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Local constitutive relations, i.e. a weak spatial dispersion, are usually considered in the effective description of metamaterials. However, they are often insufficient and effects due to a nonlocality, i.e. a strong spatial dispersion, are encountered. Recently (K. Mnasri *et al.*, arXiv:1705.10969), a generic form for a nonlocal constitutive relation has been introduced that could accurately describe the bulk properties of a metamaterial in terms of a dispersion relation. However, the description of functional devices made from such nonlocal metamaterials also requires the identification of suitable interface conditions. In this contribution, we derive the interface conditions for such nonlocal metamaterials.

I. INTRODUCTION

Metamaterials greatly enhance our abilities to control the propagation of electromagnetic waves. They derive their properties from unit cells, called the meta-atoms, that interact with light in quite a peculiar way. Metamaterials attracted attention as meta-atoms can be perceived that show, e.g. a strong magnetic response. As the properties of the meta-atom translate to the properties of the entire material, novel materials came in reach that possess a magnetic dispersion. Such materials are of fundamental scientific interest and unlock simultaneously a plethora of applications. Exemplarily, we can mention perfect lenses [1], cloaks [2], broadband anti-reflection coatings [3, 4], or directional antennas [5]; but there exist many more examples.

A key requirement to link the actual metamaterial made from structured unit cells to a conceptual homogenous material is the assignment of effective properties. Of course, describing the light propagation through a device that has metamaterial constituents can always be done while solving the full Maxwell problem, but this is impractical for larger structures and indeed quite tedious. Moreover, just as we do not consider ordinary materials in the context of electrodynamics as being composed of atoms, it is the philosophy of an actual metamaterial to discuss it in terms of emerging properties, i.e. effective material properties. Then, the metamaterial is considered as homogenous.

The procedure of assigning effective properties to a metamaterial is performed within a parameter retrieval. The effective parameters then shall allow to describe the light propagation through the metamaterial with just the same accuracy as a full wave solution would do. Technically, there are different ways of assigning effective properties [6–14]. Basic averaging procedures [9] that consider the metamaterial as an ensemble of polarisable dipoles do the job if the metamaterial is strongly diluted and the polarizability of the meta-atoms is not too strong. For most metamaterials made of strongly scattering meta-atoms that are densely packed, retrieval procedures were developed that rely on considering reflection and transmission from a slab [15–19] or that average the fields in the unit cells [9, 20]. Also, retrievals based on the dispersion relation and the wave impedance of the Bloch modes were suggested [21–24] or by dual field interpolation in Ref. 25. However, how such a retrieval is done is a rather technical question. At the heart, the philosophy of all these methods is to postulate a specific constitutive relation the metamaterial shall obey and then to link the parameters of these constitutive relations to parameters that can be measured or simulated.

In all these aforementioned retrieval methods, local constitutive relations were considered. Under such assumption, the electric field $\vec{\mathrm{E}}(\mathbf{r}, t)$ and the magnetizing field $\vec{\mathrm{H}}(\mathbf{r}, t)$ are linked locally to the electric displacement field $\vec{\mathrm{D}}(\mathbf{r}, t)$ and the magnetic field $\vec{\mathrm{B}}(\mathbf{r}, t)$. It has been shown that to observe a local magnetic response for a given metamaterial it has to have actually an electric response that corresponds to weak spatial dispersion [26–28]. The weak spatial dispersion is captured by a constitutive relation where the electric displacement depends on the second spatial derivative of the electric field at the considered point in space. In frequency space it reads as

$$D_i(\mathbf{r}, \omega) = \epsilon_{ij}(\omega) E_j(\mathbf{r}, \omega) + a_{ijkl}(\omega) \partial_k \partial_l E_j(\mathbf{r}, \omega). \tag{1}$$

A possible first derivative is linked to bi-anisotropic effects and shall not be considered here, i.e. we consider central-symmetric metamaterials. Therefore, it is omitted. By exploiting a suitable gauge transformation, this weak spatial dispersion in the electric response can be recast to appear as a local magnetic response. This, however, is only possible if the following assumption holds

$$a_{ijkl}(\omega) \partial_j \partial_k E_l(\mathbf{r}, \omega) \stackrel{!}{=} (\nabla \times [\alpha \nabla \times \mathbf{E}(\mathbf{r}, \omega)])_i. \tag{2}$$

This assumption is not valid for centro-symmetric systems but is a prerequisite to end up with local constitutive relations; hence it has been assumed to be valid. This is quite practical as for a local medium the interface conditions are known. These interface conditions, in general, express how the different fields involved behave at an interface from one medium to another. For a local medium we require that the tangential component of the electric and the magnetizing field and the normal component of the electric displacement and the magnetic field are continuous. Having interface conditions available is of fundamental importance to describe the light propagation through any kind of device that is made from a metamaterial. We need to know them as soon as the metamaterial is no longer infinitely extended in all spatial directions.

However, it got more and more obvious that such a local description is quite limited for two reasons. Foremost, it is easy to verify that the assumption postulated in expression 2 is not always justified. For a centro-symmetric metamaterial, a symmetry frequently encountered, additional terms of the form (cf. Ch. 3.2 in Ref 29)

$$\sum_{j \in \{x, y, z\}} \partial_j \left(a_{jjjj} \partial_j \mathbf{E} \right) \tag{3}$$

appear that are strictly required to be zero if the local description shall be valid. However, this cannot be guaranteed *per se*. Moreover, there is clear evidence that it is not justified to truncate the Taylor expansion in the constitutive

relation after the second term, which essentially leads to Eqn. 1. Whereas local constitutive relations strictly lead to elliptical or hyperbolic dispersion relations, the numerically computed dispersion relation of selected metamaterials possesses such properties only in quite a limited reciprocal space, i.e. for small values of k_x and k_y when considering the z-axis as the principle propagation direction. To accurately describe the dispersion relation of metamaterials requires to take higher order terms in the aforementioned Taylor expansion into account. As we concentrate on centro-symmetric materials where all odd orders vanish, the next higher order terms would be the fourth order term. Nonlocal materials, in general, are characterized by constitutive relations where the response is a functional of a set of fields over the material or of derivatives of the field. Such nonlocal materials are sometimes also refereed to as nonsimple [30].

Advanced constitutive relations that mitigate both problems were suggested only recently [31]. They constitute an immediate extension to the constitutive relations considered in Eqn. 1 and read as:

$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon(\omega)\mathbf{E}(\mathbf{r}, \omega) + \nabla \times (\alpha(\omega)\nabla \times \mathbf{E}(\mathbf{r}, \omega)) + \sum_{j \in \{x,y,z\}} \partial_j \left(a_{jjjj}\partial_j \mathbf{E}\right) + \nabla \times \nabla \times (\gamma(\omega)\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega))$$

(4)

where a tensorial notation has been introduced and the exact meaning of the quantities will be explained in detail below. Note that for the last term an assumption comparable to the assumption in expression 2 has been enforced. In particular, we require that

$$\sum_{j,k,l,m,n} a_{ijklmn}(\omega)\partial_j \partial_k \partial_l \partial_m E_n(\mathbf{r}, \omega) \stackrel{!}{=} (\nabla \times \nabla \times [\alpha \nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega)])_i$$

(5)

holds. Only with such an assumption the dispersion relation will be of the same order as the dispersion relation that emerges when considering the term in expression 3 as the extension. Therefore, they appear on equal footing. Please also note, in what follows we will use the notations β_j instead of a_{jjjj}. It has to be stressed that also other nonlocal constitutive relations where explored in the context of metamaterials but they frequently assume a specific geometry [32–36]. Also, additional boundary conditions have to be considered to solve the interface problems. These additional boundary conditions are usually introduced on phenomenological grounds [37, 38] and were controversially discussed in Refs. 39–41.

Recently [31], we could show that the dispersion relation of a referential metamaterial can be much better described with the constitutive relation of Eqn. 4. However, this analysis only considered the bulk properties, and the characteristics of the eigenmodes were discussed. To fully capitalise on such constitutive relations, we need to equip them with suitable interface conditions to describe the optical response from basic functional elements made from such metamaterials. Potentially the simplest example for such functional element is a slab and we wish to know how light is reflected and transmitted from such a slab.

In this contribution, we derive the urgently needed interface conditions from first principles by relying on a generalized (weak) formulation of Maxwell's equations. These interface conditions allow to find the Fresnel equations for the reflection and transmission of a plane wave and for both polarizations from a slab. These Fresnel equations are also derived in this contribution. The paper is made to be as self-sustained as possible, which explains its mathematical attitude.

The manuscript is structured such that in Section II we detail Maxwell's equations, the constitutive relations we are looking at, and the considered geometry. In Section III we discuss their generalized solutions. In Section IV we derive the associated interface conditions and in Section V we outline the Fresnel equations to compute the reflection and transmission from a slab of such a homogenous metamaterial characterised by nonlocal constitutive relations. The interface conditions we derive in Section IV are valid under specific smoothness assumptions. However, they remain valid also without these smoothness assumptions in a generalised sense which requires the concept of traces. Therefore, the last section is devoted to this concept of the traces. We conclude on our work in a devoted finalizing section. Finally, in an Appendix we present a short overview on the concept of generalized functions.

Prior starting with the derivation, we introduce some notations that will be used throughout the paper. Below, Ω is an *open* domain in $\mathbb{R}^n$.

- By $\mathbf{r} = (x, y, z)$ we denote points in $\mathbb{R}^3$ (spatial variable), by $t \in \mathbb{R}$ we denote a time variable.

- $\mathbb{R}_+^3 := \{\mathbf{r} \in \mathbb{R}^3 : z > 0\}$, $\mathbb{R}_-^3 := \{\mathbf{r} \in \mathbb{R}^3 : z < 0\}$, $\Gamma = \{\mathbf{r} \in \mathbb{R}^3 : z = 0\}$. They denote the different half-spaces above and below the interface we consider.

- ∇, $\nabla\cdot$, $\nabla\times$ are, respectively, the gradient, divergence, and curl with respect to $\mathbf{r}$.

- $^\mathrm{T}$ is the operation of transposition.

- $\mathbf{n} = (0,0,1)^\mathrm{T}$ is the unit normal vector on Γ.

- For $\Psi : \Gamma \to \mathbb{C}$ we set $\nabla_\Gamma \Psi := (\partial_x \Psi, \partial_y \Psi, 0)^\mathrm{T}$.

- For $\mathbf{\Phi} = (\mathbf{\Phi}_x, \mathbf{\Phi}_y, \mathbf{\Phi}_z) : \Gamma \to \mathbb{C}^3$ we set $\nabla_\Gamma \cdot \mathbf{\Phi} := \partial_x \mathbf{\Phi}_x + \partial_y \mathbf{\Phi}_y$.

- $\mathsf{C}^m(\Omega)$, $m \leq \infty$, is the space of functions with continuous derivatives up to order m on Ω.

- $\mathsf{C}^m(\overline{\Omega})$, $m \leq \infty$, is the space of all restrictions of functions in $\mathsf{C}^m(\mathbb{R}^n)$ to $\overline{\Omega}$.

- $\mathsf{C}_0^m(\Omega)$ is the space of functions $f \in \mathsf{C}^m(\Omega)$ having compact support in Ω (i.e., $\mathrm{supp}(f) := \overline{\{x :\ f(x) \neq 0\}}$ is a compact set contained in Ω).

- $\mathsf{D}'(\Omega)$ is the space of generalized scalar functions on Ω (i.e., linear continuous functionals acting on $\mathsf{C}_0^\infty(\Omega)$; see Appendix).

- $\mathsf{L}^2(\Omega)$ is the space of measurable functions $\Psi : \Omega \to \mathbb{C}$ satisfying $\int_\Omega |\Psi(\mathbf{r})|^2 \, \mathrm{d}\mathbf{r} < \infty$.

- $\mathsf{H}^1(\Omega)$ is the space of measurable functions $\Psi : \Omega \to \mathbb{C}$ such that $\Psi \in \mathsf{L}^2(\Omega)$ and each component of $\nabla\Psi$ belongs to $\mathsf{L}^2(\Omega)$. Hereinafter, differential operations are understood in the generalized sense.

