Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 921 (2017) 796–804

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Magnetostatic-field screening induced by small black holes

Slava Emelyanov

Institute for Theoretical Physics, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany

Received 10 March 2017; received in revised form 21 June 2017; accepted 22 June 2017
Available online 28 June 2017
Editor: Stephan Stieberger

Abstract

We find within the framework of quantum electrodynamics that there exists screening effect of static magnetic field that is induced by small evaporating black holes.

© 2017 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP[3].

1. Introduction

By this paper, we continue our study of various physical imprints of small black holes in local electromagnetic phenomena [1–3]. The small black holes we have been considering possess the mass M from the range 10^{10} g $\lesssim M \ll 10^{16}$ g which might have formed through the gravitational collapse at early stages of the universe evolution [4]. This corresponds to the Hawking temperature T_H [5] that is much larger than the electron rest energy m_e. As a consequence, the thermal-like term in the electron 2-point function is not exponentially suppressed by the Boltzmann factor $\exp(-m_e/T_H)$ as it holds $m_e/T_H \ll 1$. This means that the electron appears to be effectively massless. This leads to more or less sizeable quantum effects whenever a small black hole is sufficiently close to a detector.

We employ our recent results obtained in [6] to derive the Feynman propagator $S(x, x')$ of a massless Dirac field in the far-horizon region of a small black hole. This is essentially given by the ordinary Minkowski propagator $S_M(x, x')$ plus a thermal-like singularity-free correction

E-mail address: viacheslav.emelyanov@kit.edu.

http://dx.doi.org/10.1016/j.nuclphysb.2017.06.018
0550-3213/© 2017 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP[3].

$\Delta S(x, x')$ decreasing in the spatial infinity as $(r_H/R)^2$, where R is a radial distance to the black-hole centre and $r_H = 2M$ the size of the event horizon. Although it asymptotically vanishes, the correction $\Delta S(x, x')$ is, nevertheless, physically relevant as being responsible for the evaporation effect of black holes [6].

We found in [2] that local black-hole manifestations in the electromagnetic phenomena are characterized by an effective (gauge invariant) photon mass and Debye-like screening of the electrostatic field of a point-like charge. Therefore, it turns out that the quantum vacuum in the presence of small black holes shows locally up properties which are usually attributable to a many-particle system. Specifically, it resembles a hot electron-positron plasma. The purpose of this paper is to show that there also exists the shielding effect of the magnetostatic field. A similar effect can occur in the hot electron-positron plasma, but with anisotropic distribution of the constituent particles in momentum space (like in QCD for the anisotropic quark-gluon plasma [7,8]).

Throughout this paper the fundamental constants are set to $c = G = k_{\rm B} = \hbar = 1$, unless stated otherwise.

2. Screening of magnetostatic field

2.1. Fermion Feynman propagator

We derived in [6] the scalar 2-point function $W(x, x')$ in the presence of Schwarzschild black hole formed through the gravitational collapse. This can be exploited to obtain the fermion propagator. Specifically, the Feynman propagator $S(x, x')$ of a massless fermion in the far-horizon region ($R \gg r_H$) reads

$$S(x, x') = S_M(x, x') + \Delta S(x, x'), \tag{1}$$

where

$$S_M(x, x') \approx \int \frac{d^4p}{(2\pi)^4}\frac{i\not{p}}{p^2 + i\varepsilon}\exp(-ip\Delta x) \tag{2}$$

with $\Delta x \equiv x - x'$ and

$$\Delta S(x, x') \approx -2\pi g_R \int \frac{d^4p}{(2\pi)^4}\frac{\delta(p^2)}{e^{\beta|p_0|}+1}\,\bar{\not{p}}\exp(-i\bar{p}\Delta x) \quad \text{with} \quad \bar{p}^\mu = (p_0, p_0\mathbf{n}), \tag{3}$$

where $\beta = 1/T_H$ is the inverse Hawking temperature, $\mathbf{n} \equiv \mathbf{R}/R$ is the radial unit vector and

$$g_R \equiv \frac{27}{16}\left(\frac{r_H}{R}\right)^2. \tag{4}$$

It should be emphasised that $\Delta S(x, x')$ solves the field equation up to the terms vanishing as $1/R^3$ at spatial infinity. The correction $\Delta W(x, x')$ to $W_M(x, x')$ found in [6] satisfies the scalar field equation in the limit $\mathbf{x}' \to \mathbf{x}$ only. Therefore, $\Delta S(x, x')$ is a more general result (see Appendix A for further details).

