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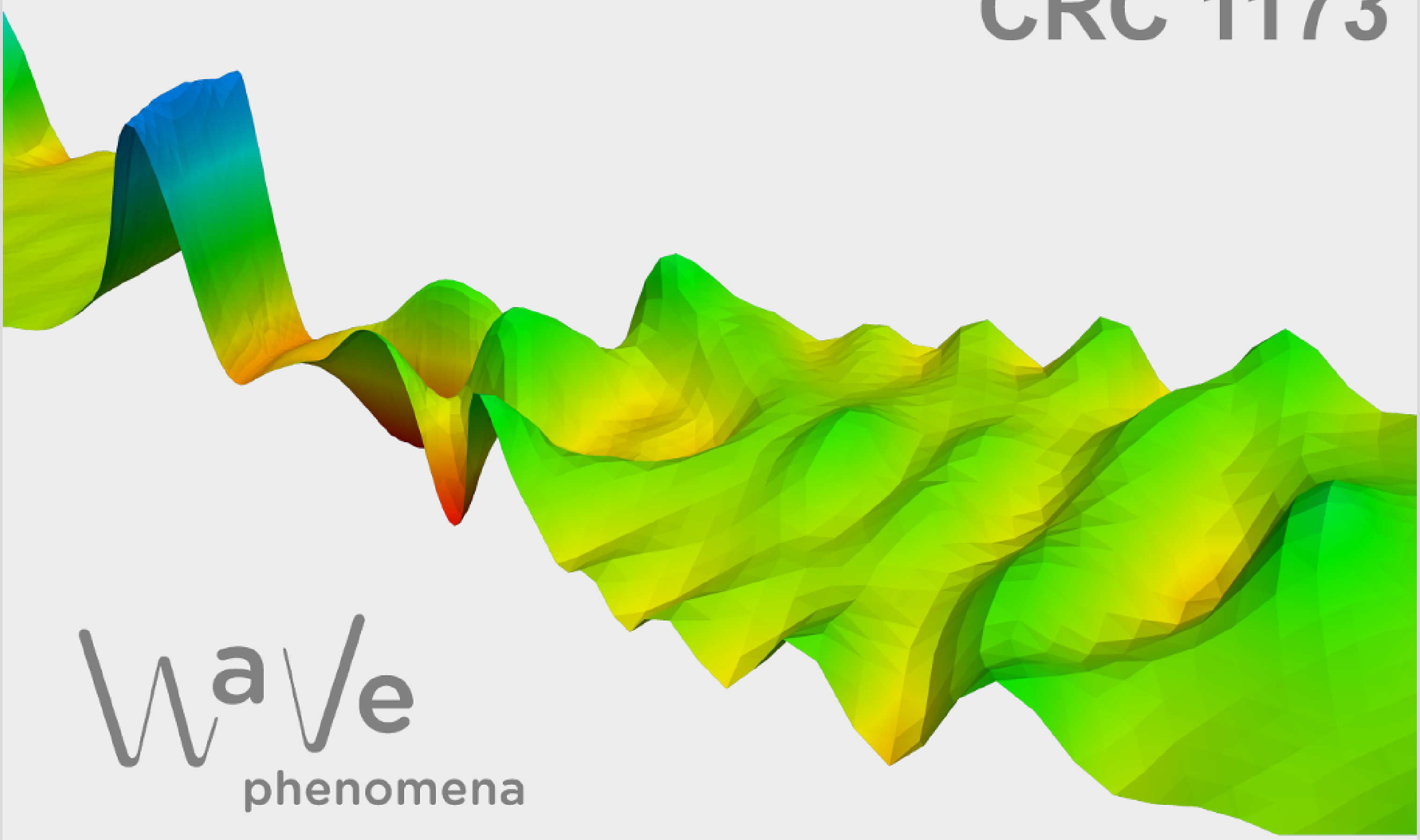

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ERROR ANALYSIS OF AN ADI SPLITTING SCHEME FOR THE INHOMOGENOUS MAXWELL EQUATIONS

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ABSTRACT. In this paper we investigate an alternating direction implicit (ADI) time integration scheme for the Maxwell equations with sources, currents and conductivity. We show its stability and efficiency. The main results establish that the scheme converges in a space similar to H^{-1} with order two to the solution of the Maxwell system. Moreover, the divergence conditions in the system are preserved in H^{-1} with order one.

1. INTRODUCTION

The Maxwell equations are the foundation of the electro-magnetic theory and one of the basic PDEs in physics. They form a large coupled system of six time-depending scalar equations in three space dimensions and thus pose considerable difficulties to the numerical treatment already in the linear case. Explicit methods like finite differences on the Yee grid [25] are efficient, but to avoid instabilities one is restricted to small time step sizes, cf. [24]. On the other hand, stable implicit methods for time integration can lead to very large linear systems to be solved in every time step. Around the year 2000 the very efficient and unconditionally stable alternating direction implicit (ADI) scheme was introduced in [20] and [26] for problems on a cuboid with isotropic material laws. In this scheme one splits the curl operator into the partial derivatives with a plus and a minus sign, see (1.4), and then applies the implicit-explicit Peaceman-Rachford method to the two subsystems, cf. (1.5). In [20] and [26] it was observed that the resulting implicit steps essentially decouple into one-dimensional problems which makes the algorithm very fast, see also Proposition 4.6 of [13] as well as (4.3) and (4.4) below. There are energy-conserving variants of the ADI splitting, see e.g. [4], [5], [11], [18], not discussed here. We refer to [13], [14] and [15] for further references about the numerical treatment of the Maxwell system.

Despite its importance, there exists very little rigorous error analysis of the ADI scheme in the literature, and the available results only cover systems without resistancy, currents and charges. For a variant of the scheme, in [5] error estimates have been shown for solutions in C^6, see also [4] and [11] for two space dimensions. The paper [13] (co-authored by one of the present authors) establishes second order convergence in L^2 for the Maxwell system on a cuboid with

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the boundary conditions of a perfect conductor. Here the initial data belong to H^3 and satisfy appropriate compatibility conditions, whereas the coefficients are contained in $W^{2,3} \cap W^{1,\infty}$. We stress that the scheme is of second order classically and that the needed degree of regularity in [13] is natural for a splitting in the highest derivatives. As in our paper, the results of [13] are concerned with the time integration on the PDE level and do not treat the space discretization. Based on these and our investigations, we expect that one can develop an error analysis for the full discretization in the future, cf. [14] and [15].

In this work we study the complete Maxwell system with conductivity, currents and charges for Lipschitz coefficients and data in H^2. Compared to [13], we thus have to modify both the scheme and the functional analytic setting for the Maxwell equations, see (1.4), (1.5) and (2.4). We establish the stability of the scheme in L^2 and H^1, and that it converges of second order in H^{-1}, roughly speaking, which is the natural level of regularity for our data. Moreover, the scheme preserves the divergence conditions (1.1c) and (1.2) of the Maxwell system in H^{-1} up to order τ. To our knowledge, such preservation results have only be shown for C^6–solutions in the case without charges and in two space dimensions for a related scheme in [4], see also [5] for three space dimensions.

We want to approximate the electric and magnetic fields $\mathbf{E}(t,x) \in \mathbb{R}^3$ and $\mathbf{H}(t,x) \in \mathbb{R}^3$ satisfying the Maxwell equations

$$\partial_t \mathbf{E}(t) = \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H}(t) - \frac{1}{\varepsilon}(\sigma \mathbf{E}(t) + \mathbf{J}(t)) \qquad \text{in } Q,\ t \geq 0, \tag{1.1a}$$

$$\partial_t \mathbf{H}(t) = -\frac{1}{\mu} \operatorname{curl} \mathbf{E}(t) \qquad \text{in } Q,\ t \geq 0, \tag{1.1b}$$

$$\operatorname{div}(\varepsilon \mathbf{E}(t)) = \rho(t), \qquad \operatorname{div}(\mu \mathbf{H}(t)) = 0 \qquad \text{in } Q,\ t \geq 0, \tag{1.1c}$$

$$\mathbf{E}(t) \times \nu = 0, \qquad \mu \mathbf{H}(t) \cdot \nu = 0 \qquad \text{on } \partial Q,\ t \geq 0, \tag{1.1d}$$

$$\mathbf{E}(0) = \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0 \qquad \text{in } Q, \tag{1.1e}$$

on the cuboid Q, where $\nu(x)$ is the outer unit normal at $x \in \partial Q$. Here the initial fields in (1.1e), the current density $\mathbf{J}(t,x) \in \mathbb{R}^3$, the permittivity $\varepsilon(x) > 0$, the permeability $\mu(x) > 0$ and the conductivity $\sigma(x) \geq 0$ are given for $x \in G$ and $t \geq 0$. We treat the conditions (1.1d) of a perfectly conducting boundary. As noted in Proposition 2.3, the charge density $\rho(t,x) \in \mathbb{R}$ depends on the data and (if $\sigma \neq 0$) on the solution via

$$\rho(t) = \operatorname{div}(\varepsilon \mathbf{E}(t)) = \operatorname{div}(\varepsilon \mathbf{E}_0) - \int_0^t \operatorname{div}(\sigma \mathbf{E}(s) + \mathbf{J}(s))\, \mathrm{d}s, \qquad t \geq 0. \tag{1.2}$$

Throughout, we assume that the material coefficients satisfy

$$\varepsilon, \mu, \sigma \in W^{1,\infty}(Q, \mathbb{R}), \qquad \varepsilon, \mu \geq \delta \quad \text{for a constant } \delta > 0, \qquad \sigma \geq 0. \tag{1.3}$$

For the initial fields and the current density we require regularity of second order and certain compatibility conditions in our theorems.

In Section 2 we present the solution theory for (1.1). In presence of conductivity, currents and charges, one has to work in the space X_{div} of fields in L^2 satisfying the magnetic conditions $\mu \mathbf{H} \cdot \nu = 0$ and $\operatorname{div}(\mu \mathbf{H}) = 0$ from (1.1) as well as the regularity $\operatorname{div}(\varepsilon \mathbf{E}) \in L^2(Q)$ of the charges, see (2.4). The last condition also enters in the norm of X_{div}. (If σ, $\mathbf{J}$ and ρ vanish, it is replaced by the equation $\operatorname{div}(\varepsilon\mathbf{E}) = 0$, see e.g. [13].) The electric boundary condition is included in the domain of the Maxwell operator M from (2.3) governing (1.1a) and (1.1b). It is crucial for our analysis that the domain of the part of M in X_{div} embeds into $H^1(Q)^6$, see Proposition 2.2.

In the ADI method one splits curl $= C_1 - C_2$ and $M = A + B$, where we put

$$A = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & \frac{1}{\varepsilon}C_1 \\ \frac{1}{\mu}C_2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & -\frac{1}{\varepsilon}C_2 \\ -\frac{1}{\mu}C_1 & 0 \end{pmatrix} \quad \text{with}$$
$$C_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

The domains of the split operators A and B are described after (3.1). Let $\tau > 0$ and $t_n := n\tau \le T$ for $n \in \mathbb{N}$. The n-th step of the scheme is given by

$$w_{n+1} = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)w_n - \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0)\big]. \tag{1.5}$$

Here we modify an approach developed in [22] for a different situation. Note that the conductivity σ is included into in the maps A and B, whereas the current density $\mathbf{J}$ is added to the scheme.

In Section 3 we analyze the operators A and B and their adjoints, showing in particular that they generate contraction semigroups (possibly up to a shift). In $L^2(Q)^6$ we can proceed as in [13], but we also have to work in the closed subspace Y of $H^1(Q)^6$ equipped with the boundary conditions in (1.1d), which leads to substantial new difficulties. Proposition 3.6 then yields the main estimates needed for the error analysis of (1.5). We point out that the domains of the parts A_Y and B_Y of A and B in Y provide a very convenient framework for the analysis of the ADI scheme, see Sections 4 and 6. In Section 4 we explain the efficiency of the scheme. Based on the results of Section 3, its stability in L^2 and H^1 is shown in Theorem 4.2. These estimates should lead to its unconditional stability independent of the mesh size of a spatial discretization. Moreover, if σ, $\mathbf{J}$ and ρ vanish, we obtain a modified energy equality for the scheme.

Our main results are proved in the final two sections. By Theorem 5.1, the error of the scheme is bounded in Y^* by $c(1+T)^2 \mathrm{e}^{\kappa T}\tau^2$ times certain second order norms of $\mathbf{E}_0$, $\mathbf{H}_0$ and $\mathbf{J}$, where c and κ only depend on the quantities in (1.3). To show this core fact, we adopt arguments from [12] and [22] to derive the (rather lengthy) error formula (5.6). It is formulated in a weak sense with test functions in Y which allows us to work in the present degree of regularity of the data. To estimate the error, one then uses the above mentioned Propositions 2.2 and 3.6. In the final Theorem 6.1 we prove a similar first order bound in H^{-1} for the error concerning the discrete analogues of the divergence conditions $\operatorname{div}(\mu\mathbf{H}) = 0$ and (1.2). Besides Proposition 3.6, this proof is based on the surprisingly simple exact formula (6.3) for this error, which follows from the structural properties of the scheme.

Concluding, we discuss possible extensions of our work. For data in H^3 one can establish variants of Theorems 5.1 and 6.1 for the L^2–norms of the errors. For these results one has to develop a rather intricate regularity theory in H^2

for the Maxwell equation and for the split operators A and B. See [8] and our companion paper [9], where one can also find numerical experiments. The scheme loses its efficiency for non-isotropic (or nonlinear) material laws so that it does not make sense to study matrix-valued coefficients ε, μ, or σ. On the other hand, numerically one could implement the scheme also on a union of cuboids (e.g., an L-shaped domain). Since such domains are not convex anymore, the domain of the Maxwell operator only embeds into $H^\alpha(Q)$ for some $\alpha \in (1/2, 1)$, see e.g. [3], which should lead to reduced convergence orders. Further technical difficulties arise since some of the arguments in Section 3 heavily depend on the structure of a cuboid. These questions shall be investigated in a later paper. In [14] and [15], an error analysis was given for the full discretization of the Maxwell system (with $\sigma = 0$), using the discontinuous Galerkian method and a locally implicit time integration scheme. We expect that one can treat the full discretization for the ADI scheme combining methods in these and our papers.

2. Basic results on the Maxwell system

We first collect notation and basic results used throughout this paper. By c we denote a generic constant which may depend only on Q and on the constants from (1.3); i.e., on δ, $\|\varepsilon\|_{W^{1,\infty}}$, $\|\mu\|_{W^{1,\infty}}$, or $\|\sigma\|_{W^{1,\infty}}$. We write I for the identity operator and $v \cdot w$ for the Euclidean inner product in $\mathbb{R}^m$.

Let X and Y be Banach spaces. On the intersection $X \cap Y$ we use the norm $\|z\|_X + \|z\|_Y$. The symbol $Y \hookrightarrow X$ means that Y is continuously embedded into X, and $X \cong Y$ that they are isomorphic. The duality pairing between X and its dual X^* is denoted by $\langle x^*, x\rangle_{X^*,X}$ or by $\langle x, x^*\rangle_{X,X^*}$ for $x \in X$ and $x^* \in X^*$; and the scalar product by $(\cdot \mid \cdot)_X$ if X is a Hilbert space. In the latter case, a dense embedding $Y \hookrightarrow X$ implies that $X \hookrightarrow Y^*$, where $x \in X \cong X^*$ acts on Y via $\langle x, y\rangle_{Y^*,Y} = (x \mid y)_X$ for $y \in Y \hookrightarrow X$.

Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators from X to Y, and $\mathcal{B}(X) = \mathcal{B}(X, X)$. The domain $D(L)$ of a linear operator L is always equipped with the graph norm $\|\cdot\|_L$ of L. If $Y \hookrightarrow X$, the part L_Y of L in Y is given by $D(L_Y) = \{y \in Y \cap D(L) \mid Ly \in Y\}$ and $L_Y y = Ly$. For two operators L and G in X, the product LG is defined on $D(LG) = \{x \in D(G) \,|\, Gx \in D(L)\}$.

Let λ belong to the resolvent set of a closed operator L in X. We occasionally need the *extrapolation space* $X_{-1} = X_{-1}^L$ of L; i.e., the completion of X with respect to the norm given by $\|x\|_{-1} = \|(\lambda I - L)^{-1}x\|_X$. One then has a continuous extension $L_{-1} : X \to X_{-1}$ whose resolvent operators extend those of L. If L generates a C_0–semigroup $T(\cdot)$ on X, then L_{-1} is the generator of the semigroup $T_{-1}(\cdot)$ of extensions to X_{-1}. This procedure can be iterated, providing $L_{-2} : X_{-1} \to X_{-2}$. If X is reflexive, then X_{-1}^L can be identified with the dual space of $D(L^*)$. See Section V.1.3 in [2] or Section II.5a in [10].

We use the standard Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and open subsets $\Omega \subseteq \mathbb{R}^m$, where $W^{0,p}(\Omega) = L^p(\Omega)$. For $s \in (0, \infty) \setminus \mathbb{N}$ we define the Slobodeckij spaces $W^{s,p}(\Omega)$ by real interpolation, see Section 7.57 in [1] or [19]. Moreover, we set $W^{-s,p}(\Omega) = W_0^{s,p'}(\Omega)^*$ for $s \geq 0$ and $p \in (1, \infty)$, where $p' = p/(p-1)$ and the subscript 0 denotes the closure of test functions in the respective norm. We are mostly interested in the case $H^s(\Omega) := W^{s,2}(\Omega)$.

In this paper we work on the cuboid $Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+) \subseteq \mathbb{R}^3$ with (Lipschitz) boundary $\Gamma = \partial Q$. For $|s| < 1$ we use the spaces $H^s(\Gamma)$ at the boundary, see Section 2.5 of [21]. We write

$$\Gamma_j^\pm = \{x \in \overline{Q} \mid x_j = a_j^\pm\} \qquad \text{and} \qquad \Gamma_j = \Gamma_j^- \cup \Gamma_j^+,$$

for $j \in \{1, 2, 3\}$ and d_Q for the smallest side length of Q.

Our analysis of the Maxwell system takes place in the space $X = L^2(Q)^6$ with the weighted inner product

$$\big((u, v) \mid (\varphi, \psi)\big)_X := \int_Q \big(\varepsilon u \cdot \varphi + \mu v \cdot \psi\big)\, \mathrm{d}x$$

for $(u, v), (\varphi, \psi) \in X$. The square of the induced norm $\|\cdot\|_X$ is twice the physical energy of the fields $(\mathbf{E}, \mathbf{H})$, and because of (1.3) it is equivalent to the usual L^2–norm. We further use the Hilbert spaces

$$H(\mathrm{curl}, Q) = \{u \in L^2(Q)^3 \mid \mathrm{curl}\, u \in L^2(Q)^3\}, \qquad \|u\|_{\mathrm{curl}}^2 = \|u\|_{L^2}^2 + \|\mathrm{curl}\, u\|_{L^2}^2,$$
$$H(\mathrm{div}, Q) = \{u \in L^2(Q)^3 \mid \mathrm{div}\, u \in L^2(Q)\}, \qquad \|u\|_{\mathrm{div}}^2 = \|u\|_{L^2}^2 + \|\mathrm{div}\, u\|_{L^2}^2.$$

Theorems 1 and 2 in Section IX.A.1.2 of [6] provide the following facts. The space of restrictions to Q of test functions on $\mathbb{R}^3$ is dense in $H(\mathrm{curl}, Q)$ and $H(\mathrm{div}, Q)$. The tangential trace $u \mapsto u \times \nu|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ has a unique continuous extension $\mathrm{tr}_t : H(\mathrm{curl}, Q) \to H^{-1/2}(\Gamma)^3$, and $H_0(\mathrm{curl}, Q)$ is the kernel of tr_t in $H(\mathrm{curl}, Q)$. We also have the integration by parts formula

$$\int_Q \mathrm{curl}\, u \cdot v\, \mathrm{d}x = \int_Q u \cdot \mathrm{curl}\, v\, \mathrm{d}x - \langle \mathrm{tr}_t\, u, v\rangle_{H^{-1/2}(Q)^3 \times H^{1/2}(Q)^3} \tag{2.1}$$

for all $u \in H(\mathrm{curl}, Q)$ and $v \in H^1(Q)^3$. Similarly, the normal trace $u \mapsto (u \cdot \nu)|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ has a unique continuous extension $\mathrm{tr}_n : H(\mathrm{div}, Q) \to H^{-1/2}(\Gamma)$. Moreover, Section 2.4 and 2.5 of [21] provide the continuous and surjective trace operator $\mathrm{tr} : H^1(Q) \to H^{1/2}(\Gamma)$, which is the extension of the map $f \mapsto f|_\Gamma$ defined on $C(\overline{Q}) \cap H^1(Q)$. Its kernel is the space $H_0^1(Q)$.

We also have to deal with cases of partial regularity. For instance, take a function $f \in L^2(Q)$ with $\partial_1 f \in L^2(Q)$. We set $Q_1 = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. A representative of f belongs to $H^1((a_1^-, a_1^+), L^2(Q_1))$, and thus possesses traces to the rectangles $\Gamma_1^\pm = \{a_1^\pm\} \times Q_1$ whose norms in $L^2(\Gamma_1^\pm)$ are bounded by $c(\|f\|_{L^2(Q)} + \|\partial_1 f\|_{L^2(Q)})$. In this way, we obtain trace operators $\mathrm{tr}_{\Gamma_j^\pm}$ and tr_{Γ_j} for $j \in \{1, 2, 3\}$. They coincide in $L^2(\Gamma_j^\pm)$, respectively $L^2(\Gamma_j)$, with the respective restrictions of $\mathrm{tr}\, f$ if $f \in H^1(Q)$. We usually write $u_1 = 0$ on Γ_2 instead of $\mathrm{tr}_{\Gamma_2}(u_1) = 0$, and so on. The following lemma will often be used to check boundary conditions.

Lemma 2.1. *For some $j, k \in \{1, 2, 3\}$ with $k \neq j$, let $f \in L^2(Q)$ satisfy $\partial_j f, \partial_k f, \partial_{jk} f \in L^2(Q)$ and $f = 0$ on Γ_j. We then have $\partial_k f = 0$ on Γ_j.*

Proof. We only consider Γ_1^-, $j = 1$, and $k = 2$ since the other cases are treated analogously. By the above observations, for a.e. $(x_2, x_3) \in Q_1$ the map $f(\cdot, x_2, x_3)$ is contained in $H^1(a_1^-, a_1^+)$, and we have $f(x_1, x_2, x_3) =$

$\int_{a_1^-}^{x_1} \partial_1 f(t, x_2, x_3)\,\mathrm{d}t$. Similarly, $\partial_{12} f(x_1, \cdot, \cdot)$ is an element of $L^2(Q_1)$ for a.e. $x_1 \in (a_1^-, a_1^+)$. Using the definition of weak derivatives, one checks that

$$\partial_2 f(x) = \int_{a_1^-}^{x_1} \partial_{12} f(t, x_2, x_3)\,\mathrm{d}t \qquad \text{for a.e. } x \in Q. \tag{2.2}$$

For each integer $n > d_Q^{-1}$, there is a smooth cut-off function χ_n on $[a_1^-, a_1^+]$ such that χ_n takes values in $[0,1]$, vanishes on $J_n = [a_1^-, a_1^- + 1/(2n)]$, is equal to 1 on $[a_1^- + 1/n, a_1^+]$, and $|\chi_n'|$ is bounded by cn. We then define $f_n(x) = \chi_n(x_1) f(x)$ for $x \in Q$. By dominated convergence, the functions $\partial_2 f_n = \chi_n \partial_2 f$ converge to $\partial_2 f$ in $L^2(Q)$ as $n \to \infty$. In the derivative $\partial_{12} f_n = \chi_n' \partial_2 f + \chi_n \partial_{12} f$, the second summand tends to $\partial_{12} f$ in $L^2(Q)$ again by Lebesgue's theorem. Let S_n be the support of χ_n. Using formula (2.2) and Hölder's inequality, we deduce

$$\begin{aligned}
\|\chi_n' \partial_2 f\|_{L^2}^2 &\le \int_{S_n} \int_{Q_1} c^2 n^2 \, |\partial_2 f(x_1, x_2, x_3)|^2 \,\mathrm{d}(x_2, x_3)\,\mathrm{d}x_1 \\
&\le c^2 n \sup_{x_1 \in S_n} \int_{Q_1} |\partial_2 f(x_1, x_2, x_3)|^2 \,\mathrm{d}(x_2, x_3) \\
&\le c^2 \sup_{x_1 \in S_n} \int_{a_1^-}^{x_1} \int_{Q_1} |\partial_{12} f(t, x_2, x_3)|^2 \,\mathrm{d}(x_2, x_3)\,\mathrm{d}t \\
&= c^2 \int_{a_1^-}^{a_1^- + 1/n} \int_{Q_1} |\partial_{12} f(t, x_2, x_3)|^2 \,\mathrm{d}(x_2, x_3)\,\mathrm{d}t \ \longrightarrow\ 0
\end{aligned}$$

as $n \to \infty$. The functions $\partial_2 f_n$ thus tend to $\partial_2 f$ in $H^1((a_1^-, a_1^+), L^2(Q_1))$ so that $\mathrm{tr}_{\Gamma_1^-}(\partial_2 f_n) = 0$ converges to the trace of $\partial_2 f$ in $L^2(\Gamma_1^-)$. □

On X we define the Maxwell operator

$$M = \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon}\,\mathrm{curl} \\ -\frac{1}{\mu}\,\mathrm{curl} & 0 \end{pmatrix}, \qquad D(M) = H_0(\mathrm{curl}, Q) \times H(\mathrm{curl}, Q). \tag{2.3}$$

This domain includes the electric boundary condition. To encode the magnetic boundary and divergence conditions in (1.1) and the regularity of the charge density $\rho = \mathrm{div}(\varepsilon u)$, we introduce the subspace

$$\begin{aligned}
X_{\mathrm{div}} &:= \{(u, v) \in X \mid \mathrm{div}(\mu v) = 0,\ \ \mathrm{tr}_n(\mu v) = 0,\ \ \mathrm{div}(\varepsilon u) \in L^2(Q)\} \\
&= \{(u, v) \in X \mid \mathrm{div}(\mu v) = 0,\ \ \mathrm{tr}_n\, v = 0,\ \ \mathrm{div}\, u \in L^2(Q)\}.
\end{aligned} \tag{2.4}$$

The above constraints are understood in $H^{-1}(Q)$, respectively $H^{-1/2}(\Gamma)$. The second equation in (2.4) follows from (1.3) by Remark 3.3 in [13] and because of $\mathrm{div}(\varepsilon u) = \nabla \varepsilon \cdot u + \varepsilon\, \mathrm{div}\, u$. In the same way one sees that $\mathrm{div}\, v$ belongs to $L^2(Q)$ if $(u, v) \in X_{\mathrm{div}}$. Equipped with the norm given by

$$\|(u, v)\|_{X_{\mathrm{div}}}^2 := \|(u, v)\|_X^2 + \|\mathrm{div}(\varepsilon u)\|_{L^2}^2\,,$$

X_{div} is a Hilbert space since the maps $\mathrm{div} : L^2(Q)^3 \to H^{-1}(Q)$ and $\mathrm{tr}_n : H(\mathrm{div}, Q) \to H^{-1/2}(\Gamma)$ are continuous.

The part of M in X_{div} is denoted by M_{div}. We actually have

$$D(M_{\mathrm{div}}^k) = D(M^k) \cap X_{\mathrm{div}} \tag{2.5}$$

for $k \in \mathbb{N}$. To show this claim, let $(u, v) \in D(M) \cap X_{\text{div}}$. As in Proposition 3.5 of [13] (for the case $\sigma = 0$) one infers that $M(u, v)$ satisfies the magnetic conditions in X_{div}. Moreover, according to assumption (1.3) the function

$$-\operatorname{div}(\varepsilon(M(u,v))_1) = \operatorname{div}(\sigma u) = \nabla(\sigma\varepsilon^{-1})\, \varepsilon u + \sigma\varepsilon^{-1} \operatorname{div}(\varepsilon u)$$

belongs to $L^2(Q)$, and thus $M(u, v)$ to X_{div}. Hence, (u, v) is contained in $D(M_{\text{div}})$, and (2.5) is shown for $k = 1$. By induction, (2.5) follows for all $k \in \mathbb{N}$.

The spaces $H(\text{curl}, Q)$ and $H(\text{div}, Q)$ contain rather irregular L^2–functions, e.g. from the kernels of curl and div. Nevertheless, their intersection embeds into $H^1(Q)^3$ if one assumes that either the tangential or the normal trace is 0. See Theorem 2.17 in [3], for instance.

Proposition 2.2. *The domain $D(M_{\text{div}})$ is continuously embedded into $H^1(Q)^6$, where the embedding constant only depends on the constants in* (1.3). *Moreover, in the sense of the traces $\operatorname{tr}_{\Gamma_j}$ the fields $(\boldsymbol{E}, \boldsymbol{H}) \in D(M_{\text{div}})$ satisfy*

$$\begin{aligned} E_2 &= E_3 = 0, & H_1 &= 0 & &\text{on } \Gamma_1,\\ E_1 &= E_3 = 0, & H_2 &= 0 & &\text{on } \Gamma_2,\\ E_1 &= E_2 = 0, & H_3 &= 0 & &\text{on } \Gamma_3. \end{aligned}$$

Proof. Let $w = (\mathbf{E}, \mathbf{H}) \in D(M_{\text{div}})$. The functions $\operatorname{curl} \mathbf{H} = -\mu(Mw)_2$ and $\operatorname{curl} \mathbf{E} = \varepsilon(Mw)_1 + \sigma\mathbf{E}$ then belong to $L^2(Q)^3$. As noted above, also $\operatorname{div} \mathbf{E}$ and $\operatorname{div} \mathbf{H}$ are contained in $L^2(Q)$. The asserted embedding then follows from Theorem 2.17 of [3]. We thus obtain $\operatorname{tr}_t \mathbf{E} = \nu \times \operatorname{tr} \mathbf{E}$ and $\operatorname{tr}_n \mathbf{H} = \nu \cdot \operatorname{tr} \mathbf{H}$ which yields the second assertion. □

We collect the main properties of the Maxwell operators and solve (1.1). Some of these results are contained in Section XVII.B.4 in [7], for instance. Actually, the proposition is true on any Lipschitz domain Q with the same proof.

