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Research Article

Singular Solutions to a (3 + 1)-D Protter-Morawetz Problem for Keldysh-Type Equations

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We study a boundary value problem for (3 + 1)-D weakly hyperbolic equations of Keldysh type (problem PK). The Keldysh-type equations are known in some specific applications in plasma physics, optics, and analysis on projective spaces. Problem PK is not well-posed since it has infinite-dimensional cokernel. Actually, this problem is analogous to a similar one proposed by M. Protter in 1952, but for Tricomi-type equations which, in part, are closely connected with transonic fluid dynamics. We consider a properly defined, in a special function space, generalized solution to problem PK for which existence and uniqueness theorems hold. It is known that it may have a strong power-type singularity at one boundary point even for very smooth right-hand sides of the equation. In the present paper we study the asymptotic behavior of the generalized solutions of problem PK at the singular point. There are given orthogonality conditions on the right-hand side of the equation, which are necessary and sufficient for the existence of a generalized solution with fixed order of singularity.

In memory of Professor Cathleen Morawetz (1923–2017)

1. Statement of the Problem

For $m \in \mathbb{R},\ 0 < m < 2$, we consider the equation

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \qquad (1)$$

in the domain

$$\Omega_m := \left\{(x,t): 0 < t < t_0,\ \frac{2}{2-m}t^{(2-m)/2} < |x| < 1 - \frac{2}{2-m}t^{(2-m)/2}\right\}, \qquad (2)$$

where $(x,t) := (x_1,x_2,x_3,t) \in \mathbb{R}^4$, $|x| = \sqrt{x_1^2+x_2^2+x_3^2}$ and $t_0 = ((2-m)/4)^{2/(2-m)}$.

The region Ω_m (see Figure 1) is bounded by the ball $\Sigma_0 := \{(x,t): t = 0,\ |x| < 1\}$ centered at the origin $O = (0,0,0,0)$ and by two characteristic surfaces of (1):

$$\Sigma_1^m := \left\{(x,t): 0 < t < t_0,\ |x| = 1 - \frac{2}{2-m}t^{(2-m)/2}\right\},$$
$$\Sigma_2^m := \left\{(x,t): 0 < t < t_0,\ |x| = \frac{2}{2-m}t^{(2-m)/2}\right\}. \qquad (3)$$

In our case $(0 < m < 2)$, the hyperplane $\{t = 0\}$ is tangential to the characteristics Σ_1^m and Σ_2^m.

In the given domain, (1) is hyperbolic, with parabolic power-type degeneration at $\Sigma_0 \subset \{t = 0\}$; that is, we have a weakly hyperbolic equation of Keldysh type.

We study the following boundary value problem.

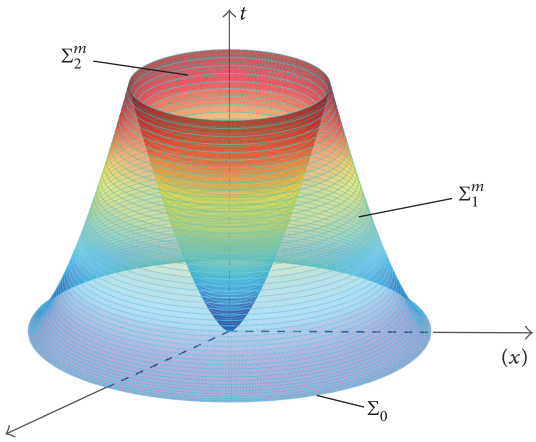

Figure 1: Region Ω_m.

Problem PK. Find a solution to (1) in Ω_m which satisfies the boundary conditions:

$$u|_{\Sigma_1^m} = 0, \qquad t^m u_t \longrightarrow 0 \quad \text{as } t \longrightarrow +0. \tag{4}$$

The adjoint problem to PK is as follows.

Problem PK*. Find a solution to the self-adjoint equation (1) in Ω_m which satisfies the boundary conditions:

$$u|_{\Sigma_2^m} = 0, \qquad t^m u_t \longrightarrow 0 \quad \text{as } t \longrightarrow +0. \tag{5}$$

2. The Main Results

Problem PK is not well-posed. Actually, the adjoint homogeneous problem PK* has infinitely many classical solutions.

In order to give their exact representation, for $k, n \in \mathbb{N} \cup \{0\}$, let us introduce the following functions:

$$E_{k,m}^n(|x|, t) := \sum_{i=0}^{k} A_i^k |x|^{-n+2i-1} \cdot \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m}\right)^{n-k-i-m/(4-2m)}, \tag{6}$$

where

$$A_i^k := (-1)^i \frac{(k-i+1)_i (n-k-i+(4-3m)/(4-2m))_i}{i!\,(n+1/2-i)_i} \tag{7}$$

with $(a)_i := \Gamma(a+i)/\Gamma(a)$, which gives $(a)_i = a(a+1)\cdots(a+i-1)$, for $i \in \mathbb{N}$, and $(a)_0 = 1$.

Further, let us denote by $Y_n^s(x)$, $n \in \mathbb{N} \cup \{0\}$, $s = 1, 2, \ldots, 2n+1$ the three-dimensional spherical functions. They are usually defined on the unit sphere $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, but for convenience of our discussions we extend them out of S^2 radially, keeping the same notation for the extended functions:

$$Y_n^s(x) := Y_n^s\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^3 \setminus \{0\}. \tag{8}$$

In this paper, we prove the following lemma.

Lemma 1. *For all $m \in \mathbb{R}$, $0 < m < 2$, $k, n \in \mathbb{N} \cup \{0\}$, $n \geq N(m,k)$ and $s = 1, 2, \ldots, 2n+1$, the functions*

$$v_{k,m}^{n,s}(x,t) := \begin{cases} E_{k,m}^n(|x|, t)\, Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O \end{cases} \tag{9}$$

are classical solutions of the homogeneous problem PK*.

It is easy to see that a necessary condition for the existence of a classical solution of problem PK is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v_{k,m}^{n,s}(x,t)$. Indeed,

$$\begin{aligned} \int_{\Omega_m} v_{k,m}^{n,s}(x,t) f(x,t)\, dx\, dt &= \int_{\Omega_m} v_{k,m}^{n,s}(x,t) L_m[u](x,t)\, dx\, dt \\ &= \int_{\Omega_m} L_m\left[v_{k,m}^{n,s}\right](x,t)\, u(x,t)\, dx\, dt = 0. \end{aligned} \tag{10}$$

This means that an infinite number of orthogonality conditions $\mu_{k,m}^{n,s} = 0$ with

$$\mu_{k,m}^{n,s} := \int_{\Omega_m} v_{k,m}^{n,s}(x,t) f(x,t)\, dx\, dt \tag{11}$$

must be fulfilled.

