

# Static Output-Feedback Control of Markov Jump Linear Systems without Mode Observation

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**Abstract**—In this paper, we address infinite-horizon optimal control of Markov Jump Linear Systems (MJLS) via static output feedback. Because the jump parameter is assumed not to be observed, the optimal control law is nonlinear and intractable. Therefore, we assume the regulator to be linear. Under this assumption, we first present sufficient feasibility conditions for static output-feedback stabilization of MJLS with non-observed mode in the mean square sense in terms of linear matrix inequalities (LMIs). However, these conditions depend on the particular state-space representation, i.e., a coordinate transform can make the LMIs feasible, while the original LMIs are infeasible. To avoid the issues with the ambiguity of the state-space representation, we therefore present an iterative algorithm for the computation of the regulator gain. The algorithm is shown to converge if the MJLS is stabilizable via mode-independent static output feedback. However, convergence of the algorithm is not sufficient for stability of the closed loop, which requires an additional stability check after the regulator gains have been computed. A numerical example demonstrates the application of the presented results.

## I. INTRODUCTION

Although many systems can be described by linear dynamical models, there are systems whose dynamics are subject to abrupt changes or jumps that cannot be captured by smooth continuous-valued differential or difference equations. Thus, a convenient approach is to model these jumps as a discrete-valued state, denoted as the *mode* of the system. In literature, such systems are referred to as *Hybrid Systems* [1]. In many applications, the discrete-valued state can be modeled as a Markov chain that is independent of the continuous-valued dynamics. These systems are then referred to as *Markov Jump Systems* (MJS) or as *Markov Jump Linear Systems* (MJLS), if the continuous-valued dynamics are linear [2]. The interaction between the continuous-valued and discrete-valued dynamics allows to model component failures [3], [4], economic processes [5], [6], or networked control systems [7], [8]. More applications of MJLS are discussed in [2].

Even though an MJLS is a nonlinear system due to the discontinuities in the dynamics, the optimal Linear Quadratic (LQ) and Linear Quadratic Gaussian (LQG) control laws are linear in the state and even in its estimate if the mode is perfectly observed [9], [10]. However, if the mode cannot be observed, there is a dual effect and the separation between control and

estimation does not hold [11]. This yields a nonlinear control law whose derivation is not tractable due to the curse of dimensionality [12]. Thus, suboptimal control strategies are of interest, particularly strategies that are closed-loop.

An important class of suboptimal control laws for MJLS without mode observation are the linear optimal regulators [3], [4], [6], [13], [14]. These regulators are derived under the assumption that the control law is linear in the state or output feedback. By doing so, it is possible to reformulate the considered cost function in terms of the closed-loop dynamics. This reformulated cost function is then minimized with respect to the regulator gains. Do Val et al. applied the described procedure to finite-horizon control of MJLS without process noise via state-feedback [6]. The authors assume time-variant regulator gains that are computed using a variational approach. This work was extended to MJLS with process noise in [3]. A finite-horizon control approach for MJLS without process noise and a constant regulator gain is presented in [4], where the authors also evaluate different optimization algorithms that can be used for the minimization of the cost function. In [14], we derived an infinite-horizon algorithm for state-feedback average cost per stage control of stochastic MJLS. To this end, we formulate a nonlinear optimization problem that is solved using Lagrange multipliers and an iterative procedure.  $H_2$  control of MJLS with clustered mode observations is addressed in [13], where the case of no mode observation is recovered when the cluster of observed modes is empty. The regulator gain for a particular observation structure is obtained via a Linear Matrix Inequality (LMI). Our regulator gain computation algorithm from [14] is an alternative computation method to [13]. The authors of [15] address the same problem as we do in this paper, i.e., output-feedback control without mode observation. However, they assume a time-variant controller, whose gains are computed using a variational approach similar to that in [6], [4]. Finally, a very related work on infinite-horizon static output feedback is presented in [16]. However, the authors of [16] consider MJLS dynamics without process noise.

Other approaches to control of MJLS without mode observation are, for example, methods based on separation assumption that implement a filter for the mode [17], [18] or model-predictive methods, see, e.g., [19]. Furthermore, research also concentrates on systems where the mode is available only with a delay [20] and systems with partially-known transition matrices [21], [22].

In this work, we generalize our previous results from [14], where we only considered the state-feedback control problem.

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The contribution of this paper consists of (i) the derivation of sufficient LMI conditions for mean square stabilizability of MJLS with non-observed mode via static output feedback and (ii) the presentation of an iterative algorithm that allows to compute a regulator gain for static output-feedback control of MJLS without mode observation that minimizes an infinite-horizon cost function. The algorithm is given as a recursion whose fixed point yields the regulator gain. However, convergence of the algorithm is not sufficient for stability. Thus, it is necessary to check stability of the closed-loop system after the regulator gain has been computed. We demonstrate this issue in a numerical example.

*Outline.* The remainder of the paper is organized as follows. In the next section, we formulate the considered problem. The main results of the paper are presented in Sec. III. We provide a numerical example that demonstrates the proposed algorithm in Sec. IV and conclude the paper in Sec. V.

