Transmission Eigenvalues for Periodic Media

**Von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie
zur Erlangung des akademischen Grades eines
Dr. rer. nat.**

genehmigte Dissertation

von

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Datum der Einreichung: 20.09.2017

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Tag der mündlichen Prüfung: 18.10.2017

Preface

This work concerns the existence and discreteness of transmission eigenvalues for acoustic scattering problems for periodic media in $\mathbb{R}^2$.

The interior transmission eigenvalue problem is a boundary value problem for a coupled set of equations defined on the support of the scattering object. Its eigenvalues, the transmission eigenvalues, appear in the study of scattering by inhomogeneous media and are closely related to non-scattering waves. They play an important role in inverse scattering theory, more precisely in target identification and nondestructive testing.

However, not only the physical point of view is interesting. The interior transmission eigenvalue problem is also a challenging mathematical problem, because it is a non-selfadjoint eigenvalue problem. It cannot be treated by the theory of eigenvalue problems for elliptic operators. However, some helpful analysis to deal with this problem has been established which will be used as a basis for some parts of this thesis.

The interior transmission eigenvalue problem is of major interest and a lot of research is still beeing done. This thesis should at least make a small contribution to it. It is organised as follows.

Chapter 1 contains a section about the history of transmission eigenvalues and a rough description of the physical background of acoustic scattering problems for penetrable scattering objects. Furthermore, a detailed explanation of periodic media is included.

In Chapter 2, we present some basic tools we will need in this work.

Two different scattering problems for periodic media, will be introduced in Chapter 3. We will then consider the case of non-scattering incident fields. In both cases, this will lead us to the study of the interior transmission eigenvalue problem.

The main results of this thesis are contained in Chapter 4. Here, the expressions 'interior transmission eigenvalue problem' and 'transmission eigenvalue' will be introduced in detail. Furthermore, Chapter 4 contains some results for bounded domains and some analysis that will be adopted for the purpose of this work. An example motivates, why it is interesting to consider the interior transmission eigenvalue problem for periodic media. Finally, we present some results on the existence and discretenes of the transmission eigenvalues for the two scattering problems. The work concludes by presenting some results about complex transmission eigenvalues for periodic media.

Acknowledgements

First of all, I wish to thank my advisor Prof. Dr. Andreas Kirsch for his valuable advice during my work on this thesis. Moreover, as a mother of three (and soon of four) children, I am very thankful for his understanding and his support. I am also very thankful for many helpful discussions (not only about mathematics) with PD Dr. Frank Hettlich and PD Dr. Tilo Arens.

Lastly, I would like to thank my family, Thomas, Lia, Sora and Deon, for their support, their love and of course for their understanding for having such a busy wife and mother.

Karlsruhe, September 2017

Contents

1 Introduction

This work will study transmission eigenvalues for acoustic scattering problems for periodic media in $\mathbb{R}^2$ and focus on the existence and discreteness of transmission eigenvalues. Also, we will present some results on complex transmission eigenvalues for periodic media. To arouse interest in this topic, we will start with an abstract of the history of transmission eigenvalues.

1.1 The History of Transmission Eigenvalues

Although nowadays the transmission eigenvalue problem is an area of significant interest, the history of transmission eigenvalues started rather slowly.

In 1986, Andreas Kirsch studied the injectivity of the farfield operator [13]. In his paper, the transmission problem and the transmission eigenvalues were mentioned for the first time. Soon after this work, David Colton and Peter Monk [10] discovered that transmission eigenvalues for spherically stratified media form at most a discrete set. Proving the existence of transmission eigenvalues was difficult due to the non-selfadjointness of the transmission eigenvalue problem. Actually, for the next 20 years, only the discreteness of transmission eigenvalues was studied. Other than that, they were more or less ignored. This is because sampling methods for reconstructing the support of an inhomogeneous medium [16] fail if the interrogation frequency corresponds to a transmission eigenvalue. Hence, transmission eigenvalues were to be avoided and since they form at most a discrete set, this result was sufficient.

In 2007 Fioralba Cakoni, David Colton and Peter Monk showed that transmission eigenvalues could be used to obtain material properties of the scattering object from farfield data [1]. Then, transmission eigenvalues suddenly became very interesting. The question of existence was answered for the first time in 2008, by John Silvester and Lassi Päivarinta [25]. They showed the existence of at least one transmission eigenvalue provided that the contrast in the medium is large enough. In 2010, it was then proven by Fioralba Cakoni, Drosses Gintides and Houssem Haddar, that there exists an infinite discrete set of transmission eigenvalues, under the assumption that the contrast does not change sign and is bounded away from zero [2].

The interest in transmission eigenvalues has increased since then.
For some new results we would like to mention [15], where Andreas Kirsch and Hayk Asatryan studied in 2014 the interior transmission eigenvalue problem for a spherically-symmetric domain with anisotropic medium and a cavity. In general,

existence of an infinite set of transmission eigenvalues for regions with cavities is an open problem.
Also, a new integral equation formulation to compute transmission eigevalues for constant refractive index has been established by Fioralba Cakoni and Rainer Kress in 2017 [5].
The numerical calculation of interior transmission eigenvalues for anisotropic media in two dimensions is considered in [19] by Andreas Kleefeld and David Colton.

Nevertheless, the transmission eigenvalue problem is still a resarch subject of major interest in inverse scattering theory with many open problems. For a list of open problems, see for example [4].

In this thesis, we will study transmission eigenvalues for periodic media and focus on the existence and discreteness. Also, some results about complex transmission eigenvalues will be presented.

1.2 Physical Background

When talking about the interior transmission eigenvalue problem, we should not forget, where the problem comes from. Acoustic scattering problems form the basis of this work. It therefore makes sense to roughly explain the physical background of acoustic scattering problems before we go into details. To give it a meaning, we will depict the three-dimensional case. However, we will later only study the two-dimensional case as a simplification.

Let us describe the physical background for acoustic scattering problems in $\mathbb{R}^3$. For more information, see for example [9]. These problems refer to the scattering of an acoustic incident field at some scattering object, which is embedded in some background medium in $\mathbb{R}^3$. We will consider penetrable objects, that means the incident field can propagate inside the obstacle. The function $n(x) = c_0^2/c^2(x) \in \mathbb{C}$, $x \in \mathbb{R}^3$ is called the refraction index. Here, the local speed of sound is denoted by c and c_0 is the speed of sound in air. We assume the background medium consists of air, that means n is equal to one there and it is $n > 1$ inside the scatterer. The wave number k is given by $k = \omega/c_0$, where ω denotes the frequency of the incident wave. The physical properties of the material of the scattering object are such that a wave propagates inside the medium with refraction index $n > 1$, that means it satisfies $\Delta u + k^2 n u = 0$ inside and $\Delta u + k^2 u = 0$ outside the medium. On the boundary, which is assumed to be Lipschitz, some transmission conditions are valid.

We will consider periodic media, which we will describe in the following chapter. Furthermore, we will restrict ourselves to the two-dimensional case from now on.

1.3 Periodic Media

To describe periodic media, we need to consider the contrast $q \in L^\infty(\mathbb{R}^2)$, which is defined as $q(x) = n(x) - 1$, $x \in \mathbb{R}^2$. The contrast is assumed to be periodic in sense of the definition below.

Definition 1.1 ***Periodicity***

- *Let $\phi : \mathbb{R}^2 \mapsto \mathbb{C}$ and e_1 denote the first unit vector in $\mathbb{R}^2$. We call ϕ to be periodic with period $p > 0$ (in x_1-direction) if*

$$\phi(x_1 + p, x_2) = \phi(x_1, x_2) \qquad \textit{for almost all } x_1, x_2 \in \mathbb{R}.$$

- *We call a set $S \subseteq \mathbb{R}^2$ periodic with period $p > 0$ (in x_1-direction) if its indicator function Id_S is periodic with the same period p, that is, $(x_1+p, x_2) \in S$, if, and only if, $(x_1, x_2) \in S$. In particular, if ϕ is periodic with period $p > 0$ (in x_1-direction), then $\operatorname{supp}\phi$ is periodic with period p.*

We will only consider periodicity in x_1-direction. That means, whenever we talk about periodicty, we mean periodicity with respect to x_1.

We are now going to specify periodic media in detail. To this end, let $q \in L^\infty(\mathbb{R}^2)$ be a periodic contrast with some period p in x_1-direction. Let $\Omega \subset \mathbb{R}^2$ be an open periodic set with finite extension in x_2-direction, such that $\operatorname{supp} q = \overline{\Omega}$. We furthermore require Ω to be a Lipschitz set of the form

$$\Omega \;=\; \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, f(x_1) < x_2 < g(x_1)\big\} \tag{1.1}$$

with periodic Lipschitz functions $f, g : \mathbb{R} \to \mathbb{R}$. Additionally, we restrict ourselves to the cases, when

$$\max_{x_1 \in \mathbb{R}} f(x_1) \;<\; \min_{x_1 \in \mathbb{R}} g(x_1).$$

We call Ω a periodic medium. Figure 1 illustrates the situation.

Assumption 1.2

- *Whenever we talk about a periodic medium, respectively periodic contrast $q \in L^\infty(\mathbb{R}^2)$, we assume that there exist $h_1, h_2 \in \mathbb{R}$ with $h_1 < h_2$ such that $q(x) = 0$ for almost all $x \in \mathbb{R}^2$ with $x_2 < h_1$ or $h_2 < x_2$.*

- *We can assume q, and hence Ω, to be 2π-periodic. Indeed, if q is periodic with period $p \neq 2\pi$, we apply a simple change of variables. We define a function $\tilde{q} := q\left(\frac{p}{2\pi}\right)$. Then $\tilde{q}$ is 2π-periodic, since*

$$\tilde{q}(y + 2\pi) \;=\; q\left(\frac{p}{2\pi}(y + 2\pi)\right) \;=\; q\left(\frac{p}{2\pi}y + p\right) \;=\; q\left(\frac{p}{2\pi}y\right) \;=\; \tilde{q}(y).$$

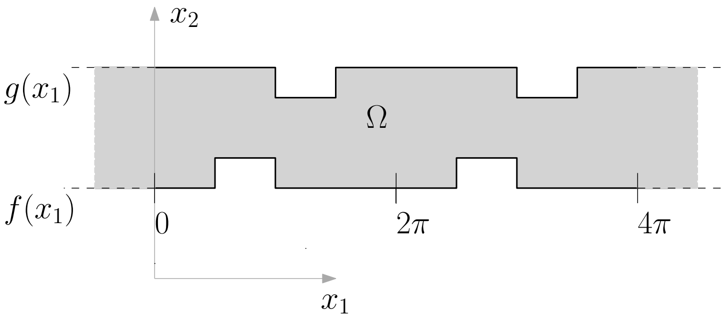

Figure 1: The periodic medium Ω.

- *We assume q to be real-valued such that $q_* \leq q \leq q^*$ for some $q_* > 0$ and $q^* < \infty$ almost everywhere inside of Ω.*

In this work we will make use of a cell Ω_μ, $\mu \in \mathbb{Z}$ fixed, defined as

$$\Omega_\mu \;:=\; \{x \in \Omega, x_1 \in [\mu 2\pi, (\mu+1)2\pi]\}, \tag{1.2}$$

see Figure 2. The dotted lines symbolize the part of the boundary which does not belong to Ω.

Remark 1.3 *Note that Ω_μ is neither closed nor open.*

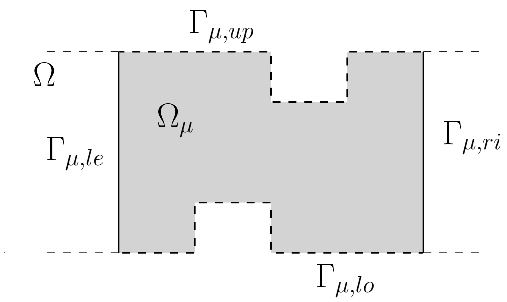

Figure 2: The cell Ω_μ.

For simplicity, we denote the upper and the lower part of the boundary by

$$\Gamma_{\mu,lo} \;:=\; \{x \in \overline{\Omega}_\mu : x_1 \in [\mu 2\pi, (\mu+1)2\pi],\; x_2 = f(x_1)\}$$

and

$$\Gamma_{\mu,up} \;:=\; \{x \in \overline{\Omega}_\mu : x_1 \in [\mu 2\pi, (\mu+1)2\pi],\; x_2 = g(x_1)\}$$

and the left and right part of the boundary by

$$\Gamma_{\mu,le} \; := \; \{x \in \overline{\Omega}_\mu : x_1 = \mu 2\pi, \; f(2\pi\mu) \leq x_2 \leq g(2\pi\mu)\}$$

and

$$\begin{aligned}
\Gamma_{\mu,ri} \; &:= \; \{x \in \overline{\Omega}_\mu : x_1 = (\mu+1)2\pi, \\
&\qquad f(2\pi(\mu+1)) \leq x_2 \leq g(2\pi(\mu+1))\} \\
&\overset{(*)}{=} \; \{x \in \overline{\Omega}_\mu : x_1 = (\mu+1)2\pi, \; f(2\pi\mu) \leq x_2 \leq g(2\pi\mu)\}.
\end{aligned}$$

Note that $(*)$ is due to the 2π-periodicity. We further define

$$\Gamma_\mu \; := \; \Gamma_{\mu,up} \cup \Gamma_{\mu,lo}.$$

In the same way, we use the notation Γ_{lo}, Γ_{up} and $\Gamma = \Gamma_{up} \cup \Gamma_{lo}$, when we talk about an unbounded (with respect to x_1) periodic medium Ω.

1.4 Quasi-periodicity

Additionally to periodic functions we will consider quasi-periodic functions.

Definition 1.4 ***Quasi-periodicity***
Let $p,\ \alpha \in \mathbb{R}$. A function $\psi : \mathbb{R} \mapsto \mathbb{C}$ is called quasi-periodic with period p and phase-shift α, if

$$\psi(t+p) \; = \; e^{ip\alpha}\psi(t).$$

We will mainly consider quasi-periodicity with phase-shift $\alpha \in \mathbb{R}$ and period $p = 2\pi$ with respect to x_1. We then use the expression *α-quasi-periodicity* as explained below.

Definition 1.5 ***α-Quasi-periodicity***
A function $\phi : \mathbb{R}^2 \mapsto \mathbb{C}$ is called α-quasi-periodic (in x_1-direction) with parameter $\alpha \in \mathbb{R}$, if

$$\phi(x_1 + 2\pi, x_2) \; = \; e^{i\alpha 2\pi}\phi(x_1, x_2), \tag{1.3}$$

for almost all $x \in \mathbb{R}^2$.

Note that for $\alpha = 0$, α-quasi-periodicity equals 2π-periodicity.

In the following, we omit the expression ‘in x_1-direction’, whenever this is clear.

Remark 1.6
α-quasi-periodicity can analogously be defined by the following:

A function ϕ is called α-quasi-periodic, if $x_1 \mapsto e^{-i\alpha x_1}\phi(x_1, x_2)$ is 2π-periodic for every x_2.

This is because $e^{-i\alpha x_1}\phi(x_1, x_2)$ is 2π-periodic if, and only if, $e^{-i\alpha(x_1+2\pi)}\phi(x_1 + 2\pi, x_2) = e^{-i\alpha x_1}\phi(x_1, x_2)$, which is eqivalent to (1.3).

Notation 1.7 *We denote by M_α, $\alpha \neq 0$, the operator of multiplication by $e^{-i\alpha x_1}$, that means*

$$M_\alpha \phi(x) \;=\; e^{-i\alpha x_1}\phi(x).$$

Taking the derivative with respect to x_1 on both sides of (1.3), one easily sees that if ϕ is α-quasi-periodic, the function $\frac{\partial \phi}{\partial x_1}$ is α-quasi-periodic as well.

2 Basic Tools

2.1 Basic Function Spaces

We consider a periodic medium Ω and a cell Ω_μ, $\mu \in \mathbb{Z}$ fixed, as described in Chapter 1.3. By construction (see (1.2)), Ω_μ is neither closed nor open. To generalize this situation, we consider some set D with $\overline{D} = \overline{\mathcal{D}}$ for some Lipschitz domain $\mathcal{D}$. This is useful to define the Sobolev spaces in this chapter.

Throughout this chapter we assume $\alpha \in \mathbb{R}$. Restrictions on this will be made when needed. The following function spaces are taken as a basis.

$$\begin{aligned}
C^\infty(\overline{D}) &= \{u : \overline{D} \to \mathbb{C} : \ u \text{ is infinitely often differentiable} \\
&\qquad \text{in } \mathring{D} \text{ and all derivatives can be extended} \\
&\qquad \text{continuously to } \overline{D}\}\,, \\
C_0^\infty(D) &= \{u \in C^\infty(\overline{D}) : \text{supp } u \subset \mathring{D}\}\,, \\
C_{0,\alpha}^\infty(\Omega_\mu) &= \{u \in C^\infty(\overline{\Omega_\mu}) : \text{supp } u \subset \Omega_\mu\,, \\
&\qquad e^{i\alpha 2\pi n} D^m u(x_1, x_2) = D^m u(x_1 + 2\pi n, x_2), \\
&\qquad \text{for all } x \in \Omega_\mu,\ n \in \mathbb{Z} \text{ and } m \in \mathbb{N}^2\},
\end{aligned} \tag{2.1}$$

where $m = (m_1, m_2) \in \mathbb{N}_0^2$ is a multiindex. Here, the notation

$$D^m u \ := \ \frac{\partial^{|m|} u}{\partial^{m_1} x_1 \, \partial^{m_2} x_2},$$

where $|m| = m_1 + m_2$ is used.

For $\alpha = 0$ the space $C_{0,\alpha}^\infty(\Omega_\mu)$ denotes the 2π-periodic functions on Ω. We call this space $C_{0,0}^\infty(\Omega_\mu)$.

Remark 2.1 *By* supp $u \subset \Omega_\mu$*, the case when* supp u *touches the left and right part of the boundary of* Ω_μ *(*$\Gamma_{\mu,le}$ *and* $\Gamma_{\mu,ri}$*), is included, see Figure 3. This is because* $\Gamma_{\mu,le}$ *and* $\Gamma_{\mu,ri}$ *belong to* Ω_μ*.*

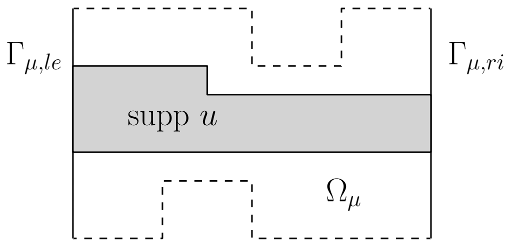

Figure 3: The support of a function u touching the left and right part of the boundary.

Before introducing Sobolev spaces, we should clearly define weak derivatives.

Definition 2.2 *Let $m = (m_1, m_2) \in \mathbb{N}_0^2$ be fixed.*

a) For $u \in L^2(D)$, f is called the weak derivative of u of order m, if there exists a function $f \in L^2(D)$ such that

$$\int_D f(x)\overline{\varphi(x)}\,dx \;=\; (-1)^{|m|}\int_D u(x)\mathsf{D}^m\overline{\varphi(x)}\,dx,$$

for all $\varphi \in C_0^\infty(D)$. We write $f = \mathsf{D}^m u$.

b) For $u \in L^2(\Omega_\mu)$, f is called the weak derivative of u of order m and parameter α, if there exists a function $f \in L^2(\Omega_\mu)$ such that

$$\int_{\Omega_\mu} f(x)\overline{\varphi(x)}\,dx \;=\; (-1)^{|m|}\int_{\Omega_\mu} u(x)\mathsf{D}^m\overline{\varphi(x)}\,dx,$$

for all $\varphi \in C_{0,\alpha}^\infty(\Omega_\mu)$. We write $f = \mathsf{D}^m u$.

Note that we consider α-quasiperiodic test functions in part b) of the definition above.

Now we define the following Sobolev spaces of first and second order.

Definition 2.3 *The Sobolev space $H^p(D)$, $p = 1, 2$, is defined by*

$$H^p(D) \;=\; \{u \in L^2(D) : \mathsf{D}^m u \in L^2(D),\ \text{for all } |m| \leq p\}.$$

The inner product reads

$$(u,v)_{H^p(D)} \;=\; \sum_{|m|\le p} (\mathsf{D}^m u, \mathsf{D}^m v)_{L^p(D)} \tag{2.2}$$

and $\|\cdot\|_{H^p(D)}$ *is the corresponding norm. The derivatives have to be understood as in Definition 2.2 a).*

Definition 2.4 *a) The space* $H_0^p(D)$ *is defined as the closure of* $C_0^\infty(D)$ *with respect to the norm* $\|\cdot\|_{H^p(D)}$, $p = 1,2$.

b) The space $H_{0,0}^p(\Omega_\mu)$ *is defined as the closure of* $C_{0,0}^\infty(\Omega_\mu)$ *with respect to the norm* $\|\cdot\|_{H^p(\Omega_\mu)}$, $p = 1,2$, *where the norm* $\|\cdot\|_{H^p(\Omega_\mu)}$ *is defined analogously to (2.2) with derivatives as in Definition 2.2 b).*
The space $H_{0,0}^p(\Omega_\mu)$ *is called the Sobolev space of* 2π*-periodic functions on* Ω_μ *with vanishing traces.*

With help of the operator M_α we define the corresponding Sobolev space of α-quasi-periodic functions on Ω_μ, $\alpha \neq 0$, with vanishing traces.

Definition 2.5 *The space* $H_{0,\alpha}^p(\Omega_\mu)$, $\alpha \neq 0$, *is defined as*

$$H_{0,\alpha}^p(\Omega_\mu) \;=\; \left\{ u \in L^2(\Omega_\mu):\; M_\alpha u \in H_{0,0}^p(\Omega_\mu) \right\}.$$

We call this space the Sobolev space of α*-quasi-periodic functions on* Ω_μ *with vanishing traces.*

Lemma 2.6 *Let* $\tilde{H}_{0,\alpha}^p(\Omega_\mu)$, $\alpha \neq 0$, *be the space defined as the closure of* $C_{0,\alpha}^\infty(\Omega_\mu)$ *with respect to the norm* $\|\cdot\|_{H^p(\Omega_\mu)}$, $p = 1,2$. *Then*

$$\tilde{H}_{0,\alpha}^p(\Omega_\mu) \;=\; H_{0,\alpha}^p(\Omega_\mu).$$

Proof: To obtain the space $\tilde{H}_{0,\alpha}^p(\Omega_\mu)$, $\alpha \neq 0$, we consider the space $C_{0,0}^\infty(\Omega_\mu)$ and shift it to $C_{0,\alpha}^\infty(\Omega_\mu)$ by applying M_α^{-1} to the functions in $C_{0,0}^\infty(\Omega_\mu)$. We then take the closure with respect to the norm $\|\cdot\|_{H^p(\Omega_\mu)}$, $p = 1,2$. Since the operator M_α of multiplication is continuous and thus commutates with the limit, the space $\tilde{H}_{0,\alpha}^p(\Omega_\mu)$ coinsides with $H_{0,\alpha}^p(\Omega_\mu)$, where we first take the closure of $C_{0,0}^\infty(\Omega_\mu)$, obtain $H_{0,0}^p(\Omega_\mu)$ and then apply the operator M_α^{-1} to the functions in $H_{0,0}^p(\Omega_\mu)$. $\square$

We keep Lemma 2.6 in mind, when using density arguments for α-quasi-periodic function spaces.

Remark 2.7 *We understand the Sobolev spaces $H^p_{0,\alpha}(\Omega_\mu)$, $p=1,2$, in a different way from $H^p_0(\Omega_\mu)$. Roughly speaking, here, $H^1_{0,\alpha}(\Omega_\mu)$ implies that the trace u is zero only on the upper and lower part of the boundary, $H^2_{0,\alpha}(\Omega_\mu)$ means that the trace $\frac{\partial u}{\partial \nu}$ vanishes there as well. In contrast to $H^p_0(\Omega_\mu)$, we do not have this property on the left and right part of the boundary, but periodicity.*

Definition 2.8 *a) We equip $H^2_0(D)$ with the inner product $(\!(u,v)\!)_{H^2(D)} \;:=\; \int_D \frac{1}{q}\Delta u \Delta \overline{v}\,dx$ and the corresponding norm $|\!|\!| \cdot |\!|\!|_{H^2(D)}$.*

b) For $\alpha \in \mathbb{R}$, we equip $H^2_{0,\alpha}(\Omega_\mu)$ with the inner product $(\!(u,v)\!)_{H^2(\Omega_\mu)} := \int_{\Omega_\mu} \frac{1}{q}\Delta u \Delta \overline{v}\,dx$ and the corresponding norm $|\!|\!| \cdot |\!|\!|_{H^2(\Omega_\mu)}$. The derivatives have to be understood as in Definition 2.2 part b).

On $H^2_0(D)$ the two norms, $|\!|\!| \cdot |\!|\!|_{H^2(D)}$ and $\| \cdot \|_{H^2(D)}$ are equivalent, see [16], Theorem 4.13.

We will show that the two norms are equivalent on $H^2_{0,\alpha}(\Omega_\mu)$ as well. To this end, we need two tools, the Friedrich inequality and integration by parts.

Lemma 2.9 ***Friedrich Inequality***
For any cell Ω_μ there exists $c > 0$ with

$$\|u\|_{L^2(\Omega_\mu)} \;\leq\; c\,\|\nabla u\|_{L^2(\Omega_\mu)}$$

for all $u \in H^1_{0,\alpha}(\Omega_\mu)$.

Proof: The idea of this proof is taken from Theorem 4.15 from [17]. Let for fixed $\mu \in \mathbb{Z}$

$$Q_\mu := [\mu 2\pi, (\mu+1)2\pi] \times (b_1, b_2)]$$

with $b_1, b_2 \in \mathbb{R}$ such that $b_1 < \min_{x_1 \in \mathbb{R}} f(x_1)$ and $b_2 > \max_{x_1 \in \mathbb{R}} g(x_1)$, where f and g denote the upper and the lower part of the boundary of Ω as described in (1.1). Let $u \in C^\infty_{0,\alpha}(\Omega_\mu)$ and extend u by zero to Q_μ. It holds

$$\begin{aligned} u(x) \;=\; u(x_1,x_2) \;&=\; u(x_1,b_1) \;+\; \int_{b_1}^{x_2} \frac{\partial u}{\partial x_2}(x_1,t)\,dt \\ &=\; \int_{b_1}^{x_2} \frac{\partial u}{\partial x_2}(x_1,t)\,,dt \end{aligned}$$

because $u(x_1,b_1)=0$. By Cauchy-Schwarz,

$$\begin{aligned} |u(x)|^2 \;&\leq\; (x_2-b_1)\int_{b_1}^{x_2} \left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2 dt \\ &\leq\; (b_2-b_1)\int_{b_1}^{b_2} \left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2 dt. \end{aligned}$$

We conclude

$$\begin{aligned}\int_{b_1}^{b_2} |u(x)|^2 dx_2 &\leq \int_{b_1}^{b_2} (b_2 - b_1) \int_{b_1}^{b_2} \left| \frac{\partial u}{\partial x_2}(x_1, t) \right|^2 dt\, dx_2 \\ &= (b_2 - b_1)^2 \int_{b_1}^{b_2} \left| \frac{\partial u}{\partial x_2}(x_1, t) \right|^2 dt\end{aligned}$$

and hence

$$\begin{aligned}\|u\|^2_{L^2(\Omega_\mu)} &= \|u\|^2_{L^2(Q_\mu)} \\ &= \int_{2\pi\mu}^{2\pi(\mu+1)} \int_{b_1}^{b_2} |u(x_1, x_2)|^2 dx_2\, dx_1 \\ &\leq (b_2 - b_1)^2 \int_{2\pi\mu}^{2\pi(\mu+1)} \int_{b_1}^{b_2} \left| \frac{\partial u}{\partial x_2}(x_1, t) \right|^2 dt\, dx_1 \\ &\leq (b_2 - b_1)^2 \|\nabla u\|^2_{L^2(Q_\mu)} \\ &= (b_2 - b_1)^2 \|\nabla u\|^2_{L^2(\Omega_\mu)}\end{aligned}$$

Since $C^\infty_{0,\alpha}(\Omega_\mu)$ is dense in $H^1_{0,\alpha}(\Omega_\mu)$, the latter holds true for all $u \in H^1_{0,\alpha}(\Omega_\mu),\ \alpha \in \mathbb{R}$.

□

Lemma 2.10 ***Partial Integration on $H^1_{0,\alpha}(\Omega_\mu)$.***
Let Ω_μ be a cell of Ω. Then it holds

$$\int_{\Omega_\mu} \nabla u\, \overline{v} \;+\; u\, \nabla \overline{v}\, dx \;=\; 0,$$

for all $u, v \in H^1_{0,\alpha}(\Omega_\mu)$.

Proof: We know from the Theorem of Gauß (see [17] for example) that for all $u, v \in C^\infty_{0,\alpha}(\Omega_\mu)$, $j = 1, 2$,

$$\int_{\Omega_\mu} (\partial_j u)\, \overline{v}\, dx \;=\; -\int_{\Omega_\mu} u\, (\partial_j \overline{v})\, dx \;+\; \int_{\partial\Omega_\mu} u\, \overline{v}\, \nu_j\, ds,$$

where ν denotes the outward pointing unit normal vector. Due to the zero boundary condition on the upper and the lower part of the boundary, the boundary integral reduces to

$$\int_{\partial\Omega_\mu} u\, \overline{v}\, \nu\, ds \;=\; \int_{\Gamma_{\mu,le}} u\, \overline{v}\, \nu\, ds \;+\; \int_{\Gamma_{\mu,ri}} u\, \overline{v}\, \nu\, ds,$$

and vanishes, because the integral on the left part of the boundary reads

$$\int_{\Gamma_{\mu,le}} u\,\overline{v}\,\nu\;ds \;=\; \int_{f(2\pi\mu)}^{g(2\pi\mu)} u(2\pi\mu,x_2)\,\overline{v}(2\pi\mu,x_2)\,\nu_{le}\,ds,$$

where $\nu_{le} = (-1,0)^\top$ denotes the outward pointing unit normal vector on $\Gamma_{\mu,le}$. The integral on the right part of the boundary reads

$$\begin{aligned}\int_{\Gamma_{\mu,ri}} u\,\overline{v}\,\nu\;ds \;&=\; \int_{f(2\pi\mu)}^{g(2\pi\mu)} e^{-i\alpha x_1} u(2\pi\mu,x_2)\,e^{i\alpha x_1}\overline{v}(2\pi\mu,x_2)\,\nu_{ri}\,ds \\ &=\; \int_{f(2\pi\mu)}^{g(2\pi\mu)} u(2\pi\mu,x_2)\,\overline{v}(2\pi\mu,x_2)\,\nu_{ri}\,ds,\end{aligned}$$

where $\nu_{ri} = (1,0)^\top$. Since $\nu_{le} = -\nu_{ri}$,

$$\int_{\Omega_\mu} (\partial_j u)\,\overline{v}\,dx \;+\; \int_{\Omega_\mu} u\,(\partial_j\overline{v})\,dx \;=\; 0,$$

for all $u,v \in C^\infty_{0,\alpha}(\Omega_\mu)$, $j = 1,2$.

