Semilinear and quasilinear stochastic evolution equations in Banach spaces

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"Jetzt gibts nur eins. Seil über d'Schulder und da Wage zie'e! Ziehen!"

-Christian Streich, 2013

Abstract

In this thesis, we investigate the Cauchy problem for the quasilinear stochastic evolution equation

$$\begin{cases} u(t) = [-A(u(t))u(t) + f(t)]\,\mathrm{d}t + B(u(t))dW(t), & t \in [0,T], \\ u(0) = u_0 \end{cases}$$

in a Banach space X.

In the first part of the thesis, we concentrate on the parabolic situation, i.e. we assume that $-A(u(t))$ is for every t a generator of an analytic semigroup and that $A(u(t))$ has a bounded H^∞-calculus. Under a local Lipschitz assumption on $u \mapsto A(u)$ we prove existence and uniqueness of a local strong solution up to a maximal stopping time that can be characterised by a blow-up alternative. We apply our local well-posedness result to a second order parabolic partial differential equation on $\mathbb{R}^d$, to a generalised Navier-Stokes equation describing non-Newtonian fluids and to a convection-diffusion equation on a bounded domain with Dirichlet, Neumann or mixed boudary conditions. In the last situation, we can even show that the solution exists globally.

In the second part of the thesis, we go to a special hyperbolic situation. We look at a Maxwell equation on a domain D with perfect conductor boundary condition in chiral media with a nonlinear retarded material law, i.e. we consider

$$A(u)u(t) = -Mu(t) + |u(t)|^q u(t) - \int_0^t G(t-s)u(s)\,\mathrm{d}s\,.$$

Here, $M(u_1,u_2) = (\operatorname{curl} u_2, -\operatorname{curl} u_1)^T$ is the Maxwell operator on $L^2(D)^3 \times L^2(D)^3$. To solve this equation we apply a refined version of the monotonicity approach using the spectral multipliers of the Hodge-Laplace operator, which is a componentwise Laplace operator with boundary conditions comparable to those of M^2. We show existence and uniqueness of a weak solution u in the sense of partial differential equations and under stronger assumptions we prove that u is a strong solution, i.e. $Mu(t,x)$ exists almost surely for almost all $t \in [0,T]$ and $x \in D$.

Contents

Introduction

Most of the laws of nature are modelled by time dependent partial differential equations and in many real world problems from engineering, physics and chemistry these equations are highly nonlinear. This poses severe difficulties for the mathematical investigation and in many cases it is even unclear whether a unique solution exists or not.

In this thesis, we focus on quasilinear and semilinear equations. A partial differential equation for a quantity u is quasilinear if it is linear in the top order derivatives of u and it is semilinear if the coefficients in front of the top order derivatives of u are independent of u. A typical example is the reaction-diffusion equation

$$\begin{cases} \partial_t u(t,x) &= \sum_{i,j=1}^d a_{ij}\big(t,x,u(t,x),\nabla u(t,x)\big)\partial_i\partial_j u(t,x) + F(u(t,x)), \quad t \in [0,T],\ x \in D, \\ u(0,x) &= u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x \in D, \end{cases}$$

on a domain $D \subset \mathbb{R}^d$. It is linear in the second order derivatives $\partial_i\partial_j u$ but nonlinear in u and ∇u. In the special case $a_{ij} = a_{ij}(t,x)$ the above equation is semilinear.

As usual in the context of time dependent partial differential equations, we formulate the problem as an ordinary differential equation of first order in time on a Banach space X which contains the spatial dependency. This yields

$$\text{(Q)} \begin{cases} u'(t) &= -A(u(t))u(t) + F(u)(t), \quad t \in [0,T], \\ u(0) &= u_0. \end{cases}$$

Here, $A(u(t))$ is a linear spatial differential operator for every t and $F(u)$ is a nonlinear term that only depends on lower order derivatives. In the literature $A(u(t))u(t)$ is called the quasilinear part of the equation, whereas $F(u)$ is the semilinear part. A common choice for X is $L^q(D)$ or the distributional space $W^{-1,q}(D)$ if one is interested in weak solutions.

In the past, it turned out that for the well-posedness theory one has to distinguish parabolic and hyperbolic equations that require completely different approaches. We define these notions in the same way as Kato in [58]. We call (Q) parabolic if $-A(u(t))$ is a generator of an analytic semigroup for every t. Otherwise, we say that (Q) is of hyperbolic type.

Parabolic quasilinear equations have been studied for more than 30 years using strong linearisation techniques relying on the solvability of non-autonomous equations under certain Hölder continuity assumptions (see e.g. [74], [103]) or relying on maximal L^p-regularity (see e.g. [4], [5], [22], [87]). It is quite remarkable that due to parabolic smoothing it was possible to develop a theory that covers many examples at once. A good overview about the abstract theory and many descriptive examples can be found in [87]. Here, the authors consider (Q) with $A(u(t)) = -\sum_{i,j=1}^{d} a_{ij}\big(t,x,u(t,x),\nabla u(t,x)\big)\partial_i\partial_j$ and $A(u(t)) = -\operatorname{div}\big(a(u(t))\nabla u(t)\big)$ for several types of boundary condition. For equations of hyperbolic type, the situation is completely different. Here, the solution strategy highly depends on the equation itself and on the boundary condition.

In addition to the highly nonlinear behaviour many applications also have some uncertainty concerning the external sources or the precise behaviour of the medium. To include this randomness into the model, researches beginning with Itô in 1946 (see [50], [51]) replaced the original partial differential equation by an equation that is perturbed with white noise. This is the time derivative of a Brownian motion $\beta : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ on a probability space Ω. However, since β is not differentiable in time, we formally write

$$\begin{cases} \mathrm{d}u(t) &= \big[-A(u(t))u(t) + F(u)(t)\big]\,\mathrm{d}t + \big[B(u)(t) + b(t)\big]\,\mathrm{d}\beta(t), \quad t \in [0,T], \\ u(0) &= u_0. \end{cases} \tag{1}$$

Here one distinguishes between the additive noise b which is a random force and the multiplicative noise $B(u)$ perturbing the medium. This equation is interpreted as an integral equation in the Banach space X, i.e. u is a solution of (1) if and only if

$$u(t) - u(0) = \int_0^t -A(u(s))u(t) + F(u)(s)\,\mathrm{d}s + \int_0^t B(u)(s) + b(s)\,\mathrm{d}\beta(s)$$

almost surely for all $t \in [0,T]$ in X. Here, the stochastic integral is a Banach space valued Itô-integral in the sense of [100].

The main object of research in this thesis is (1) either in the parabolic or in the hyperbolic situation. As in the deterministic setting we treat the parabolic case abstractly in a general way, whereas in the hyperbolic case a general theory seems out of reach and we focus on a special hyperbolic Maxwell equation.

First, we briefly describe our results in the parabolic setting and we compare them to the existing literature. We develop an abstract theory for well-posedness of quasilinear stochastic parabolic evolution equations up to a maximal stopping time τ. Furthermore, we apply our abstract results to (1) with the elliptic operators $A(u) = -\sum_{i,j=1}^{N} a_{ij}(u)\partial_i\partial_j$ and $A(u) = -\operatorname{div}(a(u)\nabla)$ on $\mathbb{R}^d$ and on a bounded domain $D \subset \mathbb{R}^d$ with mixed boundary conditions. If we restrict us to Dirichlet boundary conditions, we can show that under additional assumptions the solution does not explode and exists on the whole interval $[0,T]$. This improves the result of Hofmanova and Zhang in [46]. Moreover, we give an application to fluid dynamics and prove well-posedness of a generalized stochastic Navier-Stokes equation for non-Newtonian fluids.

Special quasilinear stochastic parabolic equations have been extensively studied in the literature in case of monotone coefficients (see e.g. [12,40,64,84]). In the same spirit is [73], where the authors extend the results to locally monotone coefficients. Existence and uniqueness of (1) with $A(u) = -\operatorname{div}(a(u)\nabla)$ was proved by Hofmanova and Zhang in [46]. However, as far as we know, there is no abstract theory comparing to the state of knowledge in the deterministic parabolic situation.

In the hyperbolic case, we focus on a semilinear Maxwell equation in chiral media on a domain D and we show existence and uniqueness of a weak solution u, i.e. u solves the equation in the sense of distributions. Under additional assumptions we can even show that u is a strong solution. The equation is motivated by [88] in Chapter 2 and 7. We choose $A(u(t)) = M$ with the Maxwell operator

$$M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \operatorname{curl} u_2 \\ -\operatorname{curl} u_1 \end{pmatrix}$$

for three dimensional vector fields u_1, u_2 and we impose the perfect conductor boundary condition $u_1 \times \nu = 0$ on ∂D. As semilinear part we choose

$$F(u)(t) = -|u(s)|^q u(s) + \int_0^t G(t-r)u(r)\,\mathrm{d}r + J(t)$$

with a power-type nonlinearity that describes the optical response, a nonlocal dispersive memory term and an external current J. We end up with

$$\begin{cases} du(t) &= \big[Mu(t) - |u(t)|^q u(t) + \int_0^t G(t-r)u(r)\,\mathrm{d}r + J(t)\big]\,\mathrm{d}t + \big[B(u)(t) + b(t)\big]\,\mathrm{d}\beta(t), \\ u(0) &= u_0. \end{cases}$$

In a deterministic setting there are good results for nonlinear Maxwell equations. Especially of interest is the well-posedness of (1) with $A(u(t)) = \kappa(u(t))^{-1}M$ with a positive definite matrix function $\kappa : \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$. This models a material of Kerr type, i.e. with a polarisation P given by $P(E) = |E|^2 E$. The corresponding equation was studied by Müller on $\mathbb{R}^3$ (see [80]) and by Spitz on a domain with perfect conductor boundary condition. They used smooth initial data and made use of the fact that in a deterministic setting the time regularity increases with increasing space regularity. However, their technique is not available in a stochastic setting, since solutions of stochastic differential equations are only Hölder continuous of order $\beta < \frac{1}{2}$ and one has to find a different approach. For this reasons we focus on the semilinear equation from above with $\kappa(u) = I$ and develop new tools to study nonlinear Maxwell equations in a stochastic setting. As far as we know, there are no comparable well-posedness results. One reason might be that in the absence of Strichartz estimates for $(e^{tM})_{t\in\mathbb{R}}$, even local solvability is a tricky issue. Moreover, there is no embedding of the form $D(M) \hookrightarrow L^p$ that helps to control the nonlinearity. Therefore, our research can be seen as a start for the analysis of nonlinear stochastic Maxwell equations.

Sketch of our approach to quasilinear parabolic equations

Our goal is to prove existence and uniqueness of a strong solution up to a maximal stopping time τ of the quasilinear parabolic stochastic evolution equation

$$\text{(QSEE)}\begin{cases} du(t) &= \big[-A(u(t))u(t)+F(u)(t)\big]\,\mathrm{d}t+\sum_{k=1}^{\infty}B_k(u)(t)\,\mathrm{d}\beta_k(t), \quad t\in[0,\tau),\\ u(0) &= u_0\end{cases}$$

in $L^p(0,\tau;E)$ for $p>2$ and for a UMD Banach space E of type 2, in $\gamma(0,\tau;E)$ for a UMD Banach space E and in $L^q(U;L^p(0,\tau))$ for $p,q>2$. Here $(\beta_k)_k$ is a sequence of independent Brownian motions on a probability space Ω. We develop a framework with maximal regularity estimates for the deterministic and the stochastic convolution as input and a unified well-posedness theory in all the three spaces as output.

In the following, we explain our ideas for the construction of a strong solution using the space $L^p(0,\tau,E)$ with $E=L^q(U)$ for some $q>2$. The core strategy remains unchanged in the additional settings.

Van Neerven, Veraar and Weis investigated in [96,97] maximal regularity estimates for the stochastic convolution. Together with the well-known deterministic maximal regularity this leads to a well-posedness theory for semilinear stochastic evolution equations of the form

$$\text{(SEE)}\begin{cases} du(t) &= \big[-\Lambda u(t)+G(u)(t)\big]\,\mathrm{d}t+\sum_{k=1}^{\infty}B_k(u)(t)\,\mathrm{d}\beta_k(t), \quad t\in[0,T],\\ u(0) &= u_0.\end{cases}$$

Here, $(\Lambda(\omega))_{\omega\in\Omega}$ is a family of closed and densely defined operators with common domain E^1 such that almost all $\Lambda(\omega)$ have a bounded H^∞-calculus with bound independent of $\omega\in\Omega$. Moreover, the initial value $u_0:\Omega\to(E,E^1)_{1-1/p,p}$ is strongly $\mathcal{F}_0$-measurable. We consider Lipschitz continuous nonlinearities $G:[0,T]\times E^1\to L^q(U)$ and $B_k:[0,T]\times E^1\to[E,E^1]_{\frac{1}{2}}$ with small enough Lipschitz constant.

Now, we are in a position to briefly describe our main assumptions for the quasilinear theory and our strategy. We assume that the domain of the operators $A(z)$, $z\in(E,E^1)_{1-1/p,p}$, is constant, i.e. there is a Banach space E^1 such that $D(A(z))=E^1$ for every $z\in\big(E,E^1\big)_{1-1/p,p}$. Moreover, we demand A to be globally Lipschitz continuous, i.e. there exists $L>0$ such that

$$\|A(z)-A(y)\|_{B(E^1,E)}\le L\|z-y\|_{(E,E^1)_{1-1/p,p}}$$

for every $y,z\in(E,E^1)_{1-1/p,p}$ and we assume that the operators $A(z)$ have a bounded H^∞-calculus with bound independent of z. As a first step, we consider

$$\widetilde{F}_1(u(t))=\theta_\lambda\Big(\sup_{s\in[0,t]}\|u(s)-u_0\|_{(E,E^1)_{1-1/p,p}}+\|u\|_{L^p(0,t;E^1)}\Big)\big(A(u(t))-A(u_0)\big)u(t),$$

where $\theta_\lambda:[0,\infty)\to[0,1]$, $\lambda>0$, is a Lipschitz continuous cut-off function such that $\theta_\lambda\equiv 1$ on $[0,\lambda]$ and $\theta_\lambda\equiv 0$ on $[2\lambda,\infty)$. This means that as long as $u(t)$ is close enough to u_0

and $\|u\|_{L^p(0,t;E^1)}$ is small, we have $A(u(t))u(t) = A(u_0)u(t) + \widetilde{F}_1(u(t))$. We prove that $\widetilde{F}_1$ has a Lipschitz constant proportional to λ and thus, choosing λ small enough, satisfies the assumptions needed to solve (SEE) with $\Lambda = A(u_0)$ and $G(u) = -\widetilde{F}_1(u)+F(u)$. The solution u of (SEE) exists on $[0,T]$. However, u solely solves (QSEE) on the random interval $[0,\tau_1]$, where τ_1 is a stopping time given by

$$\tau_1 = \inf\big\{t \in [0,T] : \|u(t) - u_0\|_{(E,E^1)_{1-1/p,p}} + \|u\|_{L^p(0,t;E^1)} > \lambda\big\}.$$

Since the interval $[0,\tau_1]$ on which u solves (QSEE) might be larger than $[0,\tau_1]$ we have to extend it to a maximal interval $[0,\tau)$. We know that the set of stopping times σ such that there exists a unique solution u on $[0,\sigma]$ is non-empty, since τ_1 is contained in this set. We show that the essential supremum $\tau : \Omega \to [0,T]$ of this set exists and that τ is also a stopping time. Moreover, we prove that τ is maximal and satisfies

$$\mathbb{P}\Big\{\tau < T, \|u\|_{L^p(0,\tau;E^1)} < \infty, u : [0,\tau) \to (E,E^1)_{1-1/p,p} \text{ is uniformly continuous}\Big\} = 0.$$

This condition implies that it is sufficient for global existence to show pathwise uniform continuity of u as a function with values in $(E,E^1)_{1-1/p,p}$ and $\|u\|_{L^p(0,\tau;E^1)} < \infty$ almost surely. Finally, we extend this result to quasilinearities $u \mapsto A(u)$ that are Lipschitz continuous on every ball in $(E,E^1)_{1-1/p,p}$ with a localisation technique.

Sketch of our approach to the nonlinear Maxwell equation with retarded material law

The other problem we address is the nonlinear stochastic Maxwell equation

$$\begin{cases} du(t) &= \big[Mu(t) - |u(t)|^q u(t) + (G*u)(t) + J(t)\big]\,\mathrm{d}t + [b(t) + \sum_{k=1}^\infty B_k(u)(t)]\,\mathrm{d}\beta_k(t), \\ u(0) &= u_0 \end{cases} \tag{2}$$

for a $6d$ vector field $u = (u_1,u_2)$ with the retarded material law $(G*u)(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s$ and the perfect conductor boundary condition $u_1 \times \nu = 0$ on ∂D. We consider a domain $D \subset \mathbb{R}^3$, obviously in the case $D = \mathbb{R}^3$ the boundary condition drops. Here, $(\beta_k)_k$ is a sequence of independent Brownian motions on a probability space Ω. At first, we show that (2) has a unique weak solution

$$u \in L^2(\Omega; C(0,T;L^2(D)))^6 \cap L^{q+2}(\Omega \times [0,T] \times D)^6. \tag{3}$$

This is done in two steps. First, we use a version of the Galerkin method from Röckner and Prévot (see [85]) to solve (2) in the special case $G \equiv 0$ and make use of the monotone structure of the nonlinearity. The novelty is that we are able to deal with the term Mu, despite the fact that $u \notin D(M)$. Afterwards, we include the retarded material law with Banach's fixed point theorem.

The proof of the existence and uniqueness of a strong solution that additionally satisfies

$$Mu \in L^2(\Omega; L^\infty(0,T;L^2(D)))^6 + L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$$

is more tricky. Again, we start with $G \equiv 0$ and add a non-trivial G at the very end. In a deterministic setting, one would try to estimate $\|u'(t)\|_{L^2(D)^6}$ and then use (3) to control Mu. However, solutions of stochastic differential equations are not differentiable in time. The first idea was to derive an estimate for $\|Mu(t) - |u(t)|^q u(t) + J(t)\|^2_{L^2(D)^6}$ with Gronwall's Lemma, but we failed since the Itô-formula for this quantity contains the term

$$\|D_{vv}(|v|^q v)(u(t))\big(B_k(u(t)), B_k(u(t))\big)\|^2_{L^2(D)^6}, \tag{4}$$

we could not estimate properly. Hence, we choose the noise $\sum_{j=1}^N \big(b_j(t) + iB_j u(t)\big) d\beta_j(t)$ and use the rescaling transformation

$$y(t) = u(t)e^{-i\sum_{j=1}^N B_j\beta_j(t)}$$

to get rid of the multiplicative noise and to avoid the difficulties from (4). The arising equation has the form

$$\text{(TSEE)}\begin{cases} dy(t) &= [My(t) - |y(t)|^q y(t) + A(t)y(t) + \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^N \widetilde{b}_i(t)\,\mathrm{d}\beta_i(t), \\ u(0) &= u_0, \end{cases}$$

where $A(t)$ is a nonautonomous operator having random coefficients, $\widetilde{J}$ is a new current and $\widetilde{b}$ is a new additive noise. For $\psi \in C_c^\infty(D)$ with $\operatorname{supp}\psi \subset [0,2]$ and $\psi = 1$ on $[0,1]$, we define the spectral multipliers $S_n = \psi(-2^{-n}\Delta_H)$ and $P_n = \mathbf{1}_{[0,2^n]}(-\Delta_H)$. Here, Δ_H is the Hodge-Laplacian on L^p that is the component-wise Laplace operator with domain

$$\Big\{(u_1,u_2) \in L^p(D)^6 : \operatorname{curl} u_1, \operatorname{curl} u_2, \operatorname{curl}\operatorname{curl} u_1, \operatorname{curl}\operatorname{curl} u_2 \in L^p(D)^3, \operatorname{div} u_1 \in W_0^{1,p}(D),$$
$$\operatorname{div} u_2 \in W^{1,p}(D), u_1 \times \nu = 0, u_2 \cdot \nu = 0, \operatorname{curl} u_2 \times \nu = 0 \text{ on } \partial D\Big\}.$$

We show that P_n, S_n are self-adjoint on $L^2(D)^6$ and commute with both Δ_H and M. Further, we have $\|S_n u\|_{L^p(D)^6} \le C\|u\|_{L^p(D)^6}$ with a constant $C > 0$ depending on p, but not on u and n. Note that such an estimate is not applicable for P_n in a general situation. We point out that the uniform L^p-boundedness of $(S_n)_n$ is a consequence of the fact that the semigroup generated by Δ_H satisfies generalised Gaussian bounds. The deep connection between Δ_H and M originates from the formula

$$-\Delta = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div},$$

which implies $\Delta_H = M^2$ in the range of the Helmholtz projection P_H and $M^2 = 0$ in the range of $I - P_H$.

We truncate (TSEE) with a refined Faedo-Galerkin approach, i.e. we solve

$$\begin{cases} dy_n(t) &= P_n[My_n(t) - |y_n(t)|^q y_n(t) + A(t)y_n(t) + \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^N S_n\widetilde{b}_i(t)\,\mathrm{d}\beta_i(t), \\ y_n(0) &= S_n u_0. \end{cases}$$

P_n and S_n reduce the problem to an ordinary stochastic differential equation that can be solved easily. Afterwards, we estimate

$$\big\|P_n M y_n(t) - P_n|y_n(t)|^q y_n(t) + P_n A(t)y_n(t) + P_n\widetilde{J}(t)\big\|^2_{L^2(D)^6}$$

using Itô's formula, the monotone structure of the equation and the properties of P_n and S_n. This yields an estimate for My_n that is uniform in n. Finally, we pass to the limit again using the monotonicity of the nonlinearity and undo the transformation to get a strong solution u of (2) such that $Mu(t,x)$ exists almost surely for almost all $t \in [0,T]$ and $x \in D$.

Outline of this thesis

This thesis is organised as the follows. Chapter 1 contains an overview over the most important definitions and theorems that are used frequently. In particular, we recall some facts about vector calculus and the related Sobolev, Besov and Triebel-Lizorkin spaces. Then, we collect important stochastic concepts like adaptivity and stopping times. We introduce γ-radonifying operators and sketch the construction of stochastic integrals in UMD Banach spaces. A short overview over sectorial operators and their functional calculi ends this chapter.

In Chapter 2, we develop a framework with maximal regularity estimates for the deterministic and the stochastic convolution as input and a well-posedness theory for quasilinear parabolic stochastic evolution equations as output. In Section 2.1, we recall three maximal regularity concepts for stochastic evolution equations from the literature. The first one is maximal regularity in the space $L^p(0,T;E)$ for a UMD Banach space E of type 2 and some $p > 2$. Here, we follow [96]. Next, we consider maximal regularity in $\gamma(0,T;E)$ for some UMD Banach space E in the same way as [98]. Moreover, we treat maximal regularity in spaces of the form $L^q(U;L^p(0,T))$ for $U \subset \mathbb{R}^d$ which was treated in [8]. To get more flexibility for applications we slightly generalise the last approach by allowing fractional spaces of the form $\Lambda^\beta\big(L^q(U;L^p(0,T))\big)$ for some densely defined and invertible operator Λ on $L^q(U)$ that has an $\mathcal{R}_p$-bounded H^∞-calculus. With these estimates for the stochastic and the deterministic convolution, we show existence and uniqueness of a strong solution of the equation

$$\begin{cases} du(t) &= \big[-Au(t) + F(u)(t)\big]\,\mathrm{d}t + \sum_{k=1}^\infty B_k(u)\,\mathrm{d}\beta_k(t), \quad t \in (\sigma,T], \\ u(\sigma) &= u_\sigma \end{cases}$$

in Section 2.2. Here, our strategy is a version of the argumentation in [96]. Our contribution is that we treat the three settings from above in a unified way, that we allow a random initial time σ and that we allow F and B to be memory terms with the Volterra property. This means that the restriction $F(u)_{|[\sigma,\tilde{\sigma}]}$ only depends on $u_{|[\sigma,\tilde{\sigma}]}$. These novelties will be essential in the treatment of the quasilinear equation

$$\begin{cases} du(t) &= \big[-A(u(t))u(t) + F(u)(t)\big]\,\mathrm{d}t + \sum_{k=1}^\infty B_k(u)(t)\,\mathrm{d}\beta_k(t), \quad t \in [0,\tau), \\ u(0) &= u_0 \end{cases}$$

in Section 2.3. Here, we follow the strategy we sketched above simultaneously in our three settings. First, we show existence and uniqueness of a strong solution with a globally Lipschitz continuous quasilinearity in Subsection 2.3.1 and we generalise this to a locally Lipschitz continuous quasilinearity in Subsection 2.3.2.

In Chapter 3, we apply our abstract results to quasilinear parabolic stochastic equations. In these examples, we benefit from the extensive literature about elliptic operators, their regularity properties and their functional calculi. In Section 3.1, we show that our theory covers the most straightforward parabolic example $A(u(t)) = -\sum_{i,j=1}^{d} a_{ij}(u(t), \nabla u(t))\partial_i\partial_j$. In Section 3.2, we investigate the reaction-diffusion equation

$$\begin{cases} du(t) &= \big[\operatorname{div}(a(u(t))\nabla u(t)) + F(u)(t)\big]\,\mathrm{d}t + \sum_{k=1}^{\infty} B_k(u)(t)\,\mathrm{d}\beta_k(t), \quad t \in [0,\tau), \\ u(0) &= u_0 \end{cases}$$

and we proof existence and uniqueness of a weak solution u in the sense of partial differential equations, which means that the equation holds in a distributional sense. If we restrict ourselves to a bounded domain with Dirichlet boundary conditions, we can even show that u exists on the whole time interval $[0, T]$. Our last example in Section 3.3 is inspired from fluid dynamics and we treat non-Newtonian fluids in a stochastic setting.

In Chapter 4, we consider the nonlinear Maxwell equation with retarded material law. In Section 4.1, we show the existence of a sequence of orthogonal projections $(P_n)_n$ and of related operators $(S_n)_n$ that commute with M and that satisfy $S_n x \to x$ for $n \to \infty$ with convergence in L^p for every $x \in L^p$. We construct them with a spectral multiplier theorem for the Hodge-Laplacian and we exploit the deep connection between the Hodge-Laplacian, the Helmholtz-projection and M^2 to show the commutation property. In Section 4.2, we use P_n and S_n to show existence and uniqueness of a solution u in the distributional sense. Under stronger assumptions, we show in Section 4.3 that the solution is more regular, i.e. $Mu(t, x)$ exists almost surely for almost all $t \in [0, T]$ and $x \in [0, T]$. We end this chapter with a comparison to the results in the literature and with some remarks about the more general situation with non-trivial electric permittivity ε and magnetic permeability μ.

In Appendix A, we present a byproduct of our research. We prove a theorem about boundedness of pseudo-differential operators on Banach spaces with a rough symbol that has a special structure which allows us to apply square function estimates. In particular, we show that given a UMD Banach space X the operator

$$Lf(t) = A(t)\int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s$$

which arises in the context of maximal regularity for nonautonomous deterministic evolution equations is bounded on $L^p(0, T; X)$ for all $p \in (1, \infty)$. Here, we just require that $t \mapsto A(t)$ is measurable in time.

CHAPTER 1

Preliminaries

The purpose of this section is to provide a short overview over the basic tools and notations used in this thesis. For most of the proofs and further details, we give references to the literature.

Before we start, we fix some notation. Given normed spaces X and Y, $\mathcal{B}(X,Y)$ denotes the set of all linear and bounded operators from X to Y. We write $C(a,b;X)$ for the space of uniformly continuous functions on $[a,b]$ with values in X equipped with its usual norm. Given $U \subset \mathbb{R}^d$ and a measure μ on U, $L^q(U,\mu;X)$ is the space of strongly measurable $f : U \to X$ such that

$$\|f\|_{L^q(U,\mu;X)} := \Big(\int_U \|f(x)\|_X^q \,\mathrm{d}\mu(x) \Big)^{\frac{1}{q}} < \infty$$

with the obvious variation in the case $q = \infty$. The ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x,r) := \{y \in X : \ \|x-y\|_X < r\}$.

1.1. Vector calculus and related function spaces

In this section, we introduce the differential operators ∇, div and curl in a weak setting and we provide some trace theorems. Throughout this section, let $D \subset \mathbb{R}^d$ be a Lipschitz domain. In the context of curl, we always choose $d = 3$. We define the cross product $a \times b$ by

$$a \times b := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^T$$

for $a, b \in \mathbb{C}^3$ and for smooth functions $f : D \to \mathbb{C}$, $g : D \to \mathbb{C}^d$ and $h : D \to \mathbb{C}^3$, we set

$$\begin{aligned}
\operatorname{grad} f &:= \nabla f := (\partial_1 f, \dots, \partial_d f)^T, \\
\Delta f &:= \sum_{j=1}^{d} \partial_j^2 f, \\
\operatorname{div} g &:= \sum_{i=1}^{d} \partial_i g_i, \\
\operatorname{curl} h &:= (\partial_2 h_3 - \partial_3 h_2, \partial_3 h_1 - \partial_1 h_3, \partial_1 h_2 - \partial_2 h_1)^T.
\end{aligned}$$

Note that curl and Δ are formally symmetric operators, whereas the formal adjoint of grad is $-\operatorname{div}$, i.e. we have

$$\begin{aligned}
\int_D \nabla f_1(x) \cdot f_2(x) \,\mathrm{d}x &= -\int_D f_1(x) \operatorname{div} f_2(x) \,\mathrm{d}x, \\
\int_D \operatorname{curl} g_1(x) \cdot g_2(x) \,\mathrm{d}x &= \int_D g_1(x) \cdot \operatorname{curl} g_2(x) \,\mathrm{d}x, \\
\int_D \Delta h_1(x) h_2(x) \,\mathrm{d}x &= \int_D h_1(x) \Delta h_2(x) \,\mathrm{d}x
\end{aligned}$$

for all smooth f_j, g_j and h_j, $j = 1, 2$, that are compactly supported in D. This leads us to derivatives in a weak sense. For given $u \in L^1_{\mathrm{loc}}(D)$ with

$$\int_D u(x) \operatorname{div} \phi(x) \,\mathrm{d}x = -\int_D f(x) \cdot \phi(x) \,\mathrm{d}x$$

for some $f \in L^1_{\mathrm{loc}}(D)^d$ and for all $\phi \in C_c^\infty(D)^d$, we say that ∇u exists in the weak sense and $\nabla u := f$. For given $v \in L^1_{\mathrm{loc}}(D)^d$ with

$$\int_D v(x) \cdot \nabla \phi(x) \,\mathrm{d}x = -\int_D g(x) \phi(x) \,\mathrm{d}x$$

for some $g \in L^1_{\mathrm{loc}}(D)$ and for all $\phi \in C_c^\infty(D)$, we say that $\operatorname{div} v$ exists in the weak sense and $\operatorname{div} v := g$. Finally, for given $w \in L^1_{\mathrm{loc}}(D)^3$ with

$$\int_D w(x) \cdot \operatorname{curl} \phi(x) \,\mathrm{d}x = \int_D h(x) \cdot \phi(x) \,\mathrm{d}x$$

for some $h \in L^1_{\mathrm{loc}}(D)^3$ and for all $\phi \in C_c^\infty(D)^3$, we say that $\operatorname{curl} w$ exists in the weak sense and $\operatorname{curl} w := h$. One can show that the classical derivative and the weak derivative of smooth functions coincide and that this concept is a proper generalisation. As a next step, we define Sobolev spaces that are associated with grad, div and curl. For $q \in [1, \infty]$, we set

$$\begin{aligned}
W^{1,q}(D) &:= \big\{ u \in L^q(D) : \ \nabla u \text{ exists weakly with } \nabla u \in L^q(D)^d \big\}, \\
W^q(\operatorname{div})(D) &:= \big\{ v \in L^q(D)^d : \ \operatorname{div} v \text{ exists weakly with } \operatorname{div} v \in L^q(D) \big\}, \\
W^q(\operatorname{curl})(D) &:= \big\{ w \in L^q(D)^3 : \ \operatorname{curl} w \text{ exists weakly with } \operatorname{curl} w \in L^q(D)^3 \big\}.
\end{aligned}$$

These spaces are Banach spaces equipped with the usual graph norm. In the same way, we define higher order Sobolev spaces using higher order weak derivatives. However, since we just need Sobolev spaces of order 2 in this thesis, we solely introduce

$$W^{2,q}(D) := \big\{ u \in W^{1,q}(D) : \ \nabla(\partial_i u) \text{ exists weakly with } \nabla(\partial_i u) \in L^q(D)^d, i = 1, \dots, d \big\},$$

equipped with the norm

$$\|u\|_{W^{2,q}(D)} := \sum_{i,j=1}^{d} \|\partial_i \partial_j u\|_{L^q(D)} + \|\nabla u\|_{L^q(D)^d} + \|u\|_{L^q(D)}.$$

Throughout this thesis, we use several well-known identities from vector-calculus.

Lemma 1.1.1. *Let $d = 3$ and let $u : D \to \mathbb{C}$ and $v, w : D \to \mathbb{C}^3$ be smooth. The following identities hold true.*

a) $\operatorname{curl} \nabla u = 0$.

b) $\operatorname{div} \operatorname{curl} v = 0$.

c) $\operatorname{curl} \operatorname{curl} v = \nabla \operatorname{div} v - (\Delta v_1, \Delta v_2, \Delta v_3)^T$.

d) $\operatorname{div}(v \times w) = \operatorname{curl} w \cdot v - \operatorname{curl} v \cdot w$.

e) $\operatorname{div}(uv) = u \operatorname{div} v + v \cdot \nabla u$.

In particular, these identities can be used to define traces for functions in $W^p(\operatorname{div})(D)$ and $W^p(\operatorname{curl})(D)$. To motivate this procedure, we use Gauß Divergence Theorem to get

$$\begin{aligned}\int_{\partial D} \phi \cdot (f \times \nu) \, \mathrm{d}\sigma &= \int_{\partial D} \nu \cdot (\phi \times f) \, \mathrm{d}\sigma = \int_D \operatorname{div}(\phi \times f)(x) \, \mathrm{d}x \\ &= \int_D \operatorname{curl} f(x) \cdot \phi(x) \, \mathrm{d}x - \int_D f(x) \cdot \operatorname{curl} \phi(x) \, \mathrm{d}x.\end{aligned}$$

for $f, \phi \in C^1(\overline{D})^3$ and

$$\begin{aligned}\int_{\partial D} \psi(g \cdot \nu) \, \mathrm{d}\sigma &= \int_D \operatorname{div}(\psi g)(x) \, \mathrm{d}x \\ &= \int_D \nabla \psi(x) \cdot g(x) \, \mathrm{d}x + \int_D \psi(x) \operatorname{div} g(x) \, \mathrm{d}x.\end{aligned}$$

for $g \in C^1(\overline{D})^d$ and $\psi \in C^1(\overline{D})$. This leads to the following definition.

Definition 1.1.2. *For $g \in W^p(\operatorname{div})(D)$, we say $g \cdot \nu = 0$ on ∂D, if*

$$\int_D \nabla \psi(x) \cdot g(x) \, \mathrm{d}x + \int_D \psi(x) \operatorname{div} g(x) \, \mathrm{d}x = 0$$

for all $\psi \in C^1(\overline{D})$. Last but not least, for $d = 3$ and $f \in W^p(\operatorname{curl})(D)$, we say $f \times \nu = 0$ on ∂D, if

$$\int_D \operatorname{curl} f(x) \cdot \phi(x) \, \mathrm{d}x - \int_D f(x) \cdot \operatorname{curl} \phi(x) \, \mathrm{d}x = 0$$

for all $\phi \in C^1(\overline{D})$.

As a consequence, we can define the subspaces

$$\begin{aligned}W^p(\operatorname{div}, 0)(D) &:= \{u \in W^p(\operatorname{div})(D) : \; u \cdot \nu = 0 \text{ on } \partial D\} \\ W^p(\operatorname{curl}, 0)(D) &:= \{u \in W^p(\operatorname{curl})(D) : \; u \times \nu = 0 \text{ on } \partial D\}.\end{aligned}$$

For Lipschitz domains D with compact boundary, one can show that $W^p(\mathrm{div},0)(D)$ is the closure of $C_c^\infty(D)^3$ in $W^p(\mathrm{div})(D)$ and that $W^p(\mathrm{curl})(D)$ is the closure of $C_c^\infty(D)^3$ in $W^p(\mathrm{curl})(D)$. This can be found amongst others in [80], Theorem 2.21 and Theorem 2.23.

In Chapter 3, we also consider parabolic equations with mixed boundary conditions. Hence, it will be necessary not only to introduce the subspace of functions in $W^{1,p}(D)$ that vanish on the boundary, but also the subspace of functions that vanish on a part of the boundary.

Definition 1.1.3. *Let $\Gamma \subset \partial D$ be open in the topology of ∂D. For $q \in [1,\infty]$, we define $W_\Gamma^{1,q}(D)$ as the completion of*

$$C_\Gamma^\infty(D) := \big\{\phi|_D : \phi \in C_c^\infty(\mathbb{R}^d) \text{ and } \operatorname{supp}(\phi) \cap (\partial D \setminus \Gamma) = \emptyset\big\}$$

with respect to the norm $\|\phi\|_{W_\Gamma^{1,q}(D)} := \|\nabla\phi\|_{L^q(D)} + \|\phi\|_{L^q(D)}$.

Since every smooth function $f \in W_\Gamma^{1,q}(D)$ satisfies $f|_{\partial D\setminus\Gamma} = 0$, $\partial D \setminus \Gamma$ is understood as the Dirichlet part. In the special cases $\Gamma = \partial D$ and $\Gamma = \emptyset$, we write $W^{1,q}(D) := W_{\partial D}^{1,q}(D)$ and $W_0^{1,q}(D) := W_\emptyset^{1,q}(D)$. The first of these notations is justified, since $C^\infty(D) \cap L^q(D)$ is dense in $W^{1,q}(D)$ (see e.g. [2], Theorem 3.22). As we discuss operators of the form $u \mapsto \operatorname{div}(a\nabla u)$ with domain $W_\Gamma^{1,q}(D)$ we have to introduce the range of this operator, which is the space $W_\Gamma^{-1,q}(D)$. It is defined as the dual space of $W_\Gamma^{1,\frac{q}{q-1}}(D)$ with respect to the standard L^2-duality, which means that

$$\langle u,v\rangle_{\left(W_\Gamma^{-1,q}(D),W_\Gamma^{1,\frac{q}{q-1}}(D)\right)} = \int_D u(x)v(x)\,\mathrm{d}x$$

if $u \in W_\Gamma^{-1,q}(D) \cap L^q(D)$ and $v \in W_\Gamma^{1,\frac{q}{q-1}}(D)$.

Finally, we introduce Besov spaces and Triebel-Lizorkin spaces of positive order. Let $s > 0$ and $p,q \in [1,\infty]$. We start with the special case $D = \mathbb{R}^d$ since in this case, we are able to use the Fourier transform for a neat definition. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \geq 0$, $\operatorname{supp}(\phi) \subset \{\frac{1}{2} \leq |x| \leq 2\}$ and with $\sum_{j=-\infty}^\infty \phi(2^{-j}x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$. We set $\phi_j := \phi(2^{-j}\cdot)$ and $\phi_0 := 1 - \sum_{j=1}^\infty \phi_j$. Every ϕ_j can be associated with an operator $\phi_j(\partial)$ given by

$$\phi_j(\partial)f := \mathcal{F}^{-1}\big(\xi \mapsto \phi_j(\xi)(\mathcal{F}f)(\xi)\big).$$

Given $p,q \in [1,\infty]$, $s \geq 0$ and $f \in C_c^\infty(\mathbb{R}^d)$, we define

$$\|f\|_{B_{q,p}^s(\mathbb{R}^d)} := \|\phi_0(\partial)f\|_{L^q(\mathbb{R}^d)} + \Big(\sum_{j=1}^\infty \|2^{sj}\phi_j(\partial)f\|_{L^q(\mathbb{R}^d)}^p\Big)^{1/p}$$

$$\|f\|_{F_{q,p}^s(\mathbb{R}^d)} := \|\phi_0(\partial)f\|_{L^q(\mathbb{R}^d)} + \Big\|\Big(\sum_{j=1}^\infty |2^{sj}\phi_j(\partial)f|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}$$

with the usual modification if $p = \infty$. We now define $B_{q,p}^s(\mathbb{R}^d)$ and $F_{q,p}^s(\mathbb{R}^d)$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norms $\|\cdot\|_{B_{q,p}^s(\mathbb{R}^d)}$ and $\|\cdot\|_{F_{q,p}^s(\mathbb{R}^d)}$ respectively. In Chapter 3, we also need Besov spaces on a bounded domain D. We set

$$\|f\|_{B_{q,p}^s(D)} := \inf\big\{\|g\|_{B_{q,p}^s(\mathbb{R}^d)} : g \in B_{q,p}^s(\mathbb{R}^d) \text{ and } g = f \text{ a.e. in } D\big\}$$

and we define $B^s_{q,p}(D)$ as the completion of $C^\infty(\overline{D})$ with respect to this norm.

1.2. Basic stochastic theory

Throughout this thesis, let $(\Omega, \mathfrak{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions, i.e. $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and the filtration is right-continuous. We start with the definition of adaptivity for operator valued processes.

Definition 1.2.1. *Given a Banach space X and a Hilbert space H, a stochastic process $W : \Omega \times [0,T] \to \mathcal{B}(H;X)$ is called adapted, if the random variable $W(t)h : \Omega \to X$ is strongly $\mathcal{F}_t$-measurable for all $t \in [0,T]$ and all $h \in H$. If additionally $Wh : \Omega \times [0,t] \to X$ is for all $h \in H$ strongly $\mathcal{F}_t \otimes \mathrm{Borel}(0,t)$-measurable, W is called progressively measurable.*

It is easy to see that any progressively measurable process is adapted. However, if W has almost surely continuous paths, the converse also holds true. We want to remark that one often makes this definition with $H = \mathbb{C}$ and with the identification $\mathcal{B}(\mathbb{C}, E) = E$. We need these more general notions for the definition of the stochastic integral in Section 1.3. Next, we introduce the Brownian motion relative to the filtration $\mathbb{F}$.

Definition 1.2.2. *An $\mathbb{F}$-adapted process $\beta : \Omega \times [0,\infty) \to \mathbb{R}$ is called Brownian motion relative to $\mathbb{F}$, if the following conditions are satisfied.*

a) *$\beta(0) = 0$ almost surely.*

b) *For almost all $\omega \in \Omega$, the paths $t \mapsto \beta(\omega, t)$ are continuous.*

c) *For $0 \leq s < t$, the increment $\beta(t) - \beta(s)$ is a Gaussian random variable with mean 0 and variance $t - s$.*

d) *For $0 \leq s < t$, the increment $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s$.*

It will be necessary to stop a stochastic process when it leaves certain balls around the initial value. However, this time will differ from path to path and therefore we introduce stopping times. $\tau : \Omega \to [0,T]$ is called $\mathbb{F}$-stopping time, if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0,T]$. By the right-continuity, this is equivalent to $\{\tau < t\} \in \mathcal{F}_t$. The σ-algebra

$$\mathcal{F}_\tau = \big\{A \in \mathfrak{F} : A \cap \{t \leq \tau\} \in \mathcal{F}_t \ \forall t \in [0,T]\big\}$$

is called σ-algebra of τ-past and can be interpreted as the knowledge of an observer at the random moment τ. The following well-known result will be used frequently. The proof can be found e.g. in [60], Lemma 9.21 and Lemma 9.23.

Proposition 1.2.3. *$\mathcal{F}_\tau$ is a σ-algebra and satisfies the following properties.*

a) *If $\tau = t$ almost surely for some $t \in [0,T]$, we have $\mathcal{F}_\tau = \mathcal{F}_t$.*

b) *Given another $\mathbb{F}$-stopping time σ, we have $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$. In particular, if $\tau \leq \sigma$ almost surely, we have the inclusion $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$.*

c) If $(X(\cdot,t))_{t\in[0,T]}$ is a progressively measurable process with respect to $\mathbb{F}$, the random variable $X_\tau(\omega) := X(\omega,\tau(\omega))$ is $\mathcal{F}_\tau$-measurable.

Throughout this thesis, we will use the notation

$$\Lambda \times [\tau,\mu) := \big\{(\omega,t)\in\Lambda\times[0,T]:\ t\in[\tau(\omega),\mu(\omega))\big\}$$

for some $\Lambda\subset\Omega$ and some stopping times τ,μ with $\tau\le\mu$ almost surely. Closed and open random intervals are defined similarly. If we call a process u defined on $\Omega\times[\tau,\mu]$ adapted or progressively measurable, we mean that $u\mathbf{1}_{[\tau,\mu]}$ is adapted or progressively measurable as process on $[0,T]$.

In the next Lemma, we show that an exit time of a stochastic process is a stopping time. Results of these type are well-known, but we give a proof for convenience of the reader.

Lemma 1.2.4. *Let $X:\Omega\times[0,T]\to\mathbb{R}_{\ge0}$ be an $\mathbb{F}$-adapted process with almost surely continuous paths, σ an $\mathbb{F}$-stopping time with values in $[0,T]$ and $\lambda>0$. If we define*

$$\widetilde{\sigma} = \inf\big\{t\in[0,T-\sigma] : X(t+\sigma)>\lambda\big\}\wedge T,$$

then $\sigma+\widetilde{\sigma}$ is also an $\mathbb{F}$-stopping time.

Proof. Since $\mathbb{F}$ is right-continuous, it is sufficient to prove $\{\sigma+\widetilde{\sigma}<t\}\in\mathcal{F}_t$ for given $t\in[0,T]$. We start with

$$\big\{\sigma+\widetilde{\sigma}<t\big\} = \bigcup_{q_1,q_2\in\mathbb{Q}_{\ge0},q_1+q_2<t}\{\sigma<q_1,\widetilde{\sigma}<q_2\} \tag{1.2.1}$$

and prove that the sets $\{\sigma<q_1,\widetilde{\sigma}<q_2\}$ are contained in $\mathcal{F}_t$. For fixed $q_1,q_2\in\mathbb{Q}_{\ge0}$ with $q_1+q_2<t$, the definition of $\widetilde{\sigma}$ and the pathwise continuity of $t\mapsto X_t$ yield

$$\{\widetilde{\sigma}<q_2\} = \bigcup_{s\in[0,q_2)}\{X_{\sigma+s}>\lambda\} = \bigcup_{q\in[0,q_2)\cap\mathbb{Q}}\{X_{\sigma+q}>\lambda\}.$$

Thus, we have

$$\big\{\sigma<q_1,\widetilde{\sigma}<q_2\big\} = \bigcup_{q\in[0,q_2)\cap\mathbb{Q}}\Big(\{\sigma<q_1\}\cap\{X_{\sigma+q}>\lambda\}\Big).$$

Moreover, Proposition 1.2.3 implies $\{X_{\sigma+q}>\lambda\}\in\mathcal{F}_{\sigma+q}$ and since $\{\sigma<q_1\}\in\mathcal{F}_{q_1}$ in any case by definition of stopping times, we conclude

$$\big\{\sigma<q_1,\widetilde{\sigma}<q_2\big\}\in\bigcup_{q\in[0,q_2)\cap\mathbb{Q}}\big(\mathcal{F}_{q_1}\cap\mathcal{F}_{\tau+q}\big)\subset\mathcal{F}_{q_1+q_2}\cap\mathcal{F}_{\sigma+q_2}\subset\mathcal{F}_{\min(q_1+q_2,\sigma+q_2)}\subset\mathcal{F}_{q_1+q_2}.$$

Hence, the claimed result follows by (1.2.1). $\square$

In Chapter 2, we construct a local solution of a quasilinear stochastic differential equation up to an eventually small stopping time and we want to continue it to a solution on a maximal random interval. Therefore, we need to maximise the set

$$\big\{\tau:\Omega\to[0,T]\mid \tau \text{ is an } \mathbb{F}\text{-stopping time and there exists a solution on } [0,\tau]\big\}.$$

However, we cannot take the pointwise supremum for every fixed ω, since the supremum over uncountably many stopping times is not necessarily a stopping time any more. To overcome this difficulty, we introduce the essential supremum of a family of random variables.

Definition 1.2.5. *Let Λ be a family of real-valued random variables on Ω. Then,* $\operatorname{ess\,sup} \Lambda$ *is a random variable on Ω that satisfies the following properties.*

a) For all $X \in \Lambda$, we have $X \leq \operatorname{ess\,sup} \Lambda$ almost surely.

b) If Y is a random variable with $Y \geq X$ almost surely for all $X \in \Lambda$, then we also have $Y \geq \operatorname{ess\,sup} \Lambda$ almost surely.

From this definition it is apparent that two different essential supremums coincide up to a set of measure zero. However, it is not clear whether $\operatorname{ess\,sup} \Lambda$ exists. It turns out the following theorem is sufficient for our purpose.

Theorem 1.2.6. *Let Λ be a nonempty family of nonnegative and bounded random variables on Ω. In this case,* $\operatorname{ess\,sup} \Lambda$ *exists and if Λ is additionally closed under pairwise maximisation, there exists a sequence $(X_n)_n \subset \Lambda$ with $X_{n+1} \geq X_n$ almost surely and* $\lim_{n\to\infty} X_n = \operatorname{ess\,sup} \Lambda$ *almost surely.*

Proof. The proof can be found in [57], Theorem $A.3$. □

In particular, this theorem implies that the essential supremum of a set of $[0, T]$-valued stopping times that is closed under pairwise maximisation is again a stopping time.

1.3. γ-radonifying operators and stochastic integration in UMD Banach spaces

First, we introduce two important notions from the geometry of Banach spaces.

Definition 1.3.1. *Let Y be a Banach space and $(r_n)_n$ an independent sequence of Rademacher random variables. We make the following definition.*

a) Y has type $p \in [1, \infty)$, if there exists $C > 0$ such that

$$\Big(\mathbb{E}\|\sum_{j=1}^{N} r_j x_j\|_Y^p\Big)^{\frac{1}{p}} \leq C\Big(\sum_{j=1}^{N} \|x_n\|_Y^p\Big)^{\frac{1}{p}}$$

for all finite sequences $(x_j)_{j=1}^N$.

b) Y has the UMD property, if for all $p \in (1, \infty)$, there exists a constant $C > 0$ only depending on p and Y, such that the following holds. Whenever $(f_n)_{n=1}^N$ is a finite *martingale, then for all scalars* $|\varepsilon_n| = 1$, $n = 1, \ldots, N$, *we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n (f_n - f_{n-1}) \Big\|_Y^p \leq C \mathbb{E}\Big\| \sum_{n=1}^{N} (f_n - f_{n-1}) \Big\|_Y^p.$$

Note that Hilbert spaces or Banach spaces that are isomorphic to closed subspaces of $L^q(U;\mu)$, $q > 2$ are of type 2 and have the UMD property.

Next, we define the γ-spaces. Let $\widetilde{H}$ a separable Hilbert space with orthonormal basis $(h_n)_{n\in\mathbb{N}}$, Y a Banach space and $(\gamma_n)_{n\in\mathbb{N}}$ a sequence of independent standard-Gaussian distributed random variables. The Banach space $\gamma(\widetilde{H};Y)$ of γ-radonifying operators is the closure of

$$\{T : \widetilde{H} \to Y \text{ linear and of finite rank}\}$$

with respect to the norm

$$\|T\|_{\gamma(\widetilde{H};Y)} = \Big(\mathbb{E}\| \sum_{n=1}^{\infty} \gamma_n T h_n \|_Y^2\Big)^{1/2}$$

Note that the norm is independent of the choice of the orthonormal basis. In the special case $Y = L^q(O,\mu)$ for some $q \in (1,\infty)$, $\gamma(\widetilde{H};Y)$ is isomorphic to $L^q(U;\widetilde{H})$ via the isomorphism $L^q(U;\widetilde{H}) \ni f \mapsto T_f \in \gamma(\widetilde{H};Y)$, where T_f is defined by

$$T_f(h)(x) := \langle f(x), h\rangle_H$$

for $h \in \widetilde{H}$ and $x \in U$. The equivalence of $\|T_f\|_{\gamma(\widetilde{H};Y)} \simeq \|f\|_{L^q(O;\widetilde{H})}$ can be shown easily by the Kahane-Khintchine inequality

$$\Big(\mathbb{E}\| \sum_{n=1}^{\infty} \gamma_n f_n \|_Y^q\Big)^{1/q} \simeq_q \mathbb{E}\| \sum_{n=1}^{\infty} \gamma_n f_n \|_Y$$

for $q \in [1,\infty)$. Throughout this thesis, we shortly write $\gamma(a,b;Y) := \gamma(L^2(a,b);Y)$ and $\gamma([a,b]\times H;Y) := \gamma(L^2(a,b;H);Y)$. For further details about γ-radonifying operators, we refer to the survey paper of Van Neerven (see [94]).

Before we introduce to the stochastic integral, we need a slightly different version of the adaptivity, we defined in Definition 1.2.1 for stochastic processes.

Definition 1.3.2. *Let* $p \in [1,\infty)$. *A finite linear combination of processes* $G : \Omega \times [0,T] \times H \to Y$ *of the form*

$$G = \mathbf{1}_{(s,t]\times B} \langle \cdot, y\rangle_H x$$

with $B \in \mathcal{F}_s$, $y \in H$ *and* $x \in Y$ *is called elementary adapted. For given* $\mathbb{F}$*-stopping times* τ, μ *with* $0 \leq \tau \leq \mu \leq T$ *almost surely, a process* $G : \Omega \to \gamma([\tau,\mu]\times H;Y)$ *is called strongly adapted, if there exists a sequence of elementary adapted processes* $(G_n)_n$ *with* $G_n \mathbf{1}_{[\tau,\mu]} \to G\mathbf{1}_{[\tau,\mu]}$ *in probability in* $\gamma([\tau,\mu]\times H;Y)$ *for* $n \to \infty$. *Moreover, for* $p \in [1,\infty)$, *we set*

$$L^0_{\mathbb{F}}(\Omega;\gamma([\tau,\mu]\times H;Y)) := \big\{G : \Omega \to \gamma([\tau,\mu]\times H;Y) : G \text{ is strongly adapted} \big\},$$
$$L^p_{\mathbb{F}}(\Omega;\gamma([\tau,\mu]\times H;Y)) := L^0_{\mathbb{F}}(\Omega;\gamma([\tau,\mu]\times H;Y)) \cap L^p(\Omega;\gamma([\tau,\mu]\times H;Y)).$$

At first this looks like a completely new concept of adaptivity. However, by [99], Proposition 5.6 and the remark below this result, any adapted $X : \Omega \times [\tau, \mu] \to \gamma(H; X)$ in the sense of Definition 1.2.1 that additionally satisfies $X \in L^p(\Omega; \gamma([\tau, \mu] \times H; X))$ is strongly adapted. Next, we sketch the construction of the stochastic integral in a UMD Banach space Y. We introduce the stochastic integral in the Itô sense with respect to a cylindrical Brownian motion.

Definition 1.3.3. *Given a Hilbert space H, a cylindrical Brownian motion is a bounded linear operator $W : L^2(0, T; H) \to L^2(\Omega)$ with the following properties.*

a) For all $f \in L^2(0, T; H)$, the random variable $W(f)$ is centred Gaussian.

b) For all $t \in [0, T]$ and $f \in L^2(0, T; H)$ supported in $[0, t]$, $W(f)$ is $\mathcal{F}_t$-measurable.

c) For all $t \in [0, T]$ and $f \in L^2(0, T; H)$ supported in $(t, T]$, $W(f)$ is independent of $\mathcal{F}_t$.

d) We have $\mathbb{E}(W(f) \cdot W(g)) = \langle f, g\rangle_{L^2(0,T;H)}$ for all $f, g \in L^2(0, T; H)$.

An example of an $L^2(0, T; H)$-cylindrical Brownian motion is a family $(\beta_n)_{n\in\mathbb{N}}$ of independent real valued Brownian motions together with $H = l^2(\mathbb{N})$ and W uniquely determined by the formula $W(\mathbf{1}_{(0,t]}e_n) = \beta_n(t)$, $n \in \mathbb{N}$, where $(e_n)_n$ is the sequence of the standard unit vectors in $l^2(\mathbb{N})$.

For an elementary adapted processes $G : \Omega \times \mathbb{R}_{\geq 0} \times H \to Y$ of the form

$$G = \mathbf{1}_{(s,t]\times B}\langle \cdot, y\rangle_H x$$

with $B \in \mathcal{F}_s$, $y \in H$ and $x \in Y$, we can define the stochastic integral via

$$I(G) := \int_0^T G \,\mathrm{d}W := \mathbf{1}_B W(\mathbf{1}_{(s,t]}h)x \in X$$

and we can extend it to finite linear combinations of such processes. Van Nerven, Veraar and Weis proved in [100] the following two-sided estimate for this stochastic integral.

Theorem 1.3.4. *Let Y be a UMD Banach space and G be an elementary adapted processes in $\gamma(H; Y)$. Then, for all $p \in (1, \infty)$ one has the Itô-isomorphism*

$$\|I(G)\|_{L^p(\Omega;Y)} \simeq_p \|G\|_{L^p(\Omega;\gamma([0,T]\times H;Y))}.$$

In particular, the stochastic integral can be continued to a linear and bounded operator

$$I : L^p_{\mathbb{F}}(\Omega; \gamma([0, T] \times H; Y)) \to L^p(\Omega; Y).$$

In this thesis, we also deal with adapted integrands in $L^p(\Omega \times [0, T]; \gamma(H; Y))$ for $p > 2$. Here, we restrict ourselves to UMD Banach spaces of type 2 (details about type and cotype of Banach spaces can be found in [82]), for which the embeddings

$$L^p(0, T; \gamma(H; Y)) \hookrightarrow L^2(0, T; \gamma(H; Y)) \hookrightarrow \gamma([0, T] \times H; Y) \tag{1.3.1}$$

are bounded. Consequently, the stochastic integral $I(G)$ is also defined for adapted processes $G \in L^p(\Omega \times [0,T]; \gamma(H;Y))$. Also relevant is stochastic integration for adapted integrands in $L^r(\Omega; L^q(U; L^p(0,T; l^2(\mathbb{N}))))$ for $p, q > 2$, $r \in (1,\infty)$ and some measure space (U,μ). In this case, we can use

$$\begin{aligned} L^r(\Omega; L^q(U; L^p(0,T; l^2(\mathbb{N})))) \hookrightarrow & L^r(\Omega; L^q(U; L^2([0,T]; l^2(\mathbb{N})))) \\ = & L^r(\Omega; \gamma([0,T] \times l^2(\mathbb{N}); L^q(U))) \end{aligned}$$

to define the stochastic integral.

1.4. Sectorial operators and functional calculus

1.4.1. $\mathcal{R}$-boundedness and H^∞-calculus

Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of independent Rademacher random variables on a probability space $(\widetilde{\Omega}, \mathcal{A}, \widetilde{\mathbb{P}})$, i.e. $\widetilde{\mathbb{P}}(r_n = 1) = \widetilde{\mathbb{P}}(r_n = -1) = \frac{1}{2}$. Given the Banach spaces X and Y, a family $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$-bounded if there exists $C > 0$, such that

$$\mathbb{E}\| \sum_{j=1}^N r_j T_j x_j \|_Y^2 \le C\, \mathbb{E}\| \sum_{j=1}^N r_j x_j \|_X^2$$

for all $(T_j)_{j=1}^N \subset \mathcal{T}$ and $(x_j)_{j=1}^N \subset X$ with C independent of $N \in \mathbb{N}$. The least possible constant C will be called $\mathcal{R}$-bound of $\mathcal{T}$ or shortly $\mathcal{R}(\mathcal{T})$. Note that every R-bounded family is uniformly bounded in $\mathcal{B}(X,Y)$, whereas the converse holds only if X, Y are Hilbert spaces. For details, we refer to [21], [28] and [70].

An operator A with domain $D(A)$ is called sectorial of angle $\theta \in (0, \pi/2)$ on a Banach space Y, if it is closed, densely defined, injective and it has a dense range. Moreover, we require that its spectrum is contained in the sector $\Sigma_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}$ and that the set

$$\left\{\lambda R(\lambda, A) : \lambda \notin \Sigma_\phi\right\} \tag{1.4.1}$$

is for all $\phi \in (\theta, \pi)$ bounded in $\mathcal{B}(X)$ and the bound only depends on ϕ. In this case, $-A$ generates a holomorphic semigroup on E. If the set in (1.4.1) is even $\mathcal{R}$-bounded, one says that A is $\mathcal{R}$-sectorial.

For any holomorphic function f on Σ_ϕ, $\phi > \theta$, satisfying the growth estimate $|f(z)| \le C\frac{|z|^\delta}{1+|z|^{2\delta}}$ for some $\delta > 0$, the integral

$$f(A) = \frac{1}{2\pi i} \int_{\Sigma_\phi} f(z) R(z,A)\, \mathrm{dz}$$

exists. This integral defines a functional calculus for functions with the growth estimate from above. We say that A has a bounded $H^\infty(\Sigma_\phi)$-calculus, if there exists $C > 0$ such that

$$\|f(A)\|_{\mathcal{B}(E)} \le C\|f\|_\infty \tag{1.4.2}$$

is satisfied for all these functions. The least constant $C > 0$ will be called bound of the H^∞-calculus. In this case, the functional calculus $f \mapsto f(A)$ can be extended to any bounded holomorphic function on Σ_ϕ and (1.4.2) remains true. Moreover, if X is UMD, the boundedness of the H^∞-calculus of A particularly implies that A is $\mathcal{R}$-sectorial. Details on sectorial operators, $\mathcal{R}$-sectorial operators and the functional calculus can be found amongst others in [44] and [70]. A list of operators having a bounded H^∞-calculus can be found in [96], Example 3.2.

1.4.2. $\mathcal{R}_p$-boundedness and $\mathcal{R}_p$-bounded H^∞-calculus

Throughout this section, let $p \in (1, \infty)$, $U \subset \mathbb{R}^d$ and let μ be a σ-finite measure on U. Moreover, we shortly write $L^p(U) := L^p(U, \mu)$. In Chapter 3, we discuss quasilinear stochastic partial differential equations in spaces of the form $L^p(U; L^q(0,T))$. In his thesis, [8], Markus Antoni found out that to estimate the deterministic and the stochastic convolution in this space, one needs the notion of $\mathcal{R}_p$-boundedness and of an $\mathcal{R}_p$-bounded H^∞-calculus. In what follows, we just want to sketch these concepts. For more details, we refer to the mentioned thesis and to [65] and [69].

A family $\mathcal{T} \subset \mathcal{B}(L^q(U))$ is called $\mathcal{R}_p$-bounded if there exists $C > 0$ such that

$$\big\|\big(\sum_{j=1}^N |T_j x_j|^p\big)^{1/p}\big\|_{L^q(U)} \le C \big\|\big(\sum_{j=1}^N |x_j|^p\big)^{1/p}\big\|_{L^q(U)}$$

for all $(T_j)_{j=1}^N \subset \mathcal{T}$ and $(x_j)_{j=1}^N \subset L^q(U; l^p(\mathbb{N}))$, where C is independent of $N \in \mathbb{N}$. The least possible constant C will be called $\mathcal{R}_p$-bound of $\mathcal{T}$ or shortly $\mathcal{R}_p(\mathcal{T})$. The notion of $\mathcal{R}_2$-boundedness coincides with the notion of $\mathcal{R}$-boundedness we introduced in the previous section. However, it is important to note that a single operator is $\mathcal{R}$-bounded, but it is not necessarily $\mathcal{R}_p$-bounded (see e.g. [32], Chapter 8). Nevertheless, many famous operators from harmonic analysis, like the Riesz transform or the Hilbert transform are $\mathcal{R}_p$-bounded. For us, the most important application of $\mathcal{R}_p$-boundedness is the following result on pointwise multipliers in $L^q(U; L^p(0,T))$.

Proposition 1.4.1. *Let $S : [0,T] \to \mathcal{B}(L^q(U))$, such that $t \mapsto S(t)x$ is for all $x \in L^q(U)$ strongly measurable. Then, the set $\mathcal{T} = \{S(t) : t \in [0,T]\}$ is $\mathcal{R}_p$-bounded if and only if there exists $C > 0$ with*

$$\Big(\int_U \Big(\int_0^T \big|S(t)f(x,t)\big|^p \,\mathrm{d}t\Big)^{\frac{q}{p}} \mathrm{d}\mu(x)\Big)^{\frac{1}{q}} \le C\Big(\int_U \Big(\int_0^T \big|f(x,t)\big|^p \,\mathrm{d}t\Big)^{\frac{q}{p}} \mathrm{d}\mu(x)\Big)^{\frac{1}{q}} \tag{1.4.3}$$

for all $f \in L^q(U; L^p(0,T))$. In this case, the least possible C in (1.4.3) *is given by $\mathcal{R}_p(\mathcal{T})$.*

Proof. The proof can be found in [65], Proposition 2.12. □

From [56], Theorem 5.3 and Corollary 5.4, we know that if an operator A on $L^q(U)$ has a bounded $H^\infty(\Sigma_\theta)$-calculus, then for each $\theta' > \theta$ the set

$$\big\{f(A) : \|f\|_{H^\infty(\Sigma_{\theta'})} \le 1\big\}$$

is $\mathcal{R}$-bounded. However, we cannot replace $\mathcal{R}$- by $\mathcal{R}_p$-boundedness. Here, we need a new concept. We say that a sectorial operator A on $L^q(U)$ has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0, \frac{\pi}{2})$, if the set

$$\left\{f(A) : \|f\|_{H^\infty(\Sigma_\theta)} \leq 1\right\}$$

is $\mathcal{R}_p$-bounded.

Proposition 1.4.1 implies that an $\mathcal{R}_p$-bounded operator S on $L^q(U)$ can be extended to a bounded operator $L^q(U; L^p(0,T))$. Moreover, we can extend a closed operator $A : D(A) \to L^q(U)$ to a closed operator A on $L^q(U; L^p(0,T))$. At this point we made an abuse of notation, since we should distinguish between A and its extension. In detail, this extension procedure is discussed in [8], section 2.4. We just want to point out that the extension of a closed and densely defined operator is also closed and densely defined. Moreover, if A has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\theta)$-calculus, its extension to $L^q(U; L^p(0,T))$ has a bounded $H^\infty(\Sigma_\theta)$-calculus (see [8], Theorem 2.4.5).

Finally, we define the so called generalised Triebel-Lizorkin spaces associated to an operator A on $L^q(U)$. Here, we do not discuss these spaces in full generality, but we restrict us to the cases we need. Hence, we just discuss $p, q \in (1,\infty)$ and we assume that A has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta \in (0, \pi/2)$. The generalised Triebel-Lizorkin spaces were first introduced in [65] and their connection to parabolic stochastic partial differential equations in $L^q(U)$ was detected by [8].

Definition 1.4.2. *Let A be a closed and densely defined operator on $L^q(U)$ with $0 \in \rho(A)$ that has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\theta)$-calculus and let $\alpha \geq 0$, $p, q \in (1,\infty)$. We set*

$$\|f\|_{F^\alpha_{A,q,p}} := \Big(\int_U \Big(\int_0^\infty \left|t^{1-\alpha} A e^{-tA} f\right|^p \frac{dt}{t} \Big)^{\frac{q}{p}} \, \mathrm{d}\mu \Big)^{1/q}$$

and we define the generalised Triebel-Lizorkin space $F^\alpha_{A,q,p}$ by

$$F^\alpha_{A,q,p} := \left\{ f \in L^q(U) : \ \|f\|_{F^\alpha_{A,q,p}} < \infty \right\}.$$

The name can be explained if we choose $A = -\Delta$ on $L^q(\mathbb{R}^d)$. Then, the spaces $F^\alpha_{A,q,p}$ coincide with the classical Triebel-Lizorkin spaces $F^{2\alpha}_{q,p}(\mathbb{R}^d)$ defined in section 1.1 (see e.g. [92], Theorem 3). In section 2.5 in [8], Antoni characterised $F^\alpha_{A,q,p}$ as an interpolation space between $L^q(U)$ and $D(A)$ with a new interpolation method he called l^q-interpolation. We don't go into detail here, but we want to highlight a nice characterisation of $F^{1-1/p}_{q,p,A}$ as a trace space.

Proposition 1.4.3. *Let A be a closed and densely defined operator on $L^q(U)$ with $0 \in \rho(A)$ that has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\theta)$-calculus and let $p, q \in (1,\infty)$. Then*

$$\begin{aligned} \|x\|_{\mathrm{TR}} = \inf \Big\{ \|w'\|_{L^q(U;L^p(0,T))} + \|Aw\|_{L^q(U;L^p(0,T))} \ \Big| \ & w(0) = x, \ w \in L^q(U; W^{1,p}(0,T)) \\ & \text{and } Aw \in L^q(U; L^p(0,T)) \Big\} \end{aligned}$$

defines an equivalent norm on $F^{1-1/p}_{q,p,A}$*. In particular, given* $w \in L^q(U; W^{1,p}(0,T))$*, such that we also have* $Aw \in L^q(U; L^p(0,T))$*, we are able to evaluate* w *at time* $t \in [0,T]$ *and we have* $w(t) \in F^{1-1/p}_{q,p,A}$*.*

Proof. The proof is a combination of [8], Proposition 2.5.3 and Theorem 2.5.4. □

Unfortunately, solutions of stochastic evolution equations are not differentiable in time. However, we still want to evaluate them at a fixed time t. This will be guaranteed by the following embedding result.

Lemma 1.4.4. *Let* A *be a closed and densely defined operator on* $L^q(U)$ *with* $0 \in \rho(A)$ *that has an* $\mathcal{R}_p$*-bounded* $H^\infty(\Sigma_\theta)$*-calculus. Moreover, let* $p, q \in (1,\infty)$ *and* $\alpha \in (\frac{1}{p}, 1 + \frac{1}{p})$*. Then the embedding*

$$L^q(U; W^{\alpha,q}(0,T)) \cap \big\{w : \ Aw \in L^q(U; L^p(0,T))\big\} \hookrightarrow C(0,T; F^{\alpha-1/p}_{q,p,A})$$

is continuous.

Proof. The proof is a combination of Theorem 2.5.4 and Theorem 2.5.9 in [8]. □

1.4.3. Functional calculus via L^{p_0}-L^{p_1}-off-diagonal estimates

Many elliptic operators in both divergence and nondivergence form on $L^p(U) := L^p(U,\mu)$ have an $\mathcal{R}_p$-bounded H^∞-calculus. In a pioneering work Kunstmann and Ullmann established this property by showing L^{p_0}-L^{p_1}-off-diagonal estimates for the semigroup generated by these operators (see [69]). For this result, we additionally need that (U,d) is a metric space of homogeneous type, i.e. there exists $C > 1$ and $D > 0$, such that

$$\mu\big(B(x,\lambda r)\big) \le C\lambda^D \mu\big(B(x,r)\big)$$

for all $x \in U$ and $\lambda, r > 0$. Moreover, we define the annuli

$$A_k(x,r) := B(x,(k+1)r) \setminus B(x,kr)$$

for $k \in \mathbb{N}$. Then, their result reads as the follows.

Theorem 1.4.5. *Let* $1 \le p_0 < 2 < p_1 \le \infty$ *and* $\omega_0 \in (0, \frac{\pi}{2})$*. Let* A *be a closed and densely defined operator on* $L^2(U)$*, such that* A *has a bounded* $H^\infty(\Sigma_{\omega_0})$*-calculus. Moreover, we assume that the analytic semigroup generated by* $-A$ *satisfies for all* $\theta \in (\omega_0, \pi)$ *the off-diagonal estimates*

$$\begin{aligned}
\big\|\mathbf{1}_{A_k(x,|\lambda|^{\frac12})} e^{-\lambda A} \mathbf{1}_{B(x,|\lambda|^{\frac12})}\big\|_{\mathcal{B}(L^{p_0}(U),L^{p_1}(U))} &\le C_\theta \mu\big(B(x,|\lambda|^{\frac12})\big)^{\frac{1}{p_1}-\frac{1}{p_0}} (1+k)^{-\kappa_\theta} \\
\big\|\mathbf{1}_{B(x,|\lambda|^{\frac12})} e^{-\lambda A} \mathbf{1}_{A_k(x,|\lambda|^{\frac12})}\big\|_{\mathcal{B}(L^{p_0}(U),L^{p_1}(U))} &\le C_\theta \mu\big(B(x,|\lambda|^{\frac12})\big)^{\frac{1}{p_1}-\frac{1}{p_0}} (1+k)^{-\kappa_\theta}
\end{aligned}$$

for some $C_\theta > 0$*,* $\kappa_\theta > \max\{\frac{1}{p_0} + d(1-\frac{1}{p_1}), 1 - \frac{1}{p_1} + \frac{d}{p_0}\}$ *and for all* $x \in U$*,* $k \in \mathbb{N}_0$ *and* $\lambda \in \Sigma_{\frac{\pi}{2}-\theta}$*. Then, for all* $p, q \in (p_0, p_1)$ *and* $\alpha \in (\omega_0, \pi)$*, the operator* A *has an* $\mathcal{R}_p$*-bounded*

$H^\infty(\Sigma_\alpha)$-calculus on $L^q(U)$ with bound depending on $C_\theta, \kappa_\theta, p, q, \omega_0$ and on the bound of the H^∞-calculus on $L^2(U)$.

Proof. This statement can be found in [69], Theorem 2.3. The explicit dependence of the constants is not mentioned in this theorem. However, the main tool for the proof of this result is Proposition 2.5 in the same article and in this result, the dependency of the constants is discussed. Hence, we get the claimed dependency by closely looking at the proof of Theorem 2.3. □

A detailed list of elliptic operators satisfying these L^{p_0}-L^{p_1}-off-diagonal bounds can be found in [69], Section 3 and in [8], Section 2.3, Example A and B.

So far, we discussed operators A having several versions of a holomorphic functional calculus. However, for some operators on $L^q(U)$, we can extend the functional calculus and define $f(A) : L^q(U) \to L^q(U)$ for smooth $f : [0,\infty) \to \mathbb{R}$. For $A = -\Delta$ on $L^q(\mathbb{R}^d)$ such a spectral calculus was developed by Hörmander in [47]. Over the years, there were many generalisations of this result. The most recent versions of such a spectral calculus are due to Kunstmann and Uhl (see [68],[67]) and to Kriegler and Weis (see [61], [62]). We present an application of their results that is sufficient for our purpose. We restrict us to $U \subset \mathbb{R}^d$ equipped with the Lebesgue measure and with the Euclidean metric.

Proposition 1.4.6. *Assume that A is a nonnegative self-adjoint operator on $L^2(U)$ and assume that there exist constants $c, C > 0$ such that*

$$\left\|\mathbf{1}_{B(x,t^{\frac{1}{2}})} e^{-tA} \mathbf{1}_{B(y,t^{\frac{1}{2}})}\right\|_{\mathcal{B}(L^{p_0}(U), L^{\frac{p_0}{p_0-1}}(U))} \leq C t^{\frac{d}{2}(1-\frac{2}{p_0})} exp\big(-c\tfrac{|x-y|^2}{t^{\frac{1}{2}}}\big)$$

for all $x, y \in U$ and all $t > 0$. Then, given $f \in C_c^\infty([0,2))$ with $0 \leq f \leq 1$ and $f \equiv 1$ on $[0,1]$, the operators $S_n = f(2^{-n}A)$ are bounded on $L^p(U)$ for $p_0 < p < \frac{p_0}{p_0-1}$ and $\sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{B}(L^p(U))} < \infty$.

Proof. We apply Theorem 2.3 in [68]. We choose $\omega \in C_c^\infty(\frac{1}{4}, 1)$ such that $0 \leq \omega \leq 1$ and $\sum_{k\in\mathbb{Z}} \omega(2^{-l}x) = 1$ for all $x \in \mathbb{R} \setminus \{0\}$. It is sufficient to show that

$$K := \sup_{n\in\mathbb{N}} \sup_{l\in\mathbb{Z}} \|x \mapsto \omega(x) f(2^{l-n}x)\|_{C^s(\mathbb{R})} = \sup_{l\in\mathbb{Z}} \|x \mapsto \omega(x) f(2^l x)\|_{C^s(\mathbb{R})} < \infty$$

for $s \in \mathbb{N}$, $s \geq \frac{d}{2}$. Then, the quoted theorem implies

$$\sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{B}(L^p(U))} \leq C_p(K+1)$$

for all $p \in (p_0, \frac{p_0}{p_0-1})$. Since f and ω are bounded by 1, $\|x \mapsto \omega(x) f(2^l x)\|_{C(\mathbb{R})} \leq 1$. The derivative is given by $\omega' f(2^l\cdot) + 2^l \omega f'(2^l\cdot)$. However, due to the assumptions on the support of f and ω, this term is only nontrivial on

$$(\tfrac{1}{4}, 1) \cap [0, 2^{-l+1}) = \begin{cases} (\frac{1}{4}, \frac{1}{2}), & \text{for } l = 2, \\ (\frac{1}{4}, 1), & \text{for } l \leq 1. \end{cases}$$

This yields

$$\|x \mapsto \omega(x)f(2^l x)\|_{C^1(\mathbb{R})} \leq 1 + \|\omega'\|_\infty \|f\|_\infty + 4\|\omega\|_\infty \|f'\|_\infty,$$

which is an estimate independent of $l \in \mathbb{Z}$. Higher order derivatives can be estimated in the same way. This yields the claimed result. $\square$

CHAPTER 2

Parabolic stochastic evolution equations via maximal regularity

In this chapter, we provide a unified approach to the well-posedness of semilinear and quasilinear parabolic evolution equations. We develop a framework with maximal regularity estimates for the deterministic and the stochastic convolution as input and a theory about quasilinear equations as output. The maximal regularity estimates are based on the work of Van Neerven, Veraar and Weis in [96] and [98] and Antoni in [8]. Our unified approach to semilinear equations is orientated to Van Neerven's, Veraar's and Weis' approach in [96] and [98] and contains beside the different presentation few generalisations, whereas the theory about local well-posedness of quasilinear equations is completely new.

As before, let $(\Omega, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and W be a cylindrical Brownian motion in a Hilbert space H. Throughout this section, let τ be a stopping time with respect to $\mathbb{F}$ and $\mathcal{F}_\tau$ the corresponding σ-Algebra of τ-past.

Before we start, we shortly sketch our approach. Given a family of semigroups with random dependency $(e^{-tA(\omega)})_{\omega\in\Omega, t\geq 0}$ and with generators $(-A(\omega))_{\omega\in\Omega}$ all having the same domain E^1, such that $\omega \mapsto A(\omega)x$ is for all $x \in E^1$ strongly $\mathcal{F}_\tau$-measurable, the mild solution to the linear equation

$$\begin{cases} du(\omega,t) &= \big[-A(\omega)u(\omega,t) + f(\omega,t)\big]dt + b(\omega,t)\,\mathrm{d}W(t), \quad t \in [\tau(\omega), T], \\ u(\omega,\tau(\omega)) &= u_\tau(\omega) \end{cases}$$

is formally given by

$$u(\omega,t) = e^{-(t-\tau(\omega))A(\omega)}u_\tau(\omega) + \int_{\tau(\omega)}^t e^{-(t-s)A(\omega)}f(\omega,s)\,\mathrm{d}s + \int_{\tau(\omega)}^t e^{-(t-s)A(\omega)}b(\omega,s)\,\mathrm{d}W(s)$$

as long as $t \geq \tau(\omega)$. Therefore, to study the regularity properties of this mild solution, one has to derive regularity properties for both the deterministic convolution

$$(e^{-(\cdot)A} * f)_\tau(\omega,t) := \int_{\tau(\omega)}^t e^{-(t-s)A(\omega)}f(\omega,s)\,\mathrm{d}s := \int_0^t e^{-(t-s)A(\omega)}f(\omega,s)\mathbf{1}_{s>\tau(\omega)}\,\mathrm{d}s \quad (2.0.1)$$

and the stochastic convolution

$$(e^{-(\cdot)A} \diamond b)_\tau(\omega, t) := \int_\tau^t e^{-(t-s)A} b(\cdot, s) \,\mathrm{d}W(s)(\omega) := \int_0^t e^{-(t-s)A} b(\cdot, s) \mathbf{1}_{s>\tau} \,\mathrm{d}W(s)(\omega) \tag{2.0.2}$$

for $t \geq \tau$. These operators are at first only defined for simple functions f on $\Omega \times [\tau, \infty)$ and for adapted simple functions b on $\Omega \times [\tau, \infty) \times H$. However, we can extend them by density to operators on larger spaces. This will be the content of section 2.1. In section 2.2, we use these results to investigate the well-posedness of the semilinear equation

$$\text{(SEE)} \begin{cases} du(\omega, t) &= \big[-A(\omega, t)u(\omega, t) + F(u)(\omega, t)\big]dt + B(u)(\omega, t) \,\mathrm{d}W(t), \;\; t \in (\tau(\omega), T] \\ u(\omega, \tau(\omega)) &= u_\tau(\omega), \end{cases}$$

that starts at the random time τ with strongly $\mathcal{F}_\tau$-measurable initial data u_τ. Here, we allow F and B to be memory terms that have the Volterra property, which means that given a stopping time $\widetilde{\tau}$ with $0 \leq \tau \leq \widetilde{\tau} \leq T$ almost surely the functions $F(u)\mathbf{1}_{[\tau,\widetilde{\tau}]}$ and $B(u)\mathbf{1}_{[\tau,\widetilde{\tau}]}$ only depend on $u\mathbf{1}_{[\tau,\widetilde{\tau}]}$. We solve this equation with an iterative application of the contraction mapping theorem on $[(\tau + n\kappa) \wedge T, (\tau + (n+1)\kappa) \wedge T]$ for a small enough $\kappa > 0$. Quasilinear stochastic equations of the form

$$\text{(QSEE)} \begin{cases} du(\omega, t) &= \big[-A(u(\omega, t))u(\omega, t) + F(u)(\omega, t)\big]dt + B(u)(\omega, t) \,\mathrm{d}W(t), \quad t > 0, \\ u(\omega, 0) &= u_0(\omega), \end{cases}$$

will be discussed in section 2.3. We reduce (QSEE) on small random intervals $[\tau_1, \tau_2]$ to an equation of the form (SEE) with a nonlinear memory term and solve this with the theory we derived before. Then, we put the solutions on all the random intervals together to a solution up to a maximal stopping time that is characterised by a blow-up alternative.

