows that $r_\Delta(G) \leqslant 4$. Indeed, for any 4-regular graph F with girth at least $m+1$ we have $F \to P_m$ as follows. Considering any 2-edge-coloring of F, we see that since F has average degree 4 at least one color class has average degree at least 2, i.e., contains a cycle. Since girth$(F) \geqslant m+1$, this monochromatic cycle has length at least $m+1$, and thus contains P_m. □

Proof of 3.6(c). Table 3.1 shows all non-trivial connected graphs on at most five vertices. Recall that $H_{t,d}$ is a graph on $t+1$ vertices such that one vertex has degree d and the other vertices induce a copy of K_t. Let $S = \{C_4, P_4+e, C_4+$

$e, C_5, K_{2,3}, \mathcal{D}_4, W_4\}$. Observe that any connected graph on at most five vertices which is not in S satisfies the conditions of Theorem 3.6(a) or 3.6(b) and thus is Ramsey isolated; Table 3.1 also indicates proper colorings for the graphs which satisfy the conditions of Theorem 3.6(a). It remains to prove that each graph in S is Ramsey isolated. We consider the graphs in S grouped according to their Ramsey number. Let $S_1 = \{C_4, P_4+e, C_4+e\}$, $S_2 = \{C_5\}$, $S_3 = \{K_{2,3}, \mathcal{D}_4\}$, and $S_4 = \{W_4\}$. Here S_1 contains the graphs from S of Ramsey number 6, S_2 the graph of Ramsey number 9, S_3 those of Ramsey number 10, and S_4 the graph of Ramsey number 18, see [42, 84].

First of all we consider $G \in S_1$. Consider a connected graph H which is not isomorphic to G. If $|V(H)| \geqslant 6$, then $r(H) > 6$ and hence $H \overset{R}{\not\sim} G$. Indeed, if H is a star then coloring the edges of a copy of C_6 in K_6 red and all other edges blue does not yield a monochromatic copy of H. If H is not a star, then color a copy of $K_{1,5}$ in K_6 red and all other edges blue. Then the red edges form a star and the blue connected subgraph contains only five vertices, so the coloring has no monochromatic copy of H. So assume that $|V(H)| \leqslant 5$. If $H \notin S_1$ we have $G \overset{R}{\not\sim} H$. Indeed either $H \in S \setminus S_1$ and $r(H) \neq r(G)$, or $H \notin S$ and H is Ramsey isolated by Theorem 3.6(a) or 3.6(b). So it remains to distinguish the graphs in S_1 from each other. We have $H_{5,4} \not\to C_4$ and $H_{5,4} \not\to C_4 + e$ due to the coloring given in Figure 3.19 and $H_{5,2} \to P_4 + e$ by Lemma 3.2.15. In particular $H_{5,4} \to P_4 + e$ and thus $P_4+e \overset{R}{\not\sim} C_4$ and $P_4+e \overset{R}{\not\sim} C_4+e$. Finally $r_\delta(C_4) = 3$ [69] and $r_\delta(C_4+e) = 1$ [67] (for the latter see a remark in the conclusion of [67]). Thus $C_4 \overset{R}{\not\sim} C_4 + e$ and hence G is Ramsey isolated.

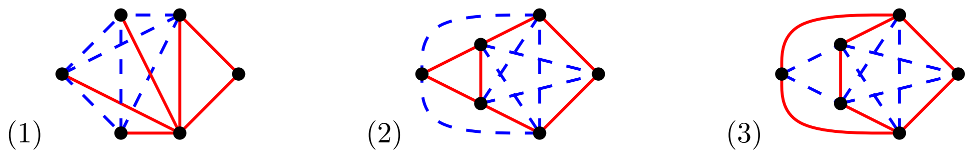


Figure 3.19: A coloring of $H_{5,2}$ without a monochromatic copy of P_5 (1), a coloring of $H_{5,4}$ without a monochromatic copy of C_4 (2), and a coloring of $H_{5,4}$ without a monochromatic copy of K_3 (3).

Next consider $G \in S_2$, i.e., $G = C_5$ and a connected graph H which is not isomorphic to G. If $|V(H)| \leqslant 5$ we have $G \overset{R}{\not\sim} H$, because $H \in S \setminus S_2$ and hence $r(H) \neq r(G)$ or $H \notin S$ (and H is Ramsey isolated by Theorem 3.6(a) or 3.6(b)). If H is bipartite then $G \overset{R}{\not\sim} H$ by Observation 3.4. If $|V(H)| \geqslant 6$ and H is not bipartite, then $r(H) > 10$. Indeed color the edges of K_{10} with two vertex disjoint red copies of K_5 and all other edges blue. Then each connected component of the red subgraph has five vertices and the blue subgraph is bipartite. In particular there is no monochromatic copy of H. We conclude that $G \overset{R}{\not\sim} H$, so G is Ramsey isolated.

Next consider $G \in S_3$ and a connected graph H which is not isomorphic to G. Since $K_{2,3}$ is bipartite but $\mathcal{D}_4$ is not, the two graphs in S_3 are not Ramsey equivalent by Observation 3.4. If $|V(H)| \leqslant 5$ then $G \overset{R}{\not\sim} H$, because either $H \in S_3 \setminus \{G\}$, or $r(H) \neq r(G)$, or $H \notin S$. So assume $|V(H)| \geqslant 6$. Then $K_{2,3} \overset{R}{\not\sim} H$ by Lemma 3.2.14. If H is bipartite then $\mathcal{D}_4 \overset{R}{\not\sim} H$ by Observation 3.4. If H is not bipartite then $\mathcal{D}_4 \overset{R}{\not\sim} H$, since $r(H) > 10 = r(\mathcal{D}_4)$ as argued above (when considering S_2). Altogether G is Ramsey isolated.

Finally consider $G \in S_4$, i.e., $G = W_4$, and a connected graph H which is not isomorphic to G. If $|V(H)| \leqslant 5$ we have $G \overset{R}{\not\sim} H$, because $H \notin S_4$ and hence $r(H) \neq r(G)$ or $H \notin S$. If $|V(H)| \geqslant 6$ then $W_4 \overset{R}{\not\sim} H$ by Lemma 3.2.13. Hence G is Ramsey isolated. □

Proof of Theorem 3.7 (Trees)

(a) Assume first that Conjecture 3.1 is true. Let T_k and T_ℓ be trees on k and ℓ vertices respectively, $k < \ell$. Note that $\mathrm{ex}(n, T_\ell) \geqslant \frac{\ell-2}{2}n - \ell^2$. Indeed, just take $\lfloor \frac{n}{\ell-1} \rfloor$ vertex disjoint copies of $K_{\ell-1}$. Then

$$\mathrm{ex}(n, T_k) \leqslant \tfrac{k-1-\epsilon}{2}n = \sqrt{n}\left(\tfrac{k-1-\epsilon}{2}\sqrt{n}\right) < \sqrt{n}\left(\tfrac{\ell-2}{2}\sqrt{n} - \ell^2\right) \leqslant \sqrt{n}\,\mathrm{ex}(\sqrt{n}, T_\ell),$$

for sufficiently large n. Thus $\mathrm{ex}(n, T_k) < \mathrm{ex}(\sqrt{n}, T_\ell)\sqrt{n}$ and Lemma 3.2.9 implies that $T_k \overset{R}{\not\sim} T_\ell$.

(b) Now, we shall prove the second statement of Theorem 3.7 without assuming the validity of Conjecture 3.1. Let T_k be a balanced tree on k vertices and T_ℓ be any tree on $\ell \geqslant k+1$ vertices. Let G be a k-regular graph of girth at least k, which is known to exist [131]. We construct a bipartite k-regular graph B of girth at least k from G by taking for each v in G two vertices v_1, v_2 in B and for every edge uv in G the edges u_1v_2 and u_2v_1 in B. Finally, let $F = L(B)$ be the line graph of B. We shall show that $F \not\to T_\ell$ and $F \to T_k$.

As B is bipartite, $F = L(B)$ is an edge disjoint union of two graphs F_1 and F_2 on the same vertex set, each is a vertex disjoint union of copies of K_k, where each clique in F_i corresponds to a set of edges incident to a vertex in the ith partite set of B, $i = 1, 2$. Note that a clique in F_1 intersects a clique in F_2 by at most one vertex and that each vertex in F belongs to two cliques, one from F_1 and one from F_2.

Coloring F_1 red and F_2 blue gives no monochromatic copy of T_ℓ since each monochromatic connected component has $k < \ell$ vertices. Thus $F \not\to T_\ell$.

Next, we show that $F \to T_k$. Let vw be an edge of T_k such that the components of $T_k - vw$ rooted at v and w have order at most $\lceil \frac{k+1}{2} \rceil$. Consider any edge-coloring of F with colors red and blue. Note that $|V(F)| = |E(B)| = \frac{k}{2}|V(B)|$ and $|E(F)| = \binom{k}{2}|V(B)| = (k-1)|V(F)|$. Hence there are at least $\frac{k-1}{2}|V(F)|$ red edges or at least $\frac{k-1}{2}|V(F)|$ blue edges. (Note that Conjecture 3.1, if true, would imply that there is a red or blue copy of T_k, independent of the girth of B and whether T_k is balanced.) Assume without loss of generality that there are at least $\frac{k-1}{2}|V(F)|$ red edges. Consider the red subgraph G_r of F. If each subgraph of G_r contains a vertex of degree less than $\frac{k-1}{2}$, then G_r contains less than $\frac{k-1}{2}|V(G_r)| = \frac{k-1}{2}|V(F)|$ edges, a contradiction. It follows that there is a subgraph G of G_r with $\delta(G) \geqslant \lceil\frac{k-1}{2}\rceil$ and $|E(G)| \geqslant \frac{k-1}{2}|V(G)|$, and so $\Delta(G) \geqslant k-1$. If $\Delta(G) = k-1$, then G is $(k-1)$-regular and we can embed T_k into G greedily. So without loss of generality we have $\Delta(G) \geqslant k$.

Let x be a vertex of maximum degree in G, i.e., $\deg_G(x) \geqslant k$. It follows that x has incident red edges in both corresponding maximum cliques C_1, C_2 in F. Without loss of generality x has at least $\lceil\frac{k-1}{2}\rceil$ incident red edges in C_1. We embed v onto x, w onto a neighbor of x in G_r in C_2 and all neighbors of v different from w onto neighbors of x in G_r in C_1. Now we can greedily embed the subtrees $T_1, \ldots, T_a$ of $T_k - v$ with their roots at the designated vertices in G_r. Say T_1 is the subtree rooted at w. As $\delta(G) \geqslant \lceil\frac{k-1}{2}\rceil \geqslant |V(T_1)| - 1 = \sum_{i=2}^{a}|V(T_i)|$ and B has girth greater than k, the embeddings of T_1 and $\bigcup_{i=2}^{a} T_i$ are in disjoint sets of cliques. It follows that $F \to T_k$. □

Proof of Theorem 3.8 (Multicolor Ramsey numbers)

(a) We prove the first part of the theorem. Let $m = r(G, G, F)$. Consider a 3-edge-coloring c of K_m without a red or blue copy of H and without a green copy of F, which exists as $m < r(H, H, F)$. Let Γ denote the graph obtained from K_m by removing all green edges under c. Thus $\Gamma \not\to H$ due to the coloring c restricted to Γ. But $\Gamma \to G$, since any 2-edge-coloring of Γ without a monochromatic copy of G can be extended by the green edges of c to an edge-coloring of K_m without a red or blue copy of G and without a green copy of F.

(b) We prove the second statement by induction on k with $k = 2$ being obvious.

Let Γ be a graph such that $\Gamma \xrightarrow{k} G$, but $\Gamma \stackrel{k}{\not\to} H$, $k \geqslant 3$. Let c be a k-edge-coloring of Γ with no monochromatic copy of H. Let a graph Γ' be obtained from Γ by deleting the edges of color 1. We have that $\Gamma' \stackrel{k-1}{\not\to} H$ since c restricted to Γ' is a $(k-1)$-coloring with no monochromatic copy of H. We claim that $\Gamma' \xrightarrow{k-1} G$, which, if true, gives $G \stackrel{R}{\not\sim}_{k-1} H$ and by induction $G \stackrel{R}{\not\sim} H$, as desired.

Let us assume for the sake of contradiction that $\Gamma' \stackrel{k-1}{\not\to} G$, i.e., there is a $(k-1)$-edge-coloring c' of Γ' without a monochromatic copy of G. We see that there is a copy of G in color 1 of c, otherwise the coloring c'' of Γ that is the same as c' on Γ' and that colors all other edges with color 1 has no monochromatic copy of G, a contradiction to the fact that $\Gamma \xrightarrow{k} G$. Repeating the argument above to all colors in c, we see that each of them contains G. More generally, we see that any edge-coloring of Γ with k colors avoiding a monochromatic copy of H must have a monochromatic copy of G in each color. However, since $G \subseteq H$, Γ' has no monochromatic copy of H under c', and hence the coloring c'' of Γ has no monochromatic copy of H. Thus c'' must have a monochromatic copy of G in each color, however there is no monochromatic copy of G in any of the colors $2, \ldots, k$, a contradiction. □

Proof of Theorem 3.9 (Ramsey numbers cycles plus edges)

(a) First we consider even cycles C_{2n} for $n \geqslant 3$. Then $r(C_{2n}) = 3n - 1$ [90]. Let $N = 3n - 1$. Color K_N with a red copy of K_{2n} and all other edges blue. Clearly there is no red copy of G because G has at least $2n + 1$ vertices and is connected. The blue subgraph consists of a copy of $K_{2n,n-1}$ where all edges within the part on $n - 1$ vertices are blue as well. Hence the blue subgraph does not contain a copy of C_{2n}. Therefore $r(G) > N = r(C_{2n})$.

Next we consider odd cycles C_{2n+1} for $n \geqslant 2$. Then $r(C_{2n+1}) = 4n + 1$ [90]. Let $N = 4n + 1$. Color K_N with a red copy of a vertex disjoint union K_{2n+1} and K_{2n} and all other edges blue. Then the red subgraph does not contain a copy of G because G is connected and has $2n + 2$ vertices. The blue subgraph is bipartite and hence does not contain a copy of G because G contains an odd cycle. Hence $r(G) > N = r(C_{2n+1})$.

(b) First we consider C_4. Color K_6 with a red copy of K_4 and all other edges blue. It easy to see that there is no red copy of $C_4 + K_2$. Hence $r(C_4 + K_2) > 6 = r(C_4)$.

Now consider even cycles C_{2n} for $n \geqslant 3$. Then $r(C_{2n}) = 3n - 1$ [90]. Let $N = 3n - 1$. Color K_N with a red copy of K_{2n} and all other edges blue. Clearly there is no red copy of $C_{2n} + K_2$. The blue subgraph consists of a copy of $K_{2n,n-1}$ where all edges within the part on $n - 1$ vertices are blue as well. Hence the blue subgraph does not contain a copy of C_{2n}. Therefore $r(C_{2n} + K_2) > N = r(C_{2n})$.

Next suppose that $G = C_{2n+1} + kK_2$ for some $k > \frac{n}{3}$. Then $r(C_{2n+1}) = 4n+1$ [90]. Let $N = 4n + 1$, $a = 2n + 1 + \lfloor \frac{2}{3}n \rfloor$, and $b = 4n + 1 - a = 2n - \lfloor \frac{2}{3}n \rfloor = n + \lceil \frac{n}{3} \rceil$. Color K_N with a red copy of K_a and all other edges blue. Then the red subgraph does not contain a copy of G because G has $2n + 1 + 2k > a$ vertices and no isolated vertex. The blue subgraph consists of a copy of $K_{a,b}$ where all edges within the part on b vertices are blue as well. Let B denote the part with b vertices. Each copy of C_{2n+1} in the blue subgraph has at least $n + 1$ vertices in B. Hence there are at most $\lceil \frac{n}{3} \rceil - 1 < k$ independent blue edges independent from any blue copy of C_{2n+1}, since each such edge has at least one vertex in B. Thus there is no blue copy of $G = C_{2n+1} + kK_2$. Therefore $r(G) > N = r(C_{2n+1})$.

Finally suppose that $G = C_{2n+1} + kK_2$ for some k, $0 \leqslant k \leqslant \frac{n}{3}$. In this case we shall show that $r(G) = r(C_{2n+1}) = N$. Consider a 2-edge-coloring of K_N. We will prove that there is a monochromatic copy of $G = C_{2n+1} + kK_2$ by induction on k.

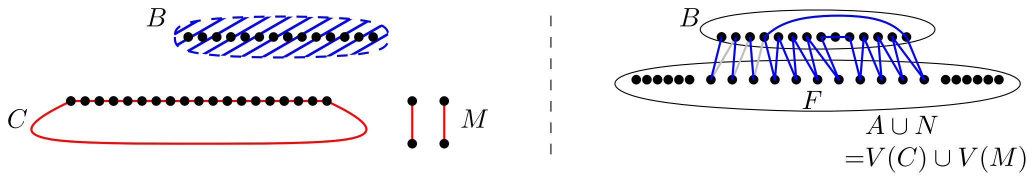


Figure 3.20: A red copy of $C_{19} + 2K_2$ in a coloring of K_{37} without red $C_{19} + 3K_2$. All edges induced by the set B are blue (left). On the right side a feasible set P of 11 blue peaks exists and thus there is a blue copy of $C_{19} + 3K_2$ with some edges within B as indicated.

By assumption there is a monochromatic copy of $C_{2n+1} = C_{2n+1} + 0K_2$. This gives a basis for $k = 0$.

Suppose that $k \geqslant 1$. Inductively there is a monochromatic, say red, copy of $C_{2n+1} + (k-1)K_2$. Let C denote the red copy of C_{2n+1} and let M denote the set of $k-1$ independent red edges, independent from C, in some red copy of $C_{2n+1} + (k-1)K_2$. Further let $A = V(C)$, $N = V(M)$, and let B denote all vertices not in A or N. Then $|B| = 2n - 2(k-1)$. For the remaining proof we assume that there is no red copy of G. We shall prove that there is a blue copy of G. Clearly all edges within B are blue. See Figure 3.20 (left) for an illustration in case $n = 9$, $k = 3$. We call a blue copy of P_3 (a path on 3 vertices) a *blue peak*, if its middle vertex is in $A \cup N$ and the other two vertices are in B. Similarly we call a red copy of P_3 a *red peak*, if its middle vertex is in B and the other two vertices are in A. We call a set of blue peaks *feasible* if their union is a path forest.

Assume first that there is a set of $4k-1$ feasible blue peaks. We claim that there is a blue copy of G. Let F denote the path forest formed by the union of these $4k-1$ feasible blue peaks. Observe that if e is an edge in a union of feasible peaks and e is incident to a leaf (a vertex of degree 1) in the union of the peaks, then the endpoints of e are contained in exactly one of the peaks. Hence, we can choose a set M' of k independent edges from F, such that there is a set P' of $3k-1$ feasible peaks left, each peak in P' not incident to any edge in M' (iteratively choose an edge e incident to a leaf in F and remove the peak containing e). Let F' denote the union of peaks in P'. Then the number of vertices in the union of F' and B which are not incident to edges of M' is $|V(F') \setminus B| + |B| - |M'| = |P'| + |B| - |M'| = 2n + 1$. Recall that each edge within B is blue and that $|P'| = 3k-1 \leqslant n-1$. Hence F' and the vertices in B not incident to edges in M' form a blue copy of C_{2n+1} (connect the paths in F' and possibly some isolated vertices from $B \setminus V(M')$ to a cycle with some edges within B). Hence there is a blue copy of $G = C_{2n+1} + kK_2$. See Figure 3.20 (right) for an illustration in case $n = 9$, $k = 3$.

