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Absence of π^2 terms in physical anomalous dimensions in DIS: Verification and resulting predictions

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ABSTRACT

We study the higher-order corrections to structure functions in inclusive deep-inelastic scattering (DIS) in massless perturbative QCD, in the context of the conjectured absence of even-n values of the Riemann zeta-function ζ_n, i.e., of powers of π^2, in Euclidean physical quantities. We provide substantial additional support for this conjecture by demonstrating that it holds, as far as it can be tested by the results of diagram computations, for the physical anomalous dimensions of structure functions at the fourth and fifth order in the strong coupling constant α_s. The conjecture is then employed to predict hitherto unknown ζ_4 and ζ_6 contributions to the anomalous dimensions for parton distributions and to the coefficient functions for the longitudinal structure function F_L.

As far as they are presently known, the anomalous dimensions – i.e., the even-N or odd-N Mellin moments of the splitting functions – for the scale dependence (evolution) of parton distributions can be expressed in terms of rational numbers and integer-n values ζ_n of the Riemann ζ-function. The same holds for the N-space coefficient functions for inclusive lepton–hadron deep-inelastic scattering (DIS) via the exchange of a boson with space-like four-momentum q, i.e., $q^2 \equiv -Q^2 < 0$. Three-loop calculations at $N \leq 14$ of these quantities were performed in refs. [1–3]. The corresponding all-N expressions were derived in refs. [4] for the anomalous dimensions (see also ref. [5] for the helicity-dependent case) and in refs. [6] for the most important structure functions in DIS.

It is an old observation that the above 'spacelike' quantities do not include terms linear in $\zeta_2 = \pi^2/6$.[1] Terms with ζ_2^2 or $\zeta_4 = 2/5\,\zeta_2^2 = \pi^4/90$ do occur in the three-loop coefficient functions in DIS (with the exception of the longitudinal structure function F_L) [1–3] and the four-loop anomalous dimensions [9–11], which together define the next-to-next-to-next-to-leading order (N^{3}LO) approximation in renormalization-group improved perturbation theory. The corresponding N^{4}LO quantities, i.e., the four-loop coefficient functions and five-loop anomalous dimensions, include contributions with ζ_4 and $\zeta_6 = 8/35\,\zeta_2^3 = \pi^6/945$ [11–13].

Already about 20 years ago, the absence of also ζ_4 in the perturbative expansion of spacelike (Euclidean) physical quantities was referred to as an empirical rule [2]. In the standard $\overline{\rm MS}$ renormalization scheme, however, this rule is violated at N^{4}LO by the scalar quark and gluon correlators that enter the hadronic decays of the Higgs boson [14]. Very recently, it has been demonstrated in ref. [15] that the ζ_4 terms in the above quantities vanish after transforming the coupling constant to the C-scheme introduced in ref. [16]. This highly non-trivial cancellation can occur since the five-loop beta function of QCD and its gauge-group generalizations include ζ_4 with most colour factors [17]. It has thus been conjectured that all even-n ζ-values, i.e., all powers of π^2, are absent from all spacelike physical quantities in massless perturbative QCD in this scheme [15].

The factorization-scheme dependent anomalous dimensions and coefficient functions can be combined to form physical anomalous dimensions for structure functions in DIS. In this manner, the above 'no-π^2 conjecture' can be **(a)** supported (or falsified) by quantities not considered in this context so far, and **(b)** used to predict new results for higher-order coefficients in the perturbation series. The latter possibility was, in fact, already mentioned (but not followed up) in ref. [2]. In the present letter, we perform both steps at N^{3}LO and N^{4}LO for the non-singlet structure func-

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[1] This does not hold for the moments of the splitting functions for fragmentation distributions [7] and the coefficient functions for semi-inclusive e^+e^- annihilation via a boson with a timelike four-momentum. It also does not hold for 'unnatural' (non-OPE) moments of DIS quantities such as the odd moments of the photon-exchange F_2 [8].

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tions $F_{2,\rm ns}$ and F_3 and for the flavour-singlet system (F_2, F_ϕ). Here F_ϕ is the structure function for DIS via the exchange of a scalar that (like the Higgs boson in the limit of a heavy top quark and n_f effectively massless flavours [18]) couples directly only to gluons [19]. We finally also address physical anomalous dimensions for F_L.

The physical anomalous dimensions K can be obtained by considering the scaling violations $dF/d\ln Q^2$ of the (vector of) N-space structure functions F, using the evolution equations for the parton distributions q and then expressing these in terms of the structure functions, viz

$$\frac{dF}{d\ln Q^2} = \frac{d}{d\ln Q^2}(Cq) = \frac{dC}{d\ln Q^2}q - C\gamma q$$
$$= \left(\beta\,\frac{dC}{da_{\rm s}} - C\gamma\right)C^{-1}F \equiv KF\,. \quad (1)$$

Here C represents the coefficient functions, γ the anomalous dimensions (we use the standard convention $\gamma = -P$ for their relation to the moments of the splitting functions), $a_{\rm s} \equiv \alpha_{\rm s}/4\pi$ the reduced strong coupling constant and β the beta function of QCD and its generalizations. We have suppressed here, as in many instances below, the dependence of C, γ and K on N for brevity. Our notation (and normalization) for the expansion coefficients of these quantities is

$$C = 1 + \sum_{\ell=1} a_{\rm s}^\ell\, c^{(\ell)}\,, \quad \{\gamma, K\} = \sum_{\ell=1} a_{\rm s}^\ell\,\{\gamma, K\}^{(\ell-1)}\,. \quad (2)$$