## II. PROBLEM FORMULATION AND BASIC CONCEPTS

We consider the MJLS with dynamics

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_{\theta_k} \mathbf{x}_k + \mathbf{B}_{\theta_k} \mathbf{u}_k + \mathbf{H}_{\theta_k} \mathbf{w}_k, \\ \mathbf{y}_k &= \mathbf{C}_{\theta_k} \mathbf{x}_k,\end{aligned}\quad (1)$$

where  $\mathbf{x}_k \in \mathbb{R}^n$  denotes the system state,  $\mathbf{u}_k \in \mathbb{R}^m$  the control input,  $\mathbf{y}_k \in \mathbb{R}^p$  the output, and  $\mathbf{w}_k \in \mathbb{R}^q$  is an independent and identically distributed stationary second-order noise with mean  $\mathbf{0}$  and covariance  $\mathbf{I}$ . The matrices  $\mathbf{A}_{\theta_k}$ ,  $\mathbf{B}_{\theta_k}$ ,  $\mathbf{H}_{\theta_k}$ , and  $\mathbf{C}_{\theta_k}$  are selected at each time step  $k$  from time-invariant sets  $\{\mathbf{A}_1, \dots, \mathbf{A}_M\}$ ,  $\{\mathbf{B}_1, \dots, \mathbf{B}_M\}$ ,  $\{\mathbf{H}_1, \dots, \mathbf{H}_M\}$ , and  $\{\mathbf{C}_1, \dots, \mathbf{C}_M\}$ , according to the value of the random variable  $\theta_k \in \{1, 2, \dots, M\}$ ,  $M \in \mathbb{N}$ , referred to as mode. The dynamics of  $\theta_k$  are modeled as a time-homogeneous regular Markov chain [23] with transition probability matrix  $\mathbf{T} = (p_{ij})_{M \times M}$ ,  $p_{ij} = P(\theta_{k+1} = j | \theta_k = i)$ . Finally, we assume that the matrices  $\mathbf{C}_{\theta_k}$  have full row rank.

For system (1), we will design a static output-feedback control law of the form

$$\mathbf{u}_k = \mathbf{L} \mathbf{y}_k \quad (2)$$

that is independent of the initial condition  $\{\mathbf{x}_0, \theta_0\}$ , and the modes  $\theta_k$  for  $k > 0$ , i.e., the mode is not observed. The control law shall minimize the infinite-horizon cost function

$$\mathcal{J} = \lim_{K \rightarrow \infty} \frac{1}{K} \mathbb{E} \left\{ \sum_{k=0}^{K-1} [\mathbf{x}_k^\top \mathbf{Q}_{\theta_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R}_{\theta_k} \mathbf{u}_k] \right\}, \quad (3)$$

where the cost matrices  $\mathbf{Q}_{\theta_k}$  and  $\mathbf{R}_{\theta_k}$  are positive semidefinite and positive definite, respectively, and the expectation  $\mathbb{E}\{\cdot\}$  is taken with respect to  $\mathbf{w}_k$  and  $\theta_k$ . In (3), the costs are averaged over the horizon length  $K$  that goes to infinity. By plugging (2) into (3), we can summarize the considered problem as

$$\begin{aligned}\inf_{\mathbf{L}} \lim_{K \rightarrow \infty} \frac{1}{K} \mathbb{E} \left\{ \sum_{k=0}^{K-1} \mathbf{x}_k^\top (\mathbf{Q}_{\theta_k} + \mathbf{C}_{\theta_k}^\top \mathbf{L}^\top \mathbf{R}_{\theta_k} \mathbf{L} \mathbf{C}_{\theta_k}) \mathbf{x}_k \right\} \\ \text{s. t. } \mathbf{x}_{k+1} = (\mathbf{A}_{\theta_k} + \mathbf{B}_{\theta_k} \mathbf{L} \mathbf{C}_{\theta_k}) \mathbf{x}_k + \mathbf{H}_{\theta_k} \mathbf{w}_k.\end{aligned}\quad (4)$$

In the next section, we first provide sufficient feasibility conditions for the optimization problem (4). Then, we derive necessary optimality conditions and provide an iterative algorithm for evaluation of these conditions.

## III. MAIN RESULTS

In this section, we present the main results of the paper. First, feasibility of (4) can be checked using the following theorem.

### Theorem 1 (Sufficient Feasibility Condition)

The optimization problem (4) is feasible, if there exist matrices  $\{\mathbf{W}_1, \dots, \mathbf{W}_M\}$ ,  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_M\}$ ,  $\mathbf{F}$ ,  $\mathbf{M}$ , and  $\mathbf{G}$  such that the LMIs

$$\begin{aligned}\begin{bmatrix} \mathbf{W}_i & \mathbf{D}_i \mathbf{G} + \mathbf{E}_i \mathbf{F} \mathbf{C}_i \\ \mathbf{G}^\top \mathbf{D}_i^\top + \mathbf{C}_i^\top \mathbf{F}^\top \mathbf{E}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ \begin{bmatrix} \mathbf{Y}_i - \mu_\infty^i \mathbf{H}_i \mathbf{H}_i^\top & \mathbf{A}_i \mathbf{G} + \mathbf{B}_i \mathbf{F} \mathbf{C}_i \\ \mathbf{G}^\top \mathbf{A}_i^\top + \mathbf{C}_i^\top \mathbf{F}^\top \mathbf{B}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ \mathbf{M} \mathbf{C}_i = \mathbf{C}_i \mathbf{G} \quad i = 1, \dots, M\end{aligned}$$