Now let P denote a maximal feasible set of blue peaks. We shall show that $|P| \geqslant 4k-1$. To this end we use the following facts which can be verified by a

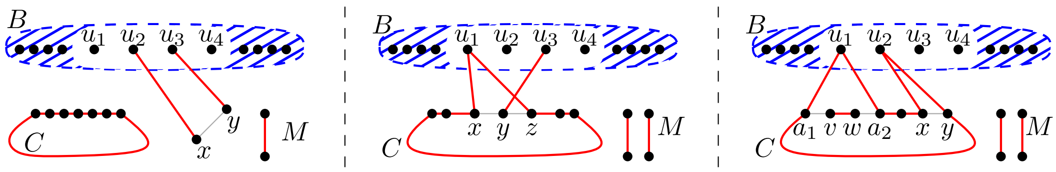


Figure 3.21: Two independent red edges between some edge xy in M and B (left), three consecutive vertices along C_7 with a red peak and an independent red edge (middle), and two disjoint red peaks, one of which connecting two vertices of distance exactly 3 on C_7 (right). In each case there is a red copy of $C_7 + 3K_2$.

simple case analysis using the fact that the absence of a blue peak forces two red incident edges at the vertices in $A \cup N$.

Claim 1. *In any copy of $K_{2,3}$ between 2 vertices from $A \cup N$ and 3 vertices from B there are 2 independent red edges or there is a blue peak.*

Claim 2. *In any copy of $K_{3,3}$ between some subset $X \subseteq A$ and three vertices from B either there is a blue peak, or for each pair of vertices in X there is a red peak containing this pair of vertices and a red edge independent from this peak.*

Claim 3. *There is a red or a blue peak in any copy of $K_{2,3}$ between 2 vertices from A and 3 vertices from B.*

We proceed with the proof of the theorem. Consider the graph Γ formed by the union of all peaks from P and all vertices from B. Then each connected component of Γ is a path with an odd number of vertices or an isolated vertex. Let $\mathcal{C}$ denote the set of connected components of Γ and let $t = |\mathcal{C}|$. Then $|B| = \sum_{K \in \mathcal{C}} \lceil |V(K)|/2 \rceil = t + \sum_{K \in \mathcal{C}} \lfloor |V(K)|/2 \rfloor = t + |P|$ and thus $t = 2n - 2k + 2 - |P|$. Thus $|P| \geqslant 4k - 1$ and we are done or $t \geqslant 2n - 2k + 2 - (4k - 2) \geqslant 6k - 6k + 4 = 4$. Assume the latter case holds, that is, $t \geqslant 4$. Let $u_1, u_2, u_3, u_4 \in B$ denote distinct vertices from different connected components of Γ each of degree at most 1 in Γ. Then there is no blue peak containing two of these vertices and a vertex from $A \cup N$ not contained in any peak in P, since adding this peak to P would yield a larger set of feasible blue peaks. We shall use this fact to prove that $|P| \geqslant 4k - 1$ again. More precisely we shall find $k - 1$ peaks in P with a vertex from N and $3k$ peaks in P with a vertex from A.

Consider a (red) edge xy in M and the copy K of $K_{2,3}$ with parts $\{x, y\}$ and $\{u_1, u_2, u_3\}$. Then there are no two independent red edges in K, since there is a red copy of $C_{2n+1} + kK_2$ otherwise (replace xy with the two independent red edges). See Figure 3.21 (left). Hence K contains a blue peak by Claim 1. Thus at least one of x and y is already contained in a peak from P as argued above (since P is maximal). This shows that there are at least $k - 1$ peaks in P containing vertices from N (but not from A). It remains to find $3k$ peaks in P with vertices from A.

Let A' denote the vertices from A not contained in any peak in P. We shall show that $|A'| \leqslant n+1$ and hence that there are at least $n \geqslant 3k$ vertices from A contained in peaks from P and we are done. To this end we shall prove that no three vertices in A' are consecutive along C (that is, do not form a path in C) and that either at most two pairs of vertices from A' are adjacent on C or there are no vertices from A' at distance exactly 3 along C.

For the sake of contradiction assume that there is a set $\{x, y, z\} \subseteq A'$ of three distinct vertices which are consecutive along C (in some order). Without loss of generality assume that y is the middle vertex among $\{x, y, z\}$ along C. Consider the copy K of $K_{3,3}$ with parts $\{x, y, z\}$ and $\{u_1, u_2, u_3\}$. Since there is no blue peak in K (as P is maximal) there is a red peak Q in K containing x and z and there is a red edge e incident to y independent from Q by Claim 2. Replacing y in C by the vertex in Q not in A yields a red copy of C_{2n+1}. See Figure 3.21 (middle). Moreover e is independent from this cycle and from the $k-1$ red edges in M. Hence there is a red copy of $G = C_{2n+1} + kK_2$, a contradiction. This shows that no three vertices from A' are consecutive along C.

We call two vertices from A' *friends* if they are neighbors in C. If there are at most two pairs of friends in A', then $|A'| \leqslant 2 + \lfloor (2n-1)/2 \rfloor = n+1$ and we are done. So suppose that there are three distinct pairs of friends in A'. For the sake of contradiction assume that there are a_1, $a_2 \in A'$ of distance exactly 3 on C and let v and w denote the vertices between a_1 and a_2 in C. Then there is a pair of friends $\{x, y\}$ which is disjoint from $\{a_1, a_2, v, w\}$, since no three vertices from A' are consecutive along C. Consider the copy of $K_{2,3}$ between $\{a_1, a_2\}$ and $\{u_1, u_2, u_3\}$. By Claim 3 there is a red peak containing a_1 and a_2 in this copy. Without loss of generality assume that this peak contains u_1. Then consider the copy of $K_{2,3}$ between $\{x, y\}$ and $\{u_2, u_3, u_4\}$. Again by Claim 3 there is another (vertex disjoint) red peak containing x and y. See Figure 3.21 (right). Without loss of generality assume that this peak contains u_2. Replacing v and w in C with u_1 and u_2 yields a red C_{2n+1}. Moreover vw is a red edge disjoint from this cycle and from the $k-1$ red edges in M. Hence there is a red copy of $G = C_{2n+1} + kK_2$, a contradiction. This shows that no two vertices from A' are of distance 3 on C.

Consider four consecutive vertices on C_{2n+1}. Then at most two of these vertices are contained in A' (since no three from A' are consecutive and no two from A' at distance exactly 3). Hence $|A'| \leqslant n+1$. Thus in both cases there are at least $n \geqslant 3k$ vertices from A contained in peaks from P. Therefore $|P| \geqslant 4k-1$ and there is a blue copy of G as argued above. Altogether $r(G) = r(C_{2n+1})$. $\square$

3.4 Small Distinguishing Graphs

For a pair of Ramsey non-equivalent graphs G and H let $r_{\mathrm{dn}}(G,H)$ denote the *Ramsey distinguishing number* defined as the smallest number of vertices among all graphs distinguishing G and H. Clearly $r_{\mathrm{dn}}(G,H) \geqslant \min(r(G), r(H))$. Recall that $H_{t,d}$ is a graph on $t+1$ vertices such that one vertex has degree d and the other vertices induce a copy of K_t. We shall refer to the non-trivial connected graphs on at most five vertices by the names given in Table 3.1. Let

$$\begin{aligned}\mathcal{A} = \{&\{C_4, C_4+e\}, \{C_5, \mathcal{T}_4\}, \{\mathcal{D}_1, \mathcal{D}_2\}, \{\mathcal{D}_3, \mathcal{D}_4\},\\ &\{H_{3,2}, \mathcal{D}_1\}, \{H_{3,2}, \mathcal{D}_2\}, \{K_4, H_{4,1}\}, \{K_4, H_{4,2}\}, \{H_{4,1}, H_{4,2}\}\},\\ \mathcal{B} = \{&\{K_3, C_4\}, \{K_3, C_4+e\}, \{K_{2,3}, H_{3,2}\},\\ &\{K_{2,3}, \mathcal{D}_1\}, \{K_{2,3}, \mathcal{D}_2\}, \{K_{2,3}, \mathcal{D}_3\}, \{K_{2,3}, \mathcal{D}_4\}\}.\end{aligned}$$

We shall show that all of the $\binom{31}{2} = 465$ pairs of non-isomorphic connected graphs on at most five vertices that are not in $\mathcal{A}$ are distinguished by a "small" graph. For all 449 such pairs $\{G,H\}$ that are not in $\mathcal{A} \cup \mathcal{B}$ we give a distinguishing graph on $\min\{r(G), r(H)\}$ vertices, which is best-possible. On the other hand there are pairs $\{G,H\}$ with $r_{\mathrm{dn}}(G,H) > \min\{r(G), r(H)\}$. We do not have small distinguishing graphs for pairs in $\mathcal{A}$.

Proposition 3.4.1. *For each pair $\{G,H\}$ of non-isomorphic connected graphs on at most five vertices that is not in $\mathcal{A} \cup \mathcal{B}$ we have*

$$r_{\mathrm{dn}}(G,H) = \min(r(G), r(H)).$$

Moreover $10 \leqslant r_{\mathrm{dn}}(G,H) \leqslant 16$ *for* $\{G,H\} \in \mathcal{B} \setminus \{\{K_3, C_4\}, \{K_3, C_4+e\}\}$*, and* $r_{\mathrm{dn}}(K_3, C_4)$*,* $r_{\mathrm{dn}}(K_3, C_4+e)$*,* $r_{\mathrm{dn}}(C_4, C_4+e) > \min\{r(G), r(H)\}$*, and* $r_{\mathrm{dn}}(K_3, C_4)$*,* $r_{\mathrm{dn}}(K_3, C_4+e) \leqslant 10$.

Proof. First of all note that two graphs G, H of different Ramsey number are distinguished by K_n where $n = \min\{r(G), r(H)\}$. It remains to consider pairs of connected graphs on at most five vertices of the same Ramsey number. Hence, we need to consider the following sets of graphs corresponding to Ramsey number 6, 7, 9, 10, and 18 respectively.

Ramsey number 6: $\{K_3, K_{1,3}, C_4, P_5, P_4+e, C_4+e\}$. We have $K_{1,5} \to K_{1,3}$ (pigeonhole principle) but $K_{1,5} \not\to K_3, C_4, P_5, P_4+e, C_4+e$ ($K_{1,5}$ does not contain these), $H_{5,2} \to P_4+e$ (Lemma 3.2.15) but $H_{5,2} \not\to K_3, C_4, P_5, C_4+e$ (Figure 3.19), $H_{5,4} \to P_5$ (Lemma 3.2.17) but $H_{5,4} \not\to K_3, C_4, C_4+e$ (Figure 3.19(2),(3)), $K_{5,5} \to C_4, C_4+e$ (Lemma 3.2.18) but $K_{5,5} \not\to K_3$ (since K_3 not bipartite). Note that $\{C_4, C_4+e\} \in \mathcal{A}$ and $\{K_3, C_4\}$, $\{K_3, C_4+e\} \in \mathcal{B}$. Note further that $H_{5,4} \not\to K_3, C_4, C_4+e$ (Figure 3.19(2),(3)) and $K_6 \to K_3, C_4, C_4+e$ implies that there

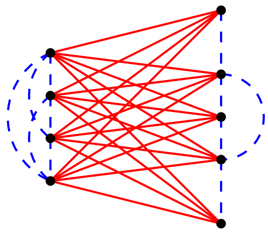


Figure 3.22: A coloring of $\mathcal{K}'$ without a monochromatic copy of $\mathcal{T}_3$.

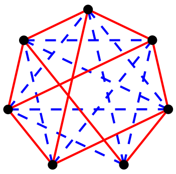


Figure 3.23: A coloring of $H_{6,5}$ without a monochromatic copy of $K_{1,4}$.

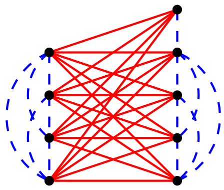


Figure 3.24: A coloring of $H_{8,5}$ without monochromatic copies of $\mathcal{T}_4$ and C_5.

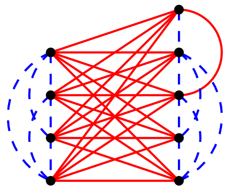


Figure 3.25: A coloring of $H_{8,6}$ without a monochromatic copy of $\mathcal{T}_5$.

is no graph on at most 6 vertices that distinguishes any pair from $\{K_3, C_4, C_4 + e\}$. Therefore $r_{\text{dn}}(K_3, C_4)$, $r_{\text{dn}}(K_3, C_4 + e)$, $r_{\text{dn}}(C_4, C_4 + e) \geqslant 7$.

Ramsey number 7: $\{H_{3,1}, K_{1,4}\}$. We have $H_{6,5} \rightarrow H_{3,1}$ (Lemma 3.2.19) but $H_{6,5} \not\rightarrow H_{3,1}$ (Figure 3.23).

Ramsey number 9: $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, C_5, \mathcal{T}_4, \mathcal{T}_5\}$. Consider the graphs $\mathcal{K}$ and $\mathcal{K}'$ given in Figure 3.8. The graph $\mathcal{K}'$ is obtained from K_7 by adding two independent vertices of degree 5 such that these two vertices have exactly 4 common neighbors. We have $\mathcal{K} \rightarrow \mathcal{T}_1, \mathcal{T}_3$ (Lemma 3.2.21, 3.2.23) but $\mathcal{K} \not\rightarrow \mathcal{T}_2, \mathcal{T}_5$ (by Lemma 3.2.16, since $\Delta(\mathcal{K}) = 7$), $\mathcal{K}' \rightarrow \mathcal{T}_1$ (Lemma 3.2.27) but $\mathcal{K}' \not\rightarrow \mathcal{T}_3$ (Figure 3.22), $H_{8,5} \rightarrow \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ (Lemma 3.2.25) but $H_{8,5} \not\rightarrow \mathcal{T}_4, \mathcal{T}_5, C_5$ (Figures 3.24, 3.25), $H_{8,6} \rightarrow C_5, \mathcal{T}_4$ (Lemma 3.2.29) but $H_{8,6} \not\rightarrow \mathcal{T}_5$ (Figure 3.25). Note that $\{C_5, \mathcal{T}_4\} \in \mathcal{A}$.

Ramsey number 10: $\{H_{3,2}, K_{2,3}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$. We have $K_{3,13} \rightarrow K_{2,3}$ ([69]) but the other graphs are not bipartite, $H_{9,6} \rightarrow H_{3,2}, \mathcal{D}_1, \mathcal{D}_2$ (Lemma 3.2.31) but $H_{9,6} \not\rightarrow \mathcal{D}_3, \mathcal{D}_4$ (Figures 3.26, 3.27). Note that each pair within the set $\{H_{3,2}, \mathcal{D}_1, \mathcal{D}_2\}$ is in $\mathcal{A}$ and $\{\mathcal{D}_3, \mathcal{D}_4\} \in \mathcal{A}$.

Ramsey number 18: $\{K_4, H_{4,1}, H_{4,2}\}$. All pairs of these graphs are in $\mathcal{A}$. □

3.5 Conclusions

This chapter addresses Ramsey equivalence of graphs and gives a negative answer to the question of Fox *et al.* [67] “Are there two connected non-isomorphic graphs that are Ramsey equivalent?” (Question 1.2) for wide families of graphs determined

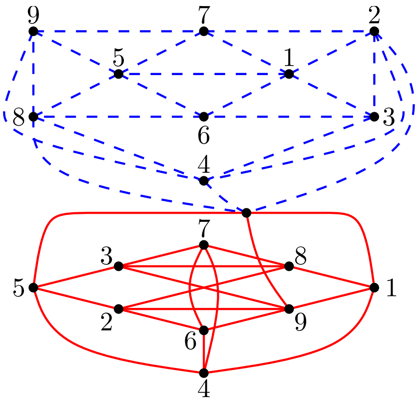


Figure 3.26: An edge-coloring of $H_{9,6}$ without a monochromatic copy of $\mathcal{D}_3$ is obtained by identifying vertices of the same label.

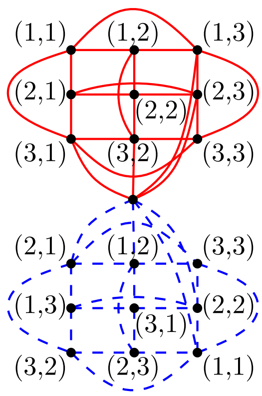


Figure 3.27: An edge-coloring of $H_{9,8}$ without a monochromatic copy of $\mathcal{D}_4$ is obtained by identifying vertices of the same label.

by so-called "clique splitting" properties and chromatic number. In particular, Theorem 3.6 gives an infinite family of graphs that are not Ramsey equivalent to any other connected graphs. This extends the only such known family consisting of all cliques, paths, and stars.

In Theorems 3.3 and 3.5 we can replace the clique number with the negative of the odd girth and mimic the proofs to obtain similar statements. This might also hold for other "sufficiently nice" parameters.

There are many questions that remain open in this area. Even the following weaker question is very far from being understood.

Question 3.2. *Which parameters ρ are Ramsey distinguishing, that is, for which ρ does $\rho(G) \neq \rho(H)$ imply that $G \overset{R}{\not\sim} H$?*

Here, we show that the chromatic number is very likely to be such a distinguishing parameter by proving this implication for graphs satisfying some additional properties. Interestingly enough, it is not clear but most likely not true that $\chi(G) \neq \chi(H)$ implies that $r_\chi(G) \neq r_\chi(H)$. Indeed, $r_\chi(K_4) = 18$, since $r(K_4) = 18$ [76] and any graph F with $\chi(F) \leqslant 17$ can be colored without a monochromatic copy of K_4 based on a coloring of K_{17} without a monochromatic copy of K_4 (see Figure 1.7). On the other hand, the positive answer to the Burr-Erdős-Lovász-Conjecture [146] shows that there is a 5-chromatic graph G with $r_\chi(G) = 4^2 + 1 = 17$. So $\chi(K_4) < \chi(G)$ but $r_\chi(K_4) > r_\chi(G)$. We believe that the following question has a positive answer.

Question 3.3. *Are there infinitely many pairs of graphs G, H with $\chi(G) \neq \chi(H)$ and $r_\chi(G) = r_\chi(H)$?*

For any set of pairwise Ramsey equivalent graphs known yet, there is a unique minimal element. For example any graph that is Ramsey equivalent to K_t contains a copy of K_t [116]. However, this observation might be just due to the fact that cliques are studied most. We pose the following question.

Question 3.4. *Is there a pair G, H of Ramsey equivalent graphs such that for each common subgraph F of G and H ($F \subseteq G$ and $F \subset H$) we have $F \overset{R}{\not\sim} G$ and $F \overset{R}{\not\sim} H$?*

We also address relations between other types of Ramsey numbers and Ramsey equivalence. For instance Theorem 3.8 gives results in terms of multicolor Ramsey numbers. The following questions are open.

Question 3.5. *Let F, G, and H be graphs with $r(G, F) \neq r(H, F)$. Does this imply $G \overset{R}{\not\sim} H$?*

Question 3.6. *If $G \overset{R}{\not\sim}_k H$ for some $k \geqslant 2$, then $G \overset{R}{\not\sim} H$?*

We have a positive answer to the last question only when G is a subgraph of H (Theorem 3.8). Positive answers to Question 3.6 and the following question would imply a negative answer to our main question of Fox *et al.* (Question 1.2).

Question 3.7. *Let G and H denote non-isomorphic connected graphs. Is there an integer k such that $r_k(G) \neq r_k(H)$?*

For example, we see that $r(P_4 + e) = r(K_{1,3}) = 6$, and $r_k(P_4 + e) > 2k + 2 = r_k(K_{1,3})$, for odd $k \geqslant 3$ with $\frac{k+1}{2}$ even (see Lemma 3.2.32 and [34]). A particular interesting case to consider are cliques with a pendant edge (that is, graphs $H_{t,1}$). It is known that $r(K_t) = r(H_{t,1})$ for each $t \geqslant 4$ [30]. We think that this also holds for more colors (an opinion shared by Irving [87]). Interestingly $17 = r_3(K_3) = r_3(H_{3,1})$ [144] and $r_4(K_3) < r_4(H_{3,1})$ if and only if $r_4(K_3) = 51$ (where it is known that $51 \leqslant r_4(K_3) \leqslant 62$) [136].

Cliques play a special role in Ramsey theory and received particular attention in Ramsey equivalence. Still, it is not clear for what graphs is a clique a minimal Ramsey graph. Specifically we have the following question.