- $\mathbf{C}^m(\Omega)$ (resp. $\mathbf{C}^m(\overline{\Omega})$, $\mathbf{C}_0^m(\Omega)$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$) is the space of vector-functions $\mathbf{\Phi} : \Omega \to \mathbb{C}^3$ with components being in $\mathsf{C}^m(\Omega)$ (resp. $\mathsf{C}^m(\overline{\Omega})$, $\mathsf{C}_0^m(\Omega)$, $\mathsf{L}^2(\Omega)$, $\mathsf{H}^1(\Omega)$).

- $\mathbf{D}'(\Omega)$ is the space of generalized vector functions on Ω (i.e., linear functionals acting on $\mathbf{C}_0^\infty(\Omega)$).

- $\mathsf{L}_{\mathrm{loc}}^2(\Omega)$ (resp. $\mathbf{L}_{\mathrm{loc}}^2(\Omega)$, $\mathsf{H}_{\mathrm{loc}}^1(\Omega)$, $\mathbf{H}_{\mathrm{loc}}^1(\Omega)$) is the space of (scalar or vector-valued) functions belonging to $\mathsf{L}^2(\widehat{\Omega})$ (resp. $\mathbf{L}^2(\widehat{\Omega})$, $\mathsf{H}^1(\widehat{\Omega})$, $\mathbf{H}^1(\widehat{\Omega})$) for each bounded subdomain $\widehat{\Omega} \subset \Omega$.

- $\mathbf{I}$ is the identity (3×3)-matrix .

II.   MAXWELL'S EQUATIONS AND THE CONSIDERED CONSTITUTIVE RELATIONS

We assume that the upper half-space $\mathbb{R}_+^3$ is occupied by vacuum, while the lower subspace $\mathbb{R}_-^3$ is occupied by a homogeneous metamaterial. We want to derive the interface conditions for this situation. Recall that Γ is the interface between them.

As usual, we denote by $\vec{\mathrm{E}}(\mathbf{r}, t)$, $\vec{\mathrm{B}}(\mathbf{r}, t)$, $\vec{\mathrm{D}}(\mathbf{r}, t)$ and $\vec{\mathrm{H}}(\mathbf{r}, t)$ the electric field, the magnetic field, the electric displacement field, and the magnetizing field. Assuming the absence of external charges and currents, we conclude that they obey Maxwell's equations (in Gaussian units) in $\mathbb{R}^3 \times \mathbb{R}$:

$$\nabla \times \vec{\mathrm{E}} + \frac{1}{c}\frac{\partial \vec{\mathrm{B}}}{\partial t} = 0, \quad \nabla \times \vec{\mathrm{H}} - \frac{1}{c}\frac{\partial \vec{\mathrm{D}}}{\partial t} = 0, \quad \nabla \cdot \vec{\mathrm{D}} = 0, \ \nabla \cdot \vec{\mathrm{B}} = 0.$$

We are interested in monochromatic waves, i.e., waves of the form

$$(\vec{\mathrm{E}}, \vec{\mathrm{B}}, \vec{\mathrm{D}}, \vec{\mathrm{H}}) = (\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}) \exp(-i\omega t),$$

where $\omega > 0$ is a frequency, and the vector functions $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ depend only on the spatial variable $\mathbf{r}$. Plugging this ansatz into Maxwell's equations we arrive at

$$\nabla \times \mathbf{E} - \frac{i\omega}{c}\mathbf{B} = 0, \quad \nabla \times \mathbf{H} + \frac{i\omega}{c}\mathbf{D} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{6}$$

Equations 6 must be supplemented by constitutive relations linking the "induced" fields $\mathbf{D}, \mathbf{H}$ with the "physical" fields $\mathbf{E}, \mathbf{B}$:

$$\mathbf{D} = \mathbf{E} + \mathbf{P}(\mathbf{E}), \qquad \mathbf{H} = \mathbf{B} - \mathbf{M}(\mathbf{H}),$$

where $\mathbf{P}$ and $\mathbf{M}$ are the polarization and the magnetization, respectively. Note that here we are concerned only with centro-symmetric metamaterials where electro-magnetic coupling effects are not encountered. Therefore, the polarization depends only on $\mathbf{E}$ and the magnetization depends only on $\mathbf{H}$. The consideration of non-centro-symmetric materials, of course, can be integrated into the discussion but it does not affect our conclusions.

We can assume that $\mathbf{M} \equiv 0$. This simply suggests that all the natural materials we are considering are non-magnetic in their intrinsic response. The magnetic effects we possibly observe always emerge from a nonlocal electric response. Under such assumption we easily get from Eqns. 6 the following equation for $\mathbf{E}$:

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \left(\mathbf{E} + \mathbf{P}(\mathbf{E}) \right). \tag{7}$$

We assume that the upper half space is filled by vacuum. Hence, for $z > 0$ one has $\mathbf{P} \equiv 0$, while for $z < 0$ (i.e., in the metamaterial) we assume the following constitutive relations that resemble those already mentioned in Eqn. 4:

$$\mathbf{P}(\mathbf{E}) = (\varepsilon - 1)\mathbf{E} + \nabla \times \alpha \left(\nabla \times \mathbf{E} \right) + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \left(\nabla \times \nabla \times \mathbf{E} \right). \tag{8}$$

Here $\varepsilon, \alpha, \beta^j, \gamma : \overline{\mathbb{R}^3_-} \to \mathbb{C}^{3 \times 3}$ are matrix-functions with C^∞-smooth and bounded entries. They depend on the frequency, i.e. the material properties are dispersive. The frequency dependency is suppressed here to simplify the notation. As we consider homogenous metamaterials, they do not depend on the spatial coordinates. Thus, the considered wave equation can be rewritten as

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \left(\tilde{\varepsilon} \mathbf{E} + \nabla \times \tilde{\alpha} \left(\nabla \times \mathbf{E} \right) + \sum_{j \in \{x,y,z\}} \partial_j \left(\tilde{\beta}^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \tilde{\gamma} \left(\nabla \times \nabla \times \mathbf{E} \right) \right), \tag{9}$$

where

$$\tilde{\varepsilon} = \begin{cases} \varepsilon, & z < 0, \\ 1, & z > 0, \end{cases} \quad \tilde{\alpha} = \begin{cases} \alpha, & z < 0, \\ 0, & z > 0, \end{cases} \quad \tilde{\beta}^j = \begin{cases} \beta^j, & z < 0, \\ 0, & z > 0, \end{cases} \quad \tilde{\gamma} = \begin{cases} \gamma, & z < 0, \\ 0, & z > 0. \end{cases}$$

III. GENERALISED SOLUTIONS

Up to now our considerations were rather formal. Now it is time to ask: in which sense has Eqn.9 to be understood? Since the matrix-functions $\tilde{\varepsilon}, \tilde{\alpha}, \tilde{\beta}^j, \tilde{\gamma}$ are only piecewise continuous, we are not allowed to regard the differential expression on the right-hand-side of Eqn. 9 in the classical sense. The natural idea is then to treat this equation in a suitable generalised sense. From these considerations we can extract already specific requirements concerning the different functions, in particular concerning the spaces in which they are defined. This is helpful in the following sections as interface conditions can be found that require the mere analysis of the space in which these functions are living.

Recall (see Appendix for more details) that the notation $\mathbf{D}'(\mathbb{R}^3)$ stands for the space of generalised functions (distributions), i.e., linear functionals acting continuously on $\mathbf{C}_0^\infty(\mathbb{R}^3)$. For example, for $\mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3)$, $\nabla \times \mathbf{E} \in \mathbf{D}'(\mathbb{R}^3)$ is defined by the action

$$(\nabla \times \mathbf{E})[\boldsymbol{\Phi}] := \int_{\mathbb{R}^3} \mathbf{E} \cdot (\nabla \times \boldsymbol{\Phi}) \, \mathrm{d}\mathbf{r}, \ \boldsymbol{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3),$$

which mimics partial integration. Also higher derivatives of $\mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3)$ can be defined analogously, but the discontinuous function $\tilde{\alpha}$ on the right-hand-side of Eqn. 9 causes difficulties in mimicing partial integration (which would be no problem if $\tilde{\alpha}$ was a C^1-function). We can solve this difficulty by requiring the additional regularity condition $\nabla \times \mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3)$. Under this assumption the product $\tilde{\alpha}(\nabla \times \mathbf{E})$ is a (regular) generalized function, which is defined by $\alpha(\nabla \times \mathbf{E})$ in $\mathbb{R}^3_-$ and 0 in $\mathbb{R}^3_+$ and now we are able to define its generalized $\nabla \times$-derivative. Actually we require even $\nabla \times \mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3)$, in order to treat the *left-hand-side* of 11 appropriately when defining generalized solutions. Note that the property $\nabla \times \mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3)$ is not guaranteed by the requirement that $\nabla \times \mathbf{E}$ is in $\mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3_+)$ and in $\mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3_-)$, but also (some kind of) continuity of the tangential component of $\mathbf{E}$ at the interface is needed.

Analogous considerations for the other terms in Eqn. 9 lead to the following natural regularity assumptions:

$$\mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3), \quad \nabla \times \mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3), \quad \mathbf{E} \in \mathbf{H}_{\mathrm{loc}}^1(\mathbb{R}^3_-), \quad \nabla \times \nabla \times \mathbf{E} \in \mathbf{L}_{\mathrm{loc}}^2(\mathbb{R}^3_-). \tag{10}$$

Now, Eqn. 9 is understood as an equality in $\mathbf{D}'(\mathbb{R}^3)$. By the definition of the generalized derivatives (see Appendix), mimicing partial integration in all occuring terms, it is equivalent to

$$\forall \mathbf{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{\Phi}) \, \mathrm{d}\mathbf{r} = \frac{\omega^2}{c^2} \int\limits_{\mathbb{R}^3_+} \mathbf{E} \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{r} + \frac{\omega^2}{c^2} \int\limits_{\mathbb{R}^3_-} \Big(\varepsilon \mathbf{E} \cdot \mathbf{\Phi}$$

$$+ \alpha \, (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{\Phi}) - \sum_{j \in \{x,y,z\}} \beta^j \partial_j \mathbf{E} \cdot \partial_j \mathbf{\Phi} + \gamma \, (\nabla \times \nabla \times \mathbf{E}) \cdot (\nabla \times \nabla \times \mathbf{\Phi}) \Big) \, \mathrm{d}\mathbf{r}. \quad (11)$$

The vector-function $\mathbf{E} : \mathbb{R}^3 \to \mathbb{C}$ is said to be a *generalized* (or *weak*) solution to Eqn. 9 if it meets the regularity properties given in conditions 10 and satisfies Eqn. 11.

Note that one can introduce another definition of the generalized solution in which the requirement $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ is omitted: we say that $\mathbf{E} : \mathbb{R}^3 \to \mathbb{C}$ a *very weak solution* to 9 if $\mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\mathbf{E} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3_-)$ (implying also $\nabla \times \mathbf{E} \in \mathbf{L}^2(\mathbb{R}^3_-)$), $\nabla \times \nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-)$ and $\forall \mathbf{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3) \int\limits_{\mathbb{R}^3} \mathbf{E} \cdot (\nabla \times \nabla \times \mathbf{\Phi}) \, \mathrm{d}\mathbf{r} = \mathrm{RHS}_{11}$, where RHS_{11} denotes the right-hand-side of equality 11. Evidently, if $\mathbf{E}$ is a weak solution to 9 then it is also a very weak solution (apparently the opposite is not true). In this paper we focus on weak solutions since for local constitutive relations (i.e., $\alpha = \gamma = \beta^j = 0$) they satisfy classical interface conditions $[\mathbf{E} \times n]$, $[\varepsilon \mathbf{E} \cdot n]$, $[(\nabla \times \mathbf{E}) \times n]$, $[(\nabla \times \mathbf{E}) \cdot n] = 0$ on Γ (here $[\dots]$ stands for the jump of the enclosed quantity across Γ) – this follows immediately from our results below.

We also remark that analogous considerations can be performed for $\mathbf{L}^1$-type spaces. Nevertheless, we prefer to deal with $\mathbf{L}^2$ functions and spaces, since we expect that the $\mathbf{L}^2$ setting yields more benefits in subsequent research (in particular, since $\mathbf{L}^2$ is a Hilbert space).

