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On the averaging in strongly damped systems: the general approach and its application to asymptotic analysis of the Sommerfeld effect

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Abstract

Passage through resonance of an unbalanced rotor mounted on a strongly damped oscillating system that is excited by an asynchronous motor of limited power is investigated using an averaging procedure for a partially strongly damped system. The approach is closely related to a singular perturbation technique. It is demonstrated that the system's dynamics can be reduced on the slow manifold to a single first-order differential equation predicting both the stationary and transient solutions of the original system. Furthermore, the technique provides insight into the system's dynamics, enabling an outline of two simple methods to avoid capture into resonance. The first method is based on an active compensation of the small averaged vibrational torque, and the second method is a passive mechanical device reducing vibration amplitudes. The approximate results obtained were compared with the direct numerical simulations of the full system and demonstrate very good accuracy everywhere except in the vicinity of the bifurcation point. In this vicinity, the asymptotically long time interval is not sufficient for predicting capture into resonance.

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1. Introduction

In considering asymptotic procedures for nonlinear dynamics, the focus is typically on systems that are close to an integrable system [1]. However, systems with complete or partial strong damping are usually found in applications,

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especially in control systems. The strong recent trend toward light-weight designs makes mechanical structures extremely sensitive to any type of vibration excitation. Therefore, tightly focused damping has to be applied to reduce vibration amplitudes to a certain acceptable level without significantly decreasing the efficiency of a mechanism as a whole. The usual situation in which the vibration amplitudes have to be reduced is the classical Sommerfeld effect, which was first described by Arnold Sommerfeld [2] and later explained by Ilya Blekhman [3]. The main effect described by Sommerfeld concerns the possibility of capture into the resonance of an unbalanced rotor mounted on an oscillating carrier system and driven by an induction engine. The explanation given by Blekhman was based on an investigation of the stability of periodic solutions using the Poincaré method. Many important mathematical results have subsequently been achieved with respect to capture into resonance and scattering on the resonance in weakly damped systems. The fundamental results by Neishtadt [4,5] and further investigations by Pechenev [6,7] have to be mentioned here. The significant complexity of this analysis is connected to the appearance of the so-called “equivalent pendulum”, which has a frequency magnitude on the order of $\varepsilon^{1/2}$, where ε is the small parameter. However, Blekhman [8] already suggested an alternative heuristic approach allowing for the reduction of the averaged dynamics of the system to only one first-order equation. However, this approach replaces the equations of motion for the oscillatory part of the system using the corresponding pure forced solution obtained under the assumption of the steady state speed in the rotational part of the system.

The objectives of this paper are as follows:

- to provide a simple justification for the analysis methods of the Sommerfeld effect in the case of strong damping;
- to demonstrate the close relationship of these methods to systems with singular perturbations [9];
- to demonstrate the efficiency of the averaged equations for the design of active strategies enabling reliable passage through resonance; and
- to suggest a simple passive device enabling avoidance of capture into resonance.

The paper is structured as follows. The basic approach for the asymptotic analysis of a strongly damped system is briefly outlined in section 2. The approach is applied in section 3 to analyze passage through resonance in the simplest possible Sommerfeld-type system with a strongly damped oscillatory part. The obtained approximate results are compared to the direct numerical simulations of the full system and demonstrate very good accuracy everywhere except in the vicinity of the bifurcation point. Then, two simple methods are outlined to avoid capture into resonance (section 4). The first method is based on active compensation of the averaged vibrational torque, and the second method is a passive mechanical device that reduces vibration amplitudes. Finally, the primary results are summarized in section 5.

2. Averaging in strongly damped systems

A very simple approach for the asymptotic analysis of partially damped dynamic systems has been suggested by Fidlin and Thomsen [10]. This method is generalized in the present paper for the case of the damped part of the system that depends significantly on the weakly damped slow variables.

