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Anomalous Lorentz and CPT violation from a local Chern–Simons-like term in the effective gauge-field action

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Abstract

We consider four-dimensional chiral gauge theories defined over a spacetime manifold with topology $\mathbb{R}^3 \times S^1$ and periodic boundary conditions over the compact dimension. The effective gauge-field action is calculated for Abelian $U(1)$ gauge fields $A_\mu(x)$ which depend on all four spacetime coordinates (including the coordinate $x^4 \in S^1$ of the compact dimension) and have vanishing components $A_4(x)$ (implying trivial holonomies in the 4-direction). Our calculation shows that the effective gauge-field action contains a local Chern–Simons-like term which violates Lorentz and CPT invariance. This result is established perturbatively with a generalized Pauli–Villars regularization and nonperturbatively with a lattice regularization based on Ginsparg–Wilson fermions.

1. Introduction

It has been shown [1] that chiral gauge theories over a manifold with an appropriate nontrivial topology necessarily have an anomalous violation of Lorentz and CPT invariance. Two direct follow-up papers on this CPT anomaly have appeared in Refs. [2,3] and a review has been pre-

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sented in Ref. [4] which also contains a brief discussion of the well-known CPT theorem and ways how this theorem can be circumvented.

The existence of the CPT anomaly for four-dimensional gauge chiral theories over the spacetime manifold $M = \mathbb{R}^3 \times S^1$ was established in Refs. [1,3] for a special class of background gauge fields, namely gauge-field configurations which are independent of the compact coordinate $x^4 \in S^1$ and have a vanishing component A_4. The question arises how the anomaly manifests itself for more general gauge-field configurations which have a nontrivial dependence on the compact x^4 coordinate.

It will be shown, in the present article, that the anomaly manifests itself by a *local* Chern–Simons-like term in the effective gauge-field action and this term is known to violate Lorentz and CPT invariance [5–7]. Our result will be established with two regularization methods, an extended version of the generalized Pauli–Villars regularization [8] for a perturbative calculation and the lattice regularization based on Ginsparg–Wilson fermions [9–13] for a nonperturbative calculation.

The outline of this article is as follows. In Sec. 2, we describe the theoretical setup of the problem and establish our notation. As said, the calculation will be done both perturbatively and nonperturbatively, with appropriate regularization methods.

In Sec. 3, we establish Lorentz and CPT violation with a perturbative approach. In Sec. 3.1, we start from the effective gauge-field action for a left-handed chiral fermion. This effective action is then perturbatively expanded and rendered finite with an extended version of the generalized Pauli–Villars regularization. In Sec. 3.2, we perform, for an Abelian $U(1)$ gauge group, the one-loop calculation of the effective gauge-field action to quadratic order and obtain a local Chern–Simons-like term. In Sec. 3.3, we show explicitly that the calculated Chern–Simons-like term violates Lorentz and CPT invariance in four spacetime dimensions.

In Sec. 4, we establish the existence of Lorentz and CPT violation with a nonperturbative approach. In Sec. 4.1, we recall the lattice setup and introduce some further notation. In Sec. 4.2, we review chiral $U(1)$ gauge theory on the lattice. The fermion action on a regular hypercubic lattice is written down and the integration measure is defined. The action of the discrete transformations on the link variable is also given. In Sec. 4.3, we discuss the effective gauge-field action on the lattice and its behavior under a CPT transformation. In Sec. 4.4, we show that the effective action is not invariant under CPT transformation, considering both relevant cases (an odd or even integer $N \equiv L/a$, with L the length of the x^4 circle and a the lattice spacing). In Sec. 4.5, we calculate the expression for the CPT-anomaly in the continuum limit ($a \to 0$).

In Sec. 5, we highlight some important points of our calculations. In Sec. 6, finally, we offer some concluding remarks.

The present article is, by necessity, rather technical. A first impression can be obtained from Secs. 2, 3.3, and 6.

2. Setup of the problem

The chiral gauge theory to be considered is defined over the following four-dimensional spacetime manifold:

$$M = \mathbb{R}^3 \times S^1 , \tag{2.1a}$$

with noncompact coordinates

$$x^1, x^2, x^3 \in \mathbb{R} , \tag{2.1b}$$

and compact coordinate

$$x^4 \in [0, L]\,. \tag{2.1c}$$

Initially, the spacetime metric is taken to be the Euclidean flat metric,

$$g_{\mu\nu}(x) = [\mathrm{diag}(1,\ 1,\ 1,\ 1)]_{\mu\nu}\,. \tag{2.2}$$

At the end of the calculation, we shall make the Wick rotation from Euclidean metric signature to Lorentzian metric signature, with x^1 or x^2 or x^3 (but not x^4) taken to correspond to the time coordinate t.

We are considering chiral gauge theories that are free of gauge anomalies. Specifically, we take the chiral gauge theory with the following non-Abelian gauge group and representation of left-handed fermions:

$$G\ = SO(10)\,, \tag{2.3a}$$

$$R_L = 3 \times [\mathbf{16}]\,, \tag{2.3b}$$

which contains the $SU(3) \times SU(2) \times U(1)$ Standard Model with 3 families of fermions (and three singlet left-handed antineutrinos).

Most of our calculations are, however, performed for a chiral $U(1)$ gauge theory consisting of a single gauge boson A and 48 left-handed fermions with $U(1)$ charges q_f, for $f = 1, \ldots, 48$. Specifically, the Abelian gauge group and the left-handed fermion representation (i.e., the set of left-handed charges q_f in units of e, the absolute value of the electron charge) are given by:

$$G\ = U(1)\,, \tag{2.4a}$$

$$\begin{aligned} R_L = 3 \times \Bigg[& 6 \times \left(\frac{1}{3}\right) + 3 \times \left(-\frac{4}{3}\right) + 3 \times \left(\frac{2}{3}\right) \\ & + 2 \times (-1) + 1 \times (2) + 1 \times (0) \Bigg]\,. \end{aligned} \tag{2.4b}$$

This particular chiral $U(1)$ gauge theory can be embedded in the $SU(2) \times U(1)$ electroweak theory of the Standard Model with $U(1)$ hypercharge $Y \equiv 2\,Q - 2\,T_3$ (the electron has charge $Q = -e$ and the positron has $Q = +e$). The further embedding in the "safe" $SO(10)$ group with left-handed representation (2.3b) explains why the perturbative gauge anomalies cancel out in the chiral $U(1)$ gauge theory considered,

$$\sum_{f=1}^{48} (q_f)^3 = 0\,, \tag{2.5}$$

for the charges q_f as given by (2.4b). For later use, we also give another sum:

$$\sum_{f=1}^{48} (q_f)^2 = F\, e^2\,, \tag{2.6a}$$

$$F = 3 \times \left[\frac{40}{3}\right] = 40\,. \tag{2.6b}$$

Other chiral $U(1)$ gauge theories give, in general, a different value for the numerical factor F.

The gauge and fermion fields are assumed to be periodic in the x^4 coordinate,

$$A_\mu(\vec{x},\, x^4+L) = A_\mu(\vec{x},\, x^4)\,, \tag{2.7a}$$

$$\psi(\vec{x},\, x^4+L) = \psi(\vec{x},\, x^4)\,, \tag{2.7b}$$

$$\overline{\psi}(\vec{x},\, x^4+L) = \overline{\psi}(\vec{x},\, x^4)\,, \tag{2.7c}$$

with

$$\vec{x} \equiv (x^1,\, x^2,\, x^3)\,. \tag{2.8}$$

Another assumption about the gauge fields is as follows:

$$A_i(x) = A_i(\vec{x},\, x^4)\,, \quad \text{for } i = 1,\, 2,\, 3\,, \tag{2.9a}$$

$$A_4(x) = 0\,. \tag{2.9b}$$

Such gauge fields can be obtained by a gauge transformation if the original gauge fields with $A_4 \neq 0$ have trivial holonomies,

$$h_4(\vec{x}) \equiv \exp\left[\int_0^L dx^4\, A_4(\vec{x},\, x^4)\right] = 1\,. \tag{2.10}$$

This Abelian holonomy $h_4(\vec{x})$ is a gauge-invariant quantity (see the last paragraph of Sec. 3.2).

The background gauge fields A_i are considered to have local support in $\mathbb{R}^3$. Specifically, take a ball $B^3 \in \mathbb{R}^3$ with a large fixed radius R. The gauge fields $A_i(x)$, for $i = 1,\, 2,\, 3$, are assumed to vanish on the boundary of the ball and outside of it,

$$A_i(\vec{x},\, x^4) = 0\,, \quad \text{for } |\vec{x}|^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 \geq R^2\,. \tag{2.11}$$

In general, Latin spacetime indices i, j, k, l, etc. run over the coordinate labels 1, 2, 3, and Greek spacetime indices μ, ν, ρ, etc. over the labels 1, 2, 3, 4. Repeated coordinate (and internal) indices are summed over. Throughout, natural units are used with $\hbar = c = 1$.

The problem, now, is to investigate, for the setup considered, the invariance of the effective gauge-field action $\Gamma[A]$ under Lorentz and CPT transformations. In Secs. 3 and 4, the effective action $\Gamma[A]$ is calculated by integrating out the fermions using, respectively, a perturbative and a nonperturbative method. The CPT anomaly is then established if we can show that this effective action changes under a CPT transformation of the background gauge field, $\Gamma[A^{\text{CPT}}] \neq \Gamma[A]$.

The actual calculation of Sec. 3 is performed first for a single left-handed fermion ψ with unit $U(1)$ charge, $q = e$. Only the final result (3.48) is extended to all chiral fermions of the theory (2.4). The same procedure is followed in Sec. 4.

3. Perturbative approach

3.1. Theory and regularization

Let us start with the action of a left-handed chiral fermion,

$$\begin{aligned} S\left[\overline{\psi},\, \psi,\, A\right] &= \int_M d^4x\; \mathcal{L}\left[\overline{\psi_L},\, \psi_L,\, A\right] \\ &= \int_M d^4x\; i\, \overline{\psi_L}\, \gamma^\mu (\partial_\mu + e\, A_\mu)\, \psi_L\,, \end{aligned} \tag{3.1}$$

where A_μ is the anti-Hermitian $U(1)$ gauge field, e the dimensionless electric charge of the fermion ψ, and $\psi_L \equiv \frac{1}{2}(1+\gamma_5)\,\psi$ the left-handed projection of the four-component Dirac spinor ψ. The γ^μ are the 4×4 Dirac matrices and $\overline{\psi} \equiv \psi^\dagger\,\gamma^4$. The Hermitian chirality matrix γ_5 has $\{\gamma_5,\,\gamma^\mu\}=0$ and $(\gamma_5)^2=\mathbb{1}_4$.

In this article, we set out to calculate the effective action of the gauge fields for the setup as described in Sec. 2. In the vacuum, there are virtual fermion–antifermion pairs which interact with the classical background gauge field. The effective action $\Gamma[A]$ is a functional which takes these interactions into account. Incidentally, the functional $\Gamma[A]$ considered here is not the complete effective action as there are also contributions from the photonic sector such as the classical Maxwell term, but our focus is solely on the contributions of the virtual fermions.

In Feynman's Euclidean path integral formalism, the functional $\Gamma[A]$ is obtained by integrating out the fermionic degrees of freedom,

$$\exp(-\Gamma[A]) = \int \mathcal{D}\overline{\psi_L}(x)\mathcal{D}\psi_L(x)\,\exp\left(-\int_M d^4x\,\mathcal{L}\left[\,\overline{\psi_L},\psi_L,A\right]\right), \tag{3.2}$$

which, loosely speaking, equals the root of the determinant of the operator $\gamma^\mu(\partial_\mu + e\,A_\mu)$. This operator has, however, an unbounded spectrum, so that the determinant is infinite. The expression (3.2) thus needs to be regularized.

Finding a manifestly gauge-invariant regularization is not straightforward. One possibility is given by the generalized Pauli–Villars regularization as discussed by Frolov and Slavnov [8], which involves an infinite set of bosonic and fermionic Pauli–Villars-type fields Ψ_s, for $s \in \mathbb{Z}/\{0\}$, with standard (Lorentz-invariant) Dirac-type mass terms $m_s\,\overline{\Psi}_s\,\Psi_s$. We will, however, extend this regularization, in order to be sensitive to anomalous Lorentz violation. In fact, we will introduce another infinite set of bosonic and fermionic Pauli–Villars-type fields ψ_r, for $r \in \mathbb{Z}/\{0\}$, with Lorentz-violating mass terms $M_r\,\psi_r^\dagger\,\psi_r$.

Specifically, the regularized Lagrange density for the chiral $U(1)$ gauge theory including both infinite sets of Pauli–Villars-type fields reads as follows:

$$\begin{aligned}\mathcal{L}_{\text{full reg. th.}} &= \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{LI-gen-PV}} + \mathcal{L}_{\text{LV-gen-PV}}\\ &= i\,\overline{\psi_0}(x)\,\gamma^\mu\left(\partial_\mu + e\,A_\mu\right)\psi_0(x)\\ &\quad + \sum_{s\neq 0}\left[i\,\overline{\Psi}_s(x)\,\gamma^\mu\left(\partial_\mu + e\,A_\mu\right)\Psi_s(x) - m_s\,\overline{\Psi}_s(x)\,\Psi_s(x)\right]\\ &\quad + \sum_{r\neq 0}\left[i\,\overline{\psi_r}(x)\,\gamma^\mu\left(\partial_\mu + e\,A_\mu\right)\psi_r(x) - M_r\,\psi_r^\dagger(x)\,\psi_r(x)\right],\end{aligned} \tag{3.3}$$

with regulator masses,

$$m_s = m\,|s|\,, \tag{3.4a}$$

$$M_r = M\,r^2\,, \tag{3.4b}$$

$$M \gg m\,. \tag{3.4c}$$

The ultraheavy regulator masses M_r violate Lorentz invariance, but can have effects on the low-energy physics in the case of an anomaly. The reason for demanding a quadratic r-dependence in (3.4b), compared to the linear s-dependence in (3.4a), will be explained in Sec. 3.2. Strictly speaking, we do not need the inequality (3.4c) for the present calculation, but it has been included, in order to make sure that possible Lorentz-violating quantum effects are subdominant compared to Lorentz-invariant quantum effects.