To avoid this we consider solutions to this problem in a generalized sense. In the present paper we study the case $\mathbf{0} < \mathbf{m} < \mathbf{4/3}$ and we use the following definition of a generalized solution of problem PK.

Definition 2 (see [1]). We call a function $u(x,t)$ a generalized solution of problem PK in Ω_m, $0 < m < 4/3$, for (1), if

(1) $u, u_{x_j} \in C(\overline{\Omega}_m \setminus O)$, $j = 1, 2, 3$, $u_t \in C(\overline{\Omega}_m \setminus \overline{\Sigma}_0)$;

(2) $u|_{\Sigma_1^m} = 0$;

(3) for each $\varepsilon \in (0,1)$ there exists a constant $C(\varepsilon) > 0$, such that

$$|u_t(x,t)| \leq C(\varepsilon)\, t^{-3m/4} \quad \text{in } \Omega_m \cap \{|x| > \varepsilon\}; \tag{12}$$

(4) the identity

$$\int_{\Omega_m} \left\{ t^m u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - f v \right\} dx_1 dx_2 dx_3 dt = 0 \tag{13}$$

holds for all v from

$$V_m := \left\{ v(x,t) : v \in C^2\left(\overline{\Omega}_m\right),\ v|_{\Sigma_2^m} = 0,\ v \equiv 0 \text{ in a neighborhood of } O \right\}. \quad (14)$$

We mention that the inequality (12) restricts the generalized solution's function space to a class which is smaller than it is allowed by the second boundary condition in (4).

Note that Definition 2 allows the generalized solution to have some singularity at the point O. The results of the present paper show that indeed there exist such singular solutions to this problem.

In our recent paper [1], we proved the following results on the existence and uniqueness of a generalized solution of problem PK.

Theorem 3 (see [1]). *If $m \in (0, 4/3)$, then there exists at most one generalized solution of problem PK in Ω_m.*

Theorem 4 (see [1]). *Let $m \in (0, 4/3)$. Suppose that the right-hand side function $f(x,t)$ is fixed as a "harmonic polynomial" of order l with $l \in \mathbf{N} \cup \{0\}$:*

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f_n^s(|x|, t)\, Y_n^s(x) \quad (15)$$

and $f \in C^1(\overline{\Omega}_m)$. Then there exists an unique generalized solution $u(x,t)$ of problem PK in Ω_m and it has the following form:

$$u(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u_n^s(|x|, t)\, Y_n^s(x). \quad (16)$$

In this paper, we derive an asymptotic formula concerning the behavior of the singularities of the generalized solution.

Theorem 5. *Let $m \in (0, 4/3)$ and the right-hand side function $f \in C^1(\overline{\Omega}_m)$ has the form (15). Then the unique generalized solution $u(x,t)$ of problem PK on the characteristic surface*

$$\Sigma_2^m = \left\{ (x,t) : 0 < |x| < \frac{1}{2},\ t = \tau(|x|) := \left(2^{-1}(2-m)|x|\right)^{2/(2-m)} \right\} \quad (17)$$

has the following expansion at point O:

$$u(x, \tau(|x|)) = \sum_{p=0}^{l} G_p(x)\, |x|^{-p-1} + G(x)\, |x|^{-m/(4-2m)}, \quad (18)$$

where

(i) *the function $G \in C^1(0 < |x| < 1/2)$ and satisfies the a priori estimate*

$$|G(x)| \le C \max_{\overline{\Omega}_m} |f| \quad (19)$$

with a constant $C > 0$ independent of f;

(ii) *the functions $G_p(x)$ have the following structure:*

$$G_p(x) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c_{k,m}^{p+2k,s}\, \mu_{k,m}^{p+2k,s}\, Y_{p+2k}^s(x), \quad p = 0, 1, \ldots, l, \quad (20)$$

where $c_{k,m}^{p+2k,s} \neq 0$ are constants independent of $f(x,t)$.

Corollary 6. *Suppose that at least one of the constants $\mu_{k,m}^{p+2k,s}$ in (20) is different from zero. Then for the corresponding function $G_p(x)$ there exists a vector $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$, such that $G_p(\alpha\varrho) \to c_{p,\alpha} = const \neq 0$ as $\varrho \to +0$. This means that the order of singularity of $u(x,t)$ will be no smaller than $p+1$.*

Corollary 7. *Let the conditions of Theorem 5 be fulfilled and in addition $f(x,t)$ satisfies the orthogonality conditions:*

$$\int_{\Omega_m} v_{k,m}^{n,s}(x,t)\, f(x,t)\, dx\, dt = 0 \quad (21)$$

for all $n = 0, 1, \ldots, l$; $k = 0, 1, \ldots, [n/2]$ and $s = 1, 2, \ldots, 2n+1$. Then the unique generalized solution $u(x,t)$ of problem PK fulfills the a priori estimate on Σ_2^m:

$$|u(x, \tau(|x|))| \le C \left(\max_{\overline{\Omega}_m} |f| \right) |x|^{-m/(4-2m)}, \quad (22)$$

where C is a positive constant independent of $f(x,t)$.

Actually, Theorem 5 gives the asymptotic behavior of the singular solutions of problem PK on Σ_2^m. It clarifies the significance of the orthogonality conditions (21): for fixed indexes (n,k,s), the corresponding condition (21) "controls" one power-type singularity. We mention here that some of the orthogonality conditions (21) involve functions $v_{k,m}^{n,s}(x,t)$, which are not classical solutions of problem PK* (see the proof of Lemma 1).

3. History of the Problem and Motivation

It is well-known that different boundary value problems (BVPs) for mixed-type equations have important applications in transonic gas dynamics (see Bers [2], Morawetz [3], and Kuz'min [4]). After a space symmetry assumption, the transonic potential flows in fluid dynamics are described in the hodograph plane by two-dimensional BVPs for the Chaplygin equation:

$$K(t)\, u_{xx} - u_{tt} = 0, \quad (23)$$

where $tK(t) > 0$ for $t \neq 0$. The Chaplygin equation (23) is elliptic in the subsonic half-plane $t < 0$ and hyperbolic in the supersonic half-plane $t > 0$.