Let $\mathbf{E}$ be a generalized solution to Eqn. 9. It is easy to show, just by taking $\mathbf{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3_+)$ in Eqn. 11, that

$$\nabla \times \nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+) \tag{12}$$

and

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \mathbf{E} \quad \text{for almost all } x \in \mathbb{R}^3_+. \tag{13}$$

"Almost all" means that the set consisting of points $x \in \mathbb{R}^3_+$ at which the property expressed in Eqn. 13 fails has a Lebesque measure zero.

Similarly, taking $\mathbf{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3_-)$, we conclude that

$$\nabla \times \alpha \, (\nabla \times \mathbf{E}) + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \, (\nabla \times \nabla \times \mathbf{E}) \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-) \tag{14}$$

and

$$\nabla \times \nabla \times \mathbf{E} = \frac{\omega^2}{c^2} \Big(\varepsilon \mathbf{E} + \nabla \times \alpha \, (\nabla \times \mathbf{E})$$

$$+ \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \, (\nabla \times \nabla \times \mathbf{E}) \Big) \quad \text{for almost all } x \in \mathbb{R}^3_-. \tag{15}$$

Note that the regularity property given in expression 14 means that the sum of the generalized functions that appear in this expression are in $\mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-)$, but it does not imply that the individual generalized functions belong to $\mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-)$.

Now that we have analyzed generalized solutions outside the interface Γ, we can derive the actual interface conditions in the next section.

IV. DERIVATION OF THE INTERFACE CONDITIONS

Let $\mathbf{E}$ be a generalised solution to Eqn. 9. We denote by $\mathbf{E}_+$ and $\mathbf{E}_-$ the restrictions of $\mathbf{E}$ to $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$, respectively. Also, we additionally assume in this section that $\mathbf{E}$ is smooth in each half-space, namely

$$\mathbf{E}_+ \in C^4(\overline{\mathbb{R}^3_+}), \ \mathbf{E}_- \in C^4(\overline{\mathbb{R}^3_-}). \tag{16}$$

These additional smoothness conditions are indeed satisfied if the coefficients α, β^s, γ are smooth, at least in the case where the differential Eqn. 9 is elliptic (e.g., if γ is positive definite); see [42]. The interface conditions we are going to derive in this section under the additional smoothness assumption expressed in 16 also remain valid when only our conditions 10 are satisfied. But then they hold only in some generalized sense which needs the concept of traces. This will be outlined in Section VI.

Below in volume integrals we will use the notation $\mathbf{E}$, while in integrals over Γ we deal with $\mathbf{E}_+$ and $\mathbf{E}_-$. In the following we distinguish between the main interface conditions and two alternative interface conditions. These alternative interface conditions do not contain any further information and indeed can be derived from the main interface conditions. However, their documentation seems useful as these alternative interface conditions look simpler. This may make them more suitable for use in some specific situations.

A. Main interface conditions

In the following we want to prove that if $\mathbf{E}$ is a generalized solution of Eqn. 9 and satisfies the regularity assumptions 16, then $\mathbf{E}$ satisfies the following interface conditions on Γ:

$$(\mathbf{E}_+ - \mathbf{E}_-) \times \mathbf{n} = 0, \tag{C_1}$$

$$(\nabla \times \mathbf{E}_+ - \nabla \times \mathbf{E}_-) \times \mathbf{n} + \frac{\omega^2}{c^2}(\alpha\nabla \times \mathbf{E}_- + \nabla \times \gamma\nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} - \frac{\omega^2}{c^2}(\mathrm{I} - \mathbf{n}\mathbf{n}^{\mathrm{T}})\beta^z\partial_z\mathbf{E}_- = 0, \tag{C_2}$$

$$(\gamma\nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} = 0, \tag{C_3}$$

$$(\beta^z\partial_z\mathbf{E}_-) \cdot \mathbf{n} = 0. \tag{C_4}$$

Conversely, if $\mathbf{E}$ satisfies 16, solves 13 in $\mathbb{R}^3_+$, solves 15 in $\mathbb{R}^3_-$ and the conditions C_1-C_4 are fulfilled, then $\mathbf{E}$ is a generalized solution to 9. Please note, these interface conditions are one of the central results from our contributions. In the following we prove each of them.

1. Proof of the first interface condition

Since $\mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ we have

$$\forall \mathbf{\Phi} \in \mathbf{C}^\infty_0(\mathbb{R}^3) : \int_{\mathbb{R}^3} \mathbf{E} \cdot (\nabla \times \mathbf{\Phi})\, \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^3} (\nabla \times \mathbf{E}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{r}. \tag{17}$$

On the other hand, by integrating by parts in each half-space we obtain

$$\int_{\mathbb{R}^3} (\nabla \times \mathbf{E}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^3_-} (\nabla \times \mathbf{E}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{r} + \int_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{r}$$

$$= \int_{\mathbb{R}^3_-} \mathbf{E} \cdot (\nabla \times \mathbf{\Phi})\, \mathrm{d}\mathbf{r} + \int_{\mathbb{R}^3_+} \mathbf{E} \cdot (\nabla \times \mathbf{\Phi})\, \mathrm{d}\mathbf{r} + \int_{\Gamma} (\mathbf{E}_+ \times \mathbf{n} - \mathbf{E}_- \times \mathbf{n}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{s}$$

$$= \int_{\mathbb{R}^3} \mathbf{E} \cdot (\nabla \times \mathbf{\Phi})\, \mathrm{d}\mathbf{r} + \int_{\Gamma} (\mathbf{E}_+ \times \mathbf{n} - \mathbf{E}_- \times \mathbf{n}) \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{s}, \tag{18}$$

where $\mathrm{d}\mathbf{s} = \mathrm{d}x\, \mathrm{d}y$ is the area of the surface element on Γ. Since $\mathbf{\Phi}$ is arbitrary, we obtain from Eqns. 17 and 18 the first interface conditions C_1.

2. Proof of the second interface condition

To proof the second interface condition we decompose each integral in Eqn. 11 in a sum of integrals over $\mathbb{R}^3_-$ and $\mathbb{R}^3_+$ and integrate by parts in such a way that all the derivatives shift from $\mathbf{\Phi}$ to $\mathbf{E}$. Then, moving all volume integrals to the left-hand-side and all integrals over Γ to the right-hand-side, we get

$$
\int_{\mathbb{R}^3} (\nabla \times \nabla \times \mathbf{E}) \cdot \mathbf{\Phi} \, d\mathbf{r} - \frac{\omega^2}{c^2} \int_{\mathbb{R}^3_+} \mathbf{E} \cdot \mathbf{\Phi} \, d\mathbf{r} - \frac{\omega^2}{c^2} \int_{\mathbb{R}^3_-} \Big(\varepsilon \mathbf{E}
$$

$$
+ \nabla \times \alpha \nabla \times \mathbf{E} + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \left(\nabla \times \nabla \times \mathbf{E}_- \right) \Big) \cdot \mathbf{\Phi} \, d\mathbf{r}
$$

$$
= \int_{\Gamma} \left(\left(\nabla \times \mathbf{E}_+ - \nabla \times \mathbf{E}_- + \frac{\omega^2}{c^2} \left(\alpha \nabla \times \mathbf{E}_- + \nabla \times \gamma \nabla \times \nabla \times \mathbf{E}_- \right) \right) \times \mathbf{n} \right) \cdot \mathbf{\Phi} \, d\mathbf{s}
$$

$$
+ \frac{\omega^2}{c^2} \int_{\Gamma} \left((\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} \right) \cdot (\nabla \times \mathbf{\Phi}) \, d\mathbf{s} - \frac{\omega^2}{c^2} \int_{\Gamma} \left(\beta^z \partial_z \mathbf{E}_- \right) \cdot \mathbf{\Phi} \, d\mathbf{s}. \quad (19)
$$

Due to Eqns. 13 and 15, the left-hand-side of Eqn. 19 vanishes and thus Eqn. 19 can be rewritten as follows:

$$
\int_{\Gamma} \left(\left(\nabla \times \mathbf{E}_+ - \nabla \times \mathbf{E}_- + \frac{\omega^2}{c^2} \left(\alpha \nabla \times \mathbf{E}_- + \nabla \times \gamma \nabla \times \nabla \times \mathbf{E}_- \right) \right) \times \mathbf{n} - \frac{\omega^2}{c^2} (I - \mathbf{n}\mathbf{n}^{\mathrm{T}}) \beta^z \partial_z \mathbf{E}_- \right) \cdot \mathbf{\Phi} \, d\mathbf{s}
$$

$$
+ \frac{\omega^2}{c^2} \int_{\Gamma} \left((\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} \right) \cdot (\nabla \times \mathbf{\Phi}) \, d\mathbf{s} - \frac{\omega^2}{c^2} \int_{\Gamma} \left(\mathbf{n}\mathbf{n}^{\mathrm{T}} \beta^z \partial_z \mathbf{E}_- \right) \cdot \mathbf{\Phi} \, d\mathbf{s} = 0. \quad (20)
$$

Now, we choose the function $\mathbf{\Phi}$ of the form

$$
\mathbf{\Phi}(\mathbf{r}) = (\Phi_1(x,y)\eta(z), \ \Phi_2(x,y)\eta(z), \ 0)^{\mathrm{T}}, \quad (21)
$$

where $\Phi_1, \Phi_2 \in \mathsf{C}_0^\infty(\mathbb{R}^2)$, $\eta \in \mathsf{C}_0^\infty(\mathbb{R})$, moreover $\eta|_{\{|z|<\delta\}} = 1$ for some $\delta > 0$. One has: $\nabla \times \mathbf{\Phi} = (-\Phi_2 \eta', \ \Phi_1 \eta', \ (\partial_x \Phi_2 - \partial_y \Phi_1)\eta)^{\mathrm{T}}$, whence

$$
\nabla \times \mathbf{\Phi}|_\Gamma = (0, \ 0, \ \partial_x \Phi_2 - \partial_y \Phi_1)^{\mathrm{T}},
$$

and, hence, the second integral in Eqn. 20 vanishes. Since the z-component of $\mathbf{\Phi}$ is equal to zero, the third integral in Eqn. 20 also vanishes. Thus

$$
\int_{\Gamma} \left(\left(\nabla \times \mathbf{E}_+ - \nabla \times \mathbf{E}_- + \frac{\omega^2}{c^2} \left(\alpha \nabla \times \mathbf{E}_- + \nabla \times \gamma \nabla \times \nabla \times \mathbf{E}_- \right) \right) \times \mathbf{n} \right.
$$

$$
\left. - \frac{\omega^2}{c^2} (I - \mathbf{n}\mathbf{n}^{\mathrm{T}}) \beta^z \partial_z \mathbf{E}_- \right) \cdot (\Phi_1, \Phi_2, 0)^{\mathrm{T}} \, d\mathbf{s} = 0. \quad (22)
$$

Finally, since the functions Φ_1 and Φ_2 are arbitrary, we conclude from Eqn. 22 the interface condition C_2.

3. Proof of the third interface condition

To proof the third interface condition we take the function $\mathbf{\Phi}$ of the form

$$
\mathbf{\Phi}(\mathbf{r}) = (\Phi_2(x,y)z\eta(z), \ -\Phi_1(x,y)z\eta(z), \ 0)^{\mathrm{T}}. \quad (23)
$$

As before, $\Phi_1, \Phi_2 \in \mathsf{C}_0^\infty(\mathbb{R}^2)$, $\eta \in \mathsf{C}_0^\infty(\mathbb{R})$, and $\eta|_{\{|z|<\delta\}} = 1$ for some $\delta > 0$. We get $\nabla \times \mathbf{\Phi} = ((\eta + z\eta')\Phi_1, \ (\eta + z\eta')\Phi_2, \ (-\partial_x \Phi_1 - \partial_y \Phi_2)z\eta)^{\mathrm{T}}$ and hence

$$
\mathbf{\Phi}|_\Gamma = 0, \quad \nabla \times \mathbf{\Phi}|_\Gamma = (\Phi_1, \ \Phi_2, \ 0)^{\mathrm{T}}. \quad (24)
$$

We plug this function $\mathbf{\Phi}$ into Eqn. 20. Due to 24, all integrals in Eqn. 20 vanish except the second one. Thus one gets:

$$
\int_{\Gamma} \left((\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} \right) \cdot (\Phi_1, \Phi_2, 0)^{\mathrm{T}} \, d\mathbf{s} = 0. \quad (25)
$$

Since Φ_1 and Φ_2 are arbitrary, Eqn. 25 implies the third interface condition C_3.

