Local Wellposedness of Nonlinear Maxwell Equations

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1
Introduction

Electromagnetic waves can be found everywhere. Since their discovery in 1888 by Heinrich Hertz, they led to countless technological innovations. The underlying theory of electromagnetism has thus not only become a classical theory in physics, but builds the cornerstone for all the advances in optics and electrical engineering that have been made in the last 130 years.

The theory of electromagnetism originated in the early 19th century. Extensive experimental studies by Hans Christian Ørsted, André-Marie Ampère, Michael Faraday and others led to several phenomenological discoveries and corresponding mathematical theories. Combined with theoretical considerations by James Clerk Maxwell, the theory of electromagnetism could finally be cast in the form

$$\begin{aligned} \varepsilon_0 \partial_t \boldsymbol{E} &= \frac{1}{\mu_0} \operatorname{curl} \boldsymbol{B} - \boldsymbol{J}, & \operatorname{div} \boldsymbol{E} &= \frac{1}{\varepsilon_0} \rho, \\ \partial_t \boldsymbol{B} &= -\operatorname{curl} \boldsymbol{E}, & \operatorname{div} \boldsymbol{B} &= 0. \end{aligned}$$

Here $\boldsymbol{E}$ denotes the electric field, $\boldsymbol{B}$ the magnetic field, and ε_0 and μ_0 the vacuum permittivity respectively permeability. The field $\boldsymbol{J}$ denotes the current density and the quantity ρ the charge density. This set of partial differential equations is known as microscopic Maxwell's equations or Maxwell's equations of the vacuum. We refer to [Rau14] and [Sha73] for an historical overview of the genesis of Maxwell's equations.

In the presence of a material, the microscopic Maxwell's equations are in principal still valid - however one has to take every single atom into account. It is hopeless to treat the arising system. One therefore wants to describe the response of a material on external electric and magnetic fields on a macroscopic level. To that purpose, the displacement field $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and the magnetizing field $\boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M}$ are introduced, where $\boldsymbol{P}$ denotes the polarization and $\boldsymbol{M}$ the magnetization. Polarization and magnetization contain the material response. For example the polarization accommodates the electric field generated by a macroscopic bound charge in the material which arises from tiny displacements of charges due to an external electric field. The electric fields $\boldsymbol{D}$ and $\boldsymbol{E}$ and the magnetic fields $\boldsymbol{B}$ and $\boldsymbol{H}$ are then described by the macroscopic Maxwell's equations

$$\begin{aligned} \partial_t \boldsymbol{D} &= \operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, & \operatorname{div} \boldsymbol{D} &= \rho, \\ \partial_t \boldsymbol{B} &= -\operatorname{curl} \boldsymbol{E}, & \operatorname{div} \boldsymbol{B} &= 0, \end{aligned} \tag{1.1}$$

also called Maxwell's equations in matter.

For applications one also has to consider Maxwell's equations on domains $G \subseteq \mathbb{R}^3$. System (1.1) then has to be equipped with suitable boundary conditions. One of the most relevant boundary conditions are those of the perfect conductor. Maxwell's equations themselves imply that the tangential components of the electric field $\boldsymbol{E}$ and the normal component of the magnetic field $\boldsymbol{B}$ have to be continuous across the boundary, see [DL90a]. If one assumes that the material on one side of the boundary is a perfect conductor, one obtains the so called perfectly conducting boundary conditions

$$\boldsymbol{E} \times \nu = 0, \quad \boldsymbol{B} \cdot \nu = 0 \quad \text{on } \partial G,$$

where ν denotes the outer unit normal vector of ∂G. If we combine the Maxwell system (1.1) with the perfectly conducting boundary conditions and suitable initial conditions, we arrive at the initial boundary value problem

$$\begin{aligned}
\partial_t \boldsymbol{D} &= \operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, & \operatorname{div} \boldsymbol{D} &= \rho, & &\text{for } x \in G, & t &\geq t_0, \\
\partial_t \boldsymbol{B} &= -\operatorname{curl} \boldsymbol{E}, & \operatorname{div} \boldsymbol{B} &= 0, & &\text{for } x \in G, & t &\geq t_0, \\
\boldsymbol{E} \times \nu &= 0, & \boldsymbol{B} \cdot \nu &= 0, & &\text{for } x \in \partial G, & t &\geq t_0, \\
\boldsymbol{E}(t_0) &= \boldsymbol{E}_0, & \boldsymbol{B}(t_0) &= \boldsymbol{B}_0, & &\text{for } x \in G,
\end{aligned} \tag{1.2}$$

for an initial time $t_0 \in \mathbb{R}$. We point out that the electric fields $\boldsymbol{E}(t,x)$ and $\boldsymbol{D}(t,x)$, the magnetic fields $\boldsymbol{H}(t,x)$ and $\boldsymbol{B}(t,x)$, and the current density $\boldsymbol{J}(t,x)$ depend on time and space and take values in $\mathbb{R}^3$. Similarly, the charge density $\rho(t,x)$ depends on time and space and takes values in $\mathbb{R}$.

System (1.2) has to be complemented by constitutive relations between the electric fields and the magnetic fields. As mentioned above, polarization and magnetization consider the reaction of the material to external electric and magnetic fields. The material response however depends on these external fields. Choosing the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ as state variables and setting $\varepsilon_0 = \mu_0 = 1$ for convenience, we obtain that $\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{P}(\boldsymbol{E}, \boldsymbol{H})$ and $\boldsymbol{B} = \boldsymbol{H} + \boldsymbol{M}(\boldsymbol{E}, \boldsymbol{H})$.

The actual form of the constitutive relations, the so called material laws, is a question of modeling. Several kinds of material laws have been considered in the literature. In the so called retarded material laws the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ depend also on the past of $\boldsymbol{E}$ and $\boldsymbol{H}$, see [BF03] and [RSY12] for instance. In dynamical material laws there are additional evolution equations for the polarization or magnetization, see e.g. [AH03], [DS12], [Joc05], or [JMR96].

In this work we concentrate on the instantaneous material laws. Here the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ are given as *local* functions of $\boldsymbol{E}$ and $\boldsymbol{H}$, i.e., we assume that there are functions $\theta_1, \theta_2 \colon G \times \mathbb{R}^6 \to \mathbb{R}^3$ such that $\boldsymbol{D}(t,x) = \theta_1(x, \boldsymbol{E}(t,x), \boldsymbol{H}(t,x))$ and $\boldsymbol{B}(t,x) = \theta_2(x, \boldsymbol{E}(t,x), \boldsymbol{H}(t,x))$. The most prominent example is the so called Kerr nonlinearity, where

$$\boldsymbol{P} = \vartheta |\boldsymbol{E}|^2 \boldsymbol{E}, \quad \boldsymbol{M} = 0, \tag{1.3}$$

and $\vartheta \colon G \to \mathbb{R}^{3\times 3}$. We further make the ansatz $\boldsymbol{J} = \boldsymbol{J}_0 + \sigma_1(\boldsymbol{E}, \boldsymbol{H})\boldsymbol{E}$, where $\boldsymbol{J}_0$ is an external current density and σ_1 denotes the conductivity. If we insert these material laws into (1.2) and formally differentiate, we obtain

$$(\partial_t \boldsymbol{D}, \partial_t \boldsymbol{B}) = \partial_y \theta(x, \boldsymbol{E}, \boldsymbol{H}) \partial_t (\boldsymbol{E}, \boldsymbol{H}) = (\operatorname{curl} \boldsymbol{H} - \boldsymbol{J}, -\operatorname{curl} \boldsymbol{E})$$

for the evolutionary part of (1.2), where $\partial_y \theta$ denotes the derivative with respect to the second variable of $\theta(x,y) = (\theta_1(x,y), \theta_2(x,y))$. The arising resulting equation is a first order quasilinear hyperbolic system, and it is thus natural to reformulate it in the language of first order systems. To that purpose, we first introduce the matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -J_j \\ J_j & 0 \end{pmatrix} \tag{1.4}$$

for $j = 1, 2, 3$. Observe that $\sum_{j=1}^{3} J_j \partial_j = \operatorname{curl}$. Writing χ for $\partial_y \theta$, $f = (-\boldsymbol{J}_0, 0)$, $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$, and using $u = (\boldsymbol{E}, \boldsymbol{H})$ as new variable, we finally obtain

$$\chi(u)\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + \sigma(u) u = f. \tag{1.5}$$

Under weak regularity assumptions we will show in Chapter 7 that a solution of (1.5) preserves the divergence conditions in (1.2) over time, so that these conditions only impose a restriction on the initial value. Similarly, if a solution of (1.5) satisfies the first part of the boundary conditions $\boldsymbol{E} \times \nu = 0$ on $(t_0, T) \times \partial G$ and the second part at the initial time, i.e., $\boldsymbol{B}(t_0) \cdot \nu = 0$ on ∂G, then it satisfies the second part $\boldsymbol{B} \cdot \nu = 0$ on $(t_0, T) \times \partial G$. We refer to Lemma 7.25 for the precise statement. Defining the matrix

$$B = \begin{pmatrix} 0 & \nu_3 & -\nu_2 & 0 & 0 & 0 \\ -\nu_3 & 0 & \nu_1 & 0 & 0 & 0 \\ \nu_2 & -\nu_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

on ∂G, we can cast system (1.2) into the first order quasilinear hyperbolic initial boundary value problem

$$\begin{cases} \chi(u)\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + \sigma(u) u = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \quad t \in J; \\ u(t_0) = u_0, & x \in G; \end{cases} \tag{1.6}$$

plus additional conditions on the initial value. Here $J = (t_0, T)$ is an open interval. We also included an inhomogeneous boundary value. On the one hand, inhomogeneous boundary conditions are interesting from the mathematical point of view, on the other hand they also have physical relevance, see [DL90a]. We make the further assumption that χ is symmetric and at least locally positive definite. This is of course a restriction on the material laws we can treat. However, the most important examples arising from Kerr-like nonlinearities are included. The advantage of this assumption is that the system (1.6) becomes symmetric, simplifying crucial parts of the theory. Without the positive definiteness assumption all of the available theory breaks down. We further note that the results for first order systems in the literature, even in the linear case, assume at least symmetrizability of the system.

The initial value problem on the full space (without boundary conditions) corresponding to (1.6) has been solved by Kato in [Kat75] in a more general setting, relying on previous results in [Kat70] and [Kat73]. Kato first freezes a function $\hat{u}$ in the nonlinearities and then studies the corresponding linear problem

$$\begin{cases} \chi(\hat{u})\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + \sigma(\hat{u}) u = f, & x \in \mathbb{R}^3, \quad t \in J; \\ u(t_0) = u_0, & x \in \mathbb{R}^3. \end{cases}$$

He establishes a priori estimates for the solution of the linearized problem in suitable norms so that he can apply a fixed point argument to obtain a solution of the quasilinear problem. His method works in an abstract functional analytic setting, which requires however the existence of an isomorphism between certain function spaces fitting to the linearized problem. It is unlikely that it is possible to construct function spaces which incorporate the perfectly conducting boundary conditions such that an isomorphism as required by Kato's theory still exists, cf. [Mül14].

We thus follow a different strategy. We still freeze a function $\hat{u}$ in the nonlinearities and study the arising linear initial boundary value problem

$$\begin{cases} \chi(\hat{u})\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + \sigma(\hat{u})u = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \quad t \in J; \\ u(t_0) = u_0, & x \in G; \end{cases} \tag{1.7}$$

aiming at a fixed point argument for the solution of the quasilinear problem. But to derive the a priori estimates needed for this fixed point argument, we use energy techniques.

Energy techniques have been proven to be a flexible and powerful tool in the theory of first order hyperbolic systems. We refer to the monograph [BGS07] for an overview of the state of the art. Energy techniques work in L^2-based spaces, but already on the L^2-level they require coefficients in $W^{1,\infty}$. Accordingly, the available theory for the linear initial boundary value problem (1.7) requires coefficients in $W^{1,\infty}(J \times G)$ and yields solutions in $C(\overline{J}, L^2(G))$, see [Ell12] for the precise statement. In view of the fixed point argument we want to apply, we thus have to bridge the gap between $C(\overline{J}, L^2(G))$ and $W^{1,\infty}(J \times G)$. This is done via Sobolev's embedding (on domains G where this embedding theorem is valid). If a solution of (1.7) belongs to $C(\overline{J}, H^s(G)) \cap C^1(\overline{J}, H^{s-1}(G))$ for $s > \frac{3}{2} + 1$, it is contained in $W^{1,\infty}(J \times G)$. If χ and σ are reasonably regular, then also $\chi(\hat{u})$ and $\sigma(\hat{u})$ are contained in $W^{1,\infty}(J \times G)$ if $\hat{u}$ belongs to $W^{1,\infty}(J \times G)$. We are therefore led to look for solutions in $H^s(G)$ with $s > \frac{5}{2}$. We stress that the requirement of this relatively high regularity is not a result of the specific techniques we want to apply, but has been a long standing assumption in the theory of quasilinear systems. It is also imposed by Kato for the initial value problem in [Kat75] and has not been weakened since that time, see for instance [Ali09], [BCD11], and [Sog13] for more recent treatises.

While the aforementioned sources all treat the initial value problem, there are less methods available for initial boundary value problems on domains. To the best of our knowledge, all results concerning quasilinear initial boundary value problems work in Sobolev spaces of integer regularity, see e.g. [BGS07] and [LMSTYZ01]. Hence, we will construct solutions of the quasilinear system (1.6) in $H^m(G)$ for $m \in \mathbb{N}$ with $m \geq 3$. In fact, we are mainly interested in the lowest regularity regime possible, which is $m = 3$. But it turns out that we can handle all $m \in \mathbb{N}$ with $m \geq 3$ by the same methods so that we derive a satisfying regularity theory for (1.6) simultaneously. Although $C(\overline{J}, H^m(G)) \cap C^1(\overline{J}, H^{m-1}(G))$ embeds into $W^{1,\infty}(J \times G)$ for $m \geq 3$, the techniques we are going to apply to solve (1.7) require that its solution has the same amount of regularity in time as in space. We thus introduce the function spaces

$$G_m(J \times G) = \bigcap_{j=0}^{m} C^j(\overline{J}, H^{m-j}(G)) \tag{1.8}$$

for all $m \in \mathbb{N}_0$, where J is an open interval and $G \subset \mathbb{R}^3$ a domain, see also [BGS07], [LMSTYZ01], and [RM74]. Defining the function $e_{-\gamma} \colon t \mapsto e^{-\gamma t}$, we equip the space $G_m(J \times G)$ with the family of time-weighted norms

$$\|v\|_{G_{m,\gamma}(J \times G)} = \max_{j=0,\ldots,m} \|e_{-\gamma} \partial_t^j v\|_{L^2(J \times G)}$$

for all $\gamma \geq 0$. In the case $\gamma = 0$, we also write $\|v\|_{G_m(J \times G)}$ instead of $\|v\|_{G_{m,0}(J \times G)}$.

To work in the spaces $G_m(J \times G)$ with $m \geq 3$ of course requires to control the solution of (1.7) in this space in view of our fixed point argument. In the case of our linearized Maxwell system (1.7) this is a highly delicate task because this problem has a *characteristic boundary*, i.e., the so called *boundary matrix*

$$A(\nu) = \sum_{j=1}^{3} A_j^{\mathrm{co}} \nu_j$$

on ∂G is singular, see Remark 3.6. In order to explain the drawback of a singular boundary matrix, we assume that $G = \mathbb{R}^3_+$ for the moment. Then the boundary matrix is simply $-A_3^{\mathrm{co}}$. The derivation of higher order a priori estimates of the linear problem

$$\begin{cases} A_0 \partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + Du = f, \qquad x \in \mathbb{R}^3_+, \qquad t \in J; \\ \qquad\qquad\qquad\qquad Bu = g, \qquad x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ \qquad\qquad\qquad\quad u(t_0) = u_0, \qquad x \in \mathbb{R}^3_+; \end{cases} \tag{1.9}$$

relies on the following idea. The derivative of a solution of (1.9) with respect to t, x_1, and x_2 again solves (1.9) with suitably adapted data, which allows to apply the basic L^2-estimate to the derivative. Controlling the error terms arising from the adapted data then yields higher order a priori estimates. However, this approach can only work for derivatives in tangential variables as we do not have any information about the boundary value of a derivative in normal direction. If the boundary is noncharacteristic, i.e., the coefficient in front of ∂_3 is invertible, one can express the derivative of a solution in the normal direction by derivatives in tangential directions and the solution itself. This explicit representation yields estimates for all derivatives of a solution.

In the characteristic case it is however unclear how to obtain control over the derivative in normal direction. It is thus not surprising that much less is known about characteristic than about noncharacteristic initial boundary value problems, cf. [BGS07]. Majda and Osher show with an explicit example that a loss of regularity may happen in characteristic problems, i.e., that the solution is less smooth than the data, see Section B.3 in [MO75]. In such a case we also have a loss of derivatives in the a priori estimates, which makes it impossible to close the fixed point argument. It is thus a key step in our strategy to prove that this loss of regularity does not occur for the Maxwell system (1.9).

A first attempt to develop a general theory for linear characteristic initial boundary value problems was made in [MO75]. Besides existence and the energy estimate on the L^2-level, also a priori estimates of higher order are studied there for a certain family of boundary conditions. However, the perfectly conducting boundary conditions are not covered by these results, see Proposition 2.2 and the discussion thereafter in [MO75]. A different approach is taken in [Gué90]. Results for the quasilinear problem (1.6) are provided there. But they require high regularity (at least $H^6(G)$) and are given in Sobolev-like spaces incorporating weights in the normal direction. In [Ohk81] a structural assumption on the coefficients of the linear problem is introduced in order to avoid a loss of regularity in normal direction. This result is applied in [Ohk89] to solve a quasilinear system under these structural conditions. But quasilinear Maxwell's equations are not covered by these results unless the material law is diagonal, i.e., the matrix function χ has only entries on the diagonal. This condition is not even satisfied in the basic examples of Kerr-like nonlinearities. In [PZ95] the authors concentrate on Maxwell's equations (1.6). They use different boundary conditions (belonging to the class considered in [MO75] in the linear case) than the perfectly conducting ones. Moreover, only the existence of a solution is claimed there, see also [CE11].

Somehow surprisingly, the physically highly relevant quasilinear Maxwell system (1.2) with perfectly conducting boundary has not yet been treated and even the basic questions on local existence and uniqueness are still open. We will close this gap by providing a complete local wellposedness theory. We will prove that

(i) the system (1.6) has a unique maximal solution u in $\bigcap_{j=0}^m C^j((T_-, T_+), H^{m-j}(G))$ for all $m \in \mathbb{N}$ with $m \geq 3$ provided the data are sufficiently regular and compatible with the material law,

(ii) finite existence time can be characterized by blowup in the Lipschitz-norm,

(iii) the solution depends continuously on the data.

We refer to Theorem 7.23 for the precise statement. We point out that this theorem is local in nature. The derivation of global properties for (1.6) is a highly nontrivial task. In particular, it is already known that global existence cannot be expected for all data. Blow-up examples in the Lipschitz-norm are given in [Maj84]. On different domains and with different boundary conditions than we consider, blow-up examples in the $H(\mathrm{curl})$-norm are provided in [DNS16].

The proof of the local wellposedness theorem requires several steps. In Chapter 2 we collect rather technical preparations, which are however fundamental in the following chapters. Section 2.1 explains how to interpret the boundary term Bu in (1.9). When we derive the a priori estimates, we start from an $L^2(\mathbb{R}^3_+)$-solution on the half-space so that we cannot simply apply the standard trace operator to u. Indeed, the solution u itself need not have a trace on $\partial\mathbb{R}^3_+$. We show how one can still make sense of Bu on the boundary. The trace concept for this term is developed in detail since we are also interested in further properties of this trace operator. In later chapters we need to know if it commutes with derivatives, mollifiers, and integration in time. These questions are also addressed in this section.

When we outlined our strategy above, we only introduced the spaces $G_m(J \times G)$ where we look for our solutions. We did not specify which properties the coefficients in the linearized problem (1.9) need to possess. This will be done in Section 2.2. We introduce function spaces for the coefficients which are tailored for the application in a fixed point argument. However, these spaces are not standard so that we have to prove several properties ourselves, e.g. bilinear estimates and estimates for the inverse if it exists. We note that these function spaces allow for quite a precise analysis. This approach might be laborious at times but it yields more general results than available in the literature even in the noncharacteristic case.

At the end of that section we also explain the compatibility conditions for the linearized problem (1.9). These are necessary conditions on the coefficients and the data so that a solution of higher regularity can exist. Roughly speaking, the compatibility conditions arise since for a $G_m(J \times G)$-solution of (1.9) higher order time derivatives of the solution still have a trace on $\{t = 0\} \times \partial G$, see Lemma 2.31.

In Chapter 3 we derive the desired a priori estimates for the linearized problem. To that purpose, we work on the half-space $G = \mathbb{R}^3_+$ with the idea that a localization procedure will transfer our results on $\mathbb{R}^3_+$ to more general domains. However, the localization requires to treat (1.9) with variable coefficients. We are thus led to study the problem

$$\begin{cases} A_0 \partial_t u + \sum_{j=1}^{3} A_j \partial_j u + D u = f, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B u = g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ u(t_0) = u_0, & x \in \mathbb{R}^3_+; \end{cases} \tag{1.10}$$

where $\sum_{j=1}^3 A_j \partial_j$ is a variable coefficient Maxwell operator in the sense that it has a structural similarity with the standard Maxwell operator $\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j$.

In a first step we show that derivatives in tangential directions of (1.10) again solve this system with modified data. We identify this data and provide estimates in the corresponding norms. For the a priori estimates we then use a basic L^2-estimate and existence result in [Ell12]. Differentiating in time and in spatially tangential directions and applying the L^2-estimate to these derivatives yields a priori estimates for the derivatives in tangential directions of a solution. In a key step we next derive an a priori estimate for the derivative in normal direction as we explained above. This is done in Lemma 3.11. There we crucially exploit the structure of the variable coefficient Maxwell operator. Once we obtained the estimate in normal direction, it only remains to set up an iterative scheme to deduce the full higher order a priori estimates.

Since [Ell12] only treats the L^2-level, we also have to establish the existence of solutions of (1.10) in higher regularity. Due to the characateristic boundary this is a difficult task, performed in Chapter 4. In the noncharacteristic case, one can rely on regularization in tangential directions since derivatives in normal direction can be expressed by the ones in tangential directions and lower order terms. The lack of such a representation of derivatives in normal direction complicates the problem heavily. We proceed in several steps, using different techniques in normal direction, spatially tangential directions, and in time, which also have to be intertwined in a subtle way. In a first step we apply a mollifier in all spatial variables, use commutator estimates from the paradifferential calculus, and exploit our a priori estimates. However, as we mollify over the half-space, it is crucial to avoid a loss of regularity across the boundary. To regularize in the spatially tangential variables, we apply a mollifier in x_1- and x_2-directions, exploit our a priori estimates and employ commutator estimates for a family of norms which has proven to be highly suitable for the regularization of boundary value problems, see [Hoe76] and [BGS07]. However, in the characteristic case it is crucial to avoid commutator terms involving a derivative in normal direction. We will show that this is possible due to the structure of the variable coefficients Maxwell operator.

The a priori estimates show that we cannot gain regularity in time by means of a mollifier. We therefore formally differentiate (1.10) in time, expecting that a solution of the differentiated problem is a candidate for the time derivative of the original solution. This approach leads to a loss of regularity in the coefficients so that we obtain regularity in time only under an additional smoothness assumption on the coefficients in a first step. To get rid of this extra assumption, we approximate the coefficients by smoother ones and make use of the a priori estimates once more. We note that this approach is quite delicate since we also have to approximate the data in such a way that the tuples consisting of approximating coefficients and data still satisfy the compatibility conditions. Finally, we obtain a full differentiability theorem, which tells us that the solution has the expected amount of regularity if the data is regular and compatible. We refer to Theorem 4.13 for the precise statement. We note that several of the problems we have to face in the regularization process are due to the fact that we only assume minimal smoothness for our coefficients. In the literature, problem (1.10) is often considered with C^∞-coefficients or coefficients of the form $\chi(v)$ for a C^∞-function χ. Some difficulties then simply disappear and different techniques are available which are not employable in our setting, cf. [BGS07]. We will make further comments on this point in Chapter 4 and Chapter 7.

In Chapter 5 we transfer the results from Chapters 3 and 4 from the half-space to more general domains. We are able to treat domains which do not have a C^∞-boundary and also certain domains with an unbounded boundary. Although the ideas for the localization are canonical, its execution is quite technical and lengthy. We refer to Chapter 5 for a discussion of the difficulties one has to face. In Chapter 6 we prove that the solution of the linearized problem (1.10) on domains has finite propagation speed, a typical feature of hyperbolic equations. Most authors establish this property only on the full space, see e.g. [Eva10], [BCD11], or [BGS07]. In [CP82] the finite propagation speed is shown for initial boundary value problems under an additional structural assumption. We will follow the ideas of [BCD11], where a weighted energy estimate is derived in order to prove the finite propagation speed property. It turns out that this approach is well adaptable to our initial boundary value problem.

In Chapter 7 we finally turn to the nonlinear problem (1.6). Working with instantaneous material laws in spaces $H^m(G)$ with $m \geq 3$ requires a higher order chain rule for compositions of the form $\chi(u)$. We establish this so called Faá di Bruno's formula and corresponding estimates in the needed function spaces in Section 7.1. The a priori estimates and the regularity result then allow us to perform a fixed point argument which yields local existence of solutions of (1.6) in $G_m((t_0, t_0 + \tau), G)$ for a small time step τ if the data is sufficiently regular and compatible. By standard techniques we extend this local solution to a maximal one. The fixed point argument also implies a first blow-up condition in the $H^m(G)$-norm. In Section 7.3 we improve this blow-up condition by characterizing the finite existence time by blowup in the Lipschitz norm. The proof requires a refined analysis of solutions of the nonlinear problem. To that purpose, we have to localize the solution of (1.6) again to the half-space. There we apply Moser-type inequalities in order to show that the (localized) solution is bounded in the $H^m(\mathbb{R}^3_+)$-norm if it is bounded in the $W^{1,\infty}(\mathbb{R}^3_+)$ norm. Finally, we address the continuous dependance of the solution on the data. The main difficulty here is that the quasilinear nature of (1.10) implicates a loss of derivatives when one estimates the difference of two solutions. The proof thus relies on a tricky splitting of the problem. In one of the arising subproblems we can again apply a regularization technique which compensates for this loss of regularity. In the second subproblem we exploit the structure of the variable coefficients Maxwell operator and ideas from the proof of the a priori estimates in normal direction in order to control all arising error terms.

2
Preliminaries

In Chapters 3 to 7 we will need various tools, whose proofs are quite technical and which would disturb the line of argument in these sections. We therefore present them here.

In the first part we introduce the concept of trace which is used in the following. It is based on the trace theorem for H_{div}. We then show that the trace operator commutes with differentiation in tangential direction, integration in time, and convolution in tangential spatial directions.

The second part of this section is concerned with the regularity of the coefficients of the initial boundary value problem (1.9). We introduce suitable function spaces and prove approximation results for them. At the end, we further show various product estimates adapted to the products between coefficient and solution.

Throughout let $t_0, T \in \mathbb{R}$, $t_0 < T$, $J = (t_0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Then $\partial\Omega = ((t_0,T) \times \mathbb{R}^2 \times \{0\}) \cup (\{t_0\} \times \overline{\mathbb{R}^3_+}) \cup (\{T\} \times \overline{\mathbb{R}^3_+})$. Set $\Gamma = (t_0,T) \times \mathbb{R}^2 \times \{0\} \subseteq \partial\Omega$. We will often identify Γ with the chart $(t_0,T) \times \mathbb{R}^2$.

2.1 The trace operator

In the following it will be useful to approximate a function v by smooth ones in such a way that certain derivatives of the approximating functions also approximate the corresponding derivatives of v.

Lemma 2.1. *Let $k \in \mathbb{N}$. Let $1 \leq p, q < \infty$.*

(i) Let Λ_i be a linear differential operator on $\mathbb{R}^3$ with constant coefficients given by

$$\Lambda_i = \sum_{l=1}^{3} c_{i,l} \partial_l^{k_{i,l}} + d_i,$$

with $d_i, c_{i,l} \in \mathbb{R}$, $k_{i,l} \in \mathbb{N}$ for $l \in \{1,2,3\}$ and $i \in \{1,\ldots,k\}$. We set $\Lambda v = \sum_{i=1}^k \Lambda_i v_i$ for every $v \in \mathcal{D}'(\mathbb{R}^3_+)^k$. We further define the space

$$H(\mathbb{R}^3_+, \Lambda, p, q) = \{v \in L^p(\mathbb{R}^3_+)^k \colon \Lambda v \in L^q(\mathbb{R}^3_+)\}.$$

Then there exists a family of linear operators

$$T_\varepsilon \colon L^1_{\mathrm{loc}}(\mathbb{R}^3_+) \to C^\infty(\mathbb{R}^3)^k$$

with $T_\varepsilon v \in H(\mathbb{R}^3_+, \Lambda, p, q)$, $T_\varepsilon v \to v$ in $L^p(\mathbb{R}^3_+)$ and $\Lambda T_\varepsilon v \to \Lambda v$ in $L^q(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ for all $v \in H(\mathbb{R}^3_+, \Lambda, p, q)$ and Λ as above. Moreover, $\Lambda T_\varepsilon v = T_\varepsilon \Lambda v$ on $\mathbb{R}^3_+$ for all $\varepsilon > 0$, $v \in H(\mathbb{R}^3_+, \Lambda, p, q)$, and Λ as above.

(ii) Let Λ_i^t *be a first order linear differential operator on* $\mathbb{R}^4$ *with constant coefficients given by*

$$\Lambda_i^t = \sum_{l=0}^{3} c_{i,l}\partial_l + d_i,$$

with $d_i, c_{i,l} \in \mathbb{R}$ *for* $l \in \{0,\dots,3\}$ *and* $i \in \{1,\dots,k\}$. *Set* $\Lambda^t v = \sum_{i=1}^k \Lambda_i^t v_i$ *for all* $v \in \mathcal{D}'(\Omega)^k$ *and*

$$H(\Omega,\Lambda^t,p) = \{v \in (L^p(\Omega))^k : \Lambda^t v \in L^p(\Omega)\}.$$

Then there is a family of linear operators

$$T_\varepsilon^t : L^1_{\mathrm{loc}}(\Omega) \to C^\infty(\mathbb{R}^4)^k$$

with $T_\varepsilon^t v \in H(\Omega,\Lambda^t,p)$, $T_\varepsilon^t v \to v$ *in* $L^p(\Omega)$ *and* $\Lambda^t T_\varepsilon^t v \to \Lambda^t v$ *in* $L^p(\Omega)$ *as* $\varepsilon \to 0$ *for all* $v \in H(\Omega,\Lambda^t,p)$ *and* Λ^t *as above. If all the operators* Λ_i^t *do not contain a time derivative, i.e.,* $c_{i,0} = 0$ *for all* $i \in \{1,\dots,k\}$, *we also have* $\Lambda^t T_\varepsilon^t v = T_\varepsilon^t \Lambda^t v$ *for all* $\varepsilon \in (0, \frac{1}{9}T)$, $v \in H(\Omega,\Lambda^t,p)$, *and* Λ^t *as above.*

Proof. (i) Let $Z : L^1_{loc}(\mathbb{R}^3_+) \to L^1_{loc}(\mathbb{R}^3)$ denote the zero-extension. We further define $\tau_\varepsilon : L^1_{loc}(\mathbb{R}^3_+) \to L^1_{loc}(\mathbb{R}^2 \times (-\varepsilon,\infty))$ by

$$(\tau_\varepsilon v)(x_1,x_2,x_3) = v(x_1,x_2,x_3+\varepsilon) \qquad \text{for almost all } x \in \mathbb{R}^3_+ \tag{2.1}$$

for each $\varepsilon \in \mathbb{R}$. With a slight abuse of notation we also write τ_ε for the translation operator on $L^1_{loc}(\mathbb{R}^3)$ defined by formula (2.1) for each $\varepsilon \in \mathbb{R}$. Let ρ be the kernel of a standard mollifier over $\mathbb{R}^3$. As usual we set $\rho_\varepsilon = \varepsilon^{-3}\rho(\varepsilon^{-1}\cdot)$ for all $\varepsilon > 0$ and $\tilde\rho(x) = \rho(-x)$ for all $x \in \mathbb{R}^3$. We then define

$$T_\varepsilon : (L^1_{loc}(\mathbb{R}^3_+))^k \to C^\infty(\mathbb{R}^3), \quad v \mapsto (\rho_\varepsilon * (\tau_{2\varepsilon} Z v_1), \dots, \rho_\varepsilon * (\tau_{2\varepsilon} Z v_k))$$

for all $\varepsilon > 0$. Clearly, $T_\varepsilon v \in (C^\infty(\mathbb{R}^3) \cap L^p(\mathbb{R}^3))^k$ for all $v \in (L^p(\mathbb{R}^3_+))^k$. As $\tau_{2\varepsilon}$ is strongly continuous on $L^p(\mathbb{R}^3)$, we further deduce

$$\begin{aligned}\|T_\varepsilon v - v\|_{L^p(\mathbb{R}^3_+)} &\le \|\rho_\varepsilon * (\tau_{2\varepsilon} Z v) - Zv\|_{L^p(\mathbb{R}^3)} \\ &\le \|\rho_\varepsilon * (\tau_{2\varepsilon} Z v - Zv)\|_{L^p(\mathbb{R}^3)} + \|\rho_\varepsilon * Zv - Zv\|_{L^p(\mathbb{R}^3)} \\ &\le \|\tau_{2\varepsilon} Z v - Zv\|_{L^p(\mathbb{R}^3)} + \|\rho_\varepsilon * Zv - Zv\|_{L^p(\mathbb{R}^3)} \longrightarrow 0 \end{aligned} \tag{2.2}$$

as $\varepsilon \to 0$.

Fix a differential operator Λ and take $v \in H(\mathbb{R}^3_+, \Lambda, p, q)$. By assumption, there is a function $v_\Lambda \in L^q(\mathbb{R}^3_+)$ with $v_\Lambda = \Lambda v$ on $\mathbb{R}^3_+$, i.e.,

$$\langle \Lambda v, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+)\times\mathcal{D}(\mathbb{R}^3_+)} = \sum_{i=1}^k \int_{\mathbb{R}^3_+} v_i \Lambda_i^* \varphi dx = \int_{\mathbb{R}^3_+} v_\Lambda \varphi dx = \langle v_\Lambda, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+)\times\mathcal{D}(\mathbb{R}^3_+)}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^3_+)$. Let $\varphi \in C_c^\infty(\mathbb{R}^3_+)$. Since Λ is a differential operator with constant coefficients, the same is true for its adjoint Λ^*. We then obtain

$$\langle \Lambda T_\varepsilon v, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+)\times\mathcal{D}(\mathbb{R}^3_+)} = \sum_{i=1}^k \langle \rho_\varepsilon * (\tau_{2\varepsilon} Z v_i), \Lambda_i^* Z\varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}$$

$$= \sum_{i=1}^k \langle Zv_i, \tau_{-2\varepsilon}(\tilde\rho_\varepsilon * \Lambda_i^* Z\varphi)\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = \sum_{i=1}^k \int_{\mathbb{R}^3_+} v_i \Lambda_i^* (\tau_{-2\varepsilon}(\tilde\rho_\varepsilon * Z\varphi)) dx. \tag{2.3}$$

Since $\operatorname{supp}\varphi \subseteq \mathbb{R}^3_+$, the support of $\tilde\rho_\varepsilon * \varphi$ is contained in $\mathbb{R}^2 \times (-\varepsilon,\infty)$. Hence, $\operatorname{supp}\tau_{-2\varepsilon}(\tilde\rho_\varepsilon * \varphi) \subseteq \mathbb{R}^3_+$, i.e., $\tau_{-2\varepsilon}(\tilde\rho_\varepsilon * \varphi)$ belongs to $C_c^\infty(\mathbb{R}^3_+)$. We thus deduce

$$\langle \Lambda T_\varepsilon v, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+)\times\mathcal{D}(\mathbb{R}^3_+)} = \sum_{i=1}^k \int_{\mathbb{R}^3_+} v_i \Lambda_i^* (\tau_{-2\varepsilon}(\tilde\rho_\varepsilon * Z\varphi)) dx$$

$$= \int_{\mathbb{R}^3_+} \Lambda v(\tau_{-2\varepsilon}(\tilde{\rho}_\varepsilon * Z\varphi))dx = \langle \rho_\varepsilon * (\tau_{2\varepsilon} Z\Lambda v), Z\varphi\rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)}$$
$$= \langle T_\varepsilon \Lambda v, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+)\times\mathcal{D}(\mathbb{R}^3_+)}, \tag{2.4}$$

i.e., $\Lambda T_\varepsilon v = T_\varepsilon \Lambda v$ on $\mathbb{R}^3_+$. The convergence of $T_\varepsilon \Lambda v$ to Λv in $L^q(\mathbb{R}^3_+)$ now follows as in (2.2).

(ii) Fix a differential operator Λ^t and take $v \in H(\Omega, \Lambda^t, p)$. Note that this particularly implies the existence of a function $v_{\Lambda^t} \in L^p(\Omega)$ with

$$\langle \Lambda^t v, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \sum_{i=1}^k \int_\Omega v_i \Lambda_i^{t^*} \varphi d(t,x) = \int_\Omega v_{\Lambda^t}\varphi dx = \langle v_{\Lambda^t}, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$$

for all $\varphi \in C_c^\infty(\Omega)$.

Next take two functions $\theta_1, \theta_2 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\theta_1 \subseteq [-1, \frac{2}{3}T]$ and $\operatorname{supp}\theta_2 \subseteq [\frac{1}{3}T, T+1]$ and $\theta_1 + \theta_2 = 1$ on $[0, T]$. We further define the sets

$$\Omega_+ = \{(t,x) \in \mathbb{R}^4 \colon t > 0,\, x_3 > 0\}$$
$$\Omega_- = \{(t,x) \in \mathbb{R}^4 \colon t < T,\, x_3 > 0\}.$$

Let Z denote the zero extension from Ω to $\mathbb{R}^4$. We first show that $\Lambda^t(\theta_1 Zv)$ belongs to $L^p(\Omega_+)$, where $(\theta_1 Zv)(t,x) = \theta_1(t)(Zv)(t,x)$ for all $(t,x) \in \mathbb{R}^4$. To that purpose, let $\varphi \in C_c^\infty(\Omega_+)$. Exploiting that Λ^t is a first order differential operator, we infer

$$\langle \Lambda^t(\theta_1 Zv), \varphi\rangle_{\mathcal{D}'(\Omega_+)\times\mathcal{D}(\Omega_+)} = \sum_{i=1}^k \langle Zv_i, \theta_1 \Lambda_i^{t^*}\varphi\rangle_{\mathcal{D}'(\Omega_+)\times\mathcal{D}(\Omega_+)}$$
$$= \sum_{i=1}^k \int_\Omega v_i(\Lambda_i^{t^*}(\theta_1\varphi) + c_{i,0}(\partial_t\theta_1)\varphi)d(t,x).$$

Since $\theta_1\varphi$ belongs to $C_c^\infty(\Omega)$, we obtain

$$\langle \Lambda^t(\theta_1 Zv), \varphi\rangle_{\mathcal{D}'(\Omega_+)\times\mathcal{D}(\Omega_+)} = \int_\Omega v_{\Lambda^t}\theta_1\varphi d(t,x) + \sum_{i=1}^k c_{i,0}\int_\Omega v_i\partial_t\theta_1\varphi d(t,x)$$
$$= \int_{\Omega_+} \Big(\theta_1 Zv_{\Lambda^t} + \sum_{i=1}^k c_{i,0}\partial_t\theta_1 Zv_i\Big)\varphi d(t,x) = \Big\langle \theta_1 Zv_{\Lambda^t} + \sum_{i=1}^k c_{i,0}\partial_t\theta_1 Zv_i, \varphi\Big\rangle_{\mathcal{D}'(\Omega_+)\times\mathcal{D}(\Omega_+)}.$$

We conclude that $\Lambda^t(\theta_1 Zv) = \theta_1 Zv_{\Lambda^t} + \sum_{i=1}^k c_{i,0}\partial_t\theta_1 Zv_i$ on Ω_+. In particular, $\Lambda^t(\theta_1 Zv) \in L^p(\Omega_+)$.

Analogously, one derives that $\Lambda^t(\theta_2 Zv) = \theta_2 Zv_{\Lambda^t} + \sum_{i=1}^k c_{i,0}\partial_t\theta_2 Zv_i$ on Ω_- and $\Lambda^t(\theta_2 Zv) \in L^p(\Omega_-)$.

We next define the translation operators

$$\tau_{1,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4) \to L^1_{loc}(\mathbb{R}^4), \quad (\tau_{1,\varepsilon} w)(t,x) = w(t+\varepsilon, x_1, x_2, x_3+\varepsilon),$$
$$\tau_{2,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4) \to L^1_{loc}(\mathbb{R}^4), \quad (\tau_{2,\varepsilon} w)(t,x) = w(t-\varepsilon, x_1, x_2, x_3+\varepsilon),$$

and the regularization operators

$$T_{1,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4)^k \to C^\infty(\mathbb{R}^4)^k, \quad w \mapsto \rho_\varepsilon * (\tau_{1,2\varepsilon}(\theta_1 w)),$$
$$T_{2,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4)^k \to C^\infty(\mathbb{R}^4)^k, \quad w \mapsto \rho_\varepsilon * (\tau_{2,2\varepsilon}(\theta_2 w)),$$

for all $\varepsilon > 0$. Finally, we set

$$T_\varepsilon^t = T_{1,\varepsilon} + T_{2,\varepsilon}.$$

Clearly, T_ε^t maps $L^p(\Omega)^k$ into $L^p(\mathbb{R}^4)^k$. As in (2.2) it follows that $T_{1,\varepsilon}w \to \theta_1 w$ in $L^p(\Omega_+)$ and $T_{2,\varepsilon}w \to \theta_2 w$ in $L^p(\Omega_-)$ as $\varepsilon \to 0$ for all $w \in L^p(\Omega)^k$. Consequently, $T_\varepsilon^t w$ converges to $(\theta_1 + \theta_2)w = w$ in $L^p(\Omega)$ as $\varepsilon \to 0$, where we used that $\theta_1 + \theta_2 = 1$ on Ω.

The same arguments as in (2.3) and (2.4) yield

$$\Lambda^t T_{1,\varepsilon} v = \rho_\varepsilon * (\tau_{1,2\varepsilon} \Lambda^t(\theta_1 Z v)) = T_{1,\varepsilon} v_{\Lambda^t} + \rho_\varepsilon * \Big(\tau_{1,2\varepsilon} \sum_{i=1}^k c_{i,0} \partial_t \theta_1 Z v_i\Big) \quad \text{on } \Omega_+, \tag{2.5}$$

$$\Lambda^t T_{2,\varepsilon} v = \rho_\varepsilon * (\tau_{2,2\varepsilon} \Lambda^t(\theta_2 Z v)) = T_{2,\varepsilon} v_{\Lambda^t} + \rho_\varepsilon * \Big(\tau_{2,2\varepsilon} \sum_{i=1}^k c_{i,0} \partial_t \theta_2 Z v_i\Big) \quad \text{on } \Omega_-, \tag{2.6}$$

for all $\varepsilon \in (0, \frac{1}{9}T)$. Since $\Lambda^t v = v_{\Lambda^t}$ belongs to $L^p(\Omega)$ we infer as in (2.2) again

$$\Lambda^t T_{1,\varepsilon} v \longrightarrow \theta_1 v_{\Lambda^t} + \sum_{i=1}^k c_{i,0} \partial_t \theta_1 v_i,$$
$$\Lambda^t T_{2,\varepsilon} v \longrightarrow \theta_2 v_{\Lambda^t} + \sum_{i=1}^k c_{i,0} \partial_t \theta_2 v_i,$$

in $L^p(\Omega)$ as $\varepsilon \to 0$. Employing that $\theta_1 + \theta_2 = 1$ on Ω and thus $\partial_t \theta_1 + \partial_t \theta_2 = 0$ on Ω, we obtain

$$\Lambda^t T_\varepsilon v = \Lambda^t T_{1,\varepsilon} v + \Lambda^t T_{2,\varepsilon} v \longrightarrow \theta_1 \Lambda^t v + \theta_2 \Lambda^t v = \Lambda^t v$$

in $L^p(\Omega)$ as $\varepsilon \to 0$. Finally, if $c_{i,0} = 0$ for all $i \in \{1, \dots, k\}$, we have

$$\Lambda^t T_\varepsilon v = \Lambda^t T_{1,\varepsilon} v + \Lambda^t T_{2,\varepsilon} v = T_{1,\varepsilon} \Lambda^t v + T_{2,\varepsilon} \Lambda^t v = T_\varepsilon \Lambda^t v$$

on Ω for all $\varepsilon \in (0, \frac{1}{9}T)$. □

To define the trace properly, we introduce the following spaces.

Definition 2.2. *We define*

$$H(\operatorname{div}_t, \Omega) = \Big\{(q_0, \dots, q_3) \in L^2(\Omega)^4 \colon \operatorname{div}_t q = \sum_{j=0}^3 \partial_j q_j \in L^2(\Omega)\Big\},$$
$$H(\operatorname{div}_t, \Omega)_3 = \{v \in L^2(\Omega) \colon \exists\, q \in H(\operatorname{div}_t, \Omega) \text{ with } q_3 = v\},$$

and we equip these spaces with the norms

$$\|v\|_{H(\operatorname{div}_t,\Omega)} = \Big(\|v\|_{L^2(\Omega)}^2 + \|\operatorname{div}_t v\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}},$$
$$\|v\|_{H(\operatorname{div}_t,\Omega)_3} = \inf_{q \in V} \|q\|_{H(\operatorname{div}_t,\Omega)},$$

where V contains all functions q from $H(\operatorname{div}_t, \Omega)$ with $q_3 = v$.

Since the space $H(\operatorname{div}_t, \Omega)_3$ is not standard, we decided to give detailed proofs of two main properties of this space. We will show that $H(\operatorname{div}_t, \Omega)_3$ is complete and that $C_c^\infty(\overline{\Omega})$ is dense in $H(\operatorname{div}_t, \Omega)_3$.

Lemma 2.3. *The space $H(\operatorname{div}_t, \Omega)_3$ is complete and $C_c^\infty(\overline{\Omega})$ is dense in $H(\operatorname{div}_t, \Omega)_3$.*

Proof. It is easily seen that

$$Y = \{q \in H(\operatorname{div}_t, \Omega) \colon q_3 = 0\}$$

is a closed subspace of $H(\mathrm{div}_t,\Omega)$. The quotient space $H(\mathrm{div}_t,\Omega)/Y$ is thus complete. Let $Q\colon H(\mathrm{div}_t,\Omega)\to H(\mathrm{div}_t,\Omega)/Y$, $u\mapsto u+Y$ be the quotient map. We denote Qu also by $\hat{u}$ for every $u\in H(\mathrm{div}_t,\Omega)$.

If $\hat{u}=\hat{v}$ for two functions $u,v\in H(\mathrm{div}_t,\Omega)$, then $u-v$ belongs to Y implying $u_3=v_3$. The map

$$J\colon H(\mathrm{div}_t,\Omega)/Y\to H(\mathrm{div}_t,\Omega)_3;\quad \hat{u}\mapsto u_3,$$

is thus well-defined. Clearly, J is linear and bijective. Moreover,

$$\begin{aligned}\|J\hat{u}\|_{H(\mathrm{div}_t,\Omega)_3}&=\|u_3\|_{H(\mathrm{div}_t,\Omega)_3}=\inf_{y\in Y}\|u-y\|_{H(\mathrm{div}_t,\Omega)}=\|\hat{u}\|_{H(\mathrm{div}_t,\Omega)/Y}\\&\le\|u\|_{H(\mathrm{div}_t,\Omega)}\end{aligned}\tag{2.7}$$

for all $\hat{u}\in H(\mathrm{div}_t,\Omega)_3$, where we used that for all $q\in H(\mathrm{div}_t,\Omega)$ with $q_3=u_3$ we have $q-u\in Y$. This means that J is an isometric isomorphism from $H(\mathrm{div}_t,\Omega)/Y$ to $H(\mathrm{div}_t,\Omega)_3$. We conclude that $H(\mathrm{div}_t,\Omega)_3$ is a Banach space.

To show the density result, let $v\in H(\mathrm{div}_t,\Omega)_3$. Take a function u in $H(\mathrm{div}_t,\Omega)$ with $u_3=v$. Lemma 2.1 (ii) gives a sequence $(\varphi_n)_n$ in $C_c^\infty(\overline{\Omega})^4$ with $\varphi_n\to u$ in $H(\mathrm{div}_t,\Omega)$ as $n\to\infty$. From (2.7) we obtain that

$$\|v-\varphi_{n,3}\|_{H(\mathrm{div}_t,\Omega)_3}=\|u_3-\varphi_{n,3}\|_{H(\mathrm{div}_t,\Omega)_3}\le\|u-\varphi_n\|_{H(\mathrm{div}_t,\Omega)}\longrightarrow 0$$

as $n\to\infty$. This shows that $C_c^\infty(\overline{\Omega})$ is dense in $H(\mathrm{div}_t,\Omega)_3$. □

We next note that $H^s(\partial\Omega)$ is well-defined for all $s\in[0,1]$ by Definition 13.5.7 in [TW09] since $\partial\Omega$ is a Lipschitz boundary which can be covered by finitely many charts. An inspection of the proof of Theorem 1.4.2.4 in [Gri85] yields the following result.

Lemma 2.4. *For $s\in(0,\frac{1}{2}]$ the space $C_c^\infty(\Gamma)$ is dense in $H^s(\Gamma)$. In particular, the zero extension of a function from $H^{\frac{1}{2}}(\Gamma)$ to $\partial\Omega$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$.*

We note that Lemma 2.4 further allows us to identify $H^{\frac{1}{2}}(\Gamma)$ with a closed subspace of $H^{\frac{1}{2}}(\partial\Omega)$ via the zero extension.

With these preparations at hand we can prove that functions from $H(\mathrm{div}_t,\Omega)_3$ have a trace on Γ. The idea is to use the trace theorem for $H(\mathrm{div}_t,\Omega)$-functions and to exploit the special structure of $\partial\Omega$.

Lemma 2.5. *There exists a unique linear and continuous trace operator*

$$\mathrm{Tr}\colon H(\mathrm{div}_t,\Omega)_3\to H^{-\frac{1}{2}}(\Gamma),$$

which extends the mapping

$$C_c^\infty(\overline{\Omega})\to C_c^\infty(\overline{\Gamma}),\quad \varphi\mapsto\varphi_{|\Gamma}.$$

Moreover, the Green's formula

$$\begin{aligned}\langle\psi_{|\Gamma},\mathrm{Tr}\,q_3\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}&=-\int_\Omega\mathrm{div}_t(\psi q)\,d(t,x)\\&=-\int_\Omega\nabla_t\psi\cdot q\,d(t,x)-\int_\Omega\psi\,\mathrm{div}_t\,q\,d(t,x)\end{aligned}$$

is valid for all $q\in H(\mathrm{div}_t,\Omega)$ and $\psi\in C_c^\infty(\overline{\Omega})$ with $\mathrm{supp}\,\psi_{|\partial\Omega}\Subset\Gamma$.

Proof. Let $\varphi\in C_c^\infty(\overline{\Omega})$. Then $\varphi_{|\Gamma}=\varphi(\cdot,0)$ belongs to $C_c^\infty(\overline{\Gamma})$. Let ν be the unit outer normal of $\partial\Omega$. Note that $\nu_{|\Gamma}=-e_3$. Let $q\in C^\infty(\overline{\Omega})^4\cap H(\mathrm{div}_t,\Omega)$ with $q_3=\varphi$. We obtain $\varphi_{|\Gamma}=-\nu\cdot q_{|\Gamma}$. Since $H^{1/2}(\Gamma)$ is a subspace of $H^{1/2}(\partial\Omega)$, we infer that

$$\|\varphi_{|\Gamma}\|_{H^{-1/2}(\Gamma)}=\|q_{3|\Gamma}\|_{H^{-1/2}(\Gamma)}=\|\nu\cdot q_{|\Gamma}\|_{H^{-1/2}(\Gamma)}\le\|\nu\cdot q_{|\partial\Omega}\|_{H^{-1/2}(\partial\Omega)}$$

$$\leq C\|q\|_{H(\mathrm{div}_t,\Omega)}, \tag{2.8}$$

where we used the standard trace theorem for $H(\mathrm{div}_t,\Omega)$, see e.g. Theorem 1 on page 204 in [DL90b] (and also Theorem 1 on page 279 of [DL90b]).

Next take $q \in H(\mathrm{div}_t,\Omega)$ with $q_3 = \varphi$ and set $\tilde{\varphi} = (0,0,0,\varphi)$. Let Y be as defined in the proof of Lemma 2.3. By this proof there is a function $y \in Y$ with $q - \tilde{\varphi} = y$. Since $y \in H(\mathrm{div}_t,\Omega)$, Lemma 2.1 and the construction of the operators T_ε therein show that there is a family $\{y_\varepsilon\}_{\varepsilon>0} \subseteq Y \cap (C^\infty(\overline{\Omega}))^4$ such that $y_\varepsilon \to y$ in $H(\mathrm{div}_t,\Omega)$ as $\varepsilon \to 0$. We thus deduce from (2.8) that

$$\|\varphi_{|\Gamma}\|_{H^{-1/2}(\Gamma)} \leq C\|\tilde{\varphi} + y_\varepsilon\|_{H(\mathrm{div}_t,\Omega)}$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain

$$\|\varphi_{|\Gamma}\|_{H^{-1/2}(\Gamma)} \leq C\|\tilde{\varphi} + y\|_{H(\mathrm{div}_t,\Omega)} = C\|q\|_{H(\mathrm{div}_t,\Omega)}.$$

Since $q \in H(\mathrm{div}_t,\Omega)$ with $q_3 = \varphi$ was arbitrary, we can take the infimum over all such q. This leads to

$$\|\varphi_{|\Gamma}\|_{H^{-1/2}(\Gamma)} \leq C\|\varphi\|_{H(\mathrm{div}_t,\Omega)_3}.$$

We conclude that the restriction of $C_c^\infty(\overline{\Omega})$-functions to Γ is continuous from the space $H(\mathrm{div}_t,\Omega)_3$ to $H^{-1/2}(\Gamma)$. As $C_c^\infty(\overline{\Omega})$ is dense in $H(\mathrm{div}_t,\Omega)_3$ by Lemma 2.3, there exists a unique continuous extension Tr.

Take $\psi \in C_c^\infty(\overline{\Omega})$ with $\operatorname{supp}\psi_{|\partial\Omega} \Subset \Gamma$. Let $\varphi \in C_c^\infty(\overline{\Omega})^4$. Since $\psi_{|\partial\Omega\setminus\Gamma} = 0$, Gauß' Theorem gives

$$\begin{aligned}\langle \psi_{|\Gamma}, \operatorname{Tr}\varphi_3\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} &= \int_\Gamma \psi\,\varphi_3\,d\sigma = -\int_{\partial\Omega}\psi\,\varphi\cdot\nu\,d\sigma = -\int_\Omega \mathrm{div}_t(\psi\,\varphi)\,d(t,x)\\ &= -\int_\Omega \nabla_t\psi\cdot\varphi\,d(t,x) - \int_\Omega \psi\,\mathrm{div}_t\,\varphi\,d(t,x).\end{aligned}$$

For $q \in H(\mathrm{div}_t,\Omega)$ we take a sequence $(\varphi_n)_n$ in $C_c^\infty(\overline{\Omega})^4$ converging to q in $H(\mathrm{div}_t,\Omega)$. Then

$$\int_\Omega \nabla_t\psi\cdot\varphi_n\,d(t,x) + \int_\Omega \psi\,\mathrm{div}_t\,\varphi_n\,d(t,x) \longrightarrow \int_\Omega \nabla_t\psi\cdot q\,d(t,x) + \int_\Omega \psi\,\mathrm{div}_t\,q\,d(t,x)$$

as $n \to \infty$. The continuity of the coordinate map $P_3\colon H(\mathrm{div}_t,\Omega) \to H(\mathrm{div}_t,\Omega)_3$, $q \mapsto q_3$ further shows that $\varphi_{n,3}$ tends to q_3 in $H(\mathrm{div}_t,\Omega)_3$ as $n \to \infty$. Consequently, $\operatorname{Tr}\varphi_{n,3}$ converges to $\operatorname{Tr} q_3$ in $H^{-1/2}(\Gamma)$ as $n \to \infty$, so that we arrive at

$$\langle \psi_{|\Gamma}, \operatorname{Tr} q_3\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = -\int_\Omega \nabla_t\psi\cdot q\,d(t,x) - \int_\Omega \psi\,\mathrm{div}_t\,q\,d(t,x). \qquad \square$$

For smooth functions, the order of taking the trace and a derivative in a tangential direction does not matter. The following corollary shows that this result extends in a certain sense to $H(\mathrm{div}_t,\Omega)_3$.

Corollary 2.6. *Let $v \in H(\mathrm{div}_t,\Omega)_3$. Assume that also $\partial_j v$ belongs to $H(\mathrm{div}_t,\Omega)_3$ for an index $j \in \{0,1,2\}$, where $\partial_0 = \partial_t$. Then the distributional derivative of* $\operatorname{Tr} v$ *in direction e_j belongs to $H^{-1/2}(\Gamma)$ with*

$$\partial_j \operatorname{Tr} v = \operatorname{Tr}(\partial_j v).$$

Proof. We first note that in this proof we will identify the C^∞-manifold $\Gamma = (0,T)\times\mathbb{R}^2\times\{0\}$ with the image of its chart $(0,T)\times\mathbb{R}^2$. Due to Lemma 2.4 it is moreover clear that $H^{-1/2}(\Gamma)$ is continuously embedded in $\mathcal{D}'(\Gamma)$ via

$$\langle \rho, u\rangle_{\mathcal{D}(\Gamma)\times\mathcal{D}'(\Gamma)} = \langle \rho, u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

for $\rho \in C_c^\infty(\Gamma) = \mathcal{D}(\Gamma)$ and $u \in H^{-1/2}(\Gamma)$.

Since v belongs to $H(\operatorname{div}_t, \Omega)_3$ there is a function $q \in H(\operatorname{div}_t, \Omega)$ with $q_3 = v$. Analogously, $\partial_j v \in H(\operatorname{div}_t, \Omega)_3$ implies the existence of a function $r \in H(\operatorname{div}_t, \Omega)$ with $r_3 = \partial_j v$. Applying Lemma 2.1, we see that $T_\varepsilon q$ belongs to $(C^\infty(\overline{\Omega}))^4 \cap H(\operatorname{div}_t, \Omega)$ for all $\varepsilon > 0$ and $T_\varepsilon q \to q$ in $H(\operatorname{div}_t, \Omega)$ as $\varepsilon \to 0$. It follows that $T_\varepsilon v$ is contained in $C^\infty(\overline{\Omega}) \cap H(\operatorname{div}_t, \Omega)_3$ and $T_\varepsilon v \to v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$. In the same way we deduce that $T_\varepsilon \partial_j v \to \partial_j v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$.

In the case $j \in \{1, 2\}$, we then infer that

$$\partial_j \operatorname{Tr}(T_\varepsilon v) = \partial_j T_\varepsilon v(\cdot, 0) = \operatorname{Tr}(\partial_j T_\varepsilon v) = \operatorname{Tr}(T_\varepsilon \partial_j v) \tag{2.9}$$

for all $\varepsilon > 0$, where we used that $T_\varepsilon v \in (C^\infty(\overline{\Omega}))^4$ and that T_ε commutes with ∂_j by Lemma 2.1 (ii). Letting $\varepsilon \to 0$, the right-hand side of (2.9) tends to $\operatorname{Tr}(\partial_j v)$ in $H^{-1/2}(\Gamma)$ since $T_\varepsilon(\partial_j v) \to \partial_j v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$ and Tr is continuous on $H(\operatorname{div}_t, \Omega)_3$. On the left-hand side of (2.9), the function $\operatorname{Tr}(T_\varepsilon v)$ converges to $\operatorname{Tr} v$ in $H^{-1/2}(\Gamma)$ since $T_\varepsilon v \to v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$. We conclude that $\partial_j \operatorname{Tr}(T_\varepsilon v)$ tends to $\partial_j \operatorname{Tr} v$ in $\mathcal{D}'(\Gamma)$. The continuous embedding of $H^{-1/2}(\Gamma)$ into $\mathcal{D}'(\Gamma)$ thus leads to $\partial_j \operatorname{Tr} v = \operatorname{Tr}(\partial_j v)$ in $\mathcal{D}'(\Gamma)$ and therefore $\partial_j \operatorname{Tr} v$ belongs to $H^{-1/2}(\Gamma)$ and the previous equality is also valid in this space.

It remains to consider the case $j = 0$. Formulas (2.5) and (2.6) from the proof of Lemma 2.1, for the differential operator ∂_t and $k = 1$ in each component, yield

$$\partial_t T_{j,\varepsilon} q = T_{j,\varepsilon} \partial_t q + \rho_\varepsilon * (\tau_{j,2\varepsilon} \partial_t \theta_j Z q)$$

on Ω_+ respectively Ω_- for $j \in \{1, 2\}$ and all $\varepsilon > 0$. With the same arguments as in the proof of Lemma 2.1 one can now show that $\operatorname{div}_t \rho_\varepsilon * (\tau_{j,2\varepsilon} \partial_t \theta_j Z q)$ belongs to $L^2(\Omega_+)$ respectively $L^2(\Omega_-)$ and that

$$\rho_\varepsilon * (\tau_{1,2\varepsilon} \partial_t \theta_1 Z q) + \rho_\varepsilon * (\tau_{2,2\varepsilon} \partial_t \theta_2 Z q) \longrightarrow 0$$

in $H(\operatorname{div}_t, \Omega)$ as $\varepsilon \to 0$. We conclude that

$$\rho_\varepsilon * (\tau_{1,2\varepsilon} \partial_t \theta_1 Z v) + \rho_\varepsilon * (\tau_{2,2\varepsilon} \partial_t \theta_2 Z v) \longrightarrow 0$$

in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$ and therefore

$$\operatorname{Tr}(\rho_\varepsilon * (\tau_{1,2\varepsilon} \partial_t \theta_1 Z v) + \rho_\varepsilon * (\tau_{2,2\varepsilon} \partial_t \theta_2 Z v)) \longrightarrow 0$$

in $H^{-1/2}(\Gamma)$ as $\varepsilon \to 0$. Analogous to (2.9), we next infer

$$\begin{aligned} \partial_t \operatorname{Tr}(T_\varepsilon v) &= \partial_t T_\varepsilon v(\cdot, 0) = \operatorname{Tr}(\partial_t T_\varepsilon v) \\ &= \operatorname{Tr}(T_\varepsilon \partial_t v) + \operatorname{Tr}(\rho_\varepsilon * (\tau_{1,2\varepsilon} \partial_t \theta_1 Z v) + \rho_\varepsilon * (\tau_{2,2\varepsilon} \partial_t \theta_2 Z v)) \end{aligned}$$

for all $\varepsilon > 0$. The rest of the proof is the same as in the case $j \in \{1, 2\}$. □

In Chapter 4 we examine the regularity of solutions. A crucial tool in this context are mollifiers in spatial tangential variables. To apply them effectively in Chapter 4, we need to be able to commute them with the trace operator.

We start by fixing some notation. Let χ be the kernel of a standard mollifier over $\mathbb{R}^2$, i.e., χ is a nonnegative function in $C_c^\infty(\mathbb{R}^2)$, positive on $B(0,1)$ with integral one. Set $\chi_\varepsilon = \varepsilon^{-2} \chi(\varepsilon^{-1} \cdot)$. The convolution operator over $\mathbb{R}^2$ with kernel χ_ε is given by

$$J_\varepsilon v = \chi_\varepsilon * v \tag{2.10}$$

for all $v \in \mathcal{S}'(\mathbb{R}^2)$. With a slight abuse of notation, we also denote the convolution in spatial tangential variables over $\Omega = J \times \mathbb{R}^3_+$ respectively $J \times \mathbb{R}^2 \cong \Gamma$ by J_ε; i.e.,

$$J_\varepsilon v(t,x) = \chi_\varepsilon *_{\mathrm{ta}} v(t,x) = \int_{\mathbb{R}^2} v(t,(x_1,x_2) - y, x_3) \chi_\varepsilon(y) dy \tag{2.11}$$

for all $(t,x) \in \Omega$, where $v \in L^1_{loc}(\Omega)$, and

$$J_\varepsilon v(t,x') = \chi_\varepsilon *_{\text{ta}} v(t,x') = \int_{\mathbb{R}^2} v(t,x'-y)\chi_\varepsilon(y)dy \tag{2.12}$$

for all $(t,x') \in J \times \mathbb{R}^2$, where $v \in L^1_{loc}(J \times \mathbb{R}^2)$.

At least for sufficiently smooth functions, the convolution in tangential spatial variables does not effect the boundary values. We want to show that this is still true for functions in $H(\text{div}_t, \Omega)_3$, i.e., the operators J_ε and Tr commute on $H(\text{div}_t, \Omega)_3$. To that purpose, we first have to extend the operator J_ε to $H^{-1/2}(\Gamma)$. We therefore introduce the formal adjoint

$$J_\varepsilon^* w = \tilde{\chi}_\varepsilon *_{\text{ta}} w$$

for $w \in L^2(J \times \mathbb{R}^2)$ and $\tilde{\chi}_\varepsilon(x_1,x_2) = \chi_\varepsilon(-x_1,-x_2)$ for all $(x_1,x_2) \in \mathbb{R}^2$. The next lemma shows that J_ε^* maps $H^{1/2}(\Gamma)$ into itself.

Lemma 2.7. *Let $\varepsilon > 0$. Then J_ε^* maps $H^{1/2}(\Gamma)$ continuously into itself.*

Proof. We first note that

$$\mathcal{F}_3(\tilde{\chi}_\varepsilon *_{\text{ta}} f)(\tau,\xi) = (\mathcal{F}_2\tilde{\chi}_\varepsilon)(\xi)(\mathcal{F}_3 f)(\tau,\xi)$$

for all $f \in \mathcal{S}(\mathbb{R}^3)$ and $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^2$, where $\mathcal{F}_2$ denotes the Fourier transform over $\mathbb{R}^2$ and $\mathcal{F}_3$ means the Fourier transform over $\mathbb{R}^3$. By continuity, this equality extends to all $f \in L^2(\mathbb{R}^3)$.

Next take $w \in H^{1/2}(J \times \mathbb{R}^2)$ and denote its zero-extension to $\mathbb{R}^3$ by W. Then W belongs to $H^{1/2}(\mathbb{R}^3)$ by Lemma 2.4 and $\tilde{\chi}_\varepsilon *_{\text{ta}} w = \tilde{\chi}_\varepsilon *_{\text{ta}} W$ on $J \times \mathbb{R}^2$. Consequently,

$$\begin{aligned}
\|J_\varepsilon^* w\|^2_{H^{1/2}(J\times\mathbb{R}^2)} &= \|\tilde{\chi}_\varepsilon *_{\text{ta}} W\|^2_{H^{1/2}(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} (1+|(\tau,\xi)|^2)^{1/2} |\mathcal{F}_3(\tilde{\chi}_\varepsilon *_{\text{ta}} W)(\tau,\xi)|^2 d(\tau,\xi) \\
&= \int_{\mathbb{R}^3} (1+|(\tau,\xi)|^2)^{1/2} |\mathcal{F}_2\tilde{\chi}_\varepsilon(\xi)|^2 |\mathcal{F}_3 W(\tau,\xi)|^2 d(\tau,\xi) \\
&\le \|\mathcal{F}_2\tilde{\chi}_\varepsilon\|^2_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^3} (1+|(\tau,\xi)|^2)^{1/2} |\mathcal{F}_3 W(\tau,\xi)|^2 d(\tau,\xi) \\
&= \|\mathcal{F}_2\tilde{\chi}_\varepsilon\|^2_{L^\infty(\mathbb{R}^2)} \|W\|^2_{H^{1/2}(\mathbb{R}^3)} = \|\mathcal{F}_2\tilde{\chi}_\varepsilon\|^2_{L^\infty(\mathbb{R}^2)} \|w\|^2_{H^{1/2}(J\times\mathbb{R}^2)}.
\end{aligned}$$

We conclude that J_ε^* maps $H^{1/2}(J \times \mathbb{R}^2)$ continuously into itself. □

Lemma 2.7 allows us to extend the operators J_ε to $H^{-1/2}(\Gamma)$ by duality. We set

$$\langle J_\varepsilon v, \psi \rangle = \langle v, J_\varepsilon^* \psi \rangle \tag{2.13}$$

for all $v \in H^{-1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$.

Finally, we can show that the trace operator Tr commutes with J_ε on $H(\text{div}_t, \Omega)_3$.

Lemma 2.8. *Let $v \in H(\text{div}_t, \Omega)_3$ and let $\varepsilon > 0$. Then*

$$\text{Tr}\, J_\varepsilon v = J_\varepsilon \, \text{Tr}\, v. \tag{2.14}$$

Proof. As usual we identify Γ with $J \times \mathbb{R}^2$. Let $\varphi \in C_c^\infty(\overline{\Omega})$. Then

$$\begin{aligned}
(\text{Tr}\, J_\varepsilon \varphi)(t,x_1,x_2) = (J_\varepsilon\varphi)(t,x_1,x_2,0) &= \int_{\mathbb{R}^2} \chi_\varepsilon(y)\varphi(t,x_1-y_1,x_2-y_2,0)dy \\
&= \int_{\mathbb{R}^2} \chi_\varepsilon(y)(\text{Tr}\,\varphi)(t,x_1-y_1,x_2-y_2)dy \\
&= (J_\varepsilon \,\text{Tr}\,\varphi)(t,x_1,x_2)
\end{aligned} \tag{2.15}$$

for all $(t, x_1, x_2) \in J \times \mathbb{R}^2$. For $v \in H(\operatorname{div}_t, \Omega)_3$ we find a sequence $(\varphi_n)_n$ in $C_c^\infty(\overline{\Omega})$ converging to v by Lemma 2.3. Since J_ε is continuous on $H(\operatorname{div}_t, \Omega)_3$ and Tr maps $H(\operatorname{div}_t, \Omega)_3$ continuously into $H^{-1/2}(\Gamma)$, we obtain $\operatorname{Tr} J_\varepsilon \varphi_n \to \operatorname{Tr} J_\varepsilon v$ in $H^{-1/2}(\Gamma)$ as $n \to \infty$. Moreover, $\operatorname{Tr} \varphi_n \to \operatorname{Tr} v$ in this space and as J_ε is continuous on $H^{-1/2}(\Gamma)$ by Lemma 2.7 and (2.13), we obtain $J_\varepsilon \operatorname{Tr} \varphi_n \to J_\varepsilon \operatorname{Tr} v$ in $H^{-1/2}(\Gamma)$ as $n \to \infty$. The assertion is then a consequence of (2.15). □

In Chapter 4 we also have to integrate solutions of (3.2) in time in order to gain regularity in time. We are therefore also interested in the traces of such integrals. But before stating the appropriate result, we again have to introduce some notation.

Definition 2.9. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$ and $J = (t_0, T)$. Let $d \in \mathbb{N}$, and $U \subseteq \mathbb{R}^d$. We define*

$$\begin{aligned} &I_{J\times U} \colon L^2(J \times U) \to L^2(J \times U), \\ &(I_{J\times U}\, v)(t,x) = \int_{t_0}^t v(s,x)ds \quad \textit{for almost all } (t,x) \in J \times U. \end{aligned}$$

The next lemma shows that the above operator is not only well-defined, but also linear and bounded.

Lemma 2.10. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$ and $J = (t_0, T)$. Let $d \in \mathbb{N}$, and $U \subseteq \mathbb{R}^d$. The operator $I_{J\times U}$ is linear and continuous both on $L^2(J \times U)$ and on $H^1(J \times U)$.*

Proof. The operator $I_{J\times U}$ is clearly linear. Minkowski's inequality further shows

$$\begin{aligned} \|I_{J\times U}\, v\|_{L^2(J\times U)} &= \Big\| \int_J \chi_{[t_0,t]}(s) v(s,x) ds \Big\|_{L^2_{t,x}(J\times U)} \\ &\le \int_J \|\chi_{[t_0,t]}(s) v(s,x)\|_{L^2_{t,x}(J\times U)} ds \le (T - t_0) \|v\|_{L^2(J\times U)} \end{aligned} \tag{2.16}$$

for all $v \in L^2(J \times U)$.

By Fubini's theorem, we have $\partial_j I_{J\times U} v = I_{J\times U} \partial_j v$ for all $v \in H^1(J \times U)$ and $j \in \{1, \dots, d\}$. We thus obtain the estimate $\|\partial_j I_{J\times U} v\|_{L^2(J\times U)} \le (T - t_0)\|\partial_j v\|_{L^2(J\times U)}$ from (2.16) for all $v \in H^1(J \times U)$ and $j \in \{1, \dots, d\}$. Moreover, we have $\partial_t I_{J\times U} v = v$ for all $v \in H^1(J \times U)$ so that the assertion follows. □

If we want to commute the trace operator with the integral, we first have to make sense of the integral on $H^{-1/2}(\Gamma)$.

Corollary 2.11. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$ and $J = (t_0, T)$. Identify Γ with $J \times \mathbb{R}^2$. Then the operator I_Γ introduced in Definition 2.9 extends uniquely to a linear continuous operator on $H^{-1/2}(\Gamma)$, which we still denote by I_Γ.*

Proof. By interpolation, see Theorem 1.4.3.5 in [Gri85], we infer from Lemma 2.10 that I_Γ maps $H^{1/2}(\Gamma)$ continuously into itself. Analogously, one obtains that the operator

$$\begin{aligned} &\tilde{I}_\Gamma \colon L^2(J \times \Gamma) \to L^2(J \times \Gamma), \\ &(\tilde{I}_\Gamma v)(t,x) = \int_t^T v(s,x)ds \quad \text{for almost all } (t,x) \in J \times U; \end{aligned}$$

is continuous on $H^{1/2}(\Gamma)$. Since $\tilde{I}_\Gamma$ is the adjoint of I_Γ on $L^2(\Gamma)$, and $H^{1/2}(\Gamma)$ is densely imbedded in $L^2(\Gamma)$, we infer that the extension by duality, i.e.,

$$\langle I_\Gamma v, \psi \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \langle v, \tilde{I}_\Gamma \psi \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$$

for all $v \in H^{-1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$, is unique. □

After these preparations we are now ready to prove that the trace operator and the integral in time commute.

Corollary 2.12. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$ and $J = (t_0, T)$. Let $v \in H(\operatorname{div}_t, \Omega)_3$ with $I_\Omega v \in H(\operatorname{div}_t, \Omega)_3$. Then*

$$\operatorname{Tr} I_\Omega v = I_\Gamma \operatorname{Tr} v,$$

where I_Ω is the integral operator from Definition 2.9 and I_Γ the integral operator from Corollary 2.11.

Proof. As always we identify Γ with its chart $(t_0, T) \times \mathbb{R}^2$. Let T_ε be the operator defined in Lemma 2.1 (i) for all $\varepsilon > 0$ and let Z denote the zero-extension of a function defined on $\mathbb{R}^3_+$ to $\mathbb{R}^3$.

I) Fix $\varepsilon > 0$. We start by showing that div_t and T_ε commute on $H(\operatorname{div}_t, \Omega)$, where the operator T_ε is applied pointwise in time.

So let $w \in H(\operatorname{div}_t, \Omega)$ and $\psi \in C_c^\infty(\Omega)$. Let $\tilde{\rho}(x) = \rho(-x)$ for all $x \in \mathbb{R}^3$. We compute

$$\begin{aligned}
&\int_\Omega T_\varepsilon w \cdot \nabla_t \psi d(t,x) = \int_J \int_{\mathbb{R}^3} \rho_\varepsilon * (\tau_{2\varepsilon} Z w) \cdot \nabla_t Z\psi \, dx dt \\
&= \int_J \int_{\mathbb{R}^3} \tau_{2\varepsilon} Z w \cdot \tilde{\rho}_\varepsilon * (\nabla_t Z\psi) \, dx dt = \int_J \int_{\mathbb{R}^3} \tau_{2\varepsilon} Z w \cdot \nabla_t (\tilde{\rho}_\varepsilon * Z\psi) \, dx dt \\
&= \int_J \int_{\mathbb{R}^3_+} w \cdot \nabla_t \, \tau_{-2\varepsilon} (\tilde{\rho}_\varepsilon * Z\psi) \, dx dt = - \int_J \int_{\mathbb{R}^3_+} \operatorname{div}_t w \, \tau_{-2\varepsilon} (\tilde{\rho}_\varepsilon * Z\psi) \, dx dt \\
&= - \int_\Omega T_\varepsilon (\operatorname{div}_t w) \, \psi \, d(t,x),
\end{aligned}$$

where we used that $\tau_{-2\varepsilon}(\tilde{\rho}_\varepsilon * Z\psi)$ belongs to $C_c^\infty(\Omega)$ in the third line. We conclude that $T_\varepsilon w$ belongs to $H(\operatorname{div}_t, \Omega)$ and $\operatorname{div}_t T_\varepsilon w = T_\varepsilon \operatorname{div}_t w$.

II) Let $q \in H(\operatorname{div}_t, \Omega)$ with $q_3 = v$ and $r \in H(\operatorname{div}_t, \Omega)$ with $r_3 = I_\Omega v$. Step I) and Lemma 2.1 (i) show that $T_\varepsilon q \to q$ and $T_\varepsilon r \to r$ in $H(\operatorname{div}_t, \Omega)$ as $\varepsilon \to 0$. We conclude that $T_\varepsilon v$ and $T_\varepsilon I_\Omega v$ converge to v respectively $I_\Omega v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$. Fubini's theorem further yields

$$\begin{aligned}
T_\varepsilon I_\Omega w(t,x) &= \int_{\mathbb{R}^3} \Big(\int_{t_0}^t (\tau_{2\varepsilon} Z w)(s,y) ds \Big) \rho_\varepsilon(x-y) dy \\
&= \int_{t_0}^t \int_{\mathbb{R}^3} \rho_\varepsilon(x-y) (\tau_{2\varepsilon} Z w)(s,y) dy ds = \int_{t_0}^t T_\varepsilon w(s,x) ds = I_\Omega T_\varepsilon w(t,x)
\end{aligned}$$

for all $w \in L^2(\Omega)$ and $(t,x) \in \Omega$, i.e.,

$$T_\varepsilon I_\Omega v = I_\Omega T_\varepsilon v$$

on Ω for all $\varepsilon > 0$.

Next observe that $T_\varepsilon w$ belongs to $L^2(J, H^1(\mathbb{R}^3_+))$ for all $w \in L^2(\Omega)$. It is moreover easy to check - via the definition of the weak derivative and Fubini's theorem - that $\partial_j I_\Omega w = I_\Omega \partial_j w$ for all $w \in L^2(J, H^1(\mathbb{R}^3_+))$ and $j \in \{1,2,3\}$. Therefore, both $I_\Omega T_\varepsilon q$ and $T_\varepsilon q$ belong to $L^2(J, H^1(\mathbb{R}^3_+))$.

Let tr be the standard trace operator from $H^1(\mathbb{R}^3_+)$ to $H^{1/2}(\partial \mathbb{R}^3_+)$. With a slight abuse of notation we also denote by tr the operator which maps $L^2(J, H^1(\mathbb{R}^3_+))$ to $L^2(J, H^{1/2}(\partial \mathbb{R}^3_+))$ defined by $\operatorname{tr}(u)(t) = \operatorname{tr}(u(t))$ for almost all $t \in J$. If a function u belongs to $L^2(J, H^1(\mathbb{R}^3_+)) \cap H(\operatorname{div}_t, \Omega)_3$, step I) and Lemma 2.1 (i) imply that $T_\varepsilon u$ converges to u both in $L^2(J, H^1(\mathbb{R}^3_+))$ and $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$. As the operators tr and Tr coincide on smooth functions, we obtain that the operators tr and Tr coincide on $L^2(J, H^1(\mathbb{R}^3_+)) \cap H(\operatorname{div}_t, \Omega)_3$. Exploiting that tr is a continuous operator from $H^1(\mathbb{R}^3_+)$ to $H^{1/2}(\partial \mathbb{R}^3_+)$, we thus infer

$$\operatorname{Tr}(T_\varepsilon I_\Omega v) = \operatorname{tr}(T_\varepsilon I_\Omega v) = \operatorname{tr}(I_\Omega T_\varepsilon v) = \operatorname{tr} \Big(\int_{t_0}^t T_\varepsilon v(s) ds \Big) = \int_{t_0}^t \operatorname{tr}(T_\varepsilon v(s)) ds$$

$$= \int_{t_0}^{t} \operatorname{tr}(T_\varepsilon v)(s)ds = I_\Gamma \operatorname{tr}(T_\varepsilon v) = I_\Gamma \operatorname{Tr}(T_\varepsilon v) \tag{2.17}$$

for all $\varepsilon > 0$. The functions $T_\varepsilon I_\Omega v$ converge to $I_\Omega v$ in $H(\operatorname{div}_t, \Omega)_3$ as $\varepsilon \to 0$, so that the continuity of Tr implies

$$\operatorname{Tr}(T_\varepsilon I_\Omega v) \longrightarrow \operatorname{Tr} I_\Omega v$$

in $H^{-1/2}(\Gamma)$. On the other hand, $T_\varepsilon v$ tends to v in $H(\operatorname{div}_t, \Omega)_3$ and thus $\operatorname{Tr}(T_\varepsilon v) \to \operatorname{Tr} v$ in $H^{-1/2}(\Gamma)$. As I_Γ is a continuous operator on this space, we obtain

$$I_\Gamma \operatorname{Tr}(T_\varepsilon v) \longrightarrow I_\Gamma \operatorname{Tr} v$$

as $\varepsilon \to 0$. The identity in (2.17) thus implies the assertion. □

We next state that the trace operator also commutes with the multiplication with C^∞-functions.

Lemma 2.13. *Let $\varphi \in C^\infty(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$. Then φv is an element of $H(\operatorname{div}_t, \Omega)_3$ and*

$$\operatorname{Tr}(\varphi v) = \varphi(\cdot, 0) \operatorname{Tr} v \tag{2.18}$$

for all $v \in H(\operatorname{div}_t, \Omega)_3$.

Proof. The first part of the assertion is a direct consequence of the assumption that φ belongs to $W^{1,\infty}(\overline{\Omega})$. Identity (2.18) clearly holds for $v \in C_c^\infty(\overline{\Omega})$. Since $C_c^\infty(\overline{\Omega})$ is dense in $H(\operatorname{div}_t, \Omega)_3$, the assertion then follows. □

We have now developed the concept of a trace for functions in $H(\operatorname{div}_t, \Omega)_3$. In the remaining part of this section we will show how this leads to a trace operator for weak solutions of certain first order partial differential equations.

Remark 2.14. Let $A_0, \ldots, A_3 \in (W^{1,\infty}(\Omega))^{n\times n}$ be symmetric, $D \in (L^\infty(\Omega))^{n\times n}$, and let $f \in (L^2(\Omega))^n$. Define the differential operator

$$L = A_0 \partial_t + \sum_{j=1}^{3} A_j \partial_j + D. \tag{2.19}$$

By a *weak solution* of $Lu = f$ we mean a function $u \in (L^2(\Omega))^n$ with

$$\int_\Omega f \cdot \varphi \, dx = \int_\Omega u \cdot L^* \varphi \, dx = -\sum_{j=0}^{3} \int_\Omega u \cdot \partial_j (A_j \varphi) \, dx + \int_\Omega u \cdot D^T \varphi \, dx \tag{2.20}$$

for all $\varphi \in (H_0^1(\Omega))^n$. Note that Lu is at first defined in $(H^{-1}(\Omega))^n$ and (2.20) says that

$$\langle Lu, \varphi \rangle_{H^{-1}\times H_0^1} = \int_\Omega u \cdot L^* \varphi \, dx = \int_\Omega f \cdot \varphi \, dx = \langle f, \varphi \rangle_{H^{-1}\times H_0^1}$$

for all $\varphi \in H_0^1(\Omega)$. We conclude that $Lu = f$ in $L^2(\Omega)$, in particular $Lu \in L^2(\Omega)$.

Next observe that Lu can be equivalently written as

$$Lu = \sum_{j=0}^{3} \partial_j (A_j u) - \sum_{j=0}^{3} \partial_j A_j u + Du.$$

Hence,

$$\sum_{j=0}^{3} \partial_j (A_j u) = f + \sum_{j=0}^{3} \partial_j A_j u - Du \in L^2(\Omega). \tag{2.21}$$

For all $k \in \{1, \ldots, n\}$ we define $q_j^k = (A_j u)_k$ for $j \in \{0, \ldots, 3\}$. From (2.21) we deduce that $q^k \in H(\operatorname{div}_t, \Omega)$ for all $k \in \{1, \ldots, n\}$. In particular, $q_3^k = (A_3 u)_k$ belongs to $H(\operatorname{div}_t, \Omega)_3$ for all $k \in \{1, \ldots, n\}$. The trace of each component of $A_3 u$ on Γ is therefore well-defined in the sense of Lemma 2.5. ◇

We use our observations from Remark 2.14 for the following definition.

Definition 2.15. *Take $n \in \mathbb{N}$. Let $A_0, \dots, A_3 \in W^{1,\infty}(\Omega)^{n\times n}$ be symmetric, $D \in L^\infty(\Omega)^{n\times n}$, and let $u, f \in L^2(\Omega)^n$. Assume that u is a weak solution of $Lu = f$, where L is defined by* (2.19)*. We then define the trace of $A_3 u$ on Γ by*

$$\mathrm{Tr}(A_3 u) = (\mathrm{Tr}(A_3 u)_1, \dots, \mathrm{Tr}(A_3 u)_n)$$

in $H^{-1/2}(\Gamma)^n$.

In the following chapters we will study boundary conditions on Γ which are *conservative* in the sense of [Ell12], i.e., there are matrices B and C such that $2A_3 = C^T B + B^T C$ and the boundary condition is given by $Bu = g$ on Γ. This structural assumption yields a matrix M with $B = MA_3$ so that we can make sense of the term Bu on Γ. We will make this notion more precise in Section 3.2. For the moment, we take these considerations as motivation for the following definition.

Definition 2.16. *Let $A_0, A_1, A_2, A_3 \in W^{1,\infty}(\Omega)^{n\times n}$ be symmetric and $D \in L^\infty(\Omega)^{n\times n}$. Assume that there are matrices $B \in W^{1,\infty}(\Omega)^{k\times n}$ and $M \in W^{1,\infty}(\Omega)^{k\times n}$ for natural numbers k and n. Take $u, f \in L^2(\Omega)^6$. Suppose that u is a weak solution of $Lu = f$. We define the trace of Bu on Γ by*

$$\mathrm{Tr}(Bu) = M \cdot \mathrm{Tr}(A_3 u).$$

Remark 2.17. For a solution u in $G_1(\Omega)$ we can define the trace of u itself on Γ. There is even more than one way to do so. Since $u \in G_1(\Omega)$, each component of u belongs to $H(\mathrm{div}_t, \Omega)_3$ and the trace $\mathrm{Tr}\, u_k$ exists in $H^{-1/2}(\Gamma)$ in the sense of Lemma 2.5 for $k \in \{1, \dots, 6\}$. However, the most natural way to define the trace of u is arguably the following. We set $(\mathrm{Tr}_1 u)(t) = \mathrm{tr}(u(t))$ for all $t \in \overline{J}$ and $u \in C(\overline{J}, H^1(\mathbb{R}^3_+))$, where tr is the usual trace operator from $H^1(\mathbb{R}^3_+)$ to $H^{1/2}(\partial\mathbb{R}^3_+)$ applied componentwise. This defines Tr_1 as a mapping from $C(\overline{J}, H^1(\mathbb{R}^3_+))$ to $C(\overline{J}, H^{1/2}(\partial\mathbb{R}^3_+))$. In particular, also the traces

$$\mathrm{Tr}_1(Bu) = M_{|\Gamma} \cdot \mathrm{Tr}_1(A_3 u) = M \cdot A_3 \,\mathrm{Tr}_1 u = B \,\mathrm{Tr}_1 u$$

are defined in a natural way. However, our solution concept in Definition 3.1 will (and has to) consider the trace of Bu in the sense of Definition 2.16. So the natural question arises if these two trace operators coincide on $G_1(\Omega)$.

This is indeed true and can be seen by the following argument. Let $v \in G_1(\Omega)$. Let $\tilde{v}$ be a continuous extension of v in $C(\mathbb{R}, H^1(\mathbb{R}^3_+))$ which is zero outside some compact subset of $\mathbb{R}$. Let φ be the kernel of a standard mollifier over $\mathbb{R}$ and ψ be the kernel of a standard mollifier over $\mathbb{R}^3$. Note that $\varphi\psi$ then forms the kernel of a mollifier over $\mathbb{R}^4$. We define the family $\{v_\varepsilon\}_{\varepsilon>0}$ by

$$v_\varepsilon(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \tilde{v}(t-s, x-y)\varphi_\varepsilon(s)\psi_\varepsilon(y)\,dy\,ds$$

for all $\varepsilon > 0$. Then v_ε belongs to $G_1(\Omega)$ for all $\varepsilon > 0$. Moreover,

$$\begin{aligned} v_\varepsilon(t,x) - \tilde{v}(t,x) &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\tilde{v}(t-s, x-y) - \tilde{v}(t,x))\varphi_\varepsilon(s)\psi_\varepsilon(y)\,dy\,ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\tilde{v}(t-s, x-y) - \tilde{v}(t,x-y))\varphi_\varepsilon(s)\psi_\varepsilon(y)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^3} (\tilde{v}(t, x-y) - \tilde{v}(t,x))\psi_\varepsilon(y)\,dy \end{aligned}$$

for all $(t,x) \in \mathbb{R}^4$ and $\varepsilon > 0$. Standard properties of mollifiers show that the second integral converges to zero in $C(\overline{J}, H^1(\mathbb{R}^3_+))$, where we also exploit that $\overline{J}$ is compact.

Employing Minkowsi's inequality and the translation invariance of Lebesgue measure, we obtain for the first term

$$\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^3} (\tilde{v}(t-s, x-y) - \tilde{v}(t, x-y)) \varphi_\varepsilon(s) \psi_\varepsilon(y) dy ds \Big\|_{H^1_x(\mathbb{R}^3_+)}$$
$$\leq \int_{\mathbb{R}} \|\tilde{v}(t-s) - \tilde{v}(t)\|_{H^1(\mathbb{R}^3_+)} \varphi_\varepsilon(s) ds$$

for all $t \in \mathbb{R}$. Since $\tilde{v}$ is continuous on an open set containing $\overline{J}$, we infer that the term on the right-hand side converges to zero uniformly in $t \in \overline{J}$. We have thus shown that $v_\varepsilon \to v$ in $C(\overline{J}, H^1(\mathbb{R}^3_+))$ as $\varepsilon \to 0$.

We further note that $v_\varepsilon \in C^\infty(\Omega)$ so that the definitions of the trace operators Tr and tr yield

$$(\operatorname{Tr}_1 v_\varepsilon)(t, x_1, x_2) = (\operatorname{tr} v_\varepsilon(t))(x_1, x_2) = v_\varepsilon(t, x_1, x_2, 0) = \operatorname{Tr} v_\varepsilon(t, x_1, x_2)$$

for all $t \in J$, $(x_1, x_2) \in \mathbb{R}^2$, and $\varepsilon > 0$. The continuity of Tr_1 yields that $\operatorname{Tr}_1 v_\varepsilon$ tends to $\operatorname{Tr}_1 v$ in $C(\overline{J}, H^{1/2}(\mathbb{R}^2))$. Since this space is continuously embedded in $H^{-1/2}(\Gamma)$, cf. Remark 2.19, we further infer that $\operatorname{Tr}_1 v_\varepsilon$ converges to $\operatorname{Tr}_1 v$ in $H^{-1/2}(\Gamma)$ as $\varepsilon \to 0$.

On the other hand, $C(\overline{J}, H^1(\mathbb{R}^3_+))$ is continuously embedded in $H(\operatorname{div}_t, \Omega)_3$, implying $v_\varepsilon \to v$ in $H(\operatorname{div}_t, \Omega)_3$ as $n \to \infty$. We therefore obtain that $\operatorname{Tr} v_\varepsilon$ tends to $\operatorname{Tr} v$ in $H^{-1/2}(\Gamma)$. As a result, $\operatorname{Tr}_1 v = \operatorname{Tr} v$ in $H^{-1/2}(\Gamma)$, i.e., the two traces coincide in $G_1(\Omega)$. ◇

In the following corollary we transfer the properties of the trace operator Tr which we have shown in Corollary 2.6, Lemma 2.8, and Corollary 2.12 to $\operatorname{Tr}(Bu)$, when u is a weak solution of $Lu = f$.

Corollary 2.18. *Let $k, n \in \mathbb{N}$ and let $A_0, A_1, A_2, A_3 \in W^{1,\infty}(\Omega)^{n \times n}$ be symmetric and $D \in L^\infty(\Omega)^{n \times n}$. Assume that there are matrices $B \in W^{1,\infty}(\Omega)^{k \times n}$ and $M \in W^{1,\infty}(\Omega)^{k \times n}$ such that $B = MA_3$. Let $u \in (L^2(\Omega))^n$ with $Lu \in L^2(\Omega)^n$.*

(i) Assume additionally that $B, M \in W^{2,\infty}(\Omega)^{k \times n}$, $A_3 \in W^{2,\infty}(\Omega)$ and that also u belongs to $H^1(\Omega)^n$ and $L\partial_j u$ to $L^2(\Omega)^n$ for an index $j \in \{0,1,2\}$. Then the distributional derivative $\partial_j \operatorname{Tr}(Bu)$ exists in $H^{-1/2}(\Gamma)^k$ and

$$\partial_j \operatorname{Tr} Bu = \operatorname{Tr} B \partial_j u + \operatorname{Tr}_1 \partial_j B u.$$

(ii) Let $\{J_\varepsilon\}_{\varepsilon > 0}$ be the mollifier introduced in (2.10) to (2.12). Assume that also $LJ_\varepsilon u$ belongs to $L^2(\Omega)^n$ for a parameter $\varepsilon > 0$. Then $\operatorname{Tr}(BJ_\varepsilon u)$ exists in $H^{-1/2}(\Gamma)^k$ and

$$\operatorname{Tr} BJ_\varepsilon u = \operatorname{Tr}((BJ_\varepsilon - J_\varepsilon B)u) + J_\varepsilon \operatorname{Tr} Bu.$$

(iii) Assume A_3 and M are time-independent and that also $LI_\Omega u$ belongs to $L^2(\Omega)^n$. Then $BI_\Omega u$ has a trace in $H^{-1/2}(\Gamma)^k$ and

$$\operatorname{Tr} I_\Omega Bu = I_\Gamma \operatorname{Tr} Bu.$$

(iv) Let $\varphi \in C_c^\infty(\overline{\Omega})$. Then

$$\operatorname{Tr} B(\varphi u) = \varphi(\cdot, 0) \operatorname{Tr} Bu.$$

Proof. In Remark 2.14 we have seen that the functions q^k defined by $q^k_l = (A_l u)_k$ for $l \in \{0, \ldots, 3\}$ and $k \in \{1, \ldots, 6\}$ belong to $H(\operatorname{div}_t, \Omega)$. In particular, $(A_3 u)_k$ is contained in $H(\operatorname{div}_t, \Omega)_3$ for all $k \in \{1, \ldots, 6\}$.

(i) Applying the same argument to $\partial_j u$, we infer that also $\tilde{q}^k$ defined by $\tilde{q}^k_l = (A_l \partial_j u)_k$ are elements of $H(\operatorname{div}_t, \Omega)$ for all $k \in \{1, \ldots, 6\}$. Since $\partial_j A_3 u$ is contained in

$H^1(\Omega)$, we infer that $\partial_j(A_3u)_k = (A_3\partial_j u)_k + (\partial_j A_3 u)_k$ belongs to $H(\mathrm{div}_t, \Omega)_3$ for all $k \in \{1, \dots, 6\}$. Corollary 2.6 thus yields that the distributional derivative of $\mathrm{Tr}(A_3u)_k$ belongs to $H^{-1/2}(\Gamma)$ and

$$\partial_j \mathrm{Tr}(A_3u)_k = \mathrm{Tr}(A_3\partial_j u + \partial_j A_3 u)_k = \mathrm{Tr}(A_3\partial_j u) + \mathrm{Tr}_1(\partial_j A_3)\,\mathrm{Tr}_1(u),$$

where we also write Tr_1 for the restriction of an $W^{1,\infty}(\Omega)$-function to Γ. Therefore, by Definition 2.15 and since derivatives in tangential derivatives commute with the restriction to the boundary of $W^{2,\infty}(\Omega)$-functions, we obtain

$$\begin{aligned}\partial_j \mathrm{Tr}(Bu) &= \mathrm{Tr}_1(\partial_j M)\,\mathrm{Tr}_1(A_3u) + \mathrm{Tr}_1(M)\,\mathrm{Tr}(A_3\partial_j u) + \mathrm{Tr}_1(M)\,\mathrm{Tr}_1(\partial_j A_3)\,\mathrm{Tr}_1(u)\\ &= \mathrm{Tr}(B\partial_j u) + \mathrm{Tr}_1(\partial_j B)\,\mathrm{Tr}_1(u).\end{aligned}$$

The assertions of (i) now follow.

(iii) As in the proof of part (i) we deduce that $\tilde{q}^k$ defined by $\tilde{q}^k_l = (A_l I_\Omega q)_k$ belongs to $H(\mathrm{div}_t, \Omega)$ for all $k \in \{1, \dots, 6\}$ so that $(A_3 I_\Omega u)_k$ is contained in $H(\mathrm{div}_t, \Omega)_3$ for all $k \in \{1, \dots, 6\}$. Corollary 2.12 therefore implies that $(A_3 I_\Omega u)_k$ has a trace in $H^{-1/2}(\Gamma)$ and

$$\mathrm{Tr}(A_3 I_\Omega u)_k = \mathrm{Tr}\, I_\Omega (A_3u)_k = I_\Gamma\, \mathrm{Tr}(A_3u)_k$$

for all $k \in \{1, \dots, 6\}$. With Definition 2.15 and 2.16, we finally obtain

$$\begin{aligned}\mathrm{Tr}(BI_\Omega u) &= M \cdot \mathrm{Tr}(A_3 I_\Omega u) = M \cdot (\mathrm{Tr}(A_3 I_\Omega u)_k)_{k=1,\dots,6} = M \cdot (I_\Gamma\, \mathrm{Tr}(A_3u)_k)_{k=1,\dots,6}\\ &= M \cdot I_\Gamma\, \mathrm{Tr}(A_3u) = I_\Gamma M \cdot \mathrm{Tr}(A_3u) = I_\Gamma\, \mathrm{Tr}(Bu),\end{aligned}$$

and thus the assertion.

(ii) and (iv) These assertions follows in the same way as the ones in (i) and (iii), using Lemma 2.8 respectively Lemma 2.13. ☐

At the end of this section we show that $H^{1/2}(\Gamma)$ is continuously embedded into $L^2(J, H^{1/2}(\mathbb{R}^2))$. This implies that $L^2(J, H^{-1/2}(\mathbb{R}^2))$ is continuously embedded in $H^{-1/2}(\Gamma)$, showing that the regularity result for A_3u in [Ell12], where u is a solution of a certain initial boundary value problem, is indeed an improvement of what we know from Corollary 2.18.

Remark 2.19. Let $d \in \mathbb{N}$ and $J \subseteq \mathbb{R}$ be an interval. Throughout we denote the isometric isomorphism from $L^p(\mathbb{R}, L^p(\mathbb{R}^d))$ onto $L^p(\mathbb{R}^{d+1})$ by $\mathcal{I}$ for $1 \le p < \infty$.

I) In the following we will need that for an element $f \in L^1(\mathbb{R}, L^1(\mathbb{R}^d))$ we have

$$\Big(\int_{\mathbb{R}} f(t)dt\Big)(x) = \int_{\mathbb{R}} (\mathcal{I}f)(t,x)dt$$

for almost all $x \in \mathbb{R}^d$, where the integral on the left-hand side is an $L^1(\mathbb{R}^d)$-valued Bochner-integral, whereas $(\mathcal{I}f)(\cdot, x)$ belongs to $L^1(\mathbb{R})$ for almost all $x \in \mathbb{R}^d$ by Fubini's theorem. Since we are not aware of a reference in the literature, we give the proof for this identity here.

To that purpose, let f be a simple function on $\mathbb{R}$ with values in $L^1(\mathbb{R}^d)$. This means that we find finitely many disjoint intervals I_j of finite length, $j \in \{1, \dots, m\}$, such that $f = \sum_{j=1}^m \chi_{I_j} g_j$, where $g_j \in L^1(\mathbb{R}^d)$ for $1 \le j \le m$. For this f we have

$$\Big(\int_{\mathbb{R}} f(t)dt\Big)(x) = \Big(\sum_{j=1}^m |I_j|\, g_j\Big)(x) = \sum_{j=1}^m |I_j|\, g_j(x)$$

for almost all $x \in \mathbb{R}^d$, where $|I_j|$ denotes the length of the interval I_j. On the other hand,

$$\int_{\mathbb{R}} (\mathcal{I}f)(t,x)\, dt = \sum_{j=1}^m g_j(x)\, |I_j|$$

for almost all $x \in \mathbb{R}^d$, so that the claim is true for simple functions.

Next take $f \in L^1(\mathbb{R}, L^1(\mathbb{R}^d))$. Let $(f_n)_n$ be a sequence of simple functions converging to f in $L^1(\mathbb{R}, L^1(\mathbb{R}^d))$. The integrals $\int_{\mathbb{R}} f_n(t)\,dt$ thus converge to $\int_{\mathbb{R}} f(t)\,dt$ in $L^1(\mathbb{R}^d)$, and, after excluding a subsequence, we obtain pointwise convergence almost everywhere in $\mathbb{R}^d$. Moreover, $(\mathcal{I}f_n)_n$ converges to $\mathcal{I}f$ as $n \to \infty$ in $L^1(\mathbb{R}^{d+1})$ so that Fubini's theorem gives that

$$\int_{\mathbb{R}} (\mathcal{I}f_n)(t,\cdot)\,dt \longrightarrow \int_{\mathbb{R}} (\mathcal{I}f)(t,\cdot)\,dt$$

in $L^1(\mathbb{R}^d)$. Excluding another subsequence, we obtain that the above convergence holds also pointwise almost everywhere. The claimed identity is therefore valid for almost all $x \in \mathbb{R}^d$.

II) We next show that

$$\mathcal{I}\mathcal{F}_x(\mathcal{F}_t f) = \mathcal{F}_{t,x}(\mathcal{I}f)$$

for all $f \in L^2(\mathbb{R}, L^2(\mathbb{R}^d))$, where $\mathcal{F}_t$ is the $L^2(\mathbb{R}^d)$-valued Fourier transform over $\mathbb{R}$, $\mathcal{F}_x$ is the Fourier transform on $L^2(\mathbb{R}^d)$ (applied pointwise), and $\mathcal{F}_{t,x}$ is the Fourier transform over $\mathbb{R}^{d+1}$.

To show this claim we start once more with a simple function f, i.e., $f = \sum_{j=1}^m \chi_{I_j} g_j$ for functions $g_j \in L^2(\mathbb{R}^d)$, disjoint intervals I_j of finite length, and an index $m \in \mathbb{N}$. For each j we take a sequence $(g_{j,n})_n$ in $C_c^\infty(\mathbb{R}^d)$ with $g_{j,n} \longrightarrow g_j$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. We define the functions f_n by $f_n = \sum_{j=1}^m \chi_{I_j} g_{j,n}$ for all $n \in \mathbb{N}$. Due to step I) we then have

$$\begin{aligned}(\mathcal{F}_t(\chi_{I_j} g_{j,n})(\tau))(x) &= \Big(\int_{\mathbb{R}} e^{-it\cdot\tau}(\chi_{I_j} g_{j,n})(t)\,dt\Big)(x) = \int_{\mathbb{R}} e^{-it\cdot\tau}\mathcal{I}(\chi_{I_j} g_{j,n})(t,x)\,dt \\ &= \int_{\mathbb{R}} e^{-it\cdot\tau}\chi_{I_j}(t) g_{j,n}(x) dt = \widehat{\chi}_{I_j}(\tau) g_{j,n}(x)\end{aligned}$$

for all $\tau \in \mathbb{R}$ and for almost all $x \in \mathbb{R}^d$. In particular, $\mathcal{F}_t(\chi_{I_j} g_{j,n})(\tau) = \widehat{\chi}_{I_j}(\tau) g_{j,n}$ in $L^2(\mathbb{R}^d)$ for all $\tau \in \mathbb{R}$. Hence,

$$\begin{aligned}\mathcal{F}_x((\mathcal{F}_t f_n)(\tau))(\xi) &= \mathcal{F}_x\Big(\sum_{j=1}^m \widehat{\chi}_{I_j}(\tau) g_{j,n}\Big)(\xi) = \sum_{j=1}^m \widehat{\chi}_{I_j}(\tau)\mathcal{F}_x(g_{j,n})(\xi) \\ &= \sum_{j=1}^m \int_{\mathbb{R}} e^{-it\cdot\tau}\chi_{I_j}(t)\,dt \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g_{j,n}(x)\,dx \\ &= \sum_{j=1}^m \int_{\mathbb{R}^{1+d}} e^{-i(t,x)\cdot(\tau,\xi)}\mathcal{I}(\chi_{I_j} g_{j,n}) d(t,x) = \mathcal{F}_{t,x}(\mathcal{I}f_n)(\tau,\xi)\end{aligned}$$

for all $(\tau,\xi) \in \mathbb{R}^{1+d}$. This implies that $\mathcal{I}(\mathcal{F}_x(\mathcal{F}_t f_n)) = \mathcal{F}_{t,x}(\mathcal{I}f_n)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, the functions f_n converge to f in $L^2(\mathbb{R}, L^2(\mathbb{R}^d))$ and therefore

$$\mathcal{I}(\mathcal{F}_x(\mathcal{F}_t f)) = \lim_{n\to\infty} \mathcal{I}(\mathcal{F}_x(\mathcal{F}_t f_n)) = \lim_{n\to\infty} \mathcal{F}_{t,x}(\mathcal{I}f_n) = \mathcal{F}_{t,x}(\mathcal{I}f)$$

in $L^2(\mathbb{R}^{1+d})$. Approximating a general $f \in L^2(\mathbb{R}, L^2(\mathbb{R}^d))$ by a sequence of simple functions and using the continuity of the Fourier transforms on the corresponding spaces then proves the assertion.

III) Finally, let $u \in H^{1/2}(\Gamma)$. We identify Γ with its chart $J \times \mathbb{R}^2$ again. Let $U \in H^{1/2}(\mathbb{R}^3)$ with $U = u$ on $J \times \mathbb{R}^2$. To simplify the notation we identify the representant of $\mathcal{I}^{-1}U$ in $L^2(\mathbb{R}, L^2(\mathbb{R}^2))$ with U. Using the result from step II), we compute

$$\|u\|^2_{L^2(J,H^{1/2}(\mathbb{R}^2))} \le \|U\|^2_{L^2(\mathbb{R},H^{1/2}(\mathbb{R}^2))} = \|\mathcal{F}_t U\|^2_{L^2(\mathbb{R},H^{1/2}(\mathbb{R}^2))}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \|\mathcal{F}_t U(\tau)\|_{H^{1/2}}^2 \, d\tau = \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\mathcal{F}_x(\mathcal{F}_t U(\tau))(\xi)|^2 (1 + |\xi|^2)^{\frac{1}{2}} \, d\xi d\tau \\
&\leq \int_{\mathbb{R}^3} |\mathcal{F}_{t,x} U(\tau, \xi)|^2 (1 + |\tau|^2 + |\xi|^2)^{\frac{1}{2}} \, d(\tau, \xi) = \|U\|_{H^{1/2}(\mathbb{R}^3)}^2.
\end{aligned}
$$

Taking the infimum over all $U \in H^{1/2}(\mathbb{R}^3)$ with $U = u$ on $J \times \mathbb{R}^2$ and using that $H^{1/2}(\overline{\Gamma}) = H^{1/2}(\Gamma)$ (see Theorem 1.4.3.1 in [Gri85]), we obtain

$$
\|u\|_{L^2(J, H^{1/2}(\mathbb{R}^2))}^2 \leq \|u\|_{H^{1/2}(\Gamma)}^2.
$$

This implies that $H^{1/2}(\Gamma)$ is continuously embedded in $L^2(J, H^{1/2}(\mathbb{R}^2))$. Identifying L^2 with its dual, we finally conclude that $L^2(J, H^{-1/2}(\mathbb{R}^2))$ is continuously embedded in $H^{-1/2}(\Gamma)$. ◇

2.2 Function spaces

In this section we first introduce various function spaces and corresponding norms which play a crucial role in the following. We have already seen the spaces $G_k(J \times G)$ for $k \in \mathbb{N}$ in the introduction, where G is an open subset of $\mathbb{R}^3$. In these spaces we will construct our solutions. In this context we will also need the function spaces

$$
\begin{aligned}
\tilde{G}_k(J \times G) &:= \bigcap_{j=0}^{k} W^{j,\infty}(J, H^{k-j}(G)), \\
H^k_{\mathrm{ta}}(J \times G) &:= \{v \in L^2(J \times G) \colon \partial^\alpha v \in L^2(J \times G) \text{ for all } \alpha \in \mathbb{N}_0^4 \text{ with } |\alpha| \leq k \\
&\qquad \text{and } \alpha_3 = 0\}, \\
H^k_{\mathrm{ta}}(G) &:= \{v \in L^2(G) \colon \partial^\alpha v \in L^2(G) \text{ for all } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq k \text{ and } \alpha_3 = 0\}.
\end{aligned}
$$

In the following we will mainly work in the spaces $L^2(J \times G)^6, H^k(J \times G)^6, G_k(J \times G)^6$, and so on. However, when it is clear from the context if a function is scalar or vector valued, we will simply write $v \in L^2(J \times G)$ instead of $v \in L^2(J \times G)^6$ and analogously for the other function spaces. We equip the spaces $G_k(J \times G)$ respectively $\tilde{G}_k(J \times G)$ with the norms

$$
\|v\|_{G_k(J \times G)} = \max_{0 \leq |\alpha| \leq k} \|\partial^\alpha v\|_{L^\infty(J, L^2(G))} \qquad (v \in \tilde{G}_k(J \times G))
$$

for all $k \in \mathbb{N}$. We further introduce the norms

$$
\|v\|_{H^k_{\mathrm{ta}}(J \times G)} = \Big(\sum_{\substack{0 \leq |\alpha| \leq k \\ \alpha_3 = 0}} \|\partial^\alpha v\|_{L^2(J \times G)}^2 \Big)^{1/2} \qquad (v \in H^k_{\mathrm{ta}}(J \times G))
$$

and analogously

$$
\|v\|_{H^k_{\mathrm{ta}}(G)} = \Big(\sum_{\substack{0 \leq |\alpha| \leq k \\ \alpha_3 = 0}} \|\partial^\alpha v\|_{L^2(G)}^2 \Big)^{1/2} \qquad (v \in H^k_{\mathrm{ta}}(G))
$$

on $H^k_{\mathrm{ta}}(J \times G)$ respectively $H^k_{\mathrm{ta}}(G)$ for $k \in \mathbb{N}$.

Let $e_{-\gamma} \colon \mathbb{R} \to \mathbb{R}$ be defined by $e^{-\gamma t}$ for all $t \in \mathbb{R}$. We will also use the weighted norms

$$
\begin{aligned}
\|v\|_{L^2_\gamma(J \times G)} &= \|e_{-\gamma} v\|_{L^2(J \times G)} \qquad (v \in L^2(J \times G)), \\
\|v\|_{H^k_\gamma(J \times G)} &= \Big(\sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha v\|_{L^2_\gamma(J \times G)}^2 \Big)^{1/2} \qquad (v \in H^k(J \times G)),
\end{aligned}
$$

$$\|v\|_{G_{k,\gamma}(J\times G)} = \max_{0\le|\alpha|\le k} \|e_{-\gamma}\partial^\alpha v\|_{L^\infty(J,L^2(G))} \qquad (v \in \tilde{G}_k(J\times G)),$$

and analogously for $\|v\|_{H^k_{\mathrm{ta},\gamma}(J\times G)}$. Observe that the weighted norms are all equivalent to the unweighted ones since the interval J is bounded. Moreover, we have $\|v\|_{L^2_\gamma(J\times G)} \le \|v\|_{L^2(J\times G)}$ for all $v \in L^2(J\times G)$ and the analogous estimate is true for the other norms.

Up to now we introduced the function spaces from which we take the data and where we expect the solutions. For a thorough study of the coefficients, we will also need the following spaces.

Definition 2.20. *Let $m, k \in \mathbb{N}$, $\eta > 0$, and $a \in \mathbb{R}^{k\times k}$. We set*

$$\begin{aligned}
F_{m,k}(J\times G) &:= \{A \in W^{1,\infty}(J\times G)^{k\times k} \colon \partial^\alpha A \in (L^\infty(J, L^2(G)))^{k\times k} \text{ for all } \alpha \in \mathbb{N}_0^4 \\
&\qquad\qquad \text{with } 1 \le |\alpha| \le m\},\\
F_{m,k,\eta}(J\times G) &:= \{A \in F_{m,k}(J\times G) \colon A(t,x) \in \mathrm{Sym}(k) \text{ for all } (t,x) \in J\times G \text{ and}\\
&\qquad\qquad A(t,x) \text{ is positive definite with } A(t,x) \ge \eta\\
&\qquad\qquad \text{for all } (t,x) \in J\times G\},\\
F^{\mathrm{cp}}_{m,k}(J\times G) &:= \{A \in F_{m,k}(J\times G) \colon \text{there is a compact subset } K \text{ of } \overline{J\times G}\\
&\qquad\qquad \text{such that } A \text{ is constant on } \overline{J\times G} \setminus K\},\\
F^{\mathrm{cp},a}_{m,k}(J\times G) &:= \{A \in F_{m,k}(J\times G) \colon \text{there is a compact subset } K \text{ of } \overline{J\times G} \text{ with}\\
&\qquad\qquad A = a \text{ on } \overline{J\times G} \setminus K\},\\
F^{\mathrm{c}}_{m,k}(J\times G) &:= \{A \in F_{m,k}(J\times G) \colon \lim_{|(t,x)|\to\infty} A(t,x) \text{ exists}\},\\
F^{\mathrm{c},a}_{m,k}(J\times G) &:= \{A \in F_{m,k}(J\times G) \colon \lim_{|(t,x)|\to\infty} A(t,x) = a\},\\
F^{\mathrm{i}}_{m,k,\eta}(J\times G) &:= F_{m,k,\eta}(J\times G) \cap F^{\mathrm{i}}_{m,k}(J\times G), \quad i \in \{\mathrm{cp}, \mathrm{c}\},\\
F^{\mathrm{i},a}_{m,k,\eta}(J\times G) &:= F_{m,k,\eta}(J\times G) \cap F^{\mathrm{i},a}_{m,k}(J\times G), \quad i \in \{\mathrm{cp}, \mathrm{c}\},\\
F^0_{m,k}(G) &:= \{A_0 \in L^\infty(G)^{k\times k} \colon \partial^\alpha A_0 \in L^2(G)^{k\times k} \text{ for all } \alpha \in \mathbb{N}_0^3\\
&\qquad\qquad \text{with } 1 \le |\alpha| \le m\}.
\end{aligned}$$

We equip all these spaces with the norms

$$\|A\|_{F_m(\Omega)} = \max\{\|A\|_{W^{1,\infty}(\Omega)}, \max_{1\le|\alpha|\le m} \|\partial^\alpha A\|_{L^\infty(J,L^2(G))}\} \qquad (A \in F_{m,k}(\Omega)),$$

respectively

$$\|A_0\|_{F^0_m(\mathbb{R}^3_+)} = \max\{\|A_0\|_{L^\infty(\mathbb{R}^3_+)}, \max_{1\le|\alpha|\le m} \|\partial^\alpha A_0\|_{L^2(\mathbb{R}^3_+)}\} \qquad (A_0 \in F^0_{m,k}(\mathbb{R}^3_+)).$$

In fact, we will only need the cases $k = 6$ and $k = 1$. If it is clear from the context whether we mean $k = 6$ or $k = 1$, we drop the index k to streamline the notation.

We go on with a crucial approximation result for elements from the spaces above.

Lemma 2.21. *Let $m, k \in \mathbb{N}$. Take an open interval $J \subseteq \mathbb{R}$ and set $\Omega = J \times \mathbb{R}^3_+$. Choose $A \in F_{m,k}(\Omega)$. Then there exists a family $\{A_\varepsilon\}_{\varepsilon>0}$ in $C^\infty(\overline{\Omega})$ with*

(i) $\partial^\alpha A_\varepsilon \in F_{m,k}(\Omega)$ for all $\alpha \in \mathbb{N}_0^4$ and $\varepsilon > 0$,

(ii) $\|A_\varepsilon\|_{W^{1,\infty}(\Omega)} \le C\|A\|_{W^{1,\infty}(\Omega)}$ and $\|\partial^\alpha A_\varepsilon\|_{L^\infty(J,L^2(\mathbb{R}^3_+))} \le C\|A\|_{F_m(\Omega)}$ for all multiindices $1 \le |\alpha| \le m$ and $\varepsilon > 0$,

(iii) $A_\varepsilon \to A$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$, and

(iv) $A_\varepsilon(0) \to A(0)$ in $L^\infty(\mathbb{R}^3_+)$ and $\partial^\alpha A$ and $\partial^\alpha A_\varepsilon$ have a representative in the space $C(\overline{J}, L^2(\mathbb{R}^3_+))$ with $\partial^\alpha A_\varepsilon(0) \to \partial^\alpha A(0)$ in $L^2(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ for all $\alpha \in \mathbb{N}_0^4$ with $0 < |\alpha| \le m - 1$.

If A additionally belongs to $F^{\mathrm{cp}}_{m,k}(\Omega)$, $F^{\mathrm{c}}_{m,k}(\Omega)$, $F_{m,k,\eta}(\Omega)$ for an $\eta > 0$, or the intersection of two of these spaces, then the same is true for A_ε for all $\varepsilon > 0$.

Proof. Without loss of generality we assume that $J = (0, T)$ for a time $T > 0$.

I) Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ and $\rho \in C_c^\infty(\mathbb{R})$ be nonnegative functions with integral 1 and support in the unit ball. As usual we set $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$ for all $x \in \mathbb{R}^3$ and $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$ for all $t \in \mathbb{R}$ for all $\varepsilon > 0$. Let $\theta_1, \theta_2 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\theta_1 \subseteq [-1, \frac{2}{3}T]$ and $\operatorname{supp}\theta_2 \subseteq [\frac{1}{3}T, T+1]$ such that θ_1 equals 1 in a neighborhood of 0, θ_2 equals 1 in a neighborhood of T, and $\theta_1 + \theta_2 = 1$ on $[0, T]$. Take $\varepsilon_0 \in (0, \frac{1}{3}T)$ so small that $\theta_1(t+\varepsilon) + \theta_2(t-\varepsilon) > 0$ for all $t \in \overline{J}$ and $\varepsilon \in (0, 2\varepsilon_0]$, $\theta_1 = 1$ on $B(0, 2\varepsilon_0)$ and $\theta_2 = 1$ on $B(T, 2\varepsilon_0)$. As in the proof of Lemma 2.1, we define the translation operators

$$\tau_{1,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4) \to L^1_{loc}(\mathbb{R}^4), \quad (\tau_{1,\varepsilon} w)(t,x) = w(t+\varepsilon, x_1, x_2, x_3+\varepsilon),$$
$$\tau_{2,\varepsilon} \colon L^1_{loc}(\mathbb{R}^4) \to L^1_{loc}(\mathbb{R}^4), \quad (\tau_{2,\varepsilon} w)(t,x) = w(t-\varepsilon, x_1, x_2, x_3+\varepsilon).$$

Then there is a constant C independent of ε such that

$$\partial_t^l \frac{1}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2} \leq C \tag{2.22}$$

on $\overline{J}$ for all $l \in \{0, \dots, m-1\}$ and $\varepsilon \in (0, 2\varepsilon_0)$. Moreover, we have

$$\partial_t^l \frac{1}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2} \longrightarrow 0$$

in $L^\infty(J)$ as ε tends to 0 for all $l \in \{1, \dots, m-1\}$.