Consider the following system:

$$\dot{x} = \varepsilon X\left(x, y, t\right),\ \dot{y} = K\left(x\right) y + \varepsilon Y\left(x, y, t\right),\ x \in D \subset \mathbb{R}^n,\ \varepsilon << 1 \tag{1}$$

Assume that the logarithmic norm of the matrix $K\left(x\right)$ is strongly negative, i.e.,

$$\mu\left(K\right) = \lim_{h \to 0^+} \frac{\left\| E + hK \right\| - 1}{h} = \max\left\{ spectrum\left[\frac{1}{2}\left(K + K^T \right) \right] \right\} = K_0 < 0 \text{ for all } x \in D \tag{2}$$

Then, subject to the usual assumptions [9], the solution of this system can be obtained in the following form:

$$x = \sum_{i=0}^{\infty} \varepsilon^i \xi_i(\varepsilon t) + \sum_{i=0}^{\infty} \varepsilon^{i+1} \zeta_i(t); \; \|\zeta_i(t)\| \le C_i e^{-k_i t}, \, k_i > 0$$
$$y = \sum_{i=0}^{\infty} \varepsilon^{i+1} \gamma_i(\varepsilon t) + \sum_{i=0}^{\infty} \varepsilon^i \eta_i(t); \; \|\eta_i(t)\| \le D_i e^{-m_i t}, \, m_i > 0 \tag{3}$$

Limiting these solutions to the first terms, simple first-order approximations can be obtained, which enables the estimation of the exact solutions with a small mistake within the asymptotically long time interval:

$$\dot{\xi}_0 = \varepsilon \left\langle X(\xi_0, \eta_0, t) \right\rangle_t$$
$$\eta_0 = e^{K(\xi_0)t} y_0, \, \xi_0(0) = x_0 \tag{4}$$
$$\|x - \xi_0\| = O(\varepsilon), \, 0 \le t < O(\varepsilon^{-1})$$

Note that equations (1) are closely related to a singularly perturbed system. Through the use of simple rescaling of the independent variable the system can be shown to be the following:

$$x' = X(x, y, \varepsilon\tau)$$
$$\varepsilon y' = K(x) y + \varepsilon Y(x, y, \varepsilon\tau) \tag{5}$$
$$t = \varepsilon\tau$$

3. Analyzing the Sommerfeld effect as a partially strongly damped system

This approach will be used in this section for analyzing the passage through the resonance in a classical Sommerfeld system.

3.1. System under consideration and the governing equations

Consider the simplest system describing the interaction of an unbalanced rotor mounted on an oscillatory system with one degree of freedom with an asynchronous electric motor of limited power (see Fig. 1).

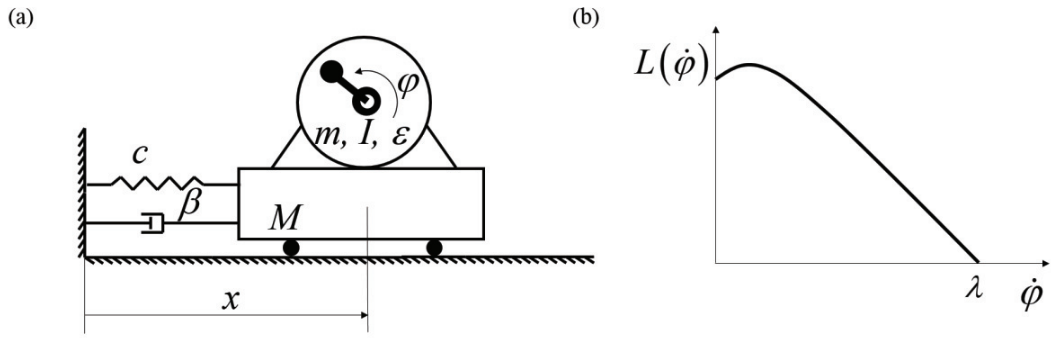

Fig. 1. (a) The simplest oscillating system interacting with an inertial exciter of limited power; (b) the torque characteristics of the induction motor.