4. Proof of the fourth interface condition

Finally, due to C_2, C_3, the first two integrals in Eqn. 20 vanish for any arbitrary function $\mathbf{\Phi}$, whence one gets

$$\int_\Gamma \left(\mathbf{n}\mathbf{n}^{\mathrm{T}}\beta^z\partial_z\mathbf{E}_-\right)\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{s} = 0, \tag{26}$$

and, consequently, we arrive at the last condition C_4.

5. Further remarks

Conversely, let now $\mathbf{E}$ satisfy requirement 16, solve Eqn. 13 in $\mathbb{R}^3_+$, solve Eqn. 15 in $\mathbb{R}^3_-$, and let C_1–C_4 hold. We have to show that $\mathbf{E}$ is a generalized solution to Eqn. 9.

Indeed, it follows from the assumption 16 that $\mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\mathbf{E} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3_-)$ and $\nabla \times \nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-)$. Moreover, due to C_1, $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$. Thus, conditions 10 are satisfied.

Integrating by parts we get for an arbitrary $\mathbf{\Phi} \in \mathbf{C}^\infty_0(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} (\nabla \times \mathbf{E})\cdot(\nabla \times \mathbf{\Phi})\,\mathrm{d}\mathbf{r} - \frac{\omega^2}{c^2}\int_{\mathbb{R}^3_+}\mathbf{E}\cdot\mathbf{\Phi}\,\mathrm{d}\mathbf{r} - \frac{\omega^2}{c^2}\int_{\mathbb{R}^3_-}\Big(\varepsilon\mathbf{E}\cdot\mathbf{\Phi} + \alpha\,(\nabla \times \mathbf{E})\cdot(\nabla \times \mathbf{\Phi})$$

$$- \sum_{j\in\{x,y,z\}}\beta^j\partial_j\mathbf{E}\cdot\partial_j\mathbf{\Phi} + \gamma\,(\nabla \times \nabla \times \mathbf{E})\cdot(\nabla \times \nabla \times \mathbf{\Phi})\Big)\,\mathrm{d}\mathbf{r} = \mathrm{LHS}_{19} - \mathrm{RHS}_{19}. \tag{27}$$

where LHS_{19} (resp. RHS_{19}) is the expression standing in the left-hand-side (resp. right-hand-side) of Eqn. 19. Due to 13-15, $\mathrm{LHS}_{19} = 0$ and, due to conditions C_2-C_4, $\mathrm{RHS}_{19} = 0$. Therefore, Eqn. 11 holds true and, consequently, $\mathbf{E}$ is a generalized solution to Eqn. 9.

With that we have been offering proofs for the main interface conditions. In the following, we formulate two alternative interface conditions that can be used as well, but they are not fundamental since they follow from the definition of the generalized solution and the main interface conditions.

B. Alternative interface conditions

Let $\mathbf{E}$ be a generalized solution to Eqn. 9 satisfying the regularity assumptions 16. Then $\mathbf{E}$ meets the interface conditions C_1-C_4. The fulfilment of C_1-C_4 is also a sufficient condition for being a generalized solution if 13 and 15 hold – see the statement after conditions C_1-C_4. In this subsection, we derive two alternative interface conditions on Γ. They appear slightly simpler and are, therefore, of practical use in further research.

1. First alternative interface condition

Let $\Psi \in \mathsf{C}^\infty_0(\mathbb{R}^3)$ be an arbitrary function. Since $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\nabla \cdot (\nabla \times \mathbf{E}) = 0$ and $\nabla \times (\nabla\Psi) = 0$ one has

$$0 = -\int_{\mathbb{R}^3}(\nabla\cdot(\nabla\times\mathbf{E}))\Psi\,\mathrm{d}\mathbf{r} = -\int_{\mathbb{R}^3_+}(\nabla\cdot(\nabla\times\mathbf{E}))\Psi\,\mathrm{d}\mathbf{r} - \int_{\mathbb{R}^3_-}(\nabla\cdot(\nabla\times\mathbf{E}))\Psi\,\mathrm{d}\mathbf{r}$$

$$= \int_\Gamma ((\nabla\times\mathbf{E}_+ - \nabla\times\mathbf{E}_-)\cdot\mathbf{n})\Psi\,\mathrm{d}\mathbf{s} + \int_{\mathbb{R}^3_+}(\nabla\times\mathbf{E})\cdot(\nabla\Psi)\,\mathrm{d}\mathbf{r} + \int_{\mathbb{R}^3_-}(\nabla\times\mathbf{E})\cdot(\nabla\Psi)\,\mathrm{d}\mathbf{r}$$

$$= \int_\Gamma ((\nabla\times\mathbf{E}_+ - \nabla\times\mathbf{E}_-)\cdot\mathbf{n})\Psi\,\mathrm{d}\mathbf{s} + \int_\Gamma ((\mathbf{E}_+ - \mathbf{E}_-)\times\mathbf{n})\cdot\nabla\Psi\,\mathrm{d}\mathbf{s} + \int_{\mathbb{R}^3}\mathbf{E}\cdot(\nabla\times(\nabla\Psi))\,\mathrm{d}\mathbf{r}$$

$$\stackrel{C_1}{=} \int_\Gamma ((\nabla\times\mathbf{E}_+ - \nabla\times\mathbf{E}_-)\cdot\mathbf{n})\Psi\,\mathrm{d}\mathbf{s},$$

whence we get the following interface condition on Γ:

$$(\nabla \times \mathbf{E}_+ - \nabla \times \mathbf{E}_-) \cdot \mathbf{n} = 0. \tag{C$_5$}$$

2. Second alternative interface condition

For materials governed by local constitutive relations (i.e., $\alpha = \gamma = \beta^j = 0$) one has also the interface condition

$$(\mathbf{E}_+ - \varepsilon \mathbf{E}_-) \cdot \mathbf{n} = 0, \tag{28}$$

which follows from the fact that $\tilde{\varepsilon} \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ and, due to Eqn. 9, $\nabla \cdot (\tilde{\varepsilon} \mathbf{E}) = 0$.

Let us now derive an analogue of the interface condition in Eqn. 28 for our nonlocal model. For this purpose, we plug into Eqn. 11 the function $\mathbf{\Phi}$ of the form

$$\mathbf{\Phi} = \nabla \Psi,$$

where Ψ is a smooth compactly supported scalar function. Under this choice $\nabla \times \mathbf{\Phi} = 0$ and Eqn. 11 becomes

$$\int_{\mathbb{R}^3_+} \mathbf{E} \cdot \nabla \Psi \, d\mathbf{r} + \int_{\mathbb{R}^3_-} \left(\varepsilon \mathbf{E} \cdot \nabla \Psi - \sum_{j \in \{x,y,z\}} \beta^j \partial_j \mathbf{E} \cdot \partial_j \nabla \Psi \right) d\mathbf{r} = 0. \tag{29}$$

Let us additionally assume that $\mathrm{supp} \Psi \subset \mathbb{R}^3_-$; integrating by part in Eqn. 29 we get:

$$\forall \Psi \in \mathsf{C}^\infty_0(\mathbb{R}^3_-) : \int_{\mathbb{R}^3_-} \left(\nabla \cdot \left(\varepsilon \mathbf{E} + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) \right) \right) \Psi \, d\mathbf{r} = 0,$$

whence, since $\mathsf{C}^\infty_0(\mathbb{R}^3_-)$ is dense in $\mathsf{L}^2(\mathbb{R}^3_-)$,

$$\nabla \cdot \left(\varepsilon \mathbf{E} + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) \right) = 0 \ \text{ in } \mathbb{R}^3_-. \tag{30}$$

Similarly, by using $\Psi \in \mathsf{C}^\infty_0(\mathbb{R}^3_+)$, we obtain:

$$\nabla \cdot \mathbf{E} = 0 \ \text{ in } \mathbb{R}^3_+. \tag{31}$$

Now, we take an arbitrary $\Psi \in \mathsf{C}^\infty_0(\mathbb{R}^3)$ in Eqn. 29 and integrate by parts in each half-space $\mathbb{R}^3_\pm$. We get:

$$\int_{\mathbb{R}^3_+} (\nabla \cdot \mathbf{E}) \Psi \, d\mathbf{r} + \int_{\mathbb{R}^3_-} \nabla \cdot \left(\varepsilon \mathbf{E} + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) \right) \Psi \, d\mathbf{r}$$

$$+ \int_\Gamma \left(\left(\mathbf{E}_+ - \varepsilon \mathbf{E}_- - \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E}_- \right) \right) \cdot \mathbf{n} \right) \Psi \, d\mathbf{s} + \int_\Gamma \beta^z \partial_z \mathbf{E}_- \cdot \nabla \Psi \, d\mathbf{s} = 0. \tag{32}$$

In this equality the integrals over $\mathbb{R}^3_\pm$ vanish due to Eqn. 30 and Eqn. 31. Moreover, in view of interface condition C_4, the last integral in Eqn. 32 can be rewritten as follows:

$$\int_\Gamma \beta^z \partial_z \mathbf{E}_- \cdot \nabla \Psi \, d\mathbf{s} = \int_\Gamma (\mathrm{I} - \mathbf{n}\mathbf{n}^\mathrm{T}) \beta^z \partial_z \mathbf{E}_- \cdot \nabla_\Gamma \Psi \, d\mathbf{s} = - \int_\Gamma \nabla_\Gamma \cdot (\beta^z \partial_z \mathbf{E}_-) \Psi \, d\mathbf{s}. \tag{33}$$

Since $\Psi \in \mathsf{C}^\infty_0(\mathbb{R}^3)$ is an arbitrary function we conclude from Eqns. 32-33 the following interface condition on Γ:

$$\left(\mathbf{E}_+ - \varepsilon \mathbf{E}_- - \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E}_- \right) \right) \cdot \mathbf{n} - \nabla_\Gamma \cdot (\beta^z \partial_z \mathbf{E}_z) = 0. \tag{C$_6$}$$

That closes this section. In Section VI we will clarify how to treat the interface conditions in the general case, i.e., when the generalized solution does not have the additional regularity expressed in condition 16.

We will use these interface conditions to derive Fresnel equations for the transmission and reflection of a plane wave from a slab in the next section.

V. FRESNEL FORMULAE

In this section, we apply the interface conditions obtained above to the problem of light propagation through a slab of metamaterial.

The geometry of the pertinent problem hence is defined as

$$\Omega = \left\{ \mathbf{r} \in \mathbb{R}^3 : \ -\delta < z < 0 \right\}, \quad \Omega_- = \left\{ \mathbf{r} \in \mathbb{R}^3 : \ z < -\delta \right\}, \quad \Omega_+ = \left\{ \mathbf{r} \in \mathbb{R}^3 : \ z > 0 \right\},$$

where $\delta > 0$. We assume that Ω is filled with a metamaterial, which is governed by the constitutive relations expressed in Eqn. 8. Moreover, additionally we assume that the coordinate system of the laboratory corresponds to the coordinate system of the principle axis of the metamaterial. The materials considered here are centro-symmetric. Therefore, the material tensors are considered to be diagonal. In essence, this means that the unit cells are aligned to the slab we consider. In this coordinate system the material properties read as

$$
\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}, \quad
\alpha = \begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_z \end{pmatrix}, \quad
\gamma = \begin{pmatrix} \gamma_x & 0 & 0 \\ 0 & \gamma_y & 0 \\ 0 & 0 & \gamma_z \end{pmatrix},
$$

$$
\beta^x = \begin{pmatrix} \beta_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
\beta^y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
\beta^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_z \end{pmatrix}.
$$

$$(34)$$

Note that this is not an essential condition but it helps us to keep the expressions sufficiently simple. The remaining space (that is $\Omega_- \cup \Omega_+$) is occupied by vacuum.

We also denote

$$\Gamma_- = \{ \mathbf{r} \in \partial\Omega : \ z = -\delta \}, \quad \Gamma_+ = \{ \mathbf{r} \in \partial\Omega : \ z = 0 \}.$$

Now, assume that we have an incident plane wave impinging on the the slab from Ω_+. A part of this wave will be reflected, the other part will be transmitted through the slab to Ω_-. Our goal is to find the amplitudes of these reflected and transmitted waves, in other words, we want to derive Fresnel-type formulae. Even though discussed here for a plane wave, an arbitrary illumination can always be written as a superposition of plane waves. Therefore, the plane wave assumption is by no means a limitation.