We then set

$$A_\varepsilon = (\rho_\varepsilon \varphi_\varepsilon) * \frac{\tau_{1,2\varepsilon}(\theta_1 A) + \tau_{2,2\varepsilon}(\theta_2 A)}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where we identify A with its zero extension to $\mathbb{R}^4$.

Then A_ε is an element of $C^\infty(\overline{\Omega})$ for all $\varepsilon \in (0, \varepsilon_0)$ and (i) and (ii) are satisfied. For (iii) we note that $W^{1,\infty}(\Omega)$ equals the space of functions which are Lipschitz continuous on Ω as Ω is convex. In particular, A belongs to $BUC(\Omega)$. We thus obtain that

$$A_\varepsilon \longrightarrow \frac{\theta_1 A + \theta_2 A}{\theta_1 + \theta_2} = A$$

in $L^\infty(\Omega)$ as $\varepsilon \to 0$. The first part of (iv) now also follows. For the remaining assertion take $\alpha \in \mathbb{N}_0^4$ with $0 < |\alpha| \leq m-1$. Then $\partial^\alpha A \in L^\infty(J, L^2(\mathbb{R}^3_+))$ and $\partial_t \partial^\alpha A \in L^\infty(J, L^2(\mathbb{R}^3_+))$ so that $\partial^\alpha A$ has a representative in $C(\overline{J}, L^2(\mathbb{R}^3_+))$ with which we identify $\partial^\alpha A$ in the following.

Let $\zeta > 0$. As $\overline{J}$ is compact, $\partial^\alpha A$ is uniformly continuous on $\overline{J}$ with values in $L^2(\mathbb{R}^3_+)$. Hence, there exists a number $\delta_1 > 0$ such that

$$\|\partial^\alpha A(t_1) - \partial^\alpha A(t_2)\|_{L^2(\mathbb{R}^3_+)} \leq \frac{\zeta}{4} \tag{2.23}$$

for all $t_1, t_2 \in \overline{J}$ with $|t_1 - t_2| \leq \delta_1$.

Moreover, the translation operator $\tau_y \colon v \mapsto v(\cdot - y)$ is strongly continuous on $L^2(\mathbb{R}^3_+)$. We thus find a number $\delta_2 > 0$ such that

$$\|\tau_y \partial^\alpha A(0) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)} \leq \frac{\zeta}{4}$$

for all $y \in \mathbb{R}^3$ with $|y| \leq \delta_2$. Set $\delta = \min\{\delta_1, \delta_2\}$. For $(s, y) \in \mathbb{R}^4$ with $|(s,y)| < \delta$ we then obtain

$$\|\tau_y \partial^\alpha A(s) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)}$$

$$\leq \|\tau_y \partial^\alpha A(s) - \tau_y \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)} + \|\tau_y \partial^\alpha A(0) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)}$$
$$\leq \|\partial^\alpha A(s) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)} + \frac{\zeta}{4} \leq \frac{\zeta}{2}, \tag{2.24}$$

where we also used that τ_y is contractive on $L^2(\mathbb{R}^3_+)$ when the functions from this space are identified with their zero extension.

Employing that θ_1 equals 1 and θ_2 equals 0 on $B(0, 2\varepsilon_0)$, we derive that

$$\partial^\alpha A_\varepsilon(0) = (\varphi_\varepsilon \rho_\varepsilon) * (\tau_{1,2\varepsilon} \partial^\alpha A)(0)$$

for all $\varepsilon \in (0, \varepsilon_0)$. We set $\varepsilon_1 = \min\{\frac{\delta}{3}, \varepsilon_0\}$ and denote a point $x \in \mathbb{R}^3$ by (x', x_3). Using Minkowski's inequality and that the support of ρ_ε respectively φ_ε is contained in $B(0, \varepsilon)$, we infer

$$\begin{aligned}
&\|\partial^\alpha A_\varepsilon(0) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)} \\
&= \Big\| \int_{\mathbb{R}^4} (\partial^\alpha A(-s + 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon) - \partial^\alpha A(0, x)) \rho_\varepsilon(s) \varphi_\varepsilon(y) d(s, y) \Big\|_{L^2_x(\mathbb{R}^3_+)} \\
&\leq \int_{\mathbb{R}^4} \|\partial^\alpha A(-s + 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon) - \partial^\alpha A(0, x' - y', x_3 - y_3 + 2\varepsilon)\|_{L^2_x(\mathbb{R}^3_+)} \\
&\qquad \cdot \rho_\varepsilon(s) \varphi_\varepsilon(y) d(s, y) \\
&\quad + \int_{\mathbb{R}^4} \|\partial^\alpha A(0, x' - y', x_3 - y_3 + 2\varepsilon) - \partial^\alpha A(0, x)\|_{L^2_x(\mathbb{R}^3_+)} \rho_\varepsilon(s) \varphi_\varepsilon(y) d(s, y) \\
&\leq \int_{\mathbb{R}} \|\partial^\alpha A(-s + 2\varepsilon) - \partial^\alpha A(0)\|_{L^2(\mathbb{R}^3_+)} \rho_\varepsilon(s) ds + \frac{\zeta}{2} \int_{\mathbb{R}^3} \varphi_\varepsilon(y) dy \\
&\leq \frac{\zeta}{2} \int_{\mathbb{R}} \rho_\varepsilon(s) ds + \frac{\zeta}{2} = \zeta
\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. The last assertion in (iv) thus follows.

II) If A additionally belongs to $F^{\mathrm{cp}}_{m,k}(\Omega)$, there is a compact set $K \subseteq \overline{\Omega}$ and a matrix $a \in \mathbb{R}^{k \times k}$ such that $A = a$ on $\Omega \setminus K$. Define the compact set

$$K' = (K + \overline{B}(0, (2\sqrt{2} + 1)\varepsilon_1)) \cap \overline{\Omega}.$$

Let $(t, x) \in \Omega \setminus K'$ and $\varepsilon \in (0, \varepsilon_1)$. We then have that

$$(B((t + 2\varepsilon, x', x_3 + 2\varepsilon), \varepsilon) \cup B((t - 2\varepsilon, x', x_3 + 2\varepsilon), \varepsilon)) \subseteq \mathbb{R}^4 \setminus K$$

for all $\varepsilon \in (0, \varepsilon_1)$. We particularly infer that

$$(t - s + 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon), (t - s - 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon) \in \mathbb{R}^4 \setminus K$$

for all $(s, y) \in B(0, \varepsilon)$. We obtain

$$\frac{\tau_{1,2\varepsilon}(\theta_1 A) + \tau_{2,2\varepsilon}(\theta_2 A)}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2}(t - s, x - y) = \frac{\theta_1(t - s + 2\varepsilon)a + \theta_2(t - s - 2\varepsilon)a}{\theta_1(t - s + 2\varepsilon) + \theta_2(t - s - 2\varepsilon)} = a$$

and hence

$$A_\varepsilon(t, x) = \int_{\mathbb{R}^4} \rho_\varepsilon(s) \varphi_\varepsilon(y) \frac{\tau_{1,2\varepsilon}(\theta_1 A) + \tau_{2,2\varepsilon}(\theta_2 A)}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2}(t - s, x - y) d(s, y) = a.$$

III) Now let $F^{\mathrm{c}}_{m,k}(\Omega)$ and $a \in \mathbb{R}^{k \times k}$ with $\lim_{|(t,x)| \to \infty} A(t, x) = a$. Let $\zeta > 0$. Then there is a compact set $K \subseteq \overline{\Omega}$ such that $|A(t, x) - a| < \zeta$ for all $(t, x) \in \Omega \setminus K$. We define K' as in step II) and infer that

$$|A(t - s + 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon) - a| < \zeta, \quad |A(t - s - 2\varepsilon, x' - y', x_3 - y_3 + 2\varepsilon) - a| < \zeta$$

for all $(t,x) \in \Omega \setminus K'$, $(s,y) \in B(0,\varepsilon)$, and $\varepsilon \in (0,\varepsilon_1)$. We thus arrive at

$$\begin{aligned} |A_\varepsilon(t,x) - a| &\leq \int_{\mathbb{R}^4} \rho_\varepsilon(s)\varphi_\varepsilon(y) \frac{\tau_{1,2\varepsilon}(\theta_1|A-a|) + \tau_{2,2\varepsilon}(\theta_2|A-a|)}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2}(t-s,x-y)d(s,y) \\ &< \zeta \end{aligned}$$

for all $(t,x) \in \Omega \setminus K'$ and $\varepsilon \in (0,\varepsilon_1)$. We conclude that $A_\varepsilon(t,x)$ converges to a as $|(t,x)| \to \infty$ for all $\varepsilon \in (0,\varepsilon_1)$.

IV) Finally, we assume that $A \in F_{m,k,\eta}(\Omega)$ for an $\eta > 0$. Then

$$\xi^T A_\varepsilon(t,x)\xi = \int_{\mathbb{R}^4} \rho_\varepsilon(s)\varphi_\varepsilon(y) \frac{\tau_{1,2\varepsilon}(\theta_1\xi^T A\xi) + \tau_{2,2\varepsilon}(\theta_2\xi^T A\xi)}{\tau_{1,2\varepsilon}\theta_1 + \tau_{2,2\varepsilon}\theta_2}(t-s,x-y)d(s,y) \geq \eta$$

for all $\xi \in \mathbb{R}^k$ with $|\xi| = 1$, $(t,x) \in \Omega$, and $\varepsilon \in (0,\varepsilon_1)$. □

For the treatment of quasilinear equations bilinear respectively multilinear estimates are an indispensable tool. The next lemma provides the most basic results in this direction. In a certain sense, one might think of it as an extension of the well-known fact that the Sobolev space $H^m(\mathbb{R}^d)$ is an algebra if $m > \frac{d}{2}$, see e.g. Theorem 4.38 in [AF09]. In addition to this algebra property, we also need to deal with products involving a factor with smoothness parameter less than $\frac{d}{2}$ and we have to provide the bilinear estimates in the norms corresponding to the function spaces $G_m(\Omega)$ respectively $F_m^0(\mathbb{R}^3_+)$. However, the proof is elementary as it only combines the Sobolev embedding theorem with Hölder's inequality.

Lemma 2.22. *Let $J \subseteq \mathbb{R}$ be an open interval and let $G \subseteq \mathbb{R}^3$ be a domain with a uniform C^2-boundary (see e.g. Definition 2.24). Take $m_1, m_2 \in \mathbb{N}$ with $m_1 \geq m_2$ and $m_1 \geq 2$ and a parameter $\gamma \geq 0$.*

(i) Let $k \in \{0,\ldots,m_1\}$, $f \in \tilde{G}_{m_1-k}(J\times G)$, and $g \in \tilde{G}_k(J\times G)$. Then $fg \in \tilde{G}_0(J\times G)$ and

$$\|fg\|_{G_{0,\gamma}(J\times G)} \leq C\min\{\|f\|_{G_{m_1-k}(J\times G)}\|g\|_{G_{k,\gamma}(\Omega)}, \|f\|_{G_{m_1-k,\gamma}(J\times G)}\|g\|_{G_k(J\times G)}\}.$$

(ii) Let $f \in \tilde{G}_{m_1}(J\times G)$ and $g \in \tilde{G}_{m_2}(J\times G)$. Then $fg \in \tilde{G}_{m_2}(J\times G)$ and

$$\begin{aligned} \|fg\|_{G_{m_2,\gamma}(J\times G)} \leq C\min\{&\|f\|_{G_{m_1}(J\times G)}\|g\|_{G_{m_2,\gamma}(J\times G)}, \\ &\|f\|_{G_{m_1,\gamma}(J\times G)}\|g\|_{G_{m_2}(J\times G)}\}. \end{aligned}$$

(iii) Let $f \in F_{m_1}(J\times G)$ and $g \in G_{m_2}(J\times G)$. Then $fg \in G_{m_2}(J\times G)$ and

$$\|fg\|_{G_{m_2,\gamma}(J\times G)} \leq C\|f\|_{F_{m_1}(J\times G)}\|g\|_{G_{m_2,\gamma}(J\times G)}.$$

(iv) Let $f \in F_{m_1}(J\times G)$ and $g \in F_{m_2}(J\times G)$. Then $fg \in F_{m_2}(J\times G)$ and

$$\|fg\|_{F_{m_2}(J\times G)} \leq C\|f\|_{F_{m_1}(J\times G)}\|g\|_{F_{m_2}(J\times G)}.$$

(v) Let $k \in \{0,\ldots,m_1\}$, $f \in H^{m_1-k}(G)$, and $g \in H^k(G)$. Then $fg \in L^2(G)$ and

$$\|fg\|_{L^2(G)} \leq C\|f\|_{H^{m_1-k}(G)}\|g\|_{H^k(G)}.$$

(vi) Let $f \in H^{m_1}(G)$ and $g \in H^{m_2}(G)$. Then $fg \in H^{m_2}(G)$ and

$$\|fg\|_{H^{m_2}(G)} \leq C\|f\|_{H^{m_1}(G)}\|g\|_{H^{m_2}(G)}.$$

(vii) Let $f \in F^0_{m_1}(G)$, $g \in H^{m_2}(G)$. Then $fg \in H^{m_2}(G)$ and

$$\|fg\|_{H^{m_2}(G)} \leq C\|f\|_{F^0_{m_1}(G)}\|g\|_{H^{m_2}(\mathbb{R}^3_+)}.$$

Proof. We first note that the regularity assumption on the boundary of G implies the usual Sobolev embeddings, see e.g. Theorem 4.12 in [AF09].

(i) Let $k = 0$. By Sobolev's embedding, the function f belongs to $\tilde{G}_{m_1}(J \times G) \hookrightarrow L^\infty(J, H^2(G)) \hookrightarrow L^\infty(\Omega)$ so that the product fg is contained in $\tilde{G}_0(J \times G)$ and satisfies

$$\begin{aligned}
&\|fg\|_{G_{0,\gamma}(J\times G)} \leq \|f\|_{L^\infty(J\times G)} \|g\|_{G_{0,\gamma}(J\times G)} \leq C\|f\|_{G_{m_1}(J\times G)} \|g\|_{G_{0,\gamma}(J\times G)}, \\
&\|fg\|_{G_{0,\gamma}(J\times G)} \leq \|e_{-\gamma} f\|_{L^\infty(J\times G)} \|g\|_{G_0(J\times G)} \leq C\|f\|_{G_{m_1,\gamma}(J\times G)} \|g\|_{G_0(J\times G)}.
\end{aligned}$$

In the same way one shows the assertion in the case $k = m_1$. In the case $k \in \{1, \ldots, m_1 - 1\}$ the functions f and g belong to $G_1(J \times G)$. Hölder's inequality and Sobolev's embedding thus yield

$$\begin{aligned}
\|fg\|_{G_{0,\gamma}(J\times G)} &= \sup_{t\in J} \|e^{-\gamma t} f(t) g(t)\|_{L^2(G)} \leq \sup_{t\in J} (\|e^{-\gamma t} f(t)\|_{L^6(G)} \|g(t)\|_{L^3(G)}) \\
&\leq C \sup_{t\in J} \|e^{-\gamma t} f(t)\|_{H^1(G)} \sup_{t\in J} \|g(t)\|_{H^1(G)} \leq C\|f\|_{G_{1,\gamma}(J\times G)} \|g\|_{G_1(J\times G)} \\
&\leq C\|f\|_{G_{m_1-k,\gamma}(J\times G)} \|g\|_{G_k(J\times G)}.
\end{aligned}$$

Interchanging the role of f and g, the assertion then follows.

(ii) Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m_2$. We have

$$\partial^\alpha(fg) = \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f\, \partial^{\alpha-\beta} g. \tag{2.25}$$

Fix $\beta \leq \alpha$. The function $\partial^\beta f$ belongs to $\tilde{G}_{m_1-|\beta|}(\Omega)$ whereas

$$\partial^{\alpha-\beta} g \in \tilde{G}_{m_2-|\alpha-\beta|}(\Omega) = \tilde{G}_{m_2-|\alpha|+|\beta|}(\Omega) \hookrightarrow \tilde{G}_{|\beta|}(\Omega).$$

Assertion (ii) now follows from (i).

(iii) and (iv) are proven as (ii) combined with straightforward considerations for zeroth and first order derivatives.

(v) and (vi) are shown as (i) and (ii).

(vii) This fact follows easily from (vi). □

In the following chapters we will study partial differential equations where the coefficient in front of the time derivative is uniformly positive definite. The inverse of this coefficient A_0 thus exists and we have to deal with expressions involving this inverse, in particular when it is evaluated at zero. The next lemma tells us that $A_0(0)^{-1}$ is as smooth as $A_0(0)$ and it provides us with estimates of $A_0(0)^{-1}$ in terms of $A_0(0)$.

Lemma 2.23. *Let $J \subseteq \mathbb{R}$ be an open interval and $\Omega = J \times \mathbb{R}^3_+$. Take $m, k \in \mathbb{N}$ with $m \geq 3$, $\eta > 0$, and $A_0 \in F_{m,k,\eta}(\Omega)$. Choose $t_0 \in \overline{J}$. Then $A_0(t_0)^{-1}$ belongs to $F^0_{m-1,k}(\mathbb{R}^3_+)$,*

$$\partial^\alpha A_0(t_0)^{-1} = -A_0(t_0)^{-1} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(t_0) \partial^{\alpha-\beta} A_0(t_0)^{-1} \tag{2.26}$$

for all $\alpha \in \mathbb{N}_0^3$ with $0 < |\alpha| \leq m - 1$, and

$$\begin{aligned}
&\|A_0(t_0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \leq C(\eta), \\
&\|A_0(t_0)^{-1}\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq C(\eta)(1 + \|A_0(t_0)\|_{F^0_{m-1}(\mathbb{R}^3_+)})^{m-2} \|A_0(t_0)\|_{F^0_{m-1}(\mathbb{R}^3_+)}
\end{aligned} \tag{2.27}$$

if $m \geq 2$. If $\tilde{A}_0$ is another element of $F_{m,k,\eta}(\Omega)$, we have

$$\begin{aligned}
&\|A_0(t_0)^{-1} - \tilde{A}_0(t_0)^{-1}\|_{F^0_{m-1}(\mathbb{R}^3_+)} \\
&\leq C(\eta)\big(1 + \|A_0(t_0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} + \|\tilde{A}_0(t_0)\|_{F^0_{m-1}(\mathbb{R}^3_+)}\big)^{m-1} \|A_0(t_0) - \tilde{A}_0(t_0)\|_{F^0_{m-1}(\mathbb{R}^3_+)}.
\end{aligned} \tag{2.28}$$

Proof. Without loss of generality we assume that $J = (0,T)$ for a time $T > 0$ and that $t_0 = 0$.

I) Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\Lambda_1 \in F^0_{n+1}(\mathbb{R}^3_+), \Lambda_2 \in F^0_n(\mathbb{R}^3_+)$. Let $\alpha \in \mathbb{N}^3_0$ with $0 < |\alpha| \leq n+1$. If $\beta = \alpha$, we have

$$\begin{aligned}\|\partial^\beta \Lambda_1 \partial^{\alpha-\beta}\Lambda_2\|_{L^2(\mathbb{R}^3_+)} &= \|\partial^\alpha \Lambda_1 \Lambda_2\|_{L^2(\mathbb{R}^3_+)} \leq \|\partial^\alpha \Lambda_1\|_{L^2(\mathbb{R}^3_+)} \|\Lambda_2\|_{L^\infty(\mathbb{R}^3_+)} \\ &\leq \|\Lambda_1\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|\Lambda_2\|_{F^0_n(\mathbb{R}^3_+)}.\end{aligned}$$

If $0 < \beta < \alpha$, we obtain that $\partial^\beta \Lambda_1 \in H^{n+1-|\beta|}(\mathbb{R}^3_+)$ and $\partial^{\alpha-\beta}\Lambda_2 \in H^{n-|\alpha|+|\beta|}(\mathbb{R}^3_+)$. Lemma 2.22 (vi) applied with $k = n+1-|\beta|$ and

$$m_1 = 2n + 1 - |\alpha| \geq n \geq 2$$

then yields

$$\begin{aligned}&\|\partial^\beta \Lambda_1 \partial^{\alpha-\beta}\Lambda_2\|_{L^2(\mathbb{R}^3_+)} \leq C\|\partial^\beta \Lambda_1\|_{H^{n+1-|\beta|}(\mathbb{R}^3_+)} \|\partial^{\alpha-\beta}\Lambda_2\|_{H^{2n+1-|\alpha|-(n+1-|\beta|)}(\mathbb{R}^3_+)} \\ &\leq C\|\Lambda_1\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|\partial^{\alpha-\beta}\Lambda_2\|_{H^{n-|\alpha-\beta|}(\mathbb{R}^3_+)} \leq C\|\Lambda_1\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|\Lambda_2\|_{F^0_n(\mathbb{R}^3_+)}.\end{aligned}$$

Combining the two estimates, we arrive at

$$\Big\| \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \Lambda_1 \partial^{\alpha-\beta}\Lambda_2 \Big\|_{L^2(\mathbb{R}^3_+)} \leq C\|\Lambda_1\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|\Lambda_2\|_{F^0_n(\mathbb{R}^3_+)}.$$

II) We next observe that since $A_0(t,x)$ is positive definite with $A_0(t,x) \geq \eta$ for all $(t,x) \in \Omega$, we obtain $|A_0(t,x)^{-1}| \leq 1/\eta$ for all $(t,x) \in \Omega$, where $|\cdot|$ denotes the matrix-norm induced by the euclidean norm on $\mathbb{R}^k$. We thus obtain

$$\|A_0^{-1}\|_{L^\infty(\Omega)} \leq \frac{C}{\eta} \quad \text{and} \quad \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \leq \frac{C}{\eta}. \tag{2.29}$$

We fix a sequence $(A_{0,l})_l$ in $F^{\mathrm{c}}_{m,k,\eta}(\Omega) \cap C^\infty(\overline{\Omega})$ with

$$\|A_{0,l}(0) - A_0(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \longrightarrow 0$$

as $l \to \infty$, which exists by Lemma 2.21. Since $A_{0,l}$ is classically differentiable, we can apply Leibniz' formula - which is also valid for matrix valued functions - to deduce

$$\begin{aligned}0 = \partial^\alpha (A_{0,l} A_{0,l}^{-1}) &= \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_{0,l} \partial^{\alpha-\beta} A_{0,l}^{-1}, \\ \partial^\alpha A_{0,l}^{-1} &= -A_{0,l}^{-1} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_{0,l} \partial^{\alpha-\beta} A_{0,l}^{-1}\end{aligned} \tag{2.30}$$

for all $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3_0 \setminus \{0\}$. We further know that $\|A_{0,l}^{-1}\|_{L^\infty} \leq C/\eta$ since $A_{0,l} \in F_{m,k,\eta}(\Omega)$ by Lemma 2.21.

III) We will show inductively that for all $n \in \{1, \ldots, m-1\}$ the function $A_0(0)^{-1}$ belongs to $F^0_n(\mathbb{R}^3_+)$, formula (2.26) is valid for all $\alpha \in \mathbb{N}^3_0$ with $|\alpha| = n$, and estimates (2.27) and (2.28) hold for n.

We start with $n = 1$. First note that

$$\begin{aligned}&\|\Lambda_1^{-1} - \Lambda_2^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\Lambda_2^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|\Lambda_1 - \Lambda_2\|_{L^\infty(\mathbb{R}^3_+)} \|\Lambda_1^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \\ &\leq C(\eta) \|\Lambda_1 - \Lambda_2\|_{L^\infty(\mathbb{R}^3_+)}\end{aligned} \tag{2.31}$$

for all $\Lambda_1, \Lambda_2 \in F^0_{m-1}(\mathbb{R}^3_+)$ with $\Lambda_1, \Lambda_2 \geq \eta$. We thus deduce

$$\|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \leq C(\eta) \|A_{0,l}(0) - A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \longrightarrow 0$$

as $l \to \infty$ and

$$\|A_0(0)^{-1} - \tilde{A}_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \leq C(\eta)\|A_0(0) - \tilde{A}_0(0)\|_{L^\infty(\mathbb{R}^3_+)}.$$

Now take $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 1$. Then

$$\begin{aligned}
&\|\partial^\alpha A_{0,l}(0)^{-1} + A_0(0)^{-1}\partial^\alpha A_0(0)\, A_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&= \|A_{0,l}(0)^{-1}\partial^\alpha A_{0,l}(0)\, A_{0,l}(0)^{-1} - A_0(0)^{-1}\partial^\alpha A_0(0)\, A_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\leq \|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)}\|\partial^\alpha A_{0,l}(0)\|_{L^2(\mathbb{R}^3_+)}\|A_{0,l}(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \\
&\qquad + \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)}\|\partial^\alpha A_{0,l}(0) - \partial^\alpha A_0(0)\|_{L^2(\mathbb{R}^3_+)}\|A_{0,l}(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \\
&\qquad + \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)}\|\partial^\alpha A_0(0)\|_{L^2(\mathbb{R}^3_+)}\|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \\
&\leq C(\eta)(1 + \|A_{0,l}(0)\|_{F_1^0(\mathbb{R}^3_+)} + \|A_0(0)\|_{F_1^0(\mathbb{R}^3_+)})\|A_{0,l}(0) - A_0(0)\|_{F_1^0(\mathbb{R}^3_+)} \qquad (2.32)
\end{aligned}$$

Letting $l \to \infty$, we obtain

$$\partial^\alpha A_{0,l}(0)^{-1} \longrightarrow -A_0(0)^{-1}\partial^\alpha A_0(0)\, A_0(0)^{-1} \quad \text{in } L^2(\mathbb{R}^3_+).$$

We conclude $A_0(0)^{-1} \in F_1^0(\mathbb{R}^3_+)$ with

$$\begin{aligned}
\partial^\alpha A_0(0)^{-1} &= -A_0(0)^{-1}\partial^\alpha A_0(0)\, A_0(0)^{-1}, \\
\|A_0(0)^{-1}\|_{F_1^0(\mathbb{R}^3_+)} &\leq C(\eta)\|A_0(0)\|_{F_1^0(\mathbb{R}^3_+)}.
\end{aligned}$$

Replacing $A_{0,l}(0)$ by $\tilde{A}_0(0)$ in (2.32), we further obtain

$$\begin{aligned}
&\|\partial^\alpha A_0(0)^{-1} - \partial^\alpha \tilde{A}_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\eta)(1 + \|A_0(0)\|_{F_1^0(\mathbb{R}^3_+)} + \|\tilde{A}_0(0)\|_{F_1^0(\mathbb{R}^3_+)})\|A_0(0) - \tilde{A}_0(0)\|_{F_1^0(\mathbb{R}^3_+)}.
\end{aligned}$$

Consequently, the claim is true for $n = 1$. Moreover, as $\partial^\alpha A_0(0) \in H^1(\mathbb{R}^3_+) \hookrightarrow L^6(\mathbb{R}^3_+)$, we deduce from (2.26) that also $\partial^\alpha A_0(0)^{-1}$ belongs to $L^6(\mathbb{R}^3_+)$ for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 1$.

Next consider the case $n = 2$. Take $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$. Let Λ_1 be an element of $\operatorname{span}\{A_0(0), A_{0,l}(0) \colon l \in \mathbb{N}\}$ and let Λ_2 belong to $\operatorname{span}\{A_0(0)^{-1}, A_{0,l}(0)^{-1} \colon l \in \mathbb{N}\}$. Using Sobolev's embedding and Hölder's inequality, we infer

$$\begin{aligned}
&\Big\| \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \Lambda_1 \partial^{\alpha-\beta}\Lambda_2 \Big\|_{L^2(\mathbb{R}^3_+)} \\
&\leq \|\partial^\alpha \Lambda_1\|_{L^2(\mathbb{R}^3_+)}\|\Lambda_2\|_{L^\infty(\mathbb{R}^3_+)} + \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \|\partial^\beta \Lambda_1\|_{L^3(\mathbb{R}^3_+)}\|\partial^{\alpha-\beta}\Lambda_2\|_{L^6(\mathbb{R}^3_+)} \\
&\leq C\|\Lambda_1\|_{F_2^0(\mathbb{R}^3_+)}\Big(\|\Lambda_2\|_{L^\infty(\mathbb{R}^3_+)} + \sum_{|\beta|=1} \|\partial^\beta \Lambda_2\|_{L^6(\mathbb{R}^3_+)}\Big).
\end{aligned}$$

Applying this estimate three times and using formula (2.26) for first order derivatives in combination with

$$\begin{aligned}
&\sum_{|\beta|=1} \|A_{0,l}(0)^{-1}\partial^\beta A_{0,l}(0) A_{0,l}(0)^{-1}\|_{L^6(\mathbb{R}^3_+)} \leq \|A_{0,l}(0)^{-1}\|^2_{L^\infty(\mathbb{R}^3_+)} \sum_{|\beta|=1} \|\partial^\beta A_{0,l}(0)\|_{L^6(\mathbb{R}^3_+)} \\
&\leq C(\eta) \sum_{|\beta|=1} \|\partial^\beta A_{0,l}(0)\|_{H^1(\mathbb{R}^3_+)} \leq C(\eta)\|A_{0,l}\|_{F_2^0(\mathbb{R}^3_+)},
\end{aligned}$$

we deduce

$$\Big\|\partial^\alpha A_{0,l}(0)^{-1} + A_0(0)^{-1} \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1}\Big\|_{L^2(\mathbb{R}^3_+)}$$

$$
\begin{aligned}
&= \Big\| A_{0,l}(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_{0,l}(0) \partial^{\alpha-\beta} A_{0,l}(0)^{-1} \\
&\qquad\qquad - A_0(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1} \Big\|_{L^2(\mathbb{R}^3_+)} \\
&\le C \|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} C(\eta) \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\quad + C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_{0,l}(0) - A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} C(\eta) \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\quad + C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\qquad \cdot \Big(\|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} + \sum_{|\beta|=1} \|\partial^\beta A_{0,l}(0)^{-1} - \partial^\beta A_0(0)^{-1}\|_{L^6(\mathbb{R}^3_+)} \Big) \\
&\le C(\eta) \|A_{0,l}(0)\|^2_{F_2^0(\mathbb{R}^3_+)} \|A_{0,l}(0) - A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \\
&\quad + C(\eta) \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} \|A_{0,l}(0) - A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\quad + C(\eta) \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} \Big(C(\eta) \|A_{0,l}(0) - A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \\
&\qquad + C(\eta)(1 + \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}) \|A_{0,l}(0) - A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}) \Big) \\
&\le C(\eta)(1 + \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)})^2 \|A_{0,l}(0) - A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}, \qquad (2.33)
\end{aligned}
$$

where we also employed the estimate

$$
\begin{aligned}
&\|\partial^\beta A_{0,l}(0)^{-1} - \partial^\beta A_0(0)^{-1}\|_{L^6(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_{0,l}(0)\|_{F_2^0(\mathbb{R}^3_+)} + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}) \|A_{0,l}(0) - A_0(0)\|_{F_2^0(\mathbb{R}^3_+)},
\end{aligned}
$$

for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = 1$, which follows as in (2.32) by replacing $L^2(\mathbb{R}^3_+)$ by $L^6(\mathbb{R}^3_+)$ and exploiting Sobolev's inequality.

Letting l to infinity in (2.33), we conclude that $\partial^\alpha A_0(0)^{-1} \in L^2(\mathbb{R}^3_+)$ and

$$
\partial^\alpha A_0(0)^{-1} = A_0(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$. Replacing $A_{0,l}(0)$ by $A_0(0)$ in (2.33), we also derive

$$
\begin{aligned}
&\|\partial^\alpha A_0(0)^{-1} - \partial^\alpha \tilde{A}_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} + \|\tilde{A}_0(0)\|_{F_2^0(\mathbb{R}^3_+)})^2 \|A_0(0) - \tilde{A}_0(0)\|_{F_2^0(\mathbb{R}^3_+)}
\end{aligned}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$. In combination with (2.28) for $n = 1$ we obtain

$$
\begin{aligned}
&\|A_0(0) - \tilde{A}_0(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} + \|\tilde{A}_0(0)\|_{F_2^0(\mathbb{R}^3_+)})^2 \|A_0(0) - \tilde{A}_0(0)\|_{F_2^0(\mathbb{R}^3_+)}.
\end{aligned}
$$

Finally, the same arguments as in (2.33) yield

$$
\begin{aligned}
&\|\partial^\alpha A_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \le C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} C(\eta) \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)} \\
&\le C(\eta) \|A_0(0)\|^2_{F_2^0(\mathbb{R}^3_+)}
\end{aligned}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$ and thus in combination with (2.27) for $n = 1$

$$
\|A_0(0)^{-1}\|_{F_2^0(\mathbb{R}^3_+)} \le C(\eta)(1 + \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}) \|A_0(0)\|_{F_2^0(\mathbb{R}^3_+)}.
$$

This shows the assertion for $n = 2$.

Now assume that we have shown the claim for an index $n \in \{2, \ldots, m-2\}$. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = n+1$. The induction hypothesis implies that $A_0(0)^{-1} \in F_n^0(\mathbb{R}^3_+)$ and that (2.27) and (2.28) hold for n. Step I) therefore yields

$$
\begin{aligned}
&\Big\|\partial^\alpha A_{0,l}(0)^{-1} + A_0(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1}\Big\|_{L^2(\mathbb{R}^3_+)} \\
&= \Big\| A_{0,l}(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_{0,l}(0) \partial^{\alpha-\beta} A_{0,l}(0)^{-1} \\
&\qquad\qquad - A_0(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1}\Big\|_{L^2(\mathbb{R}^3_+)} \\
&\le C \|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_{0,l}(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|A_{0,l}(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} \\
&\quad + C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_{0,l}(0) - A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|A_{0,l}(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} \\
&\quad + C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|A_{0,l}(0)^{-1} - A_0(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} \qquad (2.34) \\
&\le C(\eta)(1 + \|A_{0,l}(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} + \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)})^{n+1} \|A_{0,l}(0) - A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)}
\end{aligned}
$$

for all $l \in \mathbb{N}$. We proceed as in the cases $n = 1$ and $n = 2$. Letting $l \to \infty$, we infer that $\partial^\alpha A_0(0)^{-1}$ belongs to $L^2(\mathbb{R}^3_+)$ with

$$
\partial^\alpha A_0(0)^{-1} = -A_0(0)^{-1} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_0(0) \partial^{\alpha-\beta} A_0(0)^{-1}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = n+1$. Replacing $A_{0,l}$ by $\tilde{A}_0$ in (2.34) further gives

$$
\begin{aligned}
&\|\partial^\alpha A_0(0)^{-1} - \partial^\alpha \tilde{A}_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} + \|\tilde{A}_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)})^{n+1} \|A_0(0) - \tilde{A}_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)}
\end{aligned}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = n+1$. In combination with the induction hypothesis we thus arrive at

$$
\begin{aligned}
&\|A_0(0)^{-1} - \tilde{A}_0(0)^{-1}\|_{F^0_{n+1}(\mathbb{R}^3_+)} \\
&\le \|A_0(0)^{-1} - \tilde{A}_0(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} + \sum_{|\alpha|=n+1} \|\partial^\alpha A_0(0)^{-1} - \partial^\alpha \tilde{A}_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} + \|\tilde{A}_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)})^{n+1} \|A_0(0) - \tilde{A}_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)}.
\end{aligned}
$$

Employing the same arguments as in (2.34), we estimate

$$
\begin{aligned}
&\|\partial^\alpha A_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \le C \|A_0(0)^{-1}\|_{L^\infty(\mathbb{R}^3_+)} \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} \|A_0(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} \\
&\le C(\eta) \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)} (1 + \|A_0(0)\|_{F^0_n(\mathbb{R}^3_+)})^{n-1} \|A_0(0)\|_{F^0_n(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)})^n \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)}
\end{aligned}
$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = n+1$ and thus

$$
\begin{aligned}
\|A_0(0)^{-1}\|_{F^0_{n+1}(\mathbb{R}^3_+)} &\le \|A_0(0)^{-1}\|_{F^0_n(\mathbb{R}^3_+)} + \sum_{|\alpha|=n+1} \|\partial^\alpha A_0(0)^{-1}\|_{L^2(\mathbb{R}^3_+)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)})^n \|A_0(0)\|_{F^0_{n+1}(\mathbb{R}^3_+)}.
\end{aligned}
$$

All the assertions now follow. □

We already mentioned in the introduction that solutions of (1.10) of higher regularity lead to compatibility conditions at the boundary. These compatibility conditions appear in the half-space case as well as on domains. We want to treat both cases simultaneously in the following. To that purpose, we first have to clarify which kind of domains we want to consider.

We will treat domains with a uniform C^{m+2}-boundary. Since there are slightly different definitions of C^m-boundaries in the literature, we first make our notion of a uniform C^m-boundary precise, see Paragraph 4.10 in [AF09] and Sections 1.2 and 1.3 in [Gri85].

Definition 2.24. *Let $m \in \mathbb{N}$. A domain $G \subseteq \mathbb{R}^d$ satisfies the* uniform C^m-regularity condition *if there exists a locally finite open cover $(U_i)_{i\in\mathbb{N}}$ of ∂G and corresponding functions $\varphi_i \in C^m(U_i)$ which are bijections onto $B(0,1)$ such that $\psi_i = \varphi_i^{-1} \in C^m(B(0,1))$ for all $i \in \mathbb{N}$ and the following conditions are satisfied.*

(i) There is a natural number N such that for all $\Lambda \subseteq \mathbb{N}$ with $|\Lambda| \geq N$ we have $\bigcap_{i\in\Lambda} U_i = \emptyset$.

(ii) There exists a number $\delta > 0$ such that $G_\delta := \{x \in G\colon \operatorname{dist}(x, \partial G) < \delta\}$ is contained in $\bigcup_{i=1}^{\infty} \psi_i(B(0,1/2))$.

(iii) For each $i \in \mathbb{N}$ we have $\varphi_i(U_i \cap G) = \{y \in B(0,1)\colon y_d > 0\} =: B(0,1)_+$.

(iv) There is a constant $M_1 > 0$ such that

$$\begin{aligned} |\partial^\alpha \varphi_{i,j}(x)| &\leq M_1 \qquad \text{for all } x \in U_i, \\ |\partial^\alpha \psi_{i,j}(y)| &\leq M_1 \qquad \text{for all } y \in B(0,1), \end{aligned} \tag{2.35}$$

for all $j \in \{1,\dots,d\}$, $i \in \mathbb{N}$, and $\alpha \in \mathbb{N}_0^d$ with $0 < |\alpha| \leq m$.

If a domain G satisfies the uniform C^m-regularity condition, we also say that the domain has a *uniform C^m-boundary*. .

On domains with a uniform C^m-boundary we can define Sobolev spaces via localization. If we further assume that there is a smooth partition of unity $(\theta_i)_{i\in\mathbb{N}}$ for ∂G subordinate to the covering $(U_i)_{i\in\mathbb{N}}$ such that all derivatives of the functions θ_i are bounded, we can also construct a trace operator which has the same properties as the one on the half-space. It is hard to find a fitting reference in the literature, since many authors restrict themselves to bounded domains respectively domains with a bounded boundary, see e.g. [Gri85], [Neč12], and [TW09]. In [AF09] a trace operator on a general uniform C^m-boundary is constructed. However, this operator only takes values in $L^q(\partial G)$. The same is true for [Tan97]. In [Bro61] the author works on general uniform C^m-domains but only deals with Sobolev spaces of integer regularity. Nevertheless, this article is a good reference as it actually performs the localization argument and one can see how the arguments transfer to the case of fractional Sobolev spaces.

In particular, the work [Bro61] shows that the properties of a uniform C^m-boundary are strong enough to allow the same constructions as performed in [TW09] and [Gri85] in the case of a bounded domain. The only difference is that in the bounded domain case there are only finitely many charts while we have to deal with infinitely many. The local finiteness of the covering provided by Definition 2.24 and the assumption of the existence of a partition of unity as described above are a sufficient replacement for the finiteness of the covering. We also refer to Chapter 5 for an example of a localization procedure on a domain with a uniform C^m-regular boundary.

Definition 2.25. *Let $m \in \mathbb{N}$ with $m \geq 2$ and let $G \subseteq \mathbb{R}^d$ be a domain satisfying the uniform C^m-regularity condition. Let $s \in [0, m]$. Take a covering $(U_i)_{i\in\mathbb{N}}$ and corresponding chart maps $(\varphi_i)_{i\in\mathbb{N}}$ as in Definition 2.24. We define the Sobolev space $H^s(\partial G)$ as the set of all functions $g \in L^2(\partial\Gamma)$ such that the functions $g \circ \varphi_i^{-1}$ belong to $H^s(\varphi_i(U_i \cap \partial G))$ and*

$$\|g\|_{H^s(\partial G)}^2 := \sum_{i=1}^{\infty} \|g \circ \varphi_i^{-1}\|_{H^s(\varphi_i(U_i\cap\partial G))}^2 < \infty.$$

As usual, this definition is independent of the concrete covering $(U_i, \varphi_i)_{i\in\mathbb{N}}$ and taking another one leads to an equivalent norm.

As mentioned above, for the construction of the trace operator we make an additional assumption, which is concerned with a suitable partition of unity. A localization procedure then gives the following result.

Lemma 2.26. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let G be a domain with a uniform C^m-boundary. Let $(U_i)_{i\in\mathbb{N}}$ be a covering of ∂G as in Definition 2.24. Assume that there is a smooth partition of unity $(\theta_i)_{i\in\mathbb{N}}$ for ∂G subordinate to $(U_i)_{i\in\mathbb{N}}$ such that there is a constant C with*

$$|\partial^\alpha \theta_i(x)| \leq C$$

for all $x \in U_i$ and $i \in \mathbb{N}$. Then there is a continuous trace operator $\mathrm{tr}_{\partial G}$ which maps $H^k(G)$ continuously into $H^{k-\frac{1}{2}}(\partial G)$ for all $k \in \{1, \ldots, m\}$. Moreover, the operator $\mathrm{tr}_{\partial G}$ extends the mapping $C_c^\infty(\overline{G}) \to C^m(\partial G), \varphi \mapsto \varphi_{|\partial G}$.

Lemma 2, Lemma 3, and Lemma 4 in [Bro61] show that the assumption of the existence of such a partition of unity is inessential. A suitable replacement for this partition of unity can in fact be constructed for every uniform C^m-boundary. However, we will need this partition of unity in Chapter 5 for the localization procedure of an initial boundary value problem. The assumption of the existence of such a partition is therefore included in the definition of a tame uniform C^m-boundary, see Definition 5.4. Even more assumptions are included there, whose benefit are not apparent yet, so that we postponed the definition of a tame uniform C^m-boundary to Chapter 5. Since we will only consider domains with a tame uniform C^m-boundary in Chapter 5, we can also restrict ourselves to such domains here. For the moment it is enough to know that every domain with a uniform C^m-boundary possesses a smooth partition of unity for its boundary as in Lemma 2.26.

In later sections it will be convenient to have the statement of Lemma 2.23 not only on the half-space but also on domains. To that purpose, we first remark that we can transfer the result from Lemma 2.21 from the half-space to domains.

Remark 2.27. Let $m, k \in \mathbb{N}$, $J \subseteq \mathbb{R}$ be an open interval, and $G \subseteq \mathbb{R}^3$ be a domain with a uniform $C^{\max\{m,2\}}$-boundary. Take $A \in F^{\mathrm{c}}_{m,k}(J \times G)$. Then the assertion of Lemma 2.21 still holds with $\mathbb{R}^3_+$ replaced by G.

The proof of this statement is reduced to Lemma 2.21 via a localization procedure, cf. the proof of Theorem 5.6. The assumption that A has a limit at infinity is introduced to account for the fact that the domain G may have an unbounded boundary so that infinitely many charts may be necessary to cover it. ◇

Since in the proof of Lemma 2.23 the assumption that the underlying spatial domain is the half-space $\mathbb{R}^3_+$ was only used to apply Lemma 2.21, we obtain the following corollary.

Corollary 2.28. *Let $m, k \in \mathbb{N}$ with $m \geq 3$ and $\eta > 0$. Take an open interval $J \subseteq \mathbb{R}$ and a domain $G \subseteq \mathbb{R}^3$ with a uniform $C^{\max\{m,2\}}$-boundary. Pick $A_0, \tilde{A}_0 \in F^{\mathrm{c}}_{m,k,\eta}(J \times G)$. Then the assertions of Lemma 2.23 still hold with $\mathbb{R}^3_+$ replaced by G.*

We now return to the compatibility conditions. They appear since we can both differentiate the differential equation and the boundary condition in (1.10) with respect to time. The former yields a formula for $\partial_t^p u(0)$ only involving the coefficients and the data, while the latter prescribes the trace of $\partial_t^p u(0)$ at the boundary of $\mathbb{R}^3_+$ respectively G. Therefore, coefficients and data have to be compatible. These "higher order initial values" will be ubiquitous in the following. Hence, it is reasonable to introduce a precise notation, which clarifies their dependencies on the coefficients and data.

Definition 2.29. *Let $m \in \mathbb{N}$ with $\tilde{m} = \max\{m, 3\}$, $J \subset \mathbb{R}$ be an interval, and $G \subset \mathbb{R}^3$ be a domain with a tame uniform $C^{\max\{m,2\}}$-boundary. We define inductively for all $p \in \{0, \ldots, m\}$ the operators*

$$S_{G,m,p} \colon J \times F_{\tilde{m},\mathrm{pd}}(J \times G) \times (F_{\tilde{m}}(J \times G))^4 \times H^m(J \times G) \times H^m(G) \to H^{m-p}(G),$$

$$
\begin{aligned}
&S_{G,m,0}(t_0, A_0, A_1, A_2, A_3, D, f, u_0) = u_0, \\
&S_{G,m,p}(t_0, A_0, A_1, A_2, A_3, D, f, u_0) \\
&= A_0(t_0)^{-1}\Big(\partial_t^{p-1} f(t_0) - \sum_{j=1}^{3} A_j \partial_j S_{m,p-1}(t_0, A_0, A_1, A_2, A_3, D, f, u_0) \\
&\qquad - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(t_0) S_{m,p-l}(t_0, A_0, A_1, A_2, A_3, D, f, u_0) \\
&\qquad - \sum_{j=1}^{3}\sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l A_j(t_0) \partial_j S_{m,p-1-l}(t_0, A_0, A_1, A_2, A_3, D, f, u_0) \\
&\qquad - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l D(t_0) S_{m,p-1-l}(t_0, A_0, A_1, A_2, A_3, D, f, u_0)\Big), \qquad (2.36)
\end{aligned}
$$

where $F_{m,\mathrm{pd}}(J \times G) = \bigcup_{\eta>0} F_{m,\eta}(J \times G)$.

Observe that these operators indeed map into $H^{m-p}(G)$ by Lemma 2.33 below. We want to make the motivation before Definition 2.29 precise. To that purpose, we first have to say with which boundary data we are going to work. In Section 3.2 we will see that we have to treat initial boundary value problems which incorporate the loss of half a derivative from the boundary to the interior. We thus make the following definition.

Definition 2.30. *Let $m \in \mathbb{N}_0$, $J \subseteq \mathbb{R}$ be an open interval, and $G \subseteq \mathbb{R}^3$ be a domain with a tame uniform $C^{\max\{m,2\}}$-boundary. We define*

$$E_m(J \times \partial G) = \bigcap_{j=0}^{m} H^j(J, H^{m+\frac{1}{2}-j}(\partial G))$$

and equip this space with the family of norms

$$\|g\|_{E_{m,\gamma}(J\times\partial G)} = \max_{0\le j\le m} \|\partial_t^j g\|_{L^2_\gamma(J,H^{m+1/2-j}(\partial G))} = \max_{0\le j\le m} \|e_{-\gamma}\partial_t^j g\|_{L^2(J,H^{m+1/2-j}(\partial G))}$$

for all $\gamma \ge 0$.

We point out that in the case $G = \mathbb{R}^3_+$ the space $E_m(J \times \partial\mathbb{R}^3_+)$ consists of those functions $g \in L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$ for which all derivatives in time and spatially tangential directions up to order m belong to $L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$. The next lemma makes the motivation before Definition 2.29 precise.

Lemma 2.31. *Let $\eta > 0$, $m, k, l \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take an open interval $J \subseteq \mathbb{R}$ and a domain $G \subseteq \mathbb{R}^3$ with a tame uniform $C^{\max\{m,2\}}$-boundary. Choose $A_0 \in F_{\tilde{m},k,\eta}(J \times G)$, symmetric $A_1, A_2, A_3 \in F_{\tilde{m},k}(J \times G)$, and $D \in F_{\tilde{m},k}(J \times G)$. Let $B \in W^{m+1,\infty}(J \times G)^{l\times k}$. Pick data $f \in H^m(\Omega)^k$, $g \in E_m(J \times \partial G)^l$, and $u_0 \in H^m(\mathbb{R}^3_+)^k$. Assume that there is a solution u of (1.10) on the domain G, i.e., a solution of (5.1), that belongs to $G_m(J \times G)^k$. Then*

$$\partial_t^p u(t) = S_{G,m,p}(t, A_0, \ldots, A_3, D, f, u(t))$$

and

$$\mathrm{tr}_{\partial G}(B S_{G,m,p}(t, A_0, \ldots, D, f, u(t))) := B\, \mathrm{tr}_{\partial G}\, S_{G,m,p}(t, A_0, \ldots, D, f, u(t)) = \partial_t^p g(t)$$

for all $t \in \overline{J}$ and $p \in \{0, \ldots, m-1\}$.

Proof. After differentiating the differential equation in (1.10) with the domain $\mathbb{R}^3_+$ replaced by G $p-1$ times with respect to time, we note that all appearing terms are still continuous with respect to time. Inserting t, solving for $\partial_t^p u(t)$, and performing an induction on p yields $\partial_t^p u(t) = S_{G,m,p}(t, A_0, \ldots, A_3, D, f, u(t))$ for all $t \in \overline{J}$ and $p \in \{0, \ldots, m\}$.

If $p \in \{0, \ldots, m-1\}$, the regularity of u and g allows us to differentiate the boundary condition in (1.10) p times with respect to t, leading to $B \operatorname{tr}_{\partial G} \partial_t^p u = \partial_t^p g$ on $J \times \partial G$. Since $B \operatorname{tr}_{\partial G} \partial_t^p u$ is still continuous with respect to time, we obtain

$$B \operatorname{tr}_{\partial G} S_{G,m,p}(t, A_0, \ldots, A_3, D, f, u(t))) = B \operatorname{tr}_{\partial G} \partial_t^p u(t) = \partial_t^p g(t)$$

for all $p \in \{0, \ldots, m-1\}$. □

The previous lemma applied with $t = t_0$ shows that the identity

$$B \operatorname{tr}_{\partial G} S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)) = \partial_t^p g(t_0)$$

is a necessary condition for the existence of a $G_m(\Omega)$-solution of (1.10). Note that this is a compatibility condition for the coefficients and the data. The natural question at this point is, whether this condition is also sufficient for the existence of a $G_m(J \times G)$-solution. We will answer it positively in Chapter 4.

Definition 2.32. *Let $J \subseteq \mathbb{R}$ be an open interval and $G \subseteq \mathbb{R}^3$ be a domain with a tame uniform $C^{\max\{m,2\}}$-boundary. Pick $m, k, l \in \mathbb{N}$ and set $\tilde{m} = \max\{m, 3\}$. Take $A_0 \in F_{\tilde{m},k,\mathrm{pd}}(J \times G)$, symmetric $A_1, A_2, A_3 \in F_{\tilde{m},k}(J \times G)$, $D \in F_{\tilde{m},k}(J \times G)$, and $B \in W^{m+1,\infty}(J \times G)^{l \times k}$. Choose data $f \in H^m(J \times G)^k$, $g \in E_m(J \times \partial G)^l$, and $u_0 \in H^m(G)^k$. We say that the tupel $(t_0, A_0, \ldots, A_3, D, B, f, g, u_0)$ fulfills the* linear compatibility conditions of order m *if*

$$\operatorname{tr}_{\partial G}(B S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)) = \partial_t^p g(t_0) \quad \text{for } 0 \le p \le m-1. \tag{2.37}$$

As mentioned above the operators $S_{G,m,p}$ will be omnipresent in the following sections. It is therefore essential to have good estimates for them. The next lemma shows that $S_{G,m,p}$ maps into $H^{m-p}(G)$ as claimed and that the $H^{m-p}(G)$-norm can be estimated by suitable norms of the coefficients and the data evaluated at t_0.

Lemma 2.33. *Let $J \subseteq \mathbb{R}$ be an interval and let $t_0 \in \overline{J}$. Take $\eta > 0$, $m \in \mathbb{N}$, and set $\tilde{m} := \max\{m, 3\}$. Let $G \subseteq \mathbb{R}^3$ be a domain with a tame uniform $C^{\max\{m,2\}}$-boundary. Pick $r_0 > 0$. Choose $A_0 \in F_{\tilde{m},\eta}(J \times G)$, symmetric $A_1, A_2, A_3 \in F_{\tilde{m}}(J \times G)$, and $D \in F_{\tilde{m}}(J \times G)$ with*

$$\begin{aligned} &\|A_i(t_0)\|_{F^0_{\tilde{m}-1}(G)} \le r_0, \quad \|D(t_0)\|_{F^0_{\tilde{m}-1}(G)} \le r_0, \\ &\max_{1 \le j \le m-1} \|\partial_t^j A_i(t_0)\|_{H^{\tilde{m}-1-j}(G)} \le r_0, \quad \max_{1 \le j \le m-1} \|\partial_t^j D(t_0)\|_{H^{\tilde{m}-1-j}(G)} \le r_0 \end{aligned}$$

for all $i \in \{0, \ldots, 3\}$. Take $f \in H^m(J \times G)$ and $u_0 \in H^m(G)$. Then the function $S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)$ is contained in $H^{m-p}(G)$ for all $p \in \{0, \ldots, m\}$. Moreover, there exist constants

$$C_{m,p} = C_{m,p}(\eta, r_0) > 0 \tag{2.38}$$

such that

$$\begin{aligned} &\|S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)\|_{H^{m-p}(G)} \\ &\qquad \le C_{m,p}\Big(\sum_{j=0}^{p-1} \|\partial_t^j f(t_0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big) \end{aligned}$$

for $0 \le p \le m$.

Proof. Without loss of generality we assume $t_0 = 0$. Observe that in the case $p = 0$ there is nothing to show. To streamline the notation, we further write $S_{m,p}$ for $S_{G,m,p}(0, A_0, \dots, A_3, D, f, u_0)$.

I) Let $m \in \mathbb{N}$ and $p = 1$. We then have $f(0) \in H^{m-1}(G)$, $\partial_j u_0 \in H^{m-1}(G)$ for $j = 1, 2, 3$, and $D(0)u_0 \in H^{m-1}(G)$ by Lemma 2.22 (vii). Part (vii) of the same lemma and Corollary 2.28 therefore yield $S_{m,1} \in H^{m-1}(G)$ and

$$\begin{aligned}
\|S_{m,1}\|_{H^{m-1}(G)} &\le C\|A_0(0)^{-1}\|_{F^0_{\tilde m-1}(G)} \Big\| f(0) - \sum_{j=1}^3 A_j(0)\partial_j u_0 - D(0)u_0\Big\|_{H^{m-1}(G)} \\
&\le C(\eta)(1 + \|A_0(0)\|_{F^0_{\tilde m-1}(G)})^{\tilde m-1} \\
&\qquad \cdot \big(\|f(0)\|_{H^{m-1}(G)} + C(r_0)\|u_0\|_{H^m(G)} + \|D(0)\|_{F^0_{\tilde m-1}(G)}\|u_0\|_{H^{m-1}(G)}\big) \\
&\le C(\eta, r_0)(1 + \|A_0(0)\|_{F^0_{\tilde m-1}(G)})^{\tilde m-1}(1 + \|D(0)\|_{F^0_{\tilde m-1}(G)}) \\
&\qquad \cdot \big(\|f(0)\|_{H^{m-1}(G)} + \|u_0\|_{H^m(G)}\big).
\end{aligned}$$

The assertion is thus true for all pair $(m, 1)$, $m \in \mathbb{N}$. In particular, we are done in the case $m = 1$.

II) In this step we consider $m \ge 3$ (implying $m = \tilde m$) and $p \in \{2, \dots, m-1\}$. Assume that the assertion has been shown for all $1 \le k < p$. The function $S_{m,p-1}$ then belongs to $H^{m-(p-1)}(G)$ and thus $\partial_j S_{m,p-1}$ is contained in $H^{m-p}(G)$ with

$$\begin{aligned}
\Big\|\sum_{j=1}^3 A_j(0)\partial_j S_{m,p-1}\Big\|_{H^{m-p}(G)} &\le C(r_0)\|S_{m,p-1}\|_{H^{m-p+1}(G)} \\
&\le C(r_0) \cdot C_{m,p-1}\Big(\sum_{j=0}^{p-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big). \qquad (2.39)
\end{aligned}$$

Next let $l \in \{1, \dots, p-1\}$. If $m - 1 - l < 2$, we have $l > m - 3$ and hence

$$m - p + l > 2m - p - 3 \ge 2m - (m-1) - 3 = m - 2 \ge 1,$$

i.e., $m-p+l \ge 2$. Further note that $m-p+l > m-p$ and $m-1-l \ge m-p$. As $\partial_t^l A_0(0) \in H^{m-1-l}(G)$ and $S_{m,p-l} \in H^{m-(p-l)}(G)$, Lemma 2.22 (vi) shows that $\partial_t^l A_0(0)S_{m,p-l} \in H^{\min\{m-p+l, m-1-l\}}(G) \hookrightarrow H^{m-p}(G)$. From the induction hypothesis, we infer that

$$\begin{aligned}
&\|\partial_t^l A_0(0)S_{m,p-l}\|_{H^{m-p}(G)} \le C\|\partial_t^l A_0(0)\|_{H^{m-1-l}(G)}\|S_{m,p-l}\|_{H^{m-p+l}(G)} \\
&\quad \le C\|\partial_t^l A_0(0)\|_{H^{m-1-l}(G)} C_{m,p-l}(\eta, r_0)\Big(\sum_{j=0}^{p-l-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\Big\|\sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l A_0(0)S_{m,p-l}\Big\|_{H^{m-p}(G)} \\
&\le C \max_{1\le l\le p-1}\|\partial_t^l A_0(0)\|_{H^{m-1-l}(G)} \max_{1\le l\le p-1}\|S_{m,p-l}\|_{H^{m-p+l}(G)} \\
&\le C(\eta, r_0)\Big(\sum_{j=0}^{p-2}\|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big). \qquad (2.40)
\end{aligned}$$

Analogously, one obtains

$$\begin{aligned}
&\Big\|\sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l D(0)S_{m,p-l-1}\Big\|_{H^{m-p}(G)} \\
&\le C(1 + \|D(0)\|_{F^0_{m-1}(G)})(1 + \max_{1\le l\le p-1}\|\partial_t^l D(0)\|_{H^{m-1-l}(G)}) \max_{1\le l\le p-1}\|S_{m,p-l-1}\|_{H^{m-p+l+1}(G)}
\end{aligned}$$

$$\leq C(\eta, r_0)\Big(\sum_{j=0}^{p-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big). \tag{2.41}$$

Lemma 2.22 (vii), Corollary 2.28, and (2.39) to (2.41) yield

$$\begin{aligned}
\|S_{m,p}\|_{H^{m-p}(G)} &\leq \|A_0(0)^{-1}\|_{F^0_{\tilde{m}-1}(G)} \Big\| \partial_t^{p-1} f(0) - \sum_{j=1}^{3} A_j \partial_j S_{m,p-1} \\
&\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(0) S_{m,p-l} - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l D(0) S_{m,p-1-l} \Big\|_{H^{m-p}(G)} \\
&\leq C(\eta)(1 + \|A_0(0)\|_{F^0_{\tilde{m}-1}(G)})^{\tilde{m}-1} \Big(\|\partial_t^{p-1} f(0)\|_{H^{m-p}(G)} \\
&\quad + C(\eta, r_0)\Big(\sum_{j=0}^{p-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big)\Big) \\
&\leq C_{m,p}(\eta, r_0)\Big(\sum_{j=0}^{p-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big).
\end{aligned}$$

III) It only remains to show the assertion for the pair (m, m) for all $m \geq 2$. To this purpose we first note that

$$\begin{aligned}
\Big\| \sum_{j=1}^{3} A_j \partial_j S_{m,m-1} \Big\|_{L^2(G)} &\leq C \|S_{m,m-1}\|_{H^1(G)} \\
&\leq C \cdot C_{m,m-1}\Big(\sum_{j=0}^{m-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big)
\end{aligned}$$

due to the first two steps. For $l \in \{1, \dots, m-1\}$ the function $\partial_t^l A_0(0)$ belongs to $H^{\tilde{m}-1-l}(G)$ while $S_{m,m-l}$ is an element of $H^{m-(m-l)}(G) = H^l(G)$ due to the previous steps. Lemma 2.22 (v) thus shows

$$\begin{aligned}
&\|\partial_t^l A_0(0) S_{m,m-l}\|_{L^2(G)} \leq C \|A_0(0)\|_{H^{\tilde{m}-l-1}(G)} \|S_{m,m-l}\|_{H^l(G)} \\
&\leq C(\eta, r_0)\Big(\sum_{j=0}^{m-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big),
\end{aligned}$$

where we also used step I) respectively step II). For $l \in \{0, \dots, m-1\}$ one obtains analogously

$$\begin{aligned}
&\|\partial_t^l D(0) S_{m,m-l-1}\|_{L^2(G)} \leq C \|D(0)\|_{H^{\tilde{m}-l-1}(G)} \|S_{m,m-l-1}\|_{H^l(G)} \\
&\leq C(\eta, r_0)\Big(\sum_{j=0}^{m-2} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big).
\end{aligned}$$

Finally, we deduce as in step II) that

$$\|S_{m,m}\|_{L^2(G)} \leq C_{m,m}(\eta, r_0)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big). \qquad \square$$

At several places in the following it will be important to know that there exists a function v in $G_m(J \times G)$ with prescribed k-th time derivative at time 0 in $H^{m-k}(G)$ for all $k \in \{0, \dots, m\}$. We will show in the next lemma that such a function v exists. The proof is a slight adaption of the proof of Theorem 2.5.7 in [Hoe76], where the existence of an $H^m(\mathbb{R}^n_+)$-function with prescribed boundary trace of the kth normal derivative in $H^{m-k-1/2}(\mathbb{R}^{n-1})$ for $0 \leq k \leq m-1$ is shown.

Lemma 2.34. *Let $m \in \mathbb{N}$ and let $G \subseteq \mathbb{R}^3$ either equal $\mathbb{R}^3$ or be a domain with a tame uniform $C^{\max\{m,2\}}$-boundary. Take $h_k \in H^{m-k}(G)$ for all $k \in \{0, \ldots, m\}$. Then there is a function $u \in G_m(\mathbb{R} \times G)$ with $\partial_t^k u(0) = h_k$ for all $k \in \{0, \ldots, m\}$ and there is a constant $C = C(m)$ such that*

$$\|u\|_{G_m(\mathbb{R}\times G)} \leq C \sum_{k=0}^{m} \|h_k\|_{H^{m-k}(G)}.$$

Proof. Let $g_k \in \mathcal{S}(\mathbb{R}^3)$ for $k \in \{0, \ldots, m\}$ and $\psi \in C_c^\infty(\mathbb{R})$ such that ψ equals 1 in a neighborhood of 0. We define the function v by

$$v(t,x) = \mathcal{F}^{-1}\Big(\psi((1+|\cdot|^2)^{1/2}t) \sum_{k=0}^{m} \hat{g}_k \frac{t^k}{k!}\Big)(x) \tag{2.42}$$

for all $(t,x) \in \mathbb{R}^4$, where $\mathcal{F}$ denotes the spatial Fourier transform. Since the mapping

$$(t,\xi) \mapsto \psi((1+|\xi|^2)^{1/2}t) \sum_{k=0}^{m} \hat{g}_k(\xi) \frac{t^k}{k!}$$

belongs to $\mathcal{S}(\mathbb{R}^4)$, also the inverse spatial Fourier transform is an element of $\mathcal{S}(\mathbb{R}^4)$. The dominated convergence theorem further yields

$$\partial_t^k v(0) = g_k \tag{2.43}$$

for all $k \in \{0, \ldots, m\}$.

The crucial step is to show the estimate

$$\|v\|_{G_m(\mathbb{R}\times\mathbb{R}^3)} \leq C \sum_{k=0}^{m} \|g_k\|_{H^{m-k}(\mathbb{R}^3)}. \tag{2.44}$$

To this purpose we take $j \in \{0, \ldots, m\}$ and compute

$$\begin{aligned}
&\|\partial_t^j v\|^2_{L^\infty(\mathbb{R}, H^{m-j}(\mathbb{R}^3))} = \sup_{t\in\mathbb{R}} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-j} |\mathcal{F}(\partial_t^j v)(t,\xi)|^2 d\xi \\
&\leq C \sup_{t\in\mathbb{R}} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-j} \sum_{k=0}^{m} \Big|\partial_t^j \Big(\psi((1+|\xi|^2)^{1/2}t)\hat{g}_k(\xi)\frac{t^k}{k!}\Big)\Big|^2 d\xi \\
&\leq C \sum_{k=0}^{m} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-j-k} |\hat{g}_k(\xi)|^2 |\partial_t^j(\psi((1+|\xi|^2)^{1/2}t)((1+|\xi|^2)^{1/2}t)^k)|^2 d\xi \\
&\leq C \sum_{k=0}^{m} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} |\hat{g}_k(\xi)|^2 \sup_{t\in\mathbb{R}} |\partial_t^j(\psi(t)t^k)|^2 d\xi \\
&= C \sum_{k=0}^{m} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} |\hat{g}_k(\xi)|^2 d\xi = C \sum_{k=0}^{m} \|g_k\|^2_{H^{m-k}(\mathbb{R}^3)}.
\end{aligned}$$

Taking the maximum over all $j \in \{0, \ldots, m\}$, we arrive at (2.44).

The assumptions on the domain G imply that there exists a total extension operator E for G, see paragraph 4.11 in [AF09] and Theorem VI.3.1.5 in [Ste70]. (In the case $G = \mathbb{R}^3$ we set $E = \mathrm{id}$.) Then the functions Eh_k belong to $H^{m-k}(\mathbb{R}^3)$. We take sequences $(g_n^k)_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^3)$ such that

$$g_n^k \longrightarrow Eh_k$$

in $H^{m-k}(\mathbb{R}^3)$. Let $v_n \in \mathcal{S}(\mathbb{R}^4)$ be the function constructed in (2.42) for the tuple $(g_n^0, \ldots, g_n^m)$ for all $n \in \mathbb{N}$. Using the linearity of the construction (2.42) and estimate (2.44), we deduce that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $G_m(\mathbb{R}\times\mathbb{R}^3)$. It thus converges to a function u in $G_m(\mathbb{R}\times\mathbb{R}^3)$. In particular, we have

$$\partial_t^j u(0) = \lim_{n\to\infty} \partial_t^j v_n(0) = \lim_{n\to\infty} g_n^j = Eh_j$$

for all $j \in \{0, \ldots, m\}$. The restriction of u to $\mathbb{R}\times G$ has all the asserted properties. □

3

A priori estimates for the linearized problem

For given coefficients $A_0, \dots, A_3 \in W^{1,\infty}(\Omega)$ and $D \in L^\infty(\Omega)$ we define the first order linear differential operator $L = L(A_0, \dots, A_3, D)$ by

$$L(A_0, \dots, A_3, D)u = A_0 \partial_t u + \sum_{j=1}^{3} A_j \partial_j u + Du. \tag{3.1}$$

We consider the corresponding linear initial boundary value problem

$$\begin{cases} A_0 \partial_t u + \sum_{j=1}^{3} A_j \partial_j u + Du = f, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bu = g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ u(0) = u_0, & x \in \mathbb{R}^3_+; \end{cases} \tag{3.2}$$

where B is a suitable matrix function. In this section we estimate an a priori given solution of this problem in the norm of $G_m(\Omega)$ by the inhomogeneity f and the initial value u_0 in corresponding norms. We start with a result from [Ell12] which yields such an estimate for $m = 0$ and also establishes the existence and uniqueness of such a solution. We then show that the tangential derivatives of a solution of higher regularity again solve (3.17) with suitable coefficients, inhomogeneities, and initial values involving also lower derivatives of the solution. Iteratively, we can then derive a priori estimates for tangential derivatives of solutions.

Because of the boundary condition in (3.2), there is no hope that the above procedure could also work for the normal derivative. If the boundary is noncharacteristic, i.e., the boundary matrix is invertible, one can express the normal derivative of the solution by the solution itself and its spatial tangential and time derivatives. However, in our case the boundary is characteristic so that we need another technique. We exploit that the differential operator L in fact encodes Maxwell's equations, i.e., the special structure of A_1, A_2 and A_3, to get the estimate for the normal derivative. Finally, an iteration process yields the a priori estimates in $G_m(\Omega)$ for all $m \in \mathbb{N}$.

We start by giving the solution concept for this initial boundary value problem.

Definition 3.1. *Let $J \subseteq \mathbb{R}$ be an interval, $t_0 \in \overline{J}$, and $\Omega = J \times \mathbb{R}^3_+$. Take an inhomogeneity $f \in L^2(\Omega)$, a boundary value $g \in L^2(J, H^{1/2}(\mathbb{R}^3_+))$, and an initial value $u_0 \in L^2(\mathbb{R}^3_+)$. A solution of the linear initial boundary value problem* (3.2) *is a function $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ with*

(i) $Lu = f$ in the weak sense (2.20)*, i.e.,*

$$\langle Lu, \varphi \rangle_{H^{-1} \times H^1_0} = \langle f, \varphi \rangle_{H^{-1} \times H^1_0} = \langle f, \varphi \rangle_{L^2 \times L^2} \qquad \text{for all } \varphi \in H^1_0(\Omega),$$

(ii) $\mathrm{Tr}(Bu) = g$ *on* $J \times \partial\mathbb{R}^3_+$,

(iii) and $u(t_0) = u_0$,

where the trace $\mathrm{Tr}(Bu)$ *is defined in Definition 2.16.*

3.1 Properties of regular solutions

We next note that a weak solution u, which additionally belongs to $G_1(\Omega)$, solves (3.2) also in the strong sense.

Remark 3.2. Assume that a solution u of (3.2) belongs to $G_1(\Omega)$. Then Lu is an element of $L^2(\Omega)$. An integration by parts yields that $Lu = f$ in $L^2(\Omega)$ where the derivatives exist in $L^2(\Omega)$. By Remark 2.17 we also have

$$g = \mathrm{Tr}(Bu) = M \cdot \mathrm{Tr}(A_3 u) = M \cdot \mathrm{Tr}_1(A_3 u) = B \cdot \mathrm{Tr}_1 u.$$

The function u therefore solves (3.2) also in the strong sense, i.e., the derivatives exist in $L^2(\Omega)$ and we obtain the boundary value of Bu by means of the standard trace operator. ◇

The above result is the fundament for our a priori estimates. For the iteration scheme explained in the introduction of this section, it is crucial to know that the derivatives again solve a certain equation.

Lemma 3.3. *Let* $J \subseteq \mathbb{R}$ *be an interval and set* $\Omega = J \times \mathbb{R}^3_+$. *Take coefficients* $A_0, \dots, A_3, D \in W^{1,\infty}(\Omega)$ *and an index* $k \in \{0, \dots, 3\}$. *Choose* $f \in L^2(\Omega)$ *with* $\partial_k f \in L^2(\Omega)$ *and* $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ *such that* $\partial_j u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ *for all* $j \in \{0, \dots, 3\}$ *with* $\partial_k A_j \neq 0$. *Assume that* u *solves* $L(A_0, \dots, A_3, D)u = f$ *in the weak sense, i.e.,*

$$\langle L(A_0, \dots, A_3, D)u, \varphi\rangle_{H^{-1}\times H^1_0} = \langle f, \varphi\rangle_{H^{-1}\times H^1_0}$$

for all $\varphi \in H^1_0(\Omega)$. *Suppose that also* $\partial_k u$ *belongs to* $L^2(\Omega)$. *Then the function* $\partial_k u$ *solves* $L(A_0, \dots, A_3, D)v = f_{1,k}$ *in the weak sense, where*

$$f_{1,k} = \partial_k f - \sum_{j=0}^{3} \partial_k A_j \partial_j u - \partial_k D u$$

belongs to $L^2(\Omega)$.

Proof. Let $j \in \{0, \dots, 3\}$ with $\partial_k A_j \neq 0$. Since $\partial_j u$ and $\partial_k u$ belong to $L^2(\Omega)$, we obtain that both $\partial_j \partial_k u$ and $\partial_k \partial_j u$ belong to $H^{-1}(\Omega)$. As both functions coincide in $\mathcal{D}'(\Omega)$, they are equal in $H^{-1}(\Omega)$.

In the following, we abbreviate $L(A_0, \dots, A_3, D)$ by L. Exploiting that the product of a $W^{1,\infty}(\Omega)$-function with an $H^1_0(\Omega)$-function again belongs to $H^1_0(\Omega)$, we compute

$$\begin{aligned}
\langle L\partial_k u, \varphi\rangle_{H^{-1}\times H^1_0} &= \sum_{j=0}^{3} \langle \partial_j \partial_k u, A_j^T \varphi\rangle_{H^{-1}\times H^1_0} + \langle \partial_k u, D^T \varphi\rangle_{H^{-1}\times H^1_0} \\
&= -\sum_{j=0}^{3} \langle \partial_j u, \partial_k (A_j^T \varphi)\rangle_{L^2\times L^2} - \langle u, \partial_k (D^T \varphi)\rangle_{L^2\times L^2} \\
&= -\langle Lu, \partial_k \varphi\rangle_{H^{-1}\times H^1_0} - \sum_{j=0}^{3} \langle \partial_k A_j \partial_j u, \varphi\rangle_{L^2\times L^2} - \langle \partial_k D u, \varphi\rangle_{L^2\times L^2} \\
&= \langle \partial_k f - \sum_{j=0}^{3} \partial_k A_j \partial_j u - \partial_k D u, \varphi\rangle_{L^2\times L^2} = \langle f_{1,k}, \varphi\rangle_{L^2\times L^2}
\end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$. We conclude that $L\partial_k u = f_{1,k}$ in $H^{-1}(\Omega)$. □

In the following, it will be useful to have a higher order analogue of the above result.

Lemma 3.4. *Let $T' > 0$, $T \in (0, T')$, and $\Omega = (0,T) \times \mathbb{R}^3_+$. Pick $m \in \mathbb{N}$ and set $\tilde{m} = \max\{m, 3\}$. Take $A_0, \ldots, A_3, D \in F_{\tilde{m}}(\Omega)$, $f \in H^m(\Omega)$, and $u \in G_m(\Omega)$. Assume that the function u solves $L(A_0, \ldots, A_3, D)u = f$ in the weak sense. Pick $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and define the function f_α by*

$$f_\alpha = \partial^\alpha f - \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial^{\alpha-\beta} \partial_j u - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta D \partial^{\alpha-\beta} u.$$

Take $r > 0$ such that $\|A_i\|_{F_{\tilde{m}}(\Omega)} \leq r$ and $\|D\|_{F_{\tilde{m}}(\Omega)} \leq r$ for all $i \in \{0, \ldots, 3\}$. Then f_α belongs to $H^{m-|\alpha|}(\Omega)$ and $\partial^\alpha u$ is a weak solution of $L(A_0, \ldots, A_3, D)v = f_\alpha$. Moreover, there is a constant $C_m = C_m(r, T')$ with

$$\|f_\alpha\|_{H^{m-|\alpha|}_\gamma(\Omega)} \leq \|f\|_{H^m_\gamma(\Omega)} + C_m \|u\|_{G_{m,\gamma}(\Omega)}$$

for all $\gamma \geq 0$.

Proof. I) Fix $j \in \{0, \ldots, 3\}$. We first note that $\partial^\beta A_j$ is an element of $\tilde{G}_{\tilde{m}-|\beta|}(\Omega)$ and $\partial_j \partial^{\alpha-\beta} u$ one of $\tilde{G}_{m-|\alpha|+|\beta|-1}(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| \in \{1, \ldots, |\alpha|\}$. In order to establish that their product belongs to $\tilde{G}_{m-|\alpha|}(\Omega)$, we have to distinguish several cases. Fix $\beta \in \mathbb{N}_0^4$ with $|\beta| \in \{1, \ldots, |\alpha|\}$ and $\gamma \geq 0$.

In the case $|\beta| \leq \tilde{m} - 2$ we have

$$\min\{m - |\alpha| + |\beta| - 1, \tilde{m} - |\beta|\} \geq m - |\alpha|$$

and $\tilde{m} - |\beta| \geq 2$ so that Lemma 2.22 (ii) gives $\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u \in \tilde{G}_{m-|\alpha|}(\Omega)$ and

$$\begin{aligned}\|\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|,\gamma}(\Omega)} &\leq C \|\partial^\beta A_j\|_{G_{\tilde{m}-|\beta|}(\Omega)} \|\partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|+|\beta|-1,\gamma}(\Omega)} \\ &\leq Cr \|u\|_{G_{m,\gamma}(\Omega)}.\end{aligned}$$

If $|\beta| = \tilde{m}$, we infer

$$3 \leq \tilde{m} = |\beta| \leq |\alpha| \leq m$$

so that $m = |\alpha|$ and

$$m - |\alpha| + |\beta| - 1 \geq 2.$$

Hence, Lemma 2.22 (ii) again yields that $\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u$ is contained in $\tilde{G}_{m-|\alpha|}(\Omega)$ with

$$\begin{aligned}\|\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|,\gamma}(\Omega)} &\leq C \|\partial^\beta A_j\|_{G_{\tilde{m}-|\beta|}(\Omega)} \|\partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|+|\beta|-1,\gamma}(\Omega)} \\ &\leq Cr \|u\|_{G_{m,\gamma}(\Omega)}.\end{aligned}$$

It remains to consider $|\beta| = \tilde{m} - 1$. (Note that then $m \geq 2$.) If in this case $|\alpha| = |\beta|$ and $m \geq 3$, we once more have that $m - |\alpha| + |\beta| - 1 \geq 2$ and Lemma 2.22 (ii) applies again. If $|\alpha| = |\beta|$ and $m = 2$, then both $\partial^\beta A_j$ and $\partial_j \partial^{\alpha-\beta} u$ belong to $\tilde{G}_1(\Omega)$. Therefore, Lemma 2.22 (i) shows that $\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u$ is an element of $\tilde{G}_0(\Omega) = \tilde{G}_{m-|\alpha|}(\Omega)$ and

$$\begin{aligned}&\|\partial^\beta A_j \partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|,\gamma}(\Omega)} \leq C \|\partial^\beta A_j\|_{G_1(\Omega)} \|\partial_j \partial^{\alpha-\beta} u\|_{G_{m-1,\gamma}(\Omega)} \\ &\leq C \|\partial^\beta A_j\|_{G_{\tilde{m}-|\beta|}(\Omega)} \|\partial_j \partial^{\alpha-\beta} u\|_{G_{m-|\alpha|+|\beta|-1,\gamma}(\Omega)} \leq Cr \|u\|_{G_{m,\gamma}(\Omega)}.\end{aligned}$$

Finally, if $|\alpha| > |\beta|$, we deduce that $m \geq 3$ and that both $\partial^\beta A_j$ and $\partial_j \partial^{\alpha-\beta} u$ are an element of $\tilde{G}_1(\Omega)$. We then proceed as above.

We conclude that $\sum_{j=1}^3 \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha-\beta} u$ belongs to $\tilde{G}_{m-|\alpha|}(\Omega)$. Since $\partial^{\alpha-\beta} u \in \tilde{G}_{m-|\alpha|+|\beta|}(\Omega) \hookrightarrow \tilde{G}_{m-|\alpha|+|\beta|-1}(\Omega)$ for $0 < \beta \leq \alpha$, also the function

$\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta D\partial^{\alpha-\beta}u$ is contained in $\tilde{G}_{m-|\alpha|}(\Omega)$. We have thus shown that f_α belongs to $H^{m-|\alpha|}(\Omega)$ with

$$\|f_\alpha\|_{H_\gamma^{m-|\alpha|}(\Omega)} \leq \|f\|_{H_\gamma^m(\Omega)} + CrT^{1/2}\|u\|_{G_{m,\gamma}(\Omega)}$$

for all $\gamma \geq 0$.

II) As u is the weak solution of (1.6) on J and belongs to $G_1(\Omega)$, the function also solves $L(A_0,\ldots,A_3,D)u = f$ in the strong sense by Remark 3.2. It particularly fulfills

$$A_0\partial_t u + \sum_{j=1}^{3} A_j\partial_j u + Du = f \tag{3.3}$$

on Ω. All appearing factors possess weak derivatives in $\tilde{G}_0(\Omega)$ up to order $m-1$. We can thus apply the product rule to infer

$$\begin{aligned}\sum_{j=0}^{3}\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta A_j\partial_j\partial^{\alpha-\beta}u + \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta D\partial^{\alpha-\beta}u &= \partial^\alpha f,\\ A_0\partial_t\partial^\alpha u + \sum_{j=1}^{3}A_j\partial_j\partial^\alpha u + D\partial^\alpha u &= f_\alpha\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \in \{1,\ldots,m-1\}$.

Next let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$. Take $\alpha' \in \mathbb{N}_0^4$ and $k \in \{0,1,2,3\}$ such that $\alpha = \alpha' + e_k$. Observe that $\partial^{\alpha'}u \in G_1(\Omega)$ solves

$$L(A_0,\ldots,A_3,D)v = f_{\alpha'}$$

in the weak sense, $\partial_j\partial^{\alpha'}u$ belongs to $L^2(\Omega)$ for all $j \in \{0,\ldots,3\}$, and $\partial_k f_{\alpha'}$ is an element of $L^2(\Omega)$ as $f_{\alpha'} \in H^1(\Omega)$ by step I). Lemma 3.3 thus implies that $\partial^\alpha u$ weakly solves

$$L(A_0,\ldots,A_3,D)v = f_{\alpha',k},$$

where we set

$$f_{\alpha',k} = \partial_k f_{\alpha'} - \sum_{j=0}^{3}\partial_k A_j\partial_j\partial^{\alpha'}u - \partial_k D\partial^{\alpha'}u. \tag{3.4}$$

We claim that $f_{\alpha',k} = f_\alpha$. To that purpose, we first note that

$$\begin{aligned}&\{\beta\in\mathbb{N}_0^4\colon 0<\beta\leq\alpha\} = \{\beta\in\mathbb{N}_0^4\colon 0<\beta\leq\alpha';\ \beta_k=0\}\\ &\quad\cup\{\beta\in\mathbb{N}_0^4\colon 0<\beta\leq\alpha';\ 1\leq\beta_k\leq\alpha_k-1\}\cup\{\beta\in\mathbb{N}_0^4\colon 0<\beta\leq\alpha;\ \beta_k=\alpha_k\}\\ &= \{\beta\in\mathbb{N}_0^4\colon 0<\beta\leq\alpha';\ \beta_k=0\}\cup\{\beta\in\mathbb{N}_0^4\colon e_k<\beta\leq\alpha'\}\\ &\quad\cup\{e_k\}\cup\{\beta\in\mathbb{N}_0^4\colon e_k<\beta\leq\alpha;\ \beta_k=\alpha_k\},\end{aligned} \tag{3.5}$$

where all unions are disjoint. We now take a look at the sum

$$\sum_{j=0}^{3}\sum_{0<\beta\leq\alpha'}\binom{\alpha'}{\beta}\partial^\beta A_j\partial_j\partial^{\alpha'-\beta}u$$

appearing in $f_{\alpha'}$. Since every summand is the product of functions possessing weak derivatives of first order in $\tilde{G}_0(\Omega)$, we compute

$$\partial_k\Big(\sum_{j=0}^{3}\sum_{0<\beta\leq\alpha'}\binom{\alpha'}{\beta}\partial^\beta A_j\partial_j\partial^{\alpha'-\beta}u\Big) + \sum_{j=0}^{3}\partial_k A_j\partial_j\partial^{\alpha'}u$$

$$
\begin{aligned}
&= \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^{\beta+e_k} A_j \partial_j \partial^{\alpha-(\beta+e_k)} u \\
&\quad + \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u + \sum_{j=0}^{3} \partial_k A_j \partial_j \partial^{\alpha'} u \\
&= \sum_{j=0}^{3} \sum_{e_k<\beta\leq\alpha} \binom{\alpha'}{\beta - e_k} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u \\
&\quad + \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u + \sum_{j=0}^{3} \partial^{e_k} A_j \partial_j \partial^{\alpha-e_k} u \\
&= \sum_{j=0}^{3} \sum_{e_k<\beta\leq\alpha'} \left(\binom{\alpha'}{\beta - e_k} + \binom{\alpha'}{\beta} \right) \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u \\
&\quad + \sum_{j=0}^{3} \sum_{\substack{e_k<\beta\leq\alpha \\ \beta_k=\alpha_k}} \binom{\alpha'}{\beta - e_k} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u + \sum_{j=0}^{3} \sum_{\substack{0<\beta\leq\alpha' \\ \beta_k=0}} \binom{\alpha'}{\beta} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u \\
&\quad + \sum_{j=0}^{3} \left(\binom{\alpha'}{e_k} + 1 \right) \partial^{e_k} A_j \partial_j \partial^{\alpha-e_k} u,
\end{aligned}
$$

where we used that $\binom{\alpha'}{\beta} = 0$ if $\beta \in \mathbb{N}_0^4 \setminus \{0\}$ does not fulfill $0 < \beta \leq \alpha'$ in the last step. Observe that

$$
\begin{aligned}
\binom{\alpha'}{\beta - e_k} + \binom{\alpha'}{\beta} &= \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \left(\binom{\alpha'_k}{\beta_k - 1} + \binom{\alpha'_k}{\beta_k} \right) = \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha'_k + 1}{\beta_k} \\
&= \prod_{i=0}^{3} \binom{\alpha_i}{\beta_i} = \binom{\alpha}{\beta}
\end{aligned}
$$

for all $\beta \in \mathbb{N}_0$ with $e_k < \beta \leq \alpha'$,

$$
\binom{\alpha'}{\beta - e_k} = \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha'_k}{\alpha_k - 1} = \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha_k}{\alpha_k} = \binom{\alpha}{\beta}
$$

for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and $\beta_k = \alpha_k$,

$$
\binom{\alpha'}{\beta} = \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha'_k}{0} = \prod_{\substack{i=0 \\ i\neq k}}^{3} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha_k}{0} = \binom{\alpha}{\beta}
$$

for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and $\beta_k = 0$, and

$$
\binom{\alpha'}{e_k} + 1 = \alpha'_k + 1 = \alpha_k = \binom{\alpha}{e_k}.
$$

We then conclude via (3.5) that

$$
\begin{aligned}
&\partial_k \Big(\sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^{\beta} A_j \partial_j \partial^{\alpha'-\beta} u \Big) + \sum_{j=0}^{3} \partial_k A_j \partial_j \partial^{\alpha'} u \\
&= \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\beta} A_j \partial_j \partial^{\alpha-\beta} u.
\end{aligned}
$$

Analogously, one derives

$$\partial_k\Big(\sum_{0<\beta\le\alpha'}\binom{\alpha'}{\beta}\partial^\beta D\partial^{\alpha'-\beta}u\Big)+\partial_k D\partial^{\alpha'}u=\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta D\partial^{\alpha-\beta}u.$$

Since also $\partial_k\partial^{\alpha'}f=\partial^\alpha f$, we arrive at

$$f_{\alpha',k}=f_\alpha. \tag{3.6}$$

We have thus shown that $\partial^\alpha u$ is a solution of

$$L(A_0,\dots,A_3,D)v=f_\alpha$$

in the weak sense for all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m$. □

The previous lemma shows that for a $G_m(\Omega)$-solution u of $Lu=f$ the derivative $\partial^\alpha u$ solves $L\partial^\alpha u=f_\alpha$ in the weak sense for all multiindices α with $|\alpha|\le m$. If we differentiate only in tangential variables, we expect that the derivative even solves the initial boundary value problem (3.2) with suitably adapted boundary and initial value. In order to verify this conjecture, we have to study the trace of $B\partial^\alpha u$ for all multiindices α with $|\alpha|\le m$ and $\alpha_3=0$.

Lemma 3.5. *Let $T'>0$, $T\in(0,T')$, and $\Omega=(0,T)\times\mathbb{R}^3_+$. Pick $m\in\mathbb{N}$ and set $\tilde m=\max\{m,3\}$. Take coefficients $A_0,\dots,A_2,D\in F_{\tilde m}(\Omega)$, $A_3\in W^{m+1,\infty}(\Omega)$, and $B\in W^{m+1,\infty}(\Omega)$. Assume that there is a matrix $M\in W^{m+1,\infty}(\Omega)$ such that $B=MA_3$. Choose $f\in H^m(\Omega)$ and $g\in E_m(J\times\partial\mathbb{R}^3_+)$. Suppose that there is a function $u\in H^m(\Omega)$ which solves $L(A_0,\dots,A_3,D)u=f$ in the weak sense and which satisfies $\mathrm{Tr}(Bu)=g$. Pick $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m$ and $\alpha_3=0$. Define the function*

$$g_\alpha=\partial^\alpha g-\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{tr}(\partial^\beta B\,\partial^{\alpha-\beta}u).$$

Then g_α belongs to $L^2(J,H^{1/2}(\partial\mathbb{R}^3_+))$ and $\mathrm{Tr}(B\partial^\alpha u)=g_\alpha$. Moreover, we can estimate

$$\|g_\alpha\|_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}\le\|g\|_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)}+C\|B\|_{W^{m+1,\infty}(\Omega)}\|u\|_{H^m_\gamma(\Omega)}$$

for all $\gamma\ge 0$.

Proof. Take a multiindex $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m$ and $\alpha_3=0$. In the case $|\alpha|\le m-1$ we note that $\partial^\alpha(Bu)$ belongs to $H^1(\Omega)$ so that we can exploit that Tr coincides with the standard trace operator tr applied pointwise in time (cf. Remark 2.17). Since the standard trace operator commutes with derivatives in tangential directions, we obtain

$$\begin{aligned}\partial^\alpha g=\mathrm{tr}(\partial^\alpha(Bu))&=\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{tr}(\partial^\beta B\partial^{\alpha-\beta}u),\\ \mathrm{tr}(B\partial^\alpha u)&=g_\alpha.\end{aligned}$$

Here we exploited that also $\partial^\beta B\partial^{\alpha-\beta}u$ is contained in $H^1(\Omega)$ for all $0\le\beta\le\alpha$.

Next assume that $|\alpha|=m$. Pick $\alpha'\in\mathbb{N}_0^4$ and $j\in\{0,1,2\}$ such that $\alpha=\alpha'+e_j$. The first step shows that $\mathrm{Tr}(B\partial^{\alpha'}u)=\mathrm{tr}(B\partial^{\alpha'}u)=g_{\alpha'}$. The assumptions and Lemma 3.4 further imply that $\partial^{\alpha'}u$ belongs to $H^1(\Omega)$ and that $L(A_0,\dots,A_3,D)\partial^\alpha u$ is contained in $L^2(\Omega)$. Corollary 2.18 (i) thus yields that

$$\begin{aligned}\partial_j\,\mathrm{Tr}(B\partial^{\alpha'}u)&=\mathrm{Tr}(B\partial^\alpha u)+\mathrm{tr}(\partial_jB\partial^{\alpha'}u),\\ \mathrm{Tr}(B\partial^\alpha u)&=\partial_jg_{\alpha'}-\mathrm{tr}(\partial_jB\partial^{\alpha'}u).\end{aligned}$$

The analysis of the binomial coefficients in the proof of Lemma 3.4, see (3.5) to (3.6), now shows that

$$g_\alpha=\partial_jg_{\alpha'}-\mathrm{tr}(\partial_jB\partial^{\alpha'}u)=\mathrm{Tr}(B\partial^\alpha u).$$

Take again a multiindex $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and $\alpha_3 = 0$ and fix $\gamma \geq 0$. To derive the estimate for g_α we first note that $\partial^\alpha g$ is contained in $E_0(J \times \partial\mathbb{R}^3_+)$ and $\|\partial^\alpha g\|_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} \leq \|g\|_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)}$. Moreover, the function $\partial^\beta B \partial^{\alpha-\beta} u$ is an element of $H^1(\Omega)$ and hence,

$$\|\partial^\beta B\partial^{\alpha-\beta}u\|_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} \leq C\|\partial^\beta B\partial^{\alpha-\beta}u\|_{L^2_\gamma(J,H^1(\mathbb{R}^3_+))} \leq C\|B\|_{W^{m+1,\infty}(\Omega)}\|u\|_{H^m_\gamma(\Omega)}$$

for all $0 < \beta \leq \alpha$. The assertion thus follows. □

3.2 First order a priori estimates

We now begin to derive the desired a priori estimates for regular solutions of the linear initial boundary value problem (3.2). Our starting point is an existence and uniqueness result in L^2 from [Ell12], which also yields a basic a priori estimate in L^2. The results from Section 3.1 allow us to apply this zeroth order estimate to tangential derivatives of a more regular solution, leading to a priori estimates in the tangential variables. Since the problem (3.2) has a characteristic boundary, it is then crucial to show that we can also control the normal derivative of a regular solution. Here we heavily rely on the structure of the Maxwell equations. The combination of the estimates for tangential and normal derivatives finally yields a full first order a priori estimate.

In the derivation of a priori estimates without a loss of derivatives we exploit the fact that we study a class of generalized linearized Maxwell equations on the half space due to the localization procedure. This means that the coefficients in front of the spatial derivatives posess a certain structure which resembles the structure of the curl operator. We make this idea more precise by setting

$$\begin{aligned} F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\Omega) = \Big\{ &A \in F^{\mathrm{cp}}_{m,6}(\Omega) \cap W^{m+1,\infty}(\Omega)^{6\times 6} \colon \exists \mu_1,\mu_2,\mu_3 \in F^{\mathrm{cp}}_{m,1}(\Omega) \cap W^{m+1,\infty}(\Omega) \\ &\text{such that } A = \sum_{j=1}^3 A^{\mathrm{co}}_j \mu_j \Big\}, \\ F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega) = \Big\{ &A \in F^{\mathrm{cp}}_{m,6}(\Omega) \cap W^{m+1,\infty}(\Omega)^{6\times 6} \colon \exists \mu_1,\mu_2,\mu_3 \in F^{\mathrm{cp}}_{m,1}(\Omega) \cap W^{m+1,\infty}(\Omega) \\ &\text{such that } A = \sum_{j=1}^3 A^{\mathrm{co}}_j \mu_j \text{ and } \exists k \in \{1,2,3\} \text{ with } |\mu_k| \geq \tau \text{ on } \Omega \Big\}. \end{aligned}$$

Observe that all elements of $F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\Omega)$ and $F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$ are symmetric.

We first remark that if the boundary matrix A_3 of the initial boundary value problem (3.2) belongs to $F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\Omega)$ and does not vanish anywhere, then it satisfies the structural assumption made on its spectrum on page 1925 in [Ell12].

Remark 3.6. Let $\alpha \in \mathbb{R}^3$ and $A = \sum_{i=1}^3 \alpha_i A^{\mathrm{co}}_i$. A straightforward computation shows that

$$\det(\lambda I - A) = \lambda^2(\lambda + |\alpha|)^2(\lambda - |\alpha|)^2$$

for all $\lambda \in \mathbb{C}$. Hence, if α does not equal 0, the matrix A has exaxtly 2 positive and 2 negative eigenvalues, counted with multiplicities, and 0 is a repeated eigenvalue of multiplicity 2. ◇

We will assume in the following that the boundary conditions in (3.2) are *conservative* in the sense of [Ell12], i.e., that for nowhere vanishing $A_3 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$ the matrix B belongs to $W^{1,\infty}(J \times \mathbb{R}^3_+)^{2\times 6}$, constant outside of a compact set, and there exists a matrix $C \in W^{1,\infty}(J \times \mathbb{R}^3_+)^{2\times 6}$, constant outside of a compact set, such that

$$A_3 = \mathrm{Re}(C^T B) = \frac{1}{2}\Big(C^T B + B^T C\Big)$$

on $J \times \partial\mathbb{R}^3_+$. For later reference, we introduce the notation

$$\mathcal{BC}_G(A_3) = \{B \in F_0^{\mathrm{cp}}(J \times G) \colon \exists C \in F_0(J \times G) \text{ such that } A_3 = \operatorname{Re}(C^T B) \text{ on } J \times G\} \tag{3.7}$$

for all nowhere vanishing $A_3 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$ and domains $G \subseteq \mathbb{R}^3$. When we study more regular solutions of (3.2), we also need more regular coefficients. We therefore also define

$$\mathcal{BC}^m_G(A_3) = \{B \in W^{m+1,\infty}_{\mathrm{cp}}(J \times G) \colon \exists C, M \in W^{m+1,\infty}_{\mathrm{cp}}(J \times G) \text{ such that } A_3 = \operatorname{Re}(C^T B) \text{ and } B = MA_3 \text{ on } J \times G\} \tag{3.8}$$

for all $m \in \mathbb{N}$ and domains $G \subseteq \mathbb{R}^3$, where $W^{m+1,\infty}_{cp}(J \times G)$ contains those functions in $W^{m+1,\infty}(J \times G)$ which are constant outside of a compact subset of $\overline{J \times G}$.

Proposition 5.1 of [Ell12] and its proof then give the following result.

Lemma 3.7. *Let $\eta > 0$ and $r \geq r_0 > 0$. Take $A_0 \in F^{\mathrm{cp}}_{0,6,\eta}(\Omega)$, $A_1, A_2, A_3 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$ with $\|A_i\|_{W^{1,\infty}(\Omega)} \leq r$ and $\|A_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$ for all $i \in \{0, \dots, 3\}$, and $A_3(t,x) \neq 0$ for all $(t,x) \in \Omega$. Let $D \in L^\infty(\Omega)$ with $\|D\|_{L^\infty(\Omega)} \leq r$ and $B \in \mathcal{BC}_{\mathbb{R}^3_+}(A_3)$ with $\|B\|_{W^{1,\infty}(\Omega)} \leq r_0$. Let $f \in L^2(\Omega)$, $g \in L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$, and $u_0 \in L^2(\mathbb{R}^3_+)$. Then (3.2) has a unique solution u in $C(\overline{J}, L^2(\mathbb{R}^3_+))$, and there exists a number $\gamma_0 = \gamma_0(\eta, r, A_3) \geq 1$ such that*

$$\begin{aligned}
&\sup_{t \in J} \|e^{-\gamma t} u(t)\|^2_{L^2(\mathbb{R}^3_+)} + \gamma \|u\|^2_{L^2_\gamma(\Omega)} \\
&\leq C_{0,0} \|u_0\|^2_{L^2(\mathbb{R}^3_+)} + C_{0,0} \|g\|^2_{L^2_\gamma(J, H^{1/2}(\partial\mathbb{R}^3_+))} + C_0 \frac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega)}
\end{aligned} \tag{3.9}$$

for all $\gamma \geq \gamma_0$, where $C_0 = C_0(\eta, r, A_3)$ and $C_{0,0} = C_{0,0}(\eta, r_0, A_3)$.

Observe that the first term on the left hand side is nothing else but the $G_{0,\gamma}$-norm of u.

For the proof of the higher order tangential a priori estimates we need a variant of the standard trace theorem with modified Sobolev norms.

Lemma 3.8. *Let $v \in H^1(\mathbb{R}^3_+)$. We then have the estimate*

$$\|\operatorname{tr} v\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)} \leq 4\kappa \|v\|^2_{H^1_{\mathrm{ta}}(\mathbb{R}^3_+)} + \frac{1}{\kappa} \|\partial_3 v\|^2_{L^2(\mathbb{R}^3_+)} \tag{3.10}$$

for all weights $\kappa > 0$, where $\operatorname{tr} \colon H^1(\mathbb{R}^3_+) \to H^{1/2}(\partial\mathbb{R}^3_+)$ denotes the usual trace operator.

Proof. First take v from $C^\infty_c(\mathbb{R}^3)$ and let $\mathcal{F}_2$ denote the two dimensional Fourier transform in x_1- and x_2-direction. We then compute via Hölder's and Young's inequality

$$\begin{aligned}
&\|v(\cdot,0)\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{1}{2}} |\mathcal{F}_2 v(\cdot,0)|^2(\xi) d\xi \\
&= -\int_0^\infty \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{1}{2}} \partial_3((\mathcal{F}_2 v)(\xi, x_3))^2 d\xi dx_3 \\
&= -2\int_0^\infty \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{1}{2}} (\mathcal{F}_2 v)(\xi, x_3) \partial_3(\mathcal{F}_2 v)(\xi, x_3) d\xi dx_3 \\
&\leq 2\Big(\int_0^\infty \int_{\mathbb{R}^2} (1 + |\xi|^2) |\mathcal{F}_2 v(\xi, x_3)|^2 d\xi dx_3\Big)\Big(\int_0^\infty \int_{\mathbb{R}^2} |\partial_3 \mathcal{F}_2 v(\xi, x_3)|^2 d\xi dx_3\Big) \\
&= 2\|v\|_{H^1_{\mathrm{ta}}(\mathbb{R}^3_+)} \|\partial_3 v\|_{L^2(\mathbb{R}^3_+)} \leq 4\kappa \|v\|^2_{H^1_{\mathrm{ta}}(\mathbb{R}^3_+)} + \frac{1}{\kappa} \|\partial_3 v\|^2_{L^2(\mathbb{R}^3_+)}
\end{aligned}$$

for all $\kappa > 0$. Since $C^\infty_c(\overline{\mathbb{R}^3_+})$, (i.e., the restriction of $C^\infty_c(\mathbb{R}^3)$-functions to $\mathbb{R}^3_+$) are dense in $H^1(\mathbb{R}^3_+)$, the assertion now follows. □

We now start to derive the desired a priori estimates. In a first step, we give estimates for the tangential derivatives of a solution.

Lemma 3.9. *Let $\eta > 0$ and $r \geq r_0 > 0$. Let $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, $T' > 0$, $T \in (0, T')$, $J = (0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Take $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2, A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$ with $A_3(t,x) \neq 0$ for all $(t,x) \in \Omega$, $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ with*

$$\|A_i\|_{F_{\tilde{m}}(\Omega)} \leq r, \quad \|D\|_{F_{\tilde{m}}(\Omega)} \leq r,$$
$$\max\{\|A_i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1 \leq j \leq m-1} \|\partial_t^j A_i(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)}\} \leq r_0,$$
$$\max\{\|D(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1 \leq j \leq m-1} \|\partial_t^j D(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)}\} \leq r_0,$$
$$\|B\|_{W^{\tilde{m}+1,\infty}(J \times \partial\mathbb{R}^3_+)} \leq r_0$$

for all $i \in \{0, \ldots, 3\}$. Choose $f \in H^m_{\mathrm{ta}}(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Assume that the solution u of (3.2) belongs to $G_m(\Omega)$. Then there exists $\gamma_m = \gamma_m(\eta, r, \gamma_{3.7;0}) \geq 1$ such that

$$\sum_{\substack{|\alpha| \leq m \\ \alpha_3 = 0}} \|\partial^\alpha u\|^2_{G_{0,\gamma}(\Omega)} + \gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} \leq C_{m,0}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big)$$
$$+ C_{m,0}\|g\|^2_{E_{m,\gamma}(J \times \partial\mathbb{R}^3_+)} + \frac{C_m}{\gamma}\Big(\|f\|^2_{H^m_\gamma(\Omega)} + \|u\|^2_{G_{m,\gamma}(\Omega)}\Big), \tag{3.11}$$

for all $\gamma \geq \gamma_0$, where $C_m = C_m(\eta, r, T', C_{3.7;0})$, and $C_{m,0} = C_{m,0}(\eta, r_0, C_{3.7;0,0})$. Here $\gamma_{3.7;0}$, $C_{3.7;0}$, and $C_{3.7;0,0}$ denote the corresponding constants from Lemma 3.7.

Proof. Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and $\alpha_3 = 0$. Lemma 3.4 yields $L(A_0, \ldots, A_3, D)\partial^\alpha u = f_\alpha$ in $H^{-1}(\Omega)$ with

$$f_\alpha = \partial^\alpha f - \sum_{j=0}^{3} \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial^{\alpha-\beta} \partial_j u - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta D \partial^{\alpha-\beta} u.$$

We further obtain from Lemma 2.31 that

$$\partial^\alpha u(0) = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\mathbb{R}^3_+,m,\alpha_0}(0, A_0, \ldots, A_3, D, f, u_0) =: u_{0,\alpha}.$$

Finally, Lemma 3.5 shows that

$$\mathrm{Tr}(B\partial^\alpha u) = \partial^\alpha g - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{tr}(\partial^\beta B \partial^{\alpha-\beta} u) =: g_\alpha.$$

We conclude that $\partial^\alpha u$ is a solution of the initial boundary value problem

$$\begin{cases} L(A_0, \ldots, A_3, D)v = f_\alpha, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bv = g_\alpha, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = u_{0,\alpha}, & x \in \mathbb{R}^3_+. \end{cases} \tag{3.12}$$

We note that f_α is an element of $H^{m-|\alpha|}(\Omega)$ with

$$\|f_\alpha\|_{H^{m-|\alpha|}_\gamma(\Omega)} \leq \|f\|_{H^m_\gamma(\Omega)} + C_{3.4,m}\|u\|_{G_{m,\gamma}(\Omega)} \tag{3.13}$$

by Lemma 3.4, where $C_{3.4,m} = C_{3.4,m}(r, T')$ denotes the constant from Lemma 3.4. Lemma 2.33 further yields that $u_{0,\alpha}$ belongs to $H^{m-|\alpha|}(\mathbb{R}^3_+)$ and

$$\|u_{0,\alpha}\|_{H^{m-|\alpha|}(\mathbb{R}^3_+)} \leq C_{2.33;m,|\alpha|}\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}\Big), \tag{3.14}$$

where $C_{2.33;m} = C_{2.33;m}(\eta, r_0)$ is the constant from Lemma 2.33. We next estimate g_α. To that purpose, we pick a multiindex $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and we observe that $\partial^\beta B$ belongs to $W^{1,\infty}(J \times \partial\mathbb{R}^3_+)$ and $\partial^{\alpha-\beta} u$ to $H^1(\Omega)$. Hence, $\operatorname{tr} \partial^{\alpha-\beta} u$ is an element of $E_0(J \times \partial\mathbb{R}^3_+)$ and therefore $\operatorname{Tr}(\partial^\beta B \partial^{\alpha-\beta} u)$ is contained in $E_0(J \times \partial\mathbb{R}^3_+)$. Let $\kappa > 0$ be a parameter to be chosen below. Lemma 3.8 applied with weight $\kappa\gamma$ thus yields

$$\begin{aligned}\| \operatorname{tr}(\partial^\beta B \partial^{\alpha-\beta} u)\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} &\leq \|B\|^2_{W^{m+1,\infty}(J\times\partial\mathbb{R}^3_+)} \|\partial^{\alpha-\beta} u\|^2_{L^2_\gamma(J,H^{1/2}(\mathbb{R}^3_+))} \\ &\leq 4r_0^2 \kappa\gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + r_0^2 \frac{1}{\kappa\gamma} \|\partial_3 \partial^{\alpha-\beta} u\|^2_{L^2_\gamma(\Omega)}\end{aligned}$$

for all $\gamma > 0$. Consequently, we obtain

$$\|g_\alpha\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} \leq 2\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + C(m) r_0^2 (\kappa\gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \frac{1}{\kappa\gamma}\|u\|^2_{H^m_\gamma(\Omega)}) \quad (3.15)$$

for all $\gamma > 0$, where we also used that the trace operator commutes with tangential derivatives.

Since $\partial^\alpha u$ solves the initial boundary value problem (3.12), we can apply estimate (3.9) to $\partial^\alpha u$ and then insert estimates (3.13) to (3.15) to deduce

$$\begin{aligned}&\|\partial^\alpha u\|^2_{G_{0,\gamma}(\Omega)} + \gamma \|\partial^\alpha u\|^2_{L^2_\gamma(\Omega)} \\ &\leq C_{0,0} \|u_{0,\alpha}\|^2_{L^2(\mathbb{R}^3_+)} + C_{0,0}\|g_\alpha\|^2_{L^2_\gamma(J,H^{1/2}(\partial\mathbb{R}^3_+))} + C_0 \frac{1}{\gamma}\|f_\alpha\|^2_{L^2_\gamma(\Omega)} \\ &\leq \tilde{C}_{m,0}\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \tilde{C}_{m,0}\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\ &\quad + \tilde{C}_m \kappa\gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \tilde{C}_m \frac{1}{\gamma}\frac{\kappa+1}{\kappa}\|u\|^2_{G_{m,\gamma}(\Omega)} + \tilde{C}_m \frac{1}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}\end{aligned}$$

for all $\gamma \geq \gamma_0$, where $\gamma_0(\eta, r, A_3) = \gamma_{3.7;0}(\eta, r, A_3)$ is the corresponding number from Lemma 3.7 and where $\tilde{C}_{m,0} = \tilde{C}_{m,0}(\eta, r_0, C_{3.7;0,0})$ and $\tilde{C}_m = \tilde{C}_m(\eta, r, T', C_{3.7;0})$ denote constants with the described dependancies which may change from line to line. Summing over all multiindices $\alpha \in \mathbb{N}_0^4$ with $\alpha_3 = 0$ and $|\alpha| \leq m$, we thus arrive at

$$\begin{aligned}&\sum_{\substack{\alpha\in\mathbb{N}_0^4\\ \alpha_3=0}} \|\partial^\alpha u\|^2_{G_{0,\gamma}(\Omega)} + \gamma\|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} \\ &\leq \tilde{C}_{m,0}\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \tilde{C}_{m,0}\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\ &\quad + \tilde{C}_m \kappa\gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \tilde{C}_m \frac{1}{\gamma}\frac{\kappa+1}{\kappa}\|u\|^2_{G_{m,\gamma}(\Omega)} + \tilde{C}_m \frac{1}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)} \qquad (3.16)\end{aligned}$$

for all $\gamma \geq \gamma_0$. We fix the constant $\tilde{C}_{3.16;m} = \tilde{C}_{3.16;m}(\eta, r, T', C_{3.7;0})$ appearing on the right hand side of (3.16) and we set $\kappa = (2\tilde{C}_{3.16;m})^{-1}$. We obtain

$$\begin{aligned}&\sum_{\substack{\alpha\in\mathbb{N}_0^4\\ \alpha_3=0}} \|\partial^\alpha u\|^2_{G_{0,\gamma}(\Omega)} + \gamma\|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} \\ &\leq \tilde{C}_{3.16;m,0}\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \tilde{C}_{3.16;m,0}\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\ &\quad + \frac{\gamma}{2}\|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \tilde{C}_{3.16;m}\frac{1}{\gamma}(1 + 2\tilde{C}_{3.16;m})\|u\|^2_{G_{m,\gamma}(\Omega)} + \tilde{C}_{3.16;m}\frac{1}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}\end{aligned}$$

for all $\gamma \geq \gamma_0$ and the assertion thus follows. □

Remark 3.10. If $m = 1$ in the previous lemma, the proof shows that it is enough to demand that the coefficients belong to $W^{1,\infty}(\Omega)$ and the matrix B to $\mathcal{BC}^1_{\mathbb{R}^3_+}(A_3)$. Also the constants then only depend on the corresponding $W^{1,\infty}(\Omega)$-, $L^\infty(\mathbb{R}^3_+)$-, respectively $W^{2,\infty}(J \times \partial\mathbb{R}^3_+)$-norms. ◇

The above procedure only works in tangential directions because differentiation in the normal direction does not preserve the boundary condition. Since the boundary matrix A_3 is not invertible, we neither obtain the normal derivative from the equation itself. Instead, we will use the structure of the Maxwell equations to get an estimate for the normal derivative.

We consider the initial value problem

$$\begin{cases} L(A_0, \dots, A_3, D)u = f, & x \in \mathbb{R}^3_+, \qquad t \in J; \\ \qquad\qquad u(0) = u_0, & x \in \mathbb{R}^3_+. \end{cases} \tag{3.17}$$

In the spirit of Definition 3.1, we define a solution of (3.17) to be a function $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ with $u(0) = u_0$ in $L^2(\mathbb{R}^3_+)$ and $Lu = f$ in $H^{-1}(\Omega)$. In the iteration and regularization process it will be important that we do not impose a boundary condition in (3.17) and the next lemma.

For the formulation of Lemma 3.11 below we also need the following notion. Take $A_1, A_2, A_3 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$. The definition of this space then implies that there are functions $\mu_{lj} \in F^{\mathrm{cp}}_{0,1}(\Omega)$ such that

$$A_j = \sum_{l=1}^{3} A_l^{\mathrm{co}} \mu_{lj}$$

for all $j \in \{1, 2, 3\}$. We set

$$M = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix},$$

where μ denotes the 3×3-matrix $(\mu_{lj})_{lj}$, and define

$$\mathrm{Div}(A_1, A_2, A_3)h = \Big(\sum_{k=1}^{3} (M^T \nabla h)_{kk}, \sum_{k=1}^{3} (M^T \nabla h)_{(k+3)k} \Big)$$

for all $h \in L^2(\mathbb{R}^3_+)$.

Lemma 3.11. *Let $T' > 0$, $\eta, \tau > 0$, $\gamma \geq 1$, and $r \geq r_0 > 0$. Pick $T \in (0, T']$ and set $J = (0, T)$ and $\Omega = J \times \mathbb{R}^3_+$. Take $A_0 \in F^{\mathrm{cp}}_{0,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{0,\mathrm{coeff},\tau}(\Omega)$, and $D \in F^{\mathrm{cp}}_0(\Omega)$ with*

$$\|A_i\|_{W^{1,\infty}(\Omega)} \leq r, \quad \|D\|_{W^{1,\infty}(\Omega)} \leq r,$$
$$\|A_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0, \quad \|D(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$$

for all $i \in \{0, \dots, 3\}$. Choose $f \in G_0(\Omega)$ with $\mathrm{Div}(A_1, A_2, A_3)f \in L^2(\Omega)$ and $u_0 \in H^1(\mathbb{R}^3_+)$. Let u solve (3.17) with initial value u_0 and inhomogeneity f. Assume that $u \in C^1(\overline{J}, L^2(\mathbb{R}^3_+)) \cap C(\overline{J}, H^1_{\mathrm{ta}}(\mathbb{R}^3_+)) \cap L^\infty(J, H^1(\mathbb{R}^3_+))$. Then u belongs to $G_1(\Omega)$ and there are constants $C_{1,0} = C_{1,0}(\eta, \tau, r_0) \geq 1$ and $C_1 = C_1(\eta, \tau, r, T') \geq 1$ such that

$$\|\nabla u\|^2_{G_{0,\gamma}(\Omega)} \leq e^{C_1 T} \Big((C_{1,0} + T C_1) \Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f\|^2_{G_{0,\gamma}(\Omega)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big)$$
$$+ \frac{C_1}{\gamma} \| \mathrm{Div}(A_1, A_2, A_3) f \|^2_{L^2_\gamma(\Omega)} \Big). \tag{3.18}$$

If f additionally belongs to $H^1(\Omega)$, we get

$$\|\nabla u\|^2_{G_{0,\gamma}(\Omega)} \leq e^{C_1 T} \Big((C_{1,0} + T C_1) \Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big)$$

$$+ \frac{C_1}{\gamma} \|f\|^2_{H^1_\gamma(\Omega)}\Big). \tag{3.19}$$

Finally, if f merely belongs to $L^2(\Omega)$ with $\operatorname{Div}(A_1, A_2, A_3)f \in L^2(\Omega)$, we still have

$$\|\nabla u\|^2_{L^2_\gamma(\Omega)} \leq e^{C_1 T}\Big((C_{1,0} + TC_1)\Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{L^2_\gamma(\Omega)} + \|f\|^2_{L^2_\gamma(\Omega)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big)$$
$$+ \frac{C_1}{\gamma} \|\operatorname{Div}(A_1, A_2, A_3)f\|^2_{L^2_\gamma(\Omega)}\Big). \tag{3.20}$$

Proof. I) We prepare the main part of the proof by showing that $A\nabla u$ has a weak time derivative in $L^\infty(J, H^{-1}(\mathbb{R}^3_+))$, where $(\nabla u)_{kj} = \partial_j u_k$, A is a function from $W^{1,\infty}(\Omega)$, and the time derivative is taken componentwise.

To prove this claim, we take $a \in W^{1,\infty}(\Omega)$ and $v \in C^1(\overline{J}, L^2(\mathbb{R}^3_+)) \cap L^\infty(J, H^1(\mathbb{R}^3_+))$. Let $\varphi \in H^1_0(\mathbb{R}^3_+)$, $\psi \in C_c^\infty(J)$, and $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^3_+)$. Then $t \mapsto a(t)\varphi$ maps J into $H^1_0(\mathbb{R}^3_+)$. Moreover, $a\varphi$ and $\partial_j(a\varphi)$ belong to $L^2(\Omega) \cong L^2(J, L^2(\mathbb{R}^3_+))$. Via cutoff and mollification we deduce that $t \mapsto a(t)\varphi$ is strongly measurable from J to $H^1(\mathbb{R}^3_+)$. We conclude that $a\varphi$ belongs to $L^\infty(J, H^1_0(\mathbb{R}^3_+))$. Analogously, we deduce that $\partial_t a\varphi$ is an element of $L^\infty(J, L^2(\mathbb{R}^3_+))$. Using Remark 2.19, we compute

$$\int_{\mathbb{R}^3_+} \Big(\int_J a(t)\varphi \partial_t \psi(t) dt\Big)(x)\tilde{\varphi}(x) dx = \int_{\mathbb{R}^3_+} \int_J a(t,x)\partial_t\psi(t)\varphi(x)\tilde{\varphi}(x) dt dx$$
$$= -\int_{\mathbb{R}^3_+} \int_J \partial_t a(t,x)\psi(t)\varphi(x)\tilde{\varphi}(x) dx = \int_{\mathbb{R}^3_+} \Big(-\int_J \partial_t a(t)\varphi\psi(t) dt\Big)(x)\tilde{\varphi}(x) dx,$$

where we used that $\psi\varphi \in H^1_0(\Omega)$ and $a\tilde{\varphi} \in H^1(\Omega)$. As $C_c^\infty(\mathbb{R}^3_+)$ is dense in $L^2(\mathbb{R}^3_+)$, we conclude

$$\int_J a(t)\varphi\partial_t\psi(t) dt = -\int_J \partial_t a(t)\varphi\psi(t) dt$$

in $L^2(\mathbb{R}^3_+)$. Therefore, $a\varphi$ has a weak time derivative in $L^\infty(J, L^2(\mathbb{R}^3_+))$ and $\partial_t(a\varphi) = \partial_t a\varphi$.