are feasible, where  $\mu_\infty^i = \lim_{k \rightarrow \infty} P(\theta_k = i)$  is the limit distribution of the Markov chain  $\{\theta_k\}$  and

$$\mathbf{D}_i = \begin{bmatrix} \mathbf{Q}_i^{\frac{1}{2}} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{E}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_i^{\frac{1}{2}} \end{bmatrix}, \quad \mathcal{E}_i(\mathbf{Y}) = \sum_{j=1}^M p_{ij} \mathbf{Y}_j, \quad (5)$$

where  $\mathbf{Q}_i^{\frac{1}{2}}$  and  $\mathbf{R}_i^{\frac{1}{2}}$  are obtained using, e.g., the Cholesky decomposition<sup>1</sup>. The mean square stabilizing regulator gain is given by  $\mathbf{L} = \mathbf{F} \mathbf{M}^{-1}$ .

**Proof 1** To derive the LMI from Theorem 1, we convert the infinite-horizon LQG problem of minimizing (3) subject to (1) into an  $H_2$  problem by introducing the performance output

$$\mathbf{z}_k = \mathbf{D}_{\theta_k} \mathbf{x}_k + \mathbf{E}_{\theta_k} \mathbf{u}_k,$$

where  $\mathbf{D}_{\theta_k}$  and  $\mathbf{E}_{\theta_k}$  are as in (5) so that  $\mathbf{D}_{\theta_k}^\top \mathbf{E}_{\theta_k} = \mathbf{0}$  holds. From [13], we have that the minimum of the  $H_2$  norm of an MJLS

$$\begin{aligned}\mathbf{x}_{k+1} &= \tilde{\mathbf{A}}_{\theta_k} \mathbf{x}_k + \mathbf{H}_{\theta_k} \mathbf{w}_k, \\ \mathbf{z}_k &= \tilde{\mathbf{D}}_{\theta_k} \mathbf{x}_k\end{aligned}$$

with non-observed mode is the solution of

$$\inf_{\mathbf{P}_i} \sum_{i=1, \dots, M} \text{trace} \left[ \tilde{\mathbf{D}}_i \mathbf{P}_i \tilde{\mathbf{D}}_i^\top \right] \quad (6)$$

$$\text{s. t. } \mathbf{P}_i > \sum_{j=1}^M \tilde{\mathbf{A}}_j \mathbf{P}_j \tilde{\mathbf{A}}_j^\top + \mu_\infty^j \mathbf{H}_j \mathbf{H}_j^\top, \quad i = 1, \dots, M.$$

According to [13], problem (6) can be bounded from above with

$$\begin{aligned}\inf_{\mathbf{W}_i, \mathbf{Y}_i, \mathbf{G}} \sum_{i=1}^M \text{trace} [\mathbf{W}_i] \\ \text{s. t. } \begin{bmatrix} \mathbf{W}_i & \tilde{\mathbf{D}}_i \mathbf{G} \\ \mathbf{G}^\top \tilde{\mathbf{D}}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ \begin{bmatrix} \mathbf{Y}_i - \mu_\infty^i \mathbf{H}_i \mathbf{H}_i^\top & \tilde{\mathbf{A}}_i \mathbf{G} \\ \mathbf{G}^\top \tilde{\mathbf{A}}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0}.\end{aligned} \quad i = 1, \dots, M.$$

<sup>1</sup>As can be seen later, matrices  $\mathbf{Y}_i$  can be interpreted as the costs-to-go associated with the current system state and the operator  $\mathcal{E}_i(\mathbf{Y})$  are the backwards evolution of these costs.

Now, substituting  $\tilde{\mathbf{A}}_i = \mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i$  and  $\tilde{\mathbf{D}}_i = \mathbf{D}_i + \mathbf{E}_i \mathbf{L} \mathbf{C}_i$ , we obtain the Bilinear Matrix Inequality (BMI)

$$\begin{aligned} & \inf_{\mathbf{W}_i, \mathbf{Y}_i, \mathbf{G}_i, \mathbf{L}} \sum_{i=1}^M \text{trace}[\mathbf{W}_i] \\ \text{s. t. } & \begin{bmatrix} \mathbf{W}_i & \mathbf{D}_i \mathbf{G} + \mathbf{E}_i \mathbf{L} \mathbf{C}_i \mathbf{G} \\ \mathbf{G}^\top \mathbf{D}_i^\top + \mathbf{G}^\top \mathbf{C}_i^\top \mathbf{L}^\top \mathbf{E}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ & \begin{bmatrix} \mathbf{Y}_i - \underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top & \mathbf{A}_i \mathbf{G} + \mathbf{B}_i \mathbf{L} \mathbf{C}_i \mathbf{G} \\ \mathbf{G}^\top \mathbf{A}_i^\top + \mathbf{G}^\top \mathbf{C}_i^\top \mathbf{L}^\top \mathbf{B}_i^\top & \mathbf{G} + \mathbf{G}^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ & i = 1, \dots, M. \end{aligned} \quad (7)$$