The regulator fields Ψ_s in (3.3) are unrestricted four-component Dirac fields, whereas the regulator fields ψ_r, including the original massless field $\psi_0 \equiv \psi_L$, are chiral four-component Dirac fields, obeying the condition

$$\psi_r \equiv \frac{1}{2}\,(1+\gamma_5)\,\psi_r\,, \quad \text{for } r \in \mathbb{Z}\,. \tag{3.5}$$

The fields have, moreover, the following Grassmann parities:

$$\varepsilon(\Psi_s) = (-1)^{s+1}\,, \quad \text{for } s \in \mathbb{Z}/\{0\}\,, \tag{3.6a}$$

$$\varepsilon(\psi_r) = (-1)^{r+1}\,, \quad \text{for } r \in \mathbb{Z}\,. \tag{3.6b}$$

For the purpose of searching for anomalous Lorentz violation, we only need to consider the chiral fields ψ_r, as will be explained in Sec. 3.2.

We now take the Weyl representation of the 4×4 Dirac gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \widetilde{\sigma}^\mu \\ \widetilde{\sigma}^{\mu\dagger} & 0 \end{pmatrix}, \qquad \gamma_5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{3.7}$$

with $\widetilde{\sigma}^\mu \equiv (\sigma^m,\, i\,\mathbb{1}_2)$ in terms of the 2×2 Pauli spin matrices σ^m and the 2×2 identity matrix $\mathbb{1}_2$. As said before, ψ_0 with $M_0 = 0$ in (3.3) corresponds to the original four-component chiral field ψ_L and, for the Weyl representation (3.7) with diagonal γ_5, can be written as

$$\psi_0 = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}, \tag{3.8}$$

where ξ_0 is an anticommuting two-component spinor field. The $r \neq 0$ fields ψ_r in (3.3) constitute an infinite set of Pauli–Villars fields with Grassmann parities (3.6b) and regulator masses (3.4b). Each chiral regulator field ψ_r $(r \neq 0)$ can also be written as

$$\psi_r = \begin{pmatrix} \xi_r \\ 0 \end{pmatrix}, \tag{3.9}$$

with a two-component field ξ_r having the Grassmann parity (i.e., loop-factor in Feynman diagrams)

$$\varepsilon(\xi_r) = (-1)^{r+1}\,, \quad \text{for } r \in \mathbb{Z}\,. \tag{3.10}$$

With the above definitions, the truncated regularized theory is given by

$$\begin{aligned} \mathcal{L}_{\text{trunc. reg. th.}} &= \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{LV-gen-PV}} \\ &= \sum_{r=-\infty}^{\infty} \Big[i\, \xi_r^\dagger(x)\, \sigma^\mu \left(\partial_\mu + e\,A_\mu\right) \xi_r(x) - M_r\, \xi_r^\dagger(x)\, \xi_r(x) \Big], \end{aligned} \tag{3.11}$$

with $\sigma^\mu \equiv (i\sigma^m,\, \mathbb{1}_2)$ and M_r from (3.4b).

In order to prepare for the calculation of the next subsection, we define

$$\widetilde{\gamma}^1 \equiv i\,\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \widetilde{\gamma}^2 \equiv i\,\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.12a}$$

$$\widetilde{\gamma}^3 \equiv i\,\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \widetilde{\gamma}^4 \equiv \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.12b}$$

and rewrite the standard Weyl action from (3.11) as

$$\mathcal{I}_0 = \int d^4x\, \mathcal{L}_{\text{chiral}} = \int d^4x\, i\, \xi_0^\dagger(x)\, \widetilde{\gamma}^\mu (\partial_\mu + e\, A_\mu)\, \xi_0\,, \tag{3.13}$$

where ξ_0 is the two-component spinor field. A similar action holds for the chiral regulator fields ξ_r $(r \neq 0)$,

$$\begin{aligned}\mathcal{I}_{\text{reg}} &= \int d^4x\, \mathcal{L}_{\text{LV-gen-PV}} \\ &= \int d^4x \sum_{r\neq 0} \Big[i\, \xi_r^\dagger(x)\, \widetilde{\gamma}^\mu (\partial_\mu + e\, A_\mu)\, \xi_r - M_r\, \xi_r^\dagger\, \xi_r \Big].\end{aligned} \tag{3.14}$$

The 2×2 matrices $\widetilde{\gamma}^\mu$ in (3.13) and (3.14) obey the following relation:

$$\widetilde{\gamma}^i\, \widetilde{\gamma}^j = \tilde{g}^{ij}\, \mathbb{1} - \epsilon^{ijk}\, \widetilde{\gamma}_k\,, \tag{3.15}$$

with the three-dimensional Euclidean flat metric $\tilde{g}^{ij} = [\text{diag}(-1,-1,-1)]^{ij}$ and the totally antisymmetric Levi–Civita symbol ϵ^{ijk}, normalized by $\epsilon^{123} = 1$. From (3.15), we have that the anti-commutator of the $\widetilde{\gamma}^i$ matrices has precisely the same structure as the one of Dirac matrices in $\mathbb{R}^3$, namely, $\{\widetilde{\gamma}^i, \widetilde{\gamma}^j\} = 2\, \tilde{g}^{ij}\, \mathbb{1}$. This is, in fact, the reason for using these matrices $\widetilde{\gamma}^\mu$, as will become clear in Sec. 3.2. Note, however, that the matrices $\widetilde{\gamma}^\mu$ do not satisfy the properties of Dirac gamma matrices in four-dimensional spacetime, because $\widetilde{\gamma}^4$ does not anti-commute with the other $\widetilde{\gamma}^i$ matrices. In our calculations, we shall only use relation (3.15).

For standard Minkowski spacetime without compactification of the x^4 coordinate, we expand the gauge field A_μ in Fourier modes as follows:

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, A_\mu(p), \tag{3.16}$$

and write down the vacuum-polarization kernel

$$\pi^{ij}(p) = \int \frac{d^4k}{(2\pi)^4}\, \text{tr}\Big[\widetilde{\gamma}^i\, S(k)\, \widetilde{\gamma}^j\, S(k+p) \Big]. \tag{3.17}$$

In our case, where the x^4 coordinate is compactified, we make the following replacements:

$$\int d^4x \to \int_0^L dx^4 \int_{\mathbb{R}^3} d^3x \tag{3.18a}$$

and

$$\int \frac{d^4p}{(2\pi)^4} \to \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}. \tag{3.18b}$$

The Fourier expansion of the gauge field A_μ is now given by

$$A_\mu(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3}\, e^{2\pi i n x^4/L}\, e^{i\vec{p}\cdot\vec{x}}\, A_\mu(p_n), \tag{3.19}$$

with the following definitions:

$$p_n \equiv (\vec{p},\ \rho_n)\,, \tag{3.20a}$$

$$\rho_n \equiv 2\pi n/L\,, \tag{3.20b}$$

$$p_n^2 \equiv |\vec{p}|^2 + (\rho_n)^2\,. \tag{3.20c}$$

3.2. Calculation

The expression for the perturbatively-expanded effective gauge-field action in three spacetime dimensions with one compactified coordinate has been given in Ref. [14]; see, in particular, Eqs. (22)–(26) of that article. For the action (3.13) with the replacement (3.18a), we have four spacetime dimensions with one compactified coordinate. Adopting a similar procedure as the one of Ref. [14], we write down the physically relevant factor in the perturbatively-expanded effective gauge-field action,

$$\Gamma[A] = -i\,\frac{e^2}{2}\,\frac{1}{L}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}\,A_i(-p_n)\,\pi^{ij}(p_n)\,A_j(p_n) + O(e^3), \tag{3.21}$$

with the unregularized vacuum-polarization kernel

$$\pi^{ij}(p_n)\Big|^{\text{(unreg.)}} = \frac{1}{L}\sum_{m=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\Big[\widetilde{\gamma}^i\,S(k_m)\,\widetilde{\gamma}^j\,S(k_m+p_n)\Big]. \tag{3.22}$$

The propagator $S(k_m)$ is defined as:

$$S(k_m) \equiv \frac{1}{\widetilde{\gamma}^i k_i + \widetilde{\gamma}^4 k_{4m}} = \frac{\widetilde{\gamma}^i\,k_i - \widetilde{\gamma}^4\,k_{4m}}{(\widetilde{\gamma}^i k_i)^2 - k_{4m}^2} = -\frac{\widetilde{\gamma}^i\,k_i - \widetilde{\gamma}^4\,k_{4m}}{(k_i)^2 + k_{4m}^2}. \tag{3.23}$$

The ultraviolet divergences of the anomalous terms in (3.22) are regularized by the infinite set of Pauli–Villars-type fields $\xi_r(x)$, for $r \neq 0$, from (3.14). The infrared divergences are regularized by imposing antiperiodic boundary conditions for the $\xi_r(x)$ fields ($r \in \mathbb{Z}$) on the surface of a large ball B^3, where the gauge fields $A_i(x)$ vanish according to (2.11).

For a particular Fourier mode n of the background gauge field, the regularized two-point function is proportional to the following expression:

$$\pi^{ij}(p_n)\Big|^{\text{(reg.)}}$$
$$= \sum_{r=-\infty}^{\infty}(-1)^r\,\frac{1}{L}\sum_{m=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\,\frac{\mathrm{tr}\Big[\widetilde{\gamma}^i(\not{k}+M_r)\widetilde{\gamma}^j(\not{k}+\not{p}+M_r)\Big]}{\Big(k_m^2+M_r^2\Big)\Big((k_m+p_n)^2+M_r^2\Big)}, \tag{3.24}$$

with the short-hand notation $\not{p} \equiv \widetilde{\gamma}^i\,p_i - \widetilde{\gamma}^4\,p_{4n}$ for the matrices (3.12), which are Dirac gamma matrices in three spacetime dimensions but not in four. The factor $(-1)^r$ in (3.24) comes from the Grassmann parity (3.10) of the fields and M_r is given by (3.4b). From now on, we drop the superscript 'reg.' as the regularization is manifest from having the sum over r.

Introducing the Feynman parameter x and changing the momentum variable k_μ to l_μ, with $l_i \equiv k_i + x\,p_i$ and $l_4 \equiv k_4$, we rewrite the expression for the vacuum-polarization kernel (3.24) as

$$
\begin{aligned}
\pi^{ij}(p_n) = &\sum_{r=-\infty}^{\infty} (-1)^r \int_0^1 dx \, \frac{1}{L} \sum_{m=-\infty}^{\infty} \int \frac{d^3 l}{(2\pi)^3} \\
&\times \mathrm{tr}\Big[\widetilde{\gamma}^i \left(\widetilde{\gamma}^k \, l_k - x \, \widetilde{\gamma}^k \, p_k - \omega_m + M_r\right) \widetilde{\gamma}^j \\
&\cdot \left(\widetilde{\gamma}^k \, l_k + (1-x) \, \widetilde{\gamma}^k \, p_k - \omega_m - \rho_n + M_r\right)\Big] \left(|\vec{l}|^2 + \Delta\right)^{-2},
\end{aligned} \tag{3.25}
$$

with p_n, ρ_n, and ρ_n^2 from (3.20) and the further definitions

$$
l_m \equiv (\vec{l}, \, \omega_m) \, , \tag{3.26a}
$$

$$
\omega_m \equiv 2\pi m / L \, , \tag{3.26b}
$$

$$
\Delta \equiv (\omega_m + x\rho_n)^2 + x(1-x) \, p_n^2 + M_r^2 \, . \tag{3.26c}
$$

The odd powers of the l_i in the numerator of (3.25) vanish by symmetry reasons. The term in (3.25) with an odd number of p_n momenta in the numerator of the integrand is written as

$$
\begin{aligned}
\widetilde{T}^{ij}(p_n) = &\sum_{r=-\infty}^{\infty} (-1)^r \frac{1}{L} \sum_{m=-\infty}^{\infty} (-\omega_m + M_r) \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \\
&\times \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k] \, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4] \, \rho_n}{(|\vec{l}|^2 + \Delta)^2} \, .
\end{aligned} \tag{3.27}
$$

Part of the above equation still gives rise to a finite L-independent term with an even number of p_n momenta,

$$
\begin{aligned}
&\frac{1}{L} \sum_{m=-\infty}^{\infty} (-\omega_m) \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r \, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k] \, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4] \, \rho_n}{(|\vec{l}|^2 + \Delta)^2} \\
&\propto \left(\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k] \, \rho_n \, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4] \, \rho_n \rho_n\right),
\end{aligned} \tag{3.28}
$$

and we are left with the following term with an odd number of p_n momenta:

$$
T^{ij}(p_n) = \frac{1}{L} \sum_{m=-\infty}^{\infty} \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r \, M_r \, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k] \, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j] \, \rho_n}{(|\vec{l}|^2 + \Delta)^2} \, , \tag{3.29}
$$

where we have taken care to move the r sum inwards as it must be performed first.