Since $v \in C^1(\overline{J}, L^2(\mathbb{R}^3_+))$, the function ∇v belongs to $C^1(\overline{J}, H^{-1}(\mathbb{R}^3_+))$ and thus $\partial_t \nabla v$ to $C(\overline{J}, H^{-1}(\mathbb{R}^3_+)) \hookrightarrow L^1(J, H^{-1}(\mathbb{R}^3_+))$. Combined with $\nabla v \in L^\infty(J, L^2(\mathbb{R}^3_+))$, $a\varphi \in L^\infty(J, H^1_0(\mathbb{R}^3_+))$ and $\partial_t(a\varphi) \in L^\infty(J, L^2(\mathbb{R}^3_+)) \hookrightarrow L^1(J, L^2(\mathbb{R}^3_+))$, we can apply a variant of Theorem II.5.12 in [BF13] to deduce that $\langle \nabla v, a\varphi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}$ has a weak time derivative given by

$$\partial_t \langle \nabla v, a\varphi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} = \langle \partial_t\nabla v, a\varphi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} + \langle \nabla v, \partial_t a\varphi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}.$$

We then infer

$$\Big\langle \int_J a(t)\nabla v(t)\partial_t\psi(t) dt, \varphi\Big\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}$$
$$= \int_J \langle \nabla v(t), a(t)\varphi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}\partial_t\psi(t) dt$$
$$= -\int_J \Big(\langle \partial_t\nabla v(t), a(t)\varphi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} + \langle \nabla v(t), \partial_t a(t)\varphi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}\Big)\psi(t) dt$$
$$= -\int_J \langle a(t)\partial_t\nabla v(t) + \partial_t a(t)\nabla v(t), \varphi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}\psi(t) dt$$
$$= \Big\langle -\int_J \Big(a(t)\partial_t\nabla v(t) + \partial_t a(t)\nabla v(t)\Big)\psi(t) dt, \varphi\Big\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)},$$

where we used the canonical embedding of $L^2(\mathbb{R}^3_+)$ into $H^{-1}(\mathbb{R}^3_+)$ via

$$\langle \partial_t a(t)\nabla v(t), \varphi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle \partial_t a(t)\nabla v(t), \varphi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}.$$

As a result, $a\nabla v$ has the weak time derivative

$$\partial_t(a\nabla v) = a\partial_t \nabla v + \partial_t a \nabla v$$

in $L^\infty(J, H^{-1}(\mathbb{R}^3_+))$.

II) For the assertion of the lemma it is enough to show that $\partial_3 u$ belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+))$ and that inequalities (3.18) to (3.20) hold.

By the definition of the spaces $F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$ and $F^{\mathrm{cp}}_{0,\mathrm{coeff},\tau}(\Omega)$ there are functions $\mu_{lj} \in F^{\mathrm{cp}}_{0,1}(\Omega)$ for $l, j \in \{1,2,3\}$ and an index $i \in \{1,2,3\}$ with

$$|\mu_{i3}(t,x)| \geq \tau \quad \text{for all } (t,x) \in \Omega$$

such that

$$A_j = \sum_{l=1}^{3} A_l^{\mathrm{co}} \mu_{lj}$$

for all $j \in \{1,2,3\}$. Moreover,

$$A_l^{\mathrm{co}} = \begin{pmatrix} 0 & -J_l \\ J_l & 0 \end{pmatrix} \text{ with } J_{l;mn} = -\varepsilon_{lmn} \tag{3.21}$$

for all $l, m, n \in \{1,2,3\}$, where ε_{lmn} denotes the Levi-Civita symbol, i.e.,

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \\ -1 & \text{if } (i,j,k) \in \{(3,2,1),(2,1,3),(1,3,2)\}, \\ 0 & \text{else.} \end{cases}$$

We set

$$M = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix},$$

where μ denotes the 3×3-matrix $(\mu_{lj})_{lj}$. Applying step I), we can take componentwise the weak time derivative of $M^T A_0 \nabla u$ and we obtain

$$\begin{aligned}
&\partial_t(M^T A_0 \nabla u) = \partial_t M^T A_0 \nabla u + M^T \partial_t A_0 \nabla u + M^T A_0 \partial_t \nabla u \\
&= \partial_t M^T A_0 \nabla u + M^T \partial_t A_0 \nabla u + M^T A_0 \nabla \partial_t u \\
&= \partial_t M^T A_0 \nabla u + M^T \partial_t A_0 \nabla u + M^T A_0 \nabla \Big(A_0^{-1}\Big(f - \sum_{j=1}^{3} A_j \partial_j u - Du\Big)\Big) \\
&= \partial_t M^T A_0 \nabla u + M^T \partial_t A_0 \nabla u + M^T A_0 \nabla A_0^{-1}\Big(f - \sum_{j=1}^{3} A_j \partial_j u - Du\Big) \\
&\quad + M^T \nabla f - M^T \sum_{j=1}^{3} \nabla A_j \partial_j u - M^T \nabla D u - M^T D \nabla u - M^T \sum_{j=1}^{3} A_j \nabla \partial_j u
\end{aligned} \tag{3.22}$$

in $L^\infty(J, H^{-1}(\mathbb{R}^3_+))$. Here we set

$$((\nabla A_0^{-1})g)_{jk} := \sum_{l=1}^{6} \partial_k A_{0;jl}^{-1} g_l$$

and analogously for A_j with $j \in \{1,2,3\}$ and D. We also use the formula

$$\begin{aligned}
(\nabla(A_0^{-1}g))_{jk} &= \partial_k (A_0^{-1}g)_j = \sum_{l=1}^{6} \partial_k (A_{0;jl}^{-1} g_l) = \sum_{l=1}^{6} (\partial_k A_{0;jl}^{-1} g_l + A_{0;jl}^{-1} \partial_k g_l) \\
&= (\nabla A_0^{-1} g)_{jk} + (A_0^{-1} \nabla g)_{jk},
\end{aligned}$$

which follows for any $\mathbb{R}^6$-valued L^2-function g from the Leibniz rule in $H^{-1}(\Omega)$ for the product of a Lipschitz-function with an L^2-function. We abbreviate

$$\Lambda := \partial_t M^T A_0 \nabla u + M^T \partial_t A_0 \nabla u + M^T A_0 \nabla A_0^{-1}\Big(f - \sum_{j=1}^{3} A_j \partial_j u - Du\Big) + M^T \nabla f - M^T \sum_{j=1}^{3} \nabla A_j \partial_j u - M^T \nabla D u - M^T D \nabla u. \tag{3.23}$$

We further compute

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{kk} = \sum_{j,k=1}^{3}\sum_{l,p=1}^{6} M^T_{kl} A_{j;lp} \partial_k \partial_j u_p = \sum_{j,k,n=1}^{3}\sum_{l,p=1}^{6} M^T_{kl} A^{\mathrm{co}}_{n;lp} \mu_{nj} \partial_k \partial_j u_p = \sum_{j,k,l,n=1}^{3}\sum_{p=1}^{6} \mu_{lk} A^{\mathrm{co}}_{n;lp} \mu_{nj} \partial_k \partial_j u_p,$$

using that $M_{lk} = 0$ for all $(l,k) \in \{4,5,6\} \times \{1,2,3\}$. Formula (3.21) thus leads to

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{kk} = \sum_{j,k,l,n,p=1}^{3} \varepsilon_{nlp} \mu_{lk} \mu_{nj} \partial_k \partial_j u_{p+3}. \tag{3.24}$$

Interchanging the indices l and n as well as k and j, we arrive at

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{kk} = \sum_{j,k,l,n,p=1}^{3} \varepsilon_{lnp} \mu_{nj} \mu_{lk} \partial_j \partial_k u_{p+3} = -\sum_{j,k,l,n,p=1}^{3} \varepsilon_{nlp} \mu_{lk} \mu_{nj} \partial_k \partial_j u_{p+3}. \tag{3.25}$$

Equations (3.24) and (3.25) yield

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{kk} = 0. \tag{3.26}$$

Analogously, we derive

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{(k+3)k} = \sum_{j,k,n=1}^{3}\sum_{l,p=1}^{6} M^T_{(k+3)l} A^{\mathrm{co}}_{n;lp} \mu_{nj} \partial_k \partial_j u_p = \sum_{j,k,l,n=1}^{3}\sum_{p=1}^{6} M_{(l+3)(k+3)} A^{\mathrm{co}}_{n;(l+3)p} \mu_{nj} \partial_k \partial_j u_p,$$

because $M_{l(k+3)} = 0$ for all $(l,k) \in \{1,2,3\} \times \{1,2,3\}$. Exploiting (3.21) again and arguing as before, we infer

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{(k+3)k} = -\sum_{j,k,l,n,p=1}^{3} \varepsilon_{nlp} \mu_{lk} \mu_{nj} \partial_k \partial_j u_p = -\sum_{j,k,l,n,p=1}^{3} \varepsilon_{lnp} \mu_{nj} \mu_{lk} \partial_j \partial_k u_p = \sum_{j,k,l,n,p=1}^{3} \varepsilon_{nlp} \mu_{lk} \mu_{nj} \partial_k \partial_j u_p,$$

$$\sum_{k=1}^{3}\Big(M^T \sum_{j=1}^{3} A_j \nabla \partial_j u\Big)_{(k+3)k} = 0. \tag{3.27}$$

In view of (3.23), equation (3.22) simplifies to

$$\sum_{k=1}^{3} \partial_t (M^T A_0 \nabla u)_{kk} = \sum_{k=1}^{3} \Lambda_{kk},$$
$$\sum_{k=1}^{3} \partial_t (M^T A_0 \nabla u)_{(k+3)k} = \sum_{k=1}^{3} \Lambda_{(k+3)k}.$$

An integration in $H^{-1}(\mathbb{R}^3_+)$ from 0 to t then leads to the identities

$$\sum_{k=1}^{3} (M^T A_0 \nabla u)_{kk}(t) = \sum_{k=1}^{3} (M^T A_0 \nabla u)_{kk}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{kk}(s) ds,$$
$$\sum_{k=1}^{3} (M^T A_0 \nabla u)_{(k+3)k}(t) = \sum_{k=1}^{3} (M^T A_0 \nabla u)_{(k+3)k}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{(k+3)k}(s) ds$$

for all $t \in \overline{J}$. The integrands on the right-hand sides are also integrable with values in $L^2(\mathbb{R}^3_+)$, implying that the integrals exist in $L^2(\mathbb{R}^3_+)$ and the equalities hold in $L^2(\mathbb{R}^3_+)$ for all $t \in \overline{J}$. We recall that the k-th row respectively the k-th column of a matrix M' are denoted by $M'_{k\cdot}$ respectively $M'_{\cdot k}$. We set

$$\begin{aligned} F_7(t) &= \sum_{k=1}^{3} (M^T A_0 \nabla u)_{kk}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{kk}(s) ds - \sum_{k=1}^{2} (M^T A_0)_{k\cdot} \partial_k u(t), \\ F_8(t) &= \sum_{k=1}^{3} (M^T A_0 \nabla u)_{(k+3)k}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{(k+3)k}(s) ds - \sum_{k=1}^{2} (M^T A_0)_{(k+3)\cdot} \partial_k u(t) \end{aligned} \tag{3.28}$$

for all $t \in \overline{J}$. Moreover, we put

$$(F_1, \dots, F_6)^T = f - \sum_{j=0}^{2} A_j \partial_j u - Du. \tag{3.29}$$

The function $F = (F_1, \dots, F_8)^T$ then belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+))$. Introducing the matrix

$$\hat{M} = \begin{pmatrix} A_3 \\ (M^T A_0)_{3\cdot} \\ (M^T A_0)_{6\cdot} \end{pmatrix} \in F_0(\Omega)^{8 \times 6},$$

we obtain

$$\hat{M} \partial_3 u = F. \tag{3.30}$$

For the convenience of the reader, we note that

$$\hat{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & \mu_{33} & -\mu_{23} \\ 0 & 0 & 0 & -\mu_{33} & 0 & \mu_{13} \\ 0 & 0 & 0 & \mu_{23} & -\mu_{13} & 0 \\ 0 & -\mu_{33} & \mu_{23} & 0 & 0 & 0 \\ \mu_{33} & 0 & -\mu_{13} & 0 & 0 & 0 \\ -\mu_{23} & \mu_{13} & 0 & 0 & 0 & 0 \\ M^T_{3l} A_{0;l1} & M^T_{3l} A_{0;l2} & M^T_{3l} A_{0;l3} & M^T_{3_l} A_{0;l4} & M^T_{3l} A_{0;l5} & M^T_{3l} A_{0;l6} \\ M^T_{6l} A_{0;l1} & M^T_{6l} A_{0;l2} & M^T_{6l} A_{0;l3} & M^T_{6_l} A_{0;l4} & M^T_{6l} A_{0;l5} & M^T_{6l} A_{0;l6} \end{pmatrix},$$

where summation over the index l (from 1 to 6) is implicitly assumed. By hypothesis, there is an index $j \in \{1, 2, 3\}$ such that

$$|\mu_{j3}(t, x)| \geq \tau \tag{3.31}$$

for all $(t,x) \in \Omega$. We assume that $j = 3$. The other cases are treated analogously. We multiply $\hat{M}$ with the matrices

$$G_1 = \begin{pmatrix} \mu_{33}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{33}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_{33}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{33}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.32}$$

and

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_{13} & -\mu_{23} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_{13} & \mu_{23} & 1 & 0 & 0 \\ -M_{3l}^T A_{0;l5} & -M_{3l}^T A_{0;l4} & 0 & -M_{3l}^T A_{0;l2} & -M_{3l}^T A_{0;l1} & 0 & 1 & 0 \\ -M_{6l}^T A_{0;l5} & -M_{6l}^T A_{0;l4} & 0 & -M_{6l}^T A_{0;l2} & -M_{6l}^T A_{0;l1} & 0 & 0 & 1 \end{pmatrix}. \tag{3.33}$$

It follows

$$G_2 G_1 \hat{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -\mu_{33}^{-1}\mu_{23} \\ 0 & 0 & 0 & 1 & 0 & -\mu_{33}^{-1}\mu_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\mu_{33}^{-1}\mu_{23} & 0 & 0 & 0 \\ 1 & 0 & -\mu_{33}^{-1}\mu_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 & 0 & \alpha_{36} \\ 0 & 0 & \alpha_{63} & 0 & 0 & \alpha_{66} \end{pmatrix}$$

with the numbers

$$\alpha_{33} = \mu_{33}^{-1} \sum_{j=1}^{3} \sum_{l=1}^{6} M_{3l}^T A_{0;lj} \mu_{j3} = \mu_{33}^{-1} M_{3\cdot}^T A_0 M_{\cdot 3},$$

$$\alpha_{36} = \mu_{33}^{-1} \sum_{j=1}^{3} \sum_{l=1}^{6} M_{3l}^T A_{0;l(j+3)} \mu_{j3} = \mu_{33}^{-1} M_{3\cdot}^T A_0 M_{\cdot 6},$$

$$\alpha_{63} = \mu_{33}^{-1} \sum_{j=1}^{3} \sum_{l=1}^{6} M_{6l}^T A_{0;lj} \mu_{j3} = \mu_{33}^{-1} M_{6\cdot}^T A_0 M_{\cdot 3},$$

$$\alpha_{66} = \mu_{33}^{-1} \sum_{j=1}^{3} \sum_{l=1}^{6} M_{6l}^T A_{0;l(j+3)} \mu_{j3} = \mu_{33}^{-1} M_{6\cdot}^T A_0 M_{\cdot 6}.$$

Let $\xi \in \mathbb{R}^2$ with $|\xi| = 1$. We then estimate

$$\begin{aligned} \xi^T \mu_{33} \begin{pmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{pmatrix} \xi &= (0,0,\xi_1,0,0,\xi_2) M^T A_0 M (0,0,\xi_1,0,0,\xi_2)^T \\ &\geq \eta |M(0,0,\xi_1,0,0,\xi_2)^T|^2 = \eta |\mu_{\cdot 3}|^2 |\xi|^2 \geq \eta \mu_{33}^2. \end{aligned} \tag{3.34}$$

Due to (3.31) the function μ_{33} does not change its sign so that (3.31) further implies that

$$\xi^T \begin{pmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{pmatrix} \xi \geq \eta \tau$$

or

$$\xi^T \begin{pmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{pmatrix} \xi \leq -\eta\tau,$$

i.e., the matrix

$$\begin{pmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{pmatrix} \tag{3.35}$$

is either positive or negative definite. Hence, it has an inverse β satisfying

$$\|\beta\|_{L^\infty(\Omega)} \leq C(\eta, \tau).$$

Introducing the matrices

$$G_3 = \begin{pmatrix} I_{6\times 6} & 0 \\ 0 & \beta \end{pmatrix} \tag{3.36}$$

and

$$G_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_{33}^{-1}\mu_{23} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mu_{33}^{-1}\mu_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \mu_{33}^{-1}\mu_{23} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \mu_{33}^{-1}\mu_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.37}$$

we compute

$$G_4 G_3 G_2 G_1 \hat{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =: \tilde{M}. \tag{3.38}$$

We further point out that

$$\|G_4 G_3 G_2 G_1\|_{L^\infty(\Omega)} \leq C(\eta, \tau)(1 + c_0)^3$$

with the constant

$$c_0 = \max\{\max_{j=0,\dots,3} \|A_j\|_{L^\infty(\Omega)}, \|D\|_{L^\infty(\Omega)}\}.$$

Equation (3.30) and (3.38) yield

$$\tilde{M}\partial_3 u = G_4 G_3 G_2 G_1 F. \tag{3.39}$$

Since the matrices G_i belong to $C(\overline{J}, L^\infty(\mathbb{R}^3_+))$ and F is contained in $C(\overline{J}, L^2(\mathbb{R}^3_+))$, we infer that $\partial_3 u$ is contained in $C(\overline{J}, L^2(\mathbb{R}^3_+))$ and

$$\|\partial_3 u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(\eta, \tau)(1 + c_0)^3 \|F(t)\|_{L^2(\mathbb{R}^3_+)} \tag{3.40}$$

for all $t \in \overline{J}$. To estimate $\|F(t)\|_{L^2(\mathbb{R}^3_+)}$ we first note that

$$\begin{aligned} \|F(t)\|_{L^2(\mathbb{R}^3_+)} &\leq \|(F_1, \dots, F_6)^T(t)\|_{L^2(\mathbb{R}^3_+)} + \|(F_7, F_8)^T(t)\|_{L^2(\mathbb{R}^3_+)} \qquad (3.41) \\ &\leq \|f(t)\|_{L^2(\mathbb{R}^3_+)} + c_0 \sum_{j=0}^{2} \|\partial_j u(t)\|_{L^2(\mathbb{R}^3_+)} + c_0 \|u(t)\|_{L^2(\mathbb{R}^3_+)} + \|(F_7, F_8)^T(t)\|_{L^2(\mathbb{R}^3_+)} \end{aligned}$$

for all $t \in \overline{J}$. Applying Minkowski's inequality, we further deduce

$$\|(F_7, F_8)^T(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(r_0)\|u_0\|_{H^1(\mathbb{R}^3_+)} + c_0 \sum_{k=1}^{2} \|\partial_k u(t)\|_{L^2(\mathbb{R}^3_+)}$$
$$+ C(\eta, r) \int_0^t (\|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} + \|u(s)\|_{L^2(\mathbb{R}^3_+)} + \|\operatorname{Div} f(s)\|_{L^2(\mathbb{R}^3_+)} + \|f(s)\|_{L^2(\mathbb{R}^3_+)}) ds$$

for all $t \in \overline{J}$, where we abbreviate $\operatorname{Div}(A_1, A_2, A_3)$ by Div. This estimate, (3.40) and (3.41), lead to the inequality

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\eta, \tau)(1 + c_0)^3 \Big(\|f(t)\|_{L^2(\mathbb{R}^3_+)} + c_0 \sum_{j=0}^{2} \|\partial_j u(t)\|_{L^2(\mathbb{R}^3_+)} + c_0 \|u(t)\|_{L^2(\mathbb{R}^3_+)} \\
&\quad + C(r_0)\|u_0\|_{H^1(\mathbb{R}^3_+)} + C(\eta, r) \int_0^t (\|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} + \|u(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\qquad + \|\operatorname{Div} f(s)\|_{L^2(\mathbb{R}^3_+)} + \|f(s)\|_{L^2(\mathbb{R}^3_+)}) ds \Big)
\end{aligned} \tag{3.42}$$

for all $t \in \overline{J}$. Next we fix a number $\gamma \geq 1$. Using Hölder's inequality, we infer

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\eta, \tau)(1 + c_0)^3 \Big(e^{\gamma t} \|f\|_{G_{0,\gamma}(\Omega)} + c_0 e^{\gamma t} \sum_{j=0}^{2} \|\partial_j u\|_{G_{0,\gamma}(\Omega)} + c_0 e^{\gamma t} \|u\|_{G_{0,\gamma}(\Omega)} \\
&\quad + C(r_0)\|u_0\|_{H^1(\mathbb{R}^3_+)} + C(\eta, r) \Big(\int_0^t e^{2\gamma s} ds \Big)^{1/2} \Big(\|u\|_{L^2_\gamma(\Omega)} + \|\operatorname{Div} f\|_{L^2_\gamma(\Omega)} + \|f\|_{L^2_\gamma(\Omega)} \Big) \\
&\quad + C(\eta, r) \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} ds \Big) \\
&\leq C(\eta, \tau)(1 + c_0)^3 \Big(e^{\gamma t} \|f\|_{G_{0,\gamma}(\Omega)} + c_0 e^{\gamma t} \sum_{j=0}^{2} \|\partial_j u\|_{G_{0,\gamma}(\Omega)} + c_0 e^{\gamma t} \|u\|_{G_{0,\gamma}(\Omega)} \\
&\quad + C(r_0)\|u_0\|_{H^1(\mathbb{R}^3_+)} + C(\eta, r) \frac{1}{\sqrt{\gamma}} e^{\gamma t} \Big(\|u\|_{L^2_\gamma(\Omega)} + \|\operatorname{Div} f\|_{L^2_\gamma(\Omega)} + \|f\|_{L^2_\gamma(\Omega)} \Big) \\
&\quad + C(\eta, r) \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} ds \Big) \\
&=: g(t) + C(\eta, r) \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} ds
\end{aligned}$$

for all $t \in \overline{J}$. Since the function g increases in t, Gronwall's inequality yields

$$\begin{aligned}
\|\nabla u(t)\|_{L^2(\mathbb{R}^3_+)} &\leq C(\eta, \tau, r_0)(1 + c_0)^4 e^{\gamma t} \Big(\|f\|_{G_{0,\gamma}(\Omega)} + \sum_{j=0}^{2} \|\partial_j u\|_{G_{0,\gamma}(\Omega)} + \|u\|_{G_{0,\gamma}(\Omega)} \\
&\quad + \|u_0\|_{H^1(\mathbb{R}^3_+)} + C(\eta, r) \frac{1}{\sqrt{\gamma}} \Big(\|u\|_{L^2_\gamma(\Omega)} + \|\operatorname{Div} f\|_{L^2_\gamma(\Omega)} + \|f\|_{L^2_\gamma(\Omega)} \Big) \Big) e^{C(\eta, r) t}, \\
\|\nabla u\|_{G_{0,\gamma}(\Omega)} &\leq C(\eta, \tau, r_0)(1 + c_0)^4 \Big(\|f\|_{G_{0,\gamma}(\Omega)} + \sum_{j=0}^{2} \|\partial_j u\|_{G_{0,\gamma}(\Omega)} + \|u\|_{G_{0,\gamma}(\Omega)} \\
&\quad + \|u_0\|_{H^1(\mathbb{R}^3_+)} + C(\eta, r)(\|u\|_{G_{0,\gamma}(\Omega)} + \|f\|_{G_{0,\gamma}(\Omega)}) \\
&\quad + C(\eta, r) \frac{1}{\sqrt{\gamma}} \|\operatorname{Div} f\|_{L^2_\gamma(\Omega)} \Big) e^{C(\eta, r) T}
\end{aligned} \tag{3.43}$$

for all $t \in \overline{J}$. Since $\partial_t A_0$ belongs to $L^\infty(\Omega)$, we obtain

$$\begin{aligned}\|A_0\|_{L^\infty(\Omega)} &= \Big\|A_0(0) + \int_0^t \partial_t A_0(s)ds\Big\|_{L^\infty(\Omega)} \leq \|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} + T\|A_0\|_{W^{1,\infty}(\Omega)} \\ &\leq r_0 + Tr.\end{aligned}$$

We argue analogously for A_j with $j \in \{1, 2, 3\}$ and D, which yields $c_0 \leq r_0 + Tr$.

To conclude (3.18), we write u as

$$u(t) = u(0) + \int_0^t \partial_t u(s)ds$$

in $L^2(\mathbb{R}^3_+)$ using that u belongs to $C^1(\overline{J}, L^2(\mathbb{R}^3_+))$. Minkowski's and Hölder's inequality then imply

$$\begin{aligned}\|u\|_{G_{0,\gamma}(\Omega)} &\leq \|u(0)\|_{L^2(\mathbb{R}^3_+)} + \sup_{t\in J}\Big(e^{-\gamma t}\int_0^t \|\partial_t u(s)\|_{L^2(\mathbb{R}^3_+)}ds\Big) \\ &\leq \|u_0\|_{L^2(\mathbb{R}^3_+)} + \sup_{t\in J}\Big(e^{-\gamma t}\Big(\int_0^t e^{2\gamma s}ds\Big)^{1/2}\Big(\int_0^t e^{-2\gamma s}\|\partial_t u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds\Big)^{1/2}\Big) \\ &\leq \|u_0\|_{L^2(\mathbb{R}^3_+)} + \frac{1}{\sqrt{2\gamma}}\|\partial_t u\|_{L^2_\gamma(\Omega)}, \\ \|u\|^2_{G_{0,\gamma}(\Omega)} &\leq 2\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + \frac{1}{\gamma}T\|\partial_t u\|^2_{G_{0,\gamma}(\Omega)}. \end{aligned} \tag{3.44}$$

Plugging this inequality into (3.43), the assertion (3.18) follows. If f additionally belongs to $H^1(\Omega)$, we argue as in (3.44) for the function f. We infer

$$\|f\|^2_{G_{0,\gamma}(\Omega)} \leq 2\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)}. \tag{3.45}$$

Inserting this estimate into (3.18) and adapting $C_{1,0}$ and C_1, the estimate (3.19) follows.

Now assume that f only belongs to $L^2(\Omega)$ with $\operatorname{Div} f \in L^2(\Omega)$. Then estimate (3.42) is still valid for almost all $t \in J$. We square (3.42), multiply with the exponential $e_{-2\gamma}$, and integrate from 0 to t. Also applying Hölder's inequality in the second estimate, we derive

$$\begin{aligned}&\int_0^t e^{-2\gamma s}\|\nabla u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds \\ &\leq C(\eta,\tau)(1+c_0)^6\Big(\int_0^t e^{-2\gamma s}\|f(s)\|^2_{L^2(\mathbb{R}^3_+)}ds + c_0^2\sum_{j=0}^2\int_0^t e^{-2\gamma s}\|\partial_j u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds \\ &\quad + c_0^2\int_0^t e^{-2\gamma s}\|u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds + C(r_0)\int_0^t e^{-2\gamma s}ds\|u_0\|^2_{H^1(\mathbb{R}^3_+)} \\ &\quad + C(\eta,r)\int_0^t e^{-2\gamma s}\Big(\int_0^s (\|\nabla u(s')\|_{L^2(\mathbb{R}^3_+)} + \|u(s')\|_{L^2(\mathbb{R}^3_+)} + \|f(s')\|_{L^2(\mathbb{R}^3_+)} \\ &\qquad + \|\operatorname{Div} f(s')\|_{L^2(\mathbb{R}^3_+)})ds'\Big)^2 ds\Big) \\ &\leq C(\eta,\tau)(1+c_0)^6\Big(\|f\|^2_{L^2_\gamma(\Omega)} + c_0^2\sum_{j=0}^2\|\partial_j u\|^2_{L^2_\gamma(\Omega)} + c_0^2\|u\|^2_{L^2_\gamma(\Omega)} + C(r_0)\frac{1}{\gamma}\|u_0\|^2_{H^1(\mathbb{R}^3_+)} \\ &\quad + C(\eta,r)\int_0^t e^{-2\gamma s}\Big(\int_0^s e^{2\gamma s'}ds'\Big)\Big(\int_0^s e^{-2\gamma s'}(\|\nabla u(s')\|^2_{L^2(\mathbb{R}^3_+)} + \|u(s')\|^2_{L^2(\mathbb{R}^3_+)} \\ &\qquad + \|f(s')\|^2_{L^2(\mathbb{R}^3_+)} + \|\operatorname{Div} f(s')\|^2_{L^2(\mathbb{R}^3_+)})ds'\Big)ds\Big)\end{aligned}$$

$$\le C(\eta,\tau)(1+c_0)^6\Big(\|f\|^2_{L^2_\gamma(\Omega)} + c_0^2\sum_{j=0}^2\|\partial_j u\|^2_{L^2_\gamma(\Omega)} + c_0^2\|u\|^2_{L^2_\gamma(\Omega)} + C(r_0)\frac{1}{\gamma}\|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big)$$
$$+ C(\eta,\tau,r)\frac{1}{\gamma}\Big(T\|u\|^2_{L^2_\gamma(\Omega)} + T\|f\|^2_{L^2_\gamma(\Omega)} + T\|\operatorname{Div} f\|^2_{L^2_\gamma(\Omega)}\Big)$$
$$+ C(\eta,\tau,r)\frac{1}{\gamma}\int_0^t\int_0^s e^{-2\gamma s'}\|\nabla u(s')\|^2_{L^2(\mathbb{R}^3_+)}ds'\,ds$$

for all $t\in\overline{J}$. Gronwall's inequality thus yields

$$\int_0^t e^{-2\gamma s}\|\nabla u(s)\|^2_{L^2(\mathbb{R}^3_+)}ds$$
$$\le C(\eta,\tau,r_0)(1+c_0)^8\Big(\|f\|^2_{L^2_\gamma(\Omega)} + \sum_{j=0}^2\|\partial_j u\|^2_{L^2_\gamma(\Omega)} + \|u\|^2_{L^2_\gamma(\Omega)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big)e^{C(\eta,\tau,r)t}$$
$$+ C(\eta,\tau,r)\frac{1}{\gamma}\Big(T\|u\|^2_{L^2_\gamma(\Omega)} + T\|f\|^2_{L^2_\gamma(\Omega)} + T\|\operatorname{Div} f\|^2_{L^2_\gamma(\Omega)}\Big)e^{C(\eta,\tau,r)t}$$

for all $t\in\overline{J}$. We insert the time $t=T$ in this estimate and exploit again that $c_0\le r_0+Tr$. Moreover, we deduce similar to (3.44) that

$$\|u(t)\|_{L^2(\mathbb{R}^3_+)} \le \|u_0\|_{L^2(\mathbb{R}^3_+)} + \int_0^t\|\partial_t u(s)\|_{L^2(\mathbb{R}^3_+)}ds \quad (t\in\overline{J}),$$
$$\|u\|^2_{L^2_\gamma(\Omega)} \le 2\int_0^T e^{-2\gamma t}\|u_0\|^2_{L^2(\mathbb{R}^3_+)}dt + 2\int_0^T e^{-2\gamma t}\Big(\int_0^t\|u(s)\|_{L^2(\mathbb{R}^3_+)}ds\Big)^2 dt$$
$$\le 2T'\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + T'\|\partial_t u\|^2_{L^2_\gamma(\Omega)}.$$

We employ this estimate and inequality (3.44) and we adapt $C_{1,0}$ and C_1 to conclude (3.20). □

We can now combine Lemmas 3.7, 3.9, and 3.11 to the following corollary.

Corollary 3.12. *Let $T'>0$, $\eta,\tau>0$, and $r\ge r_0>0$. Pick $T\in(0,T']$ and set $J=(0,T)$ and $\Omega=J\times\mathbb{R}^3_+$. Take $A_0\in F^{\mathrm{cp}}_{0,\eta}(\Omega)$, $A_1,A_2\in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\Omega)$, $A_3\in F^{\mathrm{cp}}_{0,\mathrm{coeff},\tau}(\Omega)$, and $D\in F^{\mathrm{cp}}_0(\Omega)$ with $\|A_i\|_{W^{1,\infty}(\Omega)}\le r$, $\|D\|_{W^{1,\infty}(\Omega)}\le r$, $\|A_i(0)\|_{L^\infty(\mathbb{R}^3_+)}\le r_0$, and $\|D(0)\|_{L^\infty(\mathbb{R}^3_+)}\le r_0$ for all $i\in\{0,\dots,3\}$. Choose a function $B\in\mathcal{BC}^1_{\mathbb{R}^3_+}(A_3)$ with $\|B\|_{W^{2,\infty}(J\times\partial\mathbb{R}^3_+)}\le r_0$. Let $f\in H^1(\Omega)$, $g\in E_1(J\times\partial\mathbb{R}^3_+)$, and $u_0\in H^1(\mathbb{R}^3_+)$. Assume that the solution u of (3.2) belongs to $G_1(\Omega)=C^1(\overline{J},L^2(\mathbb{R}^3_+))\cap C(\overline{J},H^1(\mathbb{R}^3_+))$. Then there is a number $\gamma_1=\gamma_1(\eta,\tau,r,T',\gamma_{3.7;0})\ge1$ such that*

$$\|u\|^2_{G_{1,\gamma}(\Omega)} \le (C_{1,0}+TC_1)e^{C_1T}\Big(\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big)$$
$$+ C_1e^{C_1T}\frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)},$$

for all $\gamma\ge\gamma_1$, where $C_{1,0}=C_{1,0}(\eta,\tau,r_0,C_{3.7;0,0})\ge1$ and $C_1=C_1(\eta,\tau,r,T',C_{3.7;0})\ge 1$. Here the constants $\gamma_{3.7;0}$, $C_{3.7;0,0}$, and $C_{3.7;0}$ are the constants from Lemma 3.7.

Proof. Applying Lemma 3.11, i.e., estimate (3.19), and estimate (3.45) with f replaced by u, we first deduce that

$$\|u\|^2_{G_{1,\gamma}(\Omega)} \le \sum_{j=0}^2\|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|u\|^2_{G_{0,\gamma}(\Omega)} + \|\partial_3 u\|^2_{G_{0,\gamma}(\Omega)}$$
$$\le \Big(1+\frac{T}{\gamma}\Big)\sum_{j=0}^2\|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + 2\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + C_1'e^{C_1'T}\frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)}$$

$$+ (C'_{1,0} + TC'_1)e^{C'_1 T}\Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big)$$

for all $\gamma \geq 1$, where $C'_{1,0} = C'_{1,0}(\eta, \tau, r_0)$ and $C'_1 = C'_1(\eta, \tau, r, T')$ are the corresponding constants from Lemma 3.11. Remark 3.10 next implies

$$\begin{aligned}\|u\|^2_{G_{1,\gamma}(\Omega)} &\leq (\tilde{C}_{1,0} + T\tilde{C}_1)e^{\tilde{C}_1 T}\Big(\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big) + \tilde{C}_1 e^{\tilde{C}_1 T}\frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)} \\ &\quad + (\tilde{C}_{1,0} + T\tilde{C}_1)e^{\tilde{C}_1 T}\Big(C''_{1,0}\Big(\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big) \\ &\qquad + C''_{1,0}\|g\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \frac{C''_1}{\gamma}\Big(\|f\|^2_{H^1_\gamma(\Omega)} + \|u\|^2_{G_{1,\gamma}(\Omega)}\Big)\Big)\end{aligned}$$

for all $\gamma \geq \gamma_0$, where $\gamma_0 = \gamma_0(\eta, r, \gamma_{3.7;0})$, $C''_{1,0} = C''_{1,0}(\eta, r_0, C_{3.7;0,0})$, and $C''_1 = C''_1(\eta, r, C_{3.7;0})$ are the corresponding constants from Remark 3.10, while the constants $\tilde{C}_1 = \tilde{C}_1(\eta, \tau, r, T', C_{3.7;0})$ and $\tilde{C}_{1,0} = \tilde{C}_{1,0}(\eta, \tau, r_0, C_{3.7;0,0})$ may change from line to line. Choosing $\gamma_1 = \gamma_1(\eta, \tau, r, T', \gamma_{3.7;0,0})$ so large that

$$\gamma_1 \geq \gamma_0 \qquad \text{and} \qquad \gamma_1 \geq 2(\tilde{C}_{1,0} + T'\tilde{C}_1)e^{\tilde{C}_1 T'}C''_1,$$

we arrive at

$$\begin{aligned}\|u\|^2_{G_{1,\gamma}(\Omega)} &\leq (\tilde{C}_{1,0} + T\tilde{C}_1)e^{C_1 T}\Big(\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big) \\ &\quad + \tilde{C}_1 e^{\tilde{C}_1 T}\frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)}\end{aligned}$$

for all $\gamma \geq \gamma_1$. □

3.3 Higher order a priori estimates

The a priori estimates of higher order now follow by an iteration process. Performing this iteration, the operators $S_{\mathbb{R}^3_+,m,p}$ will appear at several places. Since the underlying spatial domain $\mathbb{R}^3_+$ is fixed in this section, we suppress it in our notation and only write $S_{m,p}$ for $S_{\mathbb{R}^3_+,m,p}$.

Theorem 3.13. *Let $T' > 0$, $\eta, \tau > 0$, and $r \geq r_0 > 0$. Pick $T \in (0, T']$ and set $J = (0,T)$, and $\Omega = J \times \mathbb{R}^3_+$. Let $m \in \mathbb{N}$ and $\tilde{m} = \max\{m, 3\}$. Choose $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, and $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$ with*

$$\begin{aligned}&\|A_i\|_{F_{\tilde{m}}(\Omega)} \leq r, \quad \|D\|_{F_{\tilde{m}}(\Omega)} \leq r, \\ &\max\{\|A_i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1}\|\partial_t^j A_i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \leq r_0, \\ &\max\{\|D(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1}\|\partial_t^j D(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \leq r_0\end{aligned}$$

for all $i \in \{0, \ldots, 3\}$. Take $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ with $\|B\|_{W^{\tilde{m}+1,\infty}(J\times\partial\mathbb{R}^3_+)} \leq r_0$. Let $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Assume that the solution u of (3.2) belongs to $G_m(\Omega)$. Then there is a number $\gamma_m = \gamma_m(\eta, \tau, r, T', \gamma_{3.7;0}) \geq 1$ such that

$$\begin{aligned}\|u\|^2_{G_{m,\gamma}(\Omega)} &\leq (C_{m,0} + TC_m)e^{mC_1 T}\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\ &\qquad + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}\end{aligned}$$

for all $\gamma \geq \gamma_m$, where $C_m = C_m(\eta, \tau, r, T', C_{3.7;0}) \geq 1$, $C_{m,0} = C_{m,0}(\eta, \tau, r_0, C_{3.7;0,0}) \geq 1$, and $C_1 = C_1(\eta, \tau, r, T', C_{3.7;0})$ is the constant from Corollary 3.12. Here the constants $\gamma_{3.7;0}$, $C_{3.7;0,0}$, and $C_{3.7;0}$ are the constants from Lemma 3.7.

Proof. We prove the assertion inductively. To this purpose we observe that Corollary 3.12 shows that the assertion holds for $m = 1$ with the constants γ_1, C_1, and $C_{1,0}$ from Corollary 3.12. Now assume that $m \geq 2$ and that the assertion has been shown for $1 \leq l \leq m-1$.

Let $p \in \{0,1,2\}$. As in (3.12) we deduce that $\partial_p u$ solves (3.2) with differential operator $L(A_0, \dots, A_3, D)$, inhomogeneity $f_{1,p}$, boundary value $g_{1,p}$, and initial value $\partial_p u_0$, where

$$f_{1,p} = \partial_p f - \sum_{i=0}^{3} \partial_p A_i \partial_i u - \partial_p D u, \tag{3.46}$$

$$g_{1,p} = \partial_p g - \mathrm{Tr}(\partial_p B u), \tag{3.47}$$

$$\partial_0 u_0 = S_{m,1}(0, A_0, \dots, A_3, D, f, u_0).$$

Note that $f_{1,p}$ belongs to $H^{m-1}(\Omega)$ by Lemma 3.4. We further observe that $\partial_p B$ is an element of $W^{m,\infty}(\Omega)$, while the trace of u is contained in $E_{m-1}(J \times \partial\mathbb{R}^3_+)$ as $u \in G_m(\Omega)$. Consequently, the function $g_{1,p}$ belongs to $E_{m-1}(J \times \partial\mathbb{R}^3_+)$. Since $A_0 \in F^{\mathrm{cp}}_{\tilde m,\eta}(\Omega)$, $A_1, A_2, A_3 \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$, $f \in H^m(\Omega)$, and $u_0 \in H^m(\mathbb{R}^3_+)$, Lemma 2.33 yields that $S_{m,1}(0, A_0, \dots, A_3, D, f, u_0)$ is contained in $H^{m-1}(\mathbb{R}^3_+)$. The induction hypothesis with $l = m-1$ therefore gives

$$\begin{aligned} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} &\leq (C_{m-1,0} + TC_{m-1}) e^{(m-1)C_1 T} \Big(\sum_{j=0}^{m-2} \|\partial_t^j f_{1,p}(0)\|^2_{H^{m-2-j}(\mathbb{R}^3_+)} \\ &\quad + \|g_{1,p}\|^2_{E_{m-1,\gamma}(J \times \partial\mathbb{R}^3_+)} + \|\partial_p u_0\|^2_{H^{m-1}(\mathbb{R}^3_+)} \Big) + \frac{C_{m-1}}{\gamma} e^{(m-1)C_1 T} \|f_{1,p}\|^2_{H^{m-1}_\gamma(\Omega)} \end{aligned} \tag{3.48}$$

for all $\gamma \geq \gamma_{m-1}$.

We next estimate the terms appearing on the right-hand side of (3.48). To that purpose, let $j \in \{0, \dots, m-2\}$. We observe that

$$\begin{aligned} \|\partial_t^j \partial_p f(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} &\leq \max\{\|\partial_t^j f(0)\|_{H^{m-1-j}(\mathbb{R}^3_+)}, \|\partial_t^{j+1} f(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)}\} \\ &\leq \max_{0 \leq l \leq m-1} \|\partial_t^l f(0)\|_{H^{m-1-l}(\mathbb{R}^3_+)}. \end{aligned}$$

Moreover,

$$\begin{aligned} \partial_t^j (\partial_p A_0 \partial_t u)(0) &= \sum_{l=0}^{j} \binom{j}{l} \partial_t^l \partial_p A_0(0) \partial_t^{j-l+1} u(0) \\ &= \sum_{l=0}^{j} \binom{j}{l} \partial_t^l \partial_p A_0(0) S_{m,j+1-l}(0, A_0, \dots, A_3, D, f, u_0). \end{aligned}$$

Since the function $\partial_t^l \partial_p A_0(0)$ belongs to $H^{\tilde m - l - 2}(\mathbb{R}^3_+)$ and $S_{m,j+1-l}$ to $H^{m-j+l-1}(\mathbb{R}^3_+)$ by Lemma 2.33, Lemma 2.22 (v) in the case $j = m-2$ and Lemma 2.22 (vi) in the case $j < m-2$ show

$$\begin{aligned} &\|\partial_t^l \partial_p A_0(0) S_{m,j+1-l}(0, A_0, \dots, A_3, D, f, u_0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} \\ &\leq C \|\partial_t^l \partial_p A_0(0)\|_{H^{\tilde m - 2 - l}(\mathbb{R}^3_+)} \|S_{m,j+1-l}(0, A_0, \dots, A_3, D, f, u_0)\|_{H^{m-j+l-1}(\mathbb{R}^3_+)} \\ &\leq C_{2.33;m,j-l+1}(\eta, r_0) r_0 \Big(\sum_{k=0}^{j-l} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)} \Big), \end{aligned}$$

where we also applied Lemma 2.33 in the last line. We thus infer

$$\|\partial_t^j (\partial_p A_0 \partial_t u)(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} \leq C(\eta, r_0) \Big(\sum_{k=0}^{j} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)} \Big).$$

Analogously, we deduce

$$\|\partial_t^j(\partial_p A_i \partial_i u)(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} \leq C(\eta, r_0)\Big(\sum_{k=0}^{j} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}\Big),$$

$$\|\partial_t^j(\partial_p Du)(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} \leq C(\eta, r_0)\Big(\sum_{k=0}^{j} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}\Big)$$

for all $i \in \{1, 2, 3\}$. In view of (3.46), we arrive at

$$\|\partial_t^j f_{1,p}(0)\|_{H^{m-2-j}(\mathbb{R}^3_+)} \leq C(\eta, r_0)\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}\Big). \quad (3.49)$$

Lemma 3.4 next yields

$$\|f_{1,p}\|_{H^{m-1}_\gamma(\Omega)} \leq \|f\|_{H^m_\gamma(\Omega)} + C_{3.4;m}\|u\|_{G_{m,\gamma}(\Omega)} \quad (3.50)$$

for all $\gamma > 0$. The term $\|\partial_p u_0\|_{H^{m-1}(\mathbb{R}^3_+)}$ is dominated by $\|u_0\|_{H^m(\mathbb{R}^3_+)}$ in the case $p \in \{1, 2\}$, whereas in the case $p = 0$ we use Lemma 2.33 to obtain

$$\begin{aligned}\|\partial_0 u_0\|_{H^{m-1}(\mathbb{R}^3_+)} &= \|S_{m,1}(0, A_0, \ldots, A_3, D, f, u_0)\|_{H^{m-1}(\mathbb{R}^3_+)} \\ &\leq C_{2.33;m,1}(\|f(0)\|_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}) \quad (3.51)\end{aligned}$$

with $C_{2.33;m,1} = C_{2.33;m,1}(\eta, r_0)$ from Lemma 2.33. To estimate $g_{1,p}$ in the norm of $E_{m-1}(J \times \partial\mathbb{R}^3_+)$, we take a multiindex $\alpha \in \mathbb{N}_0^4$ with $\alpha_3 = 0$ and $|\alpha| \leq m - 1$. For any multiindex $\beta \leq \alpha$ the function $\partial^\beta \partial_p B$ then belongs to $W^{1,\infty}(\Omega)$ and $\partial^{\alpha-\beta} u$ to $H^1(\Omega)$. We infer that $\operatorname{tr} \partial^{\alpha-\beta} u$ is an element of $E_0(J \times \partial\mathbb{R}^3_+)$ and therefore $\operatorname{Tr}(\partial^\beta \partial_p B \partial^{\alpha-\beta} u)$ is contained in $E_0(J \times \partial\mathbb{R}^3_+)$. Lemma 3.8 thus yields

$$\begin{aligned}\|\operatorname{Tr}(\partial^\beta \partial_p B \partial^{\alpha-\beta} u)\|^2_{E_{0,\gamma}(J \times \partial\mathbb{R}^3_+)} &\leq \|B\|^2_{W^{m+1,\infty}(J \times \partial\mathbb{R}^3_+)} \|\partial^{\alpha-\beta} u\|^2_{L^2_\gamma(J, H^{1/2}(\partial\mathbb{R}^3_+))} \\ &\leq r_0^2\Big(\gamma\|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \frac{1}{\gamma}\|\partial_3 \partial^{\alpha-\beta} u\|^2_{L^2_\gamma(\Omega)}\Big)\end{aligned}$$

for all $\gamma > 0$. Consequently, we obtain

$$\begin{aligned}&\|g_{1,p}\|^2_{E_{m-1,\gamma}(J \times \partial\mathbb{R}^3_+)} \quad (3.52) \\ &\leq 2\|\partial_p g\|^2_{E_{m-1,\gamma}(J \times \partial\mathbb{R}^3_+)} + 2 \sum_{|\alpha| \leq m-1, \alpha_3 = 0} \|\partial^\alpha \operatorname{Tr}(\partial_p B u)\|^2_{L^2_\gamma(J, H^{1/2}(\partial\mathbb{R}^3_+))} \\ &\leq C\|g\|^2_{E_{m,\gamma}(J \times \partial\mathbb{R}^3_+)} + C r_0^2\Big(\gamma\|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \frac{1}{\gamma}\Big(\sum_{p=0}^{2} \|\partial_p u\|^2_{H^{m-1}_\gamma(\Omega)} + \|u\|^2_{H^{m-1}_\gamma(\Omega)}\Big)\Big)\end{aligned}$$

for all $\gamma > 0$, where we also used that the trace operator commutes with tangential derivatives.

We insert the estimates (3.49) to (3.52) into (3.48) and combine it with the induction hypothesis to infer

$$\begin{aligned}&\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} \\ &\leq (C_{m-1,0} + TC_{m-1})e^{(m-1)C_1 T} \sum_{p=0}^{2} \Big(\sum_{j=0}^{m-2} (\|\partial_t^j f_{1,p}(0)\|^2_{H^{m-2-j}(\mathbb{R}^3_+)} + \|\partial_t^j f(0)\|^2_{H^{m-2-j}(\mathbb{R}^3_+)}) \\ &\quad + \|g_{1,p}\|^2_{E_{m-1,\gamma}(J \times \partial\mathbb{R}^3_+)} + \|g\|^2_{E_{m-1,\gamma}(J \times \partial\mathbb{R}^3_+)} + \|\partial_p u_0\|^2_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^{m-1}(\mathbb{R}^3_+)}\Big)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{C_{m-1}}{\gamma} e^{(m-1)C_1 T} \Big(\sum_{p=0}^{2} \|f_{1,p}\|^2_{H^{m-1}_\gamma(\Omega)} + \|f\|^2_{H^{m-1}_\gamma(\Omega)} \Big) \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{(m-1)C_1 T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \\
&\quad + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + \gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} + \frac{1}{\gamma}\Big(\sum_{p=0}^{2} \|\partial_p u\|^2_{H^{m-1}_\gamma(\Omega)} + \|u\|^2_{H^{m-1}_\gamma(\Omega)} \Big)\Big) \\
&\quad + \frac{\tilde{C}_m}{\gamma} e^{(m-1)C_1 T} \Big(\|f\|^2_{H^m_\gamma(\Omega)} + \sum_{i=0}^{3} \|\partial_i u\|^2_{G_{m-1,\gamma}(\Omega)} + \|u\|^2_{G_{m-1,\gamma}(\Omega)} \Big)\Big) \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{(m-1)C_1 T} \Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \\
&\quad + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + \gamma \|u\|^2_{H^m_{\gamma,\mathrm{ta}}(\Omega)} \Big) + \frac{\tilde{C}_m}{\gamma} e^{(m-1)C_1 T} \|f\|^2_{H^m_\gamma(\Omega)} \\
&\quad + \tilde{C}_m \frac{1}{\gamma} \Big(\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} + \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \Big)
\end{aligned}
$$

for all $\gamma \geq \gamma_{m-1}$, where $\tilde{C}_{m,0} = \tilde{C}_{m,0}(\eta, \tau, r_0, C_{3.7;0,0})$ and $\tilde{C}_m = \tilde{C}_m(\eta, \tau, r, T', C_{3.7;0})$ denote constants which may change from line to line. Employing estimate (3.11), we find

$$
\begin{aligned}
&\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{(m-1)C_1 T} \Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\
&\quad + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) + \frac{\tilde{C}_m}{\gamma} e^{(m-1)C_1 T} \|f\|^2_{H^m_\gamma(\Omega)} \\
&\quad + \tilde{C}_m \frac{1}{\gamma} \Big(\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} + \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \Big)
\end{aligned} \tag{3.53}
$$

for all $\gamma \geq \gamma_{m-1}$. We consequently find a number $\tilde{\gamma}_m = \tilde{\gamma}_m(\eta, \tau, r, T', \gamma_{3.7;0})$ such that

$$
\begin{aligned}
&\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G_{m-1,\gamma}(\Omega)} \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{(m-1)C_1 T} \Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \\
&\quad + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) + \frac{\tilde{C}_m}{\gamma} e^{(m-1)C_1 T} \|f\|^2_{H^m_\gamma(\Omega)} + \tilde{C}_m \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)}
\end{aligned} \tag{3.54}
$$

for all $\gamma \geq \tilde{\gamma}_m$.

It only remains to control the $G_{0,\gamma}(\Omega)$-norm of $\partial_3^m u$. To this purpose, we compute

$$
\begin{aligned}
\partial_3^{m-1} L u &= \partial_3^{m-1} \Big(\sum_{k=0}^{3} A_k \partial_k u + Du \Big) \\
&= \sum_{j=0}^{m-1} \binom{m-1}{j} \Big(\sum_{k=0}^{3} \partial_3^j A_k \, \partial_3^{m-1-j} \partial_k u + \partial_3^j D \, \partial_3^{m-1-j} u \Big) \\
&= \sum_{j=1}^{m-1} \binom{m-1}{j} \Big(\sum_{k=0}^{3} \partial_3^j A_k \, \partial_k \partial_3^{m-1-j} u + \partial_3^j D \, \partial_3^{m-1-j} u \Big) + L \partial_3^{m-1} u,
\end{aligned}
$$

where we employed Lemma 2.22. We conclude that $\partial_3^{m-1} u$ solves the initial value problem

$$\begin{cases} Lv = f_{m,3}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ v(0) = \partial_3^{m-1} u_0, & x \in \mathbb{R}^3_+; \end{cases}$$

where $f_{m,3} := \partial_3^{m-1} f - \sum_{0<j\leq m-1} \binom{m-1}{j} (\sum_{k=0}^3 \partial_3^j A_k \, \partial_k \partial_3^{m-1-j} u + \partial_3^j D \, \partial_3^{m-1-j} u)$. Since

$$\partial_3^j A_k, \, \partial_3^j D \in \tilde{G}_{\tilde{m}-j}(\Omega),$$
$$\partial_k \partial_3^{m-1-j} u, \, \partial_3^{m-1-j} u \in G_{m-1-(m-1-j)}(\Omega) = G_j(\Omega)$$

for all $k \in \{0, \ldots, 3\}$ and $j \in \{1, \ldots, m-1\}$, Lemma 2.22 (ii) yields that

$$\partial_3^j A_k \partial_k \partial_3^{m-1-j} u, \, \partial_3^j D \partial_3^{m-1-j} u \in \tilde{G}_{\min\{j, \tilde{m}-j\}}(\Omega) \hookrightarrow \tilde{G}_1(\Omega)$$

for all $k \in \{0, \ldots, 3\}$, $0 < j \leq m-1$, and $\gamma > 0$. Hence, $f_{m,3}$ belongs to $H^1(\Omega)$. Moreover, Lemma 2.22 (ii) allows us to estimate

$$\begin{aligned} \|\partial_3^j A_k \partial_k \partial_3^{m-1-j} u\|_{G_{1,\gamma}(\Omega)} &\leq \|\partial_3^j A_k \partial_k \partial_3^{m-1-j} u\|_{G_{\min\{j,\tilde{m}-j\},\gamma}(\Omega)} \\ &\leq C \|\partial_3^j A_k\|_{G_{\tilde{m}-j}(\Omega)} \|\partial_k \partial_3^{m-1-j} u\|_{G_{j,\gamma}(\Omega)} \\ &\leq C \|A_k\|_{F_{\tilde{m}}(\Omega)} \|\partial_k u\|_{G_{m-1,\gamma}(\Omega)}, \\ \|\partial_3^j D \partial_3^{m-1-j} u\|_{G_{1,\gamma}(\Omega)} &\leq C \|D\|_{F_{\tilde{m}}(\Omega)} \|u\|_{G_{m-1,\gamma}(\Omega)} \end{aligned}$$

for all $k \in \{0, \ldots, 3\}$, $0 < j \leq m-1$, and $\gamma > 0$. We conclude that

$$\begin{aligned} \|f_{m,3}\|_{H^1_\gamma(\Omega)} &\leq \|f\|_{H^m_\gamma(\Omega)} + C\sqrt{T} r \Big(\sum_{k=0}^3 \|\partial_k u\|_{G_{m-1,\gamma}(\Omega)} + \|u\|_{G_{m-1,\gamma}(\Omega)} \Big), \\ \|f_{m,3}\|^2_{H^1_\gamma(\Omega)} &\leq C \|f\|^2_{H^m_\gamma(\Omega)} + CTr^2 \Big(\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{k=0}^2 \|\partial_k u\|^2_{G_{m-1,\gamma}(\Omega)} \Big) \\ &\quad + CTr^2 \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \end{aligned} \tag{3.55}$$

for all $\gamma > 0$. As $\partial_3^j A_k(0)$ is an element of $H^{\tilde{m}-1-j}(\mathbb{R}^3_+)$ and $\partial_3^{m-1-j} \partial_k u(0)$ of $H^j(\mathbb{R}^3_+)$, Lemma 2.22 (v) yields

$$\begin{aligned} &\|\partial_3^j A_k(0) \partial_3^{m-1-j} \partial_k u(0)\|_{L^2(\mathbb{R}^3_+)} \\ &\leq C \|\partial_3^j A_k(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)} \|\partial_3^{m-1-j} \partial_k u(0)\|_{H^j(\mathbb{R}^3_+)} \\ &\leq C \|\partial_3^j A_k(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)} (\|\partial_3^{m-1-j} S_{m,1}(0, A_0, \ldots, A_3, D, f, u_0)\|_{H^j(\mathbb{R}^3_+)} \\ &\qquad + \sum_{p=1}^2 \|\partial_3^{m-1-j} \partial_p u_0\|_{H^j(\mathbb{R}^3_+)}) \\ &\leq C \|A_k(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} (\|S_{m,1}(0, A_0, \ldots, A_3, D, f, u_0)\|_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}) \\ &\leq C C_{2.33;m,1} r_0 (\|f(0)\|_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|_{H^m(\mathbb{R}^3_+)}) \end{aligned}$$

for all $k \in \{0, \ldots, 3\}$ and $j \in \{1, \ldots, m-1\}$, where $C_{2.33;m,1} = C_{2.33;m,1}(\eta, r_0)$ is the constant from Lemma 2.33. Analogously, we obtain

$$\|\partial_3^j D(0) \partial_3^{m-1-j} u(0)\|_{L^2(\mathbb{R}^3_+)} \leq C r_0 \|u_0\|_{H^m(\mathbb{R}^3_+)}$$

for all $j \in \{1, \ldots, m-1\}$. We thus infer

$$\|f_{m,3}(0)\|^2_{L^2(\mathbb{R}^3_+)} \leq \tilde{C}_0 (\|f(0)\|^2_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}) \tag{3.56}$$

with a constant $\tilde{C}_0 = \tilde{C}_0(\eta, r_0)$.

We recapitulate that the function $\partial_3^{m-1} u \in G_1(\Omega)$ solves (3.17) with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity $f_{m,3} \in H^1(\Omega)$, and initial value $\partial_3^{m-1} u_0 \in H^1(\mathbb{R}^3_+)$. So Lemma 3.11 tells us that

$$\begin{aligned}
&\|\partial_3 \partial_3^{m-1} u\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq (C_{1,0} + TC_1) e^{C_1 T} \Big(\sum_{j=0}^{2} \|\partial_j \partial_3^{m-1} u\|^2_{G_{0,\gamma}(\Omega)} + \|f_{m,3}(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|\partial_3^{m-1} u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big) \\
&\qquad + \frac{C_1}{\gamma} e^{C_1 T} \|f_{m,3}\|^2_{H^1_\gamma(\Omega)}
\end{aligned}$$

for all $\gamma \geq 1$. Combined with (3.55) and (3.56) the above inequality implies

$$\begin{aligned}
&\|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{C_1 T} \Big(\|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{j=0}^{2} \|\partial_j u\|^2_{G_{m-1,\gamma}(\Omega)} + \|f(0)\|^2_{H^{m-1}(\mathbb{R}^3_+)} \\
&\qquad + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) + \frac{\tilde{C}_m}{\gamma} (\|f\|^2_{H^m_\gamma(\Omega)} + \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)}) \qquad (3.57)
\end{aligned}$$

for all $\gamma \geq 1$. We then use (3.57) to estimate

$$\begin{aligned}
\|u\|^2_{G_{m,\gamma}(\Omega)} &\leq \|u\|^2_{G_{m-1,\gamma}(\Omega)} + \sum_{k=0}^{2} \|\partial_k u\|^2_{G_{m-1,\gamma}(\Omega)} + \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{C_1 T} \Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{m-1,\gamma}(\Omega)} + \|u\|^2_{G_{m-1,\gamma}(\Omega)} \Big) + \frac{\tilde{C}_m}{\gamma} \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \\
&\quad + (\tilde{C}_{m,0} + T\tilde{C}_m) e^{C_1 T} \Big(\|f(0)\|^2_{H^{m-1}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) + \frac{\tilde{C}_m}{\gamma} \|f\|^2_{H^m_\gamma(\Omega)}
\end{aligned}$$

for all $\gamma \geq 1$. Together with (3.54), it follows

$$\begin{aligned}
\|u\|^2_{G_{m,\gamma}(\Omega)} &\leq (\tilde{C}_{m,0} + T\tilde{C}_m) e^{mC_1 T} \Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J \times \partial\mathbb{R}^3_+)} \\
&\qquad + \|u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) + \frac{\tilde{C}_m}{\gamma} \Big(\|f\|^2_{H^m_\gamma(\Omega)} + \|\partial_3^m u\|^2_{G_{0,\gamma}(\Omega)} \Big) \qquad (3.58)
\end{aligned}$$

for all $\gamma \geq \tilde{\gamma}_m$. We define $C_{m,0} = C_{m,0}(\eta, \tau, r_0, C_{3.7;0,0})$, $C_m = C_m(\eta, \tau, r, T', C_{3.7;0})$, and $\gamma_m = \gamma_m(\eta, \tau, r, T', \gamma_{3.7;0})$ by

$$C_{m,0} = 2\tilde{C}_{3.58;m,0}, \quad C_m = 2\tilde{C}_{3.58;m}, \quad \gamma_m = \max\{\tilde{\gamma}_m, 2\tilde{C}_{3.58;m}\},$$

where $\tilde{C}_{3.58;m,0} = \tilde{C}_{3.58;m,0}(\eta, \tau, r_0, C_{3.7;0,0})$ and $\tilde{C}_{3.58;m} = \tilde{C}_{3.58;m}(\eta, \tau, r, T', C_{3.7;0})$ are the corresponding constants from the right-hand side of (3.58). Consequently, we have $\tilde{C}_{3.58;m}\gamma^{-1} \leq \frac{1}{2}$ for all $\gamma \geq \gamma_m$, so that we conclude

$$\|u\|^2_{G_{m,\gamma}(\Omega)} \le (C_{m,0} + TC_m)e^{mC_1T}\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{H^{m-1-k}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}$$

for all $\gamma \ge \gamma_m$. □

4

Regularity of the solution of the linearized problem

In this section we establish that for all $m \in \mathbb{N}$ the solution of (3.2) with data $u_0 \in H^m(\mathbb{R}^3_+)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $f \in H^m(\Omega)$ indeed belongs to $G_m(\Omega)$ if data and coefficients satisfy the compatibility conditions. At the same time we expand the class of allowed coefficients from $F^{\mathrm{cp}}_m(\Omega)$ to $F^{\mathrm{c}}_m(\Omega)$.

As one might expect, the proofs which lead to this result involve several regularization steps. The idea is, roughly speaking, that a regularized solution of (3.2) still solves (3.2) with modified data. The a priori estimates from Chapter 3 can then be applied to this regularized solution and they will eventually lead to convergence of a sequence of regularized solutions to the original solution in a higher order norm.

However, we cannot simply apply a standard mollifier since convolution in x_3-direction would violate the boundary condition. Analogously, convolution in time causes serious problems due to the shift of the initial value. In fact, this shift prevents the convergence of the approximating sequence so that we cannot gain regularity in this way.

We will therefore use another approach to obtain regularity in time (see Lemma 4.7 below). The regularity in space then follows in two steps. First we use a mollifier in the spatial tangential variables. Then, having the regularity in all tangential variables, we apply a mollifier in all space variables and employ the estimates for the solution of the initial value problem (3.17).

4.1 Regularity in space

In this section we show that regularity of the solution in time implies regularity in space. The intuitive idea for that purpose is to apply a mollifier in spatial variables and apply our a priori estimates to the regularized solution. However, since we are treating a characteristic problem, there are several difficulties. As mentioned in the introduction, in the noncharacteristic case it is enough to mollify in spatial tangential variables as every appearing derivative in normal direction can be expressed by derivatives in tangential directions and lower order terms. The lack of such an explicit representation makes it necessary to mollify also in the normal direction. It is then crucial to avoid a loss of regularity across the boundary.

In order to regularize in spatially tangential variables, we introduce a family of norms which is highly suitable for that task. This family of norms has successfully been applied to gain regularity in noncharacteristic problems. Dealing with a characteristic problem, we have to avoid normal derivatives in the arising commutator terms. We will show that the structure of the variable coefficients Maxwell operator allows us to do so.

We start by reducing the question of spatial regularity to the question of regularity in x_1- and x_2-direction, cf. Lemma 3.11. To that purpose we apply a mollifier M_ε in all spatial directions to a solution of (3.2) with regular data. However, this may lead to a loss of regularity across the boundary. We therefore shift the complete problem in negative x_3-direction. By means of our a priori estimate from Lemma 3.11 and commutator estimates from the paradifferential caluculus, we then obtain additional regularity for the restrictions of u to a family of subsets of $\mathbb{R}^3_+$. In a second step we show that this result is enough to infer that the function u has the desired regularity in x_3-direction.

Lemma 4.1. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, and $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$. Pick $f \in H^m(\Omega)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Let u be a solution of the linear initial value problem (3.17) with differential operator $L = L(A_0, \dots, A_3, D)$, inhomogeneity f, and initial value u_0. Assume that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$.*

Take $k \in \{1, \dots, m\}$ and a multiindex $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. Suppose that $\partial^\beta u$ is contained in $G_0(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_3 \leq k - 1$. Then $\partial^\alpha u$ is an element of $G_0(\Omega)$.

Proof. I) We have to start with several preparations. Let $\rho \in C_c^\infty(\mathbb{R}^3)$ be a positive function with $\int_{\mathbb{R}^3} \rho(x)dx = 1$ and $\operatorname{supp} \rho \subseteq B(0,1)$. We denote the convolution operator with kernel $\rho_\varepsilon = \varepsilon^{-3}\rho(\varepsilon^{-1}\cdot)$ by M_ε for all $\varepsilon > 0$, where the convolution is taken over $\mathbb{R}^3$. We further define

$$E_\tau v(x) = v(x_1, x_2, x_3 + \tau) \tag{4.1}$$

for all $v \in L^1_{loc}(\mathbb{R}^3_+)$, $\tau > 0$, and for almost all $x \in \mathbb{R}^2 \times (-\tau, \infty)$. Clearly, E_τ belongs to $\mathcal{L}(W^{l,p}(\mathbb{R}^3_+), W^{l,p}(\mathbb{R}^2 \times (-\tau, \infty)))$ and

$$\partial^{\tilde{\alpha}} E_\tau v = E_\tau \partial^{\tilde{\alpha}} v$$

for all $\tilde{\alpha} \in \mathbb{N}_0^4$ with $|\tilde{\alpha}| \leq l$, $l \in \mathbb{N}_0$, $1 \leq p \leq \infty$, and $\tau > 0$. If $v \in L^1_{loc}(\mathbb{R}^3)$, we further define $E_\tau v$ by formula (4.1) for all $\tau \in \mathbb{R}$.

Let Z_U denote the operator which maps each $L^1_{loc}(U)$-function to its zero-extension on $\mathbb{R}^3$, where U is a subdomain of $\mathbb{R}^3$. In the following it will always be clear to which domain U we refer so that we will drop the index U. Let $v \in L^2(\mathbb{R}^3_+)$ and $\psi \in H^1_0(\mathbb{R}^3_+)$. Observe that

$$\begin{aligned}
&\langle E_\delta v, \psi \rangle_{H^{-1}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)} \\
&= \int_{\mathbb{R}^3_+} v(x_1, x_2, x_3 + \delta)\psi(x)dx = \int_{\mathbb{R}^2 \times (\delta, \infty)} v(x)\psi(x_1, x_2, x_3 - \delta)dx \\
&= \int_{\mathbb{R}^3_+} v(x)(Z\psi)(x_1, x_2, x_3 - \delta)dx = \langle v, E_{-\delta} Z\psi \rangle_{H^{-1}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)}
\end{aligned}$$

for all $\delta > 0$. Note moreover that $RE_{-\delta}Z$ maps $H^1_0(\mathbb{R}^3_+)$ continuously into itself, where R is the restriction to $\mathbb{R}^3_+$. We therefore define - with a small abuse of notation - the map

$$\begin{aligned}
&E_\delta \colon H^{-1}(\mathbb{R}^3_+) \to H^{-1}(\mathbb{R}^3_+), \\
&\langle E_\delta v, \psi \rangle_{H^{-1}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)} = \langle v, E_{-\delta} Z\psi \rangle_{H^{-1}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)} \quad \text{for all } \psi \in H^1_0(\mathbb{R}^3_+)
\end{aligned}$$

for all $\delta > 0$. Since partial derivatives commute with $E_{-\delta}Z$ on $H^1_0(\mathbb{R}^3_+)$, we deduce the equality

$$\partial_j E_\delta v = E_\delta \partial_j v \tag{4.2}$$

for all $v \in L^2(\mathbb{R}^3_+)$ and $\delta > 0$.

We next take a closer look on the convolution operator M_ε, which is defined for functions in $L^1_{loc}(\mathbb{R}^3)$. We want to extend this operator in a sense to functions in $L^1_{loc}(\mathbb{R}^3_+)$. To that purpose, let $0 < \varepsilon < \delta$. For functions v in $L^1_{loc}(\mathbb{R}^3_+)$ we will employ the regularization

$$RM_\varepsilon E_\delta Zv = (M_\varepsilon E_\delta Zv)_{|\mathbb{R}^3_+}.$$

Usually, it is clear from the context whether we consider $M_\varepsilon E_\delta Zv$ as a function on $\mathbb{R}^3_+$ or $\mathbb{R}^3$ and we will not write down the restriction to $\mathbb{R}^3_+$ explicitly in these cases.

It is easy to see that if v has a weak derivative in $\mathbb{R}^3_+$, then also $M_\varepsilon E_\delta Zv$ has a weak derivative in $\mathbb{R}^3_+$ and

$$\partial_j M_\varepsilon E_\delta v = M_\varepsilon E_\delta \partial_j v$$

for all $j \in \{1,2,3\}$.

We define $\tilde\rho$ by $\tilde\rho(x) = \rho(-x)$ for all $x \in \mathbb{R}^3$. The convolution operator with kernel $\tilde\rho_\varepsilon$ is denoted by $\tilde M_\varepsilon$ for all $\varepsilon > 0$. Let $v \in L^2(\mathbb{R}^3_+)$ and $\psi \in H^1_0(\mathbb{R}^3_+)$. Let $0 < \varepsilon < \delta$. We then compute

$$\begin{aligned}
&\langle M_\varepsilon E_\delta v, \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \int_{\mathbb{R}^3_+} M_\varepsilon E_\delta v(x)\psi(x)dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho_\varepsilon(x-y)E_\delta Zv(y)Z\psi(x)dydx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde\rho_\varepsilon(y-x)Z\psi(x)dxE_\delta Zv(y)dy \\
&= \int_{\mathbb{R}^3} \tilde M_\varepsilon Z\psi(y)E_\delta Zv(y)dy = \int_{\mathbb{R}^2\times(-\delta,\infty)} \tilde M_\varepsilon Z\psi(y)v(y+\delta\, e_3)dy \\
&= \int_{\mathbb{R}^3_+} (\tilde M_\varepsilon Z\psi)(y-\delta\, e_3)v(y)dy = \langle v, E_{-\delta}\tilde M_\varepsilon Z\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}.
\end{aligned}$$

As above, $E_{-\delta}\tilde M_\varepsilon Z$ maps $H^1_0(\mathbb{R}^3_+)$ continuously into itself. Hence, the operator

$$\begin{aligned}
&M_\varepsilon E_\delta \colon H^{-1}(\mathbb{R}^3_+) \to H^{-1}(\mathbb{R}^3_+), \\
&\langle M_\varepsilon E_\delta v, \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle v, E_{-\delta}\tilde M_\varepsilon Z\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}
\end{aligned} \tag{4.3}$$

continuously extends the map $M_\varepsilon E_\delta$ which was initially defined on $L^2(\mathbb{R}^3_+)$. We deduce the identity

$$\partial_j M_\varepsilon E_\delta v = M_\varepsilon \partial_j E_\delta v = M_\varepsilon E_\delta \partial_j v$$

by duality for all $j \in \{1,2,3\}$ and $v \in L^2(\mathbb{R}^3_+)$ using that the partial derivative commutes with $\tilde M_\varepsilon$, $E_{-\delta}$, and Z on $H^1_0(\mathbb{R}^3_+)$.

We further note that for $A \in W^{1,\infty}(\mathbb{R}^3_+)$ and $v \in H^{-1}(\mathbb{R}^3_+)$ we have

$$\begin{aligned}
&\langle (E_\delta A)E_\delta v, \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle E_\delta v, (E_\delta A)\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} \\
&= \langle v, E_{-\delta}Z((E_\delta A)\psi)\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle v, AE_{-\delta}Z\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} \\
&= \langle Av, E_{-\delta}Z\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle E_\delta(Av), \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}
\end{aligned}$$

for all $\psi \in H^1_0(\mathbb{R}^3_+)$, i.e.,

$$(E_\delta A)E_\delta v = E_\delta(Av) \tag{4.4}$$

in $H^{-1}(\mathbb{R}^3_+)$.

II) Let $0 < \varepsilon < \delta$. Set $\alpha' = \alpha - e_3 \in \mathbb{N}_0^4$ and note that $|\alpha'| = m-1$ and $\alpha'_3 = k-1$. In particular, $\partial^{\alpha'}u$ belongs to $G_0(\Omega)$. Due to the mollifier the function $M_\varepsilon E_\delta \partial^{\alpha'}u$ belongs to $C^1(\overline{J}, H^1(\mathbb{R}^3_+)) \hookrightarrow G_1(\Omega)$, $M_\varepsilon E_\delta \partial^{\alpha'}u_0$ is an element of $H^1(\mathbb{R}^3_+)$, $L(E_\delta A_0, \ldots, E_\delta A_3, E_\delta D)M_\varepsilon E_\delta \partial^{\alpha'}u$ is contained in $G_0(\Omega)$, and

$$\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3)L(E_\delta A_0, \ldots, E_\delta A_3, E_\delta D)M_\varepsilon E_\delta \partial^{\alpha'}u$$

in $L^2(\Omega)$. We want to apply estimate (3.18) from Lemma 3.11 with differential operator $L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D)$ to $M_\varepsilon E_\delta \partial^{\alpha'} u$. To that purpose, we have to deal with the terms

$$\|L(E_\delta A_0, \dots E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)}$$

and

$$\|\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) L(E_\delta A_0, \dots E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u\|_{L^2_\gamma(\Omega)}$$

for $\gamma \geq 1$. We fix such a parameter γ and compute

$$\begin{aligned}
&f_{\alpha'}^{\delta,\varepsilon} := L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u \qquad (4.5)\\
&= L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u - M_\varepsilon (L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D) E_\delta \partial^{\alpha'} u)\\
&\quad + M_\varepsilon E_\delta \Big(\partial^{\alpha'} f - \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha'-\beta} u - \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta D \partial^{\alpha'-\beta} u \Big)\\
&= \sum_{i=0}^{3} ((E_\delta A_i) M_\varepsilon \partial_i - M_\varepsilon ((E_\delta A_i)\partial_i) E_\delta \partial^{\alpha'} u + ((E_\delta D) M_\varepsilon - M_\varepsilon (E_\delta D)) E_\delta \partial^{\alpha'} u\\
&\quad + M_\varepsilon E_\delta \Big(\partial^{\alpha'} f - \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha'-\beta} u - \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta D \partial^{\alpha'-\beta} u \Big),
\end{aligned}$$

where we exploited the results from step I) and Lemma 3.4. We point out that $\partial_t \partial^{\alpha'} u$ is an element of $G_0(\Omega)$ as u is contained in $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$. Therefore, $\partial_t E_\delta \partial^{\alpha'} u$ and $E_\delta A_0 \partial_t E_\delta \partial^{\alpha'} u$ map the compact interval $\overline{J}$ continuously into $L^2(\mathbb{R}^3_+)$, implying that the functions $M_\varepsilon \partial_t E_\delta \partial^{\alpha'} u$ and $M_\varepsilon((E_\delta A_0)\partial_t E_\delta \partial^{\alpha'} u)$ converge to $\partial_t E_\delta \partial^{\alpha'} u$ respectively $(E_\delta A_0)\partial_t E_\delta \partial^{\alpha'} u$ in $G_0(\Omega)$. We thus obtain that

$$\|(E_\delta A_0) M_\varepsilon \partial_t E_\delta \partial^{\alpha'} u - M_\varepsilon((E_\delta A_0)\partial_t E_\delta \partial^{\alpha'} u)\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \qquad (4.6)$$

as $\varepsilon \to 0$. Analogously, we derive

$$\|(E_\delta D) M_\varepsilon \partial^{\alpha'} u - M_\varepsilon (E_\delta D) \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \qquad (4.7)$$

as $\varepsilon \to 0$. For the remaining commutator terms we employ estimates for the commutator of a $W^{1,\infty}$-function with a mollifier. Take $j \in \{1,2,3\}$. To match the assumptions of these commutator estimates, we now extend the coefficient A_j by reflection at $\partial\mathbb{R}^3_+$ to a function in $W^{1,\infty}(\mathbb{R}^3)$ which we still denote by A_j. We then note - as $E_\delta Z \partial^{\alpha'} u(t)$ is an element of $L^2(\mathbb{R}^3)$ - that

$$(E_\delta A_j) M_\varepsilon \partial_j E_\delta Z \partial^{\alpha'} u(t) - M_\varepsilon((E_\delta A_j)\partial_j E_\delta Z \partial^{\alpha'} u(t)) \qquad (4.8)$$

defines an element of $H^{-1}(\mathbb{R}^3)$, whose restriction to $\mathbb{R}^3_+$ coincides with

$$(E_\delta A_j) M_\varepsilon \partial_j E_\delta \partial^{\alpha'} u(t) - M_\varepsilon((E_\delta A_j)\partial_j E_\delta \partial^{\alpha'} u(t))$$

in $H^{-1}(\mathbb{R}^3_+)$ for all $t \in \overline{J}$. But on $\mathbb{R}^3$ we can apply Theorem C.14 of [BGS07], which tells us that the difference in (4.8) is contained in $L^2(\mathbb{R}^3)$. In particular, its restriction to $\mathbb{R}^3_+$ belongs to $L^2(\mathbb{R}^3_+)$ and we derive

$$\begin{aligned}
&\|(E_\delta A_j) M_\varepsilon \partial_j E_\delta \partial^{\alpha'} u(t) - M_\varepsilon((E_\delta A_j)\partial_j E_\delta \partial^{\alpha'} u(t))\|_{L^2(\mathbb{R}^3_+)}\\
&\quad \leq \|(E_\delta A_j) M_\varepsilon \partial_j E_\delta Z \partial^{\alpha'} u(t) - M_\varepsilon((E_\delta A_j)\partial_j E_\delta Z \partial^{\alpha'} u(t))\|_{L^2(\mathbb{R}^3)}\\
&\quad \leq C \|E_\delta A_j\|_{W^{1,\infty}(\mathbb{R}^3)} \|E_\delta Z \partial^{\alpha'} u(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad \leq C \, \|A_j\|_{W^{1,\infty}(\mathbb{R}^3_+)} \, \|\partial^{\alpha'} u(t)\|_{L^2(\mathbb{R}^3_+)}, \qquad (4.9)\\
&\lim_{\varepsilon\to 0} \|(E_\delta A_j) M_\varepsilon \partial_j E_\delta Z \partial^{\alpha'} u(t) - M_\varepsilon((E_\delta A_j)\partial_j E_\delta Z \partial^{\alpha'} u(t))\|_{L^2(\mathbb{R}^3_+)} = 0 \qquad (4.10)
\end{aligned}$$

from Theorem C.14 in [BGS07] for all $t \in \overline{J}$. Replacing $u(t)$ by $u(t) - u(s)$ in (4.9) for $t, s \in \overline{J}$, the continuity of u on the compact interval $\overline{J}$ and a standard compactness argument yield that the convergence in (4.10) is also uniform in t, i.e.,

$$\|(E_\delta A_j) M_\varepsilon \partial_j E_\delta \partial^{\alpha'} u - M_\varepsilon((E_\delta A_j)\partial_j E_\delta \partial^{\alpha'} u)\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \tag{4.11}$$

as $\varepsilon \to 0$. Next we take $j = 0$ and note that $\partial_t u$ is contained in $G_{m-1}(\Omega)$ by assumption. Consequently, $\partial_t \partial^{\alpha'-\beta} u$ is an element of $G_{|\beta|}(\Omega)$ while $\partial^\beta A_0$ belongs to $\tilde{G}_{\tilde{m}-|\beta|}(\Omega) \hookrightarrow G_{\tilde{m}-1-|\beta|}(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha'$. Lemma 2.22 (i) thus shows that

$$\sum_{0<\beta\leq\alpha'} \binom{\alpha}{\beta} \partial^\beta A_0 \partial_t \partial^{\alpha'-\beta} u \in G_0(\Omega).$$

We deduce

$$\sum_{0<\beta\leq\alpha'} \binom{\alpha}{\beta} \partial^\beta D \partial^{\alpha'-\beta} u \in G_0(\Omega)$$

in the same way. Now take $j \in \{1, 2, 3\}$. Then $\partial^\beta A_j$ belongs to $G_{\tilde{m}-|\beta|}(\Omega)$ and $\partial_j \partial^{\alpha'-\beta} u$ is an element of $G_{|\beta|-1}(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha'$. Lemma 2.22 (i) applies again and it yields that

$$\sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha'-\beta} u \in G_0(\Omega).$$

Since f has a representative in $G_{m-1}(\Omega)$, we conclude that the term

$$f_{\alpha'} := \partial^{\alpha'} f - \sum_{j=0}^{3} \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha'-\beta} u - \sum_{0<\beta\leq\alpha'} \binom{\alpha'}{\beta} \partial^\beta D \partial^{\alpha'-\beta} u$$

is an element of $C(\overline{J}, L^2(\mathbb{R}^3_+))$. Since $\overline{J}$ is compact, we infer as above that

$$M_\varepsilon E_\delta f_{\alpha'} \longrightarrow E_\delta f_{\alpha'}$$

in $G_0(\Omega)$ as $\varepsilon \to 0$. Combining this fact with (4.5), (4.6), (4.7), and (4.11), we thus arrive at

$$\|f^{\delta,\varepsilon}_{\alpha'} - E_\delta f_{\alpha'}\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \tag{4.12}$$

as $\varepsilon \to 0$.

To deal with the term $\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u$, we fix the functions $\mu_{lj} \in F^{\mathrm{cp}}_{\tilde{m},1}(\Omega) \cap W^{m+1,\infty}(\mathbb{R}^3_+)$ with

$$A_j = \sum_{l=1}^{3} A^{\mathrm{co}}_l \mu_{lj}$$

for all $j \in \{1, 2, 3\}$ which exist by the definition of $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$ respectively $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$. We set

$$\tilde{\mu} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

and compute

$$\begin{aligned}
&(E_\delta \tilde{\mu})^T \nabla L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u \\
&= (E_\delta \tilde{\mu})^T \nabla \Big(\sum_{j=0}^{3} (E_\delta A_j) \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u \Big) + (E_\delta \tilde{\mu})^T \nabla ((E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u)
\end{aligned}$$

$$= \sum_{j=0}^{3} (E_\delta \tilde{\mu})^T (E_\delta \nabla A_j) \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u + (E_\delta \tilde{\mu})^T (E_\delta \nabla D) M_\varepsilon E_\delta \partial^{\alpha'} u \tag{4.13}$$

$$+ E_\delta(\tilde{\mu}^T A_0) \nabla M_\varepsilon E_\delta \partial_t \partial^{\alpha'} u + E_\delta(\tilde{\mu}^T D) \nabla M_\varepsilon E_\delta \partial^{\alpha'} u + \sum_{j=1}^{3} E_\delta(\tilde{\mu}^T A_j) \nabla \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u,$$

where we again exploited the results from step I). We set

$$\Lambda^{\delta,\varepsilon} = \sum_{j=0}^{3} (E_\delta \tilde{\mu})^T (E_\delta \nabla A_j) \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u + (E_\delta \tilde{\mu})^T (E_\delta \nabla D) M_\varepsilon E_\delta \partial^{\alpha'} u$$
$$+ E_\delta(\tilde{\mu}^T A_0) \nabla M_\varepsilon E_\delta \partial_t \partial^{\alpha'} u + E_\delta(\tilde{\mu}^T D) \nabla M_\varepsilon E_\delta \partial^{\alpha'} u$$

and note that (3.26) and (3.27) show

$$\sum_{k=1}^{3} \Big(\sum_{j=1}^{3} E_\delta(\tilde{\mu}^T A_j) \nabla \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u \Big)_{kk} = 0,$$
$$\sum_{k=1}^{3} \Big(\sum_{j=1}^{3} E_\delta(\tilde{\mu}^T A_j) \nabla \partial_j M_\varepsilon E_\delta \partial^{\alpha'} u \Big)_{(k+3)k} = 0.$$

We thus obtain that

$$\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) L(E_\delta A_0, \ldots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u = \Big(\sum_{k=1}^{3} \Lambda^{\delta,\varepsilon}_{kk}, \sum_{k=1}^{3} \Lambda^{\delta,\varepsilon}_{(k+3)k} \Big). \tag{4.14}$$

We rewrite $\Lambda^{\delta,\varepsilon}$ in the form

$$\Lambda^{\delta,\varepsilon} = \sum_{j=0}^{3} [E_\delta(\tilde{\mu}^T \nabla A_j), M_\varepsilon] \partial_j E_\delta \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T \nabla D), M_\varepsilon] E_\delta \partial^{\alpha'} u$$
$$+ [E_\delta(\tilde{\mu}^T A_0), M_\varepsilon] \nabla E_\delta \partial_t \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T D), M_\varepsilon] \nabla E_\delta \partial^{\alpha'} u$$
$$+ M_\varepsilon E_\delta \Big(\sum_{j=0}^{3} \tilde{\mu}^T \nabla A_j \partial_j \partial^{\alpha'} u + \tilde{\mu}^T \nabla D \partial^{\alpha'} u + \tilde{\mu}^T A_0 \nabla \partial_t \partial^{\alpha'} u + \tilde{\mu}^T D \nabla \partial^{\alpha'} u \Big).$$

We introduce the function

$$\tilde{f}_{\alpha'} = \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T A_0) \nabla \partial^{\alpha'-\beta} \partial_t u + \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T D) \nabla \partial^{\alpha'-\beta} u$$
$$+ \sum_{j=0}^{3} \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \nabla A_j) \partial^{\alpha'-\beta} \partial_j u + \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \nabla D) \partial^{\alpha'-\beta} u.$$

As u and $\partial_t u$ are contained in $C(\overline{J}, H^{m-1}(\mathbb{R}^3_+))$, Lemma 2.22 implies that the function $\tilde{f}_{\alpha'}$ is an element of $L^2(\Omega)$. With this definition at hand, we write $\Lambda^{\delta,\varepsilon}$ in the form

$$\Lambda^{\delta,\varepsilon} = \sum_{j=0}^{3} [E_\delta(\tilde{\mu}^T \nabla A_j), M_\varepsilon] \partial_j E_\delta \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T \nabla D), M_\varepsilon] E_\delta \partial^{\alpha'} u$$
$$+ [E_\delta(\tilde{\mu}^T A_0), M_\varepsilon] \nabla E_\delta \partial_t \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T D), M_\varepsilon] \nabla E_\delta \partial^{\alpha'} u$$
$$+ \partial^{\alpha'} M_\varepsilon E_\delta \Big(\sum_{j=0}^{3} \tilde{\mu}^T \nabla A_j \partial_j u + \tilde{\mu}^T \nabla D u + \tilde{\mu}^T A_0 \nabla \partial_t u + \tilde{\mu}^T D \nabla u \Big) - M_\varepsilon E_\delta \tilde{f}_{\alpha'}$$
$$= \sum_{j=0}^{3} [E_\delta(\tilde{\mu}^T \nabla A_j), M_\varepsilon] \partial_j E_\delta \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T \nabla D), M_\varepsilon] E_\delta \partial^{\alpha'} u$$

$$
\begin{aligned}
&+ [E_\delta(\tilde{\mu}^T A_0), M_\varepsilon]\nabla E_\delta \partial_t \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T D), M_\varepsilon]\nabla E_\delta \partial^{\alpha'} u + \partial^{\alpha'} M_\varepsilon E_\delta(\tilde{\mu}^T \nabla f) \\
&- M_\varepsilon E_\delta \tilde{f}_{\alpha'} - \sum_{j=1}^3 \partial^{\alpha'} M_\varepsilon E_\delta(\tilde{\mu}^T A_j \nabla \partial_j u) \\
=:{}& \tilde{\Lambda}^{\delta,\varepsilon} - \sum_{j=1}^3 \partial^{\alpha'} M_\varepsilon E_\delta(\tilde{\mu}^T A_j \nabla \partial_j u).
\end{aligned}
$$

The cancellation properties of the differential operator established in (3.26) and (3.27) imply

$$
\Big(\sum_{k=1}^3 \Lambda^{\delta,\varepsilon}_{kk}, \sum_{k=1}^3 \Lambda^{\delta,\varepsilon}_{(k+3)k}\Big) = \Big(\sum_{k=1}^3 \tilde{\Lambda}^{\delta,\varepsilon}_{kk}, \sum_{k=1}^3 \tilde{\Lambda}^{\delta,\varepsilon}_{(k+3)k}\Big).
$$

In view of (4.14), we conclude that

$$
\begin{aligned}
&\mathrm{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) L(E_\delta A_0, \ldots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u \\
&= \Big(\sum_{k=1}^3 \tilde{\Lambda}^{\delta,\varepsilon}_{kk}, \sum_{k=1}^3 \tilde{\Lambda}^{\delta,\varepsilon}_{(k+3)k}\Big).
\end{aligned} \tag{4.15}
$$

Since $\partial_t \partial^{\alpha'} u$ and $\partial^{\alpha'} u$ belong to $C(\overline{J}, L^2(\mathbb{R}^3_+))$ and ∇A_0 and ∇D are contained in $L^\infty(\Omega)$, we have

$$
[E_\delta(\tilde{\mu}^T \nabla A_0), M_\varepsilon]\partial_t E_\delta \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T \nabla D), M_\varepsilon] E_\delta \partial^{\alpha'} u \longrightarrow 0 \tag{4.16}
$$

in $L^2(\Omega)$ as ε tends to zero. Exploiting that ∇A_j, A_0, D, and $\tilde{\mu}$ belong to $W^{1,\infty}(\Omega)$ for $j \in \{1,2,3\}$, that $\partial^{\alpha'} u$ and $\partial_t \partial^{\alpha'} u$ are elements of $C(\overline{J}, L^2(\mathbb{R}^3_+))$, and arguing as in (4.9) and (4.10), we also infer

$$
\begin{aligned}
&\sum_{j=1}^3 [E_\delta(\tilde{\mu}^T \nabla A_j), M_\varepsilon]\partial_j E_\delta \partial^{\alpha'} u + [E_\delta(\tilde{\mu}^T A_0), M_\varepsilon]\nabla E_\delta \partial_t \partial^{\alpha'} u \\
&\quad + [E_\delta(\tilde{\mu}^T D), M_\varepsilon]\nabla E_\delta \partial^{\alpha'} u \longrightarrow 0
\end{aligned} \tag{4.17}
$$

in $L^2(\Omega)$ as $\varepsilon \to 0$. We recall that $\tilde{f}_{\alpha'}$ belongs to $L^2(\Omega)$ and $\tilde{\mu}^T \nabla f$ to $H^{m-1}(\Omega)$. The definition of $\tilde{\Lambda}^{\delta,\varepsilon}$, (4.15), (4.16), and (4.17) thus imply that

$$
\begin{aligned}
&\mathrm{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) L(E_\delta A_0, \ldots, E_\delta A_3, E_\delta D) M_\varepsilon E_\delta \partial^{\alpha'} u \\
&\longrightarrow E_\delta \Big(\sum_{k=1}^3 (\tilde{f}_{\alpha'} + \partial^{\alpha'}(\tilde{\mu}^T \nabla f))_{kk}, \sum_{k=1}^3 (\tilde{f}_{\alpha'} + \partial^{\alpha'}(\tilde{\mu}^T \nabla f))_{(k+3)k}\Big) =: E_\delta f_{\mathrm{div},\alpha'}
\end{aligned} \tag{4.18}
$$

in $L^2(\Omega)$ as $\varepsilon \to 0$.

We point out that we have shown among other things that $f^{\delta,\varepsilon}_{\alpha'}$ belongs to $G_0(\Omega)$ and $\mathrm{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) f^{\delta,\varepsilon}_{\alpha'}$ is contained in $L^2(\Omega)$ for all $0 < \varepsilon < \delta$. Moreover, $M_\varepsilon E_\delta \partial^{\alpha'} u_0 \in H^1(\mathbb{R}^3_+)$ and $M_\varepsilon E_\delta \partial^{\alpha'} u$ belongs to $C^1(\overline{J}, L^2(\mathbb{R}^3_+)) \cap C(\overline{J}, H^1(\mathbb{R}^3_+)) = G_1(\Omega)$ for all $0 < \varepsilon < \delta$.

Next take $\eta, r > 0$ with $A_0 \geq \eta$, $\|A_i\|_{W^{1,\infty}(\Omega)} \leq r$, and $\|D\|_{W^{1,\infty}(\Omega)} \leq r$ for all $i \in \{0, \ldots, 3\}$. Note that we particularly have $\|A_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r$ and $\|D(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r$ for $i \in \{0, \ldots, 3\}$.

Now let $\delta > 0$ and take $n_\delta \in \mathbb{N}$ with $\frac{1}{n_\delta} < \delta$. Fix a number $\gamma \geq 1$ and define the constant $C' = C'(\eta, r, T)$ by

$$
C' = \Big(C_{3.11;1,0} + T C_{3.11;1} + \frac{C_{3.11;1}}{\gamma}\Big) e^{C_{3.11;1} T} \tag{4.19}
$$

where $C_{3.11;1,0} = C_{3.11;1,0}(\eta, r)$ and $C_{3.11;1} = C_{3.11;1}(\eta, r, T)$ are the corresponding constants from Lemma 3.11. Observe that $M_\varepsilon E_\delta \partial^{\alpha'} u$ solves the initial value problem (3.17) with differential operator $L(E_\delta A_0, \dots, E_\delta A_3, E_\delta D)$, inhomogeneity $f_{\alpha'}^{\delta,\varepsilon}$ and initial value $M_\varepsilon E_\delta u_0$ for each $\varepsilon \in (0, \delta)$. Moreover,

$$\|E_\delta A_i\|_{W^{1,\infty}(\Omega)} \le r, \qquad \|E_\delta D\|_{W^{1,\infty}(\Omega)} \le r,$$
$$\|E_\delta A_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \le r, \qquad \|E_\delta D(0)\|_{L^\infty(\mathbb{R}^3_+)} \le r,$$

for all $\delta > 0$ and $i \in \{0, \dots, 3\}$. Lemma 3.11 thus shows

$$\begin{aligned} &\|\nabla(M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u - M_{\frac{1}{k}} E_\delta \partial^{\alpha'} u)\|^2_{G_{0,\gamma}(\Omega)} \\ &\le C'\Big(\sum_{j=0}^{2} \|(M_{\frac{1}{n}} - M_{\frac{1}{k}})\partial_j E_\delta \partial^{\alpha'} u\|^2_{G_{0,\gamma}(\Omega)} + \|f_{\alpha'}^{\delta,\frac{1}{n}} - f_{\alpha'}^{\delta,\frac{1}{k}}\|^2_{G_{0,\gamma}(\Omega)} \\ &\quad + \|\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3)(f_{\alpha'}^{\delta,\frac{1}{n}} - f_{\alpha'}^{\delta,\frac{1}{k}})\|^2_{L^2_\gamma(\Omega)} + \|(M_{\frac{1}{n}} - M_{\frac{1}{k}})E_\delta \partial^{\alpha'} u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big), \end{aligned} \tag{4.20}$$

for all $n, k \in \mathbb{N}$ with $n, k \ge n_\delta$. We will next show that the right-hand side of (4.20) converges to 0 as $n, k \to \infty$.

As $\partial_j M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u_0 = M_{\frac{1}{n}} E_\delta \partial_j \partial^{\alpha'} u_0$ for all $j \in \{1, 2, 3\}$ on $\mathbb{R}^3_+$ and $Z\partial_j \partial^{\alpha'} u_0 \in L^2(\mathbb{R}^3)$, we infer

$$M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u_0 \longrightarrow E_\delta \partial^{\alpha'} u_0 \tag{4.21}$$

in $H^1(\mathbb{R}^3_+)$ as $n \to \infty$. We remark that the translation operator E_δ is crucial here since otherwise we could not commute the derivative with the mollifier in $L^2(\mathbb{R}^3_+)$. We highlight this fact only at this place but of course the translation operator is always essential when we commute mollifier and derivative on the half-space.

Analogously, we have $\partial_j E_\delta \partial^{\alpha'} u = E_\delta \partial_j \partial^{\alpha'} u$ on $\mathbb{R}^3_+$ for all $j \in \{0, 1, 2, 3\}$. As $E_\delta \partial_j \partial^{\alpha'} u$ belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+))$ for $j \in \{0, 1, 2\}$, the set $\{E_\delta Z \partial_j \partial^{\alpha'} u(t) \colon t \in \overline{J}\}$ is compact in $L^2(\mathbb{R}^3_+)$. Therefore, the functions $M_{\frac{1}{n}} E_\delta \partial_j \partial^{\alpha'} u$ converge to $E_\delta \partial_j \partial^{\alpha'} u$ in $L^2(\mathbb{R}^3_+)$ uniformly in $t \in \overline{J}$ as $n \to \infty$, so that

$$\sum_{j=0}^{2} \|M_{\frac{1}{n}} \partial_j E_\delta \partial^{\alpha'} u - \partial_j E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \tag{4.22}$$

as $n \to \infty$.

The same argument shows that

$$\|M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u - E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \tag{4.23}$$

as $n \to \infty$.

The formulas (4.20) to (4.22) as well as (4.12) and (4.18) imply that the sequence $(\nabla M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u)_{n \ge n_\delta}$ is a Cauchy sequence in $G_0(\Omega)$. In (4.23) we have seen that $(M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u)_{n \ge n_\delta}$ converges to $E_\delta \partial^{\alpha'} u$ in $G_0(\Omega)$. We conclude that $E_\delta \partial^{\alpha'} u$ belongs to $C(\overline{J}, H^1(\mathbb{R}^3_+))$ and that

$$\|\nabla M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u - \nabla E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)} \longrightarrow 0 \tag{4.24}$$

as $n \to \infty$ for all $\delta > 0$.

As in (4.20), Lemma 3.11 yields with $\gamma = 1$

$$\begin{aligned} \|\nabla M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u\|^2_{G_{0,\gamma}(\Omega)} &\le C'\Big(\sum_{j=0}^{2} \|M_{\frac{1}{n}} \partial_j E_\delta \partial^{\alpha'} u\|^2_{G_{0,\gamma}(\Omega)} + \|f_{\alpha'}^{\delta,\frac{1}{n}}\|^2_{G_{0,\gamma}(\Omega)} \\ &\quad + \|\operatorname{Div}(E_\delta A_1, E_\delta A_2, E_\delta A_3) f_{\alpha'}^{\delta,\frac{1}{n}}\|^2_{L^2_\gamma(\Omega)} + \|M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u_0\|^2_{H^1(\mathbb{R}^3_+)}\Big), \end{aligned} \tag{4.25}$$

for all $n \in \mathbb{N}$ with $n^{-1} \leq \delta$, where the constant C' was introduced in (4.19). Recall that $f_{\alpha'}^{\delta,1/n}$ was defined in (4.5). In (4.24) we have seen that $(\nabla M_{\frac{1}{n}} E_\delta \partial^{\alpha'} u)_n$ converges to $\nabla E_\delta \partial^{\alpha'} u$ in $G_0(\Omega)$ as $n \to \infty$. In the limit $n \to \infty$, the estimate (4.25) thus leads to

$$\|\nabla E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)}^2 \leq C' \Big(\sum_{j=0}^{2} \|\partial_j E_\delta \partial^{\alpha'} u\|_{G_{0,\gamma}(\Omega)}^2 + \|E_\delta f_{\alpha'}\|_{G_{0,\gamma}(\Omega)}^2 + \|E_\delta f_{\mathrm{div},\alpha'}\|_{L^2_\gamma(\Omega)}^2 + \|E_\delta \partial^{\alpha'} u_0\|_{H^1(\mathbb{R}^3_+)}^2 \Big), \tag{4.26}$$

where we also employed (4.12), (4.18), and (4.21) to (4.24).

III) We next show that $\partial^{\alpha'} u(t)$ is an element of $H^1(\mathbb{R}^3_+)$ for all $t \in \overline{J}$. Note that we only have to prove that $\partial_3 \partial^{\alpha'} u(t)$ belongs to $L^2(\mathbb{R}^3_+)$ for this claim. We abbreviate $\mathbb{R}^2 \times (\delta, \infty)$ by $\mathbb{R}^3_\delta$ and denote the restriction operator to $\mathbb{R}^3_\delta$ by R_δ for all $\delta > 0$. In the next step we show that $R_\delta u(t)$ belongs to $H^1(\mathbb{R}^3_\delta)$ for all $\delta > 0$.

Fix $\delta > 0$ and $t \in \overline{J}$. Let $\varphi \in C_c^\infty(\mathbb{R}^3_\delta)$. We compute

$$\begin{aligned} \int_{\mathbb{R}^3_\delta} R_\delta \partial^{\alpha'} u(t,x) \partial_3 \varphi(x) dx &= \int_{\mathbb{R}^3_+} E_\delta \partial^{\alpha'} u(t,x) \partial_3 E_\delta \varphi(x) dx \\ = -\int_{\mathbb{R}^3_+} \partial_3 E_\delta \partial^{\alpha'} u(t,x) E_\delta \varphi(x) dx &= -\int_{\mathbb{R}^3_\delta} E_{-\delta} Z \partial_3 E_\delta \partial^{\alpha'} u(t,x) \varphi(x) dx, \end{aligned}$$

using that $E_\delta \varphi \in C_c^\infty(\mathbb{R}^3_+)$. It follows

$$\partial_3 R_\delta \partial^{\alpha'} u(t) = E_{-\delta} Z \partial_3 E_\delta \partial^{\alpha'} u(t) \in L^2(\mathbb{R}^2 \times (\delta, \infty)) \tag{4.27}$$

as $\partial_3 E_\delta \partial^{\alpha'} u(t) \in L^2(\mathbb{R}^3_+)$.

Next pick $\overline{\delta} > \delta$ and $\varphi \in C_c^\infty(\mathbb{R}^3_{\overline{\delta}})$. We compute

$$\begin{aligned} \int_{\mathbb{R}^3_{\overline{\delta}}} R_{\overline{\delta}} \partial_3 R_\delta \partial^{\alpha'} u(t,x) \varphi(x) dx &= \int_{\mathbb{R}^3_\delta} \partial_3 R_\delta \partial^{\alpha'} u(t,x) Z \varphi(x) dx \\ = -\int_{\mathbb{R}^3_\delta} R_\delta \partial^{\alpha'} u(t,x) Z \partial_3 \varphi(x) dx &= -\int_{\mathbb{R}^3_{\overline{\delta}}} R_{\overline{\delta}} \partial^{\alpha'} u(t,x) \partial_3 \varphi(x) dx \\ = \int_{\mathbb{R}^3_{\overline{\delta}}} \partial_3 R_{\overline{\delta}} \partial^{\alpha'} u(t,x) \varphi(x) dx, \end{aligned}$$

where we exploited that $\mathrm{supp}(\varphi) \Subset \mathbb{R}^3_{\overline{\delta}}$. Since $\varphi \in C_c^\infty(\mathbb{R}^3_{\overline{\delta}})$ was arbitrary, we conclude $\partial_3 R_{\overline{\delta}} \partial^{\alpha'} u(t) = \partial_3 R_\delta \partial^{\alpha'} u(t)$ on $\mathbb{R}^3_{\overline{\delta}}$. In particular, we can define the function $v(t) \in L^1_{loc}(\mathbb{R}^3_+)$ by setting

$$v(t,x) = \partial_3 R_\delta \partial^{\alpha'} u(t,x) \quad \text{for all } x \in \mathbb{R}^3_\delta \text{ and } \delta > 0.$$

Take $\varphi \in C_c^\infty(\mathbb{R}^3_+)$. Fix a number $\tau > 0$ with $\mathrm{dist}(\mathrm{supp}(\varphi), \partial \mathbb{R}^3_+) > \tau$, i.e., $\mathrm{supp}(\varphi) \Subset \mathbb{R}^3_\tau$. We then deduce

$$\begin{aligned} \int_{\mathbb{R}^3_+} \partial^{\alpha'} u(t,x) \partial_3 \varphi(x) dx &= \int_{\mathbb{R}^3_\tau} R_\tau \partial^{\alpha'} u(t,x) \partial_3 \varphi(x) dx = -\int_{\mathbb{R}^3_\tau} \partial_3 R_\tau \partial^{\alpha'} u(t,x) \varphi(x) dx \\ &= -\int_{\mathbb{R}^3_\tau} v(t,x) \varphi(x) dx = -\int_{\mathbb{R}^3_+} v(t,x) \varphi(x) dx. \end{aligned}$$

This means that $\partial_3 \partial^{\alpha'} u(t) = v(t) \in L^1_{loc}(\mathbb{R}^3_+)$.

We further note that $Z \partial_3 R_\delta \partial^{\alpha'} u(t)$ converges pointwise almost everywhere on $\mathbb{R}^3_+$ to $v(t) = \partial_3 \partial^{\alpha'} u(t)$ as $\delta \to 0$. Using (4.27), we further infer

$$\|Z \partial_3 R_\delta \partial^{\alpha'} u(t)\|_{L^2(\mathbb{R}^3_+)}^2 = \int_{\mathbb{R}^3_\delta} |E_{-\delta} Z \partial_3 E_\delta \partial^{\alpha'} u(t,x)|^2 dx = \int_{\mathbb{R}^3_+} |\partial_3 E_\delta \partial^{\alpha'} u(t,x)|^2 dx$$

$$= \|\partial_3 E_\delta \partial^{\alpha'} u(t)\|^2_{L^2(\mathbb{R}^3_+)}. \tag{4.28}$$

Let $(\delta_n)_n$ be a null-sequence. Fatou's lemma, (4.28), and (4.26) then imply

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} |\partial_3 \partial^{\alpha'} u(t,x)|^2 dx = \int_{\mathbb{R}^3_+} \liminf_{n\to\infty} |Z\partial_3 R_{\delta_n} \partial^{\alpha'} u(t,x)| dx \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^3_+} |Z\partial_3 R_{\delta_n} \partial^{\alpha'} u(t,x)|^2 dx = \liminf_{n\to\infty} \|\partial_3 E_{\delta_n} \partial^{\alpha'} u(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\leq e^{2\gamma T} \liminf_{n\to\infty} \|\nabla E_{\delta_n} \partial^{\alpha'} u\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq C' e^{2\gamma T} \Big(\sum_{j=0}^{2} \|\partial_j \partial^{\alpha'} u\|^2_{G_{0,\gamma}(\Omega)} + \|f_{\alpha'}\|^2_{G_{0,\gamma}(\Omega)} + \|f_{\mathrm{div},\alpha'}\|^2_{L^2_\gamma(\Omega)} + \|\partial^{\alpha'} u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big) \\
&=: K_u^2 < \infty,
\end{aligned}$$

where we used that $\partial_j \partial^{\alpha'} u, \partial^{\alpha'} u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ for $j \in \{0,1,2\}$, $f_{\alpha'} \in G_0(\Omega)$, $f_{\mathrm{div},\alpha'} \in L^2(\Omega)$, and $\partial^{\alpha'} u_0 \in H^1(\mathbb{R}^3_+)$. We conclude that $\partial_3 \partial^{\alpha'} u(t)$ belongs to $L^2(\mathbb{R}^3_+)$ with $\|\partial_3 \partial^{\alpha'} u(t)\|_{L^2(\mathbb{R}^3_+)} \leq K_u$ for all $t \in \overline{J}$.

We further point out that $R_\delta \partial_3 \partial^{\alpha'} u(t) = R_\delta v(t) = \partial_3 R_\delta \partial^{\alpha'} u(t)$. This fact implies that

$$|Z\partial_3 R_\delta \partial^{\alpha'} u(t)| \leq |\partial_3 \partial^{\alpha'} u(t)|$$

on $\mathbb{R}^3_+$. As $Z\partial_3 R_\delta \partial^{\alpha'} u(t)$ tends to $\partial_3 \partial^{\alpha'} u(t)$ pointwise almost everywhere on $\mathbb{R}^3_+$, the dominated convergence theorem shows that

$$Z\partial_3 R_\delta \partial^{\alpha'} u(t) \longrightarrow \partial_3 \partial^{\alpha'} u(t)$$

in $L^2(\mathbb{R}^3_+)$ as $\delta \to 0$.

Since $\partial_3 E_\delta Z \partial^{\alpha'} u$ belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+))$ for all $\delta > 0$, one can argue as in (4.28) to deduce that $Z\partial_3 R_\delta \partial^{\alpha'} u$ is also continuous on $\overline{J}$ with values in $L^2(\mathbb{R}^3_+)$ and thus strongly measurable. Hence, $\partial_3 \partial^{\alpha'} u$ is the pointwise limit of strongly measurable functions and therefore itself strongly measurable on J with values in $L^2(\mathbb{R}^3_+)$. As a result, $\partial_3 \partial^{\alpha'} u$ and thus $\nabla \partial^{\alpha'} u$ belong to $L^\infty(J, L^2(\mathbb{R}^3_+))$. We then obtain via Lemma 3.11 that $\partial^{\alpha'} u$ is contained in $C(\overline{J}, H^1(\mathbb{R}^3_+))$. □

The regularization in spatially tangential variables below will be performed in two steps. In a first one, regularity is only obtained in $L^2(\Omega)$ for purely tangential derivatives. It is important to note that the techniques from the proof of Lemma 4.1 imply that in this case *all* derivatives up to highest order belong to $L^2(\Omega)$, i.e., that the solution is contained in $H^m(\Omega)$. This result then allows us to infer that all tangential derivatives up to highest order are contained in $G_0(\Omega)$.

Corollary 4.2. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, and $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$. Pick $f \in H^m(\Omega)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Let u be a solution of (3.17) with differential operator $L = L(A_0, \ldots, A_3, D)$, inhomogeneity f, and initial value u_0. Assume that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$.*

Take $k \in \{1, \ldots, m\}$ and a multiindex $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. Suppose that $\partial^\beta u$ is contained in $L^2(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_3 \leq k-1$. Then $\partial^\alpha u$ is an element of $L^2(\Omega)$.

Proof. We only have to make small adaptions to the proof of Lemma 4.1. In step II) of that proof we replace the a priori estimate (3.18) from Lemma 3.11 by estimate (3.20). The arguments from step II) then yield that $E_\delta \partial^{\alpha'} u$ is an element of $L^2(J, H^1(\mathbb{R}^3_+))$. Integrating over the time-space domain in step III) of the proof of Lemma 4.1, we derive that $\partial^{\alpha'} u$ belongs to $L^2(J, H^1(\mathbb{R}^3_+))$. □

For the regularization in spatial tangential variables, we first introduce the family of weighted norms

$$\|v\|^2_{H^s_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} = \int_{\mathbb{R}_+}\int_{\mathbb{R}^2} (1+|\xi|^2)^{s+1}(1+|\delta\xi|^2)^{-1}|(\mathcal{F}_2 v)(\xi,x_3)|^2 d\xi dx_3 \tag{4.29}$$

for all $s \in \mathbb{R}$ and $\delta > 0$, where $\mathcal{F}_2$ denotes the Fourier transform in x_1- and x_2-direction and v belongs to $H^s_{\mathrm{ta}}(\mathbb{R}^3_+)$, see Section 2.4 in [Hoe76]. As in the unweighted case we have of course the identity

$$\|v\|^2_{H^{s+1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} = \|v\|^2_{H^s_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \sum_{j=1}^{2} \|\partial_j v\|^2_{H^s_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \tag{4.30}$$

for all $s \in \mathbb{R}$ and $\delta > 0$. We further note that the definition directly implies

$$\|v\|_{H^s_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \le \|v\|_{H^{s+1}_{\mathrm{ta}}(\mathbb{R}^3_+)}$$

for all $v \in H^{s+1}_{\mathrm{ta}}(\mathbb{R}^3_+)$, $s \in \mathbb{R}$, and $\delta > 0$.

We further take a function $\chi \in C^\infty_c(\mathbb{R}^2)$ such that $\mathcal{F}_2\chi(\xi) = O(|\xi|^{m+1})$ as $\xi \to 0$ and $\mathcal{F}_2\chi(t\xi) = 0$ for all $t \in \mathbb{R}$ implies $\xi = 0$, cf. [Hoe76]. As usual we set $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(x/\varepsilon)$ for all $x \in \mathbb{R}^2$ and $\varepsilon > 0$ and denote the convolution in spatial tangential variables with χ_ε by J_ε, i.e.,

$$J_\varepsilon v(x) = \chi_\varepsilon *_{\mathrm{ta}} v(x) = \int_{\mathbb{R}^2} \chi(y) v(x_1 - y_1, x_2 - y_2, x_3) dy$$

for all $v \in L^2(\mathbb{R}^3_+)$.

One of the advantages to work with the weighted norms from (4.29) is that one can reduce the task of showing that a function v from $H^s_{\mathrm{ta}}(\mathbb{R}^3_+)$ belongs to $H^{s+1}_{\mathrm{ta}}(\mathbb{R}^3_+)$ to finding a uniform bound in $\delta > 0$ for the $H^s_{\mathrm{ta},\delta}(\mathbb{R}^3_+)$-norms. The following properties of this family of weighted norms can be found in (2.4.4), Theorem 2.4.1, Theorem 2.4.2, Theorem 2.4.5, and Theorem 2.4.6 in [Hoe76].

Lemma 4.3. *Let $s \in [0,m]$, $v \in H^{s-1}_{\mathrm{ta}}(\mathbb{R}^3_+)$, and let $A \in C^\infty(\mathbb{R}^3_+)$ be constant outside of a compact subset of $\overline{\mathbb{R}^3_+}$.*

(i) Assume that there is a constant C, independent of δ, such that

$$\|v\|_{H^{s-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \le C$$

for all $\delta > 0$ in a neighborhood of 0. Then v belongs to $H^s_{\mathrm{ta}}(\mathbb{R}^3_+)$.

(ii) There exist constants c and C, independent of δ and v, such that

$$\begin{aligned} c\|v\|^2_{H^{s-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} &\le \|v\|^2_{H^{s-1}_{\mathrm{ta}}(\mathbb{R}^3_+)} + \int_0^1 \|J_\varepsilon v\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2s-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\ &\le C\|v\|^2_{H^{s-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \end{aligned}$$

for all $\delta \in (0,1)$.

(iii) There is a constant C, independent of δ and v, such that

$$\int_0^1 \|AJ_\varepsilon v - J_\varepsilon(Av)\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2s-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \le C\|v\|^2_{H^{s-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)}$$

for all $\delta \in (0,1)$.

We note that Hörmander states the commutator estimate only for coefficients from the Schwartz space. The proof of Theorem 2.4.2 in [Hoe76] however also works for smooth coefficients which are constant outside of a compact set.

In order to prove regularity in the spatially tangential variables, we will derive a uniform bound in δ for the norm $\|u\|_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)}$ of the solution u. In view of Lemma 4.3 (ii), we study the initial boundary value problem solved by $J_\varepsilon u$ and apply our a priori estimates to it. Since J_ε only mollifies in x_1- and x_2-direction, we experience a loss of derivatives in the commutator terms involving a derivative in normal direction. It is unclear how to avoid this loss. To overcome this problem, we therefore transform the initial value problem (3.2) to one with a constant boundary matrix. In this modified problem no commutator terms involving a derivative in x_3-direction appear. Moreover, regularity of the solution of the modified problem transfers to the solution of the original problem.

Lemma 4.4. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, $T > 0$, $J = (0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Take coefficients $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$ and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. We further assume that these coefficients and a function M as in the definition of $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ belong to $C^\infty(\overline{\Omega})$. Let u be the weak solution of (3.2) with differential operator $L(A_0, \dots, A_3, D)$, and data $f \in H^m_{\mathrm{ta}}(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m_{\mathrm{ta}}(\mathbb{R}^3_+)$. Suppose that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$. Pick a multiindex $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_0 = \alpha_3 = 0$. Then $\partial^\alpha u$ is an element of $C(\overline{J}, L^2(\mathbb{R}^3_+))$.*

Proof. In this proof it is crucial to avoid normal derivatives. We will therefore not study the differential operator $L = L(A_0, \dots, A_3, D)$ but instead a suitably transformed operator $\tilde{L} = L(\tilde{A}_0, \dots, \tilde{A}_3, \tilde{D})$.

I) The definition of $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$ respectively $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$ yields functions $\mu_{ij} \in F^{\mathrm{cp}}_{\tilde{m},1}(\Omega) \cap W^{\tilde{m}+1,\infty}(\Omega)$ for $i, j \in \{1, 2, 3\}$ such that

$$A_j = \sum_{i=1}^{3} A_i^{\mathrm{co}} \mu_{ij}$$

for all $j \in \{1, 2, 3\}$ and there is an index $k \in \{1, 2, 3\}$ such that

$$|\mu_{k3}| \geq \tau$$

on $\mathbb{R}^3_+$. We assume that $k = 3$ and that $\mu_{33} \geq \tau$ on $\mathbb{R}^3_+$. The other cases are treated analogously. We introduce the matrices

$$\hat{G}_r = \frac{1}{\sqrt{\mu_{33}}} \begin{pmatrix} 1 & 0 & \mu_{13} \\ 0 & 1 & \mu_{23} \\ 0 & 0 & \mu_{33} \end{pmatrix} \quad \text{and} \quad \hat{A}_3 = \begin{pmatrix} 0 & \mu_{33} & -\mu_{23} \\ -\mu_{33} & 0 & \mu_{13} \\ \mu_{23} & -\mu_{13} & 0 \end{pmatrix}. \tag{4.31}$$

Note that

$$\hat{G}_r^T \hat{A}_3 \hat{G}_r = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & \hat{A}_3 \\ -\hat{A}_3 & 0 \end{pmatrix}. \tag{4.32}$$

Setting

$$G_r = \begin{pmatrix} \hat{G}_r & 0 \\ 0 & \hat{G}_r \end{pmatrix},$$

we thus obtain

$$G_r^T A_3 G_r = \begin{pmatrix} 0 & \hat{G}_r^T \hat{A}_3 \hat{G}_r \\ -\hat{G}_r^T \hat{A}_3 \hat{G}_r & 0 \end{pmatrix} =: \tilde{A}_3 = A_3^{\mathrm{co}}. \tag{4.33}$$

We point out that $\tilde{A}_3$ is constant. Moreover, both $\hat{G}_r$ and its inverse

$$\hat{G}_r^{-1} = \sqrt{\mu_{33}} \begin{pmatrix} 1 & 0 & -\mu_{13}\mu_{33}^{-1} \\ 0 & 1 & -\mu_{23}\mu_{33}^{-1} \\ 0 & 0 & \mu_{33}^{-1} \end{pmatrix}$$

belong to $W^{\tilde{m}+1,\infty}(\Omega)$. Hence, the same is true for G_r and G_r^{-1}. In particular, if we show that $\partial^\alpha(G_r^{-1}u)$ belongs to $G_0(\Omega)$, it follows that also $\partial^\alpha u$ is contained in $G_0(\Omega)$ as we already know that u is an element of $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$.