For the non-singlet cases, eq. (1) is simplified by $C\gamma C^{-1} = \gamma$, thus only the N^{n}LO anomalous dimensions $\gamma^{(n)}$ (together with the coefficient functions at this and all previous orders) enter the physical anomalous dimensions at N^{n}LO. In this case one has $K = -\gamma$ for those colour factors, such as C_F^4 at order $a_{\rm s}^4$, that cannot be generated by multiplying the beta function and powers of the coefficient functions. For the system (F_2, F_ϕ), the quantities F, C, γ and K represent the matrices

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,\rm q} & C_{2,\rm g} \\ C_{\phi,\rm q} & C_{\phi,\rm g} \end{pmatrix},$$
$$K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_{\rm qq} & \gamma_{\rm qg} \\ \gamma_{\rm gq} & \gamma_{\rm gg} \end{pmatrix}. \quad (3)$$

The non-singlet expansion coefficients of K have been written down to order $a_{\rm s}^5$, the maximal order of the present study, in eqs. (2.7) and (2.8) of ref. [20] for the choice $\mu_{\rm R}^2 = Q^2$ of the renormalization scale already employed in eq. (1). The generalization to $\mu_{\rm R}^2 \neq Q^2$, see eq. (2.9) of ref. [20], does not provide new information. Corresponding matrix-notation results for systems like (F_2, F_ϕ) have been given to $a_{\rm s}^4$ in eq. (2.25) of ref. [21]; their extension to the $a_{\rm s}^5$ terms is straightforward.

Only a small part of the complete expressions is required for the ζ_4 and ζ_6 contributions. Consequently, the ζ_4 parts of the $a_{\rm s}^4$ (N^{3}LO) physical anomalous dimensions considered for now take the simple form

$$\widetilde{K}_{2,\rm ns}^{(3)} = -\widetilde{\gamma}_{\rm ns}^{+(3)} - 3\beta_0\,\tilde{c}_{2,\rm ns}^{(3)}\,, \qquad \widetilde{K}_3^{(3)} = -\widetilde{\gamma}_{\rm ns}^{-(3)} - 3\beta_0\,\tilde{c}_3^{(3)} \quad (4)$$

and

$$\widetilde{K}_{22}^{(3)} = -\widetilde{\gamma}_{\rm qq}^{(3)} - 3\beta_0\,\tilde{c}_{2,\rm q}^{(3)} - \gamma_{\rm gq}^{(0)}\,\tilde{c}_{2,\rm g}^{(3)} + \gamma_{\rm qg}^{(0)}\,\tilde{c}_{\phi,\rm q}^{(3)}\,,$$
$$\widetilde{K}_{2\phi}^{(3)} = -\widetilde{\gamma}_{\rm qg}^{(3)} - 3\beta_0\,\tilde{c}_{2,\rm g}^{(3)} - \gamma_{\rm qg}^{(0)}\left(\tilde{c}_{2,\rm q}^{(3)} - \tilde{c}_{\phi,\rm g}^{(3)}\right)$$
$$- \tilde{c}_{2,\rm g}^{(3)}\left(\widetilde{\gamma}_{\rm qq}^{(3)} - \widetilde{\gamma}_{\rm gg}^{(3)}\right),$$
$$\widetilde{K}_{\phi 2}^{(3)} = -\widetilde{\gamma}_{\rm gq}^{(3)} - 3\beta_0\,\tilde{c}_{\phi,\rm q}^{(3)} - \gamma_{\rm gq}^{(0)}\left(\tilde{c}_{2,\rm q}^{(3)} - \tilde{c}_{\phi,\rm g}^{(3)}\right)$$
$$- \tilde{c}_{\phi,\rm q}^{(3)}\left(\widetilde{\gamma}_{\rm qq}^{(3)} - \widetilde{\gamma}_{\rm gg}^{(3)}\right),$$
$$\widetilde{K}_{\phi\phi}^{(3)} = -\widetilde{\gamma}_{\rm gg}^{(3)} - 3\beta_0\,\tilde{c}_{\phi,\rm g}^{(3)} - \gamma_{\rm qg}^{(0)}\,\tilde{c}_{\phi,\rm q}^{(3)} + \gamma_{\rm gq}^{(0)}\,\tilde{c}_{2,\rm g}^{(3)} \quad (5)$$

in terms of the four-loop anomalous dimensions and three-loop coefficient functions. The latter are completely known [6,21]. Here and below a tilde above a quantity indicates the coefficient of ζ_4. If the no-π^2 conjecture is correct, then the left-hand sides of eqs. (4) and eqs. (5) vanish for all N.

Diagram calculations of the N^{3}LO non-singlet anomalous dimensions up to $N = 16$ (and all-N expressions in the limit of a large number of colours n_c) have been presented in ref. [10]. These results are sufficient for determining the all-N expressions for $\widetilde{\gamma}_{\rm ns}^{\pm(3)}(N)$, which are found to read

$$\widetilde{\gamma}_{\rm ns}^{\pm(3)} = 8C_F(C_A - C_F)\beta_0$$
$$\times\left[9(C_A - 2C_F)\left(\frac{5}{4} - (\eta + \eta^2)(\pm 1)^N + 2S_{-2}\right)\right.$$
$$\left. - 3n_f\left(\frac{3}{2} + \eta - 2S_1\right)\right] \quad (6)$$

with $\eta \equiv D_0 - D_1 \equiv 1/N - 1/(N+1)$. Here and below the argument N of all harmonic sums $S_{\vec{w}}(N)$ [22] is suppressed for brevity. For $\widetilde{\gamma}_{\rm ns}^{+(3)}$, the anomalous dimension for flavour differences of quark–antiquark sums, eq. (6) is valid at even N. For its quark–antiquark difference counterpart $\widetilde{\gamma}_{\rm ns}^{-(3)}$ it provides the odd-N values with, of course, $\widetilde{\gamma}_{\rm ns}^{-(3)}(N=1) = 0$ as required by fermion number conservation. Using eq. (6) in eq. (4) we indeed find

$$\widetilde{K}_{2,\rm ns}^{(3)} = \widetilde{K}_3^{(3)} = 0 \quad \text{for all even/odd } N\,. \quad (7)$$