2.1. Maximal regularity for the deterministic and the stochastic convolution

In this section, we will discuss maximal regularity estimates in three different spaces. We will consider the spaces $L^p(\Omega \times [0, T]; E)$ and $L^r(\Omega; \gamma(0, T; E))$ for a Banach space E and $L^r(\Omega; L^q(U; L^p(0, T)))$. Each of these maximal regularity concepts has its own advantages. For the maximal γ-regularity, we can assume E to be UMD, whereas we additionally need that E is of type 2 in the first setting. For maximal regularity in $L^r(\Omega; L^q(U; L^p(0, T)))$, the assumptions on the operator A are most restrictive, since we have to assume A to have an $\mathcal{R}_p$-bounded H^∞-calculus and not only an ordinary H^∞-calculus as in the other cases. However, especially for $p > q \geq 2$ this approach yields stronger results, since the space $L^q(U; L^p(0, T))$ is smaller than $L^p(0, T; L^q(U))$ and $\gamma(0, T; L^q(U)) = L^q(U; L^2(0, T))$.

2.1.1. Maximal L^p-regularity of the deterministic and the stochastic convolution in Banach spaces of type 2

In this section, we discuss the regularity of the stochastic and the deterministic convolution based on the work of Van Neerven, Veraar and Weis in [96]. Additionally, we prove lower order estimates that will be needed in what follows. Although, they are standard, we give the details for convenience. This setting will be called [TT]. We make the following assumptions.

[TT1] Let $p \in (2,\infty)$ and let E, E^1 be UMD Banach spaces of type 2 and with a dense embedding $E^1 \hookrightarrow E$. Moreover, let the family

$$\{J_\delta : \delta > 0\} \subset B\big(L^p(\Omega \times (0,\infty); \gamma(H;E)), L^p(\Omega \times (0,\infty); E)\big)$$

defined by

$$J_\delta b(t) := \delta^{-1/2} \int_{(t-\delta)\vee 0}^{t} b(s)\,\mathrm{d}W(s)$$

be R-bounded.

[TT2] Let $A : \Omega \to \mathcal{B}(E^1, E)$ be such that $\omega \mapsto A(\omega)x$ is for all $x \in E^1$ strongly $\mathcal{F}_\tau$-measurable and such that $0 \in \rho(A(\omega))$ for almost all $\omega \in \Omega$. Moreover, we assume that $A(\omega)$ is for almost all $\omega \in \Omega$ closed with $D(A(\omega)) = E^1$, i.e there exists $M > 0$, such that we have

$$M^{-1}\|x\|_{E^1} \le \|A(\omega)x\|_E \le M\|x\|_{E^1}$$

for almost all $\omega \in \Omega$ and all $x \in E^1$. Further, $A(\omega)$ has for almost all $\omega \in \Omega$ a bounded H^∞-calculus of angle $\eta \in [0, \pi/2)$ with

$$\|\Psi(A(\omega))\|_{\mathcal{B}(E)} \le M\|\Psi\|_{H^\infty(\Sigma_\eta)}$$

for all $\Psi \in H^\infty(\Sigma_\eta)$. Here, all the occurring constants are independent of $\omega \in \Omega$.

Note that the requirement that not only E but also E^1 is a UMD space of type 2 is not restrictive, since by [TT2] they are isomorphic and both the UMD property and the type of E are stable under isomorphisms. In particular, the interpolation spaces $[E, E^1]_{\frac{1}{2}}$ and $(E, E^1)_{1-1/p,p}$ also inherit the UMD property from E.

We start with maximal regularity estimates of the deterministic convolution $\big(e^{-(\cdot)A} * f\big)_\tau$. So far, we can exclude the dependence on ω and argue pathwise. The following purely deterministic theorem is sufficient for our purpose.

Theorem 2.1.1. *Let $\tau \ge 0$. The on $L^p(\tau,\infty;E)$ well-defined deterministic convolution*

$$(e^{-(\cdot)A} * f)_\tau(t) = \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s$$

satisfies $(e^{-(\cdot)A} * f)_\tau \in L^p(\tau,\infty;E^1)$ and

$$\|(e^{-(\cdot)A} * f)_\tau\|_{L^p(\tau,\infty;E^1)} + \|(e^{-(\cdot)A} * f)_\tau\|_{C(\tau,\infty;(E,E^1)_{1-1/p,p})} \le C_{\mathrm{MRD}}\|f\|_{L^p(\tau,\infty;E)}.$$

Here, $C_{\mathrm{MRD}} > 0$ depends on p, E, η and M.

Proof. The proof can be found in [102], Theorem 3.4. Note that our assumption on the $H^\infty(\Sigma_\eta)$-calculus implies the $\mathcal{R}$-sectoriality of A required in this theorem. □

In the sequel, it will be very helpful to know that the constant in a lower order estimate of the deterministic convolution becomes smaller, if one reduces the size of the considered interval.

Proposition 2.1.2. *Let $\tau \geq 0$ and $\kappa > 0$. We have*

$$\|(e^{-(\cdot)A} * f)_\tau\|_{L^p(\tau,\tau+\kappa;E)} \leq C\kappa\|f\|_{L^p(\tau,\tau+\kappa;E)}$$

for all $f \in L^p(\tau, \tau+\kappa; E)$ with a constant $C > 0$ only depending on $\sup_{t\in[\tau,\tau+\kappa]} \|e^{-tA}\|_{\mathcal{B}(E)}$.

Proof. Using Hölder's inequality and the boundedness of $e^{-(\cdot)A}$, we estimate

$$\Big\| \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s \Big\|_E \lesssim \kappa^{1-1/p}\|f\|_{L^p(\tau,\tau+\kappa;E)}.$$

Taking the L^p-norm implies the claimed result. □

Before we turn to the stochastic convolution, we want to mention a well-known trace estimate for $[\tau,\infty) \ni t \mapsto e^{-(t-\tau)A}u_\tau$.

Proposition 2.1.3. *Let $\tau \geq 0$. Then, there exists a constant $C > 0$, such that*

$$\|t \mapsto e^{-(t-\tau)A}u_\tau\|_{L^p(\tau,\infty;E^1)} + \|t \mapsto e^{-(t-\tau)A}u_\tau\|_{C(\tau,\infty;(E,E^1)_{1-1/p,p})} \leq C\|u_\tau\|_{(E,E^1)_{1-1/p,p}}$$

for all $u_\tau \in (E, E^1)_{1-1/p,p}$.

Proof. The estimate

$$\begin{aligned}\|t \mapsto Ae^{-(t-\tau)A}u_0\|_{L^p(\tau,\infty;E)} + \|t \mapsto e^{-(t-\tau)A}u_0\|_{C(\tau,\infty;(E,D(A))_{1-1/p,p})}&\\ \leq C\|u_0\|_{(E,D(A))_{1-1/p,p}}&\end{aligned}$$

is well-known and can be found e.g. in [75]. The claimed result then follows from the equivalence of the norms $\|A\cdot\|_E$ and $\|\cdot\|_{E^1}$. □

Next, we provide estimates for the stochastic convolution. Thus, from now on, let τ be an $\mathbb{F}$-stopping time. At first, we have to make sure that the stochastic integral in (2.0.2) is well-defined, because it is not immediately clear that the integrand

$$(\omega, s) \mapsto e^{-(t-s)A(\omega)}b(\omega, s)\mathbf{1}_{\tau(\omega)<s\leq t}$$

is adapted to $\mathbb{F}$.

Lemma 2.1.4. *Let $b : \Omega\times[\tau,\infty) \to \gamma(H;E)$ be $\mathbb{F}$-adapted with $b \in L^p(\tau,T;\gamma(H;E))$ almost surely. Then, the random variable*

$$\omega \mapsto e^{-(t-s)A(\omega)}b(\omega,s)\mathbf{1}_{t\geq s>\tau(\omega)}$$

is for all $0 \leq s \leq t$ strongly $\mathcal{F}_s-$measurable.

Proof. Since $\omega \mapsto A(\omega)x$ is for all $x \in E^1$ strongly $\mathcal{F}_\tau$-measurable, $\omega \mapsto R(\lambda, A(\omega))x$ is for all $x \in E$ strongly $\mathcal{F}_\tau$-measurable. Since the identity

$$e^{-tA(\omega)}x = \lim_{n\to\infty} \left(\tfrac{n}{t}R(\tfrac{n}{t}, A(\omega))\right)^n x$$

holds true for all $x \in E$ and $\omega \to e^{-tA(\omega)}x$ is also for all $x \in E$ and $t \geq 0$ strongly $\mathcal{F}_\tau$-measurable as pointwise limit of strongly $\mathcal{F}_\tau$-measurable functions.

Now, we prove that for fixed $s \leq t$ the map $\omega \mapsto e^{-(t-s)A(\omega)}x\mathbf{1}_{s>\tau(\omega)}$ is strongly $\mathcal{F}_s$-measurable. Indeed, for every Borel set $B \subset E$ and $x \in E$ we have

$$\{e^{-(t-s)A}x\mathbf{1}_{s>\tau} \in B\} = \{0 \in B, s \leq \tau\} \cup \{e^{-(t-s)A}x \in B, s > \tau\}.$$

Since the filtration $\mathbb{F}$ is right-continuous, we have both $\{s \leq \tau\} \in \mathcal{F}_s$ and $\{s > \tau\} \in \mathcal{F}_s$. Thus, we obtain $\{0 \in B, s \leq \tau\} \in \mathcal{F}_s$ and hence, Proposition 1.2.3 yields

$$\{e^{-(t-s)A}x \in B, s > \tau\} \in \mathcal{F}_\tau \cap \mathcal{F}_s = \mathcal{F}_{\tau\wedge s} \subset \mathcal{F}_s.$$

Last but not least, we conclude that

$$\omega \mapsto e^{-(t-s)A(\omega)}b(\omega,s)\mathbf{1}_{s>\tau(\omega)}$$

is strongly $\mathcal{F}_s$-measurable in the sense of Definition 1.2.1 as composition of strongly $\mathcal{F}_s$-measurable functions. □

Now we are in the position to state the maximal L^p-regularity result and the maximal inequality for the stochastic integral.

Theorem 2.1.5. *The stochastic convolution*

$$(e^{-(\cdot)A} \diamond b)_\tau(t) = \int_0^t e^{-(t-s)A}b(\cdot,s)\mathbf{1}_{(\tau,\infty)}(s)\,\mathrm{d}W(s)$$

is well-defined for all adapted $b \in L^p_{\mathbb{F}}(\Omega\times[\tau,\infty),\gamma(H;E))$ and we have

$$\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^p(\Omega\times(\tau,\infty);E^1)} + \big(\mathbb{E}\sup_{t\in[0,\infty)} \|(e^{-(\cdot)A} \diamond b)_\tau\|^p_{(E,E^1)_{1-1/p,p}}\big)^{1/p}$$
$$\leq C_{\mathrm{MRS}}\|b\|_{L^p(\Omega\times(\tau,\infty),\gamma(H;[E,E^1]_{\frac{1}{2}}))}$$

for all $b \in L^p_{\mathbb{F}}(\Omega\times[\tau,\infty),\gamma(H;[E,E^1]_{\frac{1}{2}}))$. Here, the constant $C_{\mathrm{MRS}} > 0$ only depends on E,p,η and M.

Proof. By Lemma 2.1.4, the stochastic convolution is well-defined. The proof follows the line of the proof of Theorem 1.1 in [97]. There, the result is only shown for $E = L^q(\mu)$, but it extends to the general situation under the additional assumption that $(J_\delta)_{\delta>0}$ is R-bounded. This extension is discussed in [96], Proposition 3.5. □

As for the deterministic convolution, we want to derive a lower order estimate that improves if one lessens the size of the considered interval.

Proposition 2.1.6. *Let $\kappa > 0$ and $T > 0$. Then, we have*

$$\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^p(\Omega\times[\tau,(\tau+\kappa)\wedge T];E)} \leq C\kappa^{1/2}\|b\|_{L^p(\Omega\times[\tau,(\tau+\kappa)\wedge T],\gamma(H;E))}$$

for all $b \in L^p(\Omega \times [\tau, (\tau+\kappa) \wedge T], \gamma(H;E))$ with a constant $C > 0$ only depending on E and the bound of $e^{-(\cdot)A}$.

Proof. Since E has type 2, the space $L^2(0,t;\gamma(H;E))$ embeds into $\gamma([0,t]\times H;E)$ (see e.g. [99], page 11). Thus, the Itô isomorphism (see Theorem 1.3.4) yields

$$\begin{aligned}\|(e^{-(\cdot)A} \diamond b)_\tau(t)\|_{L^p(\Omega,E)} &\simeq \|s \mapsto e^{-(t-s)A}b(s)\mathbf{1}_{\tau\leq s\leq(\tau+\kappa)\wedge t}\|_{L^p(\Omega;\gamma([0,T]\times H;E))}\\ &\lesssim \|s \mapsto e^{-(t-s)A}b(s)\mathbf{1}_{\tau\leq s\leq(\tau+\kappa)\wedge t}\|_{L^p(\Omega;L^2(0,T;\gamma(H;E)))}\end{aligned}$$

for all $t \in [\tau, (\tau+\kappa)\wedge T]$. Since e^{-tA} is bounded on E uniformly in t, we have

$$\begin{aligned}&\|t \mapsto (e^{-(\cdot)A} \diamond b)_\tau(\cdot,t)\|_{L^p(\Omega\times[\tau,(\tau+\kappa)\wedge T];E))}\\ &\qquad\lesssim \|(t,s) \mapsto b(s)\mathbf{1}_{\tau\leq s\leq(\tau+\kappa)\wedge t}\|_{L^p(\Omega\times[0,T],L^2(0,T;\gamma(H;E)))}\\ &\qquad\leq \kappa^{1/2-1/p}\|(t,s) \mapsto b(s)\mathbf{1}_{\tau\leq s\leq(\tau+\kappa)\wedge t}\|_{L^p(\Omega\times[0,T]^2;\gamma(H;E))}\\ &\qquad\leq \kappa^{1/2}\|b\|_{L^p(\Omega\times[0,T];\gamma(H;E))}.\end{aligned}$$

Here, we used Hölder's inequality and Fubini. This closes the proof. □

2.1.2. Maximal γ-regularity of the deterministic and stochastic convolution in UMD Banach spaces

In this section, we discuss the regularity of the stochastic and the deterministic convolution in a space of γ-radonifying operators based on the work of Van Neerven, Veraar and Weis in [98]. Additionally, we prove lower estimates that will be needed later on. Although they are standard, we give the details for convenience. In what follows, this setting will be called [GM]. Throughout this section, we make the following assumptions.

[GM1] Let $r \in (1,\infty)$ and let E, E^1 be UMD Banach spaces with property-(α) and a dense embedding $E^1 \hookrightarrow E$.

[GM2] We assume the mapping $A : \Omega \to \mathcal{B}(E^1, E)$ to be strongly $\mathcal{F}_\tau$-measurable such that $D(A(\omega)) = E^1$ for almost all $\omega \in \Omega$, i.e there exists $M > 0$, such that we have

$$M^{-1}\|x\|_{E^1} \leq \|A(\omega)x\|_E \leq M\|x\|_{E^1}$$

for almost all $\omega \in \Omega$ and all $x \in E^1$. Moreover, we assume $0 \in \rho(A(\omega))$ for almost all $\omega \in \Omega$ and that $A(\omega)$ has for almost all $\omega \in \Omega$ a bounded H^∞-calculus of angle $\eta \in [0, \pi/2)$ with

$$\|\Psi(A(\omega))\|_{\mathcal{B}(E)} \leq M\|\Psi\|_{H^\infty(\Sigma_\eta)}$$

for all $\Psi \in H^\infty(\Sigma_\eta)$. Here, all the occurring constants are independent of $\omega \in \Omega$.

We start with maximal regularity estimates of the deterministic convolution $\left(e^{-(\cdot)A} * f\right)_\tau$. So far, we can ignore the dependence on ω and argue pathwise. The following purely deterministic theorem is sufficient for our purpose.

Theorem 2.1.7. *Let $\tau \geq 0$. Then, the for simple functions $f : \Omega \times [\tau, \infty) \to E$ well-defined deterministic convolution*

$$(e^{-(\cdot)A} * f)_\tau(t) = \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s$$

satisfies $(e^{-(\cdot)A} * f)_\tau \in \gamma(\tau, \infty; E^1)$ and

$$\|(e^{-(\cdot)A} * f)_\tau\|_{\gamma(\tau,\infty;E^1)} + \|(e^{-(\cdot)A} * f)\|_{C(\tau,\infty;[E,E^1]_{\frac{1}{2}})} \leq C_{\mathrm{MRD}}\|f\|_{\gamma(\tau,\infty;E)}.$$

Here, the constant $C_{\mathrm{MRD}} > 0$ depends on p, E, η and M. Hence, we can extend $f \mapsto (e^{-(\cdot)A} * f)_\tau$ to a bounded operator from $\gamma(\tau, \infty; E)$ to $\gamma(\tau, \infty; E^1)$.

Proof. The proof can be found in [98], Theorem 3.3. Note that our assumption on the $H^\infty(\Sigma_\eta)$-calculus implies the γ-sectoriality of A which is required in this theorem. □

Again, it will be very helpful to know that the constant in a lower order estimate of the deterministic convolution becomes smaller, if one reduces the size of the considered interval.

Proposition 2.1.8. *Let $\tau, \kappa \geq 0$. We have*

$$\|(e^{-(\cdot)A} * f)_\tau\|_{\gamma(\tau,\tau+\kappa;E)} \leq C\kappa\|f\|_{\gamma(\tau,\tau+\kappa;E)}$$

for all $f \in \gamma(\tau, \tau + \kappa; E)$ with a constant $C > 0$ only depending on M.

Proof. Let $f \in C_c^\infty(\tau, \tau + \kappa; E)$ and $g \in C_c^\infty(\tau, \tau + \kappa; E^{'})$. Then, $t \mapsto \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s$ is also an E-valued function and we can estimate

$$\begin{aligned}
&\Big|\int_\tau^{\tau+\kappa} \Big\langle \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s, g(t)\Big\rangle_{(E,E')}\,\mathrm{d}t\Big| \\
&\qquad = \Big|\int_\tau^{\tau+\kappa}\int_\tau^{\tau+\kappa} \big\langle e^{-(t-s)A} f(s)\mathbf{1}_{\tau\leq s\leq t\leq \tau+\kappa}, g(t)\big\rangle_{(E,E')}\,\mathrm{d}s\,\mathrm{d}t\Big| \\
&\qquad \leq \|(t,s) \mapsto e^{-(t-s)A} f(s)\mathbf{1}_{\tau\leq s\leq t\leq\tau+\kappa}\|_{\gamma([\tau,\tau+\kappa]^2;E)}\|(t,s) \mapsto g(t)\|_{\gamma([\tau,\tau+\kappa]^2;E')}.
\end{aligned}$$

In the last inequality, we used the finite cotype of E and the corresponding γ-Hölder inequality (see Corollary 5.5 in [55]). By assumption the $A(\omega)$ have a bounded H^∞-calculus with ω-independent bound. Thus, Remark 7.1 in [55] implies that the operators $A(\omega)$ are

γ-sectorial with an ω-independent bound and in particular $(e^{-tA(\omega)})_{t\geq 0}$ is γ-bounded. Moreover, γ-Fubini (see Proposition 3.14 in [94]) for spaces with property-α yields

$$\begin{aligned}\|(t,s)\mapsto e^{-(t-s)A}f(s)\mathbf{1}_{\tau\leq s\leq t\leq\tau+\kappa}\|_{\gamma([\tau,\tau+\kappa]^2;E)} &\lesssim \|(t,s)\mapsto f(s)\|_{\gamma([\tau,\tau+\kappa]^2;E)}\\ &\simeq \|t\mapsto \big(s\mapsto f(s)\big)\|_{\gamma(\tau,\tau+\kappa;\gamma(\tau,\tau+\kappa;E))}\\ &= \kappa^{1/2}\|f\|_{\gamma(\tau,\tau+\kappa;E)}\end{aligned}$$

and similarly

$$\|(t,s)\mapsto g(t)\|_{\gamma([\tau,\tau+\kappa]^2;E)} \lesssim \kappa^{1/2}\|g\|_{\gamma(\tau,\tau+\kappa;E')}.$$

Since $C_c^\infty(\tau,\tau+\kappa;E)$ and $C_c^\infty(\tau,\tau+\kappa;E')$ are dense in the spaces $\gamma(\tau,\tau+\kappa;E)$ and $\gamma(\tau,\tau+\kappa;E^{'})$ respectively, we proved the claimed result. □

Before we turn to the stochastic convolution, we provide a trace estimate for $[\tau,\infty)\ni t\mapsto e^{-(t-\tau)A}u_\tau$.

Proposition 2.1.9. *Let $\tau\geq 0$. Then, there exists $C>0$, such that*

$$\|t\mapsto e^{-(t-\tau)A}u_\tau\|_{\gamma(\tau,\infty;E^1)} + \|t\mapsto e^{-(t-\tau)A}u_\tau\|_{C(\tau,\infty;[E,E^1]_{\frac12})} \leq C\|u_\tau\|_{[E,E^1]_{\frac12}}$$

for all $u_\tau\in[E,E^1]_{\frac12}$.

Proof. The estimate

$$\|t\mapsto Ae^{-(t-\tau)A}u_0\|_{\gamma(\tau,\infty;E)} + \|t\mapsto e^{-(t-\tau)A}u_0\|_{C(\tau,\infty;[E,D(A)]_{\frac12})} \leq C\|u_0\|_{[E,D(A)]_{\frac12}}$$

can be found in [98], Theorem 3.8. The claimed result then follows from the equivalence of the norms $\|A\cdot\|_E$ and $\|\cdot\|_{E^1}$. □

As in Lemma 2.1.4, we can show that the integrand of the stochastic convolution

$$(\omega,s)\mapsto e^{-(t-s)A(\omega)}b(\omega,s)\mathbf{1}_{\tau(\omega)<s\leq t}$$

is strongly adapted to $\mathbb{F}$ in the sense of Definition 1.3.2. Hence, we are in the position to state the maximal γ-regularity result and the maximal inequality for the stochastic integral.

Theorem 2.1.10. *The stochastic convolution*

$$(e^{-(\cdot)A}\diamond b)_\tau(t) = \int_0^t e^{-(t-s)A}b(\cdot,s)\mathbf{1}_{(\tau,\infty)}(s)\,\mathrm{d}W(s)$$

is for all adapted $b\in L^r_{\mathbb{F}}(\Omega;\gamma([\tau,\infty)\times H;E))$ well-defined and we have

$$\begin{aligned}\|(e^{-(\cdot)A}\diamond b)_\tau\|_{L^r(\Omega;\gamma(\tau,\infty;E^1))} + \big(\mathbb{E}\sup_{t\in[\tau,\infty)}\|(e^{-(\cdot)A}\diamond b)_\tau\|^r_{[E,E^1]_{\frac12}}\big)^{1/r}&\\ \leq C_{\mathrm{MRS}}\|b\|_{L^r(\Omega;\gamma([\tau,\infty)\times H;[E,E^1]_{\frac12}))}&\end{aligned}$$

with a constant $C_{\mathrm{MRS}}>0$ only depending on p,η and M.

Proof. The proof follows the lines of the proof of Proposition 4.3 in [98]. □

As for the deterministic convolution, we want to derive a lower order estimate that improves if one lessens the size of the considered interval.

Proposition 2.1.11. *Let $\kappa > 0$. Then, we have*

$$\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;\gamma(\tau,\tau+\kappa;E))} \leq C\kappa^{1/2}\|b\|_{L^r(\Omega;\gamma([\tau,\tau+\kappa]\times H;E))}$$

for all $b \in L^r(\Omega;\gamma([\tau,(\tau+\kappa)\wedge T]\times H;E))$ with a constant $C>0$ only depending on E and on M.

Proof. Since E has property-(α), we can apply γ-Fubini (see Proposition 3.14 in [94]) and the Itô-isomorphism (see Theorem 1.3.4) to obtain

$$\begin{aligned}
&\|t \mapsto (e^{-(\cdot)A} \diamond b)_\tau(t)\mathbf{1}_{[\tau,\tau+\kappa]}(t)\|_{L^r(\Omega;\gamma(0,\infty;E))} \\
&\qquad \simeq_E \|t \mapsto (e^{-(\cdot)A} \diamond b)_\tau(\cdot,t)\mathbf{1}_{[\tau,\tau+\kappa]}(t)\|_{\gamma(0,\infty;L^r(\Omega,E))} \\
&\qquad \simeq \|(t,s) \mapsto e^{-(t-s)A}b(s)\mathbf{1}_{\tau<s\leq t\leq\tau+\kappa}\|_{L^r(\Omega;\gamma((0,\infty)\times(0,\infty)\times H;E))}.
\end{aligned}$$

By assumption $A(\omega)$ has a bounded H^∞-calculus with ω-independent bound. Thus, Remark 7.1 in [55] implies that the operators $A(\omega)$ are γ-sectorial with an ω-independent bound. In particular, the set $\left\{e^{-tA(\omega)} : t \in (0,\infty)\right\} \subset \mathcal{B}(E)$ is γ-bounded and therefore we can estimate

$$\begin{aligned}
&\|(t,s) \mapsto e^{-(t-s)A}b(\omega,s)\mathbf{1}_{\tau<s\leq t\leq\tau+\kappa}\|_{\gamma((0,\infty)\times(0,\infty)\times H;E)} \\
&\qquad\qquad \lesssim \|(t,s) \mapsto b(s)\mathbf{1}_{\tau<s\leq t\leq\tau+\kappa\wedge T}\|_{\gamma((0,\infty)\times(0,\infty)\times H;E)} \\
&\qquad\qquad \lesssim \kappa^{1/2}\|b\mathbf{1}_{[\tau,(\tau+\kappa)\wedge T]}\|_{\gamma((0,\infty)\times H;E)}
\end{aligned}$$

almost surely, which yields the claimed result. □

2.1.3. Maximal $L^q(U;L^p(0,T))$-regularity of the deterministic and stochastic convolution

In this section, we discuss the regularity of the stochastic and the deterministic convolution based on the work of Antoni in [8]. In the maximal regularity settings above, we just had to fix a state space E and a constant domain E^1 for the operators $A(\omega)$. To adapt this flexibility to Antoni's approach, which was only developed in the space $L^q(U;L^p(0,T))$, we introduce a scale of possible state spaces and possible domains. This will be done by choosing the state space as a fractional domain of an operator Λ that has an $\mathcal{R}_p$-bounded H^∞-calculus on $L^q(U;L^p(0,T))$. In what follows, this setting will be called [LQ]. We make the following assumptions.

[LQ1] Let $r \in (1,\infty)$, $q \in (1,\infty)$, $p \in [2,\infty)$, $U \subset \mathbb{R}^d$ and let μ be a σ-finite measure on U. We choose $H = l^2(\mathbb{N})$ and $W(t) = \sum_{j=1}^\infty e_k\beta_k(t)$ with a sequence $(\beta_k)_k$ of independent

Brownian motions relative to $\mathbb{F}$ and with unit vectors $(e_k)_k \subset l^2(\mathbb{N})$. Moreover, let Λ be a closed and densely defined operator on $L^q(U) := L^q(U,\mu)$ with $0 \in \rho(\Lambda)$ that has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_{\widetilde{\eta}})$-calculus for some $\widetilde{\eta} \in (0, \frac{\pi}{2})$. For given $\alpha \in (\frac{1}{p}, 1]$, let $E^\alpha := D(\Lambda^\alpha)$ and denote by $E^{\alpha-1}$ the extrapolation space of $L^q(U)$ equipped with the norm $\|\Lambda^{\alpha-1} \cdot \|_{L^q(U)}$.

[LQ2] We assume the mapping $A : \Omega \to \mathcal{B}(E^\alpha, E^{\alpha-1})$ to be strongly $\mathcal{F}_\tau$-measurable. Moreover, the operators $A(\omega)$ are for almost all $\omega \in \Omega$ closed, densely defined with $0 \in \rho(A(\omega))$ and have an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\eta)$ calculus with

$$\mathcal{R}_p\Big(\big\{\psi(A(\omega)) : \|\psi\|_{H^\infty(\Sigma_\eta)} \leq 1\big\} \subset \mathcal{B}(L^q(U))\Big) \leq M$$

for some $M > 0$ and $\eta \in (0, \pi/2)$ independent of $\omega \in \Omega$. Moreover, the operators $\Lambda^\alpha A^{-\alpha}$, $A^\alpha \Lambda^{-\alpha}$, $A^{\alpha-1}\Lambda^{1-\alpha}$, $\Lambda^{\alpha-1}A^{1-\alpha}$ are almost surely $\mathcal{R}_p$-bounded with ω independent bounds.

As we have seen in section 1.4.2, Λ can be extended to an invertible operator $\Lambda_{p,a,b}$ on $L^q(U; L^p(a,b))$ that has a bounded $H^\infty(\Sigma_{\widetilde{\eta}})$-calculus. In what follows, we write $E^\alpha(a,b) := D(\Lambda^\alpha_{p,a,b})$ and $E^{\alpha-1}(a,b)$ for the extrapolation space of $L^q(U; L^p(a,b))$ with respect to $\|\Lambda^{\alpha-1}_{p,a,b} \cdot \|_{L^q(U;L^p(a,b))}$.

Due to Proposition 1.4.1, the assumptions on the $\mathcal{R}_p$-boundedness from [LQ2] imply the equivalences of the norms $\|A^\alpha \cdot \|_{L^q(U;L^p(a,b))} \simeq \| \cdot \|_{E^\alpha(a,b)}$ and $\|A^{\alpha-1} \cdot \|_{L^q(U;L^p(a,b))} \simeq \| \cdot \|_{E^{\alpha-1}(a,b)}$ almost surely with ω-independent estimates. Since both A and Λ particularly have $\mathcal{R}_p$-bounded imaginary powers, we also get the norm equivalence $\|A^\theta \cdot \|_{L^q(U;L^p(a,b))} \simeq \|\Lambda^\theta_{p,a,b} \cdot \|_{L^q(U;L^p(a,b))}$ for all $\theta \in [\alpha - 1, \alpha]$ and that the operators $\Lambda^\theta A^{-\theta}$ and $A^\theta \Lambda^{-\theta}$ are $\mathcal{R}_p$-bounded.

Note that the restriction $\alpha \in (\frac{1}{p}, 1]$ is necessary, as for the proof of the trace estimates of the deterministic and the stochastic convolution, we need to apply the embedding

$$\big\{\Lambda^\alpha u \in L^q(U; L^p(a,b))\big\} \cap L^q(U; W^{\alpha,p}(a,b)) \hookrightarrow C(a,b; F^{\alpha-\frac{1}{p}}_{\Lambda,q,p})$$

from Lemma 1.4.4, which only holds true for $\alpha > \frac{1}{p}$. Throughout this section, we frequently use the estimate

$$\|\Gamma f\|_{L^q(U;L^p(a,b))} \leq \mathcal{R}_p(\Gamma)\|f\|_{L^q(U;L^p(a,b))}$$

for any $\mathcal{R}_p$-bounded operator Γ from Proposition 1.4.1 without explicitly mentioning it.

We start with an estimate for the deterministic convolution $\big(e^{-(\cdot)A} * f\big)_\tau$. Here, we can ignore the dependence on ω and argue pathwise. The following purely deterministic theorem is sufficient for our purpose.

Theorem 2.1.12. *Let $0 \leq \tau \leq T$ and let $f : \Omega \times [\tau, T] \to E^\alpha$ be a simple function. Then, the deterministic convolution*

$$(e^{-(\cdot)A} * f)_\tau(t) = \int_\tau^t e^{-(t-s)A} f(s)\,\mathrm{d}s$$

is well-defined, satisfies $(\Lambda^\alpha e^{-(\cdot)A} * f)_\tau \in L^q(U; L^p(\tau,T))$*. Additionally, we have*

$$\|\Lambda^\alpha (e^{-(\cdot)A} * f)_\tau\|_{L^q(U;L^p(\tau,T))} + \|(e^{-(\cdot)A} * f)_\tau\|_{C(\tau,T;F^{\alpha-1/p}_{\Lambda,q,p})} \le C_{\mathrm{MRD}} \|\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,T))}.$$

Here, the constant $C_{\mathrm{MRD}} > 0$ *only depends on* p, q*,* η*,* M *and the* $\mathcal{R}_p$*-bounds of* $\Lambda^\alpha A^{-\alpha}$*,* $A^{1-\alpha}\Lambda^{\alpha-1}$ *and* $A^{\alpha-1}\Lambda^{1-\alpha}$*. Hence, we are able to extend* $f \mapsto (e^{-(\cdot)A} * f)_\tau$ *to a bounded operator from* $E^{\alpha-1}(\tau,T)$ *to* $E^\alpha(\tau,T)$*.*

Proof. The proof of the inequality

$$\|A(e^{-(\cdot)A} * f)_\tau\|_{L^q(U;L^p(\tau,T))} + \|A^{1-\alpha}(e^{-(\cdot)A} * f)_\tau\|_{L^q(U;W^{\alpha,p}(\tau,T))} \le \widetilde{C}_{\mathrm{MRD}} \|f\|_{L^q(U;L^p(\tau,T))}$$

can be found in [8], Theorem 3.3.9. The claimed estimate in $E^\alpha(\tau,T)$ then follows from

$$\begin{aligned}
&\|\Lambda^\alpha (e^{-(\cdot)A} * f)_\tau\|_{L^q(U;L^p(\tau,T))} \\
&\qquad = \|\Lambda^\alpha A^{-\alpha} A^\alpha (e^{-(\cdot)A} * A^{1-\alpha} A^{\alpha-1} f)_\tau\|_{L^q(U;L^p(\tau,T))} \\
&\qquad \le \mathcal{R}_p\big(\Lambda^\alpha A^{-\alpha}\big) \widetilde{C}_{\mathrm{MRD}} \|A^{\alpha-1}\Lambda^{1-\alpha}\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,T))} \\
&\qquad \le \mathcal{R}_p\big(\Lambda^\alpha A^{-\alpha}\big) \widetilde{C}_{\mathrm{MRD}} \mathcal{R}_p\big(A^{\alpha-1}\Lambda^{1-\alpha}\big) \|\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,T))}.
\end{aligned}$$

In the same way, we get

$$\begin{aligned}
\|(e^{-(\cdot)A} * f)_\tau\|_{L^q(U;W^{\alpha,p}(\tau,T))} &\le \widetilde{C}_{\mathrm{MRD}} \|A^{\alpha-1} f\|_{L^q(U;L^p(\tau,T))} \\
&= \widetilde{C}_{\mathrm{MRD}} \mathcal{R}_p\big(A^{\alpha-1}\Lambda^{1-\alpha}\big) \|\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,T))}
\end{aligned}$$

for $f \in E^{\alpha-1}(\tau,T)$. Hence, we can apply Lemma 1.4.4 to get the claimed trace estimate in $F^{\alpha-1/p}_{\Lambda,q,p}$. □

In the sequel, it will very helpful to know that the constant in a lower order estimate of the deterministic convolution becomes smaller, if one reduces the size considered interval.

Proposition 2.1.13. *Let* $0 \le \tau \le T$ *and* $\kappa > 0$*, such that* $\tau + \kappa \le T$*. Then, we have*

$$\|(e^{-(\cdot)A} * f)_\tau\|_{E^{\alpha-1}(\tau,\tau+\kappa)} \le C\kappa \|f\|_{E^{\alpha-1}(\tau,\tau+\kappa)}$$

for all $f \in E^{\alpha-1}(\tau,\tau+\kappa)$ *with a constant* $C > 0$ *only depending on* M *and the* $\mathcal{R}_p$*-bound of* $\Lambda^{\alpha-1}A^{1-\alpha}$ *and* $A^{\alpha-1}\Lambda^{1-\alpha}$*, but not on* κ*.*

Proof. From [8], Proposition 3.3.1, we get

$$\|(e^{-(\cdot)A} * f)_\tau\|_{L^q(U;L^p(\tau,\tau+\kappa))} \le C\kappa \|f\|_{L^q(U;L^p(\tau,\tau+\kappa))}.$$

This together with the $\mathcal{R}_p$-boundedness of $\Lambda^{\alpha-1}A^{1-\alpha}$ and $A^{\alpha-1}\Lambda^{1-\alpha}$ yields

$$\begin{aligned}
&\|\Lambda^{\alpha-1}(e^{-(\cdot)A} * f)_\tau\|_{L^q(U;L^p(\tau,\tau+\kappa))} \\
&\qquad = \|\Lambda^{\alpha-1}A^{-\alpha+1}(e^{-(\cdot)A} * A^{\alpha-1} f)_\tau\|_{L^q(U;L^p(\tau,\tau+\kappa))} \\
&\qquad \le \mathcal{R}_p\big(\Lambda^{\alpha-1}A^{-\alpha+1}\big) C\kappa \|A^{\alpha-1}\Lambda^{1-\alpha}\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,\tau+\kappa))} \\
&\qquad \le \mathcal{R}_p\big(\Lambda^{\alpha-1}A^{1-\alpha}\big) C\kappa \mathcal{R}_p\big(A^{\alpha-1}\Lambda^{1-\alpha}\big) \|\Lambda^{\alpha-1} f\|_{L^q(U;L^p(\tau,\tau+\kappa))}.
\end{aligned}$$

□

Before we turn to the stochastic convolution, we give a trace estimate for $t \mapsto e^{-(t-\tau)A}u_0$.

Proposition 2.1.14. *Let $0 < \tau \le T$. Then, there exists $C > 0$, such that*

$$\|t \mapsto \Lambda^\alpha e^{-(t-\tau)A}u_0\|_{L^q(U;L^q(\tau,T))} + \|t \mapsto e^{-(t-\tau)A}u_0\|_{C(\tau,T;F^{1-1/p}_{\Lambda,q,p})} \le C\|u_0\|_{F^{\alpha-1/p}_{\Lambda,q,p}}$$

for all $u_0 \in F^{\alpha-1/p}_{\Lambda,q,p}$.

Proof. The estimate

$$\|t \mapsto A^\alpha e^{-(t-\tau)A}u_0\|_{L^q(U;L^q(\tau,T))} + \|t \mapsto e^{-(t-\tau)A}u_0\|_{C(\tau,T;F^{\alpha-1/p}_{A,q,p})} \le C\|u_0\|_{F^{\alpha-1/p}_{A,q,p}}.$$

is a combination of [8], Proposition 3.2.12 and [65], Theorem 4.25. The claimed result then follows from $\|A^\alpha \cdot\|_{L^q(U;L^p(\tau,T))} \simeq \|\cdot\|_{E^\alpha(\tau,T)}$ and $\|A^{\alpha-1} \cdot\|_{L^q(U;L^p(\tau,T))} \simeq \|\cdot\|_{E^{\alpha-1}(\tau,T)}$. $\square$

From now on let τ be an $\mathbb{F}$-stopping time with $0 \le \tau \le T$ almost surely. As in Lemma 2.1.4, we can show that the integrand of the stochastic convolution

$$(\omega, s) \mapsto e^{-(t-s)A(\omega)}b(\omega, s)\mathbf{1}_{\tau(\omega)<s\le t}$$

is adapted to $\mathbb{F}$. Hence, we are in the position to state the maximal regularity result and the maximal estimate for the stochastic integral.

Theorem 2.1.15. *The stochastic convolution*

$$(e^{-(\cdot)A} \diamond b)_\tau(t) = \int_0^t e^{-(t-s)A}b(s)\mathbf{1}_{(\tau,T)}(s)\,\mathrm{d}W(s)$$

is well-defined for all adapted $b \in L^r(\Omega; L^q(U; L^p(\tau, T; l^2)))$ and we have

$$\begin{aligned}\|\Lambda^\alpha(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,T)))} &+ \big(\mathbb{E} \sup_{t\in[\tau,T)} \|(e^{-(\cdot)A} \diamond b)_\tau\|^r_{F^{\alpha-1/p}_{\Lambda,q,p}}\big)^{1/r} \\ &\le C_{\mathrm{MRS}}\|\Lambda^{\alpha-\frac{1}{2}}b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}\end{aligned}$$

with a constant $C_{\mathrm{MRS}} > 0$ only depending on p, η, M and the $\mathcal{R}_p$-bounds of $\Lambda^\alpha A^{-\alpha}$ and $A^{\alpha-1/2}\Lambda^{1/2-\alpha}$. Hence, we can extend the stochastic convolution $f \mapsto (e^{-(\cdot)A} \diamond b)_\tau$ to a bounded operator from $\big\{\Lambda^{\alpha-\frac{1}{2}}b \in L^r_{\mathbb{F}}(\Omega; L^q(U; L^p(\tau, T; l^2)))\big\}$ to the space $L^r(\Omega; E^\alpha(\tau, T)) \cap L^r(\Omega; C(\tau, T; F^{\alpha-\frac{1}{p}}_{\Lambda,q,p}))$.

Proof. Following the proof of [8], Theorem 3.4.10, we get

$$\begin{aligned}\|A^{1/2}(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,T)))} &+ \|A^{1/2-\sigma}(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;W^{\sigma,p}(\tau,T)))} \\ &\le \widetilde{C}_{\mathrm{MRS}}\|b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}\end{aligned} \tag{2.1.1}$$

for $\sigma \in (0, \frac{1}{2})$ and b with $A^{\frac{1}{2}}b \in L^r(\Omega; L^q(U; L^p(\tau, T; l^2)))$. The estimate in the space $L^r(\Omega; E^\alpha(\tau, T))$ then follows from

$$\begin{aligned}&\|\Lambda^\alpha(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,T)))} \\ &\quad= \|\Lambda^\alpha A^{-\alpha}A^\alpha(e^{-(\cdot)A} \diamond A^{1/2-\alpha}A^{\alpha-1/2}b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,T)))} \\ &\quad\le \mathcal{R}_p\big(\Lambda^\alpha A^{-\alpha}\big)\widetilde{C}_{\mathrm{MRS}}\|A^{\alpha-1/2}\Lambda^{1/2-\alpha}\Lambda^{\alpha-1/2}b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))} \\ &\quad\le \mathcal{R}_p\big(\Lambda^\alpha A^{-\alpha}\big)\widetilde{C}_{\mathrm{MRS}}\mathcal{R}_p\big(A^{\alpha-1/2}\Lambda^{1/2-\alpha}\big)\|\Lambda^{\alpha-1/2}b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}.\end{aligned}$$

In the same way, for $\sigma \in (0, \frac{1}{2})$, we get

$$\begin{aligned}\|(e^{-(\cdot)A} \diamond b)_\tau\|&_{L^r(\Omega;L^q(U;W^{\sigma,p}(\tau,T)))} \\ &\leq \widetilde{C}_{\mathrm{MRS}} \mathcal{R}_p\big(A^{\sigma-1/2}\Lambda^{1/2-\sigma}\big) \|\Lambda^{\sigma-1/2} b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}\end{aligned} \tag{2.1.2}$$

as a consequence of (2.1.1). To derive the trace estimate in $F^{\alpha-1/p}_{\Lambda,q,p}$, we have to distinguish the cases $\alpha \in (\frac{1}{p}, \frac{1}{2})$ and $\alpha \in [\frac{1}{2}, 1]$. In the first case, (2.1.1) implies

$$\begin{aligned}\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;W^{\alpha,p}(\tau,T)))} &\leq \widetilde{C}_{\mathrm{MRS}} \|A^{\alpha-1/2} b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))} \\ &\leq \mathcal{R}_p\big(A^{\alpha-1/2}\Lambda^{1/2-\alpha}\big) \|\Lambda^{\alpha-1/2} b\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}\end{aligned}$$

and the embedding from Lemma 1.4.4 yields the claimed result. If on the other hand $\alpha \in [\frac{1}{2}, 1]$, choose $\varepsilon \in (0, \frac{1}{2} - \frac{1}{p})$. We use the same embedding and (2.1.2) to get

$$\begin{aligned}&\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;C(\tau,T;F^{\alpha-1/p}_{\Lambda,q,p}))} = \|\Lambda^{\alpha-1/2+\varepsilon}(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;C(\tau,T;F^{\frac{1}{2}-\varepsilon-\frac{1}{p}}_{\Lambda,q,p}))} \\ &\quad\leq C\|\Lambda^\alpha(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,T)))} + C\|\Lambda^{\alpha-\frac{1}{2}+\varepsilon}(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;W^{\frac{1}{2}-\varepsilon,p}(\tau,T)))} \\ &\quad\leq C\Big(\mathcal{R}_p\big(\Lambda^\alpha A^{-\alpha}\big)\widetilde{C}_{\mathrm{MRS}}\mathcal{R}_p\big(A^{\alpha-\frac{1}{2}}\Lambda^{\frac{1}{2}-\alpha}\big) \\ &\qquad\qquad + \mathcal{R}_p\big(A^{\alpha-\frac{1}{2}}\Lambda^{\frac{1}{2}-\alpha}\big)\Big)\|\Lambda^{\alpha-\frac{1}{2}} f\|_{L^r(\Omega;L^q(U;L^p(\tau,T;l^2)))}.\end{aligned}$$

This closes the proof. □

As for the deterministic convolution, we want to derive a lower order estimate that improves if one lessens the size of the considered interval.

Proposition 2.1.16. *Let $\kappa > 0$ and $T > 0$. Then, we have*

$$\|\Lambda^{\alpha-1}(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,(\tau+\kappa)\wedge T)))} \leq C\kappa^{1/2}\|\Lambda^{\alpha-1} b\|_{L^r(\Omega;L^q(U;L^p(\tau,(\tau+\kappa)\wedge T;l^2)))}$$

for all $b \in L^r(\Omega; L^q(U; L^p(\tau, (\tau+\kappa) \wedge T; l^2)))$ with a constant $C > 0$ only depending on M and the $\mathcal{R}_p$-bound of $\Lambda^{1-\alpha}A^{\alpha-1}$ and $A^{\alpha-1}\Lambda^{1-\alpha}$.

Proof. From Proposition 3.4.1 in [8], we get

$$\|(e^{-(\cdot)A} \diamond b)_\tau\|_{L^r(\Omega;L^q(U;L^p(\tau,(\tau+\kappa)\wedge T)))} \leq C\kappa^{1/2}\|b\|_{L^r(\Omega;L^q(U;L^p(\tau,(\tau+\kappa)\wedge T)))}$$

Using the ω-independent boundedness of $\mathcal{R}_p\big(\Lambda^{1-\alpha}A^{\alpha-1}\big)$ and $\mathcal{R}_p\big(A^{\alpha-1}\Lambda^{1-\alpha}\big)$, the result is an immediate consequence of this estimate. □

2.2. Semilinear parabolic stochastic evolution equations

In this section, we show well-posedness of semilinear parabolic stochastic evolution equations in different spaces based on the known maximal regularity estimates for the stochastic and the deterministic convolution. Since, these arguments are independent of the underlying setting, we work simultaneously in $L^p(0,T;E)$, $\gamma(0,T;E)$ with a UMD Banach space E and in $L^q(U;L^p(0,T))$. In every single of these spaces, there are results for semilinear stochastic equations (see [8], [96], and [98]). However, we not only give a unified approach to these equations, we also make slight generalisations. We start the equation at an $\mathbb{F}$-stopping time τ with given initial date $u_\tau : \Omega \to \mathrm{TR}$ that is strongly $\mathcal{F}_\tau$-measurable. Here, TR is the trace space in our theory. It differs from setting to setting and will be introduced later on. Moreover, we allow not only nonlinearities that are pointwise Lipschitz continuous, but also nonlinearities that are Lipschitz continuous with respect to the norm of the maximal regularity space. This allows us to deal with nonlinear memory terms, which will be crucial, when we apply these results to quasilinear equations. Although our results are not to much different from the originals, we still give most of the proofs to convince the reader of the validity of our changes and to highlight the common structure of all three approaches.

As before, let $(\Omega,\mathbb{P})$ be a probability space with filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and let W be a cylindrical Brownian motion in a Hilbert space H. Moreover, let $T>0$ and let τ be an $\mathbb{F}$-stopping time with $0\leq\tau\leq T$ almost surely.

We consider the stochastic evolution equation

$$\text{(SEE)}\begin{cases} du(t) = \big[-Au(t)+F(u)(t)+f(t)\big]dt + \big[B(u)(t)+b(t)\big]\,\mathrm{d}W(t), \quad t\in(\tau,T] \\ u(\tau)=u_\tau, \end{cases} \tag{2.2.1}$$

on a generalized interval $\Omega\times[\tau,T] := \{(\omega,t)\in\Omega\times[0,T] : \tau(\omega)\leq t\leq T\}$.

The general framework consists of a Banach space X with an extension to the timeline $X(a,b)$ for some interval $(a,b)\subset[0,T]$. Moreover, we have a maximal regularity space $X^1(a,b)$, a space $X_H^{\frac{1}{2}}(a,b)$ in which the stochastic part of the equations lives and a trace space TR. We will choose them in such a way that the solution u of (SEE) always satisfies $u\in X^1(\tau,T)$, $Au\in X(\tau,T)$, $B(u)\in X_H^{\frac{1}{2}}(\tau,T)$ and $u\in C(\tau,T;\mathrm{TR})$ almost surely. Additionally, r will be our integrability exponent with respect to Ω, i.e. for given $u_\tau\in L^r(\Omega;\mathrm{TR})$, we want to show that the solution u satisfies $u\in L^r(\Omega;X^1(\tau,T)\cap C(\tau,T;\mathrm{TR}))$.

To give an impression about the possible spaces, let $A=\Delta$ and $X=L^q(\mathbb{R}^d)$. Then, both $X(a,b)=L^p(a,b;L^q(\mathbb{R}^d))$ and $X(a,b)=L^q(\mathbb{R}^d;L^p(a,b))$ are possible choices with the corresponding $X^1(a,b)=L^p(a,b;W^{2,q}(\mathbb{R}^d))$ or $X^1(a,b)=W^{2,q}(\mathbb{R}^d;L^p(a,b))$. Here, TR is given by $B^{2-2/p}_{q,p}(\mathbb{R}^d)$ and $F^{2-2/p}_{q,p}(\mathbb{R}^d)$ respectively. Equations in these spaces will be discussed in depth in chapter 3.

In every setting discussed in the previous section, we will choose these abstract spaces individually. We fix the notation in the following way.

[TT] Assume [TT1] and [TT2] from section 2.1.1. In this setting, we define $X:=E$, $X(a,b):=L^p(a,b;E)$, $X^1(a,b):=L^p(a,b;E^1)$, $X_H(a,b):=L^p(a,b;\gamma(H;E))$ and $X_H^{\frac{1}{2}}(a,b):=L^p(a,b;\gamma(H;[E,E^1]_{\frac{1}{2}}))$. The trace space TR is the real interpolation space $(E;E^1)_{1-1/p,p}$.

[GM] Assume [GM1] and [GM2] from section 2.1.2. We set $X := E$, $X(a,b) := \gamma(a,b;E)$, $X^1(a,b) := \gamma(a,b;E^1)$, $X_H(a,b) := \gamma([a,b]\times H;E)$, $X_H^{\frac{1}{2}}(a,b) := \gamma([a,b]\times H;[E,E^1]_{\frac{1}{2}})$. The trace space TR is the complex interpolation space $[E;E^1]_{\frac{1}{2}}$.

[LQ] Assume [LQ1] and [LQ2] from section 2.1.3. Here, we set $X^\alpha(a,b) := D(\Lambda^\alpha_{p,a,b})$, $X^{\alpha-1}(a,b) = \Lambda^{1-\alpha}_{p,a,b}\big(L^q(U;L^p(a,b))\big)$, $X_H(a,b) := \Lambda^{1-\alpha}_{p,a,b}\big(L^q(U;L^p(a,b;l^2(\mathbb{N})))\big)$ and $X_H^{\frac{1}{2}}(a,b) := \Lambda^{\frac{1}{2}-\alpha}_{p,a,b}\big(L^q(U;L^p(a,b;l^2(\mathbb{N})))\big)$. The trace space TR is the interpolation space $F^{\alpha-1/p}_{\Lambda,q,p}$ in the sense of Definition 1.4.2.

The other assumptions are similar in every setting and can be formulated universally. However, before we can make them precise, we need to know what $L^r_{\mathbb{F}}(\Omega;X^i(\tau,\mu))$ for $i=0,1$ and $L^r_{\mathbb{F}}(\Omega;X_H^{\frac{1}{2}}(\tau,\mu))$ actually mean for given $\mathbb{F}$-stopping times τ,μ with $0\le\tau\le\mu\le T$ almost surely. In the setting [GM], this is obvious by Definition 1.3.2. In [TT], we use the same definition with the choice $Y=E^i$ for $i=0,1$ and in [LQ], we take $H=l^2(\mathbb{N})$ and $Y=\Lambda^{1-\alpha-i}\big(L^q(U)\big)$. This last choice makes sense because of

$$\Lambda^{1-\alpha-i}_{p,\tau,\mu}\Big(L^q(U;L^p(\tau,\mu))\Big)\hookrightarrow\gamma(\tau,\mu;\Lambda^{1-\alpha-i}(L^q(U)))$$

for $i=0,1$, which is a consequence of $\gamma(\tau,\mu;L^q(U))=L^q(U;L^2(a,b))$. Now we can present the universal assumptions.

[S3] $u_\tau:\Omega\to\mathrm{TR}$ is a strongly τ-measurable.

[S4] For any $\mathbb{F}$-stopping time μ with $\tau\le\mu\le T$ almost surely, the mapping

$$F:\Big\{u\in L^0_{\mathbb{F}}(\Omega;X^1(\tau,\mu)\cap C(\tau,\mu;\mathrm{TR})):u(\tau)=u_\tau\text{ a.s.}\Big\}\to L^0_{\mathbb{F}}(\Omega;X(\tau,\mu))$$

is a Volterra map, i.e. for a given $\mathbb{F}$-stopping time $\widetilde{\tau}$ with $\tau\le\widetilde{\tau}\le\mu$ almost surely, the restriction $F(u)_{|[\tau,\widetilde{\tau}]}$ only depends on $u_{|[\tau,\widetilde{\tau}]}$. This means that we have $F(u)\mathbf{1}_{[\tau,\widetilde{\tau}]}=F(v)\mathbf{1}_{[\tau,\widetilde{\tau}]}$ almost surely, whenever $u\mathbf{1}_{[\tau,\widetilde{\tau}]}=v\mathbf{1}_{[\tau,\widetilde{\tau}]}$ almost surely. Moreover, there exist an $\mathcal{F}_\tau$-measurable $\rho:\Omega\to[0,\infty)$ and constants $L_F^{(i)},\widetilde{L}_F,C_F^{(i)}\ge0$, $i=1,2$, such that F is of linear growth, i.e.

$$\|F(\phi_1)\|_{X(\tau,\mu)}\le\rho+C_F^{(1)}\|\phi_1\|_{X^1(\tau,\mu)}+C_F^{(2)}\|\phi_1\|_{C(\tau,\mu;\mathrm{TR})}$$

and Lipschitz continuous, i.e.

$$\begin{aligned}&\|F(\phi_1)-F(\phi_2)\|_{X(\tau,\mu)}\\&\qquad\le L_F^{(1)}\|\phi_1-\phi_2\|_{X^1(\tau,\mu)}+\widetilde{L}_F\|\phi_1-\phi_2\|_{X(\tau,\mu)}+L_F^{(2)}\|\phi_1-\phi_2\|_{C(\tau,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1,\phi_2\in L^0_{\mathbb{F}}(X^1(\tau,\mu)\cap C(\tau,\mu;\mathrm{TR}))$ with $\phi_1(\tau)=\phi_2(\tau)=u_\tau$ almost surely with constants independent of μ and $\omega\in\Omega$.

[S5] For any $\mathbb{F}$-stopping time μ with $\tau\le\mu\le T$ almost surely, the mapping

$$B:\Big\{u\in L^0_{\mathbb{F}}(\Omega;X^1(\tau,\mu)\cap C(\tau,\mu;\mathrm{TR})):u(\tau)=u_\tau\text{ a.s.}\Big\}\to L^0_{\mathbb{F}}(\Omega;X_H^{\frac{1}{2}}(\tau,\mu))$$

is a Volterra map, i.e. for a given $\mathbb{F}$-stopping time $\widetilde{\tau}$ with $\tau\le\widetilde{\tau}\le\mu$ almost surely, the restriction $B(u)_{|[\tau,\widetilde{\tau}]}$ only depends on $u_{|[\tau,\widetilde{\tau}]}$. This means that we have $B(u)\mathbf{1}_{[0,\widetilde{\tau}]}=$

$B(v)\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely, whenever $u\mathbf{1}_{[0,\widetilde{\tau}]} = v\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely. Moreover, there exist an $\mathcal{F}_\tau$-measurable $\rho : \Omega \to [0,\infty)$ and constants $L_B^{(i)}, \widetilde{L}_B, C_B^{(i)} \geq 0$, $i = 1,2$, such that B is of linear growth, i.e.

$$\|B(\phi_1)\|_{X_H^{\frac{1}{2}}(\tau,\mu)} \leq \rho + C_B^{(1)}\|\phi_1\|_{X^1(\tau,\mu)} + C_B^{(2)}\|\phi_1\|_{C(\tau,\mu;\mathrm{TR})}$$

and Lipschitz continuous, i.e.

$$\begin{aligned}&\|B(\phi_1) - B(\phi_2)\|_{X_H^{\frac{1}{2}}(\tau,\mu)}\\ &\quad\leq L_B^{(1)}\|\phi_1 - \phi_2\|_{X^1(\tau,\mu)} + \widetilde{L}_B\|\phi_1 - \phi_2\|_{X(\tau,\mu)} + L_B^{(2)}\|\phi_1 - \phi_2\|_{C(\tau,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1, \phi_2 \in L^0_{\mathbb{F}}(X^1(\tau,\mu) \cap C(\tau,\mu;\mathrm{TR}))$ with $\phi_1(\tau) = \phi_2(\tau) = u_\tau$ almost surely with constants independent of μ and $\omega \in \Omega$.

[S6] We assume $f \in L^r_{\mathbb{F}}(\Omega; X(\tau,T))$ and $b \in L^r_{\mathbb{F}}(\Omega; X_H^{\frac{1}{2}}(\tau,T))$.

We want to remark that it might be possible that $\tau(\omega) = T$ for some $\omega \in \Omega$. For these ω we don't need any assumptions on F and B, since $\left(e^{-(\cdot)A} * (F(\cdot,u) + f)\right)\big)_\tau(\omega,T)$ and $\left(e^{-(\cdot)A} \diamond (B(\cdot,u) + b)\right)\big)_\tau(\omega,T)$ vanish in this case anyway. In the next Lemma, we collect an important universal property of the spaces $X(a,b)$ and $X^1(a,b)$.

Lemma 2.2.1. *Let $u \in X^1(a,b)$. Then, $[a,b] \ni t \mapsto \|u\|_{X(a,t)}$ and $[a,b] \ni t \mapsto \|u\|_{X^1(a,t)}$ are continuous.*

Proof. Since the arguments are similar for $t \mapsto \|u\|_{X(a,t)}$ and $t \mapsto \|u\|_{X^1(a,t)}$, we just show the continuity of $[a,b] \ni t \mapsto \|u\|_{X^1(a,t)}$. In this proof, we have to distinguish the three different settings. The continuity of $t \mapsto \|u\|_{L^p(a,t;E)}$ and $t \mapsto \|\Lambda^{\alpha-1}u\|_{L^q(U;L^p(a,t))}$ is immediate by the dominated convergence theorem.

For the γ-setting, let $t \in [a,b]$ and $(t_n)_n$ be a sequence with $t_n \to t$ for $n \to \infty$. Defining $S_n : L^2(a,b) \to L^2(a,b), f \mapsto \mathbf{1}_{[a,t_n]}f$ and $S : L^2(a,b) \to L^2(a,b), f \mapsto \mathbf{1}_{[a,t]}f$, one can show $S_n g \to Sg$ for every $g \in L^2(a,b)$ with L^2-convergence. By Corollary 6.5 in [94], we get $S_n u \to Su$ in $\gamma(a,b;E)$ for $n \to \infty$ and in particular, we have

$$\|u\|_{\gamma(a,t_n;E)} = \|S_n u\|_{\gamma(a,b;E)} \to \|Su\|_{\gamma(a,b;E)} = \|u\|_{\gamma(a,t;E)},$$

for $n \to \infty$, which proves the claimed continuity. $\square$

Next, we introduce mild and strong solutions of (SEE).

Definition 2.2.2. *Let μ be another $\mathbb{F}$-stopping time with $\tau \leq \mu \leq T$ almost surely. A process $u : \Omega \times [\tau,\mu] \to X$ is called a mild solution of* (SEE) *if it is strongly measurable, adapted with $u(\tau) = u_\tau$ almost surely and*

a) both the deterministic convolution $\left(e^{-(\cdot)A} * F(u)\mathbf{1}_{[\tau,\mu]}\right)_\tau$ and the stochastic convolution $\left(e^{-(\cdot)A} \diamond B(u)\mathbf{1}_{[\tau,\mu]}\right)_\tau$ are well-defined.

b) The identity

$$u(t) = e^{-(t-\tau)A}u_\tau + \Big(e^{-(\cdot)A} * (F(u) + f)\Big)_\tau (t) + \Big(e^{-(\cdot)A} \diamond (B(u) + b)\Big)_\tau (t)$$

holds almost surely for all $t \in [\tau, \mu]$.

Usually, one says that a process u is a strong solution if it is sufficiently regular and the formula

$$u(t) - u_\tau = -\int_\tau^t Au(s)\,\mathrm{d}s + \int_\tau^t F(u)(s) + f(s)\,\mathrm{d}s + \int_\tau^t B(u)(s) + b(s)\,\mathrm{d}W(s)$$

holds almost surely for all $t \in [\tau, T]$. However, this is not possible in all of our settings. So, we have to explain what we mean with $\int_\tau^t \;\mathrm{d}s$, since $u \in X^1(\tau, T)$ does not necessarily imply that Au is integrable in time.

Definition 2.2.3. *Let* μ *be another* $\mathbb{F}$*-stopping time with* $\tau \le \mu \le T$ *almost surely. A process* $u : \Omega \times [\tau, \mu] \to E$ *is called a strong solution of* (SEE) *on* $[\tau, \mu]$ *if it is strongly measurable, strongly adapted with* $u(\tau) = u_\tau$ *almost surely, we have* $u \in X^1(\tau, \mu) \cap C(\tau, \mu; \mathrm{TR})$ *almost surely and* u *satisfies the following identities depending on the respective setting.*

[TT] The identity

$$u(t) - u_\tau = -\int_\tau^t Au(s)\,\mathrm{d}s + \int_\tau^t F(u)(s) + f(s)\,\mathrm{d}s + \int_\tau^t B(u)(s) + b(s)\,\mathrm{d}W(s)$$

holds almost surely for all $t \in [\tau, T]$ *as an equation in* E*. Here, the integral over time is an* E*-valued Bochner integral and the stochastic integral is well-defined as a consequence of Theorem 1.3.4 and* (1.3.1)*.*

[GM] The equation

$$u(t) - u_\tau = -Au(\mathbf{1}_{[\tau,t]}) + (F(u) + f)(\mathbf{1}_{[\tau,t]}) + \int_\tau^t B(u)(s) + b(s)\,\mathrm{d}W(s)$$

holds almost surely for all $t \in [\tau, T]$ *as an equation in* E*. Note that* $Au, F(u), f \in \gamma(\tau, T; E)$ *implies that they are linear operators from* $L^2(\tau, T)$ *to* E*. Moreover, the stochastic integral is well-defined as a consequence of Theorem 1.3.4.*

[LQ] The equality

$$\begin{aligned}\Lambda^{\alpha-1}u(t,x) - \Lambda^{\alpha-1}u_\tau(x) = \\ &-\int_\tau^t \Lambda^{\alpha-1}Au(s,x)\,\mathrm{d}s + \int_\tau^t \Lambda^{\alpha-1}F(u)(s,x) + \Lambda^{\alpha-1}f(s,x)\,\mathrm{d}s \\ &+\int_\tau^t \Lambda^{\alpha-1}B(u)(s,x) + \Lambda^{\alpha-1}b(s,x)\,\mathrm{d}W(s)\end{aligned}$$

holds almost surely for almost all $x \in U$ *and for all* $t \in [\tau, T]$ *as an equation in* $\mathbb{C}$*. These deterministic integrals are well-defined, since* $\Lambda^{\alpha-1}Au, \Lambda^{\alpha-1}F, \Lambda^{\alpha-1}f \in L^p(\tau, \mu; L^2(U))$ *almost surely.*

Under our assumptions the mild solution concept and the strong solution concepts coincide. In the maximal L^p-regularity setting [TT], this was shown in Proposition 4.4 in [96]. For the maximal L^p-regularity setting [GM], this result can be found in [98], Proposition 5.3. In the setting [LQ], a version of our result was proved in [8], Proposition 3.5.6. Adding the operator $\Lambda^{1-\alpha}$, one can follow the proof step by step.

Proposition 2.2.4. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [S3]-[S6] *be fulfilled. Moreover, let μ be another $\mathbb{F}$-stopping time with $\tau \leq \mu \leq T$ almost surely. A process $u : \Omega \times [\tau, \mu] \to X$ with $u(\tau) = u_\tau$ almost surely and with $u \in X^1(0, \mu) \cap C(\tau, \mu; \mathrm{TR})$ almost surely is a mild solution of* (SEE) *on $[\tau, \mu]$ if and only if u is a strong solution of* (SEE) *on $[\tau, \mu]$.*

To establish existence and uniqueness of a strong solution of (SEE) on $[\tau, T]$, we try to find a mild solution with the regularity properties we demanded in Proposition 2.2.4 via the contraction mapping theorem. It will emerge that we only get a solution on a smaller interval $[\tau, (\tau+\kappa) \wedge T]$ for some $\kappa > 0$ small enough and we then have to iterate the procedure. From now on, we write $\tau_0 := \tau$ and $\tau_n := (\tau + n\kappa) \wedge T$ for $n \in \mathbb{N}$. Clearly, τ_n is also a stopping time as sum and minimum of stopping times.

We assume that we already constructed a strongly adapted solution u on $[\tau, \tau_{n-1}]$ for some $n \in \mathbb{N}$ in the sense of Definition 2.2.3 and we want to extend u to $[\tau_{n-1}, \tau_n]$. We therefore consider the operator defined by

$$\begin{aligned} K_n\phi(t) =& e^{-(t-\tau_{n-1})A}u(\tau_{n-1}) + \Big(e^{-(\cdot)A} * (F(\phi) + f)\Big)_{\tau_{n-1}}(t) \\ &+ \Big(e^{-(\cdot)A} \diamond (B(\phi) + b)\Big)_{\tau_{n-1}}(t) \end{aligned} \tag{2.2.2}$$

almost surely for $t \in [\tau_{n-1}, \tau_n]$ and $K_n\phi(t) = u(t)$ for $t \in [\tau, \tau_{n-1})$ on the set

$$\begin{aligned} \mathcal{E}(\kappa, n) := \Big\{ \phi \in L^0_{\mathbb{F}}(\Omega; X(\tau, \tau_n)) \Big| \ & \phi = u \text{ on } \Omega \times [\tau, \tau_{n-1}],\ \phi \in L^r(\Omega; X^1(\tau_{n-1}, \tau_n)), \\ & \phi \in C(\tau_{n-1}, \tau_n; \mathrm{TR}) \text{ a.s., and } \mathbb{E} \sup_{t \in [\tau_{n-1}, \tau_n]} \|\phi(t)\|^r_{\mathrm{TR}} < \infty \Big\} \end{aligned}$$

endowed with the metric

$$\begin{aligned} \|v - w\|_{\mu,\kappa,n} :=& \big\|v - w\big\|_{L^r(\Omega; X^1(\tau_{n-1}, \tau_n))} + \mu \big\|v - w\big\|_{L^r(\Omega; X(\tau_{n-1}, \tau_n))} \\ &+ \Big(\mathbb{E} \sup_{t \in [\tau_{n-1}, \tau_n]} \|v(t) - w(t)\|^r_{\mathrm{TR}}\Big)^{1/r} \end{aligned}$$

for some $\mu > 0$.

In the following Lemma, we choose the open parameters κ and μ, such that K_n is a self-mapping contraction on $\mathcal{E}(\kappa, n)$. This is essentially a consequence of the maximal regularity estimates of the deterministic and the stochastic convolution we mentioned in the previous section. Summarizing section 2.1, there exist $C_{\mathrm{MRS}}, C_{\mathrm{MRD}} > 0$, such that we have

$$\big\|\big(e^{-(\cdot)A} \diamond g\big)_{\tau_{n-1}}\big\|_{L^r(\Omega; X^1(\tau_{n-1}, \tau_n) \cap C(\tau_{n-1}, \tau_n; \mathrm{TR}))} \leq C_{\mathrm{MRS}} \|g\|_{L^r(\Omega; X_H^{\frac{1}{2}}(\tau_{n-1}, \tau_n))} \tag{2.2.3}$$

for all $g \in L^r(\Omega; X_H^{\frac{1}{2}}(\tau_{n-1}, \tau_n))$ and

$$\left\| \left(e^{-(\cdot)A} * \widetilde{g}\right)_{\tau_{n-1}} \right\|_{L^r(\Omega; X^1(\tau_{n-1},\tau_n) \cap C(\tau_{n-1},\tau_n;\mathrm{TR}))} \le C_{\mathrm{MRD}} \|\widetilde{g}\|_{L^r(\Omega; X(\tau_{n-1},\tau_n))} \tag{2.2.4}$$

for all $\widetilde{g} \in L^r(\Omega; X(0,T))$. Moreover, we have the lower order estimate

$$\left\| \left(e^{-(\cdot)A} \diamond g\right)_{\tau_{n-1}} \right\|_{L^r(\Omega; X(\tau_{n-1},\tau_n))} \le C_1 \kappa^{1/2} \|g\|_{L^r(\Omega; X_H(\tau_{n-1},\tau_n))} \tag{2.2.5}$$

for all $g \in L^r(\Omega; X_H^{\frac{1}{2}}(\tau_{n-1}, \tau_n))$ and

$$\left\| \left(e^{-(\cdot)A} * \widetilde{g}\right)_{\tau_{n-1}} \right\|_{L^r(\Omega; X(\tau_{n-1},\tau_n))} \le C_2 \kappa \|\widetilde{g}\|_{L^r(\Omega; X(\tau_{n-1},\tau_n))} \tag{2.2.6}$$

for all $\widetilde{g} \in L^r(\Omega; X(\tau_{n-1}, \tau_n))$.

Lemma 2.2.5. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [S3]-[S6] *be fulfilled with*

$$C_{\mathrm{MRD}} L_F^{(i)} + C_{\mathrm{MRS}} L_B^{(i)} < 1$$

for $i = 1, 2$. Further, we assume $u_\tau \in L^r(\Omega, \mathrm{TR})$ and $\rho \in L^r(\Omega)$. Moreover, we fix $\mu > 0$ such that

$$\mu > C_{\mathrm{MRD}} \widetilde{L}_F + C_{\mathrm{MRS}} \widetilde{L}_B$$

and we choose $\kappa > 0$ small enough such that

$$\max\left\{ \kappa C_2 L_F^{(1)} \mu + \kappa^{\frac{1}{2}} C_1 L_B^{(1)} \mu, \kappa C_2 L_F^{(2)} \mu + \kappa^{\frac{1}{2}} C_1 L_B^{(2)} \mu, \kappa C_2 \widetilde{L}_F + \kappa^{\frac{1}{2}} C_1 \widetilde{L}_B \right\} < 1.$$

Then, the operator K_n defined in (2.2.2) *on $\mathcal{E}(\kappa, n)$ is a self-mapping contraction, i.e. we have $K_n(\mathcal{E}(\kappa, n)) \subset \mathcal{E}(\kappa, n)$ and*

$$\|K_n u - K_n v\|_{\mu,\kappa,n} \le \delta \|u - v\|_{\mu,\kappa,n}$$

for all $u, v \in \mathcal{E}(\kappa, n)$ with a constant $0 \le \delta < 1$.

Proof. For the time being we start with an arbitrary $\kappa > 0$ and $\mu > 0$ that will be chosen later on. The self-mapping property is immediate since we have $K\phi(\tau) = u_\tau$ almost surely by definition of K. Moreover, the linear growth of F and B, together with the trace estimate for $t \mapsto e^{-(t-\tau_{n-1})A} u(\tau_{n-1})$ and the maximal regularity estimates mentioned above yield

$$\begin{aligned}
\|K\phi\|_{\mu,\kappa,\phi} &\lesssim \|u(\tau_{n-1})\|_{L^r(\Omega;\mathrm{TR})} + \|F(\phi)\|_{L^r(\Omega; X(\tau_{n-1},\tau_n))} + \|B(\phi)\|_{L^r(\Omega; X_H^{\frac{1}{2}}(\tau_{n-1},\tau_n))} \\
&\lesssim \|\rho\|_{L^r(\Omega)} + \|u(\tau_{n-1})\|_{L^r(\Omega;\mathrm{TR})} + \|\phi\|_{L^r(\Omega; X^1(\tau,\tau_n))} + \|\phi\|_{L^r(\Omega; X(\tau,\tau_n))} \\
&\quad + \|\phi\|_{L^r(\Omega; C(\tau,\tau_n;\mathrm{TR}))} \\
&\lesssim \|\rho\|_{L^r(\Omega)} + \|u(\tau_{n-1})\|_{L^r(\Omega;\mathrm{TR})} + \|\phi\|_{L^r(\Omega; X^1(\tau_{n-1},\tau_n))} + \|\phi\|_{L^r(\Omega; C(\tau_{n-1},\tau_n;\mathrm{TR}))} \\
&\quad + \|u\|_{L^r(\Omega; X^1(\tau,\tau_{n-1}))} + \|u\|_{L^r(\Omega; C(\tau,\tau_{n-1};\mathrm{TR}))}.
\end{aligned} \tag{2.2.7}$$

In the last step, we used $\phi = u$ on $\Omega \times [\tau, \tau_{n-1}]$. Note that in the special case $n = 1$, this estimate reduces to

$$\|K\phi\|_{\mu,\kappa,\phi} \lesssim \|\rho\|_{L^r(\Omega)} + \|u_\tau\|_{L^r(\Omega;\mathrm{TR})} + \|\phi\|_{L^r(\Omega; X^1(\tau_{n-1},\tau_n))} + \|\phi\|_{L^r(\Omega; C(\tau_{n-1},\tau_n;\mathrm{TR}))}. \tag{2.2.8}$$

To check that K is a contraction on $\mathcal{E}(\kappa, n)$ we take $u, v \in \mathcal{E}(\kappa, n)$ and estimate the difference $\|Ku - Kv\|_{\mu,\kappa,n}$. Here, we are precise with the occurring constants to be able to choose κ and μ correctly.