II) Motivated by step I), we will study the regularity properties of the function $\tilde{u} = G_r^{-1}u$. To that purpose, set

$$\tilde{A}_j = G_r^T A_j G_r, \quad \tilde{D} = G_r^T D G_r - \sum_{j=1}^3 G_r^T A_j G_r \partial_j G_r^{-1} G_r, \quad \tilde{B} = BG_r, \quad \tilde{f} = G_r^T f,$$
$$\tilde{u}_0 = G_r^{-1} u_0, \quad \tilde{C} = CG_r, \quad \tilde{M} = MG_r^{-T},$$

for $j \in \{0, \ldots, 3\}$, where $C \in W^{\tilde{m}+1,\infty}(\mathbb{R}^3_+)^{2\times 6}$ and $M \in W^{\tilde{m}+1,\infty}(\mathbb{R}^3_+)^{2\times 6}$ are the matrices which satisfy

$$A_3 = \frac{1}{2}\Big(C^T B + B^T C\Big) \quad \text{and} \quad B = MA_3.$$

Recall that they exist since B is contained in $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. Observe that the matrices $\tilde{A}_i$ are symmetric for $i \in \{0, \ldots, 3\}$, $\tilde{A}_0 \in F^{\text{cp}}_{\tilde{m},\tilde{\eta}}(\Omega)$ for a number $\tilde{\eta} > 0$, $\tilde{A}_j \in W^{\tilde{m}+1,\infty}(\mathbb{R}^3_+)$ for all $j \in \{1,2,3\}$, $\tilde{D} \in F^{\text{cp}}_{\tilde{m}}(\Omega)$, $\tilde{B}, \tilde{C}, \tilde{M} \in W^{\tilde{m}+1,\infty}(\mathbb{R}^3_+)$, $\tilde{f} \in H^m(\Omega)$, $\tilde{u}_0 \in H^m(\mathbb{R}^3_+)$, and $\tilde{u} \in \bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$. All coefficients are constant outside of a compact set and belong to $C^\infty(\overline{\Omega})$. Furthermore, we have

$$\tilde{M}\tilde{A}_3 = MG_r^{-T}G_r^T A_3 G_r = MA_3 G_r = BG_r = \tilde{B},$$
$$\frac{1}{2}\Big(\tilde{C}^T\tilde{B} + \tilde{B}^T\tilde{C}\Big) = \frac{1}{2}G_r^T\Big(C^T B + B^T C\Big)G_r = \tilde{A}_3,$$

so that $\tilde{B}$ is contained in $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(\tilde{A}_3)$. The tupel $(L(\tilde{A}_0, \ldots, \tilde{A}_3, \tilde{D}), \tilde{B})$ thus satisfies the assumptions of Lemma 3.7. In the following we will abbreviate the differential operator $L(\tilde{A}_0, \ldots, \tilde{A}_3, \tilde{D})$ by $\tilde{L}$. We next compute

$$\begin{aligned}
\tilde{L}\tilde{u} &= \sum_{j=0}^3 \tilde{A}_j \partial_j \tilde{u} + \tilde{D}\tilde{u} = \sum_{j=0}^3 G_r^T A_j G_r G_r^{-1} \partial_j u + \sum_{j=1}^3 G_r^T A_j G_r \partial_j G_r^{-1} u \\
&\qquad + G_r^T D G_r G_r^{-1} u - \sum_{j=1}^3 G_r^T A_j G_r \partial_j G_r^{-1} G_r G_r^{-1} u \\
&= G_r^T f = \tilde{f}, \\
\operatorname{tr}(\tilde{B}\tilde{u}) &= \operatorname{tr}(BG_r G_r^{-1} u) = \operatorname{tr}(Bu) = g.
\end{aligned}$$

Since $\tilde{u}(0) = G_r^{-1}u(0) = \tilde{u}_0$, the function $\tilde{u}$ solves the linear initial boundary value problem

$$\begin{cases} \tilde{L}\tilde{u} = \tilde{f}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ \tilde{B}\tilde{u} = g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ \tilde{u}(0) = \tilde{u}_0, & x \in \mathbb{R}^3_+. \end{cases} \tag{4.34}$$

At the end of this step we point out that the differential operator $\tilde{L}$ has the big advantage to possess the boundary matrix $\tilde{A}_3$. This fact will be exploited several times in the following.

III) Note that for the assertion of the lemma it is enough to show that $\tilde{u}$ is contained in $C(\overline{J}, H^m_{\text{ta}}(\mathbb{R}^3_+))$. This will be established in two steps. First we will show that $\tilde{u}$ is an element of $L^2(J, H^m_{\mathrm{ta}}(\mathbb{R}^3_+))$. To that purpose we will apply Lemma 4.3 and the a priori estimates from Lemma 3.7.

Fix a parameter $\delta \in (0,1)$. Let $\gamma > 0$. The generic constants appearing in the following will all be indpendent of δ and γ. We further note that Lemma 4.3 will be used in almost every step in the following so that we will not cite it every time. Applying the differential operator $\tilde{L}$ to $J_\varepsilon \tilde{u}$, we obtain

$$\tilde{L} J_\varepsilon \tilde{u} = J_\varepsilon \tilde{f} + \sum_{j=0}^{2} [\tilde{A}_j, J_\varepsilon] \partial_j \tilde{u} + [\tilde{D}, J_\varepsilon]\tilde{u} \tag{4.35}$$

for all $\varepsilon \in (0,1)$. Lemma 4.3 allows us to estimate

$$\begin{aligned} \int_J e^{-2\gamma t} \int_0^1 \|[\tilde{A}_j, J_\varepsilon]\partial_j \tilde{u}(t)\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2m-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon dt \\ \leq C\|\tilde{u}\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\|\partial_t \tilde{u}\|^2_{L^2_\gamma(J, H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\ \leq C\|\tilde{u}\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\|\partial_t \tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} \end{aligned} \tag{4.36}$$

for all $j \in \{0,1,2\}$. We argue analogously for the commutator $[\tilde{D}, J_\varepsilon]\tilde{u}$. In particular, $\tilde{L} J_\varepsilon \tilde{u}$ is an element of $L^2(\Omega)$. Identity (4.35) further implies that $\tilde{A}_3 \partial_3 J_\varepsilon \tilde{u}$ belongs to $L^2(\Omega)$ so that $\tilde{A}_3 J_\varepsilon \tilde{u}$ is an element of $L^2(J, H^1(\mathbb{R}^3_+))$. We infer that the trace of $\tilde{B} J_\varepsilon \tilde{u}$ is contained in $L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$. Finally $J_\varepsilon \tilde{u}_0$ is an element of $L^2(\mathbb{R}^3_+)$ so that we can apply the a priori estimate from Lemma 3.7 to the function $J_\varepsilon \tilde{u}$. Before doing so, we use Lemma 4.3 to derive

$$\begin{aligned} &\sup_{t\in J} e^{-2\gamma t}\|\tilde{u}(t)\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \gamma\|\tilde{u}\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\ &\leq c^{-1}\sup_{t\in J} e^{-2\gamma t}\|\tilde{u}(t)\|^2_{H^{m-1}(\mathbb{R}^3_+)} + c^{-1}\gamma\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} \\ &\quad + c^{-1}\sup_{t\in J} e^{-2\gamma t}\int_0^1 \|J_\varepsilon \tilde{u}(t)\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\ &\quad + c^{-1}\gamma \int_J e^{-2\gamma t}\int_0^1 \|J_\varepsilon \tilde{u}(t)\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\ &\leq C\|\tilde{u}\|^2_{G_{m-1,\gamma}(\Omega)} + C\gamma\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} \\ &\quad + C\int_0^1 \Big(\|J_\varepsilon \tilde{u}\|^2_{G_{0,\gamma}(\Omega)} + \gamma\|J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(\Omega)}\Big)\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \end{aligned} \tag{4.37}$$

for all $\gamma > 0$. The a priori estimates from Lemma 3.7 now show that there is a constant C_0 and a number $\gamma_0 > 0$ such that

$$\begin{aligned} \|J_\varepsilon\tilde{u}\|^2_{G_{0,\gamma}(\Omega)} + \gamma\|J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(\Omega)} \leq C_0\|J_\varepsilon\tilde{u}_0\|^2_{L^2(\mathbb{R}^3_+)} + C_0\|\tilde{B}J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(J, H^{1/2}(\partial\mathbb{R}^3_+))} \\ + \frac{C_0}{\gamma}\|\tilde{L}J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(\Omega)} \end{aligned} \tag{4.38}$$

for all $\gamma \geq \gamma_0$. Fix such a parameter γ in the following. We next treat the terms appearing in (4.38).

Applying identity (4.35), Fubini's theorem and estimate (4.36), we infer

$$\begin{aligned} &\int_0^1 \|\tilde{L}J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(\Omega)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\ &\leq C\int_0^1 \Big(\|J_\varepsilon\tilde{f}\|^2_{L^2_\gamma(\Omega)} + \sum_{j=0}^2 \|[\tilde{A}_j, J_\varepsilon]\partial_j\tilde{u}\|^2_{L^2_\gamma(\Omega)} \\ &\qquad + \|[\tilde{D}, J_\varepsilon]\tilde{u}\|^2_{L^2_\gamma(\Omega)}\Big)\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \end{aligned}$$

$$
\begin{aligned}
&= C \int_J e^{-2\gamma t} \int_0^1 \Big(\|J_\varepsilon \tilde f(t)\|^2_{L^2(\mathbb{R}^3_+)} + \sum_{j=0}^2 \|[\tilde A_j, J_\varepsilon]\partial_j \tilde u(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\qquad\qquad + \|[\tilde D, J_\varepsilon]\tilde u(t)\|^2_{L^2(\mathbb{R}^3_+)}\Big) \varepsilon^{-2m-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&\le C\|\tilde f\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\|\tilde u\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\|\partial_t \tilde u\|^2_{H^{m-1}_\gamma(\Omega)} \\
&=: K_1 + C\|\tilde u\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))},
\end{aligned} \tag{4.39}
$$

where $K_1 < \infty$ and we once again employed Lemma 4.3 in the penultimate line.

IV) In this step we will treat the term

$$
\int_0^1 \|\tilde B J_\varepsilon \tilde u\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon
$$

appearing in (4.37) due to (4.38). To that purpose, we first rewrite $\tilde B J_\varepsilon \tilde u$ as

$$
\tilde B J_\varepsilon \tilde u = [\tilde B, J_\varepsilon]\tilde u + J_\varepsilon(\tilde B \tilde u) = [\tilde B, J_\varepsilon]\tilde u + J_\varepsilon \tilde g \tag{4.40}
$$

for all $\varepsilon > 0$. We will first treat the commutator. Lemma 3.8 with weight $\kappa = 1$ yields

$$
\begin{aligned}
\|[\tilde B, J_\varepsilon]\tilde u\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} &\le C\sum_{j=1}^2 \|\partial_j([\tilde B, J_\varepsilon]\tilde u)\|^2_{L^2_\gamma(\Omega)} + \|\partial_3([\tilde B, J_\varepsilon]\tilde u)\|^2_{L^2_\gamma(\Omega)} \\
&\qquad + C\|[\tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} \\
&\le C\sum_{j=1}^2 \Big(\|[\partial_j \tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} + \|[\tilde B, J_\varepsilon]\partial_j \tilde u\|^2_{L^2_\gamma(\Omega)}\Big) + C\|[\tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} \\
&\qquad + C\|[\partial_3 \tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} + C\|[\tilde B, J_\varepsilon]\partial_3 \tilde u\|^2_{L^2_\gamma(\Omega)}
\end{aligned} \tag{4.41}
$$

for all $\varepsilon > 0$. Since $\partial_j \tilde B$ is an element of $C^\infty(\Omega)$ which is constant outside of a compact set, Lemma 4.3 (iii) and Fubini's theorem yield

$$
\begin{aligned}
&\int_0^1 \|[\partial_j \tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&= \int_J e^{-2\gamma t}\int_0^1 \|[\partial_j \tilde B, J_\varepsilon]\tilde u(t)\|^2_{L^2(\mathbb{R}^3_+)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon dt \\
&\le C\int_J e^{-2\gamma t}\|\tilde u(t)\|^2_{H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} dt \\
&\le C\|\tilde u\|^2_{H^{m-1}_\gamma(\Omega)}, \\
&\int_0^1 \|[\tilde B, J_\varepsilon]\tilde u\|^2_{L^2_\gamma(\Omega)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \le C\|\tilde u\|^2_{H^{m-1}_\gamma(\Omega)}
\end{aligned} \tag{4.42}
$$

for all $j \in \{1,2,3\}$. Analogously, using that B itself is smooth and constant outside of a compact set, we derive

$$
\begin{aligned}
&\int_0^1 \|[\tilde B, J_\varepsilon]\partial_j \tilde u\|^2_{L^2_\gamma(\Omega)} \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&\le C\int_J e^{-2\gamma t}\|\partial_j \tilde u(t)\|^2_{H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} dt \\
&\le C\int_J e^{-2\gamma t}\|\tilde u(t)\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} dt = C\|\tilde u\|^2_{L^2_\gamma(J, H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))}
\end{aligned} \tag{4.43}
$$

for $j \in \{1,2\}$, where we also exploited identity (4.30). It is a bit more subtle to treat this term for $j = 3$. We recall that there is a matrix $\tilde M \in C^\infty(\Omega)$ constant outside of a compact set such that $\tilde B = \tilde M \tilde A_3$. We then infer

$$
[\tilde B, J_\varepsilon]\partial_3 \tilde u = [\tilde M, J_\varepsilon]\tilde A_3 \partial_3 \tilde u \tag{4.44}
$$

for all $\varepsilon > 0$, using that $\tilde{A}_3$ is constant. For the commutator, Lemma 4.3 and equation (4.44) now yield

$$\begin{aligned}
&\int_0^1 \|[\tilde{M}, J_\varepsilon]\tilde{A}_3\partial_3\tilde{u}\|^2_{L^2_\gamma(\Omega)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&\le C\|\tilde{A}_3\partial_3\tilde{u}(t)\|^2_{L^2_\gamma(J,H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&\le C\|\tilde{f}\|^2_{L^2_\gamma(J,H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\sum_{j=0}^{2}\|\tilde{A}_j\partial_j\tilde{u}\|^2_{L^2_\gamma(J,H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + C\|\tilde{D}\tilde{u}\|^2_{L^2_\gamma(J,H^{m-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&\le C\|\tilde{f}\|^2_{H^{m-1}_\gamma(\Omega)} + C\|\partial_t\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} + C\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&=: K_2 + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{0}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))},
\end{aligned} \tag{4.45}$$

where $K_2 < \infty$.

Combining (4.41), (4.42), (4.43), and (4.45), we arrive at

$$\begin{aligned}
&\int_0^1 \|[\tilde{B}, J_\varepsilon]\tilde{u}\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \qquad (4.46)\\
&\le C\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} + CK_2 + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} =: K_3 + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))},
\end{aligned}$$

where $K_3 < \infty$ as $\tilde{u} \in H^{m-1}(\Omega)$.

With this estimate we now control the first summand in (4.40). In order to control the term

$$\begin{aligned}
&\int_0^1 \|J_\varepsilon g\|_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&= \int_J e^{-2\gamma t}\int_0^1 \|J_\varepsilon g(t)\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon dt,
\end{aligned} \tag{4.47}$$

we note that the proof of Lemma 4.3 (ii), see Theorems 2.4.5 and 2.4.1 in Chapter II of [Hoe76], shows that

$$\int_0^1 \|J_\varepsilon v\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \le C\|v\|^2_{H^{m-1/2}_\delta(\partial\mathbb{R}^3_+)}$$

for all $v \in H^{1/2}(\partial\mathbb{R}^3_+)$ and $\delta \in (0,1)$. Consequently,

$$\begin{aligned}
&\int_J e^{-2\gamma t}\int_0^1 \|J_\varepsilon g(t)\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon dt \\
&\le C\int_J e^{-2\gamma t}\|g(t)\|^2_{H^{m-1/2}_\delta(\partial\mathbb{R}^3_+)} dt \\
&\le C\int_J e^{-2\gamma t}\|g(t)\|^2_{H^{m+1/2}(\partial\mathbb{R}^3_+)} dt \le C\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)}.
\end{aligned} \tag{4.48}$$

This inequality in combination with (4.40) and (4.46) finally yields

$$\begin{aligned}
&\int_0^1 \|\tilde{B}J_\varepsilon\tilde{u}\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \\
&\le C\|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + CK_3 + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&=: K_4 + C\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))},
\end{aligned} \tag{4.49}$$

where as usual $K_4 < \infty$.

At the end of this step we also note that

$$\int_0^1 \|J_\varepsilon\tilde{u}_0\|^2_{L^2(\mathbb{R}^3_+)}\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon \le C\|\tilde{u}_0\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \le C\|\tilde{u}_0\|^2_{H^m(\mathbb{R}^3_+)}. \tag{4.50}$$

V) We return to estimate (4.37). Inserting (4.38) into this inequality, we first obtain that

$$\begin{aligned}
&\sup_{t\in J} e^{-2\gamma t}\|\tilde{u}(t)\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \gamma\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&\leq C\|\tilde{u}\|^2_{G_{m-1,\gamma}(\Omega)} + C\gamma\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} \\
&\quad + C\int_0^1 \Big(\|J_\varepsilon \tilde{u}_0\|^2_{L^2(\mathbb{R}^3_+)} + \|\tilde{B}J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(J,H^{1/2}(\partial\mathbb{R}^3_+))} \\
&\qquad\qquad + \frac{1}{\gamma}\|\tilde{L}J_\varepsilon\tilde{u}\|^2_{L^2_\gamma(\Omega)}\Big)\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1} d\varepsilon. \qquad (4.51)
\end{aligned}$$

Next the bounds (4.39), (4.49), and (4.50) yield

$$\begin{aligned}
&\sup_{t\in J} e^{-2\gamma t}\|\tilde{u}(t)\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \gamma\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&\leq C\|\tilde{u}\|^2_{G_{m-1,\gamma}(\Omega)} + C\gamma\|\tilde{u}\|^2_{H^{m-1}_\gamma(\Omega)} + C\|\tilde{u}_0\|^2_{H^m(\mathbb{R}^3_+)} + CK_4 + \frac{CK_1}{\gamma} \\
&\quad + C\Big(1+\frac{1}{\gamma}\Big)\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \\
&\leq K_9 + C\Big(1+\frac{1}{\gamma}\Big)\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))}. \qquad (4.52)
\end{aligned}$$

where K_9 is a finite constant. We fix the generic constant $C = C_{4.52}$ on the right-hand side of (4.52) and pick a number $\gamma \geq \gamma_0$ with $C_{4.52}(1+\frac{1}{\gamma}) \leq \frac{\gamma}{2}$. Hence,

$$\sup_{t\in J} e^{-2\gamma t}\|\tilde{u}(t)\|^2_{H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \frac{\gamma}{2}\|\tilde{u}\|^2_{L^2_\gamma(J,H^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} \leq K_9. \qquad (4.53)$$

Since $\delta \in (0,1)$ was arbitrary, we can let δ to 0 in (4.53). By Lemma 4.3 (i) we thus infer that $\tilde{u}(t)$ belongs to $H^m_{\mathrm{ta}}(\mathbb{R}^3_+)$ for all $t \in \overline{J}$ and that $\tilde{u}$ is contained in $L^2(J, H^m_{\mathrm{ta}}(\mathbb{R}^3_+)) \cap L^\infty(J, H^m_{\mathrm{ta}}(\mathbb{R}^3_+))$. The fact that G_r is an element of $W^{m+1,\infty}(\mathbb{R}^3_+)$ and $u = G_r\tilde{u}$ finally implies that $\partial^\beta u$ is contained in $L^2(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_0 = \beta_3 = 0$.

VI) Applying Corollary 4.2 inductively, we infer that u and thus also $\tilde{u}$ is an element of $H^m(\Omega)$. To establish that $\tilde{u}$ belongs to $G_m(\Omega)$, we apply Lemma 3.7 again.

Fix a multiindex $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_0 = \alpha_3 = 0$. Since $\tilde{u}$ is a solution of (4.34), the proof of Lemma 3.4 implies that

$$\tilde{L}\partial^\alpha\tilde{u} = \partial^\alpha\tilde{f} - \sum_{j=0}^{2}\binom{\alpha}{\beta}\partial^\beta\tilde{A}_j\partial^{\alpha-\beta}\partial_j\tilde{u} - \sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta\tilde{D}\partial^{\alpha-\beta}\tilde{u} = \tilde{f}_\alpha,$$

where $\tilde{f}_\alpha$ belongs to $L^2(\Omega)$. Lemma 3.5 then yields that the function $\tilde{g}_\alpha$ defined by

$$\tilde{g}_\alpha = \partial^\alpha g - \sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\operatorname{tr}(\partial^\beta\tilde{B}\partial^{\alpha-\beta}\tilde{u})$$

belongs to $L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$ and $\operatorname{Tr}(\tilde{B}\partial^\alpha\tilde{u}) = \tilde{g}_\alpha$. Next consider the function $J_{\frac{1}{n}}\partial^\alpha\tilde{u}$, which belongs to $G_0(\Omega)$. As in (4.35) we compute

$$\tilde{L}J_{\frac{1}{n}}\partial^\alpha\tilde{u} = J_{\frac{1}{n}}\tilde{f}_\alpha + \sum_{j=0}^{2}[\tilde{A}_j, J_{\frac{1}{n}}]\partial_j\partial^\alpha\tilde{u}$$

for all $n \in \mathbb{N}$. As $\tilde{f}_\alpha$ is an element of $L^2(\Omega)$, we have

$$J_{\frac{1}{n}}\tilde{f}_\alpha \longrightarrow \tilde{f}_\alpha \qquad (4.54)$$

in $L^2(\Omega)$ as $n \to \infty$. Arguing as in (4.9) and (4.10), we further derive

$$\sum_{j=0}^{2} [\tilde{A}_j, J_{\frac{1}{n}}] \partial_j \partial^\alpha \tilde{u} \longrightarrow 0 \tag{4.55}$$

in $L^2(\Omega)$ as $n \to \infty$ since $\tilde{u}$ belongs to $H^m(\Omega)$. Similarly, we have

$$\tilde{B} J_{\frac{1}{n}} \partial^\alpha \tilde{u} = J_{\frac{1}{n}} \tilde{g}_\alpha + [\tilde{B}, J_{\frac{1}{n}}] \partial^\alpha \tilde{u}.$$

Differentiating the commutator further yields

$$\partial_k([\tilde{B}, J_{\frac{1}{n}}] \partial^\alpha \tilde{u}) = [\partial_k \tilde{B}, J_{\frac{1}{n}}] \partial^\alpha \tilde{u} + [\tilde{B}, J_{\frac{1}{n}}] \partial_k \partial^\alpha \tilde{u}$$

in $L^2(\Omega)$ for all $k \in \{1, 2, 3\}$, where we used $|\alpha| \geq 1$ and Theorem C.14 from [BGS07] again. The arguments from (4.9) and (4.10) therefore again show that

$$\partial_k([\tilde{B}, J_{\frac{1}{n}}] \partial^\alpha \tilde{u}) \longrightarrow 0$$

in $L^2(\Omega)$ as $n \to \infty$ for all $k \in \{1, 2, 3\}$. As $\tilde{g}_\alpha$ is contained in $E_0(J \times \partial\mathbb{R}^3_+)$, we conclude that

$$\tilde{B} J_{\frac{1}{n}} \partial^\alpha \tilde{u} \longrightarrow \tilde{g}_\alpha \tag{4.56}$$

in $E_0(J \times \partial\mathbb{R}^3_+)$ as $n \to \infty$. Since $\tilde{u}_0 \in H^m(\mathbb{R}^3_+)$, the functions $J_{\frac{1}{n}} \partial^\alpha \tilde{u}_0$ tend to $\partial^\alpha \tilde{u}_0$ in $L^2(\mathbb{R}^3_+)$ as $n \to \infty$. We now apply the a priori estimate from Lemma 3.7. This lemma gives a constant C_0 and a number $\gamma > 0$ such that

$$\begin{aligned} \|J_{\frac{1}{n}} \partial^\alpha \tilde{u} - J_{\frac{1}{k}} \partial^\alpha \tilde{u}\|^2_{G_{0,\gamma}(\Omega)} &\leq C_0 \|J_{\frac{1}{n}} \partial^\alpha \tilde{u}_0 - J_{\frac{1}{k}} \partial^\alpha \tilde{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \\ &+ C_0 \|\tilde{B} J_{\frac{1}{n}} \partial^\alpha \tilde{u} - \tilde{B} J_{\frac{1}{k}} \partial^\alpha \tilde{u}\|^2_{E_{0,\gamma}(J \times \partial\mathbb{R}^3_+)} + \frac{1}{\gamma} \|\tilde{L} J_{\frac{1}{n}} \partial^\alpha \tilde{u} - \tilde{L} J_{\frac{1}{k}} \partial^\alpha \tilde{u}\|^2_{L^2_\gamma(\Omega)} \end{aligned}$$

for all $n, k \in \mathbb{N}$. We infer from (4.54), (4.55), and (4.56) that $(J_{\frac{1}{n}} \partial^\alpha \tilde{u})_n$ is a Cauchy sequence in $G_0(\Omega)$. As $(J_{\frac{1}{n}} \partial^\alpha \tilde{u})_n$ converges to $\partial^\alpha \tilde{u}$ in $L^2(\Omega)$, we obtain that $\partial^\alpha \tilde{u}$ is an element of $G_0(\Omega)$. Using again that G_r belongs to $W^{m+1,\infty}(\Omega)$ and that $u = G_r \tilde{u}$, we arrive at $\partial^\alpha u \in G_0(\Omega)$. □

Lemma 4.4 and an inductive application of Lemma 4.1 now yields the following corollary.

Corollary 4.5. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, $T > 0$, $J = (0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Take $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. We further assume that these coefficients and a function M as in the definition of $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ belong to $C^\infty(\overline{\Omega})$. Pick data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ and assume that the solution u of (3.2) is contained in $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$. Then u belongs to $G_m(\Omega)$.*

In particular, in the special case $m = 1$ we see that spatial regularity follows from regularity in time if the coefficients are smooth. The same statement is true in for $m \geq 1$, as we show via an iterative scheme in the next sections. Hence, the main task that remains is to derive regularity in time when the data is regular.

We finish this section with a remark concerning the constants in our a priori estimates.

Remark 4.6. If A_3 is an element of $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, we can study the transformed initial boundary value problem also when deriving the a priori estimates in tangential directions. We then see that the corresponding constants do no longer depend on A_3 itself, but only on the $F_{\tilde{m}}(\Omega)$-norm of it and on the parameter τ, cf. Lemma 3.9. In view of the dependancies of the constants in Lemma 3.11 and Theorem 3.13, we conclude that the constants in our a priori estimates do not depend on A_3 but only on its norm and the parameter τ if A_3 is an element of $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$.

4.2 Regularity in time

In this section we lay the foundation for the differentiability theorem. In a first step we show how regularity of the coefficients and the data in combination with the compatibility conditions imply that the solution of (3.2) belongs to $C^1(\overline{J}, L^2(\mathbb{R}^3_+))$. In combination with Corollary 4.5 we then set up an iteration scheme to deduce regularity of higher order. This iteration process however requires additional regularity for the coefficient A_0, namely that not only A_0 but also $\partial_t A_0$ belongs to $F_m(\Omega)$. The proof of the differentiability theorem has to overcome this loss of regularity, which needs a series of additional arguments. We therefore postpone this proof to the next section.

We further note that our approach requires the coefficients in front of the spatial derivatives and the matrix B to be time independent. In our applications to the quasilinear Maxwell system (1.6) the corresponding linearized and localized problem possesses this property. We will thus assume the time independence of these coefficients from now on.

The first lemma of this section is the key step to obtain regularity in time. We study the initial boundary value problem which is solved by $\partial_t u$ *if* the function u is contained in $C^1(\overline{J}, L^2(\mathbb{R}^3_+))$. The solution v of this problem is a candidate for the time derivative of u. The compatibility conditions then allow us to identify the function $w(t) = \int_0^t v(s)ds + u(t_0)$ with u via the uniqueness result for solutions of the original initial boundary value problem.

However, the initial boundary value problem solved by v is not of the form (3.2) since it contains the primitive of v. We therefore first have to employ a fixed point argument to solve this problem locally and then we need to exploit the a priori estimates to extend this local solution to the whole interval.

Lemma 4.7. *Let $T > 0$, $J = (0,T)$, $\Omega = J \times \mathbb{R}^3$, and $\eta, \tau > 0$. Take coefficients $A_0 \in F^{\mathrm{cp}}_{3,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{3,\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_3(\Omega)$, and $B \in \mathcal{BC}^3_{\mathbb{R}^3_+}(A_3)$ such that A_1, A_2, A_3, and B are independent of time. Choose data $u_0 \in H^1(\mathbb{R}^3_+)$, $g \in E_1(J \times \partial\mathbb{R}^3_+)$, and $f \in H^1(\Omega)$. Assume that the tupel $(0, A_0, \dots, A_3, D, B, f, g, u_0)$ fulfills the compatibility conditions (2.37) of order $l = 1$. Let $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ be the weak solution of (3.2) with differential operator $L(A_0, \dots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0. Assume that $u \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ implies $u \in G_1(J' \times \mathbb{R}^3_+)$ for every open interval $J' \subseteq J$. Then u belongs to $G_1(\Omega)$.*

Proof. Take $r > 0$ such that

$$\begin{aligned} &\|A_i\|_{F_3(\Omega)} \leq r, \quad \|D\|_{F_3(\Omega)} \leq r, \\ &\max\{\|A_i(t)\|_{F^0_2(\mathbb{R}^3_+)}, \max_{1\leq j\leq 2} \|\partial_t^j A_i(t)\|_{H^{2-j}(\mathbb{R}^3_+)}\} \leq r, \\ &\max\{\|D(t)\|_{F^0_2(\mathbb{R}^3_+)}, \max_{1\leq j\leq 2} \|\partial_t^j D(t)\|_{H^{2-j}(\mathbb{R}^3_+)}\} \leq r \end{aligned} \tag{4.57}$$

for all $t \in \overline{J}$ and $i \in \{0, \dots, 3\}$. Recall that such a number exists due to Sobolev's embedding. Let $\gamma = \gamma(\eta, \tau, r, T)$ be defined by

$$\gamma = \max\{\gamma_{3.7;0}, \gamma_{3.13;1}\} \geq 1,$$

where $\gamma_{3.7;0} = \gamma_{3.7;0}(\eta, \tau, r)$ and $\gamma_{3.13;1} = \gamma_{3.13;1}(\eta, \tau, r, T)$ are the corresponding constants from Lemma 3.7 and Theorem 3.13 respectively, see also Remark 4.6. We further introduce the constant $C_0 = C_0(\eta, \tau, r, T)$ by

$$C_0 = \max\{C_{3.7;0,0}, C_{3.7;0,1}, C_{3.7;0}, C_{3.13;1}, (C_{3.13;1,0} + TC_{3.13;1})e^{C_{3.13;1}T}, C_{2.33;1,1}\} \geq 1,$$

where again $C_{3.7;0,0} = C_{3.7;0,0}(\eta, \tau, r)$, $C_{3.7;0} = C_{3.7;0}(\eta, \tau, r)$, $C_{3.13;1} = C_{3.13;1}(\eta, \tau, r, T)$, and $C_{2.33;1,1} = C_{2.33;1,1}(\eta, \tau, r)$ are the corresponding constants from Lemma 3.7, Theorem 3.13, and Lemma 2.33 respectively. Here we again made use of Remark 4.6. Finally, we set

$$R_1 = C_0 e^{2\gamma T}(\|f\|^2_{G_{0,\gamma}(\Omega)} + \|f\|^2_{H^1_\gamma(\Omega)} + \|g\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)}).$$

I) Let $t_0 \in \overline{J}$ and $u(t_0) \in H^1(\mathbb{R}^3_+)$ with $\|u(t_0)\|^2_{H^1(\mathbb{R}^3_+)} \leq R_1$. We show that there exists a time step $T_s > 0$ such that there is a function $v \in C([t_0, T'], L^2(\mathbb{R}^3_+))$ with

$$\begin{cases} L(A_0, \dots, A_3, \partial_t A_0 + D)v = \partial_t f - \partial_t D\Big(\int_{t_0}^t v(s)ds + u(t_0)\Big), & x \in \mathbb{R}^3_+, \quad t \in J'; \\ Bv = \partial_t g, & x \in \partial\mathbb{R}^3_+, \quad t \in J'; \\ v(t_0) = S_{1,1}(t_0, A_0, \dots, A_3, D, f, u(t_0)), & x \in \mathbb{R}^3_+; \end{cases} \tag{4.58}$$

where we define $T' := \min\{t_0 + T_s, T\}$ and $J' := (t_0, T')$. Recall that the function $S_{1,1}(t_0, A_0, \dots, A_3, D, f, u(t_0))$ belongs to $L^2(\mathbb{R}^3_+)$ by Lemma 2.33.

Take a number $T_s \in (0, T)$ to be fixed below and define J' and T' as above. We further set $\Omega' = J' \times \mathbb{R}^3_+$. Let $w \in C(\overline{J'}, L^2(\mathbb{R}^3_+))$. Note that $\partial_t A_0 + D$ and $\partial_t D$ still belong to $L^\infty(\Omega)$. Hence the problem

$$\begin{cases} L(A_0, \dots, A_3, \partial_t A_0 + D)v = \partial_t f - \partial_t D\Big(\int_{t_0}^t w(s)ds + u(t_0)\Big), & x \in \mathbb{R}^3_+, \quad t \in J'; \\ Bv = \partial_t g, & x \in \partial\mathbb{R}^3_+, \quad t \in J'; \\ v(t_0) = S_{1,1}(t_0, A_0, \dots, A_3, D, f, u(t_0)), & x \in \mathbb{R}^3_+, \end{cases}$$

has a unique solution $\Phi(w)$ in $C(\overline{J'}, L^2(\mathbb{R}^3_+))$ by Lemma 3.7. We next define

$$B_R = \{v \in C(\overline{J'}, L^2(\mathbb{R}^3_+)) \colon \|v\|_{G_{0,\gamma}(\Omega')} \leq R\}, \tag{4.59}$$

where $R > 0$ will be fixed below. Equipped with the metric induced by the $G_{0,\gamma}(\Omega)$-norm this is a complete metric space. Let $w \in B_R$. Employing Hölder's and Minkowski's inequality, Lemma 3.7, and the bound

$$\begin{aligned} \|S_{1,1}(t_0, A_0, \dots, A_3, D, f, u(t_0))\|^2_{L^2(\mathbb{R}^3_+)} &\leq 2C^2_{2.33;1,1}(\|f(t_0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u(t_0)\|^2_{H^1(\mathbb{R}^3_+)}) \\ &\leq 2C_0 R_1 \end{aligned}$$

from Lemma 2.33, we estimate

$$\begin{aligned} \|\Phi(w)\|^2_{G_{0,\gamma}(\Omega')} &\leq C_0 \Big\|\partial_t f - \partial_t D \int_{t_0}^t w(s)ds - \partial_t D\, u(t_0)\Big\|^2_{L^2_\gamma(\Omega')} \\ &\quad + C_0\|\partial_t g\|^2_{E_{0,\gamma}(J' \times \partial\mathbb{R}^3_+)} + C_0\|S_{1,1}(t_0, A_0, \dots, A_3, D, f, u(t_0))\|^2_{L^2(\mathbb{R}^3_+)} \\ &\leq 2C_0\|\partial_t D\|^2_{L^\infty(\Omega')}\Big\|\int_{t_0}^t w(s)ds + u(t_0)\Big\|^2_{L^2_\gamma(\Omega')} + 2C_0\|f\|^2_{H^1_\gamma(J' \times \mathbb{R}^3_+)} \\ &\quad + C_0\|g\|^2_{E_{1,\gamma}(J' \times \partial\mathbb{R}^3_+)} + 2C_0^2 R_1 \\ &\leq 2C_0 r^2 \int_{t_0}^{T'} e^{-2\gamma t}\Big(\int_{t_0}^t \|w(s)\|_{L^2(\mathbb{R}^3_+)}ds + \|u(t_0)\|_{L^2(\mathbb{R}^3_+)}\Big)^2 dt + 2(1 + C_0)R_1 \\ &\leq 4C_0 r^2 T_s \|w\|^2_{G_{0,\gamma}(\Omega')} + 4C_0(1 + r^2 T_s)R_1. \end{aligned} \tag{4.60}$$

Here we have used

$$\begin{aligned} &\int_{t_0}^{T'} e^{-2\gamma t}\Big(\int_{t_0}^t \|w(s)\|_{L^2(\mathbb{R}^3_+)}ds\Big)^2 dt \\ &\leq \sup_{t \in J'} \|e^{-\gamma t} w(t)\|^2_{L^2(\mathbb{R}^3_+)} \int_{t_0}^{T'} e^{-2\gamma t}\Big(\int_{t_0}^t e^{\gamma s} ds\Big)^2 dt \\ &= \frac{1}{\gamma} \sup_{t \in J'} \|e^{-\gamma t} w(t)\|^2_{L^2(\mathbb{R}^3_+)} \int_{t_0}^{T'} (1 - e^{\gamma(t_0 - t)})^2 dt \\ &\leq \frac{1}{\gamma} \sup_{t \in J'} \|e^{-\gamma t} w(t)\|^2_{L^2(\mathbb{R}^3_+)}(T' - t_0) \leq T_s \|w\|^2_{G_{0,\gamma}(\Omega')}, \end{aligned} \tag{4.61}$$

where the last step is true as $\gamma \geq 1$ and $T' - t_0 \leq T_s$. We now set

$$R = (12C_0R_1)^{1/2}$$

in (4.59) and choose $T_s \in (0, T)$ so small that

$$4C_0r^2T_s \leq \frac{1}{2}.$$

We point out that T_s is independent of t_0. Using (4.59), $C_0 \geq 1$, and this choice of R and T_s, we obtain from (4.60)

$$\|\Phi(w)\|^2_{G_{0,\gamma}(\Omega')} \leq \frac{R^2}{2} + \frac{R^2}{2} = R^2$$

for all $w \in B_R$, i.e., $\Phi(B_R) \subseteq B_R$. Moreover, replacing w by $w_1 - w_2$ in (4.61), we infer via Lemma 3.7

$$\begin{aligned}
&\|\Phi(w_1) - \Phi(w_2)\|^2_{G_{0,\gamma}(\Omega')} \leq C_0\Big\|\partial_t D \int_{t_0}^t (w_1(s) - w_2(s))ds\Big\|^2_{L^2_\gamma(\Omega')} \\
&\leq C_0\|\partial_t D\|^2_{L^\infty(\Omega')}T_s\|w_1 - w_2\|^2_{G_{0,\gamma}(\Omega')} \leq C_0r^2T_s\|w_1 - w_2\|^2_{G_{0,\gamma}(\Omega')} \\
&\leq \frac{1}{2}\|w_1 - w_2\|^2_{G_{0,\gamma}(\Omega')}
\end{aligned}$$

for all $w_1, w_2 \in B_R$. The contraction mapping principle thus gives a unique $v \in B_R$ with $\Phi(v) = v$ on J', i.e., v is the asserted solution of (4.58).

II) In this step we assume that $u(t_0)$ belongs to $H^1(\mathbb{R}^3_+)$ with $\|u(t_0)\|^2_{H^1(\mathbb{R}^3_+)} \leq R_1$ and that $(t_0, A_0, \dots, A_3, D, f, g, u(t_0))$ fulfills the compatibility conditions (2.37) of order one; i.e., $\mathrm{tr}(Bu(t_0)) = g(t_0)$.

Let J' be defined as in step I) and let v be the solution of (4.58) constructed in step I). We first show that A_0v has a weak time derivative in $H^{-1}(\mathbb{R}^3_+)$ on J'. Let $\psi \in H^1_0(\mathbb{R}^3_+)$ and take $\varphi \in C_c^\infty(t_0, T')$. Abbreviating

$$\tilde{f} = \partial_t f - \partial_t D\Big(\int_{t_0}^t v(s)ds + u(t_0)\Big),$$

we compute

$$\begin{aligned}
&\Big\langle \int_{J'} A_0(t)v(t)\varphi'(t)dt, \psi\Big\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \int_{J'} \langle A_0(t)v(t)\varphi'(t), \psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&= \int_{J'} \langle A_0(t)v(t), \varphi'(t)\psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&= \int_{J'} \langle v(t), \partial_t(A_0\varphi\psi)(t) + \sum_{j=1}^3 \partial_j(A_j\varphi(t)\psi)\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&\quad - \int_{J'} \langle(\partial_t A_0(t) + D(t))v(t), \varphi(t)\psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&\quad + \int_{J'} \langle v(t), -\sum_{j=1}^3 \partial_j(A_j\varphi(t)\psi)\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&\quad + \int_{J'} \langle D(t)v(t), \varphi(t)\psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt \\
&= \int_{J'} \langle -\tilde{f}(t), \varphi(t)\psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}dt + \int_{J'} \sum_{j=1}^3 \langle A_j\partial_j v(t), \varphi(t)\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}dt \\
&\quad + \int_{J'} \langle D(t)v(t), \varphi(t)\psi\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}
\end{aligned}$$

$$= \Big\langle - \int_{J'} \Big(\tilde{f}(t) - \sum_{j=1}^{3} A_j \partial_j v(t) - D(t)v(t) \Big) \varphi(t) dt, \psi \Big\rangle_{H^{-1}(\mathbb{R}^3_+) \times H^1_0(\mathbb{R}^3_+)},$$

where we used (4.58). We conclude that

$$\int_{J'} A_0(t)v(t)\varphi'(t)dt = - \int_{J'} \Big(\tilde{f}(t) - \sum_{j=1}^{3} A_j \partial_j v(t) - D(t)v(t) \Big) \varphi(t) dt$$

in $H^{-1}(\mathbb{R}^3_+)$; i.e., $A_0 v$ has a weak time derivative in $L^2(J', H^{-1}(\mathbb{R}^3_+))$ and

$$\partial_t (A_0 v) = \tilde{f} - \sum_{j=1}^{3} A_j \partial_j v - Dv.$$

In particular, Proposition II.5.11 in [BF13] yields

$$(A_0 v)(t) - (A_0 v)(t_0) = \int_{t_0}^{t} \partial_t (A_0 v)(s) ds$$

for all $t \in J'$. We set

$$w(t) = u(t_0) + \int_{t_0}^{t} v(s) ds$$

for all $t \in \overline{J'}$. Observe that w belongs to $C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$, $w(t_0) = u(t_0)$, and that the above formulas and (2.36) yield

$$\begin{aligned}
L(A_0, \ldots, A_3, D)w(t) &= (A_0 v)(t) + \int_{t_0}^{t} \Big(\sum_{j=1}^{3} A_j \partial_j v(s) \Big) ds + \sum_{j=1}^{3} A_j \partial_j u(t_0) + (Dw)(t) \\
&= \int_{t_0}^{t} \Big(\partial_t (A_0 v)(s) + \sum_{j=1}^{3} A_j \partial_j v(s) \Big) ds + (Dw)(t) + (A_0 v)(t_0) + \sum_{j=1}^{3} A_j \partial_j u(t_0) \\
&= \int_{t_0}^{t} (\tilde{f}(s) - (Dv)(s)) ds + (Dw)(t) + A_0(t_0) S_{1,1}(t_0, A_0, \ldots, A_3, D, f, u(t_0)) \\
&\quad + \sum_{j=1}^{3} A_j \partial_j u(t_0) \\
&= \int_{t_0}^{t} (\partial_t f(s) - (\partial_t D w + D \partial_t w)(s)) ds + (Dw)(t) + f(t_0) - (Du)(t_0) \\
&= f(t) - f(t_0) + (Dw)(t_0) + f(t_0) - (Dw)(t_0) = f(t)
\end{aligned}$$

for all $t \in \overline{J'}$. In particular, $L(A_0, \ldots, A_3, D)w$ belongs to $L^2(\Omega)$.

To compute the trace of Bw on $\Gamma' = J' \times \partial\mathbb{R}^3_+$, we stress that $\operatorname{Tr}(Bv) = \partial_t g$ on Γ' by (4.58). Since $(t_0, A_0, \ldots, A_3, D, f, g, u(t_0))$ fulfills the compatibility conditions of order one, Corollary 2.18 (iii) and Remark 2.17 then yield

$$\begin{aligned}
\operatorname{Tr}(Bw) &= \operatorname{Tr} \Big(B \int_{t_0}^{t} v(s) ds + Bu(t_0) \Big) = \operatorname{Tr}(B I_{\Omega'} v) + \operatorname{Tr}(Bu(t_0)) \\
&= I_{\Gamma'} \operatorname{Tr}(Bv) + \operatorname{tr}(Bu(t_0)) = \int_{t_0}^{t} \partial_t g(s) ds + g(t_0) = g(t)
\end{aligned}$$

for all $t \in \overline{J'}$, where we also exploited that $I_{\Omega'}$ is linear and that g has a continuous representative in $H^{1/2}(\partial\mathbb{R}^3_+)$ as $\partial_t g \in L^2(J', H^{1/2}(\partial\mathbb{R}^3_+))$. The function $w \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ consequently solves (3.2) on Ω' with initial value $u(t_0)$ at initial time t_0. As $H^{-1/2}(\Gamma)$ continuously imbeds into $H^{-1/2}(\Gamma')$, also the trace of Bu

on Γ' equals g. Therefore, the function u also solves (3.2) on Ω' with initial value $u(t_0)$ in t_0. The uniqueness statement in Lemma 3.7 thus yields $u = w$ on Ω'. We conclude that u is an element of $C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$. The assumptions therefore tell us that u belongs to $G_1(\Omega')$.

III) We next consider $t_0 = 0$. Since $u(0) = u_0 \in H^1(\mathbb{R}^3_+)$, $\|u_0\|^2_{H^1(\mathbb{R}^3_+)} \leq R_1$, and $(0, A_0, \ldots, A_3, D, B, f, g, u_0)$ fulfills the compatibility conditions of first order by assumption, step II) shows that u belongs to $G_1((0, T_0) \times \mathbb{R}^3_+)$, where we set $T_0 = \min\{T_s, T\}$. If $T_0 = T$ we are done. Otherwise, we apply Theorem 3.13 to obtain

$$\begin{aligned}\sup_{t \in [0,T_0]} \|e^{-\gamma t} u(t)\|^2_{H^1(\mathbb{R}^3_+)} &\leq C_0(\|f\|^2_{G_{0,\gamma}(\Omega)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)} + \|g\|^2_{E_{1,\gamma}(J \times \partial\mathbb{R}^3_+)} \\ &\qquad + \frac{1}{\gamma}\|f\|^2_{H^1_\gamma(\Omega)}) \\ &\leq e^{-2\gamma T} R_1.\end{aligned}$$

We conclude that $\|u(T_0)\|^2_{H^1(\mathbb{R}^3_+)} \leq R_1$. Moreover, $(T_0, A_0, \ldots, A_3, D, B, f, g, u(T_0))$ fulfills the compatibility conditions of first order by Lemma 2.31, i.e.,

$$\operatorname{Tr} B S_{1,0}(T_0, A_0, \ldots, A_3, D, f, u(T_0)) = g(T_0),$$

since u is a solution in $G_1(J' \times \mathbb{R}^3_+)$. We can therefore apply step II) with $t_0 = T_0$. We see that u belongs to $G_1((T_0, T_1) \times \mathbb{R}^3_+)$, with $T_1 = \min\{T, T_0 + T_s\}$. Since

$$\partial_t u_{|[0,T_0]}(T_0) = S_{1,1}(T_0, A_0, \ldots, A_3, D, f, u(T_0)) = \partial_t u_{|[T_0,T_1]}(T_0),$$

we infer $u \in G_1((0, T_1) \times \mathbb{R}^3_+)$. In this way we iterate. Since the time step T_s does not depend on t_0, we are done after finitely many steps. We conclude that u is an element of $G_1((0, T) \times \mathbb{R}^3_+)$. □

The previous result allows us to obtain iteratively higher order regularity via the "differentiated problem"

$$\begin{cases} L(A_0, \ldots, A_3, \partial_t A_0 + D)\partial_t u = \partial_t f - \partial_t D u, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B \partial_t u = \partial_t g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ \partial_t u(0) = S_{m+1,1}(0, A_0, \ldots, A_3, D, f, u_0), & x \in \mathbb{R}^3_+. \end{cases} \tag{4.62}$$

However, if we want to apply regularity results of order m to (4.62), we have to make sure that the tupel

$$(0, A_0, \ldots, A_3, \partial_t A_0 + D, B, f, \partial_t g, S_{m+1,1}(0, A_0, \ldots, A_3, D, f, u_0))$$

fulfills the compatibility conditions of order m.

We point out that this approach requires an extra regularity assumption on the coefficient A_0. The definition of the compatibility conditions respectively the operators $S_{m,p}$ require the zeroth order coefficient to belong to $F_{\tilde{m}}(\Omega)$. The differentiated problem (4.62) contains $\partial_t A_0$ in this coefficient. But $\partial_t A_0$ need not be an element of $F_{\tilde{m}}(\Omega)$ if A_0 belongs to $F_{\max\{m+1,3\}}(\Omega)$! We will therefore require that $\partial_t A_0$ is an element of $F_{\tilde{m}}(\Omega)$ in the rest of this section and derive the regularity result under this assumption. We will demonstrate in Section 4.3 how to remove this additional smoothness assumption.

It might be possible to avoid this extra assumption on $\partial_t A_0$ if one works with different spaces for the zeroth and first order coefficients. However, we do not think that this procedure leads to a simplification in the big picture as all the estimates in Section 2.2 and Chapter 3 become even lengthier. Moreover, in view of our nonlinear problem it is natural to use the same function space for the zeroth and first order coefficients. Finally, in view of the assumptions of Lemma 4.3, an additional approximation argument is needed anyway.

Lemma 4.8. *Let $J \subseteq \mathbb{R}$ be an interval, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take $A_0 \in F^{\mathrm{cp}}_{\max\{m+1,3\},\eta}(\Omega)$ with $\partial_t A_0 \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$ and $D \in F^{\mathrm{cp}}_{\max\{m+1,3\}}(\Omega)$. Let A_1, A_2, $A_3 \in F^{\mathrm{cp}}_{\max\{m+1,3\}}(\Omega)$ and $B \in \mathcal{BC}^{\max\{m+1,3\}}_{\mathbb{R}^3_+}(A_3)$ be time independent. Choose $t_0 \in \overline{J}$, $u_0 \in H^{m+1}(\mathbb{R}^3_+)$, $g \in E_{m+1}(J \times \partial\mathbb{R}^3_+)$, and $f \in H^{m+1}(\Omega)$. Assume that the tupel $(t_0, A_0, \ldots, A_3, D, f, g, u_0)$ fulfills the linear compatibility conditions* (2.37) *of order $m+1$, i.e.,*

$$\operatorname{Tr}(B S_{m+1,p}(t_0, A_0, \ldots, A_3, D, f, u_0)) = \partial_t^p g(t_0) \quad \textit{for } 0 \leq p \leq m.$$

Assume that $u \in G_m(\Omega)$ solves the initial boundary value problem (3.2) *with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0. We set $u_1 = S_{m+1,1}(t_0, A_0, \ldots, A_3, D, f, u_0)$ and $f_1 = \partial_t f - \partial_t D u$. Then the tupel*

$$(t_0, A_0, \ldots, A_3, \partial_t A_0 + D, B, f_1, \partial_t g, u_1)$$

fulfills the linear compatibility conditions (2.37) *of order m, i.e.,*

$$\operatorname{Tr}(B S_{m,p}(t_0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1)) = \partial_t^{p+1} g(t_0) \quad \textit{for } 0 \leq p \leq m-1.$$

Proof. Without loss of generality let $t_0 = 0$. Note that $u_1 \in H^m(\mathbb{R}^3_+)$ by Lemma 2.33, that $\partial_t g \in E_m(J \times \partial\mathbb{R}^3_+)$, and that $f_1 \in H^m(\Omega)$ by Lemma 2.22, as $\partial_t D$ belongs to $\tilde{G}_{\max\{m,2\}}(\Omega)$. Since also $\partial_t A_0 + D$ is an element of $F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, we infer that the function $S_{m,p}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1)$ is well-defined and that it belongs to $H^{m-p}(\mathbb{R}^3_+)$ for $0 \leq p \leq m$ from Lemma 2.33.

Observe that it is enough to show

$$S_{m+1,l+1}(0, A_0, \ldots, A_3, D, f, u_0) = S_{m,l}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) \tag{4.63}$$

for $0 \leq l \leq m-1$.

Recall that $\partial_t^p u(0) = S_{m,p}(0, A_0, \ldots, A_3, D, f, u_0)$ for all $0 \leq p \leq m$ by Lemma 2.31 because $u \in G_m(\Omega)$ solves (3.2). By definition,

$$S_{m+1,1}(0, A_0, \ldots, A_3, D, f, u_0) = u_1 = S_{m,0}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1).$$

Now assume that (4.63) is true for $0 \leq l \leq p-1$ for some $1 \leq p \leq m-1$. Using (2.36), the induction hypothesis, and the convention $\binom{n}{k} = 0$ for $k > n$, we compute

$$\begin{aligned}
&A_0(0)\, S_{m,p}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) \\
&= \partial_t^{p-1} f_1(0) - \sum_{j=1}^{3} A_j \partial_j S_{m,p-1}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) \\
&\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(0) S_{m,p-l}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) \\
&\quad - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l (\partial_t A_0 + D)(0) S_{m,p-1-l}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) \\
&= \partial_t^p f(0) - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^{l+1} D(0) \partial_t^{p-1-l} u(0) \\
&\quad - \sum_{j=1}^{3} A_j \partial_j S_{m+1,p}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(0) S_{m+1,p+1-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=0}^{p-1} \binom{p-1}{l} (\partial_t^{l+1} A_0 + \partial_t^l D)(0) S_{m+1,p-l}(0, A_0, \ldots, A_3, D, f, u_0)
\end{aligned}$$

$$
\begin{aligned}
&= \partial_t^p f(0) - \sum_{j=1}^{3} A_j \partial_j S_{m+1,p}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{p} \left(\binom{p-1}{l} + \binom{p-1}{l-1} \right) \partial_t^l A_0(0) S_{m+1,p+1-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{p} \left(\binom{p-1}{l} + \binom{p-1}{l-1} \right) \partial_t^l D(0) S_{m+1,p-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - D(0) S_{m+1,p}(0, A_0, \ldots, A_3, D, f, u_0) \\
&= \partial_t^p f(0) - \sum_{j=1}^{3} A_j \partial_j S_{m+1,p}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{p} \binom{p}{l} \partial_t^l A_0(0) S_{m+1,p+1-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&\quad - \sum_{l=0}^{p} \binom{p}{l} \partial_t^l D(0) S_{m+1,p-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&= A_0(0) S_{m+1,p+1}(0, A_0, \ldots, A_3, D, f, u_0).
\end{aligned}
$$

By induction, we conclude that

$$
S_{m,p}(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, u_1) = S_{m+1,p+1}(0, A_0, \ldots, A_3, D, f, u_0)
$$

for all $p \in \{0, \ldots, m-1\}$. The assertion thus follows. □

Corollary 4.5, Lemma 4.7, and Lemma 4.8 now allow us to set up an iteration scheme which yields the required regularity result in higher order if we assume that the coefficients are smooth. We will remove this extra condition in the next section.

Proposition 4.9. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Pick $T > 0$ and set $J = (0,T)$ and $\Omega = J \times \mathbb{R}^3_+$. Choose coefficients $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$ with $\partial_t A_0 \in F^{\mathrm{cp}}_{\max\{m-1,3\}}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. Assume that these coefficients are contained in $C^\infty(\overline{\Omega})$ and that A_1, A_2, A_3, and B are time independent. Take data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ such that the tupel $(0, A_0, \ldots, A_3, D, B, f, g, u_0)$ satisfies the compatibility conditions* (2.37) *of order m, i.e.,*

$$
\mathrm{Tr}(B S_{m,l}(0, A_0, \ldots, A_3, D, f, u_0)) = \partial_t^l g(0) \quad \text{for } 0 \le l \le m-1.
$$

Let u be the weak solution of (3.2) *with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0. Then u belongs to $G_m(\Omega)$.*

Proof. The assertion is true for $m = 1$ by Lemma 4.7 and Corollary 4.5. Now assume that we have shown the assertion for a number $m \in \mathbb{N}$. Let all the conditions be fulfilled for $m + 1$. By the induction hypothesis, the weak solution u of (3.2) belongs to $G_m(\Omega)$. Moreover, $\partial_t u$ solves (4.62), i.e.,

$$
\begin{cases}
L(A_0, \ldots, A_3, \partial_t A_0 + D)\partial_t u = \partial_t f - \partial_t D u, & x \in \mathbb{R}^3_+, \quad t \in J; \\
B \partial_t u = \partial_t g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\
\partial_t u(0) = S_{m+1,1}(0, A_0, \ldots, A_3, D, f, u_0), & x \in \mathbb{R}^3_+.
\end{cases}
$$

We again write u_1 for $S_{m+1,1}(0, A_0, \ldots, A_3, D, f, u_0)$ and f_1 for $\partial_t f - \partial_t D u$. Then u_1 is contained in $H^m(\mathbb{R}^3_+)$ by Lemma 2.33 and $\partial_t g$ belongs to $E_m(J \times \partial\mathbb{R}^3_+)$. Since $\partial_t D \in G_{\max\{m,2\}}(\Omega)$ and $u \in G_m(\Omega)$, Lemma 2.22 (ii) implies that f_1 belongs to $H^m(\Omega)$. Lemma 4.8 further shows that $(0, A_0, \ldots, A_3, \partial_t A_0 + D, f_1, \partial_t g, u_1)$ fulfills the compatibility conditions (2.37) of order m. Finally, we have $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega) \cap C^\infty(\overline{\Omega})$

with $\partial_t A_0 \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$ and $\partial_t A_0 + D \in F^{\mathrm{cp}}_{\tilde m}(\Omega) \cap C^\infty(\overline{\Omega})$ so that the induction hypothesis yields that $\partial_t u$ is an element of $G_m(\Omega)$, implying that $u \in \bigcap_{j=1}^{m+1} C^j(\overline{J}, H^{m+1-j}(\mathbb{R}^3_+))$. By Corollary 4.5, u then belongs to $G_{m+1}(\Omega)$. □

4.3 The differentiability theorem

The previous result yields the required amount of regularity for a solution under the assumption of additional regularity of the coefficients. Note that this assumption is indeed necessary to make our regularizing procedure work in higher order.

To get rid of the assumption that the coefficients are smooth, at least for all but A_3, we will approximate A_i by a smoother family $\{A_{i,\varepsilon}\}_{\varepsilon>0}$ for $i \in \{0,1,2\}$ and D by $\{D_\varepsilon\}_{\varepsilon>0}$, hoping that the corresponding solutions u_ε converge in such a sense to the original soution u that the regularity of u_ε can be transferred to u.

The first result is once more concerned with the compatibility conditions. If we approximate the coefficients A_i and D by families of smoother ones $\{A_{i,\varepsilon}\}_\varepsilon$ and $\{D_\varepsilon\}_\varepsilon$ and consider problem (3.2) with A_i replaced by $A_{i,\varepsilon}$ and D replaced by D_ε for $i \in \{0,1,2\}$, the tupels $(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, B, f, g, u_0)$ will not satisfy the compatibility conditions. However, these conditions are necessary for the corresponding solution u_ε to belong to $G_m(\Omega)$. To overcome this problem, we construct a family of initial values $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ in $H^m(\mathbb{R}^3_+)$ such that $u_{0,\varepsilon} \to u_0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ and the tupels $(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, B, f, g, u_{0,\varepsilon})$ fulfill the compatibility conditions for all $\varepsilon > 0$.

To that purpose, we derive a semi-explicit representation of the operators $S_{\mathbb{R}^3_+,m,p}$, which allows us to isolate the normal derivatives of u_0. An extension theorem then yields functions $u_{0,\varepsilon}$ with the desired properties.

As we are only working on the half-space in this section, we drop the underlying domain $\mathbb{R}^3_+$ in the notation of the operators $S_{\mathbb{R}^3_+,m,p}$ in the following.

Lemma 4.10. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, and $\tilde m = \max\{m, 3\}$. Take coefficients $A_0 \in F^{\mathrm{cp}}_{\tilde m,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde m,\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde m,\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$, and $B \in \mathcal{BC}^{\tilde m}_{\mathbb{R}^3_+}(A_3)$ and data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ which fulfill the compatibility conditions* (2.37) *of order m in $t_0 \in \overline{J}$, i.e.,*

$$\operatorname{Tr} B S_{m,l}(t_0, A_0, \ldots, A_3, D, f, u_0) = \partial_t^l g(t_0) \quad \text{for } 0 \le l \le m-1.$$

We suppose that A_1, A_2, and A_3 are time independent. Let $\{A_{i,\varepsilon}\}_{\varepsilon>0}$ and $\{D_\varepsilon\}_{\varepsilon>0}$ be the families of functions provided by Lemma 2.21 for A_i and D respectively for $i \in \{0,1,2\}$. Then there exists a number $\varepsilon_0 > 0$ and a family $\{u_{0,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ in $H^m(\mathbb{R}^3_+)$ such that the compatibility conditions for $(t_0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, B, f, g, u_{0,\varepsilon})$ hold; i.e.,

$$\operatorname{Tr} B S_{m,l}(t_0, A_{0,\varepsilon}, A_1, A_2, A_3, D_\varepsilon, f, u_{0,\varepsilon}) = \partial_t^l g(t_0) \quad \text{for } 0 \le l \le m-1,$$

and $u_{0,\varepsilon} \to u_0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.

Proof. Without loss of generality we assume $t_0 = 0$. Note that $A_{1,\varepsilon}$ and $A_{2,\varepsilon}$ are still time independent for all $\varepsilon > 0$. We set $u_{0,\varepsilon} = u_0 + h_\varepsilon$ and look for $h_\varepsilon \in H^m(\mathbb{R}^3_+)$ with $h_\varepsilon \to 0$ in $H^m(\mathbb{R}^3_+)$ such that the compatibility conditions are fulfilled. In view of Definition 2.16 and since $B = MA_3$, it is sufficient for that purpose to find h_ε with

$$A_3 S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, f, u_0 + h_\varepsilon) = A_3 S_{m,p}(0, A_0, \ldots, A_3, D, f, u_0)$$

for all $0 \le p \le m-1$ on $\partial\mathbb{R}^3_+$. To simplify the notation, we will drop the dependancy of the operators on A_3 and f in the following since they remain fixed throughout the proof.

I) Let $k \in \{0, \ldots, m-1\}$ and $\{\Lambda_\varepsilon\}_{\varepsilon>0} \subseteq H^k(\mathbb{R}^3_+)$ with $\Lambda_\varepsilon \to \Lambda$ in $H^k(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. We show that $A_{0,\varepsilon}(0)^{-1}\Lambda_\varepsilon \to A_0(0)^{-1}\Lambda$ in $H^k(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.

Recall from Lemma 2.21 that $A_{0,\varepsilon}(0)$ converges to $A_0(0)$ in $F^0_{\tilde m-1}(\mathbb{R}^3_+)$ so that Lemma 2.23 yields that there exists a constant C_1 with $\|A_{0,\varepsilon}(0)^{-1}\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)} \leq C_1$ and $A_{0,\varepsilon}(0)^{-1} \to A_0(0)^{-1}$ in $F^0_{\tilde m-1}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. Lemma 2.22 (vii) then shows that

$$\begin{aligned}\|A_{0,\varepsilon}(0)^{-1}\Lambda_\varepsilon - A_0(0)^{-1}\Lambda\|_{H^k(\mathbb{R}^3_+)} &\leq C\|A_{0,\varepsilon}(0)^{-1} - A_0(0)^{-1}\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}\|\Lambda_\varepsilon\|_{H^k(\mathbb{R}^3_+)} \\ &\quad + C\|A_0(0)^{-1}\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}\|\Lambda_\varepsilon - \Lambda\|_{H^k(\mathbb{R}^3_+)} \longrightarrow 0 \qquad (4.64)\end{aligned}$$

as $\varepsilon \to 0$.

II) In this step we assume that $u_{0,\varepsilon} \to u_0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. We then prove that $S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_{0,\varepsilon})$ tends to $S_{m,p}(0, A_0, A_1, A_2, D, u_0)$ in $H^{m-p}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ for $0 \leq p \leq m-1$.

To that purpose, we first abbreviate

$$S^0_{m,p} = S_{m,p}(0, A_0, A_1, A_2, D, u_0) \quad \text{and} \quad S^\varepsilon_{m,p} = S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_{0,\varepsilon})$$

for all $\varepsilon > 0$. Since $S^\varepsilon_{m,0} = u_{0,\varepsilon}$ converges to $u_0 = S^0_{m,0}$ in $H^m(\mathbb{R}^3_+)$ the assertion is clear for $p = 0$. Next assume that the assertion is true for $0 \leq p \leq k$ and some $k \in \{0, \ldots, m-2\}$. We establish that $S^\varepsilon_{m,k+1} \to S^0_{m,k+1}$ in $H^{m-k-1}(\mathbb{R}^3_+)$. Due to step I) it suffices to show that

$$\begin{aligned}&\sum_{j=1}^{2} A_{j,\varepsilon}\partial_j S^\varepsilon_{m,k} + A_3\partial_3 S^\varepsilon_{m,k} + \sum_{l=1}^{k}\binom{k}{l}\partial_t^l A_{0,\varepsilon}(0)S^\varepsilon_{m,k+1-l} + \sum_{l=0}^{k}\binom{k}{l}\partial_t^l D_\varepsilon(0)S^\varepsilon_{m,k-l} \\ &\longrightarrow \sum_{j=1}^{3} A_j\partial_j S^0_{m,k} + \sum_{l=1}^{k}\binom{k}{l}\partial_t^l A_0(0)S^0_{m,k+1-l} + \sum_{l=0}^{k}\binom{k}{l}\partial_t^l D(0)S^0_{m,k-l} \qquad (4.65)\end{aligned}$$

in $H^{m-k-1}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.

The induction hypothesis yields that $S^\varepsilon_{m,k}$ converges to $S^0_{m,k}$ in $H^{m-k}(\mathbb{R}^3_+)$. Since the coefficients $A_{1,\varepsilon}$ and $A_{2,\varepsilon}$ converge to A_1 respectively A_2 in $F^0_{\tilde m-1}(\mathbb{R}^3_+)$ and A_3 belongs to $F^0_{\tilde m-1}(\mathbb{R}^3_+)$, Lemma 2.22 (vii) implies that the first three summands on the left-hand side of (4.65) converge to $\sum_{j=1}^3 A_j\partial_j S^0_{m,k}$ in $H^{m-k-1}(\mathbb{R}^3_+)$.

Let $l \in \{1, \ldots, k\}$. The induction hypothesis also yields that

$$S^\varepsilon_{m,k+1-l} \longrightarrow S^0_{m,k+1-l}$$

in $H^{m-k-1+l}(\mathbb{R}^3_+) \hookrightarrow H^{m-k}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. On the other hand, $\partial_t^l A_{0,\varepsilon}(0)$ converges to $\partial_t^l A_0(0)$ in $H^{m-l-1}(\mathbb{R}^3_+) \hookrightarrow H^{m-k-1}(\mathbb{R}^3_+)$ by Lemma 2.21. Using Lemma 2.22 (vi) and arguing as in (4.64), we conclude that $\partial_t^l A_{0,\varepsilon}(0)S^\varepsilon_{m,k+1-l}$ converges to $\partial_t^l A_0(0)S^0_{m,k+1-l}$ in $H^{m-k-1}(\mathbb{R}^3_+)$. Analogously, we treat the terms in the third sum of (4.65). The claim thus follows.

III) The definition of the operators $S_{m,k}$ was given inductively. In principle, it is possible to derive an explicit representation of $S_{m,k}$. However, this would lead to unhandy expressions for the coefficients in front of the derivatives of u_0 and f. We are therefore satisfied with an "intermediate" result as we only need to know the regularity of these coefficients in the following. Take $r > 0$ such that

$$\begin{aligned}&\max\{\|A_i(0)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}, \max_{1\leq l\leq \tilde m-1}\|\partial_t^l A_i(0)\|_{H^{\tilde m-l-1}(\mathbb{R}^3_+)} : i \in \{0,\ldots,3\}\} \leq r, \\ &\max\{\|D(0)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}, \max_{1\leq l\leq \tilde m-1}\|\partial_t^l D(0)\|_{H^{\tilde m-l-1}(\mathbb{R}^3_+)}\} \leq r.\end{aligned}$$

We claim that

$$S_{m,k}(0, A_0, A_1, A_2, D, u_0) = \sum_{\substack{\alpha\in\mathbb{N}_0^3 \\ |\alpha|\leq k}} A_k^\alpha(0, A_0, A_1, A_2, D)\partial^\alpha u_0$$

$$+ \sum_{\substack{\beta\in\mathbb{N}_0^4\\ |\beta|\le k-1}} A_k^\beta(0,A_0,A_1,A_2,D)\partial^\beta f(0) \tag{4.66}$$

for $0 \le k \le m$ and certain functions

$$A_k^\alpha(0,A_0,A_1,A_2,D) \in \begin{cases} H^{\tilde m-k}(\mathbb{R}^3_+) + F^0_{\tilde m-1}(\mathbb{R}^3_+), & \text{for } |\alpha|\le k-1;\\ F^0_{\tilde m-1}(\mathbb{R}^3_+), & \text{for } |\alpha| = k;\end{cases}$$
$$A_k^\beta(0,A_0,A_1,A_2,D) \in \begin{cases} H^{\tilde m-k}(\mathbb{R}^3_+) + F^0_{\tilde m-1}(\mathbb{R}^3_+), & \text{for } |\beta|\le k-2;\\ F^0_{\tilde m-1}(\mathbb{R}^3_+), & \text{for } |\beta| = k-1;\end{cases} \tag{4.67}$$

for $k \in \{0,\dots,m-1\}$ and multi-indices $\alpha\in\mathbb{N}_0^3$ and $\beta\in\mathbb{N}_0^4$, which have the additional property that $A_k^{(0,0,k)}(0,A_0,A_1,A_2,D) = (-A_0(0)^{-1}A_3(0))^k$ for $0\le k\le m-1$. Moreover, we have

$$\begin{aligned} &\|A_k^\alpha(0,A_0,A_1,A_2,D)\|_{H^{\tilde m-k-1}(\mathbb{R}^3_+)+F^0_{\tilde m-1}(\mathbb{R}^3_+)} \le C_k, && \text{if } |\alpha|\le k-1;\\ &\|A_k^\alpha(0,A_0,A_1,A_2,D)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)} \le C_k, && \text{if } |\alpha| = k;\\ &\|A_k^\beta(0,A_0,A_1,A_2D)\|_{H^{\tilde m-k-1}(\mathbb{R}^3_+)+F^0_{\tilde m-1}(\mathbb{R}^3_+)} \le C_k, && \text{if } |\beta|\le k-2;\\ &\|A_k^\beta(0,A_0,A_1,A_2,D)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)} \le C_k, && \text{if } |\beta| = k-1;\end{aligned} \tag{4.68}$$

where $C_k = C_k(\eta,r)$ for k, α and β as above.

In the proof of this claim we use the following conventions. Throughout, α represents a multi-index in $\mathbb{N}_0^3$ and β a multi-index in $\mathbb{N}_0^4$. Moreover, as A_0, A_1, A_2, D, and u_0 are fixed in this proof, we omit also these arguments of $S_{m,k}$, A_k^α, and A_k^β.

Observe that $S_{m,0} = u_0$ is of the claimed form with $A_0^{(0,0,0)} = I$. Furthermore,

$$A_0(0)S_{m,1} = f(0) - \sum_{j=1}^3 A_j\partial_j u_0 - D(0)u_0$$

is of the form (4.66) with $A_1^{(0,0,0)} = -A_0(0)^{-1}D(0) \in F^0_{\tilde m-1}(\mathbb{R}^3_+)$, $A_1^{e_j} = -A_0(0)^{-1}A_j \in F^0_{\tilde m-1}(\mathbb{R}^3_+)$, and $A_1^{(0,0,0,0)} = A_0(0)^{-1} \in F^0_{\tilde m-1}(\mathbb{R}^3_+)$. Note that the coefficients $A_0^{(0,0,0)}$ and $A_1^{(0,0,1)}$ are of the form $A_0^{(0,0,0)} = (-A_0(0)^{-1}A_3)^0$ and $A_1^{(0,0,1)} = (-A_0(0)^{-1}A_3)^1$. This shows (4.67) for the indices $k=0$ and $k=1$. Lemma 2.23 and Lemma 2.22 further imply that also (4.68) is true for these indices.

We next assume that the claims have been shown for all indices $0\le l\le k$ for a number $k\in\{1,\dots,m-1\}$. Starting from (2.36), we then compute

$$\begin{aligned} &A_0(0)S_{m,k+1}\\ &= \partial_t^k f(0) - \sum_{j=1}^3 A_j\partial_j\Big[\sum_{|\alpha|\le k} A_k^\alpha\partial^\alpha u_0 + \sum_{|\beta|\le k-1} A_k^\beta\partial^\beta f(0)\Big]\\ &\quad - \sum_{l=1}^k \binom{k}{l}\partial_t^l A_0(0)\Big[\sum_{|\alpha|\le k+1-l} A_{k+1-l}^\alpha\partial^\alpha u_0 + \sum_{|\beta|\le k-l} A_{k+1-l}^\beta\partial^\beta f(0)\Big]\\ &\quad - \sum_{l=0}^k \binom{k}{l}\partial_t^l D(0)\Big[\sum_{|\alpha|\le k-l} A_{k-l}^\alpha\partial^\alpha u_0 + \sum_{|\beta|\le k-1-l} A_{k-l}^\beta\partial^\beta f(0)\Big]\\ &= -\sum_{j=1}^3\sum_{|\alpha|\le k} A_j\partial_j A_k^\alpha\partial^\alpha u_0 - \sum_{j=1}^3\sum_{|\alpha|\le k} A_j A_k^\alpha\partial^{\alpha+e_j}u_0 \\ &\quad - \sum_{l=1}^k\sum_{|\alpha|\le k+1-l}\binom{k}{l}\partial_t^l A_0(0)A_{k+1-l}^\alpha\partial^\alpha u_0 - \sum_{l=0}^k\sum_{|\alpha|\le k-l}\binom{k}{l}\partial_t^l D(0)A_{k-l}^\alpha\partial^\alpha u_0\end{aligned} \tag{4.69}$$

$$+ \partial_t^k f(0) - \sum_{j=1}^{3} \sum_{|\beta|\leq k-1} A_j \partial_j A_k^\beta \partial^\beta f(0) - \sum_{j=1}^{3} \sum_{|\beta|\leq k-1} A_j A_k^\beta \partial^{\beta+e_j} f(0)$$
$$- \sum_{l=1}^{k} \sum_{|\beta|\leq k-l} \binom{k}{l} \partial_t^l A_0(0) A_{k+1-l}^\beta \partial^\beta f(0) - \sum_{l=0}^{k} \sum_{|\beta|\leq k-1-l} \binom{k}{l} \partial_t^l D(0) A_{k-l}^\beta \partial^\beta f(0).$$

Here, the formula

$$\partial_j(A_k^\alpha \partial^\alpha u_0) = \partial_j A_k^\alpha \partial^\alpha u_0 + A_k^\alpha \partial^{\alpha+e_j} u_0$$

follows from Lemma 2.22, as $A_k^\alpha \in H^{\tilde{m}-k}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ and $\partial^\alpha u_0 \in H^{m-k+1}(\mathbb{R}^3_+)$ in the case $|\alpha| \leq k-1$ and $A_k^\alpha \in F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ and $\partial^\alpha u_0 \in H^{m-k}(\mathbb{R}^3_+)$ if $|\alpha| = k$. Analogously, one infers

$$\partial_j(A_k^\beta \partial^\beta f(0)) = \partial_j A_k^\beta \partial^\beta f(0) + A_k^\beta \partial^{\beta+e_j} f(0)$$

for $|\beta| \leq k-1$. In the following we use Lemma 2.22 several times without further reference.

We first point out that $\partial^{\alpha+e_j}$ with $|\alpha| = k$ and $j \in \{1,2,3\}$ are the only derivatives of order $k+1$ in (4.69). Moreover, the coefficients $A_{k+1}^{\alpha+e_j} := -A_0(0)^{-1} A_j A_k^\alpha$ belong to $F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ for all α with $|\alpha| = k$. Consequently, A_{k+1}^α is an element of $F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ for all α with $|\alpha| = k+1$. Lemma 2.23 and the induction hypothesis further yield that

$$\|A_{k+1}^\alpha\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} \leq C_{2.23}(\eta, r) r \cdot C_k(\eta, r).$$

Moreover,

$$A_{k+1}^{(0,0,k+1)} = -A_0(0)^{-1} A_3 A_k^{(0,0,k)} = (-A_0(0)^{-1} A_3)^{k+1}.$$

We further deduce that

$$A_j \partial_j A_k^\alpha \in H^{\tilde{m}-k-1}(\mathbb{R}^3_+) \quad \text{and} \quad A_j A_k^\alpha \in H^{\tilde{m}-k-1}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+) \tag{4.70}$$

since $A_k^\alpha \in H^{\tilde{m}-k}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ and $A_j \in F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$. The induction hypothesis yields

$$\|A_j \partial_j A_k^\alpha\|_{H^{\tilde{m}-k-1}(\mathbb{R}^3_+)} \leq C(r) \|A_k^\alpha\|_{H^{\tilde{m}-k}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} \leq \tilde{C}_k(\eta, r),$$
$$\|A_j A_k^\alpha\|_{H^{\tilde{m}-k-1}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} \leq C(r) \|A_k^\alpha\|_{H^{\tilde{m}-k}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} \leq \tilde{C}_k(\eta, r),$$

for $j \in \{1,2,3\}$. Now take $l \in \{1, \dots, k\}$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq k+1-l$. Then $\partial_t^l A_0(0)$ is contained in $H^{\tilde{m}-1-l}(\mathbb{R}^3_+) \hookrightarrow H^{\tilde{m}-k-1}(\mathbb{R}^3_+)$ and A_{k+1-l}^α in $H^{\tilde{m}-k-1+l}(\mathbb{R}^3_+)$ so that Lemma 2.22 (vi) and (vii) imply

$$\partial_t^l A_0(0) A_{k+1-l}^\alpha \in H^{\tilde{m}-k-1}(\mathbb{R}^3_+) \tag{4.71}$$

and

$$\|\partial_t^l A_0(0) A_{k+1-l}^\alpha\|_{H^{\tilde{m}-k-1}(\mathbb{R}^3_+)} \leq C \|\partial_t^l A_0(0)\|_{H^{\tilde{m}-1-l}(\mathbb{R}^3_+)} \|A_{k+1-l}^\alpha\|_{H^{\tilde{m}-k-1+l}(\mathbb{R}^3_+)}$$
$$\leq Cr \cdot C_{k+1-l}(\eta, r).$$

The same argument shows that

$$\partial_t^l D(0) A_{k-l}^\alpha \in H^{\tilde{m}-k-1}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+) \tag{4.72}$$

and

$$\|\partial_t^l D(0) A_{k-l}^\alpha\|_{H^{\tilde{m}-k-1}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)} \leq Cr \cdot C_{k-l}(\eta, r) \tag{4.73}$$

for $|\alpha| \leq k - l$ and $l \in \{1, \dots, k\}$. The identities (4.72) and (4.73) are also true for $|\alpha| \leq k$ and $l = 0$ as $D(0)$ belongs to $F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ and A^α_{k-l} to $H^{\tilde{m}-k+l}(\mathbb{R}^3_+)+F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$.

Since $A_0(0)^{-1}$ is an element of $F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ and the coefficients A^α_{k+1} are linear combinations of the products of $A_0(0)^{-1}$ with terms appearing in (4.70) to (4.72), Lemma 2.22 yields that A^α_{k+1} belongs to $H^{\tilde{m}-k-1}(\mathbb{R}^3_+) + F^0_{\tilde{m}-1}(\mathbb{R}^3_+)$ for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq k$. The estimates for the corresponding terms and Lemma 2.23 then imply that also (4.68) is true for $k+1$.

The assertion for the coefficients A^β_k follows analogously. This finishes the proof of the claim.

Rearranging (4.69) we can now write the operators $S_{m,p}$ as

$$S_{m,p}(0, A_0, A_1, A_2, D, u_0) = (-A_0(0)^{-1}A_3)^p \partial_3^p u_0 + \sum_{j=0}^{p-1} C_{p,p-j}(0, A_0, A_1, A_2, D)\partial_3^j u_0 + B_p(0, A_0, A_1, A_2, D)f(0), \tag{4.74}$$

where

$$C_{p,p-j}(0, A_0, A_1, A_2, D) = \sum_{\substack{\alpha\in\mathbb{N}_0^3 \\ |\alpha|\leq p, \alpha_3 = j}} A^\alpha_p(0, A_0, A_1, A_2, D)\partial^{(\alpha_1,\alpha_2,0)},$$

$$B_p(0, A_0, A_1, A_2, D)f = \sum_{\substack{\beta\in\mathbb{N}_0^4 \\ |\beta|\leq p-1}} A^\beta_p(0, A_0, A_1, A_2, D)\partial^\beta f(0)$$

for all $j \in \{0, \dots, p-1\}$, $p \in \{1, \dots, m-1\}$ and $f \in H^m(\Omega)$. Observe that $C_{p,p-j}$ is a differential operator which only involves tangential derivatives up to order $p-j$. The regularity of the coefficients stated in (4.67) and Lemma 2.22 further show that $C_{p,p-j}$ maps $H^{m-j}(\mathbb{R}^3_+)$ into $H^{m-p}(\mathbb{R}^3_+)$ for all $j \in \{0, \dots, p-1\}$ and $p \in \{1, \dots, m-1\}$. Lemma 2.22 and (4.68) moreover yield a constant $R_{p,p-j} = R_{p,p-j}(\eta, r)$ such that

$$\|C_{p,p-j}(0, A_0, A_1, A_2, D)v\|_{H^{m-p}(\mathbb{R}^3_+)} \leq R_{p,p-j}(\eta, r)\|v\|_{H^{m-j}(\mathbb{R}^3_+)} \tag{4.75}$$

for all $v \in H^{m-j}(\mathbb{R}^3_+)$, $j \in \{0, \dots, p-1\}$, and $p \in \{1, \dots, m-1\}$.

Similarly, B_p is a differential operator of order $p-1$ and (4.67) combined with Lemma 2.22 shows that B_p maps $H^m(\Omega)$ into $H^{m-p}(\mathbb{R}^3_+)$.

IV) Let $h \in H^m(\mathbb{R}^3_+)$. We compute

$$\begin{aligned} &S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0 + h) = (-A_{0,\varepsilon}(0)^{-1}A_3)^p\partial_3^p(u_0 + h) \\ &\quad + \sum_{j=0}^{p-1} C_{p,p-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon)\partial_3^j(u_0 + h) + B_p(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon)f \\ &= S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0) + (-A_{0,\varepsilon}(0)^{-1}A_3)^p\partial_3^p h \\ &\quad + \sum_{j=0}^{p-1} C_{p,p-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon)\partial_3^j h. \end{aligned} \tag{4.76}$$

Set $a_0^\varepsilon = 0$. Then $a_0^\varepsilon \in H^m(\mathbb{R}^3_+)^6$ and

$$S_{m,0}(0, A_0, A_1, A_2, D, u_0) - S_{m,0}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0) = u_0 - u_0 = 0 = a_0^\varepsilon.$$

Let $k \in \{0, \dots, m-2\}$. Assume that we have constructed functions $a_p^\varepsilon \in H^{m-p}(\mathbb{R}^3_+)^6$ such that

$$\begin{aligned} A_3\left((-A_{0,\varepsilon}(0)^{-1}A_3)^p a_p^\varepsilon\right) &= A_3\Big(S_{m,p}(0, A_0, A_1, A_2, D, u_0) - S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0)\Big) \\ &\quad - A_3\Big(\sum_{j=0}^{p-1} C_{p,p-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon)a_j^\varepsilon\Big), \end{aligned} \tag{4.77}$$

$$a_p^\varepsilon \longrightarrow 0 \quad \text{in } H^{m-p}(\mathbb{R}^3_+)^6 \text{ as } \varepsilon \to 0$$

for every $p \in \{0, \ldots, k\}$. Then the functions

$$\sum_{j=0}^{k} C_{k+1,k+1-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon) a_j^\varepsilon$$

and

$$S_{m,k+1}(0, A_0, A_1, A_2, D, u_0) - S_{m,k+1}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0)$$

belong to $H^{m-k-1}(\mathbb{R}^3_+)$. Since $A_{0,\varepsilon} \geq \eta$, $\|\partial^\alpha A_{i,\varepsilon}(0)\|_{L^2(\mathbb{R}^3_+)} \leq r$ for all $i \in \{0,1,2\}$, and $\|\partial^\alpha D_\varepsilon(0)\|_{L^2(\mathbb{R}^3_+)} \leq r$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq \tilde{m} - 1$ and $\varepsilon > 0$ by Lemma 2.21, estimate (4.75) and the induction hypothesis yield

$$\begin{aligned}
&\Big\| \sum_{j=0}^{k} C_{k+1,k+1-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon) a_j^\varepsilon \Big\|_{H^{m-k-1}(\mathbb{R}^3_+)} \\
&\leq \sum_{j=0}^{k} R_{k+1,k+1-j}(\eta, r) \|a_j^\varepsilon\|_{H^{m-j}(\mathbb{R}^3_+)} \longrightarrow 0
\end{aligned}$$

as $\varepsilon \to 0$. Moreover, step II) shows that also

$$\|S_{m,k+1}(0, A_0, A_1, A_2, D, u_0) - S_{m,k+1}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0)\|_{H^{m-k-1}(\mathbb{R}^3_+)} \longrightarrow 0$$

as $\varepsilon \to 0$. Lemma 4.11 below thus gives a number $\varepsilon_0 > 0$ and functions $a_{k+1}^\varepsilon \in H^{m-k-1}(\mathbb{R}^3_+)^6$ such that

$$\begin{aligned}
&A_3\left((-A_{0,\varepsilon}(0)^{-1} A_3)^{k+1} a_{k+1}^\varepsilon\right) \\
&= A_3\Big(S_{m,k+1}(0, A_0, A_1, A_2, D, u_0) - S_{m,k+1}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0)\Big) \\
&\quad - A_3\Big(\sum_{j=0}^{k} C_{k+1,k+1-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon) a_j^\varepsilon\Big), \\
&a_{k+1}^\varepsilon \longrightarrow 0 \quad \text{in } H^{m-k-1}(\mathbb{R}^3_+)^6 \text{ as } \varepsilon \to 0.
\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. The induction is thus finished.

We next define

$$b_p^\varepsilon := a_p^\varepsilon(\cdot, 0) \in H^{m-p-\frac{1}{2}}(\partial\mathbb{R}^3_+)$$

for $0 \leq p \leq m-1$. Since the trace operator from $H^{m-p}(\mathbb{R}^3_+)$ into $H^{m-p-\frac{1}{2}}(\partial\mathbb{R}^3_+)$ is continuous, we infer that $b_p^\varepsilon \to 0$ in $H^{m-p-\frac{1}{2}}(\partial\mathbb{R}^3_+)$ as $\varepsilon \to 0$. Theorem 2.5.7 in [Hoe76] now yields functions $h_\varepsilon \in H^m(\mathbb{R}^3_+)$ with

$$\partial_3^p h_\varepsilon(\cdot, 0) = b_p^\varepsilon \quad \text{on } \partial\mathbb{R}^3_+$$

for $0 \leq p \leq m-1$ and $\varepsilon \in (0, \varepsilon_0)$, which satisfy $h_\varepsilon \to 0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.

We set $u_{0,\varepsilon} = u_0 + h_\varepsilon$ for all $\varepsilon > 0$. Then $u_{0,\varepsilon}$ tends to u_0 in $H^m(\mathbb{R}^3_+)$. The equations (4.76) and (4.77) yield

$$\begin{aligned}
&\big(A_3 S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_{0,\varepsilon})\big)(\cdot, 0) \\
&= \Big(A_3 S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0) + A_3(-A_{0,\varepsilon}(0)^{-1} A_3)^p \partial_3^p h^\varepsilon \\
&\qquad + A_3 \sum_{j=0}^{p-1} C_{p,p-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon) \partial_3^j h^\varepsilon\Big)(\cdot, 0)
\end{aligned}$$

$$= \operatorname{tr}(A_3 S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_0)) + \operatorname{tr}(A_3(-A_{0,\varepsilon}(0)^{-1}A_3)^p)b_p^\varepsilon$$
$$+ \sum_{j=0}^{p-1} \operatorname{tr}(A_3 C_{p,p-j}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon))b_j^\varepsilon$$
$$= \big(A_3 S_{m,p}(0, A_0, A_1, A_2, D, u_0)\big)(\cdot, 0)$$

for $0 \le p \le m-1$. Since $(0, A_0, \dots, A_3, D, B, f, g, u_0)$ fulfills the compatibility conditions (2.37) of order m and $B = MA_3$, we conclude that

$$\operatorname{Tr}(BS_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_{0,\varepsilon})) = M \operatorname{Tr}(A_3 S_{m,p}(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, D_\varepsilon, u_{0,\varepsilon}))$$
$$= M \operatorname{Tr}(A_3 S_{m,p}(0, A_0, A_1, A_2, D, u_0)) = \operatorname{Tr}(BS_{m,p}(0, A_0, A_1, A_2, D, u_0)) = \partial_t^p g(0),$$

i.e., the tupels $(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, B, f, g, u_{0,\varepsilon})$ fulfill the compatibility conditions (2.37) of order m for all $\varepsilon \in (0, \varepsilon_0)$. □

In the previous proof we used that we can continuously invert $(-A_{0,\varepsilon}(0)^{-1}A_3)^p$ on the range of A_3 in a certain sense. The next lemma provides the precise statement. The proof relies on the structure of the matrix A_3 which allows us to transform it globally into its Gaussian normal form.

Lemma 4.11. *Let $m \in \mathbb{N}$ with $m \ge 3$ and $\eta, \tau > 0$. Take $A_0 \in F_{m,6,\eta}(\Omega)$ and $A_3 \in F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$. Pick $k \in \mathbb{N}$ with $k \le m-1$ and $p \in \mathbb{N}_0$. Choose $r > 0$ such that $\|A_0(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)}, \|A_3(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \le r$. Take an approximating family $\{A_{0,\varepsilon}\}_{\varepsilon>0}$ provided by Lemma 2.21. Let $\{v_{0,\varepsilon}\}_{\varepsilon>0}$ be a family of functions in $H^k(\mathbb{R}^3_+)^6$. Then there exists a number $\varepsilon_0 > 0$ and a family of functions $\{v_{p,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ in $H^k(\mathbb{R}^3_+)^6$ such that*

$$A_3(0)(A_{0,\varepsilon}(0)^{-1}A_3(0))^p v_{p,\varepsilon} = A_3(0)v_{0,\varepsilon}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and a constant $C = C(\eta, \tau, r)$ such that

$$\|v_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \le C\|v_{0,\varepsilon}\|_{H^k(\mathbb{R}^3_+)}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. I) Since A_3 belongs to $F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$, there are functions $\mu_1, \mu_2, \mu_3 \in F^{\mathrm{cp}}_{m,1}(\Omega)$ such that

$$A_3 = \sum_{j=1}^{3} A_j^{\mathrm{co}} \mu_j$$

and an index $i \in \{1, 2, 3\}$ with

$$|\mu_i| \ge \tau \tag{4.78}$$

on Ω. Without loss of generality we assume that $i = 3$. The other cases are treated analogously. Note that also

$$\|\mu(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \le Cr. \tag{4.79}$$

Due to the properties of the approximating family, we find an $\varepsilon_0 > 0$ such that

$$\|A_{0,\varepsilon}(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \le 2r \tag{4.80}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

II) We introduce the matrices

$$\hat{A}_3 = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ \mu_2 & -\mu_1 & 0 \end{pmatrix}, \quad \hat{G}_r = \mu_3^{-1} \begin{pmatrix} 0 & -1 & \mu_1 \\ 1 & 0 & \mu_2 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \hat{G}_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}.$$

Note that

$$\hat{G}_l \hat{A}_3 \hat{G}_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & \hat{A}_3 \\ -\hat{A}_3 & 0 \end{pmatrix}.$$

Setting

$$G_r = \begin{pmatrix} \hat{G}_r & 0 \\ 0 & \hat{G}_r \end{pmatrix}, \qquad G_l = \begin{pmatrix} \hat{G}_l & 0 \\ 0 & \hat{G}_l \end{pmatrix},$$

we thus infer

$$G_l A_3 G_r = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: A_3^G, \tag{4.81}$$

where G_l and G_r belong to $F^{\text{cp}}_{m,6}(\Omega)$ by (4.78) and Lemma 2.23. Moreover, we introduce the invertible matrix

$$G_p = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{so that} \quad A_3^p := A_3^G G_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Due to (4.78) and (4.79), we further obtain a constant $C_1 = C_1(\tau, r)$ such that

$$\|G_r(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} + \|G_l(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} + \|G_r^{-1}(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq C_1, \tag{4.82}$$

where we also exploited that

$$\hat{G}_r^{-1} = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad G_r^{-1} = \begin{pmatrix} \hat{G}_r^{-1} & 0 \\ 0 & \hat{G}_r^{-1} \end{pmatrix}.$$

III) We next want to show that the matrix

$$\Theta_\varepsilon = \begin{pmatrix} G_{l;3\cdot} A_{0,\varepsilon} G_{r;\cdot 3} & G_{l;3\cdot} A_{0,\varepsilon} G_{r;\cdot 6} \\ G_{l;6\cdot} A_{0,\varepsilon} G_{r;\cdot 3} & G_{l;6\cdot} A_{0,\varepsilon} G_{r;\cdot 6} \end{pmatrix}$$

is either uniformly positive or uniformly negative definite on Ω for every $\varepsilon > 0$. As μ_3 is continuous and satisfies (4.78), we first observe that μ_3 does not change sign on Ω and without loss of generality we assume that μ_3 is positive on Ω. Next take $\xi \in \mathbb{R}^2$ with $|\xi| = 1$. We then compute

$$\begin{aligned} \xi^T \Theta_\varepsilon \xi &= \mu_3^{-1}\Big((\xi_1\mu, 0)^T A_{0,\varepsilon}\big((\xi_1\mu, 0) + (0, \xi_2\mu)\big) + (0, \xi_2\mu)^T A_{0,\varepsilon}\big((\xi_1\mu, 0) + (0, \xi_2\mu)\big)\Big) \\ &= \mu_3^{-1}(\xi_1\mu, \xi_2\mu)^T A_{0,\varepsilon}(\xi_1\mu, \xi_2\mu) \geq \frac{1}{|\mu|}\eta|\mu|^2 \geq \eta\tau \end{aligned}$$

for all $\varepsilon > 0$. Here we also used that $A_{0,\varepsilon}$ is contained in $F^{\text{cp}}_{m,6,\eta}(\Omega)$ for all $\varepsilon > 0$ by Lemma 2.21. Consequently, the matrix Θ_ε is uniformly positive definite on Ω and in combination with Lemma 2.22 we infer that Θ_ε belongs to $F^{\text{cp}}_{m,2,\eta\tau}(\Omega)$ for all $\varepsilon > 0$. In particular, Θ_ε has an inverse with

$$\|\Theta_\varepsilon^{-1}(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq C_3(\eta, \tau, r) \tag{4.83}$$

for all $\varepsilon \in (0, \varepsilon_0)$ by Lemma 2.23, Lemma 2.22, (4.82), and (4.80).

IV) Let $w_0 \in H^k(\mathbb{R}^3_+)^6$. Due to step III) we can define scalar functions $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ by

$$(h_{1,\varepsilon}, h_{2,\varepsilon}) = -\Theta_\varepsilon^{-1}(0)(G_l(0)A_{0,\varepsilon}(0)w_0)_{(3,6)},$$

where we denote for any vector ζ from $\mathbb{R}^6$ by $\zeta_{(3,6)}$ the two-dimensional vector (ζ_3, ζ_6). Note that

$$\|(h_{1,\varepsilon}, h_{2,\varepsilon})\|_{H^k(\mathbb{R}^3_+)} \leq C_4(\eta, \tau, r)\|w_0\|_{H^k(\mathbb{R}^3_+)} \tag{4.84}$$

for all $\varepsilon \in (0, \varepsilon_0)$ by Lemma 2.22, (4.80), (4.82), and (4.83). We next set

$$\begin{aligned} \tilde{w}_{0,\varepsilon} &= G_l(0)(-A_{0,\varepsilon}(0))G_r(0)\Big(G_r^{-1}(0)w_0 + h_{1,\varepsilon}e_3 + h_{2,\varepsilon}e_6\Big), \\ \tilde{w}_{1,\varepsilon} &= G_r(0)G_p\tilde{w}_{0,\varepsilon} \end{aligned} \tag{4.85}$$

for all $\varepsilon \in (0, \varepsilon_0)$. We once more obtain a constant $C_4(\eta, \tau, r)$ such that

$$\|\tilde{w}_{1,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C_4(\eta, \tau, r)\|w_0\|_{H^k(\mathbb{R}^3_+)}$$

for all $\varepsilon \in (0, \varepsilon_0)$ due to Lemma 2.22, (4.80), (4.82), and (4.84). We further point out that the construction of $h_{1,\varepsilon}$, $h_{2,\varepsilon}$, and $\tilde{w}_{0,\varepsilon}$ yields

$$(\tilde{w}_{0,\varepsilon})_{(3,6)} = (G_l(0)(-A_{0,\varepsilon}(0))w_0)_{(3,6)} - \Theta_\varepsilon(0)(h_{1,\varepsilon}, h_{2,\varepsilon}) = 0 \tag{4.86}$$

for all $\varepsilon \in (0, \varepsilon_0)$. We can thus compute

$$\begin{aligned} &A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))\tilde{w}_{1,\varepsilon} \\ &= G_l(0)^{-1}A_3^G G_r(0)^{-1}(-A_{0,\varepsilon}(0)^{-1})G_l(0)^{-1}A_3^G G_r(0)^{-1}\tilde{w}_{1,\varepsilon} \\ &= G_l(0)^{-1}A_3^G G_r(0)^{-1}(-A_{0,\varepsilon}(0)^{-1})G_l(0)^{-1}A_3^p\tilde{w}_{0,\varepsilon} \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Using that

$$A_3^p\tilde{w}_{0,\varepsilon} = \tilde{w}_{0,\varepsilon} \tag{4.87}$$

due to (4.86), we further deduce

$$\begin{aligned} &A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))\tilde{w}_{1,\varepsilon} \\ &= G_l(0)^{-1}A_3^G G_r(0)^{-1}(-A_{0,\varepsilon}(0)^{-1})G_l(0)^{-1}\tilde{w}_{0,\varepsilon} \\ &= G_l(0)^{-1}A_3^G G_r(0)^{-1}w_0 + G_l(0)^{-1}A_3^G(h_{1,\varepsilon}e_3 + h_{2,\varepsilon}e_6) = A_3(0)w_0 \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. To sum up, we have shown that for each $w_0 \in H^k(\mathbb{R}^3_+)^6$ and $\varepsilon \in (0, \varepsilon_0)$, there is a function $w_\varepsilon \in H^k(\mathbb{R}^3_+)^6$ such that

$$A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))w_\varepsilon = A_3 w_0. \tag{4.88}$$

Moreover, there is a constant $C_{IV} = C_{IV}(\eta, \tau, r)$, in particular independent of ε, such that

$$\|w_\varepsilon\|_{H^k(\mathbb{R}^3_+)} \leq C_{IV}\|w_0\|_{H^k(\mathbb{R}^3_+)} \tag{4.89}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

V) To show the actual assertion, we proceed inductively. We claim that for all $p \in \mathbb{N}_0$ and $\varepsilon \in (0, \varepsilon_0)$ there is an operator $T_{p,\varepsilon} \colon H^k(\mathbb{R}^3_+)^6 \to H^k(\mathbb{R}^3_+)^6$ such that

$$A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))^p T_{p,\varepsilon}(w) = A_3(0)w \tag{4.90}$$

for all $w \in H^k(\mathbb{R}^3_+)^6$ and there is a constant $C_p = C_p(\eta, \tau, r)$ such that

$$\|T_{p,\varepsilon}(w)\|_{H^k(\mathbb{R}^3_+)} \leq C_p\|w\|_{H^k(\mathbb{R}^3_+)}. \tag{4.91}$$

Note that there is nothing to show in the case $p = 0$. Now assume that we have proven the claim for a number $p \in \mathbb{N}_0$. Fix $\varepsilon \in (0, \varepsilon_0)$ and $w \in H^k(\mathbb{R}^3_+)^6$. Step IV) applied with $w_0 = w$ yields a function $\tilde{w}_{p+1,\varepsilon} \in H^k(\mathbb{R}^3_+)^6$ with

$$A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))\tilde{w}_{p,\varepsilon} = A_3(0)w \tag{4.92}$$

and

$$\|\tilde{w}_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C_{IV}(\eta, \tau, r)\|w\|_{H^k(\mathbb{R}^3_+)}. \tag{4.93}$$

We now define $T_{p+1,\varepsilon}(w) = T_{p,\varepsilon}(\tilde{w}_{p,\varepsilon})$. Then $T_{p+1,\varepsilon}(w)$ is contained in $H^k(\mathbb{R}^3_+)^6$ and we compute

$$\begin{aligned}
&A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))^{p+1}T_{p+1,\varepsilon}(w) \\
&= A_3(0)(-A_{0,\varepsilon}(0)^{-1})A_3(0)(-A_{0,\varepsilon}(0)^{-1}A_3(0))^p T_{p,\varepsilon}(\tilde{w}_{p,\varepsilon}) \\
&= A_3(0)(-A_{0,\varepsilon}(0)^{-1})A_3(0)\tilde{w}_{p,\varepsilon} = A_3(0)w,
\end{aligned}$$

where we employed the induction hypothesis (4.90) and (4.92). Combining (4.91) with (4.93), we further obtain

$$\|T_{p+1,\varepsilon}(w)\|_{H^k(\mathbb{R}^3_+)} = \|T_{p,\varepsilon}(\tilde{w}_{p,\varepsilon})\|_{H^k(\mathbb{R}^3_+)} \leq C_p\|\tilde{w}_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C_p C_{IV}\|w\|_{H^k(\mathbb{R}^3_+)}.$$

As C_p and C_{IV} only depend on η, τ, and r, the claim now follows by induction.

The assertion of the lemma is finally proven by setting $v_{p,\varepsilon} = T_{p,\varepsilon}(v_{0,\varepsilon})$ for all $\varepsilon \in (0, \varepsilon_0)$. □

We can now establish the differentiability theorem for coefficients constant outside of a compact set. We will show that if A_3 is smooth and the other coefficients and the data are regular of order $\max\{m, 3\}$ respectively m and if they fulfill the compatibility conditions of order m, then the corresponding solution of (3.2) belongs to G_m. To prove this statement we only have to get rid of the additional regularity assumptions on the coefficients A_0, A_1, A_2, and D in Proposition 4.9. We will therefore approximate the coefficients by the smoother ones from Lemma 2.21 and the initial value by the functions provided by Lemma 4.10. The corresponding solutions u_ε belong to $G_m(\Omega)$ by Proposition 4.9. The key point of the proof is then to show that u_ε tends to u and that the G_m-regularity of u_ε passes to the limit u.

Theorem 4.12. *Let $\eta, \tau > 0$, $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, $T > 0$, $J = (0, T)$, and $\Omega = J \times \mathbb{R}^3_+$. Take coefficients $A_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. Suppose that A_1, A_2, A_3, and B are independent of time and that A_3 and a function M as in the definition of $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ belong to $C^\infty(\overline{\Omega})$. Choose data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ such that the tupel $(0, A_0, \ldots, A_3, D, B, f, g, u_0)$ satisfies the compatibility conditions* (2.37) *of order m. Then the weak solution u of* (3.2) *belongs to $G_m(\Omega)$.*

Proof. I) Let $\{A_{i,\varepsilon}\}_{\varepsilon>0}$ and $\{D_\varepsilon\}_{\varepsilon>0}$ be the families of functions given by Lemma 2.21 for A_i and D respectively for $i \in \{0, 1, 2\}$. In particular, the coefficients $A_{0,\varepsilon}$, $A_{1,\varepsilon}$, $A_{2,\varepsilon}$, and D_ε belong to $C^\infty(\overline{\Omega})$ and $\partial_t A_{0,\varepsilon}$ is contained in $F_{\tilde{m}}(\Omega)$ for each $\varepsilon > 0$. Lemma 4.10 provides a parameter $\varepsilon_0 > 0$ and a family $\{u_{0,\varepsilon}\}_{0<\varepsilon<\varepsilon_0} \subseteq H^m(\mathbb{R}^3_+)$ such that $(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, B, f, g, u_{0,\varepsilon})$ fulfill the compatibility conditions (2.37) of order m for all $\varepsilon \in (0, \varepsilon_0)$ and $u_{0,\varepsilon} \to u_0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. Let u_ε denote the weak solution of (3.2) with differential operator $L(A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon)$ and inhomogeneity f, boundary value g, and initial value $u_{0,\varepsilon}$ for each $\varepsilon \in (0, \varepsilon_0)$. By Proposition 4.9, the function u_ε belongs to $G_m(\Omega)$ for all $\varepsilon \in (0, \varepsilon_0)$. Let $r > 0$ such that

$$\|A_i\|_{F_{\tilde{m}}(\Omega)} \leq r \quad \text{and} \quad \|D\|_{F_{\tilde{m}}(\Omega)} \leq r$$

for all $i \in \{0, \dots, 3\}$. Due to Lemma 2.21 we then also have

$$\|A_{i,\varepsilon}\|_{F_{\tilde{m}}(\Omega)} \le Cr \quad \text{and} \quad \|D_\varepsilon\|_{F_{\tilde{m}}(\Omega)} \le Cr$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $i \in \{0, 1, 2\}$. Theorem 3.13 then yields a constant $C = C(\eta, \tau, r, T)$ and a number $\gamma = \gamma(\eta, \tau, r, T)$ such that

$$\|u_\varepsilon\|^2_{G_{m,\gamma}(\Omega)} \le C\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_{0,\varepsilon}\|^2_{H^m(\mathbb{R}^3_+)} + \frac{1}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}\Big) \tag{4.94}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Let (ε_n) be a sequence of positive numbers converging to zero. Then (4.94) and $u_{0,\varepsilon} \to u_0$ in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ yield that $(\partial^\alpha u_{\varepsilon_n})$ is bounded in $L^\infty(J, L^2(\mathbb{R}^3_+)) = (L^1(J, L^2(\mathbb{R}^3_+)))^*$ for each $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. Since $L^1(J, L^2(\mathbb{R}^3_+))$ is separable, the Banach-Alaoglu theorem gives a σ^*-convergent subsequence. Taking iteratively subsequences for each $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$, we obtain a subsequence, denoted by (u_n), such that the σ^*-limit u_α of $\partial^\alpha u_n$ exists for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. Lemma 3.7 and Lemma 2.21 imply that

$$\begin{aligned}\|u_n - u\|_{G_{0,\gamma}(\Omega)} &\le C(\|L(A_0, \dots, A_3, D)u_n - f\|^2_{G_{0,\gamma}(\Omega)} + \|u_{0,n} - u_0\|^2_{L^2(\mathbb{R}^3_+)}) \\ &\le C\Big(\sum_{i=0}^{2} \|A_i - A_{i,n}\|^2_{L^\infty(\Omega)} \|\partial_i u_n\|^2_{G_{0,\gamma}(\Omega)} \\ &\qquad + \|D - D_n\|^2_{L^\infty(\Omega)} \|u_n\|^2_{G_{0,\gamma}(\Omega)} + \|u_{0,n} - u_0\|^2_{L^2(\mathbb{R}^3_+)}\Big) \longrightarrow 0\end{aligned}$$

as $n \to \infty$, where we also exploited that $f = L(A_{0,n}, A_{1,n}, A_{2,n}, A_3, D_n)u_n$, (4.94), and that $(u_{0,n})_n$ is bounded in $H^m(\mathbb{R}^3_+)$. Consequently, u is equal to $u_{(0,0,0,0)}$. Looking at the distributional derivative, we further deduce

$$\langle \varphi, \partial^\alpha u \rangle = (-1)^{|\alpha|} \langle \partial^\alpha \varphi, u \rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle \partial^\alpha \varphi, u_n \rangle = \langle \varphi, u_\alpha \rangle$$

for all $\varphi \in C_c^\infty(\Omega)$. We conclude that $\partial^\alpha u \in L^\infty(J, L^2(\mathbb{R}^3_+))$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$; i.e., $u \in \tilde{G}_m(\Omega)$. It remains to remove the tilde here. To this purpose we will apply Lemma 4.7 to $\partial_t^{m-1} u$ and then iteratively Corollary 4.5 to $\partial_t^j u$.

II) Let $0 \le j \le m - 1$. Lemma 3.4 for $m - 1$, Corollary 2.18, and Lemma 2.31 show that $\partial_t^j u$ solves the initial value problem,

$$\begin{cases} L(A_0, \dots, A_3, D)\partial_t^j u = f_j, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B\partial_t^j u = \partial_t^j g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ \partial_t^j u(0) = S_{m,j}(0, A_0, \dots, A_3, D, f, u_0), & x \in \mathbb{R}^3_+; \end{cases}$$

where

$$f_j = \partial_t^j f - \sum_{l=1}^{j} \binom{j}{l} \Big(\partial_t^l A_0 \partial_t^{j+1-l} u + \partial_t^l D \partial_t^{j-l} u\Big).$$

Observe that the proof of Lemma 3.4 implies that f_j belongs to $H^{m-j}(\Omega)$.

III) To apply Lemma 4.7, we show that the tupel $(0, A_0, \dots, A_3, D, B, f_j, \partial_t^j g, u_0^j)$ fulfills the compatibility conditions (2.37) of order $m - j$, where we abbreviate the function $S_{m,j}(0, A_0, \dots, A_3, D, f, u_0)$ by u_0^j for all $0 \le j \le m$.

Let $m_1, m_2 \in \mathbb{N}_0$ with $m_2 \le m_1$ and $m_1 + m_2 \le m - 1$. We claim that

$$S_{m-m_1,m_2}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) = S_{m,m_1+m_2}(0, A_0, \dots, A_3, D, f, u_0). \tag{4.95}$$

Note that this identity implies that

$$BS_{m-m_1,m_2}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) = BS_{m,m_1+m_2}(0, A_0, \dots, A_3, D, f, u_0)$$

$$= \partial_t^{m_1+m_2} g(0) = \partial_t^{m_2}(\partial_t^{m_1} g)(0)$$

on $\partial\mathbb{R}^3_+$ for all $m_2 \leq m - m_1 - 1$, as the tupel $(0, A_0, \dots, A_3, D, B, f, g, u_0)$ fulfills the compatibility conditions of order m by assumption. We infer that (4.95) implies that the tupel $(0, A_0, \dots, A_3, D, B, f_{m_1}, \partial_t^{m_1} g, u_0^{m_1})$ fulfills the compatibility conditions (2.37) of order $m - m_1$.

Fix $m_1 \in \mathbb{N}_0$ with $m_1 \leq m-1$. We show (4.95) for all $m_2 \in \mathbb{N}_0$ with $m_1 + m_2 \leq m-1$ by induction. For $m_2 = 0$ we have

$$S_{m-m_1,0}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) = u_0^{m_1} = S_{m,m_1}(0, A_0, \dots, A_3, D, f, u_0).$$

In the case $m_1 = m-1$ there is nothing left to show so let $m_1 \leq m-2$ in the following. Again there is nothing to show if $m_2 = 0$ so we take $m_2 \geq 1$. Assume that we have shown (4.95) for all (m_1, j) with $0 \leq j \leq m_2 - 1$. Observe that $m_1 + 1 \leq m - 1$ here. Using (2.36), the fact that u is contained in $G_{m-1}(\Omega)$, Lemma 2.31, the definition of f_{m_1} and $u_0^{m_1}$, and the induction hypothesis, we then compute

$$\begin{aligned}
&A_0(0) S_{m-m_1,m_2}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) \\
&= \partial_t^{m_2-1} f_{m_1}(0) - \sum_{j=1}^{3} A_j \partial_j S_{m-m_1,m_2-1}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) \\
&\quad - \sum_{l=1}^{m_2-1} \binom{m_2-1}{l} \partial_t^l A_0(0) S_{m-m_1,m_2-l}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) \\
&\quad - \sum_{l=0}^{m_2-1} \binom{m_2-1}{l} \partial_t^l D(0) S_{m-m_1,m_2-l-1}(0, A_0, \dots, A_3, D, f_{m_1}, u_0^{m_1}) \\
&= \partial_t^{m_1+m_2-1} f(0) - \partial_t^{m_2-1}\big(\partial_t^{m_1}(A_0 \partial_t u + Du) - A_0 \partial_t^{m_1+1} u - D \partial_t^{m_1} u\big)(0) \\
&\quad - \sum_{j=1}^{3} A_j \partial_j S_{m,m_1+m_2-1}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{m_2-1} \binom{m_2-1}{l} \partial_t^l A_0(0) S_{m,m_1+m_2-l}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad - \sum_{l=0}^{m_2-1} \binom{m_2-1}{l} \partial_t^l D(0) S_{m,m_1+m_2-l-1}(0, A_0, \dots, A_3, D, f, u_0) \\
&= \partial_t^{m_1+m_2-1} f(0) - \partial_t^{m_1+m_2-1}(A_0 \partial_t u + Du)(0) \\
&\quad - \sum_{j=1}^{3} A_j \partial_j S_{m,m_1+m_2-1}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad - \sum_{l=1}^{m_2-1} \binom{m_2-1}{l} \partial_t^l A_0(0) S_{m,m_1+m_2-l}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad - \sum_{l=0}^{m_2-1} \binom{m_2-1}{l} \partial_t^l D(0) S_{m,m_1+m_2-l-1}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad + \sum_{l=0}^{m_2-1} \binom{m_2-1}{l} \partial_t^l A_0(0) S_{m,m_1+m_2-l}(0, A_0, \dots, A_3, D, f, u_0) \\
&\quad + \sum_{l=0}^{m_2-1} \binom{m_2-1}{l} \partial_t^l D(0) S_{m,m_1+m_2-l-1}(0, A_0, \dots, A_3, D, f, u_0) \\
&= \partial_t^{m_1+m_2-1} f(0) - \sum_{j=1}^{3} A_j \partial_j S_{m,m_1+m_2-1}(0, A_0, \dots, A_3, D, f, u_0)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=0}^{m_1+m_2-1} \binom{m_1+m_2-1}{l} \partial_t^l A_0(0) \partial_t^{m_1+m_2-l} u(0) \\
&- \sum_{l=0}^{m_1+m_2-1} \binom{m_1+m_2-1}{l} \partial_t^l D(0) \partial_t^{m_1+m_2-l-1} u(0) \\
&+ A_0(0) S_{m,m_1+m_2}(0, A_0, \ldots, A_3, D, f, u_0) \\
&= \partial_t^{m_1+m_2-1} f(0) - \sum_{j=1}^{3} A_j \partial_j S_{m,m_1+m_2-1}(0, A_0, \ldots, A_3, D, f, u_0) \\
&- \sum_{l=1}^{m_1+m_2-1} \binom{m_1+m_2-1}{l} \partial_t^l A_0(0) S_{m,m_1+m_2-l}(0, A_0, \ldots, A_3, D, f, u_0) \\
&- \sum_{l=0}^{m_1+m_2-1} \binom{m_1+m_2-1}{l} \partial_t^l D(0) S_{m,m_1+m_2-l-1}(0, A_0, \ldots, A_3, D, f, u_0) \\
&= A_0(0) S_{m,m_1+m_2}(0, A_0, \ldots, A_3, D, f, u_0),
\end{aligned}
$$

finishing the proof of the claim.

IV) Lemma 3.4 and step II) applied with $j = m-1$ show that $\partial_t^{m-1} u$ solves (3.2) with inhomogeneity $f_{m-1} \in H^1(\Omega)$, boundary value $\partial_t^{m-1} g \in E_1(J \times \partial\mathbb{R}^3_+)$, and initial value $u_0^{m-1} \in H^1(\mathbb{R}^3_+)$. By step III), the tupel $(0, A_0, \ldots, A_3, D, B, f_{m-1}, \partial_t^{m-1} g, u_0^{m-1})$ fulfills the compatibility conditions (2.37) of order 1. Next take an open subinterval J' of J. Assume that $\partial_t^{m-1} u$ belongs to $C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$. Arguing as in step VI) of the proof of Lemma 4.4, we infer that $\partial_t^{m-1} u$ is an element of $C(\overline{J'}, H^1_{\mathrm{ta}}(\mathbb{R}^3_+))$. (Note that the smoothness of the coefficients is not used in that step and that step II) of the same proof shows that we can assume without loss of generality that A_3 is constant.) Lemma 4.1 then implies that $\partial_t^{m-1} u$ is contained in $G_1(J' \times \mathbb{R}^3_+)$. Lemma 4.7 thus yields that $\partial_t^{m-1} u$ belongs to $C^1(\overline{J}, L^2(\mathbb{R}^3_+))$; i.e., $u \in C^m(\overline{J}, L^2(\mathbb{R}^3_+))$. The previous arguments applied with $J' = J$ now imply that $\partial_t^{m-1} u$ is an element of $G_1(\Omega)$.

Next assume that we have proven that $\partial_t^{m-k} u$ is an element of $G_k(\Omega)$ for some $k \in \{1, \ldots, m-1\}$. Then $\partial_t^{m-k-1} u$ belongs to

$$
\bigcap_{l=0}^{k} C^{l+1}(\overline{J}, H^{k-l}(\mathbb{R}^3_+)) = \bigcap_{l=1}^{k+1} C^{l}(\overline{J}, H^{k+1-l}(\mathbb{R}^3_+)).
$$

Observe that $\partial_t^{m-k-1} u$ solves (3.2) with inhomogeneity $f_{m-k-1} \in H^{k+1}(\Omega)$, boundary value $\partial_t^{m-k-1} g \in E_{k+1}(J \times \partial\mathbb{R}^3_+)$, and initial value $u_0^{m-k-1} \in H^{k+1}(\mathbb{R}^3_+)$ by Lemma 3.4 and step II). Arguing as before, i.e., applying step VI) from the proof of Lemma 4.4 to derive that $\partial_t^{m-k-1} u \in C(\overline{J}, H^{k+1}_{\mathrm{ta}}(\mathbb{R}^3_+))$ and then Lemma 4.1 to obtain that $\partial_t^{m-k-1} u \in C(\overline{J}, H^{k+1}(\mathbb{R}^3_+))$, we conclude that $\partial_t^{m-k-1} u$ is contained in $G_{k+1}(\Omega)$.

By induction we arrive at $\partial_t^{m-k} u \in G_k(\Omega)$ for all $k \in \{0, \ldots, m\}$. With $k = m$ we finally obtain $u \in G_m(\Omega)$. □

The main theorem of this section tells us that the results from Chapter 3 and Theorem 4.12 are still true if we replace the coefficients A_0 and D from $F^{\mathrm{cp}}_m(\Omega)$ with coefficients from $F^{\mathrm{c}}_m(\Omega)$.

The key observation is that the restriction to coefficients which are constant outside some compact set was only necessary to use the results from [Ell12]. Once one has established Lemma 3.7 with coefficients from $F^{\mathrm{c}}_m(\Omega)$, the results from Chapter 3 and Chapter 4 also follow for these coefficients.

Theorem 4.13. *Let $\eta, \tau > 0$, $m \in \mathbb{N}_0$, and $\tilde{m} = \max\{m, 3\}$. Choose $t_0 \in \mathbb{R}$, $T' > t_0$, and $T \in (t_0, T')$. Set $J = (t_0, T)$ and $\Omega = J \times \mathbb{R}^3_+$. Take coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$, $D \in F^{\mathrm{c}}_{\tilde{m}}(\Omega)$, and $B \in \mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$. Suppose that A_1, A_2, A_3, and B are independent of time and that A_3 and a function M*

as in the definition of $\mathcal{BC}^{\tilde m}_{\mathbb{R}^3_+}(A_3)$ *belong to* $C^\infty(\overline{\Omega})$. *Choose data* $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, *and* $u_0 \in H^m(\mathbb{R}^3_+)$ *such that the tupel* $(t_0, A_0, \ldots, A_3, D, B, f, g, u_0)$ *fulfills the compatibility conditions* (2.37) *of order* m. *We further assume that there are functions* $G_B^1 \in W^{\tilde m+1,\infty}(\mathbb{R}^3_+)^{2\times 2}$ *and* $G_B^2 \in W^{\tilde m+1,\infty}(\mathbb{R}^3_+)^{6\times 6}$ *such that* $G_B^1 B G_B^2$ *has Gaussian normal form.*

Then the linear initial boundary value problem (3.2) *has a unique weak solution* u *in* $G_m(\Omega)$. *Moreover, the statements of Lemma 3.11 and Theorem 3.13 are also true in this case.*

Proof. Without loss of generality, we assume that $t_0 = 0$ in this proof. The key step is to show the assertion for $m = 0$, i.e., Lemma 3.7 with coefficients $A_0 \in F^{\mathrm{c}}_{3,\eta}(\Omega)$ and $D \in F^{\mathrm{c}}_3(\Omega)$. So assume that $m = 0$ and observe that there are no compatibility conditions in that case. We will show the assertion by approximation. To that purpose, we provide two approximation results.