The equations of motion for this system can be written as follows:

$$I\ddot{\varphi} = L(\dot{\varphi}) + m\varepsilon\ddot{x}\sin\varphi$$
$$M\ddot{x} + \beta\dot{x} + cx = m\varepsilon\left(\dot{\varphi}^2\cos\varphi + \ddot{\varphi}\sin\varphi\right) \tag{6}$$

Here, M is the mass of the oscillating system, I is the inertia of the rotor, ε is the eccentricity of the unbalanced mass m, and L is the torque characteristic of the induction motor, which takes the resistance in the bearings into account.

Introducing appropriate non-dimensional parameters and linearizing the torque characteristics of the induction motor

$$k = \sqrt{\frac{c}{M}};\ D = \frac{\beta}{2\sqrt{cM}};\ \tau = kt;\ (\ )' = \frac{d(\ )}{d\tau};\ L(\dot{\varphi}) - R(\dot{\varphi}) = U - V\dot{\varphi}$$
$$e = \frac{m^2\varepsilon^2}{MI} << 1;\ u = \frac{U}{k^2 I} = O(e);\ v = \frac{V}{kI} = O(e);\ \lambda = \frac{u}{v} = O(1);\ x = \frac{m}{M}\varepsilon\xi \tag{7}$$

the equations (6) can be transformed to the basic form for further analysis:

$$\varphi'' = v(\lambda - \varphi') + e\xi''\sin\varphi$$
$$\xi'' + 2D\xi' + \xi = \varphi'^2\cos\varphi + \varphi''\sin\varphi \tag{8}$$

Here, the parameters e and v are assumed to be small. However, the damping D is not assumed to be small, which significantly simplifies both the analysis and the dynamical behavior of the system during passage through resonance.

3.2. Averaged equation describing the dynamics of the system on the slow manifold

To obtain the equations governing the motion on the slow manifold, equations (8) must first be solved with respect to the highest derivatives. The terms $o\left(e^2\right)$ are neglected here:

$$\varphi'' = v(\lambda - \varphi') + eg(\xi, \xi', \varphi, \varphi')$$
$$\xi'' + 2D\xi' + \xi = \varphi'^2\cos\varphi + v(\lambda - \varphi')\sin\varphi + ef(\xi, \xi', \varphi, \varphi')$$
$$g(\xi, \xi', \varphi, \varphi') = e\left(\varphi'^2\cos\varphi - 2D\xi' - \xi\right)\sin\varphi \tag{9}$$
$$f(\xi, \xi', \varphi, \varphi') = g(\xi, \xi', \varphi, \varphi')\sin\varphi$$

Now, the pure forced solution of the second equation can be identified and used for the variables transformation:

$$\varphi' = \omega$$
$$\xi = \frac{\omega^2\left(1-\omega^2\right)\cos\varphi}{\left(1-\omega^2\right)^2 + 4D^2\omega^2} + \frac{2D\omega^3\sin\varphi}{\left(1-\omega^2\right)^2 + 4D^2\omega^2} + \zeta \tag{10}$$

This leads to the following equations:

$$\begin{aligned}
&\zeta'' + 2D\zeta' + \zeta = eg_1(\varphi,\omega,\zeta,\zeta')\sin\varphi \\
&\varphi' = \omega \\
&\omega' = v(\lambda-\omega) + eg_1(\varphi,\omega,\zeta,\zeta') \\
&g_1(\varphi,\omega,\zeta,\zeta') = g\left(\varphi,\omega,\xi(\varphi,\omega,\zeta),\xi'(\varphi,\omega,\zeta,\zeta')\right)
\end{aligned} \tag{11}$$