Since we bound the initial  $H_2$  problem (6) with the BMI (7) from above, a feasible solution to this BMI is also a feasible solution to the original problem (4). Thus, feasibility of (4) can be determined by checking the feasibility of (7). However, it has been shown in [24] that solving a BMI is NP-hard and thus (7) is impractical. For this reason, we convert the BMI into an LMI by introducing additional constraints  $\mathbf{M} \mathbf{C}_i = \mathbf{C}_i \mathbf{G}$  as it has been proposed for systems with deterministic system matrices in [25]. Recall that  $\mathbf{C}_i$  are assumed to have full row rank. Consequently, it  $\mathbf{M}$  is invertible and we can write

$$\mathbf{C}_i = \mathbf{M}^{-1} \mathbf{C}_i \mathbf{G}. \quad (8)$$

Next, we define  $\mathbf{F} = \mathbf{L} \mathbf{M}$  and substitute  $\mathbf{L} = \mathbf{F} \mathbf{M}^{-1}$  in (7). Finally, using (8) concludes the proof.

**Remark 1** In case the mode is observed, the optimization problem (4) via static output feedback is feasible if there exist matrices  $\{\mathbf{W}_1, \dots, \mathbf{W}_M\}$ ,  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_M\}$ ,  $\{\mathbf{F}_1, \dots, \mathbf{F}_M\}$ ,  $\{\mathbf{M}_1, \dots, \mathbf{M}_M\}$ , and  $\{\mathbf{G}_1, \dots, \mathbf{G}_M\}$  such that the LMI

$$\begin{aligned} & \begin{bmatrix} \mathbf{W}_i & \mathbf{D}_i \mathbf{G}_i + \mathbf{E}_i \mathbf{F}_i \mathbf{C}_i \\ \mathbf{G}_i^\top \mathbf{D}_i^\top + \mathbf{C}_i^\top \mathbf{F}_i^\top \mathbf{E}_i^\top & \mathbf{G}_i + \mathbf{G}_i^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ & \begin{bmatrix} \mathbf{Y}_i - \underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top & \mathbf{A}_i \mathbf{G}_i + \mathbf{B}_i \mathbf{F}_i \mathbf{C}_i \\ \mathbf{G}_i^\top \mathbf{A}_i^\top + \mathbf{C}_i^\top \mathbf{F}_i^\top \mathbf{B}_i^\top & \mathbf{G}_i + \mathbf{G}_i^\top - \mathcal{E}_i(\mathbf{Y}) \end{bmatrix} > \mathbf{0} \\ & \mathbf{M}_i \mathbf{C}_i = \mathbf{C}_i \mathbf{G}_i \quad i = 1, \dots, M \end{aligned}$$

is feasible. The corresponding regulator gains are determined by  $\mathbf{L}_i = \mathbf{F}_i \mathbf{M}_i^{-1}$ . The proof is similar to the proof of Theorem 1. Furthermore, the case of cluster mode observations [13] can be recovered by searching for matrices  $\{\mathbf{F}_{j1}, \dots, \mathbf{F}_{jS}\}$ ,  $\{\mathbf{M}_{j1}, \dots, \mathbf{M}_{jS}\}$ , and  $\{\mathbf{G}_{j1}, \dots, \mathbf{G}_{jS}\}$ , where  $j1, \dots, jS$  are the indexes of the  $S \in \{0, 1, \dots, M\}$  observed modes.

Please note that the results of Theorem 1 show how the generally non-convex problem of static output-feedback stabilization of MJLS without mode observation can be convexified. However, the feasibility condition from Theorem 1 are derived by bounding the  $H_2$  problem that corresponds to (4) from above. Thus, the results from Theorem 1 are sufficient but not necessary. Furthermore, as pointed out in [25], the feasibility of the conditions from Theorem 1 depends on the particular state-space representation. Nevertheless, if the system parameters are deterministic, there exist similarity transformation, although they may be hard to find, such that the LMIs become feasible if the initial system is stabilizable via static output feedback.

It needs to be verified, if this property also holds for MJLS. In what follows, we first derive the necessary optimality conditions for problem (4) in the next theorem and then propose an iterative algorithm for the computation of the regulator gain  $\mathbf{L}$ , in order to avoid the issues with the ambiguity of the state-space representation.