I) We claim that there are sequences $(f_n)_n$ in $H^1(\Omega)$, $(g_n)_n$ in $E_1(J \times \partial\mathbb{R}^3_+)$, and $(u_{0,n})_n$ in $H^1(\mathbb{R}^3_+)$ such that the sequence $(f_n)_n$ converges to f in $L^2(\Omega)$, $(g_n)_n$ to g in $L^2(J, H^{1/2}(\mathbb{R}^3_+))$, and $(u_{0,n})_n$ to u_0 in $L^2(\mathbb{R}^3_+)$ as $n \to \infty$, and that there is a constant C_1 with

$$\|f_n\|_{H^1(\Omega)} \leq C_1 \cdot n, \quad \|g_n\|_{E_1(J\times\partial\mathbb{R}^3_+)} \leq C_1 \cdot n, \quad \|u_{0,n}\|_{H^1(\mathbb{R}^3_+)} \leq C_1 \cdot n^4 \tag{4.96}$$

for all $n \in \mathbb{N}$.

To prove this claim we take a function $\varphi_n \in C_c^\infty(\mathbb{R})$ with $0 \leq \varphi_n \leq 1$ and

$$\varphi_n(x_3) = 1 \text{ for } x_3 \geq \frac{4}{n^4}, \quad \varphi_n(x_3) = 0 \text{ for } x_3 \leq \frac{3}{n^4}, \quad \|\varphi_n'\|_{L^2(\mathbb{R})} \leq Cn^2$$

for every $n \in \mathbb{N}$. Observe that φ_n tends pointwise to 1 on $\mathbb{R}_+$ so that the theorem of dominated convergence implies that $\varphi_n u_0$ and $\varphi_n f$ converge to u_0 in $L^2(\mathbb{R}^3_+)$ respectively f in $L^2(\Omega)$ as $n \to \infty$. Let ρ be a positive function in $C_c^\infty(\mathbb{R})$ with integral 1 such that

$$\rho = \begin{cases} 1 & \text{on } B(0,1); \\ 0 & \text{on } B(0,2)^C. \end{cases}$$

We extend the function g by 0 outside of $J \times \partial\mathbb{R}^3_+$, $\varphi_n u_0$ by 0 outside of $\mathbb{R}^3_+$ and $\varphi_n f$ by 0 outside of Ω. We then set

$$\begin{aligned} \rho_{f,n}(t, x_1, x_2, x_3) &= n^4 \rho(nt)\, \rho(nx_1)\, \rho(nx_2)\, \rho(nx_3), \\ \rho_{g,n}(t, x_1, x_2) &= n^3 \rho(nt)\, \rho(nx_1)\, \rho(nx_2), \\ \rho_{0,n}(x_1, x_2, x_3) &= n^6 \rho(nx_1)\, \rho(nx_2)\, \rho(n^4 x_3) \end{aligned}$$

for all $n \in \mathbb{N}$ and $t, x_1, x_2, x_3 \in \mathbb{R}$. Next define

$$f_n = \rho_{f,n} * f, \quad g_n = \rho_{g,n} * g, \quad \text{and} \quad \tilde u_{0,n} = \rho_{0,n} * (\varphi_n u_0),$$

for all $n \in \mathbb{N}$. We remark that for the definition of g_n we convolve over t, x_1, and x_2 while for $\tilde u_{0,n}$ we take the convolution in x_1-, x_2-, and x_3-direction. Hence, the functions f_n belong to $H^1(\mathbb{R}\times\mathbb{R}^3)$, g_n to $H^2(\mathbb{R}\times\partial\mathbb{R}^3_+)$, and $\tilde u_{0,n}$ to $H^1(\mathbb{R}^3)$. Moreover,

$$\|\partial_j \tilde u_{0,n}\|_{L^2(\mathbb{R}^3_+)} \leq \|(\partial_j \rho_{0,n}) * (\varphi_n u_0)\|_{L^2(\mathbb{R}^3)} \leq n^4 \|3\rho^2 \partial_j \rho\|_{L^1(\mathbb{R}^3)} \|u_0\|_{L^2(\mathbb{R}^3_+)}$$

for all $n \in \mathbb{N}$ and $j \in \{1, 2, 3\}$, where we used Young's inequality. We conclude that there is a constant C_1' such that

$$\|\tilde u_{0,n}\|_{H^1(\mathbb{R}^3_+)} \leq C_1' n^4 \tag{4.97}$$

for all $n \in \mathbb{N}$. Analogously, one shows $\|f_n\|_{H^1(\Omega)} \leq C_1' n$ and $\|g_n\|_{H^2(\mathbb{R}\times\partial\mathbb{R}^3_+)} \leq C_1' n^2$. Exploiting that g is an element of $E_0(J\times\partial\mathbb{R}^3_+)$, one also obtains $\|g_n\|_{E_1(J\times\partial\mathbb{R}^3_+)} \leq C_1' n$.

Moreover, the function $\varphi_n u_0$ is supported in the set $\{x \in \mathbb{R}^3 \colon x_3 \geq 3/n^4\}$ and $\rho_{0,n}$ in $\{x \in \mathbb{R}^3 \colon x_3 \leq 1/n^4\}$. Consequently, $\rho_n * (\varphi_n u_0)$ is supported in $\{x \in \mathbb{R}^3 \colon x_3 \geq 2/n^4\}$. We thus obtain

$$\operatorname{tr} \tilde{u}_{0,n} = \tilde{u}_{0,n}(\cdot, 0) = 0 \tag{4.98}$$

for all $n \in \mathbb{N}$. The regularity assumptions on f, g, and u_0 also yield that

$$\begin{aligned} \tilde{u}_{0,n} &\longrightarrow u_0 && \text{in } L^2(\mathbb{R}^3_+), \\ g_n &\longrightarrow g && \text{in } E_0(J \times \partial\mathbb{R}^3_+), \\ f_n &\longrightarrow f && \text{in } L^2(\Omega). \end{aligned} \tag{4.99}$$

We now define the function $\tilde{h}_n$ by $(\tilde{h}_n)_i = (G_B^1 g_n)_i$ for $i \in \{1, 2\}$ and $(\tilde{h}_n)_i = 0$ for $i \in \{3, \ldots, 6\}$. Setting $h_n = G_B^2 \tilde{h}_n$, this construction gives

$$B h_n = g_n \tag{4.100}$$

for all $n \in \mathbb{N}$. As G_B^1 and G_B^2 are contained in $W^{\tilde{m}+1,\infty}(\mathbb{R}^3_+)$, we infer that

$$\begin{aligned} \|h_n\|_{H^1(J \times \partial\mathbb{R}^3_+)} &\leq C \|g_n\|_{H^1(J \times \partial\mathbb{R}^3_+)} \leq Cn, \\ \|h_n\|_{H^2(J \times \partial\mathbb{R}^3_+)} &\leq C \|g_n\|_{H^2(J \times \partial\mathbb{R}^3_+)} \leq Cn^2 \end{aligned}$$

for all $n \in \mathbb{N}$. In particular, Sobolev's embedding shows that

$$\begin{aligned} \|(1 - \varphi_n) h_n(0)\|_{H^1(\mathbb{R}^3_+)} &\leq C(1 + n^2) \|h_n(0)\|_{L^2(\partial\mathbb{R}^3_+)} + C\frac{1}{n^2} \sum_{j=1}^{2} \|\partial_j h_n(0)\|_{L^2(\partial\mathbb{R}^3_+)} \\ &\leq Cn^2 \|h_n\|_{H^2(J \times \partial\mathbb{R}^3_+)} \leq Cn^4 \end{aligned} \tag{4.101}$$

and

$$\begin{aligned} \|(1 - \varphi_n) h_n(0)\|_{L^2(\mathbb{R}^3_+)} &= \|1 - \varphi_n\|_{L^2(\mathbb{R}_+)} \|h_n(0)\|_{L^2(\partial\mathbb{R}^3_+)} \leq C\frac{1}{n^2} \|h_n\|_{H^1(J \times \partial\mathbb{R}^3_+)} \\ &\leq C\frac{1}{n} \end{aligned} \tag{4.102}$$

for all $n \in \mathbb{N}$. Finally, we set

$$u_{0,n} = \tilde{u}_{0,n} + (1 - \varphi_n) h_n(0)$$

for all $n \in \mathbb{N}$. This sequence converges to u_0 in $L^2(\mathbb{R}^3_+)$ as n tends to infinity due to (4.99) and (4.102) and it satisfies (4.96) because of (4.97) and (4.101). We finish this step by noting that our construction also yields

$$B u_{0,n} = B h_n(0) = g_n(0) \tag{4.103}$$

on $\partial\mathbb{R}^3_+$ since $\operatorname{tr} \tilde{u}_{0,n} = 0$, $\varphi_n(0) = 0$, and $B h_n = g_n$ by (4.100).

II) Let $W \in F_3^{\mathrm{c}}(\Omega)$. We will show that there exists a sequence $(W_n)_n$ in $F_3^{\mathrm{cp}}(\Omega)$ and a constant $C_2 \geq 1$ such that

$$\begin{aligned} &\|W_n\|_{W^{1,\infty}(\Omega)} \leq C_2 \|W\|_{W^{1,\infty}(\Omega)}, \\ &\|W_n - W\|_{L^\infty(\Omega)} \longrightarrow 0, \\ &\|\partial_j W_n - \partial_j W\|_{L^2(\Omega) + L^\infty(\Omega)} \longrightarrow 0, \end{aligned} \tag{4.104}$$

for all $j \in \{0, \ldots, 3\}$ and $n \in \mathbb{N}$ respectively as $n \to \infty$. Moreover, if $W \in F_{3,\eta}^{\mathrm{c}}(\Omega)$, we can choose the functions W_n in such a way that W_n belongs to $F_{3,\eta}^{\mathrm{cp}}(\Omega)$ for all $n \in \mathbb{N}$.

Since W belongs to $F_3^{\mathrm{c}}(\Omega)$, there exists a matrix w with $W(t,x) \to w$ as $|(t,x)| \to \infty$. Let $\varphi \in C_c^\infty(\mathbb{R}^4)$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B(0,1)$ and $\varphi = 0$ on $B(0,2)^C$. We define $\varphi_n = \varphi(\cdot/n)$ and then set

$$W_n = \varphi_n W + (1 - \varphi_n) w$$

for each $n \in \mathbb{N}$. Because φ_n belongs to $C_c^\infty(\mathbb{R}^4)$, the functions W_n are contained in $F_3(\Omega)$ for each $n \in \mathbb{N}$. Moreover, for $(t,x) \in \Omega \setminus \operatorname{supp} \varphi_n$ we have $W_n(t,x) = w$. Hence, $W_n \in F_3^{\mathrm{cp}}(\Omega)$ for all $n \in \mathbb{N}$. We further deduce

$$\partial_j W_n = \partial_j \varphi_n W + \varphi_n \partial_j W - \partial_j \varphi_n w \tag{4.105}$$

for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, 3\}$. As $\partial_j \varphi_n(t,x) = \frac{1}{n}(\partial_j \varphi)(t/n, x/n)$ for all $(t,x) \in \Omega$ and $n \in \mathbb{N}$, the first summand and the third summand on the right-hand side of (4.105) converge to zero in $L^\infty(\Omega)$. The theorem of dominated convergence shows that the second summand in (4.105) tends to $\partial_j W$ in $L^2(\Omega)$. We conclude that the third statement in (4.104) is true. Moreover, the functions $\partial_j W_n$ tend to $\partial_j W$ pointwise almost everywhere.

The identity (4.105) additionally implies that

$$\|\partial_j W_n\|_{L^\infty(\Omega)} \leq 2\|\partial_j \varphi\|_{L^\infty(\Omega)} \|W\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \|W\|_{W^{1,\infty}(\Omega)}$$

for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, 3\}$. We infer that there is a constant C_2 such that $\|W_n\|_{W^{1,\infty}(\Omega)} \leq C_2 \|W\|_{W^{1,\infty}(\Omega)}$ for all $n \in \mathbb{N}$.

In order to prove the remaining assertion in (4.104), we take an $\varepsilon > 0$. By the definition of $F_3^{\mathrm{c}}(\Omega)$ we can then find a compact subset Ω' of $\overline{\Omega}$ such that

$$|W(t,x) - w| < \frac{\varepsilon}{1 + \|\varphi\|_{L^\infty(\Omega)}}$$

for all $(t,x) \in \Omega \setminus \Omega'$. Fix an index $n_0 \in \mathbb{N}$ such that $\frac{1}{n}(t,x) \in B(0,1)$ for all $(t,x) \in \Omega'$ and $n \geq n_0$. Using that $\varphi = 1$ on $B(0,1)$ and thus $\varphi_n - 1 = 0$ on Ω' for all $n \geq n_0$, we infer that

$$\begin{aligned}
\|W_n - W\|_{L^\infty(\Omega)} &= \|(\varphi_n - 1)(W - w)\|_{L^\infty(\Omega)} \\
&\leq \|(\varphi_n - 1)(W - w)\|_{L^\infty(\Omega')} + \|(\varphi_n - 1)(W - w)\|_{L^\infty(\Omega \setminus \Omega')} \\
&\leq (1 + \|\varphi\|_{L^\infty(\Omega)}) \sup_{(t,x) \in \Omega \setminus \Omega'} |W(t,x) - w| \leq \varepsilon
\end{aligned}$$

for all $n \geq n_0$. Hence, the functions W_n converge to W in $L^\infty(\Omega)$ as $n \to \infty$.

Finally, assume that W is contained in $F_{3,\eta}^{\mathrm{c}}(\Omega)$. Let $\xi \in \mathbb{R}^6$ with $|\xi| = 1$. Due to the definition of $F_{3,\eta}^{\mathrm{c}}(\Omega)$ we have

$$\xi^T W(t,x) \xi \geq \eta$$

for all $(t,x) \in \Omega$. Letting $|(t,x)| \to \infty$, we then obtain

$$\xi^T w \xi \geq \eta.$$

We thus infer

$$\xi^T W_n \xi = \varphi_n \xi^T W \xi + (1 - \varphi_n) \xi^T w \xi \geq \varphi_n \eta + (1 - \varphi_n) \eta = \eta,$$

i.e., $W_n \in F_{3,\eta}^{\mathrm{cp}}(\Omega)$ for all $n \in \mathbb{N}$.

III) We fix three sequences $(f_n)_n$, $(g_n)_n$, and $(u_{0,n})_n$ as constructed in step I). We then choose two sequences $(A_{0,n})_n$ in $F_{3,\eta}^{\mathrm{cp}}(\Omega)$ and $(D_n)_n$ in $F_3^{\mathrm{cp}}(\Omega)$ as in step II) for A_0 respectively D which have the additional property that

$$\|A_{0,n} - A_0\|_{L^\infty(\Omega)} \leq \frac{1}{n^9} \quad \text{and} \quad \|D_n - D\|_{L^\infty(\Omega)} \leq \frac{1}{n^9} \tag{4.106}$$

for all $n \in \mathbb{N}$.

Take $r > 0$ with $\|A_i\|_{W^{1,\infty}(\Omega)} \leq C_2^{-1} r$ and $\|D\|_{W^{1,\infty}(\Omega)} \leq C_2^{-1} r$ for all $i \in \{0, \ldots, 3\}$. Due to (4.104) we then also have $\|A_{0,n}\|_{W^{1,\infty}(\Omega)} \leq r$ and $\|D_n\|_{W^{1,\infty}(\Omega)} \leq r$ for all $n \in \mathbb{N}$. We define the constant C_3 by

$$C_3(\eta, r) = \max\{C_{3.7;0,0}(\eta, r), C_{3.7;0}(\eta, r), C_{3.9;1,0}(\eta, r), C_{3.9;1}(\eta, r)\}$$

and γ by

$$\gamma(\eta, r) = \max\{\gamma_{3.7;0}(\eta, r), \gamma_{3.9;0}(\eta, r)\} \geq 1,$$

where $C_{3.7;0,0}$, $C_{3.7;0}$, $C_{3.9;1,0}$, $C_{3.9;1}$, $\gamma_{3.7;0}$, and $\gamma_{3.9;0}$ are the corresponding constants from Lemma 3.7 respectively Lemma 3.9, see also Remark 4.6. Sobolev's embedding further gives a constant $C_4 = C_4(T)$ such that $\sup_{t\in J} \|v(t)\|_{L^2(\mathbb{R}^3_+)} \leq C_4 \|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$.

We point out that because of (4.103) the tupel $(0, A_{0,n}, A_1, A_2, A_3, D_n, f_n, g_n, u_{0,n})$ fulfills the compatibility conditions (2.37) of order 1. Theorem 4.12 thus shows that the unique weak solution u_n of the initial boundary value problem (3.2) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$, inhomogeneity f_n, boundary value g_n, and initial value $u_{0,n}$ belongs to $G_1(\Omega)$ for all $n \in \mathbb{N}$. Applying the a priori estimate from Lemma 3.7 respectively Lemma 3.9 and Remark 3.10, we further obtain

$$\begin{aligned}
\|u_n\|^2_{L^2_\gamma(\Omega)} &\leq C_3(\|f_n\|^2_{L^2_\gamma(\Omega)} + \|g_n\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_{0,n}\|^2_{L^2(\mathbb{R}^3_+)}) \leq C_5, \\
\|\partial_t u_n\|^2_{L^2_\gamma(\Omega)} &\leq C_3(\|f_n(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_{0,n}\|^2_{H^1(\mathbb{R}^3_+)} + \|g_n\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_n\|^2_{H^1_\gamma(\Omega)}) \\
&\leq C_3(1 + C_4^2)(\|u_{0,n}\|^2_{H^1(\mathbb{R}^3_+)} + \|g_n\|^2_{E_{1,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_n\|^2_{H^1(\Omega)}) \\
&\leq 3C_1^2 C_3(1 + C_4^2) n^8
\end{aligned} \tag{4.107}$$

for all $n \in \mathbb{N}$, where the bound C_5 is due to the convergence properties of $(f_n)_n$, $(g_n)_n$, and $(u_{0,n})_n$ stated in step I). In the last step we also employed (4.96). Now take $k, n \in \mathbb{N}$ with $k \geq n$. Using the linearity of the differential operator $L(A_{0,k}, A_1, A_2, A_3, D_k)$, we deduce that $u_k - u_n$ solves (3.2) for $L(A_{0,k}, A_1, A_2, A_3, D_k)$ with inhomogeneity

$$f_k - f_n + (A_{0,n} - A_{0,k})\partial_t u_n + (D_n - D_k)u_n,$$

boundary value $g_k - g_n$, and initial value $u_{0,k} - u_{0,n}$. The a priori estimate from Lemma 3.7, (4.106), and (4.107) thus yield

$$\begin{aligned}
&\sup_{t\in J} \|u_k(t) - u_n(t)\|^2_{L^2(\mathbb{R}^3_+)} \leq e^{2\gamma T} \|u_k - u_n\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq C_3 e^{2\gamma T} \Big(\|f_k - f_n + (A_{0,n} - A_{0,k})\partial_t u_n + (D_n - D_k)u_n\|^2_{L^2_\gamma(\Omega)} \\
&\qquad\qquad + \|g_k - g_n\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_{0,k} - u_{0,n}\|^2_{L^2(\mathbb{R}^3_+)} \Big) \\
&\leq C(\eta, r, T) \Big(\|f_k - f_n\|^2_{L^2(\Omega)} + \|A_{0,n} - A_{0,k}\|^2_{L^\infty(\Omega)} \|\partial_t u_n\|^2_{L^2(\Omega)} \\
&\qquad + \|D_n - D_k\|^2_{L^\infty(\Omega)} \|u_n\|^2_{L^2(\Omega)} + \|g_k - g_n\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_{0,k} - u_{0,n}\|^2_{L^2(\mathbb{R}^3_+)} \Big) \\
&\leq C(\eta, r, T) \Big(\|f_k - f_n\|^2_{L^2(\Omega)} + \frac{n^8}{n^9} + \frac{1}{n^9} + \|g_k - g_n\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} \\
&\qquad\qquad + \|u_{0,k} - u_{0,n}\|^2_{L^2(\mathbb{R}^3_+)} \Big).
\end{aligned} \tag{4.108}$$

Since $(f_n)_n$ tends to f in $L^2(\Omega)$, $(g_n)_n$ to g in $E_0(J \times \partial\mathbb{R}^3_+)$, and $(u_{0,n})_n$ to u_0 in $L^2(\mathbb{R}^3_+)$, we conclude that $(u_n)_n$ is a Cauchy sequence in $C(\overline{J}, L^2(\mathbb{R}^3_+))$. Hence, this sequence converges to a function u in $C(\overline{J}, L^2(\mathbb{R}^3_+))$.

We next show that the function u is a weak solution of (3.2) with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0 in the sense of Definition 3.1. To that purpose, we note that the definition of the functions u_n yields

$$\langle f_n, \varphi \rangle_{L^2(\Omega)\times L^2(\Omega)} = \langle L(A_{0,n}, A_1, A_2, A_3, D_n)u_n, \varphi \rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)},$$

i.e.,

$$\int_\Omega f_n \cdot \varphi \, d(t, x) = \int_\Omega u_n \cdot L(A_{0,n}, A_1, A_2, A_3, D_n)^* \varphi \, d(t, x) \tag{4.109}$$

$$= -\int_\Omega u_n \cdot \partial_t(A_{0,n}\varphi)\, d(t,x) - \sum_{j=1}^3 \int_\Omega u_n \cdot \partial_j(A_j\varphi)\, d(t,x) + \int_\Omega u_n \cdot D_n^T\varphi\, d(t,x)$$

for all $\varphi \in H_0^1(\Omega)^6$. Fix such a function φ. Since (f_n) and $(u_n)_n$ converge to f respectively u in $L^2(\Omega)$, we infer

$$\int_\Omega f_n \cdot \varphi\, d(t,x) \longrightarrow \int_\Omega f \cdot \varphi\, d(t,x),$$
$$\sum_{j=1}^3 \int_\Omega u_n \cdot \partial_j(A_j\varphi)\, d(t,x) \longrightarrow \sum_{j=1}^3 \int_\Omega u \cdot \partial_j(A_j\varphi)\, d(t,x)$$

as $n \to \infty$. As $(\partial_t A_{0,n})_n$ is bounded in $L^\infty(\Omega)$ and converges pointwise to $\partial_t A_0$ by construction, the theorem of dominated convergence yields that $\partial_t A_{0,n}\varphi$ converges to $\partial_t A_0\varphi$ in $L^2(\Omega)$ as $n \to \infty$. The convergence of $(A_{0,n})_n$ and $(D_n)_n$ to A_0 respectively D in $L^\infty(\Omega)$ implies that also $(A_{0,n}\partial_t\varphi)_n$ and $(D_n^T\varphi)_n$ tend to $A_0\partial_t\varphi$ respectively $D\varphi$ in $L^2(\Omega)$. Letting $n \to \infty$ in (4.109), we thus obtain

$$\int_\Omega f \cdot \varphi\, d(t,x) = -\sum_{j=0}^3 \int_\Omega u \cdot \partial_j(A_j\varphi)\, d(t,x) + \int_\Omega u \cdot D^T\varphi\, d(t,x).$$

We conclude that

$$\langle L(A_0, \dots, A_3, D)u, \varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \langle f, \varphi\rangle_{L^2(\Omega)\times L^2(\Omega)} \tag{4.110}$$

for all $\varphi \in H_0^1(\Omega)^6$.

Next let $i \in \{1, \dots, 6\}$ and define the functions q_n^i and q^i by

$$q_n^i = ((A_{0,n}u_n)_i, (A_1u_n)_i, (A_2u_n)_i, (A_3u_n)_i)^T,$$
$$q^i = ((A_0u)_i, (A_1u)_i, (A_2u)_i, (A_3u)_i)^T$$

for all $n \in \mathbb{N}$. Since $(A_{0,n})$ tends to A_0 in $L^\infty(\Omega)$ and $(u_n)_n$ to u in $L^2(\Omega)$, we deduce that

$$q_n^i \longrightarrow q^i \tag{4.111}$$

in $L^2(\Omega)$ as $n \to \infty$. On the other hand, we have

$$\operatorname{div}_t q_n^i = \Big(\partial_t(A_{0,n}u_n) + \sum_{j=1}^3 \partial_j(A_ju_n)\Big)_i = \Big(f_n + \partial_t A_{0,n}u_n + \sum_{j=1}^3 \partial_j A_j u_n - D_nu_n\Big)_i$$

for all $n \in \mathbb{N}$. The same arguments as above show that $(D_nu_n)_n$ converges to Du in $L^2(\Omega)$ as $n \to \infty$. Moreover, we can estimate

$$\|\partial_t A_{0,n}u_n - \partial_t A_0 u\|_{L^2(\Omega)} \le \|\partial_t A_{0,n}(u_n - u)\|_{L^2(\Omega)} + \|(\partial_t A_{0,n} - \partial_t A_0)u\|_{L^2(\Omega)} \tag{4.112}$$

for all $n \in \mathbb{N}$. Because $(\partial_t A_{0,n})$ is bounded in $L^\infty(\Omega)$, the convergence of $(u_n)_n$ to u yields that the first summand on the right-hand side in (4.112) tends to 0. As $(\partial_t A_{0,n})_n$ furthermore tends pointwise almost everywhere to $\partial_t A_0$ the theorem of dominated convergence shows that also the second one tends to 0. Using that $(f_n)_n$ converges to f in $L^2(\Omega)$ as $n \to \infty$, we arrive at

$$\operatorname{div}_t q_n^i = \Big(f_n + \partial_t A_{0,n}u_n + \sum_{j=1}^3 \partial_j A_j u_n - D_nu_n\Big)_i$$
$$\longrightarrow \Big(f + \partial_t A_0 u + \sum_{j=1}^3 \partial_j A_j u - Du\Big)_i$$

in $L^2(\Omega)$ as $n \to \infty$. Employing the product rule in $H^{-1}(\Omega)$ and (4.110), we further infer

$$\operatorname{div}_t q^i = \partial_t A_0 u + A_0 \partial_t u + \sum_{j=1}^{3} (\partial_j A_j u + A_j \partial_j u) = f + \sum_{j=0}^{3} \partial_j A_j u - Du$$

in $H^{-1}(\Omega)$. Hence, $\operatorname{div}_t q_n^i$ and $\operatorname{div}_t q^i$ belong to $L^2(\Omega)$ and $(\operatorname{div}_t q_n^i)_n$ tends to $\operatorname{div}_t q^i$ as $n \to \infty$ in $L^2(\Omega)$. Combined with (4.111), this means that $(q_n^i)_n$ converges to q^i in $H(\operatorname{div}_t, \Omega)$ as $n \to \infty$. In particular, we see that $(A_3 u)_i$ belongs to $H(\operatorname{div}_t, \Omega)_3$ and that $(A_3 u_n)_i$ tends to $(A_3 u)_i$ in $H(\operatorname{div}_t, \Omega)_3$. Since the trace operator Tr is continuous from $H(\operatorname{div}_t, \Omega)_3$ to $H_0^{-1/2}(\Gamma)$ by Lemma 2.5, we obtain

$$\operatorname{Tr}(A_3 u_n)_i \longrightarrow \operatorname{Tr}(A_3 u)_i$$

in $H_0^{-1/2}(\Gamma)$ as $n \to \infty$, where $\Gamma = (0,T) \times \partial\mathbb{R}^3_+$. We point out that $\operatorname{Tr}(Bu_n) = g_n$ for all $n \in \mathbb{N}$ as the functions u_n are solutions of (3.2). Using Definition 2.15 and Definition 2.16, we arrive at

$$\begin{aligned}
\operatorname{Tr}(Bu) &= M \cdot \operatorname{Tr}(A_3 u) = M \cdot (\operatorname{Tr}(A_3 u)_1, \ldots, \operatorname{Tr}(A_3 u)_6) \\
&= M \cdot \lim_{n\to\infty} (\operatorname{Tr}(A_3 u_n)_1, \ldots, \operatorname{Tr}(A_3 u_n)_6) = \lim_{n\to\infty} M \cdot \operatorname{Tr}(A_3 u_n) \\
&= \lim_{n\to\infty} \operatorname{Tr}(Bu_n) = \lim_{n\to\infty} g_n = g,
\end{aligned}$$

where the limits are taken in $H^{-1/2}(\Gamma)$ and where we used that g_n converges to g in $L^2(J, H^{1/2}(\partial\mathbb{R}^3_+)) \hookrightarrow H^{-1/2}(J \times \partial\mathbb{R}^3_+)$ by step I).

Finally, we exploit that $(u_n)_n$ converges to u in $C(\overline{J}, L^2(\mathbb{R}^3_+))$. This implies

$$u(0) = \lim_{n\to\infty} u_n(0) = \lim_{n\to\infty} u_{0,n} = u_0,$$

where the limits are taken in $L^2(\mathbb{R}^3_+)$. We conclude that the function u solves (3.2) with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0 in the sense of Definition 3.1.

IV) Let a_0 denote the limit of A_0 at infinity. Then $|a_0| \leq \|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)}$. Take a radius $r_0 > 0$ such that $\|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$ and $\|B\|_{W^{1,\infty}(\mathbb{R}^3_+)} \leq r_0$. The construction of the functions $A_{0,n}$ in step II) then implies that

$$\|A_{0,n}(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$$

for all $n \in \mathbb{N}$. The properties of the approximating sequences in (4.104) further yield that

$$\|A_{0,n}\|_{W^{1,\infty}(\Omega)} \leq r \quad \text{and} \quad \|D_n\|_{L^\infty(\Omega)} \leq r$$

for all $n \in \mathbb{N}$ and the radius $r \geq r_0$ fixed at the beginning of step III). Lemma 3.7 and Remark 4.6 then show that

$$\begin{aligned}
\|u_n\|^2_{G_{0,\gamma}(\Omega)} + \gamma \|u_n\|^2_{L^2_\gamma(\Omega)} &\leq C_{3.7;0,0}(\eta, r_0)(\|u_{0,n}\|^2_{L^2(\mathbb{R}^3_+)} + \|g_n\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}) \\
&\quad + C_{3.7;0}(\eta, r)\frac{1}{\gamma}\|f_n\|^2_{L^2_\gamma(\Omega)}
\end{aligned}$$

for all $\gamma \geq \gamma_{3.7;0}(\eta, r)$ and $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain

$$\begin{aligned}
\|u\|^2_{G_{0,\gamma}(\Omega)} + \gamma \|u\|^2_{L^2_\gamma(\Omega)} &\leq C_{3.7;0,0}(\eta, r_0)(\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}) \\
&\quad + C_{3.7;0}(\eta, r)\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)}
\end{aligned}$$

for all $\gamma \geq \gamma_{3.7;0}(\eta, r)$. This is the estimate from Lemma 3.7 for u with constants of the same form as the ones in Lemma 3.7.

V) To show that u is the unique solution of the problem, it is enough to prove that problem (3.2) with homogeneous boundary and inital conditions and vanishing inhomogeneity has only the trivial solution. We note that this fact would be a consequence of the a priori estimate from step IV) if we already knew that they were true for *every* solution of (3.2) with differential operator $L(A_0, \dots, A_3, D)$. However, in step IV) we have proven these estimates only for the function u. Therefore, another approximation argument is needed.

So let $v \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ be a solution of (3.2) with inhomogeneity $f = 0$, boundary value $g = 0$, and initial value $v_0 = 0$. We extend v by zero on $(-\infty, 0)$ and continuously to the right such that $v(t) = 0$ for $t > T'$ and some $T' > T$. Then v belongs to $C(\mathbb{R}, L^2(\mathbb{R}^3_+)) \cap L^2(\Omega)$. We further set $A_0(t) = A_0(0)$ for $t \in (-\infty, 0)$ and $A_0(t) = A_0(T)$ for $t \in (T, \infty)$. Then A_0 is an element of $W^{1,\infty}(\mathbb{R} \times \mathbb{R}^3_+)$. Analogously, we extend D to $\mathbb{R} \times \mathbb{R}^3_+$.

Let $\delta > 0$ and $\varepsilon \in (0, \delta)$. We define $\tau_\delta r(t) = r(t - \delta)$ for all $t \in \mathbb{R}$ and $r \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^3_+)$. We further write $A_{0,\delta}$ respectively D_δ for $\tau_\delta A_0$ respectively $\tau_\delta D$. Let ρ_1 be the kernel of a standard mollifier over $\mathbb{R}$, i.e., $\rho_1 \in C_c^\infty(\mathbb{R})$ with $0 \leq \rho_1 \leq 1$, $\operatorname{supp} \rho_1 \subseteq B(0, 1)$, and ρ_1 has integral one. Let $\rho_{1,\varepsilon} = \varepsilon^{-1} \rho_1(\varepsilon^{-1} \cdot)$ and $J_\varepsilon r(t) = \int_{\mathbb{R}} \rho_{1,\varepsilon}(t - s) r(s) ds$ for all $t \in \mathbb{R}$, $r \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^3_+)$, and $\varepsilon > 0$. Let $(A^\delta_{0,n})_n$ and $(D^\delta_n)_n$ be two sequences as in step II) for $A_{0,\delta}$ and D_δ respectively. We point out that

$$
\begin{aligned}
\|A^\delta_{0,n}\|_{W^{1,\infty}(\Omega)} &\leq C_2 \|A_{0,\delta}\|_{W^{1,\infty}(\Omega)} \leq C_2 \|A_0\|_{W^{1,\infty}(\Omega)} \leq r, \\
\|D^\delta_n\|_{L^\infty(\Omega)} &\leq C_2 \|D_\delta\|_{L^\infty(\Omega)} \leq C_2 \|D\|_{L^\infty(\Omega)} \leq r, \\
\|A^\delta_{0,n}(0)\|_{L^\infty(\mathbb{R}^3_+)} &\leq \|A_{0,\delta}(0)\|_{L^\infty(\mathbb{R}^3_+)} = \|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0
\end{aligned}
$$

for all $n \in \mathbb{N}$. We further observe that $J_\varepsilon \tau_\delta v$ is an element of $C^1(\mathbb{R}, L^2(\mathbb{R}^3_+))$ and

$$
\begin{aligned}
L(A^\delta_{0,n}, &A_1, A_2, A_3, D^\delta_n) J_\varepsilon \tau_\delta v = (A^\delta_{0,n} - A_{0,\delta}) \partial_t J_\varepsilon \tau_\delta v + (D^\delta_n - D^\delta) J_\varepsilon \tau_\delta v \\
&+ L(A_{0,\delta}, A_1, A_2, A_3, D_\delta) J_\varepsilon \tau_\delta v \\
&= (A^\delta_{0,n} - A_{0,\delta}) \partial_t J_\varepsilon \tau_\delta v + (D^\delta_n - D^\delta) J_\varepsilon \tau_\delta v + A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon (A_{0,\delta} \partial_t \tau_\delta v) \\
&+ D_\delta J_\varepsilon \tau_\delta v - J_\varepsilon (D_\delta \tau_\delta v) + J_\varepsilon (L(A_{0,\delta}, A_1, A_2, A_3, D_\delta) \tau_\delta v) \\
&= (A^\delta_{0,n} - A_{0,\delta}) \partial_t J_\varepsilon \tau_\delta v + (D^\delta_n - D^\delta) J_\varepsilon \tau_\delta v + A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon (A_{0,\delta} \partial_t \tau_\delta v) \\
&+ D_\delta J_\varepsilon \tau_\delta v - J_\varepsilon (D_\delta \tau_\delta v) =: f_{n,\varepsilon,\delta},
\end{aligned}
$$

on Ω for all $n \in \mathbb{N}$, where we used that

$$
L(A_{0,\delta}, A_1, A_2, A_3, D_\delta) \tau_\delta v = \tau_\delta L(A_0, \dots, A_3, D) v = 0
$$

on Ω. Theorem C.14 in [BGS07] implies that $f_{n,\varepsilon,\delta}$ belongs to $L^2(\Omega)$. Moreover, $J_\varepsilon \tau_\delta v(0) = 0$ and from Corollary 2.12 we deduce that also $\operatorname{Tr}(B J_\varepsilon \tau_\delta v) = 0$. We conclude that $J_\varepsilon \tau_\delta v$ solves (3.2) with differential operator $L(A^\delta_{0,n}, A_1, A_2, A_3, D^\delta_n)$, inhomogeneity $f_{n,\varepsilon,\delta}$, boundary value 0, and initial value 0 for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \delta)$. Lemma 3.7 thus shows that

$$
\|J_\varepsilon \tau_\delta v\|^2_{G_{0,\gamma}(\Omega)} \leq C_{3.7;0}(\eta, r) \|f_{n,\varepsilon,\delta}\|^2_{L^2_\gamma(\Omega)}
$$

for all $n \in \mathbb{N}$, $\varepsilon \in (0, \delta)$, and a fixed number $\gamma \geq \gamma_{3.7;0}(\eta, r)$. Using the definition of $f_{n,\varepsilon,\delta}$, we obtain a constant $C = C(\eta, r)$ such that

$$
\begin{aligned}
&\|J_\varepsilon \tau_\delta v\|^2_{G_{0,\gamma}(\Omega)} \\
&\leq C \|A^\delta_{0,n} - A_{0,\delta}\|^2_{L^\infty(\Omega)} \|\partial_t J_\varepsilon \tau_\delta v\|^2_{L^2_\gamma(\Omega)} + C \|D^\delta_n - D_\delta\|^2_{L^\infty(\Omega)} \|J_\varepsilon \tau_\delta v\|^2_{L^2_\gamma(\Omega)} \\
&\quad + C \|A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon (A_{0,\delta} \partial_t \tau_\delta v)\|^2_{L^2_\gamma(\Omega)} + C \|D_\delta J_\varepsilon \tau_\delta v - J_\varepsilon (D_\delta \tau_\delta v)\|^2_{L^2_\gamma(\Omega)}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \delta)$. In the limit $n \to \infty$, we thus obtain

$$\begin{aligned} &\|J_\varepsilon \tau_\delta v\|^2_{G_{0,\gamma}(\Omega)} \\ &\leq C\|A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon(A_{0,\delta}\partial_t \tau_\delta v)\|^2_{L^2_\gamma(\Omega)} + C\|D_\delta J_\varepsilon \tau_\delta v - J_\varepsilon(D_\delta \tau_\delta v)\|^2_{L^2_\gamma(\Omega)} \end{aligned} \tag{4.113}$$

for all $\varepsilon \in (0, \delta)$, since $(A^\delta_{0,n})_n$ and $(D^\delta_n)_n$ converge to $A_{0,\delta}$ respectively D_δ in $L^\infty(\Omega)$. Using Fubini's theorem, we can write

$$\begin{aligned} &\|A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon(A_{0,\delta}\partial_t \tau_\delta v)\|^2_{L^2(\Omega)} \\ &= \int_{\mathbb{R}^3_+} \|A_{0,\delta}(\cdot, x) J_\varepsilon \partial_t \tau_\delta v(\cdot, x) - J_\varepsilon(A_{0,\delta}(\cdot, x)\partial_t \tau_\delta v(\cdot, x))\|^2_{L^2(0,T)} dx. \end{aligned}$$

Since $A_{0,\delta}(x)$ is Lipschitz-continuous on $\mathbb{R}$, Theorem C.14 from [BGS07] shows that

$$\begin{aligned} &\|A_{0,\delta}(\cdot, x) J_\varepsilon \partial_t \tau_\delta v(\cdot, x) - J_\varepsilon(A_{0,\delta}(\cdot, x)\partial_t \tau_\delta v(\cdot, x))\|_{L^2(0,T)} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0, \\ &\|A_{0,\delta}(\cdot, x) J_\varepsilon \partial_t \tau_\delta v(\cdot, x) - J_\varepsilon(A_{0,\delta}(\cdot, x)\partial_t \tau_\delta v(\cdot, x))\|_{L^2(0,T)} \\ &\leq C\|A_{0,\delta}(\cdot, x)\|_{W^{1,\infty}(\mathbb{R})}\|\tau_\delta v(\cdot, x)\|_{L^2(\mathbb{R})} \leq Cr\|\tau_\delta v(\cdot, x)\|_{L^2(\mathbb{R})} \qquad \text{for all } \varepsilon > 0 \end{aligned}$$

for almost all $x \in \mathbb{R}^3_+$. The theorem of dominated convergence then yields that

$$\|A_{0,\delta} J_\varepsilon \partial_t \tau_\delta v - J_\varepsilon(A_{0,\delta}\partial_t \tau_\delta v)\|^2_{L^2(\Omega)} \longrightarrow 0$$

as $\varepsilon \to 0$. Similarly, we deduce

$$\|D_\delta J_\varepsilon \tau_\delta v - J_\varepsilon(D_\delta \tau_\delta v)\|_{L^2(\Omega)} \longrightarrow 0$$

as $\varepsilon \to 0$. As $\tau_\delta v$ is an element of $C(\mathbb{R}, L^2(\mathbb{R}^3_+))$, we further obtain that $J_\varepsilon \tau_\delta v$ converges to $\tau_\delta v$ in $C(\overline{J}, L^2(\mathbb{R}^3_+))$ as $\varepsilon \to 0$. Letting ε to 0 in (4.113), we thus arrive at

$$\|\tau_\delta v\|^2_{G_{0,\gamma}(\Omega)} = 0.$$

for each $\delta > 0$. Now let $t \in (0, T)$. Then there is a number $\delta > 0$ such that $t + \delta < T$. We obtain

$$v(t) = \tau_\delta v(t + \delta) = 0.$$

Hence, $v = 0$ on $(0, T)$ and by continuity then also on $[0, T]$.

We conclude that the initial boundary value problem (3.2) with differential operator $L(A_0, \dots, A_3, D)$, inhomogeneity $f \in L^2(\Omega)$, boundary value g, and initial value $u_0 \in L^2(\mathbb{R}^3_+)$ has a unique solution in $C(\overline{J}, L^2(\mathbb{R}^3_+))$ for which estimate (3.9) is true with constants of the same form as in Lemma 3.7. We have thus shown Lemma 3.7 with a coefficient A_0 from $F^c_{3,\eta}(\Omega)$ and a coefficient D from $F^c_3(\Omega)$. Since we used the assumption that the coefficients are constant outside some compact set only to apply Lemma 3.7, we can now replace it by the assumption that the coefficients converge as $|(t, x)| \to \infty$ in all results of Chapter 3 and Chapter 4. The assertion of the theorem then follows from the corresponding results in these sections. □

5

Localization

In this part we want to perform the localization procedure for Maxwell's equations. Our goal is to transfer the local wellposedness theory from the half-space to domains. We study the linear initial boundary value problem

$$\begin{cases} A_0 \partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + Du = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \quad t \in J; \\ u(0) = u_0, & x \in G; \end{cases} \tag{5.1}$$

and derive a wellposedness theory corresponding to the one on the half-space from Chapters 3 and 4. The idea is to use local charts in order to transfer the results from the half-space to the domain.

In the localization argument we need an additional property of the covering of the boundary. On each chart we want to find one component of the unit normal vector of the boundary which is uniformly bounded from below. The next lemma shows that this can be achieved by a refinement of the covering.

Lemma 5.1. *Let $m \in \mathbb{N}$, $m \geq 2$. Let $G \subseteq \mathbb{R}^d$ satisfy the uniform C^m-regularity condition. Then there exists a locally finite open cover $(U_i)_{i\in\mathbb{N}}$ and corresponding functions $\varphi_i \in C^m(U_i)$ which are bijections onto an open set $V_i \subseteq B(0,1)$ such that $\psi_i = \varphi_i^{-1} \in C^m(V_i)$ for all $i \in \mathbb{N}$ and the following conditions are satisfied.*

(i) There is a natural number N such that for all $\Lambda \subseteq \mathbb{N}$ with $|\Lambda| \geq N$ we have $\bigcap_{i\in\Lambda} U_i = \emptyset$.

(ii) For each $i \in \mathbb{N}$ we have $\varphi_i(U_i \cap G) = \{y \in V_i : y_d > 0\} =: V_i^+$.

(iii) There is a constant $M_1 > 0$ such that

$$\begin{aligned} |\partial^\alpha \varphi_{i,j}(x)| &\leq M_1 \qquad \text{for all } x \in U_i, \\ |\partial^\alpha \psi_{i,j}(y)| &\leq M_1 \qquad \text{for all } y \in V_i \end{aligned} \tag{5.2}$$

for all $j \in \{1, \ldots, d\}$, $i \in \mathbb{N}$, and $\alpha \in \mathbb{N}_0^d$ with $0 < |\alpha| \leq m$.

(iv) There exists a number $\tau > 0$ such that for all $i \in \mathbb{N}$ there is an index $j \in \{1, \ldots, d\}$ such that

$$|\partial_j \varphi_{i,d}(x)| \geq \tau \tag{5.3}$$

for all $x \in U_i$.

Proof. Let $(\tilde{U}_i)_{i\in\mathbb{N}}$, $(\tilde{\varphi}_i)_{i\in\mathbb{N}}$, and $(\tilde{\psi}_i)_{i\in\mathbb{N}}$ be the covering respectively the corresponding transformations from Definition 2.24. The chain rule then implies that

$$I_{d\times d} = (\nabla(\tilde{\varphi}_i \circ \tilde{\psi}_i))(x) = \nabla\tilde{\varphi}_i(\tilde{\psi}_i(x)) \cdot \nabla\tilde{\psi}_i(x)$$

for all $x \in B(0,1)$, i.e.,

$$1 = |\nabla \tilde{\varphi}_{i,d}(\tilde{\psi}_i(x)) \cdot \partial_d \tilde{\psi}_i(x)| \le M_1 |\nabla \tilde{\varphi}_{i,d}(\tilde{\psi}_i(x))|$$

for all $x \in B(0,1)$ and hence

$$|\nabla \tilde{\varphi}_{i,d}(x)| \ge \frac{1}{M_1}$$

for all $x \in \tilde{U}_i$ and $i \in \mathbb{N}$. Consequently, there is a number $\tau > 0$ such that for all $i \in \mathbb{N}$ and $x \in \partial G \cap \tilde{U}_i$ there is an index $j \in \{1, \dots, d\}$ such that

$$|\partial_j \tilde{\varphi}_{i,d}(x)| \ge 2\tau.$$

We pick such an index and denote it by $j(x,i)$. We define the domains

$$U_{x,i} = \{y \in \tilde{U}_i \colon |\partial_{j(x,i)} \tilde{\varphi}_{i,d}(y)| > \tau \text{ and } \exists \gamma \in C([0,1], \tilde{U}_i) \text{ with } \gamma(0) = x, \gamma(1) = y\}$$

for every $x \in \partial G \cap \tilde{U}_i$ and $i \in \mathbb{N}$. The set $\overline{B}(0,n) \cap \partial G$ is compact for all $n \in \mathbb{N}$ and the system

$$\{U_{x,i} \colon x \in \partial G \cap \tilde{U}_i, i \in \mathbb{N}\}$$

forms an open cover of it for all $n \in \mathbb{N}$. Hence, there are a number $K(n) \in \mathbb{N}$ and finitely many points $x_{n,1}, \dots, x_{n,K(n)} \in \partial G$ and indices $i_{n,1}, \dots, i_{n,K(n)}$ such that

$$\overline{B}(0,n) \cap \partial G \subseteq \bigcup_{k=1}^{K(n)} U_{x_{n,k}, i_{n,k}}$$

for all $n \in \mathbb{N}$. We set

$$V_{x_{n,k}, i_{n,k}} = \tilde{\varphi}_{i_{n,k}}(U_{x_{n,k}, i_{n,k}}) \subseteq B(0,1), \quad \varphi_{x_{n,k}, i_{n,k}} = \tilde{\varphi}_{i_{n,k} | U_{x_{n,k}, i_{n,k}}},$$
$$\psi_{x_{n,k}, i_{n,k}} = \tilde{\psi}_{i_{n,k} | V_{x_{n,k}, i_{n,k}}}$$

for all $k \in \{1, \dots, K(n)\}$ and $n \in \mathbb{N}$. Then the system

$$\{(U_{x_{n,k}, i_{n,k}}, V_{x_{n,k}, i_{n,k}}, \varphi_{x_{n,k}, i_{n,k}}, \psi_{x_{n,k}, i_{n,k}}) \colon k \in \{1, \dots, K(n)\}, n \in \mathbb{N}\}$$

is countable and we fix an enumeration $(U_i, V_i, \varphi_i, \psi_i)_{i \in \mathbb{N}}$ of it. Observe that

$$\partial G = \bigcup_{n \in \mathbb{N}} \overline{B}(0,n) \cap \partial G \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{k=1}^{K(n)} U_{x_{n,k}, i_{n,k}} = \bigcup_{i \in \mathbb{N}} U_i,$$

i.e., $(U_i)_{i \in \mathbb{N}}$ is an open cover of ∂G. Moreover, we have $V_i = \varphi_i(U_i)$, $\varphi_i \in C^m(U_i, V_i)$, and $\psi_i = \varphi_i^{-1}$ for all $i \in \mathbb{N}$. By construction, these objects satisfy conditions (ii), (iii), and (iv).

By Definition 2.24 there is a number $\tilde{N}$ such that for all $\Lambda \subseteq \mathbb{N}$ with $|\Lambda| \ge \tilde{N}$ the intersection $\bigcap_{i \in \Lambda} \tilde{U}_i$ is empty. We claim that (i) holds with $N = \tilde{N}d - d + 1$. To see this assertion we assume that there was a subset Λ of $\mathbb{N}$ with $|\Lambda| \ge \tilde{N}d - d + 1$ and

$$\bigcap_{l \in \Lambda} U_l \neq \emptyset. \tag{5.4}$$

By construction, for all $l \in \Lambda$ there exist numbers $n_l \in \mathbb{N}$ and $k_l \in \{1, \dots, K(n_l)\}$ such that $U_l = U_{x_{n_l,k_l}, i_{n_l,k_l}}$. In particular, $\bigcap_{l \in \Lambda} \tilde{U}_{i_{n_l,k_l}}$ is not empty. Since the intersection of $\tilde{N}$ or more of the sets $\tilde{U}_i$ is empty, we obtain a set $\tilde{\Lambda} \subseteq \mathbb{N}$ with $|\tilde{\Lambda}| \le \tilde{N} - 1$ and $i_{n_l,k_l} \in \tilde{\Lambda}$ for all $l \in \Lambda$. The pigeon hole principle thus yields an index $\hat{i} \in \tilde{\Lambda}$ and a subset Λ' of Λ with $|\Lambda'| \ge d + 1$ such that $i_{n_l,k_l} = \hat{i}$ for all $l \in \Lambda'$. The pigeon hole principle now tells us that there are two indices p and q in Λ' such that

$j(x_{n_p,k_p},\hat{i}) = j(x_{n_q,k_q},\hat{i})$. However, the definition of the sets $U_{x,i}$ and (5.4) then imply that

$$U_{x_{n_p,k_p},i_{n_p,k_p}} = U_{x_{n_p,k_p},i_{n_p,k_p}} \cup U_{x_{n_q,k_q},i_{n_q,k_q}} = U_{x_{n_q,k_q},i_{n_q,k_q}}.$$

This means that two of the sets $(U_i)_{i\in\mathbb{N}}$ are identical in contradiction to the construction. □

Remark 5.2. If G is a domain as in Definition 2.24 or Lemma 5.1, the boundary of G can be descibed as a union of level sets of the functions $\varphi_{i,d}$ in the sense that

$$\partial G \cap U_i = \{x \in \mathbb{R}^d \colon \varphi_{i,d}(x) = 0\}.$$

We particularly obtain that for all $x \in \partial G$ the vector $\nabla\varphi_{i,d}(x)$ is normal to the boundary ∂G in x. ◇

In the following we will restrict ourselves to the case $d = 3$ since the problem we are considering is posed on domains in $\mathbb{R}^3$.

Let $J \subseteq \mathbb{R}$ be an open interval. In a nutshell, the idea of a localization is to transform the problem via local coordinate charts into problems on the half-space respectively full space. We will thus rely on the wellposedness results for the initial value problem

$$\begin{cases} A_0\partial_t u + \sum_{j=1}^{3} A_j\partial_j u + Du = f, & x \in \mathbb{R}^3, \quad t \in J; \\ u(0) = u_0, & x \in \mathbb{R}^3; \end{cases} \tag{5.5}$$

on the full space and of the initial boundary value problem

$$\begin{cases} A_0\partial_t u + \sum_{j=1}^{3} A_j\partial_j u + Du = f, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bu = g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ u(0) = u_0, & x \in \mathbb{R}^3_+; \end{cases} \tag{5.6}$$

on the half-space. The local wellposedness theory for the initial boundary value problem on the half-space has been developped in Chapter 3 and Chapter 4. However, we have not addressed the full space case yet. The initial value problem on the full space is of course easier to treat as all the problems posed by the characteristic boundary disappear.

Theorem 5.3. *Let $T' > 0$, $\eta > 0$, and $r \geq r_0 > 0$. Let $T \in (0, T']$, $J = (0, T)$, and $\tilde{\Omega} = J \times \mathbb{R}^3$. Let $m \in \mathbb{N}_0$ and $\tilde{m} = \max\{m, 3\}$. Let $A_0 \in F^c_{m,6,\eta}(\tilde{\Omega})$, $A_1, A_2, A_3 \in F^c_{m,6}(\tilde{\Omega})$ symmetric and $D \in F^c_{m,6}(\tilde{\Omega})$ with*

$$\begin{aligned} &\|A_i\|_{F_{\tilde{m}}(\tilde{\Omega})} \leq r, \quad \|D\|_{F_{\tilde{m}}(\tilde{\Omega})} \leq r, \\ &\max\{\|A_i(0)\|_{\tilde{F}_{\tilde{m}-1}(\mathbb{R}^3)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3)}\} \leq r_0, \\ &\max\{\|D(0)\|_{\tilde{F}_{\tilde{m}-1}(\mathbb{R}^3)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3)}\} \leq r_0 \end{aligned}$$

for all $i \in \{0, \dots, 3\}$. Let $f \in H^m(\tilde{\Omega})$ and $u_0 \in H^m(\mathbb{R}^3)$. Then the initial value problem (5.5) has a unique solution u in $G_m(\tilde{\Omega})$ and there are constants $C_m = C_m(\eta, r, T') \geq 1$, $C_{m,0} = C_{m,0}(\eta, r_0) \geq 1$, and $\gamma_m = \gamma_m(\eta, r, T') \geq 1$ such that

$$\begin{aligned} \|u\|^2_{G_{m,\gamma}(\tilde{\Omega})} \leq (C_{m,0} + TC_m)e^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3)} + \|u_0\|^2_{H^m(\mathbb{R}^3)}\Big) \\ + \frac{C_m}{\gamma} e^{mC_1T}\|f\|^2_{H^m_\gamma(\tilde{\Omega})} \end{aligned}$$

for all $\gamma \geq \gamma_m$.

Proof. An inspection of the proof of Theorem 2.8 in [BGS07] shows that the assertion of the theorem is true in the case $m = 0$. From here we proceed as in Chapter 3 and Chapter 4. We note that the proofs only simplify as all spatial directions can now be treated by the methods we used for the spatially tangential variables in the half-space case. □

We need a further assumption on the domains we are able to treat. Besides the properties that come with the uniform C^m-boundary, we will need several sequences of functions with special features. We refer to the discussion in front of Theorem 5.6 and its proof for a motivation of the following definition.

Definition 5.4. *Let $m \in \mathbb{N}$ with $m \geq 2$ and $G \subset \mathbb{R}^3$ be a domain which satisfies the uniform C^m-regularity condition. We say that the domain G has a* tame uniform C^m-boundary *if there exists an open cover $(U_i)_{i\in\mathbb{N}}$ of ∂G, corresponding functions $(\varphi_i)_{i\in\mathbb{N}}$, and open sets $(V_i)_{i\in\mathbb{N}}$ as in Lemma 5.1 which have the following additional properties, where $U_0 = G$, $\varphi_0 = \mathrm{id}_G$, and $V_0 = G$.*

(i) There is a smooth partition of unity $(\theta_i)_{i\in\mathbb{N}_0}$ subordinate to $(U_i)_{i\in\mathbb{N}_0}$ and a constant $M_2 > 0$ such that

$$|\partial^\alpha \theta_i(x)| \leq M_2$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$, $x \in U_i$, and $i \in \mathbb{N}_0$.

(ii) There is a sequence of functions $(\sigma_i)_{i\in\mathbb{N}_0}$ and a constant $M_3 > 0$ with $\sigma_i \in C_c^\infty(U_i)$, $0 \leq \sigma_i \leq 1$,

$$\sigma_i = 1 \quad \text{on} \quad \operatorname{supp} \theta_i,$$

and

$$|\partial^\alpha \sigma_i(x)| \leq M_3$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$, $x \in U_i$, and $i \in \mathbb{N}_0$.

(iii) There is a sequence of functions $(\omega_i)_{i\in\mathbb{N}_0}$ with $\omega_i \in C_c^\infty(V_i)$ for all $i \in \mathbb{N}_0$ such that $0 \leq \omega_i \leq 1$,

$$\omega_i = 1 \quad \text{on } K_i = \varphi_i(\operatorname{supp} \sigma_i)$$

for all $i \in \mathbb{N}$ and

$$|\partial^\alpha \omega_i(x)| \leq M_4$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq m$, $x \in V_i$ and $i \in \mathbb{N}_0$.

We say that the domain G has a tame uniform C^m-boundary with finitely many charts *if the above holds with $\mathbb{N}$ replaced by a finite index set $N \subseteq \mathbb{N}$.*

This definition is tailored for the localization argument. However, it looks a bit unhandy. To fill this definition with life we thus give two basic examples.

Example 5.5. Let $G \subset \mathbb{R}^3$ be a domain.

(i) Let $m \geq 2$ and let G satisfy the uniform C^m-regularity condition. If ∂G is compact, then G has a tame uniform C^m-boundary with finitely many charts.

(ii) The half-space $G = \mathbb{R}^3_+$ has a tame uniform C^m-boundary with finitely many charts for all $m \geq 2$.

Part (i) of the above example particularly shows that the definition of a tame uniform C^m-boundary shows an effect only if the boundary of G is unbounded.

We are now ready to prove existence, uniqueness, and a priori estimates of solutions of the Maxwell system (5.1). The proof relies on a localization procedure and the corresponding theorems on the full space and the half-space. Although the underlying idea is intuitive, the realization is technically quite involved and lengthy so that we want to outline its idea.

Given a C^{m+2}-domain with charts $(U_i, \varphi_i)_{i\in\mathbb{N}}$, we set $U_0 = G$ and obtain the covering $\overline{G} \subseteq \bigcup_{i\in\mathbb{N}_0} U_i$. We choose a smooth partition of unity $(\theta_i)_{i\in\mathbb{N}_0}$ subordinate to $(U_i)_{i\in\mathbb{N}_0}$. Then we study the full space problem solved by $\theta_0 u$ and the half space problems solved by $\Phi_i(\theta_i u)$ with suitably transformed coefficients, where Φ_i denotes the composition with the inverse of φ_i. The philosophy is that the arising "error terms" are of lower order and can be treated as a perturbation.

Following this strategy, the first question one has to answer is whether one can extend the coefficients to the full space respectively the transformed coefficients to the half-space such that the extended coefficients fulfill the assumptions of Theorem 5.3 respectively Theorem 4.13. It will be answered in step I) below. Moreover, we will need that the transformed coeffecients and data (which also involve the error terms) fulfill the compatibility conditions in order to solve the half-space problems. We will see in steps II) and III) that the compatibility conditions on the domain imply that the compatibility conditions for the transformed half-space problems are fulfilled. Applying Theorem 5.3 to $\theta_0 u$ and Theorem 4.13 to $\Phi_i(\theta_i u)$, we then derive the a priori estimates and thus uniqueness for (5.1) in step IV).

At this point it seems that the existence of solutions of (5.1) is straightforward to obtain. One solves the full space problem and the half-space problems derived in the steps before, applies Φ_i^{-1} to the half space solutions, and sums up. However, there are two problems. First of all, as long as we do not know that a solution of (5.1) exists, it is not clear that the solutions of the half space problems have compact support in $\varphi_i(U_i)$, which means that the sum over all half-space solutions and the full space solution may not yield a G_m-function. We have to localize once again, which in turn leads to further error terms. We will deal with them by using a fixed point argument to find a suitable inhomogeneity which neutralizes these additional error terms. The second problem is that in the transformed half-space problems and the full space problem error terms involving $\partial_j \theta_i u$ appear. These cannot be expressed in the terms of $\theta_i u$, which means that the inhomogeneities involve the solution u itself. Therefore, another fixed point argument is necessary to derive the existence of a solution of (5.1).

Theorem 5.6. *Let $T' > 0$, $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Pick $T \in (0, T']$ and set $J = (0, T)$. Take a domain $G \subset \mathbb{R}^3$ which has a tame uniform $C^{\tilde{m}+2}$-boundary.*

Choose coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},6,\eta}(J \times G)$, $D \in F^{\mathrm{c}}_{\tilde{m},6}(J \times G)$, and

$$B(t,x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.7}$$

for all $x \in \partial G$, where $\nu(x)$ denotes the outer unit normal on ∂G in x. Take radii $r \geq r_0 > 0$ such that

$$\|A_0\|_{F_{\tilde{m}}(J\times G)} \leq r, \quad \|D\|_{F_{\tilde{m}}(J\times G)} \leq r,$$
$$\max\{\|A_0(0)\|_{\tilde{F}_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_0(0)\|_{H^{\tilde{m}-j-1}(G)}\} \leq r_0,$$
$$\max\{\|D(0)\|_{\tilde{F}_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D(0)\|_{H^{\tilde{m}-j-1}(G)}\} \leq r_0.$$

Let $f \in H^m(J\times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ such that the linear compatibility conditions for the tupel $(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, B, f, g, u_0)$ of order m are fulfilled. Then the initial boundary value problem

$$\begin{cases} A_0 \partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + Du = f, & x \in G, \quad t \in J; \\ Bu = g, & x \in \partial G, \quad t \in J; \\ u(0) = u_0, & x \in G; \end{cases}$$

has a unique solution u belonging to $G_m(J \times G)$. Moreover, there are constants $C_m = C_m(\eta, r, T', G) \geq 1$, $C_{m,0} = C_{m,0}(\eta, r_0, G) \geq 1$, and $\gamma_m = \gamma_m(\eta, r, T', G) \geq 1$ such *that*

$$\|u\|^2_{G_{m,\gamma}(J\times G)} \le (C_{m,0} + TC_m)e^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\partial G)} + \|u_0\|^2_{H^m(G)}\Big) + C_m e^{mC_1T}\frac{1}{\gamma}\|f\|^2_{H^m_\gamma(J\times G)} \qquad (5.8)$$

for all $\gamma \ge \gamma_m$.

Proof. In the following we denote the standard trace operator mapping $H^1(G)$ to $H^{1/2}(G)$ by $\mathrm{tr}_{\partial G}$ and the trace operator mapping $H^1(\mathbb{R}^3_+)$ to $H^{1/2}(\partial\mathbb{R}^3_+)$ by $\mathrm{tr}_{\partial\mathbb{R}^3_+}$.

Fix a covering $(U_i)_{i\in\mathbb{N}_0}$, a sequence of sets $(V_i)_{i\in\mathbb{N}_0}$, and sequences of functions $(\varphi_i)_{i\in\mathbb{N}_0}$, $(\theta_i)_{i\in\mathbb{N}_0}$, $(\sigma_i)_{i\in\mathbb{N}_0}$, and $(\omega_i)_{i\in\mathbb{N}_0}$ as in Definition 5.4 for the tame uniform $C^{\tilde{m}+2}$-boundary of G.

I) As described in the outline of the localization procedure, we first have to extend the coefficients and the data to the half space after straightening a part of the boundary. To that purpose we abbreviate the inverse of φ_i by ψ_i, i.e.,

$$\psi_i\colon V_i \to U_i, \quad x \mapsto \varphi_i^{-1}(x);$$

and we introduce the operators

$$\Phi_i\colon L^2(U_i) \to L^2(V_i), \quad v \mapsto v\circ\psi_i;$$
$$\Phi_i^{-1}\colon L^2(V_i) \to L^2(U_i), \quad v \mapsto v\circ\varphi_i \qquad (5.9)$$

for all $i\in\mathbb{N}_0$. With a slight abuse of notation we also denote the composition with ψ_i on $L^2(J\times V_i)$ and $H^{-1}(J\times V_i)$ by Φ_i and analogously for Φ_i^{-1}. For $v\in L^2(J\times V_i^+)$ we then define the differential operator

$$\begin{aligned}
\mathcal{A}^i v &= \Phi_i\Big(A_0\partial_t + \sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j + D\Big)\Phi_i^{-1}v \\
&= \Phi_i\Big(A_0\partial_t v\circ\varphi_i + \sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j(v\circ\varphi_i) + Dv\circ\varphi_i\Big) \\
&= \Phi_i A_0\,\partial_t v + \Phi_i\Big(\sum_{j=1}^3\sum_{l=1}^3 A_j^{\mathrm{co}}\Phi_i^{-1}\partial_l v\,\partial_j\varphi_{i,l}\Big) + \Phi_i D\,v \\
&= \Phi_i A_0\,\partial_t v + \sum_{l=1}^3\Big(\sum_{j=1}^3 A_j^{\mathrm{co}}\Phi_i\partial_j\varphi_{i,l}\Big)\partial_l v + \Phi_i D\,v,
\end{aligned} \qquad (5.10)$$

where $\varphi_{i,l}$ denotes the l-th component of φ_i for all $i\in\mathbb{N}$. We therefore set

$$\tilde{A}_0^i = \Phi_i A_0, \quad \tilde{A}_l^i = \Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j\varphi_{i,l}\Big), \quad \tilde{D}^i = \Phi_i D$$

on V_i^+ for all $i\in\mathbb{N}$ and $l\in\{1,2,3\}$. Moreover, we define $\tilde{A}_0^0 = \Phi_0 A_0 = A_0$ and $\tilde{D}^0 = \Phi_0 D = D$ on U_0.

In the following we will always identify functions, that are only defined on a subset of some underlying domain, with their zero extension to that domain. Lemma 5.1 and the assumptions yield a number $z(i)\in\{1,2,3\}$ such that

$$|\partial_{z(i)}\varphi_{i,3}| \ge \tau \quad \text{on } U_i \qquad (5.11)$$

for all $i\in\mathbb{N}$. Reducing the size of τ if necessary, we can assume that τ is contained in $(0,1)$. We pick a point $y_i\in V_i$ for each $i\in\mathbb{N}$ and set

$$A_0^i = \omega_i\tilde{A}_0^i + (1-\omega_i)\eta, \quad (i\in\mathbb{N}_0), \qquad (5.12)$$

$$A_j^i = \omega_i \tilde{A}_j^i + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))A_{z(i)}^{\mathrm{co}}, \quad (i \in \mathbb{N}), \tag{5.13}$$

$$D^i = \omega_i \tilde{D}^i, \quad (i \in \mathbb{N}_0), \tag{5.14}$$

for all $j \in \{1,2,3\}$. The differential operator $\mathcal{A}^i$ extends in a natural way to a differential operator on $\mathbb{R}^3_+$ by setting

$$\mathcal{A}^i v = A_0^i \partial_t v + \sum_{j=1}^{3} A_j^i \partial_j v + D^i v$$

for all $v \in L^2(J \times \mathbb{R}^3_+)$ and $i \in \mathbb{N}$.

We recall from Remark 5.2 that $\nabla\varphi_{i,3}$ is normal to the boundary ∂G. Hence, there is a number $\kappa_i(x) \in \mathbb{R}$ such that

$$\nabla\varphi_{i,3}(x) = \kappa_i(x)\nu(x)$$

for all $x \in \partial G \cap U_i$ and $i \in \mathbb{N}$. In particular, $\kappa_i = \nabla\varphi_{i,3}\cdot\nu$ belongs to $C^{m+1}(\partial G \cap U_i, \mathbb{R})$ for all $i \in \mathbb{N}$.

We now set

$$\hat{B}^i = \omega_i \Phi_i(\kappa_i B) + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))B_{z(i)}^{\mathrm{co}},$$

on $\mathbb{R}^3$, where

$$B_1^{\mathrm{co}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2^{\mathrm{co}} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_3^{\mathrm{co}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define the function $b_{z(i)}\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$b_{z(i)} = \omega_i \Phi_i \partial_{z(i)}\varphi_{i,3} + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i)).$$

Since $\partial_{z(i)}\varphi_{i,3}$ does not change signs on U_i, estimate (5.11) implies the lower bound

$$\begin{aligned} |b_{z(i)}| &= \Big|\omega_i \Phi_i \partial_{z(i)}\varphi_{i,3} + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))\Big| \\ &= \omega_i |\Phi_i \partial_{z(i)}\varphi_{i,3}| + (1-\omega_i) \geq \tau\omega_i + 1 - \omega_i = 1-(1-\tau)\omega_i \geq \tau \end{aligned} \tag{5.15}$$

on $\mathbb{R}^3$ as $\tau \in (0,1)$. Consequently, the functions $b_{z(i)}$ and $b_{z(i)}^{-1}$ belong to $C^{m+1}(\overline{\mathbb{R}^3_+})$ and their restrictions to $\partial\mathbb{R}^3_+$ are elements of $C^{m+1}(\partial\mathbb{R}^3_+)$.

If $z(i) = 3$, we introduce the function

$$R_3^i = b_3^{1/2} \begin{pmatrix} b_3^{-1} & 0 & 0 \\ 0 & b_3^{-1} & 0 \\ \omega_i\Phi_i(\partial_1\varphi_{i,3})b_3^{-1} & \omega_i\Phi_i(\partial_2\varphi_{i,3})b_3^{-1} & 1 \end{pmatrix}$$

and compute that

$$R_3^i \hat{B}^i = b_3^{1/2} \begin{pmatrix} 0 & 1 & -\omega_i\Phi_i(\partial_2\varphi_{i,3})b_3^{-1} & 0 & 0 & 0 \\ -1 & 0 & \omega_i\Phi_i(\partial_1\varphi_{i,3})b_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: \tilde{B}_3^i$$

on $\partial\mathbb{R}^3_+$. We finally set

$$B_3^i = \begin{pmatrix} 0 & b_3^{1/2} & -\omega_i\Phi_i(\partial_2\varphi_{i,3})b_3^{-1/2} & 0 & 0 & 0 \\ -b_3^{1/2} & 0 & \omega_i\Phi_i(\partial_1\varphi_{i,3})b_3^{-1/2} & 0 & 0 & 0 \end{pmatrix} \tag{5.16}$$

and

$$\begin{aligned} C_3^i &= 2\begin{pmatrix} 0 & 0 & 0 & -b_3^{1/2} & 0 & \omega_i\Phi_i\partial_1\varphi_{i,3}b_3^{-1/2} \\ 0 & 0 & 0 & 0 & -b_3^{1/2} & \omega_i\Phi_i\partial_2\varphi_{i,3}b_3^{-1/2} \end{pmatrix}, \\ M_3^i &= \begin{pmatrix} 0 & 0 & 0 & -b_3^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_3^{-1/2} & 0 \end{pmatrix} \end{aligned} \tag{5.17}$$

on $\partial\mathbb{R}^3_+$. In the case $z(i) = 2$ respectively $z(i) = 1$ we define

$$R_2^i = b_2^{1/2}\begin{pmatrix} b_2^{-1} & 0 & 0 \\ \omega_i\Phi_i(\partial_1\varphi_{i,3})b_2^{-1} & 1 & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_2^{-1} \\ 0 & 0 & b_2^{-1} \end{pmatrix}$$

respectively

$$R_1^i = b_1^{1/2}\begin{pmatrix} 1 & \omega_i\Phi_i(\partial_2\varphi_{i,3})b_1^{-1} & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_1^{-1} \\ 0 & b_1^{-1} & 0 \\ 0 & 0 & b_1^{-1} \end{pmatrix}.$$

As above, we obtain

$$R_2^i\hat{B}^i = b_2^{1/2}\begin{pmatrix} 0 & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_2^{-1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\omega_i\Phi_i(\partial_1\varphi_{i,3})b_2^{-1} & 0 & 0 & 0 & 0 \end{pmatrix} =: \tilde{B}_2^i$$

respectively

$$R_1^i\hat{B}^i = b_1^{1/2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega_i\Phi_i(\partial_3\varphi_{i,3})b_1^{-1} & 0 & 1 & 0 & 0 & 0 \\ \omega_i\Phi_i(\partial_2\varphi_{i,3})b_1^{-1} & -1 & 0 & 0 & 0 & 0 \end{pmatrix} =: \tilde{B}_1^i$$

on $\mathbb{R}^3$. In the case $z(i) = 2$ we thus set

$$\begin{aligned} B_2^i &= \begin{pmatrix} 0 & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_2^{-1/2} & -b_2^{1/2} & 0 & 0 & 0 \\ b_2^{1/2} & -\omega_i\Phi_i(\partial_1\varphi_{i,3})b_2^{-1/2} & 0 & 0 & 0 & 0 \end{pmatrix}, \\ C_2^i &= 2\begin{pmatrix} 0 & 0 & 0 & -b_2^{1/2} & \omega_i\Phi_i(\partial_1\varphi_{i,3})b_2^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_2^{-1/2} & -b_2^{1/2} \end{pmatrix}, \\ M_2^i &= \begin{pmatrix} 0 & 0 & 0 & -b_2^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_2^{-1/2} \end{pmatrix}, \end{aligned}$$

while in the case $z(i) = 1$ we take

$$\begin{aligned} B_1^i &= \begin{pmatrix} -\omega_i\Phi_i(\partial_3\varphi_{i,3})b_1^{-1/2} & 0 & b_1^{1/2} & 0 & 0 & 0 \\ \omega_i\Phi_i(\partial_2\varphi_{i,3})b_1^{-1/2} & -b_1^{1/2} & 0 & 0 & 0 & 0 \end{pmatrix}, \\ C_1^i &= 2\begin{pmatrix} 0 & 0 & 0 & \omega_i\Phi_i(\partial_2\varphi_{i,3})b_1^{-1/2} & -b_1^{1/2} & 0 \\ 0 & 0 & 0 & \omega_i\Phi_i(\partial_3\varphi_{i,3})b_1^{-1/2} & 0 & -b_1^{1/2} \end{pmatrix}, \\ M_1^i &= \begin{pmatrix} 0 & 0 & 0 & 0 & -b_1^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_1^{-1/2} \end{pmatrix}. \end{aligned}$$

To simplify the notation, we will write B^i, $\tilde{B}^i$, C^i, M^i, and R^i in the following with a slight abuse of notation. We point out that the functions B^i, C^i, M^i, and R^i belong to $C^{\tilde{m}+1}(\mathbb{R}^3)$ and their restrictions to $\overline{\mathbb{R}^3_+}$ to $C^{\tilde{m}+1}(\overline{\mathbb{R}^3_+})$. The rank of B^i and C^i is identically 2 on $\overline{\mathbb{R}^3_+}$ and $R^i(x)$ is invertible for all $x \in \overline{\mathbb{R}^3_+}$. The inverse of R^i is as regular as R^i itself. We also have conservative boundary conditions on $\mathbb{R}^3_+$, i.e.,

$$\mathrm{Re}((C^i)^T B^i) = \frac{1}{2}(C^i)^T B^i + \frac{1}{2}(B^i)^T C^i = A_3^i \tag{5.18}$$

and

$$B^i = M^i A_3^i$$

on $\overline{\mathbb{R}^3_+}$ for all $i \in \mathbb{N}$. We conclude that B^i belongs to $\mathcal{BC}^m_{\mathbb{R}^3_+}(A_3)$ for all $i \in \mathbb{N}$. Moreover,

$$\begin{aligned} A_3^i &= \omega_i \Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \varphi_{i,3}\Big) + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))A^{\mathrm{co}}_{z(i)} \\ &= \omega_i \sum_{j \neq z(i)} A_j^{\mathrm{co}} \Phi_i \partial_j \varphi_{i,3} + b_{z(i)} A^{\mathrm{co}}_{z(i)}. \end{aligned}$$

Since $|b_{z(i)}| \geq \tau$ on $\mathbb{R}^3_+$ by (5.15), we infer that A_3^i is an element of $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff},\tau}(\Omega)$ for all $i \in \mathbb{N}$.

The construction of the coefficients further implies that $A_0^i \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $A_j^i \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$ for $j \in \{1,2\}$, and $D^i \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$.

We next fix a constant M_1 as in Lemma 5.1 and constants M_2, M_3, and M_4 as in Definition 5.4 for the tame uniform $C^{\tilde{m}+2}$-boundary of G. The construction of our extended coefficients then shows that

$$\begin{aligned} &\|A_0^i\|_{F_m(\Omega)} \leq C(M_1, M_4)\|A_0\|_{F_m(J\times G)}, \\ &\max\{\|A_0^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_0^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \\ &\quad \leq C(M_1, M_4)\max\{\|A_0(0)\|_{F^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1}\|\partial_t^j A_0(0)\|_{H^{\tilde{m}-j-1}(G)}\}, \\ &\|A_j^i\|_{F_{\tilde{m}}(\Omega)} \leq C(M_1, M_4), \\ &\|D^i\|_{F_{\tilde{m}}(\Omega)} \leq C(M_1, M_4)\|D\|_{F_m(J\times G)}, \\ &\max\{\|D^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \\ &\quad \leq C(M_1, M_4)\max\{\|D(0)\|_{F^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1}\|\partial_t^j D(0)\|_{H^{\tilde{m}-j-1}(G)}\}, \\ &\|B^i\|_{W^{\tilde{m}+1,\infty}(\Omega)} \leq C(M_1, M_4, \tau)\|B\|_{W^{\tilde{m}+1,\infty}(J\times G)} \end{aligned} \tag{5.19}$$

for all $j \in \{1,2,3\}$, where we used $|B_j^{\mathrm{co}}| \leq C|B(x)|$ for all $x \in G$, $j \in \{1,2,3\}$. We point out that the right-hand sides of these estimates are independent of i so that we find constants $R = R(M,r)$ and $R_0 = R_0(M, r_0)$ with

$$\begin{aligned} &\|A_0^i\|_{F_m(\Omega)} \leq R, \\ &\max\{\|A_0^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_0^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \leq R_0, \end{aligned} \tag{5.20}$$

$$\|A_j^i\|_{F_m(\Omega)} \leq R,, \tag{5.21}$$

$$\begin{aligned} &\|D^i\|_{F_m(\Omega)} \leq R, \\ &\max\{\|D^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\} \leq R_0, \end{aligned} \tag{5.22}$$

for all $i \in \mathbb{N}$, where we set $M = \max_{i=1,\dots,4} M_i$.

II) As outlined above, we will determine the initial value problem respectively the initial boundary value problem solved by $\Phi_i(\theta_i u)$ on $J \times \mathbb{R}^3$ respectively $J \times \mathbb{R}^3_+$ and apply Theorem 5.3 respectively Theorem 4.13 to it. In the derivation of the a priori estimates and the verification of the compatibility conditions in the existence part we then need to know how the operators $S_{G,m,p}$ and $S_{\mathbb{R}^3_+,m,p}$ are related when we insert the coefficients from step I) and suitably adapted data into $S_{\mathbb{R}^3_+,m,p}$. By "suitably adapted" we mean that we take care of the perturbation terms arising from the localization procedure. Motivated by step IV) below we therefore define

$$\begin{aligned} f^i(h,v) &= \Phi_i(\theta_i h) + \Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \theta_i v\Big) \quad \text{for all } v \in G_m(J\times G), h \in H^m(J \times G), \\ g^i &= \big((\mathrm{tr}_{\partial\mathbb{R}^3_+} R^i)\tilde{\Phi}_i(\mathrm{tr}_{\partial G}(\theta_i)\kappa_i g)\big)_{\alpha(i)}, \\ u_0^i &= \Phi_i(\theta_i u_0), \end{aligned} \tag{5.23}$$

for all $i \in \mathbb{N}_0$ respectively $i \in \mathbb{N}$, where $\alpha(i)$ denotes the 2-tuple obtained by removing $z(i)$ from $(1,2,3)$ and $\tilde{\Phi}_i$ the composition operator with the restriction of ψ_i to $U_i \cap \partial G$. Note that $f^i(h,v)$ belongs to $H^m(J\times\mathbb{R}^3)$ for all $v \in G_m(J\times G)$ and $h \in H^m(J\times G)$, g^i to $E_m(J\times\partial\mathbb{R}^3_+)$, and u_0^i to $H^m(\mathbb{R}^3_+)$ for all $i \in \mathbb{N}$.

Let $v \in G_m(J\times G)$ be a function with $\partial_t^p v(0) = S_{G,m,p}(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, f, u_0)$ for all $p \in \{0,\dots,m-1\}$, where the operators $S_{G,m,p}$ have been introduced in Definition 2.29. We abbreviate

$$\begin{aligned} S_{m,p}^i &= S_{\mathbb{R}^3_+,m,p}(0, A_0^i, \dots, A_3^i, D^i, f^i(f,v), u_0^i), \\ S_{m,p} &= S_{G,m,p}(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, f, u_0) \end{aligned} \tag{5.24}$$

for all $p \in \{0,\dots,m\}$ and $i \in \mathbb{N}$. Observe that $S_{m,p}^i$ and $S_{m,p}$ are well-defined due to the regularity of the coefficients and the data. Fix an index $i \in \mathbb{N}$. We claim that

$$S_{m,p}^i = \Phi_i(\theta_i S_{m,p}) \tag{5.25}$$

for all $p \in \{0,\dots,m\}$.

To show this assertion, we first note that

$$S_{m,0}^i = u_0^i = \Phi_i(\theta_i u_0) = \Phi_i(\theta_i S_{m,0}).$$

Next we assume that we have shown (5.25) for all $l \in \{0,\dots,p-1\}$ for some $p \in \{1,\dots,m\}$. The definition of the operators $S_{\mathbb{R}^3_+,m,p}$ then yield

$$\begin{aligned} S_{m,p}^i = A_0^i(0)^{-1}\Big[\partial_t^{p-1} f^i(f,v)(0) - \sum_{j=1}^3 A_j^i \partial_j S_{m,p-1}^i - \sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l A_0^i(0) S_{m,p-l}^i \\ - \sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l D^i(0) S_{m,p-1-l}^i\Big]. \end{aligned} \tag{5.26}$$

The induction hypothesis implies that $\operatorname{supp} S_{m,p-l}^i = \operatorname{supp} \Phi_i(\theta_i S_{m,p}) \subseteq \operatorname{supp} \Phi_i \theta_i \subseteq K_i$ for all $l \in \{1,\dots,p\}$ and thus

$$A_j^i \partial_j S_{m,p-1}^i = \tilde{A}_j^i \partial_j \Phi_i(\theta_i S_{m,p-1})$$

for all $j \in \{1,2,3\}$, as $\omega_i = 1$ on K_i. Because of

$$\partial_j(\Phi_i(\theta_i S_{m,p-1})) = (\nabla(\theta_i S_{m,p-1}))\circ\psi_i\,\partial_j\psi_i = \sum_{l=1}^3 \Phi_i(\partial_l(\theta_i S_{m,p-1}))\,\partial_j\psi_{i,l},$$

we obtain

$$A_j^i \partial_j S_{m,p-1}^i = \sum_{k=1}^3 A_k^{\mathrm{co}} \Phi_i \partial_k \varphi_{i,j} \sum_{l=1}^3 \Phi_i \partial_l(\theta_i S_{m,p-1}) \partial_j \psi_{i,l}$$

$$= \sum_{k,l=1}^{3} A_k^{\mathrm{co}} \Phi_i \partial_l (\theta_i S_{m,p-1}) \Phi_i \partial_k \varphi_{i,j} \, \partial_j \psi_{i,l}$$

for all $j \in \{1,2,3\}$. We observe that

$$\delta_{lk} = (I_{3\times 3})_{lk} = (\nabla \operatorname{id}_{U_i})_{lk} = (\nabla(\psi_i \circ \varphi_i))_{lk} = \sum_{j=1}^{3} \Phi_i^{-1} \partial_j \psi_{i,l} \, \partial_k \varphi_{i,j},$$

$$\delta_{lk} = \sum_{j=1}^{3} \Phi_i \partial_k \varphi_{i,j} \, \partial_j \psi_{i,l} \tag{5.27}$$

on V_i for all $k,l \in \{1,2,3\}$. We thus infer

$$\begin{aligned} \sum_{j=1}^{3} A_j^i \partial_j S_{m,p-1} &= \sum_{j,k,l=1}^{3} A_k^{\mathrm{co}} \Phi_i \partial_l (\theta_i S_{m,p-1}) \Phi_i \partial_k \varphi_{i,j} \, \partial_j \psi_{i,l} \\ &= \sum_{k,l=1}^{3} A_k^{\mathrm{co}} \Phi_i \partial_l (\theta_i S_{m,p-1}) \delta_{lk} = \sum_{k=1}^{3} A_k^{\mathrm{co}} \Phi_i \partial_k (\theta_i S_{m,p-1}). \end{aligned} \tag{5.28}$$

Since the support of every term in the brackets on the right hand side of (5.26) is contained in K_i and $\omega_i = 1$ on K_i, the induction hypothesis further yields

$$\begin{aligned} S_{m,p}^i &= \Phi_i A_0(0)^{-1} \Big(\Phi_i(\theta_i \partial_t^{p-1} f(0)) + \Phi_i \Big(\sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \theta_i \partial_t^{p-1} v(0) - \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j (\theta_i S_{m,p-1}) \Big) \\ &\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} \Phi_i (\partial_t^l A_0(0)) \Phi_i (\theta_i S_{m,p-l}) - \sum_{l=0}^{p-1} \binom{p-1}{l} \Phi_i (\partial_t^l D(0)) \Phi_i (\theta_i S_{m,p-1-l}) \Big) \\ &= \Phi_i \Big[\theta_i A_0(0)^{-1} \Big(\partial_t^{p-1} f(0) - \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j S_{m,p-1} - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(0) S_{m,p-l} \\ &\qquad - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l D(0) S_{m,p-1-l} \Big], \\ &= \Phi_i(\theta_i S_{m,p}), \end{aligned}$$

where we also employed that $\partial_t^{p-1} v(0) = S_{m,p-1}$. By induction, we conclude that

$$S_{m,p}^i = \Phi_i(\theta_i S_{m,p})$$

for all $p \in \{0, \dots, m\}$. In the same way, but easier, we also obtain

$$S_{\mathbb{R}^3,m,p}(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, f^0(f,v), u_0^0) = \theta_0 S_{m,p} \tag{5.29}$$

for all $p \in \{0, \dots, m\}$.

III) In this step we show that the tuple $(0, A_0^i, \dots, A_3^i, D^i, B^i, f^i(f,v), g^i, u_0^i)$ fulfills the linear compatibility conditions of order m, where v is any function in $G_m(J \times G)$ with $\partial_t^p v(0) = S_{m,p}$ for all $p \in \{0, \dots, m-1\}$. To that purpose we exploit that the tuple $(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, B, f, g, u_0)$ fulfills the compatibility conditions of order m on G by assumption, which means that

$$B \operatorname{tr}_{\partial G} S_{m,p} = \operatorname{tr}_{\partial G}(B S_{m,p}) = \partial_t^p g(0)$$

for all $p \in \{0, \dots, m-1\}$. Recall that $S_{m,p}$ and $S_{m,p}^i$ are elements of $H^{m-p}(G)$ respectively $H^{m-p}(\mathbb{R}^3_+)$. Fix a number $p \in \{0, \dots, m-1\}$. The trace operator commutes with multiplication by C_c^∞-functions and the composition with diffeomorphisms, which allows us to infer

$$\partial_t^p (\tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i) \kappa_i g))(0) = \tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i) \kappa_i \partial_t^p g(0)) = \tilde{\Phi}_i(\kappa_i B \operatorname{tr}_{\partial G}(\theta_i) \operatorname{tr}_{\partial G} S_{m,p})$$

$$= \operatorname{tr}_{\partial\mathbb{R}^3_+} \hat{B}^i \tilde{\Phi}_i \operatorname{tr}_{\partial G}(\theta_i S_{m,p}) = \operatorname{tr}_{\partial\mathbb{R}^3_+} \hat{B}^i \operatorname{tr}_{\partial\mathbb{R}^3_+}(\Phi_i(\theta_i S_{m,p})) = \operatorname{tr}_{\partial\mathbb{R}^3_+}(\hat{B}^i S^i_{m,p}).$$

Multiplying this equation with the trace of R^i, we arrive at

$$\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i) \operatorname{tr}_{\partial\mathbb{R}^3_+}(\hat{B}^i S^i_{m,p}) = \partial_t^p(\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i g))(0). \tag{5.30}$$

The $z(i)$-th coordinate on the left-hand side is zero, so that the same must hold for the right-hand side. Equation (5.30) is thus equivalent to

$$\operatorname{tr}_{\mathbb{R}^3_+}(B^i S^i_{m,p}) = \partial_t^p(\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i g))_{\alpha(i)}(0) = \partial_t^p g^i(0).$$

IV) Let u be a solution in $G_m(J \times G)$ of (5.1) with inhomgeneity f, boundary value g, and initial value u_0. In this step we derive a priori estimates for u by applying the a priori estimates from Theorem 5.3 respectively Theorem 4.13 to $\theta_0 u$ respectively $\Phi_i(\theta_i u)$ for $i \in \mathbb{N}$. To that purpose, we first note that the uniform boundedness of the functions φ_i, ψ_i, and θ_i in C^{m+2} implies that

$$\begin{aligned}
&u \in G_m(J \times G) \iff \theta_0 u \in G_m(J \times \mathbb{R}^3) \text{ and } \Phi_i(\theta_i u) \in G_m(J \times \mathbb{R}^3_+) \text{ for all } i \in \mathbb{N},\\
&f \in H^m(J \times G) \iff \theta_0 f \in H^m(J \times \mathbb{R}^3) \text{ and } \Phi_i(\theta_i u) \in H^m(J \times \mathbb{R}^3_+) \text{ for all } i \in \mathbb{N},\\
&g \in E_m(J \times \partial G) \iff \Phi_i(\theta_i g) \in E_m(J \times \partial\mathbb{R}^3_+) \text{ for all } i \in \mathbb{N}.
\end{aligned}$$

Fix an index $i \in \mathbb{N}$. Since $\operatorname{supp} \Phi_i(\theta_i u) \subseteq \operatorname{supp} \Phi_i \theta_i \subseteq K_i$, the definition of the extended coefficients and (5.10) imply

$$\begin{aligned}
&A_0^i \partial_t(\Phi_i(\theta_i u)) + \sum_{j=1}^3 A_j^i \partial_j(\Phi_i(\theta_i u)) + D^i \Phi_i(\theta_i u) = \mathcal{A}^i(\Phi_i(\theta_i u)) \qquad (5.31)\\
&= \Phi_i\Big(A_0 \partial_t(\theta_i u) + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j(\theta_i u) + D(\theta_i u)\Big) = \Phi_i(\theta_i f) + \Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \theta_i u\Big)
\end{aligned}$$

on $J \times \mathbb{R}^3_+$. We further know that $\operatorname{Tr}(Bu) = g$ on $J \times \partial G$. Employing again that the trace operator commutes with the multiplication of C_c^∞-functions and the composition with diffeomorphisms, a similar computation as in step II) shows that

$$\begin{aligned}
\operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}(\hat{B}^i \Phi_i(\theta_i u)) &= \operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}(\Phi_i(\theta_i \kappa_i B u)) = \tilde{\Phi}_i \operatorname{Tr}_{J\times\partial G}(\theta_i \kappa_i B u)\\
&= \tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i \operatorname{Tr}_{\partial G}(Bu)) = \tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i g),
\end{aligned}$$

where we also used that $\hat{B}^i = \tilde{\Phi}_i(\kappa_i B)$ on $K_i \cap \partial\mathbb{R}^3_+$. Multiplying this equation with the trace of R^i and removing the $z(i)$-th component of the result, we obtain

$$\operatorname{Tr}_{\partial\mathbb{R}^3_+}(B^i u) = \operatorname{Tr}_{\partial\mathbb{R}^3_+}(R^i \hat{B}^i \Phi_i(\theta_i u))_{\alpha(i)} = (\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i g))_{\alpha(i)} = g^i.$$

We further note that g^i is an element of $E_m(J \times \partial\mathbb{R}^3_+)$ and

$$\|g^i\|_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \le C(M_1, \tau) \|\operatorname{tr}_{\partial G}(\theta_i) g\|_{E_{m,\gamma}(J\times\partial G)} \tag{5.32}$$

for all $\gamma \ge 0$. We conclude that the function $\Phi_i(\theta_i u)$ is a $G_m(J \times \mathbb{R}^3_+)$-solution of the initial boundary value problem

$$\begin{cases}
A_0^i \partial_t v + \sum_{j=1}^3 A_j^i \partial_j v + D^i v = f^i(f, u), & x \in \mathbb{R}^3_+, \quad t \in J;\\
B^i v = g^i, & x \in \partial\mathbb{R}^3_+, \quad t \in J;\\
v(0) = u_0^i, & x \in \mathbb{R}^3_+.
\end{cases} \tag{5.33}$$

In the following it will be convenient to abbreviate $U_i \cap G$ by G_i for all $i \in \mathbb{N}_0$.

In order to apply Theorem 4.13, we need that the boundary matrix A_3 and a function M as in the definition of $\mathcal{BC}^{\tilde{m}}_{\mathbb{R}^3_+}(A_3)$ belong to $C^\infty(\overline{\Omega})$. To that purpose, we transform the initial boundary value problem to an equivalent one as described in the first two steps of the proof of Lemma 4.4 for $z(i) = 3$. The keypoint is that this procedure not only yields transformed coefficients A_1 and A_2 that belong to $F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\Omega)$ and $A_3 = A_3^{\mathrm{co}}$, but also the matrix M arising as the transform of $M^i_{z(i)}$ is constant. To see this claim, we recall that M is given by $M^i_3 G_r^{-T}$, where

$$
\begin{aligned}
G_r^{-T} &= \begin{pmatrix} \hat{G}_r^{-T} & 0 \\ 0 & \hat{G}_r^{-T} \end{pmatrix}, \\
\hat{G}_r^{-T} &= b_3^{1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\omega_i \Phi_i(\partial_1 \varphi_{i,3}) b_3^{-1} & -\omega_i \Phi_i(\partial_2 \varphi_{i,3}) b_3^{-1} & b_3^{-1} \end{pmatrix}, \\
M_3^i &= \begin{pmatrix} 0 & 0 & 0 & -b_3^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_3^{-1/2} & 0 \end{pmatrix}.
\end{aligned}
$$

We thus obtain

$$
M = M_3^i G_r^{-T} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.
$$

Moreover, it is straightforward to show that the compatibility conditions of order m for the original problem are fulfilled if and only if they are true for the transformed coefficients and data. Consequently, we can apply Theorem 4.13 to this transformed problem and then obtain a solution of the same regularity of the origianl problem via the inverse transform. Also the a priori estimates carry over to the original problem with an additional constant $C(M_1, \tau)$. In order to simplify the notation, we suppress this transform in the following but assume that the matrices A_3^i and $M^i_{z(i)}$ are constant. Theorem 4.13 in combination with (5.32) thus yields

$$
\begin{aligned}
&\|\Phi_i(\theta_i u)\|^2_{G_{m,\gamma}(\Omega)} \\
&\leq (C_{4.13,m,0} + T C_{4.13,m}) e^{m C_{4.13,1} T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f^i(f,u)(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_{m,\gamma}(J \times \partial \mathbb{R}^3_+)} \\
&\qquad\quad + \|\Phi_i(\theta_i u_0)\|^2_{H^m(\mathbb{R}^3_+)} \Big) + C_{4.13,m} e^{m C_{4.13,1} T} \frac{1}{\gamma} \|f^i(f,u)\|^2_{H^m_\gamma(\Omega)} \\
&\leq 4 (C_{4.13,m,0} + T C_{4.13,m}) e^{m C_{4.13,1} T} \Big(\sum_{j=0}^{m-1} \|\Phi_i(\theta_i \partial_t^j f(0))\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=1}^{3} \|A_k^{\mathrm{co}} \Phi_i(\partial_k \theta_i \partial_t^j u(0))\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_{m,\gamma}(J \times \partial \mathbb{R}^3_+)} + \|\Phi_i(\theta_i u_0)\|^2_{H^m(\mathbb{R}^3_+)} \Big) \\
&\quad + \frac{4 C_{4.13,m}}{\gamma} e^{m C_{4.13,1} T} \Big(\|\Phi_i(\theta_i f)\|^2_{H^m_\gamma(\Omega)} + \sum_{k=1}^{3} \|A_k^{\mathrm{co}} \Phi_i(\partial_k \theta_i u)\|^2_{H^m_\gamma(\Omega)} \Big) \\
&\leq C(M_1)(C_{4.13,m,0} + T C_{4.13,m}) e^{m C_{4.13,1} T} \Big(\sum_{j=0}^{m-1} \|\theta_i \partial_t^j f(0)\|^2_{H^{m-1-j}(G_i)} \\
&\qquad\quad + \sum_{j=0}^{m-1} \sum_{k=1}^{3} \|\partial_k \theta_i S_{m,j}\|^2_{H^{m-1-j}(G_i)} + \|\operatorname{tr}_{\partial G} \theta_i\, g\|^2_{E_{m,\gamma}(J \times \partial G)} + \|\theta_i u_0\|^2_{H^m(G_i)} \Big) \\
&\quad + C(M_1, \tau) \frac{C_{4.13,m}}{\gamma} e^{m C_{4.13,1} T} \Big(\|\theta_i f\|^2_{H^m_\gamma(J \times G_i)} + \sum_{k=1}^{3} \|\partial_k \theta_i u\|^2_{H^m_\gamma(J \times G_i)} \Big) \qquad (5.34)
\end{aligned}
$$

for all $\gamma \geq \gamma_{4.13,m}$. Here we also exploited $\partial_t^j u(0) = S_{m,j}$ for all $j \in \{0, \ldots, m-1\}$, and where $C_{4.13,m} = C_{4.13,m}(\eta, \tau, R, T')$, $C_{4.13,m,0} = C_{4.13,m,0}(\eta, \tau, R_0)$, and $\gamma_{4.13,m} = \gamma_{4.13,m}(\eta, \tau, R, T')$ are the corresponding constants from Theorem 4.13.

Using that $\operatorname{supp}\theta_0 u \subseteq \operatorname{supp}\theta_0$, we compute

$$A_0^0\partial_t(\theta_0 u) + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j(\theta_0 u) + D^0(\theta_0 u) = \theta_0 f + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\theta_0 u \tag{5.35}$$

on $J\times\mathbb{R}^3$. We conclude that $\theta_0 u$ is a $G_m(J\times\mathbb{R}^3)$-solution of the initial value problem

$$\begin{cases} A_0^0\partial_t v + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j v + D^0 v = \theta_0 f + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\theta_0 u, & x\in\mathbb{R}^3, \quad t\in J; \\ v(0) = \theta_0 u_0, & x\in\mathbb{R}^3. \end{cases} \tag{5.36}$$

Theorem 5.3 and a computation as in (5.34) thus shows

$$\begin{aligned} &\|\theta_0 u\|_{G_{m,\gamma}(\Omega)}^2 \\ &\le C(C_{5.3,m,0} + TC_{5.3,m})e^{mC_{5.3,1}T}\Big(\sum_{j=0}^{m-1}\|\theta_0\partial_t^j f(0)\|_{H^{m-1-j}(G_0)}^2 \\ &\qquad + \sum_{j=0}^{m-1}\sum_{k=1}^{3}\|\partial_k\theta_0 S_{m,j}\|_{H^{m-1-j}(G_0)}^2 + \|\theta_0 u_0\|_{H^m(G_0)}^2\Big) \\ &\quad + C\frac{C_{5.3,m}}{\gamma}e^{mC_{5.3,1}T}\Big(\|\theta_0 f\|_{H^m_\gamma(J\times G_0)}^2 + \sum_{k=1}^{3}\|\partial_k\theta_0 u\|_{H^m_\gamma(J\times G_0)}^2\Big) \end{aligned} \tag{5.37}$$

for all $\gamma \ge \gamma_{5.3}$, where $C_{5.3,m} = C_{5.3,m}(\eta, R, T')$, $C_{5.3,m,0} = C_{5.3,m,0}(\eta, R_0)$, and $\gamma_{5.3,m} = \gamma_{5.3,m}(\eta, R, T')$ are the corresponding constants from Theorem 5.3.

The monotone convergence theorem, the local finiteness of the covering $(U_i)_{i\in\mathbb{N}_0}$, and (2.35) imply that

$$\begin{aligned} \sum_{i=0}^{\infty}\|\theta_i u_0\|_{H^m(G_i)}^2 &= \sum_{i=0}^{\infty}\int_G\sum_{|\alpha|\le m}|\partial^\alpha(\theta_i u_0)(x)|^2dx \\ &= \sum_{i=0}^{\infty}\int_G\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|\partial^\beta\theta_i(x)\partial^{\alpha-\beta}u_0(x)|^2dx \\ &\le C(m, M_2)\sum_{i=0}^{\infty}\int_G\sum_{|\alpha|\le m}\chi_{U_i}(x)|\partial^\alpha u_0(x)|^2dx \\ &= C(m, M_2)\int_G\sum_{|\alpha|\le m}\sum_{i=0}^{\infty}\chi_{U_i}(x)|\partial^\alpha u_0(x)|^2dx \\ &\le C(m, M_2, N)\int_G\sum_{|\alpha|\le m}|\partial^\alpha u_0(x)|^2dx = C(m, M_2, N)\|u_0\|_{H^m(G)}^2. \end{aligned} \tag{5.38}$$

Analogously, we obtain

$$\sum_{i=0}^{\infty}\sum_{j=0}^{m-1}\|\theta_i\partial_t^j f(0)\|_{H^{m-1-j}(G_i)}^2 \le C(m, M_2, N)\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|_{H^{m-1-j}(G)}^2,$$

$$\sum_{i=0}^{\infty}\sum_{k=1}^{3}\sum_{j=0}^{m-1}\|\partial_k\theta_i S_{m,j}\|_{H^{m-1-j}(G_i)}^2 \le C(m, M_2, N)\sum_{j=0}^{m-1}\|S_{m,j}\|_{H^{m-1-j}(G)}^2,$$

$$\sum_{i=0}^{\infty}\|\theta_i f\|_{H^m_\gamma(J\times G_i)}^2 \le C(m, M_2, N)\|f\|_{H^m_\gamma(J\times G)}^2,$$

$$\sum_{i=0}^{\infty}\sum_{k=1}^{3}\|\partial_k\theta_i u\|_{H^m_\gamma(J\times G_i)}^2 \le C(m, M_2, N)\|u\|_{H^m_\gamma(J\times G)}^2,$$

$$\sum_{i=1}^{\infty} \|g^i\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} \leq C(m, M_1, M_2, \tau)\|g\|^2_{E_{m,\gamma}(J\times\partial G)} \tag{5.39}$$

for all $\gamma \geq 0$, employing also (5.32). We set $C'_m = \max\{C_{5.3,m}, C_{4.13,m}\}$ and $C'_{m,0} = \max\{C_{5.3,m,0}, C_{4.13,m,0}\}$. Equations (5.34) to (5.39) then yield

$$\begin{aligned}
&\|u\|^2_{G_{m,\gamma}(J\times G)} \\
&\leq C(N)\sum_{i=0}^{\infty}\|\theta_i u\|^2_{G_{m,\gamma}(J\times G_i)} \leq C(N,M_1)\sum_{i=0}^{\infty}\|\Phi_i(\theta_i u)\|^2_{G_{m,\gamma}(\Omega)} \\
&\leq C(N,M_1,\tau)(C'_{m,0}+TC'_m)e^{mC'_1T}\Big[\sum_{i=0}^{\infty}\Big(\sum_{j=0}^{m-1}\|\theta_i\partial_t^j f(0)\|^2_{H^{m-1-j}(G_i)} \\
&\qquad + \sum_{k=1}^{3}\|\partial_k\theta_i S_{m,j}\|^2_{H^{m-1-j}(G_i)} + \|\theta_i u_0\|^2_{H^m(G_i)}\Big) + \sum_{i=1}^{\infty}\|\operatorname{tr}_{\partial G}(\theta_i)\, g\|^2_{E_{m,\gamma}(J\times\partial G)}\Big] \\
&\quad + C(m,N,M_1)\frac{C'_m}{\gamma}e^{mC'_1T}\Big(\sum_{i=0}^{\infty}\|\theta_i f\|^2_{H^m_\gamma(J\times G_i)} + \sum_{i=0}^{\infty}\sum_{k=1}^{3}\|\partial_k\theta_i u\|^2_{H^m_\gamma(J\times G_i)}\Big) \\
&\leq C(m,N,M_1,M_2,\tau)(C'_{m,0}+TC'_m)e^{mC'_1T}\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} \\
&\qquad + \sum_{j=0}^{m-1}\|S_{m,j}\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\partial G)} + \|u_0\|^2_{H^m(G)}\Big) \\
&\quad + C(m,N,M_1,M_2)\frac{C'_m}{\gamma}e^{mC'_1T}\Big(\|f\|^2_{H^m_\gamma(J\times G)} + \|u\|^2_{H^m_\gamma(J\times G))}\Big)
\end{aligned} \tag{5.40}$$

for all $\gamma \geq \max\{\gamma_{5.3,m}, \gamma_{4.13,m}\}$. Applying Lemma 2.33, which tells us that

$$\|S_{m,p}\|_{H^{m-p}(G)} \leq C_{2.33,m,p}(\eta, r_0)\Big(\sum_{j=0}^{p-1}\|\partial_t^j f(0)\|_{H^{m-1-j}(G)} + \|u_0\|_{H^m(G)}\Big)$$

for all $p \in \{0, \ldots, m\}$. Choosing $\gamma_m = \gamma_m(\eta, \tau, N, M_1, M_2, r, T')$ large enough, we thus arrive at

$$\begin{aligned}
\|u\|^2_{G_{m,\gamma}(J\times G)} \leq (C_{m,0}+TC_m)e^{mC_1T}\Big(&\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\partial G)} \\
&+ \|u_0\|^2_{H^m(G)}\Big) + C_m e^{mC_1T}\frac{1}{\gamma}\|f\|^2_{H^m(J\times G)}
\end{aligned}$$

for all $\gamma \geq \gamma_m$. Employing that $R = R(M, r)$ and $R_0 = R_0(M, r_0)$, we also deduce that the constants $C_{m,0}$ and C_m are of the claimed form. We have thus shown the a priori estimates (5.8), which imply uniqueness of the $G_m(J \times G)$-solution of (5.1).

V) We introduce the spaces

$$\begin{aligned}
G_{m,\mathrm{iv}}(J\times G) &= \{v \in G_m(J\times G) \colon \partial_t^j v(0) = S_{m,j},\ j \in \{0,\ldots,m-1\}\}, \\
H^m_{\mathrm{iv},f}(J\times G) &= \{\tilde f \in H^m(J\times G) \colon \partial_t^j \tilde f(0) = \partial_t^j f(0),\ j \in \{0,\ldots,m-1\}\}.
\end{aligned}$$

We point out that $G_{m,\mathrm{iv}}(J \times G)$ is nonempty by Lemma 2.34 and $H^m_{\mathrm{iv},f}(J \times G)$ is nonempty as $f \in H^m_{\mathrm{iv},f}(J \times G)$. Because the time derivatives up to order $m-1$ in 0 of functions from $H^m_{\mathrm{iv},f}(J \times G)$ respectively $G_{m,\mathrm{iv}}(J \times G)$ coincide, we obtain

$$\begin{aligned}
S_{\mathbb{R}^3_+,m,p}(0, A_0^i, \ldots, A_3^i, D^i, f^i(\tilde f, \tilde v), u_0^i) &= S_{\mathbb{R}^3_+,m,p}(0, A_0^i, \ldots, A_3^i, D^i, f^i(f, v), u_0^i) \\
&= S^i_{m,p}
\end{aligned} \tag{5.41}$$

for all $\tilde f \in H^m_{\mathrm{iv},f}(J\times G)$, $v, \tilde v \in G_{m,\mathrm{iv}}(J\times G)$, $p\in\{0,\dots,m\}$, and $i\in\mathbb{N}$, cf. (5.24). Hence, step III) implies that the tuple $(0, A_0^i, \dots, A_3^i, D^i, B^i, f^i(\tilde f, v), g^i, u_0^i)$ fulfills the compatibility conditions of order m for all $\tilde f \in H^m_{\mathrm{iv},f}(J\times G)$, $v\in G_{m,\mathrm{iv}}(J\times G)$, and $i\in\mathbb{N}$. As explained in step IV), we can thus apply Theorem 4.13 which shows that the initial boundary value problem

$$\begin{cases} A_0^i \partial_t w + \sum_{j=1}^3 A_j^i \partial_j w + D^i w = f^i(\tilde f, v), & x\in\mathbb{R}^3_+, \quad t\in J; \\ B^i w = g^i, & x\in\partial\mathbb{R}^3_+, \quad t\in J; \\ w(0) = u_0^i, & x\in\mathbb{R}^3_+; \end{cases} \tag{5.42}$$

has a unique solution $\mathcal{U}^i(\tilde f, v)$ in $G_m(\Omega)$ for all $\tilde f\in H^m_{\mathrm{iv},f}(J\times G)$, $v\in G_{m,\mathrm{iv}}(J\times G)$, and $i\in\mathbb{N}$. Moreover, Theorem 5.3 gives a solution $\mathcal{U}^0(\tilde f, v)$ in $G_m(J\times\mathbb{R}^3)$ of the initial value problem

$$\begin{cases} A_0^0 \partial_t w + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j w + D^0 w = f^0(\tilde f, v), & x\in\mathbb{R}^3, \quad t\in J; \\ w(0) = u_0^0, & x\in\mathbb{R}^3; \end{cases} \tag{5.43}$$

for all such $\tilde f$ and v. We claim that there exists a function $f^* = f^*(v)$ in $H^m_{\mathrm{iv},f}(J\times G)$ such that

$$f^* + \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} \mathcal{U}^i(f^*, v) = f \tag{5.44}$$

for all $v\in G_{m,\mathrm{iv}}(J\times G)$. To prove this claim we define the operator

$$\Psi_v \colon H^m_{\mathrm{iv},f}(J\times G) \to H^m_{\mathrm{iv},f}(J\times G),$$
$$\tilde f \mapsto f - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} \mathcal{U}^i(\tilde f, v)$$

for every $v\in G_{m,\mathrm{iv}}(J\times G)$. We fix such a function v. The operator Ψ_v maps into $H^m(J\times G)$ since Φ_i^{-1} maps the $H^m(\Omega)$-function $\mathcal{U}^i(\tilde f, v)$ into $H^m(J\times U_i)$, $\partial_j\sigma_i$ has compact support in U_i, and the covering $(U_i)_{i\in\mathbb{N}_0}$ is locally finite. We further compute

$$\begin{aligned} \partial_t^p \Psi_v(\tilde f)(0) &= \partial_t^p f(0) - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} \partial_t^p \mathcal{U}^i(\tilde f, v)(0) \\ &= \partial_t^p f(0) - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} S_{\mathbb{R}^3_+,m,p}(0, A_0^i, \dots, A_3^i, D^i, f^i(\tilde f, v), u_0^i) \\ &= \partial_t^p f(0) - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} S^i_{m,p} \\ &= \partial_t^p f(0) - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \Phi_i^{-1} \Phi_i(\theta_i S_{m,p}) \\ &= \partial_t^p f(0) - \sum_{i=0}^\infty \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \sigma_i \theta_i S_{m,p} = \partial_t^p f(0) \end{aligned}$$

for all $p\in\{0,\dots,m-1\}$ and $\tilde f\in H^m_{\mathrm{iv},f}(J\times G)$, where we used (5.41), (5.25), and that σ_i equals 1 on the support of θ_i for all $i\in\mathbb{N}$. We deduce that Ψ_v indeed maps $H^m_{\mathrm{iv},f}(J\times G)$ into itself. Theorems 5.3 and 4.13 imply next

$$\|\Psi_v(f_1) - \Psi_v(f_2)\|^2_{H^m_\gamma(J\times G)} \le C(N, M_3) \sum_{i=0}^\infty \|\Phi_i^{-1}\mathcal{U}^i(f_1, v) - \Phi_i^{-1}\mathcal{U}^i(f_2, v)\|^2_{H^m_\gamma(J\times G_i)}$$

$$
\begin{aligned}
&\leq C(N, M_1, M_3)\Big(\|\mathcal{U}^0(f_1, v) - \mathcal{U}^0(f_2, v)\|^2_{H^m_\gamma(J\times\mathbb{R}^3)} + \sum_{i=1}^\infty \|\mathcal{U}^i(f_1, v) - \mathcal{U}^i(f_2, v)\|^2_{H^m_\gamma(\Omega)}\Big) \\
&\leq C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma}\sum_{i=0}^\infty \|\Phi_i(\theta_i(f_1 - f_2))\|^2_{H^m_\gamma(\Omega)} \\
&\leq C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma}\|f_1 - f_2\|^2_{H^m_\gamma(J\times G)} \qquad (5.45)
\end{aligned}
$$

for all $\gamma \geq \max\{\gamma_{5.3,m}, \gamma_{4.13,m}\}$, employing (5.39) in the last step. We set

$$\gamma^* = \max\{\gamma_{5.3,m}, \gamma_{4.13,m}, 4C_{5.45}\},$$

where $C_{5.45}$ denotes the constant on the right-hand side of (5.45). This estimate then leads to the bound

$$\|\Psi_v(f_1) - \Psi_v(f_2)\|_{H^m_\gamma(J\times G)} \leq \frac{1}{2}\|f_1 - f_2\|_{H^m_\gamma(J\times G)} \qquad (5.46)$$

for all $\gamma \geq \gamma^*$. Fixing a parameter $\gamma \geq \gamma^*$, we conclude that Ψ_v is a strictly contractive self-mapping from $(H^m_{\mathrm{iv},f}(J \times G), \|\cdot\|_{H^m_\gamma(J\times G)})$ into itself. Since the latter space is complete, Banach's fixed point theorem yields a unique function $f^* = f^*(v)$ in $H^m_{\mathrm{iv},f}(J \times G)$ satisfying equation (5.44).

We next define the operator

$$
\begin{aligned}
\mathcal{S}\colon G_{m,\mathrm{iv}}(J \times G) &\to G_{m,\mathrm{iv}}(J \times G), \\
v &\mapsto \sum_{i=0}^\infty \sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(v), v).
\end{aligned}
$$

We first check that $\mathcal{S}$ indeed maps into $G_{m,\mathrm{iv}}(J\times G)$. Since $\mathcal{U}^i(f^*(v), v)$ is an element of $G_m(\Omega)$, the function $\Phi_i^{-1}\mathcal{U}^i(f^*(v), v)$ belongs to $G_m(J \times G_i)$. Exploiting that σ_i has compact support in U_i and the covering $(U_i)_{i\in\mathbb{N}_0}$ is locally finite, we infer that $\mathcal{S}(v)$ belongs to $G_m(J\times G)$ for all $v \in G_{m,\mathrm{iv}}(J\times G)$. We further note that in analogy to (5.41) we have

$$
\begin{aligned}
&S_{\mathbb{R}^3,m,p}(0, A_0^0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D^0, f^0(\tilde{f}, \tilde{v}), u_0^0) \\
&\quad = S_{\mathbb{R}^3,m,p}(0, A_0^0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D^0, f^0(f, v), u_0^0) \qquad (5.47)
\end{aligned}
$$

for all $\tilde{f} \in H^m_{\mathrm{iv},f}(J\times G)$, $\tilde{v} \in G_{m,\mathrm{iv}}(J\times G)$, and $p \in \{0, \dots, m\}$. As $f^*(v) \in H^m_{\mathrm{iv},f}(J\times G)$, we now combine the formulas (5.41) and (5.47) with (5.25) and (5.29), as well as $\sigma_i = 1$ on $\operatorname{supp}\theta_i$ for all $i \in \mathbb{N}_0$, and compute

$$
\begin{aligned}
\partial_t^p\mathcal{S}(v)(0) &= \sum_{i=0}^\infty \sigma_i\Phi_i^{-1}\partial_t^p\mathcal{U}^i(f^*(v), v)(0) \\
&= \sigma_0 S_{\mathbb{R}^3,m,p}(0, A_0^0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D^0, f^0(f^*(v), v), u_0^0) \\
&\quad + \sum_{i=1}^\infty \sigma_i\Phi_i^{-1} S_{\mathbb{R}^3_+,m,p}(0, A_0^i, \dots, A_3^i, D^i, f^i(f^*(v), v), u_0^i) \\
&= \sigma_0\theta_0 S_{m,p} + \sum_{i=1}^\infty \sigma_i\Phi_i^{-1}\Phi_i(\theta_i S_{m,p}) = \sum_{i=0}^\infty \theta_i S_{m,p} = S_{m,p}
\end{aligned}
$$

for all $p \in \{0, \dots, m\}$ and $v \in G_{m,\mathrm{iv}}(J \times G)$. As a consequence, $\mathcal{S}(v)$ is an element of $G_{m,\mathrm{iv}}(J \times G)$ for all $v \in G_{m,\mathrm{iv}}(J \times G)$ as asserted.

We want to show that $\mathcal{S}$ has a fixed point in $G_{m,\mathrm{iv}}(J \times G)$. Observe that it only remains to prove that $\mathcal{S}$ is a strict contraction in order to apply Banach's fixed point theorem. So let $v_1, v_2 \in G_{m,\mathrm{iv}}(J \times G)$. We first note that

$$\partial_t^j f^*(v_1)(0) = \partial_t^j f(0) = \partial_t^j f^*(v_2)(0) \qquad (5.48)$$

for all $j \in \{0, \dots, m-1\}$ as both $f^*(v_1)$ and $f^*(v_2)$ belong to $H^m_{\mathrm{iv},f}(J \times G)$. Exploiting that $f^*(v_k)$ is a fixed point of Ψ_{v_k} for $k = 1, 2$, and estimate (5.46), we further derive

$$\begin{aligned}
&\|f^*(v_1) - f^*(v_2)\|_{H^m_\gamma(J\times G)} = \|\Psi_{v_1}(f^*(v_1)) - \Psi_{v_2}(f^*(v_2))\|_{H^m_\gamma(J\times G)} \\
&\le \|\Psi_{v_1}(f^*(v_1)) - \Psi_{v_1}(f^*(v_2))\|_{H^m_\gamma(J\times G)} + \|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|_{H^m_\gamma(J\times G)} \\
&\le \frac{1}{2}\|f^*(v_1) - f^*(v_2)\|_{H^m_\gamma(J\times G)} + \|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|_{H^m_\gamma(J\times G)} \qquad (5.49)
\end{aligned}$$

for all $\gamma \ge \gamma^*$. The definition of the operator Ψ_{v_2}, Theorems 5.3 and 4.13, and formulas (5.23) and (5.39) yield

$$\begin{aligned}
&\|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|^2_{H^m_\gamma(J\times G)} \\
&\le C(N, M_3) \sum_{i=0}^{\infty} \|\Phi_i^{-1}\mathcal{U}^i(f^*(v_2), v_1) - \Phi_i^{-1}\mathcal{U}^i(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times G_i)} \\
&\le C(N, M_1, M_3)\|\mathcal{U}^0(f^*(v_2), v_1) - \mathcal{U}^0(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times\mathbb{R}^3)} \\
&\quad + C(N, M_1, M_3) \sum_{i=1}^{\infty} \|\mathcal{U}^i(f^*(v_2), v_1) - \mathcal{U}^i(f^*(v_2), v_2)\|^2_{H^m_\gamma(\Omega)} \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma} \sum_{i=0}^{\infty} \Big\| \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \theta_i (v_1 - v_2)\Big\|^2_{H^m_\gamma(J\times G_i)} \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma}\|v_1 - v_2\|^2_{H^m_\gamma(J\times G)} \qquad (5.50)
\end{aligned}$$

for all $\gamma \ge \gamma^*$. We set $\gamma^{**} = \max\{\gamma^*, 8C_{5.50}\}$ and insert (5.50) into (5.49), where $C_{5.50}$ denotes the constant on the right-hand side of (5.50). We then arrive at

$$\|f^*(v_1) - f^*(v_2)\|_{H^m(J\times G)} \le \frac{1}{2}\|v_1 - v_2\|_{H^m(J\times G)} \qquad (5.51)$$

for all $\gamma \ge \gamma^{**}$.