The diagram calculations of four-loop singlet splitting functions, performed along the lines of refs. [1–3] with the Forcer program [23] for massless four-loop self-energy integrals, have been completed so far only at $N = 2$ and $N = 4$. The result of the hardest of these calculations, $\gamma_{\rm gg}^{(3)}(N=4)$, has been given in eq. (2.1) of ref. [9]; the other results will be presented in ref. [11]. Inserting these results into eq. (5), we find

$$\widetilde{K}_{22}^{(3)} = \widetilde{K}_{2\phi}^{(3)} = \widetilde{K}_{\phi 2}^{(3)} = \widetilde{K}_{\phi\phi}^{(3)} = 0 \quad (8)$$

for $N = 2$ and $N = 4$, thus verifying the no-π^2 conjecture also in the singlet sector. Imposing eq. (8) for all even N, we can now predict the complete results for all four quantities $\widetilde{\gamma}_{\rm ik}^{(3)}(N)$ in eq. (3):

$$\widetilde{\gamma}_{\rm ps}^{(3)} = 16C_F(C_A - C_F)\Big[n_f^2\left(15\eta + 10\eta^2 - 20\nu\right)$$
$$+ C_F n_f\left(138\nu - 72\nu^2 - 117\eta - 87\eta^2 - 18\eta^3\right)$$
$$+ (C_A - C_F)n_f\left(114\nu - 72\nu^2 - \frac{195}{2}\eta - 69\eta^2 - 12\eta^3\right.$$
$$\left. + [24\nu - 18\eta - 12\eta^2]S_1\right)\Big]\,, \quad (9)$$

$$\widetilde{\gamma}_{\rm qg}^{(3)} = 16(C_A - C_F)\Big[(C_A - C_F)n_f^2\Big(22D_0 - \frac{16}{3}D_{-1} - 4D_0^2$$
$$+ 6D_0^3 - 50D_1 + 10D_1^2 - 12D_1^3 + \frac{109}{3}D_2 + 16D_2^2$$
$$+ [4D_0 - 8D_1 + 8D_2]S_1\Big)$$
$$+ (C_A - C_F)^2 n_f\Big(\frac{122}{3}D_{-1} - 8D_{-1}^2 - \frac{91}{2}D_0 - 21D_0^2$$
$$- 18D_0^3 + 208D_1 - 96D_1^2 + 36D_1^3 - \frac{659}{3}D_2 - 114D_2^2$$

$$
\begin{aligned}
&- 24D_2^3 + [12D_{-1} + 13D_0 - 24D_0^2 + 46D_1 - 48D_1^2 \\
&- 91D_2 - 36D_2^2]S_1 + [24D_1 - 12D_0 - 24D_2]S_{1,1} \\
&+ [6D_0 - 12D_1 + 12D_2]S_2\Big) + C_F n_f^2\Big(\frac{49}{2}D_0 - 8D_{-1} + D_0^2 \\
&+ 6D_0^3 - 70D_1 + 20D_1^2 - 12D_1^3 + 55D_2 + 24D_2^2 \\
&+ [2D_0 - 4D_1 + 4D_2]S_1\Big) \\
&+ C_F(C_A - C_F)n_f\Big(\frac{268}{3}D_{-1} - 16D_{-1}^2 - 121D_0 - 55D_0^2 \\
&- 36D_0^3 + 515D_1 - 218D_1^2 + 72D_1^3 - \frac{1531}{3}D_2 - 252D_2^2 \\
&- 48D_2^3 + [16D_{-1} - \frac{41}{2}D_0 - 27D_0^2 + 122D_1 - 54D_1^2 \\
&- 144D_2 - 48D_2^2]S_1 + [24D_1 - 12D_0 - 24D_2]S_{1,1} \\
&+ [12D_1 - 6D_0 - 12D_2]S_{-2} + [6D_0 - 12D_1 + 12D_2]S_2\Big) \\
&+ C_F^2 n_f\Big(\frac{146}{3}D_{-1} - 8D_{-1}^2 - \frac{299}{4}D_0 - 34D_0^2 - 21D_0^3 \\
&+ \frac{629}{2}D_1 - 131D_1^2 + 42D_1^3 - \frac{854}{3}D_2 - 138D_2^2 - 24D_2^3 \\
&+ [4D_{-1} - \frac{13}{2}D_0 - 9D_0^2 + 40D_1 - 18D_1^2 - 53D_2 \\
&- 12D_2^2]S_1 + [6D_0 - 12D_1 + 12D_2]S_{-2}\Big)\Big]\,, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\gamma}_{\rm gq}^{(3)} &= 16(C_A - C_F)\Big[C_F n_f^2\Big(\frac{16}{3}D_{-1} - \frac{16}{3}D_0 + \frac{8}{3}D_1\Big) \\
&+ C_F(C_A - C_F)n_f\Big(\frac{226}{3}D_0 - \frac{85}{3}D_{-1} - 18D_0^2 + 12D_0^3 \\
&- \frac{167}{3}D_1 - 18D_1^2 - 6D_1^3 + [4D_0 - 4D_{-1} - 2D_1]S_1\Big) \\
&+ C_F(C_A - C_F)^2\Big(88D_1 + 34D_1^2 + 6D_1^3 - 25D_{-1} - 12D_{-1}^2 \\
&- 92D_0 + 50D_0^2 - 12D_0^3 - 4D_2 + [24D_0 - 24D_{-1} \\
&- 12D_1]S_{1,1} + [12D_{-1} - 12D_0 + 6D_1]S_2 \\
&+ [37D_{-1} + 12D_{-1}^2 - 20D_0 + 10D_1 + 4D_2]S_1\Big) \\
&+ C_F^2 n_f\Big(8D_{-1}^2 - 55D_{-1} + \frac{190}{3}D_0 + 8D_0^2 + 12D_0^3 - \frac{181}{6}D_1 \\
&- 5D_1^2 - 6D_1^3 + \frac{8}{3}D_2 + [8D_{-1} - 8D_0 + 4D_1]S_1\Big) \\
&+ C_F^2(C_A - C_F)\Big(15D_{-1} - 9D_0 + \frac{9}{2}D_1 + \Big[\frac{33}{2}D_1 - 24D_{-1} \\
&- 6D_0 + 18D_0^2 + 9D_1^2\Big]S_1 + [12D_0 - 12D_{-1} - 6D_1]S_2 \\
&+ [12D_{-1} - 12D_0 + 6D_1]S_{-2} + [24D_{-1} - 24D_0 \\
&+ 12D_1]S_{1,1}\Big) + C_F^3\Big(10D_{-1} + 12D_{-1}^2 + 101D_0 - 50D_0^2 \\
&+ 12D_0^3 - \frac{185}{2}D_1 - 34D_1^2 - 6D_1^3 + 4D_2 + [26D_0 - 13D_{-1} \\
&- 12D_{-1}^2 - 18D_0^2 - \frac{53}{2}D_1 - 9D_1^2 - 4D_2]S_1 \\
&+ [12D_0 - 12D_{-1} - 6D_1]S_{-2}\Big)\Big]\,, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\gamma}_{\rm gg}^{(3)} &= 16(C_A - C_F)\Big[C_F^2 n_f\Big(95\eta + 54\eta^2 + 18\eta^3 - 138\nu \\
&+ 72\nu^2 + 11S_1\Big) + (C_A - C_F)n_f^2\Big(12\nu - 8\eta - 6\eta^2 - 2S_1\Big) \\
&+ (C_A - C_F)^2 n_f\Big(44\eta + 33\eta^2 - 66\nu + 11S_1\Big) \\
&+ C_F n_f^2\Big(20\nu - 11\eta - 4\eta^2 - 2S_1\Big) \\
&+ C_F(C_A - C_F)n_f\Big(\frac{239}{2}\eta + 69\eta^2 + 12\eta^3 - 180\nu + 72\nu^2 \\
&+ [22 - 24\nu + 18\eta + 12\eta^2]S_1\Big)\Big]\,. \qquad (12)
\end{aligned}
$$