The fermion stress tensor $\langle \hat{T}^\mu_\nu \rangle$ can be computed by taking its trace with respect to the spinorial indices and using the equation $\mathrm{tr}\big(\bar{\psi}\gamma_\mu\partial_\nu\psi\big) = -\lim_{x'\to x}\mathrm{tr}\big(\gamma_\mu\partial_\nu S(x, x')\big)$. Making use of $S(x, x')$ given in Eq. (1), we find the renormalised energy-momentum tensor:

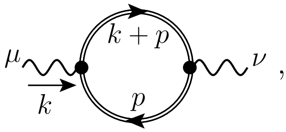
$$\langle \hat{T}^\mu_\nu \rangle \approx \frac{2}{4\pi R^2}\int_0^{+\infty}\frac{dp_0}{2\pi}\frac{p_0\Gamma_{p_0}}{e^{\beta p_0}+1}\begin{bmatrix} +1 & +1 \\ -1 & -1 \end{bmatrix} \quad \text{with} \quad \Gamma_{p_0} \equiv 27(p_0M)^2, \tag{5}$$

where the indices μ, ν run over $\{t, r\}$ and the rest elements of $\langle \hat{T}^{\mu}_{\nu} \rangle$ vanish faster than $1/R^2$ at $R \gg r_H$. This result implies that $S(x, x')$ is a correct expression of the exact propagator up to terms vanishing faster than $1/R^2$ in the far-horizon region and for points satisfying the condition $|r - r'| \ll R$.

2.2. One-loop vacuum polarisation tensor

In order to study how the presence of a small black hole can influence the local electromagnetic phenomena, one needs to compute the vacuum polarisation tensor $\Pi^{\mu\nu}(k)$. At one-loop approximation, it is given pictorially by

$$i\Pi^{\mu\nu}(k) \;=$$



, (6)

where the double line in the fermion loop refers to the propagator $S(x, x')$ that is composed of the ordinary part $S_M(x, x')$ and the correction $\Delta S(x, x')$ to it. We focus here only on that part of $\Pi^{\mu\nu}(k)$ which is induced by the presence of a small black hole. This reads

$$\Delta\Pi^{\mu\nu}(k) = -4\pi g_R e^2 \int \frac{d^4p}{(2\pi)^4} \frac{\delta(p^2)}{e^{\beta|p_0|}+1} \frac{\mathrm{tr}\big(\gamma^\mu \bar{\not{p}} \gamma^\nu (\bar{\not{p}} + \not{k})\big)}{(\bar{p}+k)^2 + i\varepsilon} \,. \tag{7}$$

This can in turn be rewritten in terms of the projection tensors $P^{\mu\nu}$ and $Q^{\mu\nu}$ introduced in [9] as follows:

$$\Delta\Pi^{\mu\nu}(k) = \pi_T(k_0, \mathbf{k}) P^{\mu\nu} + \pi_L(k_0, \mathbf{k}) Q^{\mu\nu} \,, \tag{8}$$

where we have

$$\pi_T(k_0, \mathbf{k}) = \frac{4g_R e^2}{\pi^2} \int\limits_0^{+\infty} \frac{dp\, p^3}{e^{\beta p}+1} \frac{4k_0|\mathbf{k}|\cos\theta - (k_0^2 + |\mathbf{k}|^2)(\cos^2\theta + 1)}{(k_0^2 - |\mathbf{k}|^2)^2 - 4p^2(k_0 - |\mathbf{k}|\cos\theta)^2} \,, \tag{9a}$$

$$\pi_L(k_0, \mathbf{k}) = \frac{8g_R e^2}{\pi^2} \int\limits_0^{+\infty} \frac{dp\, p^3}{e^{\beta p}+1} \frac{(k_0^2 - |\mathbf{k}|^2)(\cos^2\theta - 1)}{(k_0^2 - |\mathbf{k}|^2)^2 - 4p^2(k_0 - |\mathbf{k}|\cos\theta)^2} \tag{9b}$$

with θ being the angle between $\mathbf{k}$ and the radial unit vector $\mathbf{n}$, i.e. $\cos\theta = \mathbf{k}\cdot\mathbf{n}/|\mathbf{k}|$. The integrals in Eqs. (9) are understood as the principal value ones. It should also be stressed out that the structure of $\pi_T(k_0, \mathbf{k})$ and $\pi_L(k_0, \mathbf{k})$ significantly differs from that in the hot (isotropic) electron-positron plasma.