The first equation in (11) describes the strongly damped variable. This can easily be demonstrated by the following transformation:

$$\begin{aligned}
&\zeta = p\cos\left(\sqrt{1-D^2}t\right) + q\sin\left(\sqrt{1-D^2}t\right) \\
&\zeta' = -p\sqrt{1-D^2}\sin\left(\sqrt{1-D^2}t\right) + q\sqrt{1-D^2}\cos\left(\sqrt{1-D^2}t\right) \\
&p' = -Dp + eF\left(\varphi,\omega,p(t),q(t)\right) \\
&q' = -Dq + e\Phi\left(\varphi,\omega,p(t),q(t)\right) \\
&\varphi' = \omega \\
&\omega' = v(\lambda-\omega) + eg_2\left(\varphi,\omega,p(t),q(t)\right) \\
&g_2(\varphi,\omega,p,q) = g_1\left(\varphi,\omega,\zeta(p,q,t),\zeta'(p,q,t)\right)
\end{aligned} \tag{12}$$

The equations for p and q are strongly damped in the sense of (1) – (2), and the corresponding logarithmic norm is obviously negative. Therefore, these variables can be neglected in the first-order approximation, and only two equations remain for the rotational variables:

$$\begin{aligned}
&\varphi' = \omega \\
&\omega' = v(\lambda-\omega) + eg_2(\varphi,\omega,0,0)
\end{aligned} \tag{13}$$

By considering φ to be the new independent variable, a simple first-order equation can be obtained.

$$\frac{d\omega}{d\varphi} = v\left(\frac{\lambda}{\omega} - 1\right) + e\frac{g_2(\varphi,\omega,0,0)}{\omega} \tag{14}$$

This equation has the standard form for averaging as long as $\omega > 0$. The first-order approximation is

$$\frac{d\bar{\omega}}{d\varphi} = v\left(\frac{\lambda}{\bar{\omega}} - 1\right) - V(\bar{\omega});\ V(\bar{\omega}) = \frac{eD\bar{\omega}^4}{\left(1-\bar{\omega}^2\right)^2 + 4D^2\bar{\omega}^2} \tag{15}$$

Here, $\bar{\omega}$ is the averaged velocity of the rotor. This equation governs the slow dynamics of the rotor and enables the prediction of both its stationary behavior that depends on the parameters of the system and its transient behavior when accelerating from rest to the higher rotation speeds beyond the resonance domain. Note that this equation coincides exactly with that obtained by Blekhman [8].

3.3. Discussion of the results, comparison with the numerical simulations of the full system

Equation (15) determines, first of all, the stationary rotation speeds of the exciter and the stability, which is illustrated in Figure 2. It can be interpreted as follows. The thick black line shows the static characteristics of the asynchronous motor. The gray lines represent the vibrational torque $V(\omega)$ for different values of non-dimensional damping *D*. Depending on this parameter, there are either one or three intersections of these curves. Each of these intersections corresponds to the steady state operation point of the system. If there is only one intersection beneath the resonance at $\omega = 1$, then passage through the resonance is impossible (cf. the upper curve in Figure 2). If there are three intersections, then successful passage through the resonance or capture into the resonance depends on the initial velocity. Note that the second intersection, which is slightly beyond the resonance, is always unstable.

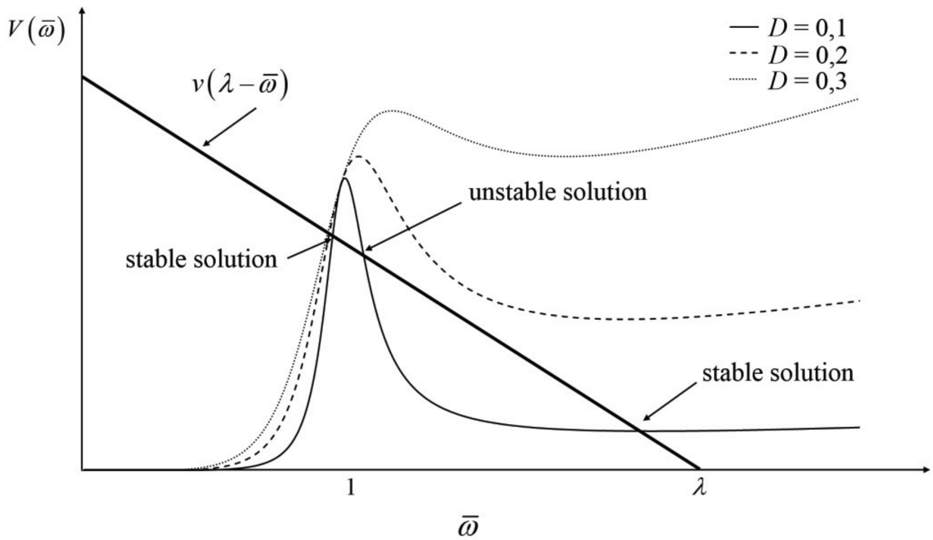

Fig. 2. Stationary regimes of the system and their stability.