### Theorem 2 (Necessary Optimality Conditions)

Consider the MJLS (1) and assume that this system is stabilizable via static output-feedback (2). Then, the optimal linear static output regulator gain that minimizes the cost function (3) is determined by the nonlinear coupled equations

$$\begin{aligned} & \sum_{i=1}^M (\mathbf{R}_i + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_\infty) \mathbf{B}_i) \mathbf{L} \mathbf{C}_i \mathbf{X}_\infty^i \mathbf{C}_i^\top \\ & + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_\infty) \mathbf{A}_i \mathbf{X}_\infty^i \mathbf{C}_i^\top = \mathbf{0}, \quad (9) \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^M p_{ij} (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_\infty^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i)^\top \\ & + \underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top - \mathbf{X}_\infty^j = \mathbf{0}, \quad (10) \end{aligned}$$

$$\begin{aligned} & -\mathbf{P}_\infty^i + \mathbf{Q}_i + \mathbf{C}_i^\top \mathbf{L}^\top \mathbf{R}_i \mathbf{L} \mathbf{C}_i \\ & + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i)^\top \mathcal{E}_i(\mathbf{P}_\infty) (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i) = \mathbf{0}, \quad (11) \end{aligned}$$

where the matrices  $\mathbf{X}_\infty^i$  and  $\mathbf{P}_\infty^i$  are positive definite.

**Proof 2** Using the notation from [2], we define the second-moment system state

$$\mathbf{X}_k^i = \mathbb{E} \{ \underline{\mathbf{x}}_k \underline{\mathbf{x}}_k^\top \mathbb{1}_{\theta_k=i} \},$$

where  $\mathbb{1}_A = 1$  if  $A$  is true and 0 otherwise. Plugging (2) into (1), we obtain the closed-loop dynamics of the second moment

$$\begin{aligned} \mathbf{X}_{k+1}^j &= \sum_{i=1}^M p_{ij} \left[ \underline{\mu}_k^i \mathbf{H}_i \mathbf{H}_i^\top \right. \\ & \left. + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_k^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i)^\top \right], \end{aligned}$$

where  $\underline{\mu}_k^i = \mathbb{P}(\theta_k = i)$  (see Proposition 3.35 in [2] for a more detailed derivation). Now, we can rewrite (4) in terms of the second moment as

$$\begin{aligned} & \inf_{\mathbf{L}} \sum_{i=1}^M \text{trace} [(\mathbf{Q}_i + \mathbf{C}_i^\top \mathbf{L}^\top \mathbf{R}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_\infty^i] \\ \text{s. t. } & \mathbf{X}_\infty^j = \sum_{i=1}^M p_{ij} \left[ \underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top \right. \\ & \left. + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_\infty^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i)^\top \right], \end{aligned}$$

where we already took the limit  $K \rightarrow \infty$  (see Ch. 4.4.3 in [2]). Introducing the positive definite Lagrange multiplier  $\{\mathbf{P}_\infty^1, \dots, \mathbf{P}_\infty^M\}$  allows us to construct the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum_{i=1}^M \text{trace} [(\mathbf{Q}_i + \mathbf{C}_i^\top \mathbf{L}^\top \mathbf{R}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_\infty^i - \mathbf{P}_\infty^i \mathbf{X}_\infty^i \\ & + \mathcal{E}_i(\mathbf{P}_\infty) (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i) \mathbf{X}_\infty^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L} \mathbf{C}_i)^\top \\ & + \underline{\mu}_\infty^i \mathcal{E}_i(\mathbf{P}_\infty) \mathbf{H}_i \mathbf{H}_i^\top]. \end{aligned}$$

From the necessary optimality conditions

$$\frac{\partial \mathcal{H}}{\partial \mathbf{L}} = \mathbf{0}, \quad \frac{\partial \mathcal{H}}{\partial \mathbf{X}_\infty^i} = \mathbf{0}, \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial \mathbf{P}_\infty^i} = \mathbf{0},$$

we obtain the set of nonlinear coupled equations (9)-(11).

Finding a solution to the equations (9)-(11) is not trivial. Thus, we propose the following iterative algorithm that converges to a solution of the equations from Theorem 2. The algorithm proceeds as follows.

*Step 1:* Set the counter  $\eta = 0$  and initialize  $\mathbf{X}_{[\eta]}^i$  and  $\mathbf{P}_{[\eta]}^i$  with random positive definite matrices.

*Step 2:* Compute  $\mathbf{L}_{[\eta]}$  either using

$$\begin{aligned} \text{vec}(\mathbf{L}_{[\eta]}) = & - \left( \sum_{i=1}^M \mathbf{C}_i \mathbf{X}_\infty^i \mathbf{C}_i^\top \otimes (\mathbf{R}_i + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_\infty) \mathbf{B}_i) \right)^\dagger \\ & \times \text{vec} \left( \sum_{i=1}^M \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_\infty) \mathbf{A}_i \mathbf{X}_\infty^i \mathbf{C}_i^\top \right), \end{aligned} \quad (12)$$

where  $\text{vec}(\cdot)$  denotes the vectorization operator,  $\mathbf{A}^\dagger$  the Moore-Penrose pseudoinverse of  $\mathbf{A}$ , and  $\otimes$  is the Kronecker product, and inverting the vectorization, or by solving the semidefinite program

$$\begin{aligned} \min \quad & \lambda \\ \text{s. t.} \quad & \begin{bmatrix} \lambda \mathbf{I} & \mathbf{L}_{[\eta]} \\ \mathbf{L}_{[\eta]}^\top & \mathbf{I} \end{bmatrix} > \mathbf{0} \\ & \sum_{i=1}^M (\mathbf{R}_i + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_{[\eta]}) \mathbf{B}_i) \mathbf{L}_{[\eta]} \mathbf{C}_i \mathbf{X}_{[\eta]}^i \mathbf{C}_i^\top \\ & \quad + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_{[\eta]}) \mathbf{A}_i \mathbf{X}_{[\eta]}^i \mathbf{C}_i^\top = \mathbf{0} \end{aligned} \quad (13)$$

that is slower but numerically more robust than (12) that involves matrix inversions.