Let now $u,v \in H^1_{0,\alpha}(\Omega_\mu)$. Then there exist sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ in $C^\infty_{0,\alpha}(\Omega_\mu)$ with

$$\|u_n - u\|_{H^1(\Omega_\mu)} \;\to\; 0 \quad\text{and}\quad \|v_n - v\|_{H^1(\Omega_\mu)} \;\to\; 0.$$

Then

$$\begin{aligned}&\left|\int_{\Omega_\mu} (\partial_j u_n)\,\overline{v_n} - (\partial_j u)\,\overline{v}\,dx\right| \\ \le\;& \left|\int_{\Omega_\mu} (\partial_j u_n)\,(\overline{v_n} - \overline{v})\,dx\right| + \left|\int_{\Omega_\mu} (\partial_j u_n - \partial_j u)\,\overline{v}\,dx\right| \\ \le\;& \left(\int_{\Omega_\mu} |\partial_j u_n|^2\;dx\right)^{1/2} \left(\int_{\Omega_\mu} |v_n - v|^2\;dx\right)^{1/2} \\ &+ \left(\int_{\Omega_\mu} |\partial_j u_n - \partial_j u|^2\;dx\right)^{1/2} \left(\int_{\Omega_\mu} |v|^2\;dx\right)^{1/2},\end{aligned}$$

$j = 1,2$, which tends to zero because

$$\begin{aligned}&\big|\|\partial_j u_n\|_{L^2(\Omega_\mu)} - \|\partial_j u\|_{L^2(\Omega_\mu)}\big| \;\le\; \|\partial_j u_n - \partial_j u\|_{L^2(\Omega_\mu)} \\ \le\;& \|u_n - u\|_{H^1(\Omega_\mu)} \;\to\; 0,\end{aligned}$$

as n tends to infinity. Analogously, it is

$$\int_{\Omega_\mu} u_n\,(\partial_j \overline{v_n})\,dx \;\to\; \int_{\Omega_\mu} u\,(\partial_j \overline{v})\,dx.$$

We obtain

$$0 \;=\; \int_{\Omega_\mu} (\partial_j u_n)\,\overline{v_n} + (\partial_j u_n)\,\overline{v_n}\,dx \;\to\; \int_{\Omega_\mu} (\partial_j u)\,\overline{v} + (\partial_j u)\,\overline{v}\,dx,$$

as n tends to infinity, and hence

$$\int_{\Omega_\mu} (\partial_j u)\,\overline{v} \;+\; (\partial_j u)\,\overline{v}\,dx \;=\; 0.$$

□

Theorem 2.11 *On $H^2_{0,\alpha}(\Omega_\mu)$, the norm $|\!|\!|\cdot|\!|\!|_{H^2(\Omega_\mu)}$ is equivalent to the norm $\|\cdot\|_{H^2(\Omega_\mu)}$. That means, there exist two positive constants $m_1, m_2 \in \mathbb{R}$ such that*

$$m_1 |\!|\!|\cdot|\!|\!|_{H^2(\Omega_\mu)} \;\leq\; \|\cdot\|_{H^2(\Omega_\mu)} \;\leq\; m_2 |\!|\!|\cdot|\!|\!|_{H^2(\Omega_\mu)}.$$

Proof: Let $u \in C^\infty_{0,\alpha}(\Omega_\mu)$. Then, with integration by parts, for $i, j = 1, 2$,

$$\begin{aligned} \int_{\Omega_\mu} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx \;&=\; -\int_{\Omega_\mu} \frac{\partial u}{\partial x_i}\, \frac{\partial^3 \overline{u}}{\partial x_i \partial x_j \partial x_j}\, dx \\ &=\; \int_{\Omega_\mu} \frac{\partial^2 u}{\partial x_i^2}\, \frac{\partial^2 \overline{u}}{\partial x_j^2}\, dx. \end{aligned}$$

Hence, it holds true that

$$2\int_{\Omega_\mu} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 dx \;=\; \int_{\Omega_\mu} \frac{\partial^2 u}{\partial x_1^2}\, \frac{\partial^2 \overline{u}}{\partial x_2^2} \;+\; \frac{\partial^2 u}{\partial x_2^2}\, \frac{\partial^2 \overline{u}}{\partial x_1^2}\, dx. \tag{2.3}$$

By a density argument, (2.3) holds true for $u \in H^2_{0,\alpha}(\Omega_\mu)$.

For q_* and q^* being the minimum and the maximum of the function q we estimate

$$
\begin{aligned}
\|\!|u|\!\|^2_{H^2(\Omega_\mu)} \;&=\; \int_{\Omega_\mu} \frac{1}{q}\, |\Delta u|^2 \, dx \;\leq\; \int_{\Omega_\mu} \frac{1}{q_*}\, |\Delta u|^2 \, dx \\
=\;& \frac{1}{q_*} \int_{\Omega_\mu} \left[\left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 u}{\partial x_2^2} \right|^2 + \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 \overline{u}}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 \overline{u}}{\partial x_1^2} \right] dx \\
\overset{(2.3)}{=}\;& \frac{1}{q_*} \int_{\Omega_\mu} \left[\left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 u}{\partial x_2^2} \right|^2 + 2 \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 \right] dx \\
=\;& \frac{1}{q_*} \int_{\Omega_\mu} \sum_{|m|=2} |\mathsf{D}^m u|^2 \; dx \\
\leq\;& \frac{1}{q_*} \int_{\Omega_\mu} \sum_{|m|\leq 2} |\mathsf{D}^m u|^2 \; dx \\
=\;& \frac{1}{q_*} \|u\|^2_{H^2(\Omega_\mu)}.
\end{aligned}
$$

On the other hand, we see with Friedrichs inequality that for $u \in C^\infty_{0,\alpha}(\Omega_\mu)$,

$$
\int_{\Omega_\mu} |u|^2 \; dx \;\leq\; c \int_{\Omega_\mu} |\nabla u|^2 \; dx,
$$

with $c > 0$, as well as for $i = 1, 2$

$$
\int_{\Omega_\mu} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \;\leq\; c' \int_{\Omega_\mu} \left| \nabla \frac{\partial u}{\partial x_i} \right|^2 dx,
$$

with $c' > 0$. Again, by a density argument this holds true for functions in $H^2_{0,\alpha}(\Omega_\mu)$.

We compute

$$
\begin{aligned}
\|u\|^2_{H^2(\Omega_\mu)} &= \int_{\Omega_\mu} \left[\sum_{|m|=2} |\mathsf{D}^m u|^2 \;+\; |\nabla u|^2 \;+\; |u|^2 \right] dx \\
&\le \int_{\Omega_\mu} \sum_{|m|=2} |\mathsf{D}^m u|^2 \; dx \;+\; c \int_{\Omega_\mu} |\nabla u|^2 \; dx \\
&= \int_{\Omega_\mu} \sum_{|m|=2} |\mathsf{D}^m u|^2 \; dx \;+\; c \int_{\Omega_\mu} \left[\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2 \right] dx \\
&\le \int_{\Omega_\mu} \sum_{|m|=2} |\mathsf{D}^m u|^2 \; dx \;+\; c' \int_{\Omega_\mu} \left[\left|\nabla\frac{\partial u}{\partial x_1}\right|^2 + \left|\nabla\frac{\partial u}{\partial x_2}\right|^2 \right] dx \\
&= \int_{\Omega_\mu} \sum_{|m|=2} |\mathsf{D}^m u|^2 \; dx \;+\; c' \int_{\Omega_\mu} \left[\left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 \;+\; \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2 \;+\; 2 \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 \right] dx \\
&\overset{(*)}{\le} c'' \int_{\Omega_\mu} |\Delta u|^2 \; dx \\
&\le c'' q^* \int_{\Omega_\mu} \frac{1}{q} \, |\Delta u|^2 \; dx \\
&= c''' |||u|||^2_{H^2(\Omega_\mu)},
\end{aligned}
$$

with constants c, c', c'', $c''' \in \mathbb{R}$. In $(*)$ we have used Cauchy-Schwarz inequality and equation (2.3). □

2.2 Embeddings

Let us now consider for fixed $\mu \in \mathbb{Z}$ a square Q_μ, as in the proof of Theorem 2.9, that is

$$Q_\mu \;=\; [\mu 2\pi, (\mu + a)2\pi] \times (b_1, b_2), \quad \mu \in \mathbb{Z} \text{ fixed},$$

with $b_1, b_2 \in \mathbb{R}$ and $a \in \mathbb{N}$ such that

$$b_1 \;<\; \min_{x_1 \in \mathbb{R}} f(x_1)\,, \quad b_2 \;>\; \max_{x_1 \in \mathbb{R}} g(x_1).$$

Furthermore, we require here that

$$b_2 - b_1 \;=\; 2\pi a, \quad a \in \mathbb{N}.$$

Here $f, g : \mathbb{R} \to \mathbb{R}$ determine the upper and lower part of the boundary of Ω.

Remark 2.12

- *Note that Q_μ as well as Ω_μ are neither closed nor open.*
- *The cell Ω_μ is contained in the square Q_μ. Figure 4 illustrates the setting. The dotted lines denote the part of the boundary which do not belong to Q_μ, respectively Ω_μ.*
- *For the following, without loss of generality, we assume $a = 1$. Indeed, if $a > 1$ we redefine Ω_μ as $\{x \in \Omega, x_1 \in (\mu 2\pi, (\mu + a)2\pi)\}$.*

It may appear easier to define Q_μ as a rectangle $\{[\mu 2\pi, (\mu + 1)2\pi] \times (b_1, b_2)\}$. The reason to postulate $b_2 - b_1 = 2\pi$ is to make the following Fourier coefficients, and hence the following definitions, more clearly arranged.

We will list some basic results from the theory of Fourier series.

Theorem 2.13 *For $u \in L^2(Q_\mu)$ the Fourier coefficients $u_n \in \mathbb{C}$ are defined by*

$$u_n = \frac{1}{4\pi^2} \int_{Q_\mu} u(x) e^{-in\cdot x} dx \quad \textit{for } n \in \mathbb{Z}^2.$$

Then $u(x) = \sum_{n \in \mathbb{Z}^2} u_n e^{in\cdot x}$ in the L^2-sense.

With Parsevals' equation, we yield for $u \in H^1(Q_\mu)$, respectively $u \in H^2(Q_\mu)$,

$$(\nabla u, \nabla v)_{L^2(Q_\mu)} = 4\pi^2 \sum_{n \in \mathbb{Z}^2} |n|^2 u_n \overline{v_n},$$

$$\sum_{|m|=2} (D^m u, D^m v)_{L^2(Q_\mu)} = 4\pi^2 \sum_{n \in \mathbb{Z}^2} \left(n_1^4 + n_1^2 n_2^2 + n_2^4\right) u_n \overline{v_n}.$$

Using this, we can now define the Sobolev space $\mathcal{H}^1_{2\pi}(Q_\mu)$ and $\mathcal{H}^2_{2\pi}(Q_\mu)$ of 2π-periodic functions on Q_μ.

Definition 2.14

$$\mathcal{H}^1_{2\pi}(Q_\mu) := \left\{ u \in L^2(Q_\mu) : \sum_{n \in \mathbb{Z}^2} \left(1 + |n|^2\right) |u_n|^2 < \infty \right\}$$

with inner product

$$(u, v)_{\mathcal{H}^1_{2\pi}(Q_\mu)} = 4\pi^2 \sum_{n \in \mathbb{Z}^2} \left(1 + |n|^2\right) u_n \overline{v_n}$$

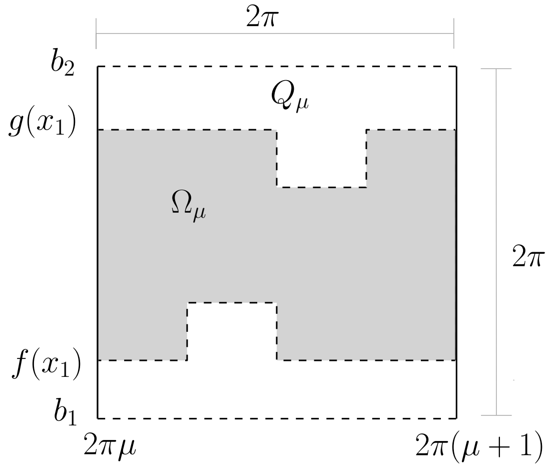

Figure 4: The setting of Ω_μ and Q_μ for $a = 1$.

and

$$\mathcal{H}^2_{2\pi}(Q_\mu) := \left\{ u \in L^2(Q_\mu) : \sum_{n\in\mathbb{Z}^2} \left(1 + |n|^2 + n_1^4 + n_1^2 n_2^2 + n_2^4\right) |u_n|^2 < \infty \right\}$$

with inner product

$$(u, v)_{\mathcal{H}^2_{2\pi}(Q_\mu)} = 4\pi^2 \sum_{n\in\mathbb{Z}^2} \left(1 + |n|^2 + n_1^4 + n_1^2 n_2^2 + n_2^4\right) u_n \overline{v_n}.$$

Furthermore, we denote the corresponding norms by $\|\cdot\|_{\mathcal{H}^1_{2\pi}(Q_\mu)}$ *and* $\|\cdot\|_{\mathcal{H}^2_{2\pi}(Q_\mu)}$*, respectively.*

Again, we use the operator M_α to define the Sobolev spaces of α-quasi-periodic functions.

Definition 2.15 *The Sobolev space of α-quasi-periodic functions $\mathcal{H}^p_\alpha(Q_\mu)$ on Q_μ is defined by setting*

$$u \in \mathcal{H}^p_\alpha(Q_\mu) \quad :\Leftrightarrow \quad M_\alpha u \in \mathcal{H}^p_{2\pi}(Q_\mu).$$

for $p = 1, 2$.

The next lemma ensures that the zero-extensions of functions of $H^p_{0,0}(\Omega_\mu)$ belong to $\mathcal{H}^p_{2\pi}(Q_\mu)$, $p = 1, 2$.

Lemma 2.16 *The extension operator*

$$E : u \mapsto \tilde{u} = \begin{cases} u & on \quad \Omega_\mu \\ 0 & on \quad Q_\mu \setminus \Omega_\mu \end{cases}$$

is linear and bounded from $H^p_{0,0}(\Omega_\mu)$ *into* $\mathcal{H}^p_{2\pi}(Q_\mu)$, $p = 1, 2$.

Proof: Before we begin to prove this lemma, we note that Ω_μ is contained in Q_μ in such a way that only the upper and lower part of the boundary lie completely in Q_μ. Therefore, a zero-extension of any function $u \in C^\infty_{0,0}(\Omega_\mu)$ belongs to $C^\infty_{0,0}(Q_\mu)$. Here, $C^\infty_{0,0}(Q_\mu)$ is defined as in (2.1), for $\alpha = 0$ and for Ω_μ replaced by Q_μ.

Knowing this, we can transmit the same idea and the same structure as in the proof of Theorem 4.11, chapter 4 from [17].

Let $u \in C^\infty_{0,0}(\Omega_\mu)$, then obviously $\tilde{u} \in C^\infty_{0,0}(Q_\mu)$. We verify for the Fourier coefficients

$$\begin{aligned} u_n &= \frac{1}{4\pi^2} \int_{Q_\mu} \tilde{u}(x) e^{-in\cdot x} dx = \frac{i}{n_j} \frac{1}{4\pi^2} \int_{Q_\mu} \tilde{u}(x) \frac{\partial}{\partial x_j} e^{-in\cdot x} dx \\ &\overset{(*)}{=} -\frac{i}{n_j} \frac{1}{4\pi^2} \int_{Q_\mu} \frac{\partial \tilde{u}(x)}{\partial x_j} e^{-in\cdot x} dx \qquad (2.4) \\ &= \frac{1}{n_j n_k} \frac{1}{4\pi^2} \int_{Q_\mu} \frac{\partial \tilde{u}(x)}{\partial x_j} \frac{\partial}{\partial x_k} e^{-in\cdot x} dx \\ &\overset{(*)}{=} \frac{-1}{n_j n_k} \frac{1}{4\pi^2} \int_{Q_\mu} \frac{\partial^2 \tilde{u}(x)}{\partial x_j \partial x_k} e^{-in\cdot x} dx, \qquad (2.5) \end{aligned}$$

for $j, k = 1, 2$, $n = (n_1, n_2) \in \mathbb{Z}^2$.

Here, $(*)$ is due to partial integration and the boundary conditions in $C^\infty_{0,0}(Q_\mu)$. To be more precise, on the upper and lower part of the boundary it holds $\tilde{u}(x) = 0$. Also, due to the 2π-periodicity, $\tilde{u}(x) \frac{\partial}{\partial x_j} e^{-in\cdot x}$, as well as $\frac{\partial \tilde{u}(x)}{\partial x_j} \frac{\partial}{\partial x_j} e^{-in\cdot x}$ take the same value on the left and the right part of the boundary.

From (2.4) and (2.5) we obtain that

$$u_n = -\frac{i}{n_j} \frac{1}{4\pi^2} \int_{Q_\mu} \frac{\partial \tilde{u}(x)}{\partial x_j} e^{-in\cdot x} dx = -\frac{i}{n_j} \hat{u}_n,$$

where $\hat{u}_n$ denote the Fourier coefficients of $\frac{\partial \tilde{u}}{\partial x_j}$ and

$$u_n = \frac{-1}{n_j n_k} \frac{1}{4\pi^2} \int_{Q_\mu} \frac{\partial^2 \tilde{u}(x)}{\partial x_j \partial x_k} e^{-in\cdot x} dx = \frac{-1}{n_j n_k} \hat{\hat{u}}_n,$$

where $\hat{\hat{u}}_n$ denote the Fourier coefficients of $\frac{\partial^2 \tilde{u}}{\partial x_j \partial x_k}$.

It holds

$$\begin{aligned}
4\pi^2 \sum_{n\in\mathbb{Z}^2} |u_n|^2 &= \|\tilde{u}\|^2_{L^2(Q_\mu)} = \|u\|^2_{L^2(\Omega_\mu)}, \\
4\pi^2 \sum_{n\in\mathbb{Z}^2} n_j^2 |u_n|^2 &= 4\pi^2 \sum_{n\in\mathbb{Z}^2} |\hat{u}_n|^2 \\
&= \left\| \frac{\partial \tilde{u}}{\partial x_j} \right\|^2_{L^2(Q_\mu)} = \left\| \frac{\partial u}{\partial x_j} \right\|^2_{L^2(\Omega_\mu)}, \\
4\pi^2 \sum_{n\in\mathbb{Z}^2} n_j^2 n_k^2 |u_n|^2 &= 4\pi^2 \sum_{n\in\mathbb{Z}^2} |\hat{\hat{u}}_n|^2 \\
&= \left\| \frac{\partial^2 \tilde{u}}{\partial x_j \partial x_k} \right\|^2_{L^2(Q_\mu)} = \left\| \frac{\partial^2 u}{\partial x_j \partial x_k} \right\|^2_{L^2(\Omega_\mu)}.
\end{aligned} \tag{2.6}$$

and hence

$$\begin{aligned}
\|u\|^2_{H^1(\Omega_\mu)} &= \|u\|^2_{L^2(\Omega_\mu)} + \|\nabla u\|^2_{L^2(\Omega_\mu)} \\
&= \|u\|^2_{L^2(\Omega_\mu)} + \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega_\mu)} + \left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(\Omega_\mu)} \\
&= 4\pi \sum_{n\in\mathbb{Z}^2} |u_n|^2 (1 + |n|^2) = \|\tilde{u}\|^2_{\mathcal{H}^1_{2\pi}(Q_\mu)},
\end{aligned}$$

as well as

$$\begin{aligned}
\|u\|^2_{H^2(\Omega_\mu)} &= \|u\|^2_{L^2(\Omega_\mu)} + \|\nabla u\|^2_{L^2(\Omega_\mu)} + \sum_{|m|=2} \|D^m u\|^2_{L^2(\Omega_\mu)} \\
&= 4\pi \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(1 + |n|^2 + n_1^4 + n_1^2 n_2^2 + n_2^4\right) = \|\tilde{u}\|^2_{\mathcal{H}^2_{2\pi}(Q_\mu)}.
\end{aligned}$$

This holds for all functions $u \in C^\infty_{0,0}(\Omega_\mu)$. Because $u \in C^\infty_{0,0}(\Omega_\mu)$ is dense in $H^2_{0,0}(\Omega_\mu)$ we conclude

$$\|u\|^2_{H^2(\Omega_\mu)} = 4\pi \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(1 + |n|^2 + n_1^4 + n_1^2 n_2^2 + n_2^4\right) = \|\tilde{u}\|^2_{\mathcal{H}^2_{2\pi}(Q_\mu)},$$

for all $u \in H^2_{0,0}(\Omega_\mu)$ and therefore E is bounded. $\square$

Theorem 2.17 *The embedding* $\tilde{I}_0 : \ H^1_{0,0}(\Omega_\mu) \hookrightarrow L^2(\Omega_\mu)$ *is compact.*

Proof: We will use the same arguments as in the proof of Theorem 4.14, Chapter 4 from [17]. Sketching the idea quickly, the proof reduces to show that the embedding $\tilde{J} : \mathcal{H}^1_{2\pi}(Q_\mu) \hookrightarrow L^2(Q_\mu)$ is compact. We are going to formulate this as a lemma subsequently. Then the composition

$$R \circ \tilde{J} \circ E : \ H^1_{0,0}(\Omega_\mu) \ \stackrel{E}{\to} \ \mathcal{H}^1_{2\pi}(Q_\mu) \ \stackrel{\tilde{J}}{\hookrightarrow} \ L^2(Q_\mu) \ \stackrel{R}{\to} \ L^2(\Omega_\mu)$$

is compact, because E and R are bounded. Here, $E : \ H^1_{0,0}(\Omega_\mu) \to \mathcal{H}^1_{2\pi}(Q_\mu)$ denotes the extension operator as in the Lemma 2.16 and $R : \ L^2(Q_\mu) \to L^2(\Omega_\mu)$ denotes the restriction operator.

For more details see [17], Theorem 4.14. □

As mentioned in the previous proof, we formulate the following lemma.

Lemma 2.18 *The embedding* $\tilde{J} : \ \mathcal{H}^1_{2\pi}(Q_\mu) \hookrightarrow L^2(Q_\mu)$ *is compact.*

For a proof see [17], Theorem 4.14. This lemma leads us to the following corollary.

Corollary 2.19 *The embedding* $J : \ \mathcal{H}^2_{2\pi}(Q_\mu) \hookrightarrow \mathcal{H}^1_{2\pi}(Q_\mu)$ *is compact.*

This can be shown analogously to the proof of the following corollary. Theorem 2.17 will also be used for this purpose.

Corollary 2.20 *The embedding* $I_0 : \ H^2_{0,0}(\Omega_\mu) \hookrightarrow H^1_{0,0}(\Omega_\mu)$ *is compact.*

Proof: We consider a bounded sequence $(u_j)_{j\in\mathbb{N}} \in H^2_{0,0}(\Omega_\mu)$. We will show that $(u_j)_{j\in\mathbb{N}} \in H^2_{0,0}(\Omega_\mu)$ has a convergent subsequence in $H^1_{0,0}(\Omega_\mu)$.

First, we see that $(u_j)_{j\in\mathbb{N}}$ and $\left(\frac{\partial u_j}{\partial x_i}\right)_{j\in\mathbb{N}}$, for $i = 1, 2$, are bounded in $H^1_{0,0}(\Omega_\mu)$ since

$$\begin{aligned} \|u_j\|^2_{H^1(\Omega_\mu)} &\leq \ \|u_j\|^2_{H^2(\Omega_\mu)} \ \leq \ c\,\|\!|u_j|\!\|^2_{H^2(\Omega_\mu)}, \\ \left\|\frac{\partial u_j}{\partial x_i}\right\|^2_{H^1(\Omega_\mu)} &\leq \ \|u_j\|^2_{H^2(\Omega_\mu)} \ \leq \ c'\,\|\!|u_j|\!\|^2_{H^2(\Omega_\mu)}, \end{aligned}$$

for $i = 1, 2$, with constants c and $c' \in \mathbb{R}$. Hence, $(u_j)_{j\in\mathbb{N}}$ as well as $\left(\frac{\partial u_j}{\partial x_i}\right)_{j\in\mathbb{N}}$ have convergent subsequences in $L^2(\Omega_\mu)$, because the embedding $H^1_{0,0}(\Omega_\mu) \hookrightarrow L^2(\Omega_\mu)$ is compact according to Theorem 2.17.

Let now
$$\partial_i^m u_j \;:=\; \frac{\partial^m u_j}{\partial x_i}, \quad m=0,1, \quad i=1,2,$$
describe these convergent subsequences with $j \in N_1 \subset \mathbb{N}$, $|N_1| = \infty$. Since $L^2(\Omega_\mu)$ is complete, $(\partial_i^m u_j)_{j\in N_1}$, is a Cauchy sequence in $L^2(\Omega_\mu)$, that means there exists N and ε such that
$$\|\partial_i^m u_{j,n_1} - \partial_i^m u_{j,n_2}\|_{L^2(\Omega_\mu)} \;\le\; \frac{\varepsilon}{\sqrt{3}} \quad \text{for all} \quad n_1, n_2 \;\ge\; N.$$
Then
$$\begin{aligned}
&\|u_{j,n_1} - u_{j,n_2}\|_{H^1(\Omega_\mu)} \\
=\;& \left(\|u_{j,n_1} - u_{j,n_2}\|^2_{L^2(\Omega_\mu)} + \|\nabla u_{j,n_1} - \nabla u_{j,n_2}\|^2_{L^2(\Omega_\mu)}\right)^{\frac{1}{2}} \\
\le\;& \left(3\frac{\varepsilon^2}{3}\right)^{\frac{1}{2}} \;=\; \varepsilon.
\end{aligned}$$
Thus, u_j, $j \in N_1$, is a Cauchy sequence in $H^1_{0,0}(\Omega_\mu)$ and, due to the completeness of $H^1_{0,0}(\Omega_\mu)$, convergent in $H^1_{0,0}(\Omega_\mu)$. □

The following corollary is a simple consequence of Theorem 2.17.

Corollary 2.21 *The embedding* $\tilde{I}_\alpha : \; H^1_{0,\alpha}(\Omega_\mu) \hookrightarrow L^2(\Omega_\mu)$ *is compact.*

Proof: Let $(u_j^\alpha)_{j\in\mathbb{N}}$ be a bounded sequence in $H^1_{0,\alpha}(\Omega_\mu)$.
Then, $(u_j^0)_{j\in\mathbb{N}} := (u_j^\alpha\, e^{-i\alpha x_1})_{j\in\mathbb{N}}$ is bounded in $H^1_{0,0}(\Omega_\mu)$. By Theorem 2.17, there exists a convergent subsequence $(u_{jk}^0)_{j,k\in\mathbb{N}}$ of $(u_j^0)_{j\in\mathbb{N}}$. Hence, there is a convergent subsequence $(u_{jk}^\alpha)_{j,k\in\mathbb{N}}$ of $(u_j^\alpha)_{j\in\mathbb{N}}$ in $L^2(\Omega_\mu)$, namely $(u_{jk}^\alpha)_{j,k\in\mathbb{N}} := (u_{jk}^0 e^{i\alpha x_1})_{j,k\in\mathbb{N}}$.

□

We have collected all the tools for a proof of the following corollary. It can be proven analogously to Corollary 2.20, we just need to use Corollary 2.21 instead of Theorem 2.17. For brevity, we will skip the proof.

Corollary 2.22 *The embedding* $I_\alpha : \; H^2_{0,\alpha}(\Omega_\mu) \hookrightarrow H^1_{0,\alpha}(\Omega_\mu)$ *is compact.*

2.3 Spectral Theory

Let H be a Hilbert space and let A be a compact normal operator. Then A has an eigensystem $(\lambda_j, \psi_j)_{j\in\mathbb{N}}$ such that

$$A\psi \;=\; \sum_{j\in\mathbb{N}} \lambda_j\,(\psi, \psi_j)\,\psi_j\,, \quad \psi \in H,$$

where $(\cdot,\cdot)$ denotes the inner product on H.

The spectrum $\sigma(A)$ of A is the set of all values $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not boundedly invertible. It is well known that the spectrum of a compact operator is the union of its eigenvalues λ_j, $j \in \mathbb{N}$, and 0, that means

$$\sigma(A) \;=\; \{0\} \;\cup\; \{\lambda_j :\; j \in \mathbb{N}\}.$$

The multiplicity of each eigenvalue is finite and the only possible accumulation point of the eigenvalues is zero.

Theorem 2.23 ***Courant min-max principle.***
Let H be a Hilbert space and let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and the corresponding norm on H. Furthermore, let $A : X \to X$ a non-negative self-adjoint compact operator, and (λ_n) denote the non-increasing sequence of the nonzero eigenvalues repeated accordingly to their multiplicity. Then

$$\lambda_1 = ||A|| = \sup\{(A\phi, \phi) : ||\phi|| = 1\}$$

and

$$\lambda_{n+1} = \inf_{\psi_1,\ldots,\psi_n\in X} \sup\{(A\phi, \phi) : \phi \perp \psi_1, \ldots, \psi_n,\; ||\phi|| = 1\}, \quad n = 1, 2, \ldots.$$

Proof: The proof can be found in [21], Theorem 15.14. □

Note that if A is a non-positive, self-adjoint compact operator, the first eigenvalue is defined as

$$\lambda_1 = \inf\{(A\phi, \phi) : ||\phi|| = 1\}$$

and is negative. Furthermore,

$$\lambda_{n+1} = \sup_{\psi_1,\ldots,\psi_n\in X} \inf\{(A\phi, \phi) : \phi \perp \psi_1, \ldots, \psi_n,\; ||\phi|| = 1\}, \quad n = 1, 2, \ldots.$$

2.4 The Floquet-Bloch Transformation

The content of this chapter is basically taken from [22]. For more details we refer to this article.

Let $\Omega := \{x \in \mathbb{R}^2 : x_2 \in (0,1)\}$ be a strip and let Ω_μ for fixed $\mu \in \mathbb{Z}$ be a cell of Ω, that means

$$\Omega_\mu \; = \; \{x \in \Omega : x_1 \in [2\pi\mu, 2\pi(\mu+1)]\}.$$

Let $\mathcal{S}$ be the Schwartz space of C_0^∞-functions decay faster than any negative power,

$$\mathcal{S} \; := \; \left\{ \phi \in C_0^\infty(\Omega) : \text{ for all } \alpha,\ \beta \in \mathbb{N}_0^2 : \sup_{x \in \Omega} \left| x^\alpha \frac{\partial^\beta \phi(x)}{\partial x^\beta} \right| < \infty \right\}.$$

Let $\Lambda := [0,1)$ and $e_1 := (1,0)^\top \in \mathbb{R}^2$. We define the Floquet-Bloch transform first from $\mathcal{S}$ into $C_0^\infty(\Omega_\mu \times \overline{\Lambda}) := \big\{ u \in C^\infty(\overline{\Omega}_\mu \times \overline{\Lambda}) :\ \operatorname{supp} u(\cdot,\alpha) \subset \Omega_\mu \text{ for all } \alpha \in \Lambda \big\}$,

$$(Tu)(x,\alpha) \; := \; \tilde{u}(x,\alpha) \; := \; \sum_{m \in \mathbb{Z}} u(x + 2\pi m e_1)\, e^{-i(x_1 + 2\pi m)\alpha}, \tag{2.7}$$

for $x \in \Omega_\mu,\ \alpha \in \Lambda$ and for $u \in \mathcal{S}$. Note that (2.7) is well defined and $(Tu)(x,\alpha) \in C_0^\infty(\Omega_\mu \times \overline{\Lambda})$, because $u(x + 2\pi m e_1)\ e^{-i(x_1+2\pi m)\alpha}$ is infinitely often differentiable with respect to both variables and u decays with any derivative.