We notice that, due to the special form of the material coefficients expressed in Eq. 34, each solution $\mathbf{E}$ of Eqn. 9 can be represented in the form

$$\mathbf{E} = \mathbf{E}^{\mathrm{TE}} + \mathbf{E}^{\mathrm{TM}},$$

where $\mathbf{E}^{\mathrm{TE}} = (E_x^{\mathrm{TE}}, 0, 0)^{\mathrm{T}}$ (transverse-electric-polarized wave) and $\mathbf{E}^{\mathrm{TM}} = (0, E_y^{\mathrm{TM}}, E_z^{\mathrm{TM}})^{\mathrm{T}}$ (transverse-magnetic-polarized wave), each of which satisfies Eqn. 9. In what follows, we treat TE- and TM-polarized incident waves separately.

In this section $\mathbf{r}$ is treated as vector-column, i.e. $\mathbf{r} = (x, y, z)^{\mathrm{T}}$.

A. TE-polarization

Assume that

$$\mathbf{E}^I = \mathbf{E}_0^I \exp(\mathrm{i}\mathbf{k}^I \cdot \mathbf{r})$$

is the incident TE-polarized plane wave. Here $\mathbf{k}^I = (k_x^I, k_y^I, k_z^I)^{\mathrm{T}}$ is the wave vector and $\mathbf{E}_0^I = (E_x^I, 0, 0)^{\mathrm{T}}$ is the amplitude vector of the incident plane wave. The wave vector in vacuum obeys the ordinary dispersion relation, i.e. $k^2 = \frac{\omega^2}{c^2}$. Moreover, in view of 31 one has $k_x^I = 0$.

Due to symmetry arguments it is reasonable to search the reflected and the transmitted fields in the same form as the incident field. Namely, the reflected field is searched in the form

$$\mathbf{E}^R = \mathbf{E}_0^R \exp(\mathrm{i}\mathbf{k}^R \cdot \mathbf{r}), \ \text{where} \ \mathbf{k}^R = (0, k_y^R, k_z^R)^{\mathrm{T}}, \ \mathbf{E}_0^R = (E_x^R, 0, 0)^{\mathrm{T}}.$$

The total field in Ω_+ is $\mathbf{E}^I + \mathbf{E}^R$. In Ω_- we have the transmitted field

$$\mathbf{E}^T = \mathbf{E}_0^T \exp(\mathrm{i}\mathbf{k}^T \cdot \mathbf{r}), \ \text{where} \ \mathbf{k}^T = (0, k_y^T, k_z^T)^{\mathrm{T}}, \ \mathbf{E}_0^T = (E_x^T, 0, 0)^{\mathrm{T}}.$$

Finally, in the slab Ω the total field has the form

$$\mathbf{E}^{\text{slab}} = \sum_{j=1}^{N} \mathbf{E}_0^j \exp(\mathrm{i}\mathbf{k}^j \cdot \mathbf{r}), \quad \mathbf{k}^j = (0, k_y^j, k_z^j)^{\text{T}}, \ \mathbf{E}_0^j = (E_x^j, 0, 0)^{\text{T}},$$

where N is the number of linearly independent eigenmodes existing in Ω. The larger number of plane waves is reminiscent to the fact that for each value of k_x and k_y multiple solutions for k_z exist at each frequency. This will be discussed below. Also, the field inside the slab is always written as a superposition of forward and backward propagating modes in the principle propagation direction. Therefore, this quite general ansatz is chosen.

Plugging the plane wave ansatzes, for example, into C_1 on Γ_+ and Γ_- we get the equation

$$E^I \mathrm{e}^{\mathrm{i}(k_x^I x + k_y^I y)} + E^R \mathrm{e}^{\mathrm{i}(k_x^R x + k_y^R y)} - \sum_{j=1}^{N} E_x^j \mathrm{e}^{\mathrm{i}(k_x^j x + k_y^j y)} = 0, \quad \sum_{j=1}^{N} E_x^j \mathrm{e}^{-\mathrm{i}k_z^j \delta} \mathrm{e}^{\mathrm{i}(k_x^j x + k_y^j y)} - E^T \mathrm{e}^{-\mathrm{i}k_z^T \delta} \mathrm{e}^{\mathrm{i}(k_x^T x + k_y^T y)} = 0$$

which hold for all $(x, y) \in \mathbb{R}^2$. As the system is translational invariant along the interface, we require all plane waves involved in the process to share the same wave vector components tangential to the interface. Hence the vectors $\mathbf{k}^I$, $\mathbf{k}^R$, $\mathbf{k}^T$ and $\mathbf{k}^j$ $(j = 1, \ldots, N)$ have the same y-components. Hereafter for the y-component of all wave vectors we use the notation k_y.

It is easy to see that $N = 4$ provided $\gamma_x \neq 0$. Indeed, plugging $\mathbf{E} = (E_x, 0, 0)^{\text{T}} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r})$ (with $k_x = 0$) into Eqn. 7 supplemented with the constitutive relations expressed in Eqn. 8, we arrive at the following dispersion relation linking ω and k for the metamaterial:

$$k_y^2 + k_z^2 = \omega^2 c^{-2} \left(\varepsilon_x + (\alpha_z k_y^2 + \alpha_y k_z^2) + \gamma_x (k_y^2 + k_z^2)^2 \right). \tag{35}$$

This is a 4th-order polynomial equation with respect to k_z, thus generically we get four eigenmodes [43]. They come in pairs and two of these eigenmodes are forward and two of these eigenmodes are backward propagating.

There are six unknowns E_x^R, E_x^T, E_x^j $(j = 1, \ldots, 4)$. On each interface Γ_+ and Γ_- we can initially impose four conditions C_1-C_4, but C_4 simply reads $0 = 0$. As a result, we have three non-trivial equations on each interface. This leaves us with a total number of six linearly independent equations which is just enough to solve uniquely for all involved amplitudes.

Plugging our plane wave ansatzes into these equations we arrive at the following linear algebraic system for $E = (E_x^R, E_x^1, E_x^2, E_x^3, E_x^4, E_x^T)^{\text{T}}$:

$$\mathcal{A}E = \mathcal{F},$$

where

$$\mathcal{A} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 0 \\ k_z^R & -A_1 & -A_2 & -A_3 & -A_4 & 0 \\ 0 & \gamma_x(\mathbf{k}^1)^2 & \gamma_x(\mathbf{k}^2)^2 & \gamma_x(\mathbf{k}^3)^2 & \gamma_x(\mathbf{k}^4)^2 & 0 \\ 0 & -\mathrm{e}^{-\mathrm{i}k_z^1\delta} & -\mathrm{e}^{-\mathrm{i}k_z^2\delta} & -\mathrm{e}^{-\mathrm{i}k_z^3\delta} & -\mathrm{e}^{-\mathrm{i}k_z^4\delta} & \mathrm{e}^{-\mathrm{i}k_z^T\delta} \\ 0 & -A_1\mathrm{e}^{-\mathrm{i}k_z^1\delta} & -A_2\mathrm{e}^{-\mathrm{i}k_z^2\delta} & -A_3\mathrm{e}^{-\mathrm{i}k_z^3\delta} & -A_4\mathrm{e}^{-\mathrm{i}k_z^4\delta} & k_z^T\mathrm{e}^{-\mathrm{i}k_z^T\delta} \\ 0 & \gamma_x(\mathbf{k}^1)^2\mathrm{e}^{-\mathrm{i}k_z^1\delta} & \gamma_x(\mathbf{k}^2)^2\mathrm{e}^{-\mathrm{i}k_z^2\delta} & \gamma_x(\mathbf{k}^3)^2\mathrm{e}^{-\mathrm{i}k_z^3\delta} & \gamma_x(\mathbf{k}^4)^2\mathrm{e}^{-\mathrm{i}k_z^4\delta} & 0 \end{pmatrix}, \quad \mathcal{F} = -E_x^I \begin{pmatrix} 1 \\ k_z^I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here $(\mathbf{k}^j)^2 := (k_y)^2 + (k_z^j)^2$, and $A_j := k_z^j \left(1 - \dfrac{\omega^2}{c^2} \left(\alpha_y + \gamma_x(\mathbf{k}^j)^2 \right) \right)$. Thus the required amplitudes E_x^R and E_x^T are determined by

$$E^R = \left[\left(\mathcal{A}^{-1}\mathcal{F} \right) \right]_1, \quad E^T = \left[\left(\mathcal{A}^{-1}\mathcal{F} \right) \right]_6,$$

where $[\cdot]_k$ denotes the k-th component of a vector. It is not hard to show that generically the matrix $\mathcal{A}$ is invertible; "generically" means that the set consisting of $\varepsilon, \alpha, \gamma, \beta^j$ for which $\det(\mathcal{A}) = 0$ has Lebesgue measure zero in the space of all admissible parameters.

Note, that the equation coming from conditions C_5 coincides with the equation coming from condition C_1 multiplied by k_y, and the equation coming from condition C_6 reads $0 = 0$.

B. TM-polarization

Again, in Ω_+ the total field is of the form $\mathbf{E}^I + \mathbf{E}^R$, where $\mathbf{E}^I = \mathbf{E}_0^I \exp(\mathrm{i}\mathbf{k}^I \cdot \mathbf{r})$ is the incident field, $\mathbf{E}^R = \mathbf{E}_0^R \exp(\mathrm{i}\mathbf{k}^R \cdot \mathbf{r})$ is the reflected field, but now the fields are TM-polarized, i.e., $\mathbf{E}_0^I = (0, E_y^I, E_z^I)^\mathrm{T}$, $\mathbf{E}_0^R = (0, E_y^R, E_z^R)^\mathrm{T}$. In Ω_- we get the transmitted field $\mathbf{E}^T = \mathbf{E}_0^T \exp(\mathrm{i}\mathbf{k}^T \cdot \mathbf{r})$, where $\mathbf{E}_0^T = (0, E_y^T, E_z^T)^\mathrm{T}$. Finally, in the slab Ω the total field has the form $\mathbf{E}^\mathrm{slab} = \sum_{j=1}^N \mathbf{E}_0^j \exp(\mathrm{i}\mathbf{k}^j \cdot \mathbf{r})$, where $\mathbf{E}_0^j = (0, E_y^j, E_z^j)^\mathrm{T}$.

Again the vectors $\mathbf{k}^I$, $\mathbf{k}^R$, $\mathbf{k}^T$ and $\mathbf{k}^j$ $(j = 1, \ldots, N)$ have the same x- and y-components. Similarly to the TE-polarization, we denote by k_y the y-component of the wave vectors, the x-components are chosen to be equal to zero.

For TM-polarization the dispersion relation reads

$$\det \begin{pmatrix} -\omega^2 c^{-2}\varepsilon_y + k_z^2 Q + \omega^2 c^{-2}\beta_y k_y^2 & -k_y k_z Q \\ -k_y k_z Q & -\omega^2 c^{-2}\varepsilon_z + k_y^2 Q + \omega^2 c^{-2}\beta_z k_z^2 \end{pmatrix} = 0, \tag{36}$$

where $Q := 1 - \omega^2 c^{-2}(\alpha_x + \gamma_z k_y^2 + \gamma_y k_z^2)$. It is easy to see that this is a 6th-order polynomial equation with respect to k_z provided $\gamma_y \neq 0$, $\beta_z \neq 0$. Thus $N = 6$. There are three forward and three backward propagating modes.

In $\Omega_+ \cup \Omega_-$ one has $\mathrm{div}\,\mathbf{E} = 0$ and consequently

$$E_y k_y + E_z k_z = 0. \tag{37}$$

In Ω one has $\mathrm{div}\left(\varepsilon\mathbf{E} + \sum_{j \in \{x,y,z\}} \partial_j(\beta^j \partial_j \mathbf{E})\right) = 0$ and consequently

$$E_y k_y(\varepsilon_y - \beta_y k_y^2) + E_z k_z(\varepsilon_z - \beta_z k_z^2) = 0. \tag{38}$$

Due to Eqns. 37-38, it is enough to determine only the z-components of the reflected and transmitted fields (except in the special case $(\varepsilon_y - \beta_y k_y^2)k_y = 0$, which we do not treat here).