In the absence of the black hole, the polarization tensor $\Pi^{\mu\nu}(k)$ has the standard non-trivial form, $\Pi^{\mu\nu}_M(k)$, and leads to the running effect of the electric charge. This part of the polarization tensor $\Pi^{\mu\nu}(k)$ starts to reveal itself at the microscopic scale that is of the order of the Compton wavelength of the electron $\lambda_e \approx 2.4\times 10^{-12}$ m. We are interested, however, in the low-energy effects which correspond to the length scale of the order of 1 m (see below). Thus, we omit $\Pi^{\mu\nu}_M(k)$ in the full polarization tensor in the sequel. The photon propagator at one-loop approximation is then given by

$$G_{\mu\nu}(k_0, \mathbf{k}) = \frac{-iP_{\mu\nu}}{k^2 - \pi_T(k_0, \mathbf{k}) + i\varepsilon} + \frac{-iQ_{\mu\nu}}{k^2 - \pi_L(k_0, \mathbf{k}) + i\varepsilon} \tag{10}$$

in the Feynman gauge, where $k^2 \equiv k_0^2 - \mathbf{k}^2$ by convention.

2.3. Spectral function and poles in photon propagator

We examine the influence of small black holes. The size of their event horizon is extremely small, i.e. $r_H \ll 1.49\times10^{-14}$ m. It means that $g_R \ll 2.5\times10^{-14}$ for $R = 1$ m and, therefore, the one-loop correction to the photon self-energy is small despite of T_H is much larger than 0.5 MeV or 6×10^9 K. Consequently, the photon dispersion relation approximately reads $k_0 \approx |\mathbf{k}|$. The non-vanishing constant value of $\pi_T(k_0, \mathbf{k})$ in the limit $|\mathbf{k}| \to k_0$ implies, however, that the pole structure of the photon propagator is slightly modified, namely we now have $k_0^2 = \mathbf{k}^2 + m_\gamma^2$ with $|\mathbf{k}| \gg m_\gamma$ (but still $T_H \gg |\mathbf{k}|$), where the effective photon mass reads

$$m_\gamma^2 = \lim_{|\mathbf{k}|\to k_0} \pi_T(k_0, \mathbf{k}) \approx \frac{1}{6} e^2 T_L^2 \quad \text{with} \quad T_L \equiv \sqrt{g_R}\, T_H \,. \tag{11}$$

Thus, although we have employed the approximate expression for the fermion propagator in [2], we re-derive our main result of that paper by using the improved propagator $S(x, x')$. It should also be mentioned that the local (L) temperature $T_L \to 0$ in the spatial infinity unlike the Hawking temperature $T_H \neq 0$, because of $\sqrt{g_R} \propto r_H/R \to 0$ for $R \to \infty$.

The physical content of the poles appearing in the photon propagator (10) can be extracted by studying the analytic properties of the propagator [10]. We find that the spectral function $\rho(k_0, \mathbf{k})$ (equalling $2\pi\varepsilon(k_0)\delta(k^2)$ in the limit $\alpha \to 0$, where α is the fine structure constant) is saturated by the transverse pole, while the longitudinal pole gives a contribution that is of the order of $m_\gamma/|\mathbf{k}| \ll 1$. This means that the transverse pole corresponds to the propagating mode, whereas the longitudinal pole does not. It appears to be analogous to the behaviour of the transverse and longitudinal mode (photon and plasmon, respectively) in the hot electron-positron plasma for $eT \ll |\mathbf{k}| \ll T$ [9,10].