In particular, certain flows around airfoils are modeled by the Guderley-Morawetz plane problem for (23) (see the monograph of Bers [2]). The domain is bounded in the elliptic half-plane by a smooth arc σ and in the hyperbolic half-plane by four characteristic segments that start from the

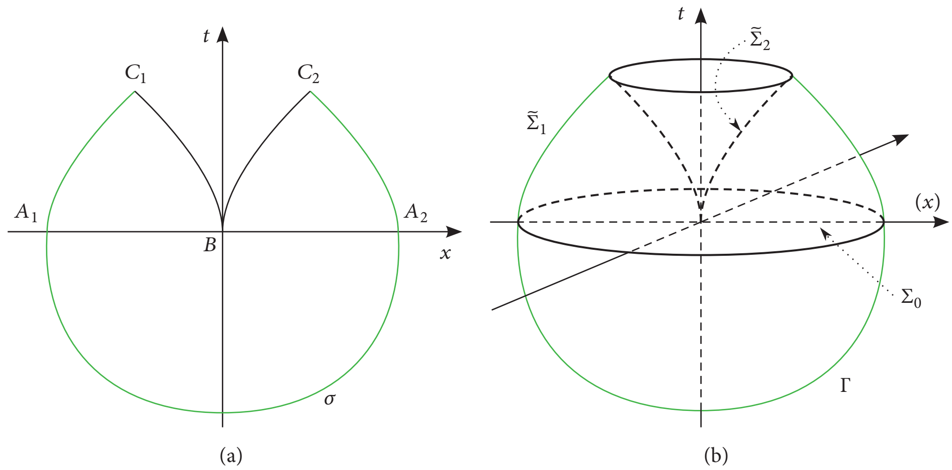

FIGURE 2: (a) Guderley-Morawetz domain. (b) Protter-Morawetz domain $\widetilde{\Omega}$.

three points A_1, A_2, and B on the sonic line; see Figure 2(a). The values of the function are prescribed along σ and along the characteristics A_1C_1 and A_2C_2. The Guderley-Morawetz problem is well studied. The existence of weak solutions and the uniqueness of a strong solution in weighted Sobolev spaces were obtained by Morawetz [5]. Lax and Phillips [6] proved that the weak solutions are strong.

Results on another BVP for the Chaplygin equation in mixed-type domain can be found in the recent paper of Liu et al. [7].

An interesting multidimensional generalization of Guderley-Morawetz problem was proposed by Protter [8, 9] for the multidimensional Chaplygin equation:

$$K(t)\,\Delta_x u - u_{tt} = K(t)\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} = f(x,t), \qquad (24)$$

where $x = (x_1, \ldots, x_N)$, $N \geq 2$. Protter considered (24) in the Protter-Morawetz domain $\widetilde{\Omega}$, which could be obtained by rotating of a symmetrical Guderley-Morawetz domain around the axis of symmetry (see Figure 2). The boundary data are prescribed on Γ in the elliptic part and on the outer characteristic surface $\widetilde{\Sigma}_1$. On the characteristic surface $\widetilde{\Sigma}_2$, data are not imposed. Aziz and Schneider [10] obtained uniqueness result for this problem, but even now there is not a single example of a nontrivial solution to the multidimensional problem (as, for example, in Lemma 1 above), neither a general existence result is known. Many difficulties and differences in comparison with the planar problems can be illustrated as well by the related problems in the hyperbolic part of the domain, also formulated by Protter.

Protter Problems. Find a solution of (24) with $K(t) = t^m$, $m \in \mathbb{R}$, $m \geq 0$ in the domain $\widetilde{\Omega} \cap \{t > 0\}$ with one of the following boundary conditions:

$$\begin{aligned} P1&: \quad u|_{\Sigma_0 \cup \widetilde{\Sigma}_1} = 0, \\ P2&: \quad u|_{\widetilde{\Sigma}_1} = 0, \ \ u_t|_{\Sigma_0} = 0. \end{aligned} \qquad (25)$$

These BVPs are multidimensional analogues of the Darboux-Goursat plane problems for the Gellerstedt equation ($m > 0$) or for the wave equation ($m = 0$). Garabedian [11] proved the uniqueness of a classical solution to problem $P1$ for the wave equation in $\mathbb{R}^4$. Popivanov and Schneider [12] showed that both problems $P1$ and $P2$ are not well-posed in the frame of classical solvability, since they have infinite-dimensional cokernels (see also Khe [13]). In [12], they suggested to study the Protter problems in the frame of generalized solutions with possible big singularities. Today it is well-known that the Protter problems have singular generalized solutions, even for smooth right-hand sides [12, 14–17]. Different aspects of Protter problems and several their generalizations (including some applications in the industrial explosion process) are studied by many authors (see Aldashev and Kim [18], Choi and Park [19], Aldashev [20], and references therein). For different statements of other related problems for mixed-type equations of the first kind, including nonlinear equations, see [21–27].

The Keldysh-type equations are another kind of mixed-type equations that also are known to play an important role in fluid mechanics, for example,

$$u_{xx} + t^m u_{tt} + a u_x + b u_t + c u = 0 \qquad (26)$$

near the line $t = 0$.

Otway [28, 29] and Lupo et al. [30] gave a statement of some 2D BVPs for elliptic-hyperbolic Keldysh-type equations with specific applications in plasma physics, including a model for analyzing the possible heating in axisymmetric cold plasmas. Čanić and Keyfitz [31] studied some plane problems for a nonlinear degenerate elliptic equation, whose solutions behave like those of a Keldysh-type equation. Such an equation arises in the modeling of a weak shock reflection at a wedge. A 2D mixed-type equation analogous in part to the Tricomi-type and the Keldysh-type equations has also been studied recently by Shuxing [32].

Keldysh [33] studied the regularity of the solutions of 2D elliptic equations of second order near the boundary, in the case when the boundary contains a segment of the line

$t = 0$. He showed that for degenerating elliptic equation (26) the formulation of the Dirichlet problem may depend on the lower order terms (the dependence is different for different values of m). Fichera [34] generalized Keldysh's results for multidimensional linear second-order equations with nonnegative characteristic form and now BVPs for them are well understood in the sense that boundary conditions should not be imposed on the whole boundary. A summary of Fichera's theory can be found in Radkevich [35, 36]. Keyfitz [37] examined whether the Fichera's classification could be extended to quasilinear equations and mentioned that contrasting behavior of the characteristics of the Tricomi and Keldysh equation (see Figures 1 and 2) may have implications, unexplored yet, for the solution of some free boundary problems arising in the fluid dynamics models.