Proposition 2.3. *Let* (1.3) *hold. Then the following assertions are true.*

a) The operators M and M_{div} generate C_0–semigroups $(\mathrm{e}^{tM})_{t\geq 0}$ on X and $(\mathrm{e}^{tM_{\text{div}}})_{t\geq 0}$ on X_{div}, respectively. Moreover, $\mathrm{e}^{tM_{\text{div}}}$ is the restriction of e^{tM} to X_{div}, and we have $\|\mathrm{e}^{tM}\|_X \leq 1$ and $\|\mathrm{e}^{tM_{\text{div}}}\|_{X_{\text{div}}} \leq c(1 + t)$ for all $t \geq 0$.

b) Let $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0)$ belong to $D(M_{\text{div}})$ and $(\boldsymbol{J}, 0)$ to $C([0, \infty), D(M_{\text{div}}))+ C^1([0, \infty), X_{\text{div}})$. There exists a unique solution $w = (\boldsymbol{E}, \boldsymbol{H})$ of (1.1) *in $C^1([0, \infty), X_{\text{div}}) \cap C([0, \infty), D(M_{\text{div}}))$ given by*

$$(\boldsymbol{E}(t), \boldsymbol{H}(t)) = \mathrm{e}^{tM_{\text{div}}}(\boldsymbol{E}_0, \boldsymbol{H}_0) - \int_0^t \mathrm{e}^{(t-s)M_{\text{div}}}\big(\tfrac{1}{\varepsilon}\boldsymbol{J}(s), 0\big)\, \mathrm{d}s \tag{2.6}$$

for $t \geq 0$. The charge density in (1.1c) *is contained in $L^2(Q)$ and satisfies*

$$\begin{aligned} \rho(t) = \operatorname{div}(\varepsilon \boldsymbol{E}(t)) &= \operatorname{div}(\varepsilon \boldsymbol{E}_0) - \int_0^t \operatorname{div}(\sigma \boldsymbol{E}(s) + \boldsymbol{J}(s))\, \mathrm{d}s \qquad (2.7)\\ &= \mathrm{e}^{-\frac{\sigma}{\varepsilon}t} \operatorname{div}(\varepsilon \boldsymbol{E}_0) - \int_0^t \mathrm{e}^{-\frac{\sigma}{\varepsilon}(t-s)} \Big(\nabla\big(\tfrac{\sigma}{\varepsilon}\big) \cdot \varepsilon \boldsymbol{E}(s) + \operatorname{div} \boldsymbol{J}(s)\Big)\, \mathrm{d}s, \quad t \geq 0. \end{aligned}$$

c) Let $w_0 \in D(M_{\text{div}}^2)$ and $(\boldsymbol{J}, 0) \in W^{2,1}([0, T], X_{\text{div}}) \cap C([0, T], D(M_{\text{div}})) =: E$ for some $T > 0$. Then w belongs to $C^2([0, T], X_{\text{div}}) \cap C^1([0, T], D(M_{\text{div}})) \cap C([0, T], D(M_{\text{div}}^2))$ with norm bounded by $c(1 + T)\big(\|w_0\|_{D(M_{\text{div}}^2)} + \|(\boldsymbol{J}, 0)\|_E\big)$.

d) The adjoint of M in X is given by $D(M^*) = D(M)$ and

$$M^* = \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & -\frac{1}{\varepsilon}\operatorname{curl} \\ \frac{1}{\mu}\operatorname{curl} & 0 \end{pmatrix}.$$

Proof. 1) If $\sigma = 0$, for instance Proposition 3.5 of [13] shows that M generates a contraction semigroup on X. Using the dissipative perturbation theorem, see Theorem III.2.7 in [10], one can extend this result to the case $\sigma \geq 0$. In the same way one shows that the operator matrix in d) with domain $D(M)$ is a generator. Because of (2.1) it is a restriction of M^*, cf. Proposition 3.5 of [13]. So assertion d) has been shown.

2) We observe that the inhomogeneity $(\frac{1}{\varepsilon}\mathbf{J}, 0)$ satisfies the same assumptions as $(\mathbf{J}, 0)$ in part b), respectively c), see (2.3)–(2.5). Let the conditions of b) be true. Corollaries 4.2.5 and 4.2.6 of [23] then provide a unique solution $w = (\mathbf{E}, \mathbf{H})$ in $C^1([0,\infty), X) \cap C([0,\infty), D(M))$ of (1.1a) and (1.1b) which also satisfies the electric boundary condition and the initial conditions. It is given by Duhamel's formula (2.6) with M instead of M_{div}.

In view of the magnetic conditions in (1.1c) and (1.1d), we introduce the closed subspace $X_{\mathrm{mag}} = \{(u, v) \in X \mid \operatorname{div}(\mu v) = 0,\ \operatorname{tr}_n(\mu v) = 0\}$ of X. As in Proposition 3.5 of [13], one sees that M maps $D(M)$ into X_{mag}. Hence, the resolvent $(\lambda I - M)^{-1}$ for $\lambda > 0$ leaves invariant X_{mag}. The same is true for the operator e^{tM} since it is the strong limit of $(\frac{n}{t}(\frac{n}{t} I - A)^{-n})^n$ in X for $t > 0$, see Corollary III.5.5 of [10]. Due to Duhamel's formula, the solution w then takes values in X_{mag} and thus solves (1.1).

Equation (1.1a) implies that $\partial_t \operatorname{div}(\varepsilon\mathbf{E}(t)) = -\operatorname{div}(\sigma\mathbf{E}(t) + \mathbf{J}(t))$ in $H^{-1}(Q)$ for $t \geq 0$, whence the first part of (2.7) follows. Writing $\sigma = \frac{\sigma}{\varepsilon}\varepsilon$, we infer

$$\begin{aligned} \partial_t \operatorname{div}(\varepsilon\mathbf{E}(t)) &= -\tfrac{\sigma}{\varepsilon}\operatorname{div}(\varepsilon\mathbf{E}(t)) - \nabla\big(\tfrac{\sigma}{\varepsilon}\big)\varepsilon\mathbf{E}(t) - \operatorname{div}\mathbf{J}(t), \\ \partial_t\big(\mathrm{e}^{\frac{\sigma}{\varepsilon}t}\operatorname{div}(\varepsilon\mathbf{E}(t)\big) &= -\mathrm{e}^{\frac{\sigma}{\varepsilon}t}\big(\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\varepsilon\mathbf{E}(t) + \operatorname{div}\mathbf{J}(t)\big) \end{aligned}$$

in $H^{-1}(Q)$. This formula leads to the second part of (2.7), and b) is established.

3) For the remaining assertions in a), we take $\mathbf{J} = 0$. Since e^{tM} is a contraction in X, we have $\|w(t)\|_X \leq \|w_0\|_X$. From (2.7) we then deduce the bound

$$\|\operatorname{div}(\varepsilon\mathbf{E}(t))\|_{L^2(Q)} \leq \|\operatorname{div}(\varepsilon\mathbf{E}_0)\|_{L^2(Q)} + ct\,\|(\mathbf{E}_0, \mathbf{H}_0)\|_X$$

and that $\operatorname{div}(\varepsilon\mathbf{E}(t))$ tends to $\operatorname{div}(\varepsilon\mathbf{E}_0)$ in $L^2(Q)$ as $t \to 0$. Thus, e^{tM} possesses a restriction $\mathrm{e}^{tM_{\mathrm{div}}}$ to X_{div}, which forms a C_0–semigroup there and is bounded by $c(1 + t)$. By Paragraph II.2.3 of [10] it is generated by M_{div}.

4) Under the assumptions of part c), we can differentiate (2.6) in X_{div} twice in t (after the substitution $r = t - s$.) Hence, w belongs to $C^2([0, T], X_{\mathrm{div}})$ and

$$w'' - w' = (M_{\mathrm{div}} - I)w' - \tfrac{1}{\varepsilon}(\mathbf{J}', 0).$$

Inverting $M_{\mathrm{div}} - I$, we thus obtain $w \in C^1([0, T], D(M_{\mathrm{div}}))$. Similarly, the equation $w' = M_{\mathrm{div}}w - \frac{1}{\varepsilon}(\mathbf{J}, 0)$ then implies that w is contained in $C([0, T], D(M_{\mathrm{div}}^2))$. These arguments also yield the asserted bound in c). □

3. The split operators

We now analyze the operators

$$A = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix} \qquad \text{with}$$
$$C_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

which are the main ingredients of our splitting algorithm. These operators are endowed with the domains

$$D(A) = \{(u, v) \in X \mid (C_1 v, C_2 u) \in X,\ \ \operatorname{tr}_{\Gamma_2} u_1 = 0,\ \ \operatorname{tr}_{\Gamma_3} u_2 = 0,\ \ \operatorname{tr}_{\Gamma_1} u_3 = 0\},$$
$$D(B) = \{(u, v) \in X \mid (C_2 v, C_1 u) \in X,\ \ \operatorname{tr}_{\Gamma_3} u_1 = 0,\ \ \operatorname{tr}_{\Gamma_1} u_2 = 0,\ \ \operatorname{tr}_{\Gamma_2} u_3 = 0\}.$$

Each domain contains one half of the electric boundary conditions in $D(M_{\mathrm{div}})$, see Proposition 2.2. These traces exist since they fit to the partial derivatives in $C_2 u$ for A and in $C_1 u$ for B. Observe that A and B map into X

$$D(A) \cap D(B) \hookrightarrow D(M) \qquad \text{and} \qquad M = A + B \quad \text{on } D(A) \cap D(B).$$

However, neither the divergence conditions nor the magnetic boundary condition for the magnetic field are included in $D(A)$ or $D(B)$. We further write

$$A_0 = A + \begin{pmatrix} \frac{\sigma}{2\varepsilon} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B_0 = B + \begin{pmatrix} \frac{\sigma}{2\varepsilon} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2}$$

with $D(A_0) = D(A)$ and $D(B_0) = D(B)$ for the parts without conductivity. As in Section 4.3 in [13], one shows the following basic integration by parts formula. Let $u, \varphi \in L^2(Q)^3$ satisfy $C_1\varphi \in L^2(Q)^2$, $C_2 u \in L^2(Q)^3$, and

$$\operatorname{tr}_{\Gamma_3} u_2 \cdot \operatorname{tr}_{\Gamma_3} \varphi_1 = 0, \qquad \operatorname{tr}_{\Gamma_1} u_3 \cdot \operatorname{tr}_{\Gamma_1} \varphi_2 = 0, \qquad \operatorname{tr}_{\Gamma_2} u_1 \cdot \operatorname{tr}_{\Gamma_2} \varphi_3 = 0.$$

(For instance, take $(u, \varphi) \in D(A)$ or $(\varphi, u) \in D(B)$.) We then have

$$(C_2 u \mid \varphi)_{L^2} = (u \mid -C_1 \varphi)_{L^2}. \tag{3.3}$$

In our splitting algorithm (4.1) we use resolvents and Cayley transforms of A and B, and those of A^* and B^* enter in the error analysis. Their properties are stated in the next proposition. Let L be a closed operator on a Banach space such that $L - \kappa I$ generates a contraction semigroup for some $\kappa \geq 0$. Then the Cayley transform

$$\gamma_\tau(L) = (I + \tau L)(I - \tau L)^{-1} \tag{3.4}$$

exists for all $\tau \in (0, 1/\kappa)$. Observe that $(I - \tau L)^{-1} = \tau^{-1}(\tau^{-1} I - L)^{-1}$.

Proposition 3.1. *a) In X we have $D(A^*) = D(A_0^*) = D(A)$ and $D(B^*) = D(B_0^*) = D(B)$, as well as $A_0^* = -A_0$, $B_0^* = -B_0$,*

$$A^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_1 \\ -\frac{1}{\mu} C_2 & 0 \end{pmatrix} \qquad \text{and} \qquad B^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_2 \\ \frac{1}{\mu} C_1 & 0 \end{pmatrix}.$$

It follows $M^* = A^* + B^*$ on $D(A^*) \cap D(B^*) \hookrightarrow D(M^*) = D(M)$.

b) The operators A, B, A^* and B^* generate C_0-semigroups of contractions on X. As a result, the resolvents $(I - \tau L)^{-1}$ and the Cayley transforms $\gamma_\tau(L)$ are contractive for all $L \in \{A, B, A^*, B^*\}$ and $\tau > 0$. Moreover, $D(M_{\mathrm{div}}) \hookrightarrow D(L)$.

Proof. Lemma 4.3 of [13] says that A_0 and B_0 are skew-adjoint on X, and hence generate a contraction semigroup. This property is inherited by A and B due to (3.2) and the dissipative perturbation Theorem III.2.7 in [10]. In the same way one shows the generator property of the operator matrices defined in part a) on the domains $D(A)$ and $D(B)$, respectively. As in Lemma 4.3 of [13], equation (3.3) implies that these operator matrices are restrictions of A^* and B^*. They are thus equal to these operators, respectively, since the right half plane belongs to all resolvent sets. The other assertions in b) then easily follow, using also Proposition 2.2. □

For our error analysis we need the restrictions of the above operators to the subspace of H^1 given by

$$Y := \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j, \;\; v_j = 0 \text{ on } \Gamma_j \text{ for all } j \in \{1,2,3\}\}.$$

We use on Y the weighted inner product

$$\big((u,v) \mid (\varphi,\psi)\big)_Y := \int_Q \Big(\varepsilon u \cdot \varphi + \mu v \cdot \psi + \varepsilon \sum_{j=1}^3 \partial_j u \cdot \partial_j \varphi + \mu \sum_{j=1}^3 \partial_j v \cdot \partial_j \psi\Big) \,\mathrm{d}x$$

with the induced norm $\|\cdot\|_Y$. Due to (1.3), this norm is equivalent to the usual norm on H^1. The continuity of the traces implies that Y is a closed subspace of $H^1(Q)^6$. We very often use that maps like $(u,v) \mapsto (\varepsilon u, \mu v)$ leave invariant Y because of (1.3). Our definitions yield the embedding

$$Y \hookrightarrow D(A) \cap D(B) \cap D(A^*) \cap D(B^*) \cap D(M) \cap D(M^*). \tag{3.5}$$

We denote by A_Y, B_Y, $(A^*)_Y$, and $(B^*)_Y$ the parts of A, B, A^*, and B^* in Y, respectively. Their domains are described in the next lemma.

Lemma 3.2. *a) We have*

$$\begin{aligned} D(A_Y) = D((A^*)_Y) &= \{(u,v) \in Y \mid (C_1 v, C_2 u) \in Y\} \\ &= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \textit{ on } \Gamma \setminus \Gamma_j, \;\; v_j = 0 \textit{ on } \Gamma_j \textit{ for } j \in \{1,2,3\}, \\ &\qquad \partial_2 u_1, \partial_3 u_2, \partial_1 u_3, \partial_3 v_1, \partial_1 v_2, \partial_2 v_3 \in H^1(Q), \\ &\qquad \partial_3 v_1 = 0 \textit{ on } \Gamma_3, \;\; \partial_1 v_2 = 0 \textit{ on } \Gamma_1, \;\; \partial_2 v_3 = 0 \textit{ on } \Gamma_2\}, \\ D(B_Y) = D((B^*)_Y) &= \{(u,v) \in Y \mid (C_2 v, C_1 u) \in Y\} \\ &= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \textit{ on } \Gamma \setminus \Gamma_j, \; v_j = 0 \textit{ on } \Gamma_j \textit{ for } j \in \{1,2,3\}, \\ &\qquad \partial_3 u_1, \partial_1 u_2, \partial_2 u_3, \partial_2 v_1, \partial_3 v_2, \partial_1 v_3 \in H^1(Q), \\ &\qquad \partial_2 v_1 = 0 \textit{ on } \Gamma_2, \; \partial_3 v_2 = 0 \textit{ on } \Gamma_3, \;\; \partial_1 v_3 = 0 \textit{ on } \Gamma_1\}. \end{aligned}$$

b) Let $(u,v), (\tilde{u},\tilde{v}) \in Y$ and $C_2 u, C_1 v, C_1 \tilde{u}, C_2 \tilde{v} \in H^1(Q)^3$. Then

$$\begin{aligned} &\partial_3 u_2 = \partial_2 u_3 = \partial_3 u_3 = \partial_3 v_1 = 0 && \textit{on } \Gamma_1, \\ &\partial_1 u_3 = \partial_3 u_1 = \partial_1 u_1 = \partial_1 v_2 = 0 && \textit{on } \Gamma_2, \\ &\partial_2 u_1 = \partial_1 u_2 = \partial_2 u_2 = \partial_2 v_3 = 0 && \textit{on } \Gamma_3; \\ &\partial_2 \tilde{u}_3 = \partial_2 \tilde{u}_2 = \partial_3 \tilde{u}_2 = \partial_2 \tilde{v}_1 = 0 && \textit{on } \Gamma_1, \\ &\partial_3 \tilde{u}_1 = \partial_3 \tilde{u}_3 = \partial_1 \tilde{u}_3 = \partial_3 \tilde{v}_2 = 0 && \textit{on } \Gamma_2, \\ &\partial_1 \tilde{u}_2 = \partial_1 \tilde{u}_1 = \partial_2 \tilde{u}_1 = \partial_1 \tilde{v}_3 = 0 && \textit{on } \Gamma_3. \end{aligned}$$

Here the respective first line in a) follows from Proposition 3.1 and (1.3). Part b) is a consequence of Lemma 2.1 and it yields the rest of assertion a). In a series of further lemmas we collect the basic properties of the above operators.