Question 3.8. *Does $r_e(H) < \binom{r(H)}{2}$ imply that $K_{r(H)}$ is not a minimal Ramsey graph for H?*

It is also not clear how small a distinguishing graph for two graphs that are not Ramsey equivalent could be. Clearly, any graph distinguishing graphs G and H has at least $\min\{r(G), r(H)\}$ vertices. We see that K_3, C_4 and $C_4 + e$ are not distinguished by any graph on $r(K_3) = r(C_4) = r(C_4 + e) = 6$ vertices (Proposition 3.4.1). As noted above it is known that for each $t \geqslant 4$ we have $r(K_t) = r(H_{t,1}) = r$ [30] but $K_r - e \not\to K_t, H_{t,1}$. So we see that K_t and $H_{t,1}$ are not distinguished by any graph on r vertices for $t \geqslant 4$. This shows that for infinitely many pairs of graphs G, H there is no distinguishing graph on $\min\{r(G), r(H)\}$ vertices. Recall that $r_{\text{dn}}(G, H)$ is the smallest order of a graph distinguishing G and H.

Question 3.9. *Is there a constant c such that for each pair of graphs G and H with $r(G) = r(H) = k$ and $G \overset{R}{\not\sim} H$ we have $r_{\mathrm{dn}}(G, H) \leqslant c\,k$?*

Finally, Theorem 3.7 shows that two trees of different order are not Ramsey equivalent provided that the Erdős-Sós-Conjecture is true or if one of the trees is balanced. We do not know whether there are two Ramsey equivalent non-isomorphic trees on the same number of vertices.

Bloom and Liebenau [16] describe the following weaker concept than Ramsey equivalence (contributed to Szabó). A pair of graphs is *Ramsey close* if their respective sets of minimal Ramsey graphs differ in at most finitely many members. Clearly any pair of Ramsey finite graphs is Ramsey close. Similar to Question 1.2 they ask whether there are two connected, Ramsey infinite, non-isomorphic graphs that are Ramsey close. The only graphs known to be Ramsey close but not Ramsey equivalent are K_3 and $K_3 + K_2$. Bloom and Liebenau conjecture that there are infinitely many such pairs of graphs. A particular interesting case to consider is given by K_t and $K_t + K_t$ [16].

4

Minimal Ordered Ramsey Graphs

4.1 Introduction and Main Results

In this chapter we initiate the study of minimal ordered Ramsey graphs. Observation 1.4 in Section 1.5 shows that for every minimal ordered Ramsey graph F of some ordered graph H the underlying (unordered) graph $\widetilde{F}$ of F is a Ramsey graph of the underlying graph $\widetilde{H}$ of H. As it turns out, $\widetilde{F}$ is not necessarily a minimal Ramsey graph of $\widetilde{H}$. Also the other way round not every Ramsey graph of $\widetilde{H}$ admits an ordering that yields an ordered Ramsey graph of H. See Figure 4.1 for examples.

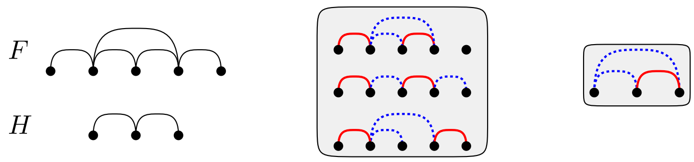


Figure 4.1: An ordered Ramsey graph F of H (left) and colorings of each proper subgraph of F showing that F is minimal (middle). We see that the underlying (unordered) graph of F is not a minimal Ramsey graph for the underlying graph of H, since $K_3 \subseteq F$ is a Ramsey graph of an unordered star on two edges. Observe that no ordering of K_3 gives an ordered Ramsey graph of H (right).

First we characterize all pairs of ordered graphs that have an ordered Ramsey graph which is a forest. Then we study the questions which pairs of ordered graphs have only finitely many minimal ordered Ramsey graphs. Our results show that the ordering of vertices heavily affects the structure of the set of ordered Ramsey graphs, at least for forests. In Section 4.2 we prove our main theorems and in Section 4.3 we give several additional results, which are not part of these theorems. In particular we give constructions of infinitely many minimal Ramsey graphs for each bonnet and some ordered matchings in that section. Concluding remarks and open questions are stated in Section 4.4.

Ramsey Forests Observe that an (unordered) graph H has a Ramsey graph which is a forest if and only if H is a star forest. For ordered graphs also the ordering affects this property. Recall that a *right star* is an ordered star with all leafs to the right of its center and a *left star* is an ordered star with all leafs to the left of its center. Further, a *monotone path* is a path $v_1 \cdots v_n$ with $v_1 < \cdots < v_n$.

Theorem 4.1. *Let H and H' be ordered graphs with at least one edge each. Then $R_<(H, H')$ contains a forest if and only if H and H' are forests and one of the following statements holds.*

(a) H or H' is a matching.

(b) For one of H or H' each component is a right star and for the other each vertex has at most one neighbor to the left.

(c) For one of H or H' each component is a left star and for the other each vertex has at most one neighbor to the right.

(d) For one of H or H' each component is a left or a right star and for the other each component is a monotone path.

Ramsey (In)Finiteness Next we present results along the lines of Theorem 1.3 from Section 1.4 and its asymmetric relatives. As for unordered graphs, we call a pair (H, H') of ordered graphs *Ramsey finite* if there are only finitely many minimal graphs in $R_<(H, H')$, and *Ramsey infinite* otherwise. Here an ordered graph $F \in R_<(H, H')$ is *minimal* if $F' \notin R_<(H, H')$ for each proper ordered subgraph F' of F. In case $H = H'$ we call H itself Ramsey finite or infinite, respectively. Clearly the following analog of Observation 1.3 holds for ordered graphs.

Observation 4.1. *A pair (H, H') of ordered graphs is Ramsey finite if and only if* $\max\{|V(F)| \mid F \in R_<(H, H'),\ F \text{ minimal}\}$ *exists, that is, the order of minimal ordered Ramsey graphs of (H, H') is bounded.*

The proof of the following theorem closely follows a recent new proof of the analogous statement for unordered graphs (Corollary 1.5 in Section 1.4) due to Nenadov and Steger [109].

Theorem 4.2. *Each ordered graph that contains a cycle is Ramsey infinite.*

Now we turn to pairs of ordered graphs where one is a forest and the other contains a cycle. In the unordered setting such a pair is Ramsey finite if and only if the forest is a matching [29, 102] (see Theorems 1.6 and 1.7 in Section 1.4). The following theorem gives partial results for ordered graphs, similar to Theorem 1.6. Recall that an ordered graph G with at least two vertices is *segmentally connected* if for any any partition $V_1 \dot\cup V_2 = V(G)$ of the vertices of G into two disjoint intervals V_1 and V_2 there is an edge with one endpoint in V_1 and the other endpoint in V_2. See Figure 4.2. Further $G \sqcup G'$ denotes the *intervally disjoint union* of ordered graphs G and G', that is, a vertex disjoint union of G and G' where all vertices of G are to left of all vertices of G'. A *monotone matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_{2i-1}u_{2i}$, $1 \leqslant i \leqslant k$.

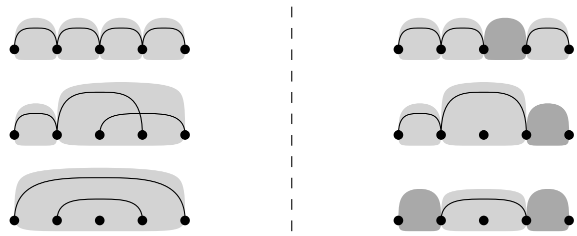


Figure 4.2: Each ordered graph on the left side is segmentally connected while each ordered graph on the right is not segmentally connected. Here each segment is marked gray, with darker empty segments. We see that a disconnected ordered graph might be segmentally connected, and each ordered graph which is not segmentally connected contains a segment without edges.

Theorem 4.3. *Let s and t be positive integers and let $H_1, \dots, H_s$, $H'_1, \dots, H'_t$ be segmentally connected ordered graphs. If (H_i, H'_j) is Ramsey finite for each $i \in [s]$ and $j \in [t]$, then $(H_1 \sqcup \dots \sqcup H_s, H'_1 \sqcup \dots \sqcup H'_t)$ is Ramsey finite.*

Corollary 4.4. *If H' is a monotone matching, then (H, H') is Ramsey finite for each ordered graph H.*

Finally we consider pairs of ordered forests. A large part of the full characterization for unordered graphs (Theorem 1.8 in Section 1.4) is due to Nešetřil and Rödl [114] who prove that each pair of (unordered) forests which are not a star forests is Ramsey infinite. Their proof is based on the fact that each pair of (unordered) forests has Ramsey graphs of arbitrarily large girth. This in turn relies on the fact that each (unordered) forest is χ-unavoidable. As we have seen in Chapter 2 (Theorem 2.1) this second fact is not true for ordered forests. We think though that the first fact holds for ordered graphs as well.

Conjecture 4.1. *For each integer t and any pair (H, H') of ordered forests there is $F \in R_<(H, H')$ with $\mathrm{girth}(F) \geqslant t$.*

If Conjecture 4.1 is true, then each pair of ordered forests where $R_<(H, H')$ does not contain a forest has minimal Ramsey graphs of arbitrarily large (but finite) girth, and hence of arbitrarily large order.

Observation 4.2. *Let (H, H') be a pair of ordered forests such that $R_<(H, H')$ contains no forest and for each integer t there is $F \in R_<(H, H')$ with $\mathrm{girth}(F) \geqslant t$. Then (H, H') is Ramsey infinite.*

Here we focus on pairs of χ-unavoidable ordered forests and defer the study of χ-avoidable ordered forests to future work, except for some small χ-avoidable ordered graphs in Section 4.3. The proof from [114] can be easily adopted for χ-unavoidable ordered forests. So Conjecture 4.1 holds for χ-unavoidable ordered forests and we have the following theorem.

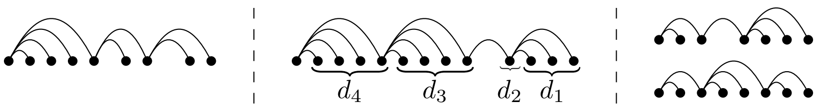


Figure 4.3: Two almost increasing right caterpillars (left, middle) and two not almost increasing right caterpillars (right).

Theorem 4.5. *If H and H' are χ-unavoidable ordered graphs and $R_<(H, H')$ contains no forest, then (H, H') is Ramsey infinite.*

This theorem leaves to consider pairs of χ-unavoidable ordered graphs where $R_<(H, H')$ contains some forest. We address such pairs of connected ordered graphs next and, again, defer the the study of such disconnected forests to future work. Recall that a *right caterpillar* is an ordered tree where each segment is a right star with at least one edge. If $S_i \preceq \cdots \preceq S_1$ are the segments of a right caterpillar H, then the *defining sequence* of H is $|E(S_1)|, \ldots, |E(S_i)|$. *Left caterpillars* are defined similarly. A left or right caterpillar with defining sequence $d_1, \ldots, d_i$ is called *almost increasing* if $i \leqslant 2$ or ($i \geqslant 3$, $d_1 \leqslant d_3$, and $d_2 \leqslant \cdots \leqslant d_i$). See Figure 4.3.

Theorem 4.6. *Let (H, H') be a Ramsey finite pair of χ-unavoidable connected ordered graphs with at least two edges. Then (H, H') is a pair of a right star and an almost increasing right caterpillar or a pair of a left star and an almost increasing left caterpillar.*

Theorem 4.7. *Let (H, H') be a pair of a right star and a right caterpillar or a pair of a left star and a left caterpillar, and let $d_1, \ldots, d_i$ be the defining sequence of the caterpillar. If either $i \leqslant 2$ or $d_1 \leqslant \cdots \leqslant d_i$, then (H, H') is Ramsey finite.*

Corollary 4.8. *A connected χ-unavoidable ordered graph is Ramsey finite if and only if it is a left or a right star.*

Unfortunately we do not resolve this case completely, see Conjecture 4.2 and the preceding discussion in Section 4.4. A summary of our results is given in Table 4.1.

H, H' cyclic	$H = H'$ ⇒ infinite		$H \neq H'$ open	
H cyclic H' forest	H' monotone matching ⇒ finite	otherwise open		
H, H' forests		H, H' χ-unavoidable & no Ramsey forest ⇒ infinite	H, H' χ-unavoidable & connected ⇒ finite iff special star & caterpillar	otherwise open

Table 4.1: Summary of results on Ramsey finiteness for ordered graphs H and H'.

4.2 Proofs of Theorems

Proof of Theorem 4.1

Lemma 4.2.1. *Let H and H' be ordered forests. Then $R_<(H, H')$ does not contain a forest in each of the following cases.*

(a) Both H and H' contain a component that is not a star.

(b) One of H or H' contains a vertex with two neighbors to the right and the other contains a vertex with two neighbors to the left.

(c) Both H and H' contain a monotone path on two edges.

(d) One of H or H' contains a monotone path on two edges and the other contains a path P on three edges.

Proof. Let F be a forest. For each of the cases we shall give a coloring of the edges of F without red copies of H and blue copies of H'.

(a) Choose a root in each component of F and color an edge red if and only if its distance to the root is odd. Then each color class forms a star forest, that is, there is neither a monochromatic copy of H nor of H'. Hence $R_<(H, H')$ contains no forest.

(b) Without loss of generality assume that H contains a vertex with two neighbors to the right and H' contains a vertex with two neighbors to the left. By induction on the number of edges of F there is a 2-coloring of its edges such that for each vertex at most one incident edge to the right is red and at most one incident edge to the left is blue. Indeed this is clear if $|E(F)| = 1$. If $|E(F)| > 1$ remove a vertex v of degree 1 from F and color the resulting forest inductively. If v is to the left of its neighbor u in F then color uv red, otherwise color it blue. This coloring contains neither red copies of H nor blue copies of H'. Hence $R_<(H, H')$ contains no forest.

(c) An induction on the number of edges of F shows that there is a 2-coloring of its edges such that for each vertex its incident edges to the left are of different color than its incident edges to the right. Indeed this is clear if $|E(F)| = 1$. If $|E(F)| > 1$ remove a vertex v of degree 1 from F and color the resulting forest inductively. Then color the edge incident to v based on the colors of edges incident to its endpoint distinct from v. This coloring contains neither red copies of H nor blue copies of H'. Hence $R_<(H, H')$ contains no forest.

(d) Without loss of generality assume that H contains a monotone path on two edges and H' contains a path P on three edges. If P contains a monotone path on two edges, then $R_<(H, H')$ contains no forest due to Case (c). Assume that P does not contain a monotone path on two edges. Color an edge e of a component T of F red if and only if the edge next to e on the (unique) path to the leftmost vertex

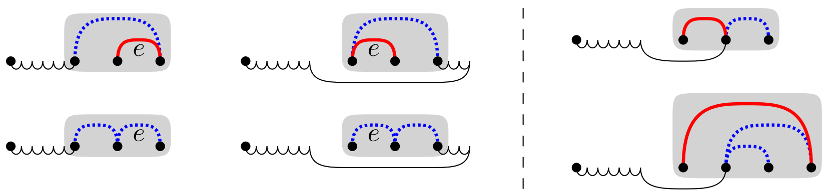


Figure 4.4: A 2-coloring of the edges of an ordered tree where an edge e is red if and only if the edge next to e on the path to the leftmost vertex forms a bend with e (left). Then there is no red monotone path on at least two edges (right, top) and no blue alternating path on at least three edges (right, bottom).

of T exists and does not form a monotone path with e (that is, forms a bend with e). Color all other edges blue. See Figure 4.4 (left). It easy to see that there is no red monotone path on two edges. Observe that the middle edge of P forms a bend with each of the other edges in P and in each copy of P in T it is the next edge on the path to the leftmost vertex for one of them. Hence one of the edges in each copy of P is red and there is no blue copy of H'. Therefore $R_<(H, H')$ contains no forest. □

Proof of Theorem 4.1. We shall prove that all pairs of ordered forests that do not have a forest as an ordered Ramsey graph are covered by Lemma 4.2.1. To this end we provide explicit constructions of ordered forests that are ordered Ramsey graphs for the remaining pairs.

Let H and H' be ordered graphs such that $R_<(H, H')$ contains a forest. Then H and H' are forests, since any monochromatic subgraph of an edge-colored forest is a forest itself. If either H or H' is a matching then Case (a) of this theorem holds. So assume that neither H nor H' is a matching. Due to Lemma 4.2.1 (a) either H or H' is a star forest. Without loss of generality assume that H is a star forest. Due to Lemma 4.2.1 (c) one of H or H' does not contain a monotone path on two edges.

First suppose that H does not contain a monotone path on two edges. Then each component of H is a left or a right star. Due to Lemma 4.2.1 (b) the following holds. If each component of H is a right star, then each vertex of H' has at most one neighbor to the left (as H is not a matching). Thus H and H' satisfy Case (b) of this theorem. Similarly, if each component of H is a left star, then each vertex of H' has at most one neighbor to the right. Thus H and H' satisfy Case (c) of this theorem. If H contains a right star on two edges as well as a left star on two two edges, then each component of H' is a monotone path. Thus H and H' satisfy Case (d) of this theorem.

Now suppose that H contains a monotone path on two edges. Then H' neither contains a monotone path on two edges nor a path on three edges due to Lemma 4.2.1 (c) and (d). Therefore each component of H' is a left or a right star. The same arguments as above show that H and H' satisfy one of the Cases (b), (c), or (d) of this theorem.

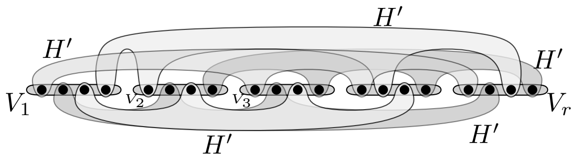


Figure 4.5: Disjoint copies of H' forming an ordered graph in $R_<(H, H')$ for some matching H.

Next consider two ordered forests H and H' that satisfy Case (a), (b), (c), or (d) of this theorem. We shall show that there is a forest in $R_<(H, H')$. First of all suppose that H and H' together contain some $t > 0$ isolated vertices. Let $\bar{H}$ and $\bar{H}'$ be obtained from H respectively H' by removing these t isolated vertices. Then there is a forest in $R_<(H, H')$ if and only if there is a forest in $R_<(\bar{H}, \bar{H}')$. Indeed, if F is an ordered forest in $R_<(H, H')$, then $F \in R_<(\bar{H}, \bar{H}')$. Suppose that $\bar{F}$ is an ordered forest in $R_<(\bar{H}, \bar{H}')$. Then we obtain an ordered forest in $R_<(H, H')$ by adding t isolated vertices to the left of all vertices in $\bar{F}$, to the right of all vertices in $\bar{F}$, as well as between any pair of consecutive vertices of $\bar{F}$. For the remaining proof we assume that neither H nor H' contains isolated vertices. We shall distinguish which of the cases of the theorem is satisfied.

(a) Without loss of generality assume that H is a matching. Consider a complete ordered graph K of order $r = r_<(H, H')$ with vertices $v_1 < \cdots < v_r$. Let $k = |V(H')|$, $m' = \binom{r}{k}$, and $m = \binom{r-1}{k-1}$. Note that K contains exactly m' copies of H' and each vertex of K is contained in exactly m copies of H' in K. We shall construct an ordered graph F that is a vertex disjoint union of m' copies of H'. For each $i \in [r]$ let $H_i^1, \ldots, H_i^m$ denote the copies of H' in K containing v_i. Choose disjoint ordered vertex sets $V_i = (v_i^1, \ldots, v_i^m)$ of size m each, $i \in [r]$. Let F denote the ordered graph with vertex set $\cup_{i=1}^r V_i$, $V_1 \prec \cdots \prec V_r$, where $v_i^j v_s^t$ is an edge in F if and only if $H_i^j = H_s^t$ and the edge $v_i v_s$ is in $H_i^j = H_s^t$, $1 \leqslant i < s \leqslant r$, $1 \leqslant j, t \leqslant m$. See Figure 4.5.

Observe that F is a vertex disjoint union of copies of H' and hence a forest. We claim that $F \to (H, H')$. Consider a 2-coloring c of the edges of F. We shall show that there is either a red copy of H or a blue copy of H'. To this end consider the edge-coloring c' of K where an edge $v_i v_s$, $1 \leqslant i < s \leqslant r$, is colored red if there is at least one red edge between V_i and V_s in F and blue otherwise. Due to the choice of K there is either a red copy of H or a blue copy of H' under c'. In either case there is a red copy of H respectively a blue copy of H' under c. Thus $F \to (H, H')$.