All these results and the fact that the solutions of the Keldysh-type equation are not differentiable at the degenerate boundary (see [38]) make it interesting to formulate and study the multidimensional Protter-Morawetz problem for Keldysh-type equations. In [39], using the exact Hardy-Sobolev inequality, we proved the uniqueness of a quasiregular solution to problem PK for equations involving lower order terms. Let us mention here that, in problem PK, unlike Tricomi case, a data on the degenerate boundary is not prescribed (similar to the elliptic case) and derivative u_t can have singularity on it, but up to the prescribed level. On the other hand, the results in [1] and the results of the present paper show some similarities between problem PK and problems $P1$, $P2$: the infinite-dimensional cokernel of the problem and the existence of generalized solutions with isolated singularities.

There are still some open questions in this area that naturally arise.

Open Problems

(1) In the case when the right-hand side function $f(x,t)$ is a "harmonic polynomial" to find additional conditions under which problem PK has a bounded solution. According to Corollary 7, when all the orthogonality conditions, which we prescribe in the present paper, are fulfilled, the generalized solution $u(x,t)$ is still allowed to have a singularity of order $\beta \in (0,1)$.

(2) To study the general case of problem PK when the right-hand side function $f(x,t)$ is a smooth function not only of the form of "harmonic polynomial" is an open problem:

- (i) Find appropriate conditions for the function $f(x,t)$ under which there exists a generalized solution.
- (ii) What kind of singularity may have the generalized solution in this case? The a priori estimate, obtained in [1], shows that when the function $f(x,t)$ is a "harmonic polynomial" the generalized solution may have at most a polynomial growth. Are there exist singular solutions with an exponential growth, as it is in the case of the Protter problems for the usual wave equation?
- (iii) To find some appropriate conditions for the function $f(x,t)$ under which problem PK has only regular, bounded, or even classical solution. Up to now such conditions for the existence of a bounded solution to Protter problems are obtained only in the case of the wave equation.

(3) To study problem PK in the more general case when $0 < m < 2$. Let us mention that the presentation of the generalized solution $u(x,t)$ from [1], which we are studying in the present paper, is valid only in the case when $m \in (0, 4/3)$. Find appropriate techniques that work for $4/3 \le m < 2$.

4. The Two-Dimensional Darboux-Goursat Problems Corresponding to Problem PK

Problem PK in the case when the right-side function $f(x,t)$ is of the form (15) can be reduced to a two-dimensional problem.

More precisely, let us look for a solution to problem PK of the form (16). Using the spherical coordinates $(r,\theta,\varphi,t) \in \mathbf{R}^4$, $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$, $r > 0$ with

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta, \tag{27}$$

and later in the characteristic coordinates

$$\xi = 1 - r - \frac{2}{2-m}t^{(2-m)/2}, \quad \eta = 1 - r + \frac{2}{2-m}t^{(2-m)/2}, \tag{28}$$

for the functions

$$U(\xi,\eta) := r(\xi,\eta)\, u_n^s(r(\xi,\eta), t(\xi,\eta)) \tag{29}$$

we obtain (see [1]) the following Darboux-Goursat problem.

Problem PK_2. Find a solution of

$$U_{\xi\eta} + \frac{\beta}{\eta-\xi}\left(U_\xi - U_\eta\right) - \frac{n(n+1)}{(2-\xi-\eta)^2}U = F(\xi,\eta) \quad \text{in } D, \tag{30}$$

satisfying the following boundary conditions:

$$U(0,\eta) = 0, \quad \lim_{\eta-\xi\to+0}(\eta-\xi)^{2\beta}\left(U_\xi - U_\eta\right) = 0, \tag{31}$$

where

$$D := \{(\xi,\eta) : 0 < \xi < \eta < 1\}, \tag{32}$$

$$F(\xi,\eta) := \frac{1}{8}(2-\xi-\eta) f_n^s(r(\xi,\eta), t(\xi,\eta)), \quad (33)$$

$$\beta := \frac{m}{2(2-m)}. \quad (34)$$

As far as we consider problem PK in the case when $m \in (0, 4/3)$ for the parameter β we have

$$0 < \beta < 1. \quad (35)$$

In conformity with Definition 2, a generalized solution of problem PK_2 is defined as follows.

Definition 8 (see [1]). We call a function $U(\xi,\eta)$ a generalized solution of problem PK_2 in D, $(0 < \beta < 1)$, if

(1) $U, U_\xi + U_\eta \in C(\overline{D} \setminus (1,1))$, $U_\xi - U_\eta \in C(\overline{D} \setminus \{\eta = \xi\})$;

(2)

$$U(0,\eta) = 0; \quad (36)$$

(3) for each $\varepsilon \in (0,1)$ there exists a constant $C(\varepsilon) > 0$, such that

$$\left|(U_\xi - U_\eta)(\xi,\eta)\right| \le C(\varepsilon)(\eta-\xi)^{-\beta} \quad \text{in } D \cap \{\xi < 1-\varepsilon\}; \quad (37)$$

(4) the identity

$$\int_D (\eta-\xi)^{2\beta} \left\{ U_\xi V_\eta + U_\eta V_\xi + \frac{2n(n+1)}{(2-\xi-\eta)^2} UV + 2FV \right\} d\xi\, d\eta = 0 \quad (38)$$

holds for all

$$V \in V^{(2)} := \left\{ V(\xi,\eta) : V \in C^2(\overline{D}),\ V(\xi,1) = 0,\ V \equiv 0 \text{ in a neighborhood of } (1,1) \right\}. \quad (39)$$

Further, using the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ associated with problem PK_2, an explicit integral representation of the generalized solution $U(\xi,\eta)$ was found. A survey of the Riemann method can be found in [40].