We start with a pathwise estimate of the deterministic convolution. Estimate (2.2.4), together with the Lipschitz continuity of F assumed in [S4] and the Volterra property of F, yield

$$\begin{aligned}
\|\big(e^{-(\cdot)A} * (F(u) - F(v))\big)_{\tau_{n-1}}\|_{X^1(\tau_{n-1},\tau_n)} &+ \|\big(e^{-(\cdot)A} * (F(u) - F(v))\big)_{\tau_{n-1}}\|_{C(\tau_{n-1},\tau_n;\mathrm{TR})} \\
&\le C_{\mathrm{MRD}}\|F(u) - F(v)\|_{X(\tau_{n-1},\tau_n)} \\
&\le C_{\mathrm{MRD}}L_F^{(1)}\|u - v\|_{X^1(\tau_{n-1},\tau_n)} + C_{\mathrm{MRD}}L_F^{(2)}\|u - v\|_{C(\tau_{n-1},\tau_n;\mathrm{TR})} \\
&\quad + C_{\mathrm{MRD}}\tilde{L}_F\|u - v\|_{X(\tau_{n-1},\tau_n)},
\end{aligned}$$

almost surely on Ω. In the same way, by (2.2.6), we obtain

$$\begin{aligned}
&\|\big(e^{-(\cdot)A} * (F(u) - F(v))\big)_{\tau_{n-1}}\|_{X(\tau_{n-1},\tau_n)} \\
&\le \kappa C_2 L_F^{(1)}\|u - v\|_{X^1(\tau_{n-1},\tau_n)} + \kappa C_2 L_F^{(2)}\|u - v\|_{C(\tau,(\tau+\kappa)\wedge T;\mathrm{TR})} + \kappa C_2\tilde{L}_F\|u - v\|_{X(\tau_{n-1},\tau_n)},
\end{aligned}$$

almost surely on Ω. This yields

$$\begin{aligned}
\|\big(e^{-(\cdot)A} * (F(u) - F(v)\big)_{\tau_{n-1}}\|_{\mu,\kappa,n} \le &\big(C_{\mathrm{MRD}}L_F^{(1)} + \kappa C_2 L_F^{(1)}\mu\big)\|u - v\|_{L^r(\Omega;X^1(\tau_{n-1},\tau_n))} \\
&+ (C_{\mathrm{MRD}}L_F^{(2)} + \kappa C_2 L_F^{(2)}\mu\big)\|u - v\|_{L^r(\Omega;C(\tau_{n-1},\tau_n;\mathrm{TR}))} \\
&+ \mu\big(C_{\mathrm{MRD}}\widetilde{L}_F\mu^{-1} + \kappa C_2\widetilde{L}_F\big)\|u - v\|_{L^r(\Omega;X(\tau_{n-1},\tau_n))}.
\end{aligned}$$

To estimate the stochastic convolution we combine the Lipschitz continuity of B assumed in [S5], the Volterra property of B and estimate (2.2.3). We get

$$\begin{aligned}
&\|\big(e^{-(\cdot)A} \diamond (B(u) - B(v))\big)_{\tau_{n-1}}\|_{L^r(\Omega;X^1(\tau_{n-1},\tau_n))\cap C(\tau_{n-1},\tau_n;\mathrm{TR})} \\
&\quad\le C_{\mathrm{MRS}}\|B(u) - B(v)\|_{L^r(\Omega;X_H^{\frac{1}{2}}(\tau_{n-1},\tau_n))} \\
&\quad\le C_{\mathrm{MRS}}L_B^{(1)}\|u - v\|_{L^r(\Omega;X^1(\tau_{n-1},\tau_n))} + C_{\mathrm{MRS}}\tilde{L}_B\|u - v\|_{L^r(\Omega;X(\tau_{n-1},\tau_n))} \\
&\qquad + C_{\mathrm{MRS}}L_B^{(2)}\big(\mathbb{E}\sup_{t\in[\tau_{n-1},\tau_n]}\|u(t) - v(t)\|_{\mathrm{TR}}^r\big)^{1/r}.
\end{aligned}$$

If we instead apply the lower order estimates, we obtain

$$\begin{aligned}
&\mu\|e^{-(\cdot)A} \diamond (B(u) - B(v))\|_{L^r(\Omega;X(\tau_{n-1},\tau_n))} \\
&\le \kappa^{1/2}C_1 L_B^{(1)}\mu\|u - v\|_{L^r(\Omega;X^1(\tau_{n-1},\tau_n))} + \kappa^{1/2}C_2\tilde{L}_B\mu\|u - v\|_{L^r(\Omega;X(\tau_{n-1},\tau_n))} \\
&\quad + \kappa^{1/2}C_1 L_B^{(2)}\mu\big(\mathbb{E}\sup_{t\in[\tau_{n-1},\tau_n]}\|u(t) - v(t)\|_{\mathrm{TR}}^r\big)^{1/r}
\end{aligned}$$

This yields

$$\begin{aligned}
&\|\big(e^{-(\cdot)A} \diamond (B(u) - B(v)\big)_{\tau_{n-1}}\|_{\mu,\kappa,n} \\
&\qquad\le\big(C_{\mathrm{MRS}}L_B^{(1)} + \kappa^{\frac{1}{2}}C_1 L_B^{(1)}\mu\big)\|u - v\|_{L^r(\Omega;X^1(\tau_{n-1},\tau_n))} \\
&\qquad\quad + (C_{\mathrm{MRS}}L_B^{(2)} + \kappa^{\frac{1}{2}}C_2 L_B^{(2)}\mu\big)\|u - v\|_{L^r(\Omega;C(\tau_{n-1},\tau_n;\mathrm{TR}))} \\
&\qquad\quad + \mu\big(C_{\mathrm{MRS}}\widetilde{L}_B\mu^{-1} + \kappa^{\frac{1}{2}}C_1\widetilde{L}_B\big)\|u - v\|_{L^r(\Omega;X(\tau_{n-1},\tau_n))}.
\end{aligned}$$

All in all, we proved $\|Ku - Kv\|_{\mu,\kappa,n} \leq \delta \|u - v\|_{\mu,\kappa,n}$, where δ is given by

$$\delta = \max \Big\{ C_{\mathrm{MRD}} L_F^{(1)} + \kappa C_2 L_F^{(1)} \mu + C_{\mathrm{MRS}} L_B^{(1)} + \kappa^{\frac{1}{2}} C_1 L_B^{(1)} \mu, \\ C_{\mathrm{MRD}} L_F^{(2)} + \kappa C_2 L_F^{(2)} \mu + C_{\mathrm{MRS}} L_F^{(2)} + \kappa^{\frac{1}{2}} C_2 L_B^{(2)} \mu, \\ C_{\mathrm{MRD}} \widetilde{L}_F \mu^{-1} + \kappa C_2 \widetilde{L}_F + C_{\mathrm{MRS}} \widetilde{L}_B \mu^{-1} + \kappa^{\frac{1}{2}} C_1 \widetilde{L}_B \Big\}.$$

To ensure $\delta < 1$ we have to choose μ and κ properly. Due to the requirement

$$C_{\mathrm{MRD}} L_F^{(i)} + C_{\mathrm{MRS}} L_B^{(i)} < 1$$

for $i = 1, 2$ some expressions are already smaller than 1. Next we fix μ such that

$$\mu > C_{\mathrm{MRD}} \widetilde{L}_F + C_{\mathrm{MRS}} \widetilde{L}_B$$

and last but not least, we choose κ small enough such that

$$\max \big\{ \kappa C_2 L_F^{(1)} \mu + \kappa^{\frac{1}{2}} C_1 L_B^{(1)} \mu, \kappa C_2 L_F^{(2)} \mu + \kappa^{\frac{1}{2}} C_1 L_B^{(2)} \mu, \kappa C_2 \widetilde{L}_F + \kappa^{\frac{1}{2}} C_1 \widetilde{L}_B \big\} < 1.$$

This closes the proof. □

Applying the Lemma from above, we can construct a strong solution of (SEE) on the random interval $[\tau, T]$ step by step. This gives the main result of this section.

Theorem 2.2.6. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [S3]-[S6] *be fulfilled with*

$$C_{\mathrm{MRS}} L_F^{(i)} + C_{\mathrm{MRS}} L_B^{(i)} < 1.$$

If we additionally assume $u_\tau \in L^r(\Omega; \mathrm{TR})$ and $\rho \in L^r(\Omega)$, there exists a unique strong solution u of

$$\text{(SEE)} \begin{cases} du(t) &= [-Au(t) + F(u)(t) + f(t)]dt + [B(u)(t) + b(t)]dW_t, \ t \in [\tau, T] \\ u(\tau) &= u_\tau, \end{cases}$$

and u has almost surely continuous paths on $[\tau, T]$ viewed as a function with values in TR. *Moreover, we have the estimate*

$$\big(\mathbb{E} \|u\|_{X^1(\tau,T)}^r + \mathbb{E} \sup_{t \in [\tau,T]} \|u(t)\|_{\mathrm{TR}}^r \big)^{1/r} \leq C \big(1 + \|u_\tau\|_{L^r(\Omega;\mathrm{TR})} + \|\rho\|_{L^r(\Omega)} \big) \tag{2.2.9}$$

for some constant $C > 0$ independent of u_τ.

Proof. First, we choose $\mu, \kappa > 0$ as in Lemma 2.2.5. Without restriction, we assume that $\kappa = T/k$ for some $k \in \mathbb{N}$. Otherwise, we choose κ slightly smaller. As before, we set $\tau_0 := \tau$ and $\tau_n := (\tau + n\kappa) \wedge T$. This choice ensures $\tau_k = T$ almost surely.

Lemma 2.2.5 yields the existence and uniqueness of a fixpoint u_1 of the operator K_1 defined in (2.2.2) in $\mathcal{E}(\kappa, 1)$. In particular, by definition of K_1, we have

$$u_1(t) = \int_\tau^t e^{-(t-s)A} (F(u_1)(s) + f(s)) \, \mathrm{d}s + \int_\tau^t e^{-(t-s)A} (B(u_1)(s) + b(s)) \, \mathrm{d}W(s) \\ + e^{-(t-\tau)A} u_\tau$$

for $t \in [\tau_0, \tau_1]$. Thus, u_1 is a mild solution on $[\tau, \tau_1]$ with almost surely continuous paths as a function with values in TR. Moreover, using the contraction property of K_1 applied to any $\psi \in \mathcal{E}(\kappa, 1)$ yields

$$\begin{aligned}\|u_1\|_{\mu,\kappa,1} &= \|K_1(u_1)\|_{\mu,\kappa,1} \le \|K_1(u_1) - K_1(\psi)\|_{\mu,\kappa,1} + \|K_1(\psi)\|_{\mu,\kappa,1} \\ &\le \delta\|u_1 - \psi\|_{\mu,\kappa,1} + C\|K_1(\psi)\|_{\mu,\kappa,1} \\ &\le \delta\|u_1\|_{\mu,\kappa,1} + \widetilde{C}_1(1 + \|u_\tau\|_{L^r(\Omega,\mathrm{TR})} + \|\rho\|_{L^r(\Omega)}).\end{aligned}$$

for some $\widetilde{C}_1 > 0$ only depending on the choice of ψ. The last estimate for $K_1(\psi)$ was shown in (2.2.8). Especially, since $\delta < 1$, we have the estimates

$$\big(\mathbb{E}\|u\|^r_{X^1(\tau,\tau_1)} + \mathbb{E}\sup_{t\in[\tau,\tau_1]}\|u(t)\|^r_{\mathrm{TR}}\big)^{1/r} \le C\big(1 + \|u_\tau\|_{L^r(\Omega;\mathrm{TR})} + \|\rho\|_{L^r(\Omega)}\big) \tag{2.2.10}$$

for some $C > 0$ independent of u_τ and ρ. We set $u := u_1$ on $\Omega \times [\tau, \tau_1]$. In the same way, we construct a strong solution $u_2 \in \mathcal{E}(\kappa, 2)$ of (SEE) on the interval $[\tau_1, \tau_2]$ with past u on $[\tau, \tau_1]$ as a fixpoint of K_2 in $\mathcal{E}(\kappa, 2)$. As above we can show that there exists $C_2 > 0$ such that

$$\begin{aligned}&\big(\mathbb{E}\|u_2\|^r_{X^1(\tau_1,\tau_2)} + \mathbb{E}\sup_{t\in[\tau_1,\tau_2]}\|u_2(t)\|^r_{\mathrm{TR}}\big)^{1/r} \\ &\le C_2\Big(1 + \|u\|_{L^r(\Omega;X^1(\tau,T))} + \|u\|_{L^r(\Omega;C(\tau,\tau_1;\mathrm{TR}))} + \|\rho\|_{L^r(\Omega)}\Big) \\ &\le CC_2\big(1 + \|u_\tau\|_{L^r(\Omega;\mathrm{TR})} + \|\rho\|_{L^r(\Omega)}\big).\end{aligned} \tag{2.2.11}$$

Here, we used (2.2.7) and (2.2.10). We set $u = u_2$ on $[\tau_1, \tau_2]$. Repeating this argument k times, we obtain a unique strong solution u_n on every interval $[\tau_{n-1}, \tau_n]$ for $n = 1, \dots, k$ as a fixed point of K_n on $\mathcal{E}(\kappa, n)$. Setting $u = u_n$ on $[\tau_{n-1}, \tau_n]$, we get a strong solution u of (SEE) on $[\tau, T]$ with the claimed regularity properties. (2.2.9) is a combination of (2.2.10), (2.2.11) and the corresponding estimates for u_n, $n = 3, \dots, k$. The uniqueness is an immediate consequence of the uniqueness of the u_n, $n = 1, \dots, k$. □

Next, we prove a very useful Lemma that ensures that if the initial data, the operators and the nonlinearities coincide on some subset of Ω of positive measure, the corresponding solutions of (SEE) also coincide on this subset.

Lemma 2.2.7. *Choose one of the three settings* [TT], [GM], [LQ], *let* $u_\tau^{(1)}, u_\tau^{(2)} \in L^r(\Omega; \mathrm{TR})$ *be strongly* $\mathcal{F}_\tau$*-measurable and set* $\Gamma := \{u_\tau^{(1)} = u_\tau^{(2)}\}$. *Moreover, let* A_1 *and* A_2 *be operator-valued random variables that satisfy* [TT2], [GM2] *or* [LQ2] *respectively and that almost surely coincide on* Γ. *Let the nonlinearities* F_j, B_j, $j = 1, 2$ *satisfy* [S4]-[S6] *with* $\rho_i \in L^r(\Omega)$ *and with*

$$C_{\mathrm{MRD}}L^{(i)}_{F_j} + C_{\mathrm{MRD}}L^{(i)}_{B_j} < 1$$

for $i, j = 1, 2,$. *Moreover, we assume that* $F_1(v\mathbf{1}_\Gamma) = F_2(v\mathbf{1}_\Gamma)$ *and* $F_1(v\mathbf{1}_\Gamma) = F_2(v\mathbf{1}_\Gamma)$ *almost surely. If the* u_i, $i = 1, 2$, *are the unique strong solutions of*

$$\begin{cases} du_i(t) &= [-A_iu_i(t) + F_i(u_i)(t) + f(t)]dt + [B_i(u_i)(t) + b(t)]\,\mathrm{d}W(t), \quad t \in [\tau, T] \\ u_i(\tau) &= u_\tau^{(i)}, \end{cases}$$

then $u_1(\omega, t) = u_2(\omega, t)$ *for almost all* $\omega \in \Gamma$ *and all* $t \in [\tau(\omega), T]$.

Proof. We choose μ, κ as in the proof of Theorem 2.2.6 and we define the stopping times τ_0 and τ_n as before. Since u_1 and u_2 are strong solutions and in particular mild solutions, we have

$$u_i(t) = e^{-(t-\tau)A_i}\widetilde{u}_i(\tau) + \big(e^{-(\cdot)A_i} * (F_i(u_i) + f)\big)_\tau + \big(e^{-(\cdot)A_i} \diamond (B_i(u_i) + b)\big)_\tau$$

for $i = 1, 2$ on $[\tau, \tau_1]$ and in particular

$$\begin{aligned} &u_1(t)\mathbf{1}_\Gamma - u_2(t)\mathbf{1}_\Gamma \\ &\quad= \big(e^{-(\cdot)A_1} * (F_1(u_1\mathbf{1}_\Gamma) - F_1(u_2\mathbf{1}_\Gamma))\big)_\tau(t) + \big(e^{-(\cdot)A_1} \diamond (B_1(u_1\mathbf{1}_\Gamma) - B_1(u_2\mathbf{1}_\Gamma))\big)_\tau(t) \end{aligned}$$

almost surely for all $t \in [\tau, \tau_1]$. Here, we made use of the fact that the initial data, the nonlinearity and that the operators coincide on Γ. Note that one can drag $\mathbf{1}_\Gamma$ into the stochastic integral, since Γ is $\mathbb{F}_\tau$-measurable and thus the integrand

$$e^{-(t-s)A_1}(B_1(u_1\mathbf{1}_\Gamma)(s) - B_1(u_2\mathbf{1}_\Gamma)(s)\mathbf{1}_{\tau < s \le t}$$

is still adapted. Using the fixed point operator K on the space $\mathcal{E}(\kappa, 1)$ from the proof of Theorem 2.2.6 and its contraction property, we obtain

$$\|u_1\mathbf{1}_\Gamma - u_2\mathbf{1}_\Gamma\|_{\mu,\kappa,1} = \|K(u_1\mathbf{1}_\Gamma)\mathbf{1}_\Gamma - K(u_2\mathbf{1}_\Gamma)\mathbf{1}_\Gamma\|_{\mu,\kappa,1} \le \delta\|u_1\mathbf{1}_\Gamma - u_2\mathbf{1}_\Gamma\|_{\mu,\kappa,1}$$

for some $\delta \in [0, 1)$. This proves $u\mathbf{1}_\Gamma = v\mathbf{1}_\Gamma$ almost surely on $[\tau, \tau_1]$. Repeating this procedure inductively as in the proof of Theorem 2.2.6 finally yields $u\mathbf{1}_\Gamma = v\mathbf{1}_\Gamma$ almost surely on $[\tau, T]$. □

As an easy application of this Lemma, we can prove existence and uniqueness of strong solutions of (SEE) with initial data u_τ that is only integrable with respect to Ω and with nonlinearities whose ρ is also only measurable. In [95], Theorem 7.1, and [89], Proposition 5.4, a similar result was proved for measurable initial data u_0. We adapt their arguments to our situation.

Corollary 2.2.8. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [S4]-[S6] *be fulfilled with*

$$C_{\mathrm{MRD}}L_F^{(i)} + C_{\mathrm{MRD}}L_B^{(i)} < 1.$$

If we don't demand anything on u_τ *and* ρ, *but to be strongly* $\mathcal{F}_\tau$*-measurable, the equation*

$$\text{(SEE)} \begin{cases} du(t) &= [-Au(t) + F(u)(t) + f(t)]dt + [B(u)(t) + b(t)]\,\mathrm{d}W(t), \quad t \in [\tau, T], \\ u(\tau) &= u_\tau \end{cases}$$

has a unique strong solution u *on* $[\tau, T]$ *with* $u \in X^1(\tau, T) \cap C(\tau, T; \mathrm{TR})$ *almost surely. However,* u *has not necessarily any integrability properties with respect to* Ω.

Proof. We define $\Gamma_k := \{\|u_\tau\|_{\mathrm{TR}} < k, \rho \le k\}$. Since both u_τ and ρ are strongly $\mathcal{F}_\tau$-measurable, we have $\Gamma_k \in \mathcal{F}_\tau$ and $\Omega = \cup_{n=1}^{\infty}\Gamma_k$. Hence, we have $u_\tau \mathbf{1}_{\Gamma_k} \in L^r(\Omega; \mathrm{TR})$ and the nonlinearities $F(u)\mathbf{1}_{\Gamma_k}$ and $B(u)\mathbf{1}_{\Gamma_k}$ satisfy [S4] and [S5]. Moreover, the $\mathcal{F}_\tau$-measurable function in the linear growth condition is given by $\rho\mathbf{1}_{\Gamma_k} \in L^r(\Omega)$. As a consequence, Theorem 2.2.6 yields a unique strong solution $u^{(k)}$ of

$$\begin{cases} du(t) = \big[-Au(t) + F(u)(t)\mathbf{1}_{\Gamma_k} + f(t)\big]dt + \big[B(u)(t)\mathbf{1}_{\Gamma_k} + b(t)\big]dWt, \quad t \in (\tau, T] \\ u(\tau) = u_\tau \mathbf{1}_{\Gamma_k}, \end{cases}$$

with $u^{(k)} \in L^r(\Omega; X^1(\tau,T) \cap C(\tau,T;\mathrm{TR})$. Further, by Lemma 2.2.7 the processes $u^{(k)}$ and $u^{(m)}$ coincide almost surely on Γ_k if $m \ge k$. Therefore, we can define the pathwise limit

$$u(\omega, \cdot) = \lim_{k\to\infty} u^{(k)}(\omega, \cdot)$$

for almost all $\omega \in \Omega$. This limit is attained after finitely many k and we have $u = u_k$ on Γ_k. Moreover, since all the $u^{(k)}$ are strongly adapted, u is also strongly adapted as almost sure limit of strongly adapted processes. Clearly, since all the u_k are strong solutions, u is also a strong solution of (SEE) and we have $u \in X^1(\tau,T) \cap C(\tau,T;\mathrm{TR})$ almost surely, because each $u^{(k)}$ has this property. It remains to prove uniqueness.

Let v be another strong solution with initial data u_τ that satisfies $v \in X^1(\tau,T)\cap C(\tau,T;\mathrm{TR})$ almost surely. Defining the $\mathbb{F}$-stopping time

$$\begin{aligned} \eta_k := &\inf\{t \in [\tau,T] : \|u\|_{X^1(\tau,t)} + \|u\|_{C(\tau,t;\mathrm{TR})} > k\} \\ &\wedge \inf\{t \in [\tau,T] : \|v\|_{X^1(\tau,t)} + \|v\|_{C(\tau,t;\mathrm{TR})} > k\} \wedge T, \end{aligned}$$

we have $\lim_{k\to\infty} \eta_k = T$ almost surely. To be precise, since $u, v \in X^1(\tau,T) \cap C(\tau,T;\mathrm{TR})$ almost surely, for almost all $\omega \in \Omega$ there exists $k(\omega)$ such that $\nu_m(\omega) = T$ for $m \ge k(\omega)$. Thus, it is sufficient to prove that u and v coincide on $[0, \eta_k]$ for all $k \in \mathbb{N}$.

Since u and v are strong solutions, they are particularly mild solutions $[\tau, T]$ and hence, we have

$$\begin{aligned} &u(t)\mathbf{1}_{[\tau,\mu_k]} - v(t)\mathbf{1}_{[\tau,\mu_k]} \\ &\qquad = \big(e^{-(\cdot)A} * (F(u) - F(v))\mathbf{1}_{[\tau,\mu_k]}\big)_\tau(t) + \big(e^{-(\cdot)A} \diamond (B(u) - B(v))\mathbf{1}_{[\tau,\mu_k]}\big)_\tau(t) \end{aligned}$$

almost surely for $t \in [\tau, T]$. The Volterra property of F and B (see [S4], [S5]) implies $F(w)\mathbf{1}_{[\tau,\mu_k]} = F(w\mathbf{1}_{[\tau,\mu_k]})\mathbf{1}_{[\tau,\mu_k]}$ and $B(w)\mathbf{1}_{[\tau,\mu_k]} = B(w\mathbf{1}_{[\tau,\mu_k]})\mathbf{1}_{[\tau,\mu_k]}$ for $w = u$ or $w = v$. Thus, we have

$$\begin{aligned} u(t)\mathbf{1}_{[\tau,\mu_k]}(t) - v(t)\mathbf{1}_{[\tau,\mu_k]}(t)& \\ =&\big(e^{-(\cdot)A} * (F(u\mathbf{1}_{[\tau,\mu_k]}) - F(v\mathbf{1}_{[\tau,\mu_k]}))\mathbf{1}_{[\tau,\mu_k]}\big)_\tau(t) \\ &+ \big(e^{-(\cdot)A} \diamond (B(u\mathbf{1}_{[\tau,\mu_k]}) - B(v\mathbf{1}_{[\tau,\mu_k]}))\mathbf{1}_{[\tau,\mu_k]}\big)_\tau(t) \end{aligned}$$

almost surely for $t \in [\tau, T]$. Using the same fixed point operator K as in (2.2.2) on the interval $[\tau, (\tau+\kappa)\wedge\eta_k]$, we have $u\mathbf{1}_{[\tau,\mu_k]} - v\mathbf{1}_{[\tau,\mu_k]} = K(u\mathbf{1}_{[\tau,\mu_k]}) - K(v\mathbf{1}_{[\tau,\mu_k]})$. As in the proof of Lemma 2.2.5, we can choose $\kappa > 0$ and $\mu > 0$, such that

$$\begin{aligned}
&\|v-w\|_{L^r(\Omega;X^1(\tau,(\tau+\kappa)\wedge\mu_k))} + \mu\|v-w\|_{L^r(\Omega;X(\tau,(\tau+\kappa)\wedge\mu_k))} \\
&\quad + \|v-w\|_{L^r(\Omega;C(\tau,(\tau+\kappa)\wedge\mu_k;\mathrm{TR}))} \\
&\leq \delta\Big(\|v-w\|_{L^r(\Omega;X^1(\tau,(\tau+\kappa)\wedge\mu_k))} + \mu\|v-w\|_{L^r(\Omega;X(\tau,(\tau+\kappa)\wedge\mu_k))} \\
&\quad + \|v-w\|_{L^r(\Omega;C(\tau,(\tau+\kappa)\wedge\mu_k;\mathrm{TR}))}\Big)
\end{aligned}$$

for some $\delta \in [0,1)$, which proves $u(t) = v(t)$ almost surely for all $t \in [\tau, (\tau+\kappa)\wedge\mu_k]$. In the same way as in the proof of Theorem 2.2.6, we can iterate this procedure and get $u(t) = v(t)$ almost surely for all $t \in [\tau, \mu_k]$. This closes the proof. □

Finally, we give an analogous result to Lemma 2.2.7 in case that u_τ and ρ are not integrable with respect to Ω.

Corollary 2.2.9. *Choose one of the settings* [TT], [GM] *or* [LQ], *let* $u_\tau^{(1)}, u_\tau^{(2)} : \Omega \to \mathrm{TR}$ *be strongly* $\mathcal{F}_\tau$ *measurable and let* A_i, F_i *and* B_i *as in Lemma 2.2.7, but with only* $\mathcal{F}_\tau$*-measurable* ρ_i *for* $i \in \{1,2\}$*. Moreover, let* u_1 *and* u_2 *be the unique strong solutions of*

$$\begin{cases} du_i(t) &= [-A_i u_i(t) + F_i(u_i)(t) + f(t)]dt + [B_i(u_i)(t) + b(t)]\,\mathrm{d}W(t), \quad t \in [\tau, T], \\ u_i(\tau) &= u_\tau^{(i)}, \end{cases}$$

for $i = 1,2$*. Then, we have* $u_1(\omega,t) = u_2(\omega,t)$ *for almost all* $\omega \in \{u_\tau^{(1)} = u_\tau^{(2)}\}$ *and for all* $t \in [\tau(\omega), T]$*.*

Proof. We define $\Gamma_k = \{\|u_\tau^{(1)}\|_{\mathrm{TR}} < k, \rho_1 < k\} \cap \{\|u_\tau^{(2)}\|_{\mathrm{TR}} < k, \rho_2 < k\}$. As we have seen in the proof of Corollary 2.2.8, we have $u_1 = u_1^{(k)}$ and $u_2 = u_2^{(k)}$ on Γ_k. Here, $u_i^{(k)} \in L^r(\Omega, X^1(\tau,T) \cap C(\tau,T;\mathrm{TR}))$ is the solution of the truncated equation

$$\begin{cases} du_i^{(k)}(t) &= [-A_i u_i^{(k)}(t) + F_i(u_i^{(k)})(t)\mathbf{1}_{\Gamma_k} + f(t)]dt + [B_i(u_i^{(k)})(t)\mathbf{1}_{\Gamma_k} + b(t)]\,\mathrm{d}W(t), \\ u_i^k(\tau) &= u_\tau^{(i)}\mathbf{1}_{\Gamma_k}. \end{cases}$$

By Lemma 2.2.7, we have $u_1^{(k)}(\omega,t) = u_2^{(k)}(\omega,t)$ for almost all $\omega \in \big\{u_\tau^{(1)} = u_\tau^{(2)}\big\} \cap \Gamma_k$ and all $t \in [\tau, T]$. Since $\cup_{n=1}^\infty \Gamma_k = \Omega$, this implies $u_1(\omega,t) = u_2(\omega,t)$ for almost all $\omega \in \{u_\tau^{(1)} = u_\tau^{(2)}\}$ and all $t \in [\tau, T]$. □

2.3. Quasilinear parabolic stochastic evolution equations

In this chapter, we consider a quasilinear stochastic evolution equation of the form

$$\text{(QSEE)} \begin{cases} du(t) &= [-A(u(t))u(t) + F(u)(t) + f(t)]\ \mathrm{d}t + [B(u)(t) + b(t)]\,\mathrm{d}W(t), \ t \in [0,T], \\ u(0) &= u_0 \end{cases}$$

for $t \in [0,T]$ with a cylindrical Brownian motion W on a Hilbert space H. Our main result will be the existence and uniqueness of a strong solution of this equation up to a maximal blow-up stopping time τ. We work in the same abstract framework as in the previous section to deal with equations in $L^p(0,T;E)$ and $\gamma(0,T;E)$ with a UMD Banach space E and in $L^q(U;L^p(0,T))$ in a unified way. As before, the general framework consists of a Banach space X with an extension to the timeline $X(a,b)$ for some interval $(a,b) \subset [0,T]$, a corresponding maximal regularity space $X^1(a,b)$ and a trace space TR. Again, we will choose these spaces in such a way that the solution u of (QSEE) always satisfies $u \in X^1(0,\tau)$, $Au \in X(0,\tau)$ and $u \in C(0,\tau;\mathrm{TR})$ almost surely.

2.3.1. Globally Lipschitz continuous quasilinearity

In the semilinear theory of the previous section, the assumptions on $\omega \mapsto A(\omega)$ were uniform with respect to ω. The application of these result gives a quasilinear theory for operators $(\omega, y) \mapsto A(\omega, y)$ with uniform assumptions with respect to ω and y and with a globally Lipschitz dependence on y.

Before we start, we present our setting in detail. We begin with the assumptions that fit to the maximal L^p-regularity estimates in type 2 Banach spaces from section 2.1.1. Again, we will denote this setting with [TT].

[TTQ1] Let $p \in (2,\infty)$, $r = p$ and E, E^1 be UMD Banach spaces with type 2 or $p = 2$ and E, E^1 Hilbert spaces. We assume the embedding $E^1 \hookrightarrow E$ to be dense and we assume that the family

$$\{J_\delta : \delta > 0\} \subset B\big(L^p(\Omega \times (0,\infty); \gamma(H;E)), L^p(\Omega \times (0,\infty); E)\big)$$

defined by

$$J_\delta b(t) := \delta^{-1/2} \int_{(t-\delta)\vee 0}^{t} b(s)\,\mathrm{d}W(s)$$

is $\mathcal{R}$-bounded.

[TT2] The mapping $A : \Omega \times (E,E^1)_{1-1/p,p} \to \mathcal{B}(E^1, E)$ is such that $\omega \mapsto A(\omega, y)x$ is strongly $\mathcal{F}_0$-measurable for all $x \in E^1$ and $y \in (E,E^1)_{1-1/p,p}$ with $0 \in \rho(A(\omega,y))$ almost surely. Moreover, we assume $D(A(\omega,y)) = E^1$, i.e

$$\|A(\omega,y)x\|_E \simeq \|x\|_{E^1}$$

for almost all $\omega \in \Omega$, all $y \in (E,E^1)_{1-1/p,p}$ and all $x \in E^1$ with estimates independent of y, x and ω.

[TTQ3] For all $y \in (E,E^1)_{1-1/p,p}$ and almost all $\omega \in \Omega$, the operators $A(\omega,y)$ are sectorial and have a bounded $H^\infty(\Sigma_\eta)$-calculus of angle $\eta \in (0,\pi/2)$, i.e.

$$\|\phi(A(\omega,y))\|_{\mathcal{B}(E)} \le C\|\phi\|_{H^\infty(\Sigma_\eta)}$$

with a constant $C > 0$ independent of ω and y.

[TTQ4] There exists $C_Q > 0$ such that for all $z, y \in (E, E^1)_{1-1/p,p}$ and almost all $\omega \in \Omega$, we have

$$\|A(\omega, z) - A(\omega, y)\|_{\mathcal{B}(E^1,E)} \le C_Q \|z - y\|_{(E,E^1)_{1-1/p,p}}.$$

In this setting, we set $X := E$, $X(a, b) := L^p(a, b; E)$, $X^1(a, b) := L^p(a, b; E^1)$ and we define $X_H^{\frac{1}{2}}(a, b) := L^p(0, T; \gamma(H; [E, E^1]_{\frac{1}{2}}))$. The trace space TR is the real interpolation space $(E; E^1)_{1-1/p,p}$.

Next, we make the assumptions that fit to the maximal γ-regularity estimates in UMD Banach spaces from section 2.1.2. Again, we will denote this setting with [GM].

[GMQ1] Let $r \in (1, \infty)$, E, E^1 UMD Banach spaces with property-(α) and a dense embedding $E^1 \hookrightarrow E$.

[GMQ2] The mapping $A : \Omega \times [E, E^1]_{\frac{1}{2}} \to \mathcal{B}(E^1, E)$ is such that $\omega \mapsto A(\omega, y)x$ is strongly $\mathcal{F}_0$-measurable for all $x \in E^1$ and $y \in [E, E^1]_{\frac{1}{2}}$ with $0 \in \rho(A(\omega, y))$ almost surely. Moreover, we assume $D(A(\omega, y)) = E^1$ almost surely, i.e we have

$$\|A(\omega, y)x\|_E \simeq \|x\|_{E^1}$$

for almost all $\omega \in \Omega$, all $y \in [E, E^1]_{\frac{1}{2}}$ and all $x \in E^1$ with estimates independent of y, x and ω.

[GMQ3] For all $y \in [E, E^1]_{\frac{1}{2}}$ and almost all $\omega \in \Omega$, the operators $A(\omega, y)$ are sectorial and have a bounded $H^\infty(\Sigma_\eta)$-calculus of angle $\eta \in (0, \pi/2)$, i.e.

$$\|\phi(A(\omega, y))\|_{\mathcal{B}(E)} \le C \|\phi\|_{H^\infty(\Sigma_\eta)}$$

with a constant $C > 0$ independent of ω and y.

[GMQ4] There exists $C_Q > 0$ such that for all $z, y \in [E, E^1]_{\frac{1}{2}}$ and almost all $\omega \in \Omega$, we have

$$\mathcal{R}\Big(\big\{A(\omega, z(t)) - A(\omega, y(t)) : t \in [a, b]\big\} \subset \mathcal{B}(E^1, E)\Big) \le C_Q \sup_{t\in[a,b]} \|z(t) - y(t)\|_{[E,E^1]_{\frac{1}{2}}}.$$

Here, we set $X := E$, $X(a, b) := \gamma(a, b; E)$, $X_H^{\frac{1}{2}}(a, b) := \gamma(0, T; \gamma(H; [E, E^1]_{\frac{1}{2}}))$ and we define $X^1(a, b) := \gamma(a, b; E^1)$. The trace space TR is the complex interpolation space $[E; E^1]_{\frac{1}{2}}$.

Next, we make the assumptions that fit to the maximal regularity estimates in the space $L^q(U; L^p(0, T))$ from section 2.1.2. In the sequel, this setting will be denoted with [LQ].

[LQQ1] Let $r \in (1, \infty)$, $p \in (2, \infty)$, $q \in (2, \infty)$, $U \subset \mathbb{R}^d$ and μ be a σ-finite measure on U. We choose $H = l^2(\mathbb{N})$ and $W(t) = \sum_{j=1}^\infty e_k\beta_k$ for an sequence $(\beta_k)_{k\in\mathbb{N}}$ of independent Brownian motions and with unit vectors $(e_k)_k \in l^2(\mathbb{N})$. Moreover, let Λ be a closed and densely defined operator on $L^q(U) := L^q(U, \mu)$ with $0 \in \rho(\Lambda)$ that has an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_{\widetilde{\eta}})$-calculus for some $\widetilde{\eta} \in (0, \frac{\pi}{2})$. For given $\alpha \in (\frac{1}{p}, 1]$, we denote $E^\alpha := D(\Lambda^\alpha)$ and $E^{\alpha-1}$ as the extrapolation space of $L^q(U)$ with the norm $\|\Lambda^{\alpha-1} \cdot\|_{L^q(U)}$.

[LQQ2] The mapping $A : \Omega \times F^{\alpha-1/p}_{\Lambda,q,p} \to \mathcal{B}(E^\alpha, E^{\alpha-1})$ is strongly $\mathcal{F}_0$-measurable. Moreover, the $A(\omega, y)$ are closed with $0 \in \rho(A(\omega, y))$ and $\Lambda^\alpha A(\omega, y)^{-\alpha}$, $A(\omega, y)^\alpha \Lambda^{-\alpha}$, $A(\omega, y)^{\alpha-1}\Lambda^{1-\alpha}$ and $\Lambda^{\alpha-1}A(\omega, y)^{1-\alpha}$ are for almost all $\omega \in \Omega$ and all $y \in F^{\alpha-1/p}_{\Lambda,q,p}$ $\mathcal{R}_p$-bounded on $L^q(U)$ with bounds independent of ω and y.

[LQQ3] For all $y \in F^{\alpha-1/p}_{\Lambda,q,p}$ and almost all $\omega \in \Omega$, the operators $A(\omega, y)$ are sectorial and have an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_\eta)$ calculus with

$$\mathcal{R}_p\Big(\big\{\psi(A(\omega, y)) : \|\psi\|_{H^\infty(\Sigma_\eta)} \le 1\big\} \subset \mathcal{B}(L^q(U))\Big) \le M$$

for some $M > 0$ and $\eta \in (0, \pi/2)$ independent of $\omega \in \Omega$ and $y \in F^{\alpha-1/p}_{\Lambda,q,p}$.

[LQQ4] There exists $C_Q > 0$ such that for all $z, y \in F^{\alpha-1/p}_{\Lambda,q,p}$ and almost all $\omega \in \Omega$, we have

$$\begin{aligned}\mathcal{R}_p\Big(\big\{\Lambda^{\alpha-1}(A(\omega, z(t)) - A(\omega, y(t)))\Lambda^{-\alpha} : t \in [a, b]\big\} \subset \mathcal{B}(L^q(U))\Big)&\\ \le C_Q \sup_{t\in[a,b]} \|z(t) - y(t)\|_{F^{\alpha-1/p}_{\Lambda,q,p}}&.\end{aligned}$$

The assumption on Λ imply that Λ can be extended to an operator $\Lambda_{p,a,b}$ on $L^q(U; L^p(a, b))$ that has a bounded $H^\infty(\Sigma_{\widetilde{\eta}})$-calculus for some angle $0 \le \widetilde{\eta} < \pi/2$. We choose $X^\alpha(a, b) := D(\Lambda^\alpha_{p,a,b})$ and $X^{\alpha-1}(a, b) = \Lambda^{1-\alpha}_{p,a,b}\big(L^q(U; L^p(a, b))\big)$. The stochastic part $B(u)+b$ is contained in the space $X^{\frac{1}{2}}_H(a, b) := \Lambda^{\frac{1}{2}-\alpha}_{p,a,b}\big(L^q(U; L^p(a, b; l^2(\mathbb{N})))\big)$. The trace space TR is the Triebel-Lizorkin space $F^{\alpha-1/p}_{\Lambda,q,p}$ in the sense of Definition 1.4.2.

The other assumptions are similar in any of the above settings and can be formulated universally.

[Q5] The initial value $u_0 : \Omega \to \mathrm{TR}$ is strongly $\mathcal{F}_0$-measurable.

[Q6] For any $\mathbb{F}$-stopping time μ with $0 \le \mu \le T$ almost surely, the mapping

$$F : L^0_{\mathbb{F}}\big(\Omega; X^1(0, \mu) \cap C(0, \mu; \mathrm{TR})\big) \to L^0_{\mathbb{F}}\big(\Omega; X(0, \mu)\big)$$

is a Volterra map, i.e. for a given $\mathbb{F}$-stopping time $\widetilde{\tau}$ with $0 \le \widetilde{\tau} \le \mu$ almost surely, the restriction $F(u)_{|[0,\widetilde{\tau}]}$ only depends on $u_{|[0,\widetilde{\tau}]}$. This means that we have $F(u)\mathbf{1}_{[0,\widetilde{\tau}]} = F(v)\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely, whenever $u\mathbf{1}_{[0,\widetilde{\tau}]} = v\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely. Moreover, there exist constants $L^{(i)}_F, \widetilde{L}_F, C^{(i)}_F \ge 0$, $i = 1, 2$ such that F is of linear growth, i.e.

$$\|F(\phi_1)\|_{X(0,\mu)} \le C^{(1)}_F \left(1 + \|\phi_1\|_{X^1(0,\mu)}\right) + C^{(2)}_F \left(1 + \|\phi_1\|_{C(0,\mu;\mathrm{TR})}\right)$$

and Lipschitz continuous, i.e.

$$\begin{aligned}&\|F(\phi_1) - F(\phi_2)\|_{X(0,\mu)}\\ &\quad \le L^{(1)}_F \|\phi_1 - \phi_2\|_{X^1(0,\mu)} + \widetilde{L}_F \|\phi_1 - \phi_2\|_{X(0,\mu)} + L^{(2)}_F \|\phi_1 - \phi_2\|_{C(0,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1, \phi_2 \in X^1(0, \mu) \cap C(0, \mu; \mathrm{TR})$ with constants independent of $\omega \in \Omega$.

[Q7] For any $\mathbb{F}$-stopping time μ with $0 \le \mu \le T$ almost surely, the mapping

$$B : L^0_{\mathbb{F}}\big(\Omega; X^1(0,\mu) \cap C(0,\mu;\mathrm{TR})\big) \to L^0_{\mathbb{F}}\big(\Omega; X_H^{\frac{1}{2}}(0,\mu)\big)$$

is a Volterra map, i.e. for a given $\mathbb{F}$-stopping time $\widetilde{\tau}$ with $0 \le \widetilde{\tau} \le \mu$ almost surely, the restriction $B(u)_{|[0,\widetilde{\tau}]}$ only depends on $u_{|[0,\widetilde{\tau}]}$. This means that we have $B(u)\mathbf{1}_{[0,\widetilde{\tau}]} = B(v)\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely, whenever $u\mathbf{1}_{[0,\widetilde{\tau}]} = v\mathbf{1}_{[0,\widetilde{\tau}]}$ almost surely. Moreover, there exist constants $L_B^{(i)}, \widetilde{L}_B, C_B^{(i)} \ge 0$, $i = 1, 2$, such that B is of linear growth, i.e.

$$\|B(\phi_1)\|_{X_H^{\frac{1}{2}}(0,\mu)} \le C_B^{(1)}\left(1 + \|\phi_1\|_{X^1(0,\mu)}\right) + C_B^{(2)}\left(1 + \|\phi_1\|_{C(0,\mu;\mathrm{TR})}\right)$$

and Lipschitz continuous, i.e.

$$\begin{aligned}&\|B(\phi_1) - B(\phi_2)\|_{X_H^{\frac{1}{2}}(0,\mu)} \\ &\quad\le L_B^{(1)}\|\phi_1 - \phi_2\|_{X^1(0,\mu)} + \widetilde{L}_B\|\phi_1 - \phi_2\|_{X(0,\mu)} + L_B^{(2)}\|\phi_1 - \phi_2\|_{C(0,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1, \phi_2 \in L^p(0,\mu;E^1) \cap C(\tau, T; \mathrm{TR})$ with constants independent of $\omega \in \Omega$.

[Q8] We assume $f \in L^r_{\mathbb{F}}(\Omega; X(0,T))$ and $b \in L^r_{\mathbb{F}}(\Omega; X_H^{\frac{1}{2}}(0,T))$.

As we have shown in section 2.1, the assumptions imply uniform maximal regularity estimates for the deterministic and the stochastic convolution in all of the three settings. There exists $C_{\mathrm{MRS}}, C_{\mathrm{MRD}} > 0$ such that for every $\mathbb{F}$-stopping time μ with $0 \le \mu \le T$ almost surely and all $y \in \mathrm{TR}$, we have

$$\left\|\left(e^{-(\cdot)A(y)} \diamond g\right)_\mu\right\|_{L^r(\Omega;X^1(\mu,T)\cap C(\mu,T;\mathrm{TR}))} \le C_{\mathrm{MRS}}\|g\|_{L^r(\Omega;X_H^{\frac{1}{2}}(\mu,T))}$$

for all $g \in L^r(\Omega; X_H^{\frac{1}{2}}(\mu,T))$ and

$$\left\|\left(e^{-(\cdot)A(y)} * \widetilde{g}\right)_\mu\right\|_{L^r(\Omega;X^1(\mu,T)\cap C(\mu,T;\mathrm{TR}))} \le C_{\mathrm{MRD}}\|\widetilde{g}\|_{L^r(\Omega;X(\mu,T))}$$

for all $\widetilde{g} \in L^r(\Omega; X(\mu,T))$. As in the semilinear case, we require small Lipschitz constants in [Q6], [Q7]. More precisely, we assume the following.

[Q9] Let the constants of [Q6] and [Q7] be small enough to ensure

$$C_{\mathrm{MRD}} L_F^{(i)} + C_{\mathrm{MRD}} L_B^{(i)} < 1$$

for $i = 1, 2$.

We define strong solutions of (QSEE) in the same way as we defined strong solutions of (SEE). The only difference is that we replace the autonomous operator by $A(\omega, u(t))$.

Definition 2.3.1. *Let μ be an $\mathbb{F}$-stopping time with $0 \le \mu \le T$ almost surely. A process $u : \Omega \times [0,\mu] \to X$ is called a strong solution of* (QSEE) *on* $[0,\mu]$ *if it is strongly measurable and strongly adapted with $u \in X^1(0,\mu) \cap C(0,\mu;\mathrm{TR})$ almost surely and if u satisfies the following identity depending on the respective setting.*

[TT]

$$u(t) - u_0 = -\int_0^t A(u(s))u(s)\,\mathrm{d}s + \int_0^t F(u)(s) + f(s)\,\mathrm{d}s + \int_0^t B(u)(s) + b(s)\,\mathrm{d}W(s)$$

almost surely for all $t \in [0,\mu]$ as an equation in E. Here, the integral over time is an E-valued Bochner integral and the stochastic integral is well-defined as a consequence of Theorem 1.3.4 and (1.3.1).

[GM]

$$u(t) - u_0 = -A(u)u(\mathbf{1}_{[\tau,t]}) + (F(u) + f)(\mathbf{1}_{[\tau,t]}) + \int_\tau^t B(u)(s) + b(s)\,\mathrm{d}W(s)$$

almost surely for all $t \in [0,\mu]$ as an equation in E. Note that $A(u)u, F(u), f \in \gamma(0,\mu;E)$ particularly means that they are linear operators from $L^2(0,\mu) \to E$. Moreover, the stochastic integral is well-defined as a consequence of Theorem 1.3.4.

[LQ]

$$\begin{aligned}\Lambda^{\alpha-1}u(t,x) - &\Lambda^{\alpha-1}\widetilde{u}(\tau,x) = \\ &-\int_0^t \Lambda^{\alpha-1}A(u)u(s,x)\,\mathrm{d}s + \int_0^t \Lambda^{\alpha-1}F(u)(s,x) + \Lambda^{\alpha-1}f(s,x)\,\mathrm{d}s \\ &+ \int_0^t \Lambda^{\alpha-1}B(u)(s,x) + \Lambda^{\alpha-1}b(s,x)\,\mathrm{d}W(s)\end{aligned}$$

holds almost surely for almost all $x \in U$ and for all $t \in [0,\mu]$ as an equation in $\mathbb{C}$. The deterministic integrals are well-defined, since $\Lambda^{\alpha-1}A(u)u, \Lambda^{\alpha-1}F(u), \Lambda^{\alpha-1}f \in L^q(U;L^p(0,\mu))$ almost surely.

Even in the deterministic case, quasilinear evolution equations do not have global solutions without further structural assumptions. Therefore, we now explain the concept of local solutions. The following definition adapts the terms Van Neerven, Veraar and Weis introduced in [95] to our situation.

Definition 2.3.2. *Let σ, σ_n, $n \in \mathbb{N}$, be $\mathbb{F}$-stopping times with $0 \le \sigma, \sigma_n \le T$ almost surely for all $n \in \mathbb{N}$.*

a) We say that $\big(u, (\sigma_n)_n, \sigma\big)$ is a local solution of (QSEE)*, if $(\sigma_n)_{n\in\mathbb{N}}$ is an increasing sequence with $\lim_{n\to\infty} \sigma_n = \sigma$ almost surely such that*

$$u \in X^1(0,\sigma_n) \cap C(0,\sigma_n;\mathrm{TR})$$

almost surely and such that u is a strong solution of (QSEE) *on $[0,\sigma_n]$ for all $n \in \mathbb{N}$.*

b) We call a local solution $\big(u, (\sigma_n)_n, \sigma\big)$ of (QSEE) *unique if every other local solution $\big(\widetilde{u}, (\widetilde{\sigma_n})_n, \widetilde{\sigma}\big)$ satisfies $\widetilde{u}(\omega,t) = u(\omega,t)$ for almost all $\omega \in \Omega$ and for all $t \in [0, \sigma \wedge \widetilde{\sigma})$.*

c) We call a local solution $\big(u, (\sigma_n)_n, \sigma\big)$ of (QSEE) *maximal unique local solution if for any other local solution $\big(\widetilde{u}, (\widetilde{\sigma_n})_n, \widetilde{\sigma}\big)$, we almost surely have $\widetilde{\sigma} \le \sigma$ and $\widetilde{u}(\omega,t) = u(\omega,t)$ for almost all $\omega \in \Omega$ and all $t \in [0,\widetilde{\sigma})$.*

If the approximating sequence σ_n is not important for a result, we shortly write (u, σ) for the local solution. In the following, we establish a well-posedness result for the quasilinear evolution equation (QSEE) up to a maximal stopping time. The next theorem is one of our main results and will be proved during this section.

Theorem 2.3.3. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [Q5]-[Q9] *be fulfilled. Then, the quasilinear stochastic evolution equation* (QSEE) *has a maximal unique local solution* $\big(u, (\tau_n)_n, \tau\big)$. *Moreover, we have*

$$\mathbb{P}\Big\{\tau < T, \|u\|_{X^1(0,\tau)} < \infty, u : [0,\tau) \to \mathrm{TR} \text{ is uniformly continuous}\Big\} = 0. \tag{2.3.1}$$

If we additionally assume $u_0 \in L^r(\Omega; \mathrm{TR})$, *the estimates*

$$\Big(\mathbb{E}\|u\|^r_{X^1(0,\tau_n)}\Big)^{1/r} \leq C^{(n)}(1 + \|u_0\|_{L^r(\Omega,\mathrm{TR})})$$

$$\bigg(\mathbb{E} \sup_{t\in[0,\tau_n]} \|u(t)\|^r_{\mathrm{TR}}\bigg)^{1/r} \leq C^{(n)}(1 + \|u_0\|_{L^r(\Omega,\mathrm{TR})})$$

hold true for all $n \in \mathbb{N}$ *and for some* $C^{(n)} > 0$ *independent of* u_0.

Note that the blow-up criterion (2.3.1) can be used to show $\tau = T$ on some paths. Indeed, if one can show $\|u\|_{X^1(0,\tau)} < \infty$ and the uniform continuity of $u : [0, \tau) \to \mathrm{TR}$ on a set $\widetilde{\Omega} \subset \Omega$, then $\tau = T$ almost surely on $\widetilde{\Omega}$. This can be seen in the following way. $\widetilde{\Omega}$ can be decomposed into a set N of measure 0 and

$$\begin{aligned}\Big(\widetilde{\Omega} \cap \{\tau = T\}\Big) &\cup \Big(\widetilde{\Omega} \cap \Big\{\tau < T, \|u\|_{X^1(0,\tau)} < \infty, u : [0,\tau) \to \mathrm{TR} \text{ is uni. cont.}\Big\}\Big) \\ &\cup \Big(\widetilde{\Omega} \cap \Big\{\tau < T, \|u\|_{X^1(0,\tau)} = \infty \text{ or } u : [0,\tau) \to \mathrm{TR} \text{ is not uni. cont.}\Big\}\Big).\end{aligned}$$

The last set has measure zero by assumption and the second set has measure zero by the blow-up criterion. Hence, we end up with $\widetilde{\Omega} = \Big(\widetilde{\Omega} \cap \{\tau = T\}\Big) \cup N$.

The proof of this theorem is rather technical and will be done in this section. As a start, we need a Lemma that brings the Lipschitz estimates in [TTQ4], [GMQ4] and [LQQ4] into the general framework.

Lemma 2.3.4. *Choose one of the settings* [TT], [GM] *or* [LQ]. *In any case, we have*

$$\|fg\|_{X(a,b)} \leq \|f\|_{L^\infty(a,b)}\|g\|_{X(a,b)}$$

for all $f \in L^\infty(a, b)$ *and all* $g \in X(a, b)$. *Moreover, there exists* $L_Q > 0$ *such that*

$$\|(A(y) - A(z))v\|_{X(a,b)} \leq L_Q \sup_{t\in[a,b]} \|y(t) - z(t)\|_{\mathrm{TR}}\|v\|_{X^1(a,b)}$$

almost surely for all $y, z \in C(a, b; \mathrm{TR})$ *and all* $v \in X^1(a, b)$ *with constants independent of* $\omega \in \Omega$ *and* $(a, b) \subset [0, T]$.

Proof. The first assertion is trivial in [TT] and [LQ], since in both cases we can drag out f from $\|fg\|_{L^p(a,b;E)}$ and $\|f\Lambda^{-\alpha}g\|_{L^q(U;L^p(a,b))}$ with the $\|\cdot\|_{L^\infty(a,b)}$ norm. By assumption [TTQ4], the second assertion is immediate in the setting [TT].

In [GM], we need a pointwise multiplier result in the space $\gamma(a,b;E)$. From [94], Theorem 5.2, we get

$$\|(A(y)-A(z))v\|_{\gamma(a,b;E)} \le \gamma\Big(\big\{A(z(t))-A(y(t)) : t\in[a,b]\big\}\subset \mathcal{B}(E^1,E)\Big)\|v\|_{\gamma(a,b;E^1)}$$

almost surely and

$$\|fg\|_{\gamma(a,b;E)} \le \|f\|_{L^\infty(a,b)}\|g\|_{\gamma(a,b;E)}.$$

The well-known fact that all $\gamma(\mathcal{T})\le\mathcal{R}(\mathcal{T})$ for all $\mathcal{T}\subset\mathcal{B}(E^1,E)$ (see e.g. [71], Theorem 1.1) and [GMQ4] complete the argument.

It remains to show the last inequality in the setting [LQ]. By Proposition 1.4.1, we get

$$\begin{aligned}
&\|\Lambda^{\alpha-1}(A(y)-A(z))v\|_{L^q(U;L^p(a,b))}\\
&\qquad=\|\Lambda^{\alpha-1}(A(y)-A(z))\Lambda^{-\alpha}\Lambda^{\alpha}v\|_{L^q(U;L^p(a,b))}\\
&\qquad\le \mathcal{R}_p\Big(\big\{\Lambda^{\alpha-1}(A(z(t))-A(y(t)))\Lambda^{-\alpha} : t\in[a,b]\big\}\subset\mathcal{B}(L^q(U))\Big)\|\Lambda^{\alpha}v\|_{L^q(U;L^p(a,b))}.
\end{aligned}$$

Together with assumption [LQQ4], this closes the proof. □

Before we start, we briefly describe our strategy. First, we prove existence and uniqueness of a strong solution u in a small ball around the initial value up to a stopping time τ_1 with the semilinear theory, we developed in section 2.2. Consequently, the set of stopping times σ such that there exists a unique solution u on $[0,\sigma]$ is non-empty and hence, the essential supremum $\tau:\Omega\to[0,T]$ of this set exists. We then show that τ is also a stopping time and that there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty}\tau_n=\tau$ almost surely. Last but not least, we derive the blow-up alternative

$$\mathbb{P}\Big\{\tau<T, \|u\|_{X^1(0,\tau)}<\infty, u:[0,\tau)\to \mathrm{TR}\ \text{ is uniformly continuous}\Big\}=0,$$

which helps us to prove that $\big(u,(\tau_n)_n,\tau\big)$ is indeed a maximal unique solution.

We begin with the definition of a cut-off function ϕ_λ that will enclose the processes in a suitable ball around the initial value. Let

$$\Phi(t)=\begin{cases}1 & \text{for } t\in[0,1]\\ -t+2 & \text{for } t\in[1,2]\\ 0 & \text{for } t\in[2,\infty)\end{cases}$$

and define $\Phi_\lambda(t):=\Phi(\frac{t}{\lambda})$ which gives us a monotonously decreasing function bounded by 1 that equals 1 on $[0,\lambda]$ and vanishes on $[2\lambda,\infty)$. Moreover, Φ_λ is Lipschitz continuous with

$$|\Phi_\lambda(t)-\Phi_\lambda(s)|\le\lambda^{-1}|t-s|$$

for all $t, s \geq 0$. Now we can define the desired cut-off function. For $u_a \in \mathrm{TR}$, $u \in C(a, b; \mathrm{TR}) \cap X^1(a, b)$ and $t \in [a, b]$, let

$$\theta_\lambda(a, t, u, u_a) := \Phi_\lambda\big(\|u\|_{X^1(a,t)} + \sup_{s\in[a,t]} \|u(s) - u_a\|_{\mathrm{TR}}\big).$$

Clearly, we have $\theta_\lambda(a, t, u, u_a) = 0$ if $\|u\|_{X^1(a,t)} + \sup_{s\in[a,t]} \|u(s) - u_a\|_{\mathrm{TR}} \geq 2\lambda$ and if on the other hand $\|u\|_{X^1(a,t)} + \sup_{s\in[a,t]} \|u(s) - u_a\|_{\mathrm{TR}} \leq \lambda$, we obtain

$$A(u(t))u(t) = A(u_a)u(t) + \theta_\lambda(a, t, u, u_a)\big(A(u(t)) - A(u_a)\big)u(t).$$

With this fact in mind, it is quite natural to consider the stochastic evolution equation

$$\begin{cases} du(t) &= \Big[-A(u_0)u(t) + \widetilde{F}_\lambda(u)(t) + f(t)\Big] \,\mathrm{d}t + [B(u)(t) + b(t)] \,\mathrm{d}W(t), \\ u(0) &= u_0, \end{cases} \tag{2.3.2}$$

where $\widetilde{F}_\lambda$ is given by

$$\widetilde{F}_\lambda(u)(t) = \theta_\lambda(0, t, u, u_0)\big(A(u_0) - A(u(t))\big)u(t) + F(u)(t).$$

Since we want to sustain the local solution to a maximal time interval, it will be necessary to consider not only the initial time zero but also, as in the previous section, an equation that begins at a $\mathbb{F}$-stopping time σ with a given past $\widetilde{u} \in X^1(0, \sigma) \cap C(0, \sigma; \mathrm{TR})$ almost surely.

The following Lemma makes sure that the nonlinearity $\widetilde{F}_\lambda$ satisfies the assumptions of Theorem 2.2.6, if one chooses λ small enough.

Lemma 2.3.5. *Choose one of the settings* [TT], [GM] *or* [LQ], *let σ, μ be $\mathbb{F}$-stopping times with $0 \leq \sigma \leq \mu \leq T$ almost surely and let $u_\sigma : \Omega \to \mathrm{TR}$ be strongly $\mathcal{F}_\sigma$-measurable. For $t \in [0, T]$, $\lambda > 0$ and $y \in L^0_{\mathbb{F}}\big(\Omega; X^1(\sigma, \mu) \cap C(\sigma, \mu, \mathrm{TR})\big)$ with $y(\sigma) = u_\sigma$, we define*

$$Q_{\lambda,\sigma}(y, u_\sigma)(t) := \begin{cases} \theta_\lambda(\sigma, t, y(t), u_\sigma)\big(A(u_\sigma) - A(y(t))\big)y(t) & , \text{if } \sigma \leq t \leq \mu, \\ 0 & , \text{if } t < \sigma. \end{cases}$$

Then, $Q_{\lambda,\sigma}$ maps

$$\Big\{y \in L^0_{\mathbb{F}}(\Omega; X^1(\sigma, \mu) \cap C(\sigma, \mu; \mathrm{TR})) : y(\tau) = u_\sigma\Big\} \to L^0_{\mathbb{F}}(\Omega; X(\sigma, \mu))$$

and has the Volterra property we introduced in [S4]. *Moreover, $Q_{\lambda,\sigma}$ is bounded i.e*

$$\|Q_{\lambda,\sigma}(u, u_\sigma)\|_{X(\sigma,\mu)} \leq 4C_Q\lambda^2$$

and Lipschitz continuous, i.e

$$\begin{aligned} &\|Q_{\lambda,\sigma}(u, u_\sigma) - Q_{\lambda,\sigma}(v, u_\sigma)\|_{X(\sigma,\mu)} \\ &\qquad \leq 6C_Q\lambda\left(\|u - v\|_{X^1(\sigma,\mu)} + \|u - v\|_{C(\sigma,\mu;\mathrm{TR})}\right) \end{aligned}$$

almost surely and for all $u, v \in L^0_{\mathbb{F}}\big(\Omega; X^1(\sigma, T) \cap C(\sigma, T; \mathrm{TR})\big)$. All in all, $Q_{\lambda,\sigma}$ satisfies the assumption [S4] *of the previous section.*

Proof. The measurability properties of $Q_{\lambda,\sigma}$ are immediate and $Q_{\lambda,\sigma}$ has the Volterra property, since both $\theta_\lambda(\sigma, t, y(t), u_\sigma)$ and $A(y(t))$ only depend on $y_{|[0,t]}$.

To prove the Lipschitz and the growth estimate we argue pathwise for fixed $\omega \in \Omega$ with $\sigma(\omega) \le \mu(\omega) \le T$. In order to keep the notation simple, we suppress the explicit dependence on ω. Let $u, v \in X^1(\sigma,\mu) \cap C(\sigma,\mu;\mathrm{TR})$ with $u(\sigma) = v(\sigma) = u_\sigma$ and define

$$\sigma_u = \inf\left\{s \in [\sigma,\mu] : \|u\|_{X^1(\sigma,s)} + \|u - u_\sigma\|_{C(\sigma,s;\mathrm{TR})} \ge 2\lambda\right\} \wedge \mu$$

and similarly

$$\sigma_v = \inf\left\{s \in [\sigma,\mu] : \|v\|_{X^1(\sigma,s)} + \|v - u_\sigma\|_{C(\sigma,s;\mathrm{TR})} \ge 2\lambda\right\} \wedge \mu.$$

The definition of $\theta_\lambda(\sigma, t, u, u_\sigma)$ ensures $Q_{\lambda,\sigma}(u, u_\sigma)(t) = 0$ for $t \ge \sigma_u$ and $Q_{\lambda,\sigma}(v, u_\sigma)(t) = 0$ for $t \ge \sigma_v$. In the following, we assume without restriction that $\sigma_u \ge \sigma_v$. First we prove the growth estimate. $\theta_\lambda \le 1$, Lemma 2.3.4 and the definition of σ_u yield

$$\begin{aligned}\|Q_{\lambda,\sigma}(u,u_\sigma)\|_{X(\sigma,\mu)} &= \|Q_{\lambda,\sigma}(u,u_\sigma)\|_{X(\sigma,\sigma_u)} \\ &\le L_Q \sup_{t\in[\sigma,\sigma_u]} \|u(t) - u_\sigma\|_{\mathrm{TR}} \|u\|_{X^1(\sigma,\sigma_u)} \le 4C_Q\lambda^2.\end{aligned}$$

For the Lipschitz estimate, we start with

$$\begin{aligned}&\|Q_{\lambda,\sigma}(u,u_\sigma) - Q_{\lambda,\sigma}(v,u_\sigma)\|_{X(\sigma,\mu)} \\ &\qquad \le \|\big(\theta_\lambda(\sigma,\cdot,u,u_\sigma) - \theta_\lambda(\sigma,\cdot,v,u_\sigma)\big)\big(A(u) - A(u_\sigma)\big)u\|_{X(\sigma,\sigma_u)} \\ &\qquad\quad + \|\theta_\lambda(\sigma,\cdot,v,u_\sigma)\big(A(u) - A(v)\big)u\|_{X(\sigma,\sigma_u)} \\ &\qquad\quad + \|\theta_\lambda(\sigma,\cdot,v,u_\sigma)\big(A(v) - A(u_\sigma)\big)(u-v)\|_{X(\sigma,\sigma_v)}.\end{aligned}$$

Note that in the last step we used $\theta_\lambda(\sigma, t, v, u_\sigma) = 0$ for $t \ge \sigma_v$. The Lipschitz continuity of θ_λ and Lemma 2.3.4 yield

$$\begin{aligned}&\left\|\big(\theta_\lambda(\sigma,\cdot,u,u_\sigma) - \theta_\lambda(\sigma,\cdot,v,u_\sigma)\big)\big(A(u) - A(u_\sigma)\big)u\right\|_{X(\sigma,\sigma_u)} \\ &\le \sup_{t\in[\sigma,\sigma_u]} \left|\theta_\lambda(\sigma,t,u,u_\sigma) - \theta_\lambda(\sigma,t,v,u_\sigma)\right| \sup_{t\in[\sigma,\sigma_u]} L_Q \sup_{t\in[\sigma,\sigma_u]} \|u(t) - u_\sigma\|_{\mathrm{TR}} \|u\|_{X^1(\sigma,\sigma_u)} \\ &\le \lambda^{-1} L_Q \sup_{s\in[\sigma,T]} \left| \|u\|_{X^1(a,s)} + \|u - u_\sigma\|_{C(\sigma,s;\mathrm{TR})} - \|v\|_{X^1(a,s)} - \|v - u_\sigma\|_{C(\sigma,s;\mathrm{TR})} \right| \\ &\qquad \sup_{t\in[\sigma,\sigma_u]} \|u(t) - u_\sigma\|_{\mathrm{TR}} \|u\|_{X^1(\sigma,\sigma_u)} \\ &\le 4C_Q\lambda \left(\|u - v\|_{X^1(\sigma,\mu)} + \|u - v\|_{C(\sigma,\mu;\mathrm{TR})}\right)\end{aligned}$$

In the last step, we used the definition of σ_u to estimate the terms not depending on the difference $u - v$. Accordingly, we derive

$$\left\|\theta_\lambda(\sigma,\cdot,v,u_\sigma)\big(A(u) - A(v)u\right\|_{X(\sigma,\sigma_u)} \le 2C_Q\lambda\|u - v\|_{C(\sigma,\mu;\mathrm{TR})}$$

and

$$\left\|\theta_\lambda(\sigma,\cdot,v,u_\sigma)\big(A(v) - A(u_\sigma)(u-v)\right\|_{X(\sigma,\sigma_v)} \le 2C_Q\lambda\|u - v\|_{X^1(\sigma,\mu)}$$

respectively. All in all, we proved

$$\left\|Q_{\lambda,\sigma}(u,u_\sigma) - Q_{\lambda,\sigma}(v,u_\sigma)\right\|_{X(0,\mu)} \leq 6C_Q\lambda \left(\|u-v\|_{X^1(\sigma,\mu)} + \|u-v\|_{C(\sigma,\mu;\mathrm{TR})}\right),$$

which is the claimed result. $\square$

Next, we construct a local solution of (QSEE) starting from a random initial time σ under the assumption that we already solved the equation on the random interval $[0,\sigma]$. We do this by solving a version of (2.3.2) with given past u and restricting the solution to a random interval on which the solution also satisfies (QSEE).

Proposition 2.3.6. *Choose one of the settings* [TT], [GM] *or* [LQ] *and let* [Q5]-[Q9] *be fulfilled. Let σ be an $\mathbb{F}$-stopping time with $0 \leq \sigma \leq T$ almost surely and u be a unique strong solution of* (QSEE) *on* $[0,\sigma]$ *with* $u \in X^1(0,\sigma) \cap C(0,\sigma,\mathrm{TR})$ *almost surely. Moreover, we assume $\lambda > 0$ to be small enough to ensure*

$$6C_Q\lambda + C_{\mathrm{MRD}}L_F^{(i)} + C_{\mathrm{MRD}}L_B^{(i)} < 1$$

for $i = 1,2$. Then, the equation

$$\begin{cases} du(t) &= [A(u(t))u(t) + F(u)(t) + f(t)]\,\mathrm{d}t + [B(u)(t) + b(t)]\,\mathrm{d}W(t) \\ u(0) &= u_0 \end{cases} \tag{2.3.3}$$

has a unique solution u on $[0,\widetilde{\sigma}]$ with $u \in X^1(0,\widetilde{\sigma}) \cap C(0,\widetilde{\sigma};\mathrm{TR})$ almost surely. Here, the $\mathbb{F}$-stopping time $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma} = \inf\left\{t \in [\sigma,T] : \|u-u_\sigma\|_{C(\sigma,t;\mathrm{TR})} + \|u\|_{X^1(\sigma,t)} > \lambda\right\} \wedge T.$$

If we additionally assume $u \in L^r(\Omega; X^1(0,\sigma)) \cap L^r(\Omega; C(0,\sigma;\mathrm{TR}))$ with

$$\big(\mathbb{E}\|u\|^r_{X^1(0,\sigma)} + \mathbb{E}\sup_{t\in[0,\sigma]}\|u(t)\|^r_{\mathrm{TR}}\big)^{1/r} \leq C\big(1 + \|u(0)\|_{L^r(\Omega,\mathrm{TR})}\big),$$

for some $C > 0$ independent of $u(0)$, we also have

$$\big(\mathbb{E}\|u\|^r_{X^1(0,\widetilde{\sigma})} + \mathbb{E}\sup_{t\in[0,\widetilde{\sigma}]}\|u(t)\|^r_{\mathrm{TR}}\big)^{1/r} \leq \widetilde{C}\big(1 + \|u(0)\|_{L^r(\Omega,\mathrm{TR})}\big)$$

for some $\widetilde{C} > 0$ independent of $u(0)$.

Proof. Let $Q_{\lambda,\sigma}$ be defined as in Lemma 2.3.5. To construct a local solution, we first consider the equation

$$\begin{cases} dw(t) &= \left[-A(u(\sigma))w(t) + F^{(1)}(w)(t) + f(t)\right]\,\mathrm{d}t + [B(w)(t) + b(t)]\,\mathrm{d}W(t),\ t \in [\sigma,T] \\ w(\sigma) &= u(\sigma), \end{cases} \tag{2.3.4}$$

where $F^{(1)}$ is given by

$$F^{(1)}(y)(t) = Q_{\lambda,\sigma}(y,u(\sigma))(t) + F\big(u\mathbf{1}_{[0,\sigma)} + y\mathbf{1}_{[\sigma,T)}\big)(t)$$

for $y \in L^0_{\mathbb{F}}\big(\Omega; X^1(\sigma,T) \cap C(\sigma,T;\mathrm{TR})\big)$ with $y(\sigma) = u(\sigma)$. Clearly, $u\mathbf{1}_{[0,\sigma)} + y\mathbf{1}_{[\sigma,T)} \in X^1(0,T) \cap C(0,T;\mathrm{TR})$ almost surely, and hence $F\big(u\mathbf{1}_{[0,\sigma)} + y\mathbf{1}_{[\sigma,T)}\big)$ is well-defined.

Lemma 2.3.5, together with [Q6], shows that $F^{(1)}$ is a Volterra mapping. Let μ be an $\mathbb{F}$-stopping time $\sigma \leq \mu \leq T$ almost surely. Then, for $y \in L^0_{\mathbb{F}}\big(\Omega; X^1(\sigma,T) \cap C(\sigma,T;\mathrm{TR})\big)$, we have the linear growth estimate

$$\begin{aligned}\|F^{(1)}(y)\|_{X(\sigma,\mu)} &\leq \|Q_{\lambda,\sigma}(y,u(\sigma))\|_{X(\sigma,T)} + \|F\big(u\mathbf{1}_{[0,\sigma)} + y\mathbf{1}_{[\sigma,T)}\big)\|_{X(\sigma,T)} \\ &\leq 4C_Q\lambda^2 + C_F^{(1)}\left(1 + \|u\|_{X^1(0,\sigma)} + \|y\|_{X^1(\sigma,\mu)}\right) \\ &\qquad + C_F^{(2)}\left(1 + \|u\|_{C(0,\sigma;\mathrm{TR})} + \|y\|_{C(\sigma,\mu;\mathrm{TR})}\right)\end{aligned}$$

almost surely and the Lipschitz continuity

$$\begin{aligned}&\|F^{(1)}(u) - F^{(1)}(v)\|_{X(0,\mu)} \\ &\quad\leq (6C_Q\lambda + L_F^{(1)})\|u-v\|_{X^1(0,\mu)} + (6C_Q\lambda + L_F^{(2)})\|u-v\|_{C(0,\mu;\mathrm{TR})} + \widetilde{L}_F\|u-v\|_{X(0,\mu)}\end{aligned}$$

almost surely. In particular, $F^{(1)}$ satisfies [S4] from the previous section with

$$\rho = 4C_Q\lambda^2 + C_F^{(1)}\left(1 + \|u\|_{X^1(0,\sigma)}\right) + C_F^{(2)}\left(1 + \|u\|_{C(0,\sigma;\mathrm{TR})}\right).$$

Note that due to the adaptivity of u on $[0,\sigma]$, ρ is $\mathcal{F}_\sigma$-measurable.

We can apply Corollary 2.2.8 and obtain a unique strong solution w of (2.3.4) on $[\sigma,T]$ with $w \in X^1(0,T) \cap C(0,T;\mathrm{TR})$ almost surely with $w(\sigma) = u(\sigma)$ almost surely. Since both $t \mapsto \|w\|_{X^1(\sigma,t)}$ and $t \mapsto \|w - u(\sigma)\|_{C(\sigma,t;\mathrm{TR})}$ are adapted and almost surely continuous (see Lemma 2.2.1),

$$\widetilde{\sigma} = \inf\left\{t \in [\sigma,T] : \|w - u(\sigma)\|_{C(\sigma,t;\mathrm{TR})} + \|w\|_{X^1(\sigma,t)} > \lambda\right\} \wedge T$$

is an $\mathbb{F}$-stopping time by Lemma 1.2.4. Moreover, for $\sigma \leq t \leq \widetilde{\sigma}$ the identity

$$Q_{\lambda,\sigma}(w,u(\sigma))(t) = \big(A(u(\sigma)) - A(w(t))\big)w(t)$$

holds. Defining $u := w$ on $[\sigma,\widetilde{\sigma}]$ finally gives us a strong solution of

$$\begin{cases} du(t) &= [A(u(t))u(t) + F(u)(t) + f(t)]\,\mathrm{d}t + [B(u)(t) + b(t)]\,\mathrm{d}W(t) \\ u(0) &= u_0 \end{cases}$$

on $[0,\widetilde{\sigma}]$. In case that $u \in L^r(\Omega, C(0,\sigma;\mathrm{TR})) \cap L^r(\Omega, X^1(0,\sigma))$ with

$$\Big(\mathbb{E}\|u\|^r_{X^1(0,\sigma)} + \mathbb{E}\sup_{t\in[0,\sigma]}\|u(t)\|^r_{\mathrm{TR}}\Big)^{1/r} \leq C\big(1 + \|u(0)\|_{L^r(\Omega,\mathrm{TR})}\big)$$

we additionally get the claimed estimate for u on $[0,\widetilde{\sigma}]$ as an immediate consequence of (2.2.9). ☐

Now, we are in the position to prove the main theorem with the following strategy. We already showed that the set Γ of stopping times $\widetilde{\tau}$ such that (QSEE) has a unique solution u

on $[0, \widetilde{\tau}]$ is non-empty. Hence, the essential supremum τ of Γ in the sense of Definition 1.2.5 exists. This τ will be our maximal stopping time that also satisfies the blow-up criterion. However, at first it is unclear, whether τ is a stopping time or not. This can be shown if Γ is closed under pairwise maximization (see Theorem 1.2.6).

Lemma 2.3.7. *If (u_1, τ_1) and (u_2, τ_2) are unique local solutions of (QSEE) with $u_i \in X^1(0, \tau_i) \cap C(0, \tau_i; \mathrm{TR})$ almost surely for $i = 1, 2$, then the equation (QSEE) has a unique solution $(u, \tau_1 \vee \tau_2)$ with $u \in X^1(0, \tau_1 \vee \tau_2) \cap C(0, \tau_1 \vee \tau_2; \mathrm{TR})$ almost surely.*

If we additionally assume $u_i \in L^r(\Omega; X^1(0, \tau_i))$ and $\mathbb{E} \sup_{t \in [0, \tau_i)} \|u(t)\|_{\mathrm{TR}}^r < \infty$ for $i = 1, 2$, we also get $u \in L^r(\Omega; X^1(0, \tau_1 \vee \tau_2))$ and $\mathbb{E} \sup_{t \in [0, \tau_1 \vee \tau_2)} \|u(t)\|_{\mathrm{TR}}^r < \infty$.

Proof. Define u by

$$u(t) = u_1(t \wedge \tau_1) + u_2(t \wedge \tau_2) - u_1(t \wedge \tau_1 \wedge \tau_2)$$

for $t \in [0, \tau_1 \vee \tau_2]$. Clearly, u is adapted as a composition of stopped adapted processes. By uniqueness, we have $u_1(t) = u_2(t)$ almost surely for every $t \in [0, \tau_1 \wedge \tau_2]$. Hence, $u = u_1$ on $\{\tau_1 > \tau_2\} \times [0, \tau_1)$ and $u = u_2$ on $\{\tau_1 \leq \tau_2\} \times [0, \tau_2)$. This proves that $(u, \tau_1 \vee \tau_2)$ is a unique solution of (QSEE) that inherits all the regularity properties from u_1 and u_2. □

Proof of Theorem 2.3.3. We define the Γ as the set of all $\mathbb{F}$-stopping times $\widetilde{\tau} : \Omega \to [0, T]$ such that there exists a unique solution $(\widetilde{u}, \widetilde{\tau})$ with $\widetilde{u} \in X^1(0, \widetilde{\tau}) \cap C(0, \widetilde{\tau}; \mathrm{TR})$ almost surely.

By Proposition 2.3.6, this set is non-empty (start with $\sigma = 0$, then the corresponding $\widetilde{\sigma}$ is in Γ). Moreover, by Lemma 2.3.7, Γ is closed under pairwise maximization, i.e. if $\tau_1, \tau_2 \in \Gamma$, we also have $\tau_1 \vee \tau_2 \in \Gamma$. Consequently, Theorem 1.2.6 yields the existence of $\tau := \operatorname{ess\,sup} \Gamma$ and of an increasing sequence of stopping times $(\tau_n)_n$ in Γ with $\tau = \lim_{n \to \infty} \tau_n$ almost surely. In particular, τ is an $\mathbb{F}$-stopping time as almost sure limit of $\mathbb{F}$-stopping times.

Each τ_n belongs to a unique solution (u_n, τ_n). This can be used to ultimately define the solution of (QSEE) on $[0, \tau)$. We set $u = u_n$ on $\Omega \times [0, \tau_n)$. Then, u is a well-defined strongly adapted process on $\Omega \times [0, \tau)$ and $\big(u, (\tau_n)_n, \tau\big)$ is a unique solution in the sense of Definition 2.3.2.

Next, we show that

$$\widetilde{\Omega} = \{\tau < T,\ \|u\|_{X^1(0,\tau)} < \infty,\ u : [0, \tau) \to \mathrm{TR} \text{ is uniformly continuous}\}$$

is a set of measure zero. Assume $\mathbb{P}(\widetilde{\Omega}) > 0$. Since u is pathwise uniformly continuous on $\widetilde{\Omega}$, we can extend u on $\widetilde{\Omega}$ to the closed interval $[0, \tau]$. Moreover, since we have $\tau_n \to \tau$ almost surely, we also have $\sup_{s \in [\tau_n, \tau)} \|u(\tau_n) - u(s)\|_{\mathrm{TR}} \to 0$ and $\|u\|_{X^1(\tau_n, \tau)} \to 0$ almost surely for $n \to \infty$ on $\widetilde{\Omega}$ by Lemma 2.2.1.

By Egorov's theorem, there exists a subset $\Lambda \subset \widetilde{\Omega}$ of positive measure such that the limits from above are uniform on Λ. In particular, there exists $N \in \mathbb{N}$ such that

$$\sup_{s \in [\tau_N(\omega), t]} \|u(\omega, \tau_N(\omega)) - u(\omega, s)\|_{\mathrm{TR}} + \|u(\omega, \cdot)\|_{X^1(\tau_N(\omega), t)} < \tfrac{\lambda}{2}$$

for all $\omega \in \Lambda$ and $t \in [\tau_N(\omega), \tau(\omega)]$, where $\lambda > 0$ is chosen in the same way as in Proposition 2.3.6. The same Proposition shows that we can sustain the unique solution u from $[0, \tau_N]$ to a unique solution $\widetilde{u}$ of (QSEE) on $[0, \widetilde{\tau}_N]$ with

$$\widetilde{\tau}_N = \inf \Big\{ t \in [\tau_N, T] : \|\widetilde{u} - u_{\tau_N}\|_{C(\tau_N, t; \mathrm{TR})} + \|\widetilde{u}\|_{X^1(\tau_N, t)} > \lambda \Big\} \wedge T.$$

By uniqueness, u and $\widetilde{u}$ coincide on $[0, \tau \wedge \widetilde{\tau}_N)$ and hence $\widetilde{\tau}_N \in \Gamma$. However, on Λ we have $\widetilde{\tau}_N > \tau$ which contradicts the definition of τ as essential supremum of Γ. All in all, we proved $\mathbb{P}(\widetilde{\Omega}) = 0$.

If $u_0 \in L^r(\Omega; \mathrm{TR})$, we replace Γ by

$$\widetilde{\Gamma} = \Big\{ \sigma \in \Gamma : \text{ the unique solution } u^{(\sigma)} \text{ corresponding to } \sigma \text{ satisfies} \\ u^{(\sigma)} \in L^r(\Omega; X^1(0, \sigma)),\ \mathbb{E} \sup_{t \in [0, \sigma)} \|u(t)\|_{\mathrm{TR}}^r < \infty \Big\}$$

and repeat the argument step by step.

It remains to prove maximality of the solution. Let $(z, (\mu_n)_n, \mu)$ be another local solution of (QSEE). By uniqueness of u, we get $z = u$ on $[0, \tau \wedge \mu)$. Assume that there is a set of positive measure $\Lambda \subset \Omega$ with $\mu > \tau$ on Λ. Then, for almost all $\omega \in \Lambda$ there exists $n = n(\omega) \in \mathbb{N}$ with $\mu_n(\omega) > \tau(\omega)$. In particular, by definition of a local solution, $u : \Lambda \times [0, \tau] \to \mathrm{TR}$ is pathwise almost surely uniformly continuous and we have $\|u\|_{X^1(0,\tau)} < \infty$ on Λ. Thus the blow-up criterion we derived above implies $\tau = T$ almost surely on Λ. But this contradicts $\mu > \tau$ on Λ, since μ is also bounded by T. Hence, we established $\mu \leq \tau$ almost surely, which is the claimed result. $\square$

We prove that if two different initial values coincide on a set of positive measure, the corresponding solutions also coincide on this set.