Finally, a third case is possible but is not shown in Figure 2. If the vibrational torque in the resonance domain is always less than the static engine torque, then passage through the resonance should occur without any difficulty and the system will achieve a stationary operation point far beyond the resonance (close to the nominal operation speed λ).

Let us now consider the transient passage though the resonance. The whole system is assumed to be at rest at the beginning. Figures 3 – 6 show a comparison of the solution to equation (15) with the results of the direct numerical simulations. All of the simulations have been performed for $D = 0,2$; $e = 0,01$ and different values of λ. Figure 3 shows the results for the case of smooth passage through the resonance. The discrepancy between the solutions is extremely small even though the transient processes in the oscillating part of the system have been neglected by setting the variables *p* and *q* to zero.

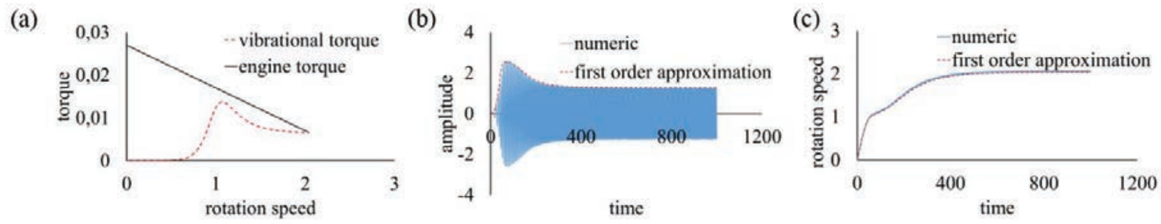

Fig. 3. Comparison of the approximate solution (dashed lines) with the direct numerical simulations (solid lines) for λ=2,7; (a) the torque diagram corresponding to Fig. 2; (b) amplitude of the transient oscillations; (c) rotation speed of the rotor.

Figure 4 shows the results for the case of capture into the resonance. Here, the discrepancy between the solutions is also extremely small.

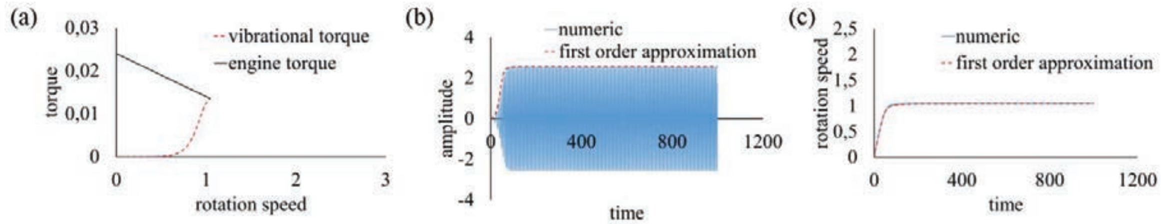

Fig. 4. Comparison of the approximate solution (dashed lines) with the direct numerical simulations (solid lines) for λ=2,4; (a) the torque diagram corresponding to Fig. 2; (b) amplitude of the transient oscillations; (c) rotation speed of the rotor.

However, the situation changes if the parameters are chosen close to the bifurcation point, i.e., to the parameters for which the static torque characteristic intersects the curve of the vibrational torque tangentially. In Figure 5, the system is still able to pass through the resonance, as predicted by equation (15). However, in Figure 6, the system passes through the resonance although the approximate equation predicts capture into resonance. The discrepancy between the predictions becomes considerably large.