*Step 3:* Compute the updates  $\mathbf{X}_{[\eta+1]}^i$  and  $\mathbf{P}_{[\eta+1]}^i$  according to

$$\begin{aligned} \mathbf{X}_{[k+1]}^j &= \sum_{i=1}^M p_{ij} \left[ \mu_\infty^i \mathbf{H}_i \mathbf{H}_i^\top \right. \\ & \quad \left. + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]} \mathbf{C}_i) \mathbf{X}_{[k]}^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]} \mathbf{C}_i)^\top \right], \\ \mathbf{P}_{[k+1]}^i &= \mathbf{Q}_i + \mathbf{C}_i^\top \mathbf{L}_{[k]}^\top \mathbf{R}_i \mathbf{L}_{[k]} \mathbf{C}_i \\ & \quad + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]} \mathbf{C}_i)^\top \mathcal{E}_i(\mathbf{P}_{[k]}) (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]} \mathbf{C}_i). \end{aligned} \quad (14)$$

*Step 4:* If the equations  $\mathbf{X}_{[\eta+1]}^i \approx \mathbf{X}_{[\eta]}^i$ ,  $\mathbf{P}_{[\eta+1]}^i \approx \mathbf{P}_{[\eta]}^i$ ; and  $\mathbf{L}_{[\eta]} \approx \mathbf{L}_{[\eta-1]}$  hold with sufficient precision, stop the algorithm. Otherwise, set  $\eta = \eta + 1$  and return to *Step 2*.

Considering the implementation of the iterative algorithm, we suggest to initialize  $\mathbf{X}_{[\eta]}^i$  and  $\mathbf{P}_{[\eta]}^i$  in *Step 1* with  $1e^{-4} \cdot \mathbf{I}$ . Although the algorithm converges for any initial positive definite values, the convergence is usually faster for the proposed value. Furthermore, computing the regulator gain  $\mathbf{L}_{[\eta]}$  using the semidefinite program (13) is computationally more stable than computing via (12). Finally, please note that the convergence of the recursion (14) is not sufficient for stability. Thus, it is necessary to check whether the closed-loop system is stable using, e.g., Corollary 2.6 from [26]. The next

theorem finalizes the theoretical contribution of this section. To this end, we will need the following lemma.

**Lemma 1** Assume that MJLS (1) is mean square stabilizable via mode-dependent static output-feedback  $\underline{u}_k = \mathbf{L}^{\theta_k} \underline{y}_k$  with observation of the mode  $\theta_k$ . Then, the recursion

$$\begin{aligned} \mathbf{X}_{[k+1]}^j &= \sum_{i=1}^M p_{ij} \left[ \mu_\infty^i \mathbf{H}_i \mathbf{H}_i^\top \right. \\ & \quad \left. + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]}^i \mathbf{C}_i) \mathbf{X}_{[k]}^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]}^i \mathbf{C}_i)^\top \right], \\ \mathbf{P}_{[k+1]}^i &= \mathbf{Q}_i + \mathbf{C}_i^\top (\mathbf{L}_{[k]}^i)^\top \mathbf{R}_i \mathbf{L}_{[k]}^i \mathbf{C}_i \\ & \quad + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]}^i \mathbf{C}_i)^\top \mathcal{E}_i(\mathbf{P}_{[k]}) (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[k]}^i \mathbf{C}_i), \end{aligned} \quad (15)$$

where  $\mathbf{L}_{[\eta]}^i$  are determined by

$$\begin{aligned} \sum_{i=1}^M (\mathbf{R}_i + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_{[\eta]}) \mathbf{B}_i) \mathbf{L}_{[\eta]}^i \mathbf{C}_i \mathbf{X}_{[\eta]}^i \mathbf{C}_i^\top \\ + \mathbf{B}_i^\top \mathcal{E}_i(\mathbf{P}_{[\eta]}) \mathbf{A}_i \mathbf{X}_{[\eta]}^i \mathbf{C}_i^\top = \mathbf{0}, \end{aligned}$$

converges to its unique solution  $(\mathbf{X}_\infty^i, \mathbf{P}_\infty^i)$  for  $\eta \rightarrow \infty$ .

**Proof 3** To show the result of Lemma 1, construct a homotopy by substituting  $\mathbf{C}_i = (1 - \alpha) \mathbf{I} + \alpha \mathbf{C}_i$ . Then, for  $\alpha = 0$ , we have a state-feedback control problem, for which the recursion (15) converges to a unique solution [2]. Now, observe that the assumption that the MJLS (1) is stabilizable via mode-dependent static output feedback ( $\alpha = 1$ ) implies that the MJLS is also stabilizable via state feedback because the latter require weaker stabilizability assumptions. Furthermore, the problems with  $\alpha < 1$  also have weaker conditions for mean square stabilizability, which implies that the number of solutions of the homotopy (15) remains constant as  $\alpha$  goes from 0 to 1.