(b) Without loss of generality assume that each component of H is a right star and each vertex in H' has at most one neighbor to the left. We shall prove that there is a forest in $R_<(H, H')$ by induction on the size of H'. If H' has only one edge, then clearly H is in $R_<(H, H')$ and we are done. So suppose that H' has at least two edges. Let e denote the edge incident to the rightmost vertex of H' (recall that

Figure 4.6: A forest F in $R_<(H, H')$ formed from disjoint copies of a forest $F' \in R_<(H, H' - e)$ with attached leafs. Here H is a forest of right stars, H' is a forest where each vertex has at most one neighbor to the left, and e is the edge incident to the rightmost vertex of H'.

H' has no isolated vertices) and let F' denote an ordered forest in $R_<(H, H' - e)$ which exists by induction. We shall construct an ordered forest F as follows. Let $v_1 < \cdots < v_n$ denote the vertices of H and let S denote the set of vertices in H that are a center of some right star (on at least one edge) in H.

Choose n disjoint ordered sets of vertices $V_1 \prec \cdots \prec V_n$, each of size $|V(F')|$. For each $i \in [n]$, with $v_i \in S$, add edges among the vertices in V_i such that V_i induces a copy of F'. For each edge v_iv_j in H, $1 \leqslant i < j \leqslant n$, add an arbitrary perfect matching between V_i and V_j. See Figure 4.6 for an illustration.

Then F is a vertex disjoint union of copies of F' with some leafs attached. In particular F is a forest. We claim that $F \in R_<(H, H')$. Consider a coloring of the edges of F. Note that each $v_i \in S$ is left endpoint of some edge v_iv_j in H and hence for each $u \in V_i$ there is an edge uv with $v \in V_j$, $j > i$. If for each $v_i \in S$ there exists $u \in V_i$ such that an edge uv is red whenever $v \in V_j$ with $j > i$, then there is a red copy of H. So assume that there is some $v_i \in S$ such that for each $u \in V_i$ there is a blue edge uv for some $v \in V_j$, $j > i$. Since V_i induces a copy of F', there is either a red copy of H or a blue copy of $H' - e$. In the latter case some edge between V_i and V_j yields a blue copy of H' in F, since e is incident to the rightmost vertex in H'. Altogether F is a forest in $R_<(H, H')$.

(c) This follows from Case (b).

(d) Without loss of generality assume that each component of H is a left or a right star and each component of H' is a monotone path. We shall prove that there is a forest in $R_<(H, H')$ by induction on the number of components of H and the size of H'. If H has only one component, then there is a forest in $R_<(H, H')$ by Case (b) or (c). If H' has only one edge, then H is a forest in $R_<(H, H')$. Suppose that H has at least two components and H' has at least two edges. Let S denote a component of H. Without loss of generality assume that S is a right star. Let e denote the edge in H' containing the rightmost vertex of H' (recall that H' has no isolated vertices). By induction there are ordered forests $A \in R_<(H - S, H')$ and $B \in R_<(H, H' - e)$. Let $a_1 < \cdots < a_n$ denote the vertices of A. Let F' consist of an intervally disjoint union of $n + 1$ copies $B_1, \ldots, B_{n+1}$ of B. Consider an ordered forest F'' that is a vertex disjoint union of F' and A where for each i, $1 \leqslant i \leqslant n$, the vertex $a_i \in V(A)$ is between B_i and B_{i+1}. See Figure 4.7 (left). Obtain an ordered forest F from F'' as follows. For each i, j with $1 \leqslant i \leqslant j \leqslant n + 1$, and each vertex u in B_i add a

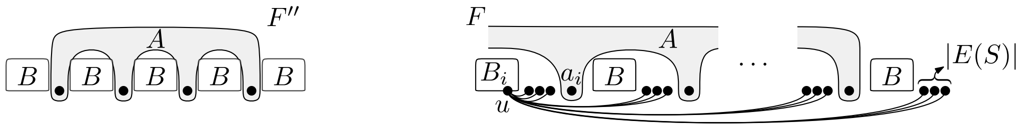


Figure 4.7: Forests $A \in R_<(H-S, H')$ and $B \in R_<(H, H'-e)$ forming a forest F in $R_<(H, H')$. Here S is a component of a forest H and e is an edge in a forest H', where each component of H is a left or a right star and each component of H' is a monotone path.

star with center u and $|E(S)|$ leafs to the right of B_j and, if $j < n+1$, to the left of a_j, such that these leafs are distinct for distinct pairs i, j, and vertices u. See Figure 4.7 (right) for an illustration.

Clearly F is a forest since F'' is a forest and all the leafs added in the last step are distinct. We claim that $F \in R_<(H, H')$. Consider a 2-coloring of the edges of F without blue copies of H'. Then there is a red copy K of $H - S$ in A. Consider some $i \in [n+1]$ such that the position of any vertex u in B_i in $K + u$ corresponds to the position of the center of S in H. If for some $u \in V(B_i)$ all the edges uv in F where v is to the right of B_i are red, then this red right star together with K contains a red copy of H. So suppose that for each vertex $u \in V(B_i)$ there is a blue edge uv where v is to the right of B_i. Then there is no blue copy of $H' - e$ in B_i, as otherwise there is a blue copy of H' since e is incident to the rightmost vertex in H'. Hence there is a red copy of H in B_i. Altogether $F \in R_<(H, H')$. □

Proof of Theorem 4.2

We shall use the hypergraph container method due to Saxton and Thomason [132] and independently Balogh *et al.* [13] to prove that (G, G) is Ramsey infinite for any ordered graph G that contains a cycle. We will follow the arguments from [109] (see also [70]) using the following result of Saxton and Thomason [133]. Considering ordered graphs instead of (unordered) graphs only affects the involved constants. Recall that the *density* of a graph is $m(G) = \max\{|E(G')|/|V(G')| \mid G' \subseteq G\}$ and that the 2-*density* is $m_2(G) = \max\{(|E(G')|-1)/(|V(G')|-2) \mid G' \subseteq G, |V(G')| \geqslant 3\}$. For a hypergraph $\mathcal{H}$ and some integer ℓ let $\Delta_\ell(\mathcal{H}) = \max\{|\mathcal{E}| \mid \mathcal{E} \subseteq E(\mathcal{H}), |\cap_{E\in\mathcal{E}} E| \geqslant \ell\}$.

Theorem 4.9 (Cor. 1.3 [133]). *For all $r \in \mathbb{N}$ and for any $\epsilon > 0$ there is $c > 0$ such that for all r-uniform hypergraphs $\mathcal{H}$ with average degree $d > 0$ and each τ, $0 < \tau \leqslant 1$, with $\Delta_\ell(\mathcal{H}) \leqslant cd\tau^{\ell-1}$, $2 \leqslant \ell \leqslant r$, the following holds. There is a function $f : 2^{V(\mathcal{H})} \to 2^{V(\mathcal{H})}$ such that for each independent set I in $\mathcal{H}$ there is $S \subseteq V(\mathcal{H})$ with*

(a) $S \subseteq I \subseteq f(S)$,

(b) $|S| \leqslant \tau|V(\mathcal{H})|$,

(c) $|E(\mathcal{H}[f(S)])| \leqslant \epsilon|E(\mathcal{H})|$.

We shall also use a corollary to the so-called Small-Subgraph-Theorem of Erdős and Rényi [59] and Bollobás [17].

Theorem 4.10 ([17],[59])**.** *Let H be a graph with at least one edge. The probability that a random graph $G(n,p)$ contains a copy of H tends to 0 (as $n \to \infty$) if $pn^{1/m(H)} \to 0$ (as $n \to \infty$).*

The following lemma is an analog of Corollary 2.2. from [109]. We include a proof for ordered graphs for completeness.

Lemma 4.2.2. *Let H be an ordered graph, $\sigma = r_<(H,H,H)$, $\epsilon = \frac{1}{4}\sigma^{-\sigma}$, $\delta = |E(H)|\frac{1}{2}\sigma^{-\sigma}$, and $n \geqslant \sigma$. If E_1, $E_2 \subseteq E(K_n)$ with E_i inducing at most $\epsilon n^{|V(H)|}$ copies of H, $i \in [2]$, then $|E_1 \cup E_2| \leqslant (1-\delta)\binom{n}{2}$.*

Proof. Let $t = |V(H)|$, $E = E_1 \cup E_2$, and $E' = E(K_n) \setminus E$. Then for each each set of σ vertices in K_n there is a copy of H with all edges in E_1, all edges in E_2, or all edges in E'. Note that each copy of H in K_n is contained in at most $n^{\sigma-t}$ such σ-sets. Thus there are at least $\binom{n}{\sigma}n^{t-\sigma}$ copies of H in K_n with all edges in one of E_1, E_2, or E'. Therefore E' induces at least $\binom{n}{\sigma}n^{t-\sigma} - 2\epsilon n^t \geqslant (\sigma^{-\sigma} - 2\epsilon)n^t = \frac{1}{2}\sigma^{-\sigma}n^t$ copies of H. Since each edge in E' is contained in at most n^{t-2} copies of H, there are at least $|E(H)|\frac{1}{2}\sigma^{-\sigma}n^t/n^{t-2} = |E(H)|\frac{1}{2}\sigma^{-\sigma}n^2 \geqslant \delta\binom{n}{2}$ edges in E'. Thus $|E| \leqslant (1-\delta)\binom{n}{2}$. □

The following lemma is implicitly contained in [109] for unordered graphs.

Lemma 4.2.3 (Ramsey Containers [109])**.** *Let H be an ordered graph. Then there are constants N_0, C, $\delta > 0$, and a function g mapping ordered graphs to ordered graphs such that for each n, $n \geqslant N_0$, and each $F \notin R_<(H)$ on vertex set $[n]$ there is an ordered graph P with*

(a) $V(P)$, $V(g(P)) \subseteq [n]$,

(b) $P \subseteq F \subseteq g(P)$,

(c) $|E(P)| \leqslant Cn^{2-1/m_2(H)}$,

(d) $|E(g(P))| \leqslant (1-\delta)\binom{n}{2}$.

Proof. Consider some (large) n and an ordered complete graph K on vertex set $[n]$. Let $\mathcal{H} = \mathcal{H}_n$ be a hypergraph with vertex set $E(K)$ that contains an edge $E \subseteq E(K)$ if and only if E forms a copy of H in K. Observe that there is a 1-1 correspondence between subsets of $V(\mathcal{H})$ and ordered graphs on vertex set $[n]$ (without isolated vertices). Let $t = |V(H)|$, $r = |E(H)|$. Then $\mathcal{H}$ is r-uniform, $|V(\mathcal{H})| = \binom{n}{2}$, $|E(\mathcal{H})| = \binom{n}{t}$, and its average degree is $d(\mathcal{H}) = \frac{r|E(\mathcal{H})|}{|V(\mathcal{H})|}$. Let δ and ϵ be given by Lemma 4.2.2 and let $c \leqslant 1$ be given by Theorem 4.9 for r and ϵ as defined here. Further let $\tau = \frac{t^t}{c}n^{-1/m_2(H)}$ and let $v(\ell) = \min\{|V(H')| \mid H' \subseteq H, |E(H')| = \ell\}$, for $\ell \in [r]$. Finally choose $N_0 \geqslant \max\{r_<(H), t\}$ such that $\frac{t^t}{c} N_0^{-1/m_2(H)} \leqslant 1$ and consider $n \geqslant N_0$. Then $d(\mathcal{H}) > 0$ and for each ℓ, $2 \leqslant \ell \leqslant r$,

$$m_2(H) = \max_{\substack{H' \subseteq H \\ |V(H')| \geqslant 3}} \frac{|E(H')|-1}{|V(H')|-2} \geqslant \max_{H' \subseteq H, |E(H')|=\ell} \frac{\ell-1}{|V(H')|-2} = \frac{\ell-1}{v(\ell)-2}, \tag{4.1}$$

$$\Delta_\ell(\mathcal{H}) \leqslant \binom{n-v(\ell)}{t-v(\ell)} \leqslant n^{t-v(\ell)} \overset{(4.1)}{\leqslant} n^{t-2-\frac{\ell-1}{m_2(H)}} \leqslant \frac{t^t \binom{n}{t}}{\binom{n}{2}} n^{-\frac{\ell-1}{m_2(H)}}$$
$$\leqslant \frac{r|E(\mathcal{H})|}{|V(\mathcal{H})|} c \left(\frac{t^t}{c}\right)^{\ell-1} n^{-\frac{\ell-1}{m_2(H)}} = cd\tau^{\ell-1}.$$

Due to Theorem 4.9 there is a function $f : 2^{V(\mathcal{H})} \to 2^{V(\mathcal{H})}$ such that for each independent set I of $\mathcal{H}$ there is $S \subseteq V(\mathcal{H})$ with $S \subseteq I \subseteq f(S)$, $|S| \leqslant \tau|V(\mathcal{H})|$, and $|E(\mathcal{H}[f(S)])| \leqslant \epsilon|E(\mathcal{H})|$. Now consider $F \notin R_<(H)$ on n vertices and a 2-coloring of the edges of F with no monochromatic copy of H. Then the color classes form independent sets I_1 and I_2 in $\mathcal{H}$. As argued above there are sets S_1, $S_2 \subseteq V(\mathcal{H})$ with $S_i \subseteq I_i \subseteq f(S_i)$, $|S_i| \leqslant \tau|V(\mathcal{H})| \leqslant \frac{t^t}{c} n^{2-1/m_2(H)}$, and $|E(\mathcal{H}[f(S_i)])| \leqslant \epsilon|E(\mathcal{H})| \leqslant \epsilon n^t$, $i = 1, 2$. Due to the latter condition and Lemma 4.2.2 we have $|f(S_1) \cup f(S_2)| \leqslant (1-\delta)n^2$. The statement of the lemma follows with $C = 2\frac{t^t}{c}$, P being the graph formed by $S_1 \cup S_2$, and $g(P)$ being the graph formed by $f(S_1) \cup f(S_2)$. □

Proof of Theorem 4.2. Let H be an ordered graph that contains a cycle. We shall prove that for each positive integer t there is an ordered graph F with $F \in R_<(H)$ and $F[V] \notin R_<(H)$ for each t-subset $V \subseteq V(F)$. More precisely we shall show that there is a constant $C' = C'(H, t)$ such that a random graph $F = G(n, p)$, with $p = C' n^{-\frac{1}{m_2(H)}}$, satisfies both conditions with probability tending to 1 as n tends to infinity. Hence there is a minimal ordered Ramsey graph contained in F on more than t vertices. Since t is arbitrary there are minimal ordered Ramsey graphs of H of arbitrarily large order and hence H is Ramsey infinite by Observation 4.1.

Fix some $t > 0$ and $C' > C$ and consider n sufficiently large such that $p \leqslant 1$. Let $\mathcal{F}$ denote the set of all ordered graphs in $R_<(H)$ with t vertices and let $\widetilde{F}$ denote the underlying (unordered) graph of F. First we shall show that with high probability $\widetilde{F}$ does not contain the underlying graph of any member of $\mathcal{F}$ as a subgraph. Consider some fixed $F' \in \mathcal{F}$. Since H contains a cycle we have $m(F') > m_2(H)$ by Lemma 1.4.1 and Observation 1.4. Therefore

$$pn^{\frac{1}{m(F')}} = n^{\frac{1}{m(F')} - \frac{1}{m_2(H)}} \longrightarrow 0 \quad (n \to \infty).$$

So with high probability $\widetilde{F}$ does not contain the underlying graph of F' as a subgraph by Theorem 4.10. In particular F does not contain F' as an ordered subgraph. This shows that with probability tending to 1 (as $n \to \infty$) F does not contain any member of $\mathcal{F}$ as a subgraph, since $\mathcal{F}$ is a finite set. So with probability tending to 1 (as $n \to \infty$) $F[V] \notin R_<(H)$ for each t-subset $V \subseteq V(F)$.

Next we shall show that with high probability $F \in R_<(H)$. Let N_0, C, $\delta > 0$ be constants and let g be a function given by Lemma 4.2.3. Let $\mathcal{P}$ denote the set of ordered graphs P with $V(P) \subseteq [n]$, $|E(P)| \leqslant Cn^{2-1/m_2(H)}$, and no isolated vertices. Note that we can assume that $|E(g(P))| \leqslant (1-\delta)\binom{n}{2}$ for each $P \in \mathcal{P}$. For sufficiently large C' the probability that there is some $P \in \mathcal{P}$ with $P \subseteq F \subseteq g(P)$ is at most

$$\begin{aligned}
\sum_{P\in\mathcal{P}} P(P \subseteq F \subseteq g(P)) \leqslant & \sum_{P\in\mathcal{P}} p^{|E(P)|}(1-p)^{\delta\binom{n}{2}} \\
= & \sum_{i=0}^{\lfloor Cn^{2-1/m_2(H)}\rfloor} \binom{\binom{n}{2}}{i} p^i (1-p)^{\delta\binom{n}{2}} \qquad (4.2)\\
\leqslant & \sum_{i=0}^{\lfloor Cn^{2-1/m_2(H)}\rfloor} \left(\frac{en^2p}{i}\right)^i e^{-p\delta\binom{n}{2}} \qquad (4.3)\\
\leqslant & \, Cn^2 \left(\frac{en^2p}{Cn^{2-1/m_2(H)}}\right)^{Cn^{2-1/m_2(H)}} e^{-p\delta\binom{n}{2}} \qquad (4.4)\\
\leqslant & \, Cn^2 \left(\frac{eC'}{C}\right)^{Cn^{2-1/m_2(H)}} e^{-(C'\delta/4)n^{2-1/m_2(H)}} \\
= & \, Cn^2 e^{\overbrace{(C+C\ln(C')-C\ln(C)-(C'\delta/4))}^{<0} n^{2-1/m_2(H)}} \\
\longrightarrow & \, 0 \ (n \to \infty).
\end{aligned}$$

Here equality 4.2 holds since for each i there are $\binom{\binom{n}{2}}{i}$ graphs on i edges in $\mathcal{P}$. Inequality 4.3 holds since $\binom{\binom{n}{2}}{i} \leqslant (en^2/i)^i$. Finally inequality 4.4 holds since for any fixed $r > 0$ the function $(r/x)^x$ of x is increasing for $0 < x \leqslant r$ (note that its derivative is $(r/x)^x(\ln(r/x) - 1)$) and $Cn^{2-1/m_2(H)} \leqslant eC'n^{2-1/m_2(H)} = en^2p$. If $n \geqslant N_0$ and $F \notin R_<(H)$, then by Lemma 4.2.3 there is some $P \in \mathcal{P}$ with $P \subseteq F \subseteq g(P)$. Thus the calculation above shows that $F \in R_<(H)$ with probability tending 1 as $n \to \infty$. Altogether we see that for sufficiently large n there is an ordered graph F with $F \in R_<(H)$ and $F[V] \notin R_<(H)$ for each t-subset $V \subseteq V(F)$. Therefore there are arbitrarily large minimal ordered Ramsey graphs of H and hence H is Ramsey infinite. □

Proof of Theorem 4.3

In order to prove that (H, H') is Ramsey finite we shall show that each minimal ordered Ramsey graph of (H, H') is a member of the following finite family of ordered graphs. Let $\mathcal{F}_s^t$ denote the set of all ordered graphs that are isomorphic to a union of ordered graphs F_i^j, $i \in [s]$, $j \in [t]$, where F_i^j is a minimal ordered Ramsey graph for (H_i, H_j), for each $i \in [s]$, $j \in [t-1]$ we have $F_i^j \prec F_i^{j+1}$, and for each $j \in [t]$, $i \in [s-1]$ we have $F_i^j \prec F_{i+1}^j$. See Figure 4.8 for an illustration in the case $s = t = 3$. For

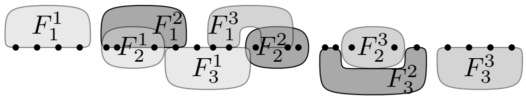


Figure 4.8: A Ramsey graph for $(H_1 \sqcup H_2 \sqcup H_3, H_1' \sqcup H_2' \sqcup H_3')$ where $F_i^j \in R_<(H_i, H_j')$, $i,\ j = 1,\ 2,\ 3$. The subgraphs of the same color correspond to constant j, the subgraphs on the same horizontal layer (above, on, respectively below the line of vertices) correspond to constant i.

some graph $A \in F_s^t$ that is isomorphic to such a union of graphs F_i^j, $i \in [s]$, $j \in [t]$, let A_i^j denote the copy of F_i^j in A. Note that $\mathcal{F}_s^t$ is finite. Let $H = H_1 \sqcup \cdots \sqcup H_s$ and $H' = H_1' \sqcup \cdots \sqcup H_t'$. We claim that $F \in R_<(H, H')$ if and only if F contains some member of $\mathcal{F}_s^t$.