Note that the boundary conditions in $C_0^\infty(\Omega_\mu \times \overline{\Lambda})$ only apply to Ω_μ.

Furthermore, $\tilde{u}$ has an extension to a C_0^∞-function on $\Omega \times \mathbb{R}$ such that $\tilde{u}$ is 2π-periodic with respect to x_1 and quasi-periodic in the second argument (see Definition (1.4)). Indeed, for $l \in \mathbb{Z}$,

$$\begin{aligned} \tilde{u}(x + 2\pi l e_1, \alpha) &= \sum_{m \in \mathbb{Z}} u(x + 2\pi(m+l)e_1)\, e^{-i(x_1 + 2\pi(m+l))\alpha} \\ &= \sum_{m \in \mathbb{Z}} u(x + 2\pi m e_1)\, e^{-i(x_1+2\pi m)\alpha} \\ &= \tilde{u}(x,\alpha). \end{aligned}$$

Also, for any $\beta \in \mathbb{Z}$, $\tilde{u}(x,\cdot)$ is quasi-periodic with period β and phase-shift $-x_1$, since

$$\begin{aligned} \tilde{u}(x, \alpha+\beta) &= \sum_{m \in \mathbb{Z}} u(x + 2\pi m e_1)\, e^{-i(x_1+2\pi m)(\alpha+\beta)} \\ &= e^{-ix_1\beta} \sum_{m \in \mathbb{Z}} u(x + 2\pi m e_1)\, e^{-i(x_1+2\pi m)\alpha} \\ &= e^{-ix_1\beta} \tilde{u}(x,\alpha). \end{aligned}$$

We can express $u \in \mathcal{S}$ by $\tilde{u} = Tu$. Let $u(\cdot, x_2) \in \mathcal{S}$. Then

$$\int_\Lambda \tilde{u}(x,\alpha)\, e^{i\alpha x_1}\, d\alpha \;=\; \sum_{m\in\mathbb{Z}} u(x + 2\pi m e_1) \int_\Lambda e^{-2\pi i m\alpha}\, d\alpha \;=\; u(x), \tag{2.8}$$

for $x \in \Omega$, because $\int_\Lambda e^{-2\pi i m\alpha} = 1$ for $m = 0$ and zero for $m \neq 0$. We will extend T to $L^2(\Omega)$ and show that T is unitary, that means $T^*T = \mathrm{Id}$. We then show subsequently that T is invertible and obtain $T^* = T^{-1}$.

For $u, v \in \mathcal{S}$,

$$\begin{aligned}
(\tilde{u}, \tilde{v})_{L^2(\Omega_\mu\times\Lambda)} &= \int_{\Omega_\mu} \Bigg[\sum_{m,l\in\mathbb{Z}} u(x+2\pi m e_1)\overline{v(x+2\pi l e_1)} \\
&\qquad\qquad \cdot \int_\Lambda e^{-i(x_1+2\pi m)\alpha} e^{i(x_1+2\pi l)\alpha}\, d\alpha \Bigg]\, dx \\
&= \int_{\Omega_\mu} \Bigg[\sum_{m\in\mathbb{Z}} u(x+2\pi m e_1)\overline{v(x+2\pi m e_1)} \Bigg]\, dx \\
&= \sum_{m\in\mathbb{Z}} \int_{\Omega_\mu} u(x+2\pi m e_1)\overline{v(x+2\pi m e_1)}\, dx \\
&= \int_\Omega u(x)\overline{v(x)}\, dx \\
&= (u,v)_{L^2(\Omega)}\,.
\end{aligned} \tag{2.9}$$

We see that $\|Tu\|_{L^2(\Omega_\mu\times\Lambda)} = \|u\|_{L^2(\Omega)}$ for all $u \in \mathcal{S}$ and therefore T has a bounded extension from $L^2(\Omega)$ into $L^2(\Omega_\mu \times \Lambda)$. Also, we conclude that $T^*T = Id$, that means, T is unitary. We are now going to show invertability of T in this space, that means $T^* = T^{-1}$. We first note that T is injective and therefore bijective on its range. Also, T is bounded. By the closed graph theorem (see for example [29], Theroem IV.4.5), we conclude that the image is closed. Therefore, it is sufficient to show that the range of T is dense in $L^2(\Omega_\mu \times \Lambda)$. Let $f \in C_0^\infty(\Omega_\mu \times \Lambda)$ and extend $f(\cdot, \alpha)$ to a 2π-periodic function with respect to x_1 in Ω. We define u by

$$u(x) \;:=\; \int_\Lambda f(x,\alpha) e^{i\alpha x_1}\, d\alpha, \quad x \in \Omega.$$

Every arbitrarily often differentiable function with compact support is in $\mathcal{S}$. Therefore, for fixed α, $f(\cdot, \alpha)$ is in $\mathcal{S}$ and we conclude by partial integration, that $u \in \mathcal{S}$.

Therefore (see (2.8)), u has a representation as

$$u(x) \;=\; \int_\Lambda \tilde{u}(x,\alpha)e^{i\alpha x_1}\,d\alpha, \quad x \in \Omega.$$

Defining $g = f - \tilde{u}$, we conclude

$$\int_\Lambda g(x,\alpha)e^{i\alpha x_1}\,d\alpha \;=\; 0, \quad x \in \Omega.$$

We will show that for fixed $\alpha \in \Lambda$, $g(x,\alpha) = 0$ for all $x \in \Omega$. This proves $f = Tu$ and that T is invertible with T^{-1} given by

$$(T^* f)(x) \;=\; (T^{-1}f)(x) \;=\; \int_\Lambda f(x,\alpha)e^{i\alpha x_1}\,d\alpha, \quad x \in \Omega, \tag{2.10}$$

where $f(\cdot,\alpha)$ has to be extended 2π-periodically into Ω.

To show that $g(x,\alpha) = 0$ for all $x \in \Omega$ and fixed α, we use the periodicity of g. We fix x and substitute $x + 2\pi m e_1$ for x with arbitrary $m \in \mathbb{Z}$. This yields

$$\begin{aligned} 0 \;=\; \int_\Lambda g(x+2\pi m e_1,\alpha)e^{i\alpha(x_1+2\pi m)}\,d\alpha \;&=\; \int_\Lambda g(x,\alpha)e^{i\alpha x_1}e^{2\pi i\alpha m}\,d\alpha \\ &=\; \int_\Lambda g_x(\alpha)e^{2\pi i\alpha m}\,d\alpha, \end{aligned} \tag{2.11}$$

with $g_x(\alpha) := g(x,\alpha)e^{i\alpha x_1}$, $\alpha \in \Lambda$.

From (2.11) we conclude that all Fourier coefficients of g_x vanish and thus $g_x = 0$ for every $x \in \Omega$. Hence, $g(x,\alpha) = 0$ for all $x \in \Omega$.

Corollary 2.24 *Let $q \in L^\infty(\mathbb{R}^2)$ be 2π-periodic. Then $T(qu) = qTu$.*

Proof: With

$$u(x) \;=\; \int_\Lambda \tilde{u}(x,\alpha)e^{i\alpha x_1}\,d\alpha, \quad x \in \Omega. \tag{2.12}$$

we conclude

$$q(x)u(x) \;=\; \int_\Lambda q(x)\tilde{u}(x,\alpha)e^{i\alpha x_1}d\alpha \;=\; (T^{-1}(q\tilde{u}))(x), \quad x \in \Omega.$$

$\square$

We define the following space

$$\begin{aligned} L^2(\Lambda, H^2_{0,0}(\Omega_\mu)) \ := \ \{f \in L^2(\Omega_\mu \times \Lambda) : f(\cdot, \alpha) \in H^2_{0,0}(\Omega_\mu) \text{ a.e. on } \Lambda, \\ \alpha \mapsto \|\!|f(\cdot,\alpha)\|\!|_{H^2(\Omega_\mu)} \text{ in } L^2(\Lambda)\}. \end{aligned}$$

We equip this space with the norm

$$\|f\|^2_{L^2(\Lambda, H^2(\Omega_\mu))} \ = \ \int_\Lambda \|\!|f(\cdot,\alpha)\|\!|^2_{H^2(\Omega_\mu)} d\alpha.$$

Also, we define

$$\begin{aligned} L^2(\Lambda, L^2(\Omega_\mu)) \ := \ \{f \in L^2(\Omega_\mu \times \Lambda) : f(\cdot, \alpha) \in L^2(\Omega_\mu) \text{ a.e. on } \Lambda, \\ \alpha \mapsto \|f(\cdot,\alpha)\|_{L^2(\Omega_\mu)} \text{ in } L^2(\Lambda)\}, \end{aligned}$$

with the norm

$$\|f\|^2_{L^2(\Lambda, L^2(\Omega_\mu))} \ = \ \int_\Lambda \|f(\cdot,\alpha)\|^2_{L^2(\Omega_\mu)} d\alpha.$$

Below, we show in Theorem 2.25 that T maps $H^2_0(\Omega)$ into $L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$. Analogously, but much simpler, one sees that T maps $L^2(\Omega)$ into $L^2(\Lambda, L^2(\Omega_\mu))$.

Theorem 2.25 *The operator T is well-defined and a bounded isomorphism from $H^2_0(\Omega)$ onto $L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$. Furthermore, it holds true that*

$$\begin{aligned} T\nabla u &= \nabla_x Tu + ie_1 \alpha Tu, \\ T\partial_{x_2}\partial_{x_1} u &= \partial_{x_2}\partial_{x_1} Tu + i\alpha T\partial_{x_2} u, \\ T\Delta u &= \Delta_x Tu + 2i\alpha T\partial_{x_1} u + \alpha^2 Tu. \end{aligned}$$

Proof: For simplicity, in this proof, we use the general H^2-norm $\|\cdot\|_{H^2(\Omega_\mu)}$. This is possible, because the norms $\|\cdot\|_{H^2(\Omega_\mu)}$ and $\|\!|\cdot\|\!|_{H^2(\Omega_\mu)}$ are equivalent on $H^2_{0,0}(\Omega_\mu)$.

Let first $u \in \mathcal{S}$. Then $Tu \in C_0^\infty(\Omega \times \mathbb{R})$ and thus from (2.12)

$$\begin{aligned} \nabla u(x) &= \int_\Lambda [\nabla_x \tilde{u}(x,\alpha) + ie_1 \alpha \tilde{u}(x,\alpha)]\, e^{i\alpha x_1}\, d\alpha, \\ \partial_{x_1}\partial_{x_2} u(x) &= \int_\Lambda \big[\partial_{x_1}\partial_{x_2}\tilde{u}(x,\alpha) + i\alpha\partial_{x_2}\tilde{u}(x,\alpha)\big] e^{i\alpha x_1}\, d\alpha \\ \Delta u(x) &= \int_\Lambda \big[\Delta_x \tilde{u}(x,\alpha) + 2i\alpha\partial_{x_1}\tilde{u}(x,\alpha) - \alpha^2\tilde{u}(x,\alpha)\big] e^{i\alpha x_1}\, d\alpha, \end{aligned}$$

for $x \in \Omega$. That means,

$$\begin{aligned}
T\nabla u &= \nabla_x Tu + ie_1\alpha Tu, && (2.13)\\
T\partial_{x_1}\partial_{x_2}u &= \partial_{x_1}\partial_{x_2}Tu + i\alpha\partial_{x_2}Tu && (2.14)\\
&\overset{(2.13)}{=} \partial_{x_1}\partial_{x_2}Tu + i\alpha T\partial_{x_2}u\\
T\Delta u &= \Delta_x Tu + 2i\alpha\partial_{x_1}Tu - \alpha^2 Tu && (2.15)\\
&\overset{(2.13)}{=} \Delta_x Tu + 2i\alpha T\partial_{x_1}u + \alpha^2 Tu. && (2.16)
\end{aligned}$$

We estimate for $u \in \mathcal{S}$,

$$\begin{aligned}
& \|Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|\nabla_x Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|\Delta_x Tu + 2\partial_{x_1}\partial_{x_2}Tu\|^2_{L^2(\Omega_\mu\times\Lambda)}\\
= \; & \|Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|T\nabla u - Tie_1\alpha u\|^2_{L^2(\Omega_\mu\times\Lambda)}\\
& + \left\| T\Delta u - T2i\alpha\partial_{x_1}u - T\alpha^2 u \; + \; 2\left(T\partial_{x_1}\partial_{x_2}u - Ti\alpha\partial_{x_2}u\right)\right\|^2_{L^2(\Omega_\mu\times\Lambda)}\\
\le \; & c_1\|Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + c_2\|T\nabla u\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|T\Delta u + T2\partial_{x_1}\partial_{x_2}u\|^2_{L^2(\Omega_\mu\times\Lambda)}\\
= \; & c_1\|u\|^2_{L^2(\Omega)} + c_2\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u + 2\partial_{x_1}\partial_{x_2}u\|^2_{L^2(\Omega)}\\
\le \; & c_3\|u\|^2_{H^2(\Omega)}, && (2.17)
\end{aligned}$$

with real constants c_1, c_2 and c_3. Now we extend u to $H^2_0(\Omega)$ and show boundedness of T from $H^2_0(\Omega)$ into $L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$,

$$\begin{aligned}
& \|Tu\|^2_{L^2(\Lambda, H^2_{0,0}(\Omega_\mu))}\\
= \; & \int_\Lambda \|Tu(\cdot,\alpha)\|^2_{H^2(\Omega_\mu)} d\alpha\\
= \; & \int_\Lambda \left[\|Tu(\cdot,\alpha)\|^2_{L^2(\Omega_\mu)} + \|\nabla_x Tu(\cdot,\alpha)\|^2_{L^2(\Omega_\mu)} + \|\Delta_x Tu + 2\partial_{x_1}\partial_{x_2}Tu\|^2_{L^2(\Omega_\mu)}\right] d\alpha\\
= \; & \|Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|\nabla_x Tu\|^2_{L^2(\Omega_\mu\times\Lambda)} + \|\Delta_x Tu \; + \; 2\partial_{x_1}\partial_{x_2}Tu\|^2_{L^2(\Omega_\mu\times\Lambda)}\\
\overset{(2.17)}{\le} \; & c_3\|u\|^2_{H^2(\Omega)}.
\end{aligned}$$

We conclude T is well defined and bounded.

To show surjectivity, we let $Tu \in L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$. In particular, this means that $Tu(\cdot,\alpha) \in H^2_{0,0}(\Omega_\mu)$ and hence $Tu(\cdot,\alpha)$, $\nabla_x Tu(\cdot,\alpha)$ and $\Delta_x Tu(\cdot,\alpha)$ are in $L^2(\Omega_\mu)$ almost everywhere on Λ.
With the identities (2.13) and (2.15) we see that $T\nabla u$ and $T\Delta u$ are in $L^2(\Omega_\mu\times\Lambda)$. Since T is invertible in this space, we conclude that u, Δu and ∇u are in $L^2(\Omega)$, and hence $u \in H^2_0(\Omega)$.

$\square$

2.5 Entire Functions

This chapter provides some basic knowledge about entire functions.

Definition 2.26 *(Entire Functions)*
A function $f : \mathbb{C} \to \mathbb{C}$, $f \not\equiv 0$, is called an entire function if it is holomorphic in the entire complex plane.

- *The maximum modulus $M(r)$ is defined by*

$$M(r) \ := \ \max_{0 \le \theta \le 2\pi} \left| f\left(re^{i\theta}\right) \right|,$$

- *the order ρ of $f(z)$ is defined by*

$$\rho \ := \ \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r},$$

- *and if $f(z)$ is of order ρ then the type τ is defined by*

$$\tau \ := \ \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}.$$

Definition 2.27 *An entire function of order $\rho = 1$ and type τ is called an entire function of exponential type τ.*

The following elementary propositions are taken from [6], page 242 and 245.

Lemma 2.28
- *Let f and g be two entire functions of exponential type at most τ. Then $f + g$ is of exponential type at most τ.*
- *Let f and g be two entire functions of exponential type τ_f and τ_g, respectively. Then the function $f \cdot g$ is an entire function of exponential type at most $\tau_f + \tau_g$.*
- *Let f and g be two entire functions of exponential type τ_f and τ_g with $\tau_f > \tau_g$. Then the function $f + g$ is an entire function of exponential type τ_f.*

Furthermore, the following holds true.

Lemma 2.29 *Let f be an entire function of exponential type. Then $g(k) := kf(k)$ is an entire function of the same exponential type.*

Proof: It holds true that

$$M_g(r) \;=\; \max_{0\le\theta\le 2\pi} \left|g\left(re^{i\theta}\right)\right| \;=\; \max_{0\le\theta\le 2\pi} r\left|f\left(re^{i\theta}\right)\right| \;=\; rM_f(r),$$

where M_r and M_f denote the maximum modulus of f and g respectively. For the order ρ_g of g it holds

$$\begin{aligned} \rho_g &= \limsup_{r\to\infty} \frac{\log\log M_g(r)}{\log r} = \limsup_{r\to\infty} \frac{\log\left(\log r + \log M_f(r)\right)}{\log r} \\ &\overset{(*)}{\le} \limsup_{r\to\infty} \frac{\log(2\log M_f(r))}{\log r} = \limsup_{r\to\infty} \frac{\log 2 + \log\log M_f(r)}{\log r} = \rho_f, \end{aligned}$$

where ρ_f denotes the order of f. Note, that in $(*)$ we have used that $M_f(r) \ge r$ for r large enough. We can assume this, because if r was growing faster than $M_f(r)$, then $\rho_f = 0$ and hence f would not be an entire function of exponential type.

On the other hand we see that for r large enough it is $\frac{\log\log(rM_f(r))}{\log r} \ge \frac{\log\log(M_f(r))}{\log r}$ and hence

$$\rho_g \;=\; \limsup_{r\to\infty} \frac{\log\log(rM_f(r))}{\log r} \;\ge\; \limsup_{r\to\infty} \frac{\log\log(M_f(r))}{\log r} \;=\; \rho_f.$$

We conclude $\rho_g = \rho_f = 1$. For the type we see that

$$\begin{aligned} \tau_g = \limsup_{r\to\infty} \frac{\log M_g(r)}{r} &= \limsup_{r\to\infty} \frac{\log r}{r} + \frac{\log M_f(r)}{r} \\ &= \limsup_{r\to\infty} \frac{\log M_f(r)}{r} = \tau_f, \end{aligned}$$

where τ_f and τ_g denote the type of f and g respectively.

Hence, f and g are exponential functions of the same type. □

The following theorem can be found in [28], page 266.

Theorem 2.30 ***Laguerre***
Let f be an entire function of order less than two that is real for real z and has only real zeros. Then the zeros of f' are also all real and are separated from each other by the zeros of f.

Corollary 2.31 *Let f be an entire function of order less than two that is real for real z. Suppose that f has infinitely many real zeros and only a finite number of complex ones. Then f' vanishes only once on each interval (z_n, z_{n+1}) formed by two consecutive real zeros of f when the interval is sufficiently far from the origin.*

For a proof see [23], Theorem 2.3. The well known Paley-Wiener Theorem can be found for example in [20], page 30.

Theorem 2.32 ***Paley-Wiener***
The entire function f is of exponential type less or equal to τ and belongs to $L^2(\mathbb{R})$, if, and only if, it has the form

$$f(z) \;=\; \int_{-\tau}^{\tau} \varphi(t) e^{izt}\, dt\,, \quad z \in \mathbb{R},$$

for some $\varphi \in L^2\,(-\tau, \tau)$. f is of type τ if φ does not vanish on a neighborhood of τ or $-\tau$.

3 Two Scattering Problems

We will describe two scattering problems for periodic media in this chapter. We will then consider the case of non-scattering incident waves, that means that the scattered field vanishes. This will lead us to a system of differential equations, which is set up inside of the periodic medium. The problem of solving this system of equations is called the interior transmission eigenvalue problem.

To set up the scattering problem, we will need to consider for $R \in \mathbb{R}_{>0}$

$$\begin{aligned} \Pi_\mu &:= \big[2\pi\mu, (\mu+1)2\pi\big] \times \mathbb{R}, \\ \Pi_{\mu,R} &:= \big[2\pi\mu, (\mu+1)2\pi\big] \times \big(-R, R\big), \end{aligned}$$

with $\mu \in \mathbb{Z}$ fixed. Also, the following additional function spaces will be useful to describe the scattering problems. For $p = 1, 2$, we define

$$\begin{aligned} H^p_{kd}(\Pi_\mu) := \{u \in H^p(\Pi_\mu) \,:\, & e^{ikd2\pi n} u(x_1, x_2) = u(x_1 + 2\pi n, x_2), \\ & \text{for all } x \in \Pi_\mu,\, n \in \mathbb{Z}\}, \end{aligned}$$

and analogously $H^p_{kd}(\Pi_{\mu,R})$. Furthermore, we define

$$H^p_{loc}(D) := \{u : D \to \mathbb{C} \,:\, u\big|_{\tilde{D}} \in H^p(\tilde{D}) \text{ for any open and bounded set } \tilde{D} \subset D\},$$

for some Lipschitz domain D, and

$$H^p_{kd,loc}(\Pi_\mu) := \{u : \Pi_\mu \to \mathbb{C} \,:\, u\big|_{\Pi_{k,R}} \in H^p_{kd}(\Pi_{\mu,R}), \text{ for any } R \in \mathbb{R}_{>0}\}.$$

3.1 Scattering Problem 1

We consider the periodic medium

$$\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, f(x_1) < x_2 < g(x_1)\big\}$$

as described before in (1.1). Let q denote the 2π-periodic contrast, that is $\operatorname{supp} q(x) = \overline{\Omega}$. Again, we denote by Γ the boundary of Ω. We first describe the scattering problem formally. To this end, we do not yet care about the regularity of the contrast or of the scattered field. The periodic medium Ω is excited by an incident plane wave u^i, which is given by

$$u^i(x) = e^{ikx\cdot\Theta},$$

where $k > 0$ denotes the wave number and

$$\Theta := (\sin\varphi, -\cos\varphi)^\top, \quad |\varphi| < \pi/2, \tag{3.1}$$

is the direction of propagation. We denote the first component of Θ by $d := \sin\varphi$, $|d| < 1$.

In general, the incident plane wave $u^i(x) = e^{ikx\cdot\Theta}$ does not share the 2π-periodicity of Ω. It rather holds

$$\begin{aligned} u^i(x_1 + 2\pi, x_2) &= \exp\left(ik(x_1 + 2\pi)\sin\varphi - ikx_2\cos\varphi\right) \\ &= \exp\left(ikx_1\sin\varphi - ikx_2\cos\varphi\right)\exp\left(ik2\pi\sin\varphi\right) \\ &= \exp\left(ikx\cdot\theta\right)\exp\left(ik2\pi\sin\varphi\right) \\ &= u^i(x)\exp\left(ik2\pi\sin\varphi\right). \end{aligned}$$

Plugging in $d = \sin\varphi$ we see

$$u^i(x_1 + 2\pi, x_2) = e^{ikd2\pi}u^i(x_1, x_2).$$

We compare this with the definition of quasi-periodicity and see that u^i is a kd-quasi-periodic function. It is $\Delta u^i(x) = \Delta e^{ikx\cdot\Theta} = -k^2 e^{ikx\cdot\Theta} = -k^2 u^i(x)$ and therefore u^i satisfies the Helmholtz equation in $\mathbb{R}^2$,

$$\Delta u^i + k^2 u^i = 0 \quad \text{in } \mathbb{R}^2.$$

The scattering problem reads, given u^i, determine the scattered field u^s and the total field $u := u^i + u^s$, which satisfies

$$\Delta u + k^2(1 + q)u = 0 \qquad \text{in } \mathbb{R}^2, \tag{3.2}$$

such that u^s satisfies the Rayleigh radiation condition, which we will introduce in a moment.

On the boundary Γ of the periodic medium the following transmission conditions are valid

$$\big[u\big]_\Gamma \;=\; 0 \quad \text{and} \quad \left[\frac{\partial u}{\partial\nu}\right]_\Gamma \;=\; 0\,, \tag{3.3}$$

where ν denotes the outward pointing unit normal vector, and $\big[f\big] := f\big|_+ - f\big|_-$.

Given that u^i satisfies the Helmholtz equation in $\mathbb{R}^2$, from (3.2) we obtain the following equation for the scattered field.

$$\Delta u^s + k^2(1 + q)u^s = -k^2 q u^i \qquad \text{in } \mathbb{R}^2. \tag{3.4}$$

Furthermore, we assume the scattered field to be kd-quasi-periodic.

As already mentioned, we require u^s to satisfy the Rayleigh radiating condition. That is, there exist $u_n^{\pm} \in \mathbb{C}$ such that

$$u^s(x) \;=\; \sum_{n\in\mathbb{Z}} u_n^{\pm} e^{i(\alpha_n x_1 \pm \beta_n x_2)}, \tag{3.5}$$

with $\alpha_n = n + kd$ and β_n are defined by

$$\beta_n = \begin{cases} \sqrt{k^2 - \alpha_n^2}, & |\alpha_n| \leq k\,, \\ i\sqrt{\alpha_n^2 - k^2}, & |\alpha_n| > k\,. \end{cases}$$

The plus and the minus sign in (3.5) refer to the cases when $x_2 > \max\{g(x_1),\, x_1 \in \mathbb{R}\}$ or $x_2 < \min\{f(x_1),\, x_1 \in \mathbb{R}\}$, respectively. Note that since we consider quasi-periodic functions, we can restrict ourselves to $x_1 \in (2\pi\mu, 2\pi(\mu+1))$ for fixed $\mu \in \mathbb{Z}$ from now on.

The Rayleigh radiation condition is recovered by a Fourier expansion with respect to x_1 of $e^{-ikdx_1}u(x)$, that means $u(x) = \sum_{n\in\mathbb{Z}} a_n(x_2)e^{ix_1\alpha_n}$. Using furthermore that u satisfies the Helmholtz equation outside of Ω, yields $a_n'' + (k^2 - \alpha_n^2)a_n = 0$ with $\alpha_n = n + kd$. Solving this and using that in $u = u^s + u^i$ the incident field is given by $e^{ikx\cdot\Theta}$, naturally leads to the Rayleigh expansion radiation condition, which ensures that the scattered field is outgoing with respect to x_2. For more details, we refer to [24].

Let us now consider a 2π-periodic contrast $q \in L^\infty(\mathbb{R}^2)$. Classically, the scattering problem is studied for $u \in H^1_{kd,loc}(\Pi_\mu)$, see for example [27]. That means, equation (3.2) is understood in the weak sense, that is

$$\int_{\Pi_\mu} \nabla u \cdot \nabla\overline{\varphi} - k^2(1+q)u\overline{\varphi}\,dx \;=\; 0, \tag{3.6}$$

for all $\varphi \in H^1_{kd}(\Pi_\mu)$ with compact support with respect to x_2, that means, there exists a compact set $D \subset \Pi_\mu$ such that φ is supported in D. All terms in (3.6) are well defined and u satisfies the first transmission condition, $u|_+ = u|_-$, by $u \in H^1_{kd,loc}(\Pi_\mu)$. However, only if a weak solution u to (3.6) is sufficiently regular, it can be interpreted as a solution to (3.2) with (3.3). In this case, the second transmission condition, $\frac{\partial u}{\partial\nu}\Big|_+ = \frac{\partial u}{\partial\nu}\Big|_-$ is well defined and included in the formulation (3.6). This can easily be seen by multiplying $\Delta u + k^2(1+q)u$ with a testfunction $\varphi \in H^1_{kd}(\Pi_\mu)$ with compact support with respect to x_2, integrating over Ω_μ and $\Pi_\mu \setminus \Omega_\mu$ and applying Green's first identity. Due to the kd-quasi-periodicity, the boundary integrals on the vertical boundaries cancel out. To finally obtain (3.6), the remaining boundary integrals also have to cancel out, which shows the second transmission condition in (3.3).

By standard elliptic regularity results, see for example [11], Theorem 8.8, a weak solution to (3.2) lies in fact in $H^2_{kd,loc}(\Pi_\mu)$. It is even a classical solution in $\mathbb{R}^2 \setminus \overline{\Omega}$ (see [11], Corollary 8.11) and hence analytic there (see [8], Theorem 3.5). For more details on the scattering problem, we refer to [27].

In this work, we are interested in non-scattering waves, that means that the scattered field u^s vanishes outside of Ω_μ, and hence $u = u^i$ in $\Pi_\mu \setminus \Omega_\mu$. The transmission conditions then yield $u = u^i$ and $\dfrac{\partial u^i}{\partial \nu} = \dfrac{\partial u}{\partial \nu}$ on Γ_μ and inside of Ω_μ it holds $\Delta u^i + k^2 u^i = 0$ and $\Delta u + k^2(1+q)u = 0$. The scattering problem above hence leads to the following problem, which is set up only inside of Ω_μ. (We will first write the system of equations formally and comment on regularity afterwards.)

We wish to find non-trivial kd-quasi-periodic solutions v and w to the system of equations,

$$\begin{aligned}
\Delta w + k^2 w &= 0 \quad \text{in } \Omega_\mu && (3.7)\\
\Delta v + k^2(1+q)v &= 0 \quad \text{in } \Omega_\mu && (3.8)\\
w &= v \quad \text{on } \Gamma_\mu && (3.9)\\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on } \Gamma_\mu. && (3.10)
\end{aligned}$$

This problem is called the *interior transmission eigenvalue problem.* The values for k such that the interior transmission eigenvalue problem has non-trivial solutions are called *transmission eigenvalues.*

Remark 3.1 *Note that k appears in the differential equation and also in the solution space.*

We will go into more details in Chapter 4.2.

Closing this subsection, we would roughly like to comment on regularities of the solutions to the interior transmission eigenvalue problem. Up to now, we have used the space $H^1_{kd,loc}(\Pi_\mu)$ as an ansatz space, which was suitable to treat the transmission conditions. However, when considering the interior transmission eigenvalue problem independently of the scattering problem, we will seek for solutions $(v, w) \in L^2(\Omega_\mu) \times L^2(\Omega_\mu)$ such that the difference $u := w - v$ is in $H^2_{0,kd}(\Omega_\mu)$. We then write the equations in the *ultra weak formulation*, that is

$$\int_{\Omega_\mu} w \left[\Delta\overline{\varphi} + k^2(1+q)\overline{\varphi}\right] dx = 0,$$

for all $\varphi \in H^2_{0,kd}(\Omega)$, and the analogous form for v. With this formulation, the two equations on the boundary, (3.9) and (3.10), are included in the Sobolev space $H^2_{0,kd}(\Omega_\mu)$, when considering $u = w - v \in H^2_{0,kd}(\Omega_\mu)$.

3.2 Scattering Problem 2

Let again $k > 0$ be the wave number. We will consider the case that a point source at some point $y \in \mathbb{R}^2$ with $y_2 > 1$ is scattered by an inhomogeneous layer $\Omega_+ = \mathbb{R} \times (0,1)$, which is contained in a strip $\Omega'_+ = \mathbb{R} \times (0, 1+h)$, for some $h > 0$, on top of a perfect conductor. The point source is defined as $\Phi_k(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$, $x \neq y$, which is the fundamental solution of the Helmholtz equation, with the Hankel function $H_0^{(1)}$ of the first kind of order zero. We enlarge Ω'_+ by choosing h big enough, such that y is contained in the strip Ω'_+.

By $\Gamma_{lo} = \mathbb{R} \times \{0\}$ we denote the lower part of the boundary of Ω_+ and Ω'_+ and by $\Gamma_{up+} = \mathbb{R} \times \{1\}$ and $\Gamma'_{up+} = \mathbb{R} \times \{1+h\}$ the upper parts, respectively.