We have eight unknowns E_z^R, E_z^T, E_z^j $(j = 1, \ldots, 6)$. On each interface Γ_+ and Γ_- we have four conditions C_1-C_4, thus, both interfaces produce totally eight equations. Plugging the plane wave ansatzes into these equations we arrive at the following system for $E = (E_z^R, E_z^1, E_z^2, E_z^3, E_z^4, E_z^5, E_z^6, E_z^T)^\mathrm{T}$: $\mathcal{A}E = \mathcal{F}$, where

$$\mathcal{A} = \begin{pmatrix}
k_z^R & \overset{j=1}{-B_1 k_z^1} & \overset{j=2}{\cdots} & \overset{j=3}{\cdots} & \overset{j=4}{\cdots} & \overset{j=5}{\cdots} & \overset{j=6}{\cdots} & 0 \\
(\mathbf{k}^R)^2 & \left(-1 + \frac{\omega^2}{c^2}\alpha_y + \frac{\omega^2}{c^2}\left(k_y^2\gamma_z + (k_z^1)^2\gamma_y\right)\right)\left(k_y^2 + (k_z^1)^2 B_1\right) & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \gamma_y k_z^1 \left(k_y^2 + (k_z^1)^2 B_1\right) & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \beta_z k_z^1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -B_1 k_z^1 \mathrm{e}^{-\mathrm{i}k_z^1\delta} & \cdots & \cdots & \cdots & \cdots & \cdots & k_z^T \mathrm{e}^{-\mathrm{i}k_z^T\delta} \\
0 & \left(-1 + \frac{\omega^2}{c^2}\alpha_y + \frac{\omega^2}{c^2}\left(k_y^2\gamma_z + (k_z^1)^2\gamma_y\right)\right)\left(k_y^2 + (k_z^1)^2 B_1\right)\mathrm{e}^{-\mathrm{i}k_z^1\delta} & \cdots & \cdots & \cdots & \cdots & \cdots & (\mathbf{k}^T)^2 \mathrm{e}^{-\mathrm{i}k_z^T\delta} \\
0 & \gamma_y k_z^1 \left(k_y^2 + (k_z^1)^2 B_1\right)\mathrm{e}^{-\mathrm{i}k_z^1\delta} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \beta_z k_z^1 \mathrm{e}^{-\mathrm{i}k_z^1\delta} & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix},$$

$$\mathcal{F} = -E_z^I \left(k_z^I, \ (\mathbf{k}^I)^2, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0\right)^\mathrm{T},$$

and $B_j := \dfrac{\varepsilon_z - (k_z^j)^2\beta_z}{\varepsilon_y - (k_y)^2\beta_y}$, $(\mathbf{k}^*)^2 := (\mathbf{k}_y)^2 + (\mathbf{k}_z^*)^2$. In the matrix $\mathcal{A}$, the entries of the columns indicated by $j = 2, 3, 4, 5, 6$ have the entries as the column with $j = 1$, but with k_z^j instead of k_z^1.

Again one can show that generically the matrix $\mathcal{A}$ is invertible. The required amplitudes E_z^R and E_z^T are determined by

$$E^R = \left[\left(\mathcal{A}^{-1}\mathcal{F}\right)\right]_1, \quad E^T = \left[\left(\mathcal{A}^{-1}\mathcal{F}\right)\right]_8.$$

We note that for TM-polarization the alternative condition C_5 reads $0 = 0$, while the equations generated by C_6 on Γ_+ and Γ_0 can be deduced from the equations generated by the main interface conditions and the dispersion relations 36.

The derivation of the Fresnel coefficients is the second major achievement of this work. They are explicitly documented in this work and can be used right away for further research.

VI.   TRACES

Let $\mathbf{E}$ be a generalized solution to Eqn. 9, i.e., 10 and 11 hold. Since $\mathbf{E}$ belongs to $\mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, it is defined up to a set of Lebesgue measure zero. The interface Γ has Lebesgue measure zero, therefore we cannot define $\mathbf{E}$ on Γ in the usual sense unless some additional assumptions are given – for example, regularity assumptions 16. In the general case (i.e., without 16) we overcome this difficulty by defining suitable restrictions of $\mathbf{E}$ and its derivatives to Γ as elements of $\mathbf{D}'(\Gamma)$ (or $\mathrm{D}'(\Gamma)$). According to the usual terminology, we will call them *traces*. Please note, that the discussion of these traces does not affect the actual interface conditions C_1-C_4 — we only have to reformulate them in some generalized sense.

Note, that the definitions of traces standing in conditions C_1 and C_5 are known (see, e.g. [44]), thus we only recall them for completeness.

A.   Traces arising in C_1

Assume for the moment that $\mathbf{E} \in \mathbf{C}^1(\overline{\mathbb{R}^3_+})$. In this case by integrating by parts we get the following equality:

$$\forall \boldsymbol{\Phi} \in \mathbf{C}^\infty_0(\mathbb{R}^3) : \int_\Gamma (\mathbf{E}_+ \times \mathbf{n}) \cdot \boldsymbol{\Phi} \, \mathrm{d}\mathbf{s} = \int_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot \boldsymbol{\Phi} \, \mathrm{d}\mathbf{r} - \int_{\mathbb{R}^3_+} \mathbf{E} \cdot (\nabla \times \boldsymbol{\Phi}) \, \mathrm{d}\mathbf{r}. \tag{39}$$

(recall that $\mathbf{n} = (0, 0, 1)^{\mathrm{T}}$). One observes that the right-hand-side of Eqn. 39 makes sense not only for $\mathbf{E} \in \mathbf{C}^1(\mathbb{R}^3_+)$, but also for less regular $\mathbf{E}$, namely $\mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ with $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$. This suggests to introduce the following definition: let $\mathbf{E}_+ \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$, $\nabla \times \mathbf{E}_+ \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ (recall that generalized solutions of Eqn. 9 possess these properties); then $\mathbf{E}_+ \times \mathbf{n}$ is an element of $\mathbf{D}'(\Gamma)$ and its action on $\boldsymbol{\Phi} \in \mathbf{C}^\infty_0(\Gamma)$ is as follows:

$$(\mathbf{E}_+ \times \mathbf{n})[\boldsymbol{\Phi}] := \int_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot \widehat{\boldsymbol{\Phi}} \, \mathrm{d}\mathbf{r} - \int_{\mathbb{R}^3_+} \mathbf{E} \cdot (\nabla \times \widehat{\boldsymbol{\Phi}}) \, \mathrm{d}\mathbf{r},$$

where $\widehat{\boldsymbol{\Phi}}$ is a function belonging to $\mathbf{C}^\infty_0(\mathbb{R}^3)$ and satisfying $\widehat{\boldsymbol{\Phi}}|_\Gamma = \boldsymbol{\Phi}$. Similarly, we define

$$(\mathbf{E}_- \times \mathbf{n})[\boldsymbol{\Phi}] := -\int_{\mathbb{R}^3_-} (\nabla \times \mathbf{E}) \cdot \widehat{\boldsymbol{\Phi}} \, \mathrm{d}\mathbf{r} + \int_{\mathbb{R}^3_-} \mathbf{E} \cdot (\nabla \times \widehat{\boldsymbol{\Phi}}) \, \mathrm{d}\mathbf{r}.$$

It is easy to see that this definition is independent of the choice of the function $\widehat{\boldsymbol{\Phi}}$. Indeed, since

$$\int_{\mathbb{R}^3_+} \mathbf{E} \cdot (\nabla \times \widehat{\boldsymbol{\Phi}}) \, \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot \widehat{\boldsymbol{\Phi}} \, \mathrm{d}\mathbf{r} \tag{40}$$

holds for all $\widehat{\boldsymbol{\Phi}} \in \mathbf{C}^\infty_0(\mathbb{R}^3_+)$, which is dense in $\mathcal{H} := \{\widehat{\boldsymbol{\Phi}} \in \mathbf{C}^\infty_0(\overline{\mathbb{R}^3_+}) : \widehat{\Phi}|_\Gamma = 0\}$, Eqn. 40 holds for $\widehat{\boldsymbol{\Phi}} \in \mathcal{H}$, and hence for $\widehat{\boldsymbol{\Phi}} = \widehat{\boldsymbol{\Phi}}_1 - \widehat{\boldsymbol{\Phi}}_2$, with $\widehat{\boldsymbol{\Phi}}_1, \widehat{\boldsymbol{\Phi}}_2 \in \mathbf{C}^\infty_0(\overline{\mathbb{R}^3_+})$ denoting two functions satisfying $\widehat{\boldsymbol{\Phi}}_1|_\Gamma = \widehat{\boldsymbol{\Phi}}_2|_\Gamma$. This, obviously, gives the desired independence.

B. Traces arising in C_2

If $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\nabla \times \nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ and $\nabla \times \nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_-)$ (due to conditions 10 and 12, generalized solutions of Eqn. 9 possess these properties), we can define the traces $(\nabla \times \mathbf{E}_\pm) \times \mathbf{n}$ in the same way as the traces $\mathbf{E}_\pm \times \mathbf{n}$:

$$((\nabla \times \mathbf{E}_+) \times \mathbf{n})[\boldsymbol{\Phi}] := \int\limits_{\mathbb{R}^3_+} (\nabla \times \nabla \times \mathbf{E}) \cdot \widehat{\boldsymbol{\Phi}} \, d\mathbf{r} - \int\limits_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot (\nabla \times \widehat{\boldsymbol{\Phi}}) \, d\mathbf{r}, \tag{41}$$

$$((\nabla \times \mathbf{E}_-) \times \mathbf{n})[\boldsymbol{\Phi}] := - \int\limits_{\mathbb{R}^3_-} (\nabla \times \nabla \times \mathbf{E}) \cdot \widehat{\boldsymbol{\Phi}} \, d\mathbf{r} + \int\limits_{\mathbb{R}^3_-} (\nabla \times \mathbf{E}) \cdot (\nabla \times \widehat{\boldsymbol{\Phi}}) \, d\mathbf{r}. \tag{42}$$

where $\widehat{\boldsymbol{\Phi}}$ is function from $\mathbf{C}^\infty_0(\mathbb{R}^3)$ satisfying $\widehat{\boldsymbol{\Phi}}|_\Gamma = \boldsymbol{\Phi}$.

Next we clarify the meaning of

$$\mathbf{G} := (\alpha \nabla \times \mathbf{E}_- + \nabla \times \gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n} - (\mathrm{I} - \mathbf{n}\mathbf{n}^{\mathrm{T}})\beta^z \partial_z \mathbf{E}_-. \tag{43}$$

The idea is to define the trace for the whole quantity $\mathbf{G}$ and not for each term separately.

Assume for the moment that $\mathbf{E}$ is smooth enough, namely condition 16 holds. Integrating by parts, one has for each $\boldsymbol{\Phi} \in \mathbf{C}^\infty_0(\mathbb{R}^3)$:

$$\int\limits_{\mathbb{R}^3_-} \left(\nabla \times \alpha \left(\nabla \times \mathbf{E} \right) + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \left(\nabla \times \nabla \times \mathbf{E} \right) \right) \cdot \boldsymbol{\Phi} \, d\mathbf{r}$$

$$= \int\limits_{\mathbb{R}^3_-} \left(\alpha \left(\nabla \times \mathbf{E} \right) \cdot \left(\nabla \times \boldsymbol{\Phi} \right) - \sum_{j \in \{x,y,z\}} \beta^j \partial_j \mathbf{E} \cdot \partial_j \boldsymbol{\Phi} + \gamma \left(\nabla \times \nabla \times \mathbf{E} \right) \cdot \left(\nabla \times \nabla \times \boldsymbol{\Phi} \right) \right) d\mathbf{r}$$

$$- \int\limits_\Gamma \mathbf{G} \cdot \boldsymbol{\Phi} \, d\mathbf{s} - \int\limits_\Gamma \left((\gamma \nabla \times \nabla \times \mathbf{E}) \times \mathbf{n} \right) \cdot (\nabla \times \boldsymbol{\Phi}) \, d\mathbf{s} + \int\limits_\Gamma \left(\mathbf{n}\mathbf{n}^{\mathrm{T}} \beta^z \partial_z \mathbf{E} \right) \cdot \boldsymbol{\Phi} \, d\mathbf{s}. \tag{44}$$

Note that the integral $\int_\Gamma \mathbf{G} \cdot \boldsymbol{\Phi} \, d\mathbf{s}$ is independent of the z-component of $\boldsymbol{\Phi}$, since $\mathbf{G} \cdot \mathbf{n} = 0$. Also we observe that the integrals over $\mathbb{R}^3_-$ exist also when the regularity condition 16 is not fulfilled, but $\mathbf{E}$ enjoys the properties expressed in conditions 10 and 14. Moreover, if we choose the function $\boldsymbol{\Phi}$ of the form given in Eqn. 21, the last two integrals in Eqn. 44 vanish.