### Theorem 3 (Convergence of the Iterative Algorithm)

The recursion (14) with (12) or (13) converges to its unique solution  $(\mathbf{X}_\infty^i, \mathbf{P}_\infty^i)$ , if the MJLS (1) is stabilizable via static output-feedback (2).

**Proof 4** The proof follows the argumentation of the proof of Theorem 3 in [27]. First, assume that  $(\overline{\mathbf{X}}_\infty^i, \overline{\mathbf{P}}_\infty^i)$  is a solution of (10)-(9) for some fixed regulator gain  $\overline{\mathbf{L}}$ . Then, the set of the solutions of (10)-(9) is not empty. Furthermore, the regulator gain  $\overline{\mathbf{L}}$  determined by each solution  $(\overline{\mathbf{X}}_\infty^i, \overline{\mathbf{P}}_\infty^i)$  stabilizes the MJLS in the mean square sense. This leaves us with necessity to show that (10)-(9) has a unique solution to which it converges. For this purpose, we define the homotopy

$$\begin{aligned} \mathbf{X}_{[\alpha, k+1]}^j &= \sum_{i=1}^M p_{ij} \left[ \underline{\mu}_\infty^i \mathbf{H}_i \mathbf{H}_i^\top \right. \\ & \quad \left. + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[\alpha, k]}^i \mathbf{C}_i) \mathbf{X}_{[\alpha, k]}^i (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[\alpha, k]}^i \mathbf{C}_i)^\top \right], \\ \mathbf{P}_{[\alpha, k+1]}^i &= \mathbf{C}_i^\top (\mathbf{L}_{[\alpha, k]}^i)^\top \mathbf{R}_i \mathbf{L}_{[\alpha, k]}^i \mathbf{C}_i + (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[\alpha, k]}^i \mathbf{C}_i)^\top \\ & \quad \times \mathcal{E}_i(\mathbf{P}_{[\alpha, k]}) (\mathbf{A}_i + \mathbf{B}_i \mathbf{L}_{[\alpha, k]}^i \mathbf{C}_i) + \mathbf{Q}_i. \end{aligned} \quad (16)$$

with  $\mathbf{L}_{[\alpha]}^i = (1 - \alpha) \overline{\mathbf{L}}_{[\eta]}^i + \alpha \mathbf{L}_{[\eta]}^i$ , where  $\mathbf{L}_{[\eta]}^i$  is the solution of (12) or (13), and  $\lim_{\eta \rightarrow \infty} \overline{\mathbf{L}}_{[\eta]}^i$  is the optimal regulator gain

		$\mathbf{T}_1, \mathbf{x}_0 = [0 \ 0]^\top$	$\mathbf{T}_1, \mathbf{x}_0 = [3 \ 0]^\top$	$\mathbf{T}_2, \mathbf{x}_0 = [0 \ 0]^\top$	$\mathbf{T}_2, \mathbf{x}_0 = [3 \ 0]^\top$
state feedback	time-invariant optimal with observed mode [2]	0.9718	24.1879	0.8280	24.5886
	time-invariant with non-observed mode [13]	0.9788	24.4780	1.0088	26.0436
	time-variant with non-observed mode [3]	0.9743	24.4031	0.8786	24.7490
	proposed	0.9730	24.4808	0.8855	25.4265
output feedback	time-variant with non-observed observed mode [15]	1.5611	26.5108	1.0351	30.6060
	proposed	1.5623	28.3532	1.0539	30.7562

TABLE I  
MEDIAN COSTS OF THE MONTE CARLO SIMULATION

for the case that the mode  $\theta_k$  is observed. From Lemma 1, we have that the recursion (16) has a unique solution for  $\alpha = 0$  to which it converges for  $\eta \rightarrow \infty$ . Because we assumed that the MJLS (1) is stabilizable via static output feedback without mode observation, i.e.,  $\alpha = 1$ , it is also stabilizable for  $\alpha = 0$  and  $\alpha < 1$  since these cases constitute a weaker stabilizability demands. From this property, we can conclude using the topological degree theory [28] that the number of solutions  $(\mathbf{X}_{\alpha,\infty}^i, \mathbf{P}_{\alpha,\infty}^i) = \lim_{\eta \rightarrow \infty} (\mathbf{X}_{[\alpha,\eta]}^i, \mathbf{P}_{[\alpha,\eta]}^i)$  remains constant as  $\alpha$  goes from 0 to 1. Thus, the initial recursion (14) with (13) or (12) for  $\alpha = 1$  has a unique solution to which it converges.

In the next section, we demonstrate the presented control law in a numerical example.

**Remark 2** As mentioned in the introduction, do Val et al. addressed  $H_2$  state-feedback control of MJLS with clustered observations in [13], where the case of no mode observation is recovered if the cluster of observed modes is empty. This problem can also be addressed with the proposed algorithm because the state-feedback control problem, i.e.,  $\mathbf{C}_{\theta_k} = \mathbf{I}$ , is a special case of the considered problem. The proposed algorithm yields regulator gains with better performance than [13] because it directly minimizes the costs (3) instead of its bound (see Sec. IV).