We reflect this setting at Γ_{lo} and obtain a strip $\Omega := \mathbb{R} \times (-1, 1)$ and we extend the point source as an odd function with respect to x_2 into the lower half space $\mathbb{R}^2_- := \mathbb{R} \times (-\infty, 0)$. This yields an additional point source $-\Phi_k(x, y^*)$ with $y^* = (y_1, -y_2)^\top$. The complete setting is illustrated in Figure 5.

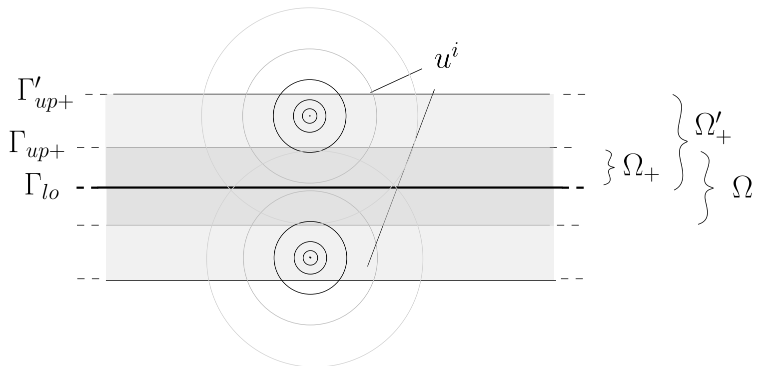

Figure 5: The Setting for Scattering Problem 2

We consider the scattering problem in the upper half space $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$. We note that due to the reflection and the odd extension, (with some modifications) similar results can be shown for the lower half space as well. Therefore, it is sufficient to consider the upper half space.

The incident wave is given by $u^i(x) := \Phi_k(x,y) - \Phi_k(x,y^*)$, $x \in \mathbb{R}^2_+$, $x \neq y$, for fixed $y \in \mathbb{R}^2_+$ and $y^* = (y_1, -y_2)^\top$ (compare with Figure 5). We note some properties of the incident field u^i. First, it is $u^i = 0$ on Γ_{lo}. Second, for $x \in \Omega_+$ and $|x_1| \geq 1$, there exist some constants c' and c'', which only depend on k and y such that

$$|u^i(x)| \;\leq\; c' \frac{x_2}{|x|^{3/2}}\,, \tag{3.11}$$

$$|\nabla u^i(x)| \;\leq\; c''\, \frac{x_2}{|x|^{3/2}}\,. \tag{3.12}$$

These estimates follow directly from the proof of Lemma 4.1 in [26]. There, it is shown that

$$|u^i(x)| \leq \frac{x_2\, y_2\, k}{|x-y^*|+|x-y|} \max_{|x-y^*|\leq s\leq|x-y|} |H_1^{(1)}(ks)|.$$

Using the asymptotic behaviour

$$H_n^{(1)}(z) \;=\; \sqrt{\frac{2}{\pi z}} e^{i(z-n\frac{\pi}{2}-\frac{\pi}{4})} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right), \quad |z|\to\infty, \quad |\arg z|\leq \pi/2,$$

for the Hankel function of order n and of the first kind, we see that $|u^i(x)| \leq \tilde{c}\frac{x_2}{|x-y|^{3/2}}$, where $\tilde{c}$ depends on k and y. Now we use, that for $|x_1|$ large enough, there exists a constant $c>0$ such that $|x-y| \geq |x|\,(1-|y|/|x|) \geq c\,|x|$. This shows that $|u^i(x)| \leq \tilde{c}\,\frac{x_2}{|x-y|^{3/2}} \leq \frac{\tilde{c}}{c}\frac{x_2}{|x|^{3/2}}$, for $|x_1|>R$, for some $R\in\mathbb{R}$. We define $\hat{c} := \max_{1\leq|x_1|\leq R}\left\{\frac{|u^i(x)|}{x_2}|x|^{3/2}\right\}$ and $c' := \max\left\{\hat{c}, \frac{\tilde{c}}{c}\right\}$. This yields $|u^i(x)| \;\leq\; c'\frac{x_2}{|x|^{3/2}}$ for $|x_1|\geq 1$, the first estimate (3.11).

For the second estimate (3.12), we first evaluate $\nabla u^i(x)$ which can be done analogously to the proof of Lemma 4.1 in [26], where $\nabla_y\left(\Phi_k(x,y)-\Phi_k(x,y^*)\right)$ is considered. Taking the gradient with respect to x simply leads to some additional minus sign. We obtain analogously to equation (4.15) of [26], that for $|x_1|$ large enough, $\nabla u^i \leq c\,\frac{x_2}{|x-y|^{3/2}}$, $c>0$. Using the same arguments as above, we obtain the second estimate (3.12).

From (3.11) and (3.12) we see that $u^i\in L^1(\Omega_+)$ as well as $\nabla u^i \in L^1(\Omega_+)$.

We are now going to describe the scattering problem. Let q be the 2π-periodic contrast. We do not yet care about the regularity of q. The incident field satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^2_+\setminus\{y\}$. The scattering problem is to determine $u^t\in H^1_{loc}(\mathbb{R}^2_+\setminus\{y\})$ which satisfies

$$\Delta u^t + k^2(1+q)u^t \;=\; 0 \quad \text{in } \mathbb{R}^2_+\setminus\{y\}, \quad u^t = 0 \text{ on } \Gamma_{lo}, \tag{3.13}$$

such that the scattered field $u^s = u^t - u^i$ is in $H^1_{loc}(\mathbb{R}^2_+)$. On Γ_{up+}, the transmission conditions (3.3) are valid for u^t.

Moreover, we need a suitable radiation condition, which we will introduce in a moment. First, we will transform the problem into an inhomogeneous equation in $H^1_{loc}(\mathbb{R}^2_+)$, to avoid dealing with singularities. To this end, we fix some $\varepsilon>0$ such that $0<2\varepsilon<\min\{y_2-1,\,1+h-y_2\}$, and choose a function $\varphi\in C^\infty(\mathbb{R}^2)$, that satisfies

$$\begin{aligned}\varphi(x) &= 0 \quad \text{for } |x-y|\leq\varepsilon,\\ \varphi(x) &= 1 \quad \text{for } |x-y|\geq 2\varepsilon.\end{aligned}$$

We then consider

$$u \;:=\; \varphi u^i + u^s \;=\; u^t + (\varphi - 1)u^i \;\in H^1_{loc}(\mathbb{R}^2_+).$$

Then it is

$$u = u^t \text{ for } |x-y| \geq 2\varepsilon \quad \text{and} \quad u = u^s \text{ for } |x-y| \leq \varepsilon.$$

In particular, $u = u^t$ in Ω_+ and u vanishes on Γ_{lo}.

We now consider $\Delta u + k^2(1+q)u$, which vanishes for $|x-y| \geq 2$. Since furthermore $u = \varphi u^i + u^s$ and u^s satisfies $\Delta u^s + k^2(1+q)u^s = \Delta u^s + k^2 u^s = 0$ outside of Ω_+ (and hence inside the disc $D := \{x \in \mathbb{R}^2_+ : \; |x-y| < 2\varepsilon\}$), we obtain

$$\begin{aligned}
& \Delta u + k^2(1+q)u \\
=\;& \big(\Delta + k^2(1+q)\big)\left(\varphi u^i\right) \\
=\;& \Delta\varphi\, u^i + 2\nabla\varphi \cdot \nabla u^i + \varphi\left(\Delta u^i + k^2(1+q)u^i\right) \\
=\;& 2\nabla\varphi \cdot \nabla u^i + \left(\Delta\varphi + \varphi k^2 q\right) u^i := f,
\end{aligned} \tag{3.14}$$

where $f \in L^2(\mathbb{R}^2_+)$ has support in the disc $D = \{x \in \mathbb{R}^2_+ : \; |x-y| < 2\varepsilon\}$.

Let now $q \in L^\infty(\mathbb{R}^2_+)$. Equation (3.14) is understood in the weak sense, that is for $u \in H^1_{loc}(\mathbb{R}^2_+)$ with $u = 0$ on Γ_{lo},

$$\int_{\mathbb{R}^2_+} \nabla u \cdot \nabla\overline{\psi} - k^2(1+q)u\,\overline{\psi}\, dx \;=\; \int_D f\overline{\psi}\, dx.$$

for any $\psi \in H^1(\mathbb{R}^2_+)$ with compact support.

The scattering problem is now undestood as to determine $u^s = u - \varphi u^i$ and with this the total field $u^t = u^s + u^i$. In the same way as in Scattering Problem 1, by choosing a weak solution u regular enough, u provides a solution to (3.14). Then, (note that $u = u^t$ for $|x-y| \geq 2\varepsilon$ and hence on the boundary Γ_{up+}) the transmission condition (3.3) make sense for u.

Furthermore, as mentioned before, to set up the scattering problem properly, a suitable radiation condition needs to be introduced. To this end, first, we require u^s to satisfies the upward propagating radiation condition (UPRC) (see [7]) that is, there exists $\phi \in L^\infty(\mathbb{R})$ with

$$u^s(x) \;=\; 2\int_{\mathbb{R}\times\{1+h\}} \phi(y)\frac{\partial}{\partial y_2}\Phi_k(x,y)ds(y)\,, \quad x_2 > 1+h.$$

Note that the integrand exists because $\frac{\partial}{\partial y_2}\Phi_k(x,\cdot) \in L^1(\Omega_+)$ for $x \in \mathbb{R}^2_+ \setminus \Omega'_+$. This can be shown analogously to (3.12).

Theorem 2.9 in [7] shows that u^i also satisfy the UPRC for $x_2 > 1+h$, because it is a radiating solution in terms of Definitions 2.5 in [7]. Therefore, we conclude that u satisfies the UPRC above Ω'_+. However, this condition is not sufficient, as one can not expect uniqueness of the solutions to the scattering problem, see Example 2.3 in [18]. We therefore need a second condition, that treats u when x_1 tends to $\pm\infty$. A proper radiation condition is introduced in [18]. We will roughly describe this condition. For more details, we refer to this article.

As shown in [18], u is of the form

$$u \;=\; u^{(1)} \;+\; u^{(2)},$$

that means

$$u^t \;=\; (1-\varphi)u^i \;+\; u^{(1)} \;+\; u^{(2)},$$

where $u^{(1)}$ is in $H^1(\mathbb{R}\times(0,1+H))$, for all $H > h$ and $u^{(2)}$ has the form

$$u^{(2)}(x) \;:=\; \psi^+(x_1)\sum_{j\in J}\sum_{l\in L_j^+} a_{j,l}^+\phi_{j,l}(x) \;+\; \psi^-(x_1)\sum_{j\in J}\sum_{l\in L_j^-} a_{j,l}^-\phi_{j,l}(x), \quad x\in\mathbb{R}^2_+.$$

Here, $\psi^\pm$ are functions such that

$$\begin{aligned} \psi^\pm(x_1) &\;\to\; 1\,, &\text{as}\quad x_1 &\;\to\; \pm\infty,\\ \psi^\pm(x_1) &\;\to\; 0\,, &\text{as}\quad x_1 &\;\to\; \mp\infty \end{aligned}$$

and $\phi_{j,l} \in H^1_{loc}(\mathbb{R}^2_+)$, $j\in J$, $l\in L_j^\pm$, with some finite index sets J and $L_j^\pm$, are surface waves, that means they are weak α_j-quasi-periodic solutions to the Helmholtz equation in $\mathbb{R}^2_+$, with some parameter $\alpha_j\in[0,1)$, $j\in J$. They are orthonomalized, equal zero on Γ_{lo} and satisfy the Rayleigh expansion condition. They are determined by some eigenvalue problem (compare with Lemma 5.6 in [18]). The subsets L_j^+ and L_j^- separate the surface waves which travel to the right and left, respectively.

We set for abbreviation,

$$u_j^\pm(x) \;:=\; \sum_{l\in L_j^\pm} a_{j,l}^\pm\phi_{j,l}(x), \quad x\in\mathbb{R}^2_+$$

for $j\in J$ and for some coefficients $a_{j,l}^\pm\in\mathbb{C}$. Then $u_j^\pm(x)\in H^1_{loc}(\mathbb{R}^2_+)$, $j\in J$, satisfy the Helmholtz equation in $\mathbb{R}^2_+$.

Let us now consider the case, that u^s vanishes above the strip $\Omega_+ = \mathbb{R}\times(0,1)$. Then, the transmission conditions yield

$$\begin{aligned} u &\;=\; u^i,\\ \frac{\partial u}{\partial x_2} &\;=\; \frac{\partial u^i}{\partial x_2}, \end{aligned}$$

on Γ_{up+}. This leads us to the following system of equations, which we formally write as

$$\begin{aligned}
\Delta u^i + k^2 u^i &= 0\,, \quad \text{in } \Omega_+, && (3.15)\\
\Delta u + k^2(1+q)u &= 0\,, \quad \text{in } \Omega_+, \qquad \text{with } u = u^{(1)} + u^{(2)} && (3.16)\\
u^i &= u \quad \text{on } \Gamma_{up,+} &&\\
\frac{\partial u^i}{\partial x_2} &= \frac{\partial u}{\partial x_2} \quad \text{on } \Gamma_{up,+} &&\\
u^i &= u \;=\; 0 \quad \text{on } \Gamma_{lo}. &&
\end{aligned}$$

The following result shows, that if u^s vanishes above the strip $\Omega = \mathbb{R} \times (0,1)$, the surface waves vanish.

Lemma 3.2 *Let u^s be the scattered field to (3.13). Then, $u^s = 0$ above the strip $\Omega = \mathbb{R} \times (0,1)$ implies that $u^{(2)} = 0$ in Ω, and hence $u = u^{(1)} \in H^1(\Omega)$.*

Proof:

If u^s vanishes above the strip Ω_+, on the boundary Γ_{up+} it holds, using the α_j-quasi-periodicity of the functions $\phi_{j,l}$, $j \in J$,

$$\begin{aligned}
& u^i(x_1 + 2\pi m, 1)\\
= \;& u(x_1 + 2\pi m, 1)\\
= \;& u^{(1)}(x_1 + 2\pi m, 1) + \psi^+(x_1 + 2\pi m) \sum_{j \in J} \sum_{l \in L_j^+} a_{j,l}^+ \phi_{j,l}(x_1 + 2\pi m, 1)\\
& \qquad + \; \psi^-(x_1 + 2\pi m) \sum_{j \in J} \sum_{l \in L_j^-} a_{j,l}^- \phi_{j,l}(x_1 + 2\pi m, 1)\\
= \;& u^{(1)}(x_1 + 2\pi m, 1) + \psi^+(x_1 + 2\pi m) \sum_{j \in J} \sum_{l \in L_j^+} a_{j,l}^+ \phi_{j,l}(x) e^{i2\pi m \alpha_j}\\
& \qquad + \; \psi^-(x_1 + 2\pi m) \sum_{j \in J} \sum_{l \in L_j^-} a_{j,l}^- \phi_{j,l}(x) e^{i2\pi m \alpha_j}, \qquad (3.17)
\end{aligned}$$

as well as

$$
\begin{aligned}
&\frac{\partial}{\partial x_2} u^i(x_1+2\pi m,1) \\
= \;& \frac{\partial}{\partial x_2} u^{(1)}(x_1+2\pi m,1) + \psi^+(x_1+2\pi m)\sum_{j\in J}\sum_{l\in L_j^+} a_{j,l}^+ \frac{\partial}{\partial x_2}\phi_{j,l}(x) e^{i2\pi m\alpha_j} \\
&\qquad + \psi^-(x_1+2\pi m)\sum_{j\in J}\sum_{l\in L_j^-} a_{j,l}^- \frac{\partial}{\partial x_2}\phi_{j,l}(x) e^{i2\pi m\alpha_j}. \quad (3.18)
\end{aligned}
$$

Equation (3.17) and (3.18) yield,

$$
\sum_{j\in J} u_j^\pm(x_1,1)e^{i2\pi m\alpha_j} \;=\; \sum_{j\in J}\sum_{l\in L_j^+} a_{j,l}^\pm \phi_{j,l}(x)e^{i2\pi m\alpha_j} \;\to\; 0\,, \tag{3.19}
$$

as well as

$$
\sum_{j\in J} \frac{\partial}{\partial x_2} u_j^\pm(x_1,1)e^{i2\pi m\alpha_j} \;=\; \sum_{j\in J}\sum_{l\in L_j^+} a_{j,l}^\pm \frac{\partial}{\partial x_2}\phi_{j,l}(x)e^{i2\pi m\alpha_j} \;\to\; 0, \tag{3.20}
$$

for $m \to \pm\infty$, because u^i and $u^{(1)}$, as well as their derivatives with respect to x_2, decay, as $|x_1| \to \infty$ and $\psi^\pm(x)$ tends to one or zero, respectively, as $|x_1| \to \infty$.

We order the distinct $\alpha_j \in (0,1)$, $j = 1,\ldots,n$ as a decreasing sequence, that means $\alpha_{j+1} < \alpha_j$ for all $j = 1,\ldots,n$. We will show that for all $n \in \mathbb{N}$, for all α_j and for all surface waves $u_j^\pm \in H^1_{loc}(\mathbb{R}^2_+)$ that satisfy (3.19) and (3.20),

$$
u_j^\pm = 0 \quad \text{and} \quad \frac{\partial}{\partial x_2} u_j^\pm = 0 \quad \text{on} \quad \Gamma_{up}.
$$

We start with showing that (3.19) implies

$$
u_j^\pm(x_1,1) = 0 \quad \text{for all } j = 1,\ldots n. \tag{3.21}
$$

For simplicity, we prove this for $m \to \infty$. Analogously, one can show the result for $m \to -\infty$. We use mathematical induction for this proof. For $n = 1$, from $u_1^+(x_1,1)e^{i2\pi m\alpha_1} \to 0$, as $m \to \infty$, we see that $u_1^+(x_1,1) = 0$. Let us now assume that the claim (3.21) holds true for one $n \in \mathbb{N}$. We will show that it also holds true for $n+1$. It is

$$
\begin{aligned}
& \sum_{j=1}^{n+1} u_j^+(x_1,1)e^{i2\pi m\alpha_j} \;\to\; 0 \\
\Leftrightarrow \quad & e^{i2\pi m\alpha_{n+1}}\left(u_{n+1}^+(x_1,1) \;+\; \sum_{j=1}^{n} u_j^+(x_1,1)e^{i2\pi m(\alpha_j-\alpha_{n+1})}\right) \to\; 0, \quad (3.22)
\end{aligned}
$$

as $m \to \infty$. We define $\alpha_j - \alpha_{n+1} =: \beta_j$. Note, that $\beta_j \in (0,1)$, because the α_j are distinct and $\alpha_{n+1} < \alpha_j$, $j = 1, \dots, n$. Then, (3.22) is satisfied if, and only if,

$$\sum_{j=1}^{n} u_j^+(x_1, 1) e^{i2\pi m\beta_j} \to -u_{n+1}^+(x_1, 1), \tag{3.23}$$

as $m \to \infty$, which yields

$$\sum_{j=1}^{n} u_j^+(x_1, 1) e^{i2\pi m\beta_j} - \sum_{j=1}^{n} u_j^+(x_1, 1) e^{i2\pi(m+1)\beta_j} \to 0,$$

as $m \to \infty$. This is satisfied if, and only if,

$$\sum_{j=1}^{n} \underbrace{u_j^+(x_1, 1)\left(1 - e^{i2\pi\beta_j}\right)}_{=:\hat{u}_j^+(x_1,1)} e^{i2\pi m\beta_j} \to 0,$$

as $m \to \infty$. From the induction hypothesis, we see that $\hat{u}_j^+(x_1, 1) = 0$ for all $j = 1, \dots, n$ and since $1 - e^{i2\pi\beta_j} \neq 0$, u_j^+ must be equal to zero for all $j = 1, \dots, n$. This shows with (3.22) that $u_{n+1}^+(x_1, 1) = 0$ as well. Additionally, we conclude with analogous arguments from (3.20), that $\frac{\partial}{\partial x_2} u_j^\pm(x_1, 1) = 0$ for all $j = 1, \dots n$.

Furthermore, $u_j^\pm$ are classical solutions to the Helmholtz equation in Ω_+ and hence analytic there (see [11], Corollary 8.11 and [8], Theorem 3.5) . We use Holmgren's Uniqueness Theorem (see for example [8], Theorem 6.12) and conclude that $u_j^\pm$ are zero inside of Ω_+. This yields that $u^{(2)} = 0$ and hence $u = u^{(1)} \in H^1(\Omega)$ inside Ω_+.

□

This lemma shows that if we assume u^s to vanish above Ω_+, we arrive at the system of equations

$$\begin{aligned}
\Delta u^i + k^2 u^i &= 0 \quad \text{in } \Omega_+, \\
\Delta u^{(1)} + k^2(1+q)u^{(1)} &= 0 \quad \text{in } \Omega_+, \\
u^i &= u^{(1)} \quad \text{on } \Gamma_{up,+} \\
\frac{\partial u^i}{\partial x_2} &= \frac{\partial u^{(1)}}{\partial x_2} \quad \text{on } \Gamma_{up,+} \\
u^i &= u^{(1)} = 0 \quad \text{on } \Gamma_{lo}.
\end{aligned}$$

Now we extend $u^{(1)}$ and u^i as odd functions with respect to x_2 to the strip $\Omega = \mathbb{R} \times (-1, 1)$. We arrive at the *interior transmission eigenvalue problem* (replacing

u^i and $u^{(1)}$ by v and w), set up inside the strip Ω, where we wish to find non-trivial solution v and w to the following system of equations.

$$\begin{aligned}
\Delta w + k^2 w &= 0 \quad \text{in } \Omega, \\
\Delta v + k^2(1+q)v &= 0 \quad \text{in } \Omega, \\
w &= v \quad \text{on } \Gamma \\
\frac{\partial w}{\partial x_2} &= \frac{\partial v}{\partial x_2} \quad \text{on } \Gamma,
\end{aligned} \tag{3.24}$$

where $\Gamma := (\mathbb{R} \times \{-1\}) \cup (\mathbb{R} \times \{1\})$ denotes the boundary of Ω. Just as for Scattering Problem 1, it makes sense to study this problem in the *ultra weak sense*. That is, we seek for solutions $(v, w) \in L^2(\Omega) \times L^2(\Omega)$ such that $v - w \in H_0^2(\Omega)$ and we understand the differential equations as

$$\int_\Omega w \left[\Delta\overline{\varphi} + k^2(1+q)\overline{\varphi}\right] dx = 0,$$

for all $\varphi \in H_0^2(\Omega)$, and the corresponding form for v. We will go into more details later in Chapter 4.2.3

Note, that neither the incident field u^i, nor $u^{(1)}$ is periodic or quasi-periodic. Nevertheless, there will be an interesting relation to α-quasi-periodic functions, which we will point out later in Chapter 4.2.3.2.

4 Transmission Eigenvalues

The main focus of this work lies in the study of transmission eigenvalues for periodic media. Since we will use some analysis for bounded domains, we will start with this case in Chapter 4.1. We will study transmission eigenvalues for Scattering Problem 1 and Scattering Problem 2 in Chapter 4.2. In Chapter 4.3 we will show that there exist complex transmission eigenvalues for some periodic medium and with some conditions on the contrast.

Although already mentioned, we will clearly define the expressions *transmission eigenvalue* and *interior transmission eigenvalue problem* for both, bounded domains and for periodic media in the following chapters.

Throuout this work we consider the contrast q to be bounded away from zero and positive. With some modifications, similar results can be obtained when considering negative contrast.

4.1 Transmission Eigenvalues for Bounded Domains

Let D be some bounded domain and let $0 < q_* \leq q \leq q* < \infty$ inside D. For $v, w \in L^2(D)$ we consider the system of equations

$$\begin{aligned} \Delta w + k^2(1+q)w &= 0 \quad \text{in } D && (4.1)\\ \Delta v + k^2 v &= 0 \quad \text{in } D && (4.2)\\ w &= v \quad \text{on } \partial D && (4.3)\\ \partial_\nu w &= \partial_\nu v \quad \text{on } \partial D. && (4.4) \end{aligned}$$

The equations (4.1) and (4.2) have to be understood in the ultra weak sense, that is

$$\begin{aligned} \int_D w\left[\Delta\overline{\varphi} + k^2(1+q)\overline{\varphi}\right] dx &= 0 && (4.5)\\ \int_D v\left[\Delta\overline{\varphi} + k^2\overline{\varphi}\right] dx &= 0 && (4.6) \end{aligned}$$

for all $\varphi \in H_0^2(D)$.

The question is, do there exist $k \in \mathbb{C}$ and non-trivial solutions $(v, w) \in L^2(D) \times L^2(D)$ such that

$$u \;:=\; w - v \;\in\; H_0^2(D) \;?$$

If so, we call $k \in \mathbb{C}$, $\mathrm{Re}\,(k) > 0$ a *transmission eigenvalue* and u the corresponding *transmission eigenfunction.* Furthermore, we use the expression *eigenpair* (v, w).

Note that the boundary conditions are already included in the space $H_0^2(D)$.

We subtract (4.5) from (4.6) and obtain

$$\int_D u\,[\Delta\,\overline{\varphi} + k^2(1+q)\,\overline{\varphi}]\,dx \;=\; k^2 \int_D v\,q\,\overline{\varphi}\,dx,$$

for all $\varphi \in H_0^2(D)$. Using Green's identities, the equation reads

$$\int_D [\Delta + k^2(1+q)]u\,\overline{\varphi}\;dx \;=\; k^2 \int_D v\,q\,\overline{\varphi}\;dx, \tag{4.7}$$

with $\varphi \in H_0^2(D)$. With a densiy argument (4.7) holds for $\varphi \in L^2(D)$.

Let us now choose $\varphi = \frac{1}{q}[\Delta + k^2]\psi$ for all $\psi \in H_0^2(D)$. We arrive at

$$\int_D \frac{1}{q}\,\left[\Delta + k^2(1+q)\right]\,u\,\left[\Delta + k^2\right]\overline{\psi}\,dx \;=\; 0, \tag{4.8}$$

for all $\psi \in H_0^2(D)$. Note that we have used (4.6) on the right hand side of the equation.

Notation 4.1 *Motivated by (4.8), for future convenience we define the sesquilinear form*

$$\begin{aligned} a_{k,q,D}(u,\psi) \;&:=\; \int_D \frac{1}{q}\left[\Delta + k^2(1+q)\right]\,u\,\left[\Delta + k^2\right]\overline{\psi}\,dx \\ &=\; \int_D \frac{1}{q}\left[\Delta u + k^2 u\right]\,\left[\Delta\overline{\psi} + k^2\overline{\psi}\right]\,dx \;+\; k^2\int_D u\left[\Delta\overline{\psi} + k^2\overline{\psi}\right]\,dx \end{aligned}$$

for u, ψ in $H_0^2(D)$. By splitting up $a_{k,q,D}$ into this sum, we can treat the part that depends on q seperately.

Lemma 4.2 *k is a transmission eigenvalue, if, and only if, there exists a non-trivial $u \in H_0^2(D)$ such that $a_{k,q,D}\,(u,\psi) = 0$ for all $\psi \in H_0^2(D)$.*

Proof: We have just seen that if k is a transmission eigenvalue relating to the corresponding eigenpair $(v, w) \in L^2(D) \times L^2(D)$ then $u = H_0^2(D)$ solves the equation $a_{k,q,D}(u,\psi) = 0$ for all $\psi \in H_0^2(D)$.

If, on the other hand, there exists a non-trivial $u \in H_0^2(D)$ such that $a_{k,q,D}(u,\psi) = 0$ for all $\psi \in H_0^2(D)$, then,

$$v := \frac{1}{k^2 q}\left[\Delta u + k^2(1+q)u\right]$$

belongs to $L^2(D)$ and satisfies

$$\int_D v\left[\Delta\overline{\psi} + k^2\overline{\psi}\right]\,dx \;=\; \int_D \frac{1}{k^2 q}\left[\Delta u + k^2(1+q)u\right]\left[\Delta\overline{\psi} + k^2\overline{\psi}\right]\,dx \overset{(4.8)}{=} 0,$$

for all $\psi \in H_0^2(D)$, the ultra weak formulation for $\Delta v + k^2 v = 0$ on $H_0^2(D)$.

Analogously, we can show that $w := u - v \in L^2(D)$ satisfies the ultra weak formulation for $\Delta w + k^2(1+q)w = 0$. □

This result is used to prove the discreteness of the transmission eigenvalues.

Lemma 4.3 *The transmission eigenvalues to the interior transmission eigenvalue problem as stated in (4.1) - (4.4) form at most a discrete set.*

We will do a similar proof of discreteness later (Lemma 4.11) and therefore skip the details here. Nevertheless, we note here that the discreteness can be shown independently of the proof of existence for transmission eigenvalues.

Theorem 4.4 *There exists a discrete set of real transmission eigenvalues to the interior transmission eigenvalue problem as stated in (4.1) - (4.4). The only possible accumulation point is infinity.*

A complete proof can be found in [3], see also [14] for the contrast beeing large enough. (The proof of Lemma 4.3 is part of this proof.) For the proof of Theorem 4.12 later we will use similar methods. To show existence of the transmission eigenvalues in Theorem 4.4 (as well as later in Theorem 4.12), the following result is used.

Lemma 4.5 *There exists a discrete set of real transmission eigenvalues for a disc B of radius R centered at zero with constant contrast $q = q_c > 0$, that means*

$$\begin{aligned} \Delta w + k^2(1+q_c)w &= 0 \quad \text{in } B && (4.9)\\ \Delta v + k^2 v &= 0 \quad \text{in } B && (4.10)\\ w &= v \quad \text{on } \partial B && (4.11)\\ \partial_\nu w &= \partial_\nu v \quad \text{on } \partial B, && (4.12) \end{aligned}$$

for $v, w \in L^2(B)$ such that $v - w \in H_0^2(B)$.

Proof: We will only show that transmission eigenvalues exist, as discreteness follows from Lemma 4.3.

Looking for solutions to the Helmholtz equation in polar coordinates leads to the Bessel differential equation. Solutions are the Bessel functions. We will choose Bessel functions of order 0 and solving the system of equations (4.9)-(4.12) leads to the determinant

$$\begin{aligned} W(k) &:= \det\begin{vmatrix} J_0(kR) & J_0(k\sqrt{1+q_c}R) \\ -J_0'(kR) & -\sqrt{1+q_c}J_0'(k\sqrt{1+q_c}R) \end{vmatrix} \\ &= J_0(kR)\sqrt{1+q_c}\,J_0'(k\sqrt{1+q_c}R) \;-\; J_0(k\sqrt{1+q_c}R)\sqrt{1+q_c}\,J_0'(kR) \\ &= -J_0(kR)\sqrt{1+q_c}\,J_1(k\sqrt{1+q_c}R) \;+\; J_0(k\sqrt{1+q_c}R)\sqrt{1+q_c}\,J_1(kR). \end{aligned}$$

If W has a zero, the system of equation (4.9) and (4.12) has non-zero solutions and hence, transmission eigenvalues exist.

We will show that there exist $k \in \mathbb{R}$ such that $W(k) = 0$. For easier understanding of the following argumentation, we note some facts about Bessel functions.

- The first zero of $J_0(k\sqrt{1+q_c}R)$ is smaller than the first zero of $J_0(kR)$.
- It is $J_0'(k) = -J_1(k)$. We will use this to have an idea if J_1 is positive or negative at some k.
- $J_0(k)$ and $J_1(k)$ do not have a common zero.
- $J_0(0) = 1$ and $J_1(0) = 0$.