The abbreviations η and D_k have been defined below eq. (6); in addition $\nu \equiv D_{-1} - D_2$ is used in eqs. (9) and (12). $\widetilde{\gamma}_{\rm qq}^{(3)}$ is obtained by adding $\widetilde{\gamma}_{\rm ns}^{+(3)}$ in eq. (6) to the pure-singlet contribution (9) which we were able to check at $N = 6$. At least at this value of N, we expect further checks in the near future. A complete determination of the all-N expressions in eqs. (9)–(12) from diagram calculations, on the other hand, would be a very (currently: too) challenging task.

It is interesting to note that all ζ_4 terms of $\gamma^{(3)}$ vanish for $C_F = C_A$, which is part of the colour-factor choice leading to an $\mathcal{N} = 1$ supersymmetric theory [24]. The same behaviour has been found (to all orders) before for the double-logarithmic large-N contributions to the off-diagonal anomalous dimensions $\gamma_{\rm qg}$ and $\gamma_{\rm gq}$ [25]. We also note that the 'diagonal' quantities (6), (9) and (12) are reciprocity respecting, see ref. [10] and references therein.

All flavour-singlet quantities in eqs. (9)–(12) include terms with $1/(N-1)^2$ which correspond to $\zeta_4 x^{-1}\ln x$ terms in the small-x expansion of the N^{3}LO splitting functions $P_{\rm ik}^{(3)}(x)$. Contributions of this x-space form are also generated, however, by terms without ζ_4 in N-space. Therefore the above results are not sufficient to obtain the ζ_4 coefficients of the next-to-next-to-leading logarithms of $P_{\rm ik}^{(3)}(x)$ in the small-x (high-energy) limit.

We now turn to the a_s^5 (N^{4}LO) contributions to the physical anomalous dimensions. Their ζ_6 terms, which we denote by $\widehat{K}^{(4)}$, are given by simple modifications of eqs. (4) and (5): on the right-hand-sides replace $\tilde{f}_{\dots}^{(3)}$ everywhere by $\hat{f}_{\dots}^{(4)}$ for $f = \gamma, c$, and replace $-3\beta_0$ everywhere by $-4\beta_0$. For the non-singlet cases, the ζ_4 terms at N^{4}LO are given by

$$
\widetilde{K}_a^{(4)} = -\widetilde{\gamma}_{\rm ns}^{\sigma\,(4)} - 3\beta_1\,\tilde{c}_a^{(3)} - 4\beta_0\left(\tilde{c}_a^{(4)} - c_a^{(1)}\,\tilde{c}_a^{(3)}\right) \qquad (13)
$$

with $\sigma = +$ for $a = 2, {\rm ns}$ and $\sigma = -$ for $a = 3$. At this order, the scheme transformation of ref. [16] includes the ζ_4 term of the five-loop beta function, see eq. (4) of ref. [15] (where this coefficient, β_4 in our notation, is denoted by β_5). Consequently, the conjecture of ref. [15] implies

$$
\beta_0\,\overline{K}_a^{(4)} = \beta_0\,\widetilde{K}_a^{(4)} + \frac{1}{3}\,\widetilde{\beta}_4\,K_a^{(0)} = 0\,. \qquad (14)
$$