2.4. Screening of static electric field

We now go over to the study of the electrostatic field $\mathbf{E} = -\nabla\varphi$ sourced by a point-like charge q in the presence of a small black hole. The electrostatic potential is given by

$$\varphi(r) = q \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k}\mathbf{x})}{\mathbf{k}^2 + \pi_L(0, \mathbf{k})} \quad \text{with} \quad \mathbf{k}\mathbf{x} = kr\cos\theta \,, \tag{12}$$

as this immediately follows from the linear response theory, where $\pi_L(0, \mathbf{k})$ must in turn be computed in the limit $\beta|\mathbf{k}| \to 0$. We find

$$\varphi(r) = \frac{q}{4\pi^2} \int_0^\infty dk k^2 \int_0^\pi d\cos\theta \, \frac{\exp(ikr\cos\theta)}{k^2 - m_\gamma^2 \tan^2\theta} \,. \tag{13}$$

To evaluate the integral in Eq. (13), we first expand the denominator of the integrand over the parameter $m_\gamma^2/\cos^2\theta$ and then integrate it order by order over the angle θ.[1] Afterwards, we rewrite the integration with respect to $|\mathbf{k}|$ to have it over $(-\infty, +\infty)$ (see Appendix B for more details). This yields

$$\varphi(r) \approx \frac{q}{4\pi r} \exp(-r/r_D) \quad \text{with} \quad r_D \equiv 1/(\gamma_L m_\gamma) \,. \tag{14}$$

[1] The point $\theta = \pi/2$ is regular as follows from $\pi_L(0, \mathbf{k})$ for $\theta = \pi/2$ and it does not contribute as can be directly shown.

Thus, we re-derive our result obtained in [2] by using the improved expression for the fermion propagator, but with the Debye-like radius r_D given by $(\gamma_L m_\gamma)^{-1}$ instead of $(\sqrt{2}m_\gamma)^{-1}$, where γ_L appears to equal $\pi/2$ (see Appendix B). This allows us to slightly enlarge the value of the maximal distance to the small black hole which should still be "visible" to a detector used in [11] for testing the Coulomb law. Specifically, the small black hole should be in the region of the size about $R_0 \approx 280$ km in order to discover the Debye-like screening of the electrostatic potential induced by that.

It appears that we can even improve the estimate of R_0 to roughly one order of magnitude if we take into account the correction $(m_\gamma r)^2 \log(m_\gamma r)$ to the exponential function in Eq. (14) which is derived in Appendix B. Specifically, this correction leads approximately to the following modified Gauss law

$$\Delta\varphi \approx -4\pi\rho + (\gamma_L m_\gamma)^2\varphi + 2m_\gamma^2 \log(m_\gamma r)\varphi \quad \text{for} \quad m_\gamma r \ll 1\,, \tag{15}$$

where ρ is a charge density. Repeating computations of [11] with this modified law, we obtain that $R_0 \approx 1.9\times 10^3$ km, where we have assumed that the size of the conducting shells in [11] is about 1 meter.

2.5. *Screening of static magnetic field*

It turns out that there exists a local shielding effect for the magnetostatic field $\mathbf{B}$ as well. This follows from the fact that $\pi_T(0,\mathbf{k}) \neq 0$ in the limit $|\mathbf{k}| \to 0$. This is in sharp contrast to the normal hot plasma, wherein $\pi_T(0,\mathbf{k}) \to 0$ in that limit. It should be mentioned that this effect does not exist for small *eternal* black holes, because $\pi_T(k_0,\mathbf{k})$ has the same structure as in the hot (isotropic) plasma and, hence, it vanishes for $|\mathbf{k}| \to 0$.

As an example, we want to consider the screening of a static magnetic field $\mathbf{B}$ sourced by the magnetic monopole of charge q_m. Introducing the magnetostatic potential φ_m, such that $\mathbf{B} = -\nabla\varphi_m$, we find

$$\varphi_m(r) = q_m \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k}\mathbf{x})}{\mathbf{k}^2 + \pi_T(0,\mathbf{k})} \quad \text{with} \quad \mathbf{k}\mathbf{x} = kr\cos\theta\,. \tag{16}$$

Computing $\pi_T(0,\mathbf{k})$ in the limit $\beta|\mathbf{k}| \to 0$ and then repeating the analysis of Sec. 2.4, we obtain

$$\varphi_m(r) \approx \frac{q_m}{4\pi r}\exp(-r/\bar{r}_D) \quad \text{with} \quad \bar{r}_D \equiv \sqrt{2}/(\gamma_T m_\gamma)\,, \tag{17}$$

where $\gamma_T \approx 0.532818$ (see Appendix B). Thus, we find that $\bar{r}_D/r_D \approx 4$. It implies that the screening of the magnetostatic potential of the monopole q_m is more effective than that of the electrostatic potential of the charge q.