Let k_1 be the first zero of $J_0(k\sqrt{1+q_c}R) = 0$ and let k_2 be the first zero of $J_0(kR) = 0$. Furthermore, we deonote by k_3 the second zero of $J_0(k\sqrt{1+q_c}R) = 0$.

There are three cases to consider.

First, let k_0 be a zero of $J_0(k\sqrt{1+q_c}R) = 0$ and of $J_0(kR) = 0$. Then we have found a zero of $W(k) = 0$.

Second, let $k_1 < k_2 < k_3$. We have illustrated the situation in Figure 6 (with $R = 1$ and $q_c = 9$). Then we choose k^* between k_2 and k_3 such that $J_0(k^*\sqrt{1+q_c}R) = J_0(k^*R)$. Then, $J_1(k^*\sqrt{1+q_c}R) < 0$ and $J_1(k^*R) > 0$. With this, we obtain

$$\begin{aligned} W(k^*) &= -J_0(k^*R)\sqrt{1+q_c}J_1(k^*\sqrt{1+q_c}R) \;+\; J_0(k^*\sqrt{1+q_c}R)\sqrt{1+q_c}J_1(k^*R) \\ &= J_0(k^*R)\left(J_1(k^*R) - \sqrt{1+q_c}J_1(k^*\sqrt{1+q_c}R)\right) \;>\; 0. \end{aligned}$$

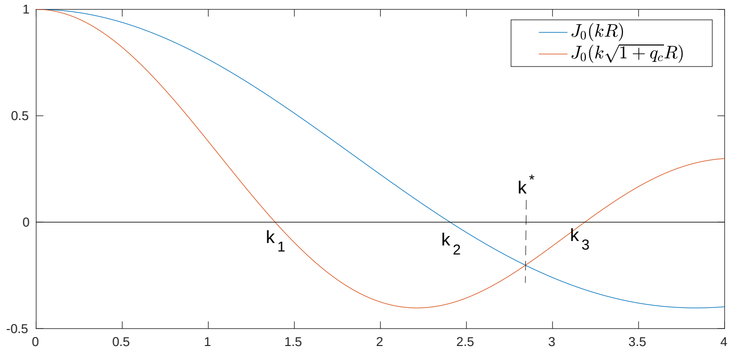

Figure 6: Second case, $k_1 < k_2 < k_3$

On the other hand, plugging k_1 into W yields

$$W(k_1) \;=\; -J_0(k_1R)\sqrt{1+q_c}J_1(k_1\sqrt{1+q_c}R) \;<\; 0,$$

because $J_0(k_1)$ and $J_1(k_1\sqrt{1+q_c}R)$ are positive.

With the mean value theorem, we conclude, that $W(k)$ has at least one zero.

Third, consider the case that $k_1 < k_3 < k_2$. For better understanding, see Figure 7. (The situation is illustrated with $R = 1$ and $q = 2$.) Then we evaluate the determinant at k_1 and at k_2 and obtain

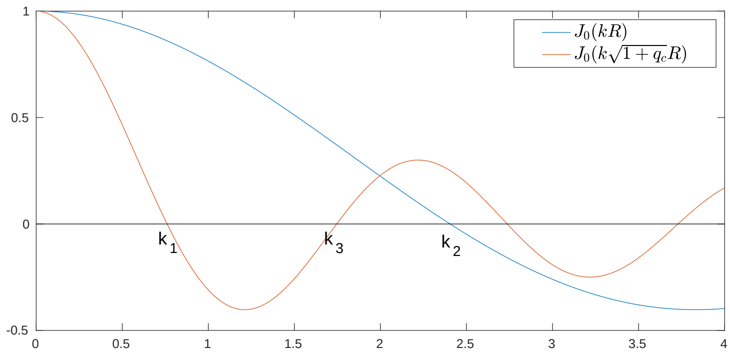

Figure 7: Third case, $k_1 < k_3 < k_2$

$$W(k_1) \;=\; -J_0(k_1R)\sqrt{1+q_c}J_1(k_1\sqrt{1+q_c}R) \;<\; 0,$$

because $J_0(k_1R) > 0$ and $J_1(k_1\sqrt{1+q_c}R) > 0$, as well as

$$W(k_2) \;=\; -J_0(k_2R)\sqrt{1+q_c}J_1(k_2\sqrt{1+q_c}R) \;>\; 0,$$

because $J_0(k_1R) > 0$ and $J_1(k_1\sqrt{1+q_c}R) < 0$.

Again, with the mean value theorem, we conclude, that $W(k)$ has at least one zero.

$\square$

Remark 4.6 *In general, transmission eigenvalues can be complex valued. This is shown for the case of sperically stratified media in [23].*

4.2 Transmission Eigenvalues for Periodic Media

Let us now define transmission eigenvalues and the interior transmission eigenvalue problem for periodic media. Basically, the definitions are analog to those for bounded domains. However, to avoid confusion with the corresponding function spaces, we will (again) define the expressions explicitely.

Let Ω be a periodic medium and let q be a 2π periodic contrast with $0 < q_* \leq q \leq q* < \infty$ inside Ω. Analogously to the case of bounded domains, we consider for $v, w \in L^2(\Omega)$ the system of equations

$$\begin{aligned} \Delta w + k^2 (1+q) w &= 0 \qquad \text{in } \Omega && (4.13)\\ \Delta v + k^2 v &= 0 \qquad \text{in } \Omega && (4.14)\\ w &= v \qquad \text{on } \Gamma && (4.15)\\ \partial_\nu w &= \partial_\nu v \quad \text{on } \Gamma, && (4.16) \end{aligned}$$

where Γ denotes the boundary of Ω. Equation (4.13) and (4.14) are again understood in the ultra weak sense, that is

$$\begin{aligned} \int_\Omega w \left[\Delta\overline{\varphi} + k^2(1+q)\overline{\varphi}\right] dx &= 0\\ \int_\Omega v \left[\Delta\overline{\varphi} + k^2\overline{\varphi}\right] dx &= 0 \end{aligned}$$

for all $\varphi \in H_0^2(\Omega)$.

We call k a transmisson eigenvalue, if there is a non-trivial solution $(v, w) \in L^2(\Omega) \times L^2(\Omega)$ such that $u := w - v$ is in $H_0^2(\Omega)$ and we call u the corresponding transmission eigenfunction.

Studying periodic media, will also lead us to the case of α-quasi-periodic solutions. In this case, due to the 2π-periodicity of Ω, it makes sense to study the system of equations on a cell Ω_μ, that means for $v, w \in L^2(\Omega_\mu)$,

$$\begin{aligned}
\Delta w + k^2\,(1+q)\,w &= 0 && \text{in } \Omega_\mu && (4.17)\\
\Delta v + k^2 v &= 0 && \text{in } \Omega_\mu && (4.18)\\
w &= v && \text{on } \Gamma_\mu && (4.19)\\
\partial_\nu w &= \partial_\nu v && \text{on } \Gamma_\mu, && (4.20)
\end{aligned}$$

where Γ_μ denotes the union of the upper and the lower part of the boundary of Ω_μ. We understand (4.17) and (4.18) in the ultra weak sense,

$$\begin{aligned}
\int_{\Omega_\mu} w\left[\Delta\overline{\varphi} + k^2(1+q)\overline{\varphi}\right] dx &= 0\\
\int_{\Omega_\mu} v\left[\Delta\overline{\varphi} + k^2\overline{\varphi}\right] dx &= 0
\end{aligned}$$

for all $\varphi \in H^2_{0,\alpha}(\Omega_\mu)$, and we call k a *transmisson eigenvalue in the α-quasi-periodic case*, if there are non-trivial solution $(v, w) \in L^2(\Omega_\mu)\times L^2(\Omega_\mu)$ such that $u^\alpha := w-v$ is in $H^2_{0,\alpha}(\Omega_\mu)$.

The following example shows that we might find different transmission eigenvalues when considering the interior transmission eigenvalue problem on a cell (a bounded domain) $\mathring{\Omega}_\mu$, $\mu \in \mathbb{Z}$ fixed, with transmission eigenfunctions in $H^2_0(\Omega_\mu)$ and when considering α-quasi-periodic solutions v, w on a 2π-periodic medium Ω. In the latter, we consider the same cell Ω_μ but with zero boundary conditions only on the upper and the lower part of Ω_μ. On the left and right part of the boundary, we have (quasi-)periodicity. The transmission eigenfunctions in this case are in $H^2_{0,\alpha}(\Omega_\mu)$.

4.2.1 An Example

Let us consider the interior transmission eigenvalue problem for a strip,

$$\Omega \;=\; \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \in (0, 2\pi)\},$$

and let the solutions of the interior transmission eigenvalue problem be 2π-periodic. A cell of Ω is defined as

$$\Omega_\mu \;=\; \{x \in \Omega : x_1 \in [\mu 2\pi, (\mu+1)2\pi]\},$$

$\mu \in \mathbb{Z}$ fixed. For the corresponding system of equations see (4.17) - (4.20). Note that Ω_μ is a square here.

Let $q \equiv 1$ in Ω and choose $k = 1$. We will show that $k = 1$ is a transmission eigenvalue. Let

$$\begin{aligned} w^0(x) &:= \cos(x_1)\cos(x_2), \\ v^0(x) &:= \cos(x_1). \end{aligned}$$

These functions satisfy

$$\begin{aligned} \Delta v^0 + k^2 v^0 &= \Delta v^0 + v^0 &= 0 \quad \text{in } \Omega_\mu \\ \Delta w^0 + k^2(1+q)w^0 &= \Delta w^0 + 2w^0 &= 0 \quad \text{in } \Omega_\mu \end{aligned}$$

as well as $w^0 - v^0 \in H^2_{0,0}(\Omega_\mu)$.

Hence, $k = 1$ is a transmission eigenvalue for this choice of Ω and q!

Note that the functions here are 2π-periodic, but we can also consider them as α-quasi-periodic functions with $\alpha = 0$.

Let us now consider the interior transmission eigenvalue problem on $\mathring{\Omega}_\mu$. We will show that in this case $k = 1$ is not a transmission eigenvalue for constant $q = q_c \geq 1$. The system of equations for the interior transmission eigenvalue problem (4.1)-(4.4) for $\mathring{\Omega}_\mu$ reads

$$\Delta w + k^2(1+q_c)w = 0 \quad \text{in } \mathring{\Omega}_\mu \tag{4.21}$$

$$\Delta v + k^2 v = 0 \quad \text{in } \mathring{\Omega}_\mu \tag{4.22}$$

$$w = v \quad \text{on } \partial\Omega_\mu \tag{4.23}$$

$$\partial_\nu w = \partial_\nu v \quad \text{on } \partial\Omega_\mu, \tag{4.24}$$

for $v, w \in L^2(\Omega_\mu)$. Note that now, the boundary data is defined on the whole boundary $\partial\Omega_\mu$ of the square $\mathring{\Omega}_\mu$.

From Lemma 4.2, we know that k is a transmission eigenvalue if, and only if, there exists a non-trivial $u = w - v \in H^2_0(\Omega_\mu)$ such that

$$0 = \int_{\Omega_\mu} \frac{1}{q_c} \left[\Delta u + k^2(1+q_c)u\right] \left[\Delta\overline{\psi} + k^2\overline{\psi}\right] dx,$$

for all $\psi \in H^2_0(\Omega_\mu)$.

Writing u as a Fourier series and using Parseval's equation, for $\psi = u$ we obtain

$$\begin{aligned} 0 &= \int_{\Omega_\mu} \frac{1}{q_c} \left|\Delta u + k^2 u\right|^2 dx + k^2 \int_{\Omega_\mu} u \left[\Delta \overline{u} + k^2 \overline{u}\right] dx \\ &= \frac{1}{q_c} \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(k^2 - |n|^2\right)^2 + k^2 \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(k^2 - |n|^2\right) \\ &= \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(k^2 - |n|^2\right) \left(\frac{1}{q_c}\left(k^2 - |n|^2\right) + k^2\right), \end{aligned} \tag{4.25}$$

where u_n, $n \in \mathbb{Z}^2$ denote the Fourier coefficients. The sum in (4.25) splits up into positive and negative terms. A term is not positive, if

$$\left(k^2 - |n|^2\right) \left(\frac{1}{q_c}\left(k^2 - |n|^2\right) + k^2\right) \le 0,$$

which is true if and only if

$$k^2 \le |n|^2 \le k^2(1 + q_c). \tag{4.26}$$

We are now going to prove the following statement.

Theorem 4.7 *Let $q_c \equiv 1$. Then $k = 1$ is not a transmission eigenvalue for $\mathring{\Omega}_\mu$.*

Proof: For $q_c = 1$ and $k^2 = 1$ (4.25) and (4.26) read

$$0 = \sum_{n\in\mathbb{Z}^2} |u_n|^2 \left(1 - |n|^2\right) \left(2 - |n|^2\right), \tag{4.27}$$

$$1 \le |n|^2 \le 2. \tag{4.28}$$

We recap that a summand of (4.27) is not positive, if (4.28) is satisfied, which is only possible for $|n|^2 = 1$ or $|n|^2 = 2$. But for these values for $|n|$,

$$\sum_{|n|^2=1} |u_n|^2 \left(1 - |n|^2\right) \left(2 - |n|^2\right) \quad \text{and} \quad \sum_{|n|^2=2} |u_n|^2 \left(1 - |n|^2\right) \left(2 - |n|^2\right)$$

vanish. For $|n|^2 > 2$ all $(1 - |n|^2)\ (2 - |n|^2)$ are strictly positive and we conclude $u_n = 0$. Also, $u_n = 0$ for $|n|^2 = 0$.

It remains to study u_n for $|n|^2 = 1$ and $|n|^2 = 2$ and check whether they are zero as well. There are eight possible tupels (n_1, n_2) to obtain $|n|^2$ equal to one or two, namely all possible combinations (n_1, n_2) with $n_1 \in \{-1, 0, 1\}$ and $n_2 \in \{-1, 0, 1\}$, except $(n_1, n_2) = (0,0)$. We will show that $u_n = 0$ for these tupels as well. To this end, we use the boundary conditions on $\partial\Omega_\mu$, that is $0 = u(x)$ and $0 = \partial_\nu u(x)$. If x_2 is on the upper part of the boundary $\Gamma_{\mu,up}$, it holds $x_2 = 2\pi$ and

$$\begin{aligned} 0 &= u(x_1, 2\pi) = \sum_{n_1\in\mathbb{Z}} \left(\sum_{n_2\in\mathbb{Z}} u_n \right) e^{in_1\cdot x_1}, \\ 0 &= \partial_{\nu_{up}} u(x_1, 2\pi) = \sum_{n_1\in\mathbb{Z}} \left(\sum_{n_2\in\mathbb{Z}} u_n \right) in_2 e^{in_1\cdot x_1}. \end{aligned}$$

Here $\nu_{up} := (0,1)^\top$ denotes the outward pointing unit normal vector on $\Gamma_{\mu,up}$. These two equations hold true if, and only if,

$$0 = \sum_{n_2\in\{-1,0,1\}} u_n \quad \text{and} \quad 0 = \sum_{n_2\in\{-1,0,1\}} u_n n_2, \tag{4.29}$$

for all $n_1 \in \{-1,0,1\}$. From this we obtain six equations,

$$\begin{aligned} n_1 = 0: \quad 0 &= \mu_{(0,1)} + \mu_{(0,0)} + \mu_{(0,-1)} = \mu_{(0,1)} + \mu_{(0,-1)} \\ 0 &= \mu_{(0,1)} - \mu_{(0,-1)} \\ n_1 = 1: \quad 0 &= \mu_{(1,0)} + \mu_{(1,1)} + \mu_{(1,-1)} \\ 0 &= \mu_{(1,1)} - \mu_{(1,-1)} \\ n_1 = -1: \quad 0 &= \mu_{(-1,0)} + \mu_{(-1,1)} + \mu_{(-1,-1)} \\ 0 &= \mu_{(-1,1)} - \mu_{(-1,-1)}. \end{aligned} \tag{4.30}$$

Considering the lower part of the boundary $\Gamma_{\mu,lo}$, that is $x_2 = 0$, we analogeously obtain the two identities (4.29) and hence the system of equations (4.30).

In the same way, the boundary conditions on the right and left part of the boundary, $\Gamma_{\mu,ri}$ and $\Gamma_{\mu,le}$, lead to two further equations, namely

$$0 = \sum_{n_1\in\{-1,0,1\}} u_n \quad \text{and} \quad 0 = \sum_{n_1\in\{-1,0,1\}} u_n n_1, \tag{4.31}$$

for all $n_2 \in \{-1,0,1\}$, which gives us

$$\begin{aligned} n_2 = 0: \quad & 0 = \mu_{(1,0)} + \mu_{(0,0)} + \mu_{(-1,0)} = \mu_{(1,0)} + \mu_{(-1,0)} \\ & 0 = \mu_{(1,0)} - \mu_{(-1,0)} \\ n_2 = 1: \quad & 0 = \mu_{(1,1)} + \mu_{(0,1)} + \mu_{(-1,1)} \\ & 0 = \mu_{(1,1)} - \mu_{(-1,1)} \\ n_2 = -1: \quad & 0 = \mu_{(1,-1)} + \mu_{(0,-1)} + \mu_{(-1,-1)} \\ & 0 = \mu_{(1,-1)} - \mu_{(-1,-1)}. \end{aligned} \tag{4.32}$$

Solving this system of equations (4.30) and (4.32) above shows that $u_n = 0$ for the tupels (n_1, n_2) of all possible combinations with $n_1 \in \{-1, 0, 1\}$ and $n_2 \in \{-1, 0, 1\}$, except $(n_1, n_2) = (0, 0)$.

We arrive at $u_n = 0$ for all $n \in \mathbb{Z}^2$ and hence, the system of equations for the interior transmission eigenvalue problem only has the trivial solution. Therefore $k = 1$ is not a transmission eigenvalue! □

4.2.2 Transmission Eigenvalues for Scattering Problem 1

In this chapter, we consider the transmission eigenvalue problem for Scattering Problem 1 as described in Chapter 3.1. That means, we are interested in a periodic medium Ω and the system of equations (4.17) - (4.20) with $\alpha := kd$. Defined analogously to Notation 4.1, we consider here

$$a_{k,q,\Omega_\mu}(u^{kd}, \psi) \;=\; \int_{\Omega_\mu} \frac{1}{q} \left[\Delta + k^2(1+q)\right] u^{kd} \left[\Delta + k^2\right] \overline{\psi}\, dx,$$

for $u^{kd}, \psi \in H^2_{0,kd}(\Omega_\mu)$. We are going to prove that there exists a discrete set of transmission eigenvalues in the kd-quasi-periodic case. To avoid dealing with function spaces depending on the wave number k we plug in $u^{kd} = u^0 e^{ikdx_1}$, where u^0 is a 2π-periodic function. We obtain for ψ replaced by ψe^{ikdx_1}

$$\begin{aligned} &a_{k,q,\Omega_\mu}(u^0 e^{ikdx_1}, \psi e^{ikdx_1}) \\ &\quad= \int_{\Omega_\mu} \frac{1}{q} \left[\Delta u^0 + 2ikd\frac{\partial u^0}{\partial x_1} + k^2\left(1+q-d^2\right)u^0\right] \\ &\qquad\qquad \cdot \left[\Delta\overline{\psi} - 2ikd\frac{\partial\overline{\psi}}{\partial x_1} + k^2\left(1-d^2\right)\overline{\psi}\right] dx, \end{aligned}$$

$u^0, \psi \in H^2_{0,0}(\Omega_\mu)$. For future convenience we define the sesquilinear form

$$\tilde{a}_{k,q,\Omega_\mu}\left(u^0, \psi\right) \;:=\; a_{k,q,\Omega_\mu}\left(u^0 e^{ikdx_1}, \psi e^{ikdx_1}\right),$$

for $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$, and rewrite $\tilde{a}_{k,q,\Omega_\mu}$ in the following form

$$\begin{aligned}
\tilde{a}_{k,q,\Omega_\mu}\left(u^0,\psi\right) &= \int_{\Omega_\mu} \frac{1}{q}\left[\Delta u^0 + 2ikd\frac{\partial u^0}{\partial x_1} + k^2\left(1+q-d^2\right)u^0\right] \\
&\qquad \cdot \left[\Delta\overline{\psi} - 2ikd\frac{\partial\overline{\psi}}{\partial x_1} + k^2\left(1-d^2\right)\overline{\psi}\right] dx \\
&= \int_{\Omega_\mu} \frac{1}{q}\left[\Delta u^0 + 2ikd\frac{\partial u^0}{\partial x_1} + k^2\left(1-d^2\right)u^0\right] \\
&\qquad \cdot \left[\Delta\overline{\psi} - 2ikd\frac{\partial\overline{\psi}}{\partial x_1} + k^2\left(1-d^2\right)\overline{\psi}\right] dx \\
&\quad + \int_{\Omega_\mu} k^2 u^0 \left[\Delta\overline{\psi} + 2ikd\frac{\partial\overline{\psi}}{\partial x_1} + k^2\left(1-d^2\right)\overline{\psi}\right] dx,
\end{aligned} \tag{4.33}$$

for $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$.

The following lemma can be shown analogously to Lemma 4.2.

Lemma 4.8 *k is a transmission eigenvalue in the kd-quasi-periodic case, if, and only if, there exists a non-trivial $u^0 \in H^2_{0,0}(\Omega_\mu)$ such that $\tilde{a}_{k,q,\Omega_\mu}\left(u^0,\psi\right) = 0$ for all $\psi \in H^2_{0,0}(\Omega_\mu)$.*

Let us now consider the sesquilinear form $\tilde{a}_{k,q,\Omega_\mu}$ again. We rewrite it in the form

$$\tilde{a}_{k,q,\Omega_\mu} = a_0 + k\tilde{a}_1 + k^2\tilde{a}_2 + k^3\tilde{a}_3 + k^4\tilde{a}_4\,,$$

where

$$a_0(u^0,\psi) = \int_{\Omega_\mu} \frac{1}{q}\Delta u^0 \Delta\overline{\psi}\, dx$$

describes the inner product $(\!(\cdot,\cdot)\!)$ on $H^2_{0,0}(\Omega_\mu)$ and the sesquilinear forms $\tilde{a}_1, \ldots \tilde{a}_4$ are defined by

$$\begin{aligned}
\tilde{a}_1\left(u^0,\psi\right) &= \int_{\Omega_\mu} \frac{1}{q}2id\frac{\partial u^0}{\partial x_1}\Delta\overline{\psi} - \frac{1}{q}2id\Delta u^0\frac{\partial\overline{\psi}}{\partial x_1}\, dx, \\
\tilde{a}_2\left(u^0,\psi\right) &= \int_{\Omega_\mu} \frac{1}{q}\left(1-d^2\right)\left(u^0\Delta\overline{\psi} + \Delta u^0\overline{\psi}\right) + \frac{1}{q}4d^2\frac{\partial u^0}{\partial x_1}\frac{\partial\overline{\psi}}{\partial x_1} + \Delta u^0\Delta\overline{\psi}\, dx \\
\tilde{a}_3\left(u^0,\psi\right) &= \int_{\Omega_\mu} \frac{1}{q}2id\left(1-d^2\right)\left(\frac{\partial u^0}{\partial x_1}\overline{\psi} - u^0\frac{\partial\overline{\psi}}{\partial x_1}\right) - 2idu^0\frac{\partial\overline{\psi}}{\partial x_1}\, dx, \\
\tilde{a}_4\left(u^0,\psi\right) &= \int_{\Omega_\mu} \frac{1}{q}\left(1-d^2\right)\left(1-d^2+q\right)u^0\overline{\psi}\, dx,
\end{aligned}$$

with $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$.

We note two properties of the sesquilinear forms. First, $\tilde{a}_l$, $l = 1, \dots, 4$ are hermitian. This is obvious for $\tilde{a}_1$ and $\tilde{a}_4$. For $\tilde{a}_2$, we see this by applying Green's second identity and for $\tilde{a}_3$, by applying Lemma 2.10.

Second, $\tilde{a}_1, \dots, \tilde{a}_4$ are bounded. We can show this by estimating the $\| \cdot \|_{L^2(\Omega_\mu)}$ norm against the $\|\!| \cdot |\!\|_{H^2(\Omega_\mu)}$ norm. We prove the boundedness of $\tilde{a}_1$ as an example. With q_* being the minimum of the function q, we obtain for $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$

$$
\begin{aligned}
&\left|\tilde{a}_1(u^0,\psi)\right| = \left| \int_{\Omega_\mu} \frac{1}{q} 2id \frac{\partial u^0}{\partial x_1} \Delta\overline{\psi} - \frac{1}{q} 2id \Delta u^0 \frac{\partial \overline{\psi}}{\partial x_1} \; dx \right| \\
&\leq \; \frac{2d}{q_*} \int_{\Omega_\mu} \left| \frac{\partial u^0}{\partial x_1} \Delta\overline{\psi} \right| + \left| \Delta u^0 \frac{\partial \overline{\psi}}{\partial x_1} \right| \; dx \\
&\leq \; \frac{2d}{q_*} \left(\left\| \frac{\partial u^0}{\partial x_1} \right\|_{L^2(\Omega_\mu)} \| \Delta \psi \|_{L^2(\Omega_\mu)} + \left\| \Delta u^0 \right\|_{L^2(\Omega_\mu)} \left\| \frac{\partial \psi}{\partial x_1} \right\|_{L^2(\Omega_\mu)} \right) \\
&\leq \; \frac{2dc}{q_*} \| u^0 \|_{H^2(\Omega_\mu)} \| \psi \|_{H^2(\Omega_\mu)} \; \leq \; \frac{2dc'}{q_*} |\!|\!| u^0 |\!|\!|_{H^2(\Omega_\mu)} |\!|\!| \psi |\!|\!|_{H^2(\Omega_\mu)},
\end{aligned}
$$

for some positive constants $c, c' \in \mathbb{R}$.

Analogously, the boundedness of $\tilde{a}_2, \tilde{a}_3$ and $\tilde{a}_4$ can be shown.

Knowing this, we can apply Riesz' representation theorem. For $\tilde{a}_l : \left(H^2_{0,0}(\Omega_\mu)\right)^2 \to \mathbb{C}$, $l = 1 \dots 4$, there exist unique bounded operators $\tilde{A}_l$ from $H^2_{0,0}(\Omega_\mu)$ into itself, such that $\tilde{a}_l(u^0, \psi) = (\!(\tilde{A}_l u^0, \psi)\!)_{H^2(\Omega_\mu)}$, for all $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$.

The operators $\tilde{A}_l$, $l = 1, \dots 4$, are all self-adjoint. Indeed, since the sesquilinear forms a_l, $l = 1, \dots 4$ are hermitian, we have

$$
\begin{aligned}
(\!(\tilde{A}_l u, \psi)\!)_{H^2(\Omega_\mu)} \;&=\; \tilde{a}_l(u,\psi) \;=\; \overline{\tilde{a}_l(\psi,u)} \;=\; \overline{(\!(\tilde{A}_l \psi, u)\!)_{H^2(\Omega)}} \\
&=\; (\!(u, \tilde{A}_l \psi)\!)_{H^2(\Omega_\mu)}.
\end{aligned}
\tag{4.34}
$$

The equation $\tilde{a}_{k,q,\Omega_\mu}(u^0, \psi) = 0$ for all $\psi \in H^2_{0,0}(\Omega_\mu)$ takes the form

$$
u^0 + k\tilde{A}_1 u^0 + k^2 \tilde{A}_2 u^0 + k^3 \tilde{A}_3 u^0 + k^4 \tilde{A}_4 u^0 \;=\; 0.
$$

Defining

$$
\tilde{A}(k) \;:=\; Id + k\tilde{A}_1 + k^2 \tilde{A}_2 + k^3 \tilde{A}_3 + k^4 \tilde{A}_4 \tag{4.35}
$$

we now state our result in the following lemma.

Lemma 4.9 *k is a transmission eigenvalue in the kd-quasi-periodic case, if, and only if, there exists a non trivial $u^0 \in H^2_{0,0}(\Omega_\mu)$ such that $\tilde{A}(k)u^0 = 0$.*

Our aim is to show the discreteness of the wavenumbers with help of the analytic Fredholm theory. To this end, we show that the inverse operator $\tilde{A}(k)^{-1}$ does not exist for at most a discrete set of values for k. In other words, we show that the equation $\tilde{A}(k)u^0 = 0$ can be solved by a non-trivial $u^0 \in H^2_{0,0}(\Omega_\mu)$ for at most a discrete set of values for k. We prove the compactness of the operators $\tilde{A}_l$, $l = 1 \dots 4$ first.

Lemma 4.10 *The operators $\tilde{A}_l$, $l = 1 \dots 4$ mapping $H^2_{0,0}(\Omega_\mu)$ into itself are compact.*

Proof: Let $(u^0_j)_{j\in\mathbb{N}}$ be a sequence that converges weakly in $H^2_{0,0}(\Omega_\mu)$ to zero. The embedding $I_0 : H^2_{0,0}(\Omega_\mu) \hookrightarrow H^1_{0,0}(\Omega_\mu)$ is a compact operator, see Corollary 2.20. We conclude $\|u^0_j\|_{H^1} \to 0$, $j \to \infty$, in particular

$$\|\nabla u^0_j\|_{L^2(\Omega_\mu)} \to 0 \quad \text{and} \quad \|u^0_j\|_{L^2(\Omega_\mu)} \to 0, \quad j \to \infty.$$

Additionally, we know $\tilde{A}_l u^0_j \rightharpoonup 0$ in $H^2_{0,0}(\Omega_\mu)$, because $\tilde{A}_l$ is bounded. Hence for $l = 1, \dots 4$,

$$\|\nabla \tilde{A}_l u^0_j\|_{L_2(\Omega_\mu)} \to 0 \quad \text{and} \quad \|\tilde{A}_l u^0_j\|_{L^2(\Omega_\mu)} \to 0, \quad j \to \infty.$$

Since $(u^0_j)_{j\in\mathbb{N}}$ and $(\tilde{A}_l u^0_j)_{j\in\mathbb{N}}$ are bounded in $H^2_{0,0}(\Omega_\mu)$, $(\Delta u^0_j)_{j\in\mathbb{N}}$ and $(\Delta \tilde{A}_l u^0_j)_{j\in\mathbb{N}}$ are bounded in $L^2(\Omega_\mu)$.

We have now collected all the tools to show the compactness of the operators. As an example we consider $\tilde{A}_1$. By similar arguments the compactness of $\tilde{A}_2$, $\tilde{A}_3$ and $\tilde{A}_4$ can be shown as well. Riesz' representation theorem, the Cauchy-Schwarz inequality and the fact that $\left|\left\|\tilde{A}_1 u^0_j\right\|\right|^2_{H^2(\Omega_\mu)} = \tilde{a}_1(u^0_j, \tilde{A}_1 u^0_j)$ lead to

$$\begin{aligned}
&\left|\left\|\tilde{A}_1 u^0_j\right\|\right|^2_{H^2(\Omega_\mu)} = \tilde{a}_1\left(u^0_j, \tilde{A}_1 u^0_j\right) \\
&= \left(\frac{1}{q} 2id \frac{\partial u^0}{\partial x_1},\ \Delta \tilde{A}_1 u^0\right)_{L^2(\Omega_\mu)} - \left(\frac{1}{q} 2id \Delta u^0, \frac{\partial \tilde{A}_1 u^0}{\partial x_1}\right)_{L^2(\Omega_\mu)} \\
&\le \frac{1}{q_*} 2d \left\|\frac{\partial u^0}{\partial x_1}\right\|_{L^2(\Omega_\mu)} \left\|\Delta \tilde{A}_1 u^0\right\|_{L^2(\Omega_\mu)} \\
&\quad + \frac{1}{q_*} 2d \left\|\Delta u^0\right\|_{L^2(\Omega_\mu)} \left\|\frac{\partial \tilde{A}_1 u^0}{\partial x_1}\right\|_{L^2(\Omega_\mu)},
\end{aligned} \tag{4.36}$$

where q_* denotes the minimum of q.