The ρ_n term in the numerator of the integrand of (3.29) ultimately gives rise to a term $\int_0^L dx^4 \int d^3x \; \delta_{ij} \, A_i \, (\partial_4 A_j)$ in the effective gauge-field action, which is a total-derivative term and vanishes due to the periodic boundary conditions (2.7). So, we are left with the following potentially CPT-violating term:

$$
T^{ij}_{\mathrm{anom}}(p_n) = \frac{1}{L} \sum_{m=-\infty}^{\infty} \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r \, M_r \, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k] \, p_k}{(|\vec{l}|^2 + \Delta)^2} . \tag{3.30}
$$

At this moment, we can mention that the other regulator fields Ψ_s from (3.3) do not contribute to this potentially anomalous term with an odd number of p_n momenta, because the trace of an odd number of Dirac matrices γ^μ vanishes. This is not the case for the trace of $\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k$, as follows from relation (3.15).

We divide the sum over m in (3.30) into two parts, namely, the sum over nonzero m and the single term $m = 0$ [this term is distinguished by having an infrared-divergent momentum integral for the $r = 0$ contribution, which is regularized by antiperiodic boundary conditions as discussed a few lines below (3.23)]. The expression then reads

$$T^{ij}_{\text{anom}}(p_n) = T^{ij}_0(p_n) + T^{ij}_{\text{rest}}(p_n)\,, \tag{3.31}$$

with

$$T^{ij}_0(p_n) = \frac{1}{L}\int \frac{d^3 l}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r\, M_r\, \frac{\text{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\, p_k}{(|\vec{l}|^2 + \Delta_0)^2}\,, \tag{3.32a}$$

$$\Delta_0 \equiv x\rho_n^2 + x(1-x)\, p_n^2 + M_r^2\,, \tag{3.32b}$$

and

$$T^{ij}_{\text{rest}}(p_n) = \frac{2}{L}\sum_{m=1}^{\infty}\int \frac{d^3 l}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r\, M_r\, \frac{\text{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\, p_k}{(|\vec{l}|^2 + \Delta)^2}\,. \tag{3.33}$$

First, consider the $m = 0$ contribution (3.32). In order to compute the sum over r, we use the following representation (defining $l \equiv |\vec{l}|$):

$$\begin{aligned} S_0 &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r M_r}{\Big(|\vec{l}|^2 + (x\rho_n)^2 + x(1-x)\, p_n^2 + M_r^2\Big)^2} \\ &= -\frac{1}{2l}\frac{\partial}{\partial l}\sum_{r=-\infty}^{\infty} \frac{(-1)^r M_r}{\Big(l^2 + (x\rho_n)^2 + x(1-x)\, p_n^2 + M_r^2\Big)} \\ &= -\frac{1}{2l}\,\frac{M}{M^2}\,\frac{\partial}{\partial l}\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{\big(\tau^2 + r^4\big)}\,, \end{aligned} \tag{3.34a}$$

with

$$\tau^2 \equiv \Big[l^2 + (x\rho_n)^2 + x\,(1-x)\, p_n^2\Big]/M^2 \equiv l^2/M^2 + \kappa\,, \tag{3.34b}$$

and the following result (for $\tau \neq 0$):

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{\tau^2 + r^4} = f(\tau)\,, \tag{3.35a}$$

$$f(\tau) \equiv \frac{\pi}{2\sqrt{\tau}}\left(\frac{\exp(i\pi/4)}{\sinh\!\big[\exp(-i\pi/4)\,\pi\,\sqrt{\tau}\,\big]} + \text{c.c.}\right). \tag{3.35b}$$

Remark that the first sum in (3.34a) contains an extra factor M_r in the numerator compared to Eq. (11) of Ref. [8] and this is the reason for demanding the r^2 behavior in the regulator masses

M_r in (3.4b). We then find the same type of $1/\sinh$ behavior in (3.35b) as in Eq. (14) of Ref. [8], which, in both cases, provides an exponential cutoff of the momentum integrals.

With result (3.35), expression (3.32) reduces to

$$T_0^{ij}(p_n) = -\frac{1}{4\pi^2 L}\,\frac{M}{M^2}\int_0^1 dx \int_0^\infty l dl\, \frac{\partial}{\partial l}\Big[f(\tau)\Big]\,\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k]\,p_k$$

$$= -\frac{1}{4\pi^2 L}\,\frac{M}{|M|}\int_0^1 dx \int_0^\infty d\,\eta\,\eta\,\frac{\partial}{\partial\eta}\Big[f(\tau)\Big]\,\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k]\,p_k\,, \tag{3.36}$$

in terms of the dimensionless variable $\eta \equiv l/|M|$. In the following, we assume positive M (the related ambiguity in the anomalous term, here by a factor $M/|M|$, is discussed further in the first paragraph of Sec. 6).

In the regularization procedure, we consider the regulator mass scale M to be much larger than a typical momentum component of the gauge field, $M^2 \gg p_n^2$, so that we can take $\kappa \equiv \big[(x\rho_n)^2 + x\,(1-x)\,p_n^2\big]/M^2 \to 0^+$ in the rest of the calculation and the x integral in (3.36) becomes trivial. Using

$$\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k] = 2\,\epsilon^{ijk}\,, \tag{3.37}$$

we then rewrite (3.36) for positive M as

$$T_0^{ij}(p_n) = -\frac{1}{2\pi^2 L}\left(\int_0^\infty d\,\eta\,\eta\,\frac{\partial}{\partial\eta}\Big[f(\eta)\Big]\right)\epsilon^{ijk}\,p_k\,. \tag{3.38}$$

The η integral in (3.38) gives a factor $\pi/2$ and the final result for the $m = 0$ sector reads

$$T_0^{ij}(p_n) = -\frac{1}{4\pi L}\,\epsilon^{ijk}\,p_k\,. \tag{3.39}$$

Now turn to the $m \neq 0$ sum (3.33),

$$T^{ij}_{\mathrm{rest}}(p_n) = \frac{1}{L}\sum_{m\neq 0}\int \frac{d^3\eta}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty}(-1)^r\,r^2\,\frac{\mathrm{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\,p_k}{\Big(|\vec{\eta}|^2 + \Delta_M\Big)^2}\,, \tag{3.40a}$$

with

$$|\vec{\eta}|^2 \equiv |\vec{l}|^2/M^2\,, \tag{3.40b}$$

and

$$\Delta_M \equiv \Big[(\omega_m + x\rho_n)^2 + x(1-x)\,p_n^2\Big]/M^2 + r^4 \sim \omega_m^2/M^2 + r^4\,, \tag{3.40c}$$

for $p_n^2/M^2 \to 0$. With large M, we can treat $\omega_m/M \equiv l_4$ as a continuous variable and rewrite (3.40a) as follows:

$$T^{ij}_{\mathrm{rest}}(p_n) = \frac{M}{2\pi}\int dl_4 \int \frac{d^3\eta}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty}(-1)^r\,r^2\,\frac{\mathrm{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\,p_k}{(\lambda^2 + r^4)^2}$$

$$= M \int \frac{d^4\lambda}{(2\pi)^4} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r\, r^2\, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k}{(\lambda^2 + r^4)^2}\,, \tag{3.41}$$

in terms of the dimensionless variable $\lambda^2 \equiv |\vec{\eta}|^2 + (l_4)^2$.

In order to compute the sum over r in (3.41), we again use the following representation:

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{(\lambda^2 + r^4)^2} = -\frac{1}{2\lambda}\frac{\partial}{\partial\lambda} \sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{(\lambda^2 + r^4)}\,, \tag{3.42}$$

where the last sum has the same form as (3.35a) and equals $f(\lambda)$ in terms of the function f defined by (3.35b). As mentioned above, the x integral in expression (3.41) is trivial and the expression reduces to

$$\begin{aligned} T^{ij}_{\text{rest}}(p_n) &= -\frac{M}{16\pi^2} \left(\int_0^\infty d\lambda\, \lambda^2\, \frac{\partial}{\partial\lambda}\Big[f(\lambda) \Big] \right) \mathrm{tr}[\widetilde{\gamma}^i\, \widetilde{\gamma}^j\, \widetilde{\gamma}^k]\, p_k \\ &= -\frac{M}{8\pi^2} \left(\int_0^\infty d\lambda\, \lambda^2\, \frac{\partial}{\partial\lambda}\Big[f(\lambda) \Big] \right) \epsilon^{ijk}\, p_k\,, \end{aligned} \tag{3.43}$$

where the last step uses (3.37). The λ integral in (3.43) gives the following factor:

$$\xi = 14\, \zeta(3)/\pi^2 \approx 1.70511\,, \tag{3.44}$$

and the final expression reads

$$T^{ij}_{\text{rest}}(p_n) = -\xi\, M\, \frac{1}{8\pi^2}\, \epsilon^{ijk}\, p_k\,. \tag{3.45}$$

Combining (3.39) and (3.45) gives the end result for the anomalous vacuum-polarization kernel (3.31),

$$T^{ij}_{\text{anom}}(p_n) = -\frac{1}{4\pi L}\, \epsilon^{ijk}\, p_k - \xi\, M\, \frac{1}{8\pi^2}\, \epsilon^{ijk}\, p_k\,, \tag{3.46}$$

with the constant ξ given by (3.44) and the regulator mass scale M entering the Pauli–Villars-type masses (3.4b). The first term in (3.46) is L-dependent and finite, whereas the second term is L-independent and divergent as the regulator mass scale M is taken to infinity. As regards the M-dependence of this second term, note that, for four-dimensional quantum electrodynamics, the vacuum polarization from the standard Pauli–Villars regularization also has an M-dependent contribution; cf. Eq. (A.6) in Ref. [8]. A suitable renormalization procedure is to subtract the same result at a reference value L_{ref} and to take $L_{\text{ref}} \to \infty$ corresponding to Minkowski spacetime (cf. Sec. 4.2 of Ref. [15]). This renormalization procedure then eliminates the second term in (3.46) and we are left with the first term only,

$$T^{ij}_{\text{anom}}(p_n)\Big|^{\text{(renorm.)}} = -\frac{1}{4\pi L}\, \epsilon^{ijk}\, p_k\,. \tag{3.47}$$

Now replace the single left-handed fermion ψ_L by the 48 left-handed fermions of the chiral $U(1)$ gauge theory (2.4), with the same regularization for each of these 48 fermions. Using (3.47), we then obtain the following local expression for the effective gauge-field action (3.21) to order e^2:

$$\mathcal{T}_{\text{anom}}^{(\text{renorm.})} = i\,F\,e^2\,\frac{1}{8\pi L}\int_0^L dx^4 \int_{\mathbb{R}^3} d^3x\;\epsilon^{ijk}\,A_i(x)\,\partial_j\,A_k(x)\,, \tag{3.48}$$

with an overall numerical factor F from (2.6b) due to the contributions of all chiral fermions of the theory (2.4). The result (3.48) gets a further factor i for spacetime metrics with Lorentzian signature and a spatial coordinate $x^4 \in S^1$ (see also the discussion of the last paragraph in Sec. 6). The local effective-action term (3.48) is the main result of the perturbative calculation.

For gauge fields $A_\mu(x)$ of local support, the term (3.48) is invariant under local Abelian gauge transformations,

$$A_\mu(x) \to A_\mu(x) + i\,\partial_\mu\,\zeta(x)\,, \tag{3.49}$$

with arbitrary real gauge parameters $\zeta(x)$ that are x^4-periodic, $\zeta(\vec{x},\,0) = \zeta(\vec{x},\,L)$. As mentioned in Sec. 2, the Abelian holonomy (2.10) is gauge-invariant under these periodic transformations. The perturbative calculation of this subsection can, in principle, be extended to the non-Abelian theory (2.3) and we expect a further cubic term in addition to the quadratic term of (3.48), in order to maintain invariance under "small" gauge transformations (see Sec. 4 in Ref. [1] for further discussion).

3.3. Lorentz and CPT violation

For arbitrary gauge fields $A_\mu(x)$ with trivial holonomies (2.10) in the chiral $U(1)$ gauge theory (2.4) with a Lorentzian metric signature, our result (3.48) gives the following term in the effective gauge-field action at the one-loop level:

$$\Gamma_{\text{anom}}[A] \;= -2\pi\,F\,e^2\,\Gamma_{\text{CS-like}}[A]\,, \tag{3.50a}$$

$$\Gamma_{\text{CS-like}}[A] \equiv \frac{1}{L}\int_0^L dx^4 \int_{\mathbb{R}^3} d^3x\;\omega_{\text{CS}}[A(\vec{x},x^4)]\,, \tag{3.50b}$$

in terms of the Chern–Simons density [16]

$$\omega_{\text{CS}}[A(\vec{x},x^4)] \equiv \frac{1}{16\pi^2}\,\epsilon^{ijk}\,A_i(\vec{x},x^4)\,\partial_j\,A_k(\vec{x},x^4)\,. \tag{3.51}$$

The numerical factor F in (3.50a) is given by (2.6b).

A topological Chern–Simons term $\Omega_{\text{CS}} = \int \omega_{\text{CS}}$ is defined only for an odd number of spacetime dimensions [16]. The action term (3.50) holds, however, in four spacetime dimensions. Hence, the qualification "Chern–Simons-like" (abbreviated as "CS-like") used in (3.50b) and elsewhere. The action term (3.50) is nontopological in the sense that it has a nontrivial dependence on the spacetime metric or vierbein (see Sec. 6.6 of Ref. [4] for further discussion and references).