Corollary 2.3.8. *Let (u_1, τ_1) and (u_2, τ_2) be the maximal unique strong solutions of* (QSEE) *to the initial values $u_0^{(1)} \in \mathrm{TR}$ and $u_0^{(2)} \in \mathrm{TR}$ respectively. Then, we have $\tau_1(\omega) = \tau_2(\omega)$ and $u_1(\omega, t) = u_2(\omega, t)$ for almost all $\omega \in \{u_0^{(1)} = u_0^{(2)}\}$ and all $t \in [0, \tau_1(\omega))$.*

Proof. We define Γ as the set of all $\mathbb{F}$-stopping times $\widetilde{\tau} : \Omega \to [0, T]$ such that the maximal unique solutions (u_1, τ_1) and (u_2, τ_2) of (QSEE) to the initial values $u_0^{(1)}$ and $u_0^{(2)}$ satisfy $u_1(\omega, t) = u_2(\omega, t)$ for almost all $\omega \in \{u_0^{(1)} = u_0^{(2)}\}$ and for all $t \in [0, \widetilde{\tau}]$.

We first show that Γ contains a stopping time that is almost surely strictly positive on $\{u_0^{(1)} = u_0^{(2)}\}$. Let $\lambda > 0$ small enough as in the proof of Proposition 2.3.6 and let

$$\sigma_i = \inf \big\{ t \in [0, \tau_i) : \|u_i\|_{X^1(0,t)} + \|u_i - u_0^{(i)}\|_{C(0,t;\mathrm{TR})} > \lambda \big\} \wedge \tau_i$$

for $i = 1, 2$. Clearly, σ_i is strictly positive. Then, u_i is a strong solution of the semilinear equation

$$\begin{cases} du_i &= \Big[A(u_0^{(i)}) u_i + F_i(u_i) + f \Big] \, \mathrm{d}t + \big[B(u_i) + b \big] \, \mathrm{d}W(t), \quad t \in [0, T] \\ u_i(0) &= u_0^{(i)}, \end{cases} \tag{2.3.5}$$

on $[0, \sigma_i]$, where $F_i(w) = Q_{\lambda,0}(w, u_i^{(0)}) + F(w)$. Clearly, on $\{u_0^{(1)} = u_0^{(2)}\}$, we have $A(u_0^{(1)}) = A(u_0^{(2)})$ almost surely and $F_1(w) = F_2(w)$ almost surely for every w. Consequently, Corollary 2.2.9 implies $u_1(\omega, t) = u_2(\omega, t)$ for almost all $\omega \in \{u_0^{(1)} = u_0^{(2)}\}$ and for all $t \in [0, \sigma_1 \wedge \sigma_2]$. In particular, we have $\sigma_1 \wedge \sigma_2 \in \Gamma$.

As in the proof of Lemma 2.3.7, we can see that Γ is closed under pairwise maximization. Thus, Theorem 1.2.6 yields the existence of $\eta := \operatorname{ess\,sup} \Gamma$ and of an increasing sequence of stopping times $(\eta_n)_n$ in Γ with $\lim_{n\to\infty} \eta_n = \eta$ almost surely. In particular, η is an $\mathbb{F}$-stopping time that is also almost surely strictly positive on $\{u_0^{(1)} = u_0^{(2)}\}$.

It remains to show $\eta = \tau_1 = \tau_2$ almost surely on $\{u_0 = v_0\}$. Assume $\eta < \tau_1 \wedge \tau_2$ on $\{u_0^{(1)} = u_0^{(2)}\}$. Then, we have $u, v \in X^1(0, \eta) \cap C(0, \eta; \mathrm{TR})$ almost surely on $\{u_0^{(1)} = u_0^{(2)}\}$ and

$$\{u_0^{(1)} = u_0^{(2)}\} = \{u_1(t) = u_2(t)\ \forall t \in [0, \eta]\} \cup N$$

for some $N \subset \Omega$ with $\mathbb{P}(N) = 0$.

Let $\lambda > 0$ as before and define

$$\widetilde{\sigma}_i = \inf \big\{ t \in [\eta, \tau_i) : \|u_i\|_{X^1(\eta,t)} + \|u_i - u_i(\eta)\|_{C(0,t;\mathrm{TR})} > \lambda \big\} \wedge \tau_i$$

for $i = 1, 2$. Then, u_i is a strong solution of the semilinear equation

$$du_i = \Big[A(u_i(\eta))u_i + \widetilde{F}_i(u_i) + f \Big]\, \mathrm{d}t + \big[\widetilde{B}_i(u_i) + b \big]\, \mathrm{d}W(t), \quad t \in [0, T] \tag{2.3.6}$$

on $[\eta, \widetilde{\sigma}_i]$ with initial data $u_i(\eta)$ and

$$\widetilde{F}_i(w) := F\big(\widetilde{u}_i \mathbf{1}_{[0,\sigma)} + w \mathbf{1}_{[\sigma,T]}\big)$$
$$\widetilde{B}_i(w) := B\big(\widetilde{u}_i \mathbf{1}_{[0,\sigma)} + w \mathbf{1}_{[\sigma,T]}\big)$$

for all $w \in L^0_{\mathbb{F}}\big(\Omega; X^1(\eta, T) \cap C(\eta, T; \mathrm{TR})\big)$ with $w(\eta) = u_i(\eta)$ almost surely. Since we have $u_1(\omega, t) = u_2(\omega, t)$ for almost all $\omega \in \{u_0^{(1)} = u_0^{(2)}\}$ and all $t \in [0, \eta]$, the $A(u_i(\eta))$, the $\widetilde{F}_i$ and the $\widetilde{B}_i$ coincide on $\{u_0^{(1)} = u_0^{(2)}\}$ and we can apply Corollary 2.2.9 to get $u_1(\omega, t) = u_2(\omega, t)$ for almost all $\omega \in \{u_0^{(1)} = u_0^{(2)}\}$ and all $t \in [0, \widetilde{\sigma}_1 \wedge \widetilde{\sigma}_2]$. However, we have $\widetilde{\sigma}_1 \wedge \widetilde{\sigma}_2 > \eta$ almost surely on $\{u_0^{(1)} = u_0^{(2)}\}$, which cannot be since η was defined as the essential supremum of Γ. This proves $\eta = \tau_1 \wedge \tau_2$ on $\{u_0^{(1)} = u_0^{(2)}\}$.

Last but not least, we show that $\tau_1 = \tau_2$ almost surely on $\{u_0^{(1)} = u_0^{(2)}\}$. Assume $\tau_2 < \tau_1$ on $\Lambda \subset \{u_0^{(1)} = u_0^{(2)}\}$ with $\mathbb{P}(\Lambda) > 0$. Since u_1 and u_2 coincide almost surely on $\{u_0^{(1)} = u_0^{(2)}\} \times [0, \tau_2)$ and u_1 has a larger time of existence on Λ, we know that $u_2 : [0, \tau_2) \to \mathrm{TR}$ is uniformly continuous and $\|u_2\|_{X^1(0,\tau_2)} < \infty$ on Λ. Hence, the blow-up criterion from Theorem 2.3.3 implies $\tau_2 = T$ almost surely on Λ, which contradicts $\tau_2 < \tau_1$ on Λ. Consequently, $\mathbb{P}(\tau_2 < \tau_1, u_0^{(1)} = u_0^{(2)}) = 0$. In the same way, we show $\mathbb{P}(\tau_1 < \tau_2, u_0^{(1)} = u_0^{(2)}) = 0$, which closes the proof. □

2.3.2. Locally Lipschitz continuous quasilinearity

In the different versions of the assumptions 2−4 in the respective settings, we assumed a uniform boundedness of the functional calculus of $A(u(t))$ and a global Lipschitz condition on

A. However, we established a local well-posedness theory only using local methods. Therefore, we can generalize our result in the next section and allow local Lipschitz continuous nonlinearities. In the same way as before, we have to distinguish our three settings. In every setting, the assumptions 2-4 are replaced by the following weaker conditions. Afterwards we shortly repeat our unified notation. We begin with the improvement in [TT].

[TTQ2*] The mapping $A : \Omega \times (E, E^1)_{1-1/p,p} \to \mathcal{B}(E^1, E)$ is such that $\omega \mapsto A(\omega, y)x$ is strongly $\mathcal{F}_0$-measurable for all $x \in E^1$ and $y \in (E, E^1)_{1-1/p,p}$ and such that $D(A(\omega, y)) = E^1$. More precisely, for every $n \in \mathbb{N}$, there exists $\mu(n) > 0$ and $C(n) > 0$, such that

$$C(n)^{-1}\|x\|_{E^1} \leq \|(\mu(n) + A(\omega, y))x\|_E \leq C(n)\|x\|_{E^1}$$

for almost all $\omega \in \Omega$, all $y \in (E, E^1)_{1-1/p,p}$ with $\|y\|_{(\mathrm{E},\mathrm{E}^1)_{\text{1-1/p,p}}} \leq n$ and all $x \in E^1$.

[TTQ3*] For all $n \in \mathbb{N}$, there exists $\mu(n), C(n) > 0$ such that the operators $\mu(n) + A(\omega, y)$ have a bounded $H^\infty(\Sigma_{\eta(n)})$-calculus of angle $\eta(n) \in (0, \pi/2)$ with

$$\|\phi(\mu(n) + A(\omega, y))\|_{\mathcal{B}(E)} \leq C(n)\|\phi\|_{H^\infty(\Sigma_\eta)}$$

for almost all $\omega \in \Omega$, for all $\phi \in H^\infty(\Sigma_{\eta(n)})$, and for all $y \in \mathrm{TR}$ with $\|y\|_{(E,E^1)_{1-1/p,p}} \leq n$.

[TTQ4*] For all $n \in \mathbb{N}$ there exist $C_Q(n) > 0$ such that

$$\|A(\omega, z) - A(\omega, y)\|_{\mathcal{B}(E^1,E)} \leq C_Q(n)\|z - y\|_{\mathrm{TR}}$$

for almost all $\omega \in \Omega$ and all $\|y\|_{(\mathrm{E},\mathrm{E}^1)_{\text{1-1/p,p}}}, \|z\|_{(\mathrm{E},\mathrm{E}^1)_{\text{1-1/p,p}}} \leq n$.

In this setting, we set $X := E$, $X(a, b) := L^p(a, b; E)$, $X^1(a, b) := L^p(a, b; E^1)$ and we define $X_H^{\frac{1}{2}}(a, b) := L^p(a, b; \gamma(H; [E, E^1]_{\frac{1}{2}}))$. The trace space TR is the real interpolation space $(E, E^1)_{1-1/p,p}$. Next, we give the refined assumptions for [GA].

[GMQ2*] The mapping $A : \Omega \times [E, E^1]_{\frac{1}{2}} \to \mathcal{B}(E^1, E)$ is such that $\omega \mapsto A(\omega, y)x$ is for all $x \in E^1$ and $y \in [E, E^1]_{\frac{1}{2}}$ strongly $\mathcal{F}_0$-measurable and such that $D(A(\omega, y)) = E^1$. More precisely, for every $n \in \mathbb{N}$, there exists $\mu(n), C(n) > 0$, such that

$$C(n)^{-1}\|x\|_{E^1} \leq \|(\mu(n) + A(\omega, y))x\|_E \leq C(n)\|x\|_{E^1}$$

for almost all $\omega \in \Omega$, all $y \in [E, E^1]_{\frac{1}{2}}$ with $\|y\|_{[E,E^1]_{\frac{1}{2}}} \leq n$ and all $x \in E^1$ with estimates independent of ω.

[GMQ3*] For all $n \in \mathbb{N}$, there exist $\mu(n), C(n) > 0$ such that the operators $\mu(n) + A(\omega, y)$ have a bounded $H^\infty(\Sigma_{\eta(n)})$-calculus of angle $\eta(n) \in (0, \pi/2)$ with

$$\|\phi(\mu(n) + A(\omega, y))\|_{\mathcal{B}(E)} \leq C(n)\|\phi\|_{H^\infty(\Sigma_\eta)}$$

for almost all $\omega \in \Omega$ and for all $\phi \in H^\infty(\Sigma_{\eta(n)})$, $y \in [E, E^1]_{\frac{1}{2}}$ with $\|y\|_{[E,E^1]_{\frac{1}{2}}} \leq n$.

[GMQ4*] For all $n \in \mathbb{N}$ there exist $C_Q(n) > 0$ such that

$$\mathcal{R}\Big(\big\{A(\omega, z(t)) - A(\omega, y(t)) : t \in [a, b]\big\} \subset \mathcal{B}(E^1, E)\Big) \leq C_Q(n) \sup_{t\in[a,b]} \|y(t) - z(t)\|_{[\mathrm{E},\mathrm{E}^1]_{1/2}}$$

for almost all $\omega \in \Omega$ and for all $y, z \in [\mathrm{E}, \mathrm{E}^1]_{1/2}$ with $\|y\|_{[E,E^1]_{1/2}}, \|z\|_{[E,E^1]_{1/2}} \leq n$.

Here, we set $X := E$, $X(a,b) := \gamma(a,b;E)$, $X_H^{\frac{1}{2}}(a,b) := \gamma(0,T;\gamma(H;[E,E^1]_{\frac{1}{2}}))$ and we define $X^1(a,b) := \gamma(a,b;E^1)$. The trace space TR is the complex interpolation space $[E;E^1]_{\frac{1}{2}}$. Last but not least, we give the refined assumptions for [LQ].

[LQQ2*] The mapping $A : \Omega \times F^{\alpha-1/p}_{\Lambda,q,p} \to \mathcal{B}(E^\alpha, E^{\alpha-1})$ is strongly $\mathcal{F}_0$-measurable. Moreover, the $A(\omega,y)$ are closed and for any $n \in \mathbb{N}$ there exists $\mu(n), C(n) > 0$, such that the operators $\Lambda^\alpha(\mu(n)+A(\omega,y))^{-\alpha}$, $(\mu(n)+A(\omega,y))^\alpha\Lambda^{-\alpha}$, $(\mu(n)+A(\omega,y))^{\alpha-1}\Lambda^{1-\alpha}$ and $\Lambda^{\alpha-1}(\mu(n)+A(\omega,y))^{1-\alpha}$ are $\mathcal{R}_p$-bounded on $L^q(U)$ with $\mathcal{R}_p$-bound $C(n)$ for almost all $\omega \in \Omega$ and all $\|y\|_{F^{\alpha-1/p}_{\Lambda,q,p}} \leq n$.

[LQQ3*] For every $n \in \mathbb{N}$, there exist $\mu(n), C(n) > 0$, such that the operators $\mu(n) + A(\omega,y)$ have an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_{\eta(n)})$ calculus $\eta(n) \in (0,\pi/2)$ that additionally satisfies

$$\mathcal{R}_p\Big(\big\{\psi(A(\omega,y)) : \|\psi\|_{H^\infty(\Sigma_{\eta(n)})} \leq 1\big\} \subset \mathcal{B}(L^q(U))\Big) \leq C(n)$$

for almost all $\omega \in \Omega$ and for all $y \in F^{\alpha-1/p}_{\Lambda,q,p}$ with $\|y\|_{F^{\alpha-1/p}_{\Lambda,q,p}} \leq n$.

[LQQ4*] For all $n \in \mathbb{N}$ there exist $C_Q(n) > 0$ such that we have

$$\mathcal{R}_p\Big(\big\{\Lambda^{\alpha-1}(A(\omega,z(t)) - A(\omega,y(t)))\Lambda^{-\alpha} : t \in [a,b]\big\} \subset \mathcal{B}(L^q(U))\Big) \leq C_Q \sup_{t\in[a,b]} \|z(t)-y(t)\|_{F^{\alpha-1/p}_{\Lambda,q,p}}$$

almost all $\omega \in \Omega$ and for all $\|y\|_{F^{\alpha-1/p}_{\Lambda,q,p}}, \|z\|_{F^{\alpha-1/p}_{\Lambda,q,p}} \leq n$.

The assumption on Λ imply that Λ can be extended to an operator $\Lambda_{p,a,b}$ on $L^q(U;L^p(a,b))$ that has a bounded $H^\infty(\Sigma_{\widetilde{\eta}})$-calculus for some angle $0 \leq \widetilde{\eta} < \pi/2$. We choose $X^\alpha(a,b) := D(\Lambda^\alpha_{p,a,b})$ and $X^{\alpha-1}(a,b) = \Lambda^{1-\alpha}_{p,a,b}\big(L^q(U;L^p(a,b))\big)$. The stochastic part $B(u)+b$ is contained in the space $X_H^{\frac{1}{2}}(a,b) := \Lambda^{\frac{1}{2}-\alpha}_{p,a,b}\big(L^q(U;L^p(a,b;l^2(\mathbb{N})))\big)$. The trace space TR is the Triebel-Lizorkin space $F^{\alpha-1/p}_{\Lambda,q,p}$ in the sense of Definition 1.4.2.

The local Lipschitz conditions on the nonlinearities are universal and can be formulated in our general framework.

[Q6*] F has the same mapping properties as in [Q6]. More precisely, for every $n \in \mathbb{N}$ there exist $L_F^{(i)}(n), \widetilde{L}_F(n), C_F^{(i)}(n) \geq 0$, $i = 1,2$, such that F is locally of linear growth, i.e.

$$\|F(\phi_1)\|_{X(0,\mu)} \leq C_F^{(1)}(n)\left(1 + \|\phi_1\|_{X^1(0,\mu)}\right) + C_F^{(2)}(n)\left(1 + \|\phi_1\|_{C(0,\mu;\mathrm{TR})}\right)$$

and locally Lipschitz continuous, i.e.

$$\begin{aligned}&\|F(\phi_1) - F(\phi_2)\|_{X(0,\mu)}\\ &\quad\leq L_F^{(1)}(n)\|\phi_1-\phi_2\|_{X^1(0,\mu)} + \widetilde{L}_F(n)\|\phi_1-\phi_2\|_{X(0,\mu)} + L_F^{(2)}(n)\|\phi_1-\phi_2\|_{C(0,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1, \phi_2 \in L^0_{\mathbb{F}}(\Omega; X^1(0,\mu) \cap C(0,\mu;\mathrm{TR}))$ with $\phi_1(0) = \phi_2(0) = u_0$ and $\sup_{t\in[0,\mu]} \|\phi_i\|_{\mathrm{TR}} \leq n$ almost surely for $i = 1,2$. The occurring constants are independent of $\omega \in \Omega$.

[Q7*] B has the same mapping properties as in [Q7]. More precisely, for every $n \in \mathbb{N}$, there exist $L_B^{(i)}(n), \widetilde{L}_B(n), C_B^{(i)}(n) \geq 0$, $i = 1, 2$, such that B is of linear growth, i.e.

$$\|B(\phi_1)\|_{X_H^{\frac{1}{2}}(0,\mu)} \leq C_B^{(1)}(n)\left(1 + \|\phi_1\|_{X^1(0,\mu)}\right) + C_B^{(2)}(n)\left(1 + \|\phi_1\|_{C(0,\mu;\mathrm{TR})}\right)$$

and Lipschitz continuous, i.e.

$$\begin{aligned}&\|B(\phi_1) - B(\phi_2)\|_{X_H^{\frac{1}{2}}(0,\mu)} \\ &\quad\leq L_B^{(1)}(n)\|\phi_1 - \phi_2\|_{X^1(0,\mu)} + \widetilde{L}_B(n)\|\phi_1 - \phi_2\|_{X(0,\mu)} + L_B^{(2)}(n)\|\phi_1 - \phi_2\|_{C(0,\mu;\mathrm{TR})}\end{aligned}$$

almost surely for all $\phi_1, \phi_2 \in L^0_{\mathbb{F}}(\Omega; X^1(0,\mu) \cap C(0,\mu;\mathrm{TR}))$ with $\phi_1(0) = \phi_2(0) = u_0$ almost surely and with $\sup_{t\in[0,\mu]} \|\phi_i\|_{\mathrm{TR}} \leq n$ almost surely for $i = 1, 2$. The occurring constants are independent of $\omega \in \Omega$.

As we have shown in Section 2.1, the assumptions imply maximal regularity estimates for the deterministic and the stochastic convolution in all of the three settings. However, since the occurring constants are uniform on balls in TR of radius n, we solely get uniform maximal regularity estimates of $A(y)$ for $\|y\|_{\mathrm{TR}} < n$. More precisely, for every $n \in \mathbb{N}$, there exists $C_{\mathrm{MRS}}(n), C_{\mathrm{MRD}}(n) > 0$ such that for any stopping time μ with $0 \leq \mu \leq T$ almost surely, we have

$$\left\|\left(e^{-(\cdot)A(y)} \diamond g\right)_\mu\right\|_{L^r(\Omega;X^1(\mu,T)\cap C(\mu,T;\mathrm{TR}))} \leq C_{\mathrm{MRS}}(n)\|g\|_{L^r(\Omega;X_H^{\frac{1}{2}}(\mu,T))}$$

for all $\|y\|_{\mathrm{TR}} \leq n$ and all $g \in L^r_{\mathbb{F}}(\Omega; X_H^{\frac{1}{2}}(\mu,T))$ and

$$\left\|\left(e^{-(\cdot)A(y)} * \widetilde{g}\right)_\mu\right\|_{L^r(\Omega;X^1(\mu,T)\cap C(\mu,T;\mathrm{TR}))} \leq C_{\mathrm{MRD}}(n)\|\widetilde{g}\|_{L^r(\Omega;X(\mu,T))}$$

for all $\|y\|_{\mathrm{TR}} \leq n$ and all $\widetilde{g} \in L^r(\Omega; X(\mu,T))$. As before, we require small Lipschitz constants in [Q6*], [Q7*]. More precisely, we assume the following.

[Q9*] Let the constants of [Q6*] and [Q7*] be small enough to ensure

$$C_{\mathrm{MRD}}(n)L_F^{(i)}(n) + C_{\mathrm{MRD}}(n)L_B^{(i)}(n) < 1$$

for every $n \in \mathbb{N}$ and for $i = 1, 2$.

Before, we start we comment on the local Lipschitz assumptions for F and B. We have to admit that [Q9*] is a lot more restrictive than [Q9]. The main difference is that even in concrete situations, it is very difficult to calculate the constants $C_{\mathrm{MRD}}(n)$ and $C_{\mathrm{MRS}}(n)$ precisely, usually one solely knows that these constants are increasing with n. In practice, this means that we can just allow decreasing sequences $L_F^{(i)}(n)$ and $L_B^{(i)}(n)$ that converge to zero and even worse, we usually do not know anything about the rate of convergence we have to require. At least, locally Lipschitz continuous lower order terms can be handled very well. We make this precise in the following proposition.

Proposition 2.3.9. *Let $\theta \in [0,1)$ and let*

$$\begin{aligned}F &: L^0_{\mathbb{F}}(\Omega; [X(0,\mu), X^1(0,\mu)]_\theta) \to L^0_{\mathbb{F}}(\Omega; X(0,\mu))\\ B &: L^0_{\mathbb{F}}(\Omega; [X(0,\mu), X^1(0,\mu)]_\theta) \to L^0_{\mathbb{F}}(\Omega; X_H^{\frac{1}{2}}(0,\mu))\end{aligned}$$

be Volterra mappings in the sense of [Q6] *and* [Q7]*. Moreover, we assume that for every* $n \in \mathbb{N}$*, there exists* $\widetilde{L}(n) \geq 0$ *such that for every* $\mathbb{F}$*-stopping time* μ *with* $0 \leq \mu \leq T$ *almost surely, we have*

$$\begin{aligned}\|F(\phi_1) - F(\phi_2)\|_{X(0,\mu)} &\leq \widetilde{L}(n)\|\phi_1 - \phi_2\|_{[X(0,\mu),X^1(0,\mu)]_\theta},\\ \|B(\phi_1) - B(\phi_2)\|_{X_H^{\frac{1}{2}}(0,\mu)} &\leq \widetilde{L}(n)\|\phi_1 - \phi_2\|_{[X(0,\mu),X^1(0,\mu)]_\theta}\end{aligned}$$

almost surely for every $\phi_1, \phi_2 \in L^0_{\mathbb{F}}(\Omega; X^1(0,\mu) \cap C(0,\mu; \mathrm{TR}))$ *with* $\phi_1(0) = \phi_2(0) = u_0$ *and* $\sup_{t\in[0,\mu]} \|\phi_i\|_{\mathrm{TR}} \leq n$ *almost surely for* $i = 1, 2$*. Then,* F *and* B *satisfy* [Q6*], [Q7*] *and* [Q9*].

Proof. For $a, b, \varepsilon > 0$, we calculate

$$ab = a\varepsilon^{1-\theta} b\varepsilon^{\theta-1} \leq (1-\theta)\varepsilon a^{\frac{1}{1-\theta}} + \theta\varepsilon^{\frac{\theta-1}{\theta}} b^{\frac{1}{\theta}}.$$

Together with the properties of complex interpolation this yields

$$\begin{aligned}\widetilde{L}(n)\|\phi_1 - \phi_2\|_{[X(0,\mu),X^1(0,\mu)]_\theta} &\leq \widetilde{L}(n)^{1-\theta}\|\phi_1 - \phi_2\|^{1-\theta}_{X(0,\mu)}\widetilde{L}(n)^\theta\|\phi_1 - \phi_2\|^\theta_{X^1(0,\mu)}\\ &\leq (1-\theta)\varepsilon\widetilde{L}(n)\|\phi_1 - \phi_2\|_{X^1(0,\mu)} + \theta\varepsilon^{\frac{\theta-1}{\theta}}\widetilde{L}(n)\|\phi_1 - \phi_2\|_{X(0,\mu)}\end{aligned}$$

for every $\varepsilon > 0$. For given $n \in \mathbb{N}$, we choose $\varepsilon > 0$ small enough such that

$$(1-\theta)\varepsilon\widetilde{L}(n)(C_{\mathrm{MRD}}(n) + C_{\mathrm{MRD}}(n)) < 1.$$

This proves the claimed result. □

In the setting [TT], $[X(0,\mu), X^1(0,\mu)]_\theta$ is given by $L^p(0,\mu; [E, E^1]_\theta)$, whereas in [GM] it equals $\gamma(0,\mu; [E, E^1]_\theta)$. In [LQ], things are more complicated. Here, it coincides with a fractional domain of the operator $\Lambda_{p,0,\mu}$, which is the extrapolation of Λ to $L^q(U; L^p(0,\mu))$. More precisely, we have $[X(0,\mu), X^1(0,\mu)]_\theta = D(\Lambda^{\alpha-1+\theta}_{p,0,\mu})$, if $\alpha - 1 + \theta \geq 0$. If on the other hand $\alpha - 1 + \theta < 0$ it equals to completion of $L^q(U; L^p(0,\mu))$ with respect to the norm $\|\Lambda^{\alpha-1+\theta}_{p,0,\mu} \cdot \|_{L^q(U;L^p(0,\mu))}$.

To construct a solution of (QSEE) for a given $\mathcal{F}_0$-measurable $u_0 : \Omega \to \mathrm{TR}$, we first investigate the truncated equation

$$\begin{cases} du(t) &= \big[-A_n(u(t))u(t) + F_n(u)(t) + f(t)\big]\,\mathrm{d}t + \big[B_n(u)(t) + b(t)\big]\,\mathrm{d}W(t),\\ u(0) &= u_0\mathbf{1}_{\Gamma_n}, \end{cases} \tag{2.3.7}$$

where $A_n(\omega, y) := A(\omega, R_n y)$, $F_n(y) := F(R_n y)$, $B_n(y) := B(R_n y)$, $\Gamma_n := \{\|u_0\|_{\mathrm{TR}} \leq \frac{n}{2}\}$. Here, the cut-off mapping $R_n : \mathrm{TR} \to \mathrm{TR}$ is defined by

$$R_n y = \begin{cases} y, & \text{if } \|y\|_{\mathrm{TR}} \leq n\\ \frac{ny}{\|y\|_{\mathrm{TR}}}, & \text{if } \|y\|_{\mathrm{TR}} > n. \end{cases} \tag{2.3.8}$$

The idea to use such a truncation to extend global Lipschitz nonlinearities to local ones was used several time in case of semilinear equations (see e.g. [19], Theorem 4.10, [89], Proposition 5.4, [95], Theorem 8.1). The following Lemma is well-known. However, since we nowhere found a proof, we give it for convenience of the reader.

Lemma 2.3.10. *Given $n \in \mathbb{N}$, the mapping $R_n : \mathrm{TR} \to \mathrm{TR}$ defined in (2.3.8) is Lipschitz, i.e.*

$$\|R_n x - R_n y\|_{\mathrm{TR}} \leq 2\|x - y\|_{\mathrm{TR}}.$$

In particular, A_n satisfies the assumptions $2-4$ of the globally Lipschitz case in any of the three settings and F_n and B_n satisfy [Q6] and [Q7] respectively.

Proof. Let $x, y \in \mathrm{TR}$. If they are both contained in ball of radius n around zero, there is nothing to prove. So we start with the case that they are both outside this ball. Then triangle inequality yields

$$\begin{aligned}\Big\|\tfrac{nx}{\|x\|_{\mathrm{TR}}} - \tfrac{ny}{\|y\|_{\mathrm{TR}}}\Big\|_{\mathrm{TR}} &\leq n\Big\|\tfrac{x}{\|x\|_{\mathrm{TR}}} - \tfrac{y}{\|x\|_{\mathrm{TR}}}\Big\|_{\mathrm{TR}} + n\Big\|\tfrac{y}{\|x\|_{\mathrm{TR}}} - \tfrac{y}{\|y\|_{\mathrm{TR}}}\Big\|_{\mathrm{TR}} \\ &\leq \tfrac{n}{\|x\|_{\mathrm{TR}}}\|x-y\|_{\mathrm{TR}} + \tfrac{n}{\|x\|_{\mathrm{TR}}}\big|\|y\|_{\mathrm{TR}} - \|x\|_{\mathrm{TR}}\big| \\ &\leq 2\|x-y\|_{\mathrm{TR}}.\end{aligned}$$

If we have $\|x\|_{\mathrm{TR}} > n$ and $\|y\|_{\mathrm{TR}} \leq n$ we estimate

$$\begin{aligned}\Big\|\tfrac{nx}{\|x\|_{\mathrm{TR}}} - y\Big\|_{\mathrm{TR}} &\leq n\Big\|\tfrac{x}{\|x\|_{\mathrm{TR}}} - \tfrac{y}{\|x\|_{\mathrm{TR}}}\Big\|_{\mathrm{TR}} + \Big\|\tfrac{ny}{\|x\|_{\mathrm{TR}}} - y\Big\|_{\mathrm{TR}} \\ &\leq \|x-y\|_{\mathrm{TR}} + \tfrac{\|y\|_{\mathrm{TR}}}{\|x\|_{\mathrm{TR}}}\big|n - \|x\|_{\mathrm{TR}}\big| \\ &\leq \|x-y\|_{\mathrm{TR}} + \big(\|x\|_{\mathrm{TR}} - n\big) \leq 2\|x-y\|_{\mathrm{TR}}.\end{aligned}$$

Since R_n maps into a ball around zero with radius n, all the local assumptions for A, F, B become to global assumptions for A_n, F_n, B_n. □

We can apply Theorem 2.3.3 to the truncated equation (2.3.7) and obtain for every $n \in \mathbb{N}$ a unique maximal local solution $\big(u_n, (\tau_{nk})_k, \tau_n\big)$. To do this, note that one can infix the spectral shift from [TTQ2*], [GAQ2*], [LQQ2*], i.e. we actually solve

$$\begin{cases} du(t) &= \Big[-\widetilde{A}_n(u(t))u(t) + \widetilde{F}_n(u)(t) + f(t)\Big]\,\mathrm{d}t + [B_n(u)(t) + b(t)]\,\mathrm{d}W(t), \\ u(0) &= u_0\mathbf{1}_{\Gamma_n}, \end{cases}$$

with $\widetilde{A}_n(u(t))u(t) = \big(\mu(n) + A_n(u(t))\big)u(t)$ and $\widetilde{F}_n(u)(t) = F_n(u)(t) + \mu(n)u(t)$. In each case, the solution u_n of the truncated equation is a solution of (QSEE) on $\Gamma_n \times [0, \sigma_n)$, where σ_n is defined by

$$\sigma_n := \tau_n \wedge \inf\big\{t \in [0, \tau_n) : \|u_n(t)\|_{\mathrm{TR}} > n\big\}. \tag{2.3.9}$$

Note that σ_n is indeed an $\mathbb{F}$-stopping time, since τ_n is one and entrance times of continuous $\mathbb{F}$-adapted processes into open sets are also stopping times by Lemma 1.2.4. In the following Lemma, we show that the sequence $(\sigma_n)_n$ increases pathwise starting from a large enough $n \in \mathbb{N}$.

Lemma 2.3.11. *There is a set $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that the sequence $\big(\sigma_n(\omega)\big)_{n\in\mathbb{N}}$ is for all $\omega \in \Omega \setminus N$ monotonously increasing beginning from some $n = n(\omega) \in \mathbb{N}$. Moreover, we have $u_k(\omega, t) = u_l(\omega, t)$ for almost all $\omega \in \Omega$, for all $l > k \geq n(\omega)$ and all $t \in [0, \sigma_k(\omega))$.*

Proof. Given $\omega \in \Omega$, choose $n = n(\omega)$ such that $\omega \in \Gamma_n$. Since $\|u_0\|_{\mathrm{TR}}$ is almost surely finite, this can be done for almost all $\omega \in \Omega$. Let $l > k \geq n$. We first prove that we have $u_k(\omega, t) = u_l(\omega, t)$ for almost all $\omega \in \Gamma_n$ and all $t \in [0, \sigma_k(\omega) \wedge \sigma_l(\omega))$. Clearly, both u_k and u_l solve

$$\begin{cases} du(t) &= [-A_l(u(t))u(t) + F_l(u)(t) + f(t)]\,\mathrm{d}t + [B_l(u)(t) + b(t)]\,\mathrm{d}W(t), \\ u(0) &= u_0\mathbf{1}_{\Gamma_n} \end{cases} \tag{2.3.10}$$

in the strong sense on $[0, \sigma_k)$ and $[0, \sigma_l)$ respectively and therefore the uniqueness result from Corollary 2.3.8 directly yields the almost sure coincidence of u_l and u_k on $\Gamma_n \times [0, \sigma_l \wedge \sigma_k)$.

To prove the pathwise monotonicity of the stopping times on Γ_n, we distinguish the cases $\Gamma_n = \Lambda_n \,\dot\cup\, \widetilde{\Lambda}_n \,\dot\cup\, \widetilde{N}$ with a null-set $\widetilde{N}$,

$$\Lambda_n = \Gamma_n \cap \{ \sup_{s\in[0,\tau_l)} \|u_l(s)\|_{\mathrm{TR}} \leq l \}$$

and

$$\widetilde{\Lambda}_n = \Gamma_n \cap \{ \sup_{s\in[0,\tau_l)} \|u_l(s)\|_{\mathrm{TR}} > l \}.$$

We have $\sigma_l = \tau_l$ on Λ_n and $\sigma_l = \inf\{t \in [0, \tau_l) : \|u_l(t)\|_{\mathrm{TR}} > l\}$ on $\widetilde{\Lambda}_n$. As an immediate consequence, we get $\sigma_k \leq \tau_l = \sigma_l$ almost surely on Λ_n, since τ_l was chosen as the maximal stopping time of a solution of (2.3.7) which coincides with the maximal time of existence of (2.3.10) on Γ_n. On $\widetilde{\Lambda}_n$, we argue differently. Here, it suffices to note that by almost sure coincidence of u_l and u_k on $\Gamma_n \times [0, \sigma_l \wedge \sigma_k)$, we have

$$\sup_{s\in[0,\sigma_k\wedge\sigma_l)} \|u_l(s)\|_{\mathrm{TR}} = \sup_{s\in[0,\sigma_k\wedge\sigma_l)} \|u_k(s)\|_{\mathrm{TR}} \leq k,$$

whereas

$$\sup_{s\in[0,\sigma_l)} \|u_l(s)\|_{\mathrm{TR}} = l.$$

Thus, we must have $\sigma_k < \sigma_l$ on $\widetilde{\Lambda}_n$. Putting these cases together, we finally proved the claimed result, namely $\sigma_k \leq \sigma_l$ almost surely on Γ_n. Last but not least, we choose N as the union of all sets of measure zero we excluded in this proof. □

We proved that $(\sigma_n)_n$ is at least for large natural numbers pathwise almost surely monotonously increasing and we know from the definition of $(\sigma_n)_n$ that the sequence is bounded by T. Therefore we can define the $\mathbb{F}$-stopping time

$$\mu = \lim_{n\to\infty} \sigma_n. \tag{2.3.11}$$

Moreover, we set

$$u(\omega, t) := \lim_{n\to\infty} u_n(\omega, t)\mathbf{1}_{\Gamma_n}\mathbf{1}_{[0,\sigma_n)}(t). \tag{2.3.12}$$

for $\omega \in \Omega$ and $t \in [0, \mu)$. Note that for given ω and t, this limit is attained after finitely many steps by Lemma 2.3.11. In particular, u is a strongly adapted process on $\Omega \times [0, \mu)$. Since the u_n are strong solutions of (QSEE) on $\Gamma_n \times [0, \sigma_n)$, u is a good candidate for a local solution of (QSEE) on $\Omega \times [0, \mu)$. We just have to find a sequence of stopping times $(\mu_n)_n$ that approximates μ such that $u \in C(0, \mu_n; \mathrm{TR}) \cap X^1(0, \mu_n)$ almost surely for all $n \in \mathbb{N}$ and such that u is a strong solution of (QSEE) on $[0, \mu_n]$. Note that σ_n does not need to have this property, since we used the maximal stopping times τ_n in the definition of σ_n and therefore, we cannot preclude that σ_n is a blow-up time on some paths.

Theorem 2.3.12. *Choose one of the settings and assume* [TTQ1], [TTQ2*]-[TTQ4*] *or* [GMQ1], [GMQ2*]-[GMQ4*] *or* [LQQ1], [LQQ2*]-[LQQ4*]. *Moreover, we assume* [Q5], [Q8] *and* [Q6*], [Q7*], [Q9*]. *Then, there is an increasing sequence of* $\mathbb{F}$*-stopping times* $(\mu_n)_n$ *with* $0 \le \mu_n \le T$ *almost surely such that* $\big(u, (\mu_n)_n, \mu\big)$ *is the maximal unique solution of*

$$\text{(QSEE)} \begin{cases} du(t) &= [-A(u(t))u(t) + F(u)(t) + f(t)]\,\mathrm{d}t + [B(u)(t) + b(t)]\,\mathrm{d}W(t) \\ u(0) &= u_0. \end{cases}$$

Moreover, we have the blow-up criterion

$$\mathbb{P}\big\{\mu < T,\ \|u\|_{X^1(0,\mu)} < \infty,\ u : [0, \mu) \to \mathrm{TR}\ \textit{is uniformly continuous}\big\} = 0.$$

Proof. First we construct the sequence of stopping times $(\mu_n)_{n\in\mathbb{N}}$. Recall the definition of σ_n in (2.3.9) and of μ in (2.3.11). If we additionally set

$$\sigma_{nk} := \tau_{nk} \wedge \inf\big\{t \in [0, \tau_n) : \|u_n\|_{\mathrm{TR}} > n\big\},$$

we have the pointwise almost sure convergences $\mu = \lim_{n\to\infty} \sigma_n$ and $\sigma_n = \lim_{k\to\infty} \sigma_{nk}$. Since the stopping times σ_n, σ_{nk} are all bounded by T, the dominated convergence theorem yields $\sigma_n \to \mu$ for $n \to \infty$ and $\sigma_{nk} \to \sigma_n$ in $L^1(\Omega)$ for $k \to \infty$. If we now choose for given $n \in \mathbb{N}$ the natural number $k(n)$ such that $\|\sigma_n - \sigma_{nk(n)}\|_{L^1(\Omega)} \le \frac{1}{n}$, we obtain $\sigma_{nk(n)} \to \sigma$ in $L^1(\Omega)$ for $n \to \infty$. Choosing a suitable subsequence still denoted by $(\sigma_{nk(n)})_{n\in\mathbb{N}}$ yields $\sigma_{nk(n)} \to \sigma$ pointwise almost surely for $n \to \infty$. Moreover, since $(\Gamma_n)_n$ is an increasing sequence with $\Omega = \cup_{n\in\mathbb{N}} \Gamma_n$, we also have $\sigma_{nk(n)} \mathbf{1}_{\Gamma_n} \to \sigma$ pointwise almost surely for $n \to \infty$. Unfortunately this sequence is not necessarily increasing anymore. Therefore, we define

$$\mu_n := \max_{i\in\{1,\dots,n\}} \sigma_{ik(i)} \mathbf{1}_{\Gamma_i}$$

and prove that $(\mu_n)_n$ is the sequence, we wanted to construct. Clearly, since $\sigma_{nk(n)}$ is an $\mathbb{F}$-stopping time for all $n \in \mathbb{N}$ and since $\Gamma_n \in \mathcal{F}_0$, μ_n is also an $\mathbb{F}$-stopping time. Furthermore the trivial bounds $\sigma_{nk(n)} \le \mu_n \le \mu$ for every $n \in \mathbb{N}$ yield $\mu_n \to \mu$ almost surely.

It remains to check that u is a strong solution of (QSEE) on $[0, \mu_n]$. It is sufficient to show that u is a strong solution of (QSEE) on $\Gamma_n \times [0, \sigma_{nk}]$ for all $n, k \in \mathbb{N}$. We have $u(\omega, t) = u_n(\omega, t)$ for almost all $\omega \in \Gamma_n$ and all $t \in [0, \sigma_n(\omega)) \supset [0, \sigma_{nk}(\omega)]$ by definition of u. Since u_n is a strong solution of the truncated equation (2.3.8) on $[0, \tau_{nk}]$ and in particular a strong solution of (QSEE) on $\Gamma_n \times [0, \sigma_{nk}]$, we conclude that u itself is a strong solution of (QSEE) on $\Gamma_n \times [0, \sigma_{nk}]$.

Next, we prove

$$\mathbb{P}\big\{\mu < T,\ \|u\|_{X^1(0,\mu)} < \infty,\ u : [0,\mu) \to \mathrm{TR} \text{ is uniformly continuous}\big\} = 0.$$

Since uniformly continuous functions on a bounded interval are always bounded, we only need to prove $\mathbb{P}(\Omega_n) = 0$ for every $n \in \mathbb{N}$, where Ω_n is given by

$$\begin{aligned}\Omega_n := \big\{&\mu < T,\ \|u\|_{X^1(0,\mu)} < \infty,\ u : [0,\mu) \to \mathrm{TR} \text{ is uniformly continuous},\\ &\|u\|_{C(0,\mu;\mathrm{TR})} \in [\tfrac{n-1}{2}, \tfrac{n}{2})\big\}.\end{aligned}$$

We first show that for almost all $\omega \in \{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$, we have $\mu(\omega) = \tau_n(\omega)$.

Clearly $\tau_n = \sigma_n$ on $\{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$. Furthermore the sequence $(\sigma_k)_{k\ge n}$ increases on the even larger set Γ_n by Lemma 2.3.11 and converges to μ. Thus we have $\tau_n \le \mu$ on $\{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$.

On the other hand, we have $\tau_n \ge \mu$ on $\{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$ since on this subset of Ω, u solves the truncated equation

$$\begin{cases} dw(t) &= [-A_n(w(t))u(t) + F_n(w)(t) + f(t)]\,\mathrm{d}t + [B_n(w)(t) + b(t)]\,\mathrm{d}W(t),\\ w(0) &= u_0\mathbf{1}_{\Gamma_n}, \end{cases} \tag{2.3.13}$$

and τ_n was defined as the maximal stopping. This finally proves $\tau_n = \mu$ on the set $\{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$ and the above argument also shows $u(\omega,t) = u_n(\omega,t)$ for almost all $\omega \in \{\|u\|_{C(0,\mu;\mathrm{TR})} \in [\frac{n-1}{2}, \frac{n}{2})\}$ and all $t \in [0, \mu(\omega))$. In conclusion, we have

$$\begin{aligned}&\mathbb{P}\big\{\mu < T,\ \|u\|_{X^1(0,\mu)} < \infty,\ u : [0,\mu) \to \mathrm{TR} \text{ is uniformly continuous},\\ &\qquad\qquad\qquad\qquad \|u\|_{C(0,\mu;\mathrm{TR})} \in [\tfrac{n-1}{2}, \tfrac{n}{2})\big\}\\ =&\mathbb{P}\big\{\tau_n < T,\ \|u_n\|_{X^1(0,\tau_n)} < \infty,\ u_n : [0,\tau_n) \to \mathrm{TR} \text{ is uniformly continuous},\\ &\qquad\qquad\qquad\qquad \|u_n\|_{C(0,\tau_n;\mathrm{TR})} \in [\tfrac{n-1}{2}, \tfrac{n}{2})\big\}\end{aligned}$$

and by Theorem 2.3.3 this quantity equals zero.

It remains to check that $\big(u, (\mu_n)_n, \mu\big)$ is a maximal unique solution. Let $\big(v, (\kappa_n)_n, \kappa\big)$ be another local solution of (QSEE). We first prove that u and v coincide on $\Omega \times [0, \mu\wedge\kappa)$. Define the sequence $(\rho_n)_n$ of $\mathbb{F}$-stopping times by

$$\rho_n := \inf\big\{t \in [0,\mu) : \|u\|_{\mathrm{TR}} > n\big\} \wedge \inf\big\{t \in [0,\kappa) : \|v\|_{\mathrm{TR}} > n\big\} \wedge \mu \wedge \kappa$$

for $n \in \mathbb{N}$. Then both u and v solve the truncated equation (2.3.13) on $\Gamma_n \times [0, \rho_n)$ and this equation is uniquely solvable up to a maximal stopping time, which implies $u(\omega,t) = v(\omega,t)$ for almost all $\omega \in \Gamma_n$ and all $t \in [0, \rho_n)$. Since $\rho_n \to \mu\wedge\kappa$ almost surely for $n \to \infty$ and $\cup_{n=1}^{\infty}\Gamma_n = \Omega \setminus \widetilde{N}$ for some set of measure zero $\widetilde{N}$, we conclude that u and v coincide on $\Omega\times[0,\mu\wedge\kappa)$. Maximality is then a consequence of the blow-up alternative we derived above.

Indeed, if we had $\kappa > \mu$ on a set of postive measure Λ, then $u : \Lambda \times [0,\mu) \to \mathrm{TR}$ would be almost surely uniformly continuous and we had $\|u\|_{X^1(0,\mu)} < \infty$ almost surely on Λ. But this would imply $\mu = T$ on Λ, which contradicts $\kappa > \mu$ on Λ, since κ is also bounded by T. □

The following corollary shows, that we can mix spatial regularity and regularity in time of the maximal unique solution $(u, (\tau_n)_n, \tau)$.

Corollary 2.3.13. *Let the assumptions from Theorem 2.3.12 be fulfilled and let $\theta \in [0, \frac{1}{2})$. Then, the maximal unique solution $(u, (\tau_n)_n, \tau)$ has the following additional regularity depending on the respective setting.*

[TT] $u \in W^{\theta,p}(0,\tau_n; [E, E^1]_{1-\theta})$ almost surely for every $n \in \mathbb{N}$.

[GM] $u \in \gamma(W^{-\theta,2}(0,\tau_n); [E, E^1]_{1-\theta})$ almost surely for every $n \in \mathbb{N}$.

[LQ] $u \in \Lambda^{\theta-\alpha}\big(L^q(U; W^{\theta,p}(0,\tau_n))\big)$ almost surely for every $n \in \mathbb{N}$.

Proof. u is a strong solution of

$$\begin{cases} du(t) &= \big[-A(u_0)u(t) + \widetilde{F}(u)(t) + f(t)\big]\,\mathrm{d}t + \big[B(u)(t) + b(t)\big]\,\mathrm{d}W(t), \quad t \in [0,\tau_n], \\ u(0) &= u_0 \end{cases} \tag{2.3.14}$$

in the sense of Definition 2.2.3 with

$$\widetilde{F}(u)(t) = \big(A(u_0) - A(u(t))\big)u(t) + F(u)(t).$$

Moreover, we have $u \in X^1(0,\tau_n) \cap C(0,\tau_n; \mathrm{TR})$ almost surely. In particular, u is also a mild solution of (2.3.14), i.e.

$$u(t) = e^{-tA(u_0)}u_0 + \Big(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u) + f)\Big)_0(t) + \Big(e^{-(\cdot)A(u_0)} \diamond (B(u) + b)\Big)_0(t)$$

almost surely for all $t \in [0,\tau_n]$. By [S5], we have $B(u) + b \in X_H^{\frac{1}{2}}(0,\tau_n)$ almost surely. Define

$$\eta_l := \inf\big\{t \in [0,\tau_n] : \ \|B(u) + b\|_{X_H^{\frac{1}{2}}(0,t)} > l\big\} \wedge \tau_n$$

and set $\Gamma_l := \{\|u_0\|_{\mathrm{TR}} \le l\}$. Then, we have $t \mapsto \mathbf{1}_{\Gamma_l} B(u)(t\wedge\eta_l) + b(t\wedge\eta_l) \in L^r(\Omega; X_H^{\frac{1}{2}}(0,\tau_n))$ and we can apply regularity results for the stochastic convolution in all the three settings. At this point, it essential that we restrict us to Γ_l, since on Γ_l the operator $A(u_0)$ has a bounded H^∞-calculus that is uniform with respect to ω. Under the assumptions for [TT], we get

$$\Big(e^{-(\cdot)A(u_0)} \diamond \mathbf{1}_{\Gamma_l}\big(B(u)(\cdot\wedge\eta_l) + b(\cdot\wedge\eta_l)\big)\Big)_0 \in L^r(\Omega; W^{\theta,p}(0,\tau_n; [E,E^1]_{1-\theta}))$$

by [96], Theorem 3.5. The analogous result for [GM] namely

$$\Big(e^{-(\cdot)A(u_0)} \diamond \mathbf{1}_{\Gamma_l}\big(B(u)(\cdot\wedge\eta_l) + b(\cdot\wedge\eta_l)\big)\Big)_0 \in L^r(\Omega; \gamma(W^{-\theta,2}(0,\tau_n); [E,E^1]_{1-\theta}))$$

can be found in [98], Theorem 3.3. For [LQ] and

$$\left(e^{-(\cdot)A(u_0)} \diamond \mathbf{1}_{\Gamma_l}\big(B(u)(\cdot \wedge \eta_l) + b(\cdot \wedge \eta_l)\big)\right)_0 \in L^r\big(\Omega; \Lambda^{\theta-\alpha}\big(L^q(U; W^{\theta,p}(0,\tau_n))\big)\big)$$

we use [8], Theorem 3.4.10. It remains to pass to the limit $l \to \infty$ to get the claimed pathwise regularity for the stochastic convolution. Here, we make use of the fact that for almost every $\omega \in \Omega$, there exists $l = l(\omega)$ such that $\omega \in \Gamma_l$ and $\eta_l = \tau_n$.

For the deterministic convolution, we can argue pathwise. However, it is important to note the estimates are not independent of $\omega \in \Omega$, since both the bound of the functional calculus and the norm of $A(u_0(\omega))$ depend on $\|u_0(\omega)\|_{\mathrm{TR}}$.

In [TT] complex interpolation together with Theorem 2.1.1 yields

$$\begin{aligned}\Big\|e^{-tA(u_0)}u_0 + &\left(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u)+f)\right)_0\Big\|_{W^{\theta,p}(0,\tau_n;[E,E^1]_{1-\theta})}\\ \lesssim &\Big\|e^{-tA(u_0)}u_0 + \left(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u)+f)\right)_0\Big\|_{W^{1,p}(0,\tau_n;E)}\\ &+ \Big\|e^{-tA(u_0)}u_0 + \left(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u)+f)\right)_0\Big\|_{L^p(0,\tau_n;E^1)}\\ \lesssim &\|u_0\|_{(E,E^1)_{1-1/p,p}} + \|\widetilde{F}(u)+f\|_{L^p(0,\tau_n;E)}.\end{aligned}$$

In [GM], Theorem 3.3 from [98] directly gives

$$\begin{aligned}\Big\|e^{-tA(u_0)}u_0 + \left(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u)+f)\right)_0\Big\|_{\gamma(W^{-\theta,2}(0,\tau_n);[E,E^1]_{1-\theta})}\\ \lesssim \|u_0\|_{[E,E^1]_{\frac{1}{2}}} + \|\widetilde{F}(u)+f\|_{\gamma(0,\tau_n;E)}\end{aligned}$$

and in [LQ], we use [8], Theorem 3.3.9, to get

$$\begin{aligned}\Big\|\Lambda^{\alpha-\theta}e^{-tA(u_0)}u_0 + \Lambda^{\alpha-\theta}\left(e^{-(\cdot)A(u_0)} * (\widetilde{F}(u)+f)\right)_0\Big\|_{L^q(U;W^{\theta,p}(0,\tau_n))}\\ \lesssim \|u_0\|_{F^{\alpha-\frac{1}{p}}_{\Lambda,q,p}} + \|\Lambda^{\alpha-1}(\widetilde{F}(u)+f)\|_{L^q(U;L^p(0,\tau_n))}.\end{aligned}$$

Hence, it remains to estimate the right hand sides. This can be done in a unified way. The constant in the following estimate depends on $\sup_{t\in[0,\tau_n]}\|u(t)\|_{\mathrm{TR}}$. We have

$$\begin{aligned}&\|u_0\|_{\mathrm{TR}} + \|(A(u_0)-A(u))u + F(u) + f\|_{X(0,\tau_n)}\\ &\quad\lesssim 1 + \|u_0\|_{\mathrm{TR}} + \sup_{t\in[0,\tau_n]}\|u(t)-u_0\|_{\mathrm{TR}}\|u\|_{X^1(0,\tau_n)} + \|u\|_{X^1(0,\tau_n)} + \|u\|_{C(0,\tau_n;\mathrm{TR})}\\ &\qquad + \|f\|_{X(0,\tau_n)}\end{aligned}$$

and the right hand side is almost surely finite, since u is a strong solution on $[0,\tau_n]$. This closes the proof. $\square$

CHAPTER 3

Examples for quasilinear parabolic stochastic evolution equations

In the following chapter, we apply the theory we developed in Chapter 3 to quasilinear stochastic partial differential equations. At first, we tread quasilinear parabolic equations in both nondivergence form, i.e. with principal part $\sum_{i,j=1}^{d} a_{ij}(\cdot, u(t), \nabla u(t))\partial_i\partial_j u(t)$, and in divergence form, i.e. with principal part $\operatorname{div}(a(u(t))\nabla u(t))$. In these examples we benefit from the extensive literature about elliptic operators, their regularity properties and their functional calculi. Applying Theorem 2.3.12, we show existence and uniqueness of a maximal unique solution $(u, (\tau_n)_n, \tau)$ in all the three settings. Using the blow-up characterisation from Theorem 2.3.12, we can even prove global well-posedness of a divergence form equation on a bounded domain with Dirichlet boundary conditions. The last example is inspired from fluid dynamics. We treat non-Newtonian fluids in a stochastic setting and derive local well-posedness.

3.1. A quasilinear parabolic equation in nondivergence form on $\mathbb{R}^d$

In this section, we discuss the most straightforward example, namely

$$\begin{cases} du(t) &= \left[\sum_{i,j=1}^{d} a_{ij}(\cdot, u(t), \nabla u(t))\partial_i\partial_j u(t) + F(u)(t)\right] \mathrm{d}t + \sum_{j=1}^{\infty} B_j(u)(t)\, \mathrm{d}\beta_j(t), \\ u(0) &= u_0 \end{cases} \tag{3.1.1}$$

on $\mathbb{R}^d$. For simplicity, we restrict us to noise perturbation with respect to an independent sequence of Brownian motions $(\beta_n)_n$ on a probability space $(\Omega, \mathbb{P})$ relative to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the usual conditions.

First, we discuss this example in the settings [TT] and [LQ]. At the end of this section, we treat it in [GM], since in this case, we will need different assumptions on the coefficients.

At first, we show existence and uniqueness of a maximal unique solution $(u, (\tau_n)_n, \tau)$ for given initial data $u_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$ or $u_0 \in F^{2-2/p}_{q,p}(\mathbb{R}^d)$. In the first case, u will be in $u \in L^p(0,\tau_n; W^{2,q}(\mathbb{R}^d))$ almost surely and in the second case, we show that u is contained in $W^{2,q}(\mathbb{R}^d; L^p(0,\tau_n))$ almost surely for every $n \in \mathbb{N}$. For the treatment of this equation, we essentially need that the operator $u \mapsto \sum_{i,j=1}^d a_{ij}(\cdot, v(t), \nabla v(t))\partial_i\partial_j u$ has the domain $W^{2,q}(\mathbb{R}^d)$ for any fixed v and that it has a bounded H^∞-calculus. At first, we specify our assumptions.

[E1] The coefficient matrix $a = (a_{ij})_{i,j=1,\dots,d} : \mathbb{R}^d \times \mathbb{C} \times \mathbb{C}^d \to \mathbb{C}^{d\times d}$ is uniformly elliptic, i.e.

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^d, y\in\mathbb{C}, z\in\mathbb{C}^d} \inf_{|\xi|=1} \operatorname{Re} \xi^T a(x,y,z)\overline{\xi} = \delta_0 > 0.$$

Moreover, a is β-Hölder continuous in the first and locally Lipschitz continuous in the second and the third component, i.e. there exists a constant $C > 0$ and for every $n \in \mathbb{N}$, there exist constants $L(n) > 0$, $\widetilde{L}(n) > 0$ depending on n such that

$$|a(x,y,\widetilde{y}) - a(\widetilde{x}, z, \widetilde{z})| \le C|x-\widetilde{x}|^\beta + L(n)|y-z| + \widetilde{L}(n)|\widetilde{y} - \widetilde{z}|$$

for all $x, \widetilde{x} \in \mathbb{R}^d$ and all $|y|, |z|, |\widetilde{y}|, |\widetilde{z}| < n$. Further, we assume $a(\cdot, 0, 0) \in L^\infty(\mathbb{R}^d)$.

[E2] We choose $p, q \in (2,\infty)$ such that $1 - 2/p > d/q$ and $r \in (1,\infty)$.

[E3] We either choose $u_0 : \Omega \to B^{2-2/p}_{q,p}(\mathbb{R}^d)$ or $u_0 : \Omega \to F^{2-2/p}_{q,p}(\mathbb{R}^d)$. In both cases, we require u_0 to be strongly $\mathcal{F}_0$-measurable.

[E4] The nonlinearities F and $(B_n)_n$ satisfy [Q6*] and [Q7*] together with [Q9*]. If $u_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$, we take the assumptions in the setting [TT], whereas we choose [LQ] if $u_0 \in F^{2-2/p}_{q,p}(\mathbb{R}^d)$. In any case, the underlying Hilbert space H is given by $l^2(\mathbb{N})$.

Note that the nonlinearities particularly fulfil [E4] if the are of lower order. This is a immediate consequence of Proposition 2.3.9. We want to apply Theorem 2.3.12 with the family of operators $A(z) = -\sum_{i,j=1}^d a_{ij}(\cdot, z, \nabla z)\partial_i\partial_j$ in the settings [TT] and [LQ]. We only have to check [TTQ2*] – [TTQ4*] and [LQQ2*] – [LQQ4*].

We first show that the operators $A(z)$ have a holomorphic functional calculus and that they have a constant domain $W^{2,q}(\mathbb{R}^d)$. For elliptic operators in nondivergence form, these results are well-known.

Lemma 3.1.1. *Let $p, q \in (1,\infty)$, $M > 0$, $0 \le \theta_0 \le \pi/2$, $\delta > 0, \beta > 0$ and $\alpha > 0$. Moreover, let*

$$Bf(x) = -\sum_{i,j=1}^d b_{ij}(x)\partial_i\partial_j f(x)$$

with uniformly elliptic coefficient matrix $b = (b_{ij})_{i,j=1,\dots,d}$, i.e.

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^d} \inf_{|\xi|=1} \operatorname{Re} \xi^T b(x)\overline{\xi} = \delta > 0$$

and with $\xi^T b(x)\overline{\xi} \in \Sigma_{\theta_0}$ for all $x \in \mathbb{R}^d$ and $\xi^d \in \mathbb{C}^d$. Moreover, b additionally satisfies

$$\sup_{x\in\mathbb{R}^d} |b_{ij}(x)| + \sup_{x,y\in\mathbb{R}^d} \frac{|b_{ij}(x) - b_{ij}(y)|}{|x-y|^\beta + |x-y|^\alpha} \le M$$

for all $i,j = 1,\dots,d$. *Then,* B *is a closed operator on* $L^q(\mathbb{R}^d)$ *with domain* $W^{2,q}(\mathbb{R}^d)$. *Moreover, given* $\theta \in (\theta_0, \pi/2)$, *there exist* $\mu > 0$ *and* $K > 0$ *only depending on* $p, q, M, \theta_0, \delta, \beta$ *and* α *such that*

$$K^{-1}\|(\mu+B)x\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} \le \|x\|_{W^{2,q}(\mathbb{R}^d;l^p(\mathbb{N}))} \le K\|(\mu+B)x\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} \tag{3.1.2}$$

for all $x \in L^q(\mathbb{R}^d; l^p(\mathbb{N}))$ *and such that*

$$\|f(\mu+B)\|_{\mathcal{B}(L^q(\mathbb{R}^d))} \le K\|f\|_{H^\infty(\Sigma_\theta)}$$

for all $f \in H^\infty(\Sigma_\theta)$.

Proof. In [6], Theorem 9.4 the authors show that elliptic operators in nondivergence form have a bounded functional calculus if the coefficients are Hölder continuous. In the meantime, there are more general versions (see e.g. [33]). However, we chose this result, since the authors discussed the precise dependencies of the constants in detail. For the estimate (3.1.2) we note that

$$\|(\mu+B)x\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} \le C\|x\|_{W^{2,q}(\mathbb{R}^d;l^p(\mathbb{N}))}$$

for some constant $C > 0$ is immediate by Lemma 2.3.4 and by the boundedness of $(b_{ij})_{ij}$. The reverse estimate is more tricky. From [39], Remark 5.6, we know

$$\|x\|_{W^{2,q}(\mathbb{R}^d;\nu)} \le \widetilde{C}\|(\mu+B)x\|_{L^q(\mathbb{R}^d;\nu)}$$

for all Muckenhoupt weights $\nu \in A_q$ and all $x \in W^{2,q}(\mathbb{R}^d;\nu)$. The authors are also precise with the constant $\widetilde{C} > 0$ depending on q, δ, M, θ and on ν in an A_q-consistent way. As a consequence, we get the vector valued inequality (3.1.2) by extrapolation. This can be found in [23], Corollary 3.12. For the precise estimate of the constant see also [38], Theorem 2.3. $\square$

In particular, $-(\mu+B)$ is the generator of a bounded analytic semigroup on $L^q(\mathbb{R}^d)$ of angle $\pi/2 - \theta_0$ which satisfies $\|e^{-(\mu+B)z}\|_{\mathcal{B}(L^q(\mathbb{R}^d))} \le M_\eta$ for some $\eta \in (0, \pi/2 - \theta_0)$, some $M_\theta > 0$ and for all $z \in \Sigma_\eta$. It is quite remarkable that in this special situation, sectoriality of $\mu + B$ implies that $e^{-t(\mu+B)}$ satisfies the generalised Gaussian estimates we need for the $\mathcal{R}_p$-bounded holomorphic functional calculus of B.

Lemma 3.1.2. *Let* $1 < q_0 < q_1 < \infty$ *and* $\eta \in (0, \pi/2 - \theta_0)$. *Then, there exist* $C > 0$ *and* $b > 0$ *only depending on* $q_0, q_1, M, \eta, \theta_0, \delta, \beta$ *and* α *such that the operator* B *introduced in Lemma 3.1.1 satisfies*

$$\|\mathbf{1}_{B(x,|z|^{-\frac{1}{2}})} e^{-zB} \mathbf{1}_{B(y,|z|^{-\frac{1}{2}})}\|_{\mathcal{B}(L^{q_0}(\mathbb{R}^d), L^{q_1}(\mathbb{R}^d))} \le C z^{-\frac{d}{2}(\frac{1}{q_0}-\frac{1}{q_1})} e^{-\mu \operatorname{Re}(z)} e^{-\frac{b|x-y|^2}{\operatorname{Re}(z)}}$$

for all $z \in \Sigma_\eta$. *Here,* $\mu > 0$ *is the same as in Lemma 3.1.1.*

Proof. This statement is due to Kunstmann in [66], Theorem 6.1. The proof is in the same article in Corollary 3.5. and consists of an interpolation between the estimates in Theorem 3.1 and an application of Lemma 3.4. In Theorem 3.1, the precise dependency of the involved constants on $q_0, q_1, M, \eta, \theta_0, \delta$ and α is mentioned. Interpolation preserves the dependency of the constants and closely inspecting the proof of Lemma 3.4, we see that C and b only depend on the parameters from above. $\square$

Now, we are in the position to check [TTQ2*], [TTQ3*] and [LQQ2*], [LQQ3*]. For the setting [LQ], we choose $\Lambda := (I-\Delta)$. Of course, $(I-\Delta)$ is an $\mathcal{R}_p$-sectorial operator on $L^q(\mathbb{R}^d)$ with $0 \in \rho(I-\Delta)$ that has an $\mathcal{R}_p$-bounded H^∞-calculus. This can be found in [69], section 3.

Proposition 3.1.3. *For all $n \in \mathbb{N}$, there exist $C(n) > 0, \eta(n) \in (0, \pi/2)$ and $\mu(n) > 0$ such that the following statements hold true.*

a) For all $u \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$ with $\|u\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)} \leq n$ the operators $\mu(n) + A(u)$ have the domain $W^{2,q}(\mathbb{R}^d)$ with

$$C(n)^{-1}\|(\mu(n) + A(u))x\|_{L^q(\mathbb{R}^d)} \leq \|x\|_{W^{2,q}(\mathbb{R}^d)} \leq C(n)\|(\mu(n) + A(u))x\|_{L^q(\mathbb{R}^d)}$$

for all $x \in W^{2,q}(\mathbb{R}^d)$ and they have a bounded $H^\infty(\Sigma_{\eta(n)})$-calculus with

$$\|f(\mu(n) + A(u))\|_{\mathcal{B}(L^q(\mathbb{R}^d))} \leq C(n)\|f\|_{H^\infty(\Sigma_{\eta(n)})}$$

for all $f \in H^\infty(\Sigma_{\eta(n)})$.

b) For all $u \in F^{2-2/p}_{q,p}(\mathbb{R}^d)$ with $\|u\|_{F^{2-2/p}_{q,p}(\mathbb{R}^d)} \leq n$ the operators $\mu(n) + A(u)$ have the domain $W^{2,q}(\mathbb{R}^d)$ with the same estimate as above. Moreover, $(I-\Delta)(\mu(n)+A(u))^{-1}$ and $(\mu(n) + A(u))(I-\Delta)^{-1}$ are $\mathcal{R}_p$-bounded with bound $C(n)$. Further, they have an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_{\eta(n)})$-calculus with

$$\mathcal{R}_p\Big(\big\{f(\mu(n) + A(u)) : \|f\|_{H^\infty(\Sigma_{\eta(n)})} \leq 1\big\} \subset \mathcal{B}(L^q(\mathbb{R}^d)\Big) \leq C(n).$$

In particular, the operators $A(u)$ satisfy [TTQ2*], [TTQ3*] *and* [LQQ2*], [LQQ3*] *respectively.*

Proof. By choice of p and q, functions in $B^{2-2/p}_{q,p}(\mathbb{R}^d)$ and in $F^{2-2/p}_{q,p}(\mathbb{R}^d)$ and their first order derivatives are α- Hölder continuous for some $\alpha > 0$. Hence, we can apply Lemma 3.1.1 and Lemma 3.1.2 to $A(u)$ and get the existence of θ_u with $0 \leq \theta_u \leq \pi/2$ and of constants $C_u > 0, \widetilde{C}_u, b_u > 0, \mu_u > 0$ such that

$$\|f(\mu_u + A(u))\|_{\mathcal{B}(L^q(\mathbb{R}^d))} \leq C_u\|f\|_{H^\infty(\Sigma_\eta)} \tag{3.1.3}$$

for some $\eta > 0$ with $\theta_u \leq \pi/2 - \eta$ and for all $f \in H^\infty(\Sigma_\eta)$ and

$$\|\mathbf{1}_{B(x,|z|^{-\frac{1}{2}})} e^{-zA(u)} \mathbf{1}_{B(y,|z|^{-\frac{1}{2}})}\|_{\mathcal{B}(L^{q_0}(\mathbb{R}^d), L^{q_1}(\mathbb{R}^d))} \leq \widetilde{C}_u z^{-\frac{d}{2}(\frac{1}{q_0}-\frac{1}{q_1})} e^{-\mu_u \operatorname{Re}(z)} e^{-\frac{b|x-y|^2}{\operatorname{Re}(z)}} \tag{3.1.4}$$

for all $z \in \Sigma_\eta$. Again, by Lemma 3.1.1, we know that $\mu_u + A(u)$ is invertible with

$$C_u^{-1}\|(\mu_u + A(u))\|_{L^q(\mathbb{R}^d)} \le \|x\|_{W^{2,q}(\mathbb{R}^d)} \le C_u\|(\mu_u + A(u))\|_{L^q(\mathbb{R}^d)}.$$

We now show that these constants do not explicitly depend on u, but on $\|u\|_{C^\alpha(\mathbb{R}^d)}$ and $\|\nabla u\|_{C^\alpha(\mathbb{R}^d)}$ for some $\alpha > 0$ and on the constants in [E1]. To do this, we have to estimate the quantities M, δ, α and θ_0 from Lemma 3.1.1 in this situation. The coefficient matrix $a(\cdot, u, \nabla u)$ is uniformly elliptic with ellipticity constant δ_0, hence we can use $\delta := \delta_0$. Moreover, by the Hölder continuity of a, u and ∇u, we get

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d} |a(x, u(x), \nabla u(x))| \\
&\qquad \le \sup_{x\in\mathbb{R}^d} |a(x, u(x), \nabla u(x)) - a(x, 0, 0)| + \|a(\cdot, 0, 0)\|_{L^\infty(\mathbb{R}^d)} \\
&\qquad \le L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)\|u\|_{L^\infty(\mathbb{R}^d)} + \widetilde{L}\big(\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\big)\|\nabla u\|_{L^\infty(\mathbb{R}^d)} + \|a(\cdot, 0, 0)\|_{L^\infty(\mathbb{R}^d)}
\end{aligned}$$

and

$$\begin{aligned}
&|a(x, u(x), \nabla u(x)) - a(y, u(y), \nabla u(y))| \\
&\le C|x-y|^\beta + L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)|u(x) - u(y)| + \widetilde{L}\big(\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\big)|\nabla u(x) - \nabla u(y)| \\
&\le C|x-y|^\beta + L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)\|u\|_{C^\alpha(\mathbb{R}^d)}|x-y|^\alpha + \widetilde{L}\big(\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\big)\|\nabla u\|_{C^\alpha(\mathbb{R}^d)}|x-y|^\alpha \\
&\lesssim_{\|u\|_{C^\alpha(\mathbb{R}^d)}, \|\nabla u\|_{C^\alpha(\mathbb{R}^d)}} |x-y|^\beta + |x-y|^\alpha.
\end{aligned}$$

Hence, we proved that M only depends on $\|u\|_{C^\alpha(\mathbb{R}^d)}$ and $\|\nabla u\|_{C^\alpha(\mathbb{R}^d)}$. It remains to investigate the dependency of the angle θ_0. Given $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, we estimate

$$\begin{aligned}
\operatorname{Im} \xi^T a(x, u(x), \nabla u(x))\overline{\xi} &\le |\xi^T a(x, u(x), \nabla u(x))\overline{\xi}| \le \|a(\cdot, u, \nabla u)\|_{L^\infty(\mathbb{R}^d)}|\xi|^2 \\
&\le \frac{\|a(\cdot, u, \nabla u)\|_{L^\infty(\mathbb{R}^d)}}{\delta_0} \operatorname{Re} \xi^T a(x, u(x), \nabla u(x))\overline{\xi}
\end{aligned}$$

which yields

$$\arg\big(\xi^T a(x, u(x), \nabla u(x))\overline{\xi}\big) \le \arctan\Big(\frac{\|a(\cdot, u, \nabla u)\|_{L^\infty(\mathbb{R}^d)}}{\delta_0}\Big)$$

for all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{C}^d$. Consequently, θ_0 only depends on $\|u\|_{L^\infty(\mathbb{R}^d)}$ and $\|\nabla u\|_{L^\infty(\mathbb{R}^d)}$.

All in all, we showed that the constants in (3.1.3) and (3.1.4) only depend on $\|u\|_{C^\alpha(\mathbb{R}^d)}$ and $\|\nabla u\|_{C^\alpha(\mathbb{R}^d)}$. However, using Sobolev embeddings, we get both $\|u\|_{C^\alpha(\mathbb{R}^d)} + \|\nabla u\|_{C^\alpha(\mathbb{R}^d)} \le \|u\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)}$ and $\|u\|_{C^\alpha(\mathbb{R}^d)} + \|\nabla u\|_{C^\alpha(\mathbb{R}^d)} \le \|u\|_{F^{2-2/p}_{q,p}(\mathbb{R}^d)}$. Hence, $C_u, \widetilde{C}_u, b_u, \theta_u, \mu_u$ do not depend precisely on u, but only on $\|u\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)}$ or $\|u\|_{F^{2-2/p}_{q,p}(\mathbb{R}^d)}$. This proves part *a*). The $\mathcal{R}_p$-bounded functional calculus in *b*) is then an immediate consequence of Theorem 1.4.5.

It remains to show that $(I-\Delta)(\mu(n)+A(u))^{-1}$ and $(\mu(n)+A(u))(I-\Delta)^{-1}$ are $\mathcal{R}_p$-bounded with bounds depending on n for all $\|u\|_{\mathrm{TR}} \le n$. This follows from (3.1.2) and the discussion about the dependence of the constants from above. □

It remains to show that our quasilinearity is locally Lipschitz with respect to the trace spaces $B^{2-2/p}_{q,p}(\mathbb{R}^d)$ and $F^{2-2/p}_{q,p}(\mathbb{R}^d)$.

Lemma 3.1.4. *For all $n \in \mathbb{N}$, there exists a constant $C_Q(n) > 0$ such that*

a) for all $y, z \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$ with norm at most n and all $v \in W^{2,q}(\mathbb{R}^d)$, we have

$$\|A(z)v - A(y)v\|_{L^q(\mathbb{R}^d)} \leq C_Q(n)\|z-y\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)}\|v\|_{W^{2,q}(\mathbb{R}^d)}.$$

b) For all $y, z \in C(a,b; F^{2-2/p}_{q,p}(\mathbb{R}^d))$ with norm at most n, we have

$$\begin{aligned}\mathcal{R}_p\Big(\big\{A(z(t)) - A(y(t))(I-\Delta)^{-1} : t \in[a,b]\big\} \subset \mathcal{B}(L^q(\mathbb{R}^d), L^q(\mathbb{R}^d))\Big)& \\ \leq C_Q(n) \sup_{t\in[a,b]} \|z(t)-y(t)\|_{F^{2-2/p}_{q,p}(\mathbb{R}^d)}&.\end{aligned}$$

In particular, [TTQ4] *and* [LPQ4] *are fulfilled.*

Proof. We just prove *b*), part *a*) follows the same lines. Let $y, z \in C(a,b; F^{2-2/p}_{q,p}(\mathbb{R}^d))$ with norm at most n, let $t_1, \cdots, t_N \in [a,b]$ and $v_1, \ldots, v_N \in W^{2,q}(\mathbb{R}^d)$. We estimate

$$\begin{aligned}&\Big\|\Big(\sum_{k=1}^N |(A(y(t_k)) - A(z(t_k)))(I-\Delta)^{-1}v|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_{i,j=1}^d \Big\|\Big(\sum_{k=1}^N |(a_{ij}(\cdot, y(t_k), \nabla y(t_k)) - a_{ij}(\cdot, z(t_k), \nabla z(t_k)))\partial_i\partial_j(I-\Delta)^{-1}v|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_{i,j=1}^d \sup_{k=1,\ldots,N} \|a_{ij}(\cdot, y(t_k), \nabla y(t_k)) - a_{ij}(\cdot, z(t_k), \nabla z(t_k))\|_{L^\infty(\mathbb{R}^d)} \\
&\qquad\qquad\qquad \Big\|\Big(\sum_{k=1}^N |\partial_i\partial_j(I-\Delta)^{-1}v_k|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_{i,j=1}^d \big(L(n)\|y-z\|_{L^\infty([a,b]\times\mathbb{R}^d)} + \widetilde{L}(n)\|\nabla y - \nabla z\|_{L^\infty([a,b]\times\mathbb{R}^d)}\big) \\
&\qquad\qquad\qquad \Big\|\Big(\sum_{k=1}^N |\partial_i\partial_j(I-\Delta)^{-1}v_k|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)} \\
&\lesssim (L(n) + \widetilde{L}(n)) \sup_{t\in[a,b]} \|y(t)-z(t)\|_{F^{2-2/p}_{q,p}(\mathbb{R}^d)} \Big\|\Big(\sum_{k=1}^N |\partial_i\partial_j(I-\Delta)^{-1}v_k|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}\end{aligned}$$

almost surely. From [49], Theorem 5.6.12 we know that $\partial_i\partial_j(I-\Delta)^{-1}$ is a bounded operator on $L^q(\mathbb{R}^d; l^p)$. This finally proves the claimed result. $\square$

Now, we are in the position to apply our abstract result to equation (3.1.1).

Theorem 3.1.5. *If* [E1]−[E4] *are fulfilled, there is a maximal unique local solution $(u, (\tau_n)_n, \tau)$ of equation* (3.1.1). *If $u_0 : \Omega \to B^{2-2/p}_{q,p}(\mathbb{R}^d)$, we have*

$$u \in L^p(0,\tau_n; W^{2,q}(\mathbb{R}^d)) \cap C(0,\tau_n; B^{2-2/p}_{q,p}(\mathbb{R}^d)) \cap W^{\theta,p}(0,\tau_n; W^{2-2\theta,q}(\mathbb{R}^d))$$

almost surely for every $\theta \in (0, \frac{1}{2})$ and for every $n \in \mathbb{N}$. Moreover, τ satisfies

$$\mathbb{P}\big\{\tau < T,\ \|u\|_{L^p(0,\tau;W^{2,q}(\mathbb{R}^d))} < \infty,\ u : [0,\tau) \to B^{2-2/p}_{q,p}(\mathbb{R}^d) \text{ is uniformly continuous}\big\} = 0.$$

If on the other hand $u_0 : \Omega \to F^{2-2/p}_{q,p}(\mathbb{R}^d)$, *we have*

$$u \in W^{2,q}(\mathbb{R}^d; L^p(0,\tau_n)) \cap C(0,\tau_n; F^{2-2/p}_{q,p}(\mathbb{R}^d)) \cap W^{2-2\theta}(\mathbb{R}^d; W^{\theta,p}(0,\tau_n))$$

almost surely for every $\theta \in (0, \frac{1}{2})$ *and for every* $n \in \mathbb{N}$. *Furthermore,* τ *satisfies*

$$\mathbb{P}\big\{\tau < T,\ \|u\|_{W^{2,q}(\mathbb{R}^d;L^p(0,\tau))} < \infty,\ u : [0,\tau) \to F^{2-2/p}_{q,p}(\mathbb{R}^d) \text{ is uniformly continuous}\big\} = 0.$$

Proof. We apply Theorem 2.3.12 and Corollary 2.3.13 in [TT] with the spaces $E = L^q(\mathbb{R}^d)$, $E^1 = W^{2,q}(\mathbb{R}^d)$ and with $(E, E^1)_{1-1/p,p} = B^{2-2/p}_{q,p}(\mathbb{R}^d)$. The assumption [TTQ1] is then satisfied straight away, whereas [TTQ2*],[TTQ3*] are checked in Proposition 3.1.3 and [TTQ4*] follows from Lemma 3.1.4.