Since $\left\|\nabla u_j^0\right\|_{L^2(\Omega_\mu)}$ and $\left\|\nabla \tilde{A}_1 u_j^0\right\|_{L^2(\Omega_\mu)}$ tend to zero, $\left\|\frac{\partial u_j^0}{\partial x_1}\right\|_{L^2(\Omega_\mu)}$ and $\left\|\frac{\partial \tilde{A}_1 u_j^0}{\partial x_1}\right\|_{L^2(\Omega_\mu)}$ tend to zero as well. Hence, both terms in (4.36) tend to zero and we arrive at $\left\|\left|\tilde{A}_1 u_j^0\right|\right\|^2_{H^2(\Omega_\mu)} \to 0$, as $j \to \infty$. Thus, $\tilde{A}_1$ is a compact operator. □

We define

$$\tilde{G}(k) := k\tilde{A}_1 + k^2\tilde{A}_2 + k^3\tilde{A}_3 + k^4\tilde{A}_4,$$

and write

$$\tilde{A}(k) = Id + \tilde{G}(k),$$

where $\tilde{G}(k)$ is a compact operator for every $k \in \mathbb{C}$. For fixed $\varepsilon > 0$ we consider the set

$$\Lambda := \{z \in \mathbb{C},\ \operatorname{Im} z \in (-\varepsilon, \varepsilon)\}.$$

Now we use the analytic Fredholm theory (see [9], Theorem 8.26) to show the discreteness of the wavenumbers. It holds either

a) $\tilde{A}(k)^{-1} = (Id + \tilde{G}(k))^{-1}$ does not exist for any $k \in \Lambda$ or

b) $\tilde{A}(k)^{-1} = (Id + \tilde{G}(k))^{-1}$ exists for all $k \in \Lambda \setminus S$, where S is a discrete subset of Λ.

With $k = 0$ we obtain $\tilde{A}(0) = \mathrm{Id}$, which is invertible. It follows that $\left(\tilde{A}(k)\right) u^0 = 0$ can only be solved by $u^0 = 0$ for all $k \in \Lambda \setminus S$, where S is a discrete set. Thus, $u^0 \neq 0$ solves $\left(\tilde{A}(k)\right) u^0 = 0$ for at most a discrete set of values for k. We have proven the lemma stated below.

Lemma 4.11 *There exists at most a discrete set of transmission eigenvalues in the kd-quasi-periodic case.*

We have not yet discussed whether transmission eigenvalues exist. This will be done in the proof of the theorem below.

Theorem 4.12 *There exists a discrete set of real transmission eigenvalues in the kd-quasi-periodic case. The only possible accumulation point is infinity.*

Proof: We consider a disc B_1 of radius 1 and a constant contrast q_c. Let k_1 be the first transmission eigenvalue to B_1 with contrast q_c. Note that k_1 exists according to Lemma 4.5. By a scaling argument, $k_\varepsilon = k_1/\varepsilon$ is the first transmission eigenvalue to a disc B_ε with contrast q_c. Indeed, let $x/\varepsilon \in B_1$, which is true if, and only if, $x \in B_\varepsilon$. Then define $v_\varepsilon(x) := v(x/\varepsilon)$, where v is a solution of the Helmholtz equation, and $w_\varepsilon(x) := w(x/\varepsilon)$, where w satisfies the equation $(\Delta + k_1^2(1+q_c))\, w = 0$.

With k_1 being the first transmission eigenvalue to B_1, we obtain

$$\Delta v_\varepsilon(x) \;=\; \frac{1}{\varepsilon^2}(\Delta v)\left(\frac{x}{\varepsilon}\right) \;=\; -\frac{k_1^2}{\varepsilon^2} v\left(\frac{x}{\varepsilon}\right) \;=\; -\frac{k_1^2}{\varepsilon^2} v_\varepsilon(x),$$

which is true if, and only if,

$$\left(\Delta + \frac{k_1^2}{\varepsilon^2}\right) v_\varepsilon(x) \;=\; 0.$$

Analogously, we can treat w and obtain

$$\left(\Delta + \frac{k_1^2}{\varepsilon^2}(1+q_c)\right) w_\varepsilon(x) \;=\; 0.$$

Defining $k_\varepsilon := \frac{k_1}{\varepsilon}$ shows that k_ε is the first transmission eigenvalue to B_ε.

For any $m > 1$, there exists $\varepsilon = \varepsilon(m)$, such that Ω_μ contains m disjoint discs B_ε^j, $j = 1, \dots m$. We call $u_j \in H_0^2(B_\varepsilon^j)$, $j = 1, \dots m$, the transmission eigenfunctions to k_ε and the disc B_ε^j.

Now we extend every u_j, $j = 1, \dots m$ by zero to Ω_μ and then $k_\varepsilon d$-quasi-periodically to the whole Ω. We call these extensions $u_j^{k_\varepsilon d}$, $j = 1 \dots m$. Due to the zero conditions on ∂B_ε^j it holds that $u_j^{k_\varepsilon d} \in H^2_{0,k_\varepsilon d}(\Omega_\mu)$. Additionally, $u_1^{k_\varepsilon d}, \dots u_m^{k_\varepsilon d}$ have disjoint supports. We know that $a_{k_\varepsilon, q_c, B_\varepsilon^j}(u_j, u_j) = 0$, for $u_j \in H_0^2(B_\varepsilon^j)$, and hence

$$a_{k_\varepsilon, q_c, \Omega_\mu}\left(u_j^{k_\varepsilon d}, u_j^{k_\varepsilon d}\right) \;=\; 0,$$

for $u_j^{k_\varepsilon d} \in H^2_{0,k_\varepsilon d}(\Omega_\mu)$, $j = 1, \dots, m$.

Furthermore, for q_* beeing the minimum of the function q (and hence constant),

$$\begin{aligned}
a_{k_\varepsilon, q, \Omega_\mu}\left(u_j^{k_\varepsilon d}, u_j^{k_\varepsilon d}\right) \;&=\; \int_{\Omega_\mu} \frac{1}{q} \left|[\Delta + k_\varepsilon]\, u_j^{k_\varepsilon d}\right|^2 \;+\; k_\varepsilon^2 u_j^{k_\varepsilon d}\left[\Delta + k_\varepsilon^2\right] u_j^{k_\varepsilon d}\, dx \\
&\leq\; \int_{\Omega_\mu} \frac{1}{q_*} \left|[\Delta + k_\varepsilon]\, u_j^{k_\varepsilon d}\right|^2 \;+\; k_\varepsilon^2 u_j^{k_\varepsilon d}\left[\Delta + k_\varepsilon^2\right] u_j^{k_\varepsilon d}\, dx \\
&=\; a_{k_\varepsilon, q_*, \Omega_\mu}\left(u_j^{k_\varepsilon d}, u_j^{k_\varepsilon d}\right) \;=\; 0,
\end{aligned}$$

for $j = 1, \dots, m$.

Again, to avoid dealing with function spaces depending on k_ε we substitute $u_j^{k_\varepsilon d} = u_j^0 e^{ik_\varepsilon d x_1}$, where $u_j^0 \in H^2_{0,0}(\Omega_\mu)$. That means,

$$\tilde{a}_{k_\varepsilon,q,\Omega_\mu}\left(u_j^0, u_j^0\right) \le 0, \quad j = 1, \dots, m. \tag{4.37}$$

We recall $\tilde{A}(k) = I + k\tilde{A}_1 + k^2\tilde{A}_2 + k^3\tilde{A}_3 + k^4\tilde{A}_4$. Note that since $\tilde{A}(0) = Id$ the spectrum $\sigma(\tilde{A}(0))$ consists of 1 only. We conclude from (4.37)

$$(\!(\tilde{A}(k_\varepsilon)\, u_j^0, u_j^0)\!)_{H^2(\Omega_\mu)} \;\le\; 0, \quad j = 1, \dots, m.$$

The min-max principle (Theorem 2.23) shows that the smallest eigenvalue of $\tilde{A}(k_\varepsilon)$ is less or equal to zero. Indeed,

$$\begin{aligned} 0 \;&\ge\; \inf\left\{(\!(\tilde{A}(k_\varepsilon)u, u)\!)_{H^2(\Omega_\mu)} :\ \|u\|_{H^2(\Omega_\mu)} = 1\right\} \\ &=\; 1 \,+\, \inf\left\{(\!(\tilde{G}(k_\varepsilon)u, u)\!)_{H^2(\Omega_\mu)} :\ \|u\|_{H^2(\Omega_\mu)} = 1\right\} \\ &=\; 1 \,+\, \lambda_{min,\tilde{G}} \;=\; \lambda_{min,\tilde{A}}, \end{aligned}$$

where $\lambda_{min,\tilde{G}}$ and $\lambda_{min,\tilde{A}}$ denote the smallest eigenvalue of $\tilde{G}(k_\varepsilon)$ and $\tilde{A}(k_\varepsilon)$, respectively.

The eigenvalues depend continuously on k. This is a simple consequence of [12], Chapter 2, §1, 6., Theorem 1.10. We conclude that there must be a $\hat{k}$ between 0 and k_ε such that $\tilde{A}(\hat{k})$ has zero as the smallest eigenvalue. This means that there exists $u \in H^2_{0,0}(\Omega_\mu)$ such that $\tilde{A}(\hat{k})u = 0$ and therefore $\hat{k}$ is a transmission eigenvalue. This shows the existence of transmission eigenvalues.

Furthermore, the $k_\varepsilon d$-quasi-periodic functions $u_1^{k_\varepsilon d}, \dots, u_m^{k_\varepsilon d}$ and therefore the 2π-periodic functions $u_1^0, \dots, u_m^0$ have disjoint supports, which means they are linearly independent and orthogonal on $H^2_{0,0}(\Omega_\mu)$. Since also $(\!(\tilde{A}(k_\varepsilon)\, u_j^0, u_j^0)\!)_{H^2(\Omega_\mu)} \le 0$ for $j = 1, \dots, m$, we conclude that $\tilde{A}(k_\varepsilon)$ is non-positive on a m-dimensional subspace U, spanned by the vectors $u_1^0, \dots, u_m^0$. With the min-max principle we have m non-positive eigenvalues, counting multiplicity. Indeed, the eigenvalues are of the form

$$\lambda_1 \;=\; \inf\left\{(\!(\tilde{A}(k_\varepsilon)\,\phi, \phi)\!)_{H^2(\Omega_\mu)} :\ \|\phi\|_{H^2(\Omega_\mu)} = 1\right\}$$

and

$$\lambda_{n+1} \;=\; \sup_{\psi_1, \dots \psi_n \in U} \inf\left\{(\!(\tilde{A}(k_\varepsilon)\,\phi, \phi)\!)_{H^2(\Omega_\mu)} :\ \phi \perp \psi_1, \dots, \psi_n,\ \|\phi\|_{H^2(\Omega_\mu)} = 1\right\},$$

with $n = 1, 2, \dots, m-1$. With the arguments above, we conclude with the mean value theorem that there exist m transmission eigenvalues inside $[0, k_\varepsilon]$.

We now let $m \to \infty$, which yields $\varepsilon(m) \to 0$. The multiplicity of each eigenvalue is finite (note that $\tilde{A}(k) = \mathrm{Id} + \tilde{G}(k)$, where $\tilde{G}(k)$ is compact). Also, since $\varepsilon \to 0$, k_ε tends to ∞. We conclude that the only possible accumulation point of the transmission eigenvalues is infinity. Indeed, if we had infinitely many identical transmission eigenvalues $\hat{k} := \hat{k}_n$, $n = 1, 2, \ldots$, that means $0 = \tilde{A}(\hat{k})u_n = \lambda_n u_n$ (where u_n and λ_n denote the corresponding eigenfunction and eigenvalue at $\hat{k} = \hat{k}_n$), then all λ_n, $n = 1, 2, \ldots$ would be zero, which is not possible.

□

4.2.3 Transmission Eigenvalues for Scattering Problem 2

In this chapter, we study the interior transmission eigenvalue problem relating to the scattering problem from Chapter 3.2. That means, we consider a strip $\Omega = \mathbb{R} \times (-1, 1)$. The goal of this chapter is to show that there exist transmission eigenvalues. That means, we are looking for eigenpairs of the form $(v, w) \in L^2(\Omega) \times L^2(\Omega)$. To show this, we first need to study transmission eigenvalues for α-quasi-periodic solutions as auxiliary problem. With the Floquet-Bloch transform we will see in Chapter 4.2.3.2 that these two problems are closely related.

4.2.3.1 α-Quasi-Periodic Solutions

Let v, w be α-quasi-periodic functions on the strip $\Omega = \mathbb{R} \times (-1, 1)$ and let $\Omega_\mu := \mu 2\pi, (\mu + 1)2\pi] \times (-1, 1)$ for fixed $\mu \in \mathbb{Z}$ describes a cell of Ω. The system of equations we consider reads

$$\begin{aligned} \Delta w + k^2(1+q)w &= 0 && \text{in } \Omega_\mu && (4.38)\\ \Delta v + k^2 v &= 0 && \text{in } \Omega_\mu && (4.39)\\ w &= v && \text{on } \Gamma_\mu && (4.40)\\ \partial_\nu w &= \partial_\nu v && \text{on } \Gamma_\mu. && (4.41) \end{aligned}$$

Note that we study this problem in the ultra weak sense. Analogously to the previous Chapter 4.2.2, we can show again, that there exists a discrete set of real transmission eigenvalues in the α-quasi-periodic case. Note, that there is no need to pass over to the 2π-periodic functions. Everything can be proven with α-quasi-periodic functions, which indeed simplifies the analysis of Chapter 4.2.2 slightly. We formulate this in the following theorem.

Theorem 4.13 *For fixed $\alpha \in [0,1)$, there exists a discrete set of real transmission eigenvalues in the α-quasi-periodic case as stated in (4.38) - (4.41). The only possible accumulation point is infinity.*

To avoid repetition, we will not include the complete proof here. Nevertheless, in the following, we will mention some parts of the proof, which we will need later. Note, that in the same way as before (Lemma 4.2) it is possible to prove the lemma below.

Lemma 4.14 *k is a transmission eigenvalue in the α-quasi-periodic case, if, and only if, there exists a non-trivial $u^\alpha \in H^2_{0,\alpha}(\Omega_\mu)$ such that $a_{k,q,\Omega_\mu}(u^\alpha, \psi) = 0$ for all $\psi \in H^2_{0,\alpha}(\Omega_\mu)$.*

We now split a_{k,q,Ω_μ} in $a_{k,q,\Omega_\mu} = a_0 + k^2 a_1 + k^4 a_2$ where

$$a_0(u^\alpha, \psi) \; = \; \int_{\Omega_\mu} \frac{1}{q} \Delta u^\alpha \Delta \psi \, dx$$

describes the inner product $(\!(\cdot,\cdot)\!)_{H^2(\Omega_\mu)}$ on $H^2_{0,\alpha}(\Omega_\mu)$ and

$$\begin{aligned} a_1(u^\alpha, \psi) \; &= \; \int_{\Omega_\mu} \frac{1}{q}(1+q) u^\alpha \Delta \overline{\psi} \; + \; \frac{1}{q} \Delta u^\alpha \overline{\psi} \, dx, \\ &= \; \int_{\Omega_\mu} \frac{1}{q} u^\alpha \Delta \overline{\psi} \; + \; u^\alpha \Delta \overline{\psi} \; + \; \frac{1}{q} \Delta u^\alpha \overline{\psi} \, dx, \qquad (4.42) \\ a_2(u^\alpha, \psi) \; &= \; \int_{\Omega_\mu} \frac{1}{q}(1+q) u^\alpha \overline{\psi}\,, \qquad (4.43) \end{aligned}$$

with $u^\alpha, \psi \in H^2_{0,\alpha}(\Omega_\mu)$. The sesquilinear forms a_1 and a_2 are hermitian (easy to see with Green's second identity) and bounded, which can be shown analogously to the previous chapter (Chapter 4.2.2). Riesz' representation theorem provides that for $a_l : H^2_{0,\alpha}(\Omega_\mu) \to \mathbb{C}$, $l = 1,2$, there exist unique bounded operators A_l from $H^2_{0,\alpha}(\Omega_\mu)$ into itself, such that

$$a_l(u^\alpha, \psi) = (\!(A_l u^\alpha, \psi)\!)_{H^2(\Omega_\mu)}, \quad l = 1,2,$$

for all $u^\alpha, \psi \in H^2_{0,\alpha}(\Omega_\mu)$. These operators are self-adjoint, since a_1 and a_2 are hermitian. The equation $a_{k,q,\Omega_\mu}(u^\alpha, \psi) = 0$ for all $\psi \in H^2_{0,\alpha}(\Omega_\mu)$ can now be written as

$$u + k^2 A_1 u + k^4 A_2 u \; = \; 0.$$

We define the operator

$$A(k) \; := \; \mathrm{Id} + k^2 A_1 + k^4 A_2 \qquad (4.44)$$

from $H^2_{0,\alpha}(\Omega_\mu)$ into itself and formulate the lemma below.

Lemma 4.15 *k is a transmission eigenvalue in the α-quasi-periodic case, if, and only if, there exists a non trivial $u^\alpha \in H^2_{0,\alpha}(\Omega_\mu)$ such that $A(k)u^\alpha = 0$.*

It can be shown analogously to Lemma 4.9.

4.2.3.2 Solutions for a Periodic Layer

We are now interested in transmission eigenvalues to the system of equations

$$\Delta w + k^2(1+q)w = 0 \quad \text{in } \Omega \tag{4.45}$$

$$\Delta v + k^2 v = 0 \quad \text{in } \Omega \tag{4.46}$$

$$w = v \quad \text{on } \Gamma \tag{4.47}$$

$$\partial_\nu w = \partial_\nu v \quad \text{on } \Gamma, \tag{4.48}$$

that means we are looking for non-trivial solutions $v, w \in L^2(\Omega)$, where Ω is the strip $\mathbb{R} \times (-1, 1)$. The equations (4.45) und (4.46) have to be understood in the ultra weak sense.

To study transmission eigenvalues we will use the results for α-quasi-periodic solutions from the previous chapter (Chapter 4.2.3.1).

We start with adopting some analysis, which is similar to the approach of the previous chapters. For this purpose, analogously to Notation 4.1, we define $a_{k,q,\Omega}(u,\psi)$ for functions on $H^2_0(\Omega)$. The following lemma holds true. We will skip the proof, as it can be done in the same way as for Lemma 4.2.

Lemma 4.16 *k is a transmission eigenvalue if, and only if, there exists a non-trivial $u \in H^2_0(\Omega)$ such that $a_{k,q,\Omega}(u,\psi) = 0$ for all $\psi \in H^2_0(\Omega)$.*

Again, we split $a_{k,q,\Omega}$ in $a_{k,q,\Omega} = a_0 + k^2 a_1 + k^4 a_2$ where $a_0(u,\psi)$ describes the inner product $(\!(\cdot,\cdot)\!)_{H^2(\Omega)}$ on $H^2_0(\Omega)$ and $a_1(\cdot,\psi)$ and $a_2(\cdot,\psi)$ are as in (4.42) and (4.43) for all $\psi \in H^2_0(\Omega)$.

There exist unique bounded operators A_l, $l = 1, 2$, from $H^2_0(\Omega)$ into itself such that $a_l(u,\psi) = (\!(A_l u, \psi)\!)_{H^2(\Omega)}$, for all $u, \psi \in H^2_0(\Omega)$. Defining $A(k)$ from $H^2_0(\Omega)$ into itself as in (4.44), we obtain the following result.

Lemma 4.17 *k is a transmission eigenvalue if, and only if, there exists a non-trivial $u \in H^2_0(\Omega)$ such that $A(k)u = 0$.*

Remark 4.18 *Note that we do not have an analogous statement about embeddings as in Corollary 2.20 or Corollary 2.22. We therefore cannot show the compactness of the operators and hence, we cannot apply the Analytic Fredholm Theory.*

For convenience, we rewrite Lemma 4.16 in the formal way, that is, k is a transmission eigenvalue if, and only if, there exists a non-trivial solution $u \in H_0^2(\Omega)$ to

$$\left[\Delta + k^2\right] \frac{1}{q} \left[\Delta + k^2(1+q)\right] u = 0 \tag{4.49}$$

on the strip Ω.

Definition 4.19 *We say k is in the resolvent set ρ of* (4.49) *if*

$$a_{k,q,\Omega}(u,\varphi) = \int_\Omega \frac{1}{q} \left[\Delta + k^2(1+q)\right] u \left[\Delta + k^2\right] \overline{\varphi}\, dx = \int_\Omega g\overline{\varphi}\, dx, \tag{4.50}$$

for all $\varphi \in H_0^2(\Omega)$, is uniquely solvable for $u \in H_0^2(\Omega)$ for all $g \in L^2(\Omega)$. Furthermore, we denote by $\zeta := \mathbb{C} \setminus \rho$ the spectrum.

This is equivalent to saying $A(k) = Id + k^2 A_1 + k^4 A_2$ is an isomorphism in $H_0^2(\Omega)$. Now consider α-quasi-periodic functions $u^\alpha \in H_{0,\alpha}^2(\Omega_\mu)$ with $\alpha \in \Lambda = [0,1)$ and rewrite Lemma 4.14 in the formal way, that is, k is a transmission eigenvalue, if, and only if, there exists a non-trivial solution $u^\alpha \in H_{0,\alpha}^2(\Omega_\mu)$ to

$$\left[\Delta + k^2\right] \frac{1}{q} \left[\Delta + k^2(1+q)\right] u^\alpha \;=\; 0 \tag{4.51}$$

on Ω_μ.

Definition 4.20 *We say k is in the resolvent set ρ_α of* (4.51) *for $\alpha \in \Lambda$, if*

$$a_{k,q,\Omega_\mu}(u^\alpha,\psi) = \int_{\Omega_\mu} \frac{1}{q} \left[\Delta + k^2(1+q)\right] u^\alpha \left[\Delta + k^2\right] \overline{\psi}\, dx = \int_{\Omega_\mu} f\overline{\psi}\, dx, \tag{4.52}$$

for all $\psi \in H_{0,\alpha}^2(\Omega_\mu)$, is uniquely solvable for $u^\alpha \in H_{0,\alpha}^2(\Omega_\mu)$ for all $f \in L^2(\Omega_\mu)$. Furthermore, we denote by $\zeta_\alpha := \mathbb{C} \setminus \rho_\alpha$, $\alpha \in \Lambda$, the spectrum for the α-quasi-periodic case.

As already mentioned, we would like to use the Floquet-Bloch transform to elaborate the relation of the set of transmission eigenvalues to (4.49) and the set of transmission eigenvalues to (4.51). To this end, we express $a_{k,q,\Omega_\mu}(u^\alpha,\psi)$ in (4.52) with 2π-periodic functions by using $u^\alpha = u^0 e^{i\alpha x_1}$. We obtain with ψ replaced by

$\psi e^{i\alpha x_1}$

$$
\begin{aligned}
& a_{k,q,\Omega_\mu}(u^0 e^{i\alpha x_1}, \psi e^{i\alpha x_1}) \\
= \;& \int_{\Omega_\mu} \frac{1}{q} \left[\Delta u^0 + 2i\alpha \frac{\partial u^0}{\partial x_1} + k^2(1+q)u^0 - \alpha^2 u^0 \right] \\
& \qquad \cdot \left[\Delta\overline{\psi} - 2i\alpha \frac{\partial \overline{\psi}}{\partial x_1} + k^2\overline{\psi} - \alpha^2\overline{\psi} \right] dx \\
=: \;& a_\alpha(u^0, \psi) \\
=: \;& a_{\alpha,0}(u^0,\psi) + k^2 a_{\alpha,1}(u^0,\psi) + k^4 a_{\alpha,2}(u^0,\psi),
\end{aligned}
$$

for $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$, where

$$
\begin{aligned}
& a_{\alpha,0}(u^0,\psi) \\
& \quad = \int_{\Omega_\mu} \frac{1}{q} \left[\Delta u^0 + 2i\alpha \frac{\partial u^0}{\partial x_1} - \alpha^2 u^0 \right] \left[\Delta\overline{\psi} - 2i\alpha \frac{\partial \overline{\psi}}{\partial x_1} - \alpha^2\overline{\psi} \right] dx, \\
& a_{\alpha,1}(u^0,\psi) \\
& \quad = \int_{\Omega_\mu} \frac{1}{q} \left[\Delta u^0 + 2i\alpha \frac{\partial u^0}{\partial x_1} - \alpha^2 u^0 \right] \overline{\psi} \\
& \qquad + \frac{1}{q}(1+q)u^0 \left[\Delta\overline{\psi} - 2i\alpha \frac{\partial \overline{\psi}}{\partial x_1} - \alpha^2\overline{\psi} \right] dx, \\
& a_{\alpha,2}(u^0,\psi) = \int_{\Omega_\mu} \frac{1}{q}(1+q)u^0\overline{\psi} dx,
\end{aligned}
$$

for $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$.

Hence, k is in the resolvent set of (4.51) if, and only if,

$$
a_\alpha(u^0,\psi) = \int_{\Omega_\mu} f\overline{\psi}\, dx, \quad \psi \in H^2_{0,0}(\Omega_\mu) \tag{4.53}
$$

is uniquely solvable for all $f \in L^2(\Omega_\mu)$. This is easy to see by replacing f by $fe^{i\alpha x_1}$ (and ψ by $\psi e^{i\alpha x_1}$) in (4.52).

Again, the sesquilinear forms $a_{\alpha,0}$, $a_{\alpha,1}$ and $a_{\alpha,2}$ are bounded and there exist unique operators $A_{\alpha,j}$, $j = 0,1,2$, from $H^2_{0,0}(\Omega_\mu)$ into itself with $a_{\alpha,j}(u^0,\psi) = (\!(A_l u^0, \psi)\!)_{H^2(\Omega_\mu)}$ for all $u^0, \psi \in H^2_{0,0}(\Omega_\mu)$, such that k is in the resolvent set of (4.51) if, and only if,

$$
A_\alpha(k) \;:=\; A_{\alpha,0} + k^2 A_{\alpha,1} + k^4 A_{\alpha,2} \tag{4.54}
$$

is an isomorphism in $H^2_{0,0}(\Omega_\mu)$.

We note that if k is in the resolvent sets of (4.49) or (4.51), respectively, $(A(k))^{-1}$ and $(A_\alpha(k))^{-1}$ exist and are bounded by the bounded inverse theorem.

Using the Floquet-Bloch transform, see Chapter 2.4, we will now show that there is a connection between $a_{k,q,\Omega}$ as in (4.50) and a_α as in (4.53). Let $u, \varphi \in H^2_0(\Omega)$ and $\tilde{u} = Tu$ and $\tilde{\varphi} = T\varphi$,

$$
\begin{aligned}
& a_{k,q,\Omega}(u,\varphi) \\
= \;& \left(\frac{1}{q}\left[\Delta + k^2(1+q)\right] u, \left[\Delta + k^2\right]\varphi\right)_{L^2(\Omega)} \\
= \;& \left(\frac{1}{q} T\left[\Delta + k^2(1+q)\right] u, T\left[\Delta + k^2\right]\varphi\right)_{L^2(\Omega_\mu\times\Lambda)} \\
= \;& \left(\frac{1}{q}\left[T\Delta + Tk^2(1+q)\right] u, \left[T\Delta + Tk^2\right]\varphi\right)_{L^2(\Omega_\mu\times\Lambda)} \\
\overset{(*)}{=} \;& \Bigg(\frac{1}{q}\left[\Delta_x Tu + 2i\alpha\partial_{x_1} Tu - \alpha^2 Tu + k^2(1+q)Tu\right], \\
& \quad \left[\Delta_x T\varphi + 2i\alpha\partial_{x_1} T\varphi - \alpha^2 T\varphi + k^2 T\varphi\right]\Bigg)_{L^2(\Omega_\mu\times\Lambda)} \\
= \;& \int_\Lambda \Bigg(\frac{1}{q}\left[\Delta_x + 2i\alpha\partial_{x_1} - \alpha^2 + k^2(1+q)\right]\tilde{u}(\cdot,\alpha), \\
& \qquad \left[\Delta_x + 2i\alpha\partial_{x_1} - \alpha^2 + k^2\right]\tilde{\varphi}(\cdot,\alpha)\Bigg)_{L^2(\Omega_\mu)} d\alpha \\
= \;& \int_\Lambda a_\alpha(\tilde{u}(\cdot,\alpha), \tilde{\varphi}(\cdot,\alpha))\, d\alpha,
\end{aligned}
\tag{4.55}
$$

where we have used the formulas from Theorem 2.25 in $(*)$. The main result is formulated in the following theorem.

Theorem 4.21 *Let $\zeta \subset \mathbb{C}$ be the spectrum as in Definition 4.19 and $\zeta_\alpha \subset \mathbb{C}$ the spectrum as in Definition 4.20 for all $\alpha \in \Lambda := [0,1)$. Then*

$$\zeta \;=\; \bigcup_{\alpha\in\Lambda} \zeta_\alpha.$$

Note that the set of transmission eigevalues in the α-quasi-periodic case equals ζ_α, due to the compactness of the operators $A_{\alpha,1}$ and $A_{\alpha,2}$ in (4.54), whereas we only know that the set of transmission eigenvalues for the periodic medium Ω is contained in ζ.

Proof:

Let first $k \in \zeta_\alpha$ for some $\alpha \in \Lambda = [0,1)$. Then, k is a transmission eigenvalues and hence there exists a non-trivial solution $u^0 \in H^2_{0,0}(\Omega_\mu)$ to $a_\alpha(u^0, \psi) = 0$ for all $\psi \in H^2_{0,0}(\Omega_\mu)$. Let $\hat{u}$ be the α-quasi-periodic extension of $u^\alpha := u^0 e^{i\alpha x_1} \in H^2_{0,\alpha}(\Omega_\mu)$ to Ω.