After these preparations, we can now estimate the difference of $\mathcal{S}(v_1)$ and $\mathcal{S}(v_2)$. Applying the a priori estimates from Theorem 5.3 respectively Theorem 4.13 once more and recalling that v_1 and v_2 belong to $G_{m,\mathrm{iv}}(J \times G)$, we infer

$$\begin{aligned}
&\|\mathcal{S}(v_1) - \mathcal{S}(v_2)\|^2_{G_{m,\gamma}(J\times G)} \\
&\le C(N, M_1, M_3)\|\mathcal{U}^0(f^*(v_1), v_1) - \mathcal{U}^0(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times\mathbb{R}^3)} \\
&\quad + C(N, M_1, M_3) \sum_{i=1}^{\infty} \|\mathcal{U}^i(f^*(v_1), v_1) - \mathcal{U}^i(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times\mathbb{R}^3_+)} \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma}\Big(\|f^0(f^*(v_1), v_1) - f^0(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times\mathbb{R}^3)} \\
&\qquad + \sum_{i=1}^{\infty} \|f^i(f^*(v_1), v_1) - f^i(f^*(v_2), v_2)\|^2_{H^m_\gamma(J\times\mathbb{R}^3_+)}\Big) \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma} \sum_{i=0}^{\infty} \Big(\|\theta_i(f^*(v_1) - f^*(v_2))\|^2_{H^m_\gamma(J\times G_i)} \\
&\qquad + \Big\| \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \theta_i (v_1 - v_2)\Big\|^2_{H^m_\gamma(J\times G_i)}\Big) \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma}\Big(\|f^*(v_1) - f^*(v_2)\|^2_{H^m_\gamma(J\times G)} + \|v_1 - v_2\|^2_{H^m_\gamma(J\times G)}\Big) \\
&\le C(m, \eta, \tau, N, M, r, T')\frac{1}{\gamma} \cdot \frac{3}{2}\|v_1 - v_2\|^2_{G_{m,\gamma}(J\times G)} \qquad (5.52)
\end{aligned}$$

for all $\gamma \geq \gamma^{**}$, where we used again (5.23), (5.39), and (5.48). We finally set $\gamma_{\mathcal{S}} = \max\{\gamma^{**}, 6C_{5.52}\}$, for the constant $C_{5.52}$ on the right-hand side of (5.52). It follows

$$\|\mathcal{S}(v_1) - \mathcal{S}(v_2)\|_{G_{m,\gamma}(J\times G)} \leq \frac{1}{2}\|v_1 - v_2\|_{G_{m,\gamma}(J\times G)}$$

for all $\gamma \geq \gamma_{\mathcal{S}}$. Banach's fixed point theorem thus yields a unique fixed point u of $\mathcal{S}$ in $G_{m,\mathrm{iv}}(J \times G)$.

VI) We claim that the fixed point u of $\mathcal{S}$ is a solution of (5.1). To verify this assertion, we first compute

$$\begin{aligned} Lu &:= A_0\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u + Du = A_0\partial_t\mathcal{S}(u) + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\mathcal{S}(u) + D\mathcal{S}(u) \\ &= \sum_{i=0}^{\infty}\sigma_i\Big(A_0\partial_t\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) + D\Phi_i^{-1}\mathcal{U}^i(f^*(u),u)\Big) \\ &\quad + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) \end{aligned}$$

on $J \times G$. Recall from (5.10) that

$$\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\Phi_i^{-1}v = \sum_{j=1}^{3}\Phi_i^{-1}(\tilde{A}_j^i\partial_j v) = \Phi_i^{-1}\Big(\sum_{j=1}^{3} A_j^i\partial_j v\Big)$$

on $\operatorname{supp}\sigma_i$ for all $v \in L^2(V_i)$. Since also $A_0 = \Phi_i^{-1}A_0^i$ and $D = \Phi_i^{-1}D^i$ on $\operatorname{supp}\sigma_i$ for all $i \in \mathbb{N}_0$, the very definition of the functions $\mathcal{U}^i(f^*(u),u)$ and (5.23) imply the equality

$$\begin{aligned} Lu &= \sum_{i=0}^{\infty}\sigma_i\Phi_i^{-1}\Big(A_0^i\partial_t\mathcal{U}^i(f^*(u),u) + \sum_{j=1}^{3} A_j^i\partial_j\mathcal{U}^i(f^*(u),u) + D^i\mathcal{U}^i(f^*(u),u)\Big) \\ &\quad + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) \\ &= \sum_{i=0}^{\infty}\sigma_i\Phi_i^{-1}f^i(f^*(u),u) + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) \\ &= \sum_{i=0}^{\infty}\sigma_i\theta_i f^*(u) + \sum_{i=0}^{\infty}\sum_{j=1}^{3}\sigma_i A_j^{\mathrm{co}}\partial_j\theta_i u + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u). \end{aligned}$$

Employing that σ_i equals 1 on the support of θ_i, that θ_i is a partition of unity, and the defining property of $f^*(u)$, i.e. (5.44), we deduce

$$\begin{aligned} Lu &= \sum_{i=0}^{\infty}\theta_i f^*(u) + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\theta_i u + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) \\ &= f^*(u) + \sum_{i=0}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u) = f. \end{aligned}$$

Since the covering $(U_i)_{i\in\mathbb{N}_0}$ is locally finite and the trace operator commutes with C_c^∞-functions, we can compute

$$\operatorname{Tr}_{J\times\partial G}(Bu) = \operatorname{Tr}_{J\times\partial G}(B\mathcal{S}(u)) = \operatorname{Tr}_{J\times\partial G}\Big(B\sum_{i=1}^{\infty}\sigma_i\Phi_i^{-1}\mathcal{U}^i(f^*(u),u)\Big)$$

$$= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G} \sigma_i \operatorname{Tr}_{J\times \partial G}(B \Phi_i^{-1} \mathcal{U}^i(f^*(u),u))$$
$$= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i) \kappa_i^{-1} \operatorname{Tr}_{J\times \partial G} \Big(\Phi_i^{-1} \big(\omega_i \Phi_i(\kappa_i B) \mathcal{U}^i(f^*(u),u) \big) \Big),$$

where we used that $\Phi_i^{-1}\omega_i = 1$ on $\operatorname{supp} \sigma_i$. The identity $\hat{B}^i = \omega_i \Phi_i(\kappa_i B)$ on $\operatorname{supp} \sigma_i$ and the definition of $\tilde{B}^i$, cf. (5.16), then yield

$$\begin{aligned} \operatorname{Tr}_{J\times\partial G}(Bu) &= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i)\kappa_i^{-1} \operatorname{Tr}_{J\times \partial G} \Big(\Phi_i^{-1} \big(\hat{B}^i \mathcal{U}^i(f^*(u),u) \big) \Big) \\ &= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i)\kappa_i^{-1} \tilde{\Phi}_i^{-1} \operatorname{Tr}_{J\times\mathbb{R}^3_+}((R^i)^{-1} \tilde{B}^i \mathcal{U}^i(f^*(u),u)). \end{aligned}$$

Since $\mathcal{U}^i(f^*(u),u)$ solves the initial boundary value problem (5.42) with the boundary value g^i defined in (5.23) for every $i \in \mathbb{N}$, we arrive at

$$\begin{aligned} \operatorname{Tr}_{J\times\partial G}(Bu) &= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i)\kappa_i^{-1} \tilde{\Phi}_i^{-1} \Big(\operatorname{tr}_{\mathbb{R}^3_+}((R^i)^{-1})\, g^i_{z(i)\to 0} \Big) \\ &= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i)\kappa_i^{-1} \tilde{\Phi}_i^{-1} \Big(\operatorname{tr}_{\mathbb{R}^3_+}((R^i)^{-1})\ \operatorname{tr}_{\mathbb{R}^3_+}(R^i) \tilde{\Phi}_i(\operatorname{tr}_{\partial G}(\theta_i)\kappa_i g) \Big) \\ &= \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\sigma_i \theta_i) g = \sum_{i=1}^{\infty} \operatorname{tr}_{\partial G}(\theta_i) g = g, \end{aligned}$$

where $g^i_{z(i)\to 0}$ denotes the vector we get by adding a zero in the $z(i)$-th component of g^i. Finally, we have

$$\begin{aligned} u(0) = \mathcal{S}(u)(0) &= \sum_{i=0}^{\infty} \sigma_i \Phi_i^{-1} \mathcal{U}^i(f^*(u),u)(0) = \sum_{i=0}^{\infty} \sigma_i \Phi_i^{-1} u_0^i = \sum_{i=0}^{\infty} \sigma_i \Phi_i^{-1} \Phi_i(\theta_i u_0) \\ &= \sum_{i=0}^{\infty} \sigma_i \theta_i u_0 = \sum_{i=0}^{\infty} \theta_i u_0 = u_0, \end{aligned}$$

where we employed that $\mathcal{U}^0(f^*(u),u)$ solves (5.43) with initial value u_0^0 and that $\mathcal{U}^i(f^*(u),u)$ solves (5.42) with initial value u_0^i for all $i \in \mathbb{N}$. We conclude that u is a solution of (5.1) in $G_m(J \times G)$. □

The above theorem provides a satisfying wellposedness theory for the linear initial boundary value problem (5.1). It also allows us to prove the uniqueness and local existence of solutions of the nonlinear Maxwell system (1.6) in Section 7.2. However, the derivation of more sophisticated properties of solutions, both in the linear and the nonlinear case, often require to return to the half-space, see Theorem 6.1, Proposition 7.20, and Lemma 7.22. We thus summarize the definition of the localized coefficients and the localized data for later reference.

Definition 5.7. *Let $T > 0$, $\eta > 0$, $m \in \mathbb{N}$, and set $\tilde{m} = \max\{m, 3\}$ and $J = (0,T)$. Take a domain $G \subset \mathbb{R}^3$ which has a tame uniform $C^{\tilde{m}+2}$-boundary. Choose coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},6,\eta}(J\times G)$, $D \in F^{\mathrm{c}}_{\tilde{m},6}(J \times G)$, and B as defined in (5.7). Choose functions $v \in G_m(J \times G)$, $h \in H^m(J\times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$.*

Fix a covering $(U_i)_{i\in\mathbb{N}_0}$, a sequence of sets $(V_i)_{i\in\mathbb{N}_0}$, and sequences of functions $(\varphi_i)_{i\in\mathbb{N}_0}$, $(\theta_i)_{i\in\mathbb{N}_0}$, $(\sigma_i)_{i\in\mathbb{N}_0}$, and $(\omega_i)_{i\in\mathbb{N}_0}$ as in Definition 5.4 for the tame uniform $C^{\tilde{m}+2}$-boundary of G. We set $\Phi_i \colon L^2(U_i) \to L^2(V_i), w \mapsto w \circ \varphi_i^{-1}$ for all $i \in \mathbb{N}_0$.

We then define the localized coefficients

$$A_0^i = A_0^i(A_0, \eta), \quad A_j^i, \quad D^i = D^i(D), \quad B^i$$

as in (5.12) *to* (5.14) *and* (5.16) *and the localized data*

$$f^i = f^i(h, v), \quad g^i(g), \quad u_0^i(u_0)$$

as in (5.23) *for all* $i \in \mathbb{N}_0$ *respectively* $i \in \mathbb{N}$.

Step IV) of the proof of Theorem 5.6 then immediately yields the following corollary.

Corollary 5.8. *In the framework of Definition 5.7, assume that a function* $u \in G_m(J \times G)$ *solves the linear initial boundary value problem* (5.1) *with inhomogeneity* $f \in H^m(J \times G)$*, boundary value* $g \in E_m(J \times \partial G)$*, and initial value* $u_0 \in H^m(G)$*. Then the function* $\Phi_i(\theta_i u) := (\theta_i u) \circ \varphi_i^{-1}$ *solves the linear initial boundary value problem*

$$\begin{cases} A_0^i \partial_t v + \sum_{j=1}^{3} A_j^i \partial_j v + D^i v = f^i(f, u), & x \in \mathbb{R}^3_+, \quad t \in J; \\ B^i v = g^i, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = u_0^i, & x \in \mathbb{R}^3_+; \end{cases}$$

for all $i \in \mathbb{N}$ *and the function* $\theta_0 u$ *solves the linear initial value problem*

$$\begin{cases} A_0^0 \partial_t v + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j v + D^0 v = f^0(f, u), & x \in \mathbb{R}^3, \quad t \in J; \\ v(0) = u_0^0, & x \in \mathbb{R}^3. \end{cases}$$

6

Finite propagation speed

One of the unifying features of hyperbolic partial differential equations is the finite propagation speed. This means that initial disturbances travel with finite speed, see Theorem 6.1 and Corollary 6.2 for the precise statements. There are several ideas to prove the finite propagation speed property. In [Eva10] the rate of change of local energies is used, whereas [BCD11] relies on weighted energy estimates. In [BGS07] so called characteristic cones are exploited. While the aforementioned sources work on the full space, mixed problems are treated in [CP82] assuming the uniform Lopatinski condition.

We will follow the approach of [BCD11]. As it turns out, the technique of weighted energy estimates is quite flexible and well adaptable to our setting. We provide two equivalent versions of the finite propagation speed property, one formulated for the backward light cone and the other for the forward one. We start with the backward version, which states that the solution is equal to zero on a backward light cone if the data vanish on it. We further express the upper bound for the propagation speed in terms of the coefficients.

As announced, the proof relies on a weighted energy estimate with a parameter dependent weight that blows up on the backward light cone as the parameter tends to infinity. Since the data vanish we can bound the weighted solution independent of the parameter which implies that the solution has to be zero on the cone.

Theorem 6.1. *Let $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, and G be a tame uniform $C^{\tilde{m}+2}$-domain. Pick $T > 0$ and set $J = (0, T)$. Take a parameter η and coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},\eta}(J \times G)$, $D \in F^{\mathrm{c}}_{\tilde{m}}(J \times G)$, and*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Fix a covering $(U_i)_{i \in \mathbb{N}_0}$ of $\overline{G}$ and corresponding diffeomorphisms $(\varphi_i)_{i \in \mathbb{N}_0}$ as in Definition 5.4. Let M be a bound for the derivatives of the functions $(\varphi_i)_{i \in \mathbb{N}_0}$ as in Lemma 5.1 (cf. Definition 5.4). Set

$$C_0 = \frac{9M^2}{\eta}.$$

Take $R > 0$ and $x_0 \in \overline{G}$. We define the backward cone $\mathcal{C}$ by

$$\mathcal{C} = \{(t, x) \in \overline{J} \times \mathbb{R}^3 \colon |x - x_0| < R - C_0\, t\}.$$

Let $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ satisfy

$$\begin{aligned} f &= 0 && \text{on } \mathcal{C} \cap (J \times G), \\ g &= 0 && \text{on } \mathcal{C} \cap (J \times \partial G), \\ u_0 &= 0 && \text{on } \mathcal{C}_{t=0} \cap G, \end{aligned}$$

where $\mathcal{C}_{t=0}$ is defined by $\mathcal{C}_{t=0} = \{x \in \mathbb{R}^3 \colon (0,x) \in \mathcal{C}\}$. Suppose that the tuple $(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, B, f, g, u_0)$ fulfills the linear compatibility conditions (2.37) *of order m. Then the unique solution $u \in G_m(J \times G)$ of the linear initial boundary value problem* (5.1) *with inhomogeneity f, boundary value g, and initial value u_0 vanishes on the cone $\mathcal{C}$, i.e.,*

$$u(t,x) = 0 \quad \textit{for almost all } (t,x) \in \mathcal{C} \cap (J \times G).$$

Proof. I) Let $\varepsilon > 0$ and set $K = C_0^{-1}$. In the main part of the proof we will need a function $\psi \in C^\infty(\mathbb{R}^3)$ with

$$-2\varepsilon + K(R - |x - x_0|) \leq \psi(x) \leq -\varepsilon + K(R - |x - x_0|) \quad \text{for all } x \in \mathbb{R}^3, \tag{6.1}$$

$$\|\nabla \psi\|_{L^\infty(\mathbb{R}^3)} \leq K. \tag{6.2}$$

For the sake of completeness, we show in this first step that such a function exists. To that purpose, we set

$$\varphi(\tau) = -\frac{3}{2}\varepsilon + K(R - |\tau|)$$

for all $\tau \in \mathbb{R}$. Let ρ be the kernel of a standard mollifier over $\mathbb{R}$. Since φ is globally Lipschitz continuous with Lipschitz constant K, we obtain

$$\begin{aligned} |\rho_\delta * \varphi(\tau) - \varphi(\tau)| &\leq \int_{\mathbb{R}} \rho_\delta(\sigma) |\varphi(\tau - \sigma) - \varphi(\tau)| d\sigma \leq K \int_{\mathbb{R}} \rho_\delta(\sigma) |\sigma| d\sigma \\ &\leq K\,\delta \int_{\mathbb{R}} \delta^{-1} \rho\Big(\frac{\sigma}{\delta}\Big) \Big|\frac{\sigma}{\delta}\Big| d\sigma = K\,\delta \int_{\mathbb{R}} \rho(\sigma)|\sigma| d\sigma \longrightarrow 0 \end{aligned}$$

uniformly in τ as $\delta \to 0$. Exploiting that φ is weakly differentiable with weak derivative $\partial_\tau \varphi(\tau) = -K \operatorname{sgn}(\tau)$, we further deduce that

$$|(\rho_\delta * \varphi)'(\tau)| = \Big| \int_{\mathbb{R}} \rho_\delta(\sigma) \partial_\tau \varphi(\tau - \sigma) d\sigma \Big| \leq K \int_{\mathbb{R}} \rho_\delta(\sigma) d\sigma = K$$

for all $\tau \in \mathbb{R}$. Choosing $\delta > 0$ small enough, we thus obtain a function $\tilde{\varphi} = \rho_\delta * \varphi \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} &-\frac{5}{3}\varepsilon + K(R - |\tau|) \leq \tilde{\varphi}(\tau) \leq -\varepsilon + K(R - |\tau|) \quad \text{for all } \tau \in \mathbb{R}, \\ &\|\tilde{\varphi}'\|_{L^\infty(\mathbb{R})} \leq K. \end{aligned}$$

Next take $\tilde{\delta} \in (0, (3K)^{-1}\varepsilon)$. We set

$$\psi(x) = \tilde{\varphi}\Big(\sqrt{\tilde{\delta}^2 + |x - x_0|^2}\Big)$$

for all $x \in \mathbb{R}^3$. We then obtain that ψ belongs to $C^\infty(\mathbb{R}^3)$ and satisfies the inequalities

$$\begin{aligned} &-2\varepsilon + K(R - |x - x_0|) \leq -\frac{5}{3}\varepsilon + K\Big(R - \sqrt{\tilde{\delta}^2 + |x - x_0|^2}\Big) \leq \psi(x), \\ &\psi(x) \leq -\varepsilon + K\Big(R - \sqrt{\tilde{\delta}^2 + |x - x_0|^2}\Big) \leq -\varepsilon + K(R - |x - x_0|) \end{aligned}$$

for all $x \in \mathbb{R}^3$ and

$$\begin{aligned} &\partial_j \psi(x) = \tilde{\varphi}'\Big(\sqrt{\tilde{\delta}^2 + |x - x_0|^2}\Big) \frac{x_j - x_{0;j}}{\sqrt{\tilde{\delta}^2 + |x - x_0|^2}}, \quad \text{for all } x \in \mathbb{R}^3, j \in \{1,2,3\}, \\ &\|\nabla \psi\|_{L^\infty(\mathbb{R}^3)} \leq K. \end{aligned}$$

Consequently, ψ has all the claimed properties.

II) Fix $\varepsilon > 0$. Take the function $\psi = \psi_\varepsilon$ from step I) and set

$$\Psi(t,x) = -t + \psi(x)$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^3$. Note that Ψ belongs to $C^\infty(\overline{J} \times \mathbb{R}^3)$.

We next want to derive a weighted energy inequality for u. To that purpose, we have to return to the half-space again. Since G has a uniform $C^{\tilde{m}+2}$ domain there are charts $(U_i)_{i\in\mathbb{N}_0}$ which form an open cover of $\overline{G}$, corresponding diffeomorphisms $(\varphi_i)_{i\in\mathbb{N}_0}$, a partition of unity $(\theta_i)_{i\in\mathbb{N}_0}$ subordinate to $(U_i)_{i\in\mathbb{N}_0}$, and a family of functions $(\omega_i)_{i\in\mathbb{N}_0}$ as in Definition 5.7. Take the operators Φ_i and the localized coefficients and data

$$\begin{aligned} A_0^i = A_0^i(A_0,\eta), \quad A_j^i, \quad D^i = D^i(D), \quad B^i, \\ f^i = f^i(f,u), \quad g^i = g^i(g), \quad u_0^i = u_0^i(u_0) \end{aligned}$$

from Definition 5.7 for all $i \in \mathbb{N}$ respectively $i \in \mathbb{N}_0$. Corollary 5.8 then shows that the localized functions $u^i := \Phi_i(\theta_i u)$ solve the linear initial boundary value problem

$$\begin{cases} A_0^i \partial_t u^i + \sum_{j=1}^3 A_j^i \partial_j u^i + D^i u^i = f^i, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B^i u^i = g^i, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ u^i(0) = u_0^i, & x \in \mathbb{R}^3_+; \end{cases} \tag{6.3}$$

for all $i \in \mathbb{N}$ and the linear intial value problem

$$\begin{cases} A_0^i \partial_t u^i + \sum_{j=1}^3 A_j^i \partial_j u^i + D^i u^i = f^i, & x \in \mathbb{R}^3, \quad t \in J; \\ u^i(0) = u_0^i, & x \in \mathbb{R}^3; \end{cases} \tag{6.4}$$

in the case $i = 0$, where we set $A_j^0 = A_j^{\mathrm{co}}$ for $j \in \{1,2,3\}$. We next define

$$\psi_i = \omega_i \cdot \Phi_i \psi$$

for all $i \in \mathbb{N}_0$. These functions belong to $C^{\tilde{m}+2}(\mathbb{R}^3)$ for all $i \in \mathbb{N}_0$, where we identify a compactly supported function with its zero extension. Set

$$\Psi_i(t,x) = -t + \psi_i(x)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and $i \in \mathbb{N}_0$. Observe that the functions Ψ_i belong to the space $C^{\tilde{m}+2}(\overline{J \times \mathbb{R}^3_+})$ for all $i \in \mathbb{N}$ and Ψ_0 is contained in $C^{\tilde{m}+2}(J \times \mathbb{R}^3)$. We introduce the functions

$$u_\tau^i = e^{\tau\Psi_i} u^i, \quad f_\tau^i = e^{\tau\Psi_i} f^i, \quad g_\tau^i = e^{\tau \operatorname{tr} \Psi_i} g^i, \quad u_{0,\tau}^i = e^{\tau\Psi_i(0,\cdot)} u_0^i$$

for each $\tau > 0$ and $i \in \mathbb{N}$ respectively $i \in \mathbb{N}_0$. We further note that

$$\begin{aligned} |\partial_j \psi_i(x)| &\leq KM \\ |\partial_j \partial_k \psi_i(x)| &\leq C(K,M,\varepsilon) \end{aligned} \tag{6.5}$$

for all $x \in \operatorname{supp} \Phi_i \theta_i$ and $j,k \in \{0,\ldots,3\}$. We drop the dependancy on K and M in the following as they remain fixed throughout the proof. Consequently, there is a constant $C = C(\tau,\varepsilon)$ such that

$$e^{\tau\Psi_i(t,x)} + \sum_{j=0}^3 |\partial_j e^{\tau\Psi(t,x)}| + \sum_{j,k=0}^3 |\partial_j\partial_k e^{\tau\Psi(t,x)}| \leq C$$

for all $(t,x) \in \overline{J} \times \operatorname{supp} \Phi_i\theta_i$ and $\tau > 0$. We thus infer that u_τ^i belongs to $G_1(\Omega)$, f_τ^i to $H^1(\Omega)$, g_τ^i to $E_1(J \times \partial\mathbb{R}^3_+)$, and $u_{0,\tau}^i$ to $H^1(\mathbb{R}^3_+)$ for all $\tau > 0$.

With this amount of regularity we can compute

$$A_0^i \partial_t u_\tau^i = A_0^i e^{\tau \Psi_i} \partial_t u^i - \tau A_0^i e^{\tau \Psi_i} u^i = e^{\tau \Psi_i}\Big(f^i - \sum_{j=1}^3 A_j^i \partial_j u^i - D^i u^i \Big) - \tau A_0^i u_\tau^i$$

$$= f_\tau^i - \sum_{j=1}^3 A_j^i \partial_j u_\tau^i + \tau \sum_{j=1}^3 \partial_j \psi_i e^{\tau \Psi_i} A_j^i u^i - D^i u_\tau^i - \tau A_0^i u_\tau^i,$$

$$A_0^i \partial_t u_\tau^i + \sum_{j=1}^3 A_j^i \partial_j u_\tau^i + D^i u_\tau^i = f_\tau^i - \tau\Big(A_0^i - \sum_{j=1}^3 \partial_j \psi A_j^i \Big) u_\tau^i \tag{6.6}$$

for all $\tau > 0$ and $i \in \mathbb{N}_0$. Moreover, we have

$$u_\tau^i(0) = e^{\tau \Psi_i(0,\cdot)} u^i(0) = e^{\tau \Psi_i(0,\cdot)} u_0^i = u_{0,\tau}^i$$

for all $i \in \mathbb{N}_0$ and

$$\operatorname{tr} B^i u_\tau^i = \operatorname{tr}(e^{\tau \Psi_i} B^i u^i) = e^{\tau \operatorname{tr} \Psi_i} \operatorname{Tr}(B^i u^i) = e^{\tau \operatorname{tr} \Psi_i} g^i = g_\tau^i \tag{6.7}$$

by Corollary 2.18 (iv) for all $\tau > 0$ and $i \in \mathbb{N}$.

Next fix $i \in \mathbb{N}$. We note that $(A_j^i \xi, \xi)_{\mathbb{R}^6 \times \mathbb{R}^6} \leq 3M|\xi|^2$ for all $j \in \{1,2,3\}$ as the spectral norm of the matrices A_j^{co} equals 1 by Remark 3.6 and the fact that A_j^{co} is symmetric for all $j \in \{1,2,3\}$. We thus deduce that $A_0^i - \sum_{j=1}^3 \partial_j \psi_i A_j^i$ is positive semidefinite on $\overline{J} \times \operatorname{supp} \Phi_i \theta_i$ since

$$\Big(\big(A_0^i - \sum_{j=1}^3 \partial_j \psi_i A_j^i\big)\xi, \xi \Big)_{\mathbb{R}^6 \times \mathbb{R}^6} \geq \eta |\xi|^2 - \sum_{j=1}^3 \|\partial_j \psi_i\|_{L^\infty(\mathbb{R}^3)} \|A_j^i\|_{L^\infty(\Omega)} |\xi|^2$$

$$\geq \eta |\xi|^2 - KM|\xi|^2 \sum_{j=1}^3 \|A_j^i\|_{L^\infty(\Omega)} \geq \eta |\xi|^2 - 9KM^2 |\xi|^2 = \eta |\xi|^2 - \eta K C_0 |\xi|^2 = 0,$$

on $\overline{J} \times \operatorname{supp} \Phi_i \theta_i$ for $\xi \in \mathbb{R}^6$. Here we used (6.5), the definition of C_0, and that $K = C_0^{-1}$. Identity (6.6) in combination with this estimate then yields

$$\begin{aligned}
&\partial_t \langle A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&= \langle \partial_t A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} + 2 \langle A_0^i \partial_t u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&= \langle \partial_t A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} + 2 \Big\langle f_\tau^i - \sum_{j=1}^3 A_j^i \partial_j u_\tau^i - D^i u_\tau^i, u_\tau^i \Big\rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&\quad - 2\tau \Big\langle \Big(A_0^i - \sum_{j=1}^3 \partial_j \psi_i A_j^i \Big) u_\tau^i, u_\tau^i \Big\rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&\leq \langle \partial_t A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} - 2 \sum_{j=1}^3 \langle A_j^i \partial_j u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&\quad - 2 \langle D^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} + 2 \langle f_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)}
\end{aligned}$$

for almost all $t \in J$ and for all $\tau > 0$. Hence,

$$\begin{aligned}
&\eta \|u_\tau^i(t)\|^2_{L^2(\mathbb{R}^3_+)} \leq \langle A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} \\
&= \langle A_0^i(0) u_\tau^i(0), u_\tau^i(0) \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)} + \int_0^t \partial_t \langle A_0^i u_\tau^i, u_\tau^i \rangle_{L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)}(s) ds \\
&\leq \|A_0^i\|_{L^\infty(\Omega)} \|u_{0,\tau}^i\|^2_{L^2(\mathbb{R}^3_+)} + (\|\partial_t A_0^i\|_{L^\infty(\Omega)} + 2\|D^i\|_{L^\infty(\Omega)}) \int_0^t \|u_\tau^i(s)\|^2_{L^2(\mathbb{R}^3_+)} ds
\end{aligned}$$

$$- 2\sum_{j=1}^{3}\int_0^t \langle A_j^i(s)\partial_j u_\tau^i(s), u_\tau^i(s)\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} ds$$
$$+ 2\int_0^t \|f_\tau^i(s)\|_{L^2(\mathbb{R}^3_+)}\|u_\tau^i(s)\|_{L^2(\mathbb{R}^3_+)} ds \tag{6.8}$$

for all $t \in \overline{J}$ and $\tau > 0$. Since $u_\tau^i \in G_1(\Omega)$, the symmetry of the matrices A_j^i and integration by parts further imply

$$\begin{aligned}
&\langle A_j^i\partial_j u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&= \langle \partial_j(A_j^i u_\tau^i), u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} - \langle \partial_j A_j^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&= -\langle A_j^i\partial_j u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} - \langle \partial_j A_j^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}, \\
&\langle A_j^i\partial_j u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} = -\frac{1}{2}\langle \partial_j A_j^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}
\end{aligned} \tag{6.9}$$

on J for $j \in \{1,2\}$ and

$$\begin{aligned}
&\langle A_3^i\partial_3 u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&= \langle \partial_3(A_3^i u_\tau^i), u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} - \langle \partial_3 A_3^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&= -\langle A_3^i\partial_3 u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} + \int_{\partial\mathbb{R}^3_+} \operatorname{tr}(A_3^i u_\tau^i)(\sigma)\operatorname{tr}(u_\tau^i)(\sigma)d\sigma \\
&\quad - \langle \partial_3 A_3^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)}, \\
&\langle A_3^i\partial_3 u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} = -\frac{1}{2}\langle \partial_3 A_3^i u_\tau^i, u_\tau^i\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} \\
&\qquad + \frac{1}{2}\int_{\partial\mathbb{R}^3_+} \operatorname{tr}(A_3^i u_\tau^i)(\sigma)\operatorname{tr}(u_\tau^i)(\sigma)d\sigma
\end{aligned} \tag{6.10}$$

on J for all $\tau > 0$. We take a constant C_1 independent of i such that

$$\frac{1}{\eta}\Big(\sum_{j=0}^{3}\|A_j^i\|_{W^{1,\infty}(\Omega)} + 2\|D^i\|_{L^\infty(\Omega)} + \frac{1}{\eta}\Big) \le C_1.$$

Inserting (6.9) and (6.10) into (6.8), we derive

$$\begin{aligned}
&\eta\|u_\tau^i(t)\|^2_{L^2(\mathbb{R}^3_+)} \\
&\le \eta\, C_1\|u_{0,\tau}^i\|^2_{L^2(\mathbb{R}^3_+)} + (\|\partial_t A_0^i\|_{L^\infty(\Omega)} + 2\|D^i\|_{L^\infty(\Omega)})\int_0^t \|u_\tau^i(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \\
&\quad + \sum_{j=1}^{3}\int_0^t \langle \partial_j A_j^i(s)u_\tau^i(s), u_\tau^i(s)\rangle_{L^2(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)} ds \\
&\quad + \eta\|f_\tau^i\|^2_{L^2(\Omega)} + \frac{1}{\eta}\int_0^t \|u_\tau^i(s)\|^2_{L^2(\mathbb{R}^3_+)} ds - \langle \operatorname{tr}(A_3^i u_\tau^i), \operatorname{tr} u_\tau^i\rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \\
&\le \eta\, C_1\|u_{0,\tau}^i\|^2_{L^2(\mathbb{R}^3_+)} + \eta\|f_\tau^i\|^2_{L^2(\Omega)} + \eta\, C_1\int_0^t \|u_\tau^i(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \\
&\quad - \langle \operatorname{tr}(A_3^i u_\tau^i), \operatorname{tr} u_\tau^i\rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)}
\end{aligned} \tag{6.11}$$

for all $t \in J$ and $\tau > 0$, where we denote $(0,t)\times\partial\mathbb{R}^3_+$ by Γ_t. In order to estimate the last term in (6.11), we recall that the boundary matrix A_3 decomposes as

$$A_3^i = \operatorname{Re}((C_{A_3}^i)^T B^i) = \frac{1}{2}(C_{A_3}^i)^T B^i + \frac{1}{2}(B^i)^T(C_{A_3}^i),$$

where $C_{A_3}^i$ is an element of $(W^{m+1,\infty}(\partial\mathbb{R}^3_+))^{2\times 6}$ which has a limit as $|(t,x)| \to \infty$ as B^i belongs to $\mathcal{BC}^m_{\mathbb{R}^3_+}(A_3)$. Employing (6.7), Corollary 2.18 (iv), and that u^i belongs to

$G_1(\Omega)$, we thus infer

$$\begin{aligned}
&\langle \operatorname{tr}(A_3^i u_\tau^i), \operatorname{tr} u_\tau^i \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} = \langle C^i_{A_3} \operatorname{tr} u_\tau^i, B^i \operatorname{tr} u_\tau^i \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \\
&= \langle C^i_{A_3} \operatorname{tr} u_\tau^i, g_\tau^i \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} = \langle e^{\tau \operatorname{tr}\Psi_i} C^i_{A_3} \operatorname{tr} u^i, e^{\tau \operatorname{tr}\Psi_i} g^i \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \\
&= \langle C^i_{A_3} \operatorname{tr} u^i, g_{2\tau}^i \rangle_{L^2(\Gamma_t)\times L^2(\Gamma_t)} \leq \|C^i_{A_3} \operatorname{tr} u^i\|_{L^2(\Gamma_t)} \|g^i_{2\tau}\|_{L^2(\Gamma_t)} \qquad (6.12)\\
&\leq C \|\operatorname{tr} u^i\|_{L^2(\Gamma_t)} \|g^i_{2\tau}\|_{L^2(\Gamma_t)} \leq C\|u^i\|_{H^1((0,t)\times\mathbb{R}^3_+)} \|g^i_{2\tau}\|_{L^2(\Gamma_t)} \leq C \|u^i\|_{H^1(\Omega)} \|g^i_{2\tau}\|_{L^2(\Gamma)}
\end{aligned}$$

for all $t \in J$ and $\tau > 0$, where Γ denotes $J \times \partial\mathbb{R}^3_+$ as usual. We point out that $\|u^i\|_{H^1(\Omega)}$ is finite by Lemma 3.11. Estimates (6.12) and (6.11) finally lead to

$$\begin{aligned}
\|u^i_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \leq C_1 \|u^i_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_\tau\|^2_{L^2(\Omega)} + \frac{C}{\eta} \|u^i\|_{H^1(\Omega)} \|g^i_{2\tau}\|_{L^2(\Gamma)} \\
+ C_1 \int_0^t \|u^i_\tau(s)\|^2_{L^2(\mathbb{R}^3_+)} ds
\end{aligned}$$

for all $t \in J$ and $\tau > 0$ so that Gronwall's lemma implies

$$\sup_{t\in J} \|u^i_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \leq \Big(C_1 \|u^i_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_\tau\|^2_{L^2(\Omega)} + \frac{C}{\eta} \|u^i\|_{H^1(\Omega)} \|g^i_{2\tau}\|_{L^2(\Gamma)}\Big) e^{C_1 T} \qquad (6.13)$$

for all $\tau > 0$. Using that $\|\Phi_i v\|_{H^1(V_i)}$ and $\|v\|_{H^1(U_i)}$ define equivalent norms (with equivalence constants independent of i), see Theorem 1.1.7 in [Maz11], and applying Young's inequality, we arrive at

$$\begin{aligned}
&\sup_{t\in J} \|u^i_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)} \leq \Big(C_1 \|u^i_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_\tau\|^2_{L^2(\Omega)} + \frac{C}{\eta} \|u^i\|_{H^1(J\times G)} \|g^i_{2\tau}\|_{L^2(\Gamma)}\Big) e^{C_1 T} \\
&\leq C(\eta, G)\Big(\|u^i_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_\tau\|^2_{L^2(\Omega)} + \frac{1}{\tau}\|\theta_i u\|^2_{H^1(J\times G)} + \tau \|g^i_{2\tau}\|^2_{L^2(\Gamma)}\Big) e^{C_1 T} \qquad (6.14)
\end{aligned}$$

for all $\tau > 0$. Analogously, but easier as we do not have to deal with the integral over the boundary, we obtain

$$\sup_{t\in J} \|u^0_\tau(t)\|^2_{L^2(\mathbb{R}^3)} \leq \Big(C_1 \|u^0_{0,\tau}\|^2_{L^2(\mathbb{R}^3)} + \|f^0_\tau\|^2_{L^2(J\times\mathbb{R}^3)}\Big) e^{C_1 T} \qquad (6.15)$$

for all $\tau > 0$.

We can now recompose estimates (6.14) and (6.15) to the desired weighted energy estimate on the domain G. We argue as in (5.38) to (5.40) in order to derive

$$\begin{aligned}
&\sup_{t\in J} \|e^{\tau\Psi(t)} u(t)\|^2_{L^2(G)} \leq C(G) \sum_{i=0}^\infty \|e^{\tau\Psi}\theta_i u\|^2_{G_0(J\times G)} \leq C(G) \sum_{i=0}^\infty \|u^i_\tau\|^2_{G_0(\Omega)} \\
&\leq C(\eta, G)\Big(\sum_{i=0}^\infty (\|u^i_{0,\tau}\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_\tau\|^2_{L^2(\Omega)} + \frac{1}{\tau}\|\theta_i u\|^2_{H^1(J\times G)}) + \sum_{i=1}^\infty \tau \|g^i_{2\tau}\|^2_{L^2(\Gamma)}\Big) e^{C_1 T} \\
&\leq C(\eta, G)\Big(\|e^{\tau\Psi(0,\cdot)} u_0\|^2_{L^2(G)} + \|e^{\tau\Psi} f\|^2_{L^2(J\times G)} + \frac{1}{\tau}\|u\|^2_{H^1(J\times G)} \\
&\qquad + \tau \|e^{2\tau \operatorname{tr}\Psi} g\|^2_{L^2(J\times\partial G)}\Big) \qquad (6.16)
\end{aligned}$$

for all $\tau > 0$.

III) In this step we show that the weighted energy estimate (6.16) leads to the convergence of $\sup_{t\in J}\|u_\tau(t)\|_{L^2(G)}$ to 0 as $\tau \to \infty$, which in turn implies that u has to vanish on the cone $\mathcal{C}$.

Take $(s, x) \in J \times G$ such that (s, x) is not contained in $\mathcal{C}$. Then $|x - x_0| \geq R - C_0 s$ which is equivalent to

$$-s + K(R - |x - x_0|) = -s + \frac{1}{C_0}(R - |x - x_0|) \leq 0.$$

In particular, we obtain

$$-s+\psi(x) \le -s-\varepsilon+K(R-|x-x_0|) \le -\varepsilon$$

and hence

$$e^{\tau\Psi(s,x)} = e^{\tau\,(-s+\psi(x))} \le e^{-\tau\varepsilon}$$

for all $\tau > 0$. On the other hand, by assumption we have $f(s,x) = 0$ for almost all $(s,x) \in \mathcal{C}$, $g(s,x) = 0$ for almost all $(s,x) \in \mathcal{C} \cap (J \times \partial G)$, and $u_0(x) = 0$ for almost all $x \in \mathcal{C}_{t=0} \cap G$. We conclude that

$$|f_\tau(s,x)| \le |f(s,x)| \text{ for all } \tau > 0 \quad \text{and} \quad |f_\tau(s,x)| \longrightarrow 0 \text{ as } \tau \to \infty$$

for almost all $(s,x) \in J \times G$. Lebesgue's dominated convergence theorem therefore yields

$$\|f_\tau\|_{L^2(J\times G)} \longrightarrow 0$$

as $\tau \to \infty$. Analogously, we deduce that

$$\tau\|g_{2\tau}\|_{L^2(J\times\partial G)} \longrightarrow 0 \quad \text{and} \quad \|u_{0,\tau}\|_{L^2(G)} \longrightarrow 0$$

as $\tau \to \infty$. The weighted energy estimate (6.16) thus gives

$$\sup_{t\in J} \|u_\tau(t)\|^2_{L^2(G)} \longrightarrow 0$$

as $\tau \to \infty$. In particular, there is a constant $C_2 \ge 0$ independent of τ such that

$$\sup_{t\in J} \|u_\tau(t)\|^2_{L^2(G)} \le C_2 \tag{6.17}$$

for all $\tau > 0$.

Now take a point (t,x) from $\mathcal{C}_{3\varepsilon}$, where the reduced cones $\mathcal{C}_\delta$ are defined by

$$\mathcal{C}_\delta = \{(t,x) \in \overline{J} \times \mathbb{R}^3 \colon |x-x_0| < R - C_0 t - C_0\delta\}$$

for all $\delta > 0$. Then

$$3\varepsilon < \frac{1}{C_0}(R-|x-x_0|) - t = K(R-|x-x_0|) - t \le -t+\psi(x)+2\varepsilon = \Psi(t,x)+2\varepsilon,$$
$$\varepsilon < \Psi(t,x).$$

Consequently, we infer

$$\int_{\mathcal{C}_{3\varepsilon}} |u(t,x)|^2 dx dt \le e^{-2\varepsilon\tau} \int_{\mathcal{C}_{3\varepsilon}} e^{2\tau\Psi(t,x)}|u(t,x)|^2 dx dt \le e^{-2\varepsilon\tau} T \sup_{t\in J} \|u_\tau(t)\|^2_{L^2(\mathbb{R}^3_+)}$$
$$\le C_2 T e^{-2\varepsilon\tau}$$

for all $\tau > 0$, where we also employed (6.17). Letting $\tau \to \infty$, we obtain

$$\int_{\mathcal{C}_{3\varepsilon}} |u(t,x)|^2 dx dt = 0$$

and thus $|u(t,x)| = 0$ for almost all $(t,x) \in \mathcal{C}_{3\varepsilon}$.

Finally, we take a sequence $(\varepsilon_n)_n$ in $(0,1)$ with $\varepsilon_n \to 0$ as $n \to \infty$. Since $u(t,x) = 0$ for almost all $(t,x) \in \mathcal{C}_{3\varepsilon_n}$ for all $n \in \mathbb{N}$, we conclude that

$$u(t,x) = 0 \quad \text{for almost all } (t,x) \in \bigcup_{n\in\mathbb{N}} \mathcal{C}_{3\varepsilon_n} = \mathcal{C}. \qquad \square$$

We also formulate the finite propagation speed property using the forward light cone, cf. [BCD11]. This version shows that if the data is supported on a forward light cone, then also the solution is supported in this cone.

Corollary 6.2. *Let $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, and G be a tame uniform $C^{\tilde{m}+2}$-domain. Pick $T > 0$ and set $J = (0, T)$. Take a parameter η and coefficients $A_0 \in F^{\mathrm{c}}_{\tilde{m},\eta}(J \times G)$, $D \in F^{\mathrm{c}}_{\tilde{m}}(J \times G)$, and*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Fix a covering $(U_i)_{i \in \mathbb{N}_0}$ of $\overline{G}$ and corresponding diffeomorphisms $(\varphi_i)_{i \in \mathbb{N}_0}$ as in Definition 5.4. Let M be a bound for the derivatives of the functions $(\varphi_i)_{i \in \mathbb{N}_0}$ as in Lemma 5.1 (cf. Definition 5.4). Set

$$C_0 = \frac{9M^2}{\eta}.$$

Let $R > 0$ and $x_0 \in \overline{G}$. We define the forward cone $\mathcal{K}$ by

$$\mathcal{K} = \{(t, x) \in \overline{J} \times \mathbb{R}^3 \colon |x - x_0| \leq R + C_0\, t\}.$$

Let $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ such that

$$\begin{aligned} f &= 0 \qquad \text{on } (J \times G) \setminus \mathcal{K}, \\ g &= 0 \qquad \text{on } (J \times \partial G) \setminus \mathcal{K}, \\ u_0 &= 0 \qquad \text{on } G \setminus \mathcal{K}_{t=0}, \end{aligned}$$

where $\mathcal{K}_{t=0}$ is defined by $\mathcal{K}_{t=0} = \{x \in \mathbb{R}^3 \colon (0, x) \in \mathcal{K}\}$. Suppose that the tuple $(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, B, f, g, u_0)$ fulfills the linear compatibility conditions (2.37) of order m. Then the unique solution $u \in G_m(J \times G)$ of the linear initial boundary value problem (5.1) with inhomogeneity f, boundary value g, and initial value u_0 is supported in the cone $\mathcal{K}$, i.e.,

$$u(t, x) = 0 \quad \text{for almost all } (t, x) \in (J \times G) \setminus \mathcal{K}.$$

Proof. We argue by contradiction and assume that u does not vanish on $(J \times G) \setminus \mathcal{K}$, i.e., there is a subset $\tilde{\mathcal{M}}_1 \subseteq (J \times G) \setminus \mathcal{K}$ of positive measure such that u is not identically 0 on $\tilde{\mathcal{M}}_1$. Since

$$(J \times G) \setminus \mathcal{K} = \bigcup_{\delta > 0} \{(t, x) \in (J \times G) \setminus \mathcal{K} \colon \operatorname{dist}((t, x), \partial\mathcal{K}) > \delta\},$$

we particularly find a set $\tilde{\mathcal{M}}_2$ of positive measure and a parameter $\delta > 0$ such that $\tilde{\mathcal{M}}_2 \subseteq (J \times G) \setminus (\mathcal{K} + B(0, \delta))$ and u does not vanish identically on $\tilde{\mathcal{M}}_2$. Employing that $J \times G \subseteq \bigcup_{n \in \mathbb{N}} [\frac{1}{n}, T - \frac{1}{n}] \times \overline{B}(x_0, n)$, we obtain an index $N \in \mathbb{N}$ with $N > R$ and a set $\mathcal{M} \subseteq ([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \cap ((J \times G) \setminus (\mathcal{K} + B(0, \delta))$ such that $\mathcal{M}$ has positive measure and $u(t, x) \neq 0$ for almost all $(t, x) \in \mathcal{M}$.

We define the family of backward cones

$$\mathcal{C}_{x', R'} = \{(t, x) \in \overline{J} \times \mathbb{R}^3 \colon |x - x'| < R' - C_0 t\}$$

for all $x' \in \mathbb{R}^3$ and $R' > 0$ and consider the system

$$\mathfrak{A} = \{\mathcal{C}_{x', N-R} \colon |x' - x_0| = N\}.$$

We claim that $\mathfrak{A}$ forms an open covering of $([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \setminus (\mathcal{K} + B(0, \delta))$. To see this assertion, take $(t, x) \in ([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \setminus (\mathcal{K} + B(0, \delta))$ and set

$$\tilde{x} = x_0 + \frac{N}{|x - x_0|}(x - x_0).$$

Using that $x \in \overline{B}(x_0, N)$ and $(t,x) \in (\mathbb{R} \times \mathbb{R}^3) \setminus (\mathcal{K} + B(0,\delta))$, we derive

$$|x - \tilde{x}| = |\,|x - x_0| - N| = N - |x - x_0| < N - (R + C_0 t) = N - R - C_0 t,$$

i.e., $(t,x) \in \mathcal{C}_{\tilde{x},N-R}$. As $|\tilde{x} - x_0| = N$, the cone $\mathcal{C}_{\tilde{x},N-R}$ belongs to $\mathfrak{A}$ and the claim follows.

The compactness of $([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \setminus (\mathcal{K} + B(0,\delta))$ then yields finitely many points $x_1, \dots, x_m$ such that $|x_i - x_0| = N$ for all $i \in \{1, \dots, m\}$ and $\{\mathcal{C}_{x_i,N-R} \colon i \in \{1, \dots, m\}\}$ covers $([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \setminus (\mathcal{K} + B(0,\delta))$. Since $\mathcal{M}$ is a subset of $([\frac{1}{N}, T - \frac{1}{N}] \times \overline{B}(x_0, N)) \setminus (\mathcal{K} + B(0,\delta))$, there is an index $l \in \{1, \dots, m\}$ such that $\mathcal{M} \cap \mathcal{C}_{x_l,N-R}$ has positive measure. However, for $(t,x) \in \mathcal{C}_{x_l,N-R}$ we have

$$|x - x_0| \geq |x_l - x_0| - |x - x_l| = N - |x - x_l| > N - (N - R - C_0 t) = R + C_0 t,$$

i.e., (t,x) belongs to $(\overline{J} \times \mathbb{R}^3) \setminus \mathcal{K}$. We conclude that f vanishes on $\mathcal{C}_{x_l,N-R} \cap (J \times G)$, g on $\mathcal{C}_{x_l,N-R} \cap (J \times \partial G)$, and u_0 on $\mathcal{C}_{t=0;x_l,N-R} \cap G$. Theorem 6.1 thus shows that the solution u vanishes on $\mathcal{C}_{x_l,N-R} \cap (J \times G)$, i.e., $u(t,x) = 0$ for almost all $(t,x) \in \mathcal{C}_{x_l,N-R}$. This contradicts $u(t,x) \neq 0$ for almost all $(t,x) \in \mathcal{M} \cap \mathcal{C}_{x_l,N-R}$ as this set has positive measure. $\square$

7

Local wellposedness of the nonlinear system

In this chapter we finally turn to the main subject of this work, the local wellposedness of the nonlinear Maxwell system (1.6). The construction of a solution of (1.6) is the first key step in this direction. We see in section 7.2 that the results from Chapter 3 and Chapter 4 allow us to apply a fixed point argument that yields the existence of such a solution.

However, we recall that the constants in the a priori estimates depend on the coefficients which take the form $\chi(u)$ in the quasilinear setting. We have to control the appearing norms of $\chi(u)$ in terms of u to make the fixed point argument work. Therefore, we need a higher order chain rule and corresponding estimates. We provide this rather technical material in section 7.1.

In subsection 7.3 we prove a refined estimate of solutions to (1.6), which allows us to provide a blow-up criterion which only depends on the Lipschitz-norm of the solution. This criterion also leads to a satisfactory regularity theory in our setting. We then deal with estimates of the difference of two solutions of (1.6). These estimates are the crucial tool to prove that the solutions of (1.6) depend continuously on the data.

7.1 Material laws

In the study of quasilinear problems one often has to control compositions $\theta(v)$ in higher regularity in terms of v. It is thus natural to consider a higher order chain rule. This so called Faá di Bruno's formula is therefore widespread in the literature, see e.g. [BGS07], [BCD11]. However, this formula is usually merely stated for smooth functions. Moreover, we are not only interested in the formula itself but also in corresponding estimates of the $F_m(\Omega)$-norm of $\theta(v)$ in terms of the $G_m(\Omega)$-norm of v. For the convenience of the reader we therefore provide detailed proofs of these results. Finally, we also show bounds for the differences $\theta(v_1) - \theta(v_2)$, which are crucial to establish the contractivity of a certain fixed point operator and the continuous dependence.

We start with the higher order chain rule for functions $\theta(v)$ and estimates for their $F_m(\Omega)$-norm. The proof is a standard iterative application of the chain and product rule combined with Lemma 2.22. We further give the proof for the slightly more general case that the functions take values in $\mathbb{R}^n$ instead of $\mathbb{R}^6$.

Throughout this section let G be a domain in $\mathbb{R}^3$ with a uniformly C^2-boundary, $\mathcal{U} \subset \mathbb{R}^n$ be a convex domain, J be an open interval, and $\Omega = J \times G$. Moreover, we denote the image of a function v by $\operatorname{im} v$.

Lemma 7.1. *Let $m, n \in \mathbb{N}$ and $\tilde{m} = \max\{m, 3\}$. Let $\mathcal{U}_1$ be a compact subset of $\mathcal{U}$.*

(i) Let $\theta \in C^m(\mathcal{U},\mathbb{R})$. *For each* $v \in \tilde{G}_{\tilde{m}}(\Omega)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$ *the function* $\theta(v)$ *belongs to the function space* $F_m(\Omega)$. *For* $l_1,\dots,l_j \in \{1,\dots,n\}$, $\gamma_1,\dots,\gamma_j \in \mathbb{N}_0^4$ *with* $|\gamma_i| \leq m$, $1 \leq j \leq |\alpha|$, *and* $\alpha \in \mathbb{N}_0^4$ *with* $1 \leq |\alpha| \leq m$ *there exist constants* $C(\alpha,j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j)$ *such that*

$$\partial^\alpha \theta(v) = \sum_{1\leq j\leq|\alpha|} \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}_0^4\setminus\{0\}\\ \sum\gamma_i=\alpha}} \sum_{l_1,\dots,l_j=1}^{n} C(\alpha,j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j) \cdot (\partial_{l_j}\cdots\partial_{l_1}\theta)(v)\prod_{i=1}^{j}\partial^{\gamma_i}v_{l_i} \tag{7.1}$$

for all $v \in \tilde{G}_{\tilde{m}}(\Omega)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$ *and* $\alpha \in \mathbb{N}_0^4$ *with* $0 < |\alpha| \leq m$. *Moreover, there exists a constant* $C(\theta,m,n,R,\mathcal{U}_1)$ *such that*

$$\|\theta(v)\|_{F_m(\Omega)} \leq C(\theta,m,n,R,\mathcal{U}_1)(1+\|v\|_{G_{\tilde{m}}(\Omega)})^{m-1}\|v\|_{G_{\tilde{m}}(\Omega)} \tag{7.2}$$

for all $v \in \tilde{G}_{\tilde{m}}(\Omega)$ *with* $\|v\|_{L^\infty(\Omega)} \leq R$ *and* $\operatorname{im} v \subseteq \mathcal{U}_1$.

(ii) Let $\theta \in C^{m-1}(\mathcal{U},\mathbb{R})$. *For all* $v \in H^{\tilde{m}-1}(G)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$ *the composition* $\theta(v)$ *belongs to* $F^0_{\tilde{m}-1}(G)$. *We further have that*

$$\partial^\alpha \theta(v) = \sum_{1\leq j\leq|\alpha|} \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}_0^3\setminus\{0\}\\ \sum\gamma_i=\alpha}} \sum_{l_1,\dots,l_j=1}^{n} C_0(\alpha,j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j) \cdot (\partial_{l_j}\cdots\partial_{l_1}\theta)(v)\prod_{i=1}^{j}\partial^{\gamma_i}v_{l_i} \tag{7.3}$$

for all $v \in H^{\tilde{m}-1}(G)$ *and* $\alpha \in \mathbb{N}_0^3$ *with* $0 < |\alpha| \leq m-1$, *and the constants*

$$C_0(\alpha,j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j) = C((0,\alpha_1,\alpha_2,\alpha_3),j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j)$$

from (i). There further exists a constant $C_0(\theta,m,n,R_0,\mathcal{U}_1)$ *such that*

$$\|\theta(v)\|_{F^0_{m-1}(G)} \leq C_0(\theta,m,n,R_0,\mathcal{U}_1)(1+\|v\|_{H^{\tilde{m}-1}(G)})^{m-1} \tag{7.4}$$

for all $v \in H^{\tilde{m}-1}(G)$ *with* $\|v\|_{L^\infty(G)} \leq R_0$ *and* $\operatorname{im} v \subseteq \mathcal{U}_1$.

(iii) Assume additionally that $m \geq 2$. *Let* $\theta \in C^m(\mathcal{U},\mathbb{R})$ *and* $r_0 > 0$. *Then there is a constant* $C(\theta,m,n,r_0,\mathcal{U}_1)$ *such that*

$$\begin{aligned}&\|\partial_t^j\theta(v)(0)\|_{H^{m-j-1}(G)}\\ &\leq C(\theta,m,n,r_0,\mathcal{U}_1)(1+\max_{0\leq l\leq j}\|\partial_t^l v(0)\|_{H^{\tilde{m}-l-1}(G)})^{m-1}\max_{0\leq l\leq j}\|\partial_t^l v(0)\|_{H^{\tilde{m}-l-1}(G)}\end{aligned}$$

for all $j \in \{1,\dots,m-1\}$ *and* $v \in \tilde{G}_{\tilde{m}}(\Omega)$ *with* $\operatorname{im} v \subseteq \mathcal{U}$, $\|v(0)\|_{L^\infty(G)} \leq r_0$ *and* $\operatorname{im} v(0) \subseteq \mathcal{U}_1$.

Proof. (i) We show the assertion by induction with respect to m. So let $m = 1$. Since $v \in \tilde{G}_{\tilde{m}}(\Omega) \hookrightarrow H^3(\Omega) \hookrightarrow L^\infty(\Omega)$ we find a sequence $(v_k)_k$ in $C_c^\infty(\overline{\Omega})$ such that $v_k \to v$ in $H^3(\Omega)$, $\|v_k\|_{L^\infty(\Omega)} \leq 2\|v\|_{L^\infty(\Omega)}$, and $v_k \to v$ pointwise almost everywhere. We further infer that $\operatorname{im} v$ is a compact subset of $\mathcal{U}$. We can thus choose a compact subset $\mathcal{U}_2$ of $\mathcal{U}$ such that, after adapting $(v_k)_k$ if necessary, $\operatorname{im} v_k \subseteq \mathcal{U}_2$ and $\operatorname{im} v \subseteq \mathcal{U}_2$ for all $k \in \mathbb{N}$. Let $j \in \{0,1,2,3\}$ and $l_1 \in \{1,\dots,n\}$. We then estimate

$$\|\theta(v_k)-\theta(v)\|_{L^\infty(\Omega)} \leq \max_{x\in\mathcal{U}_2}|\theta'(x)|\,\|v_k-v\|_{L^\infty} \longrightarrow 0,$$

$$\begin{aligned}
&\int_\Omega |(\partial_{l_1}\theta)(v_k)\partial_j v_{k,l_1} - (\partial_{l_1}\theta)(v)\partial_j v_{l_1}|^2 dx \\
&\quad \le 2\int_\Omega |\partial_{l_1}\theta(v_k)|^2\,|\partial_j v_{k,l_1} - \partial_j v_{l_1}|^2 dx + 2\int_\Omega |\partial_{l_1}\theta(v_k) - \partial_{l_1}\theta(v)|^2\,|\partial_j v_{l_1}|^2 dx \\
&\quad \le C \max_{x\in\mathcal{U}_2} |\theta'(x)|^2\,\|\partial_j v_k - \partial_j v\|^2_{L^2(\Omega)} + C\int_\Omega |\partial_{l_1}\theta(v_k) - \partial_{l_1}\theta(v)|^2\,|\partial_j v_{l_1}|^2 dx \\
&\quad \longrightarrow 0
\end{aligned}$$

as $k \to \infty$. Here we used that $\partial_{l_1}\theta$ is continuous so that $\partial_{l_1}\theta(v_k)$ converges pointwise almost everywhere to $\partial_{l_1}\theta(v)$. The theorem of dominated convergence with majorant $C\max_{x\in\mathcal{U}_2}|\theta'(x)|^2|\partial_j v_{l_1}|^2$ thus gives the above convergence of the second integral.

The first order weak derivatives of $\theta(v)$ hence exist and are given by

$$\partial_j\theta(v) = \sum_{l_1=1}^{n} (\partial_{l_1}\theta)(v)\partial_j v_{l_1}$$

for $j \in \{0,\dots,3\}$. This fact shows (7.1) for $m = 1$. Moreover, for functions v with $\operatorname{im} v \subseteq \mathcal{U}_1$, we infer

$$\begin{aligned}
\|\theta(v)\|_{L^\infty(\Omega)} &\le \max_{x\in\mathcal{U}_1} |\theta(x)|, \\
\|\partial_j\theta(v)\|_{L^\infty(\Omega)} &\le C(n) \max_{x\in\mathcal{U}_1} |\theta'(x)|\,\|\partial_j v\|_{L^\infty(\Omega)} \le C(n,\theta,\mathcal{U}_1)\,\|v\|_{G_3(\Omega)}, \\
\|\partial_j\theta(v)\|_{G_0(\Omega)} &\le C(n) \max_{x\in\mathcal{U}_1} |\theta'(x)|\,\|\partial_j v\|_{G_0(\Omega)} \le C(n,\theta,\mathcal{U}_1)\,\|v\|_{G_1(\Omega)}.
\end{aligned}$$

Hence, $\theta(v)$ belongs to $F_1(\Omega)$ and estimate (7.2) has been proved for $m = 1$.

Now assume that the assertion holds for all $k \in \{1,\dots,m\}$ and some $m \in \mathbb{N}$. We will establish that the assertion is also valid for $m+1$.

To that purpose, let $\theta \in C^{m+1}(\mathbb{R}^n,\mathbb{R})$ and take $v \in \tilde{G}_{\max\{m+1,3\}}(\Omega)$ with $\operatorname{im} v \subseteq \mathcal{U}$. Observe that formula (7.1) holds for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$ by the induction hypothesis. Therefore, take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m+1$. Choose a unit vector $e_k \in \mathbb{N}_0^4$ such that α decomposes as $\alpha = \alpha' + e_k$ with $\alpha' \in \mathbb{N}_0^4$ and $|\alpha'| \le m$. Since $\partial_k v \in \tilde{G}_{\max\{m,2\}}(\Omega)$ and $(\partial_{l_1}\theta)(v) \in F_m(\Omega)$ by the induction hypothesis for all $l_1 \in \{1,\dots,n\}$, Lemma 2.22 (ii) yields that $\partial_k\theta(v) = \sum_{l_1=1}^n (\partial_{l_1}\theta)(v)\partial_k v_{l_1}$ belongs to $\tilde{G}_m(\Omega)$. The induction hypothesis implies the identities

$$\begin{aligned}
\partial^\alpha\theta(v) &= \partial^{\alpha'} \sum_{l_{m+1}=1}^{n} (\partial_{l_{m+1}}\theta)(v)\partial_k v_{l_{m+1}} \\
&= \sum_{l_{m+1}=1}^{n} \sum_{\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta(\partial_{l_{m+1}}\theta)(v)\partial^{\alpha'-\beta}\partial_k v_{l_{m+1}} \\
&= \sum_{l_{m+1}=1}^{n} \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \Big(\sum_{1\le j\le|\beta|} \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}_0^4\setminus\{0\} \\ \sum\gamma_i=\beta}} \sum_{l_1,\dots,l_j=1}^{n} C(\beta,j,l_1,\dots,l_j,\gamma_1,\dots,\gamma_j) \\
&\qquad\qquad \cdot (\partial_{l_j}\cdots\partial_{l_1}\partial_{l_{m+1}}\theta)(v) \prod_{i=1}^{j} \partial^{\gamma_i} v_{l_i} \Big) \partial^{\alpha-\beta} v_{l_{m+1}} \\
&\quad + \sum_{l_{m+1}=1}^{n} (\partial_{l_{m+1}}\theta)(v)\partial^\alpha v_{l_{m+1}}.
\end{aligned} \tag{7.5}$$

Fix a multiindex $0 < \beta \le \alpha'$, a number $j \in \{1,\dots,|\beta|\}$, and $l_1,\dots,l_j \in \{1,\dots,n\}$, $l_{m+1} \in \{1,\dots,n\}$, and $\gamma_1,\dots,\gamma_j \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^j \gamma_i = \beta$. We then observe that

$$(l_1,\dots,l_j,l_{m+1};\gamma_1,\dots,\gamma_j,\alpha-\beta) \in I_\alpha,$$

where I_α is defined by

$$I_\alpha = \bigcup_{j=1}^{|\alpha|} \Big\{ (l_1', \ldots, l_j'; \gamma_1', \ldots, \gamma_j') \colon l_1', \ldots, l_j' \in \{1, \ldots, n\}, \gamma_1', \ldots, \gamma_j' \in \mathbb{N}_0^4 \setminus \{0\}, \\ \sum_{i=1}^{j} \gamma_i' = \alpha \Big\}.$$

As also $(l_{m+1}, \alpha) \in I_\alpha$, formula (7.1) follows for α by rearranging (7.5).

The induction hypothesis further says that $\theta(v) \in F_m(\Omega)$ and

$$\|\theta(v)\|_{F_m(\Omega)} \leq C(\theta, m, n, R, \mathcal{U}_1)(1 + \|v\|_{G_{\tilde{m}}(\Omega)})^{m-1} \|v\|_{G_{\tilde{m}}(\Omega)}$$

for all $v \in \tilde{G}_{\tilde{m}}(\Omega)$ with $\|v\|_{L^\infty(\Omega)} \leq R$ and $\operatorname{im} v \subseteq \mathcal{U}_1$.

It remains to show that $\partial^\alpha \theta(v) \in \tilde{G}_0(\Omega)$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m+1$ and to estimate these derivatives. To this purpose fix an index α and a function v as above. Take $j \in \{1, \ldots, m+1\}$ and $\gamma_1, \ldots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^{j} \gamma_i = \alpha$.

We start with the case $m = 1$. Here we have $v \in \tilde{G}_3(\Omega)$ and thus

$$\begin{aligned}
&\Big\| \sum_{l_1, \ldots, l_j = 1}^{n} (\partial_{l_j} \cdots \partial_{l_1} \theta)(v) \prod_{i=1}^{j} \partial^{\gamma_i} v_{l_i} \Big\|_{G_0(\Omega)} \\
&\leq C \sum_{l_1, \ldots, l_j = 1}^{n} \|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v)\|_{L^\infty(\Omega)} \prod_{i=1}^{j} \|\partial^{\gamma_i} v_{l_i}\|_{G_{3-|\gamma_i|}(\Omega)} \\
&\leq C \sum_{l_1, \ldots, l_j = 1}^{n} \max_{x \in \mathcal{U}_1} |(\partial_{l_j} \cdots \partial_{l_1} \theta)(x)| (1 + \|v\|_{G_{\tilde{m}}(\Omega)})^m \|v\|_{G_{\tilde{m}}(\Omega)}
\end{aligned}$$

where the first estimate is trivial if $j = 1$ and it follows from Lemma 2.22 (ii) if $j = 2$, since $3 - |\gamma_i| \geq 2$ for at least one $i \in \{1, j\}$.

In the case $m \geq 2$ there is at most one multiindex γ_i with $|\gamma_i| \geq m$ appearing in the formula (7.1). Otherwise we would have

$$m + 1 = |\alpha| = \sum_{i=1}^{j} |\gamma_i| \geq 2m \geq m + 2,$$

a contradiction. For the multiindices γ_i with $|\gamma_i| \leq m - 1$ the function $\partial^{\gamma_i} v_{l_i}$ belongs to $\tilde{G}_{m+1-|\gamma_i|}(\Omega) \hookrightarrow \tilde{G}_2(\Omega)$. A successive application of Lemma 2.22 (ii) thus yields

$$\begin{aligned}
&\Big\| \sum_{l_1, \ldots, l_j = 1}^{n} (\partial_{l_j} \cdots \partial_{l_1} \theta)(v) \prod_{i=1}^{j} \partial^{\gamma_i} v_{l_i} \Big\|_{G_0(\Omega)} \\
&\leq C \sum_{l_1, \ldots, l_j = 1}^{n} \|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v)\|_{L^\infty(\Omega)} \prod_{i=1}^{j} \|\partial^{\gamma_i} v_{l_i}\|_{G_{m+1-|\gamma_i|}(\Omega)} \\
&\leq C \sum_{l_1, \ldots, l_j = 1}^{n} \max_{x \in \mathcal{U}_1} |(\partial_{l_j} \cdots \partial_{l_1} \theta)(x)| (1 + \|v\|_{G_{m+1}(\Omega)})^m \|v\|_{G_{m+1}(\Omega)}.
\end{aligned}$$

We now take the $\tilde{G}_0(\Omega)$-norm of $\partial^\alpha \theta(v)$ in (7.1), combine the above estimates and take the maximum of all involved constants. It follows

$$\|\partial^\alpha v\|_{G_0(\Omega)} \leq C(1 + \|v\|_{G_{\max\{m+1,3\}}(\Omega)})^m \|v\|_{G_{\max\{m+1,3\}}(\Omega)}$$

for all $v \in \tilde{G}_{\max\{m+1,3\}}(\Omega)$ with $\|v\|_{L^\infty(\Omega)} \leq R$ and $\operatorname{im} v \subseteq \mathcal{U}_1$, where the constant C depends on θ, m, n, R, and $\mathcal{U}_1$. We conclude that $\theta(v)$ belongs to $F_{m+1}(\Omega)$ and (7.2) holds for $m + 1$.

(ii) The proof works in the same way as in (i). The asserted coincidence of the constants is clear in the case $|\alpha| = 1$ and then follows for the higher order ones by induction due to (7.5).

(iii) This part follows from part (i) and the techniques used there. However, since the details are not the same, we present them. Take $v \in \tilde{G}_{\tilde{m}}(\Omega)$ with $\operatorname{im} v \subseteq \mathcal{U}$, $\|v(0)\|_{L^\infty(G)} \leq r_0$, and $\operatorname{im} v(0) \subseteq \mathcal{U}_1$. Part (i) shows that

$$\partial_t^j \theta(v)(0) = \sum_{1 \leq k \leq j} \sum_{\substack{\gamma_1, \ldots, \gamma_k \in \mathbb{N}_0^4 \setminus \{0\} \\ \sum \gamma_i = (j,0,0,0)}} \sum_{l_1, \ldots, l_k = 1}^{n} C((j,0,0,0), k, l_1, \ldots, l_k, \gamma_1, \ldots, \gamma_k) \cdot (\partial_{l_k} \ldots \partial_{l_1} \theta)(v(0)) \prod_{i=1}^{k} (\partial^{\gamma_i} v_{l_i})(0)$$

for all $j \in \{1, \ldots, m-1\}$. Observe that $\partial^{\gamma_i} v_{l_i}$ belongs to $\tilde{G}_1(\Omega) \hookrightarrow G_0(\Omega)$ for all appearing γ_j and l_i so that the point-evaluation in zero is well-defined.

We start with $m = 2$. Then $\partial_t(\theta(v))(0) = \sum_{l_1=1}^n \partial_{l_1}\theta(v(0))\partial_t v_{l_1}(0)$ and we can thus estimate

$$\begin{aligned} \|\partial_t \theta(v)(0)\|_{L^2(G)} &\leq \sum_{l_1=1}^{n} \|\partial_{l_1}\theta(v(0))\|_{L^\infty(G)} \|\partial_t v(0)\|_{L^2(G)} \\ &\leq C(\theta, n, \mathcal{U}_1) \|\partial_t v(0)\|_{L^2(G)}. \end{aligned}$$

It remains to consider the case $m \geq 3$. To that purpose, we take $j \in \{1, \ldots, m-1\}$, $k \in \{1, \ldots, j\}$, $l_1, \ldots, l_k \in \{1, \ldots, n\}$, $\gamma_1, \ldots, \gamma_k \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^k \gamma_i = (j, 0, 0, 0)$.

We first note that $v(0) \in H^{\tilde{m}-1}(G)$ and $\partial_{l_k} \ldots \partial_{l_1} \theta \in C^{m-k}(\mathbb{R}^n, \mathbb{R})$ so that part (ii) yields $\partial_{l_k} \ldots \partial_{l_1}\theta(v(0)) \in F^0_{m-k}(G)$. Moreover, the function $\partial^{\gamma_i} v(0)$ is an element of $H^{m-|\gamma_i|-1}(G) \hookrightarrow H^{m-j-1}(G)$ for all $i \in \{1, \ldots, k\}$.

Take $\gamma_p \in \mathbb{N}_0^4$ with $|\gamma_p| = \max_{1 \leq i \leq k} |\gamma_i|$. If $|\gamma_p| \geq 2$, we derive from the inequality

$$|\gamma_i| + |\gamma_p| \leq j \leq m - 1$$

that $m-1-|\gamma_i| \geq 2$ for all $i \in \{1, \ldots, k\} \setminus \{p\}$. Therefore, we can apply Lemma 2.22 (vi) repeatedly and thus estimate

$$\begin{aligned} \Big\| \prod_{i=1}^{k} \partial^{\gamma_i} v_{l_i}(0) \Big\|_{H^{m-1-j}(G)} &\leq \Big\| \prod_{i=1}^{k} \partial^{\gamma_i} v_{l_i}(0) \Big\|_{H^{m-1-|\gamma_p|}(G)} \\ &\leq C(m) \prod_{i=1}^{k} \|\partial^{\gamma_i} v_{l_i}(0)\|_{H^{m-1-|\gamma_i|}(G)} \leq C(m) \max_{1 \leq l \leq j} \|\partial_t^l v_{l_i}(0)\|^k_{H^{m-1-l}(G)}. \end{aligned} \tag{7.6}$$

Next, let $\max_{1 \leq i \leq k} |\gamma_i| \leq 1$. It follows that $|\gamma_i| = 1$ for all $i \in \{1, \ldots, k\}$ and hence $k = j$. If $m \geq 4$, we infer $m-1-|\gamma_i| \geq 2$ so that Lemma 2.22 (vi) again yields (7.6). If $m = 3$ and $j = 1$, the estimate in (7.6) trivially holds. If $m = 3$ and $j = 2$, we obtain

$$\begin{aligned} \Big\| \prod_{i=1}^{k} \partial^{\gamma_i} v(0) \Big\|_{H^{m-1-j}(G)} &= \|\partial_t v_{l_1}(0) \partial_t v_{l_2}(0)\|_{L^2(G)} \leq \|\partial_t v_{l_1}\|_{L^3(G)} \|\partial_t v_{l_2}\|_{L^6(G)} \\ &\leq C \|\partial_t^l v(0)\|^2_{H^1(G)} \leq C \max_{1 \leq l \leq j} \|\partial_t^l v(0)\|^j_{H^{m-1-l}(G)}. \end{aligned}$$

So (7.6) has been established in all cases.

If $k \leq m - 2$, Lemma 2.22 (vii), (7.6) and part (ii) then imply

$$\Big\| (\partial_{l_k} \cdots \partial_{l_1} \theta)(v(0)) \prod_{i=1}^{k} (\partial^{\gamma_i} v_{l_i})(0) \Big\|_{H^{m-1-j}(G)}$$

$$
\begin{aligned}
&\leq C(m)\|(\partial_{l_k}\cdots\partial_{l_1}\theta)(v(0))\|_{F^0_{m-k}(G)}\Big\|\prod_{i=1}^{k}\partial^{\gamma_i}v(0)\Big\|_{H^{m-1-j}(G)}\\
&\leq C(m)\|(\partial_{l_k}\cdots\partial_{l_1}\theta)(v(0))\|_{F^0_{m-k}(G)}\max_{1\leq l\leq j}\|\partial_t^l v_{l_i}(0)\|^k_{H^{m-1-l}(G)}\\
&\leq C(\theta,m,n,r_0,\mathcal{U}_1)(1+\|v(0)\|_{H^{m-k}(G)})^{m-k}\max_{1\leq l\leq j}\|\partial_t^l v(0)\|^k_{H^{m-1-l}(G)}\\
&\leq C(\theta,m,n,r_0,\mathcal{U}_1)(1+\max_{0\leq l\leq j}\|\partial_t^l v(0)\|_{H^{m-1-l}(G)})^{m-1}\max_{0\leq l\leq j}\|\partial_t^l v(0)\|_{H^{m-1-l}(G)}.
\end{aligned}
$$

In the case $k=m-1$, we also have $j=m-1$ and therefore

$$
\begin{aligned}
&\Big\|(\partial_{l_k}\cdots\partial_{l_1}\theta)(v(0))\prod_{i=1}^{k}(\partial^{\gamma_i}v_{l_i})(0)\Big\|_{H^{m-1-j}(G)}\\
&=\Big\|(\partial_{l_k}\cdots\partial_{l_1}\theta)(v(0))\prod_{i=1}^{k}(\partial^{\gamma_i}v_{l_i})(0)\Big\|_{L^2(G)}\\
&\leq\|(\partial_{l_k}\cdots\partial_{l_1}\theta)(v(0))\|_{L^\infty(G)}\Big\|\prod_{i=1}^{k}(\partial^{\gamma_i}v_{l_i})(0)\Big\|_{L^2(G)}\\
&\leq C(\theta,m,n,r_0,\mathcal{U}_1)(1+\max_{1\leq l\leq j}\|\partial_t^l v(0)\|_{H^{m-1-l}(G)})^{m-1}\max_{1\leq l\leq j}\|\partial_t^l v(0)\|_{H^{m-1-l}(G)}
\end{aligned}
$$

by (7.6). □

We next establish analogous estimates for differences $\theta(v_1)-\theta(v_2)$.

Corollary 7.2. *Let $m\in\mathbb{N}$, $\tilde{m}=\max\{m,3\}$, and $\gamma\geq 0$. Let $\theta\in C^{m-1}(\mathcal{U},\mathbb{R})$ and $R>0$.*

(i) Let $v_1,v_2\in\tilde{G}_{\tilde{m}-1}(\Omega)$ with $\|v_1\|_{L^\infty(\Omega)},\|v_2\|_{L^\infty(\Omega)},\|v_1\|_{G_{\tilde{m}-1}(\Omega)},\|v_2\|_{G_{\tilde{m}-1}(\Omega)}\leq R$, and $\operatorname{im}v_1,\operatorname{im}v_2\subseteq\mathcal{U}_1$. Then there exists a constant $C=C(\theta,m,n,R,\mathcal{U}_1)$ such that

$$\|(\partial^\alpha\theta(v_1))(t)-(\partial^\alpha\theta(v_2))(s)\|_{L^2(G)}\leq C\sum_{\substack{\beta\in\mathbb{N}_0^4\\0\leq|\beta|\leq\tilde{m}-1}}\|\partial^\beta v_1(t)-\partial^\beta v_2(s)\|_{L^2(G)} \tag{7.7}$$

for almost all $t\in J$ and almost all $s\in J$ if $\alpha\in\mathbb{N}_0^4$ with $0\leq|\alpha|\leq m-2$. In the case $|\alpha|=m-1$ and $m>1$ we have the estimate

$$
\begin{aligned}
\|(\partial^\alpha\theta(v_1))(t)-(\partial^\alpha\theta(v_2))(s)\|_{L^2(G)}&\leq C\sum_{\substack{\beta\in\mathbb{N}_0^4\\0\leq|\beta|\leq\tilde{m}-1}}\|\partial^\beta v_1(t)-\partial^\beta v_2(s)\|_{L^2(G)}\\
&+C\sum_{l_1,\ldots,l_{m-1}=1}^{n}\|(\partial_{l_{m-1}}\ldots\partial_{l_1}\theta)(v_1(t))-(\partial_{l_{m-1}}\ldots\partial_{l_1}\theta)(v_2(s))\|_{L^\infty(G)}
\end{aligned} \tag{7.8}
$$

for almost all $t\in J$ and almost all $s\in J$. If θ additionally belongs to $C^m(\mathcal{U},\mathbb{R})$, the estimate (7.7) is true for almost all $t\in J$ and almost all $s\in J$ in the case $|\alpha|=m-1$. Finally, if $\alpha_0=0$, it is enough to sum in (7.7) and (7.8) over those multiindices β with $\beta_0=0$.

(ii) Let $v_1,v_2\in\tilde{G}_{\tilde{m}}(\Omega)$ with $\|v_1\|_{L^\infty(\Omega)},\|v_2\|_{L^\infty(\Omega)},\|v_1\|_{G_{\tilde{m}-1}(\Omega)},\|v_2\|_{G_{\tilde{m}-1}(\Omega)}\leq R$, $\operatorname{im}v_1,\operatorname{im}v_2\subseteq\mathcal{U}_1$, and $\theta\in C^m(\mathcal{U},\mathbb{R})$. Then the difference $\theta(v_1)-\theta(v_2)$ belongs to $\tilde{G}_m(\Omega)$ and there exists a constant $C=C(\theta,m,n,R,\mathcal{U}_1)$ such that

$$\|\theta(v_1)-\theta(v_2)\|_{G_{m-1,\gamma}(\Omega)}\leq C\|v_1-v_2\|_{G_{\tilde{m}-1,\gamma}(\Omega)}$$

for all $\gamma\geq 0$.

(iii) Let $v_1, v_2 \in H^{\tilde m}(G)$ with $\|v_1\|_{L^\infty(G)}, \|v_2\|_{L^\infty(G)}, \|v_1\|_{H^{\tilde m-1}(G)}, \|v_2\|_{H^{\tilde m -1}(G)} \leq R$, $\operatorname{im} v_1, \operatorname{im} v_2 \subseteq \mathcal{U}_1$, and $\theta \in C^m(\mathcal{U}, \mathbb{R})$. Then the difference $\theta(v_1) - \theta(v_2)$ is an element of $H^m(G)$ and there is a constant $C = C(\theta, m, n, R, \mathcal{U}_1)$ such that

$$\|\theta(v_1) - \theta(v_2)\|_{H^{m-1}(G)} \leq C \|v_1 - v_2\|_{H^{\tilde m-1}(G)}.$$

Proof. (i) Let $v_1, v_2 \in \tilde G_{\tilde m}(\Omega)$ with $\|v_1\|_{L^\infty(\Omega)}, \|v_2\|_{L^\infty(\Omega)} \leq R$, and $\operatorname{im} v_1, \operatorname{im} v_2 \subseteq \mathcal{U}_1$. We first note that there is nothing to show in the case $m = 1$. So we assume $m \geq 2$ in the following. Observe that in the case $|\alpha| = 0$ the estimate (7.7) is a consequence of the mean value theorem and Sobolev's embedding. If $|\alpha| \geq 1$, Lemma 7.1 implies the formula

$$\begin{aligned}
&\partial^\alpha \theta(v_1)(t) - \partial^\alpha \theta(v_2)(s) \\
&= \sum_{1 \leq j \leq |\alpha|} \sum_{\substack{\gamma_1, \dots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\} \\ \sum \gamma_i = \alpha}} \sum_{l_1, \dots, l_j = 1}^{n} C(\alpha, j, l_1, \dots, l_j, \gamma_1, \dots, \gamma_j) \\
&\qquad \cdot \Big((\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) \prod_{i=1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s)) \prod_{i=1}^{j} \partial^{\gamma_i} v_{2,l_i}(s) \Big) \\
&= \sum_{1 \leq j \leq |\alpha|} \sum_{\substack{\gamma_1, \dots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\} \\ \sum \gamma_i = \alpha}} \sum_{l_1, \dots, l_j = 1}^{n} C(\alpha, j, l_1, \dots, l_j, \gamma_1, \dots, \gamma_j) \\
&\qquad \cdot \Big[\Big((\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s)) \Big) \prod_{i=1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) \qquad (7.9) \\
&\quad + \sum_{k=1}^{j} (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2)(s) \prod_{i=1}^{k-1} \partial^{\gamma_i} v_{2,l_i}(s) \cdot (\partial^{\gamma_k} v_{1,l_k}(t) - \partial^{\gamma_k} v_{2,l_k}(s)) \prod_{i=k+1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) \Big]
\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^4$ with $1 \leq |\alpha| \leq m-1$ and almost all $t \in J$ and almost all $s \in J$.

We now take $\alpha \in \mathbb{N}_0^4$ with $1 \leq |\alpha| \leq m-1$, $j \in \{1, \dots, |\alpha|\}$, $\gamma_1, \dots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^j \gamma_i = \alpha$, and $l_1, \dots, l_j \in \{1, \dots, n\}$. Observe that $|\gamma_i| \leq m-1$ for $1 \leq i \leq j$.

I) Let $w_i \in H^{\tilde m - 1 - |\gamma_i|}(G)$ for all $i \in \{1, \dots, j\}$. We first claim that there is a constant $C(m)$ such that

$$\Big\| \prod_{i=1}^{j} w_i \Big\|_{L^2(G)} \leq C(m) \prod_{i=1}^{j} \|w_i\|_{H^{\tilde m - 1 - |\gamma_i|}(G)}. \qquad (7.10)$$

Without loss of generality we assume that $|\gamma_j| = \max_{1 \leq i \leq j} |\gamma_i|$. We then estimate

$$\Big\| \prod_{i=1}^{j} w_i \Big\|_{L^2(G)} \leq C \Big\| \prod_{i=1}^{j-1} w_i \Big\|_{H^{|\gamma_j|}(G)} \|w_j\|_{H^{\tilde m - 1 - |\gamma_j|}(G)} \qquad (7.11)$$

using Lemma 2.22 (v). Observe that $w_i \in H^{\tilde m - 1 - |\gamma_i|}(G) \hookrightarrow H^{|\gamma_j|}(G)$ for all $i \in \{1, \dots, j-1\}$ since $|\gamma_i| + |\gamma_j| \leq |\alpha| \leq m-1$ for $1 \leq i \leq j-1$. In the case $|\gamma_j| \geq 2$, inequality (7.11) and a successive application of Lemma 2.22 (vi) thus yield

$$\Big\| \prod_{i=1}^{j} w_i \Big\|_{L^2(G)} \leq C(m) \prod_{i=1}^{j} \|w_i\|_{H^{\tilde m - 1 - |\gamma_i|}(G)}.$$

Next assume that $|\gamma_j| \leq 1$. In the case $j = 1$ the estimate (7.10) trivially holds, while in the case $j = 2$ this inequality follows from (7.11) and $|\gamma_2| \leq \tilde m - 1 - |\gamma_1|$. If $j \geq 3$ we deduce from $m - 1 \geq j$ that $m \geq 4$. Exploiting that $|\gamma_i| \leq |\gamma_j| \leq 1$

for $1 \leq i \leq j-1$, we infer that w_i belongs to $H^{\tilde{m}-1-|\gamma_i|}(G) \hookrightarrow H^{m-2}(G)$. We can therefore again successively apply Lemma 2.22 (vi) and derive from (7.11) that

$$\Big\| \prod_{i=1}^{j} w_i \Big\|_{L^2(G)} \leq C(m) \prod_{i=1}^{j} \|w_i\|_{H^{m-1-|\gamma_i|}(G)}.$$

We have thus shown (7.10).

II) We now fix two representatives of v_1 and v_2, still denoted by v_1 and v_2 and two corresponding nullsets $N_1, N_2 \subseteq J$ such that v_1 and v_2 satisfy

$$\begin{aligned}
&\partial^{\tilde{\alpha}} v_1(t) \in H^{\tilde{m}-1-|\tilde{\alpha}|}(G), \quad \partial^{\tilde{\alpha}} v_2(s) \in H^{\tilde{m}-1-|\tilde{\alpha}|}(G), \\
&\|\partial^{\tilde{\alpha}} v_1(t)\|_{H^{\tilde{m}-1-|\tilde{\alpha}|}(G)} \leq \|v_1\|_{G_{\tilde{m}-1}(\Omega)}, \quad \|\partial^{\tilde{\alpha}} v_2(s)\|_{H^{\tilde{m}-1-|\tilde{\alpha}|}(G)} \leq \|v_2\|_{G_{\tilde{m}-1}(\Omega)}, \\
&|v(t)| \leq R, \quad |v(s)| \leq R, \quad \operatorname{im} v(t) \subseteq \mathcal{U}_1, \quad \operatorname{im} v(s) \subseteq \mathcal{U}_1
\end{aligned}$$

and (7.9) for all $t \in J \setminus N_1, s \in J \setminus N_2$ and $\tilde{\alpha} \in \mathbb{N}_0^4$ with $|\tilde{\alpha}| \leq m-1$. Step I) then shows

$$\begin{aligned}
&\Big\| \Big((\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s)) \Big) \prod_{i=1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) \Big\|_{L^2(G)} \\
&\leq \|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s))\|_{L^\infty(G)} \Big\| \prod_{i=1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) \Big\|_{L^2(G)} \\
&\leq C(m) \|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s))\|_{L^\infty(G)} \prod_{i=1}^{j} \|\partial^{\gamma_i} v_1(t)\|_{H^{\tilde{m}-1-|\gamma_i|}(G)} \\
&\leq C(\theta, m, n, R) \|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s))\|_{L^\infty(G)}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\| (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s)) \prod_{i=1}^{k-1} \partial^{\gamma_i} v_{2,l_i}(s) \cdot (\partial^{\gamma_k} v_{1,l_k}(t) - \partial^{\gamma_k} v_{2,l_k}(s)) \cdot \prod_{i=k+1}^{j} \partial^{\gamma_i} v_{1,l_i}(t) \Big\|_{L^2(G)} \\
&\leq C(m) \max_{x \in \mathcal{U}_1} |(\partial_{l_j} \cdots \partial_{l_1} \theta)(x)| \prod_{i=1}^{k-1} \|\partial^{\gamma_i} v_{2,l_i}(s)\|_{H^{\tilde{m}-1-|\gamma_i|}(G)} \\
&\qquad \cdot \|\partial^{\gamma_k} v_{1,l_k}(t) - \partial^{\gamma_k} v_{2,l_k}(s)\|_{H^{\tilde{m}-1-|\gamma_k|}(G)} \cdot \prod_{i=k+1}^{j} \|\partial^{\gamma_i} v_{1,l_i}(t)\|_{H^{\tilde{m}-1-|\gamma_i|}(G)} \\
&\leq C(\theta, m, n, R, \mathcal{U}_1) \|\partial^{\gamma_k} v_1(t) - \partial^{\gamma_k} v_2(s)\|_{H^{\tilde{m}-1-|\gamma_k|}(G)}
\end{aligned}$$

for all $t \in J \setminus N_1$, $s \in J \setminus N_2$, and $k \in \{1, \ldots, j\}$. We insert these estimates into (7.9) and take the maximum of all involved constants, obtaining inequality (7.8). If $|\alpha| \leq m-2$ or θ belongs to $C^m(\mathcal{U}, \mathbb{R})$, we exploit that

$$\begin{aligned}
&\|(\partial_{l_j} \cdots \partial_{l_1} \theta)(v_1(t)) - (\partial_{l_j} \cdots \partial_{l_1} \theta)(v_2(s))\|_{L^\infty(G)} \\
&\leq \max_{x \in \mathcal{U}_1} |(\partial_{l_j} \cdots \partial_{l_1} \theta)'(x)| \|v_1(t) - v_2(s)\|_{L^\infty(G)} \leq C(\theta, m, n, \mathcal{U}_1) \|v_1(t) - v_2(s)\|_{H^2(G)}
\end{aligned}$$

for all $t \in J \setminus N_1$ and $s \in J \setminus N_2$, which yields (7.7). Finally, we note that $\gamma_{i,0} = 0$ for all $i \in \{1, \ldots j\}$ if $\alpha_0 = 0$, implying the final assertion.

(ii) Take $\gamma \geq 0$. We observe that

$$\|\theta(v_1(t)) - \theta(v_2(t))\|_{L^2(G)} \leq \max_{x \in \mathcal{U}_1} |\theta'(x)| \|v_1(t) - v_2(t)\|_{L^2(G)} \tag{7.12}$$

for almost all $t \in J$. We further employ estimate (7.7) with $s = t$, multiply with $e^{-\gamma t}$, and take the essential supremum and the maximum over all multiindices $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m-1$. In this way we derive assertion (ii).

(iii) We set $\tilde{v}_1(t) = v_1$ and $\tilde{v}_2(t) = v_2$ for all $t \in J$. Applying part (i) with $s = t$ and (7.12) to $\tilde{v}_1$ and $\tilde{v}_2$ yields the claim. □

7.2 Local existence

In this section we construct solutions of the nonlinear initial boundary value problem (1.6) with material laws χ and σ that are C^m with $m \geq 3$ and where χ is positive definite. We will see that, similar to the linear problem, an H^m-solution of (1.6) has to fulfill certain compatibility conditions. We will also provide a variant where χ only needs to be locally positive definite. Having constructed a local solution on a small time interval, we use standard techniques to extend it to a maximal solution. Moreover, we provide a first blow-up condition in the $H^m(G)$-norm, which follows from the fixed point argument.

Throughout this section we use the following assumptions. For a given integer m the set G denotes a subdomain of $\mathbb{R}^3$ which fulfills the uniform $C^{\tilde{m}+2}$ regularity condition, where $\tilde{m} = \max\{m, 3\}$. Moreover, $\mathcal{U}$ denotes a convex subdomain of $\mathbb{R}^6$.

We first prove that solutions of (1.6) are unique. By a solution of the nonlinear problem (1.6) we mean a function u which belongs to $\bigcap_{j=0}^{m} C^j(I, H^{m-j}(G))$ with $\overline{\operatorname{im} u(t)} \subseteq \mathcal{U}$ for all $t \in I$ which solves (1.6), where $\operatorname{im} u(t)$ means the image of $u(t)$, $I \subseteq \mathbb{R}$ is an interval with $t_0 \in I$, and m is an integer with $m \geq 3$. The proof relies on the basic L^2-a priori estimate and Corollary 7.2.

Lemma 7.3. *Let $t_0 \in \mathbb{R}, T > 0$, and $J = (t_0, t_0 + T)$. Let $m \in \mathbb{N}$ with $m \geq 3$. Take $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$ and $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$. Set $\chi = \zeta_1 \tilde{\chi}$ and $\sigma = \zeta_2 \tilde{\sigma}$ and suppose that χ is symmetric and uniformly positive definite. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$. Let u_1 and u_2 be two solutions in $G_m(J \times G)$ of* (1.6) *with inhomogeneity f, boundary value g, and initial value u_0 at initial time t_0. Then $u_1 = u_2$.*

Proof. Set

$$K = \{T_0 \in J \colon u_1 = u_2 \text{ on } [t_0, T_0]\}.$$

This set is nonempty since $u_1(t_0) = u_0 = u_2(t_0)$. Let $T_1 = \sup K$. The continuity of u_1 and u_2 implies that the two functions coincide on $[t_0, T_1]$.

Since u_1 and u_2 are solutions of (1.6) and belong to $G_m(J \times G)$, there is a compact subset $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $\operatorname{im} u_1, \operatorname{im} u_2 \subseteq \mathcal{U}_1$. We now assume that T_1 is not equal to T. We then take a time $T_u \in (T_1, T]$ to be fixed below and we set $J_u = [T_1, T_u]$. We observe that u_1 and u_2 are both solutions of (1.6) in $G_m(J_u \times G)$ with inhomogeneity f, boundary value g, and initial value $u_1(T_1) = u_2(T_1)$. In particular, both functions solve the linear initial boundary value problem (3.2) with data f, g, and $u_1(T_1)$ and differential operator $L(\chi(u_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_1))$ respectively $L(\chi(u_2), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_2))$. We abbreviate these operators by $L(\chi(u_1), \sigma(u_1))$ and $L(\chi(u_2), \sigma(u_2))$ in the following. Lemma 7.1, Lemma 2.22, and Sobolev's embedding yield that $\chi(u_1)$, $\chi(u_2)$, $\sigma(u_1)$, and $\sigma(u_2)$ are elements of $F_3^{\mathrm{c}}(J \times G)$. Choose a radius $r > 0$ such that

$$\max\{\|u_1\|_{G_m(J\times G)}, \|u_2\|_{G_m(J\times G)}, \|\zeta_1\|_{F_0(J\times G)}, \|\zeta_2\|_{F_0(J\times G)}\} \leq r.$$

Lemma 7.1 and Lemma 2.22 provide a radius $R = R(\chi, \sigma, r, \mathcal{U}_1)$ such that the bounds

$$\begin{aligned}
&\max\{\|\chi(u_1)\|_{F_3(J\times G)}, \|\sigma(u_1)\|_{F_3(J\times G)}\} \leq R, \\
&\max\{\|\chi(u_1(T_1))\|_{F_2^0(G)}, \max_{1\leq j\leq 2} \|\partial_t^j \chi(u_1)(T_1)\|_{H^{m-1-j}(G)}\} \leq R, \\
&\max\{\|\sigma(u_1(T_1))\|_{F_2^0(G)}, \max_{1\leq j\leq 2} \|\partial_t^j \sigma(u_1)(T_1)\|_{H^{m-1-j}(G)}\} \leq R
\end{aligned}$$

hold true, where $\mathcal{U}_1$ is a compact subset of $\mathcal{U}$ with $\operatorname{im} u_1(t), \operatorname{im} u_2(t) \subseteq \mathcal{U}_1$ for all $t \in J_u$. We further recall that $\chi(u_1)$ is symmetric and uniformly positive definite. Therefore, Theorem 5.6 for the differential operator $L(\chi(u_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_1))$ can be applied to $u_1 - u_2$. We take $\eta = \eta(\chi) > 0$ such that $\chi \geq \eta$ and set $\gamma = \gamma_{5.6,0}(\eta, R)$, where

$\gamma_{5.6,0}$ denotes the corresponding constant from Theorem 5.6. Theorem 5.6 and its proof, Lemma 2.22, and Corollary 7.2 (ii) then show that

$$\begin{aligned}
&\|u_1 - u_2\|^2_{G_{0,\gamma}(J_u \times G)} \\
&\leq C_{5.6}(\eta, R, T, G)\|(L(\chi(u_1), \sigma(u_1))u_1 - L(\chi(u_1), \sigma(u_1))u_2)\|^2_{L^2_\gamma(J_u \times G)} \\
&= C(\chi, \sigma, r, T, G)\|f - \chi(u_1)\partial_t u_2 - \sigma(u_1)u_2 + \chi(u_2)\partial_t u_2 + \sigma(u_2)u_2 - f\|^2_{L^2_\gamma(J_u \times G)} \\
&\leq C(\chi, \sigma, r, T, G)(T_u - T_1)\|\partial_t u_2\|^2_{L^\infty(J_u \times G)}\|\chi(u_1) - \chi(u_2)\|^2_{G_{0,\gamma}(J_u \times G)} \\
&\quad + C(\chi, \sigma, r, T, G)(T_u - T_1)\|u_2\|^2_{L^\infty(J_u \times G)}\|\sigma(u_1) - \sigma(u_2)\|^2_{G_{0,\gamma}(J_u \times G)} \\
&\leq C(\chi, \sigma, r, T, G, \mathcal{U}_1)(\|\partial_t u_2\|^2_{G_2(J_u \times G)} + \|u_2\|^2_{G_2(J_u \times G)})(T_u - T_1)\|u_1 - u_2\|^2_{G_{0,\gamma}(J_u \times G)},
\end{aligned}$$

where $C_{5.6}$ is the corresponding constant from Theorem 5.6. Fixing the generic constant in the last line of the above estimate, we choose T_u so small that

$$C(\chi, \sigma, r, T, G, \mathcal{U}_1)(\|\partial_t u_2\|^2_{G_2(J_u \times G)} + \|u_2\|^2_{G_2(J_u \times G)})(T_u - T_1) \leq \frac{1}{2}.$$

Hence,

$$\|u_1 - u_2\|_{G_{0,\gamma}(J_u \times G)} = 0,$$

implying $u_1 = u_2$ on $[T_1, T_u]$ and thus on $[t_0, T_u]$. This result contradicts the definition of T_1. We conclude that $T_1 = T$, i.e., $u_1 = u_2$ on J. □

We have now collected all the tools to prove the local existence theorem. However, before doing so, we take a more precise look on the compatibility conditions. Recall that the definition of the operators $S_{G,m,p}$ in (2.36) depends on time derivatives of $\chi(u)$ and $\sigma(u)$ in t_0, where u is an element of $\tilde{G}_m(J \times G)$. However, we would like to formulate the definition of $S_{G,m,p}$ independently of u as we are going to vary u in our fixed point argument. Lemma 7.1 fortunately shows that the time derivatives of $\chi(u)$, respectively $\sigma(u)$, in t_0 only depend on χ, respectively σ, and time derivatives of u at t_0.

Definition 7.4. *Let $J \subseteq \mathbb{R}$ be an open interval, $m \in \mathbb{N}$, $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ be time independent, and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1\tilde{\chi}$, $\sigma = \zeta_2\tilde{\sigma}$, and assume that χ is symmetric and uniformly positive definite. We then define the operators*

$$S_{\chi,\sigma,G,m,p} \colon \overline{J} \times H^{\max\{m,3\}}(J \times G) \times H^{\max\{m,2\}}(G, \mathcal{U}) \to H^{m-p}(G)$$

by $S_{\chi,\sigma,G,m,0}(t_0, f, u_0) = u_0$ and then inductively

$$\begin{aligned}
S_{\chi,\sigma,G,m,p}(t_0, f, u_0) = \chi(u_0)^{-1}\Big(&\partial_t^{p-1} f(t_0) - \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j S_{\chi,\sigma,G,m,p-1}(t_0, f, u_0) \\
&- \sum_{l=1}^{p-1}\binom{p-1}{l} M_1^l(t_0, f, u_0) S_{\chi,\sigma,G,m,p-l}(t_0, f, u_0) \\
&- \sum_{l=0}^{p-1}\binom{p-1}{l} M_2^l(t_0, f, u_0) S_{\chi,\sigma,G,m,p-1-l}(t_0, f, u_0)\Big),
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
M_k^p = \sum_{1\leq j\leq p}\ \sum_{\substack{\gamma_1,\ldots,\gamma_j \in \mathbb{N}_0^4\setminus\{0\} \\ \sum\gamma_i = (p,0,0,0)}}\ \sum_{l_1,\ldots,l_j=1}^{n} & C((p,0,0,0), j, l_1, \ldots, l_j, \gamma_1, \ldots, \gamma_j) \\
&\cdot \zeta_k(\partial_{l_j}\cdots\partial_{l_1}\theta_k)(u_0)\prod_{i=1}^{j} S_{\chi,\sigma,G,m,|\gamma_i|}(t_0, f, u_0)_{l_i}
\end{aligned} \tag{7.14}$$

for $1 \leq p \leq m$, $k \in \{1, 2\}$, where $\theta_1 = \tilde{\chi}$, $\theta_2 = \tilde{\sigma}$, $M_2^0 = \tilde{\sigma}(u_0)$, and C is the constant from Lemma 7.1. By $H^{\max\{m,2\}}(G, \mathcal{U})$ we mean those functions $u_0 \in H^{\max\{m,2\}}(G)$ with $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$.

We will show in Lemma 7.7 below that the range of $S_{\chi,\sigma,G,m,p}$ is indeed contained in $H^{m-p}(G)$. Before doing so, we note that the operators $S_{\chi,\sigma,G,m,p}$ are the right objects in order to handle higher order time derivatives of solutions of the nonlinear problem (1.6).

Lemma 7.5. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, $m \in \mathbb{N}$, $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ be time independent, and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$, and assume that χ is symmetric and uniformly positive definite. Choose $B \in W^{m+1,\infty}(J \times G)$, $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ with $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$. Assume that problem (1.6) has a solution u which belongs to $G_m(J \times G)$. Then*

$$\partial_t^j u(t) = S_{\chi,\sigma,G,m,j}(t, f, u(t)) \tag{7.15}$$

for all $t \in \overline{J}$ and $j \in \{0, \dots, m\}$.

Proof. The assertion follows inductively by differentiation of (1.6) and Lemma 7.1. □

We remark that the operators $S_{\chi,\sigma,G,m,j}$ and Lemma 7.5 are the nonlinear analogues to the linear operators $S_{G,m,j}$ and Lemma 2.31.

Motivated by the previous result and in analogy to Definition 2.32 in the linear case we introduce the following notion.

Definition 7.6. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, $m, k \in \mathbb{N}$, $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ be time independent, and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$, and assume that χ is symmetric and uniformly positive definite. Choose $B \in W^{m+1,\infty}(J \times G)^{k\times 6}$, $f \in H^m(J \times G)^6$, $g \in E_m(J \times \partial G)^k$, and $u_0 \in H^m(G)^6$.*

We say that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ satisfies the nonlinear compatibility conditions of order m *if $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ and*

$$\operatorname{tr}_{\partial G}(B S_{\chi,\sigma,G,m,p}(t_0, f, u_0)) = \partial_t^p g(t_0) \quad \text{for } 0 \le p \le m-1. \tag{7.16}$$

In the next lemma we collect several crucial properties of the operators $S_{\chi,\sigma,G,m,p}$.

Lemma 7.7. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take time independent $\zeta_1, \zeta_2 \in F_{\tilde{m},6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$, and assume that χ is symmetric and uniformly positive definite. Choose data $f, \tilde{f} \in H^{\tilde{m}}(J \times G)$ and $u_0, \tilde{u}_0 \in H^{\tilde{m}}(G)$ such that $\overline{\operatorname{im} u_0}$ and $\overline{\operatorname{im} \tilde{u}_0}$ is contained in $\mathcal{U}$. Take $r > 0$ such that*

$$\sum_{j=0}^{\tilde{m}-1} \|\partial_t^j f(t_0)\|_{H^{\tilde{m}-j-1}(G)} + \|u_0\|_{H^{\tilde{m}}(G)} \le r,$$
$$\sum_{j=0}^{\tilde{m}-1} \|\partial_t^j \tilde{f}(t_0)\|_{H^{\tilde{m}-j-1}(G)} + \|\tilde{u}_0\|_{H^{\tilde{m}}(G)} \le r.$$

Then the function $S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ belongs to $H^{m-p}(G)$ and there is a constant $C_1 = C_1(\chi, \sigma, m, r, \mathcal{U}_1)$ such that

$$\|S_{\chi,\sigma,G,m,p}(t_0, f, u_0)\|_{H^{m-p}(G)} \le C_1 \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)} \Big)$$

for all $p \in \{0, \dots, m\}$, where $\mathcal{U}_1$ is a compact subset of $\mathcal{U}$ such that $\operatorname{im} u_0 \subseteq \mathcal{U}_1$. Moreover, there is a constant $C_2 = C_2(\chi, \sigma, m, r, \mathcal{U}_2)$ with

$$\begin{aligned} &\|S_{\chi,\sigma,G,m,p}(t_0, f, u_0) - S_{\chi,\sigma,G,m,p}(t_0, \tilde{f}, \tilde{u}_0)\|_{H^{m-p}(G)} \\ &\le C_2 \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde{f}(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)} \Big) \end{aligned}$$

for all $p \in \{0, \dots, m\}$, where $\mathcal{U}_2$ is a compact subset of $\mathcal{U}$ such that $\operatorname{im} u_0, \operatorname{im} \tilde{u}_0 \subseteq \mathcal{U}_2$.

Proof. As the data f, $\tilde{f}$, u_0, $\tilde{u}_0$, and t_0 are fixed in this proof, we abbreviate the operators $S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ and $S_{\chi,\sigma,G,m,p}(t_0, \tilde{f}, \tilde{u}_0)$ by $S_{\chi,\sigma,G,m,p}$ respectively $\tilde{S}_{\chi,\sigma,G,m,p}$ for all $p \in \{0, \dots, m\}$. Analogously, we write M_k^l and $\tilde{M}_k^l$ for the operators $M_k^l(t_0, f, u_0)$ and $M_k^l(t_0, \tilde{f}, \tilde{u}_0)$ from (7.14) for all $(l, k) \in \{1, \dots, m\} \times \{1, 2\} \cup \{(0, 2)\}$.

We prove the assertion by induction with respect to p. Clearly, the claim is true for $p = 0$. Now assume that the assertion has been shown for all $p' \in \{0, \dots, p-1\}$ for some $p \in \{1, \dots, m\}$.

I) We first assume that $m \geq 3$ and thus $\tilde{m} = m$. Let $l \in \{1, \dots, p-1\}$, $j \in \{1, \dots, l\}$. Take $k_1, \dots, k_j \in \{1, \dots, l\}$ with $\sum_{i=1}^j k_i = l$. Let $v_0 \in H^{m-j-1}(G)$, $v_i \in H^{m-k_i}(G)$ for $i \in \{1, \dots, j\}$, and $v_{j+1} \in H^{m-p+l}(G)$.

Let us first consider the case $p \leq m-1$. Here we have

$$\begin{aligned}(m-j-1) + \sum_{i=1}^j (m-k_i) + (m-p+l) &= m-j-1+mj-l+m-p+l \\ &= mj-j+2m-p-1 \geq mj-j+3. \end{aligned} \tag{7.17}$$

If two of the above terms in paranthesis were strictly smaller than 2, we would obtain

$$(m-j-1) + \sum_{i=1}^j (m-k_i) + (m-p+l) \leq 2 + j(m-1) = mj - j + 2$$

and thus a contradiction to (7.17). Therefore, at most one summand in (7.17) is strictly smaller than 2 and we can successively apply Lemma 2.22 (vi) to the product $\prod_{i=0}^{j+1} v_i$. It hence belongs to $H^{m-p}(G)$ and satisfies

$$\Big\| \prod_{i=0}^{j+1} v_i \Big\|_{H^{m-p}(G)} \leq C \|v_0\|_{H^{m-j-1}(G)} \prod_{i=1}^j \|v_i\|_{H^{m-k_i}(G)} \|v_{j+1}\|_{H^{m-p+l}(G)}. \tag{7.18}$$

Now assume that $p = m$. In this case we infer

$$\begin{aligned}(m-j-1) + \sum_{i=1}^j (m-k_i) + (m-p+l) &= mj-j+m-1 \\ &\geq mj-j+2. \end{aligned} \tag{7.19}$$

If three of the above summands were strictly smaller than 2, it would follow

$$\begin{aligned}(m-j-1) + \sum_{i=1}^j (m-k_i) + (m-p+l) &\leq 3 + (j-1)(m-1) = mj-j-m+4 \\ &\leq mj-j+1, \end{aligned}$$

which contradicts inequality (7.19). So at most two summands in (7.19) are strictly smaller than 2. If one of them was 0, we would get

$$(m-j-1) + \sum_{i=1}^j (m-k_i) + (m-p+l) \leq 1 + j(m-1) = mj - j + 1,$$

again a contradiction to (7.19). We conclude that if two summands n_1 and n_2 in (7.19) are strictly smaller than 2, then they are equal to 1 and so $H^{n_1}(G) = H^{n_2}(G) = H^1(G) \hookrightarrow L^4(G)$. Hölder's inequality and Lemma 2.22 (vi) then yield that $\prod_{i=0}^{j+1} v_i$ belongs to $L^2(G)$ and that this product fulfills (7.18) with $p = m$. If less than two summands in (7.19) are strictly less than 2, we obtain from Lemma 2.22 (vi) again that $\prod_{i=0}^{j+1} v_i$ belongs to $L^2(G)$ and that the estimate (7.18) holds.

Let $\theta_1 = \tilde{\chi}$ and $\theta_2 = \tilde{\sigma}$. Take an index $k \in \{1, 2\}$, $l_1, \dots, l_j \in \{1, \dots, n\}$, and $\gamma_1, \dots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^j \gamma_i = (l, 0, 0, 0)$. Then the function $\partial_{l_j} \dots \partial_{l_1} \theta_k$ is an element of $C^{m-j}(\mathcal{U}, \mathbb{R}^{6\times 6})$ and Lemma 7.1 (ii) implies that $(\partial_{l_j} \dots \partial_{l_1}\theta_k)(u_0)$ belongs to $H^{m-j}(G)$. Lemma 2.22 (vii), estimate (7.18), Lemma 7.1 (ii), Corollary 7.2 (iii), and the induction hypothesis thus yield

$$\begin{aligned}
&\Big\| \zeta_k (\partial_{l_j} \dots \partial_{l_1}\theta_k)(u_0) \prod_{i=1}^{j} S_{\chi,\sigma,G,m,|\gamma_i|;l_i}\, S_{\chi,\sigma,G,m,p-l+1-k} \Big\|_{H^{m-p}(G)} \\
&\le C \|\zeta_k\|_{F^0_{\tilde m-1}(G)} \|(\partial_{l_j} \dots \partial_{l_1}\theta_k)(u_0)\|_{H^{m-j-1}(G)} \prod_{i=1}^{j} \|S_{\chi,\sigma,G,m,|\gamma_i|;l_i}\|_{H^{m-|\gamma_i|}(G)} \\
&\qquad \cdot \|S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p+l}(G)} \\
&\le C(\chi,\sigma,m,r,\mathcal{U}_1) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)} \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\Big\| \zeta_k (\partial_{l_j} \dots \partial_{l_1}\theta_k)(u_0) \prod_{i=1}^{j} S_{\chi,\sigma,G,m,|\gamma_i|;l_i}\, S_{\chi,\sigma,G,m,p-l+1-k} \\
&\quad - \zeta_k (\partial_{l_j} \dots \partial_{l_1}\theta_k)(\tilde u_0) \prod_{i=1}^{j} \tilde S_{\chi,\sigma,G,m,|\gamma_i|;l_i}\, \tilde S_{\chi,\sigma,G,m,p-l+1-k} \Big\|_{H^{m-p}(G)} \\
&\le C \|\zeta_k\|_{F^0_{\tilde m-1}(G)} \|(\partial_{l_j} \dots \partial_{l_1}\theta_k)(u_0) - (\partial_{l_j} \dots \partial_{l_1}\theta_k)(\tilde u_0)\|_{H^{m-j-1}(G)} \\
&\qquad \cdot \prod_{i=1}^{j} \|S_{\chi,\sigma,G,m,|\gamma_i|}\|_{H^{m-|\gamma_i|}(G)} \|S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p+l}(G)} \\
&\quad + C \|\zeta_k\|_{F^0_{\tilde m-1}(G)} \sum_{q=1}^{j} \|(\partial_{l_j} \dots \partial_{l_1}\theta_k)(\tilde u_0)\|_{H^{m-j-1}(G)} \prod_{i=1}^{q-1} \|\tilde S_{\chi,\sigma,G,m,|\gamma_i|}\|_{H^{m-|\gamma_i|}(G)} \\
&\qquad \cdot \|S_{\chi,\sigma,G,m,|\gamma_q|} - \tilde S_{\chi,\sigma,G,m,|\gamma_q|}\|_{H^{m-|\gamma_q|}(G)} \\
&\qquad \cdot \prod_{i=q+1}^{j} \|S_{\chi,\sigma,G,m,|\gamma_i|}\|_{H^{m-|\gamma_i|}(G)} \|S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p+l}(G)} \\
&\quad + C \|\zeta_k\|_{F^0_{\tilde m-1}(G)} \|(\partial_{l_j} \dots \partial_{l_1}\theta_k)(\tilde u_0)\|_{H^{m-j-1}(G)} \prod_{i=1}^{j} \|\tilde S_{\chi,\sigma,G,m,|\gamma_q|}\|_{H^{m-|\gamma_i|}(G)} \\
&\qquad \cdot \|S_{\chi,\sigma,G,m,p-l+1-k} - \tilde S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p+l}(G)} \\
&\le C(\chi,\sigma,m,r,\mathcal{U}_2) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde f(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde u_0\|_{H^m(G)} \Big). \qquad (7.20)
\end{aligned}$$

In view of the definitions, we have shown the estimates

$$\begin{aligned}
&\|M_k^l\, S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p}(G)} \\
&\le C(\chi,\sigma,m,r,\mathcal{U}_1) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)} \Big), \qquad (7.21) \\
&\|M_k^l\, S_{\chi,\sigma,G,m,p-l+1-k} - \tilde M_k^l\, \tilde S_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p}(G)} \\
&\le C(\chi,\sigma,m,r,\mathcal{U}_2) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde f(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde u_0\|_{H^m(G)} \Big). \qquad (7.22)
\end{aligned}$$

It remains to look at the case $l = 0$ and $k = 2$. As above we derive

$$\|M_2^0\, S_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)} \le C \|\sigma(u_0)\|_{F^0_m(G)} \|S_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)}$$

$$\leq C(\chi, \sigma, m, r, \mathcal{U}_1)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)}\Big) \tag{7.23}$$

and

$$\begin{aligned}
&\|M_2^0 S_{\chi,\sigma,G,m,p-1} - \tilde{M}_2^0\, \tilde{S}_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)} \\
&\leq \|\sigma(u_0) - \sigma(\tilde{u}_0)\|_{H^{m-1}(G)} \|S_{\chi,\sigma,G,m,p-1}\|_{H^{m-p+1}(\mathbb{R}^3_+)} \\
&\quad + \|\sigma(\tilde{u}_0)\|_{F^0_{m-1}(G)} \|S_{\chi,\sigma,G,m,p-1} - \tilde{S}_{\chi,\sigma,G,m,p-1}\|_{H^{m-p+1}(G)} \\
&\leq C(\chi, \sigma, m, r, \mathcal{U}_2)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde{f}(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)}\Big)
\end{aligned} \tag{7.24}$$

using Lemma 2.22 (vi) and (vii), Lemma 7.1 (ii), Corollary 7.2 (iii), and the induction hypothesis.

II) Since $\tilde{\chi}$ is an element of $C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$, Lemma 2.23, Lemma 2.22 (vii), and Lemma 7.1 (ii) show that $\chi^{-1}(u_0)$ belongs to $F^0_{m-1}(G)$ and that we can estimate $\|\chi^{-1}(u_0)\|_{F^0_{m-1}(G)} \leq C(\chi, m, r, \mathcal{U}_1)$. Lemma 2.23, Lemma 2.22 (vi), the induction hypothesis, (7.21), and (7.23) thus yield

$$\begin{aligned}
&\|S_{\chi,\sigma,G,m,p}\|_{H^{m-p}(G)} \\
&\leq C\|\chi^{-1}(u_0)\|_{F^0_{m-1}(G)} \Big(\|\partial_t^{p-1} f(t_0)\|_{H^{m-p}(G)} + \sum_{j=1}^{3} \|\partial_j S_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)} \\
&\qquad + \sum_{k=1}^{2}\sum_{l=1}^{p-1} \|M_k^l\, S_{\chi,\sigma,m,p-l+1-k}\|_{H^{m-p}(G)} + \|M_2^0\, S_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)}\Big) \\
&\leq C(\chi, \sigma, m, r, \mathcal{U}_1)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)}\Big)
\end{aligned}$$

for $1 \leq p \leq m$. This estimate trivially holds in the case $p = 0$. Corollary 7.2 (iii) together with Lemma 2.23 and Lemma 2.22 (vii) further shows that

$$\|\chi^{-1}(u_0) - \chi^{-1}(\tilde{u}_0)\|_{H^{m-1}(G)} \leq C(\chi, m, r, \mathcal{U}_2)\|u_0 - \tilde{u}_0\|_{H^{m-1}(G)}.$$

Combining this estimate with the induction hypothesis and (7.21) to (7.24) and (7.13), we deduce

$$\begin{aligned}
&\|S_{\chi,\sigma,G,m,p} - \tilde{S}_{\chi,\sigma,G,m,p}\|_{H^{m-p}(G)} \\
&\leq C(\chi, \sigma, m, r, \mathcal{U}_2)\|u_0 - \tilde{u}_0\|_{H^{m-1}(G)} \\
&\quad + C(\chi, m, r, \mathcal{U}_2)\Big(\|\partial_t^{p-1} f(t_0) - \partial_t^{p-1} \tilde{f}(t_0)\|_{H^{m-p}(G)} \\
&\qquad + \sum_{j=1}^{3} \|\partial_j S_{\chi,\sigma,G,m,p-1} - \partial_j \tilde{S}_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)} \\
&\qquad + \sum_{k=1}^{2}\sum_{l=1}^{p-1} \|M_k^l\, S_{\chi,\sigma,G,m,p-l+1-k} - \tilde{M}_k^l\, \tilde{S}_{\chi,\sigma,G,m,p-l+1-k}\|_{H^{m-p}(G)} \\
&\quad + \|M_2^0\, S_{\chi,\sigma,G,m,p-1} - \tilde{M}_2^0\, \tilde{S}_{\chi,\sigma,G,m,p-1}\|_{H^{m-p}(G)}\Big) \\
&\leq C(\chi, \sigma, m, r, \mathcal{U}_2)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0) - \partial_t^j \tilde{f}(t_0)\|_{H^{m-j-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)}\Big).
\end{aligned}$$

The induction hypothesis is thus also true for the index p. By induction, the assertion now follows for $m \geq 3$.