Theorem 9 (see [1]). *Let $0 < \beta < 1$ and $F \in C^1(\overline{D})$. Then there exists one and only one generalized solution of problem PK_2 in D, which has the following integral representation at a point $(\xi_0,\eta_0) \in D$:*

$$U(\xi_0,\eta_0) = \int_0^{\xi_0} \int_\xi^{\eta_0} F(\xi,\eta)\, \Phi(\xi,\eta;\xi_0,\eta_0)\, d\eta\, d\xi, \quad (40)$$

and it satisfies the following estimates:

$$\begin{aligned} &|U(\xi,\eta)| \le K M_F \xi (2-\xi-\eta)^{-n} && \text{in } \overline{D} \setminus (1,1), \\ &\left|(U_\xi + U_\eta)(\xi,\eta)\right| \le K M_F (2-\xi-\eta)^{-n-1} && \text{in } \overline{D} \setminus (1,1), \\ &\left|U_\eta(\xi,\eta)\right| \le K M_F \xi (\eta-\xi)^{-\beta} (2-\xi-\eta)^{-n-1} && \text{in } \overline{D} \setminus \{\eta = \xi\}, \end{aligned} \quad (41)$$

where K is a positive constant and

$$M_F := \max\left\{ \max_{\overline{D}} |F|, \max_{\overline{D}} \left|F_\xi + F_\eta\right| \right\}. \quad (42)$$

The Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$, $(\xi_0,\eta_0) \in D$, which we have found in [1], can be represented as follows:

$$\Phi(\xi,\eta;\xi_0,\eta_0) = \begin{cases} \Phi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \Phi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases} \quad (43)$$

where

$$\begin{aligned} \Phi^+(\xi,\eta;\xi_0,\eta_0) &:= \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^\beta \cdot F_3(\beta, n+1, 1-\beta, -n, 1; X, Y), \\ \Phi^-(\xi,\eta;\xi_0,\eta_0) &:= \gamma \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^\beta \cdot X^{-\beta} H_2\left(\beta, \beta, -n, n+1, 2\beta; \frac{1}{X}, -Y\right), \end{aligned} \quad (44)$$

$$X = X(\xi,\eta,\xi_0,\eta_0) := \frac{(\xi_0-\xi)(\eta_0-\eta)}{(\eta-\xi)(\eta_0-\xi_0)}, \quad (45)$$

$$Y = Y(\xi,\eta,\xi_0,\eta_0) := -\frac{(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}, \quad (46)$$

$$\gamma = \frac{\Gamma(\beta)}{\Gamma(1-\beta)\Gamma(2\beta)}. \quad (47)$$

Here $F_3(a_1,a_2,b_1,b_2,c;x,y)$ is the Appell series:

$$F_3(a_1,a_2,b_1,b_2,c;x,y) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_i (b_1)_j (b_2)_i}{(c)_{i+j}\, i!\, j!} x^j y^i, \quad (48)$$

and $H_2(a_1,a_2,b_1,b_2,c;x,y)$ is the Horn series:

$$H_2(a_1,a_2,b_1,b_2,c;x,y) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a_1)_{j-i} (a_2)_j (b_1)_i (b_2)_i}{(c)_j\, i!\, j!} x^j y^i \quad (49)$$

(for basic information on the Appell and the Horn series, see [41], p. 222–228.)

According to Theorem 9, the generalized solution is allowed to have a singularity of order no greater than n at point $(1,1)$. But it is still not clear if such a singularity really exists and how it depends on the right-hand side of the equation. In the next section, we study more deeply the function $U(\xi,\eta)$, given by (40), or, more precisely, its restriction on the segment $\eta=1$, $0\le\xi<1$.

5. The Asymptotic Expansion of the Solution of Problem PK_2

Introduce the following functions:

$$\widetilde{E}_k^n(\xi,\eta) := r(\xi,\eta)\,E_{k,m}^n(r(\xi,\eta),t(\xi,\eta)) = \sum_{i=0}^{k} A_i^k \frac{(1-\xi)^{n-k-i-\beta}(1-\eta)^{n-k-i-\beta}}{2^{2i-n}(2-\xi-\eta)^{n-2i}}, \quad (50)$$

where $E_{k,m}^n(|x|,t)$ are functions (6), closely connected with the solutions of the homogeneous adjoint problem PK^*. Then we prove the following lemma.

Lemma 10. *For $k=0,\dots,[n/2]$ the following equalities hold*

$$\int_D(\eta-\xi)^{2\beta}\widetilde{E}_k^n(\xi,\eta)F(\xi,\eta)\,d\xi\,d\eta = C_m\mu_{k,m}^{n,s}, \qquad C_m = 2^{(3m-2)/(2-m)}(2-m)^{-m/(2-m)}, \quad (51)$$

where $\mu_{k,m}^{n,s}$ are the coefficients (11) and the relation between $F(\xi,\eta)$ and the Fourier coefficient f_n^s from the expansion of $f(x,t)$ is given by (33).

Proof. Denote

$$G_m := \left\{(r,t): 0<t<t_0,\ \frac{m}{2-m}t^{(2-m)/m}<r<1-\frac{m}{2-m}t^{(2-m)/m}\right\}. \quad (52)$$

Denote also by $\mathbb{Y}_n^s$ the spherical functions expressed in the spherical coordinates; that is, $Y_n^s(x)=\mathbb{Y}_n^s(\theta(x),\varphi(x))$. Then, using the orthonormality of the spherical functions on the unit sphere S^2, a direct calculation gives

$$\begin{aligned}\mu_{k,m}^{n,s} &= \int_{\Omega_m} v_{k,m}^{n,s}(x,t)\,f(x,t)\,dx\,dt \\ &= \int_0^{\pi}\int_0^{2\pi}\int_{G_m} E_{k,m}^n(r,t)\,\mathbb{Y}_n^s(\theta,\varphi) \\ &\cdot\left(\sum_{p=0}^{l}\sum_{q=1}^{2p+1} f_p^q(r,t)\,\mathbb{Y}_p^q(\theta,\varphi)\right)\sin\theta\, r^2 dr\,dt\,d\varphi\,d\theta \\ &= \int_{S^2}(\mathbb{Y}_n^s)^2(\theta,\varphi)\,dS\int_{G_m}\left(E_{k,m}^n f_n^s\right)(r,t)\,r^2 dr\,dt \\ &= 2^{(3m-2)/(m-2)}(2-m)^{m/(2-m)}\int_D(\eta-\xi)^{2\beta}\widetilde{E}_k^n(\xi,\eta) \\ &\cdot F(\xi,\eta)\,d\xi\,d\eta.\end{aligned} \quad (53)$$

The proof is complete. □

Theorem 11. *Suppose that $F\in C^1(\overline{D})$. Then the restriction $U(\xi,1)$ of the generalized solution of problem PK_2 has the following expansion on the segment $\{0\le\xi<1\}$:*

$$U(\xi,1) = \sum_{k=0}^{[n/2]} b_k^n\mu_{k,m}^{n,s}(1-\xi)^{-n+2k} + (1-\xi)^{1-\beta}g(\xi), \quad (54)$$

where $g(\xi)\in C^1([0,1))$ and $|g(\xi)|\le C\|F\|_{C(D)}$, with constants $C>0$ and $b_k^n\ne 0$ independent of F.

Proof. According to Theorem 9 the condition $F\in C^1(\overline{D})$ assures that there exists an unique generalized solution $U(\xi,\eta)$ of problem PK_2, given by (40). According to Definition 8, we see that the restriction $U(\xi,1)$ should belong to $C^1([0,1))$.