3. Concluding remarks

3.1. *Improved Wigner distribution*

We have derived the exact correction to the Minkowski part of the propagator. This is non-singular and induced by black holes in the far-horizon region. It is exact in that sense that this precisely satisfies the field equation up to the terms vanishing faster than $1/R^2$ for $R \gg r_H$. Substituting this in the definition of the Wigner distribution $\mathcal{W}(x,p)$ [6], we obtain for the massless scalar field that

$$\mathcal{W}(x,p) = \frac{1}{8\pi^2 p_0^3 R^2}\,\frac{\Gamma_{p_0}}{e^{\beta p_0}-1}\,\delta(p_0-p)\delta(p^\theta)\delta(p^\phi)\,, \tag{18}$$

where $\mathbf{p} = (p^r, p^\theta, p^\phi)$ and we have set $p^r \equiv p$. The parameter Γ_{p_0} is given in Eq. (5). This implies that the effective Wigner distribution introduced in [6] appears to be an exact result (up to the terms $1/R^n$ with $n \geq 3$).

3.2. Quantum vacuum as anisotropic hot plasma

We have found that there exists a local shielding effect for the magnetostatic field which is induced by small black holes. The analogous effect can occur in the hot plasma which is described by the one-particle distribution with the anisotropy in momentum space.

Although it is tempting to describe the local electromagnetic effects in the presence of small black holes as if the vacuum is a plasma-like medium, this analogy seems to be incomplete. Indeed, this "medium" cannot support the plasmon-like excitations which are normally attributed to the collective excitations of the plasma particles [10]. Specifically, the plasma-like frequency ω_p characterising these excitations can be computed by considering the limit $|\mathbf{k}| \to 0$ with $k_0 \sim eT_L \ll eT_H$ in $\pi_T(k_0, \mathbf{k})$ and $\pi_L(k_0, \mathbf{k})$. It turns out that ω_p for the transverse and longitudinal mode are different and depend on the angle between $\mathbf{k}$ and the radial unit vector $\mathbf{n}$. We found in [2] that the mode of the frequency $k_0 \sim eT_L$ has a wavelength which is much larger than the distance to the black-hole centre R. This kind of waves cannot be described within our approximation. At these scales, the hot-anisotropic-plasma analogy may not hold.

3.3. Modified dispersion relation of photon

We found in [2] as well as in Sec. 2.3 above that the photon dispersion relation modifies in the presence of black holes, namely photons acquire a mass term m_γ. In the far-horizon region, one has

$$m_\gamma^2 \propto +\alpha T_L^2 \begin{cases} 1\,, & T_H \gg m_e\,, \\ (m_e/T_H)^{\frac{3}{2}} \exp(-m_e/T_H)\,, & T_H \ll m_e\,, \end{cases} \tag{19}$$

which vanishes when one neglects the interaction term between the electron/positron and electromagnetic field.

In the near-horizon region, the effective photon mass squared m_γ^2 might be *negative*. Indeed, the polarization tensor can be computed within the kinetic theory by employing the one-particle distribution function and the transport equation. We found in [6] that the one-particle distribution near the event horizon is negative. This might imply that photons can come out of the event horizon [13].[2]

Acknowledgement

It is a pleasure to thank José Queiruga for discussions.

[2] Note that the flux of these positive-energy photons has a different nature in comparison with that of the Hawking radiation leading to the decrease of the event-horizon size. The former is due to various quantum processes which might occur in matter inside the horizon, whereas the latter is featureless and originates well outside black holes. Thus, this kind of photons if existent could bring us information about internal structure of black holes.