Checking this prediction and its ζ_6 counterpart $\widehat{K}_a^{(4)} = 0$ requires the four-loop coefficient functions for $F_{2,\rm ns}$ and F_3 and the corresponding five-loop splitting functions. The former quantities have been computed so far at $N \le 6$ [9,11]. The latter have been obtained, very recently, at $N = 2$ and $N = 3$ [12,13] by using the R*-operation [26] as extended to generic numerators in ref. [27] and implemented using the latest version [28] of Form [29], together with the Forcer program [23]. The leading large-n_f contributions to $\gamma_{\rm ns}^{(4)}$ and $c_{2,\rm ns}^{(4)}$, which both include ζ_4 terms, have been determined at all N in refs. [30].

We now show, by explicitly writing down all contributions to eqs. (13) and (14), that the latter relation is fulfilled for $K_{2,\rm ns}$ at $N = 2$. The corresponding verification for K_3 at $N = 3$ is completely analogous, but will be suppressed for brevity. For the same reason we do not show the (less critical, since the scheme transformation of ref. [16] is not required) verification for the ζ_6 parts of $K_{2,\rm ns}$ and K_3 at these values of N, nor the all-N verification for the ζ_4 part of $K_{2,\rm ns}$ in the large-n_f limit. The recent result for $\widetilde{\gamma}_{\rm ns}^{+(4)}(N=2)$ [12,13] is given by

$$\tilde{\gamma}_{\rm ns}^{+(4)} = \frac{1792}{9} n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_R} - \frac{512}{9} n_f \frac{d_F^{abcd}d_A^{abcd}}{n_R} + \frac{704}{3} C_F \frac{d_A^{abcd}d_A^{abcd}}{n_A}$$
$$+ \frac{128}{81} C_F n_f^4 - \frac{128}{3} C_F^2 n_f^3 + \frac{1072}{81} C_F^3 n_f^2 + \frac{21248}{81} C_F^4 n_f$$
$$- \frac{10912}{9} C_A \frac{d_F^{abcd}d_A^{abcd}}{n_R} - \frac{5632}{9} C_A n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} + \frac{2752}{81} C_A C_F n_f^3$$
$$+ \frac{20752}{27} C_A C_F^2 n_f^2 + \frac{48256}{81} C_A C_F^3 n_f - \frac{59840}{81} C_A C_F^4$$
$$- 784 C_A^2 C_F n_f^2 - \frac{114536}{27} C_A^2 C_F^2 n_f - \frac{229472}{81} C_A^2 C_F^3$$
$$+ \frac{274768}{81} C_A^3 C_F n_f + \frac{170968}{27} C_A^3 C_F^2 - \frac{221920}{81} C_A^4 C_F \,. \quad (15)$$

Here $T_f = 1/2$ has been inserted; the power of T_f for each term can be readily reconstructed. The result for QCD is obtained for $C_A = n_R = 3$, $n_A = 8$, $C_F = 4/3$, $d_A^{abcd}d_A^{abcd}/n_A = 135/8$, $d_F^{abcd}d_A^{abcd}/n_R = 5/2$ and $d_F^{abcd}d_F^{abcd}/n_R = 5/36$. The ζ_4 parts of the four-loop coefficient function $c_{2,\rm ns}^{(4)}(N=2)$ [9,11] and of the five-loop beta function β_4 [17] read

$$\tilde{c}_{2,\rm ns}^{(4)} = \frac{248}{3} \frac{d_F^{abcd}d_A^{abcd}}{n_R} + \frac{128}{3} n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} + \frac{16}{27} C_F n_f^3 - 16 C_F^2 n_f^2$$
$$- \frac{220}{27} C_F^3 n_f + \frac{1552}{27} C_F^4 + 16 C_A C_F n_f^2 + 176 C_A C_F^2 n_f$$
$$+ \frac{3592}{27} C_A C_F^3 - \frac{505}{3} C_A^2 C_F n_f - 354 C_A^2 C_F^2 + \frac{4367}{27} C_A^3 C_F \quad (16)$$

and

$$\tilde{\beta}_4 = 176 n_f \frac{d_A^{abcd}d_A^{abcd}}{n_A} - 416 n_f^2 \frac{d_F^{abcd}d_A^{abcd}}{n_A} + 128 n_f^3 \frac{d_F^{abcd}d_F^{abcd}}{n_A}$$
$$- \frac{44}{3} C_F^2 n_f^3 - 968 C_A \frac{d_A^{abcd}d_A^{abcd}}{n_A} + 2288 C_A n_f \frac{d_F^{abcd}d_A^{abcd}}{n_A}$$
$$- 704 C_A n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_A} + \frac{28}{3} C_A C_F n_f^3 + \frac{286}{3} C_A C_F^2 n_f^2 + \frac{14}{3} C_A^2 n_f^3$$
$$- \frac{236}{3} C_A^2 C_F n_f^2 - \frac{242}{3} C_A^2 C_F^2 n_f - \frac{26}{3} C_A^3 n_f^2 + \frac{451}{3} C_A^3 C_F n_f$$
$$- \frac{583}{6} C_A^4 n_f + \frac{121}{6} C_A^5 \,. \quad (17)$$