Further, we set $\eta_0=1$ in (40). Essential for the following calculations is the decomposition of $\Phi^-(\xi,\eta;\xi_0,1)$ given in Theorem A.3 which we prove in Appendix. Using (A.34) we obtain

$$\begin{aligned}U(\xi_0,1) &= \int_0^{\xi_0}\int_\xi^1 F(\xi,\eta)\,\Phi(\xi,\eta;\xi_0,1)\,d\eta\,d\xi \\ &= \int_0^{\xi_0}\int_\xi^{\xi_0} F(\xi,\eta)\,\Phi_1^-(\xi,\eta;\xi_0)\,d\eta\,d\xi \\ &+ \int_0^{\xi_0}\int_\xi^{\xi_0} F(\xi,\eta)\,\Phi_2^-(\xi,\eta;\xi_0)\,d\eta\,d\xi \\ &+ \int_0^{\xi_0}\int_{\xi_0}^{1} F(\xi,\eta)\,\Phi^+(\xi,\eta;\xi_0,1)\,d\eta\,d\xi \\ &=: J_1+J_2+J_3.\end{aligned} \quad (55)$$

According to (A.35) and Lemma 10 we have

$$\begin{aligned}J_1 &= \sum_{k=0}^{[n/2]} c_k^n(1-\xi_0)^{-n+2k} \\ &\cdot\int_0^{\xi_0}\int_\xi^{\xi_0}(\eta-\xi)^{2\beta}F(\xi,\eta)\,\widetilde{E}_{k,m}^n(\xi,\eta)\,d\eta\,d\xi \\ &= \sum_{k=0}^{[n/2]} c_k^n(1-\xi_0)^{-n+2k} \\ &\cdot\left\{\int_D(\eta-\xi)^{2\beta}F(\xi,\eta)\,\widetilde{E}_{k,m}^n(\xi,\eta)\,d\eta\,d\xi\right. \\ &\left.-\int_{\xi_0}^1\int_0^\eta(\eta-\xi)^{2\beta}F(\xi,\eta)\,\widetilde{E}_{k,m}^n(\xi,\eta)\,d\xi\,d\eta\right\}\end{aligned}$$

$$= \sum_{k=0}^{[n/2]} c_k^n (1-\xi_0)^{-n+2k} \Big\{ C_m \tilde{\mu}_{k,m}^n - \int_{\xi_0}^{1} \int_0^{\eta} (\eta-\xi)^{2\beta} F(\xi,\eta) \tilde{E}_{k,m}^n (\xi,\eta) \, d\xi \, d\eta \Big\} = \sum_{k=0}^{[n/2]} b_k^n \tilde{\mu}_{k,m}^n (1-\xi_0)^{-n+2k} + J_{1,0}, \tag{56}$$

with $b_k^n = C_m c_k^n \neq 0$.

The functions $\tilde{E}_k^n$, given by (50), can be estimated in D as follows:

$$\left|\tilde{E}_{k,m}^n (\xi,\eta)\right| \le a_k^n (1-\xi)^{-\beta} (1-\eta)^{n-2k-\beta}, \quad a_k^n = \text{const} > 0. \tag{57}$$

Then we have

$$\begin{aligned} \left|J_{1,0}\right| &\le \sum_{k=0}^{[n/2]} c_k^n (1-\xi_0)^{-n+2k} \cdot \left| \int_{\xi_0}^{1} \int_0^{\eta} (\eta-\xi)^{2\beta} F(\xi,\eta) \tilde{E}_{k,m}^n (\xi,\eta) \, d\xi \, d\eta \right| \\ &\le \|F\|_{C(D)} \sum_{k=0}^{[n/2]} a_k^n c_k^n \cdot \int_{\xi_0}^{1} \int_0^{\eta} (\eta-\xi)^{2\beta} (1-\xi)^{-\beta} (1-\eta)^{-\beta} \, d\xi \, d\eta \\ &\le k_2 \|F\|_{C(D)} (1-\xi_0)^{1-\beta}, \quad k_2 = \text{const} > 0. \end{aligned} \tag{58}$$

For J_2, using the estimate (A.36) from Theorem A.3, we obtain

$$\left|J_2\right| \le k_1 \|F\|_{C(D)} (1-\xi_0) \cdot \int_0^{\xi_0} \int_{\xi}^{\xi_0} (\xi_0-\eta)^{-\beta} (1-\eta)^{-1} \, d\eta \, d\xi. \tag{59}$$

Making a substitution $\eta = \xi + (\xi_0 - \xi)\sigma$ we compute

$$\int_{\xi}^{\xi_0} (\xi_0-\eta)^{-\beta} (1-\eta)^{-1} \, d\eta = \frac{(\xi_0-\xi)^{1-\beta}}{1-\xi} \int_0^1 (1-\sigma)^{-\beta} (1-z\sigma)^{-1} \, d\sigma \tag{60}$$

with

$$z := \frac{\xi_0-\xi}{1-\xi}. \tag{61}$$

Formula (A.11) gives

$$\int_0^1 (1-\sigma)^{-\beta} (1-z\sigma)^{-1} \, d\sigma = \frac{\Gamma(1-\beta)}{\Gamma(2-\beta)} \, {}_2F_1 (1,1,2-\beta; z) \tag{62}$$

and with (A.13) we estimate

$$\left| {}_2F_1 (1,1,2-\beta; z) \right| \le k_3 \frac{(1-\xi)^{\beta}}{(1-\xi_0)^{\beta}}, \quad k_3 = \text{const} > 0. \tag{63}$$

Applying the results (60), (62), and (63) into (59) we obtain

$$\left|J_2\right| \le k_4 \|F\|_{C(D)} (1-\xi_0)^{1-\beta}, \quad k_4 = \text{const} > 0. \tag{64}$$

According to the results from [1] (Lemmas A.1, A.2, and A.3 therein), we have an estimate

$$\left|\Phi^{+} (\xi,\eta;\xi_0,1)\right| \le k_5 (\eta-\xi_0)^{-\beta}, \quad k_5 = \text{const} > 0. \tag{65}$$

Then for J_3 we have

$$\left|J_3\right| \le k_6 \|F\|_{C(D)} (1-\xi_0)^{1-\beta}, \quad k_6 = \text{const} > 0. \tag{66}$$

Therefore (54) holds with $g(\xi) := (1-\xi)^{\beta-1}(J_{1,0} + J_2 + J_3)$. Obviously, $g(\xi) \in C^1([0,1))$, because $J_{1,0} + J_2 + J_3 = U(\xi,1) - \sum_{k=0}^{[n/2]} b_k^n \mu_{k,m}^{n,s} (1-\xi)^{-n+2k}$.