Appendix A. Scalar Feynman propagator

We have derived a correction to the Minkowski 2-point function in the far-horizon region for a massless scalar field in [6]. This correction can be written as follows

$$\Delta W(x,x') \approx +g_R \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{n_\beta(k_0)}{k_0}\,\exp(i\mathbf{k}\Delta\mathbf{x})\Big(1-\frac{i}{2}\mathbf{k}\Delta\mathbf{x}\Big)\cos\big(\bar{k}\Delta x\big)\,, \tag{A.1}$$

where $\bar{k}^\mu \equiv k_0(1,\mathbf{n})$ by definition and we have omitted cubic- and higher-order terms with respect to $\Delta\mathbf{x}$ as well as those terms which vanish faster than $1/R^2$ in the asymptotically flat region.

The correction to the scalar Feynman propagator is thus given by

$$\Delta G(x,x') \approx +2\pi g_R \int \frac{d^4k}{(2\pi)^4}\,\frac{\delta(k^2)}{e^{\beta|k_0|}-1}\Big(1-\frac{i}{2}\mathbf{k}\Delta\mathbf{x}\Big)\exp(-i\bar{k}\Delta x + i\mathbf{k}\Delta\mathbf{x})\,. \tag{A.2}$$

Bearing in mind the structure of the radial modes, we want to find a function $h \equiv h(i\mathbf{k}\Delta\mathbf{x})$ which satisfies the following conditions

$$h = 1 - \frac{i}{2}\mathbf{k}\Delta\mathbf{x} + \mathrm{O}\big((\mathbf{k}\Delta\mathbf{x})^2\big)\,, \tag{A.3a}$$

$$0 = h'' + 2h' + h\,, \tag{A.3b}$$

where the prime denotes the differentiation with respect to the argument of the function h. The second condition implies that $\Delta G(x,x')$ is a solution of the scalar field equation, i.e. $\Box\Delta G(x,x') = 0$, up to the terms vanishing faster than $1/R^2$ for the large values of R. Thus, we obtain

$$\Delta G(x,x') \approx +2\pi g_R \int \frac{d^4k}{(2\pi)^4}\,\frac{\delta(k^2)}{e^{\beta|k_0|}-1}\,\exp(-i\bar{k}\Delta x)\,. \tag{A.4}$$

This result can be directly employed to derive $\Delta S(x,x')$ for the massless Dirac field.

Appendix B. Computation of electrostatic potential

The electrostatic potential we compute here reads

$$\varphi(r) = \frac{q}{4\pi^2}\sum_{n=0}^{+\infty}\int_0^{\infty} dk\,\frac{k^2 m_\gamma^{2n}}{(k^2+m_\gamma^2)^{n+1}}\int_{-1}^{+1} dz\,\frac{\exp(ikrz)}{z^{2n}} \;\equiv\; \sum_{n=0}^{+\infty}\varphi_n(r)\,. \tag{B.1}$$

We first consider the term $n = 0$. One has

$$\varphi_0(r) = \frac{iq}{4\pi^2}\int_{-\infty}^{+\infty} dk\,\frac{k\,e^{-ikr}}{k^2+m_\gamma^2} \;=\; \frac{q}{4\pi r}\,\exp(-m_\gamma r)\,, \tag{B.2}$$

where we have chosen the contour C_∞ to evaluate the integral over k by employing the residue theorem. This contour is depicted in Fig. 1. The next term in the expansion of the potential $\varphi(r)$ reads

$$\varphi_1(r) = \frac{qm_\gamma^2}{4\pi^2}\int_{-\infty}^{+\infty} dk\,\frac{irk^3\mathrm{E}_1(ikr) - k^2 e^{-ikr}}{(k^2+m_\gamma^2)^2} \;\approx\; \frac{q}{4\pi r}(-m_\gamma r/2) \quad \text{for} \quad m_\gamma r \ll 1\,, \tag{B.3}$$

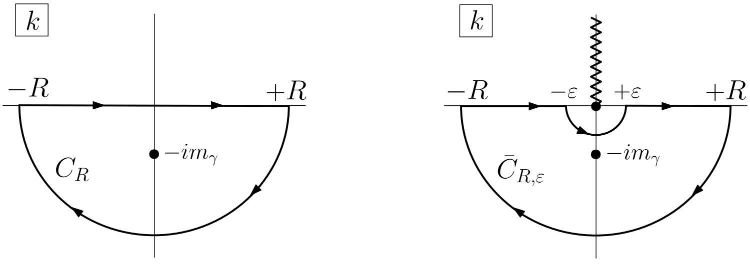


Fig. 1. The contours C_∞ and $\bar{C}_{\infty,0}$ are chosen when the integrand contains the exponential function and the exponential integral function, respectively. The residue theorem is then used to evaluate the principal value integrals over the real values of k.