Observe that the integrand of (3.50b) is proportional to $\epsilon^{\mu\nu\rho 4}\,A_\mu(x)\,\partial_\nu\,A_\rho(x)$, which has the spacetime index '4' singled-out. This term is, therefore, Lorentz noninvariant. Next, recall that the CPT transformation of an anti-Hermitian gauge field is given by [1]

$$A_\mu(x) \to A_\mu(-x)\,. \tag{3.52}$$

The term (3.50b) changes sign under a CPT transformation (3.52). The Lorentz-violating term (3.50b) is, therefore, also CPT-odd [the Lorentz-invariant Maxwell term $(\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\partial^\mu A^\nu - \partial^\nu A^\mu)$ is CPT-even].

4. Nonperturbative approach

4.1. Lattice setup

In our calculation, we consider a chiral gauge theory which is defined over a four-dimensional spacetime manifold $M = \mathbb{R}^3 \times S^1$, with noncompact coordinates $x^1, x^2, x^3 \in \mathbb{R}$ and compact coordinate $x^4 \in [0, L]$. Initially, the metric is taken to be the Euclidean flat metric $g_{\mu\nu} = [\mathrm{diag}(1, 1, 1, 1)]_{\mu\nu}$. The vierbeins (tetrads) are trivial and given by

$$e^a_\mu(x) = \delta^a_\mu\,, \tag{4.1}$$

with the Lorentz index $a = 1, 2, 3, 4$ and the Einstein index $\mu = 1, 2, 3, 4$.

We consider, in particular, chiral gauge theories that are free of gauge anomalies. As mentioned in Sec. 2, we can take the $SO(10)$ chiral gauge theory (2.3). But, in order to be sure of having a well-defined lattice gauge theory [13], we restrict ourselves to the Abelian $U(1)$ theory (2.4). The actual calculation in the rest of this section is performed for a single left-handed fermion ψ_L with unit $U(1)$ charge, $q = e$. Only the final result (4.111) is extended to all chiral fermions of the theory (2.4).

To regularize the ultraviolet divergences of this gauge theory, a rectangular hypercubic lattice with lattice spacing a is introduced,

$$(x^1, x^2, x^3, x^4) \equiv (\vec{x}, x^4) = (\vec{n}\,a, n_4\,a), \tag{4.2a}$$

with integers

$$n_1, n_2, n_3 \in [0, N'], \quad n_4 \in [0, N]\,. \tag{4.2b}$$

The fermion fields and link variables are periodic with respect to the x^4 coordinate,

$$\psi(x^1, x^2, x^3, L) = \psi(x^1, x^2, x^3, 0)\,, \tag{4.3a}$$

$$\overline{\psi}(x^1, x^2, x^3, L) = \overline{\psi}(x^1, x^2, x^3, 0)\,, \tag{4.3b}$$

$$U_\mu(x^1, x^2, x^3, L) = U_\mu(x^1, x^2, x^3, 0)\,, \tag{4.3c}$$

with $L \equiv N\,a$. For the other coordinates, the link variables are again periodic but the fermion fields are taken to be antiperiodic, for example,

$$\psi(L', x^2, x^3, x^4) = -\psi(0, x^2, x^3, x^4)\,, \tag{4.4a}$$

$$\overline{\psi}(L', x^2, x^3, x^4) = -\overline{\psi}(0, x^2, x^3, x^4)\,, \tag{4.4b}$$

$$U_\mu(L', x^2, x^3, x^4) = U_\mu(0, x^2, x^3, x^4)\,, \tag{4.4c}$$

and similarly for the other coordinates x^2 and x^3.

The assumptions (2.9) for the continuum gauge fields translate into the following conditions on the link variables of the lattice:

$$U_i(x) = U_i(x^1, x^2, x^3, x^4)\,, \quad \text{for } i = 1, 2, 3\,, \tag{4.5a}$$

$$U_4(x) = \mathbb{1}\,. \tag{4.5b}$$

As mentioned before, such link variables can be obtained by a gauge transformation only if there are trivial holonomies,

$$H_4(x^1, x^2, x^3) \equiv \prod_{\text{links}} U_4(x^1, x^2, x^3, x^4) = \mathbb{1}\,, \tag{4.6}$$

where the product runs over all U_4 links in the 4-direction at a fixed value of $\vec{x}$ (for non-Abelian gauge groups, these non-commuting matrices U_4 are ordered along the path).

The anti-Hermitian Abelian gauge field A_μ of the continuum and the $U(1)$ link variable U_μ of the lattice are related as follows [17]:

$$U_\mu(x) = \exp\left[e \int_x^{x+a\,\widehat{\mu}} dy\, A_\mu(y) \right] \approx \exp\Big[e\, a\, A_\mu(x + a\,\widehat{\mu}/2) \Big], \tag{4.7}$$

where the integration variable y in the second expression runs over a straight line between the spacetime points x and $x + a\,\widehat{\mu}$, with unit vector $\widehat{\mu}$ in the μ direction. In (4.7), e is the dimensionless electric charge of the fermion.

Recall from Sec. 2 that Latin spacetime indices i, j, k, l, etc. run over the coordinate labels 1, 2, 3, and Greek spacetime indices μ, ν, ρ, etc. over the labels 1, 2, 3, 4, and that we use natural units with $\hbar = c = 1$.

4.2. *Chiral fermions on the lattice*

4.2.1. *Ginsparg–Wilson relation*

In order to avoid the fermion-doubling problem, Wilson introduced an operator, now known as the Wilson–Dirac operator [17], which includes a term of second order in the difference operators,

$$D_W = \frac{1}{2} \sum_{\mu=1}^{4} \Big[\gamma_\mu (\nabla_\mu + \nabla_\mu^*) + s\, a \nabla_\mu \nabla_\mu^* \Big], \tag{4.8}$$

with 4×4 Dirac matrices γ_μ and a parameter s to be described below. Here, the gauge-covariant derivatives of the continuum are replaced by gauge-covariant forward and backward difference operators on the lattice,

$$\nabla_\mu \psi(x) \equiv \frac{1}{a} \Big(R[U_\mu(x)]\psi(x + a\,\widehat{\mu}) - \psi(x) \Big), \tag{4.9a}$$

$$\nabla_\mu^* \psi(x) \equiv \frac{1}{a} \Big(\psi(x) - R[U_\mu(x - a\,\widehat{\mu})]^{-1} \psi(x - a\,\widehat{\mu}) \Big), \tag{4.9b}$$

where R is a unitary representation of the gauge group.

The Wilson parameter s in (4.8) takes the values $s = \pm 1$. For definiteness, we choose

$$s = -1. \tag{4.10}$$

The s term in (4.8) breaks, however, the chiral invariance of the theory. In order to restore the chiral symmetry, Ginsparg and Wilson suggested to implement the following relation [9]:

$$D\,\gamma_5 + \gamma_5\, D = a\, D\, \gamma_5\, D\,, \tag{4.11}$$

which is known as the Ginsparg–Wilson relation.

Sixteen years after Ginsparg and Wilson proposed relation (4.11), Neuberger explicitly constructed a corresponding operator [10,11],

$$D[U] = \frac{1}{a} \Big(\mathbb{1} - V[U] \Big), \tag{4.12}$$

in terms of an appropriate unitary operator V. Apart from satisfying the Ginsparg–Wilson relation (4.11), the operator V should also be γ_5-Hermitian,

$$V^\dagger = \gamma_5\, V\, \gamma_5\,. \tag{4.13}$$

In terms of the Wilson–Dirac operator D_W from (4.8), this operator V reads

$$V = X\,(X^\dagger X)^{-1/2} = \int_{-\infty}^{\infty} \frac{dt}{\pi}\left(t^2 + X^\dagger X\right)^{-1}, \tag{4.14a}$$

$$X \equiv \mathbb{1} - a\, D_W\,. \tag{4.14b}$$

4.2.2. Lattice fermion action

The lattice fermion action with a Ginsparg–Wilson operator $D[U]$ defined by (4.12) and (4.14),

$$S_F[\overline{\psi}, \psi, U] = a^4 \sum_x\, \overline{\psi}(x) D[U]\psi(x)]\,, \tag{4.15}$$

is invariant under the following infinitesimal transformations [12]:

$$\psi(x) \to \psi(x) + \delta\psi(x)\,, \tag{4.16a}$$

$$\overline{\psi}(x) \to \overline{\psi}(x) + \delta\overline{\psi}(x)\,, \tag{4.16b}$$

with

$$\delta\psi(x) = i\varepsilon\,\gamma_5 V\,\psi(x) \equiv i\varepsilon\,\widehat{\gamma}_5\,\psi(x)\,, \tag{4.17a}$$

$$\delta\overline{\psi}(x) = i\varepsilon\,\overline{\psi}(x)\gamma_5\,, \tag{4.17b}$$

where ε is an infinitesimal parameter. The operator $\widehat{\gamma}_5$, as defined in (4.17a), is a Hermitian unitary operator with eigenvalues ± 1.

A chiral gauge theory for left-handed fermions on the lattice can be constructed by imposing the following constraints [13]:

$$\psi(x) = \widehat{P}_-\,\psi(x)\,, \tag{4.18a}$$

$$\overline{\psi}(x) = \overline{\psi}(x)\, P_+\,, \tag{4.18b}$$

with the projection operators

$$\widehat{P}_\pm \equiv \frac{1}{2}\,(1 \pm \widehat{\gamma}_5)\,, \tag{4.19a}$$

$$P_\pm \equiv \frac{1}{2}\,(1 \pm \gamma_5)\,, \tag{4.19b}$$

where $\widehat{\gamma}_5$ has been defined in (4.17a).

4.2.3. Discrete transformations

On the hypercubic spacetime lattice, there are certain symmetry transformations. Specifically, these lattice symmetries are

(i) the translations by an integer multiple of the lattice spacing a in the direction of one of the four coordinate axes,

(ii) the rotations by an integer multiple of the angle $\pi/2$ in hyperplanes spanned by two axes,
(iii) the parity transformation,
(iv) the time-reversal transformation,
(v) the charge-conjugation transformation.

We now give the parity, time-reversal, and charge-conjugation transformations for the link variable, considering the x^1 coordinate to be the time coordinate for the Lorentzian metric signature and using the notation $x = (x^1, x^2, x^3, x^4) \equiv (x^1, \widetilde{x})$. The parity-transformed link variable is

$$U_\mu{}^{\mathcal{P}}(x^1, \widetilde{x}) = \begin{cases} U_\mu^\dagger(x^1, -\widetilde{x} - a\,\widehat{\mu}), & \text{for } \mu = 2, 3, 4, \\ U_\mu(x^1, -\widetilde{x}), & \text{for } \mu = 1, \end{cases} \tag{4.20a}$$

the time-reflected link variable is

$$U_\mu{}^{\mathcal{T}}(x^1, \widetilde{x}) = \begin{cases} U_\mu^*(-x^1, \widetilde{x}), & \text{for } \mu = 2, 3, 4, \\ U_\mu^t(-x^1 - a, \widetilde{x}), & \text{for } \mu = 1, \end{cases} \tag{4.20b}$$

and the charge-conjugated link variable is

$$U_\mu{}^{\mathcal{C}}(x^1, \widetilde{x}) = U_\mu^*(x^1, \widetilde{x}). \tag{4.20c}$$

Hence, the combined CPT transformation on a link variable is given by

$$U_\mu{}^\theta(x) = U_\mu^\dagger(-x - a\,\widehat{\mu}). \tag{4.21}$$

4.2.4. Integration measure

The fermionic integration measure is the product of all integration measures at the sites of the hypercubic lattice,

$$\mathcal{D}\psi(x) = \prod_{x,\alpha} d\psi_\alpha(x), \quad \mathcal{D}\overline{\psi}(x) = \prod_{x,\alpha} d\overline{\psi}_\alpha(x), \tag{4.22}$$

with a multi-index α containing the spinor, gauge, and flavor indices.

The fermionic fields can be expanded as follows:

$$\psi(x) = \sum_j v_j(x)\, c_j, \quad \overline{\psi}(x) = \sum_k \bar{c}_k\, \bar{v}_k(x), \tag{4.23}$$

where the c_j and $\bar{c}_k$ are Grassmann-valued coefficients and the $v_j(x)$ and $\bar{v}_k(x)$ are two orthonormal bases of complex-valued spinorial functions. The integration measure is then given by

$$\mathcal{D}\psi(x) = \prod_j dc_j, \quad \mathcal{D}\overline{\psi}(x) = \prod_k d\bar{c}_k. \tag{4.24}$$

But this integration measure is not unique. Let $\mathcal{U}$ be a unitary operator which diagonalizes the operator $\widehat{\gamma}_5$,

$$\mathcal{U}^\dagger\, \widehat{\gamma}_5\, \mathcal{U} = \gamma_5, \tag{4.25}$$

where γ_5 on the right-hand side is diagonal in the Weyl representation of the Dirac gamma matrices. Then, the basis spinors v_j are

$$v_j(x) = \mathcal{U}\,\chi_j(x)\,, \tag{4.26}$$

where the χ_j form a complete canonical spinor basis and satisfy the chirality constraint

$$\widehat{P}_-\,\chi_j(x) = \chi_j(x)\,. \tag{4.27}$$

Now, $\mathcal{U}' = \mathcal{U}Q$ is also a diagonalization operator if Q has the following form:

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad Q_1^\dagger Q_1 = \mathbb{1}, \quad Q_2^\dagger Q_2 = \mathbb{1}\,, \tag{4.28}$$

where Q_1 and Q_2 are 2×2 block matrices in spinor space. If the basis vectors change as

$$v'_j(x) = \sum_i v_i(x)\,\mathcal{Q}_{ij}\,, \tag{4.29a}$$

with

$$\mathcal{Q}_{ij} \equiv a^4 \sum \chi_i^\dagger(x)\,Q\,\chi_j(x)\,, \tag{4.29b}$$

then the measure (4.24) changes by a factor $\det \mathcal{Q}$, which is a phase factor since $\mathcal{Q}$ is unitary.