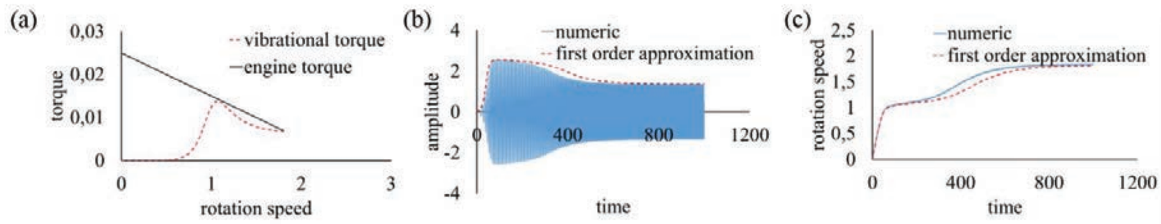

Fig. 5. Comparison of the approximate solution (dashed lines) with the direct numerical simulations (solid lines) for λ=2,5; (a) the torque diagram corresponding to Fig. 2; (b) amplitude of the transient oscillations; (c) rotation speed of the rotor.

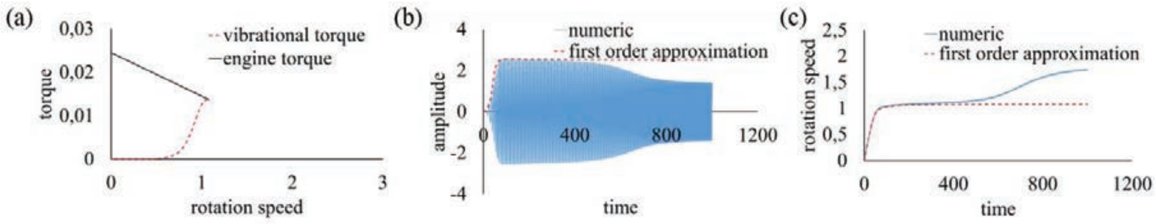

Fig. 6. Comparison of the approximate solution (dashed lines) with the direct numerical simulations (solid lines) for λ=2,45; (a) the torque diagram corresponding to Fig. 2; (b) amplitude of the transient oscillations; (c) rotation speed of the rotor.

This large discrepancy appears to contradict the main statement (4) predicting the closeness of the approximate and the exact solutions. However, this contradiction is only apparent. Statement (4) guarantees the closeness of the solutions on the asymptotically long time interval $O(1/\varepsilon)$. In the current case, the parameter is *e*=0,01, which means that a high accuracy can be expected for time ~100. This is fulfilled even in the most critical case, as shown in figure 6. The solutions diverge significantly later at times that are too large for the predictions based on averaging. Besides that, the discrepancy occurs only in the very small vicinity of the bifurcation point, which is expectable.

4. Using approximate equations to avoid capture into resonance

The approximate results obtained can be used to suggest active or passive strategies enabling avoidance of the undesired capture into resonance

4.1. Active compensation of the vibrational torque

The simplest active strategy enabling avoidance of capture into resonance is to completely compensate the vibrational torque. This compensation can be achieved by modulating the motor voltage as a function of the rotor speed:

$$u = \frac{eD\omega^5}{\left(1-\omega^2\right)^2 + 4D^2\omega^2}. \tag{16}$$

The effect is illustrated in figure 7 and shows the passage through the resonance in a system with a slowly increasing nominal rotation speed:

$$\lambda = \lambda_0 + \mu\lambda_1 t,\ \ \mu << 1. \tag{17}$$

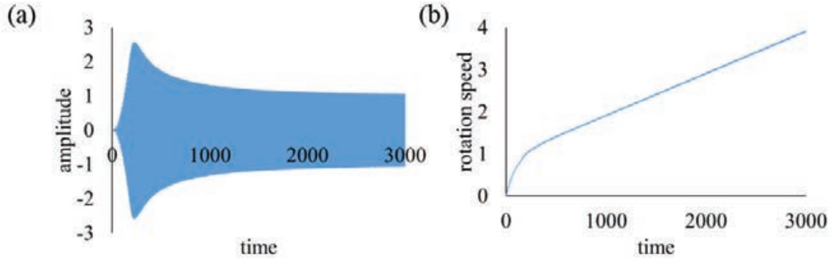

Fig. 7. Passage through resonance with a compensated vibrational torque according to (16); (a) amplitude of the transient oscillations; (b) rotation speed of the rotor.