where we have evaluated the integral with the exponential integral with the complex argument [12], $\mathrm{E}_1(z)$, by choosing the contour $\bar{C}_{\infty,0}$ shown in Fig. 1. Employing this procedure for higher values of n, we obtain

$$\varphi(r) \approx \frac{q}{4\pi r}\exp(-\gamma_L m_\gamma r)\Big(1 + (m_\gamma r)^2 \ln(m_\gamma r)\Big) \quad \text{for} \quad m_\gamma r \ll 1\,, \tag{B.4}$$

where by definition

$$\gamma_L \equiv 1 + \frac{1}{2} + \frac{1}{3{\cdot}2^3} + \frac{1}{5{\cdot}2^4} + \frac{5}{7{\cdot}2^7} + \frac{7}{9{\cdot}2^8} + \frac{3{\cdot}7}{11{\cdot}2^{10}} + \frac{3{\cdot}11}{13{\cdot}2^{11}} + \cdots \approx 1.570051\,, \tag{B.5}$$

where we have taken into account the first 26 terms in the series. Since $\pi/2 \approx 1.570796$, we conjecture that $\gamma_L = \pi/2$ exactly. For a later use, we also define

$$\gamma_T \equiv 1 - \frac{1}{2} + \frac{1}{3{\cdot}2^3} - \frac{1}{5{\cdot}2^4} + \frac{5}{7{\cdot}2^7} - \frac{7}{9{\cdot}2^8} + \frac{3{\cdot}7}{11{\cdot}2^{10}} - \frac{3{\cdot}11}{13{\cdot}2^{11}} + \cdots \approx 0.532818 \tag{B.6}$$

that holds for the first 26 terms in the series.

References

[1] S. Emelyanov, Low-energy electromagnetic radiation as an indirect probe of black-hole evaporation, Nucl. Phys. B 913 (2016) 318, arXiv:1602.01475 [hep-th].

[2] S. Emelyanov, Effective photon mass from black-hole formation, Nucl. Phys. B 919 (2017) 110, arXiv:1603.01148 [hep-th].

[3] S. Emelyanov, Quantum electromagnetic phenomena far from small evaporating black holes, in: Proceedings, New Frontiers in Black Hole Astrophysics, in: IAU Symposium, vol. 324, 2016, arXiv:1612.07637 [hep-th].

[4] S.W. Hawking, Gravitationally collapsed objects of very low mass, Mon. Not. R. Astron. Soc. 152 (1971) 75.

[5] S.W. Hawking, Black hole explosions?, Nature 248 (1974) 30;
Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199.

[6] S. Emelyanov, Quantum kinetic theory of a massless scalar model in the presence of a Schwarzschild black hole, Ann. Physik (2017), http://dx.doi.org/10.1002/andp.201700078, arXiv:1703.01674 [hep-th].

[7] C.-W. Kao, G.C. Nayak, W. Greiner, Closed-time path integral formalism and medium effects of nonequilibrium QCD matter, Phys. Rev. D 66 (2002) 034017, arXiv:hep-ph/0102153.

[8] F. Cooper, C.-W. Kao, G.C. Nayak, Infrared behavior of the gluon propagator in nonequilibrium situations, Phys. Rev. D 66 (2002) 114016, arXiv:hep-ph/0204042.

[9] H.A. Weldon, Covariant calculations at finite temperature: the relativistic plasma, Phys. Rev. D 26 (1982) 1394.

[10] M. Le Bellac, Thermal Field Theory, Cambridge University Press, 1996.

[11] E.R. Williams, J.E. Faller, H.A. Hill, New experimental test of Coulomb's law: a laboratory upper limit on the photon rest mass, Phys. Rev. Lett. 26 (1971) 721.

[12] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, 1972.
[13] S. Emelyanov, Can photon come out of black-hole horizon in QED?, arXiv:1703.05078 [hep-th].