These observations suggest the following definition: $\mathbf{G}$ is an element of $\mathbf{D}'(\Gamma)$ whose action on

$$\boldsymbol{\Phi} = (\Phi_1(x,y), \Phi_2(x,y), \Phi_3(x,y))^{\mathrm{T}} \in \mathbf{C}^\infty_0(\Gamma)$$

is defined by

$$\mathbf{G}[\boldsymbol{\Phi}] := - \int\limits_{\mathbb{R}^3_-} \left(\nabla \times \alpha \left(\nabla \times \mathbf{E} \right) + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E} \right) + \nabla \times \nabla \times \gamma \left(\nabla \times \nabla \times \mathbf{E} \right) \right) \cdot \widehat{\boldsymbol{\Phi}} \, d\mathbf{r}$$

$$+ \int\limits_{\mathbb{R}^3_-} \left(\alpha \left(\nabla \times \mathbf{E} \right) \cdot \left(\nabla \times \boldsymbol{\Phi} \right) - \sum_{j \in \{x,y,z\}} \beta^j \partial_j \mathbf{E} \cdot \partial_j \boldsymbol{\Phi} + \gamma \left(\nabla \times \nabla \times \mathbf{E} \right) \cdot \left(\nabla \times \nabla \times \widehat{\boldsymbol{\Phi}} \right) \right) d\mathbf{r}, \tag{45}$$

where $\widehat{\boldsymbol{\Phi}}(\mathbf{r}) = (\Phi_1(x,y)\eta(z), \Phi_2(x,y)\eta(z), 0)^{\mathrm{T}}$, with $\eta \in \mathsf{C}^\infty_0(\mathbb{R})$ satisfying $\eta|_{\{|z|<\delta\}} = 1$ for some $\delta > 0$. It is easy to see that this definition is independent of the function η – the proof is the same as the one for $\mathbf{E}_+ \times \mathbf{n}$.

C. Traces arising in C_3

To define the trace $(\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n}$ we use an analogous idea. This time we plug into Eqn. 44 the function $\boldsymbol{\Phi}$ of the form given in Eqn. 23. Recall that for this $\boldsymbol{\Phi}$ one has $\boldsymbol{\Phi}|_\Gamma = 0$ (and, thus, the first and the last integrals over Γ in 44 vanish), while $\nabla \times \boldsymbol{\Phi}|_\Gamma = (\Phi_1, \Phi_2, 0)$.

Taking these observations into account we define $(\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n}$ as an element of $\mathbf{D}'(\Gamma)$ whose action on $\mathbf{\Phi} = (\Phi_1(x,y), \Phi_2(x,y), \Phi_3(x,y))^{\mathrm{T}} \in \mathbf{C}_0^\infty(\Gamma)$ is defined by the right-hand-side of 45 with $\widehat{\mathbf{\Phi}}(\mathbf{r}) = (\Phi_2(x,y)z\eta(z), -\Phi_1(x,y)z\eta(z), 0)^{\mathrm{T}}$ (the functions η is as above).

D. Traces arising in C_4

Let $\mathbf{E}$ satisfy the properties given in the conditions 10 and 14. Then we define the trace $(\beta^z \partial_z \mathbf{E}_-) \cdot \mathbf{n}$ as an element of $\mathsf{D}'(\Gamma)$ whose action on $\Psi \in \mathsf{C}_0^\infty(\Gamma)$ is defined by

$$((\beta^z \partial_z \mathbf{E}_-) \cdot \mathbf{n})[\Psi] := \int\limits_{\mathbb{R}^3_-} \left(\nabla \times \alpha \left(\nabla \times \mathbf{E}\right) + \sum_{j \in \{x,y,z\}} \partial_j \left(\beta^j \partial_j \mathbf{E}\right) + \nabla \times \nabla \times \gamma \left(\nabla \times \nabla \times \mathbf{E}\right) \right) \cdot \widehat{\mathbf{\Phi}} \; \mathrm{d}\mathbf{r}$$

$$- \int\limits_{\mathbb{R}^3_-} \left(\alpha \left(\nabla \times \mathbf{E}\right) \cdot \left(\nabla \times \mathbf{\Phi}\right) - \sum_{j \in \{x,y,z\}} \beta^j \partial_j \mathbf{E} \cdot \partial_j \mathbf{\Phi} + \gamma \left(\nabla \times \nabla \times \mathbf{E}\right) \cdot \left(\nabla \times \nabla \times \widehat{\mathbf{\Phi}}\right) \right) \; \mathrm{d}\mathbf{r}$$

$$+ \left((\gamma \nabla \times \nabla \times \mathbf{E}_-) \times \mathbf{n}\right)\left[\nabla \times \widehat{\mathbf{\Phi}}|_\Gamma\right],$$

where $\widehat{\mathbf{\Phi}}(\mathbf{r}) = (0, 0, \Psi\eta)^{\mathrm{T}}$ (the functions η is as above).

The reason for such a definition is as follows. If we plug into Eqn. 44 the function $\mathbf{\Phi}(\mathbf{r}) = (0, 0, \Psi\eta)^{\mathrm{T}}$, the first integral over Γ vanishes, and the integral $\int_\Gamma \left(\mathbf{n}\mathbf{n}^{\mathrm{T}} \beta^z \partial_z \mathbf{E}\right) \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{s}$ becomes $\int_\Gamma ((\beta^z \partial_z \mathbf{E}) \cdot \mathbf{n})\Psi \, \mathrm{d}\mathbf{s}$. Moreover, the integrals over $\mathbb{R}^3_-$ exist not only for smooth $\mathbf{E}$, but also when the properties expressed in conditions 10 and 14 hold. Finally, the integral $\int_\Gamma ((\gamma \nabla \times \nabla \times \mathbf{E}) \times \mathbf{n}) \cdot (\nabla \times \mathbf{\Phi}) \, \mathrm{d}\mathbf{s}$ can be interpreted as the action of $(\gamma \nabla \times \nabla \times \mathbf{E}) \times \mathbf{n}$ on $\nabla \times \mathbf{\Phi}|_\Gamma$.

E. Traces arising in C_5

To define the trace $(\nabla \times \mathbf{E}_+) \cdot \mathbf{n}$ we use the equality

$$\int\limits_\Gamma ((\nabla \times \mathbf{E}_+) \cdot \mathbf{n}) \, \Psi \, \mathrm{d}\mathbf{s} = -\int\limits_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot (\nabla\Psi) \, \mathrm{d}\mathbf{r}, \tag{46}$$

which hold for all $\Psi \in \mathsf{C}_0^\infty(\mathbb{R}^3)$ provided $\mathbf{E} \in \mathbf{C}_0^\infty(\overline{\mathbb{R}^3_+})$. The integral in the right-hand side of Eqn. 46 exists even for less regular $\mathbf{E}$. Namely, one needs only $\nabla \times \mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ (which is again fulfilled for generalized solutions of Eqn. 9). This suggests to define $(\nabla \times \mathbf{E}_+) \cdot \mathbf{n}$ as an element of $\mathsf{D}'(\Gamma)$ acting on $\Psi \in \mathsf{C}_0^\infty(\Gamma)$ as follows:

$$((\nabla \times \mathbf{E}_+) \cdot \mathbf{n})[\Psi] := -\int\limits_{\mathbb{R}^3_+} (\nabla \times \mathbf{E}) \cdot (\nabla\widehat{\Psi}) \, \mathrm{d}\mathbf{r},$$

where $\widehat{\Psi}$ is function from $\mathsf{C}_0^\infty(\mathbb{R}^3)$ satisfying $\widehat{\Psi}|_\Gamma = \Psi$. Similarly, we define

$$((\nabla \times \mathbf{E}_-) \cdot \mathbf{n})[\Psi] := \int\limits_{\mathbb{R}^3_-} (\nabla \times \mathbf{E}) \cdot (\nabla\widehat{\Psi}) \, \mathrm{d}\mathbf{r}.$$

F. Traces arising in C_6

Let $\mathbf{E} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ and $\nabla \cdot \mathbf{E} = 0$ in $\mathsf{D}'(\mathbb{R}^3_+)$ (these conditions are fulfilled for generalized solutions of Eqn. 9). Similarly to $(\nabla \times \mathbf{E}_+) \cdot \mathbf{n}$ we define the trace $\mathbf{E}_+ \cdot \mathbf{n}$ as an element of $\mathsf{D}'(\mathbb{R}^3_+)$ acting on $\Psi \in \mathsf{C}_0^\infty(\mathbb{R}^3_+)$ as follows:

$$(\mathbf{E}_+ \cdot \mathbf{n})[\Psi] := -\int\limits_{\mathbb{R}^3_+} \mathbf{E} \cdot \nabla\widehat{\Psi} \, \mathrm{d}\mathbf{r},$$

where $\widehat{\Psi}$ is function from $\mathsf{C}_0^\infty(\mathbb{R}^3)$ satisfying $\widehat{\Psi}|_\Gamma = \Psi$.

And, finally, we examine the trace

$$W := \left(\varepsilon\mathbf{E}_- + \sum_{j\in\{x,y,z\}} \partial_j\left(\beta^j\partial_j\mathbf{E}_-\right)\right)\cdot\mathbf{n} + \nabla_\Gamma\cdot\left(\beta^z\partial_z\mathbf{E}_z\right).$$

Assuming that $\mathbf{E}$ satisfies 16 one has for each $\Psi \in \mathsf{C}_0^\infty(\mathbb{R}_-^3)$:

$$\int_\Gamma W\Psi\,\mathrm{d}\mathbf{s} = \int_\Gamma \left((\beta^z\partial_z\mathbf{E})\cdot\mathbf{n}\right)\partial_z\Psi\,\mathrm{d}\mathbf{r}$$

$$+ \int_{\mathbb{R}_-^3} \nabla\cdot\left(\varepsilon\mathbf{E} + \sum_{j\in\{x,y,z\}}\partial_j(\beta^j\partial_j\mathbf{E})\right)\Psi\,\mathrm{d}\mathbf{r} + \int_{\mathbb{R}_-^3}\left(\varepsilon\mathbf{E}\cdot\nabla\Psi - \sum_{j\in\{x,y,z\}}\beta^j\partial_j\mathbf{E}\cdot\partial_j\nabla\Psi\right)\,\mathrm{d}\mathbf{r}. \quad (47)$$

Equality 47 suggests the following definition: let $\mathbf{E}$ satisfies the regularity condition given in expressions 10, 14, and 30 holds; we define the trace W as an element of $\mathsf{D}'(\Gamma)$ acting on $\Psi \in \mathsf{C}_0^\infty(\Gamma)$ as follows:

$$W[\Psi] := ((\beta_z\partial_z\mathbf{E})\cdot\mathbf{n})\left[\partial_z\widehat{\Psi}\right] + \int_{\mathbb{R}_-^3}\left(\varepsilon\mathbf{E}\cdot\nabla\widehat{\Psi} - \sum_{j\in\{x,y,z\}}\beta^j\partial_j\mathbf{E}\cdot\partial_j\nabla\widehat{\Psi}\right)\,\mathrm{d}\mathbf{r},$$

where $\widehat{\Psi}$ is a function from $\mathsf{C}_0^\infty(\mathbb{R}^3)$ satisfying $\widehat{\Psi}|_\Gamma = \Psi$.

G. Interface conditions for non-smooth generalized solutions

Now, assume that $\mathbf{E}$ is a generalized solution of Eqn. 9. This means that $\mathbf{E}$ satisfies expressions 10, 11 and, as a result, conditions 12-15, 30-31 hold, but the fulfilment of 16 is not guaranteed, in general. Nevertheless, it is easy to see that the interface conditions C_1-C_4, C_5-C_6 still hold with traces being understood according to the above definitions — as elements of $\mathbf{D}'(\Gamma)$ or $\mathsf{D}'(\Gamma)$.