4.3. *Effective action and CPT transformation*

4.3.1. *Effective action*

As in Sec. 3, we calculate the effective gauge-field action by integrating out the chiral fermions, while maintaining gauge invariance. In lattice gauge theory, the Euclidean path integral is given by:

$$\exp(-\Gamma[U]) = \frac{1}{Z}\int \prod_x \mathcal{D}\overline{\psi}(x) \prod_x \mathcal{D}\psi(x)\;\exp\left(-S_F[\overline{\psi},\psi,U]\right), \tag{4.30}$$

where S_F is defined by (4.15). The normalization constant Z ensures that $\Gamma[\mathbb{1}] = 0$ for the constant-link-variable configuration $U_\mu(x) = \mathbb{1}$.

We Fourier expand the chiral fermionic fields as follows:

$$\psi(x) = \frac{1}{L}\sum_n \psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L}\,, \tag{4.31a}$$

$$\overline{\psi}(x) = \frac{1}{L}\sum_n \overline{\psi}_n(x^1,x^2,x^3)\,e^{-2\pi i n x^4/L}\,, \tag{4.31b}$$

where the integer n takes the values

$$-(N-1)/2 \le n \le (N-1)/2\,, \quad \text{for odd } N \ge 1\,, \tag{4.32a}$$

and

$$-(N/2)+1 \le n \le N/2\,, \quad \text{for even } N \ge 2\,, \tag{4.32b}$$

with $N = L/a$ the number of links in the compact 4-direction. The momentum component in the 4-direction is given by

$$p_4 = 2\pi n_4/L. \tag{4.33}$$

Using the Fourier expansion (4.31) of the fermionic field $\psi(x)$, we expand the operator $X(x)$, defined by (4.14b) in terms of D_W from (4.8), in the following way:

$$\begin{aligned} X(x)\,\psi(x) &= X\,\frac{1}{L}\sum_n \psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L} \\ &= \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\,X^{(n)}(x)\,\psi_n(x^1,x^2,x^3)\,, \end{aligned} \tag{4.34}$$

with

$$X^{(n)} \equiv \cos(2\pi n/N) - a\,\mathbb{D}_W - i\gamma_4\,\sin(2\pi n/N)\,, \tag{4.35}$$

and

$$\mathbb{D}_W \equiv \frac{1}{2}\sum_{i=1}^{3}\Big[\gamma_i(\nabla_i+\nabla_i^*) + sa\nabla_i\nabla_i^*\Big]. \tag{4.36}$$

This operator $\mathbb{D}_W$ still contains the standard 4×4 Dirac matrices γ_i.

For the gauge-field configurations (4.5), the operator V, defined by (4.14a), acts on the fermionic field in the following way:

$$\begin{aligned} V\psi(x) &= V\frac{1}{L}\sum_n \psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L} \\ &= \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\int_{-\infty}^{\infty}\frac{dt}{\pi}\,X^{(n)}\left(t^2+X^{(n)\dagger}X^{(n)}\right)^{-1}\psi_n(x^1,x^2,x^3) \\ &\equiv \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\,V^{(n)}(x)\,\psi_n(x^1,x^2,x^3). \end{aligned} \tag{4.37}$$

We now write the fermionic action S_F in terms of the Fourier modes from (4.31),

$$\begin{aligned} &S_F[\overline{\psi},\psi,U] = a^4\sum_x\,\overline{\psi}(x)D[U(x)]\psi(x), \\ &= \frac{1}{L^2}\sum_{m,n}a^4\sum_x\,\overline{\psi}_m(x^1,x^2,x^3)\,e^{-2\pi i m x^4/L}\,D[U(x)]\,\psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L}, \\ &= \frac{1}{L^2}\sum_{m,n}a^4\sum_x\,\overline{\psi}_m(x^1,x^2,x^3)\,e^{2\pi i(n-m)x^4/L}\,D^{(n)}[U(x)]\,\psi_n(x^1,x^2,x^3)\,, \end{aligned} \tag{4.38}$$

with the modes of the Ginsparg–Wilson operator $D^{(n)}$ defined by

$$D^{(n)} \equiv \frac{1}{a}\left(\mathbb{1}-V^{(n)}\right), \tag{4.39}$$

where $V^{(n)}$ follows from (4.35) and (4.37). In the last expression of (4.38), the quantity $e^{2\pi i n x^4/L}$ is a complex number which commutes with $D^{(n)}[U(x)]$, so that we can rewrite the above equation as follows:

$$\begin{aligned} &S_F[\overline{\psi},\psi,U] = \frac{1}{L^2}\sum_{n,m}a^4\sum_x \\ &\times\left(\overline{\psi}_m(x^1,x^2,x^3)\,e^{-2\pi i m x^4/L}\right)D^{(n)}[U(x)]\left(\psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L}\right). \end{aligned} \tag{4.40}$$

For each value of m and n, we then redefine the fermionic fields as follows:

$$\overline{\psi}_m(x^1,x^2,x^3)\,e^{-2\pi i m x^4/L} \equiv \bar{\phi}'_m(x)\,, \tag{4.41a}$$

$$\psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L} \equiv \phi'_n(x)\,, \tag{4.41b}$$

and rewrite the lattice fermion action as

$$S_F[\bar{\phi}',\phi',U] = \frac{1}{L^2}\sum_{n,m} a^4 \sum_x \bar{\phi}'_m(x)\,D^{(n)}[U(x)]\,\phi'_n(x)\,, \tag{4.42}$$

with the operators $D^{(n)}$ from (4.39).

Redefining the fermionic fields again,

$$\psi'_n(x) \equiv \frac{1}{L}\,\phi'_n(x), \quad \overline{\psi}'_m(x) \equiv \frac{1}{L}\,\bar{\phi}'_m(x)\,, \tag{4.43}$$

the final action reads

$$S_F[\overline{\psi}',\psi',U] = \sum_{m,n} a^4 \sum_x \bar{\psi}'_m(x)\,D^{(n)}[U(x)]\,\psi'_n(x)]$$

$$\equiv \sum_{m,n} S_F^{(m,n)}[\overline{\psi}'_m,\psi'_n,U]. \tag{4.44}$$

The modes $\overline{\psi}'_m$ and ψ'_n have to satisfy the following constraints:

$$\psi'_n(x) = \widehat{P}_-^{(n)}\psi'_n(x)\,, \tag{4.45a}$$

$$\overline{\psi}'_m(x) = \overline{\psi}'_m(x)P_+\,, \tag{4.45b}$$

with the usual projection operator P_+ and the modes of the projection operator $\widehat{P}_-$ given by

$$\widehat{P}_-^{(n)} = \frac{1}{2}\left(\mathbb{1} - \gamma_5\,V^{(n)}\right) \equiv \frac{1}{2}\left(\mathbb{1} - \widehat{\gamma}_5^{(n)}\right). \tag{4.46}$$

The operators $\widehat{\gamma}_5^{(n)}$ are Hermitian unitary operators. For each n, the operator $V^{(n)}$ is unitary and satisfies

$$V^{(n)\dagger} = \gamma_5 V^{(n)}\gamma_5. \tag{4.47}$$

We now expand the Fourier modes of the fermionic fields into the following series:

$$\psi'_n(x) = \sum_j v_j^{(n)}(x)\,c_j^{(n)}\,, \tag{4.48a}$$

$$\overline{\psi}'_m(x) = \sum_k \bar{c}_k^{(m)}\,\bar{v}_k^{(m)}(x)\,. \tag{4.48b}$$

Here, the $c^{(n)}$ are Grassmann-valued coefficients and the spinor functions $v_j^{(n)}(x)$ and $\bar{v}_k^{(m)}(x)$ form a complete orthogonal basis of complex-valued, (x^1, x^2, x^3)-antiperiodic, (x^4)-periodic spinors, with the following inner products:

$$\left(v_i^{(m)},v_j^{(n)}\right) \equiv a^4\sum_x v_i^{(m)\dagger}(x)\,v_j^{(n)}(x) = \delta_{ij}\,\delta_{mn}\,, \tag{4.49a}$$

$$\left(\bar{v}_k^{(m)},\bar{v}_l^{(n)}\right) \equiv a^4\sum_x \bar{v}_k^{(n)}(x)\,\bar{v}_l^{(m)\dagger}(x) = \delta_{kl}\,\delta_{mn}\,. \tag{4.49b}$$

The spinor functions $v_j^{(n)}(x)$ and $\bar{v}_k^{(m)}(x)$ have an x^4-dependence given by, respectively, $e^{2\pi i n x^4/L}$ and $e^{-2\pi i m x^4/L}$, which traces back to the definitions (4.41). With these expressions, the effective action for the gauge field can be factorized as follows:

$$\exp\left(-\Gamma[U]\right) = \prod_{m,n} \frac{1}{Z''_{m,n}} \left[\int \prod_k d\bar{c}_k^{(m)} \prod_j dc_j^{(n)} \exp\left(-\sum_{j,k} \bar{c}_k^{(m)} M_{kj}^{(m,n)} c_j^{(n)} \right) \right], \tag{4.50}$$

in terms of the matrices

$$M_{kj}^{(m,n)}[U] = a^4 \sum_x \bar{v}_k^{(m)}(x) D^{(n)}[U(x)] v_j^{(n)}(x;U)\,. \tag{4.51}$$

The constants $Z''_{m,n}$ in (4.50) normalize the integrals, so that $\Gamma[\mathbb{1}] = 0$.

After the Grassmann integrations in (4.50), we get the following expression for the effective action:

$$\Gamma[U] = -\sum_{m,n} \ln\left(\frac{1}{Z''_{m,n}} \det M_{kj}^{(m,n)}[U] \right). \tag{4.52}$$

4.3.2. Change of the effective action under CPT

Unlike the chiral gauge theory of the continuum, the chiral projector (4.19a) for the left-handed fermion in lattice chiral gauge theory depends on the link variables, as follows from the definition $\widehat{\gamma}_5[U] \equiv \gamma_5\, V[U]$. If the gauge field is CPT transformed, the basis of the chiral fermions v_j changes. This transformation affects the integration measure and the effective action is CPT noninvariant. The details are as follows.

For the link configurations as considered in (4.5), the CPT-transformed link variables are given by

$$U_4^\theta = \mathbb{1}\,, \quad U_i^\theta = U_i^\dagger(x - a\,\widehat{i}\,)\,, \tag{4.53}$$

for $i = 1, 2, 3$ and with the unit vector $\widehat{i}$ in the i-direction. Let $\mathcal{R}$ be the coordinate-reflection operator of the three coordinates $\vec{x} \equiv (x^1, x^2, x^3)$,

$$\mathcal{R} : \vec{x} \to -\vec{x}\,, \tag{4.54}$$

and let $\mathcal{R}^4$ be the coordinate-reflection operator in the fourth direction,

$$\mathcal{R}^4 : (\vec{x},\, x^4) \to (\vec{x},\, -x^4). \tag{4.55}$$

The operator $\mathbb{D}_W$, defined by (4.36), has then the following behavior under a CPT transformation:

$$\mathcal{R}\mathcal{R}^4 \gamma_5\, \mathbb{D}_W[U^\theta]\, \gamma_5 \mathcal{R}^4 \mathcal{R} = \mathbb{D}_W[U]. \tag{4.56}$$

The Ginsparg–Wilson-operator modes $D^{(m)}$ from (4.38) transform as follows:

$$\mathcal{R}\mathcal{R}^4 \gamma_5\, D^{(n)}[U^\theta]\, \gamma_5 \mathcal{R}^4 \mathcal{R} = D^{(-n)}[U]. \tag{4.57}$$

The matrices $M_{k,j}^{(m,n)}[U]$, defined by (4.51), now change as follows under the CPT transformation $U \to U^\theta$:

$$
\begin{aligned}
M_{k,j}^{(m,n)}[U^\theta] &= a^4 \sum_x \bar{v}_k^{(m)}(x)\, D^{(n)}[U^\theta(x)]\, v_j^{(n)}(x; U^\theta) \\
&= a^4 \sum_x \bar{v}_k^{(m)}(x)\, \mathcal{R}\mathcal{R}^4 \gamma_5 D^{(-n)}[U(x)] \gamma_5 \mathcal{R}^4 \mathcal{R}\, v_j^{(n)}(x; U^\theta) \\
&= \sum_{l,i} (\bar{\mathcal{Q}}_\theta^{(-m)})_{kl} \left(a^4 \sum_x \bar{v}_l^{(-m)}(x)\, D^{(-n)}[U(x)]\, v_i^{(-n)}(x; U) \right) (\mathcal{Q}_\theta^{(-n)})_{ij} \\
&= \sum_{l,i} (\bar{\mathcal{Q}}_\theta^{(-m)})_{kl}\, M_{li}^{(-m,-n)}[U]\, (\mathcal{Q}_\theta^{(-n)})_{ij}\,.
\end{aligned}
\tag{4.58}
$$

Here, the unitary matrices

$$
(\mathcal{Q}_\theta^{(-n)})_{ij} = a^4 \sum_x v_j^{(-n)\,\dagger}(\vec{x}; U)\, \gamma_5 \mathcal{R}^4 \mathcal{R}\, v_j^{(n)}(x; U^\theta)\,, \tag{4.59a}
$$

$$
(\bar{\mathcal{Q}}_\theta^{(-m)})_{kl} = a^4 \sum_x \bar{v}_k^{(m)}(x)\, \mathcal{R}\mathcal{R}^4 \gamma_5\, \bar{v}_l^{(-m)}(x)\,, \tag{4.59b}
$$

are obtained by introducing the projection operator P_+ and making use of the fact that

$$
\gamma_5\, D^{(n)} = D^{(n)}\, \widehat{\gamma}_5^{(n)}\,. \tag{4.60}
$$

With the completeness of the bases $v_j^{(n)}$ and $\bar{v}_k^{(m)}$, the summation kernels of the projection operators $\widehat{P}_-^{(n)}$ and P_+ are

$$
\widehat{P}_-^{(n)}(x, y) = \sum_i v_i^{(n)}(x; U)\, v_i^{(n)\,\dagger}(y; U) \tag{4.61a}
$$

and

$$
P_+ \frac{1}{a^4} \delta_{xy} = \sum_l \bar{v}_l^{(m)\dagger}(x)\, \bar{v}_l^{(m)}(y). \tag{4.61b}
$$

The transformation (4.58) can be absorbed by a redefinition of the fermionic variables in the multiple integral (4.50), but the integration measure picks up a Jacobian factor. Under a CPT transformation, the effective gauge-field action changes to

$$
\Gamma[U^\theta] = \Gamma[U] - \sum_{n,m'} \ln\det \left(\sum_l \left(\mathcal{Q}_\theta^{(-n)}[U] \right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(-m')} \right)_{lm} \right). \tag{4.62}
$$

The determinants of the transformation matrices $\mathcal{Q}_\theta^{(-n)}$ depend on the link variable $U_i(x)$, which opens up the possibility that the effective action is CPT noninvariant.