The proof is complete. □

From this theorem, we see that the generalized solution of problem PK_2 may have a singularity of order n and this happens in the general case: a bounded solution, or a solution with a smaller order of singularity, is possible only if some of the coefficients $\mu_{k,m}^{n,s}$ are equal to zero. This result exactly corresponds to the estimate prescribed in Theorem 9.

6. Proof of the Main Results

Now, we are ready to prove the main results stated in Section 2.

Proof of Lemma 1. First, we have obviously $v_{k,m}^{n,s}(x,t) \in C^{\infty}(\Omega_m)$.

For $n > 2k + m/(4-2m)$ it is easy to check that $v_{k,m}^{n,s} \in C(\overline{\Omega}_m \setminus O)$ and $v_{k,m}^{n,s}|_{\Sigma_2^m} = 0$.

For $n > N(m,k) := 2k + 1 + m/(2-m)$ we see that $E_{k,m}^n(|x|,t) \to 0$ as $(x,t) \to O$.

Therefore $v_{k,m}^{n,s}(x,t) \in C^{\infty}(\Omega_m) \cap C(\overline{\Omega}_m)$.

It is easy to check that for $n > N(m,k)$ the boundary conditions (5) are also satisfied.

Now, let us look for solutions of the homogeneous problem PK^* of the form (9). Passing to the spherical coordinates (27) in the homogeneous equation (1) and using that, the spherical functions satisfy the differential equation:

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} Y_n^s \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} Y_n^s + n(n+1) Y_n^s = 0 \tag{67}$$

and we see that the functions $E^n_{k,m}(r,t)$ should be solutions of

$$v_{rr} + \frac{2}{r} v_r - \left(t^m v_t\right)_t - \frac{n(n+1)}{r^2} v = 0 \quad (68)$$

in G_m. A direct calculation of the derivatives of $E^n_{k,m}(r,t)$ shows that these functions indeed satisfy (68).

The proof is complete. □

Proof of Theorem 5. Let $u(x,t)$ be the unique generalized solution of problem PK. Then it has the form (16) (see Theorem 4)

As we saw, all the functions

$$U^s_n(\xi,\eta) := r(\xi,\eta)\, u^s_n(r(\xi,\eta), t(\xi,\eta)), \quad (69)$$

$n = 0,\dots,l$, $s = 1,\dots,2n+1$, are generalized solutions of problem PK_2 with right-hand sides $F^s_n(\xi,\eta)$ given by

$$F^s_n(\xi,\eta) := \frac{1}{8}(2-\xi-\eta)\, f^s_n(r(\xi,\eta), t(\xi,\eta)). \quad (70)$$

Now, Theorem 11 states that

$$U^s_n(\xi,1) = \sum_{k=0}^{[n/2]} b^{n,s}_k \mu^{n,s}_{k,m} (1-\xi)^{-n+2k} + (1-\xi)^{1-\beta} g^s_n(\xi), \quad (71)$$

where $b^{n,s}_k$ are nonzero constant independent of F^s_n, $g^s_n(\xi) \in C^1([0,1))$ and $|g^s_n(\xi)| \le C\|F^s_n\|_{C(D)}$.

Now, using the relations (27)-(28), we make the inverse transformation from problem PK_2 to problem PK. In this way we obtain the expansion (18), (20) with

$$G(x) = 2^{1-\beta} \sum_{p=0}^{l} \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} Y^s_{p+2k}(x)\, g^s_{p+2k}(1-2|x|) \quad (72)$$

and $c^{n,s}_{k,m} = 2^{2k-n} b^{n,s}_k \neq 0$.

The assertion (i) in Theorem 5 follows from the properties of the functions $g^s_n(\xi)$ and the fact that the functions $Y^s_n(x)$ are bounded and belong to $C^1(0<|x|<1/2)$.

The proof is complete. □

Proof of Corollary 6. Let at least one of the constants $\mu^{p+2k,s}_{k,m}$ in (20)

$$G_p(x) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c^{p+2k,s}_{k,m} \mu^{p+2k,s}_{k,m} Y^s_{p+2k}(x), \quad p = 0,1,\dots,l \quad (73)$$

be different from zero. Then by the linear independence of the spherical functions Y^s_n it follows that, for the corresponding function $G_p(x)$, there exists a vector $\alpha \in S^2$, such that $G_p(\alpha) \neq 0$. But, recalling that we extend functions Y^s_n out of S^2 radially, we have that $G_p(\alpha) = G_p(\alpha\varrho) = \mathrm{const}$, $\varrho > 0$. Therefore $G_p(\alpha\varrho) \to c_{p,\alpha} = \mathrm{const} \neq 0$ as $\varrho \to +0$.

The proof is complete. □

Appendix

For $a \in \mathbb{R}$ in our calculations we use the following relations (see [42]):

$$(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)}, \quad a, a+i \neq 0,-1,-2,\dots, \quad (A.1)$$

$$(a)_i = a(a+1)\cdots(a+i-1), \quad i \in \mathbb{N}, \qquad (a)_0 = 1, \quad (A.2)$$

$$(a)_{i+j} = (a)_i (a+i)_j, \quad (A.3)$$

$$(a)_{2i} = 2^{2i} \left(\frac{a}{2}\right)_i \left(\frac{a+1}{2}\right)_i, \quad (A.4)$$

$$(a)_j = (-1)^j (1-a-j)_j. \quad (A.5)$$

Further, we recall some well-known formulae, concerning the Gauss hypergeometric series (see, e.g., [41–43]):

$${}_2F_1(a,b,c;\zeta) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{i!\,(c)_i} \zeta^i, \quad a,b,c \in \mathbb{R},\ c \neq 0,-1,-2,\dots, \quad (A.6)$$

which are also used in the computations. For $|\zeta| < 1$ the series is absolutely convergent.