Consider a 2-coloring of the edges of some $A \in \mathcal{F}_s^t$. We shall prove that there is a red copy of H or a blue copy of H' by induction on s and t. If $s = 1$, then $A = A_1^1 \sqcup \cdots \sqcup A_1^t$ where $A_1^j \in R_<(H, H_j')$ for each $j \in [t]$. If $t = 1$, then $A = A_1^1 \sqcup \cdots \sqcup A_s^1$ where $A_i^1 \in R_<(H_i, H')$ for each $i \in [s]$. In both cases it is easy to see that there is either a red copy of H or a blue copy of H'. Hence $A \in R_<(H, H')$. Suppose that $s,\ t > 1$. Let A' denote the subgraph of A formed by all subgraphs A_i^j with $(i, j) \neq (s, t)$. Then $A' \in \mathcal{F}_{s-1}^t$ and $A' \in \mathcal{F}_s^{t-1}$. By induction, A is in $R_<(H, H_1' \sqcup \cdots \sqcup H_{t-1}')$ and in $R_<(H_1 \sqcup \cdots \sqcup H_{s-1}, H')$. If there is no red copy of H and no blue copy of H' in A', then there is a red copy of $H_1 \sqcup \cdots \sqcup H_{s-1}$ and a blue copy of $H_1' \sqcup \cdots \sqcup H_{t-1}'$. Observe that $A' \prec A_s^t$. Since $A_s^t \in R_<(H_s, H_t')$ there is a red copy of H or a blue copy of H' in either case. Thus $A \in R_<(H, H')$.

Now consider an ordered graph F that does not contain any member of $\mathcal{F}_s^t$. We shall prove that $F \notin R_<(H, H')$ by induction on s and t. Consider the case $s = 1$. If $t = 1$ then F does not contain any minimal ordered Ramsey graph of $(H_1, H_1') = (H, H')$. Clearly $F \notin R_<(H, H')$. Suppose that $t > 1$ and let p denote the rightmost vertex in F such that $\{q \in V(F) \mid q > p\}$ induces a copy of some graph from $R_<(H, H_t')$ in F. Let F_ℓ and F_r denote the subgraphs of F induced by all vertices to the left respectively to the right of p. Then F_ℓ does not contain any member of $\mathcal{F}_1^{t-1}$. By induction on t there is a coloring of the edges of F_ℓ without red copies of H or blue copies of $H_1' \sqcup \cdots \sqcup H_{t-1}'$. Moreover there is a coloring of F_r without red copies of H or blue copies of H_t'. Color all remaining edges blue (that is, all edges incident to p and all edges between F_ℓ and F_r). Then there is no red copy of H since $H = H_1$ is segmentally connected. Moreover each blue copy of H_t contains some vertex q with $q \leqslant p$ and thus there is no blue copy of H'. This shows that $F \notin R_<(H, H')$. If $t = 1$ and $s > 1$, then $F \notin R_<(H, H')$ due to symmetric arguments.

So suppose that $s,\ t > 1$. If F does not contain any member of $\mathcal{F}_{s-1}^t$, then $F \notin R_<(H_1 \sqcup \cdots \sqcup H_{s-1}, H')$ by induction on s. Hence $F \notin R_<(H, H')$. So assume that F contains some member of F_{s-1}^t. Let A denote such a copy where for each

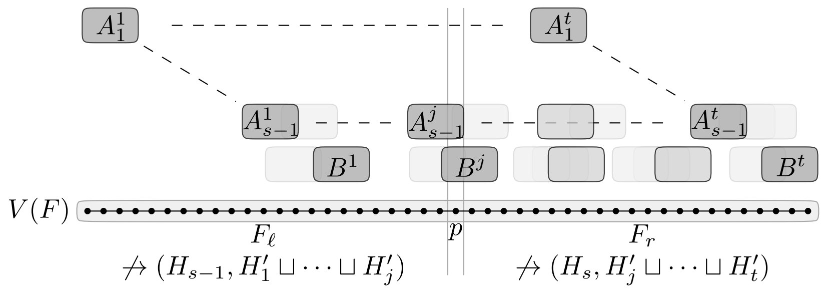


Figure 4.9: An illustration of the proof of Theorem 4.3. Here $A_i^j \in R_<(H_i, H_j')$, $B^j \in R_<(H_s, H_j')$, and F does not contain any element from $\mathcal{F}_s^t$. Therefore a combination of colorings of F_ℓ and F_r shows that $F \not\rightarrow (H_1 \sqcup \cdots \sqcup H_s, H_1' \sqcup \cdots \sqcup H_t')$.

$i \in [s-1]$ and $j \in [t]$ the rightmost vertex of A_i^j is leftmost among all such copies. Note that we can simultaneously choose such leftmost vertices for all $i \in [s-1]$ and $j \in [t]$. If F does not contain a subgraph of the form $B^1 \sqcup \cdots \sqcup B^t$ where $B^j \in R_<(H_s, H_j')$ for each $j \in [t]$, then the arguments from case $s = 1$ show that $F \notin R_<(H_s, H')$ and thus $F \notin R_<(H, H')$. Otherwise let $B = B^1 \sqcup \cdots \sqcup B^t$ denote such a subgraph of F where for each $j \in [t]$ the leftmost vertex of B^j is rightmost among all such subgraphs. Again we can simultaneously choose such rightmost vertices for all $j \in [t]$. Since $A \cup B \notin \mathcal{F}_s^t$ there is $j \in [t]$ such that B^j is not to the right of A_{s-1}^j. See Figure 4.9 for an illustration. Let p denote the rightmost vertex of A_{s-1}^j and let F_ℓ and F_r denote the subgraphs of F induced by all vertices to the left respectively to the right of p. By the choice of A the graph F_ℓ does not contain any member of F_{s-1}^j. By the choice of B the graph F_r does not contain any subgraph of the form $B_j' \sqcup \cdots \sqcup B_t'$ where $B_{j'}' \in R_<(H_s, H_{j'})$, $j \leqslant j' \leqslant t$. Therefore there is a coloring of the edges of F_ℓ without red copies of $H_1 \sqcup \cdots \sqcup H_{s-1}$ or blue copies of $H_1' \sqcup \cdots \sqcup H_j'$ by induction on s. Similarly there is a coloring of the edges of F_r without red copies of H_s or blue copies of $H_j' \sqcup \cdots \sqcup H_t'$ due to the arguments from case $s = 1$. Color all remaining edges red (that is, all edges incident to p and all edges between F_ℓ and F_r). Then each red copy of $H_1 \sqcup \cdots \sqcup H_{s-1}$ contains some vertex q with $p \leqslant q$. Hence there is no red copy of H. Similarly we see that there is no blue copy of H' since all edges incident to p and all edges between F_ℓ and F_r are red. Altogether $F \notin R_<(H, H')$. This shows that each minimal ordered Ramsey graph of (H, H') is contained in $\mathcal{F}_s^t$, since we proved in the beginning that each member of $\mathcal{F}_s^t$ is in $R_<(H, H')$. Thus (H, H') is Ramsey finite. $\square$

Proof of Theorems 4.5 and 4.6

Proof of Theorem 4.5. As H and H' are χ-unavoidable there is an integer k such that each ordered graph of chromatic number at least k contains a copy of H and a copy of H'. For each $t \geqslant 3$ let G_t be an ordered graph of chromatic number at least k^2 and girth at least t. First we prove that $G_t \in R_<(H, H')$. Consider a 2-coloring of the edges of G_t. Since $\chi(G_t) \geqslant k^2$ one of the color classes forms an ordered subgraph of chromatic number at least k. Therefore this subgraph contains a (monochromatic) copy of both of H and H'. Thus $G_t \in R_<(H, H')$.

For each $t \geqslant 3$ let G'_t be a minimal ordered Ramsey graph of (H, H') that is a subgraph of G_t. For each $t \geqslant 3$ the graph G'_t contains a cycle, since $R_<(H, H')$ contains no forest. As G_t has girth at least t, infinitely many of the graphs G'_t are not isomorphic. Thus (H, H') is Ramsey infinite. □

For the proof of Theorem 4.6 it remains to consider pairs of χ-unavoidable connected ordered graphs that are not covered by Theorem 4.5, i.e., that have a forest as an ordered Ramsey graph. Recall that such pairs are characterized by Theorem 4.1. We give explicit constructions of infinitely many ordered Ramsey graphs for these pairs using so-called determiners as building blocks. We introduce determiners and prove give explicit constructions of such ordered graphs next. The concept of (unordered) determiners is used by Burr *et al.* [33] to construct Ramsey graphs with certain properties.

Recall that $G \sqcup G'$ denotes a intervally disjoint union of ordered graphs G and G', that is, a vertex disjoint union of G and G' where all vertices of G are to left of all vertices of G'. Also recall that the concatenation $G \circ G'$ of two ordered graphs G and G' is obtained from $G \sqcup G'$ by identifying the rightmost vertex in the copy of G with the leftmost vertex in the copy of G'. Let $\vec{S}_p$ denote a right star with p edges. Given integers i, j with $1 \leqslant j \leqslant i$, and a sequence $d = d_1, d_2, \ldots$ of positive integers, let $H_i(d) = \vec{S}_{d_i} \circ \cdots \circ \vec{S}_{d_1}$ and $H_i^j(d) = \vec{S}_{d_i} \circ \cdots \circ \vec{S}_{d_j}$ be right caterpillars. For convenience let $H_0(d)$ and $H_i^{i+1}(d)$ each denote a single vertex ordered graph. Consider a right star H and a sequence $d = d_1, d_2, \ldots$ of positive integers. A *left determiner for* $(H, H_i(d))$, $i \geqslant 0$, is an ordered graph F such that

- for any 2-coloring of the edges of F without red copies of H there is a blue copy of $H_i(d)$ that contains the leftmost vertex of F, and
- there is a *good* coloring of the edges of F, that is, a 2-coloring without red copies of H or blue copies of H_{i+1} such that there is a unique blue copy of $H_i(d)$ that contains the leftmost vertex of F and this copy is induced and isolated in the blue subgraph.

A *right determiner for* $(H, H_i^j(d))$, $1 \leqslant j \leqslant i + 1$, is an ordered graph F such that

- for any 2-coloring of the edges of F without red copies of H or blue copies of $H_i(d)$ there is a blue copy of $H_i^j(d)$ that contains the rightmost vertex of F, and
- there is a *good* coloring of the edges of F, that is, a 2-coloring without red copies of H or blue copies of H_i such that there is a unique blue copy of $H_i^j(d)$ that contains the rightmost vertex of F and this copy is induced and isolated in the blue subgraph.

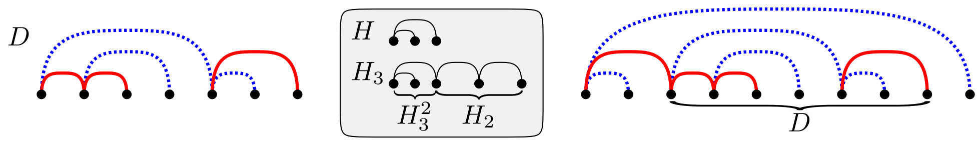


Figure 4.10: A left determiner for $(H, H_2(d))$ (left) and a right determiner for $(H, H_3^2(d))$ (right) with respective good colorings (red solid and blue dashed edges). Here H is a right star on two edges and $d_1 = d_2 = 1$ and $d_3 = 2$.

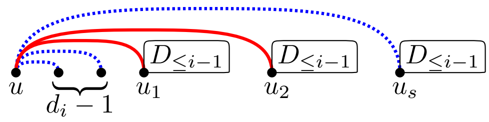


Figure 4.11: A left determiner for (H, H_i) with $s = |E(H)| = 3$, $d_i = 3$, and a good coloring of its edges. Here $D_{\leqslant i-1}$ is a left determiner for (H, H_{i-1}).

See Figure 4.10 for examples of determiners. Recall that $a \oplus_b G$ denotes the ordered graph obtained from $\vec{S}_a \sqcup (\sqcup_b G)$ by connecting the leftmost vertex of this union with the leftmost vertex of each of the b copies of G. See Figure 1.10 for an illustration.

Lemma 4.2.4. *Let H be a right star with at least one edge, let d be a sequence of positive integers, and let i, j be non-negative integers with $j \leqslant i + 1$. Then there is a left determiner for $(H, H_i(d))$ and, if $j \geqslant 2$, there is a right determiner for $(H, H_i^j(d))$.*

Proof. Let $s = |E(H)|$, $H_i = H_i(d)$, $H_i^j = H_i^j(d)$, and $d = d_1, d_2, \ldots$. First we shall construct a left determiner for (H, H_i) by induction on i. It is easy to see that a single vertex graph is left determiner for (H, H_0) and a right star on $s + d_1 - 1$ edges is a left determiner for (H, H_1). Suppose that $i \geqslant 2$ and let $D_{\leqslant i-1}$ denote a left determiner for (H, H_{i-1}), which exists by induction. Let $D = (d_i - 1) \oplus_s D_{\leqslant i-1}$, let $D_1, \ldots, D_s$ denote the copies of $D_{\leqslant i-1}$ in D, and let u_t denote the leftmost vertex of D_t, $1 \leqslant t \leqslant s$, see Figure 4.11 for an illustration.

To see that D is a left determiner for (H, H_i) consider a 2-coloring of its edges without a red copy of H. Since u is of degree $d_i + s - 1$, it is incident to at least d_i blue edges. Consider the rightmost vertex v such that the edge uv is blue. Then $v = u_t$ for some $t \in [s]$, and hence is leftmost in a blue copy of H_{i-1}. Thus there is a blue copy of H_i that contains u.

It remains to give a good coloring of the edges of D. Recall that $D_{\leqslant i-1}$ has a good coloring, that is, a 2-coloring of its edges without red copies of H or blue copies of H_i such that the blue copy of H_{i-1} that contains the leftmost vertex of $D_{\leqslant i-1}$ is induced and isolated in the blue subgraph. Color $D_1, \ldots, D_s$ according to such a coloring. Moreover color some $s - 1$ of the edges uu_t, $1 \leqslant t \leqslant s$, in red, and all other edges incident to u in blue. See Figures 4.10 and 4.11. Clearly there is no red copy of H and no blue copy of H_{i+1}. Moreover the blue copy of H_i that contains u

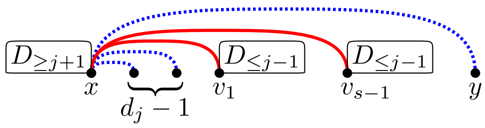


Figure 4.12: A right determiner for (H, H_i^j) with $s = |E(H)| = 3$, $d_j = 3$, and a good coloring of its edges. Here $D_{\leqslant j-1}$ is a left determiner for (H, H_{j-1}) and $D_{\geqslant j+1}$ is a right determiner for (H, H_i^{j+1}).

is induced and isolated in the blue subgraph. Hence this coloring is a good coloring of D. Thus D is a left determiner for (H, H_i).

Next we shall construct right determiners for (H, H_i^j) for i, j with $i + 1 \geqslant j \geqslant 2$. Consider a fixed $i \geqslant 1$. We shall construct a right determiner for (H, H_i^j) by induction on $i+1-j$ (that is, "from left to right"), using the already constructed left determiners. We see that a single vertex graph is a right determiner for (H, H_i^{i+1}). This forms the base case $i+1-j = 0$, that is, $j = i+1$. Suppose that $i+1-j > 0$, that is, $j \leqslant i$. Let $D_{\geqslant j+1}$ denote a right determiner for (H, H_i^{j+1}), which exists by induction, and let $D_{\leqslant j-1}$ denote a left determiner for (H, H_{j-1}) (note that $j-1 \geqslant 1$). Let $D' = (d_j-1) \oplus_{s-1} D_{\leqslant j-1}$, let x denote the leftmost vertex in D', let $D_1, \ldots, D_{s-1}$ denote the copies of $D_{\leqslant j-1}$ in D', and let v_t denote the leftmost vertex in D_t, $1 \leqslant t \leqslant s-1$. Obtain an ordered graph D from $D_{\geqslant j+1} \circ D'$ by adding a vertex y to the right of all other vertices and an edge between x and y. See Figure 4.12.

We claim that D is a right determiner for (H, H_i^j). Consider a 2-coloring of the edges of D without red copies of H or blue copies of H_i. We shall find a blue copy of H_i^j that contains y. By construction x is rightmost in a blue copy of H_i^{j+1}. Moreover x is left endpoint of at least d_j blue edges. Assume that the edge xy is red. Consider the rightmost vertex z such that the edge xz is blue. Then $z = v_t$ for some t, $1 \leqslant t \leqslant s-1$, and hence z is leftmost in a blue copy of H_{j-1}. Thus there is a blue copy of H_i, a contradiction. This shows that xy is colored blue and there is a blue copy of H_i^j that contains y.

It remains to give a good coloring of D. Recall that $D_{\geqslant j+1}$ has a good coloring, that is, a 2-coloring of its edges without red copies of H or blue copies of H_i such that the blue copy of H_i^{j+1} that contains x is induced and isolated in the blue subgraph. Similarly $D_{\leqslant j-1}$ has a good coloring. Color the leftmost copy of $D_{\geqslant j+1}$ in D and each D_t, $1 \leqslant t \leqslant s-1$, according to such colorings. Color the edges xv_t in red, $1 \leqslant t \leqslant s-1$, and all remaining edges with left endpoint x in blue. See Figures 4.10 and 4.12 for an illustration. Clearly there is no red copy of H and no blue copy of H_i (since $j \geqslant 2$). Moreover the blue copy of H_i^j that contains y is induced and isolated in the blue subgraph. Hence D is a right determiner for (H, H_i^j). □

Lemma 4.2.5. *Let G be a χ-unavoidable connected ordered graph where each vertex has at most one neighbor to the left (right). Then each segment of G is a right (left) star.*

Proof. Suppose that each vertex in G has at most one neighbor to the left. We shall show that each segment of G is a right star. Since G is χ-unavoidable and connected it is a tree and contains neither a bonnet nor a tangled path by Theorem 2.1. Hence each segment of G is a monotonically alternating tree by Theorem 2.2. Let G' be a segment of G and let u denote its leftmost vertex. Consider a neighbor v of u in G'. Then there is no edge vv' in G' with $v < v'$ since G' is monotonically alternating and thus $\chi_<(G') \leqslant 2$. If $v'v$ is an edge in G' with $v' < v$ then $u = v'$ by assumption. Hence G' is a right star. If each vertex in G has at most one neighbor to the right, then each segment is a left star due to symmetric arguments. □

Proof of Theorem 4.6. Recall that (H, H') is a Ramsey finite pair of χ-unavoidable connected ordered graphs with at least two edges each. We shall give constructions of infinitely many ordered Ramsey graphs of (H, H') using the determiners introduced above. Since H and H' are connected and χ-unavoidable both H and H' are trees. Since (H, H') is Ramsey finite there is a forest in $R_<(H, H')$ due to Theorem 4.5. Due to Theorem 4.1 and since H and H' are connected and have at least two edges each, we assume, without loss of generality, that H is a right star while each vertex of H' has at most one neighbor to the left. Since H' is a χ-unavoidable tree each segment of H' is a right star due to Lemma 4.2.5, that is, H' is a right caterpillar. Let $d = d_1, \ldots, d_i$ denote the defining sequence of $H' = H_i(d)$. Due to the two cases considered below we have that $d_2 \leqslant \cdots d_i$ and, if $i \geqslant 3$, $d_1 \leqslant d_3$. In particular H' is almost increasing. Recall that $H_t = H_t(d)$ and $H_i^{t+1} = H_i^{t+1}(d)$ are the subgraphs of $H_i(d)$ that consist of the t rightmost segments respectively the $i - t$ leftmost segments of $H_i(d)$, $0 \leqslant t \leqslant i+1$. Let $D_{\leqslant t}$ be a left determiner for (H, H_t), $0 \leqslant t < i$, and let $D_{\geqslant t}$ be a right determiner for (H, H_i^t), $2 \leqslant t \leqslant i + 1$, which exist due to Lemma 4.2.4.