Due to eqs. (4) and (7), the three-loop contribution $\tilde{c}_{2,\rm ns}^{(3)}(N=2)$ [2] can be read off from Eq. (6). For the convenience of the reader, we also recall the required one-loop quantities in the normalization used in this letter: $K_{2,\rm ns}^{(0)}(N=2) = \gamma_{\rm ns}^{(0)}(N=2) = 8/3\, C_F$, $c_{2,\rm ns}^{(1)}(N=2) = 1/3\, C_F$ and $\beta_0 = 11/3\, C_A - 2/3\, n_f$. Assembling these contributions, we arrive at eq. (14). This and the other verifications mentioned above provide substantial and highly non-trivial extra evidence for the no-π^2 conjecture in the form presented in ref. [15]. The ζ_6 coefficient of $\gamma_{\rm ns}^{+(4)}$ at $N=2$ reads

$$\hat{\gamma}_{\rm ns}^{+(4)} = \frac{800}{27} \beta_0 \Big(36 C_F^4 - 36 C_A C_F^3 - 42 C_A^2 C_F^2 + 38 C_A^3 C_F$$
$$- 48 \frac{d_F^{abcd}d_A^{abcd}}{n_R} + 18 n_f C_F^3 - 3 n_f C_A C_F^2 - 14 n_f C_A^2 C_F$$
$$+ 24 n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} \Big) \,. \quad (18)$$

The $N=4$ expressions corresponding to eqs. (15) and (18) represent the first new results for the N^{4}LO non-singlet anomalous dimensions obtained from this conjecture. They are given by

$$\tilde{\gamma}_{\rm ns}^{+(4)} = - \frac{79776202}{50625} C_A C_F^4 + \frac{6688679}{162000} C_A^2 C_F^3 + \frac{60495779}{33750} C_A^3 C_F^2$$
$$- \frac{202467481}{162000} C_A^4 C_F + \frac{6908}{15} C_F \frac{d_A^{abcd}d_A^{abcd}}{n_A}$$
$$- \frac{349624}{45} C_A \frac{d_F^{abcd}d_A^{abcd}}{n_R} + \frac{20332714}{50625} n_f C_F^4$$
$$- \frac{41674913}{81000} n_f C_A C_F^3 - \frac{330775451}{67500} n_f C_A^2 C_F^2$$
$$+ \frac{428850767}{81000} n_f C_A^3 C_F - \frac{124036}{75} n_f C_A \frac{d_F^{abcd}d_F^{abcd}}{n_R}$$
$$+ \frac{39076}{45} n_f \frac{d_F^{abcd}d_A^{abcd}}{n_R} + \frac{1704086}{10125} n_f^2 C_F^3 + \frac{8505499}{6750} n_f^2 C_A C_F^2$$
$$- \frac{4882673}{3375} n_f^2 C_A^2 C_F + \frac{11704}{25} n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_R} - \frac{2146}{25} n_f^3 C_F^2$$
$$+ \frac{139286}{2025} n_f^3 C_A C_F + \frac{1256}{405} n_f^4 C_F \,, \quad (19)$$

$$\hat{\gamma}_{\rm ns}^{+(4)} = \frac{138248}{27} C_A C_F^4 - \frac{181522}{27} C_A^2 C_F^3 + \frac{90475}{54} C_A^3 C_F^2$$
$$+ \frac{102553}{81} C_A^4 C_F + \frac{433774}{27} \frac{d_F^{abcd}d_A^{abcd}}{n_R} C_A - \frac{25136}{27} n_f C_F^4$$
$$+ \frac{136624}{27} n_f C_A C_F^3 - \frac{25495}{27} n_f C_A^2 C_F^2 - \frac{86398}{27} n_f C_A^3 C_F$$
$$+ \frac{148016}{27} n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} C_A - \frac{78868}{27} n_f \frac{d_F^{abcd}d_A^{abcd}}{n_R}$$
$$- \frac{6280}{9} n_f^2 C_F^3 + \frac{3140}{27} n_f^2 C_A C_F^2 + \frac{43736}{81} n_f^2 C_A^2 C_F$$
$$- \frac{26912}{27} n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_R} \,. \quad (20)$$

It is also possible to predict the ζ_4 and ζ_6 coefficients $\tilde{\gamma}_{\rm ij}^{(4)}$ and $\hat{\gamma}_{\rm ij}^{(4)}$ of the N^{4}LO singlet anomalous dimensions at $N=2$ and $N=4$. The prediction at $N=2$ includes a further check, since the four results must be pairwise equal due to the momentum sum rule. By evaluating the elements of K in eq. (3) as given by eq. (1), we obtain

$$\tilde{\gamma}_{\rm qq}^{(4)}(N=2) = -\tilde{\gamma}_{\rm gq}^{(4)}(N=2)$$
$$= \frac{2048}{9} n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_R} - \frac{512}{9} n_f \frac{d_F^{abcd}d_A^{abcd}}{n_R}$$
$$+ \frac{704}{3} C_F \frac{d_A^{abcd}d_A^{abcd}}{n_A} + \frac{640}{81} C_F n_f^4 - \frac{10496}{81} C_F^2 n_f^3$$
$$+ \frac{2896}{27} C_F^3 n_f^2 + \frac{1024}{3} C_F^4 n_f - \frac{10912}{9} C_A \frac{d_F^{abcd}d_A^{abcd}}{n_R}$$
$$- \frac{7040}{9} C_A n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} + \frac{6188}{81} C_A C_F n_f^3$$
$$+ \frac{142996}{81} C_A C_F^2 n_f^2 + \frac{6596}{9} C_A C_F^3 n_f - \frac{59840}{81} C_A C_F^4$$
$$- \frac{48598}{27} C_A^2 C_F n_f^2 - \frac{196028}{27} C_A^2 C_F^2 n_f - \frac{229472}{81} C_A^2 C_F^3$$
$$+ 6254 C_A^3 C_F n_f + \frac{170968}{27} C_A^3 C_F^2$$
$$- \frac{221920}{81} C_A^4 C_F \,, \quad (21)$$