In the setting [LQ], we apply the same results with the choice $\Lambda = (I - \Delta)$ and $\alpha = 0$. Here, [LQQ1] is also satisfied straight away and [LQQ2*], [LQQ3*] are checked in Proposition 3.1.3. The assumption [TTQ4*] follows from Lemma 3.1.4. This yields a solution u with

$$(I - \Delta)u \in L^q(\mathbb{R}^d; L^p(0,\tau_n))$$

almost surely for every $n \in \mathbb{N}$. Due to [49], Theorem 5.6.12 we know

$$\big\{u : (I - \Delta)u \in L^q(\mathbb{R}^d; L^p(0,\tau_n))\big\} = W^{2,q}(\mathbb{R}^d; L^p(0,\tau_n)),$$

which closes the proof. □

In particular, this theorem can be used to show that the solution $(u, (\tau_n)_n, \tau)$ is Hölder continuous in time. A Sobolev embedding yields $u \in C^\beta(0,\tau_n; W^{2-2\theta}(\mathbb{R}^d))$ if $u_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^d)$ and $u \in W^{2-2\theta}(\mathbb{R}^d; C^\beta(0,\tau_n))$ if $u_0 \in F^{2-2/p}_{q,p}(\mathbb{R}^d)$ with $\beta = \theta - \frac{1}{p}$ for all $\theta \in (\frac{1}{p}, \frac{1}{2})$.

Comparing this result with the known semilinear theory by van Neerven, Veraar and Weis in [96] and by Antoni in [8], we must admit that we cannot deal with noise of the form $B(u)(t) = \sigma \cdot \nabla u$ with $\sigma \in l^2(\mathbb{N})^d$ with a small enough norm. The reason for this is that the top-order Lipschitz constant in [Q7*] has to decrease to zero, since we have to fulfil [Q9*] and in general we cannot preclude that $C_{\mathrm{MRD}}(n)$ and $C_{\mathrm{MRS}}(n)$ increase with n and tend to infinity. In particular, an estimate of the form

$$\|\sigma \cdot \nabla u - \sigma \cdot \nabla v\|_{W^{2,q}(\mathbb{R}^d;l^2(\mathbb{N}))} \le \|\sigma\|_{l^2(\mathbb{N})}\|u - v\|_{W^{2,q}(\mathbb{R}^d)}$$

is not sufficient to fulfil [Q7*], no matter how small $\|\sigma\|_{l^2(\mathbb{N})^d}$ is.

We want to point out that the setting [GM] is also applicable to this equation if we slightly modify the quasilinear part. We choose E=$L^q(\mathbb{R}^d)$ and $E^1 = W^{2,q}(\mathbb{R}^d)$. TR is then given by $W^{1,q}(\mathbb{R}^d)$. Moreover, we have

$$\gamma(a, b; W^{k,q}(\mathbb{R}^d)) = W^{k,q}(\mathbb{R}^d; L^2(a, b)).$$

This setting has the advantage that we can choose the initial data in a larger space. Here, one possibility is to allow the coefficient matrix $(a_{ij})_{ij}$ to depend on u, but not on ∇u and to choose $q > d$. In this case, we can show the local Lipschitz estimate

$$\mathcal{R}\Big(\big\{A(z(t)) - A(y(t)) : t \in [a,b]\big\} \subset \mathcal{B}(W^{2,q}(\mathbb{R}^d), L^q(\mathbb{R}^d))\Big)$$
$$\leq C\big(\|z\|_{L^\infty([a,b]\times\mathbb{R}^d)} + \|y\|_{L^\infty([a,b]\times\mathbb{R}^d)}\big) \sup_{t\in[a,b]} \|z(t) - y(t)\|_{W^{1,q}(\mathbb{R}^d)},$$

which implies [GMQ4*], since $W^{1,q}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $q > d$. The assumptions [GMQ3*] and [GMQ4*] can be checked in the same ways as in the other settings. Here, we also make use of the embedding $W^{1,q}(\mathbb{R}^d) \hookrightarrow C^\alpha(\mathbb{R}^d)$ for some $\alpha > 0$.

Another possibility is to take the quasilinearity of the form $a_{ij}(u) = a_{i,j}(\|u\|_{W^{1,q}(\mathbb{R}^d)})$. Here, we can exploit all the advantages of the setting [GM], since we are able to solve our quasilinear equation for all $1 < q < \infty$ and not only for $q > 2$ as in the other settings.

Theorem 3.1.6. *Let the following assumptions be fulfilled.*

a) The coefficient matrix $a = (a_{ij})_{i,j=1,\ldots,d} : \mathbb{R}_{\geq 0} \to \mathbb{C}^{d\times d}$ is uniformly elliptic, i.e.

$$\operatorname*{ess\,inf}_{y\geq 0} \inf_{|\xi|=1} \operatorname{Re} \xi^T a(y)\overline{\xi} = \delta_0 > 0$$

and locally Lipschitz continuous, i.e. for every $n \in \mathbb{N}$, there exists a constant $L(n) > 0$ such that

$$|a_{i,j}(y) - a_{i,j}(z)| \leq L(n)|y - z|$$

for all $|y|, |z| < n$ and all $i, j = 1, \ldots, d$.

b) We require $u_0 : \Omega \to W^{1,q}(\mathbb{R}^d)$ to be strongly $\mathcal{F}_0$-measurable.

c) The nonlinearities F and $(B_n)_n$ satisfy [Q6*] *and* [Q7*] *together with* [Q9*].

Then, there exists a maximal unique solution $\big(u, (\tau_n)_n, \tau\big)$ of (3.1.1) *with*

$$u \in W^{2,q}(\mathbb{R}^d; L^2(0,\tau_n)) \cap C(0,\tau_n; W^{1,q}(\mathbb{R}^d))$$

almost surely for every $n \in \mathbb{N}$. Moreover, τ satisfies

$$\mathbb{P}\big\{\tau < T,\ \|u\|_{W^{2,q}(\mathbb{R}^d;L^2(0,\tau))} < \infty,\ u : [0,\tau) \to W^{1,q}(\mathbb{R}^d) \text{ is uniformly continuous}\big\} = 0.$$

Proof. In this setting, the proof is simple. Since the coefficients do not depend explicitly on x, we can apply the theory about elliptic operators with constant coefficients to

$$A(z)u = \sum_{i,j} a_{i,j}(\|z\|_{W^{1,q}(\mathbb{R}^d)})\partial_i\partial_j u$$

to get [GMQ2*] and [GMQ3*]. Amongst others this can be found in [87], Theorem 6.1.8. All the occurring constants in this result depend on the ellipticity and on the upper bound of the coefficients. In our situation, this means that the constants depend on δ_0 and $L(\|z\|_{W^{1,q}(\mathbb{R}^d)})$.

It remains to show the Lipschitz estimate required in [GMQ4*]. Let $y, z \in C(a,b;W^{1,q}(\mathbb{R}^d))$ with norm at most n, let $t_1, \cdots, t_N \in [a,b]$ and $v_1, \ldots, v_N \in W^{2,q}(\mathbb{R}^d)$. We estimate

$$\begin{aligned}
&\Big\| \Big(\sum_{k=1}^N |(A(y(t_k)) - A(z(t_k)))v|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_{i,j=1}^d \Big\| \Big(\sum_{k=1}^N |(a_{ij}(\|y(t_k)\|_{W^{1,q}(\mathbb{R}^d)}) - a_{ij}(\|z(t_k)\|_{W^{1,q}(\mathbb{R}^d)}))\partial_i\partial_j v|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_{i,j=1}^d \sup_{i,j=1\ldots,d} \sup_{k=1,\ldots,N} \big| a_{ij}(\|y(t_k)\|_{W^{1,q}(\mathbb{R}^d)}) - a_{ij}(\|y(t_k)\|_{W^{1,q}(\mathbb{R}^d)}) \big| \|v\|_{W^{2,q}(\mathbb{R}^d;l^2(\mathbb{N}))} \\
&\leq \sum_{i,j=1}^d \sup_{i,j=1\ldots,d} \sup_{k=1,\ldots,N} L(n) \big| \|y(t_k)\|_{W^{1,q}(\mathbb{R}^d)} - \|z(t_k)\|_{W^{1,q}(\mathbb{R}^d)} \big| \|v\|_{W^{2,q}(\mathbb{R}^d;l^2(\mathbb{N}))} \\
&\lesssim L(n) \sup_{t\in[a,b]} \|y(t) - z(t)\|_{W^{1,q}(\mathbb{R}^d)} \|v\|_{W^{2,q}(\mathbb{R}^d;l^2(\mathbb{N}))}.
\end{aligned}$$

This shows [GMQ4*] and hence, Theorem 2.3.12 is applicable. □

3.2. Weak solution of a quasilinear parabolic stochastic equation in divergence form

In this section, we consider a convection-diffusion equation

$$\text{(DIV)} \begin{cases} du(t) &= \big[\operatorname{div}(a(u(t))\nabla u(t)) + F(u)(t)\big]dt + B(u)(t)\,\mathrm{d}W(t)(t), \quad t \in [0,T] \\ u(0) &= u_0, \end{cases}$$

either on $D = \mathbb{R}^d$ or on a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$ with Dirichlet, Neumann or mixed boundary conditions. We aim to show existence and uniqueness of weak solutions in the sense of partial differential equations, which means that we treat this equation in $W^{-1,q}(D)$. On $\mathbb{R}^d$, both [TT] and [LQ] are applicable, whereas in the bounded domain case, we just use [TT]. The reason for this lack of generality is that we make use of the great progress within the last years concerning mixed boundary problems in $W^{-1,q}(D)$ for $q > 2$ and these tools are not deeply enough investigated in a vector valued setting, which makes it difficult to check the $\mathcal{R}_p$- boundedness assumptions needed in [LQ].

At the end of this section, we restrict ourselves to Dirichlet boundary condition and show that under a global Lipschitz assumption on the diffusion matrix $a(u)$, the solution does not explode and exists on the whole interval $[0,T]$. This generalises the work of Hofmanova and Zhang ([46]) on the torus to arbitrary bounded C^1-domains. Moreover, our method does not need initial data in the space $C^{1+\varepsilon}(\overline{D})$, but only in $(W^{-1,q}(D), W_0^{1,q}(D))_{1-1/p,p}$, which seems to be natural if one expects solutions that are pathwise in $L^p(0,T;W_0^{1,q}(D))$.

3.2.1. Local weak solution on $\mathbb{R}^d$

We aim to show existence and uniqueness of a weak solution u in the sense of partial differential equations. We will show $u \in L^p(0,\tau_n;W^{1,q}(\mathbb{R}^d)) \cap C(0,\tau_n;B_{q,p}^{1-1/p}(\mathbb{R}^d))$ for $u_0 \in$

$B^{1-1/p}_{q,p}(\mathbb{R}^d)$ which corresponds to the setting [TT]. If on the other hand $u_0 \in F^{1-1/p}_{q,p}(\mathbb{R}^d)$, our solution u will be almost surely in $W^{1,q}(\mathbb{R}^d; L^p(0,\tau_n)) \cap C(0,\tau_n; F^{1-1/p}_{q,p}(\mathbb{R}^d))$. This will be proved by using the setting [LQ]. As before, $(\tau_n)_n$ is an increasing sequence of stopping times that converges to a maximal stopping time τ almost surely. For simplicity we restrict ourselves to noise with respect to an independent sequence of Brownian motions $(\beta_n)_n$ on a probability space $(\Omega, \mathbb{P})$ relative to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We consider

$$\begin{cases} du(t) &= \big[\operatorname{div}(a(u(t))\nabla u(t)) + F(u)(t)\big]dt + \sum_{j=1}^{\infty} \big[B_j(u)(t)\big]\beta_j(t), \quad t \in [0,T], \\ u(0) &= u_0 \end{cases} \tag{3.2.1}$$

and make the following assumptions.

[L1] $a : \mathbb{C} \to \mathbb{R}^{d\times d}$ is uniformly positive definite, i.e.

$$\operatorname*{ess\,inf}_{y\in\mathbb{C}} \inf_{|\xi|=1} \xi^T a(y)\xi = \delta_0 > 0,$$

and a is locally Lipschitz continuous, i.e. for every $\alpha > 0$, there exists a constant $L(\alpha) > 0$ such that

$$|a(y) - a(z)| \leq L(\alpha)|y - z|$$

for all $|y|, |z| < \alpha$.

[L2] We choose $p, q \in (2,\infty)$ such that $1 - 2/p > d/q$.

[L3] We either choose $u_0 : \Omega \to B^{1-2/p}_{q,p}(\mathbb{R}^d)$ or $u_0 : \Omega \to F^{1-2/p}_{q,p}(\mathbb{R}^d)$. In both cases, we require u_0 to be strongly $\mathcal{F}_0$-measurable.

[L4] The nonlinearities F and $(B_n)_n$ satisfy [Q6*] and [Q7*] together with [Q9*]. If $u_0 \in B^{1-2/p}_{q,p}(\mathbb{R}^d)$, we take the assumptions in the setting [TT], whereas we choose [LQ] if $u_0 \in F^{1-2/p}_{q,p}(\mathbb{R}^d)$. In any case, the underlying Hilbert space H is given by $l^2(\mathbb{N})$.

We want to apply Theorem 2.3.12 with $A(z)u = -\operatorname{div}(a(z)\nabla u)$ in the settings [TT] and [LQ]. We have to check [TTQ2*] – [TTQ4*] and [LQQ2*] – [LQQ4*]. Our starting point are Gaussian estimates for the kernel of the semigroup generated by $\operatorname{div}(b\nabla u)$.

Lemma 3.2.1. *Let $b \in C^s(\mathbb{R}^d)^{d\times d}$ be a real-valued and uniformly positive definite matrix with $\langle b(z)\xi, \xi\rangle_{\mathbb{R}^d} \geq \delta|\xi|^2$ for every $\xi \in \mathbb{R}^d$ and every $z \in \mathbb{C}$. We define $Lf := -\operatorname{div}(b\nabla f)$. Then, the semigroup e^{-tL} is for every $t > 0$ an integral operator with kernel $k_t(x,y)$ that satisfies the Gaussian estimate $|k_t(x,y)| \leq Ct^{-\frac{d}{2}} e^{-\frac{c|x-y|^2}{t}}$ and*

$$|t^{\frac{1}{2}}\nabla k_t(x,y)| \leq \widetilde{C} t^{-\frac{d}{2}} \big(1 + \|b\|^{\frac{1}{s}}_{C^s(\mathbb{R}^d)^{d\times d}} t^{\frac{1}{2}}\big)^M e^{-\frac{c|x-y|^2}{t}}$$

for every $x, y \in \mathbb{R}^d$ and all $t > 0$. Here, the constants $c, \widetilde{C}, M > 0$ only depend on δ, d, s and $\|b\|_{C^s(\mathbb{R}^d)^{d\times d}}$, whereas the constant $C > 0$ only depends on d, δ and $\|b\|_{L^\infty(\mathbb{R}^d)^{d\times d}}$.

Proof. The gradient estimate and a slightly weaker version of the Gaussian estimate for $k_t(x,y)$ can be found in [11], Theorem 4.15. The claimed estimate for $k_t(x,y)$ can be found in [25], Theorem 6.1. Both of the theorems above are precise with the constants. □

As a consequence, we get that L has an $\mathcal{R}_p$-bounded functional calculus.

Proposition 3.2.2. *The operator L from the previous Lemma has for every $p, q \in (1, \infty)$ an $\mathcal{R}_p$-bounded functional calculus. In particular, there exists a constant $C > 0$ only depending on δ, d and $\|b\|_{L^\infty(\mathbb{R}^d)^{d\times d}}$ such that*

$$\mathcal{R}_p\Big(\big\{f(L) : \|f\|_{H^\infty(\Sigma_{\eta(n)})} \leq 1\big\} \subset B(L^q(\mathbb{R}^d))\Big) \leq C.$$

Proof. By [69], Lemma 2.2 the Gaussian estimate on k_t implies the L^1-L^∞-off-diagonal estimates for semigroup $e^{-\lambda L}$ that are required in Theorem 1.4.5. Hence, L has an $\mathcal{R}_p$-bounded functional calculus whose bound only depends on the constants in the estimate of k_t from Lemma 3.2.1. □

Again, using the Gaussian estimates from Lemma 3.2.1, one can derive a crucial property of L, namely that the Riesz transform $\nabla L^{-\frac{1}{2}}$ associated to L is $\mathcal{R}_p$-bounded.

Lemma 3.2.3. *Let L be as in the previous results. Then, $\nabla L^{-1/2}$ is an $\mathcal{R}_p$-bounded operator on $L^q(\mathbb{R}^d)$ for every $q, p \in (1, \infty)$ and $\mathcal{R}_p(\nabla L^{-1/2})$ only depends on q, p, d, $\|b\|_{C^s(\mathbb{R}^d)^{d\times d}}$ and δ. In particular, the operators $(I - \Delta)^{-\frac{1}{2}}(I + L)^{\frac{1}{2}}$, $(I - \Delta)^{\frac{1}{2}}(I + L)^{-\frac{1}{2}}$, $(I + L)^{-\frac{1}{2}}(I - \Delta)^{\frac{1}{2}}$ and $(I + L)^{\frac{1}{2}}(I - \Delta)^{-\frac{1}{2}}$ are $\mathcal{R}_p$-bounded with bounds depending on the same constants.*

Proof. The boundedness of $\nabla L^{-1/2}$ on $L^q(\mathbb{R}^d)$ is shown in [11], Theorem 5.1. The precise dependence of the constants is mentioned in Corollary 5.9 in the same article. The boundedness of of $\nabla L^{-1/2}$ on $L^q(\mathbb{R}^d; l^p(\mathbb{N}))$ for all $p, q \in (1, \infty)$ is due to [10], chapter 8. The Riesz transform is discussed therein in the remarks below Proposition 8.1. To get the result we need, you shall choose the weight $\mu = 1$.

Since $l^p(\mathbb{N})$ is UMD, we can apply [49], Proposition 5.6.3 and Theorem 5.6.11 and the vector valued boundedness of the Riesz transform to get

$$\begin{aligned}
&\|(I - \Delta)^{\frac{1}{2}}(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))}\\
&\qquad\lesssim_{p,q} \|(I + L)^{-\frac{1}{2}}f\|_{W^{1,q}(\mathbb{R}^d;l^p(\mathbb{N}))}\\
&\qquad= \|(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} + \|\nabla(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))}\\
&\qquad= \|(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} + \|\nabla L^{-\frac{1}{2}}L^{\frac{1}{2}}(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))}\\
&\qquad\leq \|(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))} + \mathcal{R}_p\big(\nabla L^{-\frac{1}{2}}\big)\|L^{\frac{1}{2}}(I + L)^{-\frac{1}{2}}f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))}\\
&\qquad\lesssim_{\delta,\|b\|_{C^s(\mathbb{R}^d)},d,p,q} \|f\|_{L^q(\mathbb{R}^d;l^p(\mathbb{N}))}
\end{aligned}$$

for every $f \in L^q(\mathbb{R}^d; l^p(\mathbb{N}))$. Note that in the last step, we used the $\mathcal{R}_p$-boundedness of the functional calculus of L from Proposition 3.2.2. As the adjoint of L is also a divergence form operator with uniformly positive definite and Hölder continuous coefficients, the same argument shows that $(I-\Delta)^{\frac{1}{2}}(I+L^*)^{-\frac{1}{2}}$ is bounded on $L^q(\mathbb{R}^d; l^p(\mathbb{N}))$ for every $p, q \in (1, \infty)$. By duality, this implies that $(I + L)^{-\frac{1}{2}}(I - \Delta)^{\frac{1}{2}}$ is bounded on $L^q(\mathbb{R}^d; l^p(\mathbb{N}))$ for every

$p,q \in (1,\infty)$. It remains to estimate $\|(I+L)^{\frac{1}{2}}(I-\Delta)^{-\frac{1}{2}}\|_{\mathcal{B}(L^q(\mathbb{R}^d;l^p(\mathbb{N})))}$. We calculate

$$\begin{aligned}
&\big\langle (I+L)^{\frac{1}{2}}(I-\Delta)^{-\frac{1}{2}}f, g\big\rangle_{L^q(\mathbb{R}^d;l^p)}\\
&\qquad=\big\langle (I+L)(I-\Delta)^{-\frac{1}{2}}f, (I+L^*)^{-\frac{1}{2}}g\big\rangle_{(L^q(\mathbb{R}^d;l^p),L^{q'}(\mathbb{R}^d;l^{p'}))}\\
&\qquad=\big\langle (I-\Delta)^{-\frac{1}{2}}f + L(I-\Delta)^{-\frac{1}{2}}f, (I+L^*)^{-\frac{1}{2}}g\big\rangle_{(L^q(\mathbb{R}^d;l^p),L^{q'}(\mathbb{R}^d;l^{p'}))}\\
&\qquad=\big\langle (I-\Delta)^{-\frac{1}{2}}f, (I+L^*)^{-\frac{1}{2}}g\big\rangle_{(L^q(\mathbb{R}^d;l^p),L^{q'}(\mathbb{R}^d;l^{p'}))}\\
&\qquad\quad+\big\langle b\nabla(I-\Delta)^{-\frac{1}{2}}f, \nabla(I+L^*)^{-\frac{1}{2}}g\big\rangle_{(L^q(\mathbb{R}^d;l^p),L^{q'}(\mathbb{R}^d;l^{p'}))},
\end{aligned}$$

which yields the estimate

$$\begin{aligned}
&\|(I+L)^{\frac{1}{2}}(I-\Delta)^{-\frac{1}{2}}\|_{\mathcal{B}(L^q(\mathbb{R}^d;l^p))}\\
&\qquad\le\|(I-\Delta)^{-\frac{1}{2}}\|_{\mathcal{B}(L^q(\mathbb{R}^d;l^p))}\|(I+L^*)^{-\frac{1}{2}}\|_{\mathcal{B}(L^{q'}(\mathbb{R}^d;l^{p'}))}\\
&\qquad\quad+\|b\|_{L^\infty(\mathbb{R}^d)}\|\nabla(I-\Delta)^{-\frac{1}{2}}\|_{\mathcal{B}(L^q(\mathbb{R}^d;l^p))}\|\nabla(I+L^*)^{-\frac{1}{2}}\|_{\mathcal{B}(L^{q'}(\mathbb{R}^d;l^{p'}))}.
\end{aligned}$$

The boundedness of $\nabla(I-\Delta)^{-\frac{1}{2}}$ is due to [49], Theorem 5.6.3 and the boundedness of $\nabla(I+L^*)^{-\frac{1}{2}}$ was shown above. The boundedness of $(I-\Delta)^{-\frac{1}{2}}$ and $(I+L^*)^{-\frac{1}{2}}$ finally follows from the $\mathcal{R}_p$-boundedness of the functional calculus (see Lemma 3.2.2). The remaining operator $(I-\Delta)^{-\frac{1}{2}}(I+L)^{\frac{1}{2}}$ can be handled with duality in a similar way as above. This proves our assertion. □

Now, we are in the position to check [TTQ2*], [TTQ3*] and [LQQ2*], [LQQ3*]. For the setting [LQ], we choose $\Lambda := (I-\Delta)$. Of course, $(I-\Delta)$ is an $\mathcal{R}_p$-sectorial operator on $L^q(\mathbb{R}^d)$ with $0\in\rho(I-\Delta)$ that has an $\mathcal{R}_p$-bounded H^∞-calculus. This can be found in [69], section 3.

Proposition 3.2.4. *For all $n\in\mathbb{N}$, there exist constants $C(n)>0$ and $\eta(n)\in(0,\pi/2)$ such that the following statements hold true.*

a) For all $u\in B^{1-2/p}_{q,p}(\mathbb{R}^d)$ with $\|u\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)}\le n$ the operators $I+A(u)$ have the domain $W^{1,q}(\mathbb{R}^d)$ with

$$C(n)^{-1}\|(I+A(u))x\|_{W^{-1,q}(\mathbb{R}^d)}\le\|x\|_{W^{1,q}(\mathbb{R}^d)}\le C(n)\|(I+A(u))x\|_{W^{-1,q}(\mathbb{R}^d)}$$

for all $x\in W^{1,q}(\mathbb{R}^d)$ and they have a bounded $H^\infty(\Sigma_{\eta(n)})$-calculus with

$$\|f(I+A(u))\|_{\mathcal{B}(W^{-1,q}(\mathbb{R}^d))}\le C(n)\|f\|_{H^\infty(\Sigma_{\eta(n)})}$$

for all $f\in H^\infty(\Sigma_{\eta(n)})$.

b) For all $u\in F^{1-2/p}_{q,p}(\mathbb{R}^d)$ with $\|u\|_{F^{1-2/p}_{q,p}(\mathbb{R}^d)}\le n$ the operators $(I-\Delta)^{\frac{1}{2}}(I+A(u))^{-\frac{1}{2}}$, $(I-\Delta)^{-\frac{1}{2}}(I+A(u))^{\frac{1}{2}}$, $(I+A(u))^{\frac{1}{2}}(I-\Delta)^{-\frac{1}{2}}$ and $(I+A(u))^{-\frac{1}{2}}(I-\Delta)^{\frac{1}{2}}$ are $\mathcal{R}_p$-bounded with bound smaller than $C(n)$. Further, they have an $\mathcal{R}_p$-bounded $H^\infty(\Sigma_{\eta(n)})$-calculus with

$$\mathcal{R}_p\Big(\big\{f(I+A(u)):\|f\|_{H^\infty(\Sigma_{\eta(n)})}\le 1\big\}\subset\mathcal{B}(L^q(\mathbb{R}^d))\Big)\le C(n).$$

In particular, the operators $A(u)$ satisfy [TTQ2*], TTQ3*] *and* [LQQ2*], [LQQ3*], *respectively.*

Proof. By choice of p and q, functions in $B^{1-2/p}_{q,p}(\mathbb{R}^d)$ and in $F^{1-2/p}_{q,p}(\mathbb{R}^d)$ are α- Hölder continuous for some $\alpha > 0$. Hence, we can apply Lemma 3.2.3 and get the $\mathcal{R}_p$ boundedness of $(I-\Delta)^{\frac{1}{2}}(I+A(u))^{-\frac{1}{2}}$, $(I-\Delta)^{-\frac{1}{2}}(I+A(u))^{\frac{1}{2}}$, $(I+A(u))^{\frac{1}{2}}(I-\Delta)^{-\frac{1}{2}}$ and $(I+A(u))^{-\frac{1}{2}}(I-\Delta)^{\frac{1}{2}}$ with bound $C_u > 0$. In particular, $(I+A(u))^{-1} : W^{-1,q}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded with bound smaller than $\widetilde{C}_u := C_u^2 \|(I-\Delta)^{-\frac{1}{2}}\|_{\mathcal{B}(L^q(\mathbb{R}^d),W^{1,q}(\mathbb{R}^d))} \|(I-\Delta)^{\frac{1}{2}}\|_{\mathcal{B}(W^{-1,q}(\mathbb{R}^d),L^q(\mathbb{R}^d))}$. This proves

$$(1 + \|a(u)\|_{L^\infty(\mathbb{R}^d)^{d\times d}})^{-1} \|(I + A(u))x\|_{W^{-1,q}(\mathbb{R}^d)} \leq \|x\|_{W^{1,q}(\mathbb{R}^d)} \leq \widetilde{C}_u \|(I + A(u))x\|_{W^{-1,q}(\mathbb{R}^d)}.$$

Moreover, by Proposition 3.2.2, there exist $c_u > 0, \eta_u \in (0, \pi/2)$ such that $A(u)$ and especially $(I + A(u))$ have an $\mathcal{R}_p$-bounded functional calculus of angle η_u with bound c_u.

Now, we show that these constants do not explicitly depend on u, but on $\|u\|_{C^\alpha(\mathbb{R}^d)}$ and on the constants in [L1]. To do this, we have to estimate the quantities δ and $\|a(u)\|_{C^\alpha(\mathbb{R}^d)}$ from Lemma 3.2.1 in our situation. The coefficient matrix $a(u)$ is uniformly elliptic with ellipticity constant δ_0, hence we can use $\delta := \delta_0$. Moreover, by the local Lipschitz continuity of a and the Hölder continuity of u, we get

$$\sup_{x\in\mathbb{R}^d} |a(u(x))| \leq \sup_{x\in\mathbb{R}^d} |a(u(x)) - a(0)| + |a(0)| \leq L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)\|u\|_{L^\infty(\mathbb{R}^d)} + |a(0)|$$

and

$$|a(u(x)) - a(u(y))| \leq L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)|u(x) - u(y)| \leq L\big(\|u\|_{L^\infty(\mathbb{R}^d)}\big)\|u\|_{C^\alpha}(\mathbb{R}^d)|x - y|^\alpha.$$

Hence, we get the $\|u\|_{C^\alpha(\mathbb{R}^d)} \leq \|u\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)}$ and $\|u\|_{C^\alpha(\mathbb{R}^d)} \leq \|u\|_{F^{1-2/p}_{q,p}(\mathbb{R}^d)}$ for some $\alpha > 0$ by applying Sobolev embeddings. In particular, the constants c_u, C_u, and $\widetilde{C}_u$ do not depend precisely on u, but only on $\|u\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)}$ and $\|u\|_{F^{1-2/p}_{q,p}(\mathbb{R}^d)}$, respectively. This closes the proof. □

It remains to show that our quasilinearity is locally Lipschitz with respect to the trace spaces $B^{1-2/p}_{q,p}(\mathbb{R}^d)$ and $F^{1-2/p}_{q,p}(\mathbb{R}^d)$.

Lemma 3.2.5. *For all $n \in \mathbb{N}$, there exists $C_Q(n) > 0$, such that the following statements hold true.*

a) For all $y, z \in B^{1-2/p}_{q,p}(\mathbb{R}^d)$ with $\|y\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)}, \|z\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)} \leq n$ and all $v \in W^{1,q}(\mathbb{R}^d)$, we have

$$\|A(z)v - A(y)v\|_{W^{-1,q}(\mathbb{R}^d)} \leq C_Q(n)\|z - y\|_{B^{1-2/p}_{q,p}(\mathbb{R}^d)}\|v\|_{W^{1,q}(\mathbb{R}^d)}.$$

b) For all $y, z \in C(a, b; F^{1-2/p}_{q,p}(\mathbb{R}^d))$ with $\|y\|_{C(a,b;F^{1-2/p}_{q,p}(\mathbb{R}^d))}, \|z\|_{C(a,b;F^{1-2/p}_{q,p}(\mathbb{R}^d))} \leq n$ we have

$$\mathcal{R}_p\Big(\big\{(I-\Delta)^{-\frac{1}{2}}\big(A(z(t)) - A(y(t))\big)(I-\Delta)^{-\frac{1}{2}} : t \in [a, b]\big\} \subset \mathcal{B}(L^q(\mathbb{R}^d))\Big)$$
$$\leq C_Q(n) \sup_{t\in[a,b]} \|z(t) - y(t)\|_{F^{1-2/p}_{q,p}(\mathbb{R}^d)}.$$

In particular, [TTQ4*] *and* [LPQ4*] *are fulfilled.*

Proof. We prove b), part a) follows the same lines. Let $y, z \in C(a,b;F_{q,p}^{1-2/p}(\mathbb{R}^d))$ with norm at most n, let $t_1, \cdots, t_N \in [a,b]$ and $v_1, \ldots, v_N \in W^{1,q}(\mathbb{R}^d)$. As $l^p(\mathbb{N})$ is UMD, Theorem 5.6.12 in [49] implies that both $(I-\Delta)^{-\frac{1}{2}}\operatorname{div}$ and $\nabla(I-\Delta)^{-\frac{1}{2}}$ are bounded on $L^q(\mathbb{R}^d;l^p)$. Thus, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k=1}^{N}|(I-\Delta)^{-\frac{1}{2}}(A(y(t_k))-A(z(t_k)))(I-\Delta)^{-\frac{1}{2}}v_k|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}\\
&\quad\lesssim_p \Big\|\Big(\sum_{k=1}^{N}|(a(y(t_k))-a(z(t_k)))\nabla(I-\Delta)^{-\frac{1}{2}}v_k|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)^d}\\
&\quad\le \sup_{k=1,\ldots,N}\|a(y(t_k))-a(z(t_k))\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\Big\|\big(\sum_{k=1}^{N}|v_k|^p\big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}\\
&\quad\le L\big(\max(\|y\|_{L^\infty([a,b]\times\mathbb{R}^d)},\|z\|_{L^\infty([a,b]\times\mathbb{R}^d)})\big)\|y-z\|_{L^\infty([a,b]\times\mathbb{R}^d)}\Big\|\big(\sum_{k=1}^{N}|v_k|^p\big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}\\
&\quad\lesssim_{d,p,q} L(n)\|y-z\|_{C(a,b;F_{q,p}^{1-2/p}(\mathbb{R}^d))}\Big\|\big(\sum_{k=1}^{N}|v_k|^p\big)^{1/p}\Big\|_{L^q(\mathbb{R}^d)}.
\end{aligned}$$

This proves the claimed result. □

Now, we are in the position to apply our abstract result to (3.2.1) and get the main result of this section.

Theorem 3.2.6. *Set* $q' := \frac{q}{q-1}$. *If* [L1] – [L4] *are fulfilled, there is a maximal unique solution* $(u,(\tau_n)_n,\tau)$ *of* (3.2.1) *that is weak in the sense of partial differential equations, i.e. the equation*

$$\begin{aligned}
\langle u(t)-u_0,\phi\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))} = &-\int_0^t\langle a(u(s))\nabla u(s),\phi\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{d}s\\
&+\int_0^t\langle F(u)(s),\phi\rangle_{(W^{-1,q}(\mathbb{R}^d),W^{1,q'}(\mathbb{R}^d))}\,\mathrm{d}s\\
&+\int_0^t\langle B(u)(s),\phi\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{dW}_s
\end{aligned}$$

holds almost surely for every $t\in[0,\tau_n]$ *and for every* $\phi\in C_c^\infty(\mathbb{R}^d)$. *If* $u_0\in B_{q,p}^{1-2/p}(\mathbb{R}^d)$, *we have*

$$u\in L^p(0,\tau_n;W^{1,q}(\mathbb{R}^d))\cap C(0,\tau_n;B_{q,p}^{1-2/p}(\mathbb{R}^d))\cap W^{\theta,p}(0,\tau_n;W^{1-2\theta,q}(\mathbb{R}^d))$$

almost surely for every $\theta\in(0,\frac{1}{2})$ *and for every* $n\in\mathbb{N}$. *Moreover,* τ *satisfies*

$$\mathbb{P}\big\{\tau<T,\ \|u\|_{L^p(0,\tau;W^{1,q}(\mathbb{R}^d))}<\infty,\ u:[0,\tau)\to B_{q,p}^{1-2/p}(\mathbb{R}^d)\text{ is uniformly continuous}\big\}=0.$$

If on the other hand $u_0\in F_{q,p}^{1-2/p}(\mathbb{R}^d)$, *we have*

$$u\in W^{1,q}(\mathbb{R}^d;L^p(0,\tau_n))\cap C(0,\tau_n;F_{q,p}^{1-2/p}(\mathbb{R}^d))\cap W^{1-2\theta}(\mathbb{R}^d;W^{\theta,p}(0,\tau_n))$$

for every $\theta \in (0, \frac{1}{2})$ and for every $n \in \mathbb{N}$. Furthermore, τ satisfies

$$\mathbb{P}\big\{\tau < T,\ \|u\|_{W^{1,q}(\mathbb{R}^d;L^p(0,\tau))} < \infty,\ u : [0,\tau) \to F^{1-2/p}_{q,p}(\mathbb{R}^d) \text{ is uniformly continuous}\big\} = 0.$$

Proof. First, we discuss the setting [TT]. We apply Theorem 2.3.12 and Corollary 2.3.13 with $E = W^{-1,q}(\mathbb{R}^d)$, $E^1 = W^{1,q}(\mathbb{R}^d)$, $\mathrm{TR} = B^{1-2/p}_{q,p}(\mathbb{R}^d)$ and $X^{\frac{1}{2}}_H(a,b) = L^p(a,b;L^q(\mathbb{R}^d;l^2(\mathbb{N})))$. The assumption [TTQ1] is then satisfied straight away, whereas [TTQ2*] and [TTQ3*] are checked in Proposition 3.2.4 and [TTQ4*] follows from Lemma 3.2.5. This yields a strong solution u in $W^{-1,q}(\mathbb{R}^d)$ with the claimed regularity properties. The claimed solution formula is immediate by testing the functionals in $W^{-1,q}(\mathbb{R}^d)$ with $\phi \in C_c^\infty(\mathbb{R}^d)$. We just use the identity

$$\big\langle \operatorname{div} a(u(s))\nabla u(s), \phi\big\rangle_{(W^{-1,q}(\mathbb{R}^d),W^{1,q'}(\mathbb{R}^d))} = -\big\langle a(u(s))\nabla u(s), \nabla\phi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}.$$

In the setting [LQ], things are more complicated. We choose $\Lambda = (I-\Delta)$ and $\alpha = \frac{1}{2}$. The assumption [LQQ1] is then satisfied straight away, whereas [LQQ2*] and [LQQ3*] are checked in Proposition 3.2.4 and [LQQ4*] follows from Lemma 3.2.5. However, Theorem 2.3.12 and Corollary 2.3.13 solely give us a solution with the claimed regularity properties that satisfies

$$\begin{aligned}(I-\Delta)^{-\frac{1}{2}}u(t,x) - &(I-\Delta)^{-\frac{1}{2}}u_0(x) = \\ &\int_0^t (I-\Delta)^{-\frac{1}{2}}\operatorname{div}(a(u)\nabla u)(s,x)\,\mathrm{d}s + \int_0^t (I-\Delta)^{-\frac{1}{2}}F(u)(s,x)\,\mathrm{d}s \\ &+ \int_0^t (I-\Delta)^{-\frac{1}{2}}B(u)(s,x)\,\mathrm{d}W(s)\end{aligned}$$

almost surely for almost all $x \in \mathbb{R}^d$ and for almost all $t \in [0,\tau_n]$. In this formula, the regularization with $(I-\Delta)^{-\frac{1}{2}}$ is needed to define the deterministic integrals over time for fixed $x \in \mathbb{R}^d$. To get rid of this regularization, we test this equation with a function $\phi \in L^{q'}(\mathbb{R}^d)$. We get

$$\begin{aligned}&\big\langle (I-\Delta)^{-\frac{1}{2}}u(t) - (I-\Delta)^{-\frac{1}{2}}u_0, \phi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))} \\ &\quad= \int_0^t \big\langle (I-\Delta)^{-\frac{1}{2}}\operatorname{div}(a(u)\nabla u)(s), \phi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{d}s \\ &\qquad+ \int_0^t \big\langle (I-\Delta)^{-\frac{1}{2}}F(u)(s), \phi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{d}s \\ &\qquad+ \int_0^t \big\langle (I-\Delta)^{-\frac{1}{2}}B(u)(s), \phi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{d}W(s)\end{aligned}$$

almost surely for all $t \in [0,\tau_n]$. This holds true for all $\phi \in L^{q'}(\mathbb{R}^d)$ and we can insert $\phi = (I-\Delta)^{\frac{1}{2}}\psi$ for some $\psi \in C_c^\infty(\mathbb{R}^d)$.

Since $(I-\Delta)^{-\frac{1}{2}}$ is self-adjoint and the adjoint of $(I-\Delta)^{-\frac{1}{2}}\operatorname{div}$ is given by $-\nabla(I-\Delta)^{-\frac{1}{2}}$, we finally end up with

$$\begin{aligned}
&\big\langle u(t)-u_0,\psi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\\
&\qquad=\int_0^t -\big\langle a(u(s))\nabla u(s),\nabla\psi\big\rangle_{L^q(\mathbb{R}^d)}\,\mathrm{d}s+\int_0^t \big\langle F(u)(s),\psi\big\rangle_{(W^{-1,q}(\mathbb{R}^d),W^{1,q'}(\mathbb{R}^d))}\,\mathrm{d}s\\
&\qquad\quad+\int_0^t \big\langle B(u)(s),\psi\big\rangle_{(L^q(\mathbb{R}^d),L^{q'}(\mathbb{R}^d))}\,\mathrm{d}W(s)
\end{aligned}$$

almost surely for every $t\in[0,\tau_n]$, which is the claimed result. □

In particular, this theorem can be used to show that the solution $(u,(\tau_n)_n,\tau)$ is Hölder continuous in time. A Sobolev embedding yields $u\in C^\beta(0,\tau_n;W^{1-2\theta}(\mathbb{R}^d))$ if $u_0\in B^{1-2/p}_{q,p}(\mathbb{R}^d)$ and $u\in W^{1-2\theta}(\mathbb{R}^d;C^\beta(0,\tau_n))$ if $u_0\in F^{1-2/p}_{q,p}(\mathbb{R}^d)$ with $\beta=\theta-\frac{1}{p}$ for all $\theta\in(\frac{1}{p},\frac{1}{2})$.

Last but not least, we want to point out that we cannot apply the setting [GM] in the same way. This is due to the fact that in this setting, we have $\mathrm{TR}=L^q(\mathbb{R}^d)$ and in particular a Sobolev embedding of the form $\mathrm{TR}\hookrightarrow L^\infty(\mathbb{R}^d)$ is not available. Thus, we need a modification of the coefficient matrix a similar to the one discussed at the end of the previous section. We could treat a coefficient matrix of the form $a(u)=a(\|u\|_{L^q(\mathbb{R}^d)})$. We get the following result with a slight modification of the proof of Theorem 3.1.6.

Theorem 3.2.7. *Assume the following assumptions.*

a) The coefficient matrix $a=(a_{ij})_{i,j=1,\dots,d}:\mathbb{R}_{\geq 0}\to\mathbb{R}^{d\times d}$ *is uniformly elliptic, i.e.*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}_{\geq 0}}\inf_{|\xi|=1}\operatorname{Re}\xi^T a(x)\overline{\xi}=\delta_0>0$$

and locally Lipschitz continuous, i.e. for every $n\in\mathbb{N}$*, there exists a constant* $L(n)>0$ *such that*

$$|a_{ij}(y)-a_{ij}(z)|\leq L(n)|y-z|$$

for all $0\leq y,z\leq n$ *and all* $i,j=1,\dots,d$*.*

b) We require $u_0:\Omega\to L^q(\mathbb{R}^d)$ *to be strongly* $\mathcal{F}_0$*-measurable.*

c) The nonlinearities F *and* $(B_n)_n$ *satisfy* [Q6*] *and* [Q7*] *together with* [Q9*].

Then, there exists a maximal unique solution $\big(u,(\tau_n)_n,\tau\big)$ *of* (3.2.1) *in* $W^{-1,q}(\mathbb{R}^d)$ *with*

$$u\in W^{1,q}(\mathbb{R}^d;L^2(0,\tau_n))\cap C(0,\tau_n;L^q(\mathbb{R}^d))$$

almost surely for every $n\in\mathbb{N}$*. Moreover,* τ *satisfies*

$$\mathbb{P}\big\{\tau<T,\ \|u\|_{W^{1,q}(\mathbb{R}^d;L^2(0,\tau))}<\infty,\ u:[0,\tau)\to L^q(\mathbb{R}^d)\text{ is uniformly continuous}\big\}=0.$$

3.2.2. Local weak solution on a bounded domain with mixed boundary conditions

In this section, we discuss the convection-diffusion equation

$$(\mathrm{DIV})\begin{cases} du(t) &= \big[\operatorname{div}(a(u(t))\nabla u(t)) + F(u)(t)\big]dt + B(u)(t)\,\mathrm{d}W(t)), \quad t \in [0,T], \\ u(0) &= u_0 \end{cases}$$

on a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, with Dirichlet, Neumann or mixed boundary conditions. In this example, we focus on the the setting [TT], since important tools needed for the setting [LQ] like the $\mathcal{R}_p$-boundedness of $\nabla(I - \operatorname{div}(a(u)\nabla))^{-\frac{1}{2}}$ are not deep enough investigated in the literature in context of bounded domains.

In this section, we work in the spaces $W^{1,q}_\Gamma(D)$ and $W^{-1,q}_\Gamma(D)$ from Definition 1.1.3 for some $\Gamma \subset \partial D$ that is open in the topology of ∂D. Then, TR will be a subspace of $B^{1-2/p}_{q,p}(D)$ that respects the boundary condition on Γ and $\partial D \setminus \Gamma$. Since we always work with $1 - 2/p > d/q$ every $u \in B^{1-2/p}_{q,p}(D)$ is continuous on $\overline{D}$. Hence, we are able to define

$$B^{1-2/p}_{q,p,\Gamma}(D) := \big\{ u \in B^{1-2/p}_{q,p}(D) : \ u|_{\partial D\setminus\Gamma} = 0 \big\}.$$

We will consider the quasilinear equation (DIV) in the space $W^{-1,q}_\Gamma(D)$ for $q \in [2,\infty)$, which means, we try to find a weak solution in the sense of partial differential equations. Remember, $\big(u,(\tau_n)_n,\tau\big)$ is a local solution of (DIV) in the setting [TT] in the sense of Definition 2.3.2 with the choice $E = W^{-1,q}_\Gamma(D)$ and $E^1 = W^{1,q}_\Gamma(D)$ if and only if the identity

$$\begin{aligned} \int_D (u(t,x) - u_0(x))\phi(x)\,\mathrm{d}x = &- \int_0^t \int_D a(u(s,x))\nabla u(s,x)\nabla\phi(x)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \langle F(u)(s), \phi\rangle_{(W^{-1,q}_\Gamma(D), W^{1,q'}_\Gamma(D))}\,\mathrm{d}s \\ &+ \int_0^t \int_D B(u)(s,x)\phi(x)\,\mathrm{d}x\,\mathrm{d}W(s) \end{aligned}$$

holds almost surely for all $t \in [0,\tau_n]$ and for all $\phi \in C^\infty_\Gamma(D)$.

At first, we look at (DIV) with a locally Lipschitz continuous diffusion matrix a. However, we have to guarantee that the operators $\operatorname{div}(a(u(\omega,t))\nabla)$ on $W^{-1,q}_\Gamma(D)$ have for almost every ω and for every t the same domain $W^{1,q}_\Gamma(D)$. In the last decades, it turned out that this property highly depends on D, its dimension and the regularity of the coefficient function. Therefore, we introduce the following notation.

Definition 3.2.8. *Let $\mu : D \to \mathbb{R}^{d\times d}$ be uniformly elliptic and uniformly continuous. Then, we define $\mathcal{T}_\mu$ as the set of all $r \in [1,\infty]$ such that the operator*

$$z \mapsto L_\mu z = -\operatorname{div}(\mu\nabla z) + z : W^{1,r}_\Gamma(D) \to W^{-1,r}_\Gamma(D)$$

is a topological isomorphism and such that the norms of L_μ and L_μ^{-1} only depend on r, the ellipticity of μ, its modulus of continuity and of $\|\mu\|_{L^\infty(D)}$.

Now, we can specify our assumptions.

[LD1] For every point $x \in \partial D$, there exists two open sets $U, V \subset \mathbb{R}^d$ and a bi-Lipschitz transformation Φ from U to V such that $x \in U$ and $\Phi(U \cap (D \cup \Gamma))$ coincides with one of the sets $\{y \in \mathbb{R}^d : |y| < 1, y_1 < 0\} \cup \{y \in \mathbb{R}^d : |y| < 1, y_1 = 0, y_2 > 0\}$ and $\{y \in \mathbb{R}^d : |y| < 1\}$.

[LD2] $a : \mathbb{C} \to \mathbb{R}^{d\times d}$ is uniformly positive definite, i.e.

$$\operatorname*{ess\,inf}_{y\in\mathbb{C}} \inf_{|\xi|=1} \xi^T a(y)\xi = \delta_0 > 0,$$

and locally Lipschitz continuous, i.e. for every $n \in \mathbb{N}$, there exists a constant $L(n) > 0$ such that

$$|a(y) - a(z)| \leq L(n)|y - z|$$

for all $|y|, |z| < n$.

[LD3] We choose $p, q \in (2, \infty)$ such that $1 - 2/p > d/q$ and $q \in \mathcal{T}_{a(z)}$ for all $z \in B^{1-\frac{2}{p}}_{q,p,\Gamma}(D)$.

[LD4] The initial value $u_0 : \Omega \to B^{1-2/p}_{q,p,\Gamma}(D)$ is a strongly $\mathcal{F}_0$- measurable random variable.

[LD5] The nonlinearities F and B satisfy [Q6*] and [Q7*] together with [Q9*] in the setting [TT] with the spaces $E = W^{-1,q}_\Gamma(D)$, $E^1 = W^{1,q}_\Gamma(D)$ and $\mathrm{TR} = B^{1-\frac{2}{p}}_{q,p,\Gamma}(D)$.

Before we proceed, we comment on our assumptions. We chose the requirement on the domain [LD1] in order to guarantee the important interpolation results

$$(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))_{1-1/p.p} = B^{1-2/p}_{q,p,\Gamma}(D), \quad [W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D)]_{1/2} = L^q(D)$$

from [42]. In particular, this representation of the real interpolation space makes sure that u_0 is in the usual space for initial values. Moreover, [LD3] implicitly contains assumptions on the boundary of D and on the coefficient function a as well, since it is impossible to ensure that

$$y \mapsto -\operatorname{div}(a(u(t))\nabla y) + y : W^{1,q}_\Gamma(D) \to W^{-1,q}_\Gamma(D)$$

is an isomorphism for all $q \in (1, \infty)$ if one just assumes [LD1] and [LD2]. Even in case of the Dirichlet Laplacian, there are counterexamples (see [52], Theorem A). In general, one only knows that a small interval $(2 - \varepsilon, 2 + \varepsilon)$ with $\varepsilon > 0$ depending on the geometry of D and Γ and on the coefficient function μ is contained in $\mathcal{T}_\mu$ (see [45], Theorem 5.6 and Remark 5.7). Nevertheless, there are several situations, in which one can fulfil [LD3]. In the following, we mention some of them.

If one assumes D to be a C^1-domain that has either pure Dirichlet ($\Lambda = \emptyset$) or pure Neumann boundary ($\Lambda = \partial D$) and one assumes μ to be a uniformly continuous coefficient function, one has $q \in \mathcal{T}_\mu$ for all $q \in (1, \infty)$. This is a classical result, which can be found in [3], section 15 or [79], page 156-157. Consequently, since we require $1 - 2/p > d/q$ and hence every $z \in B^{1-2/p}_{q,p,\Gamma}(D)$ is even Hölder continuous, we automatically have $q \in \mathcal{T}_{a(z)}$.

If D is just a Lipschitz domain with Dirichlet boundary ($\Lambda = \emptyset$) and the coefficient function μ is a symmetric, uniformly continuous matrix, then there is a $q > 3$ with $q \in \mathcal{T}_\mu$. This only helps us if $d = 2, 3$ since then it is possible to choose p large enough to ensure $1 - 2/p > d/q$. This is shown in [37], Theorem 1.1.

So far, we only gave examples for Dirichlet or Neumann boundary conditions. In case of mixed boundary conditions, we exploit the very detailed work [30]. In the case $d = 3$, the authors provide a wide range of geometries of D and Γ that permit the existence of a $q > 3$ such that $q \in \mathcal{T}_\mu$, where μ is a real scalar valued function that is uniformly continuous. Moreover, in Section 3, they provide many descriptive examples for the geometries, they allow. The following Lemma adjusts these results to our situation.

Lemma 3.2.9. *Let $D \subset \mathbb{R}^3$ and $\Gamma \subset \partial D$ satisfy Assumption 4.2 in [30] and let $a(u)$ be real and scalar valued. Then, there exists $q > 3$ such that $q \in \mathcal{T}_{a(z)}$ for every $z \in B^{1-2/p}_{q,p,\Gamma}(D)$. In particular, the norms of both $I - \operatorname{div}(a(z)\nabla)$ and $(I - \operatorname{div}(a(z)\nabla))^{-1}$ depend on the constants in Assumption 4.2, on the constants in [LD2] and on $\sup_{x \in D} |z(x)|$.*

Proof. By Theorem 4.8 in [30], there exists $q > 3$, such that $I - \operatorname{div}(a(0)\nabla)$ is a topological isomorphism from $W^{1,q}_\Gamma(D)$ to $W^{-1,q}_\Gamma(D)$. Since, we assumed a to be Lipschitz continuous and p to be large enough such that $1-2/p > 3/q$, the map $x \mapsto a(z(x))$ is Hölder continuous. In particular, the set $\{a(z) : z \in B^{1-2/p}_{q,p,\Gamma}(D) : \|z\|_{B^{1-2/p}_{q,p,\Gamma}} \leq n\}$ is compact in $C(\overline{D})$. Hence by Corollary 6.4 in [30], the map

$$\{a(z) : z \in B^{1-2/p}_{q,p,\Gamma}(D)\} \ni \mu \mapsto (I - \operatorname{div}(\mu\nabla))^{-1} \in \mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))$$

is bounded and Lipschitz continuous. In particular, this means

$$\begin{aligned}
&\|(I - \operatorname{div}(a(z)\nabla))^{-1}\|_{\mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))} \\
&\quad\leq \|(I - \operatorname{div}(a(z)\nabla))^{-1} - (I - \operatorname{div}(a(0)\nabla))^{-1}\|_{\mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))} \\
&\qquad + \|(I - \operatorname{div}(a(0)\nabla))^{-1}\|_{\mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))} \\
&\quad\leq C \sup_{x\in D} |a(z(x)) - a(0)| + \|(I - \operatorname{div}(a(0)\nabla))^{-1}\|_{\mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))} \\
&\quad\leq CL(\sup_{x\in D} |z(x)|) \sup_{x\in D} |a(z(x))| + \|(I - \operatorname{div}(a(0)\nabla))^{-1}\|_{\mathcal{B}(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|I - \operatorname{div}(a(z)\nabla)\|_{\mathcal{B}(W^{1,q}_\Gamma(D), W^{-1,q}_\Gamma(D))} &\leq 1 + \|a(z)\|_{L^\infty(D)} \\
&\leq 1 + L(\sup_{x\in D} |z(x)|) \sup_{x\in D} |z(x)| + \sup_{x\in D} |a(0)|.
\end{aligned}$$

This proves that $I - \operatorname{div}(a(z)\nabla)$ is a topological isomorphism from $W^{1,q}_\Gamma(D)$ to $W^{-1,q}_\Gamma(D)$ for every z and that the norms of both $I - \operatorname{div}(a(z)\nabla)$ and $(I - \operatorname{div}(a(z)\nabla)^{-1}$ have the required dependency on the coefficient function. $\square$

Our goal is to apply Theorem 2.3.12 to the operators

$$A(u(t))u(t) = -\operatorname{div}(a(u(t))\nabla u(t)) + u(t)$$

in the setting [TT]. In the following Lemma, we prove that $A(u(t))$ has the needed mapping properties like a timely constant domain and a bounded H^∞-calculus.

Lemma 3.2.10. *Under the assumptions* [LD1]-[LD3]*, the operators*

$$A(z)u := -\operatorname{div}(a(z)\nabla u) + u : W_\Gamma^{1,q}(D) \to W_\Gamma^{-1,q}(D)$$

are for all $z \in B_{q,p,\Gamma}^{1-2/p}(D)$ *densely defined, closed with* $0 \in \rho(A(z))$ *and have a bounded* H^∞*-calculus with bound and angle only depending on the constants* L*,* δ_0 *and on* $\|z\|_{B_{q,p,\Gamma}^{1-2/p}(D)}$*. We also have for every* $n \in \mathbb{N}$ *a constant* $C(n) > 0$ *such that the local Lipschitz estimate*

$$\|A(z) - A(y)\|_{\mathcal{B}(W_\Gamma^{1,p}(D), W_\Gamma^{-1,p}(D))} \leq C(n)\|z - y\|_{B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)}$$

holds for all $\|z\|_{B_{q,p,\Gamma}^{1-2/p}(D)}, \|y\|_{B_{q,p,\Gamma}^{1-2/p}(D)} \leq n$*. Last but not least, we have*

$$(W_\Gamma^{-1,q}(D), W_\Gamma^{1,q}(D))_{1-1/p,p} = B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)$$

and as a consequence, A *satisfies* [TTQ2*] – [TTQ4*] *from the previous chapter.*

Proof. By choice of p and q, the Sobolev embedding $B_{q,p,\Gamma}^{1-\frac{2}{p}}(D) \hookrightarrow C^l(\bar{D})$ holds true for some $l > 0$. In the sequel, we write C_J for the constant of this embedding. Given $z \in B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)$, we obtain

$$\begin{aligned}\|a(z)\|_{L^\infty(D)} &\leq \operatorname*{ess\,sup}_{x\in D} |a(z(x)) - a(0)| + |a(0)| \\ &\leq L\Big(C_J\|z\|_{B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)}\Big)C_J\|z\|_{B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)} + |a(0)|.\end{aligned}$$

In particular, the operator $A(z) : W_\Gamma^{1,q}(D) \to W_\Gamma^{-1,q}(D)$ is well-defined and bounded. Moreover, since we assumed $q \in \mathcal{T}(a(z))$, Theorem 6.5 in [31] implies that $A(z)$ with $D(A(z)) = W_\Gamma^{1,q}(D)$ is a closed operator.

By Theorem 11.5 in [9], $A(z)$ has a bounded H^∞-calculus of angle $\arctan\big(\frac{\|a(z)\|_{L^\infty(D)}}{\delta_0}\big)$ and the bound only depends on $\|a(z)\|_{L^\infty(D)}$ and δ_0 (see also [34]). Note that the critical assumption for this theorem is that $A(z)$ possesses the square root property in $L^2(D)$, i.e. the operator

$$(I - \operatorname{div}(a(z)\nabla))^{1/2} : W_\Gamma^{1,2}(D) \to L^2(D)$$

is a topolical isomorphism. This result can be found in [35], Theorem 4.1.

The claimed Lipschitz estimate for A is an immediate consequence of the Lipschitz continuity of a and a Sobolev embedding. Indeed, we have

$$\begin{aligned}\|A(z) - A(y)\|_{\mathcal{B}(W_\Gamma^{1,q}(D), W_\Gamma^{-1,q}(D))} &\lesssim \|(a(z) - a(y))\nabla\|_{\mathcal{B}(W_\Gamma^{1,q}(D), L^q(D))} \\ &\lesssim \|a(z) - a(y)\|_{L^\infty(D)} \\ &\leq C_J L(C_J n)\|z - y\|_{B_{q,p,\Gamma}^{1-\frac{2}{p}}(D)}\end{aligned}$$

for all $\|z\|_{B^{1-2/p}_{q,p,\Gamma}}(D), \|y\|_{B^{1-2/p}_{q,p,\Gamma}}(D) \leq n$.

It remains to check $(W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D))_{1-1/p,p} = B^{1-\frac{2}{p}}_{q,p,\Gamma}(D)$. By [42], Lemma 3.4, we have the identity $[W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D)]_{1/2} = L^q(D)$. Using the reiteration formula between real and complex interpolation (see e.g. [93], Theorem 1.10.3.2), it is sufficient to show

$$(L^q(D), W^{1,p}_\Gamma(D))_{1-2/p,p} = B^{1-2/p}_{q,p,\Gamma}(D).$$

This is done in [42], Remark 3.6. □

Next, we check that the spaces $W^{-1,q}_\Gamma(D)$ and $W^{1,q}_\Gamma(D)$ fit in the setting of stochastic maximal L^p-regularity.

Lemma 3.2.11. *The spaces $W^{1,q}_\Gamma(D)$ and $W^{-1,q}_\Gamma(D)$ are UMD Banach spaces with type 2. Moreover, the family of operators*

$$\{J_\delta : \delta > 0\} \subset \mathcal{B}\big(L^p(\Omega \times (0,\infty); \gamma(H; W^{-1,q}_\Gamma(D))), L^p(\Omega \times (0,\infty); W^{-1,q}_\Gamma(D))\big)$$

defined by

$$J_\delta b(t) := \delta^{-1/2} \int_{(t-\delta)\vee 0}^t b(s) \, \mathrm{d}W(s)$$

is $\mathcal{R}$-bounded. In conclusion, these spaces satisfy assumption [TTQ1] *of the previous section.*

Proof. By Lemma 3.2.10 the spaces $W^{-1,q}_\Gamma(D)$ and $W^{1,q}_\Gamma(D)$ are isomorph. In the proof of the same Lemma, we checked $[W^{-1,q}_\Gamma(D), W^{1,q}_\Gamma(D)]_{1/2} = L^q(D)$ and hence, amongst others $A(0)^{1/2}$, provides an isomorphism between $L^q(D)$ and $W^{-1,q}_\Gamma(D)$. Moreover, the type of Banach space, the UMD property and the $\mathcal{R}$-boundedness of $(J_\delta)_{\delta>0}$ are stable under isomorphisms and the UMD space $L^q(D)$ is of type 2. Noting that by [97], Theorem 3.1, the family is $\mathcal{R}$-bounded on $L^p(\Omega \times (0,\infty); \gamma(H; L^q(D)))$ completes the proof. □

Now, we are in the position to proof existence and uniqueness of a solution of (DIV) by applying Theorem 2.3.12 to the operator $A(z)y = -\operatorname{div}(a(z)\nabla y) + y$.

Theorem 3.2.12. *Let the assumptions* [LD1] – [LD5] *be satisfied. Then, there exists a maximal unique local solution $\big(u, (\tau_n)_n, \tau\big)$ of* (DIV) *in $W^{-1,q}_\Gamma(D)$ such that we have*

$$u \in L^p(0, \tau_n; W^{1,q}_\Gamma(D)) \cap C(0, \tau_n; B^{1-2/p}_{q,p,\Gamma}(D))$$

pathwise almost surely for every $n \in \mathbb{N}$. Moreover, τ satisfies

$$\mathbb{P}\big\{\tau < T,\ \|u\|_{L^p(0,\tau;W^{1,q}_\Gamma(D))} < \infty,\ u : [0,\tau) \to B^{1-2/p}_{q,p,\Gamma}(D) \text{ is uniformly continuous}\big\} = 0.$$

Proof. Writing

$$\operatorname{div}(a(z)\nabla z) + F(z) = \big(\operatorname{div}(a(z)\nabla z) - z\big) + \big(F(z) + z\big),$$

we see that we can solve the equation

$$\begin{cases} du(t) &= \Big[-A(u(t))u(t) + \widetilde{F}(u)(t)\Big]\,\mathrm{d}t + B(u)(t)\,\mathrm{d}W(t), \quad t \in [0,T] \\ u(0) &= u_0 \end{cases}$$

with $\widetilde{F}(z) := F(z) + z$ for $z \in W^{1,q}_{\Gamma}(D)$. By Lemma 3.2.10, the assumptions [TTQ2*]-[TTQ4*] are fulfilled, whereas Lemma 3.2.11 guaranties [TTQ1]. All in all, Theorem 2.3.12 yields the desired result. □

3.2.3. Global weak solution with Dirichlet boundary condition

In this section, we investigate the convection diffusion equation with Dirichlet boundary conditions ($\Gamma = \emptyset$) and we therefore restrict us to the space $W^{1,q}_{\emptyset}(D)$ that will be denoted with $W^{1,q}_0(D)$ in what follows. As usual in the literature, we write $W^{-1,q}(D)$ for $W^{-1,q}_{\emptyset}(D)$. We consider the equation

$$\text{(GDIV)} \begin{cases} du(t) = \big[\operatorname{div}(a(u(t))\nabla u(t)) + \operatorname{div}(G(u(t)))\big]dt + B(u)(t)\,\mathrm{d}W(t)), \quad t \in [0,T], \\ u(0) = u_0 \end{cases}$$

and we strengthen the assumptions in order to prove that the local solution from Theorem 3.2.12 exists on the whole interval $[0,T]$. We require:

[GD1] $D \subset \mathbb{R}^d$ is a bounded C^1-domain.

[GD2] $a : \mathbb{R} \to \mathbb{R}^{d\times d}$ is bounded and uniformly positive definite, i.e.

$$\inf_{y\in\mathbb{R}} \inf_{|\xi|=1} \xi^T a(y)\xi = \delta_0 > 0$$

and a is globally Lipschitz continuous, i.e. there exists $L > 0$ such that

$$|a(y) - a(z)| \le L|y - z|$$

for all $y, z \in \mathbb{R}$.

[GD3] We choose $p, q \in (2,\infty)$ such that $1 - 2/p > d/q$.

[GD4] The initial value $u_0 : \Omega \to B^{1-2/p}_{q,p,0}(D)$ is a strongly $\mathcal{F}_0$-measurable random variable.

[GD5] $G : \mathbb{R} \to \mathbb{R}^d$ is Lipschitz continuous, i.e. there is a constant $L_G > 0$ such that

$$|G(y) - G(z)| \le L_G|y - z|$$

for all $y, z \in \mathbb{R}$.

[GD6] The driving noise W is an l^2- cylindrical Brownian motion with the decomposition

$$W(t) = \sum_{k=1}^{\infty} e_k\beta_k(t),$$

where $(e_k)_k$ is the standard orthonormal basis of $l^2(\mathbb{N})$ and $(\beta_k)_k$ is a sequence of independent real-valued Brownian motions relative to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

[GD7] $B = (B_n)_n : \Omega \times [0,T] \times D \times \mathbb{R} \to l^2(\mathbb{N})$ is strongly measurable and $\omega \mapsto B(\omega,t,x,y)$ is for all $t \in [0,T]$, $x \in D$ and $y \in \mathbb{R}$ strongly $\mathcal{F}_t$-measurable. Furthermore, B is of linear growth, i.e.

$$\Big(\sum_{n=1}^{\infty} |B_n(\omega,t,x,y)|^2\Big)^{1/2} \le C(1+|y|)$$

and Lipschitz continuous in the last component, i.e. there is $L_B > 0$ such that

$$\Big(\sum_{n=1}^{\infty} |B_n(\omega,t,x,y) - B_n(\omega,t,x,z)|^2\Big)^{1/2} \le L_B|y-z|$$

for all $y,z \in \mathbb{R}$, $t \in [0,T]$, $x \in D$ and almost all $\omega \in \Omega$. Moreover, we assume

$$\|B_n(\omega,t,\cdot,f)\|_{\gamma(l^2(\mathbb{N});W_0^{1,2}(D))} \le C(1+\|f\|_{W_0^{1,2}(D)})$$

for all $f \in W_0^{1,2}(D)$, $t \in [0,T]$, $x \in D$ and almost all $\omega \in \Omega$.

These assumptions are strictly stronger than [LD1]-[LD5]. a is not locally, but globally Lipschitz and the nonlinearities $\operatorname{div}(G)$ and B are only of lower order. As we have already mentioned in our remarks below the assumptions of the previous section, [GD1] and [GD2] also imply $q \in \mathcal{T}_{a(z)}$ for every $z \in B^{1-2/p}_{q,p,0}(D)$ and $q \in (1,\infty)$.

All in all, Theorem 3.2.12 yields a local solution $(u,(\tau_n)_n,\tau)$ of (GDIV), i.e. an increasing sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ with $0 \le \tau_n \le T$ and $\lim_{n\to\infty}\tau_n = \tau$ almost surely and a process $u : \Omega \times [0,\tau) \to W_0^{1,q}(D)$ such that u solves (GDIV) on $[0,\tau_n]$ and

$$u \in L^p(0,\tau_n;W^{1,q}(D)) \cap C(0,\tau_n;B^{1-2/p}_{q,p,0}(D)) \tag{3.2.2}$$

almost surely for every $n \in \mathbb{N}$.

In this section, we aim to prove that we actually have $\tau = T$ almost surely. By the blow-up alternative from Theorem 3.2.12, it is sufficient to show that $u : [0,\tau) \to B^{1-2/p}_{q,p,0}(D)$ is almost surely uniformly continuous and satisfies $\|u\|_{L^p(0,\tau;W_0^{1,q}(D))} < \infty$ almost surely. However, this is not too easy, since (3.2.2) that originally comes from the abstract construction of a solution of a quasilinear equation highly depends on n and to find uniform estimates for u, we have to use the special structure of the equation.

Our first goal is to derive a uniform estimate in $L^\alpha(\Omega;L^\infty(0,\tau_n;L^\alpha(D)))$ for u for all $\alpha \in [2,\infty)$ and to do this, we need a suitable version of the Itô formula that is useful to deal with weak solutions of stochastic partial differential equations. There are several versions of the following Lemma in the literature, see e.g. [63], Theorem 2.1. or [27], Proposition A.1. The strategy is always the same. One approximates the Itô process in order to apply the classical Itô formula, rearranges the equation into the form one wants to achieve and at the end, one passes to the limit. However, as far as we know, there is no result that covers our situation. Therefore, we sketch the proof for convenience of the reader.

Lemma 3.2.13. *Let $(u, (\tau_n)_n, \tau)$ be the maximal unique local solution of* [GDIV]*. Then, for $\alpha \geq 2$, the generalized Itô formula*

$$\begin{aligned}
&\int_D |u(t,x)|^\alpha - |u_0(x)|^\alpha \,\mathrm{d}x \\
&\quad = -\int_0^t \int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}\big(a(u(s,x))\nabla u(s,x) + G(u(s,x))\big)\nabla u(s,x)\,\mathrm{d}s \\
&\quad + \sum_{k=1}^\infty \int_0^t \int_D \alpha|u(s,x)|^{\alpha-2}u(s,x)B_k(s,x,u(s,x))\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\
&\quad + \frac{1}{2}\sum_{k=1}^\infty \int_0^t \int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}B_k(s,x,u(s,x))^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

holds almost surely for all $t \in [0,\tau_n]$ and all $n \in \mathbb{N}$.

Proof. In what follows, we shortly write $F(s,x) := a(u(s,x))\nabla u(s,x) + G(u(s,x))$ and $H_k(s,x) := B_k(s,x,u(s,x))$. With this notation, we get

$$u(t) - u_0 = \int_0^t \operatorname{div} F(s)\,\mathrm{d}s + \sum_{k=1}^\infty \int_0^t H_k(s)\,\mathrm{d}\beta_k(s) \tag{3.2.3}$$

almost surely for all $t \in [0,\tau_n]$ as an equation in $W^{-1,q}(D)$. Next, we extend the functions u, u_0, F and H_k by zero to the whole space $\mathbb{R}^d$. Since we assumed Dirichlet boundary conditions, we have $u \in L^p(0,\tau_n;W^{1,q}(\mathbb{R}^d)) \cap C(0,\tau_n;B^{1-2/p}_{q,p}(\mathbb{R}^d))$ almost surely. Moreover, we define

$$\eta_l = \tau_n \wedge \inf\Big\{t \in [0,\tau_n] :\ \|u\|_{L^p(0,t;W^{1,q}(\mathbb{R}^d))} + \|u\|_{C(0,t;B^{1-2/p}_{q,p}(\mathbb{R}^d))} \geq l\Big\}.$$

Let $(\delta_m)_m \subset C_c^\infty(\mathbb{R}^d)$ be a Dirac sequence, i.e. $\delta_m \geq 0$, $\operatorname{supp}(\delta_m) \subset B(0,\frac{1}{m})$ and we have $\int_{\mathbb{R}^d} \delta_m(x)\,\mathrm{d}x = 1$ for every $m \in \mathbb{N}$. We convolute the equation (3.2.3) with δ_m and obtain

$$u_m(t,x) - u_0^{(m)}(x) = \int_0^t \operatorname{div} F_m(s,x)\,\mathrm{d}s + \sum_{k=1}^\infty \int_0^t H_k^{(m)}(s,x)\,\mathrm{d}\beta_k(s) \tag{3.2.4}$$

almost surely for every $t \in [0,\eta_l]$ and every $x \in \mathbb{R}^d$. Here $u_n, u_0^{(m)}, F_m$ and $H_k^{(m)}$ denote the convolution of the respective function u, u_0, F and H_k with δ_m. In this step, we used the well-known identity $\operatorname{div}(F) * \delta_m = \operatorname{div}(F * \delta_m)$ and the fact that by stochastic Fubini, one can interchange convolution and the stochastic integral. However, $t \mapsto u_m(t,x)$ is for every $x \in \mathbb{R}^d$ an Itô process in $\mathbb{R}$. So, we can apply Itô's formula to $|\cdot|^\alpha$ and integrate afterwards over $\mathbb{R}^d$. We obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |u_m(t,x)|^\alpha\,\mathrm{d}x &= \int_{\mathbb{R}^d} |u_0^{(m)}(x)|^\alpha\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \alpha|u_m(s,x)|^{\alpha-2}u_m(s,x)\operatorname{div} F_m(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} \alpha|u_m(s,x)|^{\alpha-2}u_m(s,x)H_k^{(m)}(s,x)\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\
&+ \frac{1}{2}\sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} \alpha(\alpha-1)|u_m(s,x)|^{\alpha-2}H_k^{(m)}(s,x)^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Further, integration by parts and $\nabla\big(|u_m(s,x)|^{\alpha-2}u_m(s,x)\big) = (\alpha-1)|u_m(s,x)|^{\alpha-2}\nabla u_m(s,x)$ yield

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u_m(t,x)|^\alpha \,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^d} |u_0^{(m)}(x)|^\alpha \,\mathrm{d}x - \int_0^t \int_{\mathbb{R}^d} \alpha(\alpha-1)|u_m(s,x)|^{\alpha-2}\nabla u_m(s,x) F_m(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\quad + \sum_{k=1}^\infty \int_{\mathbb{R}^d}\int_0^t \alpha|u_m(s,x)|^{\alpha-2}u_m(s,x)H_k^{(m)}(s,x)\,\mathrm{d}\beta_k(s)\,\mathrm{d}x \\
&\qquad\quad + \frac{1}{2}\sum_{k=1}^\infty \int_0^t\int_{\mathbb{R}^d} \alpha(\alpha-1)|u_m(s,x)|^{\alpha-2}H_k^{(m)}(s,x)^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

almost surely for all $t\in[0,\eta_l]$. Now, we can take the limit $m\to\infty$. Here, we make use of the fact that $\delta_m * f$ converges to f in $W^{1,q}(\mathbb{R}^d)$ if $f\in W^{1,q}(\mathbb{R}^d)$, in $L^\gamma(\mathbb{R}^d)$ with $1\le\gamma<\infty$ if $f\in L^\gamma(\mathbb{R}^d)$ and uniformly if $f\in C_c(\mathbb{R}^d)$. Note that as a consequence of [GD5] and [GD7], we have $F\in L^\infty(\Omega;L^p(0,\eta_l;L^q(\mathbb{R}^d)))$ and $(H_k)_k\in L^\infty(\Omega;L^p(0,\eta_l;L^q(\mathbb{R}^d;l^2(\mathbb{N}))))$. Moreover, we have $u\in L^\infty(\Omega;C([0,\eta_l]\times\mathbb{R}^d))$, which follows from $1-2/p-d/q>0$ and a Sobolev embedding. This yields

$$\begin{aligned}
|u_m|^{\alpha-2}\nabla u_m\cdot F_m &\xrightarrow{m\to\infty} |u|^{\alpha-2}\nabla u\cdot F \\
|u_m|^{\alpha-2}\sum_{k=1}^\infty (H_k^{(m)})^2 &\xrightarrow{m\to\infty} |u|^{\alpha-2}\sum_{k=1}^\infty H_k^2
\end{aligned}$$

in $L^1(\Omega\times[0,\eta_l]\times\mathbb{R}^d)$. In the same way, one shows

$$|u_m|^{\alpha-2}u_m H_k^{(m)} \xrightarrow{m\to\infty} |u|^{\alpha-2}uH_k$$

in $L^2(\Omega\times[0,\eta_l]\times\mathbb{R}^d\times\mathbb{N})$, which implies the convergence of the stochastic integral. All in all, we get

$$\begin{aligned}
&\int_D |u(t,x)|^\alpha - |u_0(x)|^\alpha\,\mathrm{d}x \\
&\quad = -\int_0^t\int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}\big(a(u(s,x))\nabla u(s,x) + G(u(s,x))\big)\nabla u(s,x)\,\mathrm{d}s \\
&\qquad + \sum_{k=1}^\infty \int_0^t\int_D \alpha|u(s,x)|^{\alpha-2}u(s,x)H_k(s,x)\,\mathrm{d}x\,\mathrm{d}\beta(s) \\
&\qquad + \frac{1}{2}\sum_{k=1}^\infty\int_0^t\int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}|H_k(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

almost surely for all $t\in[0,\eta_l]$. Since we defined $u(s,\cdot)=0, u_0=0, H(s,\cdot)=0$ and $F(s,\cdot)=0$ on $\mathbb{R}^d\setminus D$, we get the claimed identity on $[0,\eta_l]$. It remains to take $l\to\infty$. Due to $u\in L^p(0,\tau_n;W^{1,q}(D))\cap C(0,\tau_n;B^{1-2/p}_{q,p,0}(D))$ almost surely, there exists $l=l(\omega)$ such that $\eta_l(\omega)=\tau_n(\omega)$. Hence, this identity also holds true on the time interval $[0,\tau_n]$. This finishes the proof. □

The following Lemma was used several times in the literature in comparable situations (see e.g. [46], Theorem 3.1 or [27], Proposition 5.1). The difference is that we deal with Dirichlet boundary conditions, whereas the references consider periodic boundary conditions on the torus. Furthermore, we work on a random interval up to a stopping time.