#### IV. NUMERICAL EXAMPLE

In this section, we first demonstrate that convergence of the iterative algorithm from Sec. III is not sufficient for stability of the closed loop. For this reason, we compute the regulator gain using the proposed algorithm and check whether the controlled MJLS is stable using Corollary 2.6 from [26]. To this end, we need to compute the spectral radius  $\rho$  of the controlled system. If it is smaller than 1, the controlled MJLS is stable in the mean square sense.

Consider the MJLS

$$\mathbf{A}_1 = \begin{bmatrix} 1.2 & 1.2 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0.8 \\ 0 & 1 \end{bmatrix}, \mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I}, \\ \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \mathbf{H}_1 = \mathbf{H}_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$

and two different transition matrices

$$\mathbf{T}_1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \text{ and } \mathbf{T}_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

The spectral radii of the MJLS are  $\rho = 1.2970$  for  $\mathbf{T}_1$  and  $\rho = 1.3295$  for  $\mathbf{T}_2$ , respectively. Thus, the uncontrolled MJLS are unstable.

We set the observation matrices to

$$\mathbf{C}_1 = \mathbf{C}_2 = [1 \ 0].$$

The LMI from Theorem 1 is infeasible for both transition matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$ . The algorithm from Theorem 2 converges with the gains  $\mathbf{L}_1 = -0.0089$  for  $\mathbf{T}_1$  and  $\mathbf{L}_2 = -0.0122$  for  $\mathbf{T}_2$ . However, evaluation of the spectral radii yields  $\rho = 1.1463$  for  $\mathbf{L}_1$  and  $\rho = 1.1548$  for  $\mathbf{L}_2$ . Thus, the controlled MJLS are still unstable.

Now, we choose the observation matrices to

$$\mathbf{C}_1 = [1 \ 2] \text{ and } \mathbf{C}_2 = [2 \ 1].$$

With these observation matrices, the LMI from Theorem 1 is feasible for both transition matrices, i.e., the corresponding MJLS are stabilizable via mode-independent static output feedback. Computation of the regulator gains using the proposed iterative algorithm yields  $\mathbf{L}_1 = -0.6350$  for  $\mathbf{T}_1$  and  $\mathbf{L}_2 = -0.4404$  for  $\mathbf{T}_2$ . The spectral radii of the controlled MJLS are  $\rho = 0.817760$  and  $\rho = 0.987110$ , respectively.

For this setup, we conducted a Monte Carlo simulation with 1e4 runs à 100 time steps each. For comparison, we included the optimal state-feedback controller with mode observation from [2] and the time-variant output-feedback controller with non-observed mode presented by Vargas et al. in [15]. For completeness, we also compared our controller with the controller from [13] for the state-feedback scenario. The results of this simulation are depicted in Table I. The median costs for the transition matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , and the initial states  $\mathbf{x}_0^{[1]} = [0 \ 0]^\top$  and  $\mathbf{x}_0^{[2]} = [3 \ 0]^\top$  are depicted in Table I. It can be seen that the proposed time-invariant regulator gain computed using the iterative algorithm shows the same performance as the algorithm from [15] in the stationary scenario ( $\mathbf{x}_0 = \mathbf{x}_0^{[1]}$ ) and is only slightly worse in the transient scenario where the state has to be driven from  $\mathbf{x}_0 = \mathbf{x}_0^{[2]}$  to the origin. However, the proposed iterative algorithm needed only 0.025s to compute the regulator gain

for  $\mathbf{T}_1$  and  $0.03s$  for  $\mathbf{T}_2$ , while Vargas' algorithm from [3] took  $194.12s$  for  $\mathbf{T}_1$  and  $277.68s$  for  $\mathbf{T}_2$  in Matlab 2013b on a PC with Intel Core i5-3320M and 8 GB RAM. Also note that the suboptimality compared to the state-feedback controller from [2] depends on the transition matrix  $\mathbf{T}$ . Considering the state-feedback scenario, the regulator gains computed using the iterative algorithm slightly outperform the gains computed using the method from [13] presented by do Val et al.

A reference implementation of the presented algorithm is available on GitHub [29].

## V. CONCLUSION

In this paper, we addressed the problem of infinite-horizon static output-feedback optimal control of Markov Jump Linear Systems without mode observation. Because the optimal control law is nonlinear and intractable, we assumed a linear controller. Under this assumption, we first derived sufficient feasibility conditions for static output-feedback stabilization of MJLS with non-observed mode in form of LMIs. For this purpose, we converted the optimal control problem into an  $H_2$  problem and bounded the minimum of the corresponding  $H_2$  norm from above. Because the derived conditions depend on the particular state-space representation, we proposed an iterative algorithm for the computation of the regulator gain and proved its convergence. In the numerical example, we compared the performance of the regulator gain computed using the presented methods and elaborated on the fact that the convergence of the proposed regulator gain computation algorithm is not sufficient for the stability of the controlled MJLS. Thus, it is necessary either to check stability using the provided LMI conditions or the standard conditions from [26].

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