Lemma 3.3. *The operators A_Y, B_Y, $(A^*)_Y$, and $(B^*)_Y$ are closed and densely defined in Y.*

Proof. 1) The operators are closed as the parts of closed operators. For the density, we only treat A_Y since the other cases can be handled in the same way, using Proposition 3.1 for $(A^*)_Y$ and $(B^*)_Y$.

Let $(u,v) \in Y$. We approximate $u_1 =: f$ and $v_1 =: g$ in Y by functions f_n and g_n which are the first and fourth components of vectors $(u_n, v_n) \in D(A_Y)$, respectively. We use smooth cut-off functions $\chi_n^{(j)} : [a_j^-, a_j^+] \to [0,1]$ with $|(\chi_n^{(j)})'| \le cn$ that vanish on $[a_j^-, a_j^- + \frac{1}{2n}] \cup [a_j^+ - \frac{1}{2n}, a_j^+]$ and are equal to one on $[a_j^- + \frac{1}{n}, a_j^+ - \frac{1}{n}]$ for $n > (2d_Q)^{-1} =: \ell$ and $j \in \{1,2,3\}$. Also, $\rho_n^{(j)}$ is a standard C^∞-mollifier with support in $[-\frac{1}{2n}, \frac{1}{2n}]$ which acts on x_j.

2) We are looking for functions $f_n \in H^1(Q)$ with $\partial_2 f_n \in H^1(Q)$ that converge to f in $H^1(Q)$ and have zero traces on Γ_2 and Γ_3. We first set $\varphi_n = \chi_n^{(2)} f$ for $n \ge n_0$. These maps belong to $H^1(Q)$, vanish near Γ_2 and have trace 0 on Γ_3. Due to dominated convergence, the functions φ_n tend to f and $\chi_n^{(2)} \partial_j f$ to $\partial_j f$ in $L^2(Q)$ as $n \to \infty$ and for $j \in \{1,2,3\}$. As in the proof of Lemma 2.1 one shows that $((\chi_n^{(2)})' f)$ converges to 0 in $L^2(Q)$ since f vanishes on Γ_2. Summing up, (φ_n) has the limit f in $H^1(Q)$. We then extend φ_n by 0 to $\mathbb{R}^3$ and define

$$f_n = \big(\rho_n^{(2)} * \varphi_n\big)|_Q$$

for $n > \ell$. This function and $\partial_2 f_n$ belong to $H^1(Q)$, and f_n vanishes near Γ_2. For a map $h \in H^1(Q) \cap C(\overline{Q})$ with $h = 0$ on Γ_3 it is clear that $\rho_n^{(2)} * h$ is also equal to 0 on Γ_3. By approximation, we thus obtain $\operatorname{tr}_{\Gamma_3} f_n = 0$.

3) We next have to construct functions $g_n \in H^1(Q)$ with $\partial_3 g_n \in H^1(Q)$ such that $\operatorname{tr}_{\Gamma_1} g_n = 0$, $\operatorname{tr}_{\Gamma_3} \partial_3 g_n = 0$ and (g_n) has the limit g in $H^1(Q)$. Let Φ be the linear and bounded Stein extension operator that maps functions in $H^k(Q)$ to functions in $H^k(\mathbb{R}^3)$, see Theorem 5.24 in [1]. We extend g by 0 outside of Q and define

$$\psi_{n,m} = \big[\rho_n^{(2)} * \rho_n^{(3)} * \Phi(\rho_m^{(1)} * (\chi_m^{(1)} g))\big]\big|_Q$$

for all $n, m > \ell$. These maps are smooth and vanish near Γ_1 (together with their derivatives). Let $\eta > 0$. Arguing as in step 2), we can fix an index $\widetilde{m} = \widetilde{m}(\eta) > \ell$ such that

$$\Big\|\rho_{\widetilde{m}}^{(1)} * \big(\chi_{\widetilde{m}}^{(1)} g\big) - g\Big\|_{H^1} \le \eta.$$

By the properties of mollifiers, there also exists a number $\widetilde{n} = \widetilde{n}(\eta) > \ell$ with

$$\Big\|\psi_{\widetilde{n},\widetilde{m}} - \Phi\big(\rho_{\widetilde{m}}^{(1)} * \big(\chi_{\widetilde{m}}^{(1)} g\big)\big)|_Q\Big\|_{H^1} \le \eta.$$

Setting $\widehat{g} = \psi_{\widetilde{n},\widetilde{m}}$, we obtain the inequality

$$\|\widehat{g} - g\|_{H^1} \le \big(1 + \|\Phi\|_{\mathcal{B}(H^1(Q), H^1(\mathbb{R}^3))}\big)\eta.$$

In view of the needed boundary condition of $\partial_3 g_n$, we define

$$g_n(x) = \widehat{g}(x) + \int_{a_3^-}^{x_3} (\chi_n^{(3)}(t) - 1)\partial_3 \widehat{g}(x', t)\,\mathrm{d}t =: \widehat{g}(x) + r_n(x)$$

for $x = (x', x_3) \in Q$ and $n > \ell$. The functions g_n and $\partial_3 g_n = \chi_n^{(3)} \partial_3 \widehat{g}$ are contained in $H^1(Q)$. Moreover, the traces of g_n on Γ_1 and of $\partial_3 g_n$ on Γ_3 are zero by construction.

The integrand of r_n and its derivatives with respect to x_1 and x_2 are uniformly bounded by a constant, and these maps tend to 0 pointwise a.e. as $n \to \infty$. By dominated convergence, the functions r_n, $\partial_1 r_n$ and $\partial_2 r_n$ thus converge to 0 pointwise a.e. and then in $L^2(Q)$ as $n \to \infty$. The same is true for $\partial_3 r_n = (\chi_n^{(3)} - 1)\partial_3 \widehat{g}$. As a result, (g_n) has the limit g in $H^1(Q)$.

The other components of (u, v) are treated in the same way. □

We set

$$\kappa_Y = \frac{3\,\|\nabla\sigma\|_{L^\infty}}{4\delta} + \frac{3\,\|\sigma\|_{L^\infty}\,\|\nabla\varepsilon\|_{L^\infty}}{4\delta^2} + \frac{\|\nabla\varepsilon\|_{L^\infty} + \|\nabla\mu\|_{L^\infty}}{2\delta^2}. \tag{3.6}$$

Lemma 3.4. *The operators $A_Y - \kappa_Y I$, $B_Y - \kappa_Y I$, $(A^*)_Y - \kappa_Y I$, and $(B^*)_Y - \kappa_Y I$ are dissipative on Y.*

Proof. Again we only consider A_Y. Let $(u, v) \in D(A_Y)$. In view of the boundary conditions in Lemma 3.2, integration by parts yields

$$\begin{aligned}\int_Q \big(\partial_j C_1 v \cdot \partial_j u + \partial_j C_2 u \cdot \partial_j v\big)\,\mathrm{d}x &= \int_Q \big(\partial_{j2} v_3\, \partial_j u_1 + \partial_{j3} v_1\, \partial_j u_2 + \partial_{j1} v_2\, \partial_j u_3 \\ &\qquad + \partial_{j3} u_2\, \partial_j v_1 + \partial_{j1} u_3\, \partial_j v_2 + \partial_{j2} u_1\, \partial_j v_3\big)\,\mathrm{d}x \\ &= 0\end{aligned}$$

for $j \in \{1, 2, 3\}$. The above equation, (3.3) and Hölder's inequality imply

$$\begin{aligned}\big(A(u,v) \mid (u,v)\big)_Y &= \int_Q \Big(-\frac{\sigma\varepsilon}{2\varepsilon}\,|u|^2 + \frac{\varepsilon}{\varepsilon} C_1 v \cdot u + \frac{\mu}{\mu} C_2 u \cdot v - \varepsilon \sum_{j=1}^3 \partial_j\big(\tfrac{\sigma}{2\varepsilon} u\big) \cdot \partial_j u \\ &\qquad + \varepsilon \sum_{j=1}^3 \partial_j\big(\tfrac{1}{\varepsilon} C_1 v\big) \cdot \partial_j u + \mu \sum_{j=1}^3 \partial_j\big(\tfrac{1}{\mu} C_2 u\big) \cdot \partial_j v\Big)\,\mathrm{d}x \\ &= -\int_Q \frac{\sigma}{2}\,(|u|^2 + |\partial u|^2)\,\mathrm{d}x - \sum_{j=1}^3 \int_Q \big(\tfrac{\partial_j \sigma}{2} - \tfrac{\sigma \partial_j \varepsilon}{2\varepsilon}\big) u \cdot \partial_j u\,\mathrm{d}x \\ &\qquad - \sum_{j=1}^3 \int_Q \tfrac{\partial_j \varepsilon}{\varepsilon} C_1 v \cdot \partial_j u\,\mathrm{d}x - \sum_{j=1}^3 \int_Q \tfrac{\partial_j \mu}{\mu} C_2 u \cdot \partial_j v\,\mathrm{d}x \\ &\le \Big(\frac{\|\nabla\sigma\|_{L^\infty}}{4\delta} + \frac{\|\sigma\|_{L^\infty}\,\|\nabla\varepsilon\|_{L^\infty}}{4\delta^2}\Big) \int_Q \big(3\varepsilon\,|u|^2 + \varepsilon\,|\partial u|^2\big)\,\mathrm{d}x \\ &\qquad + \frac{\|\nabla\varepsilon\|_{L^\infty} + \|\nabla\mu\|_{L^\infty}}{2\delta^2} \int_Q \big(\varepsilon\,|\partial u|^2 + \mu\,|\partial v|^2\big)\,\mathrm{d}x \\ &\le \kappa_Y\,\|(u,v)\|_Y^2,\end{aligned}$$

where $|\partial u|$ and $|\partial v|$ denote the Frobenius norm of the Jacobian matrices. □

Lemma 3.5. *The operators* $(1+\kappa_Y)I - A_Y$*,* $(1+\kappa_Y)I - B_Y$*,* $(1+\kappa_Y)I-(A^*)_Y$*, and* $(1+\kappa_Y)I - (B^*)_Y$ *have dense range in* Y*.*

Proof. 1) As above we only consider $(1+\kappa_Y)I - A_Y$. We know from Lemma 3.3 that $D(A_Y)$ is dense in Y. Let $(f,g) \in D(A_Y)$. We look for fields $(u,v) \in D(A_Y)$ with $\big((1+\kappa_Y)I - A\big)(u,v) = (f,g)$; i.e.,

$$\begin{aligned}
&\big(1+\kappa_Y + \tfrac{\sigma}{2\varepsilon}\big)u_1 - \tfrac{1}{\varepsilon}\partial_2 v_3 = f_1, \qquad &&(1+\kappa_Y)v_3 - \tfrac{1}{\mu}\partial_2 u_1 = g_3,\\
&\big(1+\kappa_Y + \tfrac{\sigma}{2\varepsilon}\big)u_2 - \tfrac{1}{\varepsilon}\partial_3 v_1 = f_2, \qquad &&(1+\kappa_Y)v_1 - \tfrac{1}{\mu}\partial_3 u_2 = g_1,\\
&\big(1+\kappa_Y + \tfrac{\sigma}{2\varepsilon}\big)u_3 - \tfrac{1}{\varepsilon}\partial_1 v_2 = f_3, \qquad &&(1+\kappa_Y)v_2 - \tfrac{1}{\mu}\partial_1 u_3 = g_2.
\end{aligned} \tag{3.7}$$

We formally insert in each line the second equation into the first one, obtaining

$$\begin{aligned}
\big(\varepsilon(1+\kappa_Y) + \tfrac{\sigma}{2}\big)u_1 - \tfrac{1}{1+\kappa_Y}D_2 u_1 &= \varepsilon f_1 + \tfrac{1}{1+\kappa_Y}\partial_2 g_3 =: h_1,\\
\big(\varepsilon(1+\kappa_Y) + \tfrac{\sigma}{2}\big)u_2 - \tfrac{1}{1+\kappa_Y}D_3 u_2 &= \varepsilon f_2 + \tfrac{1}{1+\kappa_Y}\partial_3 g_1 =: h_2,\\
\big(\varepsilon(1+\kappa_Y) + \tfrac{\sigma}{2}\big)u_3 - \tfrac{1}{1+\kappa_Y}D_1 u_3 &= \varepsilon f_3 + \tfrac{1}{1+\kappa_Y}\partial_1 g_2 =: h_3.
\end{aligned} \tag{3.8}$$

Here we have set $D_j = \partial_j \frac{1}{\mu}\partial_j$ with domain

$$\begin{aligned}
D(D_j) &:= \{\varphi \in L^2(Q) \mid \partial_j\varphi \in L^2(Q),\ \ D_j\varphi \in L^2(Q),\ \ \varphi = 0 \text{ on } \Gamma_j\}\\
&= \{\varphi \in L^2(Q) \mid \partial_j\varphi \in L^2(Q),\ \ \partial_j^2\varphi \in L^2(Q),\ \ \varphi = 0 \text{ on } \Gamma_j\}.
\end{aligned}$$

for $j \in \{1,2,3\}$, where we have used (1.3). Since $(f,g) \in D(A_Y)$, the map h_j belongs to $H^1(Q)$ and satisfies $h_j = 0$ on $\Gamma \setminus \Gamma_j$, see Lemma 3.2. We also define

$$D(\partial_j) = \{\varphi \in L^2(Q) \mid \partial_j\varphi \in L^2(Q),\ \ \varphi = 0 \text{ on } \Gamma_j\}.$$

2) Let $j = 2$. We are looking for a function $w \in D(D_2)$ solving (3.8), where we put $h := h_1$. To this aim, we abbreviate

$$Lw = \big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)w - \tfrac{1}{1+\kappa_Y}\partial_2\big(\tfrac{1}{\mu}\partial_2 w\big)$$

for $w \in D(D_2)$. As in the proof of Lemma 4.3 in [13], where $\sigma = 0$ and $\kappa_Y = 0$, by means of (3.10) and the Lax–Milgram lemma we obtain a unique map w in $D(D_2)$ with $Lw = h$, and w thus satisfies (3.8). Moreover, L is invertible in X.