$$\tilde{\gamma}_{\rm qg}^{(4)}(N=2) = -\tilde{\gamma}_{\rm gg}^{(4)}(N=2)$$
$$= \frac{256}{9} n_f^2 \frac{d_F^{abcd}d_A^{abcd}}{n_A} - \frac{176}{3} n_f \frac{d_A^{abcd}d_A^{abcd}}{n_A}$$
$$- \frac{1024}{9} n_f^3 \frac{d_F^{abcd}d_F^{abcd}}{n_A} + \frac{512}{81} C_F n_f^4 - \frac{3844}{81} C_F^2 n_f^3$$
$$+ \frac{3808}{81} C_F^3 n_f^2 + \frac{6400}{81} C_F^4 n_f + \frac{5456}{9} C_A n_f \frac{d_F^{abcd}d_A^{abcd}}{n_A}$$

$$+\frac{3520}{9}C_A n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_A} - \frac{176}{81}C_A n_f^4 + \frac{4928}{81}C_A C_F n_f^3 + \frac{40348}{81}C_A C_F^2 n_f^2 + \frac{5932}{27}C_A C_F^3 n_f - \frac{910}{27}C_A^2 n_f^3 - \frac{11360}{9}C_A^2 C_F n_f^2 - \frac{136090}{81}C_A^2 C_F^2 n_f + \frac{60019}{81}C_A^3 n_f^2 + \frac{313570}{81}C_A^3 C_F n_f - \frac{204065}{81}C_A^4 n_f \quad (22)$$

and

$$\widehat{\gamma}_{qq}^{(4)}(N=2) = -\widehat{\gamma}_{gq}^{(4)}(N=2) = \frac{25600}{27}n_f \frac{d_F^{abcd}d_A^{abcd}}{n_R} - \frac{25600}{27}n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_R} - \frac{8000}{9}C_F^3 n_f^2 - \frac{3200}{3}C_F^4 n_f - \frac{140800}{27}C_A \frac{d_F^{abcd}d_A^{abcd}}{n_R} + \frac{140800}{27}C_A n_f \frac{d_F^{abcd}d_F^{abcd}}{n_R} + \frac{4000}{27}C_A C_F^2 n_f^2 + \frac{139600}{27}C_A C_F^3 n_f + \frac{35200}{9}C_A C_F^4 + \frac{56800}{81}C_A^2 C_F n_f^2 + \frac{10600}{27}C_A^2 C_F^2 n_f - \frac{35200}{9}C_A^2 C_F^3 - 4200 C_A^3 C_F n_f - \frac{123200}{27}C_A^3 C_F^2 + \frac{334400}{81}C_A^4 C_F \,, \quad (23)$$

$$\widehat{\gamma}_{qg}^{(4)}(N=2) = -\widehat{\gamma}_{gg}^{(4)}(N=2) = \frac{12800}{27}n_f^3 \frac{d_F^{abcd}d_F^{abcd}}{n_A} - \frac{12800}{27}n_f^2 \frac{d_F^{abcd}d_A^{abcd}}{n_A} - \frac{3200}{9}C_F^3 n_f^2 - \frac{3200}{9}C_F^4 n_f + \frac{70400}{27}C_A n_f \frac{d_F^{abcd}d_A^{abcd}}{n_A} - \frac{70400}{27}C_A n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_A} + 400 C_A C_F^2 n_f^2 + \frac{41200}{27}C_A C_F^3 n_f + \frac{800}{81}C_A^2 n_f^3 + \frac{1400}{9}C_A^2 C_F n_f^2 - \frac{16400}{9}C_A^2 C_F^2 n_f - \frac{7400}{27}C_A^3 n_f^2 - \frac{12100}{27}C_A^3 C_F n_f + \frac{97900}{81}C_A^4 n_f \,. \quad (24)$$

Hence the no-π^2 conjecture also passes this further five-loop check. We note that this check succeeds only due to the (ζ_4 part of the) scheme transformation of ref. [16]. This is not due to the (F_2, F_ϕ) analogue of the $\widetilde{\beta}_4$ shift (14), as the resulting contributions to the momentum sum rule cancel. Instead it arises from the need to refer to a renormalization-group invariant current [15], which is not $G^{\mu\nu}G_{\mu\nu}$ but $\beta(a_s)/a_s\, G^{\mu\nu}G_{\mu\nu}$ for the structure function F_ϕ. The resulting overall factor of a_s induces a scheme shift $\sim \widetilde{\beta}_4/\beta_0\, c^{(0)}_{\phi,g}$ of the N^{4}LO coefficient function $\tilde{c}^{(4)}_{\phi,g}$, with $c^{(0)}_{\phi,g}=1$.

Finally we step back to N^{3}LO and address the longitudinal structure function F_L. The physical anomalous dimensions for $F_{L,\rm ns}$ and the singlet system (F_2, F_L) [31], see also ref. [32], have been employed in refs. [33] to predict large-N double logarithms. It is convenient to consider $\mathcal{F}_L = F_L/(a_s\, c^{(1)}_{L,q})$ with the coefficient functions $c^{(3)}_{\lambda,\rm i} \equiv c^{(4)}_{L,\rm i}/c^{(1)}_{L,q}$ with $c^{(1)}_{L,q} = 4C_F/(N+1)$ – recall our normalization $a_s = \alpha_s/4\pi$ of the reduced coupling.