Let us check this, for example, for condition C_2. For the function $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3) \in \mathbf{C}_0^\infty(\Gamma)$ we define the function $\widehat{\mathbf{\Phi}}$ by the formula 21. One has (below we use the notation $\mathbf{G}$, which is defined in 43):

$$0 \overset{12-15}{=} \int_{\mathbb{R}_+^3}\left(\left(\nabla\times\nabla\times\mathbf{E}\right) - \frac{\omega^2}{c^2}\mathbf{E}\right)\cdot\widehat{\mathbf{\Phi}}\,\mathrm{d}\mathbf{r} + \int_{\mathbb{R}_-^3}\left(\left(\nabla\times\nabla\times\mathbf{E}\right) - \frac{\omega^2}{c^2}\Big(\varepsilon\mathbf{E}\right.$$

$$\left.+\alpha\left(\nabla\times\nabla\times\mathbf{E}\right) + \sum_{j\in\{x,y,z\}}\partial_j\left(\beta^j\partial_j\mathbf{E}\right) + \nabla\times\nabla\times\gamma\left(\nabla\times\nabla\times\mathbf{E}\right)\Big)\right)\cdot\widehat{\mathbf{\Phi}}\,\mathrm{d}\mathbf{r}$$

$$\overset{41,42,45}{=} \left((\nabla\times\mathbf{E}_+)\times\mathbf{n}\right)[\mathbf{\Phi}] - \left((\nabla\times\mathbf{E}_-)\times\mathbf{n}\right)[\mathbf{\Phi}] + \frac{\omega^2}{c^2}\mathbf{G}[\mathbf{\Phi}] + \int_{\mathbb{R}^3}(\nabla\times\mathbf{E})\cdot(\nabla\times\widehat{\mathbf{\Phi}})\,\mathrm{d}\mathbf{r}$$

$$-\frac{\omega^2}{c^2}\int_{\mathbb{R}_-^3}\left(\varepsilon\mathbf{E}\cdot\widehat{\mathbf{\Phi}} + \alpha\left(\nabla\times\mathbf{E}\right)\cdot(\nabla\times\widehat{\mathbf{\Phi}}) - \sum_{j\in\{x,y,z\}}\beta^j\partial_j\mathbf{E}\cdot\partial_j\mathbf{\Phi} + \gamma\left(\nabla\times\nabla\times\mathbf{E}\right)\cdot(\nabla\times\nabla\times\widehat{\mathbf{\Phi}})\right)\,\mathrm{d}\mathbf{r}$$

$$\overset{11}{=} \left((\nabla\times\mathbf{E}_+)\times\mathbf{n}\right)[\mathbf{\Phi}] - \left((\nabla\times\mathbf{E}_-)\times\mathbf{n}\right)[\mathbf{\Phi}] + \frac{\omega^2}{c^2}\mathbf{G}[\mathbf{\Phi}]$$

As we have shown in this section, the interface conditions as such are not affected by these considerations.

VII. CONCLUSIONS

In conclusions, we have been considering light propagation in metamaterials that are described at the effective level with nonlocal constitutive relations. Nonlocal laws have also been considered in the past but their discussion

was usually linked to a specific metamaterial geometry. Also, the interface problem has been solved by introducing additional boundary conditions on phenomenological ground. Our contribution distinguishes itself that the nonlocal constitutive relation we consider is a firm extrapolation from the previous understanding that applies to arbitrary metamaterials. The dispersion relation of such metamaterials has been already discussed in the past; and here we have equipped these dispersion relations with the necessary interface conditions.

These interface conditions are derived from first principles. They rely on the evaluation of a generalized formulation of Maxwell's equations in a small volume entailing the interface. We have been deriving four main interface conditions and also formulated two alternative conditions. They do not introduce further information but basically look simpler. This might be occasionally beneficial. For a single interface, the consideration of all interface conditions is necessary. This might sound surprising but the nonlocal metamaterial sustains multiple plane waves as eigenmodes at each given frequency. This is in contrast to a local medium where only a single plane wave is supported.

Besides the actual interface conditions, we also derived explicit expressions for the Fresnel equations that can predict reflection and transmission from a slab upon illuminating it with a plane wave. We discuss both TE and TM polarization. The Fresnel equations are documented in a convenient way and are expressed in matrix notation.

With this work, further research endeavours in the context of metamaterials are immediately possible where the physics of such nonlocal metamaterials can be explored. It starts by developing suitable retrieval procedures and the actual quantification of the nonlocality of the metamaterials. It can be extended by analysing basic optical phenomena in the presence of metamaterials with a strong nonlocality and studying potential applications that rely on such nonlocal metamaterials. Also, the development of suitable numerical tools to explore light propagation in nonlocal metamaterials is an important and timely issue. Finally, based on the general formalism, other kinds of nonlocal constitutive relations can be explored.

APPENDIX

In this appendix, we present a short overview on the concept of generalized functions. For more details we refer, e.g. to Ref. 45.

Generalized functions

Let Ω be an open set in $\mathbb{R}^n$ (in the paper, Ω is either $\mathbb{R}^3$, $\mathbb{R}^3_-$, $\mathbb{R}^3_+$ or Γ). Recall that $\mathsf{C}_0^\infty(\Omega)$ denotes the space of infinitely often differentiable functions $\Psi : \Omega \to \mathbb{C}$ with compact support $\mathrm{supp}(\Psi) := \overline{\{x : \ \Psi(x) \neq 0\}} \subset \Omega$. Note, that in textbooks concerned with generalized functions this space is usually denoted by $\mathsf{D}(\Omega)$, but we prefer the notation $\mathsf{C}_0^\infty(\Omega)$ in order to avoid possible confusions with the electric displacement field $\mathbf{D}$.

The space $\mathsf{C}_0^\infty(\Omega)$ can be equipped with a concept of convergence: a sequence $\{\Psi_k\}_{k\in\mathbb{N}}$ in $\mathsf{C}_0^\infty(\Omega)$ is said to converge to $\Psi \in \mathsf{C}_0^\infty(\Omega)$ ($\Psi_k \to \Psi$ in $\mathsf{C}_0^\infty(\Omega)$) if there is a compact set $K \subset \Omega$ containing the supports of all Ψ_k, and for each multi-index α the sequence of partial derivatives $\{\partial_\alpha \Psi_k\}$ converges uniformly to $\partial_\alpha \Psi$.

A **generalized function** (or a **distribution**) on Ω is a continuous linear functional $F : \mathsf{C}_0^\infty(\Omega) \to \mathbb{C}$, i.e., F assigns to each function $\Psi \in \mathsf{C}_0^\infty(\Omega)$ a complex number $F[\Psi]$ so that

- $F(c_1\Psi_1 + c_2\Psi_2) = c_1 F(\Psi_1) + c_2 F(\Psi_2)$ for all $\Psi_1, \Psi_2 \in \mathsf{C}_0^\infty(\Omega)$, $c_1, c_2 \in \mathbb{C}$.

- $\lim_{k\to\infty} F[\Psi_k] = F[\Psi]$ provided $\Psi_k \to \Psi$ in $\mathsf{C}_0^\infty(\Omega)$.

The space of generalized functions on Ω is denoted by $\mathsf{D}'(\Omega)$.

Examples

Recall, that $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ is the set of locally integrable functions (i.e., the functions being Lebesgue integrable over every compact subset of Ω). Any function $f \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$ yields a generalized function F_f acting on $\Psi \in \mathsf{C}_0^\infty(\Omega)$ by

$$F_f[\Psi] := \int_\Omega f(x)\Psi(x) \, \mathrm{d}x. \tag{48}$$

If $F \in \mathsf{D}'(\Omega)$ is given by 48 for some $f \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$, then F is called a **regular** generalized function.

Let $f, g \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$. Then the associated regular generalized functions F_f and F_g coincide (i.e. $F_f[\Psi] = F_g[\Psi]$, $\forall \Psi$) if and only if f and g are the same elements of $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ (i.e., they are are equal almost everywhere).

In what follows we will use the same notation for a locally integrable function and for the associated regular generalized function.

$F \in \mathsf{D}'(\Omega)$ is called **singular** if it is not a regular generalized function. An example of a singular generalized function is a *Dirac delta-function* $\delta_{x_0} \in \mathsf{D}'(\Omega)$ supported at $x_0 \in \Omega$: its acts on $\Psi \in \mathsf{C}^\infty_0(\Omega)$ by $\delta_{x_0}[\Psi] := \Psi(x_0)$.

Operations on generalized function

Linear combination

Let $F_k \in \mathsf{D}'(\Omega)$ and $c_k \in \mathbb{C}$, $k = 1, \ldots, N$. Then the generalized function $F = \sum_{k=1}^N c_k F_k$ acts on $\Psi \in \mathsf{C}^\infty_0(\Omega)$ by

$$F[\Psi] := \sum_{k=1}^N c_k F_k[\Psi].$$

Differentiation

Let $F \in \mathsf{C}^\infty_0(\Omega)$ and $\Psi \in \mathsf{C}^\infty_0(\Omega)$. Let $\alpha = (\alpha_1, \ldots \alpha_n)$ be a multiindex and $\partial_\alpha := \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dfrac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ be the associated partial differential operator. Integration by parts gives

$$\int_\Omega (\partial_\alpha F) \, \Psi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega F \, (\partial_\alpha \Psi) \, \mathrm{d}x, \tag{49}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Now, let $F \in \mathsf{D}'(\Omega)$. Equality 49 suggests to define $\partial_\alpha F$ as an element of $\mathsf{D}'(\Omega)$ acting on $\Psi \in \mathsf{C}^\infty_0(\Omega)$ by

$$(\partial_\alpha F)[\Psi] := (-1)^{|\alpha|} F[\partial_\alpha \Psi].$$

As we see, in contrast to "classical" functions, generalized functions can be differentiated infinitely often. Of course, if F is a regular generalized function, $\partial_\alpha F$ need not be regular, in general. For example, let $H(x)$ be the Heaviside step function on $\mathbb{R}$ being equal to 1 as $x > x_0$ and equal to 0 as $x \le x_0$. $H(x)$ is not differentiable on $\mathbb{R}$ due to the discontinuity at x_0. But, since $H \in \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$ we can regard it as a generalized function. It is easy to see that the generalized derivative of H is δ_{x_0}.

1. Multiplication by a smooth function

Let f be an infinitely differentiable function on Ω and $F \in \mathsf{D}'(\Omega)$. We define the product fF as an element of $\mathsf{D}'(\Omega)$ acting on $\Psi \in \mathsf{C}^\infty_0(\Omega)$ by

$$(fF)[\Psi] := F[f\Psi].$$

This definition is motivated by the commutativity of the product of scalar functions, namely

$$\int_\Omega f(x) F(x) \, \Psi(x) \, \mathrm{d}x = \int_\Omega F(x) \, f(x) \Psi(x) \, \mathrm{d}x$$

provided $F \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$.

It is clear that the above definition fails if f is not smooth since the product $f\Psi$ is no longer in $\mathsf{C}^\infty_0(\Omega)$. On the other hand if F is a regular generalized function and f is locally bounded then the product is well-defined via $(fF)[\Psi] := \int_\Omega f(x) F(x) \, \Psi(x) \, \mathrm{d}x$.

Generalized vector-functions

Recall, that $\mathbf{C}_0^\infty(\Omega)$ stands for the space of infinitely differentiable vector-functions $\mathbf{\Phi} : \Omega \to \mathbb{C}^3$ with compact support in Ω. Then we define $\mathbf{D}'(\Omega)$ as the space of continuous linear functionals acting on $\mathbf{C}_0^\infty(\Omega)$.

The definition of derivatives for $\mathbf{D}'(\Omega)$ is similar to the corresponding definition for $\mathsf{D}'(\Omega)$ – it mimics partial integration. In particular, for $\mathbf{F} \in \mathbf{D}'(\Omega)$ one has $\nabla \times \mathbf{F} \in \mathbf{D}'(\Omega)$, $\nabla \cdot \mathbf{F} \in \mathsf{D}'(\Omega)$, by the definitions

$$(\nabla \times \mathbf{F})[\mathbf{\Phi}] := \mathbf{F}[\nabla \times \mathbf{\Phi}], \quad (\nabla \cdot \mathbf{F})[\Psi] := -\mathbf{F}[\nabla\Psi].$$

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