We check that w satisfies the properties of u_1 needed for $(u,v) \in D(A_Y)$. Let $k \in \{1,2,3\}$ and $\varphi \in H_0^2(Q)$. Since $\partial_2\partial_k w = \partial_k\partial_2 w$ in $H^{-2}(Q)$ and $\partial_2 w$ belongs to $L^2(Q)$, the derivative $\partial_2\partial_k w$ is contained in $H^{-1}(Q)$ and hence $D_2\partial_k w \in H^{-2}(Q)$ by (1.3). Integrating by parts, we can thus compute

$$\begin{aligned}
&\langle L\partial_k w, \varphi\rangle_{H^{-2}\times H_0^2}\\
&\quad= \big\langle \partial_k w, \big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)\varphi\big\rangle_{H^{-1}\times H_0^1} + \frac{1}{1+\kappa_Y}\Big\langle \partial_2\partial_k w, \tfrac{1}{\mu}\partial_2\varphi\Big\rangle_{H^{-1}\times H_0^1}\\
&\quad= -\int_Q w\Big(\varphi\partial_k\big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big) + \big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)\partial_k\varphi\Big)\,\mathrm{d}x\\
&\qquad - \frac{1}{1+\kappa_Y}\int_Q \big(\big(\partial_k\tfrac{1}{\mu}\big)\partial_2\varphi + \tfrac{1}{\mu}\partial_2\partial_k\varphi\big)\,\partial_2 w\,\mathrm{d}x
\end{aligned}$$

$$= \int_Q (\partial_k h)\varphi \,\mathrm{d}x - \int_Q \big(\partial_k\big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)\big) w\varphi \,\mathrm{d}x + \frac{1}{1+\kappa_Y} \left\langle \partial_2\big((\partial_k \tfrac{1}{\mu})\partial_2 w\big), \varphi \right\rangle_{D(\partial_2)^*\times D(\partial_2)},$$

using that $H_0^2(Q) \hookrightarrow D(\partial_2)$. The above identity is true for all $\varphi \in D(\partial_2)$ because $H_0^2(Q)$ is dense in this space (cf. the proof of Lemma 3.3). We have shown

$$L\partial_k w = \partial_k h - \partial_k\big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)w + \tfrac{1}{1+\kappa_Y}\partial_2\big((\partial_k \tfrac{1}{\mu})\partial_2 w\big) =: \psi(h) \tag{3.9}$$

in $D(\partial_2)^*$. We observe that the operator L is given by the symmetric, closed, positive definite and densely defined bilinear form

$$(w, \widetilde{w}) \mapsto \Big(\big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)w, \widetilde{w}\Big)_{L^2} + \frac{1}{1+\kappa_Y}\Big(\tfrac{1}{\mu}\partial_2 w, \partial_2 \widetilde{w}\Big)_{L^2} \tag{3.10}$$

on $D(\partial_2)$. The operator L is thus self-adjoint in X and $D(\partial_2) \cong D(L^{1/2})$ by Theorems VI.2.7 and VI.2.23 in [16]. We then deduce the isomorphism $D(\partial_2)^* \cong D(L^{1/2})^*$ and that $\partial_k w = L_{-1}^{-1}\psi(h)$ belongs to $D(\partial_2) \cong D(L^{1/2})$, see Theorem V.1.4.12 in [2]. As a result, w and $\partial_2 w$ are contained in $H^1(Q)$, and we already know that $w = 0$ on Γ_2.

3) To prove that $w = 0$ on Γ_3, we approximate h by functions $h_n = \chi_n^{(3)} h$ with $n > d_Q^{-1}$, cf. step 2) of the proof of Lemma 3.3. As in that proof one sees that h_n vanishes if $|x_3 - a_3^{\pm}| \le 1/(2n)$ and that (h_n) tends to h in $H^1(Q)$ as $n \to \infty$. We take a function $\phi_n \in C_c^\infty((a_3^-, a_3^+))$ that is equal to 1 on $[a_3^- + 1/(2n), a_3^+ - 1/(2n)]$ so that $h_n = \phi_n h_n$. The function $w_n := L^{-1}h_n \in D(D_2)$ converges to w in $D(\partial_2)$. We then obtain

$$Lw_n = h_n = \phi_n h_n = \phi_n L w_n = L(\phi_n w_n),$$

and hence $w_n = \phi_n w_n$ since L is injective. Therefore w_n vanishes on Γ_3. Equation (3.9) further yields

$$\begin{aligned}
\|\partial_k(w_n - w)\|_{L^2(Q)} &= \big\|L_{-1}^{-1}\psi(h_n - h)\big\|_{L^2(Q)} \le c\,\|\psi(h_n - h)\|_{D(\partial_2)^*} \\
&\le c\Big[\|\partial_k h_n - \partial_k h\|_{L^2} + \big\|\partial_k\big((1+\kappa_Y)\varepsilon + \tfrac{\sigma}{2}\big)(w_n - w)\big\|_{L^2(Q)} \\
&\qquad + \frac{1}{1+\kappa_Y}\Big\|\partial_k\big(\tfrac{1}{\mu}\big)\partial_2(w_n - w)\Big\|_{L^2(Q)}\Big]
\end{aligned}$$

for $k \in \{1,2,3\}$. The right-hand side tends to 0 as $n \to \infty$, so that w_n converges to w in $H^1(Q)$. As a result, w has the trace 0 on Γ_3. We set $u_1 = w$.

4) In a final step we define

$$v_3 = \tfrac{1}{1+\kappa_Y} g_3 + \tfrac{1}{1+\kappa_Y}\tfrac{1}{\mu}\partial_2 u_1 \in H^1(Q),$$

compare (3.7). The trace v_3 on Γ_3 vanishes since $(f, g) \in D(A_Y)$ and $\partial_2 u_1 = 0$ on Γ_3 by Lemma 2.1. We differentiate the above equation w.r.t. x_2 and insert the equations $Lu_1 = h_1$ and (3.8). It follows

$$\partial_2 v_3 = \frac{1}{1+\kappa_Y}\partial_2 g_3 + \big((1+\kappa_Y)\varepsilon + \tfrac{1}{2}\sigma\big)u_1 - h_1 = -\varepsilon f_1 + \big(\varepsilon(1+\kappa_Y) + \tfrac{\sigma}{2}\big)u_1$$

in $L^2(Q)$; i.e., u_1 and v_3 satisfy (3.7). This equation also yields that $\partial_2 v_3$ belongs to $H^1(Q)$ and has trace 0 on Γ_2, as required in $D(A_Y)$. The other components are treated in the same way. □

We now easily derive the basic properties of our split operators on Y by means of the Lumer–Phillips theorem, see e.g. Section II.3.b in [10]. Recall the definition of κ_Y in (3.6).

Proposition 3.6. *Let $L \in \{A, B, A^*, B^*\}$. The part L_Y of L in Y generates a C_0-semigroup on Y bounded by $\mathrm{e}^{\kappa_Y t}$. The resolvent $(I - \tau L_Y)^{-1}$ is the restriction of $(I - \tau L)^{-1}$ to Y and it satisfies*

$$\left\|(I - \tau L_Y)^{-1}\right\|_{\mathcal{B}(Y)} \le \frac{1}{1 - \tau\kappa_Y}$$

for all $0 < \tau < \frac{1}{\kappa_Y}$, so that $\left\|(I - \tau L_Y)^{-1}\right\|_{\mathcal{B}(Y)} \le 2$ for all $0 < \tau \le \frac{1}{2\kappa_Y}$. The Cayley transforms are dominated by

$$\|\gamma_\tau(L_Y)\|_{\mathcal{B}(Y)} \le \mathrm{e}^{3\kappa_Y \tau}$$

for all $0 < \tau \le \tau_0$ and a constant $\tau_0 \in (0, (2\kappa_Y)^{-1}]$ only depending on κ_Y.

Proof. The generation property and the resolvent bounds follow from Lemmas 3.3–3.5 and the Lumer–Phillips theorem. The proof of Lemma 3.5 implies the asserted restriction property. Let $0 < \tau < \frac{1}{\kappa_Y}$. For $\operatorname{Re} z > 0$ we define

$$\widetilde{\gamma}_\tau(z) = \frac{1 - \tau(z - \kappa_Y)}{1 + \tau(z - \kappa_Y)}.$$

It is easily seen that

$$\sup_{\operatorname{Re} z > 0} |\widetilde{\gamma}(z)| = \frac{1 + \tau\kappa_Y}{1 - \tau\kappa_Y} \le \mathrm{e}^{3\kappa_Y \tau}$$

for all $0 < \tau \le \widetilde{\tau}$ and a constant $\widetilde{\tau} \in (0, 1/\kappa_Y)$ only depending on κ_Y. Since $L_Y - \kappa_Y I$ generates a contraction semigroup on a Hilbert space, Theorem 11.5 of [17] provides a H^∞-functional calculus for $\kappa_Y I - L_Y$ and the desired estimate

$$\|\gamma_\tau(L_Y)\|_{\mathcal{B}(Y)} = \|\widetilde{\gamma}_\tau(\kappa_Y I - L_Y)\|_{\mathcal{B}(Y)} \le \sup_{\operatorname{Re} z > 0} |\widetilde{\gamma}_\tau(z)| \le \mathrm{e}^{3\kappa_Y \tau}$$

for all $\tau \in (0, \widetilde{\tau})$, where we recall (3.4). We set $\tau_0 = \min\{\widetilde{\tau}, (2\kappa_Y)^{-1}\}$. □

4. The ADI splitting scheme

Let $\tau > 0$. We set $t_n := n\tau$ for $n \in \mathbb{N}_0$ and assume that $(\mathbf{J}(t), 0) \in D(A)$ for all $t \ge 0$. The ADI splitting scheme is given by the operators

$$\begin{aligned} S^J_{\tau,n+1}\tilde{w} &= S^{(2)}_\tau \Big[S^{(1)}_\tau \tilde{w} - \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0) \Big], \\ S^{(1)}_\tau &= (I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B) : D(B) \to D(A), \\ S^{(2)}_\tau &= (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A) : D(A) \to D(B), \end{aligned} \tag{4.1}$$

for $\tilde{w} \in D(B)$. Note that $(\frac{1}{\varepsilon}\mathbf{J}(t), 0) \in D(A)$. For $n \in \mathbb{N}_0$ and $w_0 \in D(B)$, we further write

$$(\mathbf{E}_n, \mathbf{H}_n) = w_n = S^J_{\tau,n} \cdots S^J_{\tau,1} w_0,$$

$$(\mathbf{E}_{n+1/2}, \mathbf{H}_{n+1/2}) = w_{n+1/2} = S_\tau^{(1)} S_{\tau,n}^J \cdots S_{\tau,1}^J w_0.$$

Remark 4.1. Let $w_0 \in D(B_Y)$ and $(\frac{1}{\varepsilon}\mathbf{J}(t), 0) \in D(A_Y)$ for all $t \in \mathbb{R}$. Proposition 3.6 then yields $w_n \in D(B_Y)$ and $w_{n+1/2} \in D(A_Y)$ for all $n \in \mathbb{N}_0$.

The operators $S_\tau^{(k)}$ contain implicit steps. For $\sigma = 0$ and $\mathbf{J} = 0$ it was pointed out in [20] and [26] that these steps decouple into (essentially) one dimensional problems, see also [13]. We now extend this observation to our setting. To this aim, for $\lambda \in \{\varepsilon, \mu\}$ we define the operators

$$\begin{aligned} D_\lambda^{(1)} &:= C_1 \tfrac{1}{\lambda} C_2 = \begin{pmatrix} \partial_2 \frac{1}{\lambda} \partial_2 & 0 & 0 \\ 0 & \partial_3 \frac{1}{\lambda} \partial_3 & 0 \\ 0 & 0 & \partial_1 \frac{1}{\lambda} \partial_1 \end{pmatrix} \qquad \text{and} \\ D_\lambda^{(2)} &:= C_2 \tfrac{1}{\lambda} C_1 = \begin{pmatrix} \partial_3 \frac{1}{\lambda} \partial_3 & 0 & 0 \\ 0 & \partial_1 \frac{1}{\lambda} \partial_1 & 0 \\ 0 & 0 & \partial_2 \frac{1}{\lambda} \partial_2 \end{pmatrix} \end{aligned} \tag{4.2}$$

on the domains $D(\partial_{22}) \times D(\partial_{33}) \times D(\partial_{11})$ and $D(\partial_{33}) \times D(\partial_{11}) \times D(\partial_{22})$, respectively, where $D(\partial_{kk})$ is the set of $f \in L^2(Q)$ with $\partial_{kk} f, \partial_k f \in L^2(Q)$ and $f = 0$ on Γ_k.

Let $w_0 \in D(B_Y)$ and $(\frac{1}{\varepsilon}\mathbf{J}(t), 0) \in D(A_Y)$ for all $t \in \mathbb{R}$. Remark 4.1 then yields $(\mathbf{E}_n, \mathbf{H}_n) \in D(B_Y)$ for each $n \in \mathbb{N}$. The above definitions lead to

$$\begin{aligned} \big(1 + \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+1/2} &= \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_n - \tfrac{\tau}{2\varepsilon} C_2 \mathbf{H}_n + \tfrac{\tau}{2\varepsilon} C_1 \mathbf{H}_{n+1/2}, \\ \mathbf{H}_{n+1/2} &= \mathbf{H}_n - \tfrac{\tau}{2\mu} C_1 \mathbf{E}_n + \tfrac{\tau}{2\mu} C_2 \mathbf{E}_{n+1/2}, \end{aligned}$$

with $(\mathbf{E}_{n+1/2}, \mathbf{H}_{n+1/2}) \in D(A_Y)$. Because of this regularity, we can insert the second equation into the first one and infer

$$\begin{aligned} \big((1 + \tfrac{\sigma\tau}{4\varepsilon})I - \tfrac{\tau^2}{4\varepsilon} D_\mu^{(1)}\big)\mathbf{E}_{n+1/2} &= \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_n + \tfrac{\tau}{2\varepsilon} \operatorname{curl} \mathbf{H}_n - \tfrac{\tau^2}{4\varepsilon} C_1 \tfrac{1}{\mu} C_1 \mathbf{E}_n, \\ \mathbf{H}_{n+1/2} &= \mathbf{H}_n - \tfrac{\tau}{2\mu} C_1 \mathbf{E}_n + \tfrac{\tau}{2\mu} C_2 \mathbf{E}_{n+1/2} \end{aligned} \tag{4.3}$$

in $L^2(Q)^3$, using $\operatorname{curl} = C_1 - C_2$. Observe that $\mathbf{E}_{n+1/2}$ belongs to the domain of $D_\mu^{(1)}$. Due to (4.2), the implicit part of these equations splits into (essentially) one dimensional problems; one only has to solve parameter-dependent elliptic equations in one space variable. Similarly, the second half step is rewritten as

$$\begin{aligned} \big(1 + \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+1} &= \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+\frac{1}{2}} + \tfrac{\tau}{2\varepsilon} C_1 \mathbf{H}_{n+\frac{1}{2}} - \tfrac{\tau}{2\varepsilon} C_2 \mathbf{H}_{n+1} \\ &\quad - \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})), \\ \mathbf{H}_{n+1} &= \mathbf{H}_{n+\frac{1}{2}} + \tfrac{\tau}{2\mu} C_2 \mathbf{E}_{n+\frac{1}{2}} - \tfrac{\tau}{2\mu} C_1 \mathbf{E}_{n+1} - \tfrac{\tau}{2\mu} C_2 \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})). \end{aligned}$$

As above we then obtain the essentially one-dimensional problem

$$\begin{aligned} \big((1 + \tfrac{\sigma\tau}{4\varepsilon})I - \tfrac{\tau^2}{4\varepsilon} D_\mu^{(2)}\big)\mathbf{E}_{n+1} &= \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+\frac{1}{2}} + \tfrac{\tau}{2\varepsilon} \operatorname{curl} \mathbf{H}_{n+\frac{1}{2}} \\ &\quad - \tfrac{\tau^2}{4\varepsilon} C_2 \tfrac{1}{\mu} C_2 \mathbf{E}_{n+\frac{1}{2}} - \big(1 - \tfrac{\sigma\tau}{4\varepsilon}\big)\tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})) \\ &\quad + \tfrac{\tau^3}{8\varepsilon} C_2 \tfrac{1}{\mu} C_2 \tfrac{1}{\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})), \\ \mathbf{H}_{n+1} &= \mathbf{H}_{n+\frac{1}{2}} + \tfrac{\tau}{2\mu} C_2 \mathbf{E}_{n+\frac{1}{2}} - \tfrac{\tau}{2\mu} C_1 \mathbf{E}_{n+1} \\ &\quad - \tfrac{\tau}{2\mu} C_2 \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})). \end{aligned} \tag{4.4}$$

Using the Cayley transforms $\gamma_\tau(L)$ from (3.4) and induction, we further deduce from (4.1) the closed expression of the scheme

$$w_n = (I - \tfrac{\tau}{2}B)^{-1}\gamma_\tau(A)[\gamma_\tau(B)\gamma_\tau(A)]^{n-1}(I + \tfrac{\tau}{2}B)w_0 \qquad (4.5)$$
$$\quad - (I - \tfrac{\tau}{2}B)^{-1}\sum_{k=1}^{n}[\gamma_\tau(A)\gamma_\tau(B)]^{n-k}(I + \tfrac{\tau}{2}A)\big(\tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_{k-1}) + \mathbf{J}(t_k)), 0\big).$$

Propositions 3.1 and 3.6 then easily yield the unconditional stability of the scheme in X and Y. Recall the definition of $\kappa_Y \ge 0$ in (3.6) and that of $\tau_0 > 0$ in Proposition 3.6. Both depend only on the constants in (1.3).