In the case $m \in \{1, 2\}$ the claim is shown by the same arguments, using that the data belong to $H^{\tilde{m}}(J \times G)$ respectively $H^{\tilde{m}}(G)$. □

Lemma 7.8. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m(J \times G)$ and $u_0 \in H^m(G)$ such that $\overline{\operatorname{im} u_0}$ is contained in $\mathcal{U}$. Let $r > 0$. Assume that f and u_0 satisfy*

$$\|u_0\|_{H^m(G)} \leq r, \qquad \max_{0 \leq j \leq m-1} \{\|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)}\} \leq r,$$
$$\|f\|_{G_{m-1}(J\times G)} \leq r, \quad \|f\|_{H^m(J\times G)} \leq r.$$

(i) Let $\hat{u} \in \tilde{G}_m(J \times G)$ with $\partial_t^p \hat{u}(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ for $0 \leq p \leq m-1$. Then

$$S_{G,m,p}(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0) \tag{7.25}$$

for all $p \in \{0, \ldots, m\}$.

(ii) There is a constant $C(\chi, \sigma, m, r, \mathcal{U}_1) > 0$ and a function u in $G_m(J \times G)$ with

$$\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$$

for all $p \in \{0, \ldots, m\}$ and

$$\|u\|_{G_m(J\times G)} \leq C(\chi, \sigma, m, r, \mathcal{U}_1)\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|u_0\|_{H^m(G)}\Big).$$

Here $\mathcal{U}_1$ denotes a compact subset of $\mathcal{U}$ with $\operatorname{im} u_0 \subseteq \mathcal{U}_1$.

Proof. (i) Assertion (i) follows by induction from the definition of the operators $S_{G,m,p}$ in (2.36), Lemma 7.1, and the definition of $S_{\chi,\sigma,G,m,p}$ in (7.13).

(ii) The assertion is a direct consequence of Lemma 2.34 and Lemma 7.7. □

Lemma 7.8 in particular shows, that for any $\hat{u} \in \tilde{G}_m(J \times G)$ with $\partial_t^j \hat{u}(0) = S_{\chi,\sigma,G,m,j}(t_0, f, u_0)$ for all $j \in \{0, \ldots, m-1\}$, the linear compatibility conditions (2.37) for the tuple $(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), B, f, g, u_0)$ are fulfilled if $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the nonlinear compatibility conditions (7.16).

In Lemma 2.31 we have seen that the linear compatibility conditions (2.37) are a necessary condition for the existence of a $G_m(J \times G)$-solution of (3.2). Analogously, the nonliner compatibility conditions are necessary for the existence of a $G_m(J \times G)$-solution of (1.6). For later reference, we formulate this fact as a lemma.

Lemma 7.9. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ such that $\overline{\operatorname{im} u_0}$ is contained in $\mathcal{U}$. Take $B \in W^{m+1,\infty}(G)$. Assume that there exists a $G_m(J \times G)$-solution u of (1.6) with inhomogeneity f, boundary value g, and initial value u_0 at t_0. Then*

$$\operatorname{tr}_G(B S_{\chi,\sigma,G,m,p}(t, f, u(t))) = \partial_t^p g(t)$$

for all $t \in \overline{J}$ and $0 \leq p \leq m-1$.

Proof. Recall that tr_G denotes the usual trace operator from $H^k(G)$ to $H^{k-1/2}(\partial G)$ for all $k \in \mathbb{N}$ and $(\operatorname{Tr}_{1,J\times G} u)(t) = \operatorname{tr}_G u(t)$ for all $t \in \overline{J}$ and $u \in C(\overline{J}, H^1(G))$, cf. Remark 2.17.

The definition of a solution yields that $g = \operatorname{Tr}_{1,J\times G}(Bu)$ and hence

$$g(t) = \operatorname{Tr}_{1,J\times G}(Bu)(t) = \operatorname{tr}_G B \operatorname{tr}_G u(t) = \operatorname{tr}_G B \operatorname{tr} S_{\chi,\sigma,G,m,0}(t, f, u(t))$$

for all $t \in \overline{J}$.

Next, fix an index $p \in \{1, \dots, m-1\}$. Then $\partial_t^p u$ still belongs to $G_1(J \times G)$. As time derivatives commute with $\mathrm{tr}_{\partial G}$ for smooth functions, this property extends to u for up to $m-1$ time derivatives. Hence,

$$\partial_t^p g(t) = \partial_t^p \operatorname{tr}_{\partial G}(Bu(t)) = \operatorname{tr}_{\partial G}(B\partial_t^p u(t)) = \operatorname{Tr}_{1,J\times G}(B\partial_t^p u)(t),$$

where we used that $\partial_t^p u \in G_1(J \times G)$. We infer that

$$\partial_t^p g(t) = \operatorname{Tr}_{1,J\times G}(B\partial_t^p u)(t) = B \operatorname{tr}_{\partial G}(\partial_t^p u(t)) = B \operatorname{tr} S_{\chi,\sigma,G,m,p}(t, f, u(t))$$

for all $t \in \overline{J}$, inserting that by Lemma 7.5 and assumption

$$\partial_t^p u(t) = S_{\chi,\sigma,G,m,p}(t, f, u(t)). \qquad \square$$

Finally, we can combine all the preparations and prove the desired local existence result. We apply Banach's fixed point argument. In order to show the self-mapping and the contraction property, we heavily rely on our a priori estimates. We further point out that the special structure of the constants we derived in Chapter 3 is crucial for the self-mapping property.

Theorem 7.10. *Let $t_0 \in \mathbb{R}$, $T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Let*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ with $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ such that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the nonlinear compatibility conditions (7.16) *of order m. Choose a radius $r > 0$ with*

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_m(J\times\partial G)} + \|u_0\|^2_{H^m(G)} + \|f\|^2_{H^m(J\times G)} \leq r^2,$$
$$\|\zeta_1\|_{F_m(J\times G)} + \|\zeta_2\|_{F_m(J\times G)} \leq r.$$

Take a number $\kappa > 0$ such that

$$\operatorname{dist}(\overline{\{u_0(x) \colon x \in G\}}, \partial\mathcal{U}) > \kappa.$$

Then there exists a time $\tau = \tau(\chi, \sigma, m, T, r, \kappa) > 0$ such that the nonlinear initial boundary value problem (1.6) *with inhomogeneity f, boundary value g, and initial value u_0 has a unique solution u on $[t_0, t_0 + \tau]$ which belongs to $G_m(J_\tau \times G)$, where $J_\tau = (t_0, t_0 + \tau)$.*

Proof. Without loss of generality we assume $t_0 = 0$. If $f = 0$, $g = 0$, and $u_0 = 0$, then $u = 0$ is a $G_m(J \times G)$-solution of (1.6) and it is unique by Lemma 7.3. So in the following we assume $\|f\|_{H^m(J\times G)} + \|g\|_{E_m(J\times\partial G)} + \|u_0\|_{H^m(G)} > 0$. Recall that the map $S_{\chi,\sigma,G,m,p}$ was defined in (7.13) for $0 \leq p \leq m$. Let $\tau \in (0, T]$. We set $J_\tau = (0, \tau)$ and

$$\mathcal{U}_\kappa = \{y \in \mathcal{U} \colon \operatorname{dist}(y, \partial\mathcal{U}) \geq \kappa\} \cap \overline{B}_{2C_{\mathrm{Sob}} r}(0),$$

where C_{Sob} denotes the constant for the Sobolev embedding from $H^2(G)$ into $L^\infty(G)$. Then $\mathcal{U}_\kappa$ is compact and $\overline{\operatorname{im} u_0}$ is contained in $\mathcal{U}_\kappa$.

I) Let $R > 0$. We set

$$B_R(J_\tau) := \{v \in \tilde{G}_m(J_\tau \times G) \colon \|v\|_{G_m(J_\tau\times G)} \leq R,\ \|v - u_0\|_{L^\infty(J_\tau\times G)} \leq \kappa/2$$
$$\partial_t^j v(0) = S_{\chi,\sigma,G,m,j}(0, f, u_0) \text{ for } 0 \leq j \leq m-1, \}$$

and equip it with the metric $d(v_1, v_2) = \|v_1 - v_2\|_{G_{m-1}(J_\tau \times G)}$. We first show that $B_R(J_\tau)$ is a complete metric space. Recall that $\tilde{G}_m(J_\tau \times G)$ is continuously embedded in $G_{m-1}(J_\tau \times G)$ so that $B_R(J_\tau)$ is well defined. Moreover, Lemma 7.8 (ii) shows that there is a radius $R_{7.8(ii)}(\chi, \sigma, m, r, \mathcal{U}_\kappa)$ such that $B_R(J_\tau)$ is nonempty for all $R > C_{7.8(ii)}(\chi, \sigma, m, r, \mathcal{U}_\kappa) \cdot (m+1)r$.

Let $(v_n)_n$ be a Cauchy sequence in $(B_R(J_\tau), d)$. The functions v_n then tend to v in $G_{m-1}(J_\tau \times G)$ as $n \to \infty$, and hence v satisfies $\partial_t^j v(0) = S_{\chi,\sigma,G,m,j}(0, f, u_0)$ for $0 \leq j \leq m-1$ and $\|v\|_{G_{m-1}(J_\tau \times G)} \leq R$. Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$. The sequence $(\partial^\alpha v_n)_n$ is bounded in $L^\infty(J_\tau, L^2(G)) = (L^1(J_\tau, L^2(G)))^*$. The Banach-Alaoglu Theorem thus gives a σ^*-convergent subsequence which we again denote by $(\partial^\alpha v_n)_n$. Its σ^*-limit in $L^\infty(J_\tau, L^2(G))$ is denoted by v_α. Let $\varphi \in C_c^\infty(J_\tau \times G)$. The above convergence results then imply

$$\langle \varphi, v_\alpha \rangle = \lim_{n\to\infty} \langle \varphi, \partial^\alpha v_n \rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle \partial^\alpha \varphi, v_n \rangle = (-1)^{|\alpha|} \langle \partial^\alpha \varphi, v \rangle = \langle \varphi, \partial^\alpha v \rangle$$

so that $\partial^\alpha v = v_\alpha \in L^\infty(J_\tau, L^2(G))$. In particular, v belongs to $\tilde{G}_m(J_\tau \times G)$ and

$$\|\partial^\alpha v\|_{G_0(J \times G)} = \|\partial^\alpha v\|_{L^\infty(J_\tau, L^2(G))} = \|v_\alpha\|_{L^\infty(J_\tau, L^2(G))} \leq R$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$. Finally, as $m \geq 3$, we infer

$$\begin{aligned} \|v - u_0\|_{L^\infty(J_\tau \times G)} &\leq \|v - v_n\|_{L^\infty(J_\tau \times G)} + \|v_n - u_0\|_{L^\infty(J_\tau \times G)} \\ &\leq C_{\mathrm{Sob}} \|v - v_n\|_{G_2(J_\tau \times G)} + \kappa \longrightarrow \kappa \end{aligned}$$

as $n \to \infty$, where such a constant C_{Sob} exists due to Sobolev's embedding. We conclude that v again belongs to $B_R(J_\tau)$.

II) Let $\hat{u} \in B_R(J_\tau)$. Take $\eta = \eta(\chi) > 0$ such that $\chi \geq \eta$. Then $\chi(\hat{u})$ is contained in $F_{m,\eta}^{\mathrm{c}}(J \times G)$ and $\sigma(\hat{u})$ is an element of $F_m^{\mathrm{c}}(J \times G)$ by Lemma 7.1 and Sobolev's embedding. Lemma 7.8 (i) and the assumption that $(\chi, \sigma, t_0, B, f, g, u_0)$ is compatible imply that the tuple $(\chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), B, f, g, u_0)$ fulfills the linear compatibility conditions (2.37). By Theorem 5.6 the linear initial boundary value problem (3.2) with differential operator $L(\chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}))$, inhomogeneity f, boundary value g, and initial value u_0 has a solution in $G_m(J_\tau \times G)$ which we denote by $\Phi(\hat{u})$. One thus defines a mapping Φ from $B_R(J_\tau)$ to $G_m(J_\tau \times G)$. We want to prove that Φ also maps $B_R(J_\tau)$ into $B_R(J_\tau)$ for a suitable radius R and a sufficiently small time interval J_τ.

For this purpose take numbers $\tau \in (0, T]$ and $R > C_{7.8(ii)}(\chi, \sigma, m, T, r)(m+1)r$ which will be fixed below. Let $\hat{u} \in B_R(J_\tau)$. We first note that there is a constant $C_{7.7}(\chi, \sigma, m, r, \mathcal{U}_\kappa)$ such that

$$\|S_{\chi,\sigma,G,m,p}(0, f, u_0)\|_{H^{m-p}(\mathbb{R}^3_+)} \leq C_{7.7}(\chi, \sigma, m, r, \mathcal{U}_\kappa)$$

for all $p \in \{0, \dots, m\}$ due to Lemma 7.7. Lemma 7.1 (ii) and Lemma 2.22 (vii) further provide a constant $C_{7.1(ii)}$ such that

$$\begin{aligned} \|\chi(\hat{u})(0)\|_{F_{m-1}^0(G)} &= \|\chi(u_0)\|_{F_{m-1}^0(G)} \leq C_{7.1(ii)}(\chi, m, 6, r, \mathcal{U}_\kappa), \\ \|\sigma(\hat{u})(0)\|_{F_{m-1}^0(G)} &= \|\sigma(u_0)\|_{F_{m-1}^0(G)} \leq C_{7.1(ii)}(\sigma, m, 6, r, \mathcal{U}_\kappa). \end{aligned}$$

Note that $\overline{\operatorname{im} \hat{u}}$ is contained in the compact set

$$\tilde{\mathcal{U}}_\kappa = \mathcal{U}_\kappa + \overline{B}(0, \kappa/2) \subseteq \mathcal{U}$$

as $\hat{u} \in B_R(J_\tau)$. Part (iii) of Lemma 7.1 once more combined with Lemma 2.22 (vii) yields

$$\begin{aligned} &\|\partial_t^l \chi(\hat{u})(0)\|_{H^{m-l-1}(G)} \leq C_{7.1(iii)}(\chi, m, 6, r, \mathcal{U}_\kappa)(1 + \max_{0\leq k\leq l} \|\partial_t^k \hat{u}(0)\|_{H^{m-k-1}(G)})^m \\ &= C_{7.1(iii)}(\chi, m, 6, r, \mathcal{U}_\kappa)(1 + \max_{0\leq k\leq l} \|S_{\chi,\sigma,G,m,k}(0, f, u_0)\|_{H^{m-k-1}(G)})^m \end{aligned}$$

$$\leq C_{7.1(iii)}(\chi, m, 6, r, \mathcal{U}_\kappa)(1 + C_{7.7}(\chi, \sigma, m, r, \mathcal{U}_\kappa))^m,$$
$$\|\partial_t^l \sigma(\hat{u})(0)\|_{H^{m-l-1}(G)} \leq C_{7.1(iii)}(\sigma, m, 6, r, \mathcal{U}_\kappa)(1 + C_{7.7}(\chi, \sigma, m, r, \mathcal{U}_\kappa))^m$$

for all $l \in \{1, \ldots, m-1\}$. We thus find a radius $r_0 = r_0(\chi, \sigma, m, r, \kappa)$ such that

$$\max\{\|\chi(\hat{u})(0)\|_{F^0_{m-1}(G)}, \max_{1 \leq l \leq m-1} \|\partial_t^l \chi(\hat{u})(0)\|_{H^{m-l-1}(G)}\} \leq r_0,$$
$$\max\{\|\sigma(\hat{u})(0)\|_{F^0_{m-1}(G)}, \max_{1 \leq l \leq m-1} \|\partial_t^l \sigma(\hat{u})(0)\|_{H^{m-l-1}(G)}\} \leq r_0. \tag{7.26}$$

As $\hat{u}$ belongs to $B_R(J_\tau)$, Lemma 7.1 (i) gives

$$\|\chi(\hat{u})\|_{F_m(J \times G)} \leq C_{7.1(i)}(\chi, m, 6, R, \tilde{\mathcal{U}}_\kappa)(1 + R)^m,$$
$$\|\sigma(\hat{u})\|_{F_m(J \times G)} \leq C_{7.1(i)}(\sigma, m, 6, R, \tilde{\mathcal{U}}_\kappa)(1 + R)^m.$$

We thus obtain a radius $R_1 = R_1(\chi, \sigma, m, R, \kappa)$ with

$$\|\chi(\hat{u})\|_{F_m(J \times G)} \leq R_1 \qquad \text{and} \qquad \|\sigma(\hat{u})\|_{F_m(J \times G)} \leq R_1. \tag{7.27}$$

We next define the constant $C_{m,0} = C_{m,0}(\chi, \sigma, r, \kappa)$ by

$$C_{m,0}(\chi, \sigma, r, \kappa) = C_{5.6,m,0}(\eta(\chi), r_0(\chi, \sigma, m, r, \kappa)), \tag{7.28}$$

where $C_{5.6,m,0}$ denotes the constant $C_{m,0}$ from Theorem 5.6. We will suppress the dependance of the constants on the domain G as G remains fixed. We set the radius $R = R(\chi, \sigma, m, r, \kappa)$ for $B_R(J_\tau)$ to be

$$R(\chi, \sigma, m, r, \kappa) = \max\Big\{4\sqrt{C_{m,0}(\chi, \sigma, r, \kappa)}\, r,\ C_{7.8(ii)}(\chi, \sigma, m, r, \mathcal{U}_\kappa)(m+1)r + 1\Big\}. \tag{7.29}$$

We further introduce the constants $\gamma_m = \gamma_m(\chi, \sigma, T, r, \kappa)$ and $C_m = C_m(\chi, \sigma, T, r, \kappa)$ by

$$\gamma_m = \gamma_m(\chi, \sigma, T, r, \kappa) = \gamma_{5.6,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa)), T), \tag{7.30}$$
$$C_m = C_m(\chi, \sigma, T, r) = C_{5.6,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa)), T), \tag{7.31}$$

where $\gamma_{5.6,m}$ and $C_{5.6,m}$ denote the corresponding constants from Theorem 5.6. Let

$$C_{7.2(ii)}(\theta, m, 6, R, \tilde{\mathcal{U}}_\kappa)$$

denote the corresponding constant from Corollary 7.2 (ii) for all $\theta \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$.

With these constants at hand we define the parameter $\gamma = \gamma(\chi, \sigma, m, T, r, \kappa)$ and the time step $\tau = \tau(\chi, \sigma, m, T, r, \kappa)$ by

$$\gamma = \max\Big\{\gamma_m,\ C_{m,0}^{-1} C_m\Big\}, \tag{7.32}$$
$$\tau = \min\Big\{T,\ (2\gamma + mC_{5.6,1})^{-1}\log 2,\ C_m^{-1} C_{m,0},\ (C_{\mathrm{Sob}} R)^{-1}\kappa, \tag{7.33}$$
$$[32 r^2 R^2 C_{m,0} C_{2.22}^4 (C_{7.2(ii)}^2(\chi, m, 6, R, \tilde{\mathcal{U}}_\kappa) + C_{7.2(ii)}^2(\sigma, m, 6, R, \tilde{\mathcal{U}}_\kappa))]^{-1}\Big\},$$

where $C_{2.22}$ and $C_{5.6,1}$ denote the corresponding constants from Lemma 2.22 and Theorem 5.6 respectively. Observe that γ and τ actually only depend on χ, σ, m, T, r, and κ as $C_{m,0}$, C_m, and R only depend on these quantities (see (7.28) to (7.31)). For later reference we note that the choice of γ and τ implies

$$\gamma \geq \gamma_m, \tag{7.34}$$
$$\frac{C_m}{\gamma} \leq C_{m,0}, \tag{7.35}$$

$$\tau \le T, \tag{7.36}$$
$$(2\gamma + mC_{5.6,1})\,\tau \le \log 2, \tag{7.37}$$
$$C_m\tau \le C_{m,0}, \tag{7.38}$$
$$C_{\mathrm{Sob}}R\tau \le \kappa, \tag{7.39}$$
$$4C_{m,0}C_{2.22}^4C_{7.2(ii)}^2(\theta, m, 6, R, \tilde{\mathcal{U}}_\kappa)r^2R^2\,\tau \le \frac{1}{8}, \qquad \theta \in \{\tilde{\chi}, \tilde{\sigma}\}. \tag{7.40}$$

III) Recall that $\hat{u} \in B_R(J_\tau)$ and that $\Phi(\hat{u})$ denotes the $G_m(J \times G)$-solution of (3.2) with differential operator $L(\chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}))$, inhomogeneity f, boundary value g, and initial value u_0. We want to bound $\Phi(\hat{u})$ by means of Theorem 5.6. In view of the estimates (7.26) and (7.27), the definitions of $C_{m,0}$, γ_m, and C_m in (7.28), (7.30), and (7.31), respectively, fit to the assertion of Theorem 5.6. Using also (7.34) and (7.36), we arrive at the inequality

$$\begin{aligned}\|\Phi(\hat{u})\|_{G_m(J_\tau\times G)}^2 &\le e^{2\gamma\tau}\|\Phi(\hat{u})\|_{G_{m,\gamma}(J_\tau\times G)}^2 \\ &\le (C_{m,0} + \tau C_m)e^{(2\gamma+mC_{5.6,1})\tau}\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|_{H^{m-1-j}(G)}^2 \\ &\qquad + \|g\|_{E_{m,\gamma}(J\times\partial G)}^2 + \|u_0\|_{H^m(G)}^2\Big) + \frac{C_m}{\gamma}e^{(2\gamma+mC_1)\tau}\|f\|_{H_\gamma^m(J_\tau\times G)}^2.\end{aligned}$$

Observe that

$$\|f\|_{H_\gamma^m(J_\tau\times G)}^2 \le \|f\|_{H^m(J_\tau\times G)}^2 \le \|f\|_{H^m(J\times G)}^2 \le r^2$$

and analogously $\|g\|_{E_{m,\gamma}(J\times\partial G)}^2 \le r^2$. Employing (7.35), (7.38), (7.37), and (7.29), we then deduce

$$\begin{aligned}&\|\Phi(\hat{u})\|_{G_m(J_\tau\times G)}^2 \le (C_{m,0} + C_{m,0})e^{\log 2}r^2 + C_{m,0}e^{\log 2}r^2 = 6C_{m,0}r^2 \le R^2, \\ &\|\Phi(\hat{u})\|_{G_m(J_\tau\times G)} \le R.\end{aligned}$$

Since $\Phi(\hat{u})$ belongs to $G_m(J \times G)$, Lemma 2.31 shows that

$$\partial_t^p\Phi(\hat{u})(0) = S_{G,m,p}(0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0)$$

for all $p \in \{0, \dots, m\}$. On the other hand, as an element of $B_R(J_\tau)$, the function $\hat{u}$ satisfies $\partial_t^p\hat{u}(0) = S_{\chi,\sigma,m,p}(0, f, u_0)$ for all $p \in \{0, \dots, m-1\}$. Lemma 7.8 (i) thus yields

$$\partial_t^p\Phi(\hat{u})(0) = S_{G,m,p}(0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0) = S_{\chi,\sigma,m,p}(0, f, u_0)$$

for all $p \in \{0, \dots, m-1\}$.

We further estimate

$$\begin{aligned}\|\Phi(\hat{u}) - u_0\|_{L^\infty(J_\tau\times G)} &= \Big\|\Phi(\hat{u})(0) + \int_0^t \partial_t\Phi(\hat{u})(s)ds - u_0\Big\|_{L^\infty(J_\tau\times G)} \\ &= \Big\|\int_0^t \partial_t\Phi(\hat{u})(s)ds\Big\|_{L^\infty(J_\tau\times G)} \le C_{\mathrm{Sob}}\sup_{t\in(0,\tau)}\int_0^t \|\partial_t\Phi(\hat{u})(s)\|_{H^2(G)}ds \\ &\le C_{\mathrm{Sob}}\tau\|\partial_t\Phi(\hat{u})\|_{G_2(J_\tau\times G)} \le C_{\mathrm{Sob}}\tau R \le \kappa\end{aligned}$$

for all $\hat{u} \in B_R(J_\tau)$, where we used that $\Phi(\hat{u})(0) = u_0$ for $\hat{u} \in B_R(J_\tau)$ and (7.39). We conclude that $\Phi(\hat{u})$ belongs to $B_R(J_\tau)$, i.e., Φ maps $B_R(J_\tau)$ into itself.

IV) Let $\hat{u}_1, \hat{u}_2 \in B_R(J_\tau)$. Since the functions $\chi(\hat{u}_i)$ and $\sigma(\hat{u}_i)$ belong to $F_m(J_\tau \times G)$ for $i \in \{1, 2\}$, Lemma 2.22 implies that $\chi(\hat{u}_i)\partial_t\Phi(\hat{u}_2)$ and $\sigma(\hat{u}_i)\Phi(\hat{u}_2)$ are elements of $\tilde{G}_{m-1}(J_\tau \times G) \hookrightarrow H^{m-1}(J_\tau \times G)$ for $i \in \{1, 2\}$. The function $\Phi(\hat{u}_2)$ thus fulfills

$$L(\chi(\hat{u}_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_1))\Phi(\hat{u}_2)$$

$$
\begin{aligned}
&= \chi(\hat{u}_1)\partial_t\Phi(\hat{u}_2) + \sigma(\hat{u}_1)\Phi(\hat{u}_2) - \chi(\hat{u}_2)\partial_t\Phi(\hat{u}_2) - \sigma(\hat{u}_2)\Phi(\hat{u}_2) \\
&\qquad + L(\chi(\hat{u}_2), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_2))\Phi(\hat{u}_2) \\
&= (\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2) + (\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2) + f
\end{aligned}
$$

and this function belongs to $\tilde{G}_{m-1}(J_\tau \times G) \hookrightarrow H^{m-1}(J_\tau \times G)$. We further stress that $\Phi(\hat{u}_1)(0) = u_0 = \Phi(\hat{u}_2)(0)$.

As in step III), (7.26), (7.27), (7.28), (7.30), (7.31), (7.34), and (7.36) allow us to apply Theorem 5.6 on $J_\tau \times G$ with differential operator $L(\chi(\hat{u}_1), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}_1))$ and parameter γ to obtain

$$
\begin{aligned}
&\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau\times G)} \le e^{2\gamma\tau}\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau\times G)} \\
&\le (C_{m,0} + \tau C_m)e^{(2\gamma + mC_1)\tau}\sum_{j=0}^{m-2}\|\partial_t^j(f - L\Phi(\hat{u}_2))(0)\|^2_{H^{m-2-j}(G)} \\
&\qquad + \frac{C_m}{\gamma}e^{(2\gamma+mC_1)\tau}\|f - L\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau\times G)} \\
&= (C_{m,0} + \tau C_m)e^{(2\gamma + mC_1)\tau}\sum_{j=0}^{m-2}\|\partial_t^j(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2))(0) \\
&\qquad\qquad + \partial_t^j((\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2))(0)\|^2_{H^{m-2-j}(G)} \\
&\quad + \frac{C_m}{\gamma}e^{(2\gamma+mC_1)\tau}\|(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2) + (\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau\times G)}.
\end{aligned}
$$

Lemma 2.22, Lemma 7.1, and

$$
\partial_t^l\hat{u}_1(0) = S_{\chi,\sigma,m,l}(0, f, u_0) = \partial_t^l\hat{u}_2(0)
$$

for all $l \in \{0, \ldots, m-1\}$ imply that

$$
\begin{aligned}
&\partial_t^j(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2))(0) = 0, \\
&\partial_t^j((\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2))(0) = 0
\end{aligned}
$$

for all $j \in \{0, \ldots, m-2\}$. Employing (7.37) and the triangle inequality, we then deduce

$$
\begin{aligned}
\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau\times G)} &\le 4C_m\frac{1}{\gamma}\|(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau\times G)} \\
&\quad + 4C_m\frac{1}{\gamma}\|(\sigma(\hat{u}_1) - \sigma(\hat{u}_2))\Phi(\hat{u}_2)\|^2_{H^{m-1}_\gamma(J_\tau\times G)} \\
&=: I_1 + I_2. \qquad (7.41)
\end{aligned}
$$

Before going on, we point out that we know from step II) that $\Phi(\hat{u}_2)$ is an element of $B_R(J_\tau)$ and hence

$$
\|\partial_t\Phi(\hat{u}_2)\|_{G_{m-1}(J_\tau\times G)} \le \|\Phi(\hat{u}_2)\|_{G_m(J_\tau\times G)} \le R. \qquad (7.42)
$$

We now treat the first summand. Lemma 2.22, (7.42), and Corollary 7.2 (ii) show that

$$
\begin{aligned}
I_1 &\le 4C_m\frac{1}{\gamma}\tau\|(\chi(\hat{u}_1) - \chi(\hat{u}_2))\partial_t\Phi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau\times G)} \\
&\le 4C_m\frac{1}{\gamma}\tau C_{2.22}^2\|\chi(\hat{u}_1) - \chi(\hat{u}_2)\|^2_{G_{m-1,\gamma}(J_\tau\times G)}\|\partial_t\Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau\times G)} \\
&\le 4C_m\frac{1}{\gamma}C_{2.22}^4 r^2 C_{7.2(ii)}^2(\tilde{\chi}, m, 6, R, \tilde{\mathcal{U}}_\kappa)R^2\tau\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1,\gamma}(J_\tau\times G)}.
\end{aligned}
$$

Exploiting (7.35) and (7.40), we finally arrive at

$$
I_1 \le 4C_{m,0}C_{2.22}^4 C_{7.2(ii)}^2(\tilde{\chi}, m, 6, R)r^2R^2\tau\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1,\gamma}(J_\tau\times G)}
$$

$$\leq \frac{1}{8}\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1,\gamma}(J_\tau \times G)} \leq \frac{1}{8}\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1}(J_\tau \times G)}. \tag{7.43}$$

Analogously, we obtain

$$I_2 \leq \frac{1}{8}\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1}(J_\tau \times G)}. \tag{7.44}$$

Estimates (7.41), (7.43), and (7.44) imply

$$\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|^2_{G_{m-1}(J_\tau \times G)} \leq \frac{1}{4}\|\hat{u}_1 - \hat{u}_2\|^2_{G_{m-1}(J_\tau \times G)},$$
$$\|\Phi(\hat{u}_1) - \Phi(\hat{u}_2)\|_{G_{m-1}(J_\tau \times G)} \leq \frac{1}{2}\|\hat{u}_1 - \hat{u}_2\|_{G_{m-1}(J_\tau \times G)}.$$

We conclude that Φ is a strict contraction on $B_R(J_\tau)$.

V) So step I) and (7.29) show that $(B_R(J_\tau), d)$ is a nonempty, complete metric space. Steps III) and IV) yield that Φ is a strict contractive selfmapping on $B_R(J_\tau)$. Banach's fixed point theorem thus gives a fixed point $u \in B_R(J_\tau)$, i.e., $\Phi(u) = u$. By definition of Φ, this means that u is a solution of the initial boundary value problem

$$\begin{cases} \chi(u)\partial_t u + \sum_{j=1}^{3} A_j \partial_j u + \sigma(u)u = f, & x \in G, \quad t \in J_\tau; \\ Bu = g, & x \in \partial G, \quad t \in J_\tau; \\ u(0) = u_0, & x \in G; \end{cases}$$

i.e., the function $u \in B_R(J) \subseteq G_m(J \times G)$ is a solution of the nonlinear initial boundary value problem (1.6). Lemma 7.3 shows that u is the unique solution of (1.6) on $[0, \tau]$. □

We want to point out that in the important special case where $\mathcal{U} = \mathbb{R}^6$ the assumption on the range of u_0 in Theorem 7.10 and the results before is empty, i.e., there is no assumption on the range of the initial value. The same is true for the results that will follow although we will not stress this observation every time. However, at least for our main result of this section, we want to state this special case explicitly.

Theorem 7.11. *Let $t_0 \in \mathbb{R}$, $T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathbb{R}^6, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Let*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ with $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ such that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the nonlinear compatibility conditions (7.16) *of order m. Choose a radius $r > 0$ with*

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|g\|^2_{E_m(J\times\partial G)} + \|u_0\|^2_{H^m(G)} + \|f\|^2_{H^m(J\times G)} \leq r^2,$$
$$\|\zeta_1\|_{F_m(J\times G)} + \|\zeta_2\|_{F_m(J\times G)} \leq r.$$

Then there exists a time $\tau = \tau(\chi, \sigma, m, T, r) > 0$ such that the nonlinear initial boundary value problem (1.6) *with inhomogeneity f, boundary value g, and initial value u_0 has a unique solution u on $[t_0, t_0 + \tau]$ which belongs to $G_m(J_\tau \times G)$, where $J_\tau = (t_0, t_0 + \tau)$.*

Remark 7.12. Let $t_0 \in \mathbb{R}$, $T > 0$ and $\tilde{J} = (-T + t_0, t_0)$. Let $m \geq 3$ and χ, σ, B, u_0, and κ as in Theorem 7.10. Let $f \in H^m(\tilde{J} \times G)$ and $g \in E_m(\tilde{J} \times G)$. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the nonlinear compatibility conditions (7.16). Take a radius $r > 0$ as in Theorem 7.10. Let $\tau = \tau(\chi, -\sigma, m, T, r, \kappa)$ from Theorem 7.10. We want to show that the problem

$$\begin{cases} \chi(u)\partial_t u + \sum_{j=1}^{3} A_j \partial_j u + \sigma(u) u = f, & x \in \mathbb{R}^3_+, \quad t \in \tilde{J}; \\ Bu = g, & x \in \partial\mathbb{R}^3_+, \quad t \in \tilde{J}; \\ u(t_0) = u_0, & x \in \mathbb{R}^3_+; \end{cases}$$

has a unique solution on $[-\tau + t_0, t_0]$. To that purpose, we introduce $J = (t_0, t_0 + T)$, $\tilde{A}_j = -A_j$ for $j \in \{1, 2, 3\}$, $\hat{\sigma} = -\sigma$, and we set $\tilde{f}(t) = -f(2t_0 - t)$ and $\tilde{g}(t) = g(2t_0 - t)$ for almost all $t \in J$. Observe that $\tilde{f}$ belongs to the space $H^m(J \times G)$ with $\|f\|_{H^m(J\times G)} = \|\tilde{f}\|_{H^m(\tilde{J}\times G)} \leq r$ and $\|\partial_t^j \tilde{f}(t_0)\|_{H^{m-1-j}(G)} = \|\partial_t^j f(t_0)\|_{H^{m-1-j}(G)}$ for all $j \in \{0, \ldots, m-1\}$, and $\|\tilde{g}\|_{E_m(J\times\partial G)} = \|g\|_{E_m(J\times\partial G)}$. We want to apply Theorem 7.10 to the system

$$\begin{cases} \chi(v)\partial_t v + \sum_{j=1}^{3} \tilde{A}_j \partial_j v + \hat{\sigma}(v) v = \tilde{f}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bv = \tilde{g}, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(t_0) = u_0, & x \in \mathbb{R}^3_+. \end{cases} \tag{7.45}$$

Since $\hat{\sigma}$ belongs to $C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$ and the coefficients $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ have the needed structure, it only remains to check that the tuple $(\chi, \hat{\sigma}, t_0, B, \tilde{f}, \tilde{g}, u_0)$ fulfills the compatibility conditions (7.16) of order m for the coefficients $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

Let $\tilde{S}_{\chi,\hat{\sigma},G,m,p}$ denote the operators from (7.13) associated to the coefficients $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$. Similarly, we write $\tilde{M}_1^p$ and $\tilde{M}_2^p$ for the operators from (7.14) associated to $\tilde{S}_{\chi,\hat{\sigma},m,p}$ and $\hat{\sigma}$ in the case $\tilde{M}_2^p$. We will show that

$$\tilde{S}_{\chi,\hat{\sigma},m,p}(t_0, \tilde{f}, u_0) = (-1)^p S_{\chi,\sigma,m,p}(t_0, f, u_0) \tag{7.46}$$

for all $p \in \{0, \ldots, m\}$. This assertion is clearly true in the case $p = 0$. Assuming that we have shown (7.46) for all $j \in \{0, \ldots, p-1\}$ and some $p \in \{1, \ldots, m\}$, we compute

$$\begin{aligned} \chi(u_0)\tilde{S}_{\chi,\hat{\sigma},m,p}(t_0, \tilde{f}, u_0) &= \partial_t^{p-1}\tilde{f}(t_0) - \sum_{j=1}^{3} \tilde{A}_j \partial_j \tilde{S}_{\chi,\hat{\sigma},m,p-1}(t_0, \tilde{f}, u_0) \\ &\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} \tilde{M}_1^l(t_0, \tilde{f}, u_0) \tilde{S}_{\chi,\hat{\sigma},m,p-l}(t_0, \tilde{f}, u_0) \\ &\quad - \sum_{l=0}^{p-1} \binom{p-1}{l} \tilde{M}_2^l(t_0, \tilde{f}, u_0) \tilde{S}_{\chi,\hat{\sigma},m,p-1-l}(t_0, \tilde{f}, u_0) \\ &= (-1)^p \partial_t^p f(t_0) - \sum_{j=1}^{3} (-1)\cdot(-1)^{p-1} A_j \partial_j S_{\chi,\sigma,m,p-1}(t_0, f, u_0) \\ &\quad - \sum_{l=1}^{p-1} \binom{p-1}{l} (-1)^l M_1^l(t_0, f, u_0)(-1)^{p-l} S_{\chi,\sigma,m,p-l}(t_0, f, u_0) \\ &\quad - \sum_{l=0}^{p-1} \binom{p-1}{l} (-1)^{l+1} M_2^l(t_0, f, u_0)(-1)^{p-1-l} S_{\chi,\sigma,m,p-1-l}(t_0, f, u_0) \\ &= (-1)^p S_{\chi,\sigma,m,p}(t_0, f, u_0). \end{aligned}$$

By induction, we obtain (7.46) for all $p \in \{0, \dots, m\}$. Hence,

$$\operatorname{Tr}(B\tilde{S}_{\chi,\hat{\sigma},m,p}(t_0, \tilde{f}, u_0)) = (-1)^p \operatorname{Tr}(BS_{\chi,\sigma,m,p}(t_0, f, u_0)) = (-1)^p \partial_t^p g(t_0) = \partial_t^p \tilde{g}(t_0)$$

for all $p \in \{0, \dots, m-1\}$ as the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the nonlinear compatibility conditions (7.16) of order m by assumption. We conclude that also the tuple $(\chi, \hat{\sigma}, t_0, B, \tilde{f}, \tilde{g}, u_0)$ fulfills the compatibility conditions (7.16) of order m with coefficients $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$. Theorem 7.10 thus gives a unique solution v of (7.45) on $[t_0, t_0 + \tau]$, which belongs to $G_m((t_0, t_0 + \tau) \times G)$.

We now set $u(t) = v(2t_0 - t)$ for all $t \in [-\tau + t_0, t_0]$. Then the function u belongs to $G_m((-\tau + t_0, t_0) \times G)$, $u(t_0) = v(t_0) = u_0$ and $Bu = g$ on $(-\tau + t_0, t_0) \times \partial G$. Moreover, we infer

$$\begin{aligned}
&\chi(u(t))\partial_t u(t) + \sum_{j=1}^{3} A_j \partial_j u(t) + \sigma(u(t))u(t) \\
&= -\Big(\chi(v(2t_0 - t))\partial_t v(2t_0 - t) + \sum_{j=1}^{3} \tilde{A}_j \partial_j v(2t_0 - t) + \hat{\sigma}(v(2t_0 - t))v(2t_0 - t)\Big) \\
&= -\tilde{f}(2t_0 - t) = f(t)
\end{aligned}$$

for all $t \in [-\tau + t_0, t_0]$. Consequently, the function u is a G_m-solution of (7.12) on $[-\tau + t_0, t_0]$.

A similar argument yields the uniqueness of the solution of (7.12). This means that we obtain a solution not only for times $t \geq t_0$ but also for times $t \leq t_0$. Via the same construction, the a priori estimates from Chapter 5 carry over to negative times. ◇

Below we will construct a maximally defined solution. To this purpose, we need the following lemma on the concatenation of solutions.

Lemma 7.13. *Let $m \in \mathbb{N}$ with $m \geq 3$. Pick intervals $J_1 = (t_0, t_1)$, $J_2 = (t_1, t_2)$, and $J = (t_0, t_2)$. Take time independent $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose $f \in H^m(J \times G)$ and $g \in E_m(J \times \partial G)$ and set $f_1 = f_{|J_1}$, $f_2 = f_{|J_2}$, $g_1 = g_{|J_1}$, and $g_2 = g_{|J_2}$. Let $u_0, u_1 \in H^m(G)$. Assume that there are $v_i \in G_m(J_i \times G)$ which are solutions of* (1.6) *with data f_i, g_i, and $v_i(t_{i-1}) = u_{i-1}$ for $i \in \{1, 2\}$ and $v_1(t_1) = u_1$. Then the function*

$$w(t) = \begin{cases} v_1(t) & \text{for } t \in \overline{J}_1, \\ v_2(t) & \text{for } t \in \overline{J}_2, \end{cases}$$

is a $G_m(J \times G)$-solution of (1.6) *with inhomogeneity f, boundary value g, and initial value $w(0) = u_0$.*

Proof. Since $v_i \in G_m(J_i \times G)$ for $i \in \{1, 2\}$, we only have to show that $\partial_t^j v_1(t_1) = \partial_t^j v_2(t_1)$ for $j \in \{0, \dots, m\}$ to establish that w is an element of $G_m(J \times G)$. But this follows easily from (7.15) in Lemma 7.5, as this identity tells us that

$$\partial_t^j v_1(t_1) = S_{\chi,\sigma,G,m,j}(t_1, f, v_1(t_1)) = S_{\chi,\sigma,G,m,j}(f, v_2(t_1), t_1) = \partial_t^j v_2(t_1)$$

for $j \in \{0, \dots, m\}$, where we also applied Lemma 7.5. So $w \in G_m(J \times G)$ and clearly $w(0) = u_0$. Furthermore, Lw exists in a strong sense, that means the differential operators can be applied pointwise in time (cf. Remark 2.17), and by assumption we have $Lw = f_1$ on J_1 and $Lw = f_2$ on J_2. We conclude that $Lw = f$ in $L^2(J \times G)$. Since v_1 and v_2 solve (1.6) in $G_1(J_i \times G)$, Remark 2.17 transferred to domains yields the boundary condition

$$\operatorname{Tr}_1(Bw)(t) = \operatorname{tr}(Bw(t)) = \operatorname{tr}(Bv_i(t)) = \operatorname{Tr}_1(Bv_i)(t) = g_i(t)$$

for $t \in \overline{J}_i$ and $i \in \{1, 2\}$, as $\operatorname{Tr}_1(Bv_i) = \operatorname{Tr}(Bv_i) = g_i$. Consequently, $\operatorname{Tr}_1(Bw) = g$ in $C(\overline{J}, H^{1/2}(\partial G))$ and thus $\operatorname{Tr}(Bw) = g$ in $H^{-1/2}(J \times \partial G)$ by Remark 2.17 on domains again. So the function w has all the properties of a G_m-solution of (1.6). □

We also underline that the restriction of a $G_m(J \times G)$-solution on any subinterval K of J is again a solution.

Lemma 7.14. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{c}_{m,6}(J \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m(J \times G)$, $g \in E_m(J \times \partial G)$, and $u_0 \in H^m(G)$ and assume that problem* (1.6) *has a solution u in $G_m(J \times G)$. Assume that I is an open subinterval of J and that $s_0 \in \overline{I}$. Then $u_{|I}$ is a solution of* (1.6) *on I with inhomogeneity $f_{|I}$, boundary value $g_{|I}$, and initial value $u(s_0)$ at s_0.*

Proof. Clearly, $u_{|I} \in G_m(I \times G)$ and $u_{|I}(s_0) = u(s_0)$. Since the differential operator can be applied pointwise in t, we also obtain $(Lu_{|I})(t) = (Lu)(t)$ for all $t \in I$ and thus $Lu_{|I} = f_{|I}$ in $L^2(I \times G)$. Due to Remark 2.17 transferred to domains, the trace $\mathrm{Tr}(Bu)$ is equal to $\mathrm{Tr}_1(Bu)$ so that we deduce

$$g(t) = \mathrm{Tr}_1(Bu)(t) = \mathrm{tr}(Bu(t))$$

for all $t \in \overline{J}$, in particular $\mathrm{Tr}_1(Bu_{|I}) = g(t)$ in $C(\overline{I}, H^{1/2}(\partial G))$. We conclude the identity $\mathrm{Tr}(Bu_{|I}) = g$ in $H^{-1/2}(\Gamma_I)$, where $\Gamma_I = I \times \partial G$. □

Definition 7.15. *Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{c}_{m,6}(\mathbb{R} \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m((T_1, T_2) \times G)$, $g \in E_m((T_1, T_2) \times G)$, and $u_0 \in H^m(G)$ for all $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$ and define B as in Theorem 7.10. We define*

$$T_+(m, t_0, f, g, u_0) = \sup\{\tau \geq t_0 \colon \exists\, G_m\text{-solution of (1.6) on } [t_0, \tau]\},$$
$$T_-(m, t_0, f, g, u_0) = \inf\{\tau \leq t_0 \colon \exists\, G_m\text{-solution of (1.6) on } [\tau, t_0]\}.$$

The interval $(T_-(m, t_0, f, g, u_0), T_+(m, t_0, f, g, u_0)) =: I_{max}(m, t_0, f, g, u_0)$ is called the maximal interval of existence.

The next lemma justifies the name "maximal interval of existence". It states that there is a unique G_m-solution of (1.6) on I_{max} which cannot be extended beyond this interval.

Proposition 7.16. *Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{c}_{m,6}(\mathbb{R} \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m((T_1, T_2) \times G)$, $g \in E_m((T_1, T_2) \times G)$, and $u_0 \in H^m(G)$ for all $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$ and define B as in Theorem 7.10. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions* (7.16) *of order m. Then there exists a unique maximal solution $u \in \bigcap_{j=0}^{m} C^j(I_{max}, H^{m-j}(G))$ of* (1.6) *on I_{max} which cannot be extended beyond this interval.*

Proof. For simplicity, we abbreviate $T_+ = T_+(m, t_0, f, g, u_0)$, $T_- = T_-(m, t_0, f, g, u_0)$, and $I_{max} = I_{max}(m, t_0, f, g, u_0)$. Note that we have $T_+ > t_0$ by Theorem 7.10 and $T_- < t_0$ by Remark 7.12. Take times $\tau_1, \tau_2 \in I_{max}$ with

$$T_- < \tau_1 < t_0 < \tau_2 < T_+.$$

The definition of the maximal interval of existence and Lemmas 7.13 and 7.14 yield a G_m-solution v of (1.6) on $[\tau_1, \tau_2]$. We set $u = v$ on $[\tau_1, \tau_2]$. Because of Lemma 7.3, we obtain an extension of u if we decrease τ_1 and increase τ_2 within I_{max}. This construction thus yields a function u on I_{max}. Moreover, u belongs to $\bigcap_{j=0}^{m} C^j(I_{max}, H^{m-j}(G))$. Since the differential operator and the trace can be evaluated pointwise in t, we conclude that u is a G_m-solution of (1.6) on I_{max}.

Now let J' be an interval with $I_{max} \subseteq J'$ such that there exists a function $v \in \bigcap_{j=0}^{m} C^j(J', H^{m-j}(G))$ of (1.6) on J'. The definition of I_{max} already gives $J' \subseteq [T_-, T_+]$. If $T_+ \in J'$, then v belongs to $G_m((t_0, T_+) \times G)$ and the closure of the range of $v(T_+)$ has positive distance to $\partial\mathcal{U}$. Lemma 7.9 further shows that the tuple $(\chi, \sigma, T_+, B, f, g, v(T_+))$ fulfills the compatibility conditions (7.16) of order m. Theorem 7.10 thus gives a solution v' on $[T_+, \tau]$, where $\tau > 0$. The concatenation of v and v' at T_+ is again a solution by Lemma 7.13, contradicting the definition of T_+. So T_+ does not belong to J'. Analogously, we deduce that T_- is not contained in J', implying that $J' = I_{max}$.

The uniqueness of the solution on I_{max} follows from Lemma 7.3. □

As usual the fixed point argument from Theorem 7.10 also yields a blow-up criterion. As long as the $H^m(G)$-norm of the solution remains bounded, we can extend the solution. Therefore this norm has to blow up at the maximal existence time if this time is finite.

Lemma 7.17. *Let $t_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ with $m \geq 3$. Take time independent $\zeta_1, \zeta_2 \in F^{c}_{m,6}(\mathbb{R} \times G)$ and $\tilde{\chi}, \tilde{\sigma} \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. Set $\chi = \zeta_1 \tilde{\chi}$, $\sigma = \zeta_2 \tilde{\sigma}$ and assume that χ is symmetric and uniformly positive definite. Choose data $f \in H^m((T_1, T_2) \times G)$, $g \in E_m((T_1, T_2) \times G)$, and $u_0 \in H^m(G)$ for all $T_1, T_2 \in \mathbb{R}$ with $T_1 < T_2$ and define B as in Theorem 7.10. Assume that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions* (7.16) *of order m. Let u be the maximal solution of* (1.6) *on I_{max} provided by Proposition 7.16. If $T_+ = T_+(m, t_0, f, g, u_0) < \infty$, then one of the following alternatives*

(i) $\liminf_{t \nearrow T_+} \operatorname{dist}(\overline{\{u(t,x) \colon x \in G\}}, \partial\mathcal{U}) = 0$,

(ii) $\lim_{t \nearrow T_+} \|u(t)\|_{H^m(\mathbb{R}^3_+)} = \infty$

occurs. The analogous result is true for $T_-(m, t_0, f, g, u_0)$.

Proof. Let $T_+ < \infty$ and assume that alternative (i) does not hold. This means that there exists $\kappa > 0$ such that

$$\operatorname{dist}(\overline{\{u(t,x) \colon x \in G\}}, \partial\mathcal{U}) \geq \kappa$$

for all $t \in (t_0, T_+)$. Assume that there exists a sequence $(t_n)_n$ converging from below to the maximal existence time T_+ such that $R := \sup_{n \in \mathbb{N}} \|u(t_n)\|_{H^m(\mathbb{R}^3_+)}$ is finite. Fix a time $T' > T_+$ and take a radius $r > R$ with $\|f\|_{H^m((t_0, T') \times G)} < r$. Then pick an index $N \in \mathbb{N}$ such that

$$t_N + \tau(\chi, \sigma, m, T' - t_0, r, \kappa) > T_+,$$

with $\tau = \tau(\chi, \sigma, m, T' - t_0, r, \kappa)$ from Theorem 7.10. Lemma 7.9 tells us that the tuple $(\chi, \sigma, t_n, B, f, g, u(t_n))$ fulfills the compatibility conditions (7.16). Since the distance between $\operatorname{im} u(t_N)$ and $\partial\mathcal{U}$ is larger or equal than κ, Theorem 7.10 thus gives a G_m-solution v of (1.6) with inhomogeneity f, boundary value g, and initial value $u(t_N)$ at t_N on $[t_N, t_N + \tau]$. Setting $w(t) := u(t)$ if $t \in [t_0, t_N]$ and $w(t) := v(t)$ if $t \in [t_N, t_N + \tau]$, we obtain a G_m-solution of (1.6) with data f, g, and u_0 on $[t_0, t_N + \tau]$ by Lemma 7.13. This contradicts the definition of T_+ since $t_N + \tau > T_+$. The assertion for T_- is proven analogously. □

This criterion is a direct consequence of the construction of the solution in Theorem 7.10. It will be one of the main topics of the following section to improve this result.

Remark 7.18. We want to finish this section with a remark concerning the assumptions on χ and σ. In fact, we can treat more general material laws than stated so far.

(i) In the definitions and results of this section we assumed that the functions ζ_1 and ζ_2 belong to $F_m(J \times G)$. The reason for this assumption was that we applied the bilinear estimates from Lemma 2.22 applied to $\zeta_1 \chi(v)$ respectively $\zeta_2 \sigma(v)$ for a function v from $\tilde{G}_m(J \times G)$. However, the function spaces $F_m(J \times G)$ and accordingly Lemma 2.22 were tailored for functions of the form $\theta(v)$, where θ is a C^m-function and v a $\tilde{G}_m$-function, which do not have better regularity respectively integrability properties than belonging to $F_m(J \times G)$. For the estimates of Lemma 2.22 respectively the a priori estimates and the linear theorey in Chapters 3 to 5 we could have allowed coefficients A_0 respectively D in $F_m(J \times G) + W^{m,\infty}(J \times G)$. Observe that the estimates for products involving a factor from $W^{m,\infty}(J \times G)$ are easier as we do not need any Sobolev embedding here. While we did not work this out in Chapters 3 to 5 for the sake of the clarity of the arguments, this observation allows us to treat time independent functions ζ_1 and ζ_2 from $F_m(J \times G) + W^{m,\infty}(G)$. In fact, we are mainly interested in ζ_1 and ζ_2 from $W^{m,\infty}(G)$ as we think that this is the natural assumption for applications.

(ii) We further note that the variables χ and σ appear linearly in problem (1.6) and consequently in the results of this section. Due to the triangle inequality, we can therefore also treat material laws χ and σ which are linear combinations of the functions we used so far. To make this statement more precise, we introduce

$$\mathcal{ML}^m(G,\mathcal{U}) := \{\theta \colon G \times \mathcal{U} \to \mathbb{R}^6 \,|\, \exists l \in \mathbb{N}, \zeta_1, \ldots, \zeta_l \in F^{\mathrm{c}}_{m,6}(G) + W^{m,\infty}_{\mathrm{c}}(G)^{6\times 6}, \theta_1, \ldots, \theta_l \in C^m(\mathcal{U}, \mathbb{R}^{6\times 6}) \text{ such that } \theta = \sum_{j=1}^{l} \zeta_j \theta_j\},$$

$$\mathcal{ML}^m_{\mathrm{pd}}(G,\mathcal{U}) := \{\theta \in \mathcal{ML}^m(G,\mathcal{U}) \,|\, \theta \text{ is uniformly positive definite on } G \times \mathcal{U}\}$$

for all $m \in \mathbb{N}$. Here $F^{\mathrm{c}}_{m,6}(G)$ is defined in analogy to $F^{\mathrm{c}}_{m,6}(J \times G)$ and $W^{m,\infty}_{\mathrm{c}}(G)$ denotes the space of those functions in $W^{m,\infty}(G)$ which have a limit as $|x| \to \infty$. If we replace the assumptions on χ and σ in the definitions and assumptions of this section by $\chi \in \mathcal{ML}^m_{\mathrm{pd}}(G,\mathcal{U})$ respectively $\sigma \in \mathcal{ML}^m(G,\mathcal{U})$, we obtain the same results.

7.3 Local wellposedness

In this section we continue our investigation of the nonliear system (1.6). While we concentrated on existence and uniqueness of a solution in the previous section, we complete here the local wellposedness theory of (1.6) by providing a refined blow-up criterion and showing the continuous dependance of solutions on their data.

Our first goal is to sharpen the blow-up criterion from Lemma 7.17, which depends on the $H^m(\mathbb{R}^3_+)$-norm of the solution. There are several examples of quasilinear systems, both on the full space and on domains, where the blow-up condition is given in terms of the Lipschitz-norm of the solution, see e.g. [Maj84], [BGS07], [LMSTYZ01], [BCD11], [KP83], [Kla80], and [BKM84]. Indeed we show that the limes superior of the spatial Lipschitz-norm of a solution of (1.6) has to blow up in finite time if the solution does not exist globally.

The second main topic of this section is the continuous dependance of the solutions of (1.6) on the data. We explain this notion in detail in Theorem 7.23 below. However, we already remark that, due to the quasilinear nature of (1.6) and the associated loss of regularity in $H^m(G)$, we cannot expect anything better than continuous dependance, cf. [BCD11], [MMT12], and [IT17].

Throughout this section we use the following assumptions. For a given integer m the set G denotes a subdomain of $\mathbb{R}^3$ which fulfills the uniform $C^{\tilde{m}+2}$ regularity condition, where $\tilde{m} = \max\{m, 3\}$. Moreover, $\mathcal{U}$ denotes a convex subdomain of $\mathbb{R}^6$.

The refinement of the blow-up criterion relies on an improved “a posteriori estimate” of the solution of (1.6) based on Moser type calculus inequalities. These inequalities, which were introduced by Moser in [Mos66], follow from the combination of the Gagliardo-Nirenberg estimates (see [Nir59]) with Hölder's inequality and allow to treat products of derivatives of a function effectively. Therefore, they have proven to be a powerful tool in the analysis of nonlinear partial differential equations.

As these Moser type inequalities are usually stated on the whole space or the torus (see e.g. [Maj84] and [KM81]), while we need them on domains (which are bounded in at least one coordinate direction), we provide a full proof of the version we will apply later for the convenience of the reader.

Lemma 7.19. *Let $T > 0$, $J = (0,T)$, and $n \in \mathbb{N}_0$.*

(i) Let $v, w \in L^\infty(J \times G) \cap H^n(J \times G)$. We then have

$$\|\partial^\alpha v \, \partial^{\tilde{\alpha}} w\|_{L^2(J\times G)} \leq C\Big(\|v\|_{L^\infty(J\times G)}\|w\|_{H^n(J\times G)} + \|w\|_{L^\infty(J\times G)}\|v\|_{H^n(J\times G)}\Big)$$

for all $\alpha, \tilde{\alpha} \in \mathbb{N}_0^4$ with $|\alpha| \in \{0, \ldots, n\}$ and $|\tilde{\alpha}| = n - |\alpha|$.

(ii) Let $n \in \mathbb{N}$ and $v, w \in W^{1,\infty}(J \times G)$ such that all derivatives of v and w with order between 1 and n belong to $L^2(J \times G)$. We then have

$$\|\partial^\alpha v \, \partial^{\tilde{\alpha}} w\|_{L^2(J\times G)} \leq C\Big(\|\nabla_t v\|_{L^\infty(J\times G)} \sum_{j=0}^{3} \|\partial_j w\|_{H^{n-1}(J\times G)} + \|\nabla_t w\|_{L^\infty(J\times G)} \sum_{j=0}^{3} \|\partial_j v\|_{H^{n-1}(J\times G)}\Big)$$

for all $\alpha, \tilde{\alpha} \in \mathbb{N}_0^4$ with $|\alpha| \in \{1, \ldots, n\}$ and $|\tilde{\alpha}| = n + 1 - |\alpha|$.

(iii) Let $n \in \mathbb{N}$ and $\theta \in C^n(\mathcal{U}, \mathbb{R}^{6\times 6})$. Let $v \in \tilde{G}_{\max\{n,3\}}(J \times G)$ such that there exists a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im} v \subseteq \mathcal{U}_1$. Take $R > 0$ with $\|v\|_{L^\infty(J\times G)} \leq R$. Then there is a constant $C = C(\theta, n, R, \mathcal{U}_1)$ with

$$\|\partial^\alpha \theta(v)\|_{L^2(J\times G)} \leq C\|v\|_{H^{|\alpha|}(J\times G)}$$

for all $\alpha \in \mathbb{N}_0^4 \setminus \{0\}$ with $|\alpha| \leq n$.

Proof. Due to their importance in the proof of this lemma, we recall a special case of the Gagliardo-Nirenberg estimates. Let $n \in \mathbb{N}$ and let $u \in L^\infty(\mathbb{R}^4)$ with all n-th order derivatives in $L^2(\mathbb{R}^4)$. Then

$$\|\partial^\alpha u\|_{L^{2n/l}(\mathbb{R}^4)} \leq C\|u\|_{L^\infty(\mathbb{R}^4)}^{1-l/n} \sum_{\substack{\beta\in\mathbb{N}_0^4 \\ |\beta|=n}} \|\partial^\beta u\|_{L^2(\mathbb{R}^4)}^{l/n} \tag{7.47}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = l$ and $l \in \{1, \ldots, n\}$, see [Nir59]. Let E denote Stein's extension operator, see Theorem VI.3.5 in [Ste70]. Observe that the domain $J \times G$ satisfies the minimal smoothness condition which is required for the existence of Stein's extension operator. For a function $v \in L^\infty(J \times G) \cap H^n(J \times G)$ we then obtain

$$\begin{aligned} \|\partial^\alpha v\|_{L^{2n/l}(J\times G)} &\leq \|\partial^\alpha (Ev)\|_{L^{2n/l}(\mathbb{R}^4)} \leq C\|Ev\|_{L^\infty(\mathbb{R}^4)}^{1-l/n} \sum_{\substack{\beta\in\mathbb{N}_0^4 \\ |\beta|=n}} \|\partial^\beta (Ev)\|_{L^2(\mathbb{R}^4)}^{l/n} \\ &\leq C\|v\|_{L^\infty(J\times G)}^{1-l/n}\|v\|_{H^n(J\times G)}^{l/n} \end{aligned} \tag{7.48}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = l$ and $l \in \{1, \ldots, n\}$.

(i) The assertion is clear if $n = 0$, if $|\alpha| = 0$, or if $|\alpha| = n$. So assume that $n \geq 2$ and $|\alpha| \in \{1, \ldots, n-1\}$ in the following. Employing Hölder's inequality, the Gagliardo-Nirenberg estimate in (7.48), and Young's inequality, we then derive

$$\|\partial^\alpha v \, \partial^{\tilde{\alpha}} w\|_{L^2(J\times G)} \leq \|\partial^\alpha v\|_{L^{2n/|\alpha|}(J\times G)}\|\partial^{\tilde{\alpha}} w\|_{L^{2n/|\tilde{\alpha}|}(J\times G)}$$

$$\begin{aligned}
&\leq C\|v\|_{L^\infty(J\times G)}^{1-|\alpha|/n}\|v\|_{H^n(J\times G)}^{|\alpha|/n} \cdot \|w\|_{L^\infty(J\times G)}^{1-|\tilde\alpha|/n}\|w\|_{H^n(J\times G)}^{|\tilde\alpha|/n} \\
&\leq C(\|v\|_{L^\infty(J\times G)}\|w\|_{H^n(J\times G)})^{1-|\alpha|/n}(\|w\|_{L^\infty(J\times G)}\|v\|_{H^n(J\times G)})^{|\alpha|/n} \\
&\leq C\Big((1-\tfrac{|\alpha|}{n})\|v\|_{L^\infty(J\times G)}\|w\|_{H^n(J\times G)} + \tfrac{|\alpha|}{n}\|w\|_{L^\infty(J\times G)}\|v\|_{H^n(J\times G)}\Big) \\
&\leq C\Big(\|v\|_{L^\infty(J\times G)}\|w\|_{H^n(J\times G)} + \|w\|_{L^\infty(J\times G)}\|v\|_{H^n(J\times G)}\Big).
\end{aligned}$$

(ii) There is nothing to show if $|\alpha| = 1$ or if $|\alpha| = n$. It thus only remains to consider $n \geq 3$ and $\alpha, \tilde\alpha \in \mathbb{N}_0^4$ with $|\tilde\alpha| = n + 1 - |\alpha|$ and $|\alpha| \in \{2, \ldots, n-1\}$. Then there exist $k, \tilde k \in \{0, \ldots, 3\}$ and $\alpha', \tilde\alpha' \in \mathbb{N}_0^4$ such that $\alpha = \alpha' + e_k$ and $\tilde\alpha = \tilde\alpha' + e_{\tilde k}$. Moreover, $|\alpha'| \in \{1, \ldots, n-2\}$ and

$$|\tilde\alpha'| = |\tilde\alpha| - 1 = n + 1 - |\alpha| - 1 = n - |\alpha| = n - (|\alpha'| + 1) = n - 1 - |\alpha'|.$$

Applying (i) with parameter $n-1$ to $\partial_k v$ and $\partial_{\tilde k} w$, we infer

$$\begin{aligned}
&\|\partial^\alpha v\, \partial^{\tilde\alpha} w\|_{L^2(J\times G)} = \|\partial^{\alpha'}\partial_k v\, \partial^{\tilde\alpha'}\partial_{\tilde k} w\|_{L^2(J\times G)} \\
&\leq C\Big(\|\partial_k v\|_{L^\infty(J\times G)}\|\partial_{\tilde k} w\|_{H^{n-1}(J\times G)} + \|\partial_{\tilde k} w\|_{L^\infty(J\times G)}\|\partial_k v\|_{H^{n-1}(J\times G)}\Big) \\
&\leq C\Big(\|\nabla_t v\|_{L^\infty(J\times G)}\sum_{j=0}^{3}\|\partial_j w\|_{H^{n-1}(J\times G)} + \|\nabla_t w\|_{L^\infty(J\times G)}\sum_{j=0}^{3}\|\partial_j v\|_{H^{n-1}(J\times G)}\Big).
\end{aligned}$$

(iii) Let $\alpha \in \mathbb{N}_0^4 \setminus \{0\}$ with $|\alpha| \leq n$. Since $v \in \tilde G_{\max\{n,3\}}(J \times G)$ with $\operatorname{im} v \subseteq \mathcal{U}$, Lemma 7.1 yields

$$\partial^\alpha \theta(v) = \sum_{1\leq j\leq|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_j\in\mathbb{N}_0^4\setminus\{0\}\\ \sum\gamma_i=\alpha}} \sum_{l_1,\ldots,l_j=1}^{6} C(\alpha, j, l_1, \ldots, l_j, \gamma_1, \ldots, \gamma_j) \cdot (\partial_{l_j}\cdots\partial_{l_1}\theta)(v)\prod_{i=1}^{j}\partial^{\gamma_i} v_{l_i}.$$

Taking the $L^2(J \times G)$-norm, we deduce

$$\|\partial^\alpha\theta(v)\|_{L^2(J\times G)} \leq C(\theta, n, R, \mathcal{U}_1) \sum_{1\leq j\leq|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_j\in\mathbb{N}_0^4\setminus\{0\}\\ \sum\gamma_i=\alpha}} \sum_{l_1,\ldots,l_j=1}^{6} \Big\|\prod_{i=1}^{j}\partial^{\gamma_i}v_{l_i}\Big\|_{L^2(J\times G)} \tag{7.49}$$

as $\operatorname{im} v \subseteq \mathcal{U}_1$. Take $j \in \{1, \ldots, |\alpha|\}$, $\gamma_1, \ldots, \gamma_j \in \mathbb{N}_0^4 \setminus \{0\}$ with $\sum_{i=1}^j \gamma_i = \alpha$, and $l_1, \ldots, l_j \in \{1, \ldots, 6\}$. Set $p_i = \frac{|\alpha|}{|\gamma_i|}$ for all $i \in \{1, \ldots, j\}$ and note that $\sum_{i=1}^j \frac{1}{2p_i} = \frac{1}{2}$. Hölder's inequality and the Gagliardo-Nirenberg estimate (7.48) thus imply

$$\begin{aligned}
&\Big\|\prod_{i=1}^{j}\partial^{\gamma_i}v_{l_i}\Big\|_{L^2(J\times G)} \leq \prod_{i=1}^{j}\|\partial^{\gamma_i}v_{l_i}\|_{L^{2p_i}(J\times G)} = \prod_{i=1}^{j}\|\partial^{\gamma_i}v_{l_i}\|_{L^{2|\alpha|/|\gamma_i|}(J\times G)} \\
&\leq \prod_{i=1}^{j}\Big(C\|v_{l_i}\|_{L^\infty(J\times G)}^{1-|\gamma_i|/|\alpha|}\|v_{l_i}\|_{H^{|\alpha|}(J\times G)}^{|\gamma_i|/|\alpha|}\Big) \leq C\|v\|_{L^\infty(J\times G)}^{j-1}\|v\|_{H^{|\alpha|}(J\times G)}.
\end{aligned}$$

Inserting this estimate into (7.49), we obtain the assertion. □

The next proposition is the key step for the improvement of the blow-up condition. Roughly speaking, it tells us that we control the $H^m(\mathbb{R}^3_+)$-norm of a solution of (1.6) as soon as we control its spatial Lipschitz norm. Its proof relies on the Moser type inequalities from Lemma 7.19. They allow us to estimate products of derivatives of the solution u - which are of the type we already encountered in Chapter 3 - by the product of the spatial Lipschitz norm of u and a L^2-based Sobolev norm. Gronwall's lemma and an induction process as in Chapter 3 then yield the assertion.

Proposition 7.20. *Let $m \in \mathbb{N}$ with $m \geq 3$ and $t_0 \in \mathbb{R}$. Take functions $\chi \in \mathcal{ML}^m_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^m(G,\mathcal{U})$. Let*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Choose data $u_0 \in H^m(G)$, $g \in E_m((-T,T) \times \partial G)$, and $f \in H^m((-T,T) \times G)$ for all $T > 0$ such that the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (7.16) of order m. Let u denote the maximal solution of (1.6) provided by Proposition 7.16 and Remark 7.18 on (T_-, T_+). We set

$$\omega(T) = \sup_{t \in (t_0,T)} \|u(t)\|_{W^{1,\infty}(G)}$$

for every $T \in (t_0, T_+)$. We further take $r > 0$ with

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|g\|_{E_m((t_0,T_+)\times\partial G)} + \|u_0\|_{H^m(G)} + \|f\|_{H^m((t_0,T_+)\times G)} \leq r.$$

We set $T^* = T_+$ if $T_+ < \infty$ and take any $T^* > 0$ if $T_+ = \infty$. Let $\omega_0 > 0$ and let $\mathcal{U}_1$ be a compact subset of $\mathcal{U}$.

Then there exists a constant $C = C(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1, G, T^* - t_0)$ such that

$$\|u\|^2_{G_m((t_0,T)\times G)} \leq C\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m((t_0,T)\times\partial G)} + \|f\|^2_{H^m((t_0,T)\times G)}\Big)$$

for all $T \in (t_0, T^*)$ which have the property that $\omega(T) \leq \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T]$. The analogous result is true on (T_-, t_0).

Proof. Without loss of generality we assume $t_0 = 0$. We further suppose that $\chi = \zeta_1 \tilde{\chi}$ and $\sigma = \zeta_2 \tilde{\sigma}$ where $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ are time-independent and $\tilde{\chi}$ and $\tilde{\sigma}$ belong to $C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. The general case then follows as described in Remark 7.18.

Let $\omega_0 > 0$ and $\mathcal{U}_1$ be a compact subset of $\mathcal{U}$. If $\omega(T) > \omega_0$ or if the set $\{u(t,x) \colon (t,x) \in [t_0,T] \times G\}$ is not contained in $\mathcal{U}_1$ for all $T \in (0, T^*)$, there is nothing to prove. Otherwise we fix $T' \in (0, T^*)$ with $\omega(T') \leq \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T']$. Let $T \in (0, T']$ be arbitrary and denote $(0,T) \times \mathbb{R}^3_+$ by Ω. Note that $\omega(T) \leq \omega(T') \leq \omega_0$ and $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T]$.

We pick a number $\eta = \eta(\chi) > 0$ such that $\chi \geq \eta$. Consequently, there is a constant C with

$$|\chi^{-1}(\xi)| \leq \frac{C}{\eta}$$

for all $\xi \in \mathbb{R}^6$. Since the function u solves (1.6), we infer

$$\begin{aligned} \|\partial_t u\|_{L^\infty(\Omega)} &\leq \|\chi^{-1}(u) f\|_{L^\infty(\Omega)} + \sum_{j=1}^3 \|\chi^{-1}(u) A_j^{\mathrm{co}} \partial_j u\|_{L^\infty(\Omega)} + \|\chi^{-1}(u)\sigma(u)u\|_{L^\infty(\Omega)} \\ &\leq \frac{C}{\eta}\Big(\|f\|_{L^\infty(\Omega)} + \sum_{j=1}^3 \|\partial_j u\|_{L^\infty(\Omega)} + \max_{\xi\in\mathcal{U}_1} |\sigma(\xi)| \|u\|_{L^\infty(\Omega)}\Big) \\ &\leq C(\eta, \sigma, \mathcal{U}_1)\Big(\|f\|_{H^m(\Omega)} + 3\,\omega(T) + \omega(T)\Big), \\ \|u\|_{W^{1,\infty}(\Omega)} &\leq \|\partial_t u\|_{L^\infty(\Omega)} + 4\,\omega(T) \leq C_{7.50}(\chi, \sigma, r, \omega_0, \mathcal{U}_1). \end{aligned} \tag{7.50}$$

In the following we will frequently apply (7.50) without further reference.

I) In a first step we localize the problem and transform it to the half-space as in Chapter 5. We therefore choose a covering $(U_i)_{i\in\mathbb{N}_0}$, a sequence of sets $(V_i)_{i\in\mathbb{N}_0}$, and sequences of functions $(\varphi_i)_{i\in\mathbb{N}_0}$, $(\theta_i)_{i\in\mathbb{N}_0}$, $(\sigma_i)_{i\in\mathbb{N}_0}$, and $(\omega_i)_{i\in\mathbb{N}_0}$ as in Definition 5.4 for the tame uniform $C^{\tilde{m}+2}$-boundary of G. Take the operators Φ_i and the localized coefficients and data

$$\begin{gathered} A_0^i = A_0^i(\chi(u),\eta), \quad A_j^i, \quad D^i = D^i(\sigma(u)), \quad B^i, \\ f^i = f^i(f,u), \quad g^i = g^i(g), \quad u_0^i = u_0^i(u_0) \end{gathered}$$

from Definition 5.7 for all $i \in \mathbb{N}$ respectively $i \in \mathbb{N}_0$. We further abbreviate $u^i = \Phi_i(\theta_i u)$ in the following.

Corollary 5.8 shows that the function $\Phi_i(\theta_i u)$ solves the initial boundary value problem

$$\begin{cases} L(A_0^i,\dots,A_3^i,D^i)v = f^i(f,u), & x\in\mathbb{R}^3_+, \quad t\in(0,T); \\ B^i v = g^i, & x\in\partial\mathbb{R}^3_+, \quad t\in(0,T); \\ v(0) = u_0^i, & x\in\mathbb{R}^3_+; \end{cases} \tag{7.51}$$

for all $i \in \mathbb{N}$ and the initial value problem

$$\begin{cases} L(A_0^0,A_1^{\mathrm{co}},A_2^{\mathrm{co}},A_3^{\mathrm{co}},D^0)v = f^0(f,u), & x\in\mathbb{R}^3_+, \quad t\in(0,T); \\ v(0) = u_0^0, & x\in\mathbb{R}^3_+; \end{cases} \tag{7.52}$$

in the case $i = 0$. Set

$$\begin{aligned} f_\alpha^i &= \partial^\alpha f^i(f,u) - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\Phi_i(\chi(u))\partial_t\partial^{\alpha-\beta}u^i - \sum_{j=1}^3\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta A_j^i\partial_j\partial^{\alpha-\beta}u^i \\ &\quad - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\Phi_i(\sigma(u))\partial^{\alpha-\beta}u^i \\ g_\alpha^i &= \partial^\alpha g^i - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\operatorname{tr}(\partial^\beta B^i\partial^{\alpha-\beta}u^i) \end{aligned} \tag{7.53}$$

for all $i \in \mathbb{N}_0$ respectively $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$.

Lemma 3.5 and Lemma 3.4 thus show that $\partial^\alpha\Phi_i(\theta_i u)$ solves the linear initial value problem

$$\begin{cases} L(A_0^i,\dots,A_3^i,D^i)v = f_\alpha^i, & x\in\mathbb{R}^3_+, \quad t\in(0,T); \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,\alpha_3)}\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0)), & x\in\mathbb{R}^3_+; \end{cases} \tag{7.54}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$ and moreover, if additionally $\alpha_3 = 0$, it solves the linear initial boundary value problem

$$\begin{cases} L(A_0^i,\dots,A_3^i,D^i)v = f_\alpha^i, & x\in\mathbb{R}^3_+, \quad t\in(0,T); \\ B^i v = g_\alpha^i, & x\in\partial\mathbb{R}^3_+, \quad t\in(0,T); \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,0)}\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0)), & x\in\mathbb{R}^3_+. \end{cases} \tag{7.55}$$

Here we already used that $\omega_i = 1$ on the support of $\Phi_i\theta_i$ for all $i \in \mathbb{N}_0$ and that

$$\partial_t^j u^i(0) = \Phi_i(\theta_i\partial_t^j u(0)) = \Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0))$$

for all $j \in \{0,\dots,m\}$ by Lemma 7.5.

II) We will show inductively that there are constants $C_k = C_k(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G,T^*)$ such that

$$\|\partial^\alpha u^i\|^2_{G_0(\Omega)} \le C_k\Big(\sum_{j=0}^k \|\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0))\|^2_{H^{k-j}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_k(J\times\partial\mathbb{R}^3_+)}$$

$$+ \|u^i\|_{H^m(\Omega)}^2 + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|\le m-1}} \Big(\|f^i_{\tilde\alpha}(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_{\tilde\alpha}\|^2_{H^1(\Omega)}\Big) + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|\le m}} \|f^i_{\tilde\alpha}\|^2_{L^2(\Omega)}\Big) \tag{7.56}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$, $k \in \{0, \ldots, m\}$, and $i \in \mathbb{N}$.

Applying Lemma 7.1 (i) and (iii) and exploiting Definition 5.7, we obtain a radius $R_1 = R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G)$ with

$$\begin{aligned}
&\|\chi(u)\|_{W^{1,\infty}(\Omega)} + \|A_0^i\|_{W^{1,\infty}(\Omega)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G),\\
&\|\sigma(u)\|_{W^{1,\infty}(\Omega)} + \|D^i\|_{W^{1,\infty}(\Omega)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G),\\
&\|\chi(u(0))\|_{L^\infty(\mathbb{R}^3_+)} \le \max_{\xi\in\mathcal{U}_1} |\chi(\xi)| \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G),\\
&\|\sigma(u(0))\|_{L^\infty(\mathbb{R}^3_+)} \le \max_{\xi\in\mathcal{U}_1} |\sigma(\xi)| \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G),\\
&\sum_{j=1}^3 \|A_j^i\|_{W^{m+1,\infty}(\mathbb{R}^3_+)} + \|B^i\|_{W^{m+1,\infty}(\mathbb{R}^3_+)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G),\\
&\|\zeta_1\|_{F_m(J\times G)} + \|\zeta_2\|_{F_m(J\times G)} \le R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G).
\end{aligned}$$

Set $\gamma_0 = \gamma_0(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G) = \gamma_{3.7,0}(\eta(\chi), R_1(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G)) \ge 1$, where $\gamma_{3.7,0}$ is the corresponding constant from Lemma 3.7. As u^i solves (7.51), Lemma 3.7 yields

$$\begin{aligned}
\|u^i\|^2_{G_0(\Omega)} &\le e^{2\gamma_0 T} \sup_{t\in(0,T)} \|e^{-\gamma_0 t} u^i(t)\|^2_{L^2(\mathbb{R}^3_+)}\\
&\le C_{3.7,0,0}(\eta, R_1) e^{2\gamma_0 T^*} \Big(\|u_0^i\|^2_{L^2(\mathbb{R}^3_+)} + \|g^i\|^2_{E_{0,\gamma_0}(J\times\partial\mathbb{R}^3_+)}\Big)\\
&\qquad + C_{3.7,0}(\eta, R_1) e^{2\gamma_0 T^*} \frac{1}{\gamma_0} \|f^i(f,u)\|^2_{L^2_{\gamma_0}(\Omega)}\\
&\le C_0\Big(\|u_0^i\|^2_{L^2(\mathbb{R}^3_+)} + \|g^i\|^2_{E_0(J\times\partial\mathbb{R}^3_+)} + \|f^i(f,u)\|^2_{L^2(\Omega)}\Big),
\end{aligned}$$

where $C_0 = C_0(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G, T^*)$ and $C_{3.7,0,0}$ respectively $C_{3.7,0}$ denote the corresponding constants from Lemma 3.7. This inequality shows the claim (7.56) for $k = 0$.

Let $k \in \{1, \ldots, m\}$ and assume that (7.56) has been shown for all $j \in \{0, \ldots, k-1\}$. We first claim that there are constants $C_{k,\alpha} = C_{k,\alpha}(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1, G, T^*)$ such that

$$\begin{aligned}
\|\partial^\alpha u^i\|^2_{G_0(\Omega)} &\le C_{k,\alpha}\Big(\sum_{j=0}^k \|\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0)\|^2_{H^{k-j}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_k(J\times\partial\mathbb{R}^3_+)}\\
&\quad + \|u^i\|^2_{H^m(\Omega)} + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|\le m-1}} \Big(\|f^i_{\tilde\alpha}(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_{\tilde\alpha}\|^2_{H^1(\Omega)}\Big) + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|\le m}} \|f^i_{\tilde\alpha}\|^2_{L^2(\Omega)}\Big)
\end{aligned} \tag{7.57}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$. We show (7.57) by another induction, this time with respect to α_3.

Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 = 0$. In step I) we have seen that $\partial^\alpha u^i$ solves the initial boundary value problem (7.55). Hence, Lemma 3.7 yields

$$\begin{aligned}
&\|\partial^\alpha u\|^2_{G_0(\Omega)} \le e^{2\gamma_0 T} \sup_{t\in(0,T)} \|e^{-\gamma_0 t}\partial^\alpha u(t)\|^2_{L^2(\mathbb{R}^3_+)}\\
&\le C_{3.7,0,0}(\eta, R_1) e^{2\gamma_0 T^*} \Big(\|\partial^{(0,\alpha_1,\alpha_2,0)}\partial_t^{\alpha_0} u^i(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|g^i_\alpha\|^2_{E_{0,\gamma_0}(J\times\partial\mathbb{R}^3_+)}\Big)\\
&\qquad + C_{3.7,0}(\eta, R_1) e^{2\gamma_0 T^*} \frac{1}{\gamma_0} \|f^i_\alpha\|^2_{L^2_{\gamma_0}(\Omega)}\\
&\le C_{3.7,0,0}(\eta, R_1) e^{2\gamma_0 T^*} \Big(\|\Phi_i(\theta_i S_{\chi,\sigma,G,k,\alpha_0}(0,f,u_0))\|^2_{H^{k-\alpha_0}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_k(J\times\partial\mathbb{R}^3_+)}\Big)
\end{aligned}$$

$$+ C_{3.7,0}(\eta, R_1)e^{2\gamma_0 T^*}\|f_\alpha^i\|^2_{L^2(\Omega)} + C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u^i\|^2_{H^m(\Omega)},$$

where we applied Lemma 3.5. We conclude that assertion (7.56) is valid for all multi-indices α with $|\alpha| = k$ and $\alpha_3 = 0$.

Now, assume that there is a number $l \in \{1, \dots, k\}$ such that (7.57) is true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 \in \{0, \dots, l-1\}$.

Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 = l$. The multi-index $\alpha' = \alpha - e_3$ belongs to $\mathbb{N}_0^4$ and satisfies $|\alpha'| = k-1 \leq m-1$. Due to step I), we know that $\partial^{\alpha'}u^i$ solves the initial value problem (7.54) with right-hand side $f^i_{\alpha'}$ and initial value

$$\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)}\Phi_i(\theta_i S_{\chi,\sigma,G,m,\alpha_0}(0,f,u_0)).$$

As $|\alpha'| \leq m-1$, the function $f^i_{\alpha'}$ belongs to $H^1(\Omega)$ by Lemma 3.4, the derivative of the higher order initial value $\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)}\Phi_i(\theta_i S_{\chi,\sigma,G,m,\alpha_0}(0,f,u_0))$ to $H^1(\mathbb{R}^3_+)$ by Lemma 7.7, and $\partial^{\alpha'}u^i$ to $G_1(\Omega)$. Moreover, A_0^i and D^i are elements of $W^{1,\infty}(\Omega)$, A_0^i is uniformly positive definite, A_1 and A_2 belong to $F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\Omega)$ and A_3 to $F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$. We can therefore apply Lemma 3.11. We choose $\gamma = 1$ to infer

$$\begin{aligned}
&\|\partial^\alpha u^i\|^2_{G_0(\Omega)} = \|\partial_3\partial^{\alpha'}u^i\|^2_{G_0(\Omega)} \leq e^{2T}\|\nabla\partial^{\alpha'}u^i\|^2_{G_{0,1}(\Omega)} \\
&\leq e^{C_1 T}\Big((C_{1,0} + TC_1)\Big(\sum_{j=0}^2 \|\partial_j\partial^{\alpha'}u^i\|^2_{G_{0,1}(\Omega)} + \|f^i_{\alpha'}(0)\|^2_{L^2(\mathbb{R}^3_+)}\Big) \\
&\qquad + (C_{1,0} + TC_1)\|\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)}\Phi_i(\theta_i S_{\chi,\sigma,G,m,\alpha_0}(0,f,u_0))\|^2_{H^1(\mathbb{R}^3_+)} + C_1\|f^i_{\alpha'}\|^2_{H^1_1(\Omega)}\Big) \\
&\leq e^{C_1 T^*}\Big((C_{1,0} + T^*C_1)\Big(\sum_{j=0}^2 \|\partial_j\partial^{\alpha'}u^i\|^2_{G_0(\Omega)} + \|f^i_{\alpha'}(0)\|^2_{L^2(\mathbb{R}^3_+)}\Big) \\
&\qquad + C_1\|f^i_{\alpha'}\|^2_{H^1(\Omega)} + (C_{1,0} + T^*C_1)\|\Phi_i(\theta_i S_{\chi,\sigma,G,m,\alpha_0}(0,f,u_0))\|^2_{H^{k-\alpha_0}(\mathbb{R}^3_+)}\Big),
\end{aligned} \tag{7.58}$$

where

$$\begin{aligned}
C_{1,0} &= C_{1,0}(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G) = C_{3.11,1,0}(\eta(\chi), R_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)), \\
C_1 &= C_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G,T^*) = C_{3.11,1}(\eta(\chi), R_1(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G),T^*).
\end{aligned}$$

Inserting the induction hypothesis for $\|\partial_j\partial^{\alpha'}u^i\|^2_{G_0(\Omega)}$, we obtain (7.57) for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $\alpha_3 = l$. By induction, we therefore infer that (7.57) is valid for all multiindices $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$. Our first induction thus shows that (7.56) is true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$.

We now sum over all multiindices with $|\alpha| \leq m$, which yields

$$\begin{aligned}
\sum_{\substack{\alpha\in\mathbb{N}_0^4 \\ |\alpha|\leq m}} \|\partial^\alpha u^i\|^2_{G_0(\Omega)} &\leq C_m\Big(\sum_{j=0}^m \|\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0))\|^2_{H^{m-j}(\mathbb{R}^3_+)} + \|g^i\|^2_{E_m(J\times\partial\mathbb{R}^3_+)} \\
&+ \|u^i\|^2_{H^m(\Omega)} + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|\leq m-1}} \Big(\|f^i_{\tilde\alpha}(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|f^i_{\tilde\alpha}\|^2_{H^1(\Omega)}\Big) + \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|\leq m}} \|f^i_{\tilde\alpha}\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{7.59}$$

where

$$C_m(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G,T^*) = \sum_{\substack{\alpha\in\mathbb{N}_0^4 \\ |\alpha|\leq m}} C_{|\alpha|,\alpha}(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G,T^*).$$

We obtain the estimate corresponding to (7.59) for $i = 0$ by the same methods.

III) We now turn to the estimate of $\|\partial^\alpha u\|_{G_m(J\times G)}$. To that purpose we first note that Φ_i maps $H^m(U_i)$ continuously into $H^m(V_i)$, see e.g. [Maz11] Theorem 1.1.7, and since all derivatives of the functions φ_i and ψ_i up to order $m+2$ are uniformly bounded, we also obtain

$$\|\Phi_i v\|_{H^m(V_i)} \leq C\|v\|_{H^m(U_i)} \quad \text{and} \quad \|\Phi_i^{-1} w\|_{H^m(U_i)} \leq C\|w\|_{H^m(V_i)}$$

for all $v \in H^m(U_i)$, $w \in H^m(V_i)$ and $i \in \mathbb{N}_0$. Employing that $(\theta_i)_{i\in\mathbb{N}_0}$ is a partition of unity and that the covering $(U_i)_{i\in\mathbb{N}}$ of ∂G is locally finite, we thus deduce

$$\begin{aligned} \|u\|^2_{G_m(J\times G)} &\leq C(G)\sum_{i=0}^{\infty}\|\theta_i u\|^2_{G_m(J\times G)} \leq C(G)\sum_{i=0}^{\infty}\sum_{j=0}^{m}\|\partial_t^j(\theta_i u)\|^2_{L^\infty(J,H^{m-j}(U_i))} \\ &\leq C(G)\sum_{i=0}^{\infty}\sum_{j=0}^{m}\|\partial_t^j\Phi_i(\theta_i u)\|^2_{L^\infty(J,H^{m-j}(V_i))} \leq C(G)\sum_{i=0}^{\infty}\sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}}\|\partial^\alpha u^i\|^2_{G_0(\Omega)}. \end{aligned} \tag{7.60}$$

In view of estimate (7.59), we proceed by estimating the terms on the right-hand side of (7.59). First we note that we infer as in Chapter 5, see (5.39), that

$$\sum_{i=1}^{\infty}\|g^i\|^2_{E_m(J\times\partial\mathbb{R}^3_+)} \leq C(G)\|g\|^2_{E_m(J\times\partial G)}. \tag{7.61}$$

Arguing as in (5.38), we further derive

$$\begin{aligned} &\sum_{i=0}^{\infty}\sum_{j=0}^{m}\|\Phi_i(\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0))\|^2_{H^{m-j}(\mathbb{R}^3_+)} \\ &\leq C(G)\sum_{i=0}^{\infty}\sum_{j=0}^{m}\|\theta_i S_{\chi,\sigma,G,m,j}(0,f,u_0)\|^2_{H^{m-j}(G)} \\ &\leq C(G)\sum_{j=0}^{m}\|S_{\chi,\sigma,G,m,j}(0,f,u_0)\|^2_{H^{m-j}(G)} \\ &\leq C(\chi,\sigma,m,r,\mathcal{U}_1,G)\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)}\Big), \\ &\sum_{i=0}^{\infty}\|u^i\|^2_{H^m(\Omega)} \leq C(G)\|u\|^2_{H^m(J\times G)}. \end{aligned} \tag{7.62}$$

It remains to estimate the terms involving f^i_α. We start with the $L^2(\Omega)$-norms. So take a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$. Then

$$\begin{aligned} \|\partial^\alpha f^i(f,u)\|^2_{L^2(\Omega)} &\leq \|f^i(f,u)\|^2_{H^m(\Omega)} \leq C(G)\Big\|\theta_i f + \sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j\theta_i u\Big\|^2_{H^m(J\times G)}, \\ \sum_{i=0}^{\infty}\|\partial^\alpha f^i(f,u)\|^2_{L^2(\Omega)} &\leq C(G)\|f\|^2_{H^m(J\times G)} + C(G)\|u\|^2_{H^m(J\times G)}. \end{aligned} \tag{7.63}$$

Next take $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$. We compute

$$\partial^\beta\Phi_i(\chi(u))\partial_t\partial^{\alpha-\beta}u^i = \sum_{0\leq\beta'\leq\beta}\binom{\beta}{\beta'}\partial^{\beta'}\Phi_i(\zeta_1)\partial^{\beta-\beta'}\Phi_i(\tilde{\chi}(u))\partial_t\partial^{\alpha-\beta}u^i. \tag{7.64}$$

Fix $\beta' \in \mathbb{N}_0^4$ with $0 \leq \beta' \leq \beta$. Since $u \in H^m(J\times G)$ can be approximated by smooth functions in this space, formula (7.1) extends to the composition with $\tilde{\chi}(u)$, see also

Theorem 1.1.7 in [Maz11]. Exploiting once again that all derivatives of the functions ψ_i are uniformly bounded for all $i \in \mathbb{N}$, we thus obtain

$$\|\partial^{\beta'}\Phi_i(\zeta_1)\partial^{\beta-\beta'}\Phi_i(\tilde{\chi}(u))\partial_t\partial^{\alpha-\beta}u^i\|_{L^2(\Omega)} \tag{7.65}$$
$$\leq C(G) \sum_{\substack{\tilde{\alpha}_1\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_1|\leq|\beta'|}} \sum_{\substack{\tilde{\alpha}_2\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_2|\leq|\beta-\beta'|}} \sum_{\substack{\tilde{\alpha}_3\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_3|\leq|\alpha-\beta|}} \|\Phi_i(\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}(\theta_i u))\|_{L^2(\Omega)}.$$

Employing the arguments from (5.38) once again, we derive

$$\sum_{i=0}^{\infty}\|\partial^{\beta'}\Phi_i(\zeta_1)\partial^{\beta-\beta'}\Phi_i(\tilde{\chi}(u))\partial_t\partial^{\alpha-\beta}u^i\|_{L^2(\Omega)}^2$$
$$\leq C(m,G)\sum_{i=0}^{\infty} \sum_{\substack{\tilde{\alpha}_1\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_1|\leq|\beta'|}} \sum_{\substack{\tilde{\alpha}_2\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_2|\leq|\beta-\beta'|}} \sum_{\substack{\tilde{\alpha}_3\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_3|\leq|\alpha-\beta|}} \|\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}(\theta_i u)\|_{L^2(J\times G)}^2$$
$$\leq C(m,G) \sum_{\substack{\tilde{\alpha}_1\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_1|\leq|\beta'|}} \sum_{\substack{\tilde{\alpha}_2\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_2|\leq|\beta-\beta'|}} \sum_{\substack{\tilde{\alpha}_3\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}_3|\leq|\alpha-\beta|}} \|\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)}^2. \tag{7.66}$$

Fix multiindices $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}_3$ as above. We distinguish several cases. First consider the case $\beta' = \beta$. We then have

$$\|\partial^{\tilde{\alpha}_1}\zeta_1\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \leq C(\chi,\mathcal{U}_1)\|\partial^{\tilde{\alpha}_1}\zeta_1\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)}. \tag{7.67}$$

If $|\beta| \leq m-2$, we estimate

$$\begin{aligned}\|\partial^{\tilde{\alpha}_1}\zeta_1\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} &\leq \|\partial^{\tilde{\alpha}_1}\zeta_1\|_{L^\infty(G)}\|\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \\ &\leq C\|\zeta_1\|_{F_m(J\times G)}\|\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)}\end{aligned}$$

by Sobolev's embedding. If $|\beta| \geq m-1$, Sobolev's embedding and Hölder's inequality imply

$$\begin{aligned}\|\partial^{\tilde{\alpha}_1}\zeta_1\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} &\leq C\|\partial^{\tilde{\alpha}_1}\zeta_1\|_{H^{m-|\beta|}(G)}\|\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J,H^{|\beta|+2-m}(G))} \\ &\leq C\|\zeta_1\|_{F_m(J\times G)}\|u\|_{H^m(J\times G)}.\end{aligned}$$

Combining the last two estimates with (7.67), we obtain

$$\|\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \leq C(\chi,r,\mathcal{U}_1)\|u\|_{H^m(J\times G)} \tag{7.68}$$

in the case $\beta' = \beta$. Next suppose that $\beta' < \beta$. If $|\beta'| \leq m-2$, we deduce

$$\|\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \leq \|\zeta_1\|_{F_m(J\times G)}\|\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)}.$$

If $\tilde{\alpha}_2 > 0$, Lemma 7.19 (ii), Lemma 7.1 (i), and Lemma 7.19 (iii) imply

$$\begin{aligned}&\|\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \\ &\leq C\Big(\|\nabla_t\tilde{\chi}(u)\|_{L^\infty(J\times G)}\|u\|_{H^{|\alpha|-|\beta'|}(J\times G)} \\ &\qquad + \|\nabla_t u\|_{L^\infty(J\times G)}\sum_{j=0}^{3}\|\partial_j\tilde{\chi}(u)\|_{H^{|\alpha|-|\beta'|-1}(J\times G)}\Big) \\ &\leq C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u\|_{H^{|\alpha|}(J\times G)}.\end{aligned} \tag{7.69}$$

In the case $\tilde{\alpha}_2 = 0$ this estimate clearly also holds. If $|\beta'| = m-1$, we use Sobolev's embedding and Hölder's inequality as above to derive

$$\|\partial^{\tilde{\alpha}_1}\zeta_1\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J\times G)} \leq C\|\partial^{\tilde{\alpha}_1}\zeta_1\|_{H^1(G)}\|\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)\partial_t\partial^{\tilde{\alpha}_3}u\|_{L^2(J,H^1(G))}$$

$$\le C\|\zeta_1\|_{F_m(J\times G)}\Big(\|\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}+\sum_{k=1}^{3}\|\partial_k\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}$$
$$+\sum_{k=1}^{3}\|\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_k\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}\Big).$$

The first term on the right-hand side above has already been treated in (7.69). For the second sum and the third one if $\tilde\alpha_2>0$ we again apply Lemma 7.19 (ii), Lemma 7.1 (i), and Lemma 7.19 (iii) to deduce

$$\sum_{k=1}^{3}\|\partial_k\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}+\sum_{k=1}^{3}\|\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_k\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}$$
$$\le C\Big(\|\nabla_t\tilde\chi(u)\|_{L^\infty(J\times G)}\|u\|_{H^{|\alpha|+1-|\beta'|}(J\times G)}$$
$$+\|\nabla_t u\|_{L^\infty(J\times G)}\sum_{j=0}^{3}\|\partial_j\tilde\chi(u)\|_{H^{|\alpha|-|\beta'|}(J\times G)}\Big)$$
$$\le C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u\|_{H^m(\Omega)}.$$

This estimate is again clear if $\tilde\alpha_2=0$. To sum up, we have shown that

$$\sum_{\substack{\tilde\alpha_1\in\mathbb{N}_0^4\\0\le|\tilde\alpha_1|\le|\beta'|}}\sum_{\substack{\tilde\alpha_2\in\mathbb{N}_0^4\\0\le|\tilde\alpha_2|\le|\beta-\beta'|}}\sum_{\substack{\tilde\alpha_3\in\mathbb{N}_0^4\\0\le|\tilde\alpha_3|\le|\alpha-\beta|}}\|\partial^{\tilde\alpha_1}\zeta_1\partial^{\tilde\alpha_2}\tilde\chi(u)\partial_t\partial^{\tilde\alpha_3}u\|_{L^2(J\times G)}$$
$$\le C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u\|_{H^m(J\times G)}$$

for all multiindices $0\le\beta'\le\beta$. In view of (7.64) to (7.66) we arrive at

$$\sum_{i=0}^{\infty}\|\partial^\beta\Phi_i(\chi(u))\partial_t\partial^{\alpha-\beta}u^i\|_{L^2(\Omega)}^2\le C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u\|_{H^m(J\times G)}^2 \tag{7.70}$$

for all $\beta\in\mathbb{N}_0^4$ with $0<\beta\le\alpha$ and $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m$.

Analogously, we can estimate

$$\sum_{i=0}^{\infty}\|\partial^\beta\Phi_i(\sigma(u))\partial^{\alpha-\beta}u^i\|_{L^2(\Omega)}^2\le C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\|u\|_{H^m(J\times G)}^2 \tag{7.71}$$

for all $\beta\in\mathbb{N}_0^4$ with $0<\beta\le\alpha$ and $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m$. Employing that the coefficients A_j^i have a uniform $W^{m,\infty}(\mathbb{R}^3_+)$ bound for $i\in\mathbb{N}_0$ and $j\in\{1,2,3\}$ and arguing as above, we also obtain

$$\sum_{i=0}^{\infty}\sum_{j=1}^{3}\|\partial^\beta A_j^i\partial_j\partial^{\alpha-\beta}u^i\|_{L^2(\Omega)}^2\le C(R_1,G)\|u\|_{H^m(J\times G)}^2. \tag{7.72}$$

Combining (7.63) and (7.70) to (7.72), we finally arrive at

$$\sum_{i=0}^{\infty}\|f_\alpha^i\|_{L^2(\Omega)}^2\le C(\chi,\sigma,r,\omega_0,\mathcal{U}_1,G)\Big(\|f\|_{H^m(J\times G)}^2+\|u\|_{H^m(J\times G)}^2\Big). \tag{7.73}$$

Next, we want to estimate $\|f_\alpha^i\|_{H^1(\Omega)}$ for $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le m-1$. Fix such a multi-index α. Let $k\in\{0,\dots,3\}$ and set $\alpha^k=\alpha+e_k$. In the proof of Lemma 3.4, to be more precise in (3.6), we have seen that $f_{\alpha^k}^i=f_{\alpha,k}^i$, where $f_{\alpha,k}^i$ is defined by

$$f_{\alpha,k}^i=\partial_k f_\alpha^i-\partial_k\Phi_i(\chi(u))\partial_t\partial^\alpha u^i-\sum_{j=1}^{3}\partial_k A_j^i\partial_j\partial^\alpha u^i-\partial_k\Phi_i(\sigma(u))\partial^\alpha u^i.$$

We can therefore estimate

$$\|\partial_k f^i_\alpha\|_{L^2(\Omega)} \leq \|f^i_{\alpha^k}\|_{L^2(\Omega)} + \|\partial_k \Phi_i(\chi(u))\partial_t \partial^\alpha u^i\|_{L^2(\Omega)} + \sum_{j=1}^3 \|\partial_k A^i_j \partial_j \partial^\alpha u^i\|_{L^2(\Omega)} \\ + \|\partial_k \Phi_i(\sigma(u))\partial^\alpha u^i\|_{L^2(\Omega)}.$$

The first term on the right-hand side can be estimated by (7.73), the second and the last one by (7.70) respectively (7.71). For the remaining sum we again use that the coefficients A^i_j have a uniform bound in $W^{1,\infty}(\mathbb{R}^3_+)$ before we argue as in (7.62). We thus deduce

$$\sum_{i=0}^\infty \|\partial_k f^i_\alpha\|^2_{L^2(\Omega)} \leq C(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G)\Big(\|f\|^2_{H^m(J\times G)} + \|u\|^2_{H^m(J\times G)}\Big),$$

which leads to

$$\sum_{i=0}^\infty \|f^i_\alpha\|^2_{H^1(\Omega)} = \sum_{i=0}^\infty \Big(\|f^i_\alpha\|^2_{L^2(\Omega)} + \sum_{k=0}^3 \|\partial_k f^i_\alpha\|^2_{L^2(\Omega)}\Big) \\ \leq C(\chi, \sigma, r, \omega_0, \mathcal{U}_1, G)\Big(\|f\|^2_{H^m(J\times G)} + \|u\|^2_{H^m(J\times G)}\Big). \tag{7.74}$$

It remains to estimate $\|f^i_\alpha(0)\|_{L^2(\mathbb{R}^3_+)}$ for $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m-1$. Fix such a multi-index α. The definition of f^i_α in (7.53) shows that

$$f^i_\alpha(0) = \partial^\alpha f^i(f,u)(0) - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta \Phi_i(\chi(u))\partial_t \partial^{\alpha-\beta} u^i)(0) \\ - \sum_{j=1}^3 \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A^i_j \partial_j \partial^{\alpha-\beta} u^i(0) - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta \Phi_i(\sigma(u))\partial^{\alpha-\beta} u^i)(0).$$

We have to estimate the appearing terms. As $\Phi_i(\chi(u))$ and $\Phi_i(\sigma(u))$ belong to $F_m(\Omega)$ by Lemma 7.1 (i) and Lemma 2.22, the derivatives $\partial^\beta \Phi_i(\chi(u))$ and $\partial^\beta \Phi_i(\sigma(u))$ are elements of $\tilde{G}_{m-|\beta|}(\Omega)$ and their time traces $\partial^\beta \Phi_i(\chi(u))(0)$ and $\partial^\beta \Phi_i(\sigma(u))(0)$ belong to $H^{m-|\beta|-1}(\mathbb{R}^3_+)$ for all $\beta \in \mathbb{N}_0^4$ with $0 < |\beta| \leq m-1$ and $i \in \mathbb{N}_0$. Moreover, since $f^i(f,u) \in H^m(\Omega)$ and $u^i \in G_m(\Omega)$ we obtain the relations

$$\partial^\alpha f^i(f,u)(0) \in H^{m-|\alpha|-1}(\mathbb{R}^3_+) \hookrightarrow L^2(\mathbb{R}^3_+), \\ \partial_t \partial^{\alpha-\beta} u^i(0) \in H^{m-|\alpha|+|\beta|-1}(\mathbb{R}^3_+) \hookrightarrow H^{|\beta|}(\mathbb{R}^3_+), \\ \partial^{\alpha-\beta} u^i(0) \in H^{m-|\alpha|+|\beta|}(\mathbb{R}^3_+) \hookrightarrow H^{|\beta|}(\mathbb{R}^3_+),$$

for all $\beta \in \mathbb{N}_0^4$ with $\beta \leq \alpha$. We set $k = |\alpha| \leq m-1$. Also employing Lemma 7.5, we estimate

$$\|\partial^\alpha f^i(f,u)(0)\|_{L^2(\mathbb{R}^3_+)} \leq \max_{0\leq l\leq k} \|\Phi_i(\theta_i \partial_t^l f(0))\|_{H^{k-l}(\mathbb{R}^3_+)} \\ + C\sum_{j=1}^3 \max_{0\leq l\leq k} \|\Phi_i(\partial_j \theta_i S_{\chi,\sigma,G,k,l}(0,f,u_0))\|_{H^{k-l}(\mathbb{R}^3_+)}, \\ \|\partial_t \partial^{\alpha-\beta} u^i(0)\|_{H^{|\beta|}(\mathbb{R}^3_+)} \leq \max_{1\leq l\leq k+1} \|\Phi_i(\theta_i S_{\chi,\sigma,G,k+1,l}(0,f,u_0))\|_{H^{k+1-l}(\mathbb{R}^3_+)} \\ \|\partial^{\alpha-\beta} u^i(0)\|_{H^{|\beta|}(\mathbb{R}^3_+)} \leq \max_{0\leq l\leq k} \|S_{\chi,\sigma,G,k,l}(0,f,u_0)\|_{H^{k-l}(\mathbb{R}^3_+)}.$$

In particular, the arguments from (7.64) to (7.66) allow us to estimate

$$\sum_{i=0}^\infty \|\partial^\alpha f^i(f,u)(0)\|^2_{L^2(\mathbb{R}^3_+)}$$

$$\le C(m,G) \sum_{i=0}^{\infty} \Big(\max_{0 \le l \le k} \|\theta_i \partial_t^l f(0)\|_{H^{k-l}(G)}^2 + \sum_{j=1}^{3} \max_{0 \le l \le k} \|\partial_j \theta_i S_{\chi,\sigma,G,k,l}(0,f,u_0)\|_{H^{k-l}(G)}^2 \Big)$$
$$\le C(m,G) \Big(\max_{0 \le l \le k} \|\partial_t^l f(0)\|_{H^{k-l}(G)}^2 + \max_{0 \le l \le k} \|S_{\chi,\sigma,G,k,l}(0,f,u_0)\|_{H^{k-l}(G)}^2 \Big)$$
$$\le C(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)}^2 + \|u_0\|_{H^m(G)}^2 \Big). \tag{7.75}$$

Since $\tilde{\chi}$ and $\tilde{\sigma}$ belong to $C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$, Lemma 7.1 (ii) and (iii) imply that the function $\partial^{\beta-\beta'}\tilde{\chi}(u)(0)$ belongs to $H^{m-|\beta|-1+|\beta'|}(G)$ and

$$\begin{aligned}
&\|\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)(0)\|_{H^{m-|\beta|-1+|\beta'|}(G)} \\
&\le C_{7.1}(\chi,m,6,r)(1 + \max_{0\le l\le |\beta|-|\beta'|} \|S_{\chi,\sigma,G,m,l}(0,f,u_0)\|_{H^{m-l-1}(\mathbb{R}^3_+)})^m \\
&\le C(\chi,\sigma,m,r,\mathcal{U}_1), \\
&\|\partial^{\tilde{\alpha}_2}\tilde{\sigma}(u)(0)\|_{H^{m-|\beta|-1}(G)} \le C(\chi,\sigma,m,r,\mathcal{U}_1)
\end{aligned}$$

for all $\tilde{\alpha}_2 \in \mathbb{N}_0^4$ with $|\tilde{\alpha}_2| \le |\beta-\beta'|$ and $\beta', \beta \in \mathbb{N}_0^4$ with $0 \le \beta' \le \beta$ and $0 < \beta \le \alpha$, where we used Lemma 7.5 and Lemma 7.7. We remark that $\partial^{\tilde{\alpha}_1}\zeta_{k'}$ belongs to $F_m(J\times G)$ if $\tilde{\alpha}_1 = 0$ and to $H^{m-|\beta'|}(G)$ if $\tilde{\alpha}_1 > 0$ for $k' \in \{1,2\}$ and $\tilde{\alpha}_1 \in \mathbb{N}_0^4$ with $|\tilde{\alpha}_1| \le |\beta'|$. Since $\max\{m-|\beta'|, m-|\beta|+|\beta'|-1\} \ge 2$ for all $0 \le \beta' \le \beta$, Lemma 2.22 thus yields that

$$\begin{aligned}
&\|\partial^{\tilde{\alpha}_1}\zeta_1 \partial^{\tilde{\alpha}_2}\tilde{\chi}(u)(0)\|_{H^{m-|\beta|-1}(G)} \\
&\le C\|\zeta_1\|_{F_m(J\times G)} \|\partial^{\tilde{\alpha}_2}\tilde{\chi}(u)(0)\|_{H^{m-|\beta|-1+|\beta'|}(G)} \le C(\chi,\sigma,m,r,\mathcal{U}_1).
\end{aligned}$$

Arguing analogously for σ, we arrive at

$$\begin{aligned}
&\|\partial^{\tilde{\alpha}_1}\zeta_1 \partial^{\tilde{\alpha}_2}\tilde{\chi}(u)(0)\|_{H^{m-|\beta|-1}(G)} \le C(\chi,\sigma,m,r,\mathcal{U}_1), \\
&\|\partial^{\tilde{\alpha}_1}\zeta_2 \partial^{\tilde{\alpha}_2}\tilde{\sigma}(u)(0)\|_{H^{m-|\beta|-1}(G)} \le C(\chi,\sigma,m,r,\mathcal{U}_1)
\end{aligned}$$

for all $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{N}_0^4$ with $|\tilde{\alpha}_1| \le |\beta'|$ and $|\tilde{\alpha}_2| \le |\beta-\beta'|$, where $\beta', \beta \in \mathbb{N}_0^4$ with $0 \le \beta' \le \beta$ and $0 < \beta \le \alpha$. Arguing as in (7.64) to (7.66) and employing Lemma 7.7, we thus obtain

$$\begin{aligned}
&\sum_{i=0}^{\infty} \|\partial^\beta \Phi_i(\chi(u))\partial_t \partial^{\alpha-\beta} u^i(0)\|_{L^2(\mathbb{R}^3_+)}^2 \\
&\le C(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G) \max_{1\le l\le k+1} \|S_{\chi,\sigma,G,m,l}(0,f,u_0)\|_{H^{m-l}(G)}^2 \\
&\le C(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)}^2 + \|u_0\|_{H^m(G)}^2 \Big).
\end{aligned} \tag{7.76}$$

Analogously, one infers

$$\begin{aligned}
&\sum_{i=0}^{\infty} \|\partial^\beta \Phi_i(\sigma(u))\partial^{\alpha-\beta} u^i(0)\|_{L^2(\mathbb{R}^3_+)}^2 \\
&\le C(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G) \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|_{H^{m-1-j}(G)}^2 + \|u_0\|_{H^m(G)}^2 \Big).
\end{aligned} \tag{7.77}$$

We also note that

$$\begin{aligned}
&\|\partial^\beta A_j^i \partial_j \partial^{\alpha-\beta} u^i(0)\|_{L^2(\mathbb{R}^3_+)} \le C(r,G)\|\partial_j \partial^{\alpha-\beta}\Phi_i(\theta_i u)(0)\|_{L^2(\mathbb{R}^3_+)} \\
&\le C(r,G) \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|\le|\alpha|-|\beta|+1}} \|\partial^{\tilde{\alpha}}(\theta_i u)(0)\|_{L^2(G)}
\end{aligned}$$

$$\leq C(m,r,G) \max_{0\leq l\leq m} \|\theta_i S_{\chi,\sigma,G,m,l}(0,f,u_0)\|_{H^{m-l}(G)} \tag{7.78}$$

for all $i \in \mathbb{N}_0$. The combination of (7.75) to (7.78) together with Lemma 7.7 thus yields

$$\sum_{i=0}^{\infty} \|f_\alpha^i(0)\|^2_{L^2(\mathbb{R}^3_+)} \leq C(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G)\Big(\sum_{l=0}^{m-1} \|\partial_t^l f(0)\|^2_{H^{m-1-l}(\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big). \tag{7.79}$$

We now insert (7.61) to (7.62) and (7.59) into (7.60) which leads to

$$\begin{aligned}\|u\|^2_{G_m(J\times G)} &\leq C'_m\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m(J\times\partial G)} \\ &+ \|u\|^2_{H^m(J\times G)} + \sum_{i=0}^{\infty}\sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\ |\tilde\alpha|\leq m-1}} \Big(\|f_{\tilde\alpha}^i(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|f_{\tilde\alpha}^i\|^2_{H^1(\Omega)}\Big) + \sum_{i=0}^{\infty}\sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\ |\tilde\alpha|\leq m}} \|f_{\tilde\alpha}^i\|^2_{L^2(\Omega)}\Big)\end{aligned} \tag{7.80}$$

for a constant $C'_m = C'_m(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G,T^*)$. Estimates (7.73), (7.74), and (7.79) thus finally yield

$$\begin{aligned}\|u\|^2_{G_m(J\times G)} \leq C'_m\Big(&\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m(J\times\partial G)} \\ &+ \|f\|^2_{H^m(J\times G)} + \|u\|^2_{H^m(J\times G)}\Big)\end{aligned} \tag{7.81}$$

for a constant $C'_m = C'_m(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,G,T^*)$.

Recall that the time $T\in(0,T']$ was arbitrary. The above estimate thus implies that

$$\begin{aligned}&\sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}} \|\partial^\alpha u(t)\|^2_{L^2(G)} \leq \sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}} \|\partial^\alpha u\|^2_{G_0((0,t)\times G)} \\ &\leq C'_m\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m((0,t)\times\partial G)} + \|f\|^2_{H^m((0,t)\times G)} \\ &\qquad + \int_0^t \sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}} \|\partial^\alpha u(s)\|^2_{L^2(G)} ds\Big)\end{aligned}$$

for all $t\in[0,T']$. Since the maps $t\mapsto \|f\|^2_{H^m((0,t)\times G)}$ and $t\mapsto \|g\|^2_{E_m((0,t)\times\partial G)}$ are monotonically increasing, Gronwall's inequality leads to

$$\begin{aligned}\sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}} \|\partial^\alpha u(t)\|^2_{L^2(G)} \leq C'_m\Big(&\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} \\ &+ \|g\|^2_{E_m((0,t)\times\partial G)} + \|f\|^2_{H^m((0,t)\times G)}\Big)e^{C'_m t}\end{aligned} \tag{7.82}$$

for all $t\in[0,T']$. Defining $C_m = C_m(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1,T^*) := C'_m e^{C'_m T^*}$ and taking again a fixed time $T\in(0,T']$, we particularly obtain

$$\begin{aligned}&\sum_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\leq m}} \|\partial^\alpha u(t)\|^2_{L^2(G)} \\ &\leq C_m\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(G)} + \|u_0\|^2_{H^m(G)} + \|g\|^2_{E_m((0,T)\times\partial G)} + \|f\|^2_{H^m((0,T)\times G)}\Big)\end{aligned}$$

for all $t\in[0,T]$. We conclude that the assertion of the proposition is valid. □

We point out that the above proof only leads to improved estimates of the nonlinear problem since we need to know that $A_0 = \chi(u)$ for the solution u.

We already mention that Proposition 7.20 in combination with Lemma 7.17 easily implies that the spatial Lipschitz norm of the solution has to blow up if the solution does not exist globally. However, we postpone the precise statement and the proof of this fact to Theorem 7.23 below where we formulate a complete local wellposedness theory also including the continuous dependance on the data. The following two results prepare the proof of the latter.

The difficulty in the investigation of continuous dependance arises from the loss of derivatives we experience because of the quasilinear nature of the system (1.6). By loss of derivatives we mean that if we compare two solutions to different data and look at the initial boundary value problem (3.2) solved by their difference, the right-hand side is less smooth than the solutions. It is therefore crucial to overcome this loss of regularity.

A first step in this direction is the following lemma. It is concerned with a sequence of linear initial boundary value problems and could have already been proven in Chapter 4. A similar result in the full space case can be found in Lemma 4.26 in [BCD11].

We return to the half-space and consider a sequence of coefficients $(A_n, D_n)_{n\in\mathbb{N}}$, which is bounded in $W^{1,\infty}(\Omega)$ and converges in $L^\infty(\Omega)$. We show that the corresponding sequence of solutions (u_n) of (3.2) with fixed inhomogeneity $f \in L^2(\Omega)$, boundary value $g \in L^2(J, H^{1/2}(\mathbb{R}^3_+))$, and initial value $u_0 \in L^2(\mathbb{R}^3_+)$ has a limit in $G_0(\Omega)$. The key observation in the proof is that due to the boundedness of the coefficients in $W^{1,\infty}(\Omega)$, the a priori estimates from Theorem 4.13 hold uniformly in n. Approximating f, g, and u_0 by smoother data $(f_j, g_j, u_{0,j})_{j\in\mathbb{N}}$, the corresponding smoother solutions u_n^j then tend to u_n uniformly in n.

We further note that it is favourable to work on the L^2-level here as we do not have to deal with compatibility conditions in that regularity regime. Moreover, it is easy to approximate data in L^2 by smoother ones which are also compatible since C_c^∞ is dense in L^2.

Lemma 7.21. *Let $J \subseteq \mathbb{R}$ be an open and bounded interval, $t_0 \in \overline{J}$, and $\Omega = J \times \mathbb{R}^3_+$. Let $\eta, \tau > 0$. Take coefficients $A_{0,n}, A_0 \in F^{\mathrm{c}}_{3,\eta}(\Omega)$, $A_1, A_2 \in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\Omega)$, $A_3 \in F^{\mathrm{cp}}_{3,\mathrm{coeff},\tau}(\Omega)$, and $D_n, D \in F^{\mathrm{c}}_3(\Omega)$ for all $n \in \mathbb{N}$ such that $(A_{0,n})_n$ and $(D_n)_n$ are bounded in $W^{1,\infty}(\Omega)$ and converge to A_0 respectively D in $L^\infty(\Omega)$. Pick $B \in \mathcal{BC}^3_{\mathbb{R}^3_+}(A_3)$. Suppose that A_1, A_2, A_3, and B are independent of time and that A_3 and a function M as in the definition of $\mathcal{BC}^3_{\mathbb{R}^3_+}(A_3)$ belong to $C^\infty(\overline{\Omega})$. We further assume that there are functions $G_B^1 \in W^{4,\infty}(\mathbb{R}^3_+)^{2\times 2}$ and $G_B^2 \in W^{4,\infty}(\mathbb{R}^3_+)^{6\times 6}$ such that $G_B^1 B G_B^2$ has Gaussian normal form. Choose data $u_0 \in L^2(\mathbb{R}^3_+)$, $g \in E_0(J \times \partial\mathbb{R}^3_+)$, and $f \in L^2(\Omega)$. Let u_n denote the weak solution of the linear initial boundary value problem (3.2) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$, inhomogeneity f, boundary value g, and initial value u_0 for all $n \in \mathbb{N}$ and u the weak solution of (3.2) with differential operator $L(A_0, \dots, A_3, D)$, inhomogeneity f, boundary value g, and initial value u_0 (see Theorem 4.13). Then $(u_n)_n$ converges to u in $G_0(\Omega)$.*

Proof. Without loss of generality we assume that $J = (0, T)$ for some $T > 0$ and $t_0 = 0$. Set $A_{0,0} = A_0$ and $D_0 = D$. Take $r > 0$ with $\|A_{0,n}\|_{W^{1,\infty}(\Omega)}, \|D_n\|_{W^{1,\infty}(\Omega)} \leq r$ for all $n \in \mathbb{N}_0$.