It is easy to see that any desired speed of the rotor can be achieved without any difficulties.

4.2. Passive device based on the sequential friction-spring element

A passive device based on the sequential friction-spring element has been suggested by Fidlin and Lobos [1] for limiting vibration amplitudes in both self-excited and externally excited systems. These authors demonstrated that such a system is able to cut away the resonance in a 1 DOF oscillating system using two effects simultaneously:

- Permanent switching between the eigenfrequencies of the system with a sticking and sliding friction element within each oscillation period; this switching prevents the development of large resonant amplitudes; and
- Energy dissipation in the friction element while sliding.

The parameters of the friction-spring element can be chosen optimally to obtain both effects.

The described element can also be used to reduce the resonant amplitudes in the system under consideration and, thus, to avoid the Sommerfeld effect. For this purpose the system has to be modified, as shown in Figure 8.

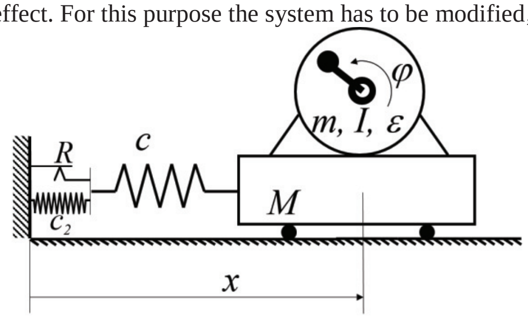

Fig. 8. The oscillating system interacting with the inertial exciter of limited power complemented with the sequential friction-spring element.

This system has been simulated numerically with the same slowly increasing nominal rotation speed determined by (17). The results are shown in Figure 9.

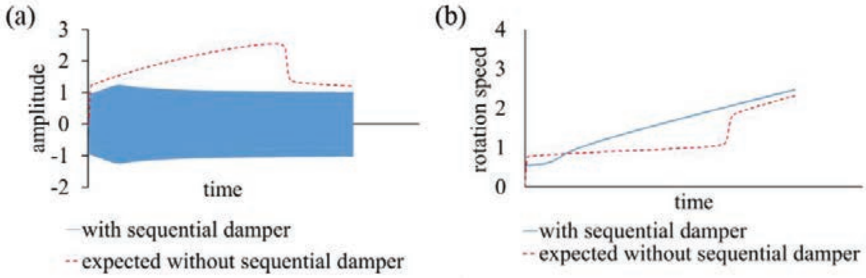

Fig. 9. Passage through the resonance in the system with the sequential friction-spring; (a) amplitude of the transient oscillations; (b) rotation speed of the rotor.

It is easy to see that this system also passes through resonance without any significant complications, although the unmodified system is expected to be captured by the resonance for a considerably long time interval. The sequential friction-spring element is active only within the resonance zone and thus does not dissipate energy as soon as passage through resonance is complete. However, the rotation speed of the rotor increases slower than in the case of active compensation and does not achieve the nominal rotation speed λ. The reason for this result is that the vibrational torque is still working in the overcritical domain and leads to the second stable solution according to figure 2.

5. Conclusions

Passage through resonance of an unbalanced rotor mounted on a strongly damped oscillating system and excited by an asynchronous motor of limited power is investigated using an averaging procedure for a partially strongly damped system. The analytical approximation coincides very well with the numerical results for both the stationary and the transient processes except for in the small parameter domain corresponding to the tangential intersection point. The approximate equations can be used for the active control of the passage through the mechanical resonance. The sequential friction-spring damper is a very effective passive solution.

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