4.4. CPT anomaly

In this subsection, we discuss the change of the effective gauge-field action under a CPT transformation. But, in order to calculate the explicit expression for the CPT-violating term, we need to know the explicit form of the bases $v_j^{(n)}$ and $\bar{v}_j^{(m)}$.

4.4.1. Basis spinors

The basis spinors for the antifermions are given by

$$\bar{v}_j^{(m)}(x) = \big(\bar{\xi}_k^{(m)}(x), 0\big)\,, \tag{4.63}$$

where $\bar{\xi}_k^{(m)}(x)$ form an orthonormal basis of two-spinors in four spacetime dimensions with the explicit x^4-dependence $e^{-2\pi i m x^4/L}$.

The basis vectors $v_j^{(n)}(x;U)$ are more difficult to obtain. We have to find unitary operators $\mathcal{U}^{(n)}$ with the property

$$\mathcal{U}^{(n)\dagger}\,\widehat{\gamma}_5^{(n)}\,\mathcal{U}^{(n)} = \gamma_5, \tag{4.64}$$

for

$$\widehat{\gamma}_5^{(n)} \equiv H^{(n)}\left(H^{(n)2}\right)^{-1/2}. \tag{4.65}$$

Here, the Hermitian operators $H^{(n)}$ are given by

$$\begin{aligned} H^{(n)} &\equiv \gamma_5\left(\overset{\star}{n} - a\,\mathbb{D}_W - i\gamma_4\,\mathring{n}\right) \\ &= \begin{pmatrix} \overset{\star}{n} + \frac{1}{2}\sum_{i=1}^{3} w_i[U] & \mathring{n} - \frac{1}{2}\sum_{i=1}^{3}\sigma_i\,t_i[U] \\ \mathring{n} + \frac{1}{2}\sum_{i=1}^{3}\sigma_i\,t_i[U] & -(\overset{\star}{n} + \frac{1}{2}\sum_{i=1}^{3} w_i[U]) \end{pmatrix}, \end{aligned} \tag{4.66a}$$

with

$$\mathring{n} \equiv \sin(2\pi n/N), \quad \overset{\star}{n} \equiv \cos(2\pi n/N) \tag{4.66b}$$

$$t_i[U] \equiv a\,(\nabla_i + \nabla_i^*), \quad w_i[U] \equiv a^2\,\nabla_i\nabla_i^*\,, \tag{4.66c}$$

The four-component basis spinors are then constructed as

$$v_j^{(n)}(x) = \mathcal{U}^{(n)}[U]\,\chi_j^{(n)}(x), \tag{4.67a}$$

with

$$\chi_j(x) = \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix}, \tag{4.67b}$$

where $\xi_j^{(n)}(x)$ form an orthonormal basis of two-spinors in four spacetime dimensions with the explicit x^4-dependence $e^{2\pi i n x^4/L}$.

For the case of an odd number N of links in the x^4 direction (assuming odd $N \geq 3$), we divide the domain of calculation into three subsets: $n<0$, $n>0$, and $n=0$. A particular property of $\widehat{\gamma}_5^{(n)}$,

$$\widehat{\gamma}_5^{(n)}\,\widetilde{\Gamma}_4 = -\widetilde{\Gamma}_4\,\widehat{\gamma}_5^{(-n)}, \tag{4.68}$$

with the definition

$$\widetilde{\Gamma}_4 \equiv i\gamma_4\gamma_5\,, \tag{4.69}$$

suggests to impose the following condition:

$$\mathcal{U}^{(-n)}[U] = \widetilde{\Gamma}_4\,\mathcal{U}^{(n)}[U]\,\widetilde{\Gamma}_4\,, \tag{4.70}$$

where the link variable U on both sides of this last equation refers to the same configuration.

4.4.2. Fixing the phases

We now obtain the required diagonalization operators for (4.64), first for nonzero n and then for $n=0$.

In the $n \neq 0$ sector, the diagonalization operator $\mathcal{U}^{(n)}$ is of the form

$$\mathcal{U}^{(n)} = \frac{1}{2}\begin{pmatrix} \mathbb{1}+W^{(n)} & \mathbb{1}-W^{(n)} \\ \mathbb{1}-W^{(n)} & \mathbb{1}+W^{(n)} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathbb{1}+Y^{(n)} & i(\mathbb{1}-Y^{(n)\dagger}) \\ i(\mathbb{1}-Y^{(n)}) & \mathbb{1}+Y^{(n)\dagger} \end{pmatrix} \cdot \begin{pmatrix} Q_1^{(n)} & 0 \\ 0 & Q_1^{(-n)} \end{pmatrix}, \tag{4.71}$$

with the unitary operators

$$W^{(n)} \equiv \left(\overset{\star}{n} - a\, D_W^{3D}\right)\left[\left(\overset{\star}{n} - a\, D_W^{3D}\right)^{\dagger}\left(\overset{\star}{n} - a\, D_W^{3D}\right)\right]^{-1/2}, \tag{4.72a}$$

$$Y^{(n)} \equiv \left[\left(\overset{\star}{n} - a\, D_W^{3D}\right)^{\dagger} W^{(n)} + i\mathring{n}\right]\left[\left(\overset{\star}{n} - a\, D_W^{3D}\right)^{\dagger}\left(\overset{\star}{n} - a\, D_W^{3D}\right) + \mathring{n}^2\right]^{-1/2}, \tag{4.72b}$$

and

$$D_W^{3D} \equiv \frac{1}{2}\sum_{i=1}^{3}\left(\sigma_i(\nabla_i + \nabla_i^*) + s a \nabla_i^* \nabla_i\right). \tag{4.73}$$

One possible choice for $Q_1^{(n)}$ is

$$Q_1^{(n)}[U] = \begin{cases} \mathbb{1}, & \text{for } n > 0, \\ W^{(n)}[U]^{\dagger}, & \text{for } n < 0. \end{cases} \tag{4.74}$$

A change of n to $-n$ gives

$$W^{(-n)} = W^{(n)}, \quad Y^{(-n)} = Y^{(n)\dagger}. \tag{4.75}$$

In the $n=0$ sector, the diagonalization operator $\mathcal{U}^{(n)}$ is of the form

$$\mathcal{U}^{(0)} = \frac{1}{2}\begin{pmatrix} \mathbb{1}+W^{(0)\dagger} & \mathbb{1}-W^{(0)} \\ -\mathbb{1}+W^{(0)\dagger} & \mathbb{1}+W^{(0)} \end{pmatrix}, \tag{4.76}$$

with $W^{(0)}$ defined by (4.72a) for $n=0$. As discussed in App. B of Ref. [3], other possible choices for $\mathcal{U}^{(0)}$ are characterized by an integer $k^{(0)} \in \mathbb{Z}$ and give an additional factor $(2\,k^{(0)}+1)$ in the final result (4.111).

4.4.3. CPT anomaly for odd $N \geq 3$

The diagonalization operators $\mathcal{U}^{(n)}[U]$ are given by (4.71) and (4.76) and the CPT-violating factor can be calculated as follows.

The operator D_W^{3D} from (4.73) transforms under CPT as

$$D_W^{3D}[U^{\theta}] = \mathcal{R}\mathcal{R}^4\, D_W^{3D}[U]^{\dagger}\, \mathcal{R}^4\mathcal{R}. \tag{4.77}$$

The operators $W^{(n)}$ and $Y^{(n)}$ transform under CPT as follows:

$$W^{(n)}[U^\theta] = \mathcal{R}\mathcal{R}^4\, W^{(n)^\dagger}[U]\, \mathcal{R}^4\mathcal{R}\,, \tag{4.78a}$$

$$Y^{(n)}[U^\theta] = \mathcal{R}\mathcal{R}^4\, W^{(n)}[U]\, Y^{(n)}[U]\, W^{(n)^\dagger}[U]\, \mathcal{R}^4\mathcal{R}\,. \tag{4.78b}$$

With the help of (4.78a) and (4.78b), we calculate the changes of the diagonalization operators $\mathcal{U}^{(n)}$ under a CPT transformation for $n < 0$, $n > 0$, and $n = 0$. The results are for $n < 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(n)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(n)}[U]\,\widetilde{\Gamma}_4 \begin{pmatrix} Y^{(n)} & 0 \\ 0 & W^{(n)}Y^{(n)^\dagger}W^{(n)^\dagger} \end{pmatrix}, \tag{4.79a}$$

for $n > 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(n)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(n)}[U]\,\widetilde{\Gamma}_4 \begin{pmatrix} W^{(n)}Y^{(n)}W^{(n)^\dagger} & 0 \\ 0 & Y^{(n)^\dagger} \end{pmatrix}, \tag{4.79b}$$

and for $n = 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(0)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(0)}[U]\widetilde{\Gamma}_4. \tag{4.79c}$$

The changed transformation matrices are for $n = 0$:

$$\begin{aligned}\left(\mathcal{Q}_\theta^{(0)}[U]\right)_{ij} &= a^4 \sum_x \chi_i^{(0)\dagger}(x)\mathcal{U}^{(0)}[U]^\dagger\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(0)}[U^\theta]\, \chi_j^{(0)}(x) \\ &= a^4 \sum_x \chi_i^{(0)\dagger}(x)\mathcal{U}^{(0)}[U]^\dagger\mathcal{U}^{(0)}[U^\theta]\, \mathcal{R}^4\mathcal{R}\gamma_5\, \chi_j^{(0)}(x)\,,\end{aligned} \tag{4.80a}$$

for $n > 0$:

$$\begin{aligned}&\left(\mathcal{Q}_\theta^{(n)}[U]\right)_{ij} \\ &= a^4 \sum_x \left(0, \xi_i^{(n)\dagger}(x)\right) \begin{pmatrix} W^{(n)}Y^{(n)}W^{(n)^\dagger} & 0 \\ 0 & Y^{(n)^\dagger} \end{pmatrix} \mathcal{R}\mathcal{R}^4\gamma_5 \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix},\end{aligned} \tag{4.80b}$$

and for $n < 0$:

$$\begin{aligned}&\left(\mathcal{Q}_\theta^{(n)}[U]\right)_{ij} \\ &= a^4 \sum_x \left(0, \xi_i^{(n)\dagger}(x)\right) \begin{pmatrix} Y^{(n)} & 0 \\ 0 & W^{(n)}Y^{(n)^\dagger}W^{(n)^\dagger} \end{pmatrix} \mathcal{R}\mathcal{R}^4\gamma_5 \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix}.\end{aligned} \tag{4.80c}$$

We shall later see that the transformation matrices for the $n < 0$ modes and the $n > 0$ modes do not contribute to the final expression of the anomalous term.

The changed transformation matrices $\bar{\mathcal{Q}}_\theta^{(m')}[U]$ are the same for all values of the Fourier index m':

$$\left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{kl} = \left(\bar{\xi}_k^{(m')}(x), 0\right)\, \mathcal{R}\mathcal{R}^4\gamma_5 \begin{pmatrix} \bar{\xi}_l^{(m')^\dagger}(x) \\ 0 \end{pmatrix}. \tag{4.81}$$

The required combinations of transformation matrices give for $n = 0$:

$$\left(\mathcal{Q}_\theta^{(0)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{lm} = -a^4 \sum_x \xi_k^{(0)\dagger}(x)\, W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\, \delta_{m'0}\,, \tag{4.82a}$$

for $n > 0$:

$$\sum_l \left(\mathcal{Q}_\theta^{(n)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{lm}$$
$$= -a^4 \sum_x \xi_k^{(n)\dagger}(x) \left(W^{(n)}[U] Y^{(n)}[U] W^{(n)}[U]^\dagger\right) \xi_m^{(n)}(x)\, \delta_{m'n}\,, \tag{4.82b}$$

and for $n < 0$:

$$\sum_l \left(\mathcal{Q}_\theta^{(n)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{lm} = -a^4 \sum_x \xi_k^{(n)\dagger}(x) Y^{(n)}[U]^\dagger \xi_m^{(n)}(x)\, \delta_{m'n}\,. \tag{4.82c}$$

For the derivation of (4.82), we have used

$$\bar{\xi}_k^{(m')} = \xi_k^{(m')\dagger}(x) \tag{4.83a}$$

and the completeness relation of the two-spinor basis $\xi_k^{(n)}(x)$,

$$\sum_k \xi_k^{(m')\dagger}(x) \xi_k^{(n)}(y) = a^{-4}\, \mathbb{1}\, \delta_{xy}\, \delta_{m'n}. \tag{4.83b}$$

Because $W^{(n)}$ and $Y^{(n)}$ are unitary, the determinant of (4.82b) for $n > 0$ is the inverse of the determinant of (4.82c) for $n < 0$, where we have used the relations (4.75). This gives

$$\prod_{n>0} \prod_{m'} \det\left(\sum_l \left(\mathcal{Q}_\theta^{(n)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{lm}\right)$$
$$\times \det\left(\sum_l \left(\mathcal{Q}_\theta^{(-n)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(m')}[U]\right)_{lm}\right) = 1. \tag{4.84}$$

We see from (4.84) that the anomalous terms arising from positive frequencies ($n > 0$) are canceled by the terms arising from negative frequencies ($n < 0$), so that only the $n = 0$ term survives. This $n = 0$ term is given by (4.82a), which effectively sets $m' = 0$.