I) We first assume that u_0 belongs to $H^1(\mathbb{R}^3_+)$, g to $E_1(J \times \partial\mathbb{R}^3_+)$, f to $H^1(\Omega)$, and that the tuples $(0, A_{0,n}, A_1, A_2, A_3, B, D_n, f, g, u_0)$ fulfill the linear compatibility conditions (2.37) of first order for each $n \in \mathbb{N}_0$. The solutions u_n and u are then contained in $G_1(\Omega)$ by Theorem 4.13. The difference $u_n - u$ further solves the linear initial boundary value problem

$$\begin{cases} L(A_{0,n}, A_1, A_2, A_3, D_n)(u_n - u) = f_n, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B(u_n - u) = 0, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ (u_n - u)(0) = 0, & x \in \mathbb{R}^3_+; \end{cases}$$

where $f_n = (A_0 - A_{0,n})\partial_t u + (D - D_n)u$ for all $n \in \mathbb{N}$. As u is an element of $G_1(\Omega)$, the right-hand side of the differential equation above belongs to $L^2(\Omega)$. Theorem 4.13 thus provides constants $\gamma = \gamma_{3.7,0}(\eta, r)$ and $C_0 = C_{3.7,0}(\eta, r)$ such that

$$\begin{aligned} \|u_n - u\|^2_{G_0(\Omega)} &\le e^{2\gamma T}\|u_n - u\|^2_{G_{0,\gamma}(\Omega)} \le C_0 e^{2\gamma T}\|(A_0 - A_{0,n})\partial_t u + (D - D_n)u\|^2_{L^2_\gamma(\Omega)} \\ &\le 2C_0 e^{2\gamma T}(\|A_{0,n} - A_0\|^2_{L^\infty(\Omega)}\|\partial_t u\|^2_{L^2_\gamma(\Omega)} + \|D_n - D\|^2_{L^\infty(\Omega)}\|u\|^2_{L^2_\gamma(\Omega)}) \end{aligned}$$

for all $n \in \mathbb{N}$. Since $A_{0,n} \to A_0$ and $D_n \to D$ in $L^\infty(\Omega)$, we conclude that the functions u_n tend to u in $G_0(\Omega)$ as $n \to \infty$.

II) We now come to the general case where u_0 belongs to $L^2(\mathbb{R}^3_+)$, g to $E_0(J \times \partial\mathbb{R}^3_+)$, and f to $L^2(\Omega)$. We recall that step I) of the proof of Theorem 4.13 shows that there are sequences of initial values $(u_{0,j})_j$ in $H^1(\mathbb{R}^3_+)$ converging to u_0 in $L^2(\mathbb{R}^3_+)$, of boundary values $(g_j)_j$ in $E_1(J \times \partial\mathbb{R}^3_+)$ converging to g in $E_0(J \times \partial\mathbb{R}^3_+)$, and of inhomogeneities $(f_j)_j$ in $H^1(\Omega)$ converging to f in $L^2(\Omega)$ such that the tuples $(0, A_{0,n}, A_1, A_2, A_3, D_n, B, f_j, g_j, u_{0,j})$ fulfill the linear compatibility conditions (2.37) of order 1 for all $n, j \in \mathbb{N}$.

Let the function u_n^j denote the weak solution of (3.2) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$, inhomogeneity f_j, boundary value g_j, and initial value $u_{0,j}$, as well as u^j the weak solution of (3.2) with differential operator $L(A_0, \ldots, A_3, D)$, inhomogeneity f_j, boundary value g_j, and initial value $u_{0,j}$ for all $n, j \in \mathbb{N}$. These solutions exist in $G_1(\Omega)$ by Theorem 4.13. Observe that $u_n^j - u_n$ solves (3.2) with differential operator $L(A_{0,n}, A_1, A_2, A_3, D_n)$, inhomogeneity $f_j - f$, boundary value $g_j - g$, and initial value $u_{0,j} - u_0$, and $u^j - u$ solves (3.2) with differential operator $L(A_0, A_1, A_2, A_3, D)$, inhomogeneity $f_j - f$, boundary value $g_j - g$, and initial value $u_{0,j} - u_0$. The a priori estimate in Theorem 4.13 thus shows

$$\begin{aligned} \|u_n^j - u_n\|^2_{G_0(\Omega)} &\le e^{2\gamma T}\|u_n^j - u_n\|^2_{G_{0,\gamma}(\Omega)} && (7.83) \\ &\le C_0\, e^{2\gamma T}\Big(\|u_{0,j} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_j - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_j - f\|^2_{L^2_\gamma(\Omega)}\Big), \\ \|u^j - u\|^2_{G_0(\Omega)} &\le e^{2\gamma T}\|u^j - u\|^2_{G_{0,\gamma}(\Omega)} && (7.84) \\ &\le C_0\, e^{2\gamma T}\Big(\|u_{0,j} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_j - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_j - f\|^2_{L^2_\gamma(\Omega)}\Big) \end{aligned}$$

for all $n, j \in \mathbb{N}$, where γ and C_0 were introduced in step I).

Let $\varepsilon > 0$. Since $(f_j)_j$ converges to f in $L^2(\Omega)$, $(g_j)_j$ to g in $E_0(J \times \partial\mathbb{R}^3_+)$, and $(u_{0,j})_j$ to u_0 in $L^2(\mathbb{R}^3_+)$, we find an index j_0 such that

$$C_0\, e^{2\gamma T}\Big(\|u_{0,j_0} - u_0\|^2_{L^2(\mathbb{R}^3_+)} + \|g_{j_0} - g\|^2_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|f_{j_0} - f\|^2_{L^2_\gamma(\Omega)}\Big) \le \frac{\varepsilon^2}{9}. \quad (7.85)$$

On the other hand, the tuple $(f_{j_0}, g_{j_0}, u_{0,j_0})$ fulfills the assumptions of step I), which therefore implies $u_n^{j_0} \to u^{j_0}$ in $G_0(\Omega)$ as $n \to \infty$. Hence, there is an index $n_0 \in \mathbb{N}$ such that

$$\|u_n^{j_0} - u^{j_0}\|_{G_0(\Omega)} \le \frac{\varepsilon}{3} \quad (7.86)$$

for all $n \ge n_0$. Combining (7.83) to (7.86), we arrive at

$$\begin{aligned} \|u_n - u\|_{G_0(\Omega)} &\le \|u_n - u_n^{j_0}\|_{G_0(\Omega)} + \|u_n^{j_0} - u^{j_0}\|_{G_0(\Omega)} + \|u^{j_0} - u\|_{G_0(\Omega)} \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all $n \ge n_0$. □

The next lemma contains the heart of the argument for the continuous dependance of solutions on the data. We prove that convergence of the data in H^m respectively E_m and of the corresponding solutions of the nonlinear problem (1.6) in G_{m-1} implies the convergence of the solutions in G_m.

The proof relies on a splitting of the highest order derivatives $\partial^\alpha u_n^i$ of the localized solutions for which we have to show convergence to $\partial^\alpha u^i$ in $G_0(\Omega)$. We write them as a sum of two terms, where the first one converges to $\partial^\alpha u^i$ due to Lemma 7.21. The second one can be estimated by Gronwall's lemma with a prefactor converging to zero, thus implying the convergence to zero of the second part. The idea of this splitting was also used in the proof of Theorem 4.24 in [BCD11].

We further remark that we consequently overcome the loss of derivatives by the regularization argument performed in the proof of Lemma 7.21, which is based on the wellposedness theory for the linear initial boundary value problem (3.2), in particular the uniqueness of solutions thereof. The initial value problem (3.17) lacks this property so that on first sight the splitting approach seems to be applicable only to purely tangential derivatives (that satisfy (3.2)). It is therefore a key observation that ideas from the proof of Lemma 3.11 allow us to reduce the estimates for general derivatives to the ones for purely tangential derivatives.

Lemma 7.22. *Let $J' \subseteq \mathbb{R}$ be an open and bounded interval and $t_0 \in \overline{J'}$. Let $m \in \mathbb{N}$ with $m \geq 3$. Take functions $\chi \in \mathcal{ML}^m_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^m(G,\mathcal{U})$. Set*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. We moreover suppose that G has a tame uniform C^{m+2}-boundary with finitely many charts. Choose $f_n, f \in H^m(J' \times G)$, $g_n, g \in E_m(J' \times \partial G)$, and $u_{0,n}, u_0 \in H^m(G)$ for all $n \in \mathbb{N}$ with

$$\|u_{0,n} - u_0\|_{H^m(G)} \longrightarrow 0, \quad \|g_n - g\|_{E_m(J' \times \partial G)} \longrightarrow 0, \quad \|f_n - f\|_{H^m(J' \times G)} \longrightarrow 0,$$

as $n \to \infty$.

We further assume that the nonlinear initial boundary value problems (1.6) with data $(t_0, f_n, g_n, u_{0,n})$ and (t_0, f, g, u_0) have solutions u_n and u on J' which belong to $G_m(J' \times G)$ for all $n \in \mathbb{N}$, that there is a compact subset $\tilde{\mathcal{U}}_1$ of $\mathcal{U}$ with $\operatorname{im} u(t) \subseteq \tilde{\mathcal{U}}_1$ for all $t \in J'$, that $(u_n)_n$ is bounded in $G_m(J' \times G)$, and that $(u_n)_n$ converges to u in $G_{m-1}(J' \times G)$.

Then the functions u_n converge to u in $G_m(J' \times G)$.

Proof. Without loss of generality we assume that $t_0 = 0$ and that $J' = (0, T')$ for a number $T' > 0$. As in the proof of Proposition 7.20 we further suppose that $\chi = \zeta_1 \tilde{\chi}$ and $\sigma = \zeta_2 \tilde{\sigma}$ where $\zeta_1, \zeta_2 \in F^{\mathrm{c}}_{m,6}(J \times G)$ are time-independent and $\tilde{\chi}$ and $\tilde{\sigma}$ belong to $C^m(\mathcal{U}, \mathbb{R}^{6\times 6})$. The general case then follows as described in Remark 7.18. For simplicity, we take $\zeta_1 = \zeta_2 = I_{6\times 6}$. The case of variable ζ_1 and ζ_2 can be treated as in Proposition 7.20. Since G has a tame uniform C^{m+2}-boundary with finitely many charts, we can cover it by finitely many charts in the localization procedure. In particular, we do not have to take care that the right-hand sides of our estimates are summable. The reduction to charts works as in Proposition 7.20 and will not be repeated here. We thus assume that $G = \mathbb{R}^3_+$ and that the coefficients are as in Definition 5.7. In particular, there is a number $\tau > 0$ such that A_3 belongs to $F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$.

Let $T \in (0, T']$, $J = (0,T)$, and $\Omega = J \times \mathbb{R}^3_+$.

Sobolev's embedding yields a constant C_S, depending on the length of the interval J' such that

$$\|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} \leq C_S \|f_n - f\|_{H^m(J' \times \mathbb{R}^3_+)} \longrightarrow 0$$

as $n \to \infty$, implying that

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} \longrightarrow 0 \tag{7.87}$$

as $n \to \infty$.

We set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $u_\infty = u$, $f_\infty = f$, $g_\infty = g$, and $u_{0,\infty} = u_0$. By assumption, (7.87), and Sobolev's embedding there is a radius $r > 0$ such that

$$\|u_n\|_{G_m(J' \times \mathbb{R}^3_+)} + \|u_n\|_{L^\infty(J' \times \mathbb{R}^3_+)} \leq r, \tag{7.88}$$

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} + \|u_{0,n}\|_{H^m(\mathbb{R}^3_+)} + \|g_n\|_{E_m(J' \times \mathbb{R}^3_+)} + \|f_n\|_{H^m(J' \times \mathbb{R}^3_+)} \leq r, \tag{7.89}$$

$$\sum_{j=1}^{3} \|A_j\|_{F_m(J' \times \mathbb{R}^3_+)} \leq r \tag{7.90}$$

for all $n \in \overline{\mathbb{N}}$. As $\operatorname{im} u(t) \subseteq \tilde{\mathcal{U}}_1$ for all $t \in J'$ and $(u_n)_n$ converges to u in $L^\infty(J \times G)$ by Sobolev's embedding, there is a compact and connected set $\mathcal{U}_1 \subseteq \mathcal{U}$ and an index n_0 such that $\operatorname{im} u_n(t) \subseteq \mathcal{U}_1$ for all $t \in J'$ and $n \geq n_0$. Without loss of generality we assume $n_0 = 1$. Lemma 7.1 (i) then shows that $\chi(u_n)$ and $\sigma(u_n)$ belong to $F_m(\Omega)$ and that there is a radius $R = R(\chi, \sigma, m, r, \mathcal{U}_1)$ with

$$\|\chi(u_n)\|_{F_m(J' \times \mathbb{R}^3_+)} + \|\sigma(u_n)\|_{F_m(J' \times \mathbb{R}^3_+)} \leq R \tag{7.91}$$

for all $n \in \overline{\mathbb{N}}$.

By assumption, the functions u_n solve the initial boundary value problem

$$\begin{cases} \chi(u_n)\partial_t u_n + \sum_{j=1}^{3} A_j \partial_j u_n + \sigma(u_n) u_n = f_n, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B u_n = g_n, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ u_n(0) = u_{0,n}, & x \in \mathbb{R}^3_+; \end{cases}$$

for all $n \in \overline{\mathbb{N}}$. Lemma 3.4 and (7.15) thus imply that the function $\partial^\alpha u_n$ solves the linear initial value problem

$$\begin{cases} L_n v = f_{\alpha,n}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,\alpha_3)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}), & x \in \mathbb{R}^3_+; \end{cases} \tag{7.92}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and $n \in \overline{\mathbb{N}}$. Due to Corollary 2.18 it also fulfills the linear initial boundary value problem

$$\begin{cases} L_n v = f_{\alpha,n}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ B v = g_{\alpha,n}, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}), & x \in \mathbb{R}^3_+; \end{cases} \tag{7.93}$$

for all $\alpha \in \mathbb{N}_0^4$ with $\alpha_3 = 0$ and $|\alpha| \leq m$ and $n \in \overline{\mathbb{N}}$. Here we set

$$\begin{aligned} L_n &= L(\chi(u_n), A_1, A_2, A_3, \sigma(u_n)), \\ f_{\alpha,n} &= \partial^\alpha f_n - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \chi(u_n) \partial^{\alpha-\beta} \partial_t u_n - \sum_{j=1}^{3} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial^{\alpha-\beta} \partial_j u_n \\ &\quad - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \sigma(u_n) \partial^{\alpha-\beta} u_n, \end{aligned}$$

$$g_{\alpha,n} = \partial^\alpha g_n - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta B \partial^{\alpha-\beta} u_n \tag{7.94}$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$ and $n \in \overline{\mathbb{N}}$.

I) Fix a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$. Lemma 3.4 also yields that $f_{\alpha,n}$ is an element of $L^2(\Omega)$ for all $n \in \overline{\mathbb{N}}$. In this step we want to estimate the difference $f_{\alpha,n} - f_{\alpha,\infty}$ in suitable norms for all $n \in \mathbb{N}$. To that purpose, we first note that

$$\begin{aligned} f_{\alpha,n} - f_{\alpha,\infty} = \partial^\alpha f_n - \partial^\alpha f - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big(&\partial^\beta \chi(u_n)(\partial^{\alpha-\beta}\partial_t u_n - \partial^{\alpha-\beta}\partial_t u) \\ &+ (\partial^\beta \chi(u_n) - \partial^\beta \chi(u))\partial^{\alpha-\beta}\partial_t u + \partial^\beta \sigma(u_n)(\partial^{\alpha-\beta} u_n - \partial^{\alpha-\beta} u) \\ &+ (\partial^\beta \sigma(u_n) - \partial^\beta \sigma(u))\partial^{\alpha-\beta} u + \sum_{j=1}^3 \partial^\beta A_j (\partial^{\alpha-\beta}\partial_j u_n - \partial^{\alpha-\beta}\partial_j u)\Big) \end{aligned} \tag{7.95}$$

for all $n \in \mathbb{N}$. We first note that the proof of Lemma 3.4 shows that

$$\begin{aligned} &\sum_{j=1}^3 \|\partial^\beta A_j (\partial^{\alpha-\beta}\partial_j u_n - \partial^{\alpha-\beta}\partial_j u)\|_{L^2_\gamma(\Omega)} \le C(r) \|u_n - u\|_{H^m(\Omega)}, \\ &\sum_{j=1}^3 \|\partial^\beta A_j \partial^{\alpha-\beta}\partial_j u_n\|_{L^2_\gamma(\Omega)} \le C(r)\|u_n\|_{H^m(\Omega)} \le C(r, T') \end{aligned} \tag{7.96}$$

for all $\gamma \ge 0$ and $n \in \mathbb{N}$.

In view of Corollary 7.2, we introduce the quantity

$$h_n(t) = \sum_{i=1}^3 \sum_{l_1,\ldots,l_m=1}^6 \|(\partial_{l_m}\ldots\partial_{l_1}\theta_i)(u_n(t)) - (\partial_{l_m}\ldots\partial_{l_1}\theta_i)(u(t))\|_{L^\infty(\mathbb{R}^3_+)}$$

for all $t \in \overline{J'}$ and $n \in \mathbb{N}$, where $\theta_1 = \chi$, $\theta_2 = \sigma$, and $\theta_3 = \chi^{-1}$. Employing that $(u_n)_n$ converges to u in $G_2(\Omega)$, we deduce that $(u_n)_n$ converges to u also uniformly. Recall that the functions u_n and u take values in $\mathcal{U}_1$ for all $n \in \mathbb{N}$. Let $l_1, \ldots, l_m \in \{1, \ldots, 6\}$. Then the functions $\partial_{l_m}\ldots\partial_{l_1}\theta_i$ are continuous and therefore uniformly continuous on the compact set $\mathcal{U}_1$ for $i \in \{1, 2, 3\}$. We conclude that $\partial_{l_m}\ldots\partial_{l_1}\theta_i(u_n)$ converges uniformly to $\partial_{l_m}\ldots\partial_{l_1}\theta_i(u)$ on $\overline{J' \times \mathbb{R}^3_+}$ for $i \in \{1, 2, 3\}$. In particular,

$$h_n(t) \longrightarrow 0 \tag{7.97}$$

for all $t \in \overline{J'}$ as $n \to \infty$ and

$$\int_0^{T'} h_n^2(t)dt \longrightarrow 0 \tag{7.98}$$

as $n \to \infty$.

We return to the task of estimating (7.95). To this aim, we observe that $\partial^\beta \chi(u_n)(s)$ belongs to $H^{m-|\beta|}(\mathbb{R}^3_+) = H^{m-1-(|\beta|-1)}(\mathbb{R}^3_+)$ and $\partial^{\alpha-\beta}\partial_t(u_n - u)(s)$ is an element of $H^{|\alpha|-|\alpha-\beta|-1}(\mathbb{R}^3_+) = H^{|\beta|-1}(\mathbb{R}^3_+)$. Lemma 2.22 (v) and Lemma 7.1 (i) thus yield

$$\begin{aligned} &\|\partial^\beta \chi(u_n)(s)\partial^{\alpha-\beta}\partial_t u_n(s)\|_{L^2(\mathbb{R}^3_+)} \\ &\le C\|\partial^\beta \chi(u_n)(s)\|_{H^{m-|\beta|}(\mathbb{R}^3_+)} \|\partial^{\alpha-\beta}\partial_t u_n(s)\|_{H^{|\beta|-1}(\mathbb{R}^3_+)} \\ &\le C\|\chi(u_n)\|_{F_m(\Omega)}\|u_n\|_{G_m(\Omega)} \le C(\chi, m, r, \mathcal{U}_1), \end{aligned} \tag{7.99}$$

as well as

$$\begin{aligned} &\|\partial^\beta \chi(u_n)(s)(\partial^{\alpha-\beta}\partial_t u_n(s) - \partial^{\alpha-\beta}\partial_t u(s))\|_{L^2(\mathbb{R}^3_+)} \\ &\le C\|\partial^\beta \chi(u_n)(s)\|_{H^{m-|\beta|}(\mathbb{R}^3_+)} \|\partial^{\alpha-\beta}\partial_t u_n(s) - \partial^{\alpha-\beta}\partial_t u(s)\|_{H^{|\beta|-1}(\mathbb{R}^3_+)} \end{aligned}$$

$$\begin{aligned}
&\leq C\|\chi(u_n)\|_{F_m(\Omega)}\|\partial^{\alpha-\beta}\partial_t u_n(s) - \partial^{\alpha-\beta}\partial_t u(s)\|_{H^{|\alpha|-|\alpha-\beta|-1}(\mathbb{R}^3_+)} \\
&\leq C(\chi, m, r, \mathcal{U}_1)\Big(\|u_n - u\|_{G_{m-1}(\Omega)} + \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=|\alpha|}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}\Big) \qquad (7.100)
\end{aligned}$$

for all $s \in J'$ and for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and $n \in \mathbb{N}$. Analogously, we note that $\partial^\beta(\chi(u_n) - \chi(u))(s)$ belongs to $H^{|\alpha|-|\beta|}(\mathbb{R}^3_+)$ and $\partial^{\alpha-\beta}\partial_t u(s)$ to $H^{m-|\alpha-\beta|-1}(\mathbb{R}^3_+) = H^{m-1-(|\alpha|-|\beta|)}(\mathbb{R}^3_+)$ for almost all $s \in J'$. Lemma 2.22 (v) thus applies again and shows

$$\begin{aligned}
&\|(\partial^\beta\chi(u_n)(s) - \partial^\beta\chi(u)(s))\partial^{\alpha-\beta}\partial_t u(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C\|\partial^\beta\chi(u_n)(s) - \partial^\beta\chi(u)(s)\|_{H^{|\alpha|-|\beta|}(\mathbb{R}^3_+)}\|\partial^{\alpha-\beta}\partial_t u(s)\|_{H^{m-|\alpha-\beta|-1}(\mathbb{R}^3_+)} \\
&\leq C \sum_{\substack{\tilde{\beta}\in\mathbb{N}_0^4 \\ 1\leq|\tilde{\beta}|\leq|\alpha|}} \|\partial^{\tilde{\beta}}\chi(u_n)(s) - \partial^{\tilde{\beta}}\chi(u)(s)\|_{L^2(\mathbb{R}^3_+)}\|u\|_{G_m(\Omega)} \\
&\leq C(\chi, m, r, \mathcal{U}_1) \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ 0\leq|\tilde{\alpha}|\leq|\alpha|}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)} + C(\chi, m, r, \mathcal{U}_1)\delta_{|\alpha|m} h_n(s) \\
&\leq C(\chi, m, r, \mathcal{U}_1)\Big(\|u_n - u\|_{G_{m-1}(\Omega)} + \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=|\alpha|}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)} + \delta_{|\alpha|m} h_n(s)\Big)
\end{aligned}$$
$$(7.101)$$

for almost all $s \in J'$ and for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and $n \in \overline{\mathbb{N}}$, where we used Corollary 7.2 (i) in the penultimate estimate and $\delta_{|\alpha|m}$ denotes the Kronecker delta. Analogously, one obtains

$$\|\partial^\beta\sigma(u_n)(s)\partial^{\alpha-\beta} u_n(s)\|_{L^2(\mathbb{R}^3_+)} \leq C(\sigma, m, r, \mathcal{U}_1) \qquad (7.102)$$

and

$$\begin{aligned}
&\|\partial^\beta\sigma(u_n)(s)(\partial^{\alpha-\beta} u_n(s) - \partial^{\alpha-\beta} u(s))\|_{L^2(\mathbb{R}^3_+)} \\
&\qquad + \|(\partial^\beta\sigma(u_n)(s) - \partial^\beta\sigma(u)(s))\partial^{\alpha-\beta} u(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\sigma, m, r, \mathcal{U}_1)\Big(\|u_n - u\|_{G_{m-1}(\Omega)} + \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=|\alpha|}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)} + \delta_{|\alpha|m} h_n(s)\Big)
\end{aligned}$$
$$(7.103)$$

for almost all $s \in J'$ and for all $\beta \in \mathbb{N}_0^4$ with $0 < \beta \leq \alpha$ and $n \in \mathbb{N}$. In view of (7.94), (7.95), and (7.96), we conclude

$$\begin{aligned}
&\|f_{\alpha,n}\|_{L^2(\Omega)} \leq C(\chi, \sigma, m, r, T', \mathcal{U}_1), \\
&\|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{L^2(\Omega)} = \int_0^T \|f_{\alpha,n}(s) - f_{\alpha,\infty}(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \\
&\leq C(\chi, \sigma, m, r, T', \mathcal{U}_1)\Big(\|f_n - f\|^2_{H^m(\Omega)} + \|u_n - u\|^2_{G_{m-1}(\Omega)} + \delta_{|\alpha|m}\int_0^T h_n^2(s) ds \\
&\qquad + \int_0^T \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big) \qquad (7.104)
\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^4$ with $0 \leq |\alpha| \leq m$ and $n \in \overline{\mathbb{N}}$. If α is a multi-index with $|\alpha| \leq m-1$, we deduce from (7.95) to (7.103) and Corollary 7.2 (ii) that

$$\|f_{\alpha,n}\|_{G_0(\Omega)} \leq C(\chi, \sigma, m, r, \mathcal{U}_1),$$

$$\|f_{\alpha,n} - f_{\alpha,\infty}\|_{G_0(\Omega)} \leq \|f_n - f\|_{G_{m-1}(\Omega)} + C(\chi, \sigma, m, r, \mathcal{U}_1)\|u_n - u\|_{G_{m-1}(\Omega)} \quad (7.105)$$

for all $n \in \overline{\mathbb{N}}$.

Finally, let $k \in \{0, \dots, 3\}$ and $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m-1$. We recall from the proof of Lemma 3.4 that

$$\partial_k f_{\alpha,n} = f_{\alpha+e_k,n} + \partial_k \chi(u_n)\partial_t \partial^\alpha u_n + \sum_{j=1}^{3} \partial_k A_j \partial_j \partial^\alpha u_n + \partial_k \sigma(u_n)\partial^\alpha u_n$$

for all $n \in \overline{\mathbb{N}}$, see (3.4) and (3.6). From inequalities (7.96) and (7.100) to (7.103) for $\tilde{\alpha} = \alpha + e_k$ and $\beta = e_k$, and (7.104) we further deduce

$$\begin{aligned} &\|f_{\alpha,n}\|_{H^1(\Omega)} \leq C(\chi, \sigma, m, r, T', \mathcal{U}_1), \\ &\|f_{\alpha,n} - f_{\alpha,\infty}\|_{H^1(\Omega)}^2 \leq C(\chi, \sigma, m, r, T', \mathcal{U}_1)\Big(\|f_n - f\|_{H^m(\Omega)}^2 + \|u_n - u\|_{G_{m-1}(\Omega)}^2 \\ &\qquad + \delta_{|\alpha| m-1} \int_0^T h_n^2(s) ds + \int_0^T \sum_{\substack{\tilde{\alpha} \in \mathbb{N}_0^4 \\ |\tilde{\alpha}| = m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds\Big) \end{aligned} \quad (7.106)$$

for all $n \in \overline{\mathbb{N}}$. We finish this step by noting that also

$$\begin{aligned} \|g_{\alpha,n} - g_{\alpha,\infty}\|_{E_{0,\gamma}(J\times\partial\mathbb{R}^3_+)}^2 \leq C(m, r, T')\Big(&\|g_n - g\|_{E_m(J\times\partial\mathbb{R}^3_+)}^2 + \|u_n - u\|_{G_{m-1}(\Omega)}^2 \\ &+ \int_0^T \sum_{\substack{\tilde{\alpha} \in \mathbb{N}_0^4 \\ |\tilde{\alpha}| = m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds\Big) \end{aligned} \quad (7.107)$$

for all $n \in \overline{\mathbb{N}}$.

II) Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = 0$. We define the functions

$$w_{0,n} = \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n})$$

for all $n \in \overline{\mathbb{N}}$. Consider the linear initial boundary value problems

$$\begin{cases} L_n v = f_{\alpha,\infty}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bv = g_{\alpha,\infty}, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = w_{0,\infty}, & x \in \mathbb{R}^3_+; \end{cases} \quad (7.108)$$

and

$$\begin{cases} L_n v = f_{\alpha,n} - f_{\alpha,\infty}, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bv = g_{\alpha,n} - g_{\alpha,\infty}, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = w_{0,n} - w_{0,\infty}, & x \in \mathbb{R}^3_+; \end{cases} \quad (7.109)$$

for all $n \in \overline{\mathbb{N}}$. Lemma 7.7 shows that the initial value $w_{0,n}$ is an element of $L^2(\mathbb{R}^3_+)$ and Lemma 3.4 yields that $f_{\alpha,n}$ belongs to $L^2(\Omega)$ for all $n \in \overline{\mathbb{N}}$. Moreover, the coefficients $\chi(u_n)$ and $\sigma(u_n)$ are Lipschitz and $\chi(u_n)$ is symmetric and uniformly positive definite for all $n \in \overline{\mathbb{N}}$ by Lemma 7.1 and the assumptions. Theorem 4.13 thus implies that the problem (7.108) has a unique solution w_n in $G_0(\Omega)$ and the problem (7.109) has a unique solution z_n in $G_0(\Omega)$ for every $n \in \overline{\mathbb{N}}$.

We point out that in the case $n = \infty$ the initial boundary value problems (7.108) and (7.93) coincide. Since the latter is solved by $\partial^\alpha u_n$ and solutions of that problem are unique by Theorem 4.13, we conclude that

$$w_\infty = \partial^\alpha u_\infty = \partial^\alpha u. \quad (7.110)$$

Furthermore, the sum $w_n + z_n$ solves the initial boundary value problem (7.93) for all $n \in \mathbb{N}$. The uniqueness assertion of Theorem 4.13 therefore gives

$$w_n + z_n = \partial^\alpha u_n \tag{7.111}$$

for all $n \in \mathbb{N}$.

Let $\eta > 0$ such that $\chi \geq \eta$. Then $\chi(u_n)$ is an element of $F^{c}_{3,\eta}(\Omega)$ for all $n \in \mathbb{N}$. We next note that $(\chi(u_n))_n$ and $(\sigma(u_n))_n$ are bounded in $W^{1,\infty}(\Omega)$ since

$$\|\chi(u_n)\|_{W^{1,\infty}(\Omega)} + \|\sigma(u_n)\|_{W^{1,\infty}(\Omega)} \leq \|\chi(u_n)\|_{F_m(\Omega)} + \|\sigma(u_n)\|_{F_m(\Omega)} \leq R,$$

as noted in (7.91). Moreover, we obtain the limits

$$\|\chi(u_n) - \chi(u)\|_{L^\infty(\Omega)} \leq \max_{\xi \in \mathcal{U}_1} |\chi'(\xi)| \|u_n - u\|_{L^\infty(\Omega)} \leq C(\chi, \mathcal{U}_1) \|u_n - u\|_{G_{m-1}(\Omega)} \longrightarrow 0, \tag{7.112}$$

$$\|\sigma(u_n) - \sigma(u)\|_{L^\infty(\Omega)} \leq \max_{\xi \in \mathcal{U}_1} |\sigma'(\xi)| \|u_n - u\|_{L^\infty(\Omega)} \leq C(\sigma, \mathcal{U}_1) \|u_n - u\|_{G_{m-1}(\Omega)} \longrightarrow 0,$$

as $n \to \infty$. Lemma 7.21 therefore tells us that

$$\|w_n - \partial^\alpha u\|_{G_0(\Omega)} = \|w_n - w_\infty\|_{G_0(\Omega)} \longrightarrow 0 \tag{7.113}$$

as $n \to \infty$.

Define $\gamma = \gamma(\chi, \sigma, m, r, T', \mathcal{U}_1) \geq 1$ by

$$\gamma = \gamma_{4.13,0}(\eta(\chi), \tau, R(\chi, \sigma, m, r, \mathcal{U}_1), T'),$$

where $\gamma_{4.13,0}$ is the corresponding constant from Theorem 4.13. This lemma applied to (7.109) then yields

$$\begin{aligned} &\|z_n\|^2_{G_0(\Omega)} \leq e^{2\gamma T} \|z_n\|^2_{G_{0,\gamma}(\Omega)} \qquad (7.114)\\ &\leq C_0 e^{2\gamma T'} \Big(\|w_{0,n} - w_{0,\infty}\|^2_{L^2(\mathbb{R}^3_+)} + \|g_{\alpha,n} - g_{\alpha,\infty}\|^2_{E_{0,\gamma}(J \times \partial\mathbb{R}^3_+)} + \frac{1}{\gamma} \|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{L^2_\gamma(\Omega)} \Big) \\ &\leq C_{7.114} \Big(\|w_{0,n} - w_{0,\infty}\|^2_{L^2(\mathbb{R}^3_+)} + \|g_{\alpha,n} - g_{\alpha,\infty}\|^2_{E_0(J \times \partial\mathbb{R}^3_+)} + \|f_{\alpha,n} - f_{\alpha,\infty}\|^2_{L^2(\Omega)} \Big), \end{aligned}$$

where

$$\begin{aligned} C_0(\chi, \sigma, m, r, T', \mathcal{U}_1) = \max\{&C_{4.13,0,0}(\eta(\chi), \tau, R(\chi, \sigma, m, r, \mathcal{U}_1), T'), \\ &C_{4.13,0}(\eta(\chi), \tau, R(\chi, \sigma, m, r, \mathcal{U}_1))\}, \end{aligned}$$

and $C_{4.13,0,0}$ and $C_{4.13,0}$ are the corresponding constants from Theorem 4.13. Note that $C_{7.114} = C_{7.114}(\chi, \sigma, m, r, T', A_3)$. We recall from (7.89) that we have chosen the radius r in such a way that

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} + \|u_{0,n}\|_{H^m(\mathbb{R}^3_+)} \leq r$$

for all $n \in \overline{\mathbb{N}}$. Lemma 7.7 thus provides a constant $C_{7.7} = C_{7.7}(\chi, \sigma, m, r, \mathcal{U}_1)$ such that

$$\begin{aligned} &\|w_{0,n} - w_{0,\infty}\|_{L^2(\mathbb{R}^3_+)} \\ &= \|\partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}) - \partial^{(0,\alpha_1,\alpha_2,0)} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|_{L^2(\mathbb{R}^3_+)} \\ &\leq \|S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}) - S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|_{H^{m-\alpha_0}(\mathbb{R}^3_+)} \\ &\leq C_{7.7} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|_{H^m(\mathbb{R}^3_+)} \Big) \end{aligned}$$

for all $n \in \mathbb{N}$. Inserting this estimate together with (7.104) and (7.107) into (7.114), we derive

$$\|z_n\|_{G_0(\Omega)}^2 \leq C_{7.115}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-j-1}(\mathbb{R}^3_+)}^2 + \|u_{0,n} - u_0\|_{H^m(\mathbb{R}^3_+)}^2 + \|g_n - g\|_{E_m(J\times\partial\mathbb{R}^3_+)}^2 + \|f_n - f\|_{H^m(\Omega)}^2 + \|u_n - u\|_{G_{m-1}(\Omega)}^2\Big) + C_{7.115}\int_0^T h_n^2(s)ds + C_{7.115}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds,$$

for all $n \in \mathbb{N}$, where we introduce a constant $C_{7.115} = C_{7.115}(\chi, \sigma, m, r, T', \mathcal{U}_1)$. We write $a_n' = a_n'(\chi, \sigma, m, r, T', \mathcal{U}_1)$ for the first part of the above right-hand side. It follows

$$\|z_n\|_{G_0(\Omega)}^2 \leq a_n' + C_{7.115}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds, \tag{7.115}$$

for all $n \in \mathbb{N}$. Observe that a_n' converges to 0 as $n \to \infty$ by our assumptions and (7.98). Formula (7.111) and inequality (7.115) imply that

$$\begin{aligned}\|\partial^\alpha u_n - \partial^\alpha u\|_{G_0(\Omega)}^2 &= \|w_n + z_n - \partial^\alpha u\|_{G_0(\Omega)}^2 \leq 2\|w_n - \partial^\alpha u\|_{G_0(\Omega)}^2 + 2\|z_n\|_{G_0(\Omega)}^2 \\ &\leq 2\|w_n - \partial^\alpha u\|_{G_0(\Omega)}^2 + 2a_n' + 2C_{7.115}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds \\ &= a_{\alpha,n} + C_{7.116}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds, \end{aligned} \tag{7.116}$$

for all $n \in \mathbb{N}$. Here we set $C_{7.116} = C_{7.116}(\chi, \sigma, m, r, T', \mathcal{U}_1)$ and note that

$$a_{\alpha,n} := a_{\alpha,n}(\chi, \sigma, m, r, T', \mathcal{U}_1) := 2\|w_n - \partial^\alpha u\|_{G_0(\Omega)}^2 + 2a_n'(\chi, \sigma, m, r, T', \mathcal{U}_1) \longrightarrow 0$$

as $n \to \infty$ by (7.113).

III) We claim that for all multiindices $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ there is a sequence $(a_{\alpha,n})_n = (a_{\alpha,n}(\chi, \sigma, m, r, T', \mathcal{U}_1))_n$ and a constant $C_\alpha = C_\alpha(\chi, \sigma, m, r, T', \mathcal{U}_1)$ such that

$$\|\partial^\alpha u_n - \partial^\alpha u\|_{G_0(\Omega)}^2 \leq a_{\alpha,n} + C_\alpha\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds \tag{7.117}$$

for all $n \in \mathbb{N}$ and

$$a_{\alpha,n} \longrightarrow 0 \tag{7.118}$$

as $n \to \infty$.

We will show this assertion by induction with respect to α_3. Observe that step II) yields that (7.117) and (7.118) are true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = 0$. Next assume that there is an index $l \in \{1, \ldots, m\}$ such that the assertion is true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = l - 1$. Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = l$. We set $\alpha' = \alpha - e_3$.

At this point the key observation is that we cannot directly apply Lemma 3.11. The reason is that this lemma was derived for a fixed differential operator and when we apply only one such operator to a difference of solutions we experience the typical loss of derivatives. Therefore, we will repeat the key step of the proof of Lemma 3.11 and apply it to the difference $\partial^{\alpha'} u_n - \partial^{\alpha'} u$ this time. We start by recalling some notation.

By the definition of $F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\Omega)$ respectively $F^{\mathrm{cp}}_{m,\mathrm{coeff},\tau}(\Omega)$ and the assumptions there are time independent functions $\mu_{lj} \in F^{\mathrm{cp}}_{m,1}(\Omega)$ for $l, j \in \{1,2,3\}$ such that

$$A_j = \sum_{l=1}^{3} A_l^{\mathrm{co}} \mu_{lj}$$

for all $j \in \{1,2,3\}$ and an index $i \in \{1,2,3\}$ with $|\mu_{i3}| \geq \tau$ on $\mathbb{R}^3_+$. Without loss of generality we assume that $i = 3$. Note that there is a constant $C = C(r)$ such that

$$\|\mu_{lj}\|_{F_m(\Omega)} \leq C \tag{7.119}$$

for all $l, j \in \{1,2,3\}$ since (7.90) is valid. We will use this estimate in the following without further reference. We set

$$M = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \quad \text{and} \quad \hat{M}_n = \begin{pmatrix} A_3 \\ (M^T \chi(u_n))_{3\cdot} \\ (M^T \chi(u_n))_{6\cdot} \end{pmatrix}$$

for all $n \in \overline{\mathbb{N}}$ as well as

$$\begin{aligned}
(F_{n;1}, \ldots, F_{n;6})^T &= f_{\alpha',n} - \chi(u_n)\partial_t \partial^{\alpha'} u_n - \sum_{j=1}^{2} A_j \partial_j \partial^{\alpha'} u_n - \sigma(u_n)\partial^{\alpha'} u_n, \\
F_{n;7}(t) &= \sum_{k=1}^{3} (M^T \chi(u_n) \nabla \partial^{\alpha'} u_n)_{kk}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{n;kk}(s) ds \\
&\quad - \sum_{k=1}^{2} (M^T \chi(u_n))_{k\cdot} \partial_k \partial^{\alpha'} u_n(t), \\
F_{n;8}(t) &= \sum_{k=1}^{3} (M^T \chi(u_n) \nabla \partial^{\alpha'} u_n)_{(k+3)k}(0) + \sum_{k=1}^{3} \int_0^t \Lambda_{n;(k+3)k}(s) ds \\
&\quad - \sum_{k=1}^{2} (M^T \chi(u_n))_{(k+3)\cdot} \partial_k \partial^{\alpha'} u_n(t),
\end{aligned} \tag{7.120}$$

where

$$\begin{aligned}
\Lambda_n &= M^T \partial_t \chi(u_n) \nabla \partial^{\alpha'} u_n + M^T \chi(u_n) \nabla \chi(u_n)^{-1} \Big(f_{\alpha',n} - \sum_{j=1}^{2} A_j \partial_j \partial^{\alpha'} u_n - \sigma(u_n)\partial^{\alpha'} u_n \Big) \\
&\quad + M^T \nabla f_{\alpha',n} - M^T \sum_{j=1}^{3} \nabla A_j \partial_j \partial^{\alpha'} u_n - M^T \nabla \sigma(u_n) \partial^{\alpha'} u_n - M^T \sigma(u_n) \nabla \partial^{\alpha'} u_n
\end{aligned}$$

for all $n \in \overline{\mathbb{N}}$, cf. (3.23), (3.28), and (3.29). Recall that

$$(\nabla A h)_{jk} = \sum_{l=1}^{6} \partial_k A_{jl} h_l$$

for any $\mathbb{R}^{6\times 6}$-valued function A and $\mathbb{R}^6$-valued function h. We then know from (3.30) that

$$\hat{M}_n \partial_3 \partial^{\alpha'} u_n = F_n \tag{7.121}$$

for all $n \in \overline{\mathbb{N}}$. We next want to estimate the difference of $F_n - F$ in $G_0(\Omega)$ for all $n \in \mathbb{N}$.

To that purpose, we first note that

$$\begin{aligned}
&\|\chi(u_n)\partial_t\partial^{\alpha'}u_n - \chi(u)\partial_t\partial^{\alpha'}u\|_{G_0(\Omega)} \\
&\leq \|(\chi(u_n)-\chi(u))\partial_t\partial^{\alpha'}u_n\|_{G_0(\Omega)} + \|\chi(u)(\partial_t\partial^{\alpha'}u_n - \partial_t\partial^{\alpha'}u)\|_{G_0(\Omega)} \\
&\leq \|\chi(u_n)-\chi(u)\|_{L^\infty(\Omega)}\|u_n\|_{G_m(\Omega)} + \|\chi(u)\|_{L^\infty(\Omega)}\|\partial^{\alpha'+e_0}u_n - \partial^{\alpha'+e_0}u\|_{G_0(\Omega)} \\
&\leq C(\chi,\sigma,m,r,\mathcal{U}_1)\|u_n-u\|_{G_{m-1}(\Omega)} + C(\chi,\mathcal{U}_1)\|\partial^{\alpha'+e_0}u_n - \partial^{\alpha'+e_0}u\|_{G_0(\Omega)} \qquad (7.122)
\end{aligned}$$

for all $n \in \mathbb{N}$, where we employed the Sobolev embedding theorem and Corollary 7.2 (ii) for the first summand and (7.91) for the second summand in the last estimate. The combination of (7.91) and the boundedness of $(u_n)_n$ in $G_m(\Omega)$ further yields a constant $C_F = C_F(R,r) = C_F(\chi,\sigma,m,r,\mathcal{U}_1)$ such that

$$\|\chi(u_n)\partial_t\partial^{\alpha'}u_n\|_{G_0(\Omega)} \leq C_F \qquad (7.123)$$

for all $n \in \overline{\mathbb{N}}$. Analogously, we obtain

$$\begin{aligned}
&\|\sigma(u_n)\partial^{\alpha'}u_n - \sigma(u)\partial^{\alpha'}u\|_{G_0(\Omega)} \\
&\leq C(\chi,\sigma,m,r,\mathcal{U}_1)\|u_n-u\|_{G_{m-1}(\Omega)} + C(\sigma,\mathcal{U}_1)\|\partial^{\alpha'}u_n - \partial^{\alpha'}u\|_{G_0(\Omega)} \\
&\leq C(\chi,\sigma,m,r,\mathcal{U}_1)\|u_n-u\|_{G_{m-1}(\Omega)} \qquad (7.124)
\end{aligned}$$

for all $n \in \mathbb{N}$ and

$$\|\sigma(u_n)\partial^{\alpha'}u_n\|_{G_0(\Omega)} \leq C_F \qquad (7.125)$$

for all $n \in \overline{\mathbb{N}}$, where we increase C_F if necessary. Exploiting the estimates (7.122) and (7.124), we infer from (7.120)

$$\begin{aligned}
\|(F_n - F)_{(1,\dots,6)}\|_{G_0(\Omega)} &\leq \|f_{\alpha',n} - f_{\alpha',\infty}\|_{G_0(\Omega)} + C(\chi,\sigma,m,r,\mathcal{U}_1)\|u_n-u\|_{G_{m-1}(\Omega)} \\
&\quad + C(\chi,\sigma,m,r,\mathcal{U}_1)\sum_{j=0}^{2}\|\partial^{\alpha'+e_j}u_n - \partial^{\alpha'+e_j}u\|_{G_0(\Omega)} \qquad (7.126)
\end{aligned}$$

for all $n \in \mathbb{N}$, where we also used that the coefficients A_j are bounded, see (7.90). Note that $|\alpha' + e_j| = m$ and $(\alpha' + e_j)_3 = l - 1$ for all $j \in \{0,1,2\}$. Applying the induction hypothesis (7.117) and also estimate (7.105), we then obtain

$$\begin{aligned}
&\|(F_n - F)_{(1,\dots,6)}\|_{G_0(\Omega)}^2 \leq C\|f_n - f\|_{G_{m-1}(\Omega)}^2 + C\|u_n - u\|_{G_{m-1}(\Omega)}^2 \\
&\quad + C\sum_{j=0}^{2}\Big(a_{\alpha'+e_j,n} + C_{\alpha'+e_j}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}}\|\partial^{\tilde\alpha}u_n(s) - \partial^{\tilde\alpha}u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds\Big) \\
&\leq C_{7.127}\|f_n - f\|_{G_{m-1}(\Omega)}^2 + C_{7.127}\|u_n - u\|_{G_{m-1}(\Omega)}^2 + C_{7.127}\sum_{j=0}^{2}a_{\alpha'+e_j,n} \\
&\quad + C_{7.127}\int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4\\|\tilde\alpha|=m}}\|\partial^{\tilde\alpha}u_n(s) - \partial^{\tilde\alpha}u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds \qquad (7.127)
\end{aligned}$$

for all $n \in \mathbb{N}$, where $C_{7.127} = C_{7.127}(\chi,\sigma,m,r,T',\mathcal{U}_1,\alpha)$. Employing (7.123) and (7.125) as well as (7.90), (7.88), and (7.91), we also deduce

$$\|(F_n)_{(1,\dots,6)}\|_{G_0(\Omega)} \leq C \qquad (7.128)$$

for all $n \in \overline{\mathbb{N}}$ and a constant $C = C(\chi,\sigma,m,r,\mathcal{U}_1)$.

It remains to treat the seventh and eigth component of $F_n - F$. In order to estimate all the appearing terms efficiently, we first prove the following auxiliary result.

IV) Let $k \in \mathbb{N}_0$, $\theta_i \in \{\chi_{jl}, \sigma_{jl}, (\chi^{-1})_{jl} \colon j, l \in \{1, \ldots, 6\}\}$ for $i \in \{1, \ldots, k\}$, and D_i be a linear differential operator of order less or equal than 1 with bounded coefficients on Ω for $i \in \{1, \ldots, k+1\}$. Let $v_n, v \in H^1(\Omega)$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\Big\| \prod_{i=1}^k D_i \theta_i(u_n)(s) D_{k+1} v_n(s) \Big\|_{L^2(\mathbb{R}^3_+)} \leq \prod_{i=1}^k \|D_i \theta_i(u_n)(s)\|_{L^\infty(\mathbb{R}^3_+)} \|D_{k+1} v_n(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C \prod_{i=1}^k \|\theta_i(u_n)\|_{W^{1,\infty}(\Omega)} \|D_{k+1} v_n(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\chi, \sigma, k, r, \mathcal{U}_1) \|D_{k+1} v_n(s)\|_{L^2(\mathbb{R}^3_+)} \qquad (7.129)
\end{aligned}$$

for almost all $s \in J$ and all $n \in \mathbb{N}$, where we employed Lemma 7.1 (i) in the last estimate. Analogously, we derive

$$\begin{aligned}
&\Big\| \prod_{i=1}^k D_i \theta_i(u_n)(s) D_{k+1} v_n(s) - \prod_{i=1}^k D_i \theta_i(u)(s) D_{k+1} v(s) \Big\|_{L^2(\mathbb{R}^3_+)} \\
&\leq \sum_{l=1}^k \Big\| \prod_{i=1}^{l-1} D_i \theta_i(u)(s) (D_l \theta_l(u_n)(s) - D_l \theta_l(u)(s)) \prod_{i=l+1}^k D_i \theta_i(u_n)(s) D_{k+1} v_n(s) \Big\|_{L^2(\mathbb{R}^3_+)} \\
&\qquad + \Big\| \prod_{i=1}^k D_i \theta_i(u)(s) (D_{k+1} v_n(s) - D_{k+1} v(s)) \Big\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\chi, \sigma, k, r, \mathcal{U}_1) \sum_{l=1}^k \|D_l \theta_l(u_n)(s) - D_l \theta_l(u)(s)\|_{L^\infty(\mathbb{R}^3_+)} \|D_{k+1} v_n(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\qquad + C(\chi, \sigma, k, r, \mathcal{U}_1) \|D_{k+1} v_n(s) - D_{k+1} v(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\chi, \sigma, k, r, \mathcal{U}_1) (\|u_n - u\|_{G_{m-1}(\Omega)} + \delta_{3m} h_n(s)) \|D_{k+1} v_n(s)\|_{L^2(\mathbb{R}^3_+)} \\
&\qquad + C(\chi, \sigma, k, r, \mathcal{U}_1) \|D_{k+1} v_n(s) - D_{k+1} v(s)\|_{L^2(\mathbb{R}^3_+)} \qquad (7.130)
\end{aligned}$$

for almost all $s \in J$ and all $n \in \mathbb{N}$, where we used Sobolev's embedding and Corollary 7.2 (i) as well as Lemma 7.1 (i) again. We further note that the estimates (7.129) and (7.130) are true for all $s \in J$ if v_n and v additionally belong to $G_1(\Omega)$.

V) We return to the task of estimating $(F_n - F)_{(7,8)}$. We start with the summand involving Λ_n. Observe that each component of Λ_n is the sum of terms whose components fit into the framework of step IV) with $k \in \{0, 1, 2\}$ and $v_n \in \{\partial^{\alpha'} u_n, f_{\alpha', n}\}$ for all $n \in \overline{\mathbb{N}}$. By means of (7.129) and (7.130), Minkowski's inequality, and (7.106), we thus deduce

$$\begin{aligned}
&\Big\| \sum_{j=1}^3 \int_0^t \begin{pmatrix} \Lambda_{jj;n}(s) \\ \Lambda_{(j+3)j;n}(s) \end{pmatrix} ds \Big\|_{G_0(\Omega)} \leq C \int_0^T \|\Lambda_n(s)\|_{L^2(\mathbb{R}^3_+)} ds \\
&\leq C(\chi, \sigma, r, \mathcal{U}_1) (\|\partial^{\alpha'} u_n\|_{H^1(\Omega)} + \|f_{\alpha', n}\|_{H^1(\Omega)}) \leq C(\chi, \sigma, m, r, T', \mathcal{U}_1), \qquad (7.131) \\
&\Big\| \sum_{j=1}^3 \int_0^t \begin{pmatrix} \Lambda_{jj;n}(s) - \Lambda_{jj;\infty}(s) \\ \Lambda_{(j+3)j;n}(s) - \Lambda_{(j+3)j;\infty}(s) \end{pmatrix} ds \Big\|_{G_0(\Omega)}^2 \\
&\leq C \Big(\int_0^T \|\Lambda_n(s) - \Lambda_\infty(s)\|_{L^2(\mathbb{R}^3_+)} ds \Big)^2 \\
&\leq C(\chi, \sigma, m, r, T', \mathcal{U}_1) \Big(\|u_n - u\|_{G_{m-1}(\Omega)}^2 + \int_0^T h_n^2(s) ds + \|f_{\alpha', n} - f_{\alpha', \infty}\|_{H^1(\Omega)}^2 \\
&\qquad\qquad + \int_0^T \sum_{\substack{\tilde{\alpha} \in \mathbb{N}_0^4 \\ |\tilde{\alpha}| = m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds \Big) \\
&\leq C(\chi, \sigma, m, r, T', \mathcal{U}_1) \Big(\|u_n - u\|_{G_{m-1}(\Omega)}^2 + \|f_n - f\|_{H^m(\Omega)}^2 + \int_0^T h_n^2(s) ds
\end{aligned}$$

$$+\int_0^T \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds\Big) \tag{7.132}$$

for all $n \in \overline{\mathbb{N}}$. Since u_n solves (1.6), Lemma 7.5 implies that

$$\partial^{\alpha'} u_n(0) = \partial^{(0,\alpha_1',\alpha_2',\alpha_3')} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0'}(t_0, f_n, u_{0,n}) =: w_{0,n}'$$

for all $n \in \overline{\mathbb{N}}$. We can apply Lemma 7.1 (i), Corollary 7.2 (ii), and Lemma 7.7 to deduce

$$\begin{aligned}
&\|(M^T\chi(u_n)\nabla\partial^{\alpha'} u_n)(0)\|_{G_0(\Omega)} = \|M^T\chi(u_{0,n})\nabla w_{0,n}'\|_{L^2(\mathbb{R}^3_+)} \\
&\leq C(\chi, r, \mathcal{U}_1)\|\partial^{(0,\alpha_1',\alpha_2',\alpha_3')} S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0'}\|_{H^1(\mathbb{R}^3_+)} \leq C(\chi, r, \mathcal{U}_1)\|S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}\|_{H^{m-\alpha_0}(\mathbb{R}^3_+)} \\
&\leq C(\chi, \sigma, m, r, \mathcal{U}_1), && (7.133) \\
&\|(M^T\chi(u_n)\nabla\partial^{\alpha'} u_n)(0) - (M^T\chi(u)\nabla\partial^{\alpha'} u)(0)\|_{G_0(\Omega)} \\
&\leq C(r)\|\chi(u_{0,n})w_{0,n}' - \chi(u_0)w_{0,\infty}'\|_{L^2(\mathbb{R}^3_+)} \leq C(\chi,\sigma,m,r,\mathcal{U}_1)(\|u_n - u\|_{G_{m-1}(\Omega)} \\
&\qquad + \|S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f_n, u_{0,n}) - S_{\chi,\sigma,\mathbb{R}^3_+,m,\alpha_0}(0, f, u_0)\|_{H^{m-\alpha_0}(\mathbb{R}^3_+)}) \\
&\leq C(\chi,\sigma,m,r,\mathcal{U}_1)\Big(\|u_n - u\|_{G_{m-1}(\Omega)} && (7.134) \\
&\qquad + \sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|_{H^m(\mathbb{R}^3_+)}\Big)
\end{aligned}$$

for all $n \in \overline{\mathbb{N}}$. Employing the same arguments as in step III) once again, see (7.122) to (7.127), we further deduce

$$\begin{aligned}
&\|M^T\chi(u_n)\partial_k\partial^{\alpha'} u_n\|_{G_0(\Omega)} \leq C(\chi,m,r,\mathcal{U}_1)\|\partial_k\partial^{\alpha'} u_n\|_{G_0(\Omega)} \leq C(\chi,m,r,\mathcal{U}_1), && (7.135) \\
&\|M^T\chi(u_n)\partial_k\partial^{\alpha'} u_n - M^T\chi(u)\partial_k\partial^{\alpha'} u\|_{G_0(\Omega)} && (7.136) \\
&\leq C\|u_n - u\|_{G_{m-1}(\Omega)}^2 + C\sum_{j=1}^2 a_{\alpha'+e_j,n} + C\int_0^T \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds
\end{aligned}$$

for all $n \in \overline{\mathbb{N}}$, $k \in \{1, 2\}$, and a constant $C = C(\chi, \sigma, m, r, T', \mathcal{U}_1, \alpha)$.

Combining now (7.131), (7.133), and (7.135) respectively (7.132), (7.134), and (7.136), we deduce

$$\begin{aligned}
&\|(F_{n;7}, F_{n;8})\|_{G_0(\Omega)} \leq C, && (7.137) \\
&\|(F_{n;7} - F_7, F_{n;8} - F_8)\|_{G_0(\Omega)} \leq C\Big(\|u_n - u\|_{G_{m-1}(\Omega)}^2 + \|f_n - f\|_{H^m(\Omega)}^2 \\
&\qquad + \sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|_{H^m(\mathbb{R}^3_+)} + \int_0^T h_n^2(s) ds \\
&\qquad + \sum_{j=1}^2 a_{\alpha'+e_j,n} + \int_0^T \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|_{L^2(\mathbb{R}^3_+)}^2 ds\Big) && (7.138)
\end{aligned}$$

for all $n \in \overline{\mathbb{N}}$ and a constant $C = C(\chi, \sigma, m, r, T', \mathcal{U}_1, \alpha)$. In view of (7.128) and (7.127) we thus arrive at

$$\begin{aligned}
&\|F_n\|_{G_0(\Omega)} \leq C, && (7.139) \\
&\|F_n - F\|_{G_0(\Omega)} \leq C\Big(\|u_n - u\|_{G_{m-1}(\Omega)}^2 + \|f_n - f\|_{H^m(\Omega)}^2
\end{aligned}$$

$$+ \sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|_{H^m(\mathbb{R}^3_+)} + \int_0^T h_n^2(s) ds$$

$$+ \sum_{j=0}^{2} a_{\alpha'+e_j,n} + \int_0^T \sum_{\substack{\tilde{\alpha}\in\mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n(s) - \partial^{\tilde{\alpha}} u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big) \tag{7.140}$$

for all $n \in \overline{\mathbb{N}}$, where $C = C(\chi, \sigma, m, r, T', \mathcal{U}_1, \alpha)$ and where we applied Sobolev's embedding theorem again.

To get from (7.121) to $\partial_3 \partial^{\alpha'} u_n$, we proceed as in the proof of Lemma 3.11. We use the matrices G_1 from (3.32) and G_4 from (3.37). Observe that

$$\|G_1\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|G_4\|_{L^\infty(\Omega)} \leq C \tag{7.141}$$

for a constant $C = C(\tau, r)$. We further introduce the matrices $G_{2,n}$ by replacing the matrix A_0 in the definition of G_2 in (3.33) by $\chi(u_n)$ for all $n \in \overline{\mathbb{N}}$. We then have

$$\|G_{2,n}\|_{L^\infty(\Omega)} \leq C\|\chi(u_n)\|_{L^\infty(\Omega)} \leq C, \tag{7.142}$$

$$\|G_{2,n} - G_{2,\infty}\|_{L^\infty(\Omega)} \leq C\|\chi(u_n) - \chi(u)\|_{L^\infty(\Omega)} \leq C\|u_n - u\|_{G_{m-1}(\Omega)} \tag{7.143}$$

for all $n \in \overline{\mathbb{N}}$ and a constant $C = C(\chi, r, \mathcal{U}_1)$, where we used the estimate from (7.112). Next define

$$\alpha_{n;jk} = \mu_{33}^{-1} M_{j\cdot}^T \chi(u_n) M_{\cdot k}$$

for all $j, k \in \{3, 6\}$ and $n \in \overline{\mathbb{N}}$. Since $\chi \geq \eta$, we obtain from (3.34) that the matrix

$$\alpha_n = \begin{pmatrix} \alpha_{n;33} & \alpha_{n;36} \\ \alpha_{n;63} & \alpha_{n;66} \end{pmatrix}$$

is either positive or negative definite with

$$\alpha_n \geq \eta\tau \quad \text{or} \quad \alpha_n \leq -\eta\tau \tag{7.144}$$

for all $n \in \overline{\mathbb{N}}$. The inverses β_n of α_n are therefore uniformly bounded and we obtain

$$\|\beta_n\|_{L^\infty(\Omega)} \leq C, \tag{7.145}$$

$$\begin{aligned} \|\beta_n - \beta_\infty\|_{L^\infty(\Omega)} &\leq \|\beta_n\|_{L^\infty(\Omega)} \|\alpha - \alpha_n\|_{L^\infty(\Omega)} \|\beta_\infty\|_{L^\infty(\Omega)} \\ &\leq C\|\chi(u_n) - \chi(u)\|_{L^\infty(\Omega)} \leq C\|u_n - u\|_{G_{m-1}(\Omega)} \end{aligned} \tag{7.146}$$

for all $n \in \overline{\mathbb{N}}$ and a constant $C = C(\eta, \tau, \chi, r, \mathcal{U}_1)$. In analogy to (3.36) we now set

$$G_{3,n} = \begin{pmatrix} I_{6\times 6} & 0 \\ 0 & \beta_n \end{pmatrix}$$

for all $n \in \overline{\mathbb{N}}$. Estimates (7.145) and (7.146) yield

$$\|G_{3,n}\|_{L^\infty(\Omega)} \leq C, \tag{7.147}$$

$$\|G_{3,n} - G_{3,\infty}\|_{L^\infty(\Omega)} \leq C\|u_n - u\|_{G_{m-1}(\Omega)} \tag{7.148}$$

for all $n \in \overline{\mathbb{N}}$ and a constant $C = C(\eta, \tau, \chi, r, \mathcal{U}_1)$. Hence, the identity

$$\tilde{M} \partial_3 \partial^{\alpha'} u_n = G_4 G_{3,n} G_{2,n} G_1 F_n \tag{7.149}$$

is valid for all $n \in \overline{\mathbb{N}}$ by (3.38) and (7.121). We thus obtain

$$\begin{aligned} &\|\partial^\alpha u_n - \partial^\alpha u\|_{G_0(\Omega)} = \|\tilde{M}\partial_3\partial^{\alpha'} u_n - \tilde{M}\partial_3\partial^{\alpha'} u\|_{G_0(\Omega)} \\ &= \|G_4 G_{3,n} G_{2,n} G_1 F_n - G_4 G_{3,\infty} G_{2,\infty} G_1 F\|_{G_0(\Omega)} \end{aligned}$$

$$\leq \|G_4\|_{L^\infty(\Omega)}\|G_1\|_{L^\infty(\Omega)}\Big(\|G_{3,n} - G_{3,\infty}\|_{L^\infty(\Omega)}\|G_{2,n}\|_{L^\infty(\Omega)}\|F_n\|_{G_0(\Omega)} + \|G_{3,\infty}\|_{L^\infty(\Omega)}\|G_{2,n} - G_{2,\infty}\|_{L^\infty(\Omega)}\|F_n\|_{G_0(\Omega)} + \|G_{3,\infty}\|_{L^\infty(\Omega)}\|G_{2,\infty}\|_{L^\infty(\Omega)}\|F_n - F_\infty\|_{G_0(\Omega)}\Big)$$

for all $n \in \mathbb{N}$. Inserting (7.139) to (7.143) as well as (7.147) and (7.148) into this estimate, we arrive at

$$\begin{aligned} &\|\partial^\alpha u_n - \partial^\alpha u\|^2_{G_0(\Omega)} \\ &\leq C(\chi, \sigma, m, r, T', \mathcal{U}_1, \alpha)\Big(\sum_{j=0}^{2} a_{\alpha'+e_j,n} + \|u_n - u\|^2_{G_{m-1}(\Omega)} + \|f_n - f\|^2_{H^m(\Omega)} \\ &\quad + \sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|^2_{H^m(\mathbb{R}^3_+)} + \int_0^{T'} h_n^2(s)ds \\ &\quad + \int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds\Big) \end{aligned} \tag{7.150}$$

for all $n \in \mathbb{N}$. By $C_\alpha = C_\alpha(\chi, \sigma, m, r, T', \mathcal{U}_1)$ we denote the constant on the right-hand side of (7.150) and we set

$$\begin{aligned} a_{\alpha,n} = C_\alpha\Big(&\sum_{j=0}^{2} a_{\alpha'+e_j,n} + \|u_n - u\|^2_{G_{m-1}(J'\times\mathbb{R}^3)} + \|f_n - f\|^2_{H^m(J'\times\mathbb{R}^3_+)} \\ &+ \sum_{j=0}^{m-1} \|\partial_t^j f_n(0) - \partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|u_{0,n} - u_0\|^2_{H^m(\mathbb{R}^3_+)} + \int_0^{T'} h_n^2(s)ds\Big) \end{aligned}$$

for all $n \in \mathbb{N}$. The assumptions, (7.87), (7.98), and the induction hypothesis (7.118) then imply that $a_{\alpha,n} = a_{\alpha,n}(\chi, \sigma, m, r, T', \mathcal{U}_1)$ converges to zero as $n \to \infty$. Due to (7.150) we conclude that (7.117) and (7.118) are true for the multi-index α.

Since the multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = l$ was arbitrary, the claims (7.117) and (7.118) hold for all such α. By induction, we thus obtain that (7.117) and (7.118) are true for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$.

We define $a_{m,n} = a_{m,n}(\chi, \sigma, m, r, T', \mathcal{U}_1)$ and $C_m = C_m(\chi, \sigma, m, r, T', \mathcal{U}_1)$ by

$$a_m = \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} a_{\tilde\alpha,n}, \qquad C_m = \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} C_{\tilde\alpha},$$

for all $n \in \mathbb{N}$. Summing (7.117) over all multiindices $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, we then get

$$\begin{aligned} \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(T) - \partial^{\tilde\alpha} u(T)\|^2_{L^2(\mathbb{R}^3_+)} &\leq \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n - \partial^{\tilde\alpha} u\|^2_{G_0(\Omega)} \\ &\leq a_{m,n} + C_m \int_0^T \sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(s) - \partial^{\tilde\alpha} u(s)\|^2_{L^2(\mathbb{R}^3_+)} ds \end{aligned}$$

for all $n \in \mathbb{N}$. Since $T \in (0, T']$ was arbitrary, Gronwall's lemma shows that

$$\sum_{\substack{\tilde\alpha\in\mathbb{N}_0^4 \\ |\tilde\alpha|=m}} \|\partial^{\tilde\alpha} u_n(T) - \partial^{\tilde\alpha} u(T)\|^2_{L^2(\mathbb{R}^3_+)} \leq a_{m,n} e^{C_m T}$$

for all $T \in [0, T']$ and $n \in \mathbb{N}$. As $(a_{m,n})_n$ converges to 0 due to (7.118), we finally arrive at

$$\sum_{\substack{\tilde{\alpha} \in \mathbb{N}_0^4 \\ |\tilde{\alpha}|=m}} \|\partial^{\tilde{\alpha}} u_n - \partial^{\tilde{\alpha}} u\|^2_{G_0(J' \times \mathbb{R}^3_+)} \leq a_{m,n} e^{C_m T'} \longrightarrow 0$$

as $n \to \infty$. Since $\|u_n - u\|_{G_{m-1}(J' \times \mathbb{R}^3_+)}$ tends to zero as $n \to \infty$, we conclude that $(u_n)_n$ converges to u in $G_m(J' \times \mathbb{R}^3_+)$. □

We now establish our main local wellposedness theorem. The first part, existence and uniqueness of a solution of (1.6), is already known from Proposition 7.16. As announced, the refined blow-up criterion follows easily from Proposition 7.20 and Lemma 7.17. As a byproduct of the characterization of finite time blowup in terms of the spatial Lipschitz norm - which is independent of m - we obtain that the maximal time of existence is independent of m, i.e., the maximal H^m existence time equals the maximal H^3 existence time. We point out that this result is crucial if one wants to approximate a solution of (1.6) by smoother ones. Of course, in that context the continuous dependance of solutions on the data is also indispensable. We give the precise statement below.

In the following we will write $B_M(x, r)$ for the ball of radius r around a point x from a metric space M.

Theorem 7.23. *Let $m \in \mathbb{N}$ with $m \geq 3$ and $t_0 \in \mathbb{R}$. Take functions $\chi \in \mathcal{ML}^m_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma \in \mathcal{ML}^m(G,\mathcal{U})$. Set*

$$B(x) = \begin{pmatrix} 0 & \nu_3(x) & -\nu_2(x) & 0 & 0 & 0 \\ -\nu_3(x) & 0 & \nu_1(x) & 0 & 0 & 0 \\ \nu_2(x) & -\nu_1(x) & 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν denotes the unit outer normal vector of ∂G. Choose data $u_0 \in H^m(G)$, $g \in E_m((-T,T) \times \partial G)$, and $f \in H^m((-T,T) \times G)$ for all $T > 0$ such that $\overline{\operatorname{im} u_0} \subseteq \mathcal{U}$ and the tuple $(\chi, \sigma, t_0, B, f, g, u_0)$ fulfills the compatibility conditions (7.16) of order m. For the maximal existence times from Definition 7.15 we then have

$$T_+ = T_+(m, t_0, f, g, u_0) = T_+(k, t_0, f, g, u_0),$$
$$T_- = T_-(m, t_0, f, g, u_0) = T_-(k, t_0, f, g, u_0)$$

for all $k \in \{3, \ldots, m\}$. The following assertions are true.

(i) There exists a unique maximal solution u of (1.6) which belongs to the function space $\bigcap_{j=0}^m C^j((T_-, T_+), H^{m-j}(G))$.

(ii) If $T_+ < \infty$, then one of the alternatives

a) the solution u leaves every compact subset of $\mathcal{U}$,

b) $\limsup_{t \nearrow T_+} \|\nabla u(t)\|_{L^\infty(G)} = \infty$,

is valid. The analogous result holds for T_-.

(iii) Let $T' \in (t_0, T_+)$ and assume that G has a tame uniform C^{m+2}-boundary with finitely many charts. Then there is a number $\delta > 0$ such that for all data $\tilde{f} \in H^m((t_0, T_+) \times G)$, $\tilde{g} \in E_m((t_0, T_+) \times \partial G)$, and $\tilde{u}_0 \in H^m(G)$ with

$$\|\tilde{f} - f\|_{H^m((t_0,T_+)\times G)} < \delta, \quad \|\tilde{g} - g\|_{E_m((t_0,T_+)\times \partial G)} < \delta, \quad \|\tilde{u}_0 - u_0\|_{H^m(\mathbb{R}^3_+)} < \delta$$

and which fulfill the compatibility conditions (7.16) of order m, we have for the maximal existence time $T_+(m, t_0, \tilde{f}, \tilde{g}, \tilde{u}_0) > T'$. We write $(M_{\chi,\sigma,m}(t_0, T_+), d)$ for the metric space

$$M_{\chi,\sigma,m}(t_0, T_+) = \{(\tilde{f}, \tilde{g}, \tilde{u}_0) \in H^m((t_0, T_+) \times G) \times E_m((t_0, T_+) \times G) \times H^m(G):$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad (\chi, \sigma, t_0, \tilde f, \tilde g, \tilde u_0) \textit{ is compatible of order } m\},\\
&d((\tilde f_1, \tilde g_1, \tilde u_{0,1}), (\tilde f_2, \tilde g_2, \tilde u_{0,2})) = \max\{\|\tilde f_1 - \tilde f_2\|_{H^m((t_0,T_+)\times G)},\\
&\qquad\qquad \|\tilde g_1 - \tilde g_2\|_{E_m((t_0,T_+)\times \partial G)}, \|\tilde u_{0,1} - \tilde u_{0,2}\|_{H^m(G)}\}.
\end{aligned}
$$

The flow map

$$
\begin{aligned}
\Psi \colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta) &\to G_m((t_0,T')\times G),\\
(\tilde f, \tilde g, \tilde u_0) &\mapsto u(\cdot; \tilde f, \tilde g, \tilde u_0),
\end{aligned}
$$

is continuous, where $u(\cdot; \tilde f, \tilde g, \tilde u_0)$ *denotes the maximal* $H^m(G)$*-solution of* (1.6) *with inhomogeneity* $\tilde f$*, boundary value* $\tilde g$*, and initial value* $\tilde u_0$*. Moreover, there is a constant* $C = C(\chi, \sigma, m, r, T_+ - t_0)$ *such that*

$$
\begin{aligned}
&\|\Psi(\tilde f_1, \tilde g_1, \tilde u_{0,1}) - \Psi(\tilde f_2, \tilde g_2, \tilde u_{0,2})\|_{G_{m-1}((t_0,T')\times G)} \le C\|\tilde f_1 - \tilde f_2\|_{H^{m-1}((t_0,T')\times G)}\\
&\quad + C\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde f_1(t_0) - \partial_t^j \tilde f_2(t_0)\|_{H^{m-j-1}(G)} + \|\tilde u_{0,1} - \tilde u_{0,2}\|_{H^m(G)}\Big) \qquad (7.151)
\end{aligned}
$$

for all $(\tilde f_1, \tilde g_1, \tilde u_{0,1}), (\tilde f_2, \tilde g_2, \tilde u_{0,2}) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta)$*. The analogous result is true for* T_-*.*

Proof. Let $k \in \{3, \dots, m-1\}$. We have $T_+ = T_+(m, t_0, f, g, u_0) \le T_+(k, t_0, f, g, u_0)$ by definition. Assume now that $T_+ < T_+(k, t_0, f, g, u_0)$. Then $T_+ < \infty$ and the maximal $H^m(G)$-solution u of (1.6), which exists on (t_0, T_+), can be extended to a $H^k(G)$-solution on $(t_0, T_+(k, t_0, f, g, u_0))$ by the definition of the maximal existence times and Lemma 7.3. In particular, the function u belongs to $G_k((t_0,T_+)\times G)$ so that

$$\sup_{t\in(t_0,T_+)} \|u(t)\|_{H^k(G)} < \infty$$

and

$$\liminf_{t\nearrow T_+} \operatorname{dist}(\overline{\{u(t,x) \colon x \in G\}}, \partial\mathcal{U}) > 0. \qquad (7.152)$$

Due to Sobolev's embedding we thus obtain that also

$$\omega_0 := \sup_{t\in(t_0,T_+)} \|u(t)\|_{W^{1,\infty}(G)} < \infty.$$

We next set $T^* = T_+(k, t_0, f, g, u_0)$ if $T_+(k, t_0, f, g, u_0) < \infty$ and we take $T^* > T_+$ otherwise. Pick a radius $r > 0$ such that

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{H^{m-j-1}(G)} + \|g\|_{E_m((t_0,T^*)\times\partial G)} + \|u_0\|_{H^m(G)} + \|f\|_{H^m((t_0,T^*)\times G)} < r.$$

Due to (7.152) and the boundedness of u there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T_+]$. Proposition 7.20 then yields

$$\sup_{t\in(t_0,T_+)} \|u(t)\|_{H^m(G)}^2 \le C_{7.20}(\chi, \sigma, G, m, r, \omega_0, \mathcal{U}_1, T^*) \cdot Cr^2.$$

But by Lemma 7.17 and (7.152) we have $\lim_{t\nearrow T_+} \|u(t)\|_{H^m(\mathbb{R}^3_+)} = \infty$ and thus a contradiction. We conclude that $T_+(k, t_0, f, g, u_0) = T_+$. The assertion for T_- is proven analogously.

(i) This is just Proposition 7.16 and Remark 7.18.

(ii) Assume that $T_+ < \infty$ and that (ii) does not hold. We then have

$$\omega_0 := \sup_{t\in(t_0,T_+)} \|u(t)\|_{W^{1,\infty}(G)} < \infty$$

and there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T_+]$. We apply Proposition 7.20 with $T^* = T_+$ to deduce

$$\|u(t)\|_{H^m(\mathbb{R}^3_+)}^2 \leq C_{7.20}(\chi, \sigma, m, r, \omega_0, \mathcal{U}_1, G, T_+ - t_0) \cdot C r^2$$

for all $t \in (t_0, T_+)$ and thus $\sup_{t \in (t_0, T_+)} \|u(t)\|_{H^m(\mathbb{R}^3_+)} < \infty$. Lemma 7.17 however shows that $\lim_{t \nearrow T_+} \|u(t)\|_{H^3(G)} = \infty$. We thus obtain a contradiction.

(iii) The difficulty in assertion (iii) is to make sure that the solutions to the data in the neighborhood we have to construct exist at least till T'. To that purpose we use an iterative scheme that allows us to apply Theorem 7.10 with the same minimal time step size in each iteration.

Recall that by Sobolev's embedding there is a constant depending only on the length of the interval $[t_0, T_+)$ such that

$$\|\tilde{f}\|_{G_{m-1}((t_0, T_+) \times G)} \leq C_S \|\tilde{f}\|_{H^m((t_0, T_+) \times G)} \tag{7.153}$$

for all $\tilde{f} \in H^m((t_0, T_+) \times G)$. Fix a time $T^* \in (T', T_+)$. We pick two radii $0 < r_0 < r < \infty$ such that

$$\begin{aligned} &\|u_0\|_{H^m(G)} + \|f\|_{G_{m-1}((t_0, T_+) \times G)} + \|f\|_{H^m((t_0, T_+) \times G)} < r_0, \\ &C_S m r_0 < r, \\ &\|u\|_{G_m((t_0, T^*) \times G)} < r. \end{aligned}$$

Moreover, there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T^*]$. Lemma 7.1 thus provides a number $\tilde{r} = \tilde{r}(\chi, \sigma, m, r, \mathcal{U}_1)$ with

$$\begin{aligned} &\|\chi(u)\|_{F_m((t_0, T^*) \times G)} + \|\sigma(u)\|_{F_m((t_0, T^*) \times G)} \leq \tilde{r}, \\ &\max\{\|\chi(u)(t_0)\|_{F^0_{m-1}(G)}, \max_{1 \leq j \leq m-1} \|\partial_t^j \chi(u)(t_0)\|_{H^{m-j-1}(G)}\} \leq \tilde{r}, \\ &\max\{\|\sigma(u)(t_0)\|_{F^0_{m-1}(G)}, \max_{1 \leq j \leq m-1} \|\partial_t^j \sigma(u)(t_0)\|_{H^{m-j-1}(G)}\} \leq \tilde{r}. \end{aligned} \tag{7.154}$$

I) Let $t' \in (t_0, T^*)$ and $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in M_{\chi, \sigma, m}(t_0, T_+)$. Assume that the solution $\tilde{u}$ of (1.6) with inhomogeneity $\tilde{f}$, boundary value $\tilde{g}$, and initial value $\tilde{u}_0$ exists on $[t_0, t']$ and thus belongs to $G_m((t_0, t') \times G)$. Pick a radius R' and a compact subset $\tilde{\mathcal{U}_1}$ of $\mathcal{U}$ such that $\|\tilde{u}\|_{G_m((t_0, t') \times G)} \leq R'$ and $\operatorname{im} u(t), \operatorname{im} \tilde{u}(t) \subseteq \tilde{\mathcal{U}_1}$ for all $t \in [t_0, t']$. Set $\tilde{T} = T_+ - t_0$. We will show that there is a constant $C = C(\chi, \sigma, m, r, R', \tilde{T}, \tilde{\mathcal{U}_1})$ such that

$$\begin{aligned} \|\tilde{u} - u\|_{G_{m-1}((t_0, t') \times G)}^2 &\leq C \|\tilde{f} - f\|_{H^{m-1}((t_0, t') \times G)}^2 + C \|\tilde{g} - g\|_{E_{m-1}((t_0, t') \times \partial G)} \\ &\quad + C\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0)\|_{H^{m-j-1}(\mathbb{R}^3_+)}^2 + \|\tilde{u}_0 - u_0\|_{H^m(\mathbb{R}^3_+)}^2\Big). \end{aligned} \tag{7.155}$$

To that purpose, we apply the linear differential operator $L(\chi(u), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u))$ to $\tilde{u} - u$. We obtain

$$L(\chi(u), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u))(\tilde{u} - u) = \tilde{f} + (\chi(u) - \chi(\tilde{u}))\partial_t \tilde{u} + (\sigma(u) - \sigma(\tilde{u}))\tilde{u} - f =: F.$$

Lemma 7.1 and Lemma 2.22 show that F is an element of $H^{m-1}((t_0, t') \times G)$. Set

$$\gamma_0 = \gamma_0(\chi, \sigma, G, m, r, \tilde{T}) = \gamma_{5.6;0}(\eta(\chi), \tilde{r}, \tilde{T}, G) \geq 1,$$

where $\chi \geq \eta(\chi) > 0$ and $\gamma_{5.6;0}$ is the corresponding constant from Thereom 5.6. This theorem then yields

$$\|\tilde{u} - u\|_{G_{m-1,\gamma}((t_0, t') \times G)}^2$$

$$\leq (C_{5.6;m,0} + \tilde{T} C_{5.6;m}) e^{m C_{5.6;1} \tilde{T}} \Big(\sum_{j=0}^{m-2} \|\partial_t^j F(t_0)\|^2_{H^{m-2-j}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^{m-1}(G)}$$

$$\|\tilde{g} - g\|^2_{E_{m-1,\gamma}((t_0,t')\times \partial G)} \Big) + \frac{C_{5.6;m}}{\gamma} e^{m C_{5.6;1}\tilde{T}} \|F\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \tag{7.156}$$

for all $\gamma \geq \gamma_0$, where

$$C_{5.6;m,0} = C_{5.6;m,0}(\eta(\chi), \tilde{r}, G), \quad C_{5.6;m} = C_{5.6;m}(\eta(\chi), \tilde{r}, \tilde{T}, G),$$
$$C_{5.6;1} = C_{5.6;1}(\eta(\chi), \tilde{r}, \tilde{T}, G)$$

are the corresponding constants from Theorem 5.6. We next apply Lemma 2.22 (ii) and then Corollary 7.2 to obtain

$$\begin{aligned}
&\|F\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \leq C\|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \\
&\quad + C\tilde{T}\|\chi(\tilde{u}) - \chi(u)\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \|\partial_t \tilde{u}\|^2_{G_{m-1}((t_0,t')\times G)} \\
&\quad + C\tilde{T}\|\sigma(\tilde{u}) - \sigma(u)\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \|\tilde{u}\|^2_{G_{m-1}((t_0,t')\times G)} \\
&\leq C\|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} + C(\chi, \sigma, m, r, R', \tilde{T}, \tilde{\mathcal{U}}_1)\|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)}.
\end{aligned} \tag{7.157}$$

Let $j \in \{0, \dots, m-2\}$. Lemma 7.1 and the definition of the M_k^l in (7.14) then show that

$$\begin{aligned}
\partial_t^j F(t_0) &= \partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0) + \sum_{l=0}^{j} \binom{j}{l} (\partial_t^l \chi(u) - \partial_t^l \chi(\tilde{u}))(t_0) \partial_t^{j+1-l} \tilde{u}(t_0) \\
&\quad + \sum_{l=0}^{j} \binom{j}{l} (\partial_t^l \sigma(u) - \partial_t^l \sigma(\tilde{u}))(t_0) \partial_t^{j-l} \tilde{u}(t_0) \\
&= \partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0) + (\chi(u_0) - \chi(\tilde{u}_0)) S_{\chi,\sigma,G,m,j+1}(t_0, \tilde{f}, \tilde{u}_0) \\
&\quad + \sum_{l=1}^{j} \binom{j}{l} (M_1^l(t_0, f, u_0) - M_1^l(t_0, \tilde{f}, \tilde{u}_0)) S_{\chi,\sigma,m,j+1-l}(t_0, \tilde{f}, \tilde{u}_0) \\
&\quad + \sum_{l=0}^{j} \binom{j}{l} (M_2^l(t_0, f, u_0) - M_2^l(t_0, \tilde{f}, \tilde{u}_0)) S_{\chi,\sigma,m,j-l}(t_0, \tilde{f}, \tilde{u}_0).
\end{aligned}$$

Lemma 7.7 and its proof (cf. (7.18) to (7.24)) now allow us to estimate

$$\begin{aligned}
&\|\partial_t^j F(t_0)\|_{H^{m-2-j}(G)} \leq \|\partial_t^j F(t_0)\|_{H^{m-1-j}(G)} \\
&\leq C(\chi, \sigma, m, r, R', \tilde{\mathcal{U}}_1) \Big(\sum_{l=0}^{m-1} \|\partial_t^l f(t_0) - \partial_t^l \tilde{f}(t_0)\|_{H^{m-l-1}(G)} + \|u_0 - \tilde{u}_0\|_{H^m(G)} \Big).
\end{aligned} \tag{7.158}$$

Inserting (7.157) and (7.158) into (7.156), we infer that there is a constant $C_{7.159} = C_{7.159}(\chi, \sigma, G, m, r, R', \tilde{T}, \tilde{\mathcal{U}}_1)$ such that

$$\begin{aligned}
&\|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)} \\
&\leq C_{7.159} \Big(\frac{1}{\gamma} \|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)(\Omega)} + \|\tilde{f} - f\|^2_{H^{m-1}_\gamma((t_0,t')\times G)} \\
&\quad + \|\tilde{g} - g\|^2_{E_{m-1,\gamma}((t_0,t')\times \partial G)} + \sum_{l=0}^{m-1} \|\partial_t^l \tilde{f}(t_0) - \partial_t^l f(t_0)\|^2_{H^{m-l-1}(\mathbb{R}^3_+)} + \|\tilde{u}_0 - u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big)
\end{aligned} \tag{7.159}$$

for all $\gamma \geq \gamma_0$. We next fix a number $\gamma = \gamma(\chi, \sigma, G, m, r, R', \tilde{T}, \tilde{\mathcal{U}}_1)$ with $\gamma \geq \gamma_0$ and $C_{7.159} \frac{1}{\gamma} \leq \frac{1}{2}$. We thus arrive at

$$\|\tilde{u} - u\|^2_{G_{m-1}((t_0,t')\times G)} \leq e^{2\gamma(t'-t_0)} \|\tilde{u} - u\|^2_{G_{m-1,\gamma}((t_0,t')\times G)}$$

$$\leq 2e^{2\gamma\tilde{T}} C_{7.159} \|\tilde{f} - f\|^2_{H^{m-1}((t_0,t')\times G)} + 2e^{2\gamma\tilde{T}} C_{7.159} \|\tilde{g} - g\|^2_{E_{m-1}((t_0,t')\times\partial G)}$$
$$+ 2e^{2\gamma\tilde{T}} C_{7.159} \Big(\sum_{l=0}^{m-1} \|\partial_t^l \tilde{f}(t_0) - \partial_t^l f(t_0)\|^2_{H^{m-l-1}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^m(G)} \Big),$$

i.e., estimate (7.155) is true.

II) Recall that $\mathcal{U}_1$ is a compact subset of $\mathcal{U}$ such that $\operatorname{im} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T^*]$. Pick a number κ such that $2\kappa < \operatorname{dist}(\mathcal{U}_1, \partial\mathcal{U})$. Take $\tau = \tau(\chi, \sigma, m, \tilde{T}, 4(m+1)r, \kappa)$ from Theorem 7.10. There is a number $N \in \mathbb{N}$ such that

$$t_0 + (N-1)\tau < T' \leq t_0 + N\tau.$$

We set $t_k = t_0 + k\tau$ for $k \in \{1, \ldots, N-1\}$. If $t_0 + N\tau < T^*$, we set $t_N = t_0 + N\tau$; else we choose any t_N from (T', T^*).

Let $0 < \delta_0 < r_0$. Take $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta_0)$. We then have

$$\begin{aligned}
\|\tilde{u}_0\|_{H^m(G)} &\leq \|u_0\|_{H^m(G)} + \|\tilde{u}_0 - u_0\|_{H^m(G)} \leq r_0 + \delta_0 < 2r_0 < 2r, \\
\|\tilde{g}\|_{E_m((t_0,T')\times\partial G)} &\leq \|g\|_{E_m((t_0,T')\times\partial G)} + \|\tilde{g} - g\|_{E_m((t_0,T')\times\partial G)} \leq r_0 + \delta_0 < 2r, \\
\|\tilde{f}\|_{H^m((t_0,T')\times G)} &\leq \|f\|_{H^m((t_0,T')\times G)} + \|\tilde{f} - f\|_{H^m((t_0,T')\times G)} \leq r_0 + \delta_0 \\
&< 2r_0 < 2r, \qquad (7.160)
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0)\|_{H^{m-1-j}(G)} &\leq m\|\tilde{f}\|_{G_{m-1}((t_0,T')\times G)} \leq C_S m \|\tilde{f}\|_{H^m((t_0,T')\times G)} \\
&< 2C_S m r_0 < 2r. \qquad (7.161)
\end{aligned}$$

So Theorem 7.10 shows that the solution $\tilde{u}$ of (1.6) with inhomogeneity $\tilde{f}$, boundary value $\tilde{g}$, and initial value $\tilde{u}_0$ at t_0 exists on $[t_0, t_1]$ and belongs to $G_m((t_0,t_1)\times G)$. Moreover, the proof of this theorem yields a radius $R = R_{7.10}(\chi, \sigma, m, \tilde{T}, 4(m+1)r, \kappa) > 4(m+1)r$, see (7.29), such that $\|\tilde{u}\|_{G_m((t_0,t_1)\times G)} \leq R$. We conclude that the flow map Ψ maps $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta_0)$ into $B_{G_m((t_0,t_1)\times G)}(0,R)$. We further deduce from step I) that there is a constant

$$C_{7.162} = C_{7.162}(\chi, \sigma, G, m, r, \tilde{T}, \kappa) = 2e^{2\gamma\tilde{T}} C_{7.159}(\chi, \sigma, G, m, r, R(\chi, \sigma, m, \tilde{T}, r, \kappa), \tilde{T})$$

such that

$$\begin{aligned}
&\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|^2_{G_{m-1}((t_0,t_1)\times G)} \\
&\leq C_{7.162} \|\tilde{f} - f\|^2_{H^{m-1}((t_0,t_1)\times G)} + C_{7.162} \|\tilde{g} - g\|^2_{E_{m-1}((t_0,t')\times\partial G)} \\
&\quad + C_{7.162} \Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(\mathbb{R}^3_+)} + \|\tilde{u}_0 - u_0\|^2_{H^m(\mathbb{R}^3_+)} \Big) \qquad (7.162)
\end{aligned}$$

for all $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta_0)$, where we denote the maximal solution of (1.6) with data $(\tilde{f}, \tilde{g}, \tilde{u}_0)$ by $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)$.

Next take a sequence $(f_n, g_n, u_{0,n})_n$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f,g,u_0),\delta_0)$ which converges to (f, g, u_0) in this space. Using that

$$\begin{aligned}
\sum_{j=0}^{m-1} \|\partial_t^j f_n(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(G)} &\leq m\|f_n - f\|_{G_{m-1}((t_0,T_+)\times G)} \\
&\leq mC_S \|f_n - f\|_{H^m((t_0,T_+)\times G)} \longrightarrow 0 \qquad (7.163)
\end{aligned}$$

as $n \to \infty$, we infer that

$$\|\Psi(f_n, g_n, u_{0,n}) - \Psi(f, g, u_0)\|_{G_{m-1}((t_0,t_1)\times G)} \longrightarrow 0$$

as $n \to \infty$. Lemma 7.22 thus shows that $(\Psi(f_n, g_n, u_{0,n}))_n$ converges to $\Psi(f, g, u_0)$ in $G_m((t_0, t_1) \times G)$. We conclude that the map

$$\Psi\colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_0) \to G_m((t_0, t_1) \times G)$$

is continuous at (f, g, u_0). In particular, there is a number $\delta_1 \in (0, \delta_0]$ such that for all data $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_1)$ we have

$$\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|_{G_m((t_0,t_1)\times G)} < r,$$
$$\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|_{L^\infty((t_0,t_1)\times G)} < \frac{\kappa}{N},$$

where we also employed Sobolev's embedding for the second estimate. To sum up, we found a radius $\delta_1 > 0$ such that $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)$ exists on $[t_0, t_1]$,

$$\begin{aligned}
&\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_1)\times G)} \\
&\leq \|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|_{G_m((t_0,t_1)\times G)} + \|\Psi(f, g, u_0)\|_{G_m((t_0,t_1)\times G)} < 2r, \\
&\operatorname{dist}(\overline{\operatorname{im}\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t_1)}, \partial\mathcal{U}) > \frac{2N-1}{N}\kappa \geq \kappa,
\end{aligned}$$

and (7.162) holds for all $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_1)$.

Now assume that there is an index $j \in \{1, \ldots, N-1\}$ and a number $\delta_j > 0$ such that $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)$ exists on $[t_0, t_j]$,

$$\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_j)\times G)} < 2r,$$
$$\operatorname{dist}(\overline{\operatorname{im}\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t)}, \partial\mathcal{U}) > \frac{2N-j}{N}\kappa$$

for all $t \in [t_0, t_j]$ and $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_j)$.

Fix such a tuple $(\tilde{f}, \tilde{g}, \tilde{u}_0)$. Then the tuple $(\chi, \sigma, t_j, B, \tilde{f}, \tilde{g}, \Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t_j))$ fulfills the nonlinear compatibility conditions (7.16) of order m by Lemma 7.9 and

$$\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t_j)\|_{H^m(G)} \leq \|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_j)\times G)} < 2r,$$
$$\operatorname{dist}(\overline{\operatorname{im}\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t_j)}, \partial\mathcal{U}) > \kappa.$$

In view of (7.161) and (7.160), Theorem 7.10 shows that the initial boundary value problem (1.6) with inhomogeneity $\tilde{f}$, boundary value $\tilde{g}$, and initial value $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)(t_j)$ at initial time t_j has a unique solution $\tilde{u}^j$ on $[t_j, t_{j+1}]$, which is bounded by R in $G_m((t_j, t_{j+1}) \times G)$. Concatenating $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)$ and $\tilde{u}^j$, we obtain a solution of (1.6) with inhomogeneity $\tilde{f}$, boundary value $\tilde{g}$, and initial value $\tilde{u}_0$ at initial time t_0 by Lemma 7.13. This means that $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)$ exists on $[t_0, t_{j+1}]$. Uniqueness of solutions of (1.6), i.e. Lemma 7.3, and Lemma 7.14 further yield $\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)_{|[t_j,t_{j+1}]} = \tilde{u}^j$ so that

$$\begin{aligned}
\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_{j+1})\times G)} &\leq \max\{\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0)\|_{G_m((t_0,t_j)\times G)}, \|\tilde{u}^j\|_{G_m((t_j,t_{j+1})\times G)}\} \\
&\leq \max\{2r, R\} \leq R.
\end{aligned}$$

We can therefore apply step I) again and we obtain as in (7.162) that

$$\begin{aligned}
&\|\Psi(\tilde{f}, \tilde{g}, \tilde{u}_0) - \Psi(f, g, u_0)\|^2_{G_{m-1}((t_0,t_{j+1})\times G)} \\
&\leq C_{7.162}\|\tilde{f} - f\|^2_{H^{m-1}((t_0,t_{j+1})\times G)} + C_{7.162}\|\tilde{g} - g\|^2_{E_{m-1}((t_0,t_{j+1})\times \partial G)} \\
&\qquad + C_{7.162}\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}(t_0) - \partial_t^j f(t_0)\|^2_{H^{m-j-1}(G)} + \|\tilde{u}_0 - u_0\|^2_{H^m(G)}\Big) \qquad (7.164)
\end{aligned}$$

for all $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_j)$. We take again a sequence $(f_n, g_n, u_{0,n})$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_j)$ converging to (f, g, u_0). Combining (7.164) with (7.163), we infer that $\Psi(f_n, g_n, u_{0,n})$ tends to $\Psi(f, g, u_0)$ as $n \to \infty$ in $G_{m-1}((t_0, t_{j+1}) \times G)$. Lemma 7.22 then implies that

$$\|\Psi(f_n, g_n, u_{0,n}) - \Psi(f, g, u_0)\|_{G_m((t_0,t_{j+1})\times G)} \longrightarrow 0$$

as $n \to \infty$. We conclude that

$$\Psi \colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_j) \to G_m((t_0, t_{j+1}) \times G)$$

is continuous at (f, g, u_0). Hence, there is $\delta_{j+1} \in (0, \delta_j]$ such that

$$\|\Psi(\tilde f, \tilde g, \tilde u_0) - \Psi(f, g, u_0)\|_{G_m((t_0,t_{j+1})\times G)} < r,$$
$$\|\Psi(\tilde f, \tilde g, \tilde u_0) - \Psi(f, g, u_0)\|_{L^\infty((t_0,t_{j+1})\times G)} < \frac{\kappa}{N}$$

for all $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_{j+1})$, where we again used Sobolev's embedding for the second estimate. We conclude that

$$\begin{aligned}
&\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_{j+1})\times G)} \\
&\le \|\Psi(\tilde f, \tilde g, \tilde u_0) - \Psi(f, g, u_0)\|_{G_m((t_0,t_{j+1})\times G)} + \|\Psi(f, g, u_0)\|_{G_m((t_0,t_{j+1})\times G)} < 2r, \\
&\operatorname{dist}(\overline{\operatorname{im}\Psi(\tilde f, \tilde g, \tilde u_0)(t)}, \partial\mathcal{U}) > \frac{2N - j - 1}{N}\kappa
\end{aligned}$$

for all $t \in [t_0, t_{j+1}]$ and $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_{j+1})$.

By induction, we thus obtain a number $\delta_N > 0$ such that $\Psi(\tilde f, \tilde g, \tilde u_0)$ exists on $[t_0, t_N]$ and

$$\|\Psi(\tilde f, \tilde g, \tilde u_0)\|_{G_m((t_0,t_N)\times G)} < 2r,$$
$$\operatorname{dist}(\overline{\operatorname{im}\Psi(\tilde f, \tilde g, \tilde u_0)(t)}, \partial\mathcal{U}) > \kappa$$

for all $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$. In particular,

$$T_+(m, t_0, \tilde f, \tilde g, \tilde u_0) > t_N > T'$$

for all $(\tilde f, \tilde g, \tilde u_0) \in B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$.

Next fix two tuples $(\tilde f_1, \tilde g_1, \tilde u_{0,1})$ and $(\tilde f_2, \tilde g_2, \tilde u_{0,2})$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$. Replacing u by $\Psi(\tilde f_2, \tilde g_2, \tilde u_{0,2})$ in step I), we deduce from (7.155) that

$$\begin{aligned}
&\|\Psi(\tilde f_1, \tilde g_1, \tilde u_{0,1}) - \Psi(\tilde f_2, \tilde g_2, \tilde u_{0,2})\|^2_{G_{m-1}((t_0,T')\times G)} \\
&\le C\|\tilde f_1 - \tilde f_2\|^2_{H^{m-1}((t_0,T')\times G)} + C\|\tilde g_1 - \tilde g_2\|^2_{E_{m-1}((t_0,T')\times G)} \\
&\quad + C\Big(\sum_{j=0}^{m-1} \|\partial_t^j \tilde f_1(t_0) - \partial_t^j \tilde f_2(t_0)\|^2_{H^{m-j-1}(G)} + \|\tilde u_{0,1} - \tilde u_{0,2}\|^2_{H^m(G)}\Big),
\end{aligned}$$

where $C = C(\chi, \sigma, m, r, \tilde T, \kappa) = C_{7.155}(\chi, \sigma, m, 2r, 2r, \tilde T, \mathcal{U}_\kappa)$ with

$$\mathcal{U}_\kappa = \{y \in \mathcal{U} \colon \operatorname{dist}(y, \partial\mathcal{U}) \ge \kappa\} \cap \overline{B}(0, 2C_{\mathrm{Sob}} r)$$

and $C_{7.155}$ is the constant from (7.155). This estimate implies (7.151). Finally, we take a sequence $(\tilde f_n, \tilde g_n, \tilde u_{0,n})_n$ in $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$ which converges to $(\tilde f_1, \tilde g_1, \tilde u_{0,1})$ in $M_{\chi,\sigma,m}(t_0, T_+)$. Employing Sobolev's inequality as in (7.153) and (7.151), we obtain that $\Psi(\tilde f_n, \tilde g_n, \tilde u_{0,n})$ tends to $\Psi(\tilde f_1, \tilde g_1, \tilde u_{0,1})$ in $G_{m-1}((t_0, T') \times G)$ as $n \to \infty$. Lemma 7.22 therefore implies that

$$\|\Psi(\tilde f_n, \tilde g_n, \tilde u_{0,n}) - \Psi(\tilde f_1, \tilde g_1, \tilde u_{0,1})\|_{G_m((t_0,T')\times G)} \longrightarrow 0$$

as $n \to \infty$. Consequently, the flow map

$$\Psi \colon B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N) \to G_m((t_0, T') \times G)$$

is continuous at $(\tilde f_1, \tilde g_1, \tilde u_{0,1})$ and thus it is continuous on $B_{M_{\chi,\sigma,m}(t_0,T_+)}((f, g, u_0), \delta_N)$. □

We note that also the nonlinear solution has finite propagation speed, cf. Chapter 6.

Remark 7.24. In the framework of Theorem 7.23 assume that the data vanish on a backward light cone or outside of a forward light cone, see Theorem 6.1 respectively Corollary 6.2 for the precise statement. Then also the solution of the nonlinear problem (1.6) vanishes on the backward respectively forward light cone. To prove this assertion we only have to interpret the function u as the solution of the linear initial boundary value problem (5.1) with coefficients $\chi(u)$ and $\sigma(u)$ and apply Theorem 6.1 respectively Corollary 6.2 to it.

With the above theorem we gave a satisfying answer to the question of local wellposedness of the quasilinear system (1.6). We now want to apply this theorem to the physical Maxwell system (1.2). In the introduction we claimed that a solution of (1.6) yields a solution of (1.2) if we impose additional conditions on the initial value. We make this assertion precise in the next lemma.

Lemma 7.25. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$. Set $J = (t_0, T)$. Assume that there exists a solution $u = (\boldsymbol{E}, \boldsymbol{H})$ in $C(\overline{J}, H^1(G)) \cap C^1(\overline{J}, L^2(G))$ of (1.6) with an inhomogeneity $f = (-\boldsymbol{J}_0, 0)$, where $\boldsymbol{J}_0 \in C(\overline{J}, H^1(G))$. Suppose that there are functions*

$$\theta \colon G \times \mathcal{U} \to \mathbb{R}^{6\times 6}, \quad \sigma_0 \colon G \times \mathcal{U} \to \mathbb{R}^{3\times 3}$$

such that $\partial_y \theta =: \chi$ belongs to $\mathcal{ML}^1_{\mathrm{pd}}(G,\mathcal{U})$ and $\sigma = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix}$ to $\mathcal{ML}^1(G,\mathcal{U})$. We further assume that $\boldsymbol{D}$ and $\boldsymbol{B}$ given by

$$(\boldsymbol{D}, \boldsymbol{B}) = \theta(x, \boldsymbol{E}, \boldsymbol{H}) \tag{7.165}$$

belong to $C(\overline{J}, H^1(G)) \cap C^1(\overline{J}, L^2(G))$. Let $\rho_0 \in L^2(G)$ and set $\boldsymbol{J} = \boldsymbol{J}_0 + \sigma_0(\boldsymbol{E}, \boldsymbol{H})$. Assume that $\operatorname{div} \boldsymbol{J}$ belongs to $C(\overline{J}, L^2(G))$. Set $\rho(t) := \rho_0 - \int_{t_0}^t \operatorname{div} \boldsymbol{J}(s) ds$ for all $t \in \overline{J}$. Then the following assertions hold.

(i) If $\operatorname{div} \boldsymbol{D}(0) = \rho_0$, then $\operatorname{div} \boldsymbol{D}(t) = \rho(t)$ for all $t \in \overline{J}$.

(ii) If $\operatorname{div} \boldsymbol{B}(0) = 0$, then $\operatorname{div} \boldsymbol{B}(t) = 0$ for all $t \in \overline{J}$.

(iii) If $\boldsymbol{E} \times \nu = 0$ on $J \times \partial G$ and $\boldsymbol{B}(0) \cdot \nu = 0$ on ∂G, then $\boldsymbol{B} \cdot \nu = 0$ on $J \times \partial G$.

Proof. Using the relations (7.165) and the fact that $\partial_y \theta = \chi$, we compute in $H^{-1}(G)$

$$\begin{aligned} \partial_t \operatorname{div} \boldsymbol{D} &= \operatorname{div} \partial_t \boldsymbol{D} = \operatorname{div}(\operatorname{curl} \boldsymbol{H} - \boldsymbol{J}) = -\operatorname{div} \boldsymbol{J}, \\ \partial_t \operatorname{div} \boldsymbol{B} &= \operatorname{div} \partial_t \boldsymbol{B} = \operatorname{div}(-\operatorname{curl} \boldsymbol{E}) = 0 \end{aligned}$$

on $\overline{J}$. If $\operatorname{div} \boldsymbol{D}(0) = \rho_0$, we thus obtain $\operatorname{div} \boldsymbol{D}(t) = \rho(t)$ for all $t \in \overline{J}$. Analogously, $\operatorname{div} \boldsymbol{B}(0) = 0$ implies $\operatorname{div} \boldsymbol{B}(t) = 0$ for all $t \in \overline{J}$.

To prove (iii), we first note that the previous computation implies that $\partial_t \boldsymbol{B}(t)$ belongs to $H(\operatorname{div}, G)$ for all $t \in \overline{J}$. Hence, this field has a normal trace in $H^{-1/2}(G)$. Using that also $\operatorname{curl} \boldsymbol{E}(t)$ belongs to $H(\operatorname{div}, G)$ for all $t \in \overline{J}$, we compute

$$\begin{aligned} &\langle \partial_t (\boldsymbol{B} \cdot \nu)(t), \varphi \rangle_{H^{-1/2}(\partial G) \times H^{1/2}(\partial G)} = \langle \partial_t \boldsymbol{B}(t) \cdot \nu, \varphi \rangle_{H^{-1/2}(\partial G) \times H^{1/2}(\partial G)} \\ &= \langle -\operatorname{curl} \boldsymbol{E}(t) \cdot \nu, \varphi \rangle_{H^{-1/2}(\partial G) \times H^{1/2}(\partial G)} \\ &= -\int_G \operatorname{div} \operatorname{curl} \boldsymbol{E}(t)\, \varphi dx - \int_G \operatorname{curl} \boldsymbol{E}(t) \cdot \nabla \varphi dx \\ &= -\int_G \boldsymbol{E}(t) \cdot \operatorname{curl} \nabla \varphi dx + \langle \boldsymbol{E}(t) \times \nu, \nabla \varphi \rangle_{H^{-1/2}(\partial G) \times H^{1/2}(\partial G)} = 0 \end{aligned}$$

for all $t \in J$ and $\varphi \in C_c^\infty(\overline{G})$. Since $C_c^\infty(\overline{G})$ is dense in $H^1(G)$ and $\operatorname{tr} H^1(G) = H^{1/2}(\partial G)$, we deduce that $\partial_t (\boldsymbol{B} \cdot \nu) = 0$ on $\overline{J} \times \partial G$. As $\boldsymbol{B}(0) \cdot \nu = 0$ on ∂G, we conclude that $\boldsymbol{B} \cdot \nu = 0$ on $J \times \partial G$. □

We conclude that if the assumptions of Theorem 7.23 are satisfied and if the initial data satisfies $\operatorname{div} \boldsymbol{D}(0) = \rho_0$, $\operatorname{div} \boldsymbol{B}(0) = 0$, and $\boldsymbol{B}(0) \cdot \nu = 0$ on ∂G, then the physical Maxwell system (1.2) has a unique maximal solution. The other statements of Theorem 7.23 are also valid.

Finally, we give some examples of material laws which are covered by Theorem 7.23. In particular, we can treat the Kerr nonlinearity as claimed in the introduction.

Example 7.26. We introduce the function $\varphi_{2n}^k \colon \mathbb{R}^k \to \mathbb{R}^k, y \mapsto |y|^{2n} y$ for each $n \in \mathbb{N}$. Then the derivatives

$$\partial_y \varphi_{2n}^k(y) = |y|^{2n} I_{k \times k} + 2n|y|^{2n-2} y y^T \tag{7.166}$$

are positive semidefinite on $\mathbb{R}^k$ for all $n \in \mathbb{N}$.

(i) Take a function $\vartheta \in W^{m,\infty}(G)^{3\times 3}$ with $\vartheta \geq 0$. Then the derivative of the function

$$\theta \colon G \times \mathbb{R}^{3+3} \to \mathbb{R}^6, \ (x, y, y') \mapsto (y + \vartheta(x)\varphi_2^3(y), y'), \tag{7.167}$$

with respecto to (y, y') belongs to $\mathcal{ML}_{\mathrm{pd}}^m(G, \mathbb{R}^6)$. We point out that θ gives rise to the Kerr nonlinearity, i.e.,

$$\boldsymbol{D} = \theta_1(x, \boldsymbol{E}, \boldsymbol{H}) = \boldsymbol{E} + \vartheta(x)|\boldsymbol{E}|^2 \boldsymbol{E}, \quad \boldsymbol{B} = \boldsymbol{H}. \tag{7.168}$$

(ii) Now take an arbitrary function $\vartheta \in W^{m,\infty}(G)^{3\times 3}$. Formula (7.166) implies that there is a radius $r > 0$ such that $\partial_{(y,y')}\theta$ belongs to $\mathcal{ML}_{\mathrm{pd}}^m(G, B_r(0))$ where $B_r(0)$ is a ball in $\mathbb{R}^6$ and θ is defined by (7.167). Hence, we can also treat the Kerr nonlinearity (7.168) if there is no lower bound on ϑ.

(iii) The Kerr nonlinearity is so popular in physics as it arises as first nonlinear approximation for the Taylor series for a general material law in which the even powers vanish for symmetry reasons. However, also higher order approximations are considered, cf. [BFLMTW07].

Take $N \in \mathbb{N}$ and functions $\vartheta_i \in W^{m,\infty}(G)^{6\times 6}$ with $\vartheta_i \geq 0$ for all $i \in \{1, \dots, N\}$. Set

$$\theta \colon G \times \mathbb{R}^6 \to \mathbb{R}^6, \ (x, y) \mapsto y + \sum_{i=1}^N \vartheta_i(x)\varphi_{2i}^6(y).$$

Then $\partial_y \theta$ is an element of $\mathcal{ML}_{\mathrm{pd}}^m(G, \mathbb{R}^6)$. If we do not assume that the functions ϑ_i have a lower bound, we still find a radius $r > 0$ such that $\partial_y \theta$ belongs to $\mathcal{ML}_{\mathrm{pd}}^m(G, B_r(0))$ as in the case of the Kerr nonlinearity. Also variants as for the Kerr nonlinearity (7.168), where only the dependance on the $\boldsymbol{E}$ or on the $\boldsymbol{H}$ field is nonlinear, are possible.

Bibliography

[AF09] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces*. Second Edition. Vol. 140. Pure and Applied Mathematics. Amsterdam: Academic Press, 2009.

[Ali09] Serge Alinhac. *Hyperbolic Partial Differential Equation*. Universitext. New York: Springer, 2009.

[AH03] H. Ammari and K. Hamdache. "Global existence and regularity of solutions to a system of nonlinear Maxwell equations". In: *J. Math. Anal. Appl.* 286.1 (2003), pp. 51 –63.

[BF03] Anatoli Babin and Alexander Figotin. "Nonlinear Maxwell Equations in Inhomogeneous Media". In: *Communications in Mathematical Physics* 241.2 (2003), pp. 519 –581.

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Vol. 343. Grundlehren der mathematischen Wissenschaften. Heidelberg: Springer, 2011.

[BKM84] J. Thomas Beale, Tosio Kato, and Andrew Majda. "Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations". In: *Communications in Mathematical Physics* 94 (1984), pp. 61 –66.

[BGS07] Sylvie Benzoni-Gavage and Denis Serre. *Multidimensional Hyperbolic Partial Differential Equations*. Oxford Mathematical Monographs. Oxford: Clarendon Press, 2007.

[BF13] Franck Boyer and Pierre Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*. Vol. 183. Applied Mathematical Sciences. New York: Springer, 2013.

[Bro61] Felix E. Browder. "On the spectral theory of elliptic differential operators. I". In: *Math. Annalen* 142 (1961), pp. 22 –130.

[BFLMTW07] K. Busch, G. von Freymann, S. Linden, S. Mingaleev, L. Tkeshelashvili, and M. Wegener. "Periodic nanostructures for photonics". In: *Phys. Reports* 444 (2007), pp. 101 –202.

[CE11] J. Cagnol and M. Eller. "Boundary regularity for Maxwell's equations with Applications to shape optimization". In: *J. Differential Equations* 250 (2011), pp. 1114 –1136.

[CP82] Jacques Chazarain and Alain Piriou. *Introduction to the theory of linear partial differential equations*. Vol. 14. Studies in mathematics and its applications. Amsterdam: North-Holland, 1982.

[DNS16] P. D'Ancona, S. Nicaise, and R. Schnaubelt. *Blow-up for nonlinear Maxwell equations*. CRC 1173-Preprint. Karlsruhe Institute of Technology, 2016.

[DL90a] Robert Dautray and Jacques-Louis Lions. *Mathematical Analysis and Numerical Methods for Science and Technology – Physical Origins and Classical Methods*. Vol. 3. Berlin: Springer, 1990.

[DL90b] Robert Dautray and Jacques-Louis Lions. *Mathematical Analysis and Numerical Methods for Science and Technology – Spectral Theory and Applications*. Vol. 3. Berlin: Springer, 1990.

[DS12] E. Dumas and F. Sueur. “Cauchy problem and quasi-stationary limit for the Maxwell-Landau-Lifschitz and Maxwell-Bloch equations”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 11.3 (2012), pp. 503 –543.

[Ell12] Matthias Eller. “On symmetric hyperbolic boundary problems with nonhomogeneous conservative boundary conditions”. In: *SIAM J. Math. Anal.* 44.3 (2012), pp. 1925 –1949.

[Eva10] Lawrence C. Evans. *Partial differential equations*. 2nd edition. Vol. 19. Graduate studies in mathematics. Providence, Rhode Island: American Mathematical Society, 2010.

[Gri85] Pierre Grisvard. *Elliptic Problems in Nonsmooth Domains*. Vol. 24. Monographs and studies in mathematics. Boston: Pitman, 1985.

[Gué90] O. Gués. “Problème mixte hyperbolique quasi-linéaire caratéristique”. In: *Comm. Partial Differential Equations* 15.5 (1990), pp. 595 –645.

[Hoe76] Lars Hoermander. *Linear Partial Differential Operators*. Fourth Edition. Vol. 116. Grundlehren der mathematischen Wissenschaften. Berlin: Springer, 1976.

[IT17] Mihaela Ifrim and Daniel Tataru. “Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation”. In: *arxiv preprint* (2017).

[Joc05] F. Jochmann. “Asymptotic behavior of the electromagnetic field for a micromagnetism equation without exchange energy”. In: *SIAM J. Math. Anal.* 37.1 (2005), pp. 276 –290.

[JMR96] J.L. Joly, G. Métivier, and J. Rauch. “Global solvability of the anharmonic oscillator model from nonlinear optics”. In: *SIAM J. Math. Anal.* 27.4 (1996), pp. 905 –913.

[Kat70] T. Kato. “Linear evolution equation of "hyperbolic" type”. In: *J. Fac. Sci. Univ. Tokyo* 17 (1970), pp. 241 –258.

[Kat73] T. Kato. “Linear evolution equation of "hyperbolic type", II”. In: *J. Math. Soc. Japan* 25 (1973), pp. 648 –666.

[Kat75] T. Kato. “The Cauchy problem for quasi-linear symmetric hyperbolic systems”. In: *Arch. Rational Mech. Anal.* 58.3 (1975), pp. 181 –205.

[Kla80] Sergiu Klainerman. “Global Existence for Nonlinear Wave Equations”. In: *Communications on Pure and Applied Mathematics* 33 (1980), pp. 43 –101.

[KM81] Sergiu Klainerman and Andrew Majda. “Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids”. In: *Communications on Pure and Applied Mathematics* 34 (1981), pp. 481 –524.

[KP83] Sergiu Klainerman and Gustavo Ponce. “Global, Small Amplitude Solutions to Nonlinear Evolutions Equations”. In: *Commun. Pure Appl. Math.* 36.1 (1983), pp. 133 –141.

[LMSTYZ01] Tai-Ping Liu, Guy Métivier, Joel Smoller, Blake Temple, Wen-An Yong, and Kevin Zumbrun. *Advances in the Theory of Shock Waves*. Vol. 47. Progress in Nonlinear Differential Equations and Their Applications. Boston: Birkhäuser, 2001.

[MO75] A. Majda and S. Osher. “Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary”. In: *Comm. Pure Appl. Math* 5 (1975), pp. 607 –675.

[Maj84] Andrew Majda. *Compressible Fluid Flow and Systems of Conservations Laws in Several Space Variables.* Vol. 53. Applied Mathematical Sciences. New York: Springer, 1984.

[MMT12] Jeremy L. Marzuola, Jason Metcalfe, and Daniel Tataru. "Quasilinear Schrödinger equations I: Small data and quadratic interactions". In: *Advances in Mathematics* 231 (2012), pp. 1151 –1172.

[Maz11] Vladimir Maz'ya. *Sobolev Spaces. with Applications to Elliptic Partial Differential Equations.* 2nd, revised and augmented Edition. Vol. 342. Grundlehren der mathematischen Wissenschaften. Berlin: Springer, 2011.

[Mos66] Jürgen Moser. "A rapidly convergent iteration method and non-linear partial differential equations - I". In: *Annali della Scuola Normale Superiore di Pisa* 20.2 (1966), pp. 265 –315.

[Mül14] Dominik Müller. *Well-posedness for a general class of quasilinear evolution equations - with applications to Maxwell's equations.* PhD-thesis. Karlsruhe: KIT, 2014.

[Neč12] Jindřich Nečas. *Direct Methods in the Theory of Elliptic Equations.* Springer Monographs in Mathematics. Heidelberg: Springer, 2012.

[Nir59] Louis Nirenberg. "On elliptic partial differential equations". In: *Annali della Scuola Normale Superiore di Pisa* 13.2 (1959), pp. 115 –162.

[Ohk81] T. Ohkubo. "Regularity of solutions to Hyperbolic mixed problems with uniformly characteristic boundary". In: *Hokkaido Mathematical Journal* 10 (1981), pp. 93 –123.

[Ohk89] T. Ohkubo. "Well posedness for quasi-linear hyperbolic mixed problems with characteristic boundary". In: *Hokkaido Mathematical Journal* 18 (1989), pp. 79 –123.

[PZ95] R. H. Picard and W. M. Zajaczkowski. "Local existence of solutions of impedance initial-boundary value problem for non-linear Maxwell equations". In: *Math. Methods Appl. Sci.* 18.3 (1995), pp. 169 –199.

[RM74] J. B. Rauch and F. J. Massey. "Differentiability of solutions to hyperbolic initial boundary value problems". In: *Trans. Amer. Math. Soc.* 189 (1974), pp. 303 –318.

[Rau14] James C. Rautio. "The long road to Maxwell's equations". In: *IEEE Spectrum* 51.12 (2014), pp. 36 –56.

[RSY12] G. F. Roach, I. G. Stratis, and A. N. Yannacopoulos. *Mathematical analysis of derterministic and stochastic problems in complex media electromagnetics.* Princeton Series in Applied Mathematics. Princeton, NJ: Princeton University Press, 2012.

[Sha73] I. S. Shapiro. "On the history of the discovery of the Maxwell equations". In: *Soviet Physics Uspehki* 15.5 (1973), pp. 651 –659.

[Sog13] C. D. Sogge. *Lectures on non-linear wave equations.* Sommerville, Mass.: International Press, 2013.

[Ste70] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton, New Jersey: Princeton University Press, 1970.

[Tan97] Hiroki Tanabe. *Functional Analytic Methods For Partial Differential Equations.* Vol. 204. Monographs and textbooks in pure and applied mathematics. New York: Marcel Dekker, Inc., 1997.

[TW09] Marius Tucsnak and George Weiss. *Observation and Control for Operator Semigroups.* Basel: Birkhäuser, 2009.

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