We will show that $k \in \zeta$. To this end, we define for fixed $\mu \in \mathbb{Z}$ the sequence

$$\Omega^n_\mu \;:=\; \{x \in \Omega,\; x_1 \in (2\pi(\mu - n), 2\pi(\mu + 1 + n)\},$$

with $n \in \mathbb{N}$. Hence, Ω^n_μ is a cell of length $(2n+1)$ times the length of Ω_μ, with respect to x_1. We choose $\psi_n \in C^\infty(\Omega)$ such that $\psi_n = 0$ for $x \notin \Omega^{n+1}_\mu$ and $\psi_n = 1$ in Ω^n_μ and such that $|\nabla\psi_n(x)| + |\Delta\psi_n(x)| \;\leq\; c$ for all x and some constant $c > 0$. Then, $u_n := \frac{1}{n^{1/3}}\psi_n \hat{u}$, $n \in \mathbb{N}$, is in $H^2_0(\Omega)$ and

$$\begin{aligned} \|\!|u_n|\!\|^2_{H^2(\Omega)} \;\geq\; c\|u_n\|^2_{H^2(\Omega)} \;\geq\; c\int_{\Omega^n_\mu} |u_n|^2 dx \\ =\; \frac{c}{n^{2/3}}\int_{\Omega^n_\mu} |\hat{u}|^2 dx \;=\; \frac{c\,(1+2n)}{n^{2/3}}\int_{\Omega_\mu} |\hat{u}|^2 dx, \end{aligned} \tag{4.56}$$

with a constant $c \in \mathbb{R}$. Note that the right hand side tends to infinity as n tends to infinity. Furthermore, for any $\varphi \in H^2_0(\Omega)$,

$$\begin{aligned} a_{k,q,\Omega}(u_n, \varphi) \;&=\; \int_\Omega \frac{1}{q}\left[\Delta + k^2(1+q)\right] u_n \left[\Delta + k^2\right] \overline{\varphi}\, dx \\ &=\; \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\left[\Delta + k^2(1+q)\right] (\psi_n\, \hat{u}) \left[\Delta + k^2\right] \overline{\varphi}\, dx. \end{aligned}$$

With the product rule we compute that $a_{k,q,\Omega}(u_n,\varphi)$ is equal to

$$\begin{aligned}
& \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\hat u\,\Delta\psi_n + \psi_n\,\Delta\hat u + 2\nabla\psi_n\cdot\nabla\hat u + k^2(1+q)\,\psi_n\,\hat u\Big]\ \big[\Delta + k^2\big]\,\overline{\varphi}\,dx \\
= \;& \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\Delta\hat u\,(\psi_n\,\Delta\overline{\varphi}) + k^2\big[(1+q)\,\hat u\,(\psi_n\,\Delta\overline{\varphi}) + \Delta\hat u\,(\psi_n\,\overline{\varphi})\big] \\
& \qquad\qquad\qquad\qquad\qquad\qquad + k^4(1+q)\hat u\,(\psi_n\,\overline{\varphi})\Big]\,dx \\
& + \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\hat u\,\Delta\psi_n\,\Delta\overline{\varphi} + 2\nabla\hat u\cdot\nabla\psi_n\,\Delta\overline{\varphi} + k^2\left(\hat u\,\Delta\psi_n\,\overline{\varphi} + 2\nabla\hat u\,\nabla\psi_n\,\overline{\varphi}\right)\Big]\,dx \\
= \;& \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\Delta\hat u\,\Delta(\psi_n\,\overline{\varphi}) + k^2\left((1+q)\,\hat u\,\Delta(\psi_n\,\overline{\varphi}) + \Delta\hat u\,(\psi_n\,\overline{\varphi})\right) \\
& \qquad\qquad\qquad\qquad\qquad\qquad + k^4(1+q)\hat u\,(\psi_n\,\overline{\varphi})\Big]\,dx \\
& + \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\hat u\,\Delta\psi_n\,\Delta\overline{\varphi} + 2\nabla\hat u\cdot\nabla\psi_n\,\Delta\overline{\varphi} - \Delta\hat u\,\Delta\psi_n\,\overline{\varphi} - 2\Delta\hat u\,\nabla\psi_n\cdot\nabla\overline{\varphi} \\
& \qquad\qquad + k^2\big(2\nabla\hat u\,\nabla\psi_n\,\overline{\varphi} - q\hat u\,\Delta\psi_n\,\overline{\varphi} - 2(1+q)\hat u\,\nabla\psi_n\cdot\nabla\overline{\varphi}\big)\Big]\,dx \\
= \;& \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\left[(\Delta + (1+q)k^2)\hat u(\Delta + k^2)(\psi_n\overline{\varphi})\right]\,dx \qquad (4.57) \\
& + \frac{1}{n^{1/3}}\int_\Omega \frac{1}{q}\Big[\hat u\,\Delta\psi_n\,\Delta\overline{\varphi} + 2\nabla\hat u\cdot\nabla\psi_n\,\Delta\overline{\varphi} - \Delta\hat u\,\Delta\psi_n\,\overline{\varphi} - 2\Delta\hat u\,\nabla\psi_n\cdot\nabla\overline{\varphi} \\
& \qquad\qquad + k^2\big(2\nabla\hat u\,\nabla\psi_n\,\overline{\varphi} - q\hat u\,\Delta\psi_n\,\overline{\varphi} - 2(1+q)\hat u\,\nabla\psi_n\cdot\nabla\overline{\varphi}\big)\Big]\,dx. \qquad (4.58)
\end{aligned}$$

We will now prove, that

$$|a_{k,q,\Omega}(u_n,\varphi)| \;\le\; \frac{c}{n^{1/3}}\left[\int_{\Omega_\mu}[u^\alpha]^2 + [\nabla u^\alpha]^2 + [\Delta u^\alpha]^2\,dx\right]^{1/2}\ \|\varphi\|_{H^2(\Omega)} \qquad (4.59)$$

for some real constant c and for all $\varphi \in H_0^2(\Omega)$.
The first integral (4.57) on the right hand side is equal to

$$\begin{aligned}
& \frac{1}{n^{1/3}}\sum_{l\in\mathbb{Z}}\int_{\Omega_\mu}\frac{1}{q(x)}\Big[\Delta u^\alpha(x) + k^2(1+q(x))u^\alpha(x)\Big] \\
& \qquad\qquad \cdot\Big[\Delta(\psi_n\overline{\varphi})(x+2\pi l e_1) + k^2(\psi_n\overline{\varphi})(x+2\pi l e_1)\Big]e^{i2\pi l\alpha}\,dx, \qquad (4.60)
\end{aligned}$$

because $\hat u$ is the α-quasi-periodic extension of $u^\alpha \in H^2_{0,\alpha}(\Omega_\mu)$. We now define

$$\begin{aligned}
\rho(x,\alpha) \;&:=\; T(\overline{\psi}_n\varphi)(x,\alpha) \\
&=\; \sum_{l\in\mathbb{Z}}\overline{\psi}_n(x+2\pi l e_1)\,\varphi(x+2\pi l e_1)\,e^{-i(x_1+2\pi l)\alpha}.
\end{aligned}$$

Here, by T we denote the Floquet-Bloch transform, a bounded isomorphism from $H_0^2(\Omega)$ onto $L^2(\Lambda, H_{0,0}^2(\Omega_\mu))$, see Chapter 2.4. Note that $\overline{\psi_n}\varphi \in H_0^2(\Omega)$. Using the formulas from Theorem 2.25, the integral (4.60) reads

$$\frac{1}{n^{1/3}} \int_{\Omega_\mu} \frac{1}{q}\Big[\Delta u^\alpha + k^2(1+q)u^\alpha\Big]\,\Big[\Delta\overline{\rho} - 2i\alpha\frac{\partial\overline{\rho}}{\partial x_1} - \alpha^2\overline{\rho} + k^2\overline{\rho}\Big]e^{-i\alpha x_1}\,dx.$$

With $u^\alpha = u^0 e^{i\alpha x_1}$, this is equal to

$$\begin{aligned} &\frac{1}{n^{1/3}} \int_{\Omega_\mu} \frac{1}{q}\Big[\Delta u^0 + 2i\alpha\frac{\partial u^0}{\partial x_1} - \alpha^2 u^0 + k^2(1+q)u^0\Big] \\ &\qquad\qquad \cdot \Big[\Delta\overline{\rho} - 2i\alpha\frac{\partial\overline{\rho}}{\partial x_1} - \alpha^2\overline{\rho} + k^2\overline{\rho}\Big]\,dx \\ =\;& \frac{1}{n^{1/3}} a_\alpha(u^0,\rho) \;=\; 0, \end{aligned}$$

because $\rho(\cdot,\alpha) \in H_{0,0}^2(\Omega_\mu)$ and u^0 is a solution to $a_\alpha(u^0,\varphi) = 0$ for all $\varphi \in H_{0,0}^2(\Omega_\mu)$.

Let us now consider the second integral (4.58) and note that due to the properties of the function $\psi_n \in C^\infty(\Omega)$ we only have to integrate over $\Omega_\mu^{n+1} \setminus \Omega_\mu^n$. We take the absolute value and obtain with Cauchy-Schwarz inequality,

$$\begin{aligned} &\frac{1}{n^{1/3}}\bigg| \int_{\Omega_\mu^{n+1}\setminus\Omega_\mu^n} \frac{1}{q}\Big[\hat{u}\,\Delta\psi_n\,\Delta\overline{\varphi} + \nabla\hat{u}\cdot\nabla\psi_n\,\Delta\overline{\varphi} - \Delta\hat{u}\,\Delta\psi_n\,\overline{\varphi} - 2\Delta\hat{u}\,\nabla\psi_n\cdot\nabla\overline{\varphi} \\ &\qquad\qquad + k^2\big(\nabla\hat{u}\,\nabla\psi_n\,\overline{\varphi} - q\hat{u}\,\Delta\psi_n\,\overline{\varphi} - 2(1+q)\hat{u}\,\nabla\psi_n\cdot\nabla\overline{\varphi}\big)\Big]\,dx\bigg| \\ \overset{(*)}{\leq}\;& \frac{\tilde{\tilde{c}}}{n^{1/3}}\left[\int_{\Omega_\mu^{n+1}\setminus\Omega_\mu^n} [\hat{u}]^2 + [\nabla\hat{u}]^2 + [\Delta\hat{u}]^2\,dx\right]^{1/2}\left[\int_{\Omega_\mu^{n+1}\setminus\Omega_\mu^n} [\varphi]^2 + [\nabla\varphi]^2 + [\Delta\varphi]^2\,dx\right]^{1/2} \\ \leq\;& \frac{\tilde{c}}{n^{1/3}}\left[\int_{\Omega_\mu^{n+1}\setminus\Omega_\mu^n} [\hat{u}]^2 + [\nabla\hat{u}]^2 + [\Delta\hat{u}]^2\,dx\right]^{1/2} \|\varphi\|_{H^2(\Omega)}, \end{aligned}$$

for real constants $\tilde{c}$ and $\tilde{\tilde{c}}$. In $(*)$ we have used that k is fixed, $|\nabla\psi_n(x)| + |\Delta\psi_n(x)| \;\leq\; c$ and $0 < q_* \leq q \leq q^* < \infty$.

The set $\Omega_\mu^{n+1} \setminus \Omega_\mu^n$ consists of $2(n+1)+1-(2n+1) = 2$ cells of the same size as Ω_μ. Since $\hat{u}$ is the α-quasi-periodic extension of u^α, we obtain

$$\begin{aligned} &\frac{\tilde{c}}{n^{1/3}}\left[\int_{\Omega_\mu^{n+1}\setminus\Omega_\mu^n} [\hat{u}]^2 + [\nabla\hat{u}]^2 + [\Delta\hat{u}]^2\,dx\right]^{1/2} \|\varphi\|_{H^2(\Omega)} \\ =\;& \frac{\tilde{c}\sqrt{2}}{n^{1/3}}\left[\int_{\Omega_\mu} [u^\alpha]^2 + [\nabla u^\alpha]^2 + [\Delta u^\alpha]^2\,dx\right]^{1/2} \|\varphi\|_{H^2(\Omega)}, \end{aligned}$$

and hence, (4.59) is proven. We note that $\frac{\tilde{c}\sqrt{2}}{n^{1/3}}\left[\int_{\Omega_\mu}[u^\alpha]^2+[\nabla u^\alpha]^2+[\Delta u^\alpha]^2\,dx\right]^{1/2}$ tends to zero as n tends to infinity, and is therefore the second integral is obviously bounded uniformly with respect to n.

We define the sequence $(g_n)_{n\in\mathbb{N}}$ in $H_0^2(\Omega)$ such that

$$a_{k,q,\Omega}(u_n,\varphi) = (\!(g_n,\varphi)\!)_{H^2(\Omega)}, \quad \text{for all } \varphi\in H_0^2(\Omega).$$

Note, that $(g_n)_{n\in\mathbb{N}}$ exists by Riesz representation theorem.

We can now show that k is in ζ. Indeed, if k was in the resolvent set, then $A(k)$ would be bounded invertible, that means $A(k)u = g$ is uniquely solvable for all $g\in H_0^2(\Omega)$ and the mapping $g\mapsto u$ is continuous. Then, on the one hand,

$$|\!|\!|u_n|\!|\!|_{H^2(\Omega)} \;\le\; c\,|\!|\!|g_n|\!|\!|_{H^2(\Omega)},$$

for a constant $c\in\mathbb{R}$ and

$$\begin{aligned} |\!|\!|g_n|\!|\!|^2_{H^2(\Omega)} &= (\!(g_n,g_n)\!)_{H^2(\Omega)} \;=\; \left|(\!(A(k)u_n,g_n)\!)_{H^2(\Omega)}\right| \;=\; |a(u_n,g_n)| \\ &\le \frac{c}{n^{1/3}}\left[\int_{\Omega_\mu}[u^\alpha]^2+[\nabla u^\alpha]^2+[\Delta u^\alpha]^2\,dx\right]^{1/2} |\!|\!|g_n|\!|\!|_{H^2(\Omega)}, \end{aligned}$$

which shows that

$$|\!|\!|g_n|\!|\!|_{H^2(\Omega)} \;\le\; \frac{c}{n^{1/3}}\left[\int_{\Omega_\mu}[u^\alpha]^2+[\nabla u^\alpha]^2+[\Delta u^\alpha]^2\,dx\right]^{1/2} \;\to\; 0,$$

as n tends to infinity. Therefore,

$$|\!|\!|u_n|\!|\!|_{H^2(\Omega)} \;\to\; 0 \quad \text{as } n \;\to\; \infty.$$

On the other hand, we have seen in (4.56), that

$$|\!|\!|u_n|\!|\!|_{H^2(\Omega)} \;\to\; \infty \quad \text{as } n \;\to\; \infty.$$

This yields a contradiction.

To show the opposite inclusion, let $k\in\mathbb{C}$ be in the resolvent set of (4.51) for all $\alpha\in\Lambda$, that is (see (4.53)),

$$a_\alpha(u^0,\psi) \;=\; (f,\psi)_{L^2(\Omega_\mu)}, \quad \text{for all } \psi\in H^2_{0,0}(\Omega_\mu), \tag{4.61}$$

is uniquely solvable for all $f\in L^2(\Omega_\mu)$ and all $\alpha\in\Lambda$.

Let now $g \in L^2(\Omega)$ and $\tilde{g} = Tg$. Note that $\tilde{g} \in L^2(\Lambda, L^2(\Omega_\mu))$.

There exists a unique $\tilde{u}(\cdot, \alpha) \in H^2_{0,0}(\Omega_\mu)$ with

$$a_\alpha(\tilde{u}(\cdot, \alpha), \psi) = (\tilde{g}(\cdot, \alpha), \psi)_{L^2(\Omega_\mu)}, \quad \text{for all } \psi \in H^2_{0,0}(\Omega_\mu), \tag{4.62}$$

because k is in the resolvent set of (4.51) for all $\alpha \in \Lambda$.

Let us assume for a moment that $\tilde{u} \in L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$. We will show this subsequently. We define $u = T^{-1}\tilde{u}$ and we know from Theorem 2.25 that $u \in H^2_0(\Omega)$. The identity (4.55) yields for $\varphi \in H^2_0(\Omega)$ and $\tilde{\varphi} = T\varphi$ that

$$\begin{aligned} a_{k,q,\Omega}(u, \varphi) &= \int_\Lambda a_\alpha(\tilde{u}(\cdot, \alpha), \tilde{\varphi}(\cdot, \alpha))\, d\alpha \\ &= \int_\Lambda (\tilde{g}(\cdot, \alpha), \tilde{\varphi}(\cdot, \alpha))_{L^2(\Omega_\mu)}\, d\alpha \;=\; (g, \varphi)_{L^2(\Omega)}. \end{aligned}$$

This proves that k is in the resolvent set of (4.49) and ends the proof.

It remains to show that for $\tilde{u}$ as in (4.62), $\tilde{u} \in L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$.

For $\psi \in H^2_{0,0}(\Omega_\mu)$ it holds

$$\begin{aligned} \left| \int_{\Omega_\mu} \tilde{g}(\cdot, \alpha)\psi\, dx \right| &\leq \|\tilde{g}(\cdot, \alpha)\|_{L^2(\Omega_\mu)} \|\psi\|_{L^2(\Omega_\mu)} \\ &\leq \|\tilde{g}(\cdot, \alpha)\|_{L^2(\Omega_\mu)} \vert\kern-0.25ex\vert\kern-0.25ex\vert \psi \vert\kern-0.25ex\vert\kern-0.25ex\vert_{H^2(\Omega_\mu)}, \end{aligned}$$

hence, Riesz representation theorem guarantees the existence of a function $\tilde{f}(\cdot, \alpha) \in H^2_{0,0}(\Omega_\mu)$ such that

$$(\tilde{g}(\cdot, \alpha), \psi)_{L^2(\Omega_\mu)} \;=\; (\!(\tilde{f}(\cdot, \alpha), \psi)\!)_{H^2(\Omega_\mu)}, \quad \text{for all } \psi \in H^2_{0,0}(\Omega_\mu). \tag{4.63}$$

We write this as (using (4.62))

$$A_\alpha(k)\tilde{u}(\cdot, \alpha) \;=\; \tilde{f}(\cdot, \alpha) \quad \text{in } H^2_{0,0}(\Omega_\mu) \quad \text{for all } \alpha \in \Lambda.$$

with $A_\alpha(k)$ as in (4.54). $A_\alpha(k)$ is invertible for all $\alpha \in \Lambda$, because k is in the resolvent set. Also, $A_\alpha(k)$ depends continuously on α. Hence, the mapping $\alpha \mapsto \|(A_\alpha(k))^{-1}\|$ is continuous.

Since $\overline{\Lambda}$ is compact, we know that there exists a positive constant c such that

$$\|(A_\alpha(k))^{-1}\| \leq c \quad \text{for all } \alpha \in \overline{\Lambda}.$$

Furthermore, to $\tilde{g} \in L^2(\Lambda,\, L^2(\Omega_\mu))$ there exists a sequence $\tilde{g}_n \in C\left(\overline{\Lambda},\, L^2(\Omega_\mu)\right)$ with $\tilde{g}_n \to \tilde{g}$ in $L^2(\Lambda,\, L^2(\Omega_\mu))$, as $n \to \infty$. Here, $C\left(\overline{\Lambda},\, L^2(\Omega_\mu)\right)$ is defined as

$$\begin{aligned} C\left(\overline{\Lambda},\, L^2(\Omega_\mu)\right) \;&:=\; \{u : \mathbb{R}^2 \to \mathbb{C} \text{ such that } u \in L^2(\Omega_\mu) \text{ almost everywhere on } \Lambda\,, \\ &\qquad \alpha \mapsto \|u(\cdot,\alpha)\|_{L^2(\Omega_\mu)} \text{ in } C(\overline{\Lambda})\}. \end{aligned}$$

Then for fixed $\alpha \in \Lambda$,

$$a_\alpha\left(\tilde{u}_n(\cdot,\alpha), \psi\right) \;=\; (\tilde{g}_n(\cdot,\alpha), \psi)_{L^2(\Omega_\mu)}, \quad \text{for all } \psi \in H^2_{0,0}(\Omega_\mu)$$

is uniquely solvable. Again, with Riesz' representation theorem there exists a sequence $\tilde{f}_n(\cdot,\alpha) \in H^2_{0,0}(\Omega_\mu)$ such that

$$(\tilde{g}_n(\cdot,\alpha), \psi)_{L^2(\Omega_\mu)} = (\!(\tilde{f}_n(\cdot,\alpha), \psi)\!)_{H^2(\Omega_\mu)}\,, \qquad \text{for all } \psi \in H^2_{0,0}(\Omega_\mu) \tag{4.64}$$

and hence,

$$a_\alpha\left(\tilde{u}_n(\cdot,\alpha), \psi\right) \;= (\!(\tilde{f}_n(\cdot,\alpha), \psi)\!)_{H^2(\Omega_\mu)},$$

for all $\psi \in H^2_{0,0}(\Omega_\mu)$. Writing again $A_\alpha(k)\tilde{u}_n(\cdot,\alpha) = \tilde{f}_n(\cdot,\alpha)$, we conclude $\tilde{u}_n(\cdot,\alpha) = A_\alpha^{-1}(k)\tilde{f}_n(\cdot,\alpha)$ and thus $\tilde{u}_n \in C\left(\overline{\Lambda}, H^2_{0,0}(\Omega_\mu)\right)$, where

$$\begin{aligned} C\left(\overline{\Lambda},\, H^2_{0,0}(\Omega_\mu)\right) \;&:=\; \{u : \mathbb{R}^2 \to \mathbb{C} \text{ such that } u \in H^2_{0,0}(\Omega_\mu) \text{ almost everywhere on } \Lambda\,, \\ &\qquad \alpha \mapsto \|u(\cdot,\alpha)\|_{H^2(\Omega_\mu)} \text{ in } C(\overline{\Lambda})\}. \end{aligned}$$

Furthermore, it is for fixed α

$$\begin{aligned} & \|\tilde{u}_n(\cdot,\alpha) - \tilde{u}(\cdot,\alpha)\|_{H^2(\Omega_\mu)} \\ =\; & \|A_\alpha^{-1}(k)\tilde{f}_n(\cdot,\alpha) - A_\alpha^{-1}(k)\tilde{f}(\cdot,\alpha)\|_{H^2(\Omega_\mu)} \\ \le\; & c\,\|\tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha)\|_{H^2(\Omega_\mu)}, \end{aligned}$$

$c \in \mathbb{R}$.

We estimate

$$\begin{aligned} \|\tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha)\|^2_{H^2(\Omega_\mu)} \;&\overset{(*)}{=}\; (\tilde{g}_n(\cdot,\alpha) - \tilde{g}(\cdot,\alpha),\, \tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha))_{L^2(\Omega_\mu)} \\ &\le\; \|\tilde{g}_n(\cdot,\alpha) - \tilde{g}(\cdot,\alpha)\|_{L^2(\Omega_\mu)}\, \|\tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha)\|_{L^2(\Omega_\mu)} \\ &\le\; \|\tilde{g}_n(\cdot,\alpha) - \tilde{g}(\cdot,\alpha)\|_{L^2(\Omega_\mu)}\, \|\tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha)\|_{H^2(\Omega_\mu)}, \end{aligned}$$

which shows,

$$\|\tilde{f}_n(\cdot,\alpha) - \tilde{f}(\cdot,\alpha)\|_{H^2(\Omega_\mu)} \;\le\; \|\tilde{g}_n(\cdot,\alpha) - \tilde{g}(\cdot,\alpha)\|_{L^2(\Omega_\mu)}.$$

In $(*)$ we have used (4.63) and (4.64).

We conclude,

$$\int_\Lambda \|\tilde{u}_n - \tilde{u}\|^2_{H^2(\Omega_\mu)} d\alpha \;\leq\; c\,\|\tilde{f}_n - \tilde{f}\|_{L^2\left(\Lambda, H^2_{0,0}(\Omega_\mu)\right)} \;\leq\; c\,\|\tilde{g}_n - \tilde{g}\|_{L^2(\Lambda, L^2(\Omega_\mu))},$$

which tends to zero as n tends to infinity. This shows that $\tilde{u} \in L^2(\Lambda, H^2_{0,0}(\Omega_\mu))$.

□

Theorem 4.21 shows that transmission eigenvalues exist for Scattering Problem 2. It would be interesting to know, if there are real intervals for k, such that k is not a transmission eigenvalue in the α-quasi periodic case for any $\alpha \in \Lambda$. That means that k is in the resolvent set ρ_α for all $\alpha \in \Lambda$ and hence in the resolvent set ρ. These intervals are called *band gaps for transmission eigenvalues*. This topic is not covered in this thesis but is, in our opinion, worth mentioning.

4.3 Complex Transmission Eigenvalues

This chapter is about some studies on complex transmission eigenvalues. We consider the transmission eigenvalue problem for Scattering Problem 1. However, we choose the periodic medium here to be a strip on top of a perfect conductor with constant contrast.

4.3.1 The Geometric Setting

We consider a strip Ω of height 1 in the upper half plane,

$$\Omega \;:=\; \{x \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \in (0,1)\}.$$

The strip is excited by an incident plane wave, which will lead us to the study of kd-quasi-periodic functions. We will make use of a cell Ω_μ, defined as

$$\Omega_\mu \;:=\; \{x \in \Omega : x_1 \in [\mu 2\pi, (\mu+1)2\pi]\},$$

for fixed $\mu \in \mathbb{Z}$. Let $q > 0$ be a constant contrast. We call the upper and the lower part of the boundary $\Gamma_{\mu,up}$ and $\Gamma_{\mu,lo}$, respectively, and we require $\Gamma_{\mu,lo}$ to be a perfect conductor. The interior transmission eigenvalue problem is to find non-trivial kd-quasi-periodic solutions $(\tilde{v}, \tilde{w}) \in L^2(\Omega_\mu) \times L^2(\Omega_\mu)$ to the system of equations,

$$\begin{aligned}
\Delta \tilde{w} + k^2 (1+q)\, \tilde{w} &= 0 && \text{in } \Omega_\mu \\
\Delta \tilde{v} + k^2 \tilde{v} &= 0 && \text{in } \Omega_\mu \\
\tilde{w} &= \tilde{v} && \text{on } \Gamma_{\mu,up} \\
\partial_\nu \tilde{w} &= \partial_\nu \tilde{v} && \text{on } \Gamma_{\mu,up} \\
\tilde{w} \ = \ \tilde{v} &= 0 && \text{on } \Gamma_{\mu,lo}.
\end{aligned}$$

We are going to show that under some conditions on the contrast there exist (real and) complex transmission eigenvalues in this case.

Remark 4.22 *Note that the existence of real transmission eigenvalues has already been shown in Chapter 4.2.2. Nevertheless, we will do some (different) analysis on this topic again, which will help us to study complex transmission eigenvalues.*

Let $k \in \mathbb{R}$. We consider the following two kd-quasi-periodic functions.

$$\begin{aligned}
\tilde{v}(x_1, x_2) &= e^{i(n+kd)x_1} v(x_2), \\
\tilde{w}(x_1, x_2) &= e^{i(n+kd)x_1} w(x_2),
\end{aligned}$$

with $n \in \mathbb{Z}$. The functions $(\tilde{v}, \tilde{w}) \in L^2(\Omega_\mu) \times L^2(\Omega_\mu)$ are solutions to the interior transmission eigenvalue problem, if, and only if, $v(x_2)$ and $w(x_2)$ satisfy the boundary conditions,

$$\begin{aligned}
w(1) &= v(1), \\
w'(1) &= v'(1), \\
w(0) &= v(0) \ = \ 0.
\end{aligned} \tag{4.65}$$

as well as

$$w''(x_2) + \left(k^2(1+q) - (n+kd)^2\right) w(x_2) = 0, \tag{4.66}$$

$$v''(x_2) + \left(k^2 - (n+kd)^2\right) v(x_2) = 0. \tag{4.67}$$

Now we choose k large enough such that for constant $q > 0$ and $n \in \mathbb{Z}$ fixed,

$$k^2(1+q) - (n+kd)^2 = k^2(1+q-d^2) - (n^2 + 2nkd) > 0$$

and

$$k^2 - (n+kd)^2 = k^2(1-d^2) - (n^2 + 2nkd) > 0.$$

Note that $d^2 < 1$ (see (3.1)).

Now, all solutions to these differential equations (4.66), (4.67) and bounary conditions (4.65) are given by

$$\begin{aligned} w(x_2) &= \alpha \sin(\sqrt{k^2(1+q)-(n+kd)^2}x_2) \\ v(x_2) &= \beta \sin(\sqrt{k^2-(n+kd)^2}x_2), \end{aligned}$$

$\alpha, \beta \in \mathbb{R}$. For convenience, we define

$$a := \sqrt{1+q-d^2} > 0 \quad \text{and} \quad b := \sqrt{1-d^2} > 0$$

as well as

$$\Lambda_a(k) := \sqrt{k^2(1+q)-(n+kd)^2} = k\sqrt{a^2-n^2/k^2-2nd/k} \tag{4.68}$$

and

$$\Lambda_b(k) := \sqrt{k^2-(n+kd)^2} = k\sqrt{b^2-n^2/k^2-2nd/k}. \tag{4.69}$$

With these definitions, it is

$$\begin{aligned} w(x_2) &= \alpha \sin(\Lambda_a(k)x_2) \\ v(x_2) &= \beta \sin(\Lambda_b(k)x_2), \end{aligned}$$

$\alpha, \beta \in \mathbb{R}$. The boundary conditions on $\Gamma_{\mu,up}$ are satisfied if α and β satisfy

$$\begin{aligned} 0 &= \beta \sin(\Lambda_b) - \alpha \sin(\Lambda_a), \\ 0 &= \beta \Lambda_b \cos(\Lambda_b) - \alpha \Lambda_a \cos(\Lambda_a). \end{aligned} \tag{4.70}$$

To find the zeros, we consider the determinant $\tilde{D}$, which is given by

$$\begin{aligned} \tilde{D} &:= \det \begin{vmatrix} \sin(\Lambda_b) & -\sin(\Lambda_a) \\ \Lambda_b \cos(\Lambda_b) & -\Lambda_a \cos(\Lambda_a) \end{vmatrix} \\ &= \Lambda_b(k)\, \sin(\Lambda_a)(k)\, \cos(\Lambda_b) \,-\, \Lambda_a\, \sin(\Lambda_b)\, \cos(\Lambda_a)\,, \end{aligned} \tag{4.71}$$

and the mapping

$$k \mapsto D(k) := \frac{k\tilde{D}(k)}{\Lambda_a \Lambda_b}.$$

It is

$$D(k) = k\left(\frac{\sin(\Lambda_b)}{\Lambda_b}\cos(\Lambda_a) - \frac{\sin(\Lambda_a)}{\Lambda_a}\cos(\Lambda_b)\right).$$

Note that $\Lambda \mapsto \frac{\sin(\Lambda)}{\Lambda}$ and $\Lambda \mapsto \cos(\Lambda)$ are even functions and hence there appear no square roots in the power series. We can therefore extend D to the entire complex plane and prove the following lemma.

Lemma 4.23 *The mapping $D : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type at most $\tau = a + b$.*

Proof: For simplification, we ignore the multiplication by k for a moment and write $D(k) = k\hat{D}(k)$. We will consider the four arising power series of $\hat{D}$ separately and apply Lemma 2.28 afterwards to show that $k \mapsto \hat{D}(k)$ is an entire function of exponential type. According to Lemma 2.29, the multiplication by k has no further influence on the order or the type, therefore we can transmit the result to D.