Theorem 4.2. *Let* (1.3) *hold,* $n \in \mathbb{N}$, $\tau \in (0, 1]$ *and* $T \ge n\tau$. *Take* $w_0 \in D(B)$ *and* $(\boldsymbol{J}, 0) \in C([0,\infty), D(A))$. *We then have*

$$\|w_n\|_{L^2} \le c\,\|w_0\|_B + cT \max_{t\in[0,T]} \|(\boldsymbol{J}, 0)\|_A\,,$$
$$\|(I - \tfrac{\tau}{2}B)w_n\|_X \le \|(I + \tfrac{\tau}{2}B)w_0\|_X + T \max_{t\in[0,T]} \|(I + \tfrac{\tau}{2}A)(\tfrac{1}{\varepsilon}\boldsymbol{J}(t), 0)\|_X\,.$$

If $0 < \tau \le \tau_0$, $w_0 \in D(B_Y)$ *and* $(\frac{1}{\varepsilon}\boldsymbol{J}, 0) \in C([0,\infty), D(A_Y))$, *we obtain*

$$\|w_n\|_{H^1} \le c\mathrm{e}^{6\kappa_Y T}\big(\|w_0\|_{B_Y} + T \max_{t\in[0,T]} \|(\tfrac{1}{\varepsilon}\boldsymbol{J}(t), 0)\|_{A_Y}\big),$$
$$\|(I - \tfrac{\tau}{2}B)w_n\|_Y \le \mathrm{e}^{6\kappa_Y T}\big(\|(I+\tfrac{\tau}{2}B_Y)w_0\|_{H^1} + T \max_{t\in[0,T]} \|(I+\tfrac{\tau}{2}A_Y)(\tfrac{1}{\varepsilon}\boldsymbol{J}(t), 0)\|_{H^1}\big).$$

The constants $c > 0$ *only depend on the constants from* (1.3).

Remark 4.3. a) In the above result we can drop the factor $\frac{1}{\varepsilon}$ in the assumptions if ε also belongs to $W^{2,3}(Q)$ since $H^1(Q) \hookrightarrow L^6(Q)$.

b) In Theorem 4.2, for $\sigma = 0$ and $\mathbf{J} = 0$ the inequality in X is actually an equality since then the operators A and B are skew-adjoint in X by Lemma 4.3 of [13], and hence their Cayley transforms are unitary in X. This can be viewed as a modified energy preservation of the scheme in the conservative case.

5. Convergence of the ADI scheme

For the semigroups from Proposition 2.3, $\tau \in (0, 1]$ and $j \in \mathbb{N}_0$, we define

$$\Lambda_{j+1}(\tau) = \frac{1}{j!\tau^{j+1}}\int_0^\tau s^j \mathrm{e}^{(\tau-s)M}\,\mathrm{d}s \quad \text{and} \quad \Lambda_{j+1}^{\mathrm{div}}(\tau) = \frac{1}{j!\tau^{j+1}}\int_0^\tau s^j \mathrm{e}^{(\tau-s)M_{\mathrm{div}}}\,\mathrm{d}s,$$

as well as $\Lambda_0(\tau) = \mathrm{e}^{\tau M}$ and $\Lambda_0^{\mathrm{div}}(\tau) = \mathrm{e}^{\tau M_{\mathrm{div}}}$. By Proposition 2.3, these operators are uniformly bounded on X, respectively X_{div}, and $\Lambda_j^{\mathrm{div}}(\tau)$ is the restriction of $\Lambda_j(\tau)$ to X_{div}. Standard semigroup theory shows that the operators leave invariant $D(M^k)$, respectively $D(M_{\mathrm{div}}^k)$, and commute with M^k, respectively M_{div}^k. For $j \ge k$ they actually map into $D(M^k)$, respectively $D(M_{\mathrm{div}}^k)$. One can further check that

$$\Lambda_j(\tau) = \frac{1}{j!}I + \tau M\Lambda_{j+1}(\tau) \quad \text{and} \quad \Lambda_j^{\mathrm{div}}(\tau) = \frac{1}{j!}I + \tau M_{\mathrm{div}}\Lambda_{j+1}^{\mathrm{div}}(\tau). \qquad (5.1)$$

Our first main result establishes the second order convergence of the ADI scheme in Y^*. According to (3.6) the number $\kappa_Y \ge 0$ only depends on the constants in (1.3), and we have $\kappa_Y = 0$ in the case of constant coefficients. We use the number $\tau_0 > 0$ from Proposition 3.6, which only depends on κ_Y.

Theorem 5.1. *Let* (1.3) *hold,* $T > 0$, $0 < \tau \le \min\{1, \tau_0\}$, $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(M_{\rm div}^2)$ *and* $(\boldsymbol{J}, 0)$ *belong to* $E := C([0,T], D(M_{\rm div})) \cap W^{2,1}([0,T], X_{\rm div})$. *Let* $w = (\boldsymbol{E}, \boldsymbol{H})$ *be the solution of* (1.1) *and* $w_n = S_{\tau,n}^J \cdots S_{\tau,1}^J w_0$ *be its approximation from* (4.1). *For all* $n\tau \le T$ *and* $(\varphi, \psi) \in Y$, *we then have*

$$|(w_n - w(n\tau)|(\varphi,\psi))_X| \le c\tau^2(1+T)^2 \mathrm{e}^{6\kappa_Y T}\big(\|w_0\|_{D(M_{\rm div}^2)} + \|(\boldsymbol{J},0)\|_E\big)\,\|(\varphi,\psi)\|_Y\,.$$

The constant $c > 0$ *only depends on the constants from* (1.3).

Proof. Proposition 2.3 yields a solution $w \in C([0,T], D(M_{\rm div}^2))$ of (1.1). Recall from Proposition 3.1 that $D(M_{\rm div}) \hookrightarrow D(A) \cap D(B)$. So the scheme (4.1) is well-defined. The properties of A, B and M contained in (3.5) and Propositions 2.2, 2.3, 3.1 and 3.6 are freely used below. We start from the Taylor expansion

$$\big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau+s), 0\big) = \Big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau) + \tfrac{s}{\varepsilon}\mathbf{J}'(n\tau) + \int_{n\tau}^{n\tau+s} (n\tau + s - r)\tfrac{1}{\varepsilon}\mathbf{J}''(r)\,\mathrm{d}r, 0\Big) \tag{5.2}$$

in $X_{\rm div}$ for $n\tau + s \in [0,T]$. This equation, Duhamel's formula (2.6) and the above definitions yield the expression

$$w((n+1)\tau) = \Lambda_0(\tau)w(n\tau) + \tau\Lambda_1(\tau)\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) + \tau^2\Lambda_2(\tau)\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau), 0\big) + R_n(\tau) \tag{5.3}$$

for the solution, where

$$R_n(\tau) = \int_0^\tau \mathrm{e}^{(\tau-s)M}\Big(\int_{n\tau}^{n\tau+s} (n\tau+s-r)\big(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0\big)\,\mathrm{d}r\Big)\,\mathrm{d}s.$$

We insert (5.2) for $s = \tau$ into the definition of $S_{\tau,n+1}^J$ from (4.1) with $\tilde{w} = S_{\tau,n}^J \cdots S_{\tau,1}^J w_0 \in D(B)$ and obtain

$$\begin{aligned} S_{\tau,n+1}^J S_{\tau,n}^I \cdots S_{\tau,1}^J w_0 = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\Big[&(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)S_{\tau,n}^I \cdots S_{\tau,1}^I w_0 \\ &+ \tau\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) + \tfrac{1}{2}\tau^2\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau), 0\big)\Big] \\ &+ (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)r_n(\tau) \end{aligned} \tag{5.4}$$

with the remainder

$$r_n(\tau) = \frac{\tau}{2}\int_{n\tau}^{(n+1)\tau} ((n+1)\tau - r)\big(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0\big)\,\mathrm{d}r.$$

In the next step we take the inner product in X of the fields $z = (\varphi, \psi) \in Y$ with the difference of (5.4) and (5.3). In the following we write A_Y^* and B_Y^* instead of $(A^*)_Y$ and $(B^*)_Y$, respectively. Putting several operators as adjoints on the side of z, we arrive at the formula

$$\begin{aligned} &\big(S_{\tau,n+1}^I \cdots S_{\tau,1}^I w_0 - w((n+1)\tau)\,\big|\,(\varphi,\psi)\big)_X \\ &= \big(S_{\tau,n}^I \cdots S_{\tau,1}^I w_0 - w(n\tau)\,\big|\,(I + \tfrac{\tau}{2}B^*)(I + \tfrac{\tau}{2}A_Y^*)(I - \tfrac{\tau}{2}A_Y^*)^{-1}(I - \tfrac{\tau}{2}B_Y^*)^{-1}z\big)_X \\ &\quad + \Big(w(n\tau)\,\big|\,\big[(I + \tfrac{\tau}{2}B^*)(I + \tfrac{\tau}{2}A_Y^*) - \Lambda_0(\tau)^*(I - \tfrac{\tau}{2}B^*)(I - \tfrac{\tau}{2}A_Y^*)\big] \\ &\qquad\qquad \cdot (I - \tfrac{\tau}{2}A_Y^*)^{-1}(I - \tfrac{\tau}{2}B_Y^*)^{-1}(\varphi,\psi)\Big)_X \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&\quad + \tau\Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau),0\big)\,\big|\,\big[(I+\tfrac{\tau}{2}A^*)(I-\tfrac{\tau}{2}B_Y^*)^{-1}-\Lambda_1(\tau)^*\big](\varphi,\psi)\Big)_X\\
&\quad + \tau^2\Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau),0\big)\,\big|\,\big[\tfrac12(I+\tfrac{\tau}{2}A^*)(I-\tfrac{\tau}{2}B_Y^*)^{-1}-\Lambda_2(\tau)^*\big](\varphi,\psi)\Big)_X\\
&\quad + \big(r_n(\tau)\,\big|\,(I+\tfrac{\tau}{2}A^*)(I-\tfrac{\tau}{2}B_Y^*)^{-1}(\varphi,\psi)\big)_X-(R_n(\tau)\,|\,(\varphi,\psi))_X\\
&=: \big(\big[S_{\tau,n}^I\cdots S_{\tau,1}^I w_0-w(n\tau)\big]\,\big|\,(I+\tfrac{\tau}{2}B^*)\gamma_\tau(A_Y^*)(I-\tfrac{\tau}{2}B_Y^*)^{-1}z\big)_X\\
&\quad + \Sigma_1(\tau)+\Sigma_2(\tau)+\Sigma_3(\tau)+\big(r_n(\tau)\,\big|\,(I+\tfrac{\tau}{2}A^*)(I+\tfrac{\tau}{2}B_Y^*)^{-1}(\varphi,\psi)\big)_X\\
&\quad - \big(R_n(\tau)\,\big|\,(\varphi,\psi)\big)_X,
\end{aligned}
$$

where we used that $I+\frac{\tau}{2}A^*$ and $(I-\frac{\tau}{2}A^*)^{-1}$ commute on $Y\hookrightarrow D(A^*)\cap D(B^*)$. We abbreviate

$$
\chi(\tau)=(I-\tfrac{\tau}{2}A_Y^*)^{-1}(I-\tfrac{\tau}{2}B_Y^*)^{-1}(\varphi,\psi)\in D(A_Y^*).
$$

Since $M^*=A^*+B^*$ on Y, we obtain

$$
\Sigma_1(\tau)=\Big(w(n\tau)\Big|\big[(I-\Lambda_0(\tau)^*)+\tfrac{\tau}{2}(I+\Lambda_0(\tau)^*)M^*+\tfrac{\tau^2}{4}(I-\Lambda_0(\tau)^*)B^*A_Y^*\big]\chi(\tau)\Big)_X.
$$

By means of (5.1) we expand

$$
\begin{aligned}
&I-\Lambda_0(\tau)^*=-\tau M^*-\tfrac12\tau^2(M^*)^2-\tau^3\Lambda_3(\tau)^*(M^*)^3 \qquad \text{on } D\big((M^*)^3\big),\\
&I+\Lambda_0(\tau)^*=2I+\tau M^*+\tau^2\Lambda_2(\tau)^*(M^*)^2 \qquad \text{on } D\big((M^*)^2\big),\\
&I-\Lambda_0(\tau)^*=-\tau\Lambda_1(\tau)^*M^* \qquad \text{on } D(M^*).
\end{aligned}
$$

Because of $Y\hookrightarrow D(B^*)$, $w(n\tau)\in D(M_{\mathrm{div}}^2)\hookrightarrow D(M^2)$ and the crucial embedding $D(M_{\mathrm{div}})\hookrightarrow D(A)\cap D(B)$ from Proposition 3.1, the above expansions yield

$$
\begin{aligned}
\Sigma_1(\tau)&=\Big\langle w(n\tau),\Big(-\tau^3\Lambda_3(\tau)_{-2}^*M_{-2}^*M_{-1}^*M^*+\tfrac{\tau^3}{2}\Lambda_2(\tau)_{-2}^*M_{-2}^*M_{-1}^*M^*\\
&\qquad\qquad -\tfrac{\tau^3}{4}\Lambda_1(\tau)_{-1}^*M_{-1}^*B^*A_Y^*\Big)\chi(\tau)\Big\rangle_{D(M^2)\times X_{-2}^{M^*}}\\
&=\tau^3\Big(\big[-M^2\Lambda_3(\tau)+\tfrac12M^2\Lambda_2(\tau)\big]w(n\tau)\,\big|\,M^*\chi(\tau)\Big)_X\\
&\quad -\tau^3\big(\tfrac14BM_{\mathrm{div}}\Lambda_1^{\mathrm{div}}(\tau)w(n\tau)\,\big|\,A^*\chi(\tau)\big)_X,
\end{aligned}
$$

where we employ the extrapolation space $X_{-2}^{M^*}\cong D(M^2)^*$, see [2] or [10]. We rewrite the term $\Sigma_2(\tau)$ as

$$
\begin{aligned}
&\tau\Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau),0\big)\,\big|\,\big[(I+\tfrac{\tau}{2}A^*)(I-\tfrac{\tau}{2}A_Y^*)-\Lambda_1(\tau)^*(I-\tfrac{\tau}{2}B^*)(I-\tfrac{\tau}{2}A_Y^*)\big]\chi(\tau)\Big)_X\\
&=\tau\Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau),0\big)\,\big|\,\big[I-\tfrac{\tau^2}{4}A^*A_Y^*-\Lambda_1(\tau)^*\big(I-\tfrac{\tau}{2}(A^*+B^*)\big)\\
&\qquad\qquad -\tfrac{\tau^2}{4}\Lambda_1(\tau)^*B^*A_Y^*\big]\chi(\tau)\Big)_X.
\end{aligned}
$$