Lemma 3.2.14. *If we assume* [GD1]-[GD7] *and additionally* $u_0 \in L^\alpha(\Omega \times D)$ *for some* $\alpha \in [2,\infty)$*, we have*

$$\big(\mathbb{E} \sup_{0\le t<\tau} \|u(t)\|_{L^\alpha(D)}^\alpha\big)^{1/\alpha} \le C_\alpha(1+\|u_0\|_{L^\alpha(\Omega\times D)})$$

with a constant $C_\alpha > 0$ *independent of* u_0*. Moreover, we have*

$$\|u\|_{L^2(\Omega\times[0,\tau);W_0^{1,2}(D))} < \infty.$$

Proof. Let

$$\eta_m = \tau \wedge \inf\Big\{t \in [0,\tau):\ \|u\|_{L^p(0,t;W^{1,q}(\mathbb{R}^d))} + \|u\|_{C(0,t;B^{1-2/p}_{q,p}(\mathbb{R}^d))} \ge m\Big\}.$$

We work on the interval $[0,\eta_m]$ and apply the Itô formula from Lemma 3.2.13. This yields

$$\begin{aligned}
&\int_D |u(x,t)|^\alpha \,\mathrm{d}x - \int_D |u_0(x)|^\alpha \,\mathrm{d}x \\
&\quad - \int_0^t \int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}\nabla u(s,x)\big(G(u(s,x)) + a(u(s,x))\nabla u(s,x)\big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \sum_{k=1}^\infty \int_0^t \int_D \alpha|u(s,x)|^{\alpha-2}u(s,x)B_k(s,x,u(s,x))\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\
&\quad + \frac{1}{2}\sum_{k=1}^\infty \int_0^t \int_D \alpha(\alpha-1)|u(s,x)|^{\alpha-2}B_k(s,x,u(s,x))^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{3.2.5}$$

almost surely for all $t \in [0,\eta_m]$. Next, we estimate $\mathbb{E}\sup_{0\le s\le t\wedge\eta_m}\int_D |u(x,s)|^\alpha \,\mathrm{d}x$ term by term beginning with the stochastic integral. Applying the scalar valued Burkholder-Davis-Gundy inequality, the assumptions on B and Hölder's inequality, we get

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t\wedge\eta_m}\Big|\sum_{k=1}^\infty \int_0^s \int_D \alpha|u(r,x)|^{\alpha-2}u(r,x)B_k(r,x,u(r,x))\,\mathrm{d}x\,\mathrm{d}\beta_k(r)\Big| \\
&\qquad \lesssim \mathbb{E}\Big(\int_0^{t\wedge\eta_m}\sum_{k=1}^\infty\Big(\int_D |u(r,x)|^{\alpha-2}u(r,x)B_k(r,x,u(r,x))\,\mathrm{d}x\Big)^2\,\mathrm{d}r\Big)^{1/2} \\
&\qquad \lesssim \mathbb{E}\Big(\int_0^{t\wedge\eta_m} \big\||u(r)|^{\frac{\alpha}{2}}\big\|_{L^2(D)}^2 \big\||u(r)|^{\frac{\alpha-2}{2}}B(r,u(r))\big\|_{L^2(D;l^2(\mathbb{N}))}^2\,\mathrm{d}r\Big)^{1/2} \\
&\qquad \lesssim \mathbb{E}\Big(\int_0^{t\wedge\eta_m} \|u(r)\|_{L^\alpha(D)}^\alpha \big\||u(r)|^{\frac{\alpha-2}{2}}(1+|u(r)|)\big\|_{L^2(D)}^2\,\mathrm{d}r\Big)^{1/2} \\
&\qquad \le \mathbb{E}\Big(\int_0^{t\wedge\eta_m} \|u(r)\|_{L^\alpha(D)}^\alpha\big(\|u(r)\|_{L^\alpha(D)}^{\alpha-2} + \|u(r)\|_{L^\alpha(D)}^\alpha\big)\,\mathrm{d}r\Big)^{1/2} \\
&\qquad \lesssim \mathbb{E}\Big(\int_0^{t\wedge\eta_m} \|u(r)\|_{L^\alpha(D)}^\alpha\big(1+\|u(r)\|_{L^\alpha(D)}^\alpha\big)\,\mathrm{d}r\Big)^{1/2} \\
&\qquad \lesssim \mathbb{E}\Big(\sup_{0\le s\le t\wedge\eta_m}\|u(r)\|_{L^\alpha(D)}^\alpha\Big)^{1/2}\Big(1+\int_0^{t\wedge\eta_m}\|u(r)\|_{L^\alpha(D)}^\alpha\,\mathrm{d}r\Big)^{1/2}.
\end{aligned}$$

Finally, the well-known estimate $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ for $a, b, \varepsilon > 0$ yields

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \Big| \sum_{k=1}^{\infty} \int_0^s \int_D \alpha |u(r,x)|^{\alpha-2} \operatorname{Re}\langle u(r,x), B_k(r,x,u(r,x)) \rangle_{\mathbb{C}} \,\mathrm{d}x \,\mathrm{d}\beta_k(r) \Big| \\
&\qquad \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \|u(r)\|_{L^\alpha(D)}^\alpha + C\Big(1 + \mathbb{E} \int_0^{t \wedge \eta_m} \|u(r)\|_{L^\alpha(D)}^\alpha \,\mathrm{d}r\Big)
\end{aligned}$$

for some constant $C > 0$. We proceed with the deterministic terms in (3.2.5). Since a is uniformly elliptic,

$$-\alpha(\alpha-1)|u(s,x)|^\alpha \nabla u(s,x) a(u(s,x)\nabla u(s,x) \leq 0$$

holds almost surely for all $s \in [0, \eta_m]$ and all $x \in D$ and the corresponding term can be dropped in an upper estimate. Moreover, the divergence theorem of Gauss and $u(t,x) = 0$ almost surely all for $t \in [0, \eta_m]$ and $x \in \partial D$ yields

$$\begin{aligned}
&\int_D \alpha(\alpha-1)|u(s,x)|^\alpha \nabla u(s,x) G(u(s,x)) \,\mathrm{d}x \\
&\qquad = \int_D \operatorname{div} \Big(\int_0^{u(t,x)} \alpha(\alpha-1)|\xi|^\alpha G(\xi) \,\mathrm{d}\xi \Big) \,\mathrm{d}x \\
&\qquad = \int_{\partial D} \Big(\int_0^{u(t,x)} \alpha(\alpha-1)|\xi|^\alpha G(s,\xi) \,\mathrm{d}\xi \Big) \nu \,\mathrm{d}\sigma(\mathrm{x}) = 0.
\end{aligned}$$

The last remaining term can be estimated with the assumptions on B. We have

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \Big| \sum_{k=1}^{\infty} \int_0^s \int_D \alpha(\alpha-1)|u(r,x)|^{\alpha-2} |B_k(r,x,u(r,x))|^2 \,\mathrm{d}x \,\mathrm{d}r \Big| \\
&\qquad \lesssim \mathbb{E} \int_0^{t \wedge \eta_m} \int_D |u(r,x)|^{\alpha-2} + |u(r,x)|^\alpha \,\mathrm{d}x \,\mathrm{d}r \\
&\qquad \lesssim 1 + \int_0^t \mathbb{E} \sup_{0 \leq s \leq r \wedge \eta_m} \int_D |u(s,x)|^\alpha \,\mathrm{d}x \,\mathrm{d}r.
\end{aligned}$$

All in all, we proved

$$\mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \int_D |u(x,s)|^\alpha \,\mathrm{d}x \lesssim 1 + \mathbb{E} \int_D |u_0(x)|^\alpha \,\mathrm{d}x + \int_0^t \mathbb{E} \sup_{0 \leq s \leq r \wedge \eta_m} \int_D |u(s,x)|^\alpha \,\mathrm{d}x \,\mathrm{d}r$$

and hence, Gronwall yields,

$$\mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \int_D |u(x,s)|^\alpha \,\mathrm{d}x \lesssim 1 + \mathbb{E} \int_D |u_0(x)|^\alpha \,\mathrm{d}x$$

for every $t \in [0,T]$ and $n \in \mathbb{N}$ with an estimate independent of m. We want to finish the proof by applying Fatou's Lemma to pass to the limit $m \to \infty$. Here, we use that $\eta_m \to \tau$ almost surely for $m \to \infty$. Note that one can interchange sup and lim inf in an upper estimate, since lim inf can be written in the form sup inf and supremums can be interchanged, whereas sup inf $\leq$ inf sup. Thus, we have

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s < \tau} \int_D |u(x,s)|^\alpha \,\mathrm{d}x &\leq \liminf_{m \to \infty} \mathbb{E} \sup_{0 \leq s \leq t \wedge \eta_m} \int_D |u(x,s)|^\alpha \,\mathrm{d}x \\
&\lesssim 1 + \liminf_{m \to \infty} \mathbb{E} \int_D |u_0(x)|^\alpha \,\mathrm{d}x \\
&= 1 + \mathbb{E} \|u_0\|_{L^\alpha(D)}^\alpha.
\end{aligned}$$

This proves the first claim. For the second claim, we have to look at (3.2.5) in the special case $\alpha = 2$. We get

$$\begin{aligned}\|u(t)\|_{L^2(D)}^2 =&\|u_0\|_{L^2(D)}^2 - 2\int_0^t \int_D \nabla u(s,x) a(u(s,x)) \nabla u(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &+ 2\sum_{k=1}^\infty \int_0^t \int_D u(s,x) B_k(s,x,u(s,x))\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\ &+ \sum_{k=1}^\infty \int_0^t \int_D B_k(s,x,u(s,x))^2\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

almost surely for all $t \in [0, \eta_m]$. Coercivity of $a(u(s,x))$ then yields

$$-\int_D \nabla u(s,x) a(u(s,x)) \nabla u(s,x)\,\mathrm{d}x \le -\delta_0 \|\nabla u(s)\|_{L^2(D)}^2.$$

As a consequence, we have

$$\begin{aligned}\delta_0 \int_0^t \|\nabla u(s)\|_{L^2(D)}^2\,\mathrm{d}s \le&\|u_0\|_{L^2(D)}^2 + 2\sum_{k=1}^\infty \int_0^t \int_D u(s,x) B_k(s,x,u(s,x))\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\ &+ \sum_{k=1}^\infty \int_0^t \int_D B_k(s,x,u(s,x))^2\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

almost surely for all $t \in [0,\tau)$ and with the estimates we did before, we get

$$\big(\mathbb{E}\|\nabla u\|_{L^2([0,\tau)\times D)}^2\,\mathrm{d}s\big)^{1/2} \lesssim (1 + \|u_0\|_{L^2(\Omega\times D)}).$$

This finishes the proof. ☐

As a consequence of these estimates, we can extend u to a pathwise continuous function with values in $L^2(D)$ on the closed interval $[0,\tau]$.

Lemma 3.2.15. *If we assume* [GD1]-[GD7] *and additionally* $u_0 \in L^2(\Omega \times D)$ *the function* $u : [0,\tau) \to L^2(D)$ *is pathwise almost surely uniformly continuous and can be extended to a continuous function on* $[0,\tau]$.

Proof. We know that u is an Itô process in $W^{-1,2}(D)$ and that we have

$$u(t) - u_0 = \int_0^t \operatorname{div}(a(u(s))\nabla u(s)) + \operatorname{div}(G(u(s)))\,\mathrm{d}s + \int_0^t B(u(s))\,\mathrm{d}W(s)$$

for every $t \in [0,\tau)$ and by Lemma 3.2.14, we have $u \in L^2(0,\tau; W_0^{1,2}(D))$ almost surely. Moreover, by [GD7] and the Itô isometry, we obtain

$$\begin{aligned}\Big\|t \mapsto \int_0^t B(u(s))\mathbf{1}_{[0,\tau)}(s)\,\mathrm{d}W(s)\Big\|_{L^2(\Omega\times[0,T];W_0^{1,2}(D))} &= \|B(u)\mathbf{1}_{[0,\tau)}\|_{L^2(\Omega\times[0,T]\times\mathbb{N};W_0^{1,2}(D))} \\ &\lesssim 1 + \|u\|_{L^2(\Omega\times[0,\tau);W_0^{1,2}(D))} < \infty\end{aligned}$$

and thus, we also have $t \mapsto \int_0^t B(u(s))\,\mathrm{d}W(s) \in L^2(0,\tau; W_0^{1,2}(D))$ almost surely. Consequently, we have

$$t \mapsto u_0 + \int_0^t \operatorname{div}(a(u(s))\nabla u(s)) + \operatorname{div}(G(u(s)))\,\mathrm{d}s \in L^2(0,\tau; W_0^{1,2}(D))$$

almost surely. On the other hand, the fundamental theorem of calculus yields

$$t \mapsto u_0 + \int_0^t \operatorname{div}(a(u(s))\nabla u(s)) + \operatorname{div}(G(u(s)))\,\mathrm{d}s \in W^{1,2}(0,\tau; W^{-1,2}(D))$$

almost surely. Since the embedding

$$W^{1,2}(0,\tau; W^{-1,2}(D)) \cap L^2(0,\tau; W_0^{1,2}(D)) \hookrightarrow C(0,\tau; L^2(D))$$

is bounded (see e.g. [90], Chapter III , Proposition 1.2), the map

$$t \mapsto u_0 + \int_0^t \operatorname{div}(a(u(s))\nabla u(s)) + \operatorname{div}(G(u(s)))\,\mathrm{d}s$$

is almost surely uniformly continuous on $[0,\tau)$ viewed as a function in $L^2(D)$. Clearly, by the Burkholder-Davies-Gundy inequality, the same holds true for the stochastic integral. This closes the proof. □

In the previous lemmas, we extended our local solution $\big(u,(\tau_n)_n,\tau\big)$ to the closed interval $[0,\tau]$ and derived estimates for u on $[0,\tau]$. As a consequence, we can apply a regularity result for quasilinear stochastic evolution equations in divergence form that yields additional regularity properties for u. It turns out that u is even pathwise Hölder continuous in space and time.

Lemma 3.2.16. *If we assume* (GD1)-(GD7) *and $u_0 \in L^m(\Omega\times D)$ for every $m \in [2,\infty)$, the process $u:\Omega\times[0,\tau]\times D \to \mathbb{R}$ is pathwise Hölder continuous in space and time. More precisely, there exists $\eta > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,\tau],x\in D} |u(t,x)| + \sup_{t,s\in[0,\tau],x,y\in D} \frac{|u(t,x)-u(s,y)|}{\max\{|t-s|^\eta, |x-y|^{2\eta}\}}\Big)^m < \infty$$

for every $m \in [2,\infty)$.

Proof. By Lemma 3.2.14 and Lemma 3.2.15, we have

$$u \in L^m(\Omega; L^\infty(0,\tau; L^m(D))) \cap L^2(\Omega\times[0,\tau); W_0^{1,2}(D))$$

for all $m \in [2,\infty)$ and $u:[0,\tau]\to L^2(D)$ is almost surely uniformly continuous. Moreover, our initial value $u_0 \in B^{1-2/p}_{q,p,0}(D)$ satisfies $u_0 = 0$ almost surely on ∂D, since we required $1-2/p > d/q$. Thus, a slight variation of [26], Theorem 2.6 implies the claimed result. The only change we need is that we investigate the equation on the random interval $[0,\tau]$ instead of $[0,T]$. However, in the proof of Theorem 2.6 one can replace T by τ without further difficulties, since they authors argue pathwise with a classical regularity result about deterministic parabolic equations by Ladyzhenskaya, Solonnikov and Uralceva (see [72], Theorem 10.1 in Chapter III). In [26], Theorem 2.6, ∂D was assumed to be smooth, but to apply Ladyzhenskaya's result, a piecewise C^1-boundary combined with the so called condition A, that is explained in [72] on page 9, is sufficient. Note that our assumption of a C^1-boundary implies this condition A. □

Finally, we can prove the main theorem of this section. We show that our local solution u is indeed a global solution that exists on the whole interval $[0,T]$. For this proof, we compare u with the solution z of a stochastic heat equation with the noise $B(u)(t)\,\mathrm{d}W(t)$. Then, we investigate the regularity properties of $u-z$, which solves a non-autonomous deterministic partial differential equation with a random parameter by applying results on maximal regularity for both the stochastic heat equation and for the arising non-autonomous equation.

Theorem 3.2.17. *If we assume* (GD1)-(GD7)*, the local solution* $\big(u,(\tau_n)_n,\tau\big)$ *of* (GDIV) *is a global solution, i.e. we have* $\tau=T$ *almost surely and the solution satisfies*

$$u\in L^p(0,T;W_0^{1,q}(D))\cap C(0,T;B_{q,p,0}^{1-2/p}(D))$$

almost surely.

Proof. We first check the theorem for $u_0\in L^\infty(\Omega;B_{q,p,0}^{1-2/p}(D))$. By Theorem 3.2.12, there exists a local solution $\big(u,(\tau_n)_n,\tau\big)$ of (GDIV) to the initial value u_0. Since we chose $1-2/p>d/q$, we have $u_0\in L^m(\Omega\times D)$ for all $m\in[2,\infty)$. As a consequence, Lemma 3.2.16 implies that $u:\Omega\times[0,\tau]\times D\to\mathbb{R}$ is pathwise almost surely uniformly continuous in space and time and $u\mathbf{1}_{[0,\tau]}\in L^m(\Omega;L^\infty(0,T;L^m(D)))$.

Next, we consider the equation

$$\begin{cases} dz(t) &= \Delta z(t)\,\mathrm{d}t + B(u)(t)\,\mathrm{d}W(t)), \quad t\in[0,\tau],\\ z(0) &= 0. \end{cases}$$

By (GD7), we have $B(u)\in L^p(\Omega\times[0,T];\gamma(l^2;L^q(D)))$. Therefore the maximal L^p-regularity result for stochastic evolution equations, Theorem 2.2.6, yields a unique solution

$$z\in L^p(\Omega\times[0,T];W_0^{1,q}(D))\cap L^p(\Omega;C(0,T;B_{q,p,0}^{1-2/p}(D))).$$

If we investigate the difference $y:=u-z$ on $[0,\tau]$, we find out that y pathwise almost surely solves the deterministic non-autonomous parabolic equation

$$\begin{cases} y'(t) &= [\operatorname{div}(a(u(t))\nabla y(t)) + \operatorname{div}(G(u(t))) + \operatorname{div}((a(u(t))-I)\nabla z(t)), \quad t\in[0,\tau]\\ y(0) &= u_0. \end{cases} \tag{3.2.6}$$

Note that any solution of this equation in $L^2(0,\tau;W_0^{1,2}(D))$ is unique by a classical result of Lions for non-autonomous evolution equations governed by forms (see e.g. [90], Chapter III, Proposition 2.3).

As a next step, we prove that this equation has deterministic maximal L^p-regularity. We estimate

$$\begin{aligned} \|\operatorname{div}(a(u(t))\nabla x) - \operatorname{div}(a(u(s))\nabla x)\|_{W^{-1,q}(D)} &\le \|(a(u(t))-a(u(s)))\nabla x\|_{L^q(D)}\\ &\le \sup_{x\in D}|a(u(t,x))-a(u(s,x))|\|x\|_{W_0^{1,q}(D)}\\ &\lesssim \sup_{x\in D}|u(t,x)-u(s,x)|\|x\|_{W_0^{1,q}(D)} \end{aligned}$$

and since u is pathwise almost surely uniformly continuous on $[0,\tau]\times D$ (see Lemma 3.2.16), the mapping $[0,\tau]\ni t\mapsto \operatorname{div}(a(u(t))\nabla)\in\mathcal{B}(W_0^{1,q}(D),W^{-1,q}(D))$ is almost surely continuous. Moreover, as we have seen in Lemma 3.2.10, the operator $\operatorname{div}(a(u(t))\nabla)$ has almost surely for fixed $t\in[0,\tau]$ a bounded H^∞-calculus on $W^{-1,q}(D)$ and its domain is given by $W_0^{1,q}(D)$. Therefore, we can apply [86], Theorem 2.5 and obtain almost surely the maximal L^p-regularity of the non-autonomous equation (3.2.6). Moreover, we both have $\operatorname{div}(G(u))\in L^p(0,\tau;W^{-1,q}(D))$ and $\operatorname{div}((a(u)-I)\nabla z)\in L^p(0,\tau;W^{-1,q}(D))$. Indeed, [GD6] and the regularity of z together with [GD2] imply

$$\|\operatorname{div}(G(u))\|_{L^p(0,\tau;W^{-1,q}(D))}\lesssim\|G(u)\|_{L^p(0,\tau;L^q(D))}\lesssim 1+\|u\|_{L^p(0,\tau;L^q(D))},$$

$$\|\operatorname{div}((a(u)-I)\nabla z)\|_{L^p(0,\tau;W^{-1,q}(D))}\lesssim\|(a(u)-I)\nabla z\|_{L^p(0,\tau;L^q(D)}\lesssim\|z\|_{L^p(0,\tau;W_0^{1,q}(D))}.$$

As a consequence of maximal regularity, we get

$$\begin{aligned}&\|y\|_{L^p(0,\tau;W_0^{1,q}(D))}+\|y\|_{C(0,\tau;B^{1-2/p}_{q,p,0}(D))}\\&\quad\le C_{\mathrm{MR}}\big(\|\operatorname{div}(G(u))\|_{L^p(0,\tau;W^{-1,q}(D))}+\|\operatorname{div}((a(u)-I)\nabla z)\|_{L^p(0,\tau;W^{-1,q}(D))}\big)\\&\quad\lesssim 1+\|u\|_{L^p(0,\tau;L^q(D))}+\|z\|_{L^p(0,\tau;W_0^{1,q}(D))}\end{aligned}$$

and thus $y\in L^p(0,\tau;W_0^{1,q}(D))\cap C(0,\tau;B^{1-2/p}_{q,p,0}(D))$ almost surely. With the unique solvability in $L^2(0,\tau;W_0^{1,2}(D))$ and $u=y+z$ one sees that u is also pathwise almost surely in the space $L^p(0,\tau;W_0^{1,q}(D))\cap C(0,\tau;B^{1-2/p}_{q,p,0}(D))$. Hence the blow-up alternative from Theorem 3.2.12 yields $\tau=T$ almost surely, which is the desired result.

Last but not least, we have to deal with arbitrary initial values $u_0:\Omega\to B^{1-2/p}_{q,p,0}(D)$. Defining $\Lambda_n:=\{\|u_0\|_{B^{1-2/p}_{q,p,0}(D)}<n\}$ and the truncated initial values $u_0^{(n)}:=u_0\mathbf{1}_{\Lambda_n}$, we can apply the result we derived above and we get global solutions u_n of (GDIV) to the initial value $u_0^{(n)}$ that pathwise almost surely satisfy $u_n\in L^p(0,T;W_0^{1,q}(D))\cap C(0,T;B^{1-2/p}_{q,p,0}(D))$. By Corollary 2.3.8, the solutions u_n and u_m coincide on $\Lambda_{n\wedge m}$ and therefore the pointwise limit $v=\lim_{n\to\infty}u_n$ is a well-defined adapted process. Moreover, since for almost all $\omega\in\Omega$ there is an $n(\omega)$ such that $v(\omega,\cdot)=u_{n(\omega)}(\omega,\cdot)$, v solves (GDIV) and has almost surely the claimed regularity. However, by maximal uniqueness of the solution (u,τ), we must have $\tau=T$ and $u(t)=v(t)$ almost surely for every $t\in[0,T]$. □

The reader may ask, why we could not prove

$$u\in L^p(\Omega\times[0,T];W_0^{1,q}(D))\cap L^p(\Omega;C(0,T;B^{1-2/p}_{q,p,0}(D)))$$

under the additional assumption $u_0\in L^p(\Omega;B^{1-2/p}_{q,p,0}(D))$. This is due to the maximal regularity result for non-autonomous deterministic equations we used. The maximal regularity constant C_{MR} highly depends on the modulus of continuity of the coefficient function which is in our case given by $a(u(\omega,t,x))$. Therefore, C_{MR} depends on the modulus of continuity of u itself, but this one differs from path to path and cannot be controlled uniformly in ω.

So, the best estimate, we can achieve is

$$\begin{aligned}\|u(\omega,\cdot)\|_{L^p(0,T;W_0^{1,q}(D))} &= \|y(\omega,\cdot)+z(\omega,\cdot)\|_{L^p(0,T;W_0^{1,q}(D))}\\ &\le CC_{\mathrm{MR}}(\omega)\big(1+\|u(\omega,\cdot)\|_{L^p(0,\tau;L^q(D))}+\|z(\omega,\cdot)\|_{L^p(0,\tau;W_0^{1,q}(D))}\big)\end{aligned}$$

for almost $\omega \in \Omega$, but it is impossible to control $\|u\|_{L^p(\Omega\times[0,T];W_0^{1,p}(D))}$ in this way. One would need a significantly stronger result on maximal L^p- regularity for non-autonomous deterministic equations with C_{MR} only depending on the upper bound and the ellipticity constant of the coefficient function $a(u(\omega,t,x))$. Unfortunately, such a result is only known for $p=2$ by a classical result of Lions and for $p\in[2-\varepsilon,2+\varepsilon]$ for some small $\varepsilon>0$ by a recent result of Disser, ter Elst and Rehberg (see [31], Proposition 6.3). This can be used to prove at least

$$u \in L^p(\Omega; L^{r_1}(0,T;W_0^{1,r_2}(D)))$$

for $r_1, r_2 \in [2-\varepsilon, 2+\varepsilon]$.

3.3. The incompressible Navier-Stokes system for generalised Newtonian fluids

We now deal with a quasilinear stochastic model in fluid dynamics. This example is inspired by Bothe and Prüss, who treated the same model in a deterministic setting (see [18]).

Throughout this section the divergence of a $d\times d$ matrix T is a vector field defined by $(\operatorname{div} T)_i = \sum_{k=1}^d \partial_k T_{ik}$ and ∇f is the Jacobian of the vector field f. We start with a universal model for fluids, namely

$$\text{(FM)}\quad \begin{cases} du(t) &= [-(u(t)\cdot\nabla)u(t)+\operatorname{div} S(t)+f(t)]\,\mathrm{d}t+[B(u)(t)-\nabla\widetilde{p}]\,\mathrm{d}W(t),\ t\in[0,T],\\ S(t) &= \widetilde{\mu}(t)-p(t)I,\ t\in[0,T],\\ \operatorname{div} u(t) &= 0, \quad t\in[0,T],\\ u(0) &= u_0. \end{cases}$$

Here, $u:[0,T]\times\mathbb{R}^d\to\mathbb{C}^d$ is the macroscopic velocity. In this model the density ρ of a perfect fluid is assumed to be constant and can therefore be chosen identically one. Together with the continuity equation

$$\partial_t\rho(t,x)+\operatorname{div} u(t,x)=0,$$

that holds in every mechanical model, this results in the restriction $\operatorname{div}(u(t))=0$. Moreover, as in every perfect fluid, the total stress tensor $S:[0,T]\times\mathbb{R}^d\to\mathbb{C}^{d\times d}$ is a sum of the viscous stress $\widetilde{\mu}:[0,T]\times\mathbb{R}^d\to\mathbb{C}^{d\times d}$ and the hydrostatic pressure pI, where p is scalar-valued.

In the following, we discuss generalised Newtonian fluids that are characterised by the assumption $\widetilde{\mu}=2\mu(|\mathcal{E}|_2^2)\mathcal{E}$, where $\mathcal{E}=\frac{1}{2}(\nabla u+\nabla u^T)$ is the symmetrised derivative of the velocity, the so called rate-of-strain tensor and $|\cdot|_2$ is the Hilbert-Schmidt norm on $\mathbb{C}^{d\times d}$. There are many examples for this model, e.g. the Ostwald-de-Waele power-law $\mu(s)=\mu_0 s^{m/2-1}$

for $m \geq 1$ and $\mu_0 > 0$, the Carreau model $\mu(s) = \mu_0(1+s)^{m/2-1}$ or the truncated Spriggs law $\mu(s) = \mu_0 s^{m/2-1}\mathbf{1}_{[s_0,\infty)}(s)$ for some $s_0 > 0$. For details about generalised Newtonian fluids, we refer to Chapter 5 in the monograph of Armstrong, Bird and Hassager ([17]). Last but not least, we would like to mention that a noise perturbation was discussed several times in case of Newtonian fluids with $\widetilde{\mu} = \mu_0\mathcal{E}$ (see e.g. [20], [76] and [96]). We want to generalise these results to the quasilinear case. However, we must admit that we cannot deal with the Kreichnan model of turbulence which is a noise perturbation of the form $(\sigma \cdot \nabla)u$ with a small enough $(\sigma_n)_n \subset l^2(\mathbb{N})$. The reason for this is the same as in Section 3.1 and we briefly described this problem below Theorem 3.1.5.

As a first step, we derive a quasilinear evolution equation from (FM). Using the product rule and $\operatorname{div}(u) = 0$, we calculate

$$
\begin{aligned}
(\operatorname{div} S)_i &= \operatorname{div}\big(\mu(|\mathcal{E}|_2^2)2\mathcal{E} - pI\big)_i = \big(\mu(|\mathcal{E}|_2^2)\operatorname{div}(2\mathcal{E}) + \mu'(|\mathcal{E}|_2^2)\nabla(|\mathcal{E}|_2^2)\cdot 2\mathcal{E}\big)_i - \partial_i p \\
&= \mu(|\mathcal{E}|_2^2)\sum_{k=1}^d \big(\partial_k\partial_i u_k + \partial_k^2 u_i\big) + \mu'(|\mathcal{E}|_2^2)\sum_{j,k,l=1}^d (\partial_l u_i + \partial_i u_l)(\partial_k u_j + \partial_j u_k)\partial_k\partial_l u_j - \partial_i p \\
&= \mu(|\mathcal{E}|_2^2)\sum_{k=1}^d \partial_k^2 u_i + \mu'(|\mathcal{E}|_2^2)\sum_{j,k,l=1}^d (\partial_l u_i + \partial_i u_l)(\partial_k u_j + \partial_j u_k)\partial_k\partial_l u_j - \partial_i p.
\end{aligned}
$$

All in all, we get the quasilinear system

$$
\begin{cases}
du(t) &= [-A(u(t))u(t) - \nabla p(t) - (u(t)\cdot\nabla)u(t) + f(t)]\,\mathrm{d}t + [B(u)(t) - \nabla\widetilde{p}]\,\mathrm{d}W(t), \\
\operatorname{div} u(t) &= 0, \\
u(0) &= u_0
\end{cases}
$$

with

$$
(A(z)u)_i = -\mu(|\tfrac{\nabla z + \nabla z^T}{2}|_2^2)\sum_{k=1}^d \partial_k^2 u_i - \mu'(|\tfrac{\nabla z + \nabla z^T}{2}|_2^2)\sum_{j,k,l=1}^d (\partial_l z_i + \partial_i z_l)(\partial_k z_j + \partial_j z_k)\partial_k\partial_l u_j.
$$

We consider this equation in $L^p(\mathbb{R}^d)^d$ and as usual in the context of fluid dynamics, we use the Helmholtz decomposition

$$
L^q(\mathbb{R}^d)^d = L^q_\sigma(\mathbb{R}^d) \oplus \nabla W^{1,q}(\mathbb{R}^d),
$$

where $L^q_\sigma(\mathbb{R}^d) = \{f \in L^q(\mathbb{R}^d)^d : \operatorname{div}(f) = 0\}$. Note that this decomposition exists for all $q \in (1,\infty)$ and induces the bounded Helmholtz projection $P : L^q(\mathbb{R}^d)^d \to L^q_\sigma(\mathbb{R}^d)$. Applying P yields the evolution equation

$$
\text{(QNS)}\begin{cases}
du(t) &= [-PA(u(t))u(t) - P(u(t)\cdot\nabla)u(t) + Pf(t)]\,\mathrm{d}t + PB(u)(t)\,\mathrm{d}W(t), \\
u(0) &= u_0
\end{cases}
$$

in $L^q_\sigma(\mathbb{R}^d)$ for the velocity u.

In the following, we use the abbreviations $B^s_{q,p,\sigma}(\mathbb{R}^d) := \{f \in B^s_{q,p}(\mathbb{R}^d)^d : \operatorname{div}(f) = 0\}$ and $W^{s,q}_\sigma(\mathbb{R}^d)^d := \{f \in W^{s,q}(\mathbb{R}^d) : \operatorname{div}(f) = 0\}$. We treat (QNS) under the following assumptions.

[QN1] Let $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be continuously differentiable such that μ and μ' are locally Lipschitz continuous, i.e. for every $n \in \mathbb{N}$ there exists $C(n) > 1$ such that

$$|\mu'(x) - \mu'(y)| + |\mu(x) - \mu(y)| \leq C(n)|x - y|$$

for all $0 \leq x, y \leq n$. Moreover, we assume $\mu(s) + 2s\mu'(s) > 0$ for all $s \geq 0$.

[QN2] We choose $p, q \in (2, \infty)$ such that $1 - 2/p > d/q$.

[QN3] The initial value $u_0 : \Omega \to B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)$ is a strongly $\mathcal{F}_0$- measurable random variable.

[QN3] The driving noise W is an l^2- cylindrical Brownian motion of the form

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t),$$

where $(e_k)_k$ is the standard orthonormal basis of l^2 and $(\beta_k)_k$ is a sequence of independent real-valued Brownian motions relative to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

[QN4] The nonlinearity $(PB_n)_n$ is chosen in such a way that it satisfies [Q7*] together with [Q9*] in the setting [TT].

[QN5] $f \in L^p(\Omega \times [0, T]; L^q(\mathbb{R}^d)^d)$ is strongly measurable and $\mathbb{F}$-adapted.

We want to apply Theorem 2.3.12 in the setting [TT] with $E = L^q_\sigma(\mathbb{R}^d)$, $E^1 = W^{2,q}_\sigma(\mathbb{R}^d)$ and $H = l^2(\mathbb{N})$. The trace space TR is then given by $(E, E^1)_{1-1/p,p} = B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)$. Due to our assumptions, [TTQ1], [Q7*] and [Q8] are directly fulfilled. We now check [Q4*], i.e. we have to prove that $PA(z)$ has for every $z \in B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)$ a bounded H^∞-calculus. In the following Proposition, we restate a result of Bothe and Prüss (see [18], proof of Theorem 4.1). Unlike Bothe and Prüss we need the precise dependence of all involved constants from z. Therefore, we need an additional argument.

Lemma 3.3.1. *We assume* [QN1] *and* [QN2]*. Then, for every $z \in B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)$, there exists $\gamma > 0, \theta \in [0, \pi/2)$ such that the operator $\gamma + PA(z)$ has the domain $W^{2,q}_\sigma(\mathbb{R}^d)$ and such that it is $\mathcal{R}$-sectorial in $L^q_\sigma(\mathbb{R}^d)$ on the sector Σ_θ. Moreover, γ, θ, the bound $C_\nu > 0$ in*

$$\mathcal{R}\Big(\big\{\lambda R\big(\lambda, \gamma + PA(z)\big)\big\} \subset \mathcal{B}(L^q_\sigma(\mathbb{R}^d))\Big) \leq C_\nu$$

for given $\nu > \theta$ and the constant $M > 0$ in

$$M^{-1}\|(\gamma + PA(z))x\|_{L^q_\sigma(\mathbb{R}^d)} \leq \|x\|_{W^{2,q}_\sigma(\mathbb{R}^d)} \leq M\|(\gamma + PA(z))x\|_{L^q_\sigma(\mathbb{R}^d)}$$

for all $x \in W^{2,q}_\sigma(\mathbb{R}^d)$ only depend on $\|z\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)}$, p, q and on the constants from [QN1].

Proof. Bothe and Prüss derive from [QN1] the strong ellipticity of $A(z)$ (see [18], page 385). In our case, $A(z)$ only depends on $\nabla z \in B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)^{d\times d}$. Thus, it is sufficient to show that given $u \in C^\alpha(\mathbb{R}^d)^{d^2} \cap L^q(\mathbb{R}^d)^{d^2}$ for some $\alpha > 0$ and a strongly elliptic operator of the form $B(u) = -\sum_{|\beta|=2} b_\beta(u)D^\beta$ with locally Lipschitz continuous coefficients $b_\beta : \mathbb{C}^{d^2} \to \mathbb{C}^{d\times d}$, the statement from above holds true with $PA(\cdot)$ replaced by $PB(\cdot)$ and θ, C_ν, M and

μ only depend on $\|u\|_\alpha$ and on $\|u\|_{L^q(\mathbb{R}^d)^{d^2}}$. The claimed result then follows directly by $\nabla z \in B_{q,p}^{1-\frac{2}{p}}(\mathbb{R}^d)^{d\times d}$ and the Sobolev embedding

$$B_{q,p}^{1-2/p}(\mathbb{R}^d)^{d\times d} \hookrightarrow C^\alpha(\mathbb{R}^d)^{d^2} \cap L^q(\mathbb{R}^d)^{d^2}$$

for some $\alpha \in (0,1)$.

Note that Bothe and Prüss prove in [18], Section 5, that $\gamma + PB(u)$ has the maximal regularity property in $L^q_\sigma(\mathbb{R}^d)$ and that the domain of $\gamma + PB(u)$ is given by $W^{2,q}_\sigma(\mathbb{R}^d)$. This implies the $\mathcal{R}$-sectoriality of $\gamma + PB(u)$ (see [102], Theorem 4.2).

We can follow the argument of Bothe and Prüss step by step, we just have to argue that the spectral shift $\widetilde{\gamma}$, the constant $\widetilde{M}$ in

$$\widetilde{M}^{-1}\|(\widetilde{\gamma} + PB(u))x\|_{L^q_\sigma(\mathbb{R}^d)} \le \|x\|_{W^{2,q}_\sigma(\mathbb{R}^d)} \le \widetilde{M}\|(\widetilde{\gamma} + PB(u))x\|_{L^q_\sigma(\mathbb{R}^d)}$$

and the maximal regularity constant of $\widetilde{\gamma} + PB(u)$ only depend on $\|u\|_\alpha$, $\|u\|_{L^q(\mathbb{R}^d)^{d^2}}$ and on the ellipticity and the local Lipschitz constants of B.

In [18], Theorem 5.1 Bothe and Prüss start with a constant coefficient elliptic operator $\widetilde{B}$. They prove that $P\widetilde{B}$ has the maximal regularity property, that the domain of $P\widetilde{B}$ is given by $W^{2,q}_\sigma(\mathbb{R}^d)$ and that all the involved estimates only depend on the bound and the ellipticity of the symbol of $\widetilde{B}$. In Corollary 5.2 in the same article, they show that one still has maximal regularity, if one perturbs $\widetilde{B}$ with functions, whose supremum is smaller than some $\eta > 0$. This η also only depends on the ellipticity and the bound of the symbol of $\widetilde{B}$.

To deal with $PB(u)$, their idea is to use the uniform continuity of $b_\beta(u)$ and convergence at infinity to choose a radius $\delta > 0$ and finitely many balls $B(x_i, \delta)$, such that we have $|b_\beta(u(x)) - b_\beta(u(x_i))| < \eta$ for all $x \in B(x_i,\delta)$ and $|b_\beta(u(x)) - b_\beta(0)| < \eta$ for $x \notin \cup_i B(x_i,\delta)$. Then, they localise the equation with a partition of unity subordinate to these balls, solve locally and put the local solutions together. Closely inspecting their proof it turns out that M, γ and the maximal regularity constant only depend on the ellipticity and the supremum of the symbol of $B(u)$ and on the number of balls needed in this argument. The ellipticity of the symbol is fixed by [QN1] and the supremum can be controlled by $\|u\|_{C^\alpha(\mathbb{R}^d)^{d\times d}}$ and $C(n)$ from [QN1]. So, it remains to estimate the number of balls by a quantity that can be controlled by $\|u\|_\alpha$ and $\|u\|_{L^q(\mathbb{R}^d)^{d^2}}$.

Fix $u \in C^\alpha(\mathbb{R}^d)^{d^2} \cap L^q(\mathbb{R}^d)^{d^2}$ and by the local Lipschitz continuity of b_β, there exists $C(\|u\|_\infty) > 1$ such that we have

$$|b_\beta(x) - b_\beta(y)| \le C(\|u\|_\infty)|x-y|$$

for all $|x|, |y| \le \|u\|_\infty$. We divide $\mathbb{R}^d$ in the two disjoint subsets $\{|u| \ge \frac{\eta}{2C(\|u\|_\infty)}\}$ and $\{|u| < \frac{\eta}{2C(\|u\|_\infty)}\}$ and we define $\delta := \left(\frac{\eta}{6\|u\|_\alpha C(\|u\|_\infty)}\right)^{1/\alpha}$. The set $\{|u| \ge \frac{\eta}{2C(\|u\|_\infty)}\}$ is closed and bounded and hence compact. Then, by Vitali's covering Lemma (see e.g. [41], Lemma 2.1.5), there are disjoint balls $(B(x_i,\delta))_{i=1,\dots,N}$ with radius δ and centre $x_i \in \{|u| \ge \frac{\eta}{2C(\|z\|_\infty)}\}$

such that

$$\{|u| \geq \tfrac{\eta}{2C(\|u\|_\infty)}\} \subset \bigcup_{i=1}^{N} B(x_i, 3\delta).$$

The balls $(B(x_i, 3\delta))_{i=1,\dots,N}$ are the sets we are looking for. Indeed, for $x \notin \cup_{i=1}^N B(x_i, 3\delta)$, we have $|u(x)| \leq \frac{\eta}{2}$ and for $x, y \in B(x_i, 3\delta)$, we have

$$|b_\beta(u(x)) - b_\beta(u(y))| \leq C(\|u\|_\infty)\|u\|_\alpha (3\delta)^\alpha \leq \tfrac{3^\alpha \eta}{6} \leq \tfrac{\eta}{2}.$$

It remains to estimate the size of N. We have

$$\cup_{i=1}^N B(x_i, \delta) \subset \Big\{|u| > \tfrac{\eta}{4C(\|u\|_\infty)}\Big\}.$$

Indeed, given $y \in B(x_i, \delta)$ there is $i = 1, \dots, N$, such that

$$|u(y)| \geq |u(x_i)| - |u(x_i) - u(y)| \geq \tfrac{\eta}{2C(\|u\|_\infty)} - \|u\|_\alpha \delta^\alpha = \tfrac{\eta}{2C(\|u\|_\infty)} - \tfrac{\eta}{6C(\|u\|_\infty)} = \tfrac{\eta}{3C(\|u\|_\infty)}.$$

Consequently, using that the $B(x_i, \delta)$ are disjoint, we get

$$C_d N \delta^d = \Big|\bigcup_{i=1}^N B(x_i, \delta)\Big| \leq \Big|\Big\{|u| > \tfrac{\eta}{4C(\|u\|_\infty)}\Big\}\Big| \leq \frac{4^q C(\|u\|_\infty)^q \|u\|^q_{L^q(\mathbb{R}^d)^{d^2}}}{\eta^q}$$

with Chebyshev's inequality, where C_d is the volume of the unit sphere in $\mathbb{R}^d$. This finally yields

$$N \leq \frac{4^q 6^{\frac{d}{\alpha}} \|u\|^q_{L^q(\mathbb{R}^d)^{d^2}} \|u\|_\alpha^{\frac{d}{\alpha}} C(\|u\|_\infty)^{\frac{d}{\alpha}+q}}{C_d \eta^{q+\frac{d}{\alpha}}},$$

which finishes the proof. □

Next, we conclude that the operators $\gamma + PA(z)$ also have a bounded H^∞-calculus. Our proof of [TTQ3*] adapts the arguments of [53], Proposition 9.5 to our situation. A key ingredient is Sneiberg's Lemma.

Lemma 3.3.2. *Let $(X_\theta)_{\theta\in(0,1)}$ and $(Y_\theta)_{\theta\in(0,1)}$ be complex interpolation scales of Banach spaces and let $S : X_\theta \to Y_\theta$ for each $\theta \in (0,1)$ be a bounded linear operator. If S is for some $\theta_0 \in (0,1)$ an isomorphism between X_{θ_0} and Y_{θ_0}, then there is a $\delta \in (0,1)$ such that S is also an isomorphism between X_μ and Y_μ for $\mu \in (\theta_0 - \delta, \theta_0 + \delta)$. In particular, $\|S^{-1}\|_{\mathcal{B}(Y_\mu, X_\mu)}$ depends on $\|S\|_{\mathcal{B}(X_\mu, Y_\mu)}$, $\|S\|_{\mathcal{B}(X_{\theta_0}, Y_{\theta_0})}$, $\|S^{-1}\|_{\mathcal{B}(Y_{\theta_0}, X_{\theta_0})}$ and $|\mu - \theta_0|$.*

A proof can be found in [101], Theorem 3.6. The precise dependence of $\|S^{-1}\|_{\mathcal{B}(Y_\mu, X_\mu)}$ on the other parameters, namely

$$\|S^{-1}\|_{\mathcal{B}(Y_\mu, X_\mu)} \leq \|S\|_{\mathcal{B}(X_\mu, Y_\mu)} \frac{\|S^{-1}\|_{\mathcal{B}(Y_{\theta_0}, X_{\theta_0})} - \|S\|_{\mathcal{B}(X_{\theta_0}, Y_{\theta_0})}|\mu - \theta_0|}{\|S\|_{\mathcal{B}(X_{\theta_0}, Y_{\theta_0})} - \|S^{-1}\|_{\mathcal{B}(Y_{\theta_0}, X_{\theta_0})}|\mu - \theta_0|}$$

for $|\mu - \theta_0|$ small enough is stated in Theorem 2.3 in the same article. The original proof is due to Sneiberg (see [91]) in Russian language.

Proposition 3.3.3. *Given $z \in B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)$, the operator*

$$u \mapsto \Lambda(z)u := \gamma u + PA(z)u : W^{2,q}_\sigma(\mathbb{R}^d) \to L^q_\sigma(\mathbb{R}^d)$$

is invertible with

$$M^{-1}\|\Lambda(z)x\|_{L^q_\sigma(\mathbb{R}^d)} \le \|x\|_{W^{2,q}_\sigma(\mathbb{R}^d)} \le M\|\Lambda(z)x\|_{L^q_\sigma(\mathbb{R}^d)},$$

for all $x \in W^{2,q}_\sigma(\mathbb{R}^d)$. Moreover, $\Lambda(z)$ has a bounded $H^\infty(\Sigma_\theta)$-calculus and the angle $\theta \in (0,\pi/2)$, the spectral shift γ, M and the constant $C > 0$ in

$$\|f(\Lambda(z))\|_{\mathcal{B}(L^q_\sigma(\mathbb{R}^d))} \le C\|f\|_\infty$$

only depend on $\|z\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)}, p, q$ and on the constants from [QN1]*. In particular, $\Lambda(z)$ satisfies* [TTQ2*] *and* [TTQ3*] *of the previous section.*

Proof. By Lemma 3.3.1 $\Lambda(z) = \gamma + PA(z)$ is invertible and $\mathcal{R}$-bounded on $L^q_\sigma(\mathbb{R}^d)$ on a sector Σ_θ, $\theta \in [0,\pi/2)$ and the occurring constants have the claimed dependency. It remains to show that $\Lambda(z)$ has a bounded H^∞-calculus.

Let $(r_n)_n$ be a sequence of independent Rademacher random variables, $\nu \in (\theta,\pi)$ and $(\lambda_j)_{j\in\mathbb{N}} \subset \Sigma_\nu$ be a dense sequence. For $\eta \in \mathbb{R}$, we define the norms

$$\begin{aligned}
\|(u_j)_j\|_{X_\eta} &:= \mathbb{E}\|\sum_{j=1}^\infty r_j u_j\|_{W^{\eta+2,q}_\sigma(\mathbb{R}^d)} + \mathbb{E}\|\sum_{j=1}^\infty r_j\lambda_j u_j\|_{W^{\eta,q}_\sigma(\mathbb{R}^d)}\\
\|(u_j)_j\|_{Y_\eta} &:= \mathbb{E}\|\sum_{j=1}^\infty r_j u_j\|_{W^{\eta,q}_\sigma(\mathbb{R}^d)}
\end{aligned}$$

and the spaces

$$\begin{aligned}
X_\eta &:= \big\{(u_j)_j \subset W^{\eta+2,q}_\sigma(\mathbb{R}^d) :\ \|(u_j)_j\|_{X_\eta} < \infty\big\}\\
Y_\eta &:= \big\{(u_j)_j \subset W^{\eta,q}_\sigma(\mathbb{R}^d) :\ \|(u_j)_j\|_{Y_\eta} < \infty\big\}.
\end{aligned}$$

Both $(X_\eta)_{\eta\in\mathbb{R}}$ and $(Y_\eta)_{\eta\in\mathbb{R}}$ form complex interpolation scales. We define the operator

$$S_\eta : X_\eta \to Y_\eta, \quad (f_j)_j \mapsto \big(\lambda_j - \Lambda(z))f_j\big)_j.$$

Due to its Hölder continuous coefficients, the operator $\Lambda(z) : W^{\eta+2,q}_\sigma(\mathbb{R}^d) \to W^{\eta,q}_\sigma(\mathbb{R}^d)$ is bounded if $|\eta| < \delta$ for some $\delta > 0$ small enough. In particular, S_η is bounded for $|\eta| < \delta$. The $\mathcal{R}$-sectoriality of $\Lambda(z)$ on $L^q_\sigma(\mathbb{R}^d)$ implies that S_0 is an isomorphism with $S_0^{-1}(u_j)_j = \big((\lambda_j - \Lambda(z))^{-1}u_j\big)_j$. By the previous Lemma, $\|S_0\|_{\mathcal{B}(X_0,Y_0)}$, $\|S_0^{-1}\|_{\mathcal{B}(Y_0,X_0)}$ only depend on the ellipticity and the Hölder norm of the coefficients and hence they are determined by $\|z\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)}$. By Sneiberg's Lemma, there exists $\beta > 0$ such that $S : X_{-\beta} \to Y_{-\beta}$ is an isomorphism and the size of β and $\|S^{-1}\|_{\mathcal{B}(Y_{-\beta},X_{-\beta})}$ depend on μ and $\|z\|_{B^{2-2/p}_{q,p}(\mathbb{R}^d)}$. Especially, we have

$$\mathbb{E}\|\sum_{j=1}^\infty r_j\lambda_j(\lambda_j - \Lambda(z))^{-1}u_j\|_{W^{-\beta,q}_\sigma(\mathbb{R}^d)} \le \|S^{-1}_{-\beta}\|_{\mathcal{B}(Y_{-\beta},X_{-\beta})}\mathbb{E}\|\sum_{j=1}^\infty r_j u_j\|_{W^{-\beta,q}_\sigma(\mathbb{R}^d)}.$$

This proves the $\mathcal{R}$-sectoriality of $\Lambda(z)$ on $W_\sigma^{-\beta,q}(\mathbb{R}^d)$ with domain $W_\sigma^{2-\beta,q}(\mathbb{R}^d)$. Indeed, let $(\widetilde{\lambda}_j)_{j=1}^N \subset \mathbb{C}\setminus\Sigma_\nu$ and $\lambda_j^{(n)} \in (\lambda_k)_k$, $n\in\mathbb{N}$, $j=1,\dots,N$, such that $\lambda_j^{(n)} \to \widetilde{\lambda}_j$ as $n\to\infty$. Then, by Fatou and the holomorphicity of the resolvent, we have

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^N r_j\widetilde{\lambda}_j R(\widetilde{\lambda}_j,\Lambda(z))f_j\Big\|_{W_\sigma^{-\beta,q}(\mathbb{R}^d)} &\le \liminf_{n\to\infty}\mathbb{E}\Big\|\sum_{j=1}^N r_j\lambda_j^{(n)} R(\lambda_j^{(n)},\Lambda(z))f_j\Big\|_{W_\sigma^{-\beta,q}(\mathbb{R}^d)}\\ &\le \|S_{-\beta}^{-1}\|_{\mathcal{B}(Y_{-\beta},X_{-\beta})}\mathbb{E}\Big\|\sum_{j=1}^N r_j f_j\Big\|_{W_\sigma^{-\beta,q}(\mathbb{R}^d)}\end{aligned}$$

for every $(f_j)_{j=1}^N \subset W_\sigma^{-\beta,q}(\mathbb{R}^d)$.

If we now apply Corollary 7.8 in [54], we get that $\Lambda(z)$ has a bounded $H^\infty(\Sigma_\eta)$ calculus on the space $\langle W_\sigma^{-\beta,p}(\mathbb{R}^d), W_\sigma^{2-\beta,p}(\mathbb{R}^d)\rangle_{\beta/2}$. Here $\langle\cdot,\cdot\rangle_\eta$ denotes Rademacher interpolation. Working through the proof of Corollary 7.8 one sees that the bound of the calculus only depends on the size of $|\beta|$ and on $\|S_{-\beta}^{-1}\|_{\mathcal{B}(Y_{-\beta},X_{-\beta})}$. It remains to identify the Rademacher interpolation space. Since the Helmholtz projection P commutes with $I-\Delta$ and $I-\Delta$ has a bounded H^∞-calculus on $W^{\alpha,p}(\mathbb{R}^d)^d$ for every $\alpha\in\mathbb{R}$, $p\in(1,\infty)$, this is also true for $P(I-\Delta)=I-\Delta$ on $W_\sigma^{\alpha,p}(\mathbb{R}^d)$. In this case, by Lemma 7.4 in [53], the Rademacher interpolation spaces and the complex interpolation spaces coincide. This finally implies

$$\langle W_\sigma^{-\beta,q}(\mathbb{R}^d), W_\sigma^{2-\beta,q}(\mathbb{R}^d)\rangle_{\beta/2} = \big(W_\sigma^{-\beta,q}(\mathbb{R}^d), W_\sigma^{2-\beta,q}(\mathbb{R}^d)\big)_{\beta/2} = L_\sigma^q(\mathbb{R}^d)$$

which completes the proof. □

It remains to check the locally Lipschitz continuity of the quasilinear part, [TTQ4*], and the locally Lipschitz continuity of the semilinear part, [Q6*]. Let $y,z\in B_{q,p,\sigma}^{2-2/p}(\mathbb{R}^d)$ with $\|y\|_{B_{q,p,\sigma}^{2-2/p}(\mathbb{R}^d)}, \|z\|_{B_{q,p,\sigma}^{2-2/p}(\mathbb{R}^d)} \le n$ and $u\in W_\sigma^{2,q}(\mathbb{R}^d)$. Recall that

$$A(z)u = -\mu(|\tfrac{\nabla z+\nabla z^T}{2}|_2^2)\sum_{k=1}^d \partial_k^2 u_i - \mu'(|\tfrac{\nabla z+\nabla z^T}{2}|_2^2)\sum_{j,k,l=1}^d(\partial_l z_i+\partial_i z_l)(\partial_k z_j+\partial_j z_k)\partial_k\partial_l u_j.$$

With the Sobolev embedding $B_{q,p,\sigma}^{2-2/p}(\mathbb{R}^d)\hookrightarrow C_b^1(\mathbb{R}^d)^d$, we estimate

$$\begin{aligned}&\|PA(y)u-PA(z)u\|_{L_\sigma^q(\mathbb{R}^d)}\\ \le&\Big(\|\mu(|\tfrac{\nabla y+\nabla y^T}{2}|_2^2)-\mu(|\tfrac{\nabla z+\nabla z^T}{2}|_2^2)\|_{L^\infty(\mathbb{R}^d)}\\ &\quad+\|\mu'(|\tfrac{\nabla y+\nabla y^T}{2}|_2^2)-\mu'(|\tfrac{\nabla z+\nabla z^T}{2}|_2^2)\|_{L^\infty(\mathbb{R}^d)}\|\nabla y\|_{L^\infty(\mathbb{R}^d)^{d\times d}}^2\\ &\quad+\|\mu'(|\tfrac{\nabla y+\nabla y^T}{2}|_2^2)\|_{L^\infty(\mathbb{R}^d)}\|\nabla y\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\|\nabla y-\nabla z\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\Big)\|u\|_{W_\sigma^{2,q}(\mathbb{R}^d)}\\ \le&C\big(\|y\|_{L_\sigma^\infty(\mathbb{R}^d)},\|z\|_{L_\sigma^\infty(\mathbb{R}^d)},\|\nabla y\|_{L^\infty(\mathbb{R}^d)^{d\times d}},\|\nabla z\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\big)\|\nabla y-\nabla z\|_{L^\infty(\mathbb{R}^d)^{d\times d}}\|u\|_{W_\sigma^{2,q}(\mathbb{R}^d)}\\ \le&C(n)\|y-z\|_{B_{q,p,\sigma}^{2-2/p}(\mathbb{R}^d)}\|u\|_{W_\sigma^{2,q}(\mathbb{R}^d)}.\end{aligned}$$

Next, we estimate the semilinear part. Let $u,v\in W_\sigma^{2,q}(\mathbb{R}^d)$. Again using a Sobolev embedding, we get

$$\begin{aligned}
&\|P(u\cdot\nabla)u - P(v\cdot\nabla)v\|_{L^q_\sigma(\mathbb{R}^d)}\\
&\qquad\le \|((u-v)\cdot\nabla)u\|_{L^q(\mathbb{R}^d)^d} + \|(v\cdot\nabla)(u-v)\|_{L^q(\mathbb{R}^d)^d}\\
&\qquad\le \|u-v\|_{L^\infty(\mathbb{R}^d)^d}\|\nabla u\|_{L^q(\mathbb{R}^d)^{d\times d}} + \|v\|_{L^\infty(\mathbb{R}^d)^d}\|\nabla u - \nabla v\|_{L^q(\mathbb{R}^d)^{d\times d}}\\
&\qquad\le \big(\|u\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)} + \|v\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)}\big)\|u-v\|_{B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)}.
\end{aligned}$$

Applying Proposition 2.3.9 we see that this estimate is sufficient to fulfil [Q6*]. All in all, we proved that $(z,u)\mapsto PA(z)u$ satisfies [TTQ4*] and $u\mapsto P(u\cdot\nabla)u$ satisfies [Q6*]. Hence, Theorem 2.3.12 can be applied to the equation

$$\text{(QNS)}\begin{cases} du(t) &= [-PA(u(t))u(t) - P(u(t)\cdot\nabla)u(t) + Pf(t)]\,\mathrm{d}t + Pg(u,\nabla u)\,\mathrm{d}W(t),\\ u(0) &= u_0.\end{cases}$$

This yields a maximal unique local strong solution $(u,(\tau_n)_n,\tau)$ of (3.3) with

$$u\in L^p(0,\tau_n;W^{2,q}_\sigma(\mathbb{R}^d))\cap C(0,\tau_n;B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d))\cap W^{\theta,p}(0,\tau_n;W^{2-2\theta,q}_\sigma(\mathbb{R}^d))$$

almost surely for every $\theta\in(0,\frac{1}{2})$ and for every $n\in\mathbb{N}$. Moreover, τ satisfies

$$\mathbb{P}\big\{\tau<T,\ \|u\|_{L^p(0,\tau;W^{2,q}_\sigma(\mathbb{R}^d))}<\infty,\ u:[0,\tau)\to B^{2-2/p}_{q,p,\sigma}(\mathbb{R}^d)\text{ is uniformly continuous}\big\}=0.$$

Finally, we want to remark that we do not know, whether we could also use the setting [LQ] or not. We would need that the operators $A(z)$ have an $\mathcal{R}_q$-bounded H^∞-calculus and we failed to show this property. The setting [GM] is not applicable in this situation, due to the fact that here TR is given by $W^{1,2}_\sigma(\mathbb{R}^d)$ and hence a Sobolev embedding of the form $\mathrm{TR}\hookrightarrow W^{1,\infty}_\sigma(\mathbb{R}^d)$ cannot hold. Consequently, we cannot show the local Lipschitz continuity of $y\mapsto PA(y)u$ in the same way as above.

CHAPTER 4

A nonlinear stochastic Maxwell equation with retarded material law

In this chapter, we consider the semilinear stochastic Maxwell equation

$$\begin{cases} du(t) &= \big[Mu(t) - |u(t)|^q u(t) + (G * u)(t) + J(t)\big]\,\mathrm{d}t + \big[B(t,u(t)) + b(t)\big]\,\mathrm{d}W(t), \\ u(0) &= u_0 \end{cases} \tag{4.0.1}$$

in $L^2(D)^6 = L^2(D)^3 \times L^2(D)^3$ driven by a cylindrical Brownian motion $W(t)$ with the retarded material law

$$(G * u)(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s$$

and the perfect conductor boundary condition $u_1 \times \nu = 0$ on ∂D. Here, the Maxwell operator is given by

$$M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \operatorname{curl} u_2 \\ -\operatorname{curl} u_1 \end{pmatrix}$$

for $3d$ vector fields u_1 and u_2. We consider a bounded domain D or the full space $D = \mathbb{R}^3$, obviously in this case the boundary condition drops.

This equation is a model for a stochastic electromagnetic system in weakly-nonlinear chiral media and was derived in Chapter 2 in [88]. It has its origin in the deterministic Maxwell system

$$\begin{cases} \partial_t(Lu(t)) &= Mu(t) + J(t), \quad t \in [0,T] \\ u(0) &= u_0 \end{cases}$$

with constitutive relation

$$Lu(t,x) = \kappa(x)u(t,x) + \int_0^t K_1(t-s,x)u(s,x)\,\mathrm{d}s + \int_0^t K_2(t-s,x)|u(s,x)|^q u(s,x)\,\mathrm{d}s.$$

This material law consists of an instantaneous part κu with a hermitian, uniformly positive definite and uniformly bounded matrix $\kappa : D \to \mathbb{C}^{6\times 6}$, a linear dispersive part $K_1 * u$ and a nonlinear dispersive part $K_2 * |u|^q u$. This power-type nonlinearity is motivated by the Kerr-Debye model. Note that in applications one would either take the nonlinearity $|u_1|^q u_1$ or

$|u_2|^q u_2$ to model either nonlinear polarisation or magnetisation. We take the two quantities to study both phenomena at once. Using the product rule, we end up with

$$\begin{cases} \kappa u' &= Mu - K_1(0)u - K_2(0)|u|^q u - (\partial_t K_1) * u - (\partial_t K_2) * |u|^q u + J, \\ u(0) &= u_0. \end{cases}$$

At this point, we introduce additional simplifications. Usually, one demands $K_1(0) : D \to \mathbb{C}^{6\times 6}$ to be bounded and positive semi-definite and $K_2(0) : D \to \mathbb{C}^{6\times 6}$ to be bounded and uniformly positive definite. But for sake of simplicity, we choose $K_1(0) \equiv 0$ and $K_2(0) \equiv I$. However, the results are unchanged by this simplification and the proofs could be adjusted easily. Next, we assume that the term $(\partial_t K_2) * |u|^q u$ can be neglected. This is typical for a weakly nonlinear medium since one assumes that both the dispersion and the nonlinear effects are weak.So, the combination satisfies $(\partial_t K_2) * |u|^q u << K_2(0)|u|^q u$. Although, this simplification seems to be motivated physically, we want to point out that our method cannot deal with such a nonlinear term since it destroys the monotone structure of the equation. Moreover, we choose $\kappa = I$. We must admit that this simplification is also necessary at this point since our methods cannot deal with coefficients so far. The problems one has to overcome if $\kappa \neq I$ are discussed in section 4.4 in detail. Setting $G := -\partial_t K_1$, we get a deterministic version of equation (4.0.1).

In many applications, there is some uncertainty concerning the external sources or the precise behaviour of the medium itself. In these cases, it is useful to model u as a random variable on a probability space Ω and to impose a stochastic noise perturbation. Here, one distinguishes between the additive noise b perturbing J and the multiplicative noise $B(u)$ perturbing the medium. A linear stochastic version of (4.0.1) was already discussed in [88], chapter 12.

However, as far as we know, there are no known results about a nonlinear stochastic Maxwell equation. One reason might be that in the absence of Strichartz estimates for $(e^{tM})_{t\in\mathbb{R}}$, even local solvability is a tricky issue. Moreover, there is no compact embedding $D(M) \hookrightarrow L^p$ that helps to control the nonlinearity. Even the deterministic version of (4.0.1) has not been treated rigorously so far. In [88], the authors profess to prove well-posedness, but their argument ignores some severe complications. Since they claim to have better deterministic results than ours, we discuss their approach in section 4.4 in detail.

4.1. The Hodge-Laplacian on a bounded C^1-domain and its spectral multipliers

In this section, we provide the spectral theory basics for our well-posedness proofs. We discuss spectral multipliers of the Hodge-Laplacian Δ_H which is the componentwise Laplace operator on $L^p(D)^6$ with boundary conditions comparable to the boundary conditions contained in $D(M^2)$. The method of finite dimensional approximation with a sequence of orthogonal projections $(P_n)_n$ is well-known in the literature about stochastic and deterministic partial differential equations. However, it turned out that we not only need $P_n x \to x$

in $L^2(D)^6$ for $n \to \infty$ for all $x \in L^2(D)^6$, but also a comparable convergence property in $L^p(D)^6$, $p \neq 2$. A sequence of orthogonal projections on L^2 that also approximates the identity in L^p can only be found in very special situations, e.g. the Fourier cut-off on the torus. Hence, we have to construct another sequence of operators $(S_n)_n$ that has the necessary convergence property in $L^p(D)^6$, $p \in (1, \infty)$, and that is not far away from being an orthogonal projection, i.e. $R(S_{n-1}) \subset R(P_n) \subset R(S_n)$ for all $n \in \mathbb{N}$. In our construction, we make use of the spectral multiplier theorems of Kunstmann and Uhl (see [68]) that work on $L^p(D)^6$. They require that the semigroup generated by Δ_H satisfies generalised Gaussian bounds. Here, we benefit from a work of Mitrea and Monniaux who already showed a version of these tricky estimates in [78].

At first, we precisely introduce Δ_H. We consider the bilinear form

$$a(u, v) = \int_D (\operatorname{curl} u)(x) \cdot (\operatorname{curl} v)(x) \,\mathrm{d}x + \int_D (\operatorname{div} u)(x)(\operatorname{div} v)(x) \,\mathrm{d}x$$

with form domain $D(a)$ either given by $V^{(1)} := W^2(\operatorname{curl}, 0)(D) \cap W^2(\operatorname{div})(D)$ or by $V^{(2)} := W^2(\operatorname{curl})(D) \cap W^2(\operatorname{div}, 0)(D)$ equipped with the norm

$$\|u\|_{V^{(i)}}^2 := \|\operatorname{curl} u\|_{L^2(D)}^2 + \|\operatorname{div} u\|_{L^2(D)}^2 + \|u\|_{L^3(D)}^2$$

for $i = 1, 2$. In both cases the form a is bilinear, symmetric and bounded. Moreover, a is coercive in the sense that

$$a(u, u) = \|u\|_{V^{(i)}}^2 - \|u\|_{L^2(D)}^2$$

for all $u \in V^{(i)}$, $i = 1, 2$. Setting

$$D(A^{(1)}) = \{u \in V^{(1)} : \operatorname{curl}\operatorname{curl} u \in L^2(D)^3,\ \operatorname{div} u \in W_0^{1,2}(D)\},$$
$$D(A^{(2)}) = \{u \in V^{(2)} : \operatorname{curl}\operatorname{curl} u \in L^2(D)^3,\ \operatorname{curl} u \times \nu = 0 \text{ on } \partial D,\ \operatorname{div} u \in W^{1,2}(D)\},$$

it turns out that a with $D(a) = V^{(1)}$ is associated with the operator

$$A^{(1)} = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div} = -\Delta$$

on the domain $D(A^{(1)})$, whereas a with $D(a) = V^{(2)}$ is associated with the operator

$$A^{(2)} = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div} = -\Delta$$

on the domain $D(A^{(2)})$. To see this, use integration by parts for curl and div and exploit the respective boundary conditions. In a more general setting, this can be found in [78], (3.17) and (3.18). By the coercivity of the corresponding forms, the operators $I + A^{(i)}$, $i = 1, 2$, are strictly positive. Moreover, the symmetry implies that they are self-adjoint on $L^2(D)^3$ (see e.g. [81], Proposition 1.24.). Since the embeddings $V^{(i)} \hookrightarrow L^2(D)^3$ are compact (see [7], Theorem 2.8), the embeddings $D(A^{(i)}) \hookrightarrow L^2(D)^3$ are also compact for $i = 1, 2$.

To simplify the notation in what follows, we combine $A^{(1)}$ and $A^{(2)}$ to a self-adjoint operator $-\Delta_H(u_1, u_2) := (A^{(1)}u_1, A^{(2)}u_2)$ for $(u_1, u_2) \in D(A^{(1)}) \times D(A^{(2)}) =: D(\Delta_H)$. In particular, the embedding $D(\Delta_H) \hookrightarrow L^2(D)^6$ is compact and $I - \Delta_H$ is positive. Hence, there exists an orthonormal basis of eigenvectors $(h_j)_{j\in\mathbb{N}}$ to the positive eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of $I-\Delta_H$ with $\lambda_j\to\infty$ for $j\to\infty$.

The next proposition shows that the semigroups generated by $-A^{(i)}$ and Δ_H satisfy generalised Gaussian estimates. We add an additional spectral shift since some of the theorems we apply in what follows require strictly positive operators.

Proposition 4.1.1. *The semigroups generated by $-(I+A^{(1)})$, $-(I+A^{(2)})$ and $-I+\Delta_H$ satisfy generalised Gaussian $(2,q)$ estimates for every $q\in[2,\infty)$, i.e. for every $q\in[2,\infty)$ there exist $C,b>0$ such that*

$$\|\mathbf{1}_{B(x,t^{\frac{1}{2}})}e^{-t(I+A^{(i)})}\mathbf{1}_{B(y,t^{\frac{1}{2}})}\|_{\mathcal{B}(L^2(D)^3,L^q(D)^3)}\le Ct^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}e^{-\frac{b|x-y|^2}{t}},\quad i=1,2,$$
$$\|\mathbf{1}_{B(x,t^{\frac{1}{2}})}e^{-t(I-\Delta_H)}\mathbf{1}_{B(y,t^{\frac{1}{2}})}\|_{\mathcal{B}(L^2(D)^6,L^q(D)^6)}\le Ct^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}e^{-\frac{b|x-y|^2}{t}}$$

for all $t>0$ and all $x,y\in D$.

Proof. In [67], the authors argue on page 239 that the semigroups generated by $-A^{(1)}$ and $-A^{(2)}$ satisfy generalised Gaussian $(2,q)$-bounds for every $q\in[2,q_D)$. Here, $q_D\in[2,\infty)$ denotes the supremum over all indexes p for which the boundary value problems

$$\begin{cases}\Delta u=f \text{ in } D,\\ \operatorname{curl}u,\ \operatorname{curl}\operatorname{curl}u\in L^p(D)^3,\ \operatorname{div}(u)\in W^{1,p}(D),\\ u\cdot\nu=0,\ \operatorname{curl}(u)\times\nu=0 \text{ on } \partial D\end{cases}$$

and

$$\begin{cases}\Delta u=f \text{ in } D,\\ \operatorname{curl}u,\ \operatorname{curl}\operatorname{curl}u\in L^p(D)^3,\ \operatorname{div}(u)\in W_0^{1,p}(D),\\ u\times\nu=0 \text{ on } \partial D\end{cases}$$

have a unique solution for given $f\in L^p(D)^3$. This argument heavily makes use of iterative resolvent estimate for the Hodge-Laplacian (see [78], section 5 and 6). By [77], Theorem 1.2 and 1.3, we know that $q_D=\infty$ since D is a C^1-domain in $\mathbb{R}^3$. Gaussian estimates are preserved under negative spectral shifts in the generator of the semigroup. Hence, also the semigroups generated by $-(I+A^{(1)})$ and $-(I+A^{(2)})$ satisfy generalised Gaussian $(2,q)$-bounds. Last but not least, we remark that by $e^{-t(I-\Delta_H)}=\binom{e^{-t(I+A^{(1)})}}{e^{-t(I+A^{(2)})}}$ these estimates also hold true for the semigroup generated by $-(I-\Delta_H)$. □

For more details about these operators, we refer to [78], where they are discussed in a more general differential geometric setting.

We define spectral multipliers with the functional calculus for self-adjoint operators on a Hilbert space that have a basis of eigenvectors. Let $\Psi\in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\Psi)\subset[\frac{1}{2},2]$ and $\sum_{l\in\mathbb{Z}}\Psi(2^{-l}x)=1$ for all $x>0$. The operators $P_n:L^2(D)^6\to L^2(D)^6$ and $S_n:L^2(D)^6\to$

$L^2(D)^6$ are defined by

$$P_n(u)x = \mathbf{1}_{[0,2^n]}(I - \Delta_H)x = \sum_{k:\lambda_k \le 2^n} \langle x, h_k\rangle_{L^2(D)^6} h_k,$$
$$S_n(u)x = \sum_{l=-\infty}^{n} \Psi(2^{-l}(I - \Delta_H))x = \sum_{k=1}^{\infty} \sum_{l=-\infty}^{n} \Psi(2^{-l}\lambda_k)\langle x, h_k\rangle_{L^2(D)^6} h_k$$

for $x \in L^2(D)^6$ and $n \in \mathbb{N}$. Note that the last sum is in fact finite since only finitely many eigenvalues of $(I - \Delta_H)$ are smaller than 2^{n+1} and hence, $\Psi(2^{-l}\lambda_k) = 0$ for all but finitely many $l \in \mathbb{Z}$ and $k \in \mathbb{N}$. The next proposition summarises the most important properties of S_n and P_n as operators on $L^2(D)^6$.

Proposition 4.1.2. *P_n and S_n satisfy the following properties.*

i) *P_n is a projection, i.e. we have $P_n^2 = P_n$ for all $n \in \mathbb{N}$.*

ii) *The operators P_n, S_n are self-adjoint with $\|P_n\|_{\mathcal{B}(L^2(D)^6)} = \|S_n\|_{\mathcal{B}(L^2(D)^6)} = 1$ for every $n \in \mathbb{N}$.*

iii) *P_n and S_m commute for every $n, m \in \mathbb{N}$.*

iv) *The ranges of P_n and S_n are finite dimensional. Moreover, we have $R(P_n), R(S_n) \subset D(M)$ for every $n \in \mathbb{N}$.*

v) *We have $R(S_{n-1}) \subset R(P_n) \subset R(S_n)$, $S_nP_n = P_n$ and $P_nS_{n-1} = S_{n-1}$ for every $n \in \mathbb{N}$.*

vi) *We have $\lim_{n\to\infty} P_nx = \lim_{n\to\infty} S_nx = x$ for every $x \in L^2(D)^6$.*

Proof. For this proof, we just need the properties of the functional calculus for the self-adjoint and positive operator $I - \Delta_H$ on the Hilbert space $L^2(D)^6$. It remains to show *iv)* and *v)*. P_n and S_n have a finite dimensional range, since only finitely many eigenvalues of $I - \Delta_H$ are smaller than 2^{n+1}. Moreover, let $y = (y_1, y_2)$ be in the range of $\mathbf{1}_{[0,2^n]}(I - \Delta_H)$ and in the range of $\sum_{l=-\infty}^{n} \Psi(2^{-l}(I - \Delta_H))$. By functional calculus, we have $y_i \in D(\Delta_H)$ and particularly $y_i \in V^{(i)}$ for $i = 1, 2$. Thus, $\operatorname{curl} y_i \in L^2(D)^3$ for $i = 1, 2$ and $y_1 \times \nu = 0$ on ∂D, which shows $(y_1, y_2) \in D(M)$. Last but not least, we note that *v)* follows by

$$\sum_{l=-\infty}^{n} \Psi(2^{-l}\cdot) = \mathbf{1}_{(0,2^n)} + \psi(2^{-n}\cdot)\mathbf{1}_{[2^n,2^{n+1})}.$$

This closes the proof. $\square$

Moreover, the operators S_n have the following property that will be crucial in what follows.

Lemma 4.1.3. *For every $p \in (1, \infty)$, the operators S_n are bounded from $L^p(D)^6$ to $L^p(D)^6$ with a bound depending on p, but not on $n \in \mathbb{N}$. Moreover, we have $S_nf \to f$ in $L^p(D)^6$ as $n \to \infty$ for all $f \in L^p(D)^6$.*

Proof. The first statement follows from the spectral multiplier theorem 5.4 in [68] as a consequence of the generalised Gaussian bounds for the semigroup generated by $I - \Delta_H$. One could also argue with the more general Theorem 7.1 in [62]. The claimed convergence property is then a special case from [61], Theorem 4.1. To apply this theorem the 0-sectoriality of $I - \Delta_H$ and the boundedness of a Mikhlin functional calculus $\mathcal{M}^\alpha$ in $L^p(D)^6$ for some $\alpha > 0$ are needed. The first property is checked in [78], Theorem 6.1, whereas the second holds true with $\alpha > 4$ by the generalised Gaussian bounds (see [61], Lemma 6.1, (3)). $\square$

Next, we introduce two different Helmholtz projections on $L^2(D)^3$. The proof for the following statement is well-known and can be found amongst others in [59], section 4.1.3.

Proposition 4.1.4. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Given $u \in L^2(D)^3$, the following decompositions hold true.*

(1) *There exists a unique $p \in W_0^{1,2}(D)$ and $\widetilde{u} \in W^2(\operatorname{div})(D)$ with $\operatorname{div}\widetilde{u} = 0$ such that $u = \widetilde{u} + \nabla p$. The corresponding operator $P_H^{(1)} : L^2(D)^3 \to L^2(D)^3, u \mapsto \widetilde{u}$ is an orthogonal projection.*

(2) *There exists a unique $p \in W^{1,2}(D)$ with $\int_D p(x)\,\mathrm{d}x = 0$ and $\widetilde{u} \in W^2(\operatorname{div}, 0)(D)$ with $\operatorname{div}\widetilde{u} = 0$ such that $u = \widetilde{u} + \nabla p$. The corresponding operator $P_H^{(2)} : L^2(D)^3 \to L^2(D)^3, u \mapsto \widetilde{u}$ is an orthogonal projection.*

In particular, $P_H(u_1, u_2) := \big(P_H^{(1)} u_1, P_H^{(2)} u_2\big)$ for $u_1, u_2 \in L^2(D)^3$ defines an orthogonal projection on $L^2(D)^6$.

The Helmholtz projection P_H is closely related to both M and Δ_H. For example, due to $\operatorname{div} P_H^{(i)} = 0$, one calculates

$$\Delta_H P_H = \begin{pmatrix} -\operatorname{curl}\operatorname{curl} P_H^{(1)} + \operatorname{grad}\operatorname{div} P_H^{(1)} \\ -\operatorname{curl}\operatorname{curl} P_H^{(2)} + \operatorname{grad}\operatorname{div} P_H^{(2)} \end{pmatrix} = \begin{pmatrix} -\operatorname{curl}\operatorname{curl} P_H^{(1)} \\ -\operatorname{curl}\operatorname{curl} P_H^{(2)} \end{pmatrix} = M^2 P_H, \tag{4.1.1}$$

which implies that $M^2 = \Delta_H$ on $D(M) \cap P_H L^2(D)^6)$. We use this connection to show some powerful commutation identities.

Lemma 4.1.5. *We have $P_H \Delta_H = \Delta_H P_H$ on $D(\Delta_H)$, $M P_H = P_H M$ on $D(M)$ and $P_n M = M P_n$, $S_n M = M S_n$ on $D(M)$.*

Proof. From [78], section 3 or from [67], Lemma 5.4 we know that $P_H^{(i)} A^{(i)} = A^{(i)} P_H^{(i)}$ for $i = 1, 2$. This shows $P_H \Delta_H = \Delta_H P_H$ on $D(\Delta_H)$ and by the properties of the functional calculus, we also have $S_n P_H = P_H S_n$ and $P_n P_H = P_H P_n$.

For the second statement, we first show that $Mu = P_H M u$ for all $u = (u_1, u_2) \in D(M)$, i.e. we have to show $P_H^{(2)} \operatorname{curl} u_1 = \operatorname{curl} u_1$ and $P_H^{(1)} \operatorname{curl} u_2 = \operatorname{curl} u_2$. Due to $\operatorname{div}\operatorname{curl} u_i = 0$ for $i = 1, 2$, we just have to show $\operatorname{curl} u_1 \cdot \nu = 0$ on ∂D for $u_1 \in W^2(\operatorname{curl}, 0)(D)$. The definition of $u_1 \times \nu = 0$ from Definition 1.1.2 *a*) together with $\operatorname{curl}\nabla = 0$ and $\operatorname{div}\operatorname{curl} = 0$ yield

$$\int_D \nabla\phi(x) \cdot \operatorname{curl} u_1(x)\,\mathrm{d}x = \int_D \operatorname{curl}\nabla\phi(x) \cdot u_1(x)\,\mathrm{d}x = 0 = \int_D \phi(x)\operatorname{div}\operatorname{curl} u_1(x)\,\mathrm{d}x$$

for every $\phi \in C^\infty(\overline{D})$, which implies $\operatorname{curl} u_1 \cdot \nu = 0$ according to Definition 1.1.2 *b*).

As a consequence of $\operatorname{curl} \nabla = 0$, we know $M(I - P_H) = 0$. All in all, we get

$$MP_H - P_H M = MP_H - M = M(P_H - I) = 0.$$

Finally, by using $M^2 = \Delta_H$ on $D(M) \cap P_H(L^2(D)^6)$ together with $M = MP_H$, we get

$$\begin{aligned} MP_n &= MP_H \mathbf{1}_{[0,2^n]}(I - \Delta_H) = M\mathbf{1}_{[0,2^n]}(I - M^2)P_H \\ &= \mathbf{1}_{[0,2^n]}(I - M^2)P_H M P_H = \mathbf{1}_{[0,2^n]}(I - \Delta_H)M = P_n M \end{aligned}$$

on $D(M)$. For $S_n M = MS_n$, one may argue analogously. □

As a consequence, we get the following density relations.

Corollary 4.1.6. $\bigcup_{n=1}^\infty R(P_n)$ *is dense in* $D(M)$ *and in* $L^p(D)^6$ *for any* $p \in (1, \infty)$.

Proof. Let $u \in D(M)$. Using the commutation property of P_n by Lemma 4.1.5 and *vi*) from Proposition 4.1.2, we get

$$\|Mu - MP_n u\|_{L^2(D)^6} = \|Mu - P_n Mu\|_{L^2(D)^6} \xrightarrow{n\to\infty} 0.$$

If on the other hand $u \in L^p(D)^6$, we get $S_n u \to u$ in $L^p(D)^6$ as $n \to \infty$ from Lemma 4.1.3. This together with Proposition 4.1.2 *v*) proves the claimed result. □

We also consider the nonlinear Maxwell equation with retarded material law (4.0.1) on the full space $\mathbb{R}^3$ and hence, we need an analogue to the P_n and S_n in this different situation. However, in the absence of boundary conditions, things are well known and far more easy. We define

$$P_n f = S_n f := \mathcal{F}^{-1}\big(\xi \mapsto \mathbf{1}_{[-2^n,2^n]}(\xi_1)\mathbf{1}_{[-2^n,2^n]}(\xi_2)\mathbf{1}_{[-2^n,2^n]}(\xi_3)\hat{f}(\xi)\big)$$

for $f \in L^2(D)^6$. As M is a differential operator, it commutes with this frequency cut-off. Moreover, P_n and S_n satisfy the same properties as in Proposition 4.1.2 expect *iv*). Further, as a consequence of the boundedness of the Hilbert transform on $L^p(\mathbb{R}^3)$, they are bounded on $L^p(\mathbb{R}^3)^6$. This finally results in an analogue to Lemma 4.1.3 and Corollary 4.1.6. For details, we refer to [41], Chapter 6.1.3. We end this section with a lemma showing the mapping properties of the projection P_n as operator from $L^2(D)^6$ to $L^p(D)^6$ and as an operator from $L^2(\mathbb{R}^3)^6$ to $L^p(\mathbb{R}^3)^6$.

Lemma 4.1.7. *Let either* D *be a bounded* C^1*-domain or* $D = \mathbb{R}^3$*. For fixed* $n \in \mathbb{N}$, $p \in [2, \infty)$ *and* $q \in (1, 2]$ *the operator* $P_n : L^q(D)^6 \to L^2(D)^6$ *and* $P_n : L^2(D)^6 \to L^p(D)^6$ *is linear and bounded with norm depending on* n.