Case 1: There is j, $1 \leqslant j \leqslant i - 2$, with $d_j > \max\{d_{j+1}, d_{j+2}\}$. We shall prove that $(H, H_i(d))$ is Ramsey infinite by constructing infinitely many Ramsey graphs. Let $a = \max\{d_{j+2}, d_{j+1}\} - 1$. Obtain a graph Γ' from $(a \oplus_{|E(H)|-1} D_{\leqslant j}) \sqcup D_{\geqslant j+3}$ by adding an edge between leftmost and rightmost vertex. Similarly obtain a graph Γ'' from $(a \oplus_{|E(H)|-1} D_{\leqslant j+1}) \sqcup D_{\geqslant j+3}$ by adding an edge between leftmost and rightmost vertex. For $n \geqslant 1$ let $\Gamma_n = D_{\geqslant j+3} \circ \Gamma'' \circ (\circ_n \Gamma') \circ D_{\leqslant i}$. See Figure 4.13 for an illustration in case $|E(H)| = 2$.

First we shall prove that $\Gamma_n \to (H, H_i)$. Consider a 2-coloring of the edges of Γ_n without red copy of H. We refer to bold and dashed edges like given in Figure 4.13, that is, an edge is dashed if its the longest edge in the copy Γ'' or in one of the copies of Γ', and an edge is bold if it has the same left endpoint as some dashed edge and its right endpoint is leftmost in a copy of $D_{\leqslant j+1}$ (in Γ'') or $D_{\leqslant j}$ (in Γ'). Observe that the left endpoint of each dashed edge is rightmost in a blue copy of H_i^{j+3} and left endpoint of at least $a + 1 \geqslant d_{j+2}$ blue edges. Hence, if a dashed edge is blue, then its right endpoint is rightmost in a blue copy of H_i^{j+2}.

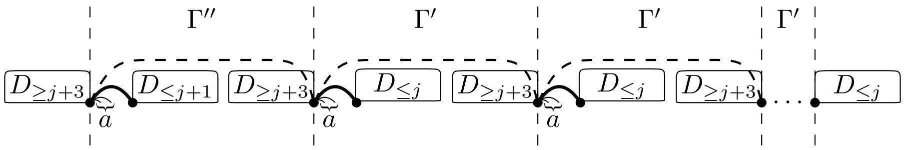


Figure 4.13: A Ramsey graph for (H, H_i) where H is a right star on two edges and H_i is a right caterpillar with at least three segments where $a = \max\{d_{j+2}, d_{j+1}\} - 1$ and $d_j > a + 1$.

First suppose that all dashed edges are blue. Consider the rightmost copy K of Γ' and its dashed edge xy. Then x is rightmost in a blue copy of H_i^{j+2} and y is leftmost in a blue copy of H_j. We see that the blue edge xy with a further blue edges in K yields a blue copy of H_i.

Now suppose that the leftmost dashed edge uv is red. Note that u is rightmost in a blue copy of H_i^{j+3}. Consider the rightmost vertex z such that the edge uz is blue. Since there are $a+1$ blue edges with left endpoint u (and uz is red), the edge uz is a bold edge. Since $a \geqslant d_{j+2} - 1$ and z is the leftmost vertex in a blue copy of $H_{\leqslant j+1}$, there is a blue copy of H_i.

Finally suppose that there is a blue dashed edge whose right endpoint w is incident to a red dashed edge. Consider the rightmost vertex z such that the edge wz is blue. Since there are $a+1$ blue edges with left endpoint w (and the dashed edge with left endpoint w is red), the edge wz is a bold edge. Recall that w is rightmost in a blue copy of H_i^{j+2}. Since $a \geqslant d_{j+1} - 1$ and z is leftmost in a blue copy of H_j, there is a blue copy of H_i. Altogether $\Gamma_n \to (H, H_i)$.

Next we shall show that each minimal Ramsey graph of (H, H_i) that is a subgraph of Γ_n contains all dashed edges, that is, contains at least $n+1$ edges. Let $\bar{\Gamma}$ be obtained from Γ_n by removing some dashed edge $\bar{e}$. We construct a coloring of $\bar{\Gamma}$ without red copies of H or blue copies of H_i as follows. Note that $\bar{\Gamma}$ consists of two connected components. First consider the component that contains the left endpoint of $\bar{e}$. Color all bold edges in this component in red, all other edges with left endpoint equal to the left endpoint of some bold edge in blue. All other edges form vertex disjoint copies of $D_{\geqslant j+3}$, $D_{\leqslant j+1}$, and $D_{\leqslant j}$. Color each of these determiners according to a good coloring. Clearly there is no red copy of H. Moreover there is no blue copy of H_i within one of the determiners. The blue edges not in one of the determiners form a right caterpillar where each segment has $a+1$ edges. Since $a + 1 < d_j$ and since the blue copy of H_i^{j+3} in in the leftmost copy of $D_{\geqslant j+3}$ is induced and isolated, there is no blue copy of H_i (but a blue copy of H_i^{j+1}).

Now consider the component that contains the right endpoint of $\bar{e}$. For each vertex p in this component that is the left endpoint of some dashed edge color this dashed edge and $|E(H)| - 2$ further edges with left endpoint p in red and all other edges with left endpoint p blue. The remaining edges form vertex disjoint copies

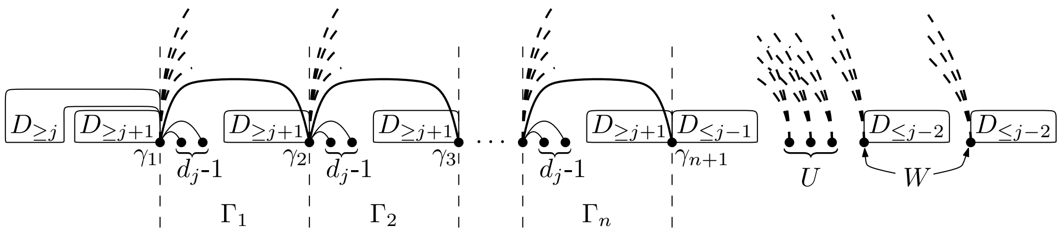


Figure 4.14: A Ramsey graph for (H, H_i) where H is a right star and H_i is a right caterpillar whose defining sequence contains $d_j < d_{j-1}$ for some j, $i \geqslant j \geqslant 3$. The dashed edges form a complete bipartite graph.

of $D_{\geqslant j+3}$ and $D_{\leqslant j}$. Color each of these determiners according to a good coloring. Clearly there is no red copy of H. Each component of the blue subgraph is contained in a copy of $D_{\geqslant j+3} \circ (a \oplus_{|E(H)|-1} D_{\leqslant j})$. As before there is no blue copy of H_i.

For each $n \geqslant 1$ choose a minimal Ramsey graph of (H, H') contained in Γ_n. The arguments before show that infinitely many of these graphs are pairwise non-isomorphic. Thus (H, H') is Ramsey infinite.

Case 2: There is j, $i \geqslant j \geqslant 3$, with $d_{j-1} > d_j$. We shall prove that (H, H_i) is Ramsey infinite by constructing infinitely many Ramsey graphs. An illustration of the following construction is given in Figure 4.14. Let Γ denote an ordered graph obtained from $\vec{S}_{d_j-1} \sqcup D_{\geqslant j+1}$ by adding an edge between the leftmost and the rightmost vertex (recall that $D_{\geqslant j+1}$ is a right determiner for (H, H_i^{j+1})). For $n \geqslant 1$ let F'_n be defined as follows. Start with a right determiner $D_{\geqslant j}$ for (H, H_i^j) and let $x < y$ denote its two rightmost vertices. Add a copy of $D_{\geqslant j+1}$ that has all its vertices to the right of x and has y as its rightmost vertex. Call the resulting graph D. To this graph D concatenate n copies $\Gamma_1, \ldots, \Gamma_n$ of Γ and a left determiner $D_{\leqslant j-1}$ for (H, H_{j-1}), one after the other in this order. Finally add $d_{j-1} - d_j > 0$ isolated vertices and an intervally disjoint union of $|E(H)| - 1$ left determiners $D_{\leqslant j-2}$ for (H, H_{j-2}) to the right of all current vertices. Altogether

$$F'_n = D \circ (\circ_n \Gamma) \circ D_{\leqslant j-1} \sqcup (\sqcup_{d_{j-1}-d_j} K_1) \sqcup (\sqcup_{|E(H)|-1} D_{\leqslant j-2}).$$

Let U be the set of isolated vertices and let W denote the set of leftmost vertices of each $D_{\leqslant j-2}$ added in the last step. We obtain an ordered graph F_n from F'_n by adding a complete bipartite graph between $U \cup W$ and the leftmost vertex of Γ_t for each $t \in [n]$.

First we shall prove that $F_n \to (H, H')$. For the sake of contradiction consider a 2-coloring of the edges of $F = F_n$ without red copies of H or blue copies of H'. Let γ_t denote the leftmost vertex of Γ_t in F, $1 \leqslant t \leqslant n$ and let γ_{n+1} be the rightmost vertex of Γ_n. We shall prove that γ_t is rightmost in a blue copy of H_i^j, $1 \leqslant t \leqslant n+1$, by induction on t. For $t = 1$ this holds since γ_1 is rightmost in a right determiner for

H_i^j. Consider $t > 1$. Since γ_{t-1} is left endpoint of $d_{j-1}+|E(H)|-1$ edges there are at least d_{j-1} blue edges with left endpoint γ_{t-1}. Consider the rightmost vertex z with $\gamma_{t-1}z$ colored blue. Since γ_{t-1} is rightmost in a blue copy of H_i^j, z is rightmost in a blue copy of H_i^{j-1}. Hence $z \notin W$, since otherwise there is a blue copy of H_i as each vertex in W is leftmost in a blue copy of H_{j-2}. Hence all edges between W and γ_{t-1} are red and, since $|W| = |E(H)|-1$, all other edges with left endpoint γ_{t-1} are blue. In particular $\gamma_{t-1}\gamma_t$ and $d_j - 1$ further edges $\gamma_{t-1}z$, with $\gamma_{t-1} < z < \gamma_t$, are blue. Since there is a blue copy of H_i^{j+1} with rightmost vertex γ_{t-1}, γ_t is rightmost is a blue copy of H_i^j. These arguments show that γ_{n+1} is rightmost in a blue copy of H_i^j. This forms a blue copy of H_i together with a blue copy of H_{j-1} in the left determiner $D_{\leqslant j-1}$ with leftmost vertex γ_{n+1}, a contradiction. Therefore $F_n \to (H, H')$.

Next we shall show that each minimal Ramsey graph of (H, H') that is a subgraph of F_n contains all edges $\gamma_t\gamma_{t+1}$, $1 \leqslant t \leqslant n$. Let $\bar{F}$ be obtained from F_n by removing the edge $\gamma_s\gamma_{s+1}$ for some s, $1 \leqslant s \leqslant n$. We construct a coloring of $\bar{\Gamma}$ without red copies of H or blue copies of H_i as follows. For each $t \leqslant s$ color all edges between γ_t and W red and all other edges with left endpoint γ_t blue. For each t, $s+1 \leqslant t \leqslant n$, color the edge $\gamma_t\gamma_{t+1}$ red and all other edges with left endpoint γ_t blue. The remaining edges are contained in an edge disjoint union of determiners and are colored according to a good coloring of each determiner. There are no red copies of H since a good coloring of a determiner has no red copy of H and each γ_t, $1 \leqslant t \leqslant n$ is left endpoint of at most $|W| = |E(H)| - 1$ red edges. Consider the unique vertex u in H_i that is contained in a copy of H_j and a copy of H_i^j. Due to the good colorings of the determiners u corresponds to one of the γ_ts in any blue copy of H_i. Observe that for each $t \geqslant s+1$ the vertex γ_t is not leftmost in a blue copy of H_j (though in a blue copy of H_{j-1}). Moreover it is not right endpoint of any blue edge. Hence there are no blue copies of H_i containing a vertex γ_t with $t \geqslant s+1$. Consider the vertices γ_t for $t \leqslant s-1$. Each of these is left endpoint of d_{j-1} blue edges, but $\gamma_t\gamma_{t+1}$ is the only such blue edge whose right endpoint has further neighbors to the right. Note that there are only $d_j - 1$ neighbors z of γ_t with $\gamma_t < z < \gamma_{t+1}$. Since $d_j < d_{j-1}$ and $j \geqslant 3$, γ_t is not leftmost in a blue copy of H_{j-1}. Therefore it is also not leftmost in a blue copy of H_j, since γ_{t+1} is leftmost in a blue copy of H_{j-1} otherwise. This shows that there is no blue copy of H_i.

For each $n \geqslant 1$ choose a minimal Ramsey graph of (H, H') contained in F_n. The arguments before show that infinitely many of these graphs are pairwise non-isomorphic. Thus (H, H') is Ramsey infinite. □

Proof of Theorem 4.7

In order to prove that (H, H') is Ramsey finite we shall show that each minimal ordered Ramsey graph of (H, H') is a member of a finite family of ordered graphs

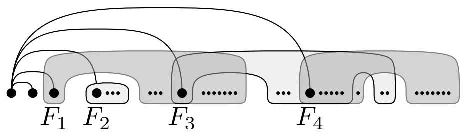


Figure 4.15: A Ramsey graph for (H, H_i) where H is a right star with four edges and H_i is a right caterpillar with $d_i = 2$. Here F_1, F_2, F_3, $F_4 \in R_<(H, H_{i-1})$ such that for any coloring without red copies of H there is a blue copy of H_{i-1} that contains the leftmost vertex of F_t, $t = 1, 2, 3, 4$. The graphs $F_1, \ldots, F_4$ might share vertices and edges as long as their leftmost vertices are mutually distinct.

defined below. Without loss of generality assume that H is a right star with s edges and H' is a right caterpillar with defining sequence $d = d_1, \ldots, d_i$ where either $i \leqslant 2$ or $d_1 \leqslant \cdots \leqslant d_i$. Recall that $H_t = H_t(d)$ is the subgraph of $H_i(d) = H'$ that consist of the t rightmost segments of $H_i(d)$, $0 \leqslant t \leqslant i$. Recursively define sets $\mathcal{F}_j$, $1 \leqslant j \leqslant i$, of ordered graphs as follows. Let $\mathcal{F}_1 = \{\vec{S}_{s+d_1-1}\}$. Consider $j > 1$. An ordered graph F is in $\mathcal{F}_j$ if and only if its leftmost vertex u has exactly $s + d_j - 1$ neighbors $v_1 < \cdots < v_{s+d_j-1}$ and there are (not necessarily disjoint) subgraphs $F_1, \ldots, F_s$ of F with $E(F - u) = \cup_{t=1}^{s} E(F_t)$, $F_t \in \mathcal{F}_{j-1}$, and v_{t+d_j-1} is leftmost in F_t, $1 \leqslant t \leqslant s$. See Figure 4.15. Note that for each $j \in [i]$ the set $\mathcal{F}_j$ is finite. We shall show that each minimal ordered Ramsey graph of (H, H_i) is in $\mathcal{F}_i$. Hence (H, H_i) is Ramsey finite.

First of all observe that for each $j \in [i]$ each graph in $\mathcal{F}_j$ is in $R_<(H, H_j)$. Even more, for each coloring of the edges of some graph $F \in \mathcal{F}_j$ without red copies of H there is a blue copy of H_j containing the leftmost vertex of F.

We consider the case $i \leqslant 2$ first. It is easy to see that $F \in R_<(H, H_1)$ if and only if F contains a copy of $\vec{S}_{s+d_1-1}$. In particular each graph in $R_<(H, H_1)$ contains some member of $\mathcal{F}_1 = \{\vec{S}_{s+d_1-1}\}$. Therefore $\vec{S}_{s+d_1-1}$ is the only minimal ordered Ramsey graph of (H, H_1). Consider an ordered graph F that does not contain any copy of some $F' \in \mathcal{F}_2$. We shall give a coloring of the edges of F without red copies of H or blue copies of H_2. Let D_1 denote the set of all vertices in F that are leftmost in a copy of some graph from $\mathcal{F}_1 = \{\vec{S}_{s+d_1-1}\}$ in F. For $u \in V(F)$ let $d_r(u)$ denote its *right degree*, that is, the number of edges uv in F with $u < v$. Note that $u \in D_1$ if and only if $d_r(u) \geqslant s + d_1 - 1$. We color the edges of F in three steps. In the first step color each edge uv, with $u < v$, red if $v \in D_1$, and there are $d_2 - 1$ vertices z with $u < z < v$. In the second step color as few further edges as possible red, such that for each $u \in V(F)$ there are at least $\min\{s - 1, d_r(u)\}$ red edges uv with $u < v$. In the last step color all yet uncolored edges blue. First assume for the sake of a contradiction that there is a blue copy of H_2. Let uv denote the longest edge incident to the leftmost vertex u in this copy. Then v is leftmost in a blue copy of H_1 and hence $d_r(v) \geqslant s + d_1 - 1$ due to the second step. In particular $v \in D_1$. Moreover there are $d_2 - 1$ vertices z with $u < z < v$. Hence uv is colored red in the first step, a contradiction as uv is blue. Next assume that there is a red copy of H. Then its

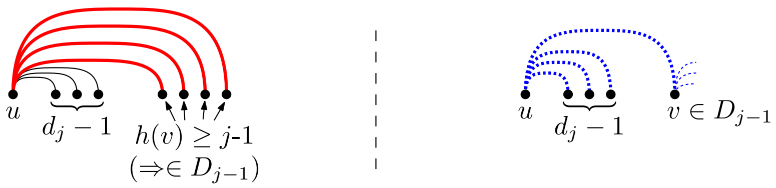


Figure 4.16: All edges uv with $u < v$, $h(u) = j - 1$, and $h(v) \geqslant j - 1$ are colored red if there are $d_j - 1$ vertices between u and v (left). Since $u \notin D_j$, u is not leftmost in a red copy of H. Moreover if u is leftmost in a blue copy of H_j, then $h(u) \geqslant j$ (and u is in D_j) since otherwise uv is colored red (right).

was created in the first step. Hence the leftmost vertex u of this red copy of H has $s + d_2 - 1$ neighbors to the right in F, the s rightmost of which are contained in D_1. Thus u is leftmost in a copy of some graph from $\mathcal{F}_2$, a contradiction. Altogether $F \notin R_<(H, H')$. This proves that a graph is in $R_<(H, H_2)$ if and only if it contains a copy of some $F' \in \mathcal{F}_2$. In particular each minimal ordered Ramsey graph of (H, H_i) is in $\mathcal{F}_i$ and hence (H, H_i) is Ramsey finite.

Next we consider the case $i \geqslant 3$. By assumption we have $d_1 \leqslant \cdots \leqslant d_i$. Observe that there is a copy of H_{j-1} in H_j that contains the leftmost vertex of H_j for each j, $2 \leqslant j \leqslant i$. Moreover, the leftmost vertex of each $F \in \mathcal{F}_j$ is contained in a copy of some $F' \in \mathcal{F}_{j-1}$ in F, $2 \leqslant j \leqslant i$. Recall that for each coloring of the edges of some graph $F \in \mathcal{F}_j$ without red copies of H there is a blue copy of H_j containing the leftmost vertex of F. Hence, for each $t \in [j]$, there is also a blue copy of H_t which contains the leftmost vertex of F under such a coloring. Consider an ordered graph F that does not contain any copy of some $F' \in \mathcal{F}_i$. We shall give a coloring of the edges of F without red copies of H or blue copies of H_i. Let $D_0 = V(F)$ and for $j \in [i]$ let D_j denote the set of all vertices in F that are leftmost in a copy of some graph from $\mathcal{F}_j$ in F. As argued above we have $\emptyset = D_i \subseteq D_{i-1} \cdots \subseteq D_1 \subseteq D_0$. For $u \in V(F)$ let $h(u)$ denote the largest j with $u \in D_j$. Color an edge uv, with $u < v$, red if and only if $h(u) \leqslant h(v)$ and there are $d_{h(u)+1} - 1$ vertices z with $u < z < v$.