Plugging in first the equation $\Lambda_1(\tau)^*=I+\tau\Lambda_2(\tau)^*M^*$ and then $\Lambda_2(\tau)^*=\frac12I+\tau\Lambda_3(\tau)^*M^*$ from (5.1), it follows

$$
\begin{aligned}
\Sigma_2(\tau)=\tau\Big\langle\big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau),0\big),\big[&-\tfrac{\tau^2}{4}A^*A_Y^*-\tau\Lambda_2(\tau)^*M^*+\tfrac{\tau}{2}M^*\\
&+\tfrac{\tau^2}{2}\Lambda_2(\tau)_{-1}^*M_{-1}^*M^*-\tfrac{\tau^2}{4}\Lambda_1(\tau)^*B^*A_Y^*\big]\chi(\tau)\Big\rangle_{D(M)\times X_{-1}^{M^*}}
\end{aligned}
$$

$$= \tau \Big\langle \big(-\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big), \big[-\tfrac{\tau^2}{4} A^* A_Y^* - \tau^2 \Lambda_3(\tau)_{-1}^* M_{-1}^* M^* + \tfrac{\tau^2}{2} \Lambda_2(\tau)_{-1}^* M_{-1}^* M^*$$
$$- \tfrac{\tau^2}{4} \Lambda_1(\tau)^* B^* A_Y^* \big] \chi(\tau) \Big\rangle_{D(M) \times X_{-1}^{M^*}}$$
$$= \tau^3 \big(\tfrac{1}{4} A \big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) \big| A^* \chi(\tau)\big)_X + \tau^3 \big(M \Lambda_3(\tau) \big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) \big| M^* \chi(\tau)\big)_X$$
$$-\tau^3 \big(\tfrac{1}{2} M \Lambda_2(\tau) \big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) \big| M^* \chi(\tau)\big)_X + \tau^3 \big(\tfrac{1}{4} B \Lambda_1^{\mathrm{div}}(\tau) \big(\tfrac{1}{\varepsilon}\mathbf{J}(n\tau), 0\big) \big| A^* \chi(\tau)\big)_X.$$

Similarly, (5.1) implies

$$\Sigma_3(\tau) = \tau^2 \Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau), 0\big) \mid \big[\tfrac{1}{2}(I + \tfrac{\tau}{2} A^*) - \Lambda_2(\tau)^* (I - \tfrac{\tau}{2} B^*)\big] (I - \tfrac{\tau}{2} B_Y^*)^{-1} z \Big)_X$$
$$= \tau^2 \Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau), 0\big) \mid \big[\tfrac{1}{2} I + \tfrac{\tau}{4} A^* - \big(\tfrac{1}{2} I + \tau \Lambda_3(\tau)^* M^*\big) + \tfrac{1}{2}\Lambda_2(\tau)^* B^* \big]$$
$$\cdot (I - \tfrac{\tau}{2} B_Y^*)^{-1} z \Big)_X$$
$$= \tau^3 \Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(n\tau), 0\big) \mid \big[\tfrac{1}{4} A^* - \Lambda_3(\tau)^* M^* + \tfrac{1}{2}\Lambda_2(\tau)^* B^* \big] (I - \tfrac{\tau}{2} B_Y^*)^{-1} z \Big)_X.$$

We recursively insert the above expressions in (5.5) and obtain (omitting the subscript Y several times)

$$\big(S_{\tau,n}^I \cdots S_{\tau,1}^I w(0) - w(n\tau) \mid (\varphi,\psi)\big)_X \tag{5.6}$$
$$= \tau^3 \sum_{k=0}^{n-1} \Big(\big[-M^2 \Lambda_3(\tau) + \tfrac{1}{2} M^2 \Lambda_2(\tau) \big] w(k\tau) \mid$$
$$M^* (I - \tfrac{\tau}{2} A^*)^{-1} \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k} (I - \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$
$$- \tau^3 \sum_{k=0}^{n-1} \Big(\tfrac{1}{4} B M_{\mathrm{div}} \Lambda_1^{\mathrm{div}}(\tau) w(k\tau) \mid A^* (I - \tfrac{\tau}{2} A^*)^{-1} \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k}$$
$$\cdot (I + \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$
$$- \tau^3 \sum_{k=0}^{n-1} \Big(\big[\tfrac{1}{4} A + \tfrac{1}{4} B \Lambda_1^{\mathrm{div}}(\tau) \big] \big(-\tfrac{1}{\varepsilon} \mathbf{J}(k\tau), 0 \big) \mid$$
$$A^* (I - \tfrac{\tau}{2} A^*)^{-1} \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k} (I + \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$
$$+ \tau^3 \sum_{k=0}^{n-1} \Big(\big[\tfrac{1}{2} M \Lambda_2(\tau) - M \Lambda_3(\tau) \big] \big(-\tfrac{1}{\varepsilon} \mathbf{J}(k\tau), 0 \big) \mid M^* (I - \tfrac{\tau}{2} A^*)^{-1}$$
$$\cdot \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k} (I - \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$
$$+ \tau^3 \sum_{k=0}^{n-1} \Big(\big(-\tfrac{1}{\varepsilon} \mathbf{J}'(k\tau), 0 \big) \mid \big[\tfrac{1}{4} A^* - \Lambda_3(\tau)^* M^* + \tfrac{1}{2} \Lambda_2(\tau)^* B^* \big]$$
$$\cdot \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k} (I - \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$
$$+ \sum_{k=0}^{n-1} \Big(r_k(\tau) \mid (I + \tfrac{\tau}{2} A^*) \big[\gamma_{\tau/2}(B)^* \gamma_{\tau/2}(A)^* \big]^{n-1-k} (I - \tfrac{\tau}{2} B^*)^{-1} (\varphi,\psi) \Big)_X$$

$$+\sum_{k=0}^{n-2}\Big(R_k(\tau)\,\big|\,(I+\tfrac{\tau}{2}B^*)\gamma_{\tau/2}(A)^*\big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-2-k}(I-\tfrac{\tau}{2}B^*)^{-1}z\Big)_X$$
$$+\big(R_{n-1}(\tau)\,\big|\,(\varphi,\psi)\big)_X.$$

The terms $r_k(\tau)$ and $R_k(\tau)$ are bounded in X by $c\tau^2\int_{k\tau}^{(k+1)\tau}\big\|(\mathbf{J}''(s),0)\big\|_X\,\mathrm{d}s$. Propositions 2.2, 2.3, 3.1 and 3.6 then imply

$$\begin{aligned}
&\big|\big(S^I_{\tau,n}\cdots S^I_{\tau,1}w_0-w(n\tau)\,\big|\,(\varphi,\psi)\big)_{L^2}\big|\\
&\le c\tau^3\mathrm{e}^{6\kappa_Y n\tau}\sum_{k=0}^{n-1}\Big(\|w(k\tau)\|_{D(M^2_{\mathrm{div}})}+\|(\mathbf{J}(k\tau),0)\|_{D(M_{\mathrm{div}})}+\big\|(\mathbf{J}'(k\tau),0)\big\|_X\\
&\qquad\qquad+\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}\big\|(\mathbf{J}''(s),0)\big\|_X\,\mathrm{d}s\Big)\,\|(\varphi,\psi)\|_{H^1}\\
&\le c\tau^2(1+T)^2\mathrm{e}^{6\kappa_Y T}\,\big(\|w_0\|_{D(M^2_{\mathrm{div}})}+\|(\mathbf{J},0)\|_E\big)\,\|(\varphi,\psi)\|_{H^1},
\end{aligned}$$

where we use $\tau\le 1$ and c only depends on the constants from (1.3). □

6. Almost preservation of the divergence conditions in H^{-1}

The solution $(\mathbf{E},\mathbf{H})$ of (1.1) fulfills the Gaussian laws (2.7) and $\operatorname{div}(\mu\mathbf{H}(t))=0$. We now show that the scheme (4.1) satisfies a discrete analogue of these divergence conditions up to an error of order τ in $H^{-1}(Q)$. We recall that the numbers $\kappa_Y\ge 0$ and $\tau_0>0$ from (3.6) and Proposition 3.6 only depend on the constants in (1.3), and that $\kappa_Y=0$ if the coefficients are constant.

Theorem 6.1. *Let* (1.3) *hold,* $T>0$, $\tau\in(0,\min\{1,\tau_0\}]$, $n\in\mathbb{N}_0$, *and* $n\tau\le T$. *Take* $w_0=(\boldsymbol{E}_0,\boldsymbol{H}_0)$ *in* $D(B_Y)$ *and* $(\frac{1}{\varepsilon}\boldsymbol{J},0)$ *in* $C([0,T],D(A_Y))\cap C^1([0,T],X)$. *Let* $w_n=(\boldsymbol{E}_n,\boldsymbol{H}_n)$ *be given by* (4.1). *We then have*

$$\begin{aligned}
&\Big\|\big(\operatorname{div}(\varepsilon\boldsymbol{E}_N),\operatorname{div}(\mu\boldsymbol{H}_N)\big)-\big(\operatorname{div}(\varepsilon\boldsymbol{E}_0),0\big)\\
&\qquad+\sum_{k=0}^{N-1}\tfrac{\tau}{2}\big(\operatorname{div}(\tfrac{\sigma}{2}\boldsymbol{E}_{k+1}+\sigma\boldsymbol{E}_{k+1/2}+\tfrac{\sigma}{2}\boldsymbol{E}_k),0\big)+\int_0^{N\tau}\big(\operatorname{div}(\boldsymbol{J}(s)),0\big)\,\mathrm{d}s\Big\|_{H^{-1}}\\
&\le c\tau\mathrm{e}^{6\kappa_Y T}\Big[\|w_0\|_{H^1}+\tau\|B_Yw_0\|_{H^1}+T\max_{t\in[0,T]}\big(\|(\boldsymbol{J}(t),0)\|_{H^1}+\tau\|A_Y(\tfrac{1}{\varepsilon}\boldsymbol{J}(t),0)\|_{H^1}\big)\Big]\\
&\qquad+c\tau\int_0^T\big\|(\boldsymbol{J}'(s),0)\big\|_{L^2}\,\mathrm{d}s
\end{aligned}$$

for a constant $c\ge 0$ *only depending on the constants in* (1.3).

In the proof, see (6.4), we will see that the integral on the left-hand side of the above inequality can be replaced by the sum

$$\sum_{k=0}^{N-1}\tfrac{\tau}{2}\big(\operatorname{div}(\mathbf{J}(t_k)+\mathbf{J}(t_{k+1})),0\big).$$

Moreover, as in Remark 4.3 one can drop the factor $\frac{1}{\varepsilon}$ in the assumption if ε also belongs to $W^{2,3}(Q)$.

Proof. We first derive a recursion formula for the divergence of w_n which is then estimated by means of Propositions 3.1 and 3.6. Remark 4.1 says that w_n belongs to $D(B_Y)$ and $w_{n+1/2}$ to $D(A_Y)$. We often use this regularity, these propositions and that $\tau \le 1$ without further notice. Take $n \in \mathbb{N}_0$ with $n+1 \le T/\tau$.

1) As before (4.3), the definition (4.1) yields

$$\begin{pmatrix} (1+\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} \\ \mathbf{H}_{n+1/2} \end{pmatrix} - \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_1\mathbf{H}_{n+1/2} \\ \frac{1}{\mu}C_2\mathbf{E}_{n+1/2} \end{pmatrix} = \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_n \\ \mathbf{H}_n \end{pmatrix} - \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_2\mathbf{H}_n \\ \frac{1}{\mu}C_1\mathbf{E}_n \end{pmatrix}$$

in Y. We insert the second line in the first one to eliminate $\mathbf{H}_{n+1/2}$ and the first line in the second one to eliminate $\mathbf{E}_{n+1/2}$, obtaining

$$\begin{aligned}\begin{pmatrix} (1+\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} \\ \mathbf{H}_{n+1/2} \end{pmatrix} &= \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_1\big[\frac{\tau}{2\mu}C_2\mathbf{E}_{n+1/2} + \mathbf{H}_n - \frac{\tau}{2\mu}C_1\mathbf{E}_n\big] \\ \frac{1}{\mu}C_2\big[\frac{\tau}{2\varepsilon}C_1\mathbf{H}_{n+1/2} + (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_n - \frac{\tau}{2\varepsilon}C_2\mathbf{H}_n\big] \end{pmatrix} \\ &\quad - \frac{\tau}{2}\begin{pmatrix} 0 \\ \frac{1}{\mu}C_2\frac{\sigma\tau}{4\varepsilon}\mathbf{E}_{n+1/2} \end{pmatrix} + \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_n \\ \mathbf{H}_n \end{pmatrix} - \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_2\mathbf{H}_n \\ \frac{1}{\mu}C_1\mathbf{E}_n \end{pmatrix}\end{aligned}$$

in $L^2(Q)^6$. Using $\operatorname{curl} = C_1 - C_2$ and the definition (4.2), we reorder the above identity and infer the first half of the recursion step

$$\begin{aligned}\begin{pmatrix} \varepsilon\mathbf{E}_{n+\frac12} - \frac{\tau^2}{4}D_\mu^{(1)}\mathbf{E}_{n+\frac12} \\ \mu\mathbf{H}_{n+\frac12} - \frac{\tau^2}{4}D_\varepsilon^{(2)}\mathbf{H}_{n+\frac12} \end{pmatrix} &= \begin{pmatrix} \varepsilon\mathbf{E}_n - \frac{\tau^2}{4}C_1\frac{1}{\mu}C_1\mathbf{E}_n \\ \mu\mathbf{H}_n - \frac{\tau^2}{4}C_2\frac{1}{\varepsilon}C_2\mathbf{H}_n \end{pmatrix} - \frac{\tau}{2}\begin{pmatrix} 0 \\ C_2\frac{\sigma\tau}{4\varepsilon}(\mathbf{E}_{n+\frac12} + \mathbf{E}_n) \end{pmatrix} \\ &\quad - \begin{pmatrix} \frac{\sigma\tau}{4}(\mathbf{E}_{n+\frac12} + \mathbf{E}_n) \\ 0 \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \operatorname{curl}\mathbf{H}_n \\ -\operatorname{curl}\mathbf{E}_n \end{pmatrix} \end{aligned} \tag{6.1}$$

in $L^2(Q)^6$. Similarly, (4.1) leads to the expression

$$\begin{aligned}&\begin{pmatrix} (1+\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1} \\ \mathbf{H}_{n+1} \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_2\mathbf{H}_{n+1} \\ \frac{1}{\mu}C_1\mathbf{E}_{n+1} \end{pmatrix} \\ &\quad= \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} \\ \mathbf{H}_{n+1/2} \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_1\mathbf{H}_{n+1/2} \\ \frac{1}{\mu}C_2\mathbf{E}_{n+1/2} \end{pmatrix} - \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})) \\ \frac{\tau}{2\mu}C_2\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})) \end{pmatrix}\end{aligned}$$

in Y. Proceeding as in the first half step, we conclude

$$\begin{aligned}\begin{pmatrix} (1+\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1} \\ \mathbf{H}_{n+1} \end{pmatrix} &= -\frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_2\big[-\frac{\tau}{2\mu}C_1\mathbf{E}_{n+1} + \mathbf{H}_{n+1/2} + \frac{\tau}{2\mu}C_2\mathbf{E}_{n+1/2}\big] \\ \frac{1}{\mu}C_1\big[-\frac{\tau}{2\varepsilon}C_2\mathbf{H}_{n+1} + (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} + \frac{\tau}{2\varepsilon}C_1\mathbf{H}_{n+1/2}\big] \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \frac{\tau}{2\mu}C_1\frac{\sigma\tau}{4\varepsilon}\mathbf{E}_{n+1} \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_2\big[\frac{\tau}{2\mu}C_2\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \\ \frac{1}{\mu}C_1\big[(1-\frac{\sigma\tau}{4\varepsilon})\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \end{pmatrix} \\ &\quad + \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} \\ \mathbf{H}_{n+1/2} \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \frac{1}{\varepsilon}C_1\mathbf{H}_{n+1/2} \\ \frac{1}{\mu}C_2\mathbf{E}_{n+1/2} \end{pmatrix} \\ &\quad - \begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})) \\ \frac{\tau}{2\mu}C_2\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1})) \end{pmatrix}\end{aligned}$$