The non-singlet case is then directly analogous to eqs. (4), hence $\widetilde{K}_{L,\rm ns}=0$ together with eq. (6) leads to an all-N prediction that we have checked against diagram calculations at $N=2$, $N=4$ and $N=6$ [9,11]. This prediction reads

$$\tilde{c}^{(4)}_{L,\rm ns} = 16 C_F^2 (C_A - C_F) D_1 \Big[6(C_A - 2C_F)\Big((\eta+\eta^2) - \frac{5}{4} - 2S_{-2}\Big) + n_f\Big(3 + 2\eta - 4S_1\Big)\Big]. \quad (25)$$

The structure of the corresponding anomalous-dimension matrix for $(F_2, \mathcal{F}_L)$ is more involved than that in eqs. (5) for (F_2, F_ϕ), since the leading-order analogue C_λ of C in eq. (3) is not given by the unit matrix, but by

$$C_\lambda^{(0)} = \begin{pmatrix} 1 & 0 \\ 1 & C^{(0)}_{\lambda,g} \end{pmatrix} . \quad (26)$$

Nevertheless, it is of course no problem to evaluate eq. (1) by symbolic manipulation to N^{3}LO accuracy also in this case. We have checked that the $(F_2, \mathcal{F}_L)$ analogues of eq. (5), here suppressed for brevity, lead to

$$\widetilde{K}^{(3)}_{22} = \widetilde{K}^{(3)}_{2L} = \widetilde{K}^{(3)}_{L2} = \widetilde{K}^{(3)}_{LL} = 0 \quad (27)$$

for $N=2$ and $N=4$. Note that K_{22} here is not the same as K_{22} in eq. (5).

The all-N forms of four-loop quantities $\tilde{c}^{(4)}_{L,\rm ps}$ and $\tilde{c}^{(4)}_{L,g}$ can be predicted by imposing eq. (27) at all even-N and using eqs. (9)–(12) for the four-loop splitting functions. In this manner we arrive at

$$\tilde{c}^{(4)}_{L,\rm ps} = 16 C_F (C_A - C_F)\Big[n_f^2 \Big(\frac{8}{3}D_{-1} - 8D_0 + 8D_1^2 + \frac{16}{3}D_2\Big) + (C_A - 2C_F) n_f \Big(4D_{-1} + 30D_0 - 12D_0^2 - 42D_1 - 18D_1^2 - 12D_1^3 + 8D_2 - 4[D_{-1} - 3D_0 + 3D_1^2 + 2D_2]S_1\Big) + C_F n_f \Big(30D_1 + 6D_1^2 + 12D_1^3 - 4D_{-1} - 18D_0 + 12D_0^2 - 8D_2\Big)\Big] \,, \quad (28)$$

$$\tilde{c}^{(4)}_{L,g} = 16(C_A - C_F)\Big[(C_A - C_F)n_f^2\Big(\frac{8}{3}D_{-1} + 26D_0 - 12D_0^2 - 72D_1 + 8D_1^2 - 24D_1^3 + \frac{130}{3}D_2 + 16D_2^2 - 8[D_1 - D_2]S_1\Big) + C_F n_f^2\Big(\frac{8}{3}D_{-1} - 10D_0 - 24D_1 + 20D_1^2 + \frac{94}{3}D_2 + 16D_2^2 - 8[D_1 - D_2]S_1\Big) + C_F(C_A - C_F)n_f\Big(117D_1 - 12D_1^2 + 36D_1^3 - 4D_{-1} - 39D_0 + 18D_0^2 - 74D_2 - 24D_2^2 + [6D_0 + 6D_1 - 12D_1^2 - 12D_2]S_1\Big) + C_F^2 n_f\Big(84D_1 - 48D_1^2 + 24D_1^3 - 4D_{-1} - 6D_0 + 12D_0^2 - 74D_2 - 24D_2^2 + [6D_0 + 6D_1 - 12D_1^2 - 12D_2]S_1\Big)\Big] \,. \quad (29)$$

The flavour-singlet quark coefficient function $\tilde{c}^{(4)}_{L,q}$ is obtained by adding $\tilde{c}^{(4)}_{L,\rm ns}$ in eq. (25) to the pure-singlet quantity (28). Since only these two coefficient functions have been determined, two of the four relations in eq. (27) are left as additional all-N checks of the no-π^2 conjecture.

To summarize: we have presented a large amount of additional evidence for the conjecture that there are no π^2 contributions to the expansion coefficients of Euclidean physical quantities in massless perturbative QCD and its generalization to a general simple compact gauge group [15]. Besides low even or odd integer-N values of the physical anomalous dimensions for the non-singlet structure functions $F_{a,\rm ns}$, $a = 2, 3, L$, and the singlet systems (F_2, F_ϕ), (F_2, F_L) at N^{3}LO and N^{4}LO – in the latter case the conjecture holds only after the scheme transformation of ref. [16], or indeed after any scheme transformation that removes the ζ_4 contributions to β_4 occurring in $\overline{\mathrm{MS}}$ [17] – these checks include four 'all-N' relations at N^{3}LO, three for even N and one for odd N.

Based on the evidence presented in ref. [15] and in this letter, this conjecture can be employed to predict new π^2 contributions to higher-order anomalous dimensions and coefficient functions. At N^{3}LO we have presented the ζ_4 contributions to the flavour-singlet splitting functions and to the coefficient functions for the longitudinal structure function F_L at all even N. Based on present four-loop Forcer [23] computations of DIS [9,11], it is possible to predict hitherto unknown ζ_4 and ζ_6 parts of N^{4}LO anomalous dimensions at $N \leq 6$. Here we have shown, for brevity, only the $N = 2$ ($N = 4$) results for the singlet (non-singlet) case. These predictions, and the no-π^2 conjecture in general, will serve as useful partial checks for very complicated future high-order computations. They may also provide input for future studies of the structure of perturbative quantum field theory.

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