To summarize, the change in the effective gauge-field action under a CPT transformation is, for odd $N \geq 3$, given by

$$\Delta\Gamma[U] \equiv \Gamma[U^\theta] - \Gamma[U] = -\ln\det\left(a^4 \sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\right), \tag{4.85}$$

with the unitary operator

$$W^{(0)}[U] = \left(\mathbb{1} - a D_W^{3D}[U]\right) \left[\left(\mathbb{1} - a D_W^{3D}[U]\right)^\dagger \left(\mathbb{1} - a D_W^{3D}[U]\right)\right]^{-1/2}. \tag{4.86}$$

4.4.4. CPT anomaly for even $N \geq 4$

For even N (equal to or larger than 4), we divide the Fourier modes n into four subsets: $-N/2 < n < 0$, $n = 0$, $0 < n < N/2$, and $n = N/2$. The case $N = 2$, for x^4-independent gauge fields, has already been discussed in Ref. [3].

Equation (4.68) is also valid for even N, as long as $n \neq N/2$. For $n = N/2$, we have

$$\widehat{\gamma}_5^{(N/2)}\, \widetilde{\Gamma}_4 = -\widetilde{\Gamma}_4\, \widehat{\gamma}_5^{(N/2)}. \tag{4.87}$$

Hence, the results from Sec. 4.4.3 can be used for $n \neq N/2$. But the $n = N/2$ diagonalization operator needs to be investigated separately.

For $n = N/2$, we have

$$\mathcal{U}^{(N/2)} = \frac{1}{2} \begin{pmatrix} \mathbb{1} + W^{(N/2)^\dagger} & \mathbb{1} - W^{(N/2)} \\ -\mathbb{1} + W^{(N/2)^\dagger} & \mathbb{1} + W^{(N/2)} \end{pmatrix}, \tag{4.88}$$

where the unitary operator $W^{(N/2)}[U]$ is defined as

$$W^{(N/2)}[U] \equiv -\left(\mathbb{1} + a D_W^{3D}[U]\right) \left[\left(\mathbb{1} + a D_W^{3D}[U]\right)^\dagger \left(\mathbb{1} + a D_W^{3D}[U]\right)\right]^{-1/2}. \tag{4.89}$$

The total change in effective gauge-field action under a CPT transformation is, for even $N \geq 4$, determined by

$$\begin{aligned} &\det\left(\sum_l \left(\mathcal{Q}_\theta^{(0)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(0)}[U]\right)_{lm}\right) \\ &\times \det\left(\sum_l \left(\mathcal{Q}_\theta^{(N/2)}[U]\right)_{kl} \left(\bar{\mathcal{Q}}_\theta^{(N/2)}[U]\right)_{lm}\right) \\ &= \det\left(a^4 \sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\right) \\ &\times \det\left(a^4 \sum_x \xi_k^{(N/2)\dagger}(x) W^{(N/2)}[U]^\dagger \xi_m^{(N/2)}(x)\right), \end{aligned} \tag{4.90}$$

with the unitary operators $W^{(0)}$ and $W^{(N/2)}$ given by, respectively, (4.86) and (4.89).

The expressions (4.85) for odd $N \geq 3$ and (4.90) for even $N \geq 4$ give the change of the effective gauge-field action under a CPT transformation according to (4.62) and are the main results of the nonperturbative lattice calculation. In order to better understand the meaning of these expressions, we consider the continuum limit of them in the next subsection.

4.5. CPT anomaly in the continuum limit

As mentioned in Sec. 4.1, we first consider an Abelian $U(1)$ gauge field coupled to a single unit-charge chiral fermion. The change in the effective gauge-field action under a CPT transformation for an odd number N of links in the 4-direction depends only on $W^{(0)}[U]$, see (4.85). For an even number N of links in the 4-direction, the corresponding change is given by (4.90).

Consider an even number N of links in the 4-direction and introduce the following short-hand notations:

$$W^{(-)^\dagger} \equiv W^{(0)^\dagger}, \quad W^{(+)^\dagger} \equiv W^{(N/2)\dagger}, \tag{4.91}$$

with

$$\begin{aligned} W^{(\pm)^\dagger} &= \mp\left(\mathbb{1} \pm a D_W^{3D}\right)^\dagger \left[\left(\mathbb{1} \pm a D_W^{3D}\right)\left(\mathbb{1} \pm a D_W^{3D}\right)^\dagger\right]^{-1/2} \\ &= -\left(D_W^{3D} \pm 1/a\right)^\dagger \left[\left(D_W^{3D} \pm 1/a\right)\left(D_W^{3D} \pm 1/a\right)^\dagger\right]^{-1/2} \end{aligned} \tag{4.92}$$

for D_W^{3D} from (4.73). The change in the effective gauge-field action is calculated from (4.90) as

$$\triangle\Gamma[U] = i\left(\mathrm{Im}\{\ln\det(D_W^{3D} - 1/a)\} + \mathrm{Im}\{\ln\det(D_W^{3D} + 1/a)\}\right) \tag{4.93a}$$

$$\equiv i\,\left(\mathrm{Im}\{\ln\det(D - m_+)\} + \mathrm{Im}\{\ln\det(D - m_-)\}\right)\,, \tag{4.93b}$$

where, in (4.93b), we have introduced further short-hand notations,

$$D \equiv D_W^{3D}\,, \quad m_+ \equiv 1/a\,, \quad m_- \equiv -(1/a)\,. \tag{4.94}$$

The first operator in (4.93a) is a Wilson–Dirac operator with positive mass $1/a$ and the second operator is a Wilson–Dirac operator with negative mass $-1/a$. Because of the antiperiodic boundary conditions in the x^1, x^2, x^3 directions, the masses for these operators are effectively increased by a contribution of order $a/(L')^2$. The values of the positive and negative effective masses are now

$$m_+^{(\mathrm{eff})} = +1/a + c_+\, a/(L')^2\,, \tag{4.95a}$$

$$m_-^{(\mathrm{eff})} = -1/a + c_-\, a/(L')^2\,, \tag{4.95b}$$

with positive constants $c_\pm$.

The vacuum-polarization kernel of the effective gauge-field action in three dimensions has been calculated in Ref. [18] to second order in the bare coupling constant e. We adopt a similar approach, in order to calculate the change in the effective action under a CPT transformation.

For this purpose, we consider an auxiliary theory of a nonchiral four-component Dirac fermion field $\Psi(x)$ with the following action over the four-dimensional lattice (4.2a):

$$S_F = -a^4 \sum_x \overline{\Psi}(x)\,[D - m]\,\Psi(x)\,, \tag{4.96}$$

where D is the operator from (4.94) and m an arbitrary mass. The corresponding effective gauge-field action $\Gamma[A]$ is given by

$$\Gamma[A] = \ln\det[D - m]\,. \tag{4.97}$$

The fermion propagator $S(x, y)_{\alpha\beta}$ from (4.96) is defined by

$$[(-D + m)S(x, y)]_{\alpha\beta} = \frac{1}{a^4}\,\delta_{\alpha\beta}\,\delta_{xy}\,. \tag{4.98}$$

In momentum space, we have

$$S(x, y) = \frac{1}{L}\sum_n \int_{-\pi/a}^{\pi/a} \frac{d^3p}{2\pi^3}\, e^{ip(\vec{x}-\vec{y})}\, e^{2\pi i n(x^4-y^4)/L}\, S_n(\vec{p})$$

$$= \frac{1}{L}\sum_n \int_{-\pi/a}^{\pi/a} \frac{d^3p}{2\pi^3}\, e^{i\,\vec{p}\cdot(\vec{x}-\vec{y})}\, e^{2\pi i n(x^4-y^4)/L}\, S(p_n)\,, \tag{4.99}$$

with, as before,

$$p_n \equiv (\vec{p},\, \rho_n)\,, \quad \rho_n \equiv 2\pi n/L\,. \tag{4.100}$$

A comment on the Fourier transforms of (4.99) is in order. The momentum steps in the fourth direction and those in the other three directions are, respectively, of order $1/L$ and $1/L'$, with

$L' \gg L$. Hence, we have kept in (4.99) the summation for the momentum in the fourth direction but used an integral for the momenta in the three other directions.

Next, define a quantity $Q(p_n)$ in such a way that

$$S(p_n) = Q(p_n)^{-1}. \tag{4.101}$$

This quantity $Q(p_n)$ is a function of $\widehat{p}_{n_\mu}$ and $\widetilde{p}_{n_\mu}$, which are defined as follows:

$$\widehat{p}_{n_\mu} \equiv \frac{2}{a} \sin\left(\frac{1}{2} a\, p_{n\,\mu}\right), \quad \widetilde{p}_{n_\mu} \equiv \frac{1}{a} \sin(a\, p_{n\,\mu}). \tag{4.102}$$

We expand the Dirac operator D in powers of the coupling constant e,

$$D = \sum_{k}^{\infty} e^k D_k\,, \tag{4.103}$$

where, for $k \geq 1$, we have

$$D_k\, \Psi(x) = \frac{(ia)^k}{2ak!} \sum_{i=1}^{3} \times [A_i(x)^k (s+\gamma_i)\Psi(x+a\,\widehat{i}\,) + (-1)^k A_i(x-a\,\widehat{i}\,)^k (s-\gamma_i)\Psi(x-a\,\widehat{i}\,)]. \tag{4.104}$$

For the effective gauge-field action, there is the following expansion in powers of the fermion charge:

$$\Gamma[A] = \sum_{k}^{\infty} e^k\, \Gamma_k[A]. \tag{4.105}$$

With the Fourier transform of the gauge field A_μ, we write the two-point function as

$$\Gamma_2[A] = -i\, \frac{1}{2}\, \frac{1}{L} \sum_{n} \int_{-\pi/a}^{\pi/a} \frac{d^3 q}{2\pi^3}\, A_i(-q_n)\, \widehat{\pi}_{ij}(q_n)\, A_j(q_n)\,, \tag{4.106}$$

where we have included the same prefactor $-i/2$ as in (3.21) and where the vacuum polarization tensor $\widehat{\pi}_{ij}(q_n)$ is now given by

$$\widehat{\pi}_{ij}(q_n) = \frac{1}{2}\, \frac{1}{L} \sum_{m} \int_{-\pi/a}^{\pi/a} \frac{d^3 p}{2\pi^3}\, [1 - T_0(q_n)] \times \mathrm{tr}\Big\{[Q\,(p_m + q_n/2)]^{-1} \partial_i\, Q(p_m)\, [Q\,(p_m - q_n/2)]^{-1} \partial_j\, Q(p_m)\Big\}. \tag{4.107}$$

The symbol $[1 - T_0(q_n)]$ in the above equation stands for a Taylor subtraction at zero momentum. Just as for the perturbative calculation of Sec. 3.2, the anomalous term originates from the $m = 0$ sector of (4.107). We now focus on this $m = 0$ sector [denoted by the superscript '(0)'] and will mention later the contribution of the $m \neq 0$ terms.

In the continuum limit, we can use the three-dimensional result from Ref. [18],

$$\widehat{\pi}_{ij}^{(0)\,(\text{cont.})}(q_n) = \lim_{a\to 0} \widehat{\pi}_{ij}^{(0)}(q_n)$$
$$= \frac{1}{L}\, A(q_n^2)\, \epsilon_{ijk}\, q_n{}^k + \frac{1}{L}\, B(q_n^2)\, (q_n^2\, \delta_{ij} - q_{n\,i}\, q_{n\,j})\,, \tag{4.108a}$$

with amplitudes $A(q_n{}^2)$ and $B(q_n{}^2)$ given by

$$A(q_n{}^2) = \frac{1}{2}\, a_0 + \frac{1}{8\pi} \int_0^1 dt \Big\{1 - m\, [m^2 + t(1-t)\, q_n{}^2]^{-1/2}\Big\}\,, \tag{4.108b}$$

$$B(q_n{}^2) = \frac{1}{4\pi} \int_0^1 dt \Big\{1 - m\, [m^2 + t(1-t)\, q_n{}^2]^{-1/2}\Big\}\,, \tag{4.108c}$$

where 'm' is the mass defined by (4.96) and not a Fourier component (for the moment, we have Fourier component $m = 0$). Henceforth, we drop the superscript '(cont.)' of (4.108a) and focus on the part with an odd number of momenta, containing the Levi-Civita symbol and the $A(q_n{}^2)$ amplitude. With the Wilson parameter $s = -1$, we have the constant $a_0 = -1/(2\pi)$. In the large negative m limit for a fixed value of $q_n{}^2$, the odd-momentum part of the polarization tensor $\widehat{\pi}_{ij}^{(0)}(q)$ vanishes, whereas, in the large positive m limit for fixed $q_n{}^2$, the odd-momentum part of the polarization tensor becomes

$$\lim_{m\to\infty} \widehat{\pi}_{ij}^{(0)\,(\text{odd-mom})}(q) = \frac{1}{L}\, \frac{a_0}{2}\, \epsilon_{ijk}\, q^k = -\frac{1}{4\pi}\, \frac{1}{L}\, \epsilon_{ijk}\, q^k\,. \tag{4.109}$$

As mentioned above, the anomalous contribution (4.109) originates from the $m = 0$ Fourier sector of (4.107). The $m \neq 0$ Fourier terms of (4.107) contribute a further term $\propto (1/a)\, \epsilon_{ijk}\, q^k$, which is L-independent and divergent in the continuum limit $a \to \infty$. Just as discussed in Sec. 3.2, this extra term can be removed by a suitable renormalization procedure.