Let us start with $\frac{\sin(\Lambda_b)}{\Lambda_b}$. It is

$$\frac{\sin(\Lambda_b(k))}{\Lambda_b(k)} = \sum_{l=0}^{\infty}(-1)^l \frac{(k^2b^2 - n^2 - 2ndk)^l}{(2l+1)!}.$$

We plug in $k = re^{i\theta}$ in the expression $k^2b^2 - n^2 - 2ndk$ and obtain

$$\begin{aligned}
\left|r^2e^{2i\theta}b^2 - n^2 - 2nre^{i\theta}d\right| &= r^2\left|e^{2i\theta}b^2 - (n/r)^2 - 2(n/r)e^{i\theta}d\right| \\
&\leq r^2\left(\left|e^{2i\theta}b^2\right| + \varepsilon(r)\right) \\
&= r^2\left(b^2 + \varepsilon(r)\right) \\
&=: \phi(r),
\end{aligned}$$

with $\varepsilon(r) \to 0$, as $r \to \infty$. Note that the square root of ϕ exists for all r. With this we obtain

$$\begin{aligned}
M(r) &:= \max_{0\leq\theta\leq 2\pi}\left|\sum_{l=0}^{\infty}(-1)^l\frac{(r^2e^{2i\theta}b^2 - n^2 - 2ndre^{i\theta})^l}{(2l+1)!}\right| \\
&\leq \max_{0\leq\theta\leq 2\pi}\sum_{l=0}^{\infty}\frac{\left|r^2e^{2i\theta}b^2 - n^2 - 2ndre^{i\theta}\right|^l}{(2l+1)!} \\
&\leq \sum_{l=0}^{\infty}\frac{\phi(r)^l}{(2l+1)!} \\
&\leq \sum_{l=0}^{\infty}\frac{\sqrt{\phi(r)}^{2l+1}}{(2l+1)!} + \sum_{l=0}^{\infty}\frac{\sqrt{\phi(r)}^{2l}}{(2l)!} \\
&= e^{\sqrt{\phi(r)}} \\
&= e^{r\sqrt{b^2+\varepsilon(r)}}, \qquad (4.72)
\end{aligned}$$

which shows that the order is at most 1, because

$$\begin{aligned}
\rho &:= \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log\log e^{r\sqrt{b^2+\varepsilon(r)}}}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log r + \log(\sqrt{b^2+\varepsilon(r)})}{\log r} \\
&= 1,
\end{aligned}$$

because $\varepsilon(r) \to 0$, as $r \to \infty$. Now we choose $j \in \mathbb{N}$ large enough, such that $\frac{j^2-n^2}{b^2} - \frac{n^2d^2}{b^4}$ is positive and we define for these j the sequence $(k_j)_{j\in\mathbb{N}}$ by

$$k_j := \frac{nd}{b^2} + i\sqrt{\frac{j^2-n^2}{b^2} - \frac{n^2d^2}{b^4}}. \tag{4.73}$$

We write $k_j = r_j e^{i\theta_j}$, with absolute value

$$r_j := |k_j| \;=\; \sqrt{\frac{j^2-n^2}{b^2}} \;\leq\; \frac{j}{b}, \tag{4.74}$$

and $\theta_j \in [0, 2\pi]$. Furthermore, from (4.73) we obtain

$$k_j^2 b^2 - n^2 - 2ndk_j = -j^2. \tag{4.75}$$

For $k_j = r_j e^{i\theta_j}$ with the corresponding pair r_j and θ_j,

$$\begin{aligned}
M(r_j) &= \max_{0\leq\theta\leq 2\pi} \left| \sum_{l=0}^{\infty} (-1)^l \frac{(r_j^2 e^{2i\theta} b^2 - n^2 - 2ndr_j e^{i\theta})^l}{(2l+1)!} \right| \\
&\geq \left| \sum_{l=0}^{\infty} (-1)^l \frac{(r_j^2 e^{2i\theta_j} b^2 - n^2 - 2ndr_j e^{i\theta_j})^l}{(2l+1)!} \right| \\
&\overset{(4.75)}{=} \sum_{l=0}^{\infty} (-1)^l \frac{(ij)^{2l}}{(2l+1)!} \;=\; \sum_{l=0}^{\infty} \frac{j^{2l}}{(2l+1)!} \\
&= \frac{1}{j}\sinh j \;=\; \frac{1}{2j}\left(e^j - e^{-j}\right) \;=\; \frac{1}{2j} e^j \left(1 - e^{-2j}\right) \\
&\geq \frac{e^j}{4j},
\end{aligned} \tag{4.76}$$

for j large enough. From this, we obtain

$$\log M(r_j) \geq j - \log 4j \geq j/2$$

for j large enough which shows for the order,

$$\begin{aligned}
\rho &= \limsup_{r\to\infty} \frac{\log\log M(r)}{\log(r)} \\
&\geq \lim_{j\to\infty} \frac{\log\log M(r_j)}{\log(r_j)} \\
&\geq \lim_{j\to\infty} \frac{\log(j/2)}{\log(r_j)} \\
&= \lim_{j\to\infty} \frac{\log j - \log 2}{\log(r_j)} \\
&\overset{(4.74)}{\geq} \lim_{j\to\infty} \frac{\log j - \log 2}{\log j - \log b} \\
&= 1.
\end{aligned}$$

We conclude $\rho = 1$. For the type τ we obtain

$$\tau := \limsup_{r\to\infty} \frac{\log M(r)}{r} \overset{(4.72)}{\leq} \limsup_{r\to\infty} \frac{\log e^{r\sqrt{b^2+\varepsilon(r)}}}{r} = b$$

as well as

$$\begin{aligned}
\tau &= \limsup_{r\to\infty} \frac{\log M(r)}{r} \overset{(4.76)}{\geq} \limsup_{j\to\infty} \frac{\log\left(e^j/(4j)\right)}{r_j} \\
&\overset{(4.74)}{\geq} \limsup_{j\to\infty} \frac{b\,(j - \log 4j)}{j} \\
&= \limsup_{j\to\infty} b - \frac{b\,\log 4j}{j} = b.
\end{aligned}$$

We conclude, that $k \mapsto \frac{\sin(\Lambda_b(k))}{\Lambda_b(k)}$ is an entire function of exponential type b. Analogously, $k \mapsto \cos(\Lambda_a(k))$ is an entire function of exponential type a and with Lemma 2.28 we conclude, that the product of $\frac{\sin(\Lambda_b)}{\Lambda_b}$ and $\cos(\Lambda_a)$ is an entire function of at most $a+b$.

Of course, everything holds true for b replaced by a and the other way around. Therefore, $\frac{\sin(\Lambda_a)}{\Lambda_a}\cos(\Lambda_b)$ is again an entire functions of exponential type at most $a+b$. Lemma 2.28 shows that $k \mapsto \hat{D}(k)$ is an entire function of type at most $a+b$. Lemma 2.29 ends the proof. □

Lemma 4.24 *Let again $a = \sqrt{1+q-d^2}$, $b = \sqrt{1-d^2}$ and Λ_a, Λ_b as in (4.68) and (4.69). Then,*

$$\Lambda_a(k) = ka - nd/a + r_a(k)$$

and

$$\Lambda_b(k) = kb - nd/b + r_b(k),$$

with some functions $r_a(k)$ and $r_b(k)$ which satisfy

$$\begin{aligned} r_a(k) &= \mathcal{O}\left(1/k\right), \\ r_b(k) &= \mathcal{O}\left(1/k\right), \end{aligned}$$

for $k \to \infty$.

Proof: We consider

$$\begin{aligned} & k\ \left(\Lambda_a(k) - ka + nd/a\right) \\ = \ & k\left(\sqrt{k^2(1+q)-(n+kd)^2} - k\sqrt{1+q-d^2} + \frac{nd}{\sqrt{1+q-d^2}}\right) \\ = \ & \frac{-n^2 - 2nkd}{\sqrt{1+q-(n/k+d)^2}+\sqrt{1+q-d^2}} + \frac{2nkd}{2\sqrt{1+q-d^2}}. \end{aligned}$$

This splits up into two parts, namely

$$\frac{-n^2}{\sqrt{1+q-(n/k+d)^2}+\sqrt{1+q-d^2}} \ \to \ \frac{-n^2}{2\sqrt{1+q-d^2}}, \quad \text{as } k \to \infty$$

and

$$2nd \cdot \left(\frac{k}{2\sqrt{1+q-d^2}} - \frac{k}{\sqrt{1+q-(n/k+d)^2}+\sqrt{1+q-d^2}}\right).$$

In the latter, let us consider the part in brackets and reduce it to a common denominator,

$$\frac{k\sqrt{1+q-(n/k+d)^2} - k\sqrt{1+q-d^2}}{2\sqrt{1+q-d^2}\left(\sqrt{1+q-(n/k+d)^2}+\sqrt{1+q-d^2}\right)}. \tag{4.77}$$

Extending the fraction by $\sqrt{1+q-(n/k+d)^2} + \sqrt{1+q-d^2}$ shows that (4.77) tends to $\frac{-nd}{4(1+q-d^2)^{3/2}}$ as k tends to infinity. This is a bounded expression and we conclude that

$$\lim_{k\to\infty} \left|k\left(\Lambda_a(k) - ka + nd/a\right)\right| \ < \ \infty.$$

Analogously, we can show

$$\Lambda_b(k) - kb + nd/b = \mathcal{O}(1/k).$$

$\square$

With this lemma, we see the following.

Lemma 4.25 *Let $r(k) = \mathcal{O}(1/k)$, $k \to \infty$. Then,*

$$\sin(r(k)) = \mathcal{O}(1/k) \quad \text{and} \quad \cos(r(k)) - 1 = \mathcal{O}(1/k). \tag{4.78}$$

Lemma 4.26 *Let again $a = \sqrt{1+q-d^2}$ and $b = \sqrt{1-d^2}$. Then,*

$$D(k) = \frac{f(k)}{ab} + \mathcal{O}(1/k), \quad k \to \infty,$$

with

$$\begin{aligned} f(k) \quad := \quad & \frac{a+b}{2} \sin\Big(k(a-b) - dn\,(1/a - 1/b)\Big) \\ - \quad & \frac{a-b}{2} \sin\Big(k(a+b) - dn\,(1/a + 1/b)\Big). \end{aligned} \tag{4.79}$$

Proof:

First we prove,

$$\sin(\Lambda_a(k)) = \sin(ka - nd/a) + \mathcal{O}(1/k), \quad k \to \infty.$$

This can be seen by replacing $\Lambda_a(k)$ by $ka - nd/a + r_a(k)$ (see Lemma 4.24) and using trigonometric identities, $\sin(u \pm v) = \sin(u)\,\cos(v) \pm \sin(v)\cos(u)$,

$$\begin{aligned} & \sin(\Lambda_a(k)) - \sin(ka - nd/a) \\ = \; & \sin(ka - nd/a + r_a(k)) - \sin(ka - nd/a) \\ = \; & \sin(ka - nd/a)\;(\cos(r_a(k)) - 1) \\ & + \cos(ka - nd/a)\sin(r_a(k)) \\ = \; & \mathcal{O}(1/k), \end{aligned}$$

where the last equality holds true because of Lemma 4.25 in the last step. With the same arguments it holds for large k

$$\cos(\Lambda_a(k)) = \cos(ka - nd/a) + \mathcal{O}(1/k).$$

Everything holds true for a replaced by b and we compute

$$\begin{aligned}
& \sin(\Lambda_b(k))\cos(\Lambda_a(k)) \\
= \; & \sin\left(kb - nd/b\right)\cos\left(ka - nd/a\right) + \mathcal{O}\left(1/k\right)
\end{aligned}$$

and

$$\begin{aligned}
& \sin(\Lambda_a(k))\cos(\Lambda_b(k)) \\
= \; & \sin\left(ka - nd/a\right)\cos\left(kb - nd/b\right) + \mathcal{O}\left(1/k\right).
\end{aligned}$$

Let us now consider the determinant $\tilde{D}$ as in (4.71). We use $\Lambda_a(k) = ka - nd/a + \mathcal{O}(1/k)$, $k \to \infty$, and the results above and plug them in. We obtain

$$\begin{aligned}
\tilde{D}(k) \; = \; & \left(kb - nd/b + \mathcal{O}\left(1/k\right)\right) \\
& \cdot \Big(\sin\left(ka - nd/a\right)\cos\left(kb - nd/b\right) + \mathcal{O}\left(1/k\right) \Big) \\
- \; & \left(ka - nd/a + \mathcal{O}\left(1/k\right)\right) \\
& \cdot \Big(\sin\left(kb - nd/b\right)\cos\left(ka - nd/a\right) + \mathcal{O}\left(1/k\right) \Big) \\
= \; & kf(k) + t(k) + \mathcal{O}\left(1/k\right),
\end{aligned}$$

$k \to \infty$, with

$$\begin{aligned}
f(k) \; = \; & b\sin(ka - nd/a)\cos(kb - nd/b) \\
- \; & a\sin(kb - nd/b)\cos(ka - nd/a)
\end{aligned} \tag{4.80}$$

and some bounded function $t(k)$. Again, using trigonometric identities, we can rewrite $f(k)$ as

$$\begin{aligned}
f(k) \; = \; & \frac{a-b}{2}\sin\left(k(a+b) - dn\left(1/a + 1/b\right)\right) \\
- \; & \frac{a+b}{2}\sin\left(k(a-b) - dn\left(1/a - 1/b\right)\right).
\end{aligned}$$

Furthermore, from Lemma 4.24 we conclude $\frac{\Lambda_a(k)}{k} - a + \frac{nd}{ak} = \mathcal{O}\left(\frac{1}{k^2}\right)$ and therefore $\frac{\Lambda_a(k)}{k} \to a$, $k \to \infty$. The same holds true for a replaced by b and we finally arrive at

$$\begin{aligned}
D(k) \; & = \; \frac{k\tilde{D}(k)}{\Lambda_a(k)\Lambda_b(k)} \; = \; \frac{k(kf(k) + t(k) + \mathcal{O}(1/k))}{\Lambda_a(k)\Lambda_b(k)} \\
& = \; \frac{f(k)}{ab} + \mathcal{O}(1/k).
\end{aligned}$$

This ends the proof.

$\square$

We extend f analytically to the entire complex plane and prove the lemma below.

Lemma 4.27 *The function $f : \mathbb{C} \to \mathbb{C}$ defined as*

$$
\begin{aligned}
f(k) \quad = \quad & \frac{a-b}{2} \sin\left(k(a+b) - dn\left(1/a + 1/b\right)\right) \\
& - \frac{a+b}{2} \sin\left(k(a-b) - dn\left(\frac{1}{a} - \frac{1}{b}\right)\right)
\end{aligned}
$$

is an entire function of at most exponential type $\tau := a + b$.

Proof: We first show for $c_1 > 0$ and $c_2 \in \mathbb{R}$ that $e^{i(kc_1+c_2)}$ is an entire function of exponential type c_1. The maximum modulus is

$$
\begin{aligned}
M(r) \quad &= \quad \max_{0 \le \theta \le 2\pi} \left| e^{i(r(\cos\theta + i\sin\theta)c_1 + c_2)} \right| \\
&= \quad \max_{0 \le \theta \le 2\pi} e^{-rc_1 \sin\theta} \\
&= \quad e^{c_1 r}.
\end{aligned}
$$

For the order, it holds true that

$$
\rho \;=\; \limsup_{r\to\infty} \frac{\log\log(e^{c_1 r})}{\log(r)} \;=\; \frac{\log(c_1 r)}{\log(r)} \;=\; 1
$$

and for the type, we have

$$
\tau \;=\; \limsup_{r\to\infty} \frac{\log(e^{c_1 r})}{r} \;=\; \frac{c_1 r}{r} \;=\; c_1.
$$

Analogously, $e^{-i(kc_1+c_2)}$ is an entire function of exponential type c_1.

We apply Lemma 2.28 to see that $\sin(kc_1 + c_2)$ is an entire function of at most type c_1. Plugging in $c_1 = a + b$ and $c_2 = dn\left(\frac{1}{a} + \frac{1}{b}\right)$ shows $\sin\left(k(a+b) - dn\left(\frac{1}{a} + \frac{1}{b}\right)\right)$ is an entire function of exponential type at most $a + b$.

In the same way, we see that $\sin\left(k(a-b) - dn\left(\frac{1}{a} - \frac{1}{b}\right)\right)$ is an entire function of exponential type at most $a - b$. Therefore, again with Lemma 2.28, f is an entire function of exponential type at most $a + b$. Note that $a + b > a - b$. $\square$

4.3.2 Existence of Complex Transmission Eigenvalues

Theorem 4.28 *Let $a = \sqrt{1+q-d^2}$ and $b = \sqrt{1-d^2}$ and let $q > 0$ be constant such that $\frac{a}{b} \notin \mathbb{N}$. Then, there exist infinitely many real and complex transmission eigenvalues.*

Proof: We will first show that there exist infinitely many real transmission eigenvalues and use this result to show that there are also infinitely many complex transmission eigenvalues. By former chapters of this work, we already know that real transmission eigenvalues exist. Nevertheless, with the following analysis on real transmission eigenvalues we can use Corollary 2.31 to prove the existence of complex transmission eigenvalues.

- Real transmission eigenvalues

Let first $k \in \mathbb{R}$. We consider $D(k) = \frac{f(k)}{ab} + \mathcal{O}(1/k)$, $k \to \infty$, as in Lemma 4.26 with $f(k)$ as in (4.79). We will show that D (and hence, the determinant $\tilde{D}$ as in (4.71)) has infinitely real zeros.

To this end we choose $k = \hat{k}_{1,l}$ such that

$$\hat{k}_{1,l}(a-b) - dn\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\pi}{2} + 2\pi l, \quad l \in \mathbb{N},$$

which implies,

$$\sin\left(\hat{k}_{1,l}(a-b) - dn\left(\frac{1}{a} - \frac{1}{b}\right)\right) = 1,$$

and hence

$$\begin{aligned} f(\hat{k}_{1,l}) &= \frac{a-b}{2}\sin\left(\hat{k}_{1,l}(a+b) - dn\left(\frac{1}{a} + \frac{1}{b}\right)\right) - \frac{a+b}{2} \\ &\leq \frac{a-b}{2} - \frac{a+b}{2} \;=\; -b \;<\; 0. \end{aligned}$$

This shows

$$D(\hat{k}_{1,l}) \;\leq\; -b/ab + \mathcal{O}(1/\hat{k}_{1,l}) \;=\; -1/a + \mathcal{O}(1/\hat{k}_{1,l}) \;<\; 0,$$

for large l. Now let $k = \hat{k}_{2,l}$ such that

$$\hat{k}_{2,l}(a-b) - dn\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{\pi}{2} + 2\pi l,$$

$l \in \mathbb{N}$. We see analogously $f(\hat{k}_{2,l}) \geq \; b \; > \; 0$ and hence

$$D(\hat{k}_{2,l}) \;\geq\; 1/a + \mathcal{O}(1/\hat{k}_{1,l}) \;>\; 0,$$

for large l.

D is a continuous function. Hence, there exists some $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$, we find $\hat{k}_l$ between $\hat{k}_{1,l}$ and $\hat{k}_{2,l}$ such that $D(\hat{k}_l) = 0$. That means, there are infinitely many real transmission eigenvalues.

- Complex transmission eigenvalues

To show that there are complex eigenvalues, we will use Corollary 2.31. Note, that f is an entire function of order 1, as we have shown in Lemma 4.27. Thus f is an entire function of order less than 2. We assume f has only a finite number of complex zeros. As seen before (in (4.80)), f can be written as

$$\begin{aligned} f(k) \;\;=\;\; & \; a\cos\left(ka - \frac{dn}{a}\right)\sin\left(kb - \frac{dn}{b}\right) \\ & - b\cos\left(kb - \frac{dn}{b}\right)\sin\left(ka - \frac{dn}{a}\right). \end{aligned} \tag{4.81}$$

We compute

$$f'(k) = -q\sin\left(ka - \frac{dn}{a}\right)\sin\left(kb - \frac{dn}{b}\right),$$

with $q = a^2 - b^2 > 0$.

The real zeros of $f'(k)$ are given by $k_{a,l} = l\pi/a + dn/a^2$ and $k_{b,l} = l\pi/b + dn/b^2$, $l \in \mathbb{Z}$. For simplicity, we omit the index l in the following and write k_a and k_b.

The idea is to show that there are infinitely many intervals formed by two consecutive zeros k_a and k_b of f', where the sign of $f(k_a)$ and $f(k_b)$ does not change, that means $f(k_a) \cdot f(k_b) > 0$. Note that there cannot exist an additional zero of f between k_a and k_b, because this would lead to an additional zero of f' between k_a and k_b. This is not possible, since k_a and k_b are two consecutive zeros of f'.

We will split the rest of the prove in two cases.

Case 1:

Let us consider the situation that k_b is a zero of $\sin(kb - dn/b)$ and not of $\sin(ka - dn/a)$. The directly following zero of f' is a zero of $\sin(ka - dn/a)$, because $a > b$. We call this zero k_a. Let us further assume that the derivatives of $\sin(kb - dn/b)$ at k_b and of $\sin(ka - dn/a)$ at k_a are negative, that means $b\cos(k_b b - dn/b) < 0$

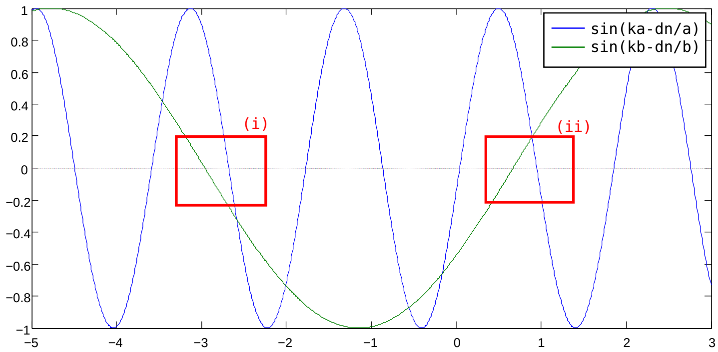

Figure 8: $\sin(ka - \frac{dn}{a})$ and $\sin(kb - \frac{dn}{b})$

and $a\cos(k_a a - dn/a) < 0$. We have illustrated the situation in (i) in Figure 8 on the left hand side.

Since the derivative of $\sin(ka - dn/a)$ in $k_a > k_b$ is negative and since there is no other zero of $\sin(ka-dn/a)$ between k_b and k_a we conclude that $\sin(k_b a-dn/a) > 0$. A similar argument shows $\sin(k_a b - dn/b) < 0$. Comparing with (4.81) shows $f(k_b) > 0$ and $f(k_a) > 0$ and therefore f does not change sign.

Assuming $b\cos(k_b b - dn/b)$ and $a\cos(k_a a - dn/a)$ to be positive yields with analogous arguments that f does not change sign. Also, changing the order of the consecutive zeros, that means $k_a < k_b$, shows the same.

Let us now consider the case that for two consecutive zeros k_a and k_b, $k_b < k_a$, of $\sin(ka - dn/a)$ and $\sin(kb - dn/b)$ respectively, the derivatives have the opposite sign, that means $b\cos(k_a b - dn/b) > 0$ and $a\cos(k_b a - dn/a) < 0$. We have illustrated the situation in (ii) in Figure 8 on the right hand side. With similar arguments as before, we conclude that $\sin(k_b a - dn/a)$ and $\sin(k_a b - dn/b)$ are positive and hence $f(k_a) < 0$ and $f(k_b) < 0$.

Again, assuming $b\cos(k_a b-dn/b) < 0$ and $a\cos(k_b a-dn/a) > 0$ as well as changing the order of the zeros shows in the same way that f does not change sign.

Note that it is not possible that the following zero of f' after k_b is a zero of $\sin(kb - dn/b)$, because $a > b$.

Case 2:

Now we consider the situation that k_1 is a common zero of $\sin(kb-dn/b)$ and also of $\sin(ka - dn/a)$. Hence, there exist l_a and $l_b \in \mathbb{Z}$ with

$$k_1 = l_a\frac{\pi}{a} + \frac{dn}{a^2} = l_b\frac{\pi}{b} + \frac{dn}{b^2}. \tag{4.82}$$

The next zero of $\sin(kb - dn/b)$ is

$$k_b \ := \ (l_b + 1)\frac{\pi}{b} + \frac{dn}{b^2}.$$

It is not possible that k_b is also a zero of $\sin(ka - dn/a)$ because otherwise there must exist some $l \in \mathbb{Z}$ such that

$$(l_b + 1)\frac{\pi}{b} + \frac{dn}{b^2} \ = \ l\frac{\pi}{a} + \frac{dn}{a^2},$$

that means with (4.82)

$$\frac{\pi}{b} + l_a\frac{\pi}{a} + \frac{dn}{a^2} \ = \ l\frac{\pi}{a} + \frac{dn}{a^2}.$$

This is satisfied if, and only if,

$$\frac{a}{b} \ = \ l - l_a \ \in \ \mathbb{N},$$

which yields a contradiction, because we excluded the case that $\frac{a}{b} \in \mathbb{N}$.

To sum up, if k_1 is a common zero of $\sin(kb - dn/b)$ and also of $\sin(ka - dn/a)$, then the next following zero of $\sin(kb - dn/b)$ provides a zero of $\sin(kb - dn/b)$ only and hence, we are back in Case 1.

Due to the periodicity of the sine and the cosine function, there are infinitely many intervals formed by two consequtive zeros of f' where the sign of f does not change. In other words, that means, there are infinitely many situations, where we find at least two zeros of f' between two zeros of f. With Corollary 2.31, this yields a contradiction to the assumption that f only has a finite number of complex transmission eigenvalues.

$\square$

Theorem 4.29 *Let a and b as in Theorem 4.28. Assume that $q > 0$ is constant such that $\frac{a}{b} \notin \mathbb{N}$. Then the complex transmission eigenvalues all lie in a strip parallel to the real axis.*

Proof: For $k \in \mathbb{C}$, from Lemma 4.27 we know that f is an entire function of type $\tau = a + b$. In Lemma 4.23 we have seen that D is also an entire function of the same type τ. Hence, by Lemma 2.28 $g(k) := D(k) - \frac{f(k)}{ab}$, for $k \in \mathbb{C}$, is an entire function of exponential type at most τ.

For real k it holds $D(k) = \frac{f(k)}{ab} + \mathcal{O}(1/k)$, $k \to \infty$, and therefore $g(k) = \mathcal{O}(1/k)$. We conclude that g belongs to $L^2(\mathbb{R})$ and hence by Paley-Wiener Theorem 2.32 there exists $\varphi \in L^2[-\tau, \tau]$ such that

$$g(k) = \int_{-\tau}^{\tau} \varphi(t) e^{ikt} dt.$$

With Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |g(k)|^2 &= \left| \int_{-\tau}^{\tau} \varphi(t) e^{ikt} dt \right|^2 \leq \|\varphi\|^2_{L^2(\mathbb{R})} \int_{-\tau}^{\tau} \left| e^{ikt} \right|^2 dt \\ &= \|\varphi\|^2_{L^2(\mathbb{R})} \int_{-\tau}^{\tau} e^{2yt} dt, \end{aligned}$$

with $y := \operatorname{Im} k$. Computing the integral

$$\int_{-\tau}^{\tau} e^{2yt} dt = \frac{1}{2y} \left[e^{2yt} \right]_{-\tau}^{\tau} = \frac{\sinh(2y\tau)}{y}$$

yields

$$|g(k)| = \left| \int_{-\tau}^{\tau} \varphi(t) e^{ikt} dt \right| \leq \|\varphi\|_{L^2(\mathbb{R})} \left(\frac{1}{|y|} \sinh(2|y|\tau) \right)^{1/2}.$$

It follows that $|g(k)| e^{-\tau |y|} \to 0$ as $|y| \to \infty$, because

$$\begin{aligned} & \left(\frac{\sinh(2|y|\tau)}{|y|} \right)^{1/2} e^{-\tau|y|} \\ = & \left(\frac{e^{2|y|\tau} - e^{-2|y|\tau}}{2|y| e^{2\tau|y|}} \right)^{1/2} = \left(\frac{1 - e^{-4|y|\tau}}{2|y|} \right)^{1/2} \to 0, \end{aligned}$$

as $|y| \to \infty$.

Let us now suppose that D has a sequence of zeros k_j, with imaginary part $|y_j| \to \infty$, as $j \to \infty$. Then it holds,

$$\frac{f(k_j)}{ab} + g(k_j) = 0, \quad \text{for every } j \in \mathbb{N}, \tag{4.83}$$

and, as we have just seen

$$|g(k_j)| e^{-\tau |y_j|} \to 0, \quad \text{as } j \to \infty. \tag{4.84}$$

We will show subsequently, that

$$\left|\frac{f(k_j)}{ab}e^{-\tau|y_j|}\right| \to \frac{a-b}{4ab} > 0, \quad \text{as } j \to \infty, \tag{4.85}$$

which yields a contradiction, because $\left|\left(\frac{f(k_j)}{ab} + g(k_j)\right) e^{-\tau|y_j|}\right|$ equals zero on the one hand (see (4.83), but tends (using (4.84) and (4.85)) to $\frac{a-b}{4ab}$ for $j \to \infty$ on the other hand.

We conclude that D cannot have an infinite number of zeros with imaginary part tending to infinity.

It is left to show the claim $\left|\frac{f(k)}{ab}e^{-\tau|y|}\right| \to \frac{a-b}{4ab} > 0$ as $|y| \to \infty$. First, we note that

$$\begin{aligned}
&\sin\big(k(a+b) - dn\,(1/a + 1/b)\,\big) \\
= \;& \frac{1}{2i}\left(e^{i\left(k(a+b)-dn\left(\frac{1}{a}+\frac{1}{b}\right)\right)} - e^{-i\left(k(a+b)-dn\left(\frac{1}{a}+\frac{1}{b}\right)\right)}\right) \\
= \;& \frac{1}{2i}\left(e^{i\left(\operatorname{Re} k(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)-\operatorname{Im} k(a+b)}\right. \\
& \qquad\qquad \left. - \, e^{-i\left(\operatorname{Re} k(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)+\operatorname{Im} k(a+b)}\right).
\end{aligned}$$

We write $k = x + iy$ and assume $y = \operatorname{Im} k > 0$ for the moment. We conclude

$$\begin{aligned}
&\sin\big(k(a+b) - dn\,(1/a + 1/b)\,\big)e^{-(a+b)|y|} \\
= \;& \frac{1}{2i}\left(e^{i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)-2y(a+b)} - e^{-i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)}\right),
\end{aligned}$$

which tends to

$$-\frac{1}{2i}\left(e^{-i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)}\right)$$

as $y \to \infty$. The limit as $y \to -\infty$ is

$$\frac{1}{2i}\left(e^{i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)}\right).$$

Analogously, we see that

$$\begin{aligned}
&\sin\big(k(a-b) - dn\,(1/a - 1/b)\,\big)e^{-(a+b)|y|} \\
= \;& \frac{1}{2i}\left(e^{i\left(x(a-b)-dn(\frac{1}{a}-\frac{1}{b})\right)-2ya} - e^{-i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})-2yb\right)}\right)
\end{aligned}$$

tends to zero as $y \to \infty$, and that

$$\begin{aligned}
& \sin\big(k(a-b) - dn\,(1/a - 1/b)\,\big)e^{-(a+b)|y|} \\
= \;& \frac{1}{2i}\left(e^{i\left(x(a-b)-dn(\frac{1}{a}-\frac{1}{b})\right)+2yb} - e^{-i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})+2ya\right)}\right)
\end{aligned}$$

tends to zero, as $y \to -\infty$. Bringing the results together shows with (4.79) that

$$\begin{aligned}
\left|\frac{f(k)}{ab}e^{-\tau|y|}\right| \;&=\; \left|\frac{f(k)}{ab}e^{-(a+b)|y|}\right| \\
= \;& \Big|\frac{a-b}{2ab}\sin\big(k(a+b) - dn\,(1/a + 1/b)\,\big)e^{-(a+b)|y|} \\
& - \frac{a+b}{2ab}\sin\big(k(a-b) - dn\,(1/a - 1/b)\,\big)e^{-(a+b)|y|}\Big|,
\end{aligned}$$

which tends to

$$\left|\frac{a-b}{4ab}\left(e^{\pm i\left(x(a+b)-dn(\frac{1}{a}+\frac{1}{b})\right)}\right)\right| \;=\; \frac{a-b}{4ab} > 0,$$

as $y \to \mp\infty$.

$\square$

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