Proof. This statement is trivial if D is bounded, since all norms on a finite dimensional space are equivalent. In the other case, it is sufficient to show that $P_n : L^q(\mathbb{R}^3)^6 \to L^2(\mathbb{R}^3)^6$ is bounded. The rest follows by duality. The Hölder and the Hausdorff-Young inequality yield

$$\begin{aligned}\|P_n f\|_{L^2(\mathbb{R}^3)^6} &= \|\xi \mapsto \mathbf{1}_{[-2^n,2^n]}(\xi_1)\mathbf{1}_{[-2^n,2^n]}(\xi_2)\mathbf{1}_{[-2^n,2^n]}(\xi_3)\hat{f}(\xi)\|_{L^2(\mathbb{R}^3)^6} \lesssim_n \|\hat{f}\|_{L^{\frac{q}{q-1}}(\mathbb{R}^3)^6} \\ &\leq \|f\|_{L^q(\mathbb{R}^3)^6},\end{aligned}$$

which finishes the proof. □

4.2. Existence and uniqueness of a weak solution

In this section, we will prove existence and uniqueness of a weak solution in the sense of partial differential equations of

$$\text{(WSEE)}\begin{cases} du(t) &= \big[Mu(t) - |u(t)|^q u(t) + (G * u)(t) + J(t)\big]\,\mathrm{d}t + B(t,u(t))\,\mathrm{d}W(t), \\ u(0) &= u_0 \end{cases}$$

for any $q > 0$. Here, we use $F(u) := |u|^q u$ and $(G * u)(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s$. This is done in two steps. First, we use a version of the Galerkin method from Röcker and Prévot (see [85]) to solve (4.0.1) in the special case $G \equiv 0$ and make use of the monotone structure of our nonlinearity. As this approach is well-known, we just discuss the different steps and concentrate on how to deal with the additional term Mu, despite the fact that $u \notin D(M)$. Afterwards, we include the retarded material law with Banach's fixed point theorem. Before we start, we explain our solution concept.

Definition 4.2.1. *We say that an adapted process $u : \Omega \times [0,T] \to L^2(D)^6$ with*

$$u \in L^2(\Omega; C(0,T; L^2(D)))^6 \cap L^{q+2}(\Omega \times [0,T] \times D)^6$$

is a weak solution of (WSEE) *if*

$$\begin{aligned}\langle u(t) - u_0, \phi\rangle_{L^2(D)^6} = &\int_0^t \big\langle -|u(s)|^q u(s) + J(s) + (G*u)(s), \phi\big\rangle_{L^2(D)^6}\,\mathrm{d}s \\ &+ \int_0^t -\big\langle u(s), M\phi\big\rangle_{L^2(D)^6}\,\mathrm{d}s + \int_0^t \big\langle B(s,u(s)), \phi\ \mathrm{d}W(s)\big\rangle_{L^2(D)^6}\end{aligned}$$

holds almost surely for all $t \in [0,T]$ and for all $\phi \in D(M) \cap L^{q+2}(D)^6$. Moreover, we call a weak solution u unique if for any other weak solution v, there exists $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that $u(\omega,t) = v(\omega,t)$ for all $\omega \in \Omega \setminus N$ and all $t \in [0,T]$.

We make the following assumptions.

[W1] Let $D \subset \mathbb{R}^3$ be a bounded C^1-domain or $D = \mathbb{R}^3$.

[W2] The initial value $u_0 : \Omega \to L^2(D)^6$ is strongly $\mathcal{F}_0$-measurable.

[W3] Let $G : \Omega \times [0,T] \to \mathcal{B}(L^2(D)^6)$ such that $x \mapsto G(t)x$ is for all $x \in L^2(D)^6$ strongly measurable and $\mathbb{F}$-adapted. Moreover, we assume

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \int_0^T \|G(\omega,t)\|_{\mathcal{B}(L^2(D)^6)}\,\mathrm{d}t < \infty.$$

[W4] Let U be a separable Hilbert space and W a U-cylindrical Brownian motion. Moreover, let $B : \Omega \times [0,T] \times L^2(D)^6 \to L^2(U, L^2(D)^6)$ be strongly measurable such that $\omega \mapsto B(\omega,t,u)$ is strongly $\mathcal{F}_t$-measurable for almost all $t \in [0,T]$ and all $u \in L^2(D)^6$. Furthermore, there exists a constant $C > 0$ such that B is of linear growth, i.e.

$$\|B(t,u)\|_{L^2(U;L^2(D)^6)} \leq C\big(1 + \|u\|_{L^2(D)^6}\big)$$

and Lipschitz continuous, i.e.

$$\|B(t,u) - B(t,v)\|_{L^2(U;L^2(D)^6)} \leq C\|u - v\|_{L^2(D)^6}$$

almost surely for almost all $t \in [0,T]$ and all $u, v \in L^2(D)^6$.

[W5] $J : \Omega \times [0,T] \to L^2(D)^6$ is strongly measurable, $\mathbb{F}$-adapted and we assume $J \in L^2(\Omega \times [0,T] \times D)^6$.

First, we need an Itô formula that is appropriate to deal with weak solutions. Our result is a version of [85], Theorem 4.2.5 that additionally allows a term with the skew-adjoint operator M in spite of the fact that our weak solution is not in $D(M)$. Our proof relies on a more straightforward regularisation technique than the original one using the spectral multipliers S_n we defined in section 4.1.

Lemma 4.2.2. *Let $X_0 \in L^2(\Omega \times D)^6$ and $Y \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6 + L^2(\Omega \times [0,T] \times D)^6$ and $Z \in L^2(\Omega \times [0,T]; L^2(U; L^2(D)^6))$ be $\mathbb{F}$-adapted. If*

$$\begin{aligned}\langle X(t), \phi\rangle_{L^2(D)^6} =& \langle X_0, \phi\rangle_{L^2(D)^6} + \int_0^t -\langle X(s), M\phi\rangle_{L^2(D)^6} + \langle Y(s), \phi\rangle_{L^2(D)^6}\,\mathrm{d}s \\ &+ \int_0^t \big\langle Z(s), \phi \ \mathrm{d}W(s)\big\rangle_{L^2(D)^6}\end{aligned} \tag{4.2.1}$$

almost surely for all $t \in [0,T]$ and all $\phi \in D(M) \cap L^{q+2}(D)^6$ and if we additionally have the regularity $X \in L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega \times [0,T] \times D)^6$, then the Itô formula

$$\begin{aligned}&\|X(t_2)\|^2_{L^2(D)^6} - \|X(t_1)\|^2_{L^2(D)^6} \\ &= \int_{t_1}^{t_2} 2\operatorname{Re}\langle X(s), Y(s)\rangle_{L^2(D)^6} + \|Z(s)\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s + 2\int_{t_1}^{t_2} \operatorname{Re}\big\langle X(s), Z(s)\,\mathrm{d}W(s)\big\rangle_{L^2(D)^6}\end{aligned} \tag{4.2.2}$$

holds almost surely for all $0 \leq t_1 \leq t_2 \leq T$. Moreover, we get the additional regularity $X \in L^2(\Omega; C(0,T; L^2(D)))^6$.

Proof. Let $0 \le t_1 \le t_2 \le T$. We plug in $\phi = S_n\Phi$ for $\Phi \in C_c^\infty(D)^6$ into the equation

$$\begin{aligned}\langle X(t_2),\phi\rangle_{L^2(D)^6} =&\langle X(t_1),\phi\rangle_{L^2(D)^6} + \int_{t_1}^{t_2} -\langle X(s), M\phi\rangle_{L^2(D)^6} + \langle Y(s),\phi\rangle_{L^2(D)^6}\,\mathrm{d}s \\ &+ \int_{t_1}^{t_2} \langle Z(s),\phi\ \mathrm{d}W(s)\rangle_{L^2(D)^6}.\end{aligned}$$

Note that by Lemma 4.1.5, S_n and M commute. Moreover, $R(S_n) \subset D(M)$. Consequently, since S_n is self-adjoint and Φ is chosen arbitrarily, we obtain

$$S_nX(t_2) - S_nX(t_1) = \int_{t_1}^{t_2} MS_nX(s) + S_nY(s)\,\mathrm{d}s + \int_{t_1}^{t_2} S_nZ(s)\,\mathrm{d}W(s)$$

almost surely. Thus, we can apply the Itô formula for Hilbert space valued processes (see e.g. [24], Theorem 4.32) to the functional $u \mapsto \|u\|^2_{L^2(D)^6}$ to get

$$\begin{aligned}&\|S_nX(t_2)\|^2_{L^2(D)^6} - \|S_nX(t_1)\|^2_{L^2(D)^6} \\ &\quad= \int_{t_1}^{t_2} 2\,\mathrm{Re}\langle S_nX(s), MS_nX(s)\rangle_{L^2(D)^6} + 2\,\mathrm{Re}\langle S_nX(s), S_nY(s)\rangle_{L^2(D)^6} \\ &\quad\quad + \|S_nZ(s)\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s + 2\int_{t_1}^{t_2} \mathrm{Re}\left\langle S_nX(s), S_nZ(s)\,\mathrm{d}W(s)\right\rangle_{L^2(D)^6}\end{aligned}$$

almost surely. Since M is skew-adjoint, the first term on the right hand side drops. In all the other terms we can pass to the limit. Thereby, we need that $S_nu \to u$ for $n \to \infty$ in $L^{q+2}(D)^6$ and $L^{\frac{q+2}{q+1}}(D)^6$ (see Lemma 4.1.3). This finally yields

$$\begin{aligned}&\|X(t_2)\|^2_{L^2(D)^6} - \|X(t_1)\|^2_{L^2(D)^6} \\ &= \int_{t_1}^{t_2} 2\,\mathrm{Re}\langle X(s), Y(s)\rangle_{L^2(D)^6} + \|Z(s)\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s + 2\int_{t_1}^{t_2} \mathrm{Re}\left\langle X(s), Z(s)\,\mathrm{d}W(s)\right\rangle_{L^2(D)^6}\end{aligned} \tag{4.2.3}$$

almost surely. Together with $X \in L^{q+2}(\Omega\times[0,T]\times D)^6 \cap L^2(\Omega\times[0,T]\times D)^6$, this identity implies $u \in L^2(\Omega; L^\infty(0,T;L^2(D)))^6$ by a classical Gronwall argument.

It remains to show the almost sure continuity in time. From (4.2.1) we know that there exists $\widetilde{\Omega} \subset \Omega$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that $t \mapsto \langle X(t),\phi\rangle_{L^2(D)^6}$ is continuous on $\widetilde{\Omega}$ for every $\phi \in D(M)\cap L^{q+2}(D)^6$. In particular, $t \mapsto X(t) \in L^2(D)^6$ is weakly continuous on $\widetilde{\Omega}$. On the other hand, by (4.2.2), there exists another set $\widetilde{\Omega}_2 \subset \widetilde{\Omega}$ such that $t \mapsto \|u(t)\|^2_{L^2}$ is continuous on $\widetilde{\Omega}_2$. Let $t \in [0,T]$ and $(t_n)_n \subset [0,T]$ with $t_n \to t$ as $n \to \infty$. As argued before, we both have $X(t_n) \to X(t)$ weakly in $L^2(D)^6$ on $\widetilde{\Omega}_2$ and $\|X(t_n)\|_{L^2(D)^6} \to \|X(t)\|_{L^2(D)^6}$ on $\widetilde{\Omega}_2$ as $n \to \infty$. This implies

$$\begin{aligned}\|X(t_n) - X(t)\|^2_{L^2(D)^6} &= \|X(t_n)\|^2_{L^2(D)^6} + \|X(t)\|^2_{L^2(D)^6} - 2\,\mathrm{Re}\left\langle X(t_n), X(t)\right\rangle_{L^2(D)^6} \\ &\xrightarrow{n\to\infty} \|X(t)\|^2_{L^2(D)^6} + \|X(t)\|^2_{L^2(D)^6} - 2\,\mathrm{Re}\left\langle X(t), X(t)\right\rangle_{L^2(D)^6} = 0\end{aligned}$$

on $\widetilde{\Omega}_2$, which proves the desired continuity. □

At first, we assume $G \equiv 0$ and solve (WSEE) without retarded material law. The reason for this simplification is that we make use of the monotone structure of the rest of the equation.

We start with a Galerkin approximation with the spectral projection P_n we defined in section 2. We investigate the truncated equation

$$\begin{cases} du_n(t) &= \big[P_n M u_n(t) - P_n F(u_n(t)) + P_n J(t)\big]\,\mathrm{d}t + P_n B(t, u_n(t))\,\mathrm{d}W(t), \\ u_n(0) &= P_n u_0 \end{cases} \tag{4.2.4}$$

in the range of P_n. To solve this equation, we derive some properties of the nonlinearity $u \mapsto F(u) = |u|^q u$ as a mapping from $L^{q+2}(D)^6$ to $L^{\frac{q+2}{q+1}}(D)^6$ with $q > 0$ and some properties of its truncation $u \mapsto F_n(u) = P_n|u|^q u$ as mapping on the range of P_n. We start with the monotonicity.

Lemma 4.2.3. *The mapping $F : L^{q+2}(D)^6 \to L^{\frac{q+2}{q+1}}(D)^6,\ u \mapsto |u|^q u$ satisfies the estimate*

$$\int_D \mathrm{Re}\,\big\langle F(v)(x) - F(u)(x), u(x) - v(x)\big\rangle_{\mathbb{C}^6}\,\mathrm{d}x \le -C\|u - v\|^{q+2}_{L^{q+2}(D)^6} \tag{4.2.5}$$

for some constant $C > 0$ and for all $u, v \in L^{q+2}(D)^6$.

Proof. Clearly, $\|F(u)\|_{L^{\frac{q+2}{q+1}}(D)^6} = \|u\|_{L^{q+2}(D)^6}$ and therefore F has the claimed mapping properties. The estimate (4.2.5) is a direct consequence of Lemma 4.4 in [29]. □

Since we often use Itô's formula, we need to know the differentiability properties of F.

Lemma 4.2.4. *The nonlinearity $F : L^{q+2}(D)^6 \to L^{\frac{q+2}{q+1}}(D)^6$, $u \mapsto |u|^q u$ is real continuously Fréchet differentiable with $\mathrm{Re}\langle F'(u)v, v\rangle_{L^2(D)^6} \ge 0$ and*

$$|F'(u)v(x)| \lesssim |u(x)|^q |v(x)|$$

for all $u, v \in L^{q+2}(D)^6$ and $x \in D$. In particular, it is locally Lipschitz continuous, i.e.

$$\|F(u) - F(v)\|_{L^{\frac{q+2}{q+1}}(D)^6} \lesssim \big(\|u\|^q_{L^{q+2}(D)^6} + \|v\|^q_{L^{q+2}(D)^6}\big)\|u - v\|_{L^{q+2}(D)^6}.$$

Moreover, if $q \in (1, \infty)$, it is twice real continuously differentiable with

$$F''(u)(v, v)(x) \lesssim |u(x)|^{q-1}|v(x)|^2$$

for all $u, v \in L^{q+2}(D)^6$ and all $x \in D$.

Proof. It is well-known that $F : L^{q+2}(D)^6 \to L^{\frac{q+2}{q+1}}(D)^6$ is real continuously Fréchet differentiable with

$$F'(u)v = q|u|^{q-2}\,\mathrm{Re}\langle u, v\rangle_{\mathbb{C}^6}\, u + |u|^q v$$

for every $u, v \in L^{q+2}(D)^6$ (see e.g. given [48], Corollary 9.3). Consequently, we also have

$$\mathrm{Re}\,\big\langle F'(u)v, v\big\rangle_{L^2(D)^6} = \int_D q|u(x)|^{q-2}\big(\mathrm{Re}\langle u(x), v(x)\rangle_{\mathbb{C}^6}\big)^2 + |u(x)|^q|v(x)|^2\,\mathrm{d}x \ge 0.$$

Moreover, we estimate

$$F'(u)v(x) \le C|u(x)|^q|v(x)|$$

for some $C > 0$. For the second derivative, we start with a formal calculation for F'' and get

$$\begin{aligned}F''(u)(v,w) =& q|u|^{q-2}\Big((q-2)|u|^{-2}\operatorname{Re}\langle u,w\rangle_{L^2(D)^6}\operatorname{Re}\langle u,v\rangle_{L^2(D)^6}u + \operatorname{Re}\langle w,v\rangle_{L^2(D)^6}u \\ &+ \operatorname{Re}\langle u,w\rangle_{L^2(D)^6}v + \operatorname{Re}\langle u,v\rangle_{L^2(D)^6}w\Big).\end{aligned}$$

For sake of readability, we do not rigorously show that $F : L^{q+2}(D)^6 \to L^{\frac{q+2}{q+1}}(D)^6$ is twice Fréchet differentiable with this derivative. However, to give an impression how to prove this, we check that last term in $F'(u)v$, namely

$$u \mapsto [v \mapsto |u|^q v] : L^{q+2}(D)^6 \to \mathcal{B}(L^{q+2}(D), L^{\frac{q+2}{q+1}}(D)^6),$$

is Fréchet differentiable with derivative $G(u)(v,w) = q|u|^{q-2}\operatorname{Re}\langle u,w\rangle_{\mathbb{C}^6}v$. Let $u,v,w \in L^{q+2}(D)^6$ with $v,w \neq 0$. Then Hölder's inequality together with the mean value theorem yields

$$\begin{aligned}&\big\||u|^q v - |u+w|^q v - G(u)(v,w)\big\|_{L^{\frac{q+2}{q+1}}(D)^6} \\ &\leq \big\||u|^q - |u+w|^q - q|u|^{q-2}\operatorname{Re}\langle u,w\rangle_{\mathbb{C}^6}\big\|_{L^{\frac{q+2}{q}}(D)^6}\|v\|_{L^{q+2}(D)^6} \\ &\lesssim \Big\|\int_0^1 \operatorname{Re}\langle |u+\theta w|^{q-2}(u+\theta w) - |u|^{q-2}u, w\rangle_{\mathbb{C}^6}\,\mathrm{d}\theta\Big\|_{L^{\frac{q+2}{q}}(D)^6}\|v\|_{L^{q+2}(D)^6} \\ &\leq \int_0^1 \big\||u+\theta w|^{q-2}(u+\theta w) - |u|^{q-2}u\big\|_{L^{\frac{q+2}{q-1}}(D)^6}\,\mathrm{d}\theta\,\|w\|_{L^{q+2}(D)^6}\|v\|_{L^{q+2}(D)^6}.\end{aligned}$$

Hence, we showed

$$\begin{aligned}&\|w\|^{-1}_{L^{q+2}(D)^6}\big\|v \mapsto |u|^q v - |u+w|^q v - G(u)(v,w)\big\|_{\mathcal{B}(L^{q+2}(D)^6, L^{\frac{q+2}{q+1}}(D)^6)} \\ &\qquad\lesssim \int_0^1 \big\||u+\theta w|^{q-2}(u+\theta w) - |u|^{q-2}u\big\|_{L^{\frac{q+2}{q-1}}(D)^6}\,\mathrm{d}\theta \qquad (4.2.6)\end{aligned}$$

for all $u, w \in L^{q+2}(D)^6$ with $w \neq 0$.

It remains to prove that this quantity tends to 0 as $w \to 0$ in $L^{q+2}(D)^6$. Let $(w_n)_n$ be a sequence in $L^{q+2}(D)^6$ with $w_n \to 0$ as $n \to \infty$ and let $(w_{n_k})_k$ be an arbitrary subsequence. Hence, there exists another subsequence, still denoted with $(w_{n_k})_k$ such that $w_{n_k} \to 0$ almost everywhere for $k \to \infty$ and such that $|w_{n_k}| \leq g$ for some $g \in L^{q+2}(D)^6$. We also have

$$|u+\theta w_{n_k}|^{q-2}(u+\theta w_{n_k}) - |u|^{q-2}u \to 0$$

almost everywhere as $k \to \infty$. Together with the bound

$$\Big||u+\theta w_{n_k}|^{q-2}(u+\theta w_{n_k}) - |u|^{q-2}u\Big| \leq |u|^{q-1} + |w_{n_k}|^{q-1} \leq |u|^{q-1} + g^{q-1},$$

for $\theta \in [0,1]$ and the fact that $u \in L^{q+2}(D)^6$, we get

$$\int_0^1 \big\||u+\theta w_{n_k}|^{q-2}(u+\theta w_{n_k}) - |u|^{q-2}u\big\|_{L^{\frac{q+2}{q-1}}(D)^6}\,\mathrm{d}\theta \to 0$$

as $k \to \infty$. All in all, this shows that the left hand side of (4.2.6) tends to 0 as $w \to 0$ and we established the Fréchet differentiability of $u \mapsto [v \mapsto |u|^q v]$ with derivative $G(u)$. The claimed estimate for $F''(u)(v,v)(x)$ is immediate. This closes the proof. $\square$

Now, we can come back to our truncated equation

$$\begin{cases} du_n(t) &= \left[P_n M u_n(t) - P_n F(u_n(t)) + P_n J(t)\right] \mathrm{d}t + P_n B(t, u_n(t))\, \mathrm{d}W(t), \\ u_n(0) &= P_n u_0. \end{cases}$$

This is a stochastic ordinary differential equation in $R(P_n) \subset L^2(D)^6$ with a locally Lipschitz nonlinearity which can be seen by Lemma 4.1.7 and Lemma 4.2.4. Indeed, let $u, v \in P_n L^2(D)^6$. Then, we can estimate

$$\begin{aligned} \|P_n F(u) - P_n F(v)\|_{L^2(D)^6} &= \|P_n F(P_n u) - P_n F(P_n v)\|_{L^2(D)^6} \\ &\lesssim_n \|F(P_n u) - F(P_n v)\|_{L^{\frac{q+2}{q+1}}(D)^6} \\ &\lesssim_n \left(\|P_n u\|^q_{L^{q+2}(D)^6} + \|P_n v\|^q_{L^{q+2}(D)^6}\right) \|P_n u - P_n v\|_{L^{q+2}(D)^6} \\ &\lesssim_n \left(\|u\|^q_{L^2(D)^6} + \|v\|^q_{L^2(D)^6}\right) \|u - v\|_{L^2(D)^6}. \end{aligned}$$

Hence, there exists an increasing sequence of stopping times $(\tau_n^{(m)})_{m\in\mathbb{N}}$ with $0 < \tau_n^{(m)} \le T$ almost surely, a stopping time $\tau_n = \lim_{m\to\infty} \tau_n^{(m)}$ and an adapted process $u_n : \Omega \times [0, \tau) \to R(P_n)$ with continuous paths that solves (4.2.4). Moreover, we have the blow-up alternative

$$\mathbb{P}\Big\{\tau_n < T, \sup_{t\in[0,\tau)} \|u_n(t)\|_{L^2(D)^6} < \infty\Big\} = 0. \tag{4.2.7}$$

The next result shows $\tau_n = T$ almost surely for every $n \in \mathbb{N}$ and a uniform estimate for u_n.

Proposition 4.2.5. *We have $\tau_n = T$ almost surely for every $n \in \mathbb{N}$ and u_n additionally satisfies*

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(D)^6} + \sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T \int_D |u_n(t,x)|^{q+2} \,\mathrm{d}x\, \mathrm{d}t < \infty.$$

Proof. Lemma 4.2.2 applied to u_n, the self-adjointness of P_n and $P_n^2 = P_n$ yield

$$\begin{aligned} &\|u_n(s)\|^2_{L^2(D)^6} - \|P_n u_0\|^2_{L^2(D)^6} \\ &= 2 \int_0^s \operatorname{Re} \left\langle u_n(r), -|u_n(r)|^q u_n(r) + J(r) \right\rangle_{L^2(D)^6} \mathrm{d}r \\ &\quad + 2 \int_0^s \operatorname{Re} \left\langle u_n(r), B(s, u_n(r))\, \mathrm{d}W(r) \right\rangle_{L^2(D)^6} + \int_0^s \|P_n B(r, u_n(r))\|^2_{\mathcal{L}^2(U; L^2(D)^6)} \mathrm{d}r \end{aligned}$$

almost surely for every $s \in [0, \tau_n^{(m)}]$. This expression simplifies to

$$\begin{aligned} &\|u_n(s)\|^2_{L^2(D)^6} + 2 \int_0^s \int_D |u_n(s,x)|^{q+2} \,\mathrm{d}x\, \mathrm{d}t - \|P_n u_0\|^2_{L^2(D)^6} \\ &\le \int_0^s 2 \operatorname{Re} \left\langle u_n(r), J(r) \right\rangle_{L^2(D)^6} + \|B(r, u_n(r))\|^2_{\mathcal{L}^2(U; L^2(D)^6)} \mathrm{d}r \\ &\quad + 2 \int_0^s \operatorname{Re} \left\langle u_n(r), B(s, u_n(r))\, \mathrm{d}W(r) \right\rangle_{L^2(D)^6} \end{aligned} \tag{4.2.8}$$

almost surely for every $s \in [0, \tau_n^{(m)}]$. Since the second term on the left hand side is positive, we can drop it for a moment. We first take the supremum over $[0, \tau_n^{(m)} \wedge t]$ for $t \in [0, T]$ and afterwards the expectation value and estimate the remaining quantities term by term. We start with the deterministic part using [W4] and [W5].

$$\begin{aligned}\mathbb{E} &\sup_{s\in[0,\tau_n^{(m)}\wedge t]} \Big| \int_0^s 2\operatorname{Re}\langle u_n(r), J(r)\rangle_{L^2(D)^6} + \|B(r,u_n(r))\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}r\Big| \\ &\qquad\qquad\lesssim 1 + \int_0^t \mathbb{E}\mathbf{1}_{s\le\tau_n^{(m)}} \|u_n(s)\|_{L^2(D)^6}\|J(s)\|_{L^2(D)^6} + \|u_n(s)\|^2_{L^2(D)^6}\,\mathrm{d}s \\ &\qquad\qquad\lesssim 1 + \int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\tau_n^{(m)}]} \|u_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s + \|J\|^2_{L^2(\Omega\times[0,T]\times D)^6}.\end{aligned}$$

The stochastic part can be estimated with the Burkholder-Davies-Gundy inequality.

$$\begin{aligned}\mathbb{E} &\sup_{s\in[0,t\wedge\tau_n^{(m)}]} \Big|\int_0^s \operatorname{Re}\langle u_n(s), B(s,u_n(s))\ \mathrm{d}W(s)\rangle_{L^2(D)^6}\Big| \\ &\le C\mathbb{E}\Big(\int_0^{\tau_n^{(m)}\wedge t} \big|\langle u_n(s), B(s,u_n(s)\rangle_{L^2(U,L^2(D)^6)}\big|^2\,\mathrm{d}s\Big)^{1/2} \\ &\le \widetilde{C}\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^{(m)}]} \|u_n(s)\|_{L^2(D)^6}\Big(1+\int_0^{t\wedge\tau_n^{(m)}} \|u(t)\|^2_{L^2(D)^6}\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big) \\ &\le \frac{1}{4}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^{(m)}]} \|u_n(s)\|^2_{L^2(D)^6} + \widetilde{C}^2\Big(1+\int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\tau_n^{(m)}]} \|u(r)\|^2_{L^2(D)^6}\,\mathrm{d}s\Big).\end{aligned}$$

Thereby, we used $ab \le \frac{1}{4}a^2 + b^2$ for all $a, b \ge 0$ in the last step. Putting these estimates together, we get

$$\begin{aligned}\mathbb{E} &\sup_{s\in[0,t\wedge\tau_n^{(m)}]} \|u_n(s)\|^2_{L^2(D)^6} \\ &\qquad\lesssim 1 + \|u_0\|^2_{L^2(D)^6} + \|J\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \int_0^t \mathbb{E}\sup_{r\in[0,s\wedge\tau_n^{(m)}]} \|u_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s\end{aligned}$$

for all $t\in[0,T]$. Consequently, Gronwall yields

$$\mathbb{E}\sup_{t\in[0,\tau_n^{(m)}]} \|u_n(t)\|^2_{L^2(D)^6} \lesssim 1 + \|J\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|u_0\|^2_{L^2(D)^6}$$

with an estimate that is independent of $n\in\mathbb{N}$. Now, we can go back to (4.2.8) and deal with the term we dropped at first. The estimate of $\mathbb{E}\sup_{t\in[0,\tau_n^{(m)}]} \|u_n(t)\|^2_{L^2(D)}$ implies

$$\mathbb{E}\int_0^{\tau_n^{(m)}} \int_D |u_n(s,x)|^{q+2}\,\mathrm{d}x\,\mathrm{d}t \lesssim 1 + \|J\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|u_0\|^2_{L^2(D)^6}$$

We use Fatou's Lemma to pass to the limit $m\to\infty$ in these estimates. Note that one can interchange sup and lim inf in an upper estimate, since lim inf can be written in the form sup inf and supremums can be interchanged, whereas sup inf $\le$ inf sup. Hence, we have

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_n)} \|u_n(t)\|^2_{L^2(D)^6} &+ \mathbb{E}\int_0^{\tau_n}\int_D |u_n(s,x)|^{q+2}\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim 1 + \|J\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|u_0\|^2_{L^2(D)^6}. \qquad (4.2.9)\end{aligned}$$

Consequently, we also have $\tau_n = T$ almost surely. Indeed, there exists $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that $\Omega \setminus \big(N \cup \{\tau_n = T\}\big)$ can be decomposed into disjoint sets

$$\Big\{\tau_n < T,\ \sup_{t\in[0,\tau_n)} \|u_n(t)\|^2_{L^2(D)^6} < \infty\Big\},\ \Big\{\tau_n < T,\ \sup_{t\in[0,\tau_n)} \|u_n(t)\|^2_{L^2(D)^6} = \infty\Big\}.$$

The first set has measure zero by (4.2.7) and the second one has measure zero since (4.2.9) implies $\sup_{t\in[0,\tau_n)} \|u_n(t)\|_{L^2(D)^6} < \infty$ almost surely. Pathwise uniform continuity on $[0,T]$ follows from Lemma 4.2.2. This closes the proof. □

In Proposition 4.2.5, we derived uniform estimates for u_n. As a consequence, Lemma 4.2.5 yields the uniform boundedness of $F(u_n)$ in $L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$. Thus, by Banach-Alaoglu, there exists $u \in L^2(\Omega; L^\infty(0,T; L^2(D)^6))$, $N \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$, $\widetilde{B} \in L^2(\Omega \times [0,T]; L^2(U; L^2(D)))^6$ and subsequences, still indexed with n such that

a) $u_n \to u$ as $n \to \infty$ in the weak* sense in $L^2(\Omega; L^\infty(0,T; L^2(D)))^6$,

b) $u_n \to u$ as $n \to \infty$ in the weak sense in $L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega \times [0,T] \times D)^6$,

c) $F(u_n) \to N$ as $n \to \infty$ in the weak sense in $L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$,

d) $B(\cdot, u_n) \to \widetilde{B}$ as $n \to \infty$ in the weak sense in $L^2(\Omega \times [0,T]; L^2(U; L^2(D)))^6$.

Since u_n is for every $n \in \mathbb{N}$ an adapted solution of the ordinary stochastic differential equation (4.2.4) in $P_n L^2(D)^6$, we have $u_n \in L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$. Consequently, since $L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$ is a closed subspace of $L^2(\Omega \times [0,T] \times D)^6$, it is also weakly closed. This implies $u \in L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$, which means that u is also adapted.

Testing (4.2.4) with $\rho\phi$ for arbitrary $\rho \in L^{q+2}(\Omega \times [0,T])$ and $\phi \in \bigcup_{n=1}^\infty R(P_n)$, the symmetry of P_n and the skew-symmetry of M yield

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle u_n(t) - u_0, \phi\rangle_{L^2(D)^6} \rho(t)\,\mathrm{d}t \\
&\quad= \mathbb{E}\int_0^T \int_0^t -\langle u_n(s), MP_n\phi\rangle_{L^2(D)^6} + \langle -F(u_n(s)) + J(s), P_n\phi\rangle_{L^2(D)^6}\,\mathrm{d}s\rho(t)\,\mathrm{d}t \\
&\qquad+ \mathbb{E}\int_0^T \int_0^t \langle B(s, u_n(s)), P_n\phi\rangle_{L^2(D)^6}\,\mathrm{d}W(s)\rho(t)\,\mathrm{d}t.
\end{aligned}$$

By weak convergence, we can pass to the limit and obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle u(t) - u_0, \phi\rangle_{L^2(D)^6} \rho(t)\,\mathrm{d}t \\
&\qquad= \mathbb{E}\int_0^T \int_0^t -\langle u(s), M\phi\rangle_{L^2(D)^6} + \langle -N(s) + J(s), \phi\rangle_{L^2(D)^6}\,\mathrm{d}s\rho(t)\,\mathrm{d}t \\
&\qquad\quad+ \mathbb{E}\int_0^T \int_0^t \langle \widetilde{B}(s), \phi\rangle_{L^2(D)^6}\,\mathrm{d}W(s)\rho(t)\,\mathrm{d}t.
\end{aligned}$$

Thereby, we used $P_n\phi = \phi$ for n large enough since $\phi \in \bigcup_{n=1}^\infty R(P_n)$ and that linear and bounded operators are also weakly continuous. Since ρ was chosen arbitrarily, we finally get

$$\begin{aligned}\langle u(t)-u_0,\phi\rangle_{L^2(D)^6}=\int_0^t -\langle u(s),M\phi\rangle_{L^2(D)^6}+\langle -N(s)+J(s),\phi\rangle_{L^2(D)^6}\,\mathrm{d}s\\ +\int_0^t\langle\widetilde{B}(s),\phi\rangle_{L^2(D)^6}\,\mathrm{d}W(s)\end{aligned}\tag{4.2.10}$$

almost surely for every $t\in[0,T]$. Hence, by density (see Lemma 4.1.6), this holds true for every $\phi\in D(M)\cap L^{q+2}(D)^6$. To show that u is a weak solution of (WSEE) with $G\equiv 0$, it remains to prove $N=F(u)$ and $\widetilde{B}=B(\cdot,u)$. This will be done by adapting a standard argument for stochastic evolution equations with monotone nonlinearities (see [85], proof of Theorem 4.2.4, page 86) to our situation. To do this, we just need an Itô formula for $\mathbb{E}e^{-Kt}\|u(t)\|^2_{L^2(D)^6}$, although $Mu(t)\notin L^2(D)^6$. Thereby, we use the abbreviation $\langle\cdot,\cdot\rangle_{L^p}$ for the duality $\big(L^p(D)^6,L^{\frac{p}{p-1}}(D)^6\big)$.

Lemma 4.2.6. *For any $K>0$, u_n and u satisfy the Itô formulae*

$$\begin{aligned}&\mathbb{E}\ e^{-Kt}\|u(t)\|^2_{L^2(D)^6}-\mathbb{E}\|u_0\|^2_{L^2(D)^6}\\ &=\mathbb{E}\int_0^t 2e^{-Ks}\operatorname{Re}\big\langle u(s),-N(s)+J(s)\big\rangle_{L^{q+2}}+e^{-Ks}\|\widetilde{B}(s)\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s\\ &\quad-\mathbb{E}\int_0^t Ke^{-Ks}\|u(s)\|^2_{L^2(D)^6}\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\ e^{-Kt}\|u_n(t)\|^2_{L^2(D)^6}-\mathbb{E}\|P_nu_0\|^2_{L^2(D)^6}\\ &=\mathbb{E}\int_0^t 2e^{-Ks}\operatorname{Re}\big\langle u_n(s),-F(u_n(s))+J(s)\big\rangle_{L^2(D)^6}+e^{-Ks}\|P_nB(s,u_n(s))\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s\\ &\quad-\mathbb{E}\int_0^t Ke^{-Ks}\|u_n(s)\|^2_{L^2(D)^6}\,\mathrm{d}s\end{aligned}$$

almost surely for all $t\in[0,T]$.

Proof. These formulae are immediate by Lemma 4.2.2, the Itô product rule and the fact that the expectation of a stochastic integral is zero. □

Proposition 4.2.7. *If we assume* [W1] – [W5]*, the equation* (WSEE) *with $G\equiv 0$ has a unique weak solution u in the sense of Definition 4.2.1.*

Proof. Throughout this proof, we write $H=L^2(D)^6$ to simplify the notation. We need to show $N=F(u)$ in $L^{\frac{q+2}{q+1}}(\Omega\times[0,T]\times D)^6$ and $\widetilde{B}=B(\cdot,u)$ in $L^2(\Omega\times[0,T];L^2(U;H))$. Let $\psi\in L^\infty(0,T)$ be nonnegative. Then, weak convergence yields

$$\begin{aligned}&\mathbb{E}\int_0^T\psi(t)\|u(t)\|^2_{L^2(D)^6}\,\mathrm{d}t\\ &\quad=\lim_{n\to\infty}\mathbb{E}\int_0^T\operatorname{Re}\big\langle\psi(t)^{\frac12}u(t),\psi(t)^{\frac12}u_n(t)\big\rangle^2_{L^2(D)^6}\,\mathrm{d}t\\ &\quad\le\Big(\mathbb{E}\int_0^T\psi(t)\|u(t)\|^2_{L^2(D)^6}\,\mathrm{d}t\Big)^{\frac12}\liminf_{n\to\infty}\Big(\mathbb{E}\int_0^T\psi(t)\|u_n(t)\|^2_{L^2(D)^6}\,\mathrm{d}t\Big)^{\frac12},\end{aligned}\tag{4.2.11}$$

which implies

$$\mathbb{E}\int_0^T \psi(t)\|u(t)\|^2_{L^2(D)^6}\,\mathrm{d}t \le \liminf_{n\to\infty}\mathbb{E}\int_0^T \psi(t)\|u_n(t)\|^2_{L^2(D)^6}\,\mathrm{d}t.$$

Let $K>0$ and $\phi \in L^2(\Omega;L^\infty(0,T;L^2(D)^6))\cap L^{q+2}(\Omega\times[0,T]\times D)^6$. Then, the Itô formula from Lemma 4.2.6 gives

$$\begin{aligned}
&\mathbb{E}e^{-Kt}\|u_n(t)\|_H^2 - \mathbb{E}\|P_n u_0\|_H^2\\
=&\mathbb{E}\int_0^t 2e^{-Ks}\operatorname{Re}\big\langle u_n(s), -F(u_n(s)) + J(s)\big\rangle_H + e^{-Ks}\|P_n B(s,u(s))\|^2_{L^2(U;H)}\,\mathrm{d}s\\
&-\mathbb{E}\int_0^t Ke^{-Ks}\|u_n(s)\|_H^2\,\mathrm{d}s\\
=&\mathbb{E}\int_0^t e^{-Ks}\Big(2\operatorname{Re}\big\langle u_n(s)-\phi(s), F(\phi(s)) - F(u_n(s))\big\rangle_H + \|B(s,u_n(s)) - B(s,\phi(s))\|^2_{L^2(U,H)}\\
&- K\|u_n(s)-\phi(s)\|_H^2 + 2\operatorname{Re}\big\langle \phi(s), F(\phi(s)) - F(u_n(s))\big\rangle_H - 2\operatorname{Re}\big\langle u_n(s), F(\phi(s)) - J(s)\big\rangle_H\\
&- \|B(s,\phi(s))\|^2_{L^2(U;H)} + 2\operatorname{Re}\big\langle B(s,u_n(s)), B(s,\phi(s))\big\rangle_{L^2(U;H)} + K\|\phi(s)\|_H^2\\
&- 2K\operatorname{Re}\big\langle u_n(s),\phi(s)\big\rangle_H\Big)\,\mathrm{d}s.
\end{aligned}$$

Next, by [W4], we can choose K large enough such that

$$\|B(s,u_n(s)) - B(s,\phi(s))\|^2_{L^2(U,H)} - K\|u_n(s)-\phi(s)\|_H^2 \le 0.$$

Together with Lemma 4.2.3, we get

$$\begin{aligned}
&\mathbb{E}e^{-Kt}\|u_n(t)\|_H^2 - \mathbb{E}\|P_n u_0\|_H^2\\
\le&\mathbb{E}\int_0^t e^{-Ks}\Big(2\operatorname{Re}\big\langle \phi(s), F(\phi(s)) - F(u_n(s))\big\rangle_H - 2\operatorname{Re}\big\langle u_n(s), F(\phi(s)) - J(s)\big\rangle_H\\
&- \|B(\phi)\|^2_{L^2(U;H)} + 2\operatorname{Re}\big\langle B(u_n), B(\phi)\big\rangle_{L^2(U;H)} + K\|\phi(s)\|_H^2 - 2K\operatorname{Re}\big\langle u_n(s),\phi(s)\big\rangle_H\Big)\,\mathrm{d}s.
\end{aligned}$$

The limit $\lim_{n\to\infty}\|P_n u(0)\|_H = \|u(0)\|_H$ (see Lemma 4.1.2) and (4.2.11) combined with Fubini and the weak convergence $u_n \to u$ in $L^{q+2}(\Omega\times[0,T]\times D)^6$ as $n\to\infty$ yield

$$\begin{aligned}
&\mathbb{E}\int_0^T \psi(t)\big(e^{-Kt}\|u(t)\|_H^2 - \|u_0\|_H^2\big)\,\mathrm{d}t\\
&\quad\le\liminf_{n\to\infty}\ \mathbb{E}\int_0^T \psi(t)\big(e^{-Kt}\|u_n(t)\|_H^2 - \|P_n u_0\|_H^2\big)\,\mathrm{d}t\\
&\quad\le\liminf_{n\to\infty}\ \mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\Big(2\operatorname{Re}\big\langle \phi(s), F(\phi(s)) - F(u_n(s))\big\rangle_H\\
&\qquad\qquad - 2\operatorname{Re}\big\langle u_n(s), F(\phi(s)) - J(s)\big\rangle_H - \|B(\phi)\|^2_{L^2(U;H)} + 2\operatorname{Re}\big\langle B(u_n), B(\phi)\big\rangle_{L^2(U;H)}\\
&\qquad\qquad + K\|\phi(s)\|_H^2 - 2K\operatorname{Re}\big\langle u_n(s),\phi(s)\big\rangle_H\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\le \mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\Big(2\operatorname{Re}\big\langle \phi(s), F(\phi(s)) - N(s)\big\rangle_{L^{q+2}}\\
&\qquad\qquad - 2\operatorname{Re}\big\langle u(s), F(\phi(s)) - J(s)\big\rangle_{L^{q+2}} - \|B(s,\phi(s))\|^2_{L^2(U;H)}\\
&\qquad\qquad + 2\operatorname{Re}\big\langle \widetilde{B}(s), B(s,\phi(s))\big\rangle_{L^2(U;H)} + K\|\phi(s)\|_H^2 - 2K\operatorname{Re}\big\langle u(s),\phi(s)\big\rangle_H\Big)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

On the other hand, by Lemma 4.2.6, we also have

$$\begin{aligned}\mathbb{E}\int_0^T &\psi(t)\big(e^{-Kt}\|u(t)\|_H^2 - \|u_0\|_H^2\big)\,\mathrm{d}t\\ =&\mathbb{E}\int_0^T \psi(t)\int_0^t 2e^{-Ks}\operatorname{Re}\big\langle u(s), -N(s)+J(s)\big\rangle_{L^{q+2}} + \|\widetilde{B}(s)\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s\\ &\qquad - \int_0^t Ke^{-Ks}\|u(s)\|^2_{L^2(D)^6}\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Inserting this equality in the left hand-side of the estimate from above, we end up with

$$\begin{aligned}0\le \mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\Big(&2\operatorname{Re}\big\langle \phi(s)-u(s), F(\phi(s))-N(s)\big\rangle_{L^{q+2}}\\ &- \|B(s,\phi(s))-\widetilde{B}(s)\|^2_{L^2(U;H)} + K\|\phi(s)-u(s)\|_H^2\Big)\,\mathrm{d}s\,\mathrm{d}t \quad (4.2.12)\end{aligned}$$

and we still have the freedom to choose a nonnegative $\psi \in L^\infty(0,T)$ and an arbitrary $\phi \in L^2(\Omega; L^\infty(0,T;L^2(D)^6)) \cap L^{q+2}(\Omega\times[0,T]\times D)^6$. At first, we choose $\phi = u$, which implies

$$0 \le -\mathbb{E}\int_0^T \phi(t)\int_0^t \|B(s,u(s))-\widetilde{B}(s)\|^2_{L^2(U;H)}\,\mathrm{d}s\,\mathrm{d}t$$

which can only hold true if $\widetilde{B}(s) = B(s,u(s))$ almost surely for almost every $s\in[0,T]$ with equality in $H = L^2(D)^6$. Next, we plug $\phi = u - \varepsilon\widetilde{\phi}v$ with $\varepsilon > 0$, $\widetilde{\phi}\in L^\infty(\Omega\times[0,T])$ and $v \in L^{q+2}(D)^6 \cap L^2(D)^6$ into (4.2.12). Then dividing both sides by ε yields

$$\begin{aligned}0\ge \mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\Big(&2\widetilde{\phi}(s)\operatorname{Re}\big\langle v, F(u(s)-\varepsilon\widetilde{\phi}(s)v)-N(s)\big\rangle_{L^{q+2}}\\ &+\varepsilon^{-1}\|B(s,u(s)-\varepsilon\widetilde{\phi}(s)v)-B(s,u(s))\|^2_{L^2(U;H)} - K\varepsilon\widetilde{\phi}(s)\|v\|_H^2\Big)\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

Next, we let $\varepsilon\to 0$. Clearly, we have

$$\varepsilon^{-1}\|B(s,u(s)-\varepsilon\widetilde{\phi}(s)v)-B(s,u(s))\|^2_{L^2(U;H)} \le \varepsilon C\widetilde{\phi}(s)\|v\|_H^2 \xrightarrow{\varepsilon\to 0} 0$$

by [W4] and as a consequence, the expression

$$\mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\Big(\varepsilon^{-1}\|B(s,u(s)-\varepsilon\widetilde{\phi}(s)v)-B(s,u(s))\|^2_{L^2(U;H)} - K\varepsilon\widetilde{\phi}\|v\|_H^2\Big)\,\mathrm{d}s\,\mathrm{d}t$$

converges to 0 as $\varepsilon\to 0$ with the dominated convergence theorem. It remains to investigate the first term. Lemma 4.2.4 yields

$$\begin{aligned}\|F(u(s)&-\varepsilon\widetilde{\phi}(s)v)-F(u)\|_{L^{\frac{q+2}{q+1}}(D)^6}\\ &\le C\varepsilon\widetilde{\phi}(s)\Big(2\|u(s)\|^q_{L^{q+2}(D)^6} + \varepsilon^q\widetilde{\phi}(s)^q\|v\|^q_{L^{q+2}(D)^6}\Big)\|v\|_{L^{q+2}(D)^6},\end{aligned}$$

which implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T &\psi(t)\int_0^t e^{-Ks}\widetilde{\phi}(s)\operatorname{Re}\big\langle v, F(u(s)-\varepsilon\widetilde{\phi}(s)v)\big\rangle_{L^{q+2}}\,\mathrm{d}s\,\mathrm{d}t\\ &=\mathbb{E}\int_0^T \psi(t)\int_0^t e^{-Ks}\widetilde{\phi}(s)\operatorname{Re}\big\langle v, F(u(s))\big\rangle_{L^{q+2}}\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

and consequently

$$0 \geq \mathbb{E} \int_0^T \psi(t) \int_0^t 2e^{-Ks}\widetilde{\phi}(s) \operatorname{Re} \left\langle v, F(u(s)) - N(s) \right\rangle_{L^{q+2}} \mathrm{d}s \, \mathrm{d}t.$$

Instead of $\widetilde{\phi}$, we could insert $-\widetilde{\phi}$. Hence, the above inequality is only true for arbitrary $\widetilde{\phi}$ if we actually have equality. With Fubini, we can rewrite it as

$$0 = \int_0^T \psi(t) \mathbb{E} \int_0^t 2e^{-Ks}\widetilde{\phi}(s) \operatorname{Re} \left\langle v, F(u(s)) - N(s) \right\rangle_{L^{q+2}} \mathrm{d}s \, \mathrm{d}t$$

for all nonnegative $\psi \in L^\infty(0,T)$, which particularly implies

$$0 = \mathbb{E} \int_0^T 2e^{-Ks}\widetilde{\phi}(s) \operatorname{Re} \left\langle v, F(u(s)) - N(s) \right\rangle_H \mathrm{d}s$$

for all $\widetilde{\phi} \in L^\infty(\Omega \times [0,T])$ and all $v \in L^{q+2}(D)^6$. Hence, we can conclude $F(u(s,x)) = N(s,x)$ almost surely for almost every $t \in [0,T]$ and $x \in D$. This shows that u is a weak solution.

It remains to prove uniqueness. Let $u, v \in L^2(\Omega; C(0,T; L^2(D)))^6 \cap L^{q+2}(\Omega \times [0,T] \times D)^6$ be weak solutions of (WSEE) to the initial value u_0. Since both u and v are weak solutions, we have

$$\begin{aligned}\langle u(t) - v(t), \phi \rangle_H = & \int_0^t -\left\langle |u(s)|^q u(s) - |v(s)|^q v(s), \phi \right\rangle_H \mathrm{d}s \\ & + \int_0^t -\left\langle u(s) - v(s), M\phi \right\rangle_H \mathrm{d}s \\ & + \int_0^t \left\langle B(s,u(s)) - B(s,v(s)), \phi \ \mathrm{d}W(s) \right\rangle_H\end{aligned}$$

almost surely for every $t \in [0,T]$ and for every $\phi \in D(M) \cap L^{q+2}(D)^6$. Applying Lemma 4.2.2 yields

$$\begin{aligned}\mathbb{E}\|u(t) - v(t)\|_H^2 = & 2 \int_0^t \mathbb{E} \left\langle -u(s) + v(s), |u(s)|^q u(s) - |v(s)|^q v(s) \right\rangle_H \\ & + \mathbb{E} \left\| B(s,u(s)) - B(s,v(s)) \right\|_{L^2(U,H)}^2 \mathrm{d}s\end{aligned}$$

almost surely for every $t \in [0,T]$. We know from Lemma 4.2.3 that the first term on the right hand-side is negative and can be dropped in an upper estimate. By the Lipschitz continuity of B, we end up with

$$\mathbb{E}\|u(t) - v(t)\|_H^2 \lesssim \int_0^t \mathbb{E}\|u(s) - v(s)\|_H^2 \, \mathrm{d}s$$

for every $t \in [0,T]$. Hence, we get $\mathbb{E}\|u(t) - v(t)\|_H^2 = 0$ for every $t \in [0,T]$ by Gronwall's Lemma. This proves the claimed uniqueness. $\square$

Finally, we add a nontrivial retarded material law G by a perturbation argument.

Theorem 4.2.8. *If we assume* [W1] – [W5], *the equation* (WSEE) *has a unique weak solution u in the sense of Definition 4.2.1.*

Proof. Let $T_0 \in (0,T]$. By Proposition 4.2.7 the equation

$$\begin{cases} du(t) &= \big[Mu(t) - F(u(t)) + (G * v)(t) + J(t)\big]\,\mathrm{d}t + B(t,u(t))\,\mathrm{d}W(t), \\ u(0) &= u_0 \end{cases}$$

has for $v \in L^2(\Omega; C(0,T_0; L^2(D)^6))$ a unique solution $u =: Kv \in L^2(\Omega; C(0,T_0; L^2(D)^6))$. Indeed, by [W3],

$$t \mapsto \int_0^t G(t-s)u(s)\,\mathrm{d}s \in L^2(\Omega \times [0,T] \times D)^6$$

and thus $G * v$ satisfies [W5]. In the following, we will show that K is a contraction on $X := L^2(\Omega; C(0,T_0; L^2(D)^6))$ if we choose $T_0 > 0$ small enough. For given $v, w \in X$, we calculate with Lemma 4.2.2 that

$$\begin{aligned} &\|Kv(s) - Kw(s)\|^2_{L^2(D)^6} \\ &\quad= \int_0^s 2\,\mathrm{Re}\,\big\langle Kv(r) - Kw(r), F(Kw(r)) - F(Kv(r)) + (G*(v-w))(r)\big\rangle_{L^2(D)^6} \\ &\qquad + \|B(r,Kv(r)) - B(r,Kw(r))\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}r \\ &\qquad + 2\int_0^t \mathrm{Re}\,\big\langle Kv(r) - Kw(r), B(r,Kv(r)) - B(r,Kw(r))\,\mathrm{d}W(r)\big\rangle_{L^2(D)^6}. \end{aligned}$$

In the following estimates, we take the supremum over $[0,t]$ for $t \in [0,T_0]$ and afterwards the expectation. We now estimate the occurring quantities term by term.

$$\begin{aligned} &\int_0^s \mathrm{Re}\big\langle Kv(r) - Kw(r), (G*(v-w))(r)\big\rangle_{L^2(D)^6}\,\mathrm{d}s \\ &\quad\le \int_0^s \frac{1}{2}\|Kv(r) - Kw(r)\|^2_{L^2(D)^6} + \frac{1}{2}\Big\|\int_0^r G(r-\lambda)(v(\lambda) - w(\lambda))\,\mathrm{d}\lambda\Big\|^2_{L^2(D)^6}\,\mathrm{d}r \\ &\quad\le \int_0^s \frac{1}{2}\sup_{\lambda\in[0,r]}\|Kv(\lambda) - Kw(\lambda)\|^2_{L^2(D)^6}\,\mathrm{d}r \\ &\qquad + \frac{T_0\|G\|^2_{L^1(0,T;\mathcal{B}(L^2(D)^6))}}{2}\sup_{\lambda\in[0,T_0]}\|v(\lambda) - w(\lambda)\|^2_{L^2(D)^6} \end{aligned}$$

for all $s \in [0,T_0]$. We can drop the contribution of F, as

$$\big\langle Kv(r) - Kw(r), F(Kw(s)) - F(Kv(s))\big\rangle_{L^2(D)^6} \le -\alpha\|Kv(r) - Kw(r)\|^{q+2}_{L^{q+2}(D)^6}$$

for all $s \in [0,T_0]$ and some $\alpha > 0$ by Lemma 4.2.3. Moreover, by [W4], we have

$$\int_0^t \|B(s,Kv(s)) - B(s,Kw(s))\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}s \le C^2\int_0^t \sup_{r\in[0,s]}\|Kv(r) - Kw(r)\|^2_{L^2(D)^6}\,\mathrm{d}s.$$

Last but not least, the Burkholder-Davies-Gundy inequality and [W4] yield

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,t]}\Big|\int_0^s \mathrm{Re}\,\big\langle Kv(r) - Kw(r), (B(r,Kv(r)) - B(r,Kw(r)))\,\mathrm{d}W(r)\big\rangle_{L^2(D)^6}\Big| \\ &\le C\mathbb{E}\Big(\int_0^t \big\|\big\langle Kv(r) - Kw(r), B(r,Kv(r)) - B(r,Kw(r))\big\rangle_{L^2(D)^6}\big\|^2_{L^2(U)}\,\mathrm{d}r\Big)^{1/2} \\ &\le C\mathbb{E}\sup_{s\in[0,t]}\|Kv(s) - Kw(s)\|_{L^2(D)^6}\Big(\int_0^t \|B(r,Kv(r)) - B(r,Kw(r))\|^2_{L^2(U;L^2(D)^6)}\,\mathrm{d}r\Big)^{1/2} \\ &\le \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]}\|Kv(s) - Kw(s)\|^2_{L^2(D)^6} + \widetilde{C}^2\int_0^t \mathbb{E}\sup_{r\in[0,s]}\|Kv(r) - Kw(r)\|^2_{L^2(D)^6}\,\mathrm{d}s. \end{aligned}$$

All in all, we derived

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|Kv(s)-Kw(s)\|^2_{L^2(D)^6}\\
&\qquad\leq \int_0^t 2(1+2\widetilde{C}^2+C^2)\mathbb{E}\sup_{\lambda\in[0,r]}\|Kv(\lambda)-Kw(\lambda)\|^2_{L^2(D)^6}\,\mathrm{d}r\\
&\qquad\quad + 2T_0\|G\|^2_{L^\infty(\Omega;L^1(0,T;\mathcal{B}(L^2(D)^6)))}\mathbb{E}\sup_{\lambda\in[0,T_0]}\|v(\lambda)-w(\lambda)\|^2_{L^2(D)^6}
\end{aligned}$$

for every $t\in[0,T_0]$. Hence, Gronwall implies

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|Kv(s)-Kw(s)\|^2_{L^2(D)^6}\\
&\leq 2T_0\|G\|^2_{L^\infty(\Omega;L^1(0,T;\mathcal{B}(L^2(D)^6)))}\Big(\mathbb{E}\sup_{\lambda\in[0,T_0]}\|v(\lambda)-w(\lambda)\|^2_{L^2(D)^6}\Big)e^{2(1+2\widetilde{C}^2+C^2)T_0}.
\end{aligned}$$

Now, we choose $T_0>0$ small enough to ensure that K is a contraction. Then, by Banach's fixed point theorem, there exists a $u_1\in L^2(\Omega;C(0,T_0;L^2(D)^6))$ solving (WSEE) on $[0,T_0]$ and from $Ku_1=u_1$ we deduce $u_1\in L^{q+2}(\Omega\times[0,T_0]\times D)^6$. Clearly, by continuity in time, we have $u_1(T_0)\in L^2(\Omega\times D)^6$ and $\omega\mapsto u_1(\omega,T_0)$ is strongly $\mathcal{F}_{T_0}$-measurable.

Next, given $v\in L^2(\Omega;C(T_0,2T_0;L^2(D)^6))$, we consider the equation

$$\begin{cases} dy=\big[My-F(y)+\int_0^{T_0}G(\cdot-s)u_1(s)\,\mathrm{d}s+\int_{T_0}^{\cdot}G(\cdot-s)v(s)\,\mathrm{d}s+J\big]\,\mathrm{d}t+B(\cdot,y)dW,\\ y(T_0)=u_1(T_0)\end{cases}$$

for $t\in[T_0,2T_0]$. By Proposition 4.2.7, we have a unique solution $y:=K_2v$. This defines an operator $K_2: L^2(\Omega;C(T_0,2T_0;L^2(D)^6))\to L^2(\Omega;C(T_0,2T_0;L^2(D)^6))$. However, K_2v-Kw_2 can be estimated in the very same way as above since the additional term $\int_0^{T_0}G(\cdot-s)u_1(s)\,\mathrm{d}s$ vanishes in this difference. As a consequence, K_2 is a contraction on $L^2(\Omega;C(T_0,2T_0;L^2(D)^6))$ and has a unique fixed point u_2. Inductively, we construct $u_n\in L^2(\Omega;C((n-1)T_0,nT_0;L^2(D)^6))$ solving

$$\begin{cases} dy(t)=\big[My(t)-F(y(t))+\int_{(n-1)T_0}^{t}G(t-s)y(s)\,\mathrm{d}s+f(t)\big]\,\mathrm{d}t+B(t,y)\,\mathrm{d}W(t),\\ y((n-1)T_0)=u_{n-1}((n-1)T_0)\end{cases}$$

with $f(t)=J(t)+\sum_{k=1}^{n-1}\int_{(k-1)T_0}^{kT_0}G(t-s)u_k(s)\,\mathrm{d}s$ and stop when $nT_0\geq T$. Finally, the process $u:=\sum_{n=1}^{\lfloor\frac{T}{T_0}\rfloor+1}u_n\mathbf{1}_{[(n-1)T_0,nT_0)}$ solves (WSEE) on $[0,T]$ and satisfies

$$u\in L^2(\Omega;C(0,T;L^2(D)^6))\cap L^{q+2}(\Omega\times[0,T]\times D)^6.$$

By construction, u is unique on every interval $[(n-1)T_0,nT_0)$, which implies uniqueness on $[0,T]$. □

4.3. Existence and uniqueness of a strong solution

In this section, we will discuss the following stochastic Maxwell equation

$$\text{(MSEE)}\begin{cases} du &=\big[Mu-|u|^q u+G*u+J\big]\,\mathrm{d}t+\sum_{j=1}^N\big[b_j+iB_ju\big]\,\mathrm{d}\beta_j,\\ u(0)&=u_0\end{cases}$$

on $L^2(D)^6$ with a monotone polynomial nonlinearity and a retarded material law. We derive existence and uniqueness of a strong solution that satisfies

$$Mu \in L^2(\Omega; L^\infty(0,T; L^2(D)^6)) + L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6.$$

As in the previous section, we start with $G \equiv 0$ and we add a nontrivial G at the very end. In a deterministic setting, one would try to estimate $\|u'(t)\|^2_{L^2(D)^6}$ and then use the equation

$$u'(t) = Mu(t) - |u(t)|^q u(t) + J(t) + \int_0^t G(t-s)u(s)\,\mathrm{d}s$$

to control Mu. However, solutions of stochastic differential equations are not differentiable in time. Our first idea to overcome this problem was to derive an estimate for

$$\|Mu(t) - |u(t)|^q u(t) + J(t)\|^2_{L^2(D)^6}$$

with Gronwall's Lemma, but we failed since the Itô formula for this quantity contains the term

$$\sum_{j=1}^N \|D_{vv}(|v|^q v)(u(t))\big(B(t,u(t)), B(t,u(t))\big)\|^2_{L^2(D)^6},$$

which we could not estimate properly. Here $B(t,u(t))$ is the abbreviation of the noise term. Hence, we had to choose the special noise $\sum_{j=1}^N \big(b_j(t) + iB_j u(t)\big)\,\mathrm{d}\beta_j(t)$ and use the rescaling transformation

$$y(t) = u(t)e^{-i\sum_{j=1}^N B_j\beta_j(t)}$$

to get rid of the multiplicative noise in the same way as Barbu and Röckner in [13] and [14] (see also [15] and [16]). The difference to our approach is that the authors have natural a priori estimates before transforming the equation and they solely transform to solve the new equation with purely deterministic techniques. Moreover, they only use multiplicative noise. We use the transformation to get better a priori estimates and we consider an equation that also has additive noise.

As in the previous section, we write $F(u) := |u|^q u$. Before we start, we explain our solution concept.

Definition 4.3.1. *A weak solution u is called strong solution of* (MSEE) *if it additionally satisfies*

$$Mu \in L^2(\Omega; L^\infty(0,T; L^2(D)^6)) + L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6.$$

Note that in case of a bounded domain $D \subset \mathbb{R}^3$, this integrability property reduces to $Mu \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$. We make the following assumptions.

[M1] Let $q \in (1,2]$ and $D \subset \mathbb{R}^3$ be a bounded C^1- domain or $D = \mathbb{R}^3$.

[M2] Let u_0 be strongly $\mathcal{F}_0$-measurable with

$$\mathbb{E}\|Mu_0\|^2_{L^2(D)^6} + \mathbb{E}\|u_0\|^{2(q+1)}_{L^{2(q+1)}(D)^6} < \infty.$$

[M3] Let $G \in L^\infty(\Omega; W^{1,1}(0,T; \mathcal{B}(L^2(D)^6)))$, such that $\omega \mapsto G(t)x$ is for all $x \in L^2(D)^6$ and all $t \in [0,T]$ strongly $\mathcal{F}_t$-measurable.

[M4] Let $J \in L^2(\Omega; W^{1,2}(0,T; L^2(D)^6))$ be $\mathbb{F}$-adapted.

[M5] Let $b_j \in L^2(\Omega; W^{1,2}(0,T; L^2(D)^6))$, $j = 1, \dots, N$, be $\mathbb{F}$-adapted. If $q \in (1,2)$, we additionally assume $b_j \in L^{\frac{2(q+2)}{2-q}}(\Omega \times [0,T] \times D)^6$ and $b_j \in L^\infty(\Omega \times [0,T] \times D)^6$ if $q = 2$. Moreover, if $q = 2$, we assume that there exists $\widetilde{n} \in \mathbb{N}$ such that we have

$$P_n\big(b_j e^{-i\sum_{l=1}^N B_l\beta_l}\big) = b_j e^{-i\sum_{l=1}^N B_l\beta_l}$$

for all $n > \widetilde{n}$. Here, we use the operator P_n defined in Section 4.1.

[M6] Let $B_j \in W^{1,\infty}(D)$ for $j = 1, \dots, N$.

At first, we assume $G \equiv 0$ and solve (MSEE) without retarded material law as in the last section. This term will be added at the very end with a perturbation argument. The reason for this simplification is that we make use of the monotone structure of the rest of the equation. We start with a rescaling transformation such that the multiplicative noise vanishes. We end up with

$$\text{(TSEE)}\begin{cases} dy(t) &= [My(t) - |y(t)|^q y(t) + A(t)y(t) + \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^N \widetilde{b}_i(t)\ \mathrm{d}\beta_i(t), \\ u(0) &= u_0, \end{cases}$$

where $A(t)$, $\widetilde{J}$ and the new additive noise $\sum_{j=1}^N \widetilde{b}_j\ \mathrm{d}\beta_j$ are given by

$$\begin{aligned} A(t,x)y(t,x) :&= \tfrac{1}{2}\sum_{j=1}^N B_j(x)^2 y(t,x) + \sum_{j=1}^N i\beta_j(t)\begin{pmatrix} \nabla B_j(x) \times y_2 \\ -\nabla B_j(x) \times y_1 \end{pmatrix}, \\ \widetilde{J}(t,x) :&= \sum_{j=1}^N \big(-ib_j(t,x)B_j(x) + J(t,x)\big)e^{-i\sum_{n=1}^N B_n(x)\beta_n(t)}, \\ \widetilde{b}_i(t,x) :&= b_i(t,x)e^{-i\sum_{j=1}^N B_j(x)\beta_j(t)} \end{aligned}$$

for $t \in [0,T]$, $x \in D$ and $i = 1, \dots, N$. First, we show that a solution of (TSEE) can be transformed to a solution of (MSEE).

Proposition 4.3.2. *An adapted stochastic process $u : \Omega \times [0,T] \to L^2(D)$ is a strong solution of* (MSEE) *with $G \equiv 0$ if and only if the adapted process $y(t) := e^{-i\sum_{l=1}^N B_l\beta_l(t)}u(t)$ satisfies*

i) $\mathbb{E}\sup_{t\in[0,T]}\|y(t)\|_{L^2(D)^6}^2 + \mathbb{E}\int_0^T\int_D |y(t,x)|^{q+2}\,\mathrm{d}x\,\mathrm{d}t < \infty$,

ii) $My + i\sum_{j=1}^N \beta_j\binom{\nabla B_j \times y_2}{-\nabla B_j \times y_1} \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6 + L^2(\Omega; L^\infty(0,T; L^2(D)^6))$

and solves the equation (TSEE).

Proof. We assume that u is a solution of (MSEE) in the sense of Definition 4.3.1 with the described regularity properties. At first, we calculate $d(e^{i\sum_{j=1}^N B_j\beta_j(t)})$ with Itô's formula and obtain

$$e^{i\sum_{j=1}^N B_j\beta_j(t)} - 1 = \sum_{j=1}^N \int_0^t iB_j e^{i\sum_{l=1}^N B_l\beta_l(s)} d\beta_n(s) - \tfrac{1}{2}\sum_{j=1}^N \int_0^t B_j^2 e^{i\sum_{l=1}^N B_l\beta_l(s)} ds.$$

Therefore, Itô's product rule yields

$$\begin{aligned}
&\langle y(t), x'\rangle_{L^2(D)^6} - \langle u_0, x'\rangle_{L^2(D)^6} = \langle u(t), e^{i\sum_{l=1}^N B_l\beta_l(t)} x'\rangle_{L^2(D)^6} - \langle u_0, x'\rangle_{L^2(D)^6}\\
&= \sum_{j=1}^N \int_0^t -\langle u(s), \tfrac{1}{2}B_j^2 e^{i\sum_{l=1}^N B_l\beta_l(s)} x'\rangle_{L^2(D)^6} + \langle b_j(s) + iB_j u(s), iB_j e^{i\sum_{l=1}^N B_l\beta_l(s)} x'\rangle_{L^2(D)^6} \,\mathrm{d}s\\
&\quad + \int_0^t \langle Mu(s) - |u(s)|^q u(s) + J(s), e^{i\sum_{l=1}^N B_l\beta_l(s)} x'\rangle_{L^2(D)^6} \,\mathrm{d}s\\
&\quad + \sum_{j=1}^N \int_0^t \langle u(s), iB_j e^{i\sum_{l=1}^N B_l\beta_l(s)} x'\rangle_{L^2(D)^6} + \langle b_j(s) + iB_j u(s), e^{i\sum_{l=1}^N B_l\beta_l(s)} x'\rangle_{L^2(D)^6} d\beta_n(s)
\end{aligned}$$

almost surely for every $x' \in C_c^\infty(D)$ and for every $t \in [0,T]$. As a consequence, we have

$$\begin{aligned}
y(t) - u_0 &= \int_0^t e^{-i\sum_{l=1}^N B_l\beta_l(s)} M\big(e^{i\sum_{l=1}^N B_l\beta_l(s)} y(s)\big) - |y(s)|^q y(s) \,\mathrm{d}s\\
&\quad + \int_0^t e^{-i\sum_{l=1}^N B_l\beta_l(s)} J + \sum_{j=1}^N \tfrac{1}{2}B_j^2 y(s) - ib_j(s) B_j e^{-i\sum_{l=1}^N B_l\beta_l(s)} \,\mathrm{d}s\\
&\quad + \sum_{n=1}^N \int_0^t b_n(s) e^{-i\sum_{l=1}^N B_l\beta_l(s)} d\beta_n(s)
\end{aligned} \tag{4.3.1}$$

almost surely for every $t \in [0,T]$. Here, we used that $u \in L^{q+2}(\Omega \times [0,T] \times D)^6$ implies $|y|^q y \in L^{q+2}(\Omega\times[0,T]\times D)^6$. Since we want to derive an equation for y, we have to commute the exponential function with M. Therefore, we compute

$$\begin{aligned}
My(t) &= M(e^{-i\sum_{l=1}^N B_l\beta_l(t)} u(t))\\
&= \begin{pmatrix} \operatorname{curl}(e^{-i\sum_{l=1}^N B_l\beta_l(t)} u_2(t)) \\ -\operatorname{curl}(e^{-i\sum_{l=1}^N B_l\beta_l(t)} u_1(t)) \end{pmatrix}\\
&= \sum_{j=1}^N -i\beta_j(t) \begin{pmatrix} \nabla B_j e^{-i\sum_{l=1}^N B_l\beta_l(t)} \times u_2(t) \\ -\nabla B_j e^{-i\sum_{l=1}^N B_l\beta_l(t)} \times u_1(t) \end{pmatrix} + \begin{pmatrix} e^{-i\sum_{l=1}^N B_l\beta_l(t)} \operatorname{curl}(u_2(t)) \\ -e^{-i\sum_{l=1}^N B_l\beta_l(t)} \operatorname{curl}(u_1(t)) \end{pmatrix}\\
&= \sum_{j=1}^N i\beta_j(t) \begin{pmatrix} -\nabla B_j \times y_2(t) \\ \nabla B_j \times y_1(t) \end{pmatrix} + e^{-i\sum_{l=1}^N B_l\beta_l(t)} Mu(t).
\end{aligned}$$

Together with $y(t) := e^{-i\sum_{l=1}^N B_l\beta_l(t)} u(t)$, this implies

$$e^{-i\sum_{l=1}^N B_l\beta_l(t)} M\big(e^{-i\sum_{l=1}^N B_l\beta_l(t)} y(t)\big) = My(t) + \sum_{j=1}^N i\beta_j(t) \begin{pmatrix} \nabla B_j \times y_2(t) \\ -\nabla B_j \times y_1(t) \end{pmatrix}.$$

Inserting this into (4.3.1) finally proves that y solves (TSEE). The other direction follows the same lines. □

We solve (TSEE) by a refined Galerkin approximation of the skew-adjoint operator M. To do this, we truncate the equation with the spectral multipliers P_n and S_{n-1} we defined in

Section 4.1 and we end up with

$$\begin{cases} dy_n(t) &= [P_n M y_n(t) - P_n F(y_n(t)) + P_n A(t) y_n(t) + P_n \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^N S_{n-1}\widetilde{b}_i(t)\ \mathrm{d}\beta_i(t), \\ y_n(0) &= S_{n-1}u_0. \end{cases} \tag{4.3.2}$$

The operator P_nM is linear and bounded and as we have shown in the previous section, the nonlinearity $P_nF : P_n(L^2(D)^6) \to P_n(L^2(D)^6)$ is locally Lipschitz continuous. Thus, we get an ordinary stochastic differential equation in $P_n(L^2(D)^6)$. Note that we had to use S_{n-1} for the truncation of the stochastic part and for the truncation of the initial data since we need to estimate these terms in $L^p(D)^6$ uniformly in n. Note that such an estimate is not available for P_n in general. In the next proposition, we derive a priori estimates for the solution exploiting the monotone structure of the equation.

Proposition 4.3.3. *The truncated equation* (4.3.2) *has for every $n \in \mathbb{N}$ a unique, pathwise continuous solution $y_n : \Omega \times [0,T] \to L^2(D)^6$ that additionally satisfies*

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} \|y_n(t)\|^2_{L^2(D)^6} + \mathbb{E}\int_0^T \|y_n(t)\|^{q+2}_{L^{q+2}(D)^6} dt \\ &\qquad \le C\Big(\|\widetilde{J}\|^2_{L^2(\Omega\times[0,T]\times D)} + \sum_{j=1}^N \|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)} + \|u_0\|^2_{L^2(D)}\Big) \end{aligned} \tag{4.3.3}$$

for some constant $C > 0$ only depending on N, T and $\sup_{j=1,\dots,N}\|B_j\|_{L^\infty(D)}$, but not on $n \in \mathbb{N}$.

Proof. First, we define the stopping time

$$\tau_m := \inf\big\{t \in [0,T] : |\beta_i(t)| > m \text{ for some } i = 1,\dots,N\big\}$$

and solve the equation

$$\begin{cases} dy_n^{(m)} &= [P_n M y_n^{(m)} - P_n F(y_n^{(m)}) + P_n A^{(m)} y_n^m + P_n \widetilde{J}]\,\mathrm{d}t + \sum_{i=1}^N S_{n-1}\widetilde{b}_i\ \mathrm{d}\beta_i, \\ u(0) &= S_{n-1}u_0, \end{cases} \tag{4.3.4}$$

where the truncated linear operator $A^{(m)}$ is given by

$$A^{(m)}(t)y(t) := \sum_{j=1}^N i\beta_j(t\wedge\tau_m)\begin{pmatrix} \nabla B_j \times y_2(t) \\ -\nabla B_j \times y_1(t)\end{pmatrix} + B_j^2 y(t).$$

By Lemma 4.2.4 and Lemma 4.1.7, this is an ordinary stochastic differential equation in the closed subspace $R(P_n) \subset L^2(D)^6$ with locally Lipschitz nonlinearity. The stopping time τ_m is necessary at this point since it ensures $\beta_j(\cdot\wedge\tau_m) \in L^\infty(\Omega\times[0,T])$. We need this truncation to be able to apply the classical results for stochastic ordinary differential equations.

There exists a stopping time $\tau^{(m,n)}$ with $0 \le \tau^{(m,n)} \le T$ almost surely, an increasing sequence of stopping times $(\tau_k^{(m,n)})_k$ with $\tau_k^{(m,n)} \to \tau^{(m,n)}$ almost surely as $k \to \infty$ and adapted processes $y_n^{(m)} : \Omega \times [0,T] \to P_nL^2(D)^6$ with

$$y_n^{(m)} \in C(0, \tau_k^{(m,n)}; L^2(D)^6)$$

almost surely such that $y_n^{(m)}$ solves (4.3.4) on $[0,\tau_k^{(m,n)}]$. Moreover, we have the blow-up alternative

$$\mathbb{P}\Big\{\tau^{(m,n)}<T,\ \sup_{t\in[0,\tau^{(m,n)})}\|y_n(t)\|_{L^2(D)^6}<\infty\Big\}=0. \tag{4.3.5}$$

To show the a priori estimate, we use the Itô formula from Lemma 4.2.2 and get

$$\begin{aligned}&\|y_n^{(m)}(t)\|^2_{L^2(D)^6}-\|S_{n-1}u_0\|^2_{L^2(D)^6}\\ &\quad=2\int_0^t \operatorname{Re}\langle y_n^{(m)}(s),-|y_n^{(m)}(s)|^q y_n^{(m)}(s)+A^{(m)}(s)y_n^{(m)}(s)+\widetilde{J}(s)\rangle_{L^2(D)^6}\,\mathrm{d}s\\ &\qquad+2\sum_{j=1}^N\int_0^t \operatorname{Re}\langle y_n^{(m)}(s),S_{n-1}\widetilde{b}_j(s)\rangle_{L^2(D)^6}\,\mathrm{d}\beta_j(s)+\sum_{j=1}^N\int_0^t\|S_{n-1}\widetilde{b}_j(s)\|^2_{L^2(D)^6}\,\mathrm{d}s.\end{aligned}$$

Using the skew-symmetry of the cross-product and the fact that both B_j and β_j are real-valued, we calculate

$$\begin{aligned}&\Big\langle y_n^{(m)}(s),i\beta_j(s\wedge\tau_m)\begin{pmatrix}\nabla B_j\times y_{n,2}^{(m)}(s)\\ -\nabla B_j\times y_{n,1}^{(m)}(s)\end{pmatrix}\Big\rangle_{L^2(D)^6}\\ &\qquad=-\Big\langle i\beta_j(s\wedge\tau_m)y_n^{(m)}(s),\begin{pmatrix}\nabla B_j\times y_{n,2}^{(m)}(s)\\ -\nabla B_j\times y_{n,1}^{(m)(s)}\end{pmatrix}\Big\rangle_{L^2(D)^6}\\ &\qquad=\Big\langle i\beta_j(s\wedge\tau_m)\begin{pmatrix}\nabla B_j\times y_{n,2}^{(m)}(s)\\ -\nabla B_j\times y_{n,1}^{(m)}(s)\end{pmatrix},y_n^{(m)}(s)\Big\rangle_{L^2(D)^6},\end{aligned}$$

which implies

$$\operatorname{Re}\Big\langle y_n^{(m)}(s),i\beta_j(s\wedge\tau_m)\begin{pmatrix}\nabla B_j\times y_{n,2}^{(m)}(s)\\ -\nabla B_j\times y_{n,1}^{(m)}(s)\end{pmatrix}\Big\rangle_{L^2(D)^6}=0$$

for all $s\in[0,\tau_k^{(n,m)}]$. Hence, the expression from above simplifies to

$$\begin{aligned}&\|y_n^{(m)}(t)\|^2_{L^2(D)^6}+2\int_0^t\int_D|y_n^{(m)}(s,x)|^{q+2}\,\mathrm{d}x\,\mathrm{d}t\\ &=\|u_0\|^2_{L^2(D)^6}+2\int_0^t\operatorname{Re}\langle y_n^{(m)}(s),\widetilde{J}(s)+\sum_{j=1}^N B_j^2y_n^{(m)}(s)\rangle_{L^2(D)^6}\,\mathrm{d}s\\ &\quad+2\sum_{j=1}^N\int_0^t\operatorname{Re}\langle y_n^{(m)}(s),S_{n-1}\widetilde{b}_j(s)\rangle_{L^2(D)^6}\,\mathrm{d}\beta_j(s)+\sum_{j=1}^N\int_0^t\|S_{n-1}\widetilde{b}_j(s)\|^2_{L^2(D)^6}\,\mathrm{d}s\end{aligned} \tag{4.3.6}$$

almost surely for all $t\in[0,\tau_k^{(m,n)}]$. Since the second term on the left-hand side is positive, we can drop it for a moment. Afterwards, we take the supremum over time and then the expectation. We estimate the remaining quantities term by term and start with the deterministic part.