in $L^2(Q)^6$. Again with (4.2) and curl $= C_1 - C_2$, this equation implies the second step of the recursion

$$\begin{pmatrix} \varepsilon\mathbf{E}_{n+1} - \frac{\tau^2}{4}D_\mu^{(2)}\mathbf{E}_{n+1} \\ \mu\mathbf{H}_{n+1} - \frac{\tau^2}{4}D_\varepsilon^{(1)}\mathbf{H}_{n+1} \end{pmatrix} \tag{6.2}$$
$$= \begin{pmatrix} \varepsilon\mathbf{E}_{n+\frac12} - \frac{\tau^2}{4}C_2\frac{1}{\mu}C_2\mathbf{E}_{n+\frac12} \\ \mu\mathbf{H}_{n+\frac12} - \frac{\tau^2}{4}C_1\frac{1}{\varepsilon}C_1\mathbf{H}_{n+\frac12} \end{pmatrix} - \begin{pmatrix} \frac{\sigma\tau}{4}(\mathbf{E}_{n+1}+\mathbf{E}_{n+\frac12}) \\ 0 \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \operatorname{curl}\mathbf{H}_{n+\frac12} \\ -\operatorname{curl}\mathbf{E}_{n+\frac12} \end{pmatrix}$$
$$+ \frac{\tau}{2}\begin{pmatrix} 0 \\ C_1\frac{\sigma\tau}{4\varepsilon}(\mathbf{E}_{n+\frac12}+\mathbf{E}_{n+1}) \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} C_2\frac{\tau}{2\mu}C_2\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})) \\ -C_1\frac{\sigma\tau}{4\varepsilon}\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})) \end{pmatrix}$$
$$- \frac{\tau}{2}\begin{pmatrix} (1-\frac{\sigma\tau}{4\varepsilon})(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})) \\ 0 \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} 0 \\ \operatorname{curl}\big(\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big) \end{pmatrix}$$

in $L^2(Q)^6$. For $\varphi \in H^1(Q)^3$ we have $0 = \operatorname{div}\operatorname{curl}\varphi = \operatorname{div} C_1\varphi - \operatorname{div} C_2\varphi$ in $H^{-1}(Q)$. Let $\lambda \in \{\varepsilon, \mu\}$. Since $D_\lambda^{(1)} = C_1\frac{1}{\lambda}C_2$ by (4.2), it follows $\operatorname{div} C_2\frac{1}{\lambda}C_2 v = \operatorname{div} D_\lambda^{(1)} v$ in $H^{-1}(Q)$ for $v \in H^1(Q)^3$ with $C_2 v \in H^1(Q)^3$, and similarly $\operatorname{div} C_1\frac{1}{\lambda}C_1 u = \operatorname{div} D_\lambda^{(2)} u$ in $H^{-1}(Q)$ for $u \in H^1(Q)^3$ with $C_1 u \in H^1(Q)^3$. From (6.2) and (6.1) we thus deduce the recursion formula

$$\begin{pmatrix} \operatorname{div}\big[\varepsilon\mathbf{E}_{n+1} - \frac{\tau^2}{4}D_\mu^{(2)}\mathbf{E}_{n+1}\big] \\ \operatorname{div}\big[\mu\mathbf{H}_{n+1} - \frac{\tau^2}{4}D_\varepsilon^{(1)}\mathbf{H}_{n+1}\big] \end{pmatrix}$$
$$= \begin{pmatrix} \operatorname{div}\big[\varepsilon\mathbf{E}_{n+\frac12} - \frac{\tau^2}{4}D_\mu^{(1)}\mathbf{E}_{n+\frac12}\big] \\ \operatorname{div}\big[\mu\mathbf{H}_{n+\frac12} - \frac{\tau^2}{4}D_\varepsilon^{(2)}\mathbf{H}_{n+\frac12}\big] \end{pmatrix} - \begin{pmatrix} \operatorname{div}[\frac{\sigma\tau}{4}(\mathbf{E}_{n+1}+\mathbf{E}_{n+\frac12})] \\ 0 \end{pmatrix}$$
$$+ \frac{\tau}{2}\begin{pmatrix} 0 \\ \operatorname{div}\big[C_1\frac{\sigma\tau}{4\varepsilon}(\mathbf{E}_{n+\frac12}+\mathbf{E}_{n+1})\big] \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} \operatorname{div}\big[C_2\frac{\tau}{2\mu}C_2\frac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \\ -\operatorname{div}\big[C_1\frac{\sigma\tau^2}{8\varepsilon^2}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \end{pmatrix}$$
$$- \frac{\tau}{2}\begin{pmatrix} \operatorname{div}\big[(1-\frac{\sigma\tau}{4\varepsilon})(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \operatorname{div}\big[\varepsilon\mathbf{E}_n - \frac{\tau^2}{4}D_\mu^{(2)}\mathbf{E}_n\big] \\ \operatorname{div}\big[\mu\mathbf{H}_n - \frac{\tau^2}{4}D_\varepsilon^{(1)}\mathbf{H}_n\big] \end{pmatrix} - \begin{pmatrix} \operatorname{div}\big[\frac{\sigma\tau}{4}\mathbf{E}_{n+1} + \frac{\sigma\tau}{2}\mathbf{E}_{n+\frac12} + \frac{\sigma\tau}{4}\mathbf{E}_n\big] \\ 0 \end{pmatrix}$$
$$- \frac{\tau}{2}\begin{pmatrix} \operatorname{div}[\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})] \\ 0 \end{pmatrix} + \frac{\tau^2}{8}\begin{pmatrix} 0 \\ \operatorname{div}\big[C_1\frac{\sigma}{\varepsilon}(\mathbf{E}_{n+1}-\mathbf{E}_n)\big] \end{pmatrix}$$
$$+ \frac{\tau^3}{16}\begin{pmatrix} \operatorname{div}\big[D_\mu^{(1)}\frac{2}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \\ -\operatorname{div}\big[C_1\frac{\sigma}{\varepsilon^2}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \end{pmatrix} + \frac{\tau^2}{8}\begin{pmatrix} \operatorname{div}\big[\frac{\sigma}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big] \\ 0 \end{pmatrix}$$

in $H^{-1}(Q)^6$. For $N \le T/\tau$ an easy induction then yields

$$\begin{pmatrix} \operatorname{div}\big[\varepsilon\mathbf{E}_N - \frac{\tau^2}{4}D_\mu^{(2)}\mathbf{E}_N\big] \\ \operatorname{div}\big[\mu\mathbf{H}_N - \frac{\tau^2}{4}D_\varepsilon^{(1)}\mathbf{H}_N\big] \end{pmatrix}$$
$$= \begin{pmatrix} \operatorname{div}\big[\varepsilon\mathbf{E}_0 - \frac{\tau^2}{4}D_\mu^{(2)}\mathbf{E}_0\big] \\ \operatorname{div}\big[\mu\mathbf{H}_0 - \frac{\tau^2}{4}D_\varepsilon^{(1)}\mathbf{H}_0\big] \end{pmatrix} - \sum_{n=0}^{N-1}\begin{pmatrix} \operatorname{div}\big[\frac{\sigma\tau}{4}\mathbf{E}_{n+1} + \frac{\sigma\tau}{2}\mathbf{E}_{n+\frac12} + \frac{\sigma\tau}{4}\mathbf{E}_n\big] \\ 0 \end{pmatrix}$$

$$+\sum_{n=0}^{N-1}\Big[\frac{\tau}{2}\begin{pmatrix}-\operatorname{div}[\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})]\\ 0\end{pmatrix}+\frac{\tau^2}{8}\begin{pmatrix}0\\ \operatorname{div}\big[C_1\frac{\sigma}{\varepsilon}(\mathbf{E}_{n+1}-\mathbf{E}_n)\big]\end{pmatrix}$$

$$+\frac{\tau^3}{16}\begin{pmatrix}\operatorname{div}\big[D_\mu^{(1)}\frac{2}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\\ -\operatorname{div}\big[C_1\frac{\sigma}{\varepsilon^2}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\end{pmatrix}+\frac{\tau^2}{8}\begin{pmatrix}\operatorname{div}\big[\frac{\sigma}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\\ 0\end{pmatrix}\Big]$$

in $H^{-1}(Q)^6$. We reorder these terms and use $\operatorname{div}(\mu\mathbf{H}_0)=0$ to derive the crucial formula for the charges of the ADI scheme

$$\begin{pmatrix}\operatorname{div}(\varepsilon\mathbf{E}_N)\\ \operatorname{div}(\mu\mathbf{H}_N)\end{pmatrix}-\begin{pmatrix}\operatorname{div}(\varepsilon\mathbf{E}_0)\\ 0\end{pmatrix} \tag{6.3}$$

$$+\frac{\tau}{2}\sum_{n=0}^{N-1}\Big[\begin{pmatrix}\operatorname{div}[\frac{\sigma}{2}\mathbf{E}_{n+1}+\sigma\mathbf{E}_{n+\frac12}+\frac{\sigma}{2}\mathbf{E}_n]\\ 0\end{pmatrix}+\begin{pmatrix}\operatorname{div}[\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})]\\ 0\end{pmatrix}\Big]$$

$$=\frac{\tau^2}{4}\begin{pmatrix}\operatorname{div}(D_\mu^{(2)}\mathbf{E}_N)\\ \operatorname{div}(D_\varepsilon^{(1)}\mathbf{H}_N)\end{pmatrix}-\frac{\tau^2}{4}\begin{pmatrix}\operatorname{div}(D_\mu^{(2)}\mathbf{E}_0)\\ \operatorname{div}(D_\varepsilon^{(1)}\mathbf{H}_0)\end{pmatrix}+\frac{\tau^2}{8}\begin{pmatrix}0\\ \operatorname{div}\big[C_1\frac{\sigma}{\varepsilon}(\mathbf{E}_N-\mathbf{E}_0)\big]\end{pmatrix}$$

$$+\sum_{n=0}^{N-1}\Big[\frac{\tau^3}{16}\begin{pmatrix}\operatorname{div}\big[D_\mu^{(1)}\frac{2}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\\ -\operatorname{div}\big[C_1\frac{\sigma}{\varepsilon^2}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\end{pmatrix}+\frac{\tau^2}{8}\begin{pmatrix}\operatorname{div}\big[\frac{\sigma}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\\ 0\end{pmatrix}\Big]$$

in $H^{-1}(Q)^6$.

2) The term for the current density $\mathbf{J}$ on the left-hand side of (6.3) can be exchanged by the corresponding integral with the asserted error since

$$\Big\|\sum_{n=0}^{N-1}\tfrac{\tau}{2}\operatorname{div}\big[\mathbf{J}(t_n)+\mathbf{J}(t_{n+1})\big]-\int_0^{N\tau}\operatorname{div}\mathbf{J}(s)\,\mathrm{d}s\Big\|_{H^{-1}} \tag{6.4}$$

$$\le\Big\|\sum_{n=0}^{N-1}\Big[\int_{t_n}^{\frac12(t_n+t_{n+1})}(\mathbf{J}(t_n)-\mathbf{J}(s))\,\mathrm{d}s+\int_{\frac12(t_n+t_{n+1})}^{t_{n+1}}(\mathbf{J}(t_{n+1})-\mathbf{J}(s))\,\mathrm{d}s\Big]\Big\|_{L^2}$$

$$\le\sum_{n=0}^{N-1}\tau\int_{t_n}^{t_{n+1}}\big\|\mathbf{J}'(r)\big\|_{L^2}\,\mathrm{d}r=\tau\int_0^T\big\|\mathbf{J}'(r)\big\|_{L^2}\,\mathrm{d}r.$$

To treat the first and main summand on the right-hand side of (6.3), we insert the closed expression (4.5) for the ADI scheme obtaining

$$\begin{pmatrix}D_\mu^{(2)}\mathbf{E}_N\\ D_\varepsilon^{(1)}\mathbf{H}_N\end{pmatrix}=-\begin{pmatrix}0 & C_2\\ C_1 & 0\end{pmatrix}B_0S_{\tau,N}^J\cdots S_{\tau,1}^J w_0=KB_0^2S_{\tau,N}^J\cdots S_{\tau,1}^J w_0$$

$$=\tfrac{2}{\tau}K(B+S)\tfrac{\tau}{2}(B_Y+S)(I-\tfrac{\tau}{2}B_Y)^{-1}\Big[\gamma_{\frac{\tau}{2}}(A_Y)\big[\gamma_{\frac{\tau}{2}}(B_Y)\gamma_{\frac{\tau}{2}}(A_Y)\big]^{N-1}(I+\tfrac{\tau}{2}B_Y)w_0$$

$$-\sum_{k=0}^{N-1}\big[\gamma_{\frac{\tau}{2}}(A_Y)\gamma_{\frac{\tau}{2}}(B_Y)\big]^k(I+\tfrac{\tau}{2}A_Y)\tfrac{\tau}{2\varepsilon}\big(\mathbf{J}(t_{N-k-1})+\mathbf{J}(t_{N-k}),0\big)\Big]$$

in $L^2(Q)^6$, where we have put

$$K=\begin{pmatrix}\varepsilon I & 0\\ 0 & \mu I\end{pmatrix}\qquad\text{and}\qquad S=B_0-B=\begin{pmatrix}\frac{\sigma}{2\varepsilon}I & 0\\ 0 & 0\end{pmatrix}.$$

Proposition 3.6 then implies

$$\frac{\tau^2}{4}\left\|\begin{pmatrix}\operatorname{div} D_\mu^{(2)}\mathbf{E}_N\\ \operatorname{div} D_\varepsilon^{(1)}\mathbf{H}_N\end{pmatrix}\right\|_{H^{-1}} \le c\tau \mathrm{e}^{6\kappa_Y N\tau}\Big[\,\|w_0\|_{H^1}+\tau\,\|B_Y w_0\|_{H^1} + N\tau\max_{t\in[0,T]}\big(\|(\mathbf{J}(t),0)\|_{H^1}+\tau\|A_Y(\tfrac{1}{\varepsilon}\mathbf{J}(t),0)\|_{H^1}\big)\Big].$$

Similarly, it follows

$$\frac{\tau^2}{4}\left\|\begin{pmatrix}\operatorname{div} D_\mu^{(2)}\mathbf{E}_0\\ \operatorname{div} D_\varepsilon^{(1)}\mathbf{H}_0\end{pmatrix}\right\|_{H^{-1}} \le c\tau^2\|B_0 w_0\|_{H^1} \le c\tau\big(\|w_0\|_{H^1}+\tau\,\|B_Y w_0\|_{H^1}\big).$$

In the same way, the remaining term of third order in (6.3) is bounded by

$$\frac{\tau^3}{16}\,\Big\|\sum_{n=0}^{N-1}\operatorname{div}\big[D_\mu^{(1)}\tfrac{2}{\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}))\big]\Big\|_{H^{-1}} \le c\tau^2 N\max_{t\in[0,T]}\big(\|(\mathbf{J}(t),0)\|_{H^1}+\tau\|A_Y(\tfrac{1}{\varepsilon}\mathbf{J}(t),0)\|_{H^1}\big).$$

The other terms in (6.3) can be estimated analogously. The assertion now follows from formula (6.3), the inequality $\tau N \le T$, and the above estimates. □

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