For the sake of a contradiction assume that there is a red copy $\bar{H}$ of H. Let u denote the leftmost vertex in $\bar{H}$ and let $j = h(u)$. For each other vertex v in $\bar{H}$ there are $d_{j+1} - 1$ vertices z with $u < z < v$, $h(v) \geqslant j$, and hence $v \in D_j$, as argued above. Thus u is leftmost in a copy of some graph from $\mathcal{F}_{j+1}$ in F, a contradiction as $h(u) = j$. See Figure 4.16 (left).

Let H_0 denote the single vertex ordered graph. Next we shall prove by induction on j, $0 \leqslant j \leqslant i$, that for each vertex u which is leftmost in a blue copy of H_j we have $u \in D_j$. This clearly holds for $j = 0$. So consider $j > 0$ and a blue copy H'' of H_j. Let uv denote the longest edge incident to the leftmost vertex u of H''. Then v is leftmost in a blue copy of H_{j-1} and hence $v \in D_{j-1}$ by induction. In particular $h(v) \geqslant j - 1$. Moreover there are $d_j - 1$ vertices z with $u < z < v$. Hence $h(u) \geqslant j$,

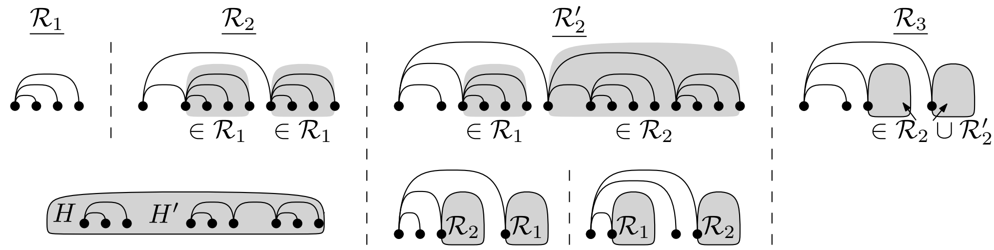


Figure 4.17: Ordered graphs in $\mathcal{R}_1$ (left), in $\mathcal{R}_2$ (middle left), in $\mathcal{R}_2'$ (middle right), and in $\mathcal{R}_3$ (right). The gray subgraphs are not necessarily disjoint.

since otherwise $h(u) \leqslant h(v)$ and $d_j - 1 \geqslant d_{h(u)+1} - 1$, and thus uv is colored red. See Figure 4.16 (right). Therefore $u \in D_j$. Since $D_i = \emptyset$ there is no blue copy of H_i.

Altogether $F \notin R_<(H, H_i)$. This proves that a graph is in $R_<(H, H_i)$ if and only if it contains a copy of some $F' \in \mathcal{F}_i$ (since each member of $\mathcal{F}_i$ is in $R_<(H, H_i)$, as argued above). Therefore, each minimal ordered Ramsey graph of (H, H') is contained in $\mathcal{F}_i$. In particular (H, H_i) is Ramsey finite. □

4.3 Further Observations and Constructions

Star and Caterpillar

The following lemma and Theorems 4.6 and 4.7 show that Conjecture 4.2 holds in case $i = 3$ and $|E(H)| = 2$. The proof of this lemma is very similar to the proof of Theorem 4.7, but there is a new kind of minimal ordered Ramsey graphs (those containing members of the sets $\mathcal{R}_2'$ in the proof, see Figure 4.17).

Lemma 4.3.1. *Let H be a right star and H' a right caterpillar with defining sequence d_1, d_2, d_3 where $d_3 \geqslant d_1 > d_2$. Then (H, H') is Ramsey finite.*

Proof. Recall that $\vec{S}_n$ denotes a right star with n edges. Recursively define families of ordered graphs as follows. Let $\mathcal{R}_1 = \{\vec{S}_{d_1+1}\}$ and let $\mathcal{R}_2$ consist of all ordered graphs F where the leftmost vertex u of F has two neighbors v such that there are $d_2 - 1$ edges uz with $u < z < v$ and v is leftmost in a copy of some member of $\mathcal{R}_1$ in F. Let $\mathcal{R}_2'$ consist of all ordered graphs F where the leftmost vertex u of F has two neighbors v_1, v_2 such that for each $i \in [2]$ the vertex v_i is leftmost in a copy G_i of a member of $\mathcal{R}_i$ and there are $d_{i+1} - 1$ edges uz with $u < z < v_i$, $z \notin \{v_1, v_2\}$. Finally let $\mathcal{R}_3$ consist of all ordered graphs F where the leftmost vertex u of F has two neighbors v such that there are $d_3 - 1$ edges uz with $u < z < v$ and v is leftmost in a copy of some member of $\mathcal{R}_2 \cup \mathcal{R}_2'$ in F. See Figure 4.17 for an illustration in case $d_1 = d_3 = 2$, $d_2 = 1$.

Note that each of the sets $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_2'$, and $\mathcal{R}_3$ has only finitely many minimal elements (under subgraph relation). Recall that $H_i = H_i(d)$, $i \in [3]$, is the right caterpillar with segments $\vec{S}_{d_i} \preceq \cdots \preceq \vec{S}_{d_1}$. Next, we establish some Ramsey properties of the members of the sets $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_2'$, $\mathcal{R}_3$. Clearly, for each $j \in [2]$ and any coloring of the edges of a member F of $\mathcal{R}_j$ without red copies of H there is a blue copy of H_j containing the leftmost vertex of F. Consider an ordered graph F in $\mathcal{R}'_2$ and a coloring of its edges without red copies of H and blue copies of H_3. We claim that there is a blue copy of H_2 which contains the leftmost vertex u of F. Let v_1 and v_2 denote neighbors of u in F such that for each $i \in [2]$ the vertex v_i is leftmost in a copy G_i of a member of $\mathcal{R}_i$ and there are $d_{i+1} - 1$ edges uz with $u < z < v_i$, $z \notin \{v_1, v_2\}$. Consider the rightmost vertex among v_1, v_2 such that uv_i is blue (at least one such edge exists since there is no red copy of H). By assumption, v_i is leftmost in a copy of G_i of some member of $\mathcal{R}_i$ and there are $d_{i+1} - 1$ neighbors z of u with $u < z < v_i$, $z \notin \{v_1, v_2\}$. Hence, there are $d_{i+1} - 1$ blue edges uz with $u < z < v_i$, no matter whether v_i is to the left or to the right of the vertex in $\{v_1, v_2\} \setminus \{v_i\}$. Since v_i is leftmost in a blue copy of H_i, as argued above, u is leftmost in a copy of H_{i+1}. By assumption there are no blue copies of H_3 and thus $i = 1$, that is, u is leftmost in a blue copy of H_2. Finally, the arguments above show that for any coloring of the edges of a member F of $\mathcal{R}_3$ without red copies of H there is a blue copy of H_3.

We claim that each graph in $R_<(H, H')$ contains a copy of a member of $\mathcal{R}_3$. Consider an ordered graph F with no copy of any member of $\mathcal{R}_3$. We shall give a coloring of the edges of F without red copies of H or blue copies of H'. Let $D_0 = V(F)$ and for $j \in [3]$ let D_j denote the set of all vertices in F that are leftmost in a copy of some graph from $\mathcal{R}_j$ in F. Further let D'_2 denote the set of all vertices in F that are leftmost in a copy of some graph from $\mathcal{R}'_2$ in F. Then $\emptyset = D_3 \subseteq D_1 \subseteq D_0$. (In contrast to the proof of Theorem 4.7 neither $D_2 \subseteq D_1$ nor $D_3 \subseteq D_2$ might hold.) Moreover $D'_2 \subseteq D_1$ since the leftmost vertex of each graph in $\mathcal{R}'_2$ has at least $d_3 + 1 \geqslant d_1 + 1$ neighbors to the right.

We color the edges of F in two steps. In the first step color each edge uv, with $u < v$, red if one of the following cases holds.

(a) $u \notin D_2 \cup D'_2$, $v \in D_1$, and there are $d_2 - 1$ edges uz with $u < z < v$.

(b) $u \notin D_3$, $v \in D_2 \cup D'_2$, and there are d_3 edges uz with $u < z < v$.

In the second step, consider each still uncolored edge uv, with $u < v$, and color it red if u is not a left endpoint of any red edge from the first step and uv is the shortest edge with left endpoint u, and color it blue otherwise. Note that, due to the second step, each vertex is either left endpoint of some red edge or not a left endpoint of any edge.

For the sake of contradiction assume that there is a red copy of H. Let $u < v_1 < v_2$ denote the vertices of such a copy. Then this copy was created in the first step of the coloring. We distinguish several cases. If uv_1 and uv_2 are colored by Case (a), then $u \notin D_2$ v_1, $v_2 \in D_1$ and there are $d_2 - 1$ edges uz with $u < z < v_1 < v_2$. Thus u is leftmost in a copy of some member of $\mathcal{R}_2$, a contradiction. If one of uv_1 or uv_2

is colored by Case (a) and the other by Case (b), then $u \notin D_2' \cup D_2$ and there is $x \in \{v_1, v_2\} \cap D_1$ such that $y \in \{v_1, v_2\} \setminus \{x\}$ is in $D_2 \cup D_2'$. So, if y is in D_2, then u is leftmost in a copy of some member of $\mathcal{R}_2'$ (note that there are d_3 edges uz with $u < z < y$ and thus $z \neq x$ for $d_3 - 1$ of them), and if y is in D_2', then $y \in D_1$ and u is leftmost in a copy of some member of $\mathcal{R}_2$. Either case yields a contradiction. Finally, if both of uv_1 and uv_2 are colored by Case (b), then $u \notin D_3$ and $v_1, v_2 \in \mathcal{R}_2 \cup \mathcal{R}_2'$. Thus u is leftmost in a copy of some member of $\mathcal{R}_3$, a contradiction. This shows that there is no red copy of H.

Next, we shall prove that for each $j \in [3]$ the leftmost vertex of some blue copy of H_j is in D_j or, if $j = 2$, in $D_j \cup D_j'$. This is clear for $j = 1$ due to the second step of the coloring. Consider the leftmost vertex u of some blue copy of H_2 and the longest edge uv incident to u in this blue copy. Then v is leftmost in a blue copy of H_1 and there are $d_2 - 1$ (blue) edges uz with $u < z < v$. Hence $v \in D_1$ as argued before. Since uv is not colored red by Case (a), $u \in D_2 \cup D_2'$ as desired. Finally, consider the leftmost vertex u of some blue copy of H_3 and the longest edge uv incident to u in this blue copy. Then v is leftmost in a blue copy of H_2, hence $v \in D_2 \cup D_2'$, and there are $d_3 - 1$ (blue) edges uz with $u < z < v$. Moreover u is leftmost in a copy of H_1, since $d_1 \leqslant d_3$, and hence $u \in D_1$. Further, there is some red edge uw with $u < w$. We distinguish several cases. If uw is colored red in the second step, then $w < v$ and there are d_3 edges uz with $u < z < v$. Hence $u \in D_3$, since uv is not colored red in Case (b) of the first step. In the remaining cases we shall find a contradiction. If uw is colored red in the first step by Case (a), then $u \notin D_2 \cup D_2'$, $w \in D_1$, and there are $d_2 - 1$ edges uz' with $u < z' < w$. Depending whether $v \in D_2$ or $v \in D_2' \subseteq D_1$, u is leftmost in a copy of some member of $\mathcal{R}_2$ or $\mathcal{R}_2'$, a contradiction. If uw is colored red in the first step by Case (b), then $u \notin D_3$ and $w \in D_2 \cup D_2'$. Hence u is leftmost in a copy of some member of $\mathcal{R}_3$, a contradiction. Altogether $u \in D_3$.

This shows that there is no blue copy of H_3, since $D_3 = \emptyset$. Therefore, each ordered graph in $R_<(H, H')$ contains a copy of some member of $\mathcal{R}_3$. Since each graph in $\mathcal{R}_3$ is in $R_<(H, H')$, as argued above, each minimal ordered Ramsey graph of (H, H') is contained in $\mathcal{R}_3$. Altogether (H, H') is Ramsey finite, since $\mathcal{R}_3$ is finite. □

Matchings

The results below show that for any ordered matching H with two edges that is not a monotone matching there is an ordered graph H' such that (H, H') is Ramsey infinite. Recall that for $k \geqslant 2$ an *all crossing k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_i u_{i+k}$, $1 \leqslant i \leqslant k$, and a *nested k-matching* is an ordered matching with vertices $u_1 < \cdots < u_{2k}$ and edges $u_i u_{2k+1-i}$, $1 \leqslant i \leqslant k$.

Lemma 4.3.2. *For each even $k \geqslant 2$ an all crossing k-matching is Ramsey infinite.*

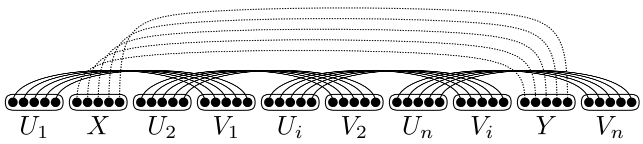


Figure 4.18: A minimal Ramsey graph F_n for an all crossing k-matching. Here $k = 6$, $n = 4$.

Proof. Let H denote an all crossing k-matching for some even $k \geqslant 2$. Further let n be a positive even integer and let X, Y, $U_1, \ldots, U_n$, and $V_1, \ldots, V_n$ denote disjoint ordered sets of vertices of size $k-1$ each, where $U_1 \prec X \prec U_2 \prec V_{n-1} \prec Y \prec V_n$, and, if $n \geqslant 3$, $U_i \prec V_{i-1} \prec U_{i+1} \prec V_i$, $2 \leqslant i \leqslant n-1$. Let F_n denote an ordered graph with vertex set $X \cup Y \cup \cup_{i \in [n]} (U_i \cup V_i)$ that consists of an all crossing $(k-1)$-matching between X and Y, as well as between U_i and V_i, $i \in [n]$. See Figure 4.18.

Consider a 2-coloring of the edges of F_n. We shall prove that there is a monochromatic copy of H. Observe that at least $\lceil (k-1)/2 \rceil$ edges between U_i and V_i are of the same color, $i \in [n]$. Let c_i denote the majority color of edges between U_i and V_i, $i \in [n]$. Further observe that all edges between $U_i \cup U_{i+1}$ and $V_i \cup V_{i+1}$ cross. If $c_i = c_{i+1}$ for some $i \in [n-1]$ then there is a monochromatic copy of H (in color c_i), since $2\lceil (k-1)/2 \rceil = k$ as k is even. So suppose that $c_i \neq c_{i+1}$ for each $i \in [n-1]$. Then $c_1 \neq c_n$, since n is even. Without loss of generality assume that there are $\lceil (k-1)/2 \rceil$ edges of color c_1 between X and Y. Since all edges between $U_1 \cup X$ and $V_1 \cup Y$ cross, there is a monochromatic copy of H in color c_1 similar as before. This shows that $F_n \in R_<(H)$.

Let F'_n be a minimal ordered Ramsey graph of H contained in F_n. We shall show that F'_n contains at least n edges. More precisely we shall show that there is an edge between U_i and V_i in F'_n for each $i \in [n]$. For the sake of contradiction assume that there are no edges between U_j and V_j in F'_n for some $j \in [n]$. We shall give a coloring without monochromatic copies of H. For each $i \in [j-1]$ color all edges between U_i and V_i red if i is even, and blue if i is odd. For each $i \in [n]$, with $i > j$, color all edges between U_i and V_i blue if i is even, and red if i is odd. Finally color all edges between X and Y red. Observe that all edges between U_1 and V_1 and all edges between U_n and V_n are blue. In particular there is no monochromatic copy of H. This shows that for each n there is a minimal ordered Ramsey graph of H with at least n edges. Thus (H, H) is Ramsey infinite. □

Lemma 4.3.3. *For each $k \geqslant 1$ a nested k-matching is Ramsey finite.*

Proof. Let H be a nested K-matching. We claim that $F \in R_<(H, H)$ if and only if F contains a nested matching with $2k-1$ edges. If F contains a nested matching with $2k-1$ edges, then clearly any 2-coloring of the edges of this matching yields a monochromatic copy of H. Hence $F \in R_<(H, H)$. Suppose that F does not contain a nested matching with $2k-1$ edges. For an edge e of F let $h(e)$ denote the largest

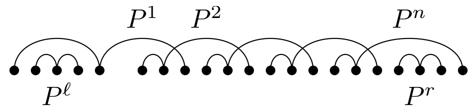


Figure 4.19: A minimal ordered Ramsey graph F_n for (H, H') where H is a nested 2-matching and H' is a monotone path on two edges. Here $n = 5$.

i such that there is a nested i-matching in F whose longest edge is e. Color an edge of F red if $h(e)$ is even and blue otherwise. Consider a monochromatic nested matching M with edges $e_1, \dots, e_n$, where e_i is longer than e_{i-1}, $1 < i \leqslant n$. Then $h(e_i) \geqslant h(e_{i-1}) + 2$ for each i, $2 \leqslant i \leqslant n$, since e_i and e_{i-1} are of the same color. Therefore $h(e_n) \geqslant h(e_1) + 2n - 2 \geqslant 2n - 1$. On the other hand $h(e_n) \leqslant 2k - 2$ and thus $n < k$. This shows that there is no monochromatic copy of H in F. Therefore a nested matching on $2k - 1$ edges is the only minimal ordered Ramsey graph for H. In particular (H, H) is Ramsey finite. □

Lemma 4.3.4. *Let H be a nested 2-matching and let H' be a monotone path on two edges. Then (H, H') is Ramsey infinite.*

Proof. For $n \geqslant 1$ let F_n denote a vertex disjoint union of $n + 2$ monotone paths P^ℓ, P^r, $P^1, \dots, P^n$ on two edges each, ordered like it is sketched in Figure 4.19. One can easily check that F_n is a minimal ordered Ramsey graph for (H, H') for each $n \geqslant 1$. Thus (H, H') is Ramsey infinite. □

Disconnected Graphs

Recall that an ordered graph H is *intervally connected* if for each nonempty interval I of vertices of H, that does not contain all vertices of H, there is an edge in H with one endpoint in I and one endpoint not in I. For an ordered graph H let $\widehat{H}$ denote the ordered graph that is a vertex disjoint union of H and a single edge uv, where u is to the left and v is to the right of all vertices in H.

Lemma 4.3.5. *If H is intervally connected and Ramsey infinite, then $\widehat{H}$ is Ramsey infinite.*

Proof. Let $v_1 < \cdots < v_n$ denote the vertices of H. Choose a smallest set S of integers from $[n-1]$ such that for each edge $v_i v_j$ of H with $i < j$ there is $p \in S$ with $i \leqslant p < j$. So S is a smallest set of positions such that each edge of H "covers" at least one position. Let F be a minimal ordered Ramsey graph of H. Obtain an ordered graph $\widehat{F}$ from H by adding an edge ab with $a < v_1 < v_n < b$ and for each $p \in S$ adding a copy F_p of F with $\{v_p\} \prec V(F_p) \prec \{v_{p+1}\}$. See Figure 4.20.

We claim that $\widehat{F}$ is a minimal ordered Ramsey graph for $\widehat{H}$. To see that $\widehat{F} \in R_<(H)$ consider a 2-coloring of the edges of $\widehat{F}$. Without loss of generality assume that ab is colored red. If all edges in the copy of H induced by $v_1, \dots, v_n$ are red, then there is a red $\widehat{H}$. Otherwise there is a blue edge $v_i v_j$ and some $p \in S$ with

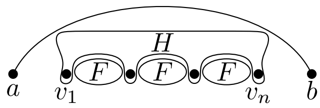


Figure 4.20: An ordered Ramsey graph for $\widehat{H}$ given some $F \in R_<(H)$.