With the results (4.108) and (4.109) obtained from the auxiliary theory (4.96), we now return to the original chiral gauge theory. The first term in (4.93) has a positive mass $m = 1/a$ and the second term has a negative mass $m = -1/a$, so that the second term does not contribute to the anomalous change in the effective gauge-field action. The anomalous change in the effective action follows solely from the first term in (4.93) and is determined by (4.109). Up till now, we have considered an even number N of links in the 4-direction. For an odd number N of links, the second term in (4.93) does not appear and the result is the same as for even N.

Changing from momentum space to position space, the first term in (4.93) gives, using (4.109), the following result up to order e^2 in the effective gauge-field action (4.106):

$$e^2\, \Gamma_2^{(\text{odd-mom.})}[A] = 2\pi i\, e^2\, \frac{1}{L} \sum_{n_4} a \int_{\mathbb{R}^3} d^3x\; \omega_{\text{CS}}[A(\vec{x}, n_4\, a)], \tag{4.110}$$

where the Chern–Simons density ω_{CS} has been defined in (3.51). The continuum limit has $a \to 0$ and $N \to \infty$, with constant product $Na = L$.

Next, change from a Euclidean metric signature to a Lorentzian metric signature and include all fermions of the chiral gauge theory (2.4), with all of these fermions treated equally on the lattice. The expression (4.110) then becomes

$$e^2\, \Gamma_2^{(\text{odd-mom.})}[A] = -F\, e^2\, \frac{2\pi}{L} \int_0^L dx^4 \int_{\mathbb{R}^3} d^3x\; \omega_{\text{CS}}[A(\vec{x}, x^4)]\,, \tag{4.111}$$

with an extra factor i for the Lorentzian metric signature and an overall numerical factor F from (2.6b) due to the contribution of all chiral fermions of the theory (2.4).

5. Discussion

In this section, we present six general remarks in order to clarify the calculations performed in Secs. 3 and 4.

First, we must explain how an apparently CPT-invariant theory has produced CPT violation. With an extended version of the generalized Pauli–Villars regularization for the perturbative calculation, the regulator masses M_r in (3.3) are the source of the Lorentz and CPT violation (these Lorentz-violating terms in the regularized action appear to be necessary in order to maintain the gauge invariance of the second-quantized theory, as discussed in Sec. 6 of Ref. [1]). With the lattice regularization for the nonperturbative calculation, the crucial observation is that the gauge-covariant diagonalization operators (4.71) and (4.76) are not CPT invariant, as shown by (4.79).

Let us expand on the CPT noninvariance of the lattice calculation. For an odd number N of links in the 4-direction, we have explicitly shown that the changes of the nonperturbative effective gauge-field action under a CPT transformation for positive n are canceled by the corresponding changes for negative n. But the $n = 0$ contribution has no counterpart to cancel its change under a CPT transformation. Specifically, the change of the $n = 0$ diagonalization operator is given by

$$\mathcal{R}\mathcal{R}^4\gamma_5\,\mathcal{U}^{(0)}[U^\theta]\,\gamma_5\mathcal{R}^4\mathcal{R} = \mathcal{U}^{(0)}[U]\begin{pmatrix} W^{(0)^\dagger} & 0 \\ 0 & W^{(0)} \end{pmatrix}, \tag{5.1}$$

where $W^{(0)^\dagger}$ acts on left-handed fermions and $W^{(0)}$ acts on right-handed fermions. The CPT transformation leads to another theory with different basis spinors [3]. This different theory can be transformed back to the original one by a redefinition of the spinors. But, then, the integration measure picks up a Jacobian factor and the effective gauge-field action $\Gamma[U]$ changes,

$$\Delta\Gamma[U] \equiv \Gamma[U^\theta] - \Gamma[U] = -\ln\det\left(a^4\sum_x \xi_k^{(0)\dagger}(x)\left(W^{(0)}[U]\right)\xi_m^{(0)}(x)\right). \tag{5.2}$$

For an even number N of links in the 4-direction, we can give the same argument as for an odd number of links. The changes in the measure for $0 < n < N/2$ are again canceled by the corresponding changes for negative n. The remaining factors are those for $n = 0$ and $n = N/2$. But the additional factor for $n = N/2$ is a lattice artefact and vanishes in the continuum limit.

Note also that the CPT anomaly vanishes for Dirac fermions with both left- and right-handed components,

$$\ln\det W^{(0)^\dagger} + \ln\det W^{(0)} = 0. \tag{5.3}$$

Second, let us discuss the conditions on the background gauge field. If the gauge fields depend upon the compactified coordinate x^4, they should not oscillate too fast with respect to the x^4 coordinate.

In the perturbative approach, we Fourier expand the gauge field A_μ in the following way:

$$A_\mu(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}\, e^{2\pi i n x^4/L}\, e^{i\vec{p}\cdot\vec{x}}\, A_\mu(p_n)\,. \tag{5.4}$$

The frequency of oscillation of A_μ with respect to x^4 is n/L. The discrete momentum corresponding to the coordinate x^4 is given by

$$\rho_n = 2\pi n/L. \tag{5.5}$$

For the generalized Pauli–Villars regularization used, the regulator mass scale M must be very much larger than the momentum component $\rho_n = 2\pi n/L$, as discussed on the lines above (3.37). Hence, the condition on the gauge fields is given by

$$n \ll M\,L\,, \tag{5.6}$$

where n controls the dimensionless oscillation frequency of the gauge field A_μ with respect to x^4 and L is the range of the compactified coordinate x^4.

In the nonperturbative approach, the 4-direction momentum ρ_n of the external gauge fields must be very small compared to the regulator scale $1/a$, in order to be able to apply the continuum expressions of Sec. 4.5. There is, then, the following condition (using $\rho_n \sim n/L$):

$$\frac{n}{L} = \frac{n}{Na} \ll \frac{1}{a} < m_+\,, \tag{5.7}$$

where m_+ is the effective mass (4.95a) for the Wilson–Dirac operator (this effective mass m_+ is similar to the Pauli–Villars regulator mass scale M of the perturbative approach). As 'n' is the frequency of oscillation of A_μ with respect to x^4, condition (5.7) is similar to condition (5.6) for the perturbative case.

Third, let us remark on the main improvements of our present calculations compared with the earlier calculations for x^4-independent background gauge fields. Recall that the perturbative calculation here used a generalized Pauli–Villars regularization method with an extra infinite set of Pauli–Villars-type fields ψ_r (with regulator masses $M_r = M\,r^2$) and maintains gauge invariance, unlike the calculation of Ref. [1] which used the standard Pauli–Villars regularization with a single set of regulator fields and a single regulator mass. In the lattice calculation here, we have explicitly obtained the diagonalization operators $\mathcal{U}^{(n)}$ and have not used an *ad-hoc* phase fixing, unlike the calculation of Ref. [3].

Fourth, let us try to understand heuristically why our new result for x^4-dependent background gauge fields is similar to the previous result for x^4-independent background gauge fields. We see, from the result (4.84), that the anomalous terms arising from the positive frequency ($n > 0$) are canceled by the terms arising from the negative frequency ($n < 0$), so that only the term corresponding to $n = 0$ contributes to the CPT violation, which also has $m' = 0$ according to (4.82a) [recall (4.62) for the definition of the Fourier modes n and m' entering the change of the effective action under CPT]. This explains why, for the case of x^4-dependent background gauge fields, we have obtained a result similar to the one for the case of x^4-independent gauge fields [1,3]. Indeed, compare (4.85) from the present paper, with a unitary operator depending on x^4-dependent gauge fields and a sum over (x^1, x^2, x^3, x^4) in the determinant, to (5.35) from Ref. [3], with essentially the same unitary operator depending on x^4-independent gauge fields and only a sum over (x^1, x^2, x^3) in the determinant.

Fifth, let us continue the heuristic discussion and comment on the absence of $\partial_4 A_i$ terms in our result. We have calculated, in the perturbative approach, the effective gauge-field action up to two-point functions (second-order in the gauge field A_μ). In this approach, the CPT-anomalous terms are independent of the momentum in the fourth direction. See, in particular, the discussion above (3.30), where the ρ_n term corresponds to the position-space partial derivative ∂_4. If we consider the non-Abelian gauge theory, the CPT-anomalous terms will involve three-point functions (third-order in the gauge field A_μ). There is then the possibility that the CPT-anomalous terms involving ∂_4 will not vanish by symmetry reasons. For the continuum limit of the lattice calculation, we have also considered only Abelian gauge fields and have expanded only up to the two-point function $\Gamma_2[A]$ (second-order in the coupling constant e). For the non-Abelian case, we expect to have higher-order contributions (notably $\Gamma_3[A]$), which may, in principle, give rise to terms involving the partial derivative ∂_4 acting on the background gauge field.

Sixth, recall that finite-temperature field theory can be described by a quantum field theory defined over a Euclidean spacetime with a compactified coordinate [17]. This Euclidean-path-integral formulation of finite-temperature field theory has the same manifold as our theory ($\mathbb{R}^3 \times S^1$), with S^1 coordinate $x^4 \in [0, L]$. The range of the compactified coordinate is determined by $L = \beta$, where β is the inverse of the temperature T (in units with $k_B = 1$). The discrete momentum components of the fermion fields (Matsubara frequencies) are given by $p_4 = (n + 1/2)\, 2\pi/\beta$, with integers $n = 0, \pm 1, \pm 2 \ldots$.

In several recent articles (see, e.g., Refs. [19,20] and references therein), calculations have been reported of a radiatively-induced Chern–Simons-like term in four-dimensional finite-temperature field theory. This temperature-dependent induced Chern–Simons-like term violates the Lorentz and CPT symmetries.

But compared to our calculation there are significant differences. Most importantly, the fermions of the finite-temperature calculations have anti-periodic boundary conditions (coming from the trace in the partition function of the finite-temperature system and having anti-commuting fields), whereas we assume a periodic spin structure over the compact dimension. In our calculation, the anomalous Chern–Simons-like term results from the zero-momentum part of the fermions, which would be absent for anti-periodic boundary conditions.

In addition, the finite-temperature calculations have an explicit Lorentz-violating term in the fermion sector with a constant b_μ (the induced Chern–Simons-like term is proportional to this constant b_μ), whereas the Lorentz violation in our calculation comes from the regulator fields. Moreover, the fermions of the finite-temperature calculations can have a mass m, whereas the original chiral fermions of our calculation are strictly massless.

As a final comment, we emphasize the importance of maintaining microcausality, also for the finite-temperature effective theory in the $T \to 0$ limit (cf. Refs. [19,21]).

6. Conclusion

For the appropriate setup of the physical system (Sec. 2), we have established perturbatively (Sec. 3) the existence of a CPT anomaly for a background gauge field A_μ which depends on the compactified x^4 coordinate and has a vanishing component A_4. We have also performed a non-perturbative calculation with a lattice regularization (Sec. 4) and have discussed the continuum limit of the lattice result. The nonperturbative result (4.111) agrees with the earlier result (3.50) obtained via the perturbative approach. (In principle, these results could have differed by an odd-integer prefactor, because, as noted in Refs. [1,3] and Sec. 4.4.2 here, there is an ambiguity in the anomalous term due to the freedom in defining the regularized theory.)

The fact that the perturbative and nonperturbative results for the CPT anomaly essentially agree is reminiscent of the Adler–Bardeen result for the triangle anomaly [22]. In this respect, note that the CPT anomaly of the perturbative calculation originates in the $m = 0$ sector of the vacuum-polarization kernel (3.22) with a linearly-diverging one-loop Feynman diagram. Still, it needs to be verified that there arise no further terms in the nonperturbative lattice calculation.

Having a possible anomalous origin of the local Chern–Simons-like term (3.50b) in the effective gauge-field action provides additional incentive to study the phenomenology of the so-called

Maxwell–Chern–Simons (MCS) theory [6]. This MCS theory contains, in the photonic sector, the standard Maxwell term and the local Chern–Simons-like term. The MCS theory can also be augmented by the addition of the standard gauge-invariant kinetic term of a Dirac spinor field (the electron–positron field).

This MCS theory appears in two varieties: one variety is parity-violating and time-reversal-invariant (corresponding to a timelike x^4 coordinate in our calculation) and the other variety is parity-conserving and time-reversal-noninvariant (corresponding to a spacelike x^4 coordinate in our calculation). Now, it is clear that our calculation for a timelike x^4 coordinate would start from a theory with closed timelike loops and such a theory is, most likely, inconsistent [23]. It has, indeed, been shown that the parity-violating (and time-reversal-invariant) variety of MCS theory is noncausal and nonunitary [24]. The parity-conserving (and time-reversal-noninvariant) variety of MCS theory appears to be well-behaved [24] and displays some interesting nonstandard effects such as photon triple-splitting [25,26] and vacuum Cherenkov radiation [26,27].

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