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t\wedge\tau_k^{(m,n)}]} \Big| \int_0^s 2\,\mathrm{Re}\,\big\langle y_n^{(m)}(r), \widetilde{J}(r) + \sum_{j=1}^N B_j^2 y_n^{(m)}(r)\big\rangle \,\mathrm{d}r + \sum_{j=1}^N \int_0^s \|S_{n-1}\widetilde{b}_j(r)\|^2_{L^2(D)^6}\,\mathrm{d}r\Big| \\
&\leq \mathbb{E}\int_0^{t\wedge\tau_k^{(m,n)}} 2\|y_n^{(m)}(r)\|_{L^2(D)^6}\|\widetilde{J}(r)\|_{L^2(D)^6} + 2N \sup_{j=1,\dots,N}\|B_j\|^2_{L^\infty(D)}\|y_n^{(m)}(r)\|^2_{L^2(D)^6}\,\mathrm{d}r \\
&\quad + \sum_{j=1}^N \|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)^6} \\
&\leq \int_0^t \mathbb{E} \sup_{r\in[0,s\wedge\tau_k^{(m,n)}]]} \|y_n^{(m)}(r)\|^2_{L^2(D)^6}\big(2N\sup_{j=1,\dots,N}\|B_j\|^2_{L^\infty(D)}+1\big)\,\mathrm{d}s + \|\widetilde{J}\|^2_{L^2(\Omega\times[0,T]\times D)^6} \\
&\quad + \sum_{j=1}^N \|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)^6}.
\end{aligned}
$$

The stochastic part can be estimated with the Burgholder-Davies-Gundy inequility. We have

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t\wedge\tau_k^{(m,n)}]} \Big|\sum_{j=1}^N \int_0^s \mathrm{Re}\,\big\langle y_n^{(m)}(s), S_{n-1}\widetilde{b}_j(s)\big\rangle_{L^2(D)^6}\,\mathrm{d}\beta_j(s)\Big| \\
&\qquad \leq C\mathbb{E}\Big(\sum_{j=1}^N\int_0^{t\wedge\tau_k^{(m,n)}} \big|\,\mathrm{Re}\langle y_n^{(m)}(s), S_{n-1}\widetilde{b}_j(s)\rangle_{L^2(D)^6}\big|^2\,\mathrm{d}s\Big)^{1/2} \\
&\qquad \leq C\mathbb{E}\sup_{s\in[0,t\wedge\tau_k^{(m,n)}]}\|y_n^{(m)}(s)\|_{L^2(D)^6}\Big(\sum_{j=1}^N\|S_{n-1}\widetilde{b}_j\|^2_{L^2([0,T]\times D)}\Big)^{1/2} \\
&\qquad \leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,t\wedge\tau_k^{(m,n)}]}\|y_n^{(m)}(s)\|^2_{L^2(D)^6} + C^2\mathbb{E}\sum_{j=1}^N\|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)}.
\end{aligned}
$$

Putting these estimates together, we get

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_k^{(m,n)}]}\|y_n^{(m)}(s)\|^2_{L^2(\Omega\times D)} \\
&\qquad \lesssim \|u_0\|^2_{L^2(D)^6} + \|\widetilde{J}\|^2_{L^2(\Omega\times[0,T]\times D)} + \sum_{j=1}^N\|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)} \\
&\qquad\quad + \big(N\sup_{j=1,\dots,N}\|B_j\|^2_{L^\infty(D)}+1\big)\int_0^t\mathbb{E}\sup_{r\in[0,s\wedge\tau_k^{(m,n)}]}\|y_n^{(m)}(r)\|^2_{L^2(D)^6}\,\mathrm{d}s.
\end{aligned}
$$

Consequently, Gronwall yields

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_k^{(m,n)}]}\|y_n^{(m)}(s)\|^2_{L^2(D)^6} \\
&\qquad \lesssim_{B_j,N,T}\Big(\|\widetilde{J}\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \sum_{j=1}^N\|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|u_0\|^2_{L^2(\Omega\times D)}\Big)
\end{aligned}
$$

for every $t \in [0,T]$. Next, we pass to the limit $k \to \infty$ with Fatou's Lemma and get

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau^{(m,n)})} \|y_n^{(m)}(t)\|_{L^2(D)^6}^2 \\
&\qquad \le \liminf_{k\to\infty} \mathbb{E} \sup_{t\in[0,\tau_k^{(m,n)}]} \|y_n^{(m)}(t)\|_{L^2(D)^6}^2 \\
&\qquad \lesssim_{B_j} \Big(\|\widetilde{J}\|_{L^2(\Omega\times[0,T]\times D)}^2 + \sum_{j=1}^N \|\widetilde{b_j}\|_{L^2(\Omega\times[0,T]\times D)}^2 + \|u_0\|_{L^2(\Omega\times D)}^2 \Big). \qquad (4.3.7)
\end{aligned}$$

Note that this bound is independent of m and n. In particular, it implies $\tau^{(m,n)} = T$ almost surely. Indeed, there exists an $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that $\Omega \setminus \big(N \cup \{\tau^{(m,n)} = T\}\big)$ can be decomposed into disjoint sets

$$\Big\{ \tau^{(m,n)} < T, \sup_{t\in[0,\tau^{(m,n)})} \|y_n^{(m)}(t)\|_{L^2(D)^6} < \infty \Big\},$$
$$\Big\{ \tau^{(m,n)} < T, \sup_{t\in[0,\tau^{(m,n)})} \|y_n^{(m)}(t)\|_{L^2(D)^6} = \infty \Big\}.$$

The first of these sets has measure zero by (4.3.5), whereas the second one has measure zero since (4.3.7) implies $\sup_{t\in[0,\tau^{(m,n)})} \|y_n^{(m)}(t)\|_{L^2(D)^6} < \infty$ almost surely. As a consequence of (4.3.6), we also get

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_D |y_n^{(m)}(s,x)|^{q+2} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad \lesssim_{B_j} \Big(\|\widetilde{J}\|_{L^2(\Omega\times[0,T]\times D)^6}^2 + \sum_{j=1}^N \|\widetilde{b_j}\|_{L^2(\Omega\times[0,T]\times D)^6}^2 + \|u_0\|_{L^2(D)}^2 \Big). \qquad (4.3.8)
\end{aligned}$$

We already know that $y_n^{(m)}$ is almost surely continuous on $[0,T)$ as a function with values in $L^2(D)^6$. The pathwise continuity up to T follows from Lemma 4.2.2.

It remains to take the limit $m \to \infty$. By uniqueness, we have $y_n^{(m)}(\omega,t) = y_n^{(k)}(\omega,t)$ for almost all $\omega \in \Omega$, all $t \in [0,\tau_m]$ and for every $k \ge m$. Moreover, for almost all $\omega \in \Omega$, there exists $m(\omega)$, such that $\tau_{m(\omega)}(\omega) = T$. Hence, the limit $y_n = \lim_{m\to\infty} y_n^{(m)}$ is well-defined, adapted and satisfies (4.3.4). Again using Fatou's Lemma yields analogous estimates to (4.3.7) and (4.3.8) for y_n. This closes the proof. $\square$

To obtain strong solutions, we need an estimate for My_n that is uniform in $n \in \mathbb{N}$. We do this in the following way. We derive an a priori estimate for

$$\Big\| P_n M y_n(t) - P_n F(y_n(t)) + P_n \sum_{j=1}^N B_j^2 y_n(t) + P_n i\beta_j(t) \begin{pmatrix} \nabla B_j \times y_{n,2}(t) \\ -\nabla B_j \times y_{n,1}(t) \end{pmatrix} + P_n \widetilde{J}(t) \Big\|_{L^2(D)^6}^2$$

and afterwards we use the estimates from Proposition 4.3.3 to get a bound for My_n. To do this, we have to show that the above quantity is an Itô process in $P_n L^2(D)^6$.

Lemma 4.3.4. *The stochastic process*

$$\Lambda_n(t) := P_n M y_n(t) - P_n F(y_n(t)) + P_n \sum_{j=1}^N B_j^2 y_n(t) + P_n i\beta_j(t) \begin{pmatrix} \nabla B_j \times y_{n,2}(t) \\ -\nabla B_j \times y_{n,1}(t) \end{pmatrix} + P_n \widetilde{J}(t)$$

is an Itô process with

$$
\begin{aligned}
d\Lambda_n =&P_n\Big[M\Lambda_n - F'(y_n)(\Lambda_n) + \sum_{j=1}^N \Big(i\beta_j\begin{pmatrix}\nabla B_j\times\Lambda_{n,2}\\ -\nabla B_j\times\Lambda_{n,1}\end{pmatrix} + B_j^2\Lambda_n\Big)\\
&-\frac{1}{2}\sum_{j=1}^N B_j^2\Big(\sum_{k=1}^N -ib_kB_k + J\Big)e^{-\sum_{l=1}^N B_l\beta_l\cdot}\\
&+\Big(\sum_{k=1}^N -i\partial_t b_kB_k + \partial_t J\Big)e^{-\sum_{l=1}^N B_l\beta_l\cdot} - \frac{1}{2}\sum_{j=1}^N F''(y_n)(S_{n-1}\widetilde{b}_j, S_{n-1}\widetilde{b}_j)\Big]\,\mathrm{d}t\\
&+\sum_{j=1}^N P_n\Big[MS_{n-1}\widetilde{b}_j - F'(y_n)(S_{n-1}\widetilde{b}_j) + \sum_{k=1}^N i\beta_k\begin{pmatrix}\nabla B_k\times S_{n-1}\widetilde{b}_{j,2}\\ -\nabla B_k\times S_{n-1}\widetilde{b}_{j,1}\end{pmatrix}\\
&+\sum_{k=1}^N B_k^2 S_{n-1}\widetilde{b}_j + i\begin{pmatrix}\nabla B_j\times y_{n,2}\\ -\nabla B_j\times y_{n,1}\end{pmatrix} - iB_j\Big(\sum_{k=1}^N -ib_kB_k + J\Big)e^{-\sum_{l=1}^N B_l\beta_l\cdot}\Big]\,\mathrm{d}\beta_j
\end{aligned}
$$

almost surely on $[0,T]$.

Proof. With Lemma 4.2.4 and Lemma 4.1.7, one shows that $P_nF(y_n)$ is an Itô process in $P_nL^2(D)^6$ with

$$
d(P_nF(y_n)) = P_n\Big[F'(y_n)(\Lambda_n) + \tfrac{1}{2}\sum_{j=1}^N F''(y_n)(S_{n-1}\widetilde{b}_j, S_{n-1}\widetilde{b}_j)\Big]dt + \sum_{j=1}^N P_nF'(y_n)S_{n-1}\widetilde{b}_j\,\mathrm{d}\beta_j.
$$

Moreover, by the product rule,

$$
P_n\widetilde{J}(t,x) = P_n\Big(\sum_{j=1}^N -ib_j(t,x)B_j(x) + J(t,x)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(t)}
$$

is an Itô process in $L^2(D)^6$ of the form

$$
\begin{aligned}
&d(P_n\widetilde{J})(t)\\
=&P_n\Big(-\frac{1}{2}\sum_{j=1}^N B_j^2\Big(\sum_{k=1}^N -ib_k(t)B_k + J(t)\Big) + \sum_{k=1}^N -i\partial_t b_k(t)B_k + \partial_t J(t)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(t)}\,\mathrm{d}t\\
&-P_n\sum_{j=1}^N\Big[iB_j\big(\sum_{k=1}^N -ib_k(t)B_k + J(t)\big)e^{-i\sum_{l=1}^N B_l\beta_l(t)}\Big]\,\mathrm{d}\beta_j.
\end{aligned}
$$

The remaining expression $\Lambda_n + P_nF(y_n) - P_n\widetilde{J}$ is a function of the Itô processes

$$
dy_n(t,x) = \Lambda_n(t)dt + S_{n-1}\sum_{j=1}^N \widetilde{b}_j\,\mathrm{d}\beta_j(t)
$$

and β_j, $j=1,\dots,N$. Hence, we can calculate $d(\Lambda_n + P_nF(y_n) - P_n\widetilde{J})$ with Itô's formula. Thereby, it is crucial that all occurring terms depend only linearly on y_n and β_j and consequently the second derivatives vanish. This finally proves the claimed result. □

Now we can derive an a priori estimate for Λ_n that is uniform in $n\in\mathbb{N}$.

Proposition 4.3.5. *The process Λ_n satisfies the estimate*

$$\mathbb{E}\sup_{t\in[0,T]}\|\Lambda_n(t)\|^2_{L^2(D)^6}\leq C\big(1+\mathbb{E}\|Mu_0\|^2_{L^2(D)^6}+\mathbb{E}\|u_0\|^2_{L^2(D)^6}+\mathbb{E}\|u_0\|^{2q+2}_{L^{2q+2}(D)^6}\big)$$

with a constant $C>0$ depending on J, b_j and B_j for $j=1,\dots,N$, but not on $n\in\mathbb{N}$.

Proof. At first, we calculate $\|\Lambda_n(t)\|^2_{L^2(D)^6}$ with the Itô formula from Lemma 4.2.2. We obtain

$$\begin{aligned}
&\|\Lambda_n(t)\|^2_{L^2(D)^6}-\|\Lambda_n(0)\|^2_{L^2(D)^6}\\
=&2\int_0^t \operatorname{Re}\Big\langle\Lambda_n(s), M\Lambda_n(s)-F'(y_n(s))(\Lambda_n(s))+\sum_{j=1}^N\Big(i\beta_j(s)\begin{pmatrix}\nabla B_j\times\Lambda_{n,2}(s)\\-\nabla B_j\times\Lambda_{n,1}(s)\end{pmatrix}+B_j^2\Lambda_n(s)\Big)\\
&\qquad-\frac12\sum_{j=1}^N B_j^2\Big(\sum_{k=1}^N -ib_k(s)B_k+J(s)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(s)}\\
&\qquad+\Big(\sum_{k=1}^N -i\partial_t b_k(s)B_k+\partial_t J(s)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(t)}\\
&\qquad-\frac12\sum_{j=1}^N F''(y_n)(S_{n-1}\widetilde{b}_j(s),S_{n-1}\widetilde{b}_j(s))\Big\rangle_{L^2(D)^6}\,\mathrm{d}s\\
&+\int_0^t\Big\|MS_{n-1}\widetilde{b}_j(s)-F'(y_n)(S_{n-1}\widetilde{b}_j(s))+\sum_{k=1}^N i\beta_k(s)\begin{pmatrix}\nabla B_k\times S_{n-1}\widetilde{b}_{j,2}(s)\\-\nabla B_k\times S_{n-1}\widetilde{b}_{j,1}(s)\end{pmatrix}\\
&\qquad+\sum_{k=1}^N B_k^2 S_{n-1}\widetilde{b}_j(s)+i\begin{pmatrix}\nabla B_j\times y_{n,2}(s)\\-\nabla B_j\times y_{n,1}(s)\end{pmatrix}\\
&\qquad-iB_j\Big(\sum_{k=1}^N -ib_k(s)B_k+J(s)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(s)}\Big\|^2_{L^2(D)^6}\,\mathrm{d}s\\
&+2\sum_{j=1}^N\int_0^t\operatorname{Re}\Big\langle\Lambda_n(s), MS_n\widetilde{b}_j(s)-F'(y_n)(S_{n-1}\widetilde{b}_j(s))+\sum_{k=1}^N i\beta_k(s)\begin{pmatrix}\nabla B_k\times S_{n-1}\widetilde{b}_{j,2}(s)\\-\nabla B_k\times S_{n-1}\widetilde{b}_{j,1}(s)\end{pmatrix}\\
&\qquad+\sum_{k=1}^N B_k^2 S_{n-1}\widetilde{b}_j(s)+i\begin{pmatrix}\nabla B_j\times y_{n,2}(s)\\-\nabla B_j\times y_{n,1}(s)\end{pmatrix}\\
&\qquad-iB_j\Big(\sum_{k=1}^N -ib_k(s)B_k+J(s)\Big)e^{-i\sum_{l=1}^N B_l\beta_l(s)}\Big\rangle_{L^2(D)^6}\,\mathrm{d}\beta_j(s).
\end{aligned}$$

As we have seen before in the proof of Proposition 4.3.3, the term

$$\operatorname{Re}\Big\langle\Lambda_n(s), M\Lambda_n(s)+\sum_{j=1}^N i\beta_j(s)\begin{pmatrix}\nabla B_j\times\Lambda_{n,2}(s)\\-\nabla B_j\times\Lambda_{n,1}(s)\end{pmatrix}\Big\rangle_{L^2(D)^6}$$

vanishes. Moreover, by Lemma 4.2.4, we have

$$-\operatorname{Re}\langle\Lambda_n(s), F(y_n(s))'\Lambda_n(s)\rangle_{L^2(D)^6}\leq 0$$

almost surely for every $s\in[0,T]$ and we can drop this term in an upper estimate. We split the remaining expression into a deterministic integral I_{det} and a stochastic integral

I_{stoch}. We take the supremum over time and afterwards the expectation. Further, we aim to control the left-hand side with Gronwall. We start with an estimate for the deterministic integral I_{det}. Using Cauchy-Schwartz and the assumptions on B_j, ∇B_j, $\partial_t b_j, J$ and $\partial_t J$ from [M4] – [M6], we get

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |I_{\mathrm{det}}(s)| \\
&\lesssim \int_0^t \|\Lambda_n(r)\|^2_{L^2(D)^6} + \sum_{j=1}^N \|\Lambda_n(r)\|_{L^2(D)^6} \|F''(y_n)(S_{n-1}\widetilde{b}_j(r), S_{n-1}\widetilde{b}_j(r)\|_{L^2(D)^6} \\
&\quad + \|MS_n\widetilde{b}_j(r)\|^2_{L^2(D)^6} + \|F'(y_n(r))(S_{n-1}\widetilde{b}_j(r))\|^2_{L^2(D)^6} + \sum_{k=1}^N \|\beta_k(r)S_{n-1}\widetilde{b}_j(r)\|^2_{L^2(D)^6} \\
&\quad + \sum_{k=1}^N \|S_{n-1}\widetilde{b}_j(r)\|^2_{L^2(D)} + \|y_n(r)\|^2_{L^2(D)}\,\mathrm{d}r.
\end{aligned}$$

The growth estimates for F' and F'' from Lemma 4.2.4 together with the uniform boundedness of S_{n-1} on $L^2(D)^6$ yield

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |I_{\mathrm{det}}(s)| \\
&\lesssim \int_0^t \|\Lambda_n(r)\|^2_{L^2(D)^6} + \sum_{j=1}^N \||y_n(r)|^{q-1}|S_{n-1}\widetilde{b}_j(r)|^2\|^2_{L^2(D)^6} + \|M\widetilde{b}_j(r)\|^2_{L^2(D)^6} + \|\widetilde{b}_j(r)\|^2_{L^2(D)} \\
&+ \||y_n(r)|^q S_{n-1}\widetilde{b}_j(r)\|^2_{L^2(D)^6} + \sum_{k=1}^N \beta_k(r)^2 \|\widetilde{b}_j(r)\|^2_{L^2(D)^6} + \|y_n(r)\|^2_{L^2(D)}\,\mathrm{d}r.
\end{aligned}$$

In the following estimate, we have to distinguish the cases $q \in (1,2)$ and $q = 2$. We start with the first one. Hölder's inequality, the fact $\beta_k \in L^\alpha(\Omega; C(0,T))$ for every $\alpha \in [2,\infty)$ and the boundedness of S_{n-1} on $L^p(D)^6$ for every $p \in (1,\infty)$ with norm independent of n yield

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |I_{\mathrm{det}}(s)| \\
&\qquad \lesssim \int_0^t \mathbb{E} \sup_{r\in[0,s]} \|\Lambda_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s + \|y_n\|^{2(q-1)}_{L^{q+2}(\Omega\times[0,T]\times D)^6} \|\widetilde{b}_j\|^4_{L^{\frac{4(q+2)}{4-q}}(\Omega\times[0,T]\times D)^6} \\
&\qquad\quad + \|M\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|y_n\|^{2q}_{L^{q+2}(\Omega\times[0,T]\times D)^6} \|\widetilde{b}_j\|^2_{L^{\frac{2(q+2)}{2-q}}(\Omega\times[0,T]\times D)^6} \\
&\qquad\quad + \|\widetilde{b}_j(s)\|^2_{L^{2+\varepsilon}(\Omega;L^2([0,T]\times D))^6} + \|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)} + \|y_n\|^2_{L^2(\Omega\times[0,T]\times D)}
\end{aligned}$$

for any $\varepsilon > 0$. In the case $q = 2$, the same argument yields

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,t]} |I_{\mathrm{det}}(t)| \lesssim &\int_0^t \mathbb{E} \sup_{r\in[0,s]} \|\Lambda_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s + \|y_n\|^2_{L^4(\Omega\times[0,T]\times D)^6} \|\widetilde{b}_j\|^4_{L^8(\Omega\times[0,T]\times D)^6} \\
&+ \|M\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)^6} + \|y_n\|^4_{L^4(\Omega\times[0,T]\times D)^6} \|S_{n-1}\widetilde{b}_j\|^2_{L^\infty(\Omega\times[0,T]\times D)^6} \\
&+ \|\widetilde{b}_j\|^2_{L^{2+\varepsilon}(\Omega;L^2([0,T]\times D))^6} + \|\widetilde{b}_j\|^2_{L^2(\Omega\times[0,T]\times D)} + \|y_n\|^2_{L^2(\Omega\times[0,T]\times D)}
\end{aligned}$$

for any $\varepsilon > 0$. At this point, we need the requirement $S_{n-1}\widetilde{b}_j = \widetilde{b}_j$ for large enough n from [M5] to get rid of S_{n-1}. Note that we already bounded $\|y_n\|_{L^{q+2}(\Omega\times[0,T]\times D)^6}$ and

$\|y_n\|_{L^2(\Omega\times[0,T]\times D)^6}$ in Proposition 4.3.3 uniformly in n. Hence, it remains to estimate the terms including $\widetilde{b}_j$. By the product rule for the curl operator, we have

$$M\widetilde{b}_j(s) = \big(Mb_j(s)\big)e^{-i\sum_{l=1}^N B_l\beta_l(s)} + \sum_{k=1}^N i\beta_k(s)\begin{pmatrix}-\nabla B_k \times b_{j,2} \\ \nabla B_k \times b_{j,1}\end{pmatrix}e^{-i\sum_{l=1}^N B_l\beta_l(s)},$$

which implies

$$\begin{aligned}\|M\widetilde{b}_j\|_{L^2(\Omega\times[0,T]\times D)} &\lesssim_{N,B_k} \|Mb_j\|_{L^2(\Omega\times[0,T]\times D)} + \sum_{k=1}^N \|\beta_k b_j\|_{L^2(\Omega\times[0,T]\times D)} \\ &\le \|Mb_j\|_{L^2(\Omega\times[0,T]\times D)} + \|b_j\|_{L^{2+\varepsilon}(\Omega;L^2([0,T]\times D))}\end{aligned}$$

for any $\varepsilon > 0$. Here, we again used the fact $\beta_k \in L^\alpha(\Omega; C(0,T))$ for every $\alpha \ge 2$ and Hölder's inequality. It remains to bound $\|b_j\|_{L^{2+\varepsilon}(\Omega;L^2([0,T]\times D))}$, but this is immediate by [M5], because we have both $b_j \in L^2(\Omega\times[0,T]\times D)^6$ and $b_j \in L^{\frac{2(q+2)}{2-q}}(\Omega\times[0,T]\times D)^6$. Altogether, we have

$$\mathbb{E}\sup_{s\in[0,t]}|I_{\mathrm{det}}(s)| \lesssim 1 + \int_0^t \mathbb{E}\sup_{r\in[0,s]}\|\Lambda_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s$$

and the estimate only depends on B_j, b_j and J but not on $n\in\mathbb{N}$. The stochastic term I_{stoch} can be controlled in the same way as in the proof of Proposition 4.3.3 with the Burkholder-Davies-Gundy inequality and the assumptions on B_j, b_j and J together with the growth estimates for F' and F''. Thus, we end up with

$$\mathbb{E}\sup_{s\in[0,t]}\|\Lambda_n(s)\|^2_{L^2(D)} \lesssim 1 + \mathbb{E}\|\Lambda_n(0)\|^2_{L^2(D)} + \int_0^t \mathbb{E}\sup_{r\in[0,s]}\|\Lambda_n(r)\|^2_{L^2(D)^6}\,\mathrm{d}s.$$

It remains to bound

$$\Lambda_n(0) = P_nMS_{n-1}u_0 - P_nF(S_{n-1}u_0) + P_n\sum_{j=1}^N B_j^2 S_{n-1}u_0 - P_n\sum_{j=1}^N ib_j(0)B_j + P_nJ(0)$$

in $L^2(\Omega\times D)^6$ independent of $n\in\mathbb{N}$. Since both b_j and J are in $L^2(\Omega; W^{1,2}(0,T;L^2(D)^6))$, the corresponding initial data $b_j(0)$ and $J(0)$ are contained in $L^2(\Omega\times D)^6$. As a consequence, the uniform boundedness of S_{n-1} on $L^p(D)^6$ for every $p\in(1,\infty)$ and of P_n on $L^2(D)^6$ yield

$$\begin{aligned}\mathbb{E}\|\Lambda_n(0)\|^2_{L^2(D)} &\lesssim 1 + \mathbb{E}\|MS_{n-1}u_0\|^2_{L^2(D)^6} + \mathbb{E}\||S_{n-1}u_0|^qS_{n-1}u_0\|^2_{L^2(D)^6} \\ &\quad + \sum_{j=1}^N \|B_j\|^2_{L^\infty(D)}\|S_{n-1}u_0\|^2_{L^2(D)^6} \\ &\lesssim 1 + \mathbb{E}\|Mu_0\|^2_{L^2(D)^6} + \mathbb{E}\|u_0\|^{2(q+1)}_{L^{2(q+1)}(D)^6} + \mathbb{E}\|u_0\|^2_{L^2(D)^6}.\end{aligned}$$

Finally, an application of Gronwall's Lemma proves the claimed result. □

In Proposition 4.3.3 and 4.3.5, we derived uniform estimates for y_n and Λ_n. As a consequence, we also get the uniform boundedness of $F(y_n)$ since

$$\|F(y_n)\|_{L^{\frac{q+2}{q+1}}(\Omega\times[0,T]\times D)} \lesssim \||y_n|^{q+1}\|_{L^{\frac{q+2}{q+1}}(\Omega\times[0,T]\times D)^6} = \|y_n\|^{q+1}_{L^{q+2}(\Omega\times[0,T]\times D)^6}.$$

By Banach-Alaoglu, there exist $y\in L^2(\Omega; L^\infty(0,T;L^2(D)))^6$, $N\in L^{\frac{q+2}{q+1}}(\Omega\times[0,T]\times D)^6$, $\Lambda\in L^2(\Omega;L^\infty(0,T;L^2(D)))^6$ and subsequences, still indexed with n such that

a) $y_n \to y$ for $n \to \infty$ in the weak* sense in $L^2(\Omega; L^\infty(0,T; L^2(D)))^6$,

b) $y_n \to y$ for $n \to \infty$ in the weak sense in $L^2(\Omega \times [0,T] \times D)^6$,

c) $F(y_n) \to N$ for $n \to \infty$ in the weak sense in $L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$,

d) $\Lambda_n \to \Lambda$ for $n \to \infty$ in the weak sense in $L^2(\Omega \times [0,T] \times D)^6$,

e) $\Lambda_n \to \Lambda$ for $n \to \infty$ in the weak* sense in $L^2(\Omega; L^\infty(0,T; L^2(D)))^6$.

Since y_n is for every $n \in \mathbb{N}$ an adapted solution of the ordinary stochastic differential equation (4.3.4) in $P_n L^2(D)^6$, we have $y_n \in L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$. Consequently, since $L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$ is a closed subspace of $L^2(\Omega \times [0,T] \times D)^6$, it is also weakly closed. This implies $y \in L^2_{\mathbb{F}}(\Omega \times [0,T] \times D)^6$, which means that y is also adapted.

In the next Lemma, we show that Λ has the correct form, that $My(t)$ exists in the sense of distributions and that we have $y_1 \times \nu = 0$ on ∂D.

Lemma 4.3.6. *The process $y : \Omega \times [0,T] \to L^2(D)^6$ additionally satisfies $y(\omega,t) \times \nu = 0$ on ∂D for almost all $\omega \in \Omega$ and $t \in [0,T]$. Moreover, we have*

$$My + \sum_{j=1}^N i\beta_j \begin{pmatrix} \nabla B_j \times y_2 \\ -\nabla B_j \times y_1 \end{pmatrix} \in L^2(\Omega; L^\infty(0,T; L^2(D)^6)) + L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6$$

and the identity

$$\Lambda = My - N + \sum_{j=1}^N B_j^2 y + i\beta_j \begin{pmatrix} \nabla B_j \times y_2 \\ -\nabla B_j \times y_1 \end{pmatrix} + \widetilde{J}$$

holds true.

Proof. Let $\phi : \Omega \times [0,T] \to \cup_{n=1}^\infty R(P_n)$ be a simple function. By weak convergence and the skew-adjointness of M, we obtain

$$\begin{aligned}
&- \langle y, M\phi \rangle_{L^2(\Omega\times[0,T]\times D)^6} \\
&= - \lim_{n\to\infty} \langle y_n, M\phi \rangle_{L^2(\Omega\times[0,T]\times D)^6} \\
&= \lim_{n\to\infty} \langle My_n, \phi \rangle_{L^2(\Omega\times[0,T]\times D)^6} \\
&= \lim_{n\to\infty} \Big\langle \Lambda_n + P_n F(y_n) - P_n \widetilde{J} - P_n \sum_{j=1}^N B_j y_n - P_n \sum_{j=1}^N i\beta_j \begin{pmatrix} \nabla B_j \times y_{n,2} \\ -\nabla B_j \times y_{n,1} \end{pmatrix}, \phi \Big\rangle_{L^2(\Omega\times[0,T]\times D)^6} \\
&= \Big\langle \Lambda + N - \widetilde{J} - \sum_{j=1}^N B_j y - \sum_{j=1}^N i\beta_j \begin{pmatrix} \nabla B_j \times y_2 \\ -\nabla B_j \times y_1 \end{pmatrix}, \phi \Big\rangle_{L^2(\Omega\times[0,T]\times D)^6}.
\end{aligned}$$

Here, we were able to drop the P_n since $P_n\phi = \phi$ for large enough n. By density of simple functions and by the density of $\cup_{n=1}^\infty R(P_n)$ in $D(M)$ and in $L^p(D)^6$ for every $p \in (1,\infty)$

(see Corollary 4.1.6), we get

$$
\begin{aligned}
&-\langle y(t), M\psi\rangle_{L^2(D)^6} \\
&\qquad = \Big\langle \Lambda(t) + N(t) - \widetilde{J}(t) - \sum_{j=1}^{N} B_j y(t) - \sum_{j=1}^{N} i\beta_j(t)\begin{pmatrix} \nabla B_j \times y_2(t) \\ -\nabla B_j \times y_1(t)\end{pmatrix}, \psi\Big\rangle_{L^2(D)^6}
\end{aligned}
\tag{4.3.9}
$$

almost surely for almost every $t \in [0,T]$ and for every $\psi \in D(M) \cap L^{q+2}(D)^6$. This holds true especially for all $\psi \in C_c^\infty(D)^6$ and hence the definition of the weak version of the curl operator in Chapter 2 yields

$$
My(t) = \Lambda(t) + N(t) - \sum_{j=1}^{N} B_j^2 y(t) - i\beta_j(t)\begin{pmatrix} \nabla B_j \times y_2(t) \\ -\nabla B_j \times y_1(t)\end{pmatrix} - \widetilde{J}(t)
$$

almost surely for almost every $t \in [0,T]$. This proves the claimed result in case that $D = \mathbb{R}^3$ since we then do not have boundary conditions. So we can assume D to be a bounded C^1-domain for the rest of the proof. We show $y_1 \times \nu = 0$ on ∂D. Note that $\psi = (0, \phi)$ with $\phi \in C^1(\overline{D})^3$ is contained in $D(M) \cap L^{q+2}(D)$. We insert this into (4.3.9) and get

$$
\begin{aligned}
-\langle y_1(t), \operatorname{curl}\phi\rangle_{L^2(D)^3} &= \langle \Lambda_2(t) + N_2(t) - \widetilde{J}_2(t) - \sum_{j=1}^{N} B_j^2 y_2(t) + i\beta_j \nabla B_j \times y_1(t), \phi\rangle_{L^2(D)^3} \\
&= \langle -\operatorname{curl} y_1(t), \phi\rangle_{L^2(D)^3}
\end{aligned}
$$

almost surely for almost every $t \in [0,T]$ and for all $\phi \in C^1(\overline{D})^3$. By definition of the tangential trace in Definition 1.1.2, this shows $y_1 \times \nu = 0$ on ∂D almost surely for almost every $t \in [0,T]$. $\square$

Consequently, we pass to the weak limit in (4.2.4) and obtain

$$
\begin{cases}
dy(t) &= [My(t) - N(t) + A(t)y(t) + \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^{N} \widetilde{b}_i(t)\ \mathrm{d}\beta_i(t), \\
y_n(0) &= u_0
\end{cases}
\tag{4.3.10}
$$

as an equation in $L^2(\Omega; L^\infty(0,T; L^2(D)^6))$. So far, we showed $y \in L^2(\Omega; L^\infty(0,T; L^2(D)^6))$. However, Lemma 4.2.2 implies the pathwise continuity of $t \mapsto y(t) \in L^2(D)^6$.

It remains to show $N(t) = F(y(t))$. But this proof is step by step the same as in Proposition 4.2.7 and uses the monotonicity of the deterministic part of the equation.

All in all, we showed that $y \in L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega; C(0,T; L^2(D)^6))$ solves

$$
\begin{cases}
dy(t) &= [My(t) - F(y(t)) + A(t)y(t) + \widetilde{J}(t)]\,\mathrm{d}t + \sum_{i=1}^{N} \widetilde{b}_i(t)\ \mathrm{d}\beta_i(t), \\
y_n(0) &= u_0
\end{cases}
\tag{4.3.11}
$$

as an equation in $L^2(\Omega; L^\infty([0,T]; L^2(D)^6))$. Transforming the equation back with Proposition 4.3.2, we get the following result.

Proposition 4.3.7. (MSEE) *with* $G \equiv 0$ *has a unique strong solution* u *satisfying*

$$u \in L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega; C(0,T; L^2(D)^6))$$

and

$$Mu \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6 + L^2(\Omega; L^\infty(0,T; L^2(D)^6)).$$

Proof. By Lemma 4.3.6, we have

$$My + \sum_{j=1}^{N} i\beta_j \begin{pmatrix} \nabla B_j \times y_2 \\ -\nabla B_j \times y_1 \end{pmatrix} \in L^2(\Omega; L^\infty(0,T; L^2(D)^6)) + L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6.$$

Consequently, we can apply Proposition 4.3.2 and obtain a solution u of (MSEE) with $G \equiv 0$. Uniqueness is immediate by Proposition 4.2.7 since our solution is also a weak solution of the equation. □

Last but not least, we want to add the term $(G * u)$, using our theory on weak solutions. This leads to the main result of this chapter.

Theorem 4.3.8. (MSEE) *has a unique solution* u *satisfying*

$$u \in L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega; C(0,T; L^2(D)^6))$$

and

$$Mu \in L^{\frac{q+2}{q+1}}(\Omega \times [0,T] \times D)^6 + L^2(\Omega; L^\infty([0,T]; L^2(D)^6)).$$

Proof. Let $u \in L^{q+2}(\Omega \times [0,T] \times D)^6 \cap L^2(\Omega; C(0,T; L^2(D)^6))$ be the unique weak solution of (MSEE) from Proposition 4.2.8. The expression $(G * u)(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s$ is differentiable in time with

$$\partial_t (G * u)(t) = G(0)u(t) + \int_0^t G'(t-s)u(s)\,\mathrm{d}s.$$

Thus, by [M3], both $(G * u)$ and $\partial_t(G * u)$ are contained in $L^2(\Omega \times [0,T] \times D)^6$. Hence, u is a solution of (MSEE) with the current $G * u + J$ that satisfies [M4]. Consequently, u has the regularity properties from Proposition 4.3.7. This closes the proof. □

4.4. Remarks and discussion

In this section, we want to compare our results to the literature and we discuss some instructive special cases of our assumptions.

First, we want to mention that Roach, Stratis and Yannacopoulus already treated our equation in the deterministic setting in [88]. They claim in Theorem 11.3.14 that

$$\begin{cases} u'(t) = \kappa^{-1} M u(t) - \kappa^{-1} |u(t)|^q u(t) + \kappa^{-1}(G * u)(t) + \kappa^{-1} J(t), \quad t \in [0,T], \\ u(0) = u_0 \end{cases}$$

has a unique strong solution $u \in L^{q+2}([0,T] \times D)^6$ with $Mu \in L^{\frac{q+2}{q+1}}([0,T] \times D)^6$ if $D \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $\kappa : D \to \mathbb{R}^{6\times 6}$ is a uniformly bounded and uniformly elliptic matrix with measurable dependence in space. Their idea is to make a Galerkin approximation with respect to an orthonormal basis $(h_n)_n$ of $W^2(\mathrm{curl}, 0)(D) \times W^2(\mathrm{curl})(D)$ that is also a basis of $L^2(D)^6$. However, besides many inaccuracies, they make two mistakes which cannot be fixed in a direct way.

Beginning from (11.12) on page 239, they derive

$$\liminf_{n\to\infty} \int_0^T \big\langle (G * u_n)(s), u_n(s) \big\rangle_{L^2(D)^6} \, \mathrm{d}s \leq \int_0^T \big\langle (G * u)(s), u(s) \big\rangle_{L^2(D)^6} \, \mathrm{d}s$$

as $n \to \infty$ as a consequence of the weak convergences of $G * u_n \to G * u$ and $u_n \to u$ in $L^2([0,T] \times D)^6$ as $n \to \infty$. However, such an argument is not available in the general situation they discuss. Maybe one can fix this with strong assumptions on the convolution kernel G (see e.g. [36]). Moreover, in their a priori estimate for the approximating problem, they implicitly use

$$\|\sum_{j=1}^n \langle u_0, h_j \rangle_{L^2(D)^6} h_j\|_{L^{2(q+1)}(D)^6} \leq C \|u_0\|_{L^{2(q+1)}(D)^6}$$

with a constant independent of $n \in \mathbb{N}$, which would mean in our notation that the norm of $P_n : L^{2(q+1)}(D)^6 \to L^{2(q+1)}(D)^6$ could be estimated independent of n. However, this is not true in general. As far as we know, such a result is only known for the Fourier basis $h_n(x) = e^{inx}$ on the torus. This is the main reason why we had to use the operators S_n that are also bounded on $L^p(D)^6$ uniformly in n.

Getting back to our result, we want to point out that the restriction to $q \in (1,2]$ only comes from the Hölder estimate

$$\|F'(y_n) S_{n-1} \widetilde{b_j}\|_{L^2(\Omega\times[0,T]\times D)^6} \leq \|y_n\|^{2q}_{L^{q+2}(\Omega\times[0,T]\times D)^6} \|S_{n-1}\widetilde{b_j}\|^2_{L^{\frac{2(q+2)}{2-q}}(\Omega\times[0,T]\times D)^6}$$

in the proof of Proposition 4.3.5. Hence, if one assumes $b_j \equiv 0$ one gets the same result as in Theorem 4.3.8 for all $q \in (1, \infty)$. In particular, this is true for the deterministic equation. Especially, we gave a proof for the theorem of Roach, Stratis and Yannacopoulus if $\kappa \equiv I$ and D is a bounded C^1-domain or $D = \mathbb{R}^3$.

Next, we want to comment on the odd-looking condition

$$P_n\big(b_i(s) e^{-i \sum_{j=1}^N B_j \beta_j(s)}\big) = b_i(s) e^{-i \sum_{j=1}^N B_j \beta_j(s)}$$

from [M5] for all $s \in [0,T]$, $i = 1, \dots, N$ and for $n \in \mathbb{N}$ large enough in case that $q = 2$. We need it in the proof of Proposition 4.3.5 for the estimate

$$\|S_n\Big(b_i(s) e^{-i \sum_{j=1}^N B_j \beta_j(s)}\Big)\|_{L^\infty(D)^6} \leq C \|b_i(s) e^{-i \sum_{j=1}^N B_j \beta_j(s)}\|_{L^\infty(D)^6}$$

with a constant independent of $n \in \mathbb{N}$. It might be possible to get this inequality without our restrictive assumption in special cases. However, we want to point out that even in the case

$D = \mathbb{R}^3$ the boundedness of S_n on $L^\infty(D)^6$ is wrong since it would imply the boundedness of the Hilbert transform on $L^\infty(D)$. If the B_j are constant, the assumption reduces to $P_n b_i(s) = b_i(s)$ for all $s \in [0,T]$. If $D = \mathbb{R}^3$, this means that the Fourier transform $\widehat{b_i}(s)$ is compactly supported on a timely independent set. In case that D is a bounded C^1-domain, this means that b_i is of the form

$$b_i(s) = \sum_{k=1}^{M} b_i^{(k)}(s) h_k, \quad s \in [0,T]$$

for some scalar-valued $b_i^{(k)} : \Omega \times [0,T] \to \mathbb{C}$. Here, $(h_k)_k$ is the sequence of eigenvectors of the Hodge-Laplacian, we introduced in section 4.1.

Last but not least, we want to discuss why we did not treat coefficients in front of the Maxwell operator. Our approach is based on the interplay of M^2, Δ_H and the Helmholtz projection P_H. In fact, we showed $M^2 = \Delta_H$ on $R(P_H)$ and $M^2 = 0$ on $N(P_H) = N(M)$. One might say that we added a self-adjoint operator $A = \left(\begin{smallmatrix} -\operatorname{grad}\operatorname{div} \\ -\operatorname{grad}\operatorname{div}\end{smallmatrix}\right)$ with $N(A) = R(P_H)$ to M^2 such that the sum, namely Δ_H, generates a semigroup having generalised Gaussian bounds. This was essential for the definition of $(S_n)_n$ and $(P_n)_n$ from Section 4.1. If we now replace M by

$$M_{\varepsilon,\mu}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \varepsilon(x)^{-1}\operatorname{curl} u_2 \\ -\mu(x)^{-1}\operatorname{curl} u_1 \end{pmatrix}$$

with the same perfect conductor boundary condition $u_1 \times \nu = 0$ on ∂D and with uniformly bounded, positive definite and Hermitian $\varepsilon, \mu : D \to \mathbb{C}^{3\times 3}$. Hence, we have

$$M_{\varepsilon,\mu}^2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\varepsilon(x)^{-1}\operatorname{curl}\mu(x)^{-1}\operatorname{curl} u_1 \\ -\mu(x)^{-1}\operatorname{curl}\varepsilon(x)^{-1}\operatorname{curl} u_2 \end{pmatrix}$$

with the boundary condition $u_1 \times \nu = 0$ and $\left(\varepsilon^{-1}\operatorname{curl} u_2\right) \times \nu = 0$ on ∂D. The operator $-M_{\varepsilon,\mu}^2$ is then positive and self-adjoint with respect to a weighted inner product on $L^2(D)^6$, namely

$$\langle v, w\rangle_{\varepsilon,\mu} := \int_D \varepsilon(x) v_1(x)\cdot w_1(x)\,\mathrm{d}x + \int_D \mu(x) v_2(x)\cdot w_2(x)\,\mathrm{d}x.$$

To adapt the our strategy from the setting with $\varepsilon, \mu \equiv I$, we would need a weighted version of the Helmholtz projection $P_{\varepsilon,\nu}$. We project orthogonally with respect to $\langle\cdot,\cdot\rangle_{\varepsilon,\mu}$ onto

$$\Big\{(u_1, u_2) \in L^2(D)^6 : \operatorname{div}(\varepsilon u_1) = 0, \operatorname{div}(\mu u_2) = 0 \text{ and } (\mu u_2)\cdot\nu = 0 \text{ on } \partial D\Big\}.$$

Analogously to A from above, we define

$$A_{\varepsilon,\mu}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\operatorname{grad}\operatorname{div}(\varepsilon u_1) \\ -\operatorname{grad}\operatorname{div}(\mu u_2) \end{pmatrix}.$$

One calculates that $A_{\varepsilon,\mu}$ is symmetric with respect to $\langle\cdot,\cdot\rangle_{\varepsilon,\mu}$. Moreover, $M_{\varepsilon,\mu}^2$, $M_{\varepsilon,\mu}^2 + A_{\varepsilon,\mu}$ and $P_{\varepsilon,\mu}$ have the same relationship as their counterparts with $\varepsilon = \mu = I$.

Hence, to follow our proof strategy, one has to show that the semigroup generated by $M_{\varepsilon,\mu}^2 + A_{\varepsilon,\mu}$ on the domain

$$\Big\{\operatorname{curl} u_1, \operatorname{curl} u_2, \operatorname{curl}\mu^{-1}\operatorname{curl} u_1, \operatorname{curl}\varepsilon^{-1}\operatorname{curl} u_2 \in L^p(D)^3, \operatorname{div}(\varepsilon u_1) \in W_0^{1,p}(D),$$
$$\operatorname{div}(\mu u_2) \in W^{1,p}(D), u_1 \times \nu = 0, (\mu u_2)\cdot\nu = 0, (\varepsilon^{-1}\operatorname{curl} u_2)\times\nu = 0 \text{ on } \partial D\Big\}$$

satisfies generalised Gaussian bounds. However, even in case of smooth ε, μ and ∂D such a result is unknown so far.

APPENDIX A

A note on pseudodifferential operators with rough symbols

In the research that led to this thesis, we also looked at nonautonomous deterministic evolution equations of the form

$$\begin{cases} u'(t) &= A(t)u(t) + f(t), \quad t \in [0,T], \\ u(0) &= u_0 \end{cases}$$

on a Banach space X with closed and densely defined operators $(A(t))_{t\in[0,T]}$. This equation is said to have maximal regularity if for given $f \in L^p(0,T;X)$, there exists a unique solution $u \in W^{1,p}(0,T;X)$ such that $t \mapsto A(t)u(t) \in L^p(0,T;X)$. A typical question in this area of research is how much regularity one has to assume on $t \mapsto A(t)$ to get maximal regularity.

One remarkable approach for commuting operators $(A(t))_{t\in[0,T]}$ is by Gallarati and Veraar (see [39]). They used the formula

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,\mathrm{d}s,$$

where $(U(t,s))_{0\le s\le t\le T}$ is the evolution family generated by $(A(t))_{t\in[0,T]}$ and proved that the operator

$$f \mapsto \int_0^{\cdot} A(\cdot)U(\cdot,s)\,\mathrm{d}s$$

is bounded on $L^p(0,T;X)$. Another approach is to use the formula

$$\begin{aligned} u(t) =& e^{-tA(t)}u_0 + \int_0^t A(t)^2 e^{-(t-s)A(t)}\big(A(t)^{-1} - A(s)^{-1}\big)A(s)u(s)\,\mathrm{d}s \\ &+ \int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s, \end{aligned}$$

that was introduced by Acquistapace and Terreni in [1]. In this setting, one mainly has to show that the operator L defined by

$$Lf(t) = \int_0^t A(t)e^{-(t-s)A(t)}f(s)\,\mathrm{d}s$$

is bounded on $L^p(0,T;X)$ and that the operator

$$(I-Q)g(t) := g(t) - \int_0^t A(t)^2 e^{-(t-s)A(t)}\big(A(t)^{-1} - A(s)^{-1}\big)g(s)\,\mathrm{d}s$$

is invertible in $L^p(0,T;X)$. This method has been successfully used by [83] where the authors proved maximal regularity under the so called Acquistapace-Terreni conditions on $t \mapsto A(t)$. Roughly speaking the Acquistapace-Terreni conditions mean that the domain $D(A(t))$ is allowed to vary in time, whereas $D(A(t)^\nu)$ is fixed for some $\nu \in (0,1]$ and $t \mapsto A(t)$ is Hölder continuous of order μ with $\nu + \mu > 1$.

To prove the boundedness of L, the authors in [83] used the representation

$$Lf(t) = \mathcal{F}^{-1}\big(\xi \mapsto A(t)R(2\pi i\xi, -A(t))\hat{f}(\xi)\big)$$

to apply their theorems on boundedness of pseudodifferential operators on UMD Banach spaces to L. However, as they have a quite general approach, they still need some Hölder regularity of $t \mapsto A(t)$. However, we found a way to show the boundedness of L if $t \mapsto A(t)$ is measurable in time solely assuming that the operators $A(t)^\varepsilon B^{-\varepsilon}$ and $A(t)^{-\varepsilon}B^{\varepsilon}$ are bounded on X for some $\varepsilon > 0$ and for some operator B that has a bounded H^∞-calculus. Unfortunately, we were not able to use this for a new theory on maximal regularity with domains varying in time because we were unable to improve the known results on $(I-Q)$. Though, we want to present our approach to the boundedness of pseudodifferential operators with a symbol that has a special structure in this thesis, since we believe it to be interesting on its own.

Proposition A.0.1. *Let $1 < p < \infty$, X be a UMD Banach space and Y Banach space. In addition, let Y^* be a closed subspace of Y' that is norming for Y. Further, we assume that $a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{B}(X,Y))$ is weakly differentiable in the second component and that $\partial_\xi a(t,\xi)$ factorises as*

$$\partial_\xi a(t,\xi) = \xi^{-1}\phi_1(\xi)T(t,\xi)\phi_2(\xi) \tag{A.0.1}$$

for almost all $t,\xi \in \mathbb{R}$, where $T(t,\xi) \in \mathcal{B}(X,Y)$ satisfies the following conditions:

i) The map $(t,\xi) \mapsto T(t,\xi)x$ is strongly measurable for all $x \in X$.

ii) The set $\{T(\cdot,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(L^p(\mathbb{R};X), L^p(\mathbb{R};Y))$ is γ-bounded.

Further we assume that $\phi_1(\xi) \in \mathcal{B}(Y)$ with $\phi_1(\xi)'(Y^*) \subset Y^*$ and $\phi_2(\xi) \in \mathcal{B}(X)$ are linear operators satisfying the square function estimates

1) $\|\phi_1(\pm\xi)'g\|_{\gamma(\mathbb{R}_{>0}, \frac{\mathrm{d}\xi}{\xi}, L^{p'}(\mathbb{R};Y^*))} \lesssim \|g\|_{L^{p'}(\mathbb{R};Y^*)}$,

2) $\|\phi_2(\pm\xi)f\|_{\gamma(\mathbb{R}_{>0}, \frac{\mathrm{d}\xi}{\xi}, L^{p}(\mathbb{R};X))} \lesssim \|f\|_{L^{p}(\mathbb{R};X)}$.

for all $f \in L^p(\mathbb{R};X)$ and $g \in L^{p'}(\mathbb{R};Y^*)$. In this case, the pseudodifferential operator

$$T_a f(t) = \mathcal{F}^{-1}(\xi \mapsto a(t,\xi)\hat{f}(\xi))(t)$$

that is well-defined on $S(\mathbb{R},X)$ extends to a bounded operator from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$.

Proof. At first, we assume that $a(t,\cdot)$ is compactly supported for almost all $t\in\mathbb{R}$. We use integration by parts to obtain

$$\begin{aligned} T_a f(t) &= -\int_{-\infty}^{\infty} \partial_\xi a(t,\xi)\int_0^\xi \hat{f}(\mu)e^{2\pi i\mu t}\,\mathrm{d}\mu\,\mathrm{d}\xi \\ &= -\int_{-\infty}^{\infty} \partial_\xi a(t,\xi) S_{\mathbf{1}_{(0,\xi)}} f(t)\,\mathrm{d}\xi \\ &= -\int_{-\infty}^{\infty} \phi_1(\xi A_1)T(t,\xi)\phi_2(\xi A_2)(S_{\mathbf{1}_{(0,\xi)}}f)(t)\frac{\mathrm{d}\xi}{\xi} \end{aligned}$$

for $f\in S(\mathbb{R},X)$, where $S_{\mathbf{1}_{(0,\xi)}}$ is the operator associated with the Fourier multiplier $\mathbf{1}_{(0,\xi)}$. Testing this expression with $g\in S(\mathbb{R},Y^*)$ yields

$$\begin{aligned} &\langle T_a f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))} \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \big\langle \phi_1(\xi)T(t,\xi)\phi_2(\xi)(S_{\mathbf{1}_{(0,\xi)}}f)(t), g(t)\big\rangle_{(Y,Y')}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{d}t \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \big\langle T(t,\xi)\phi_2(\xi)(S_{\mathbf{1}_{(0,\xi)}}f)(t), \phi_1(\xi)'g(t)\big\rangle_{(Y,Y')}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{d}t \\ &= -\int_{\mathbb{R}}\int_0^{\infty} \big\langle T(t,\xi)\phi_2(\xi)(S_{\mathbf{1}_{(0,\xi)}}f)(t), \phi_1(\xi)'g(t)\big\rangle_{(Y,Y')}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{d}t \\ &\quad -\int_{\mathbb{R}}\int_0^{\infty} \big\langle T(t,-\xi)\phi_2(-\xi)(S_{\mathbf{1}_{(0,-\xi)}}f)(t), \phi_1(-\xi)'g(t)\big\rangle_{(Y,Y')}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{d}t. \end{aligned}$$

Note that Hille's Theorem allows us to commute the integral operator $S_{\mathbf{1}_{(0,\pm\xi)}}$ and $\phi_2(\pm\xi)$ and therefore by Fubini and γ-Hölder, we have

$$\begin{aligned} &\Big|\langle T_a f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R},Y^*))}\Big| \\ &\le \sum_{j\in\{1,-1\}}\left|\int_0^\infty \big\langle T(\cdot,j\xi)\phi_2(j\xi)(S_{\mathbf{1}_{(0,j\xi)}}f), \phi_1(j\xi)'g\big\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R},Y'))}\frac{\mathrm{d}\xi}{\xi}\right| \\ &= \sum_{j\in\{1,-1\}}\left|\int_0^\infty \big\langle T(\cdot,j\xi)(S_{\mathbf{1}_{(0,j\xi)}}\phi_2(j\xi)f), \phi_1(j\xi)'g\big\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))}\frac{\mathrm{d}\xi}{\xi}\right| \\ &\le \sum_{j\in\{1,-1\}} \|T(\cdot,j\xi)(S_{\mathbf{1}_{(0,j\xi)}}\phi_2(j\xi)f)\|_{\gamma(\mathbb{R}_{\ge 0},\frac{d\xi}{\xi},L^p(\mathbb{R};Y))}\|\phi_1(j\xi)'g\|_{\gamma(\mathbb{R}_{\ge 0},\frac{d\xi}{\xi},L^{p'}(\mathbb{R};Y'))}. \end{aligned}$$

Since Y^* is a closed subspace of Y', we can use the square function estimate 1) to obtain

$$\|\phi_1(j\xi)'g\|_{\gamma(\mathbb{R}_{>0},\frac{d\xi}{\xi},L^{p'}(\mathbb{R};Y'))} = \|\phi_1(j\xi)'g\|_{\gamma(\mathbb{R}_{>0},\frac{d\xi}{\xi},L^{p'}(\mathbb{R};Y^*))} \lesssim \|g\|_{L^{p'}(\mathbb{R};Y^*)}.$$

Moreover, by Lemma 2.2 in [39], $L^{p'}(\mathbb{R},Y^*)$ is norming for $L^p(\mathbb{R};Y)$ and thus taking the supremum over all $g\in S(\mathbb{R},Y^*)$ with $\|g\|_{L^{p'}(\mathbb{R},Y^*)}=1$ which is dense in the unit sphere of $L^{p'}(\mathbb{R},Y^*)$ yields

$$\|T_a f\|_{L^p(\mathbb{R};Y)} \le \sum_{j\in\{1,-1\}} \|T(\cdot,j\xi)(S_{\mathbf{1}_{(0,j\xi)}}\phi_2(j\xi)f)\|_{\gamma(\mathbb{R}_{>0},\frac{d\xi}{\xi},L^p(\mathbb{R};Y))}. \tag{A.0.2}$$

By assumption $\{T(\cdot,\xi):\xi\in\mathbb{R}\}\subset\mathcal{B}(L^p(\mathbb{R};X),L^p(\mathbb{R};Y))$ is γ-bounded. Therefore, we can employ the theorem about γ-bounded pointwise multipliers (see [94], Theorem 5.2) in

(A.0.2) to get

$$\|T_a f\|_{L^p(\mathbb{R};Y)} \leq \sum_{j\in\{1,-1\}} \|(S_{\mathbf{1}_{(0,j\xi)}}\phi_2(j\xi)f)\|_{\gamma(\mathbb{R}_{>0},\frac{d\xi}{\xi},L^p(\mathbb{R};X))}.$$

By Discussion 3.5 in [70] we know that the family $\{S_{\mathbf{1}_{[0,\xi]}} : \xi \in \mathbb{R}\} \subset \mathcal{B}(L^p(\mathbb{R};X))$ is $\mathcal{R}$-bounded and particularly γ-bounded since UMD spaces have finite cotype (see [71], Theorem 1.1). In conclusion, we can apply the multiplier theorem once again and this results in

$$\|T_a f\|_{L^p(\mathbb{R};Y)} \lesssim \sum_{j\in\{1,-1\}} \|\phi_2(j\xi)f)\|_{\gamma(\mathbb{R}_{>0},\frac{d\xi}{\xi},L^p(\mathbb{R};X))}.$$

Using the square function estimate 2) and the density of $S(\mathbb{R};X)$ in $L^p(\mathbb{R};X)$ completes the proof for symbols with compact support in the second component.

Note that the constants in the estimates from above do not depend on the size of the support of $a(t,\cdot)$. Hence, we can deal with the general case by approximation. Let a be a symbol without compact support and $\psi \in C_c^\infty(-2,2)$ be a cut-off function taking values in $[0,1]$ with $\psi \equiv 1$ on $[-1,1]$. Define $a_n(t,\xi) := a(t,\xi)\psi(\frac{\xi}{n})$. Then a_n converges pointwise to a for $n\to\infty$ and $\|a_n(t,\xi)\|_{\mathcal{B}(X)} \leq \|a(t,\xi)\|_{\mathcal{B}(X)}$ for all $t,\xi\in\mathbb{R}$.

Thus, for $f \in S(\mathbb{R},X)$ and $g \in S(\mathbb{R},X')$, dominated convergence yields

$$\begin{aligned}\langle T_a f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \langle a(t,\xi)\hat{f}(\xi), g(t)\rangle_{(Y,Y')} e^{2\pi i t\xi}\,\mathrm{d}\xi\,\mathrm{d}t \\ &= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} \langle a_n(t,\xi)\hat{f}(\xi), g(t)\rangle_{(X,X')} e^{2\pi i t\xi}\,\mathrm{d}\xi\,\mathrm{d}t \\ &= \lim_{n\to\infty}\langle T_{a_n} f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))}.\end{aligned}$$

Using the result for compactly supported symbols, we conclude

$$\begin{aligned}|\langle T_a f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))}| &= \lim_{n\to\infty}|\langle T_{a_n} f, g\rangle_{(L^p(\mathbb{R};Y),L^{p'}(\mathbb{R};Y'))}| \\ &\lesssim \|f\|_{L^p(\mathbb{R};Y)}\|g\|_{L^p(\mathbb{R};Y')}\end{aligned}$$

for all $f \in S(\mathbb{R},X)$ and $g \in S(\mathbb{R},Y')$. This finishes the proof. □

For an application of this theorem to a concrete situation, it is important that we demand the square function estimate 1) only on a closed and norming subspace Y^* of Y', since the functions ϕ_1 and ϕ_2 are nearly always induced by an operator having a bounded H^∞-calculus. If an operator B on Y has a bounded H^∞-calculus then B' doesn't necessarily inherit this property in non-reflexive spaces. Therefore, we have to introduce the moon-dual space $Y^\#$ and the moon-dual operator $B^\#$ before we can give some examples for ϕ_1 and ϕ_2.

For a Banach space Z and an injective and sectorial operator B on Z with dense range the moon-dual space $Z^\# \subset Z'$ is defined by $Z^\# = \overline{D(B')} \cap \overline{R(B')}$ where the closure is taken in the norm of Z'. Moreover, one can define the moon-dual operator $B^\#$ as the part of B' in $Z^\#$. The advantage of this construction is that $B^\#$ is also an injective and sectorial operator with dense range. Moreover, if B has a bounded H^∞-calculus on Z, then $B^\#$ also has bounded H^∞-calculus on $Z^\#$ of the same angle. Last but not least, it is important to note that $Z^\#$ is still norming for Z. For more details and references on the moon-dual operator, we refer to appendix 15 in [70]. Note that this construction is not needed if Z is reflexive (for example if Z is UMD), since $Z^\#$ and Z' coincide in this case.

Now, we can give the most important examples for functions ϕ_1 and ϕ_2.

Example A.0.2. *Let A be a sectorial operator with $0 \in \rho(A)$ having a bounded H^∞-calculus of angle $\omega \in [0, \pi/2)$ on a Banach space Z with finite cotype and let $\phi \in H_0^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi/2)$, i.e there are constants $C > 0$ and $\alpha > 0$ such that*

$$|\phi(z)| \le C|z|^\alpha (1 + |z|)^{-2\alpha}$$

for all $z \in \Sigma_\theta$. Then, the operators $\phi_A(\xi) = \phi(|\xi|A)$ or alternatively $\phi_A(\xi) = \phi(|\xi|^{-1}A)$ satisfy $\phi_A(\xi)'(Z^\#) \subset Z^\#$ and

1) $\|\phi_A(\pm\xi)'x'\|_{\gamma(\mathbb{R}_{>0}, \frac{\mathrm{d}\xi}{\xi}; Z^\#)} \lesssim \|x'\|_{Z^\#}$

2) $\|\phi_A(\pm\xi)x\|_{\gamma(\mathbb{R}_{>0}, \frac{\mathrm{d}\xi}{\xi}; Z)} \lesssim \|x\|_Z$.

for all $x \in Z$ and $x' \in Z^\#$. In particular, the conditions 1.) and 2.) of Proposition A.0.1 are fulfilled if we choose $X = Z$ and $Y^* = Z^\#$.

Proof. The claimed result for the choice $\phi_A(\xi) = \phi(|\xi|A)$ is an immediate consequence of the square function theorem of Kalton and Weis (see [55]). If one chooses $\phi_A(\xi) = \phi(|\xi|^{-1}A)$ instead, one can apply the same estimates, since for a function $f \in \gamma(\mathbb{R}_{>0}, \frac{d\xi}{\xi}; X)$ one always has

$$\|f\|_{\gamma(\mathbb{R}_{>0}, \frac{d\xi}{\xi}, X)} = \|f((\cdot)^{-1})\|_{\gamma(\mathbb{R}_{>0}, \frac{d\xi}{\xi}, X)}.$$

This follows from the observation that $g \mapsto g((\cdot)^{-1})$ is an isometry on $L^2(\mathbb{R}_{>0}, \frac{d\xi}{\xi})$ and from Corollary 6.3 in [94]. $\square$

In concrete situations, one might use the following square function.

Example A.0.3. *Let A be a sectorial operator with $0 \in \rho(A)$ having a bounded H^∞-calculus of angle $\omega \in [0, \pi/2)$ on a Banach space Z with finite cotype. Then the operators*

$$\phi_A(\xi) = (|\xi|A)^\varepsilon (i|\xi|^{2\varepsilon} + A^{2\varepsilon})^{-1}$$
$$\widetilde{\phi}_A(\xi) = (|\xi|A)^\varepsilon (i + |\xi|^{2\varepsilon} A^{2\varepsilon})^{-1}$$

for $\xi \in \mathbb{R}$ satisfy the conditions 1) and 2) of Proposition A.0.1 if we choose $X = Z$ and $Y^* = Z^\#$.

Proof. We apply the example from above with the function $\phi(z) = z^\varepsilon(i + z^{2\varepsilon})^{-1}$ and we plug-in $|\xi|^{-1}A$ and $|\xi|A$ respectively. $\square$

Using this special square function, we can derive a corollary that can be used in a concrete situation. First, we have to introduce some notation. Given a sectorial operator A with $0 \in \rho(A)$ on a Banach space X, we define the spaces X^α for $\alpha \geq 0$ as the domain of A^α. If α is negative, we define X^α as the completion of X with respect to the norm $\|A^\alpha \cdot\|_X$.

Corollary A.0.4. *Let $1 < p < \infty$, $\varepsilon \in (0, 1/2]$ and let X be a UMD Banach space. Further let A be a sectorial operator with $0 \in \rho(A)$ having a bounded H^∞-calculus of angle $\omega \in [0, \pi/2)$ on X. Moreover, we assume that $a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{B}(X))$ is a continuously differentiable symbol in the second component and that the following sets are R-bounded with a constant not depending on t.*

i) $\{|\xi|^{1+2\varepsilon}\partial_\xi a(t,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X^\varepsilon, X^{-\varepsilon})$,

ii) $\{|\xi|\partial_\xi a(t,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X^{-\varepsilon}, X^{-\varepsilon})$,

iii) $\{|\xi|\partial_\xi a(t,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X^\varepsilon, X^\varepsilon)$,

iv) $\{|\xi|^{1-2\varepsilon}\partial_\xi a(t,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X^{-\varepsilon}, X^\varepsilon)$.

In this case, the pseudodifferential operator

$$T_a f(t) = \mathcal{F}^{-1}(\xi \mapsto a(t,\xi)\hat{f}(\xi))(t)$$

that is well-defined on $S(\mathbb{R}, X)$ extends to a bounded operator on $L^p(\mathbb{R}; X)$.

Proof. We define $\phi(\xi) = (A|\xi|)^\varepsilon(i|\xi|^{2\varepsilon} + A^{2\varepsilon})^{-1}$ for $\xi \in \mathbb{R}$. By Example A.0.3 we know that $\phi(\xi)$ satisfies the conditions of Proposition A.0.1. Moreover, we can decompose $\partial_\xi a(t,\xi)$ in the following way taking advantage of the invertibility of A.

$$\partial_\xi a(t,\xi) = \xi^{-1}\phi(\xi)\left(\xi|\xi|^{-\varepsilon}A^{-\varepsilon}(i|\xi|^{2\varepsilon} + A^{2\varepsilon})\partial_\xi a(t,\xi)|\xi|^{-\varepsilon}A^{-\varepsilon}(i|\xi|^{2\varepsilon} + A^{2\varepsilon})\right)\phi(\xi)$$

for all $t, \xi \in \mathbb{R}$. Defining

$$T(t,\xi) := \xi|\xi|^{-2\varepsilon}A^{-\varepsilon}(i|\xi|^{2\varepsilon} + A^{2\varepsilon})\partial_\xi a(t,\xi)A^{-\varepsilon}(i|\xi|^{2\varepsilon} + A^{2\varepsilon})$$

we just have to check that T fulfils the conditions in Proposition A.0.1. We calculate

$$\begin{aligned} T(t,\xi) =& A^{-\varepsilon}(\xi|\xi|^{2\varepsilon}\partial_\xi a(t,\xi))A^{-\varepsilon} + iA^{-\varepsilon}(\xi\partial_\xi a(t,\xi))A^\varepsilon \\ &+ iA^\varepsilon(\xi\partial_\xi a(t,\xi))A^{-\varepsilon} + A^\varepsilon(\xi|\xi|^{-2\varepsilon}\partial_\xi a(t,\xi))A^\varepsilon. \end{aligned}$$

Thus, the assumptions on the $\mathcal{R}$-bounded sets imply that $\{T(t,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded with a constant not depending on t. It remains to check that this implies the $\mathcal{R}$-boundedness of $\{T(\cdot,\xi) : \xi \in \mathbb{R}\} \subset \mathcal{B}(L^p(\mathbb{R}; X))$. Indeed, let $(r_k)_{k=1}^N$ be a sequence of independent Rademacher random variables, $(f_k)_{k=1}^N \subset L^p(\mathbb{R}; X)$ and $(\xi_k)_{k=1}^N \subset \mathbb{R}$. Then, the result from above and Fubini yield

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{k=1}^{N} r_k T(\cdot,\xi_k) f_k\Big\|^p_{L^p(\mathbb{R};X)} &= \int_{\mathbb{R}} \mathbb{E}\Big\|\sum_{k=1}^{N} r_k T(t,\xi_k) f_k(t)\Big\|^p_X \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left((\mathbb{E}\Big\|\sum_{k=1}^{N} r_k T(t,\xi_k) f_k(t)\Big\|^p_X)^{1/p} \right)^p \mathrm{d}t \\
&\lesssim \int_{\mathbb{R}} \left((\mathbb{E}\Big\|\sum_{k=1}^{N} r_k f_k(t)\Big\|^p_X)^{1/p} \right)^p \mathrm{d}t \\
&= \mathbb{E}\int_{\mathbb{R}} \Big\|\sum_{k=1}^{N} r_k f_k(t)\Big\|^p_X \,\mathrm{d}t = \mathbb{E}\Big\|\sum_{k=1}^{N} r_k f_k\Big\|^p_{L^p(\mathbb{R};X)}.
\end{aligned}$$

In conclusion, the claim follows by Proposition A.0.1. □

We want to apply this result to an operator that plays an important role in the proof of maximal regularity for non-autonomous parabolic evolution equations. Given a family $(A(t))_{t\in[0,T]}$ of uniformly sectorial operators on a Banach space X with $0 \in \rho(A(t))$ for all $t \in [0,T]$, we consider the operator

$$Lf(t) = A(t)\int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s. \tag{A.0.3}$$

Further, we ask how much regularity the map $t \mapsto A(t)$ must have to guarantee the boundedness of L on $L^p(\mathbb{R}, X)$ for $1 < p < \infty$. This operator was already discussed by Portal and Strkalj in [83] and by Haak and Ouhabaz in [43]. In both articles, the authors derived the pseudodifferential operator representation formula (see for example [43], (2.9))

$$Lf(t) = \mathcal{F}^{-1}\left(\xi \mapsto A(t)R(2\pi i, -A(t))\hat{f}(\xi)\right)$$

for $t \in [0,T]$ and $f \in C_c^\infty(0,T;X)$ and applied their theorems for operator valued symbol classes. Portal and Strkalj assumed Hölder continuity in the time component of the symbol and Haak and Ouhabaz who only proved their theorem in a Hilbert space setting replaced this by a quite similar Dini-continuity condition in the time component. As a consequence, in both publications the authors needed a regularity condition on $t \mapsto A(t)$ to show maximal regularity. Haak and Ouhabaz required that $A(t)$ is a associated with a coercive sesquilinear form $\mathfrak{a}(t;\cdot,\cdot) : V \times V \to \mathbb{C}$ on a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ and that there exists a non-decreasing positive function ω with

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \le \omega(t)\|u\|_V\|v\|_V$$

for all $u, v \in V$ and with $\int_0^T \frac{\omega(t)}{t}\,\mathrm{d}t < \infty$ to prove $L^p(0,T;H)$ boundedness of L. This requirement is in the same spirit as the Acquistapace-Tereni condition (see [83], section 5, (AT)) Portal and Strkalj demand on $t \mapsto A(t)$ in order to achieve $L^p(0,T;X)$-boundedness in case that X is a UMD Banach space.

Applying Corollary A.0.4 we can prove $L^p(0,T;X)$-boundedness of L only requiring measurability. Note that the assumption $0 \in \rho(A(t))$ for all $t \in [0,T]$ is not restrictive since the maximal regularity property is invariant under spectral shifts, i.e. the equation

$$\begin{cases} u'(t) &= -A(t)u(t) + f(t), \quad t \in [0,T], \\ u(0) &= u_0 \end{cases}$$

has maximal regularity if and only if

$$\begin{cases} u'(t) &= -(A(t)+\mu)u(t) + f(t), \quad t \in [0,T], \\ u(0) &= u_0 \end{cases}$$

has maximal regularity for one $\mu \in \mathbb{R}$.

Corollary A.0.5. *Let X be a UMD Banach space, $1 < p < \infty$ and $(A(t))_{t\in[0,T]}$ a uniformly $\mathcal{R}$-sectorial family of operators on X with $0 \in \rho(A(t))$ for all $t \in [0,T]$. More precisely there exists $\omega \in [0,\pi/2)$ such that $\sigma(A(t)) \subset \Sigma_\omega$ for all $t \in [0,T]$ and such that the sets*

$$\{|\lambda| R(\lambda, A(t)) : \lambda \notin \Sigma_\theta\} \subset \mathcal{B}(X) \tag{A.0.4}$$

are $\mathcal{R}$-bounded for all $\theta \in (\omega,\pi)$ and $t \in [0,T]$ with a constant depending on θ but not on t. Further we assume that there exist an invertible and sectorial operator A_0 with a bounded $H^\infty(\Sigma_\omega)$-calculus such that there is $\varepsilon > 0$ with $\|A_0^{-\varepsilon}x\|_X \simeq \|A(t)^{-\varepsilon}x\|_X$ and $\|A_0^{\varepsilon}x\|_X \simeq \|A(t)^{\varepsilon}x\|_X$ for all $x \in X$ and $t \in [0,T]$. Then, strong measurability of $t \mapsto A(t)R(2\pi i\xi, -A(t))x$ for all $x \in X$ and $\xi \in \mathbb{R}$ is sufficient for operator

$$Lf(t) = \mathcal{F}^{-1}\left(\xi \mapsto A(t)R(2\pi i\xi, -A(t))\hat{f}(\xi)\right)$$

that is well-defined on $C_c^\infty(0,T;X)$ to extend to a bounded operator on $L^p(0,T;X)$.

Proof. Defining $a(t,\xi) = A(t)R(2\pi i\xi, -A(t))$ for $t \in [0,T]$ and $a(t,\xi) = 0$ for $t \notin [0,T]$ we have $a \in L^\infty(\mathbb{R}\times\mathbb{R}, \mathcal{B}(X))$ and $\xi \mapsto a(t,\xi)$ is continuously differentiable with derivative

$$\partial_\xi a(t,\xi) = -2\pi i A(t) R(2\pi i\xi, -A(t))^2$$

for all $t \in [0,T]$. So, we just have to check the $\mathcal{R}$-boundedness of the sets $i) - iv)$ in Corollary A.0.4.

First, we have to show that for an $\mathcal{R}$- sectorial operator B on X and $\alpha \in (0,1)$ the set

$$\{|\xi|^{1-\alpha} B^\alpha R(2\pi i\xi, -B) : \xi \in \mathbb{R}\} \subset \mathcal{B}(X) \tag{A.0.5}$$

is also $\mathcal{R}$-bounded. By functional calculus, we have

$$\begin{aligned} \xi^{1-\alpha} B^\alpha R(2\pi i\xi, -B) &= \frac{1}{2\pi i}\int_\Gamma \frac{z^\alpha \xi^{1-\alpha}}{2\pi i\xi + z} R(z,B)\,\mathrm{dz} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{z^{\alpha-1}\xi^{1-\alpha}}{2\pi i\xi + z}(zR(z,B))\,\mathrm{dz} \end{aligned}$$

for all $\xi > 0$ and for some path $\Gamma = \partial\Sigma_\theta$ with $\omega < \theta < \pi/2$. Applying Corollary 2.14 in [70] yields the claimed result since the functions

$$h_\xi(z) := \frac{z^{\alpha-1}\xi^{1-\alpha}}{2\pi i\xi + z}$$

are uniformly in $L^1(\Gamma, \mathrm{dz})$ and $\{zR(z,B) : z \in \Sigma_\theta\}$ is $\mathcal{R}$-bounded by assumption. Indeed a substitution gives

$$\int_\Gamma |h_\xi(z)|\,\mathrm{d}|\mathrm{z}| = \xi^{-\alpha} \int_\Gamma \frac{|z|^\alpha}{|2\pi i + \frac{z}{\xi}|} \frac{\mathrm{d}|\mathrm{z}|}{|z|} = \int_\Gamma \frac{|z|^\alpha}{|2\pi i + z|} \frac{\mathrm{d}|\mathrm{z}|}{|z|} < \infty.$$

Now, we prove the required $\mathcal{R}$-boundedness of the following sets with a constant not depending on t.

i) $\{|\xi|^{1-2\varepsilon} A_0^\varepsilon \partial_\xi a(t,\xi) A_0^\varepsilon : \xi \in \mathbb{R}\} \subset \mathcal{B}(X)$,

ii) $\{|\xi|^{1+2\varepsilon} A_0^{-\varepsilon} \partial_\xi a(t,\xi) A_0^{-\varepsilon} : \xi \in \mathbb{R}\} \subset \mathcal{B}(X)$,

iii) $\{|\xi| A_0^\varepsilon \partial_\xi a(t,\xi) A_0^{-\varepsilon} : \xi \in \mathbb{R}\} \subset \mathcal{B}(X)$,

iv) $\{|\xi| A_0^{-\varepsilon} \partial_\xi a(t,\xi) A_0^\varepsilon : \xi \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Exemplarily, we discuss *i)* and *ii)*. The $\mathcal{R}$-boundedness of the other sets follows in the very same way. Exploiting the equivalence of the norms $\|A_0^\varepsilon \cdot\|_X$ and $\|A(t)^\varepsilon \cdot\|_X$ and accordingly $\|A_0^{-\varepsilon} \cdot\|_X$ and $\|A(t)^{-\varepsilon} \cdot\|_X$ as well as (A.0.5) with $B = A(t)$, we obtain

$$\begin{aligned}
&R\left(\{|\xi|^{1-2\varepsilon} A_0^\varepsilon \partial_\xi a(t,\xi) A_0^\varepsilon : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi|^{1-2\varepsilon} A_0^\varepsilon A(t) R(2\pi i\xi, -A(t))^2 A_0^\varepsilon : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi|^{1-2\varepsilon} A(t)^\varepsilon A(t) R(2\pi i\xi, -A(t))^2 A(t)^\varepsilon : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{A(t) R(2\pi i\xi, -A(t)) |\xi|^{1-2\varepsilon} A(t)^\varepsilon R(2\pi i\xi, -A(t)) A(t)^\varepsilon : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{A(t) R(2\pi i\xi, -A(t)) |\xi|^{1-2\varepsilon} A(t)^{2\varepsilon} R(2\pi i\xi, -A(t)) : \xi \in \mathbb{R}\}\right)\\
&\quad\lesssim 1.
\end{aligned}$$

This implies *i)*. Here, the independence from t is an immediate consequence of the independence of t in assumption (A.0.4). Analogously, we have

$$\begin{aligned}
&R\left(\{|\xi|^{1+2\varepsilon} A_0^{-\varepsilon} \partial_\xi a(t,\xi) A_0^{-\varepsilon} : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi|^{1+2\varepsilon} A_0^{-\varepsilon} A(t) R(2\pi i\xi, -A(t))^2 A_0^{-\varepsilon} : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi|^{1+2\varepsilon} A(t)^{-\varepsilon} A(t) R(2\pi i\xi, -A(t))^2 A(t)^{-\varepsilon} : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi| R(2\pi i\xi, -A(t)) |\xi|^{2\varepsilon} A(t)^{1-\varepsilon} R(2\pi i\xi, -A(t)) A(t)^{-\varepsilon} : \xi \in \mathbb{R}\}\right)\\
&\quad\simeq R\left(\{|\xi| R(2\pi i\xi, -A(t)) |\xi|^{2\varepsilon} A(t)^{1-2\varepsilon} R(2\pi i\xi, -A(t)) : \xi \in \mathbb{R}\}\right)\\
&\quad\lesssim 1.
\end{aligned}$$

□

We want to mention that this corollary also implies the L^p-boundedness of L in the autonomous case $A(t) \equiv A$. This famous result was initially shown by Lutz Weis in [102] and reads as follows.

Theorem A.0.6. *Let A be a closed, densely defined and $\mathcal{R}$-sectorial operator on a UMD Banach space X. Then, the operator*

$$Lf(t) = \int_0^t Ae^{-(t-s)A} f(s)\,\mathrm{ds}$$

that is well-defined on $C_c^\infty(0,T;D(A))$ extends to a bounded operator on $L^p(0,T;X)$ for every $p \in (1,\infty)$. In particular, the corresponding evolution equation

$$\begin{cases} u'(t) &= -Au(t)f(t), \quad t \in [0,T], \\ u(0) &= 0. \end{cases}$$

has maximal L^p-regularity for every $p \in (1,\infty)$.

Proof. The proof is a combination of [102], Theorem 3.4 and Theorem 4.2. □

Thus, our approach yields a completely new proof for this result under slightly stronger assumptions. In addition to the assumptions of Weis, our proof needs that there exists an operator A_0 with bounded $H^\infty(\Sigma_\theta)$-calculus with $\theta \in (0, \frac{\pi}{2})$ such that $\|A^\varepsilon x\|_X \simeq \|A_0^\varepsilon x\|_X$ and $\|A^{-\varepsilon} x\|_X \simeq \|A_0^{-\varepsilon} x\|_X$ for some $\varepsilon > 0$.

"Wir stehen selbst enttäuscht und sehn betroffen
Den Vorhang zu und alle Fragen offen."

-Bertolt Brecht, Der gute Mensch von Sezuan, 1943

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