$i \leqslant p < j$. Consider the copy F_p of F. Since $F \in R_<(H)$ there is a monochromatic copy of H, with vertices strictly between v_i and v_j. This monochromatic copy of H forms a monochromatic copy of $\widehat{H}$, either with v_iv_j or with ab. Hence $\widehat{F} \in R_<(\widehat{H})$.

Next we shall give for each edge e of $\widehat{F}$ a coloring of the edges of $\widehat{F} - e$ without monochromatic copy of $\widehat{H}$.

First of all observe that there is a coloring of the edges of F without red copy of $\widehat{H}$ or blue copy of H. Indeed, since F is minimal Ramsey for H one can remove a longest edge ℓ from F and color the remaining graph without monochromatic copies of H. Then coloring ℓ red clearly yields no blue copy of H. Moreover each red copy of H contains the edge ℓ. Therefore, since ℓ is a longest edge in F, there is no red copy of $\widehat{H}$.

Case 1: The edge e is in F_p for some $p \in S$. Color the edges of F_p without monochromatic copy of H and the edges of each F_q, $q \in S \setminus \{p\}$ without red copy of $\widehat{H}$ or blue copy of H. Observe that there is an edge $e = v_iv_j$ in $\widehat{F}$ such that p is the unique element of S with $i \leqslant p < j$, since S is a smallest set of positions. Color the edge e red and all remaining edges of $\widehat{F}$ blue. We claim that there is no monochromatic copy of $\widehat{H}$. Consider some copy H' of H in $\widehat{F}$. Since H' is intervally connected and not a single edge (as H is Ramsey infinite), the following holds. If H' does not contain any vertex from any F_p, $p \in S$, then $V(H') = \{v_1, \dots, v_n\}$ (it does not contain ab as it is intervally connected). If H' contains some vertex from F_q for some $q \in S$, then H' is contained in F_q (since there are no edges between F_q and the remaining graph). In the first case H' contains the red edge e and some blue edge. In the latter case H' is not monochromatic if $q = p$. If $q \neq p$ and H' is monochromatic, then H' is red. Therefore H' is clearly not part of any blue copy of $\widehat{H}$ Moreover H' is not part of any red copy of $\widehat{H}$, since there is no red copy of $\widehat{F}$ in F_q and either $q < i$ or $q \geqslant j$. Altogether there is no monochromatic copy of $\widehat{H}$.

Case 2: The edge e is induced by $\{a, b, v_1, \dots, v_n\}$. Color the edges of each F_q, $q \in S$, without red copies of $\widehat{H}$ or blue copies of H, and color all other edges blue. Clearly there is no red copy of $\widehat{H}$. Similar as above, there is at most one blue copy of H (induced by $v_1, \dots, v_n$) since H is intervally connected. Moreover this copy exists if and only if $e = ab$. hence there is no blue copy of $\widehat{H}$.

This shows that $\widehat{F}$ is a minimal Ramsey graph of $\widehat{H}$. Therefore $\widehat{H}$ is Ramsey infinite, since H has infinitely many non-isomorphic minimal ordered Ramsey graphs F and the graphs $\widehat{F}$ are distinct for distinct F. □

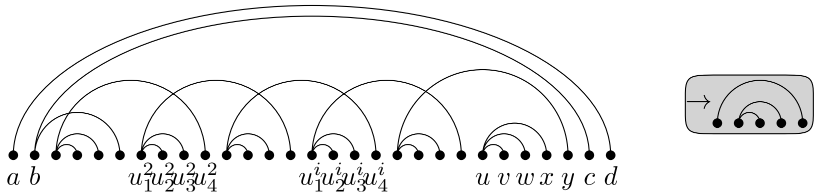


Figure 4.21: A minimal ordered Ramsey graph for $\widehat{H}$ where H is a right star with two edges.

The reverse statement of Lemma 4.3.5 is not true. A right star H on two edges is Ramsey finite due to Corollary 4.8. For $t \geqslant 1$ define an ordered graph G_t with vertex set $\{a, b, c, d, u, v, w, x, y\} \cup \bigcup_{i=1}^{t} \{u_1^i, u_2^i, u_3^i, u_4^i\}$ ordered like in Figure 4.21 and edges ad, bc, bu_1^4, $u_1^i u_2^i$, $u_1^i u_3^i$, $(1 \leqslant i \leqslant t)$, $u_1^i u_4^{i+1}$ $(1 \leqslant i < t)$, $u_t^1 y$, uv, uw, ux. One can see that G_t is a minimal ordered Ramsey graph of $\widehat{H}$ for each $t \geqslant 1$. Thus $\widehat{H}$ is Ramsey infinite.

χ-Avoidable Ordered Graphs

Lemma 4.3.6. *Each bonnet is Ramsey infinite.*

Proof. Consider a bonnet B on vertices $a < b < c < d < e$ and edges ab, ae, bc. Let $t \geqslant 1$ be an integer and consider ordered vertex sets $X = (x_1, \ldots, x_{12})$, $U_i = (u_1^i, \ldots, u_6^i)$, $1 \leqslant i \leqslant t$, and $U_{t+1} = (u_1^{t+1}, \ldots, u_5^{t+1}, v, u_6^{t+1})$. Define a graph G_t on $X \cup U_1 \cup \cdots \cup U_{t+1}$ with $X \prec U_1 \prec \cdots \prec U_{t+1}$ and edges

$$\begin{aligned}
&x_1x_2, x_1x_7, x_1x_{12}, x_3x_4, x_5x_6, x_8x_9, x_{10}x_{11},\\
&x_3u_1^1, x_3u_6^1, x_8u_1^1, x_8u_6^1,\\
&u_2^iu_3^i, u_4^iu_5^i, (1 \leqslant i \leqslant t+1),\\
&u_2^iu_1^{i+1}, u_2^iu_6^{i+1}, (1 \leqslant i \leqslant t),\\
&u_2^{t+1}v.
\end{aligned}$$

The graph G_3 is given in Figure 4.22 (top). For $t \geqslant 1$ let G_t' denote the graph obtained from G_t by removing $u_2^{t+1}, \ldots, u_5^{t+1}$, and v.

Claim. *For each $t \geqslant 1$ the edges $u_2^t u_1^{t+1}$ and $u_2^t u_6^{t+1}$ are of the same color for any 2-coloring of the edges of G_t' without monochromatic copy of B.*

Proof of Claim. We shall prove the claim by induction on t. Let c be a coloring of G_t' in red and blue without monochromatic copy of B. Consider $t = 1$. Without loss of generality assume that two edges incident to x_1 are red, say x_1x_7 and x_1x_{12}. Then the edges x_8x_9 and $x_{10}x_{11}$ are blue. Thus $x_8u_1^1$ and $x_8u_6^1$ are red, hence $u_2^1u_3^1$ and $u_4^1u_5^1$ are blue. This shows that $u_2^1u_1^2$ and $u_2u_6^2$ are both red.

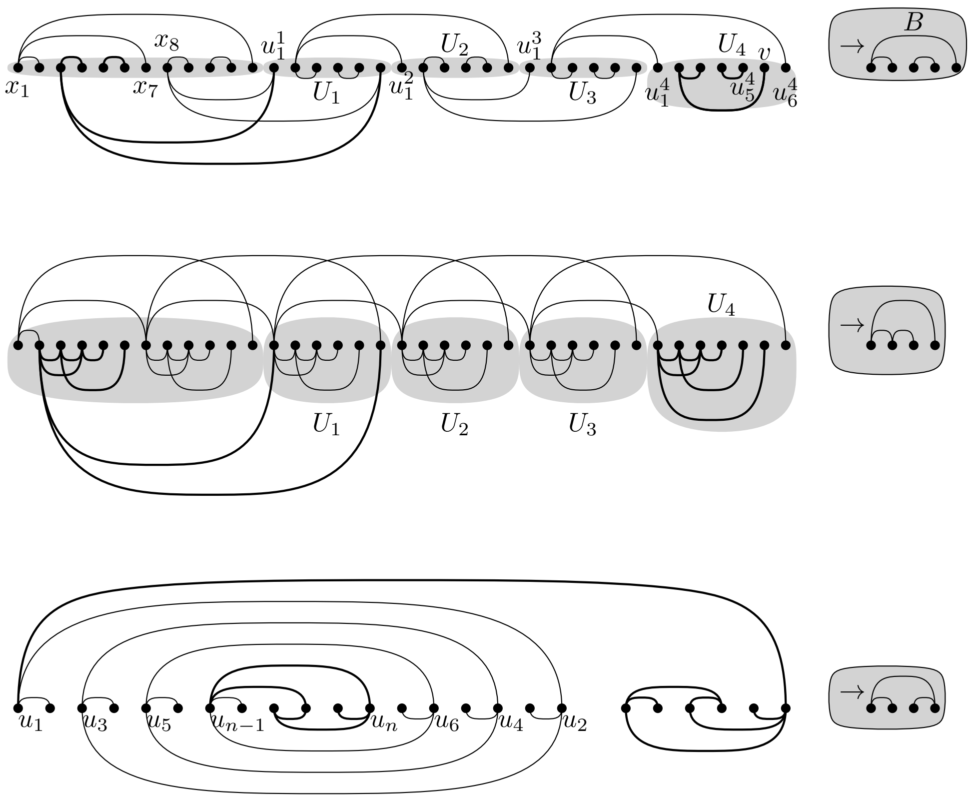


Figure 4.22: Ordered Ramsey graphs for bonnets.

Now consider $t \geqslant 2$. Observe that $x_1, \ldots, u_1^t, u_6^t$ induce a copy of G'_{t-1} in G'_t. By induction $u_1^{t-1}u_1^t$ and $u_1^{t-1}u_6^t$ are of the same color, say red. Then $u_2^t u_3^t$ and $u_4^t u_5^t$ are blue and hence $u_1^t u_1^{t+1}$ and $u_1^t u_6^{t+1}$ are red, which proves the claim. $\triangle$

Now consider a 2-coloring of the edges of G_t. By the claim either there is a monochromatic copy of B in a copy of G'_t in G_t or $u_2^t u_1^{t+1}$ and $u_2^t u_6^{t+1}$ are of the same color, say red. Consider the latter case and observe that the set $U = \{u_2^{t+1}, u_3^{t+1}, u_4^{t+1}, u_5^{t+1}, v\}$ induces a copy of B in G_t. So either one of the edges induced by U is red and there is a red copy of B, or all these edges are blue and there is a blue copy of B. This shows that $G_t \in R_<(B)$.

For each $t \geqslant 1$ let $G_t^\star$ be subgraph of G_t that is minimal Ramsey for B. We shall show that $G_t^\star$ contains the edges $u_2^i u_1^{i+1}$ for each i, $2 \leqslant i \leqslant t$. Consider $G_t - u_2^i u_1^{i+1}$ for some $i \in [t]$. If an edge e has its left endpoint in X or in U_j for some $j \leqslant i$, then color e red if has length 1 and blue otherwise. If an edge $e = xy$ has its left endpoint x in U_j for some $j > i$, then color e red if $x = u_2^j$ and $y \in \{u_3^j, u_1^{j+1}, v\}$ and blue otherwise. This coloring yields no monochromatic copies of B. This shows that $G_t^\star$ has at least $t - 1$ edges. Since t is arbitrarily large, B is Ramsey infinite.

With similar arguments infinitely many minimal ordered Ramsey graphs for the other bonnets are constructed, sketched in the middle and bottom of Figure 4.22. $\square$

4.4 Conclusions

In this chapter we study the structure of the set $R_<(H, H')$ of ordered Ramsey graphs for pairs (H, H') of ordered graphs. First of all Theorem 4.1 characterizes all such pairs (H, H') that have some forest in $R_<(H, H')$. A pair of unordered forests has a forest as a Ramsey graph if and only if one of the forests is a star forest. In contrast to this, there are pairs of ordered star forests that do not have any forest as a Ramsey graph.

Then we consider the question whether $R_<(H, H')$ contains only finitely many minimal elements. The corresponding question in the unordered setting is answered whenever $H = H'$ (Theorem 1.3), but a complete answer in the asymmetric case is known only if one of H or H' is a forest (Theorems 1.6, 1.7, 1.8, see Table 1.1 at the end of Section 1.4). Similar to the unordered setting we prove in Theorem 4.2 that any ordered graph H that contains a cycle is Ramsey infinite. Moreover Corollary 4.8 shows that a χ-unavoidable connected ordered graph H is Ramsey finite if and only if H is a star with center to the right or to the left of all its leafs. This is in contrast to the unordered setting where a connected graph is Ramsey finite if and only if it is a star with an odd number of edges (Theorem 1.3).

Intervally disjoint unions of Ramsey finite graphs are considered in Theorem 4.3. While such a union turns out to be Ramsey finite the reverse statement is open.

Question 4.1. *Let H, H_1 and H_2 denote ordered graphs such that $(H, H_1 \sqcup H_2)$ is Ramsey finite. Are both pairs (H, H_1) and (H, H_2) Ramsey finite?*

We do not consider pairs of an ordered forest and an ordered graph containing a cycle that are not handled by this theorem. It might be possible to follow the arguments of Łuczak [102] (see Theorem 1.7) to prove that all such pairs are Ramsey infinite.

Question 4.2. *Let H be an ordered graph that contains a cycle and let H' be an ordered forest that is not a monotone matching. Is (H, H') Ramsey infinite?*

Theorems 4.6 and 4.7 deal with connected χ-unavoidable graphs. The only pairs of connected χ-unavoidable ordered graphs that are not covered by these results are formed by a right (left) star and an almost increasing right (left) caterpillar with defining sequence $d_2 < d_1 \leqslant d_3 \leqslant \cdots \leqslant d_i$ for some $i \geqslant 3$. We conjecture that these pairs are Ramsey finite. In Section 4.3 we prove the following conjecture for the case $i = 3$ and $|E(H)| = 2$.

Conjecture 4.2. *Let (H, H') be a pair of χ-unavoidable connected ordered graphs with at least two edges each. Then (H, H') is Ramsey finite if and only if (H, H') is a pair of a right star and an almost increasing right caterpillar or a pair of a left star and an almost increasing left caterpillar.*

Theorem 4.7 shows that there are Ramsey finite pairs of ordered stars and ordered caterpillars of arbitrary diameter. Again this is in contrast to the unordered setting where for any Ramsey finite pair (H, H') of forests either one of H or H' is a matching or both are star forests (with additional constraints, see Theorem 1.8). So there are Ramsey infinite pairs of (unordered) graphs that admit orderings of their vertices which yield a Ramsey finite pair of ordered graphs. However this does not hold for all Ramsey infinite pairs of graphs, for example any graph that contains a cycle stays Ramsey infinite, no matter how its vertices are ordered. The other way round we conjecture that for any pair (H, H') of graphs (no matter whether Ramsey finite or infinite) there are orderings of the vertices that yield a Ramsey infinite pair of ordered graphs, unless H or H' contains only one edge.

Conjecture 4.3. *Let H and H' be (unordered) graphs with at least two edges each. Then there are orderings of the vertices of H and H' such that the pair of corresponding ordered graphs is Ramsey infinite.*

For disconnected ordered graphs we prove that any pair of an ordered graph and some monotone matching is Ramsey finite. This is similar to Theorem 1.6 stating that any pair of an unordered graph and some matching is Ramsey finite. Even more it is known that a matching is the only (unordered) graph that forms a Ramsey finite pair with all other graphs. We think that also this property carries over to monotone matchings. Some evidence for the following conjecture is given in Section 4.3 where we present results for other types of ordered matchings.

Conjecture 4.4. *For each ordered graph H that is not a monotone matching there is an ordered graph H' such that (H, H') is Ramsey infinite.*

Several cases are left open. First of all we do not handle disconnected ordered forests which are not covered by Corollary 4.4 (monotone matchings) or Theorem 4.5 (χ-unavoidable and no forest as Ramsey graph). We give some partial results in Section 4.3. An argument from [114] shows that any pair of (unordered) graphs that are not star forests is Ramsey infinite. As stated in the beginning of Section 4.1 this result is based on the fact the each (unordered) forest has Ramsey graphs of arbitrarily large girth. We retrieve this fact for χ-unavoidable ordered forests (Theorem 4.5) and conjecture that it holds for all ordered forests (see Conjecture 4.1 and Observation 4.2 in Section 1.5). One problem with χ-avoidable ordered forest is the lack of a characterization of such forests, see Chapter 2. We present results for some small χ-avoidable ordered forests in Section 4.3. As we do not have general results for χ-avoidable ordered forests we ask the following question.

Question 4.3. *Is each pair (H, H') of χ-avoidable ordered graphs Ramsey infinite?*

We think that the answer to the previous question is positive, based on the results from Section 4.3 and on Conjecture 4.1 and Observation 4.2. This leads to the following conjecture.

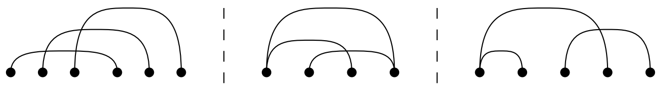


Figure 4.23: Ordered graphs that are not known to be Ramsey finite or infinite.

Conjecture 4.5. *Let (H, H') be a Ramsey finite pair of ordered forests. Then $R_<(H, H')$ contains a forest.*

We have seen in Theorem 4.6 (see also Conjecture 4.2) that the reverse statement of Conjecture 4.5 does not hold and that the family of all Ramsey finite pairs of ordered graphs might be rather diverse. Several ordered graphs which are not known to be Ramsey finite or infinite are given in Figure 4.23. Here the graph on the left is χ-unavoidable, the graph in the middle is a tangled path and hence χ-avoidable, and for the graph on the right we do not know whether it is χ-avoidable or not.

Future Work First of all one might be able to mimic the approaches in [114] and [102] for ordered graphs to identify more Ramsey infinite pairs of ordered graphs. Moreover there might be reductions like in Section 2.3 to prove that a graph is finite or infinite. For instance Theorem 4.3 and Lemma 4.3.5 (in Section 4.3) are of this kind. We are sure that the technique of signal senders and determiners can be further exploited to construct minimal ordered Ramsey graphs with desired properties. This approach is applied for graphs in [31, 33] and here for ordered graphs in the proof of Theorem 4.6. Besides constructing infinitely many minimal Ramsey graphs, this technique might be also useful to study other properties of the set of ordered Ramsey graphs. We mention here the questions for the minimum (left/right) degree, the largest bandwith, or the minimum number of pairs of crossing edges of minimal ordered Ramsey graphs, just to name a few. Of course, any question that is asked for graphs can be asked for ordered graphs.

We would like to emphasize two questions for ordered graphs. The first question is concerned with Ramsey equivalence of ordered graphs. So far we do not know any Ramsey equivalent pair of non-isomorphic ordered graphs.

Question 4.4. *Are there non-isomorphic ordered graphs G and H with $R_<(G) = R_<(H)$?*

For instance it is not clear how an analog to Observation 3.1 (on Ramsey equivalence of unions of graphs and isolated vertices) might look like. Figure 4.24 shows that an ordered K_3 is not Ramsey equivalent to any ordered graph formed by a union of K_3 and an isolated vertex. Further we observe here that for any ordered graph G and each minimal ordered Ramsey graph F of G there are colorings c_ℓ and c_r of the edges of F such that each monochromatic copy of G contains the leftmost vertex of F under c_ℓ and the rightmost vertex of F under c_r. This shows that if $G \overset{R}{\sim} H$ and $G \subseteq H$, then each copy of G in H contains the leftmost and the rightmost vertex of H.

Figure 4.24: Colorings of an ordered K_6 without monochromatic copies of $K_3 \sqcup K_1$ (left) or a monochromatic copy of K_3 with an isolated vertex between its vertices (right).

The second question is of algorithmic nature. The problem to decide for given graphs F, G, and H whether $F \to (G, H)$ is known to be coNP^{NP}-complete due to Schaefer [134] (loosely speaking, coNP^{NP} is the class of decision problems where certificates for No-instances can be verified in polynomial time by solving some problem in NP in constant time). More precisely, the problem stays coNP^{NP}-complete when H is a fixed tree and G is a complete graph (of variable order). Since the proof in [134] also uses a greedy embedding of a tree (which might fail for ordered trees as observed in the beginning of Section 2.1, see Observation 2.1), it is interesting to study the computational complexity of this problem for ordered graphs.

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