The derivatives of ${}_2F_1(a,b,c;\zeta)$ are given by

$$\frac{d^s}{d\zeta^s}\, {}_2F_1(a,b,c;\zeta) = \frac{(a)_s (b)_s}{(c)_s}\, {}_2F_1(a+s, b+s, c+s; \zeta), \quad s = 0,1,2,\dots. \quad (A.7)$$

In the special case when $c = (a+b+1)/2 \neq 0,-1,-2,\dots$ and $\zeta = 1/2$ we have

$${}_2F_1\left(a, b, \frac{a+b+1}{2}; \frac{1}{2}\right) = \begin{cases} \dfrac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\,\Gamma((b+1)/2)}, & a,b \neq -1,-3,\dots, \\ 0, & a = -1,-3,\dots \text{ or } b = -1,-3,\dots \end{cases} \quad (A.8)$$

In the case when $b = 0, -1, -2, \ldots$ and $a, b - a + 1, c \neq 0, -1, -2, \ldots$ the changing from ζ to $(1-\zeta)^{-1}$ is given by

$$\begin{aligned} {}_2F_1(a, b, c; \zeta) &= \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-\zeta)^{-b} \\ &\quad \cdot {}_2F_1\left(c-a, b, b-a+1; \frac{1}{1-\zeta}\right), \\ &\qquad \zeta \neq 1. \end{aligned} \tag{A.9}$$

The binomial series is a particular case of the hypergeometric series:

$$ {}_2F_1(a, b, a; \zeta) = (1-\zeta)^{-b}. \tag{A.10}$$

For $0 < a < c$ in the special case when $b = 0, -1, -2, \ldots$, the Euler integral representation

$$\begin{aligned} &{}_2F_1(a, b, c; \zeta) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-\zeta t)^{-b}\,dt \end{aligned} \tag{A.11}$$

is valid for each $\zeta \in \mathbb{R}$.

In the present paper we use the following estimates:

(a) If $c - a - b = 0$, then for each $\alpha > 0$ there exists a constant $c(\alpha) > 0$ such that

$$\left|{}_2F_1(a, b, c; \zeta)\right| \leq c(\alpha)(1-\zeta)^{-\alpha}. \tag{A.12}$$

(b) For the case $c - a - b < 0$ we have a constant $K > 0$ such that

$$\left|{}_2F_1(a, b, c; \zeta)\right| \leq K(1-\zeta)^{c-a-b}. \tag{A.13}$$

Here we prove some auxiliary results which we use in our calculations. First, we give two lemmas which we need for the proof of Theorem A.3.

Lemma A.1. *Let $a > 0$ and $k \in \mathbf{N} \cup \{0\}$. Then*

$$ {}_2F_1(a, -N, 2a; 2) = \begin{cases} 0, & N = 2k+1, \\ \dfrac{(1/2)_k}{(1/2+a)_k}, & N = 2k. \end{cases} \tag{A.14}$$

Proof. According to the integral representation (A.11) we have

$$\begin{aligned} &{}_2F_1(a, -N, 2a; 2) \\ &= \frac{\Gamma(2a)}{\Gamma(a)\Gamma(a)}\int_0^1 t^{a-1}(1-t)^{a-1}(1-2t)^N\,dt. \end{aligned} \tag{A.15}$$

Then for $k \in \mathbf{N} \cup \{0\}$ we have

$$ {}_2F_1(a, -2k-1, 2a; 2) = 0, \tag{A.16}$$

because the function $h(t) := t^{a-1}(1-t)^{a-1}(1-2t)^{2k+1}$ is antisymmetric in respect to the point $t = 1/2$, that is, $h(1/2 - t) = -h(1/2 + t)$.

In the case when N is an even number we proceed by the induction method. For $k = 0$ ($N = 0$, resp.), (A.14) holds obviously.

For $N = 2, 4, 6, \ldots$ from (A.15) we get

$$\begin{aligned} \frac{\Gamma(a)\Gamma(a)}{\Gamma(2a)}\,{}_2F_1(a, -N, 2a; 2) &= \frac{1}{a} \\ \cdot \int_0^1 (1-2t)^{N-1}\,d\left(t-t^2\right)^a &= \frac{2(N-1)}{a} \\ \cdot \int_0^1 t^a(1-t)^a(1-2t)^{N-2}\,dt & \\ = \frac{2(N-1)\Gamma(a+1)\Gamma(a+1)}{a\Gamma(2a+2)} & \\ \cdot {}_2F_1(a+1, 2-N, 2a+2; 2), & \end{aligned} \tag{A.17}$$

or more simply

$$\begin{aligned} &{}_2F_1(a, -N, 2a; 2) \\ &= \frac{(N-1)}{(2a+1)}\,{}_2F_1(a+1, 2-N, 2(a+1); 2). \end{aligned} \tag{A.18}$$

Our induction hypothesis is that for some $k \in \mathbf{N} \cup \{0\}$ the equality

$$ {}_2F_1(a, -2k, 2a; 2) = \frac{(1/2)_k}{(a+1/2)_k} \tag{A.19}$$

holds. But then for $k + 1$ this equality will also hold, because according to (A.18) we have

$$\begin{aligned} {}_2F_1(a, -2k-2, 2a; 2) &= \frac{(2k+1)(1/2)_k}{(2a+1)(a+3/2)_k} \\ &= \frac{(1/2)_{k+1}}{(a+1/2)_{k+1}}. \end{aligned} \tag{A.20}$$

The proof is complete. □

Lemma A.2. *Let $n, p \in \mathbf{N} \cup \{0\}$, $p \le n$, $0 < \beta < 1$ and denote*

$$Q_n^p(z) := \sum_{j=0}^{n-p} a_j b_j z^j \times {}_2F_1\left(n+p+j+1, p-n + j, p+j+1; \frac{1-z}{2}\right), \tag{A.21}$$

where

$$a_j = \frac{(\beta)_j}{(2\beta)_j\, j!}, \qquad b_j = \frac{(n+p+1)_j (p-n)_j}{(p+1)_j}. \tag{A.22}$$

Then

$$Q_n^p(z) = \begin{cases} 0, & \dfrac{(n-p+1)}{2} \in N, \\ c_{n,p}\, {}_2F_1\left(\dfrac{n+p+1}{2}, \dfrac{p-n}{2}, \dfrac{1}{2}+\beta; z^2\right), & \dfrac{(n-p+1)}{2} \notin N, \end{cases} \tag{A.23}$$

where

$$c_{n,p} = \frac{\sqrt{\pi}\,\Gamma(p+1)}{\Gamma((n+p+2)/2)\,\Gamma((p-n+1)/2)}. \tag{A.24}$$

Proof. First, we expand the function ${}_2F_1$ from (A.21) in Taylor series in powers of z:

$$\begin{aligned} {}_2F_1&\left(n+p+j+1, p-n+j, p+j+1; \frac{1-z}{2}\right) \\ &= \sum_{s=0}^{n-p-j} \frac{(n+p+j+1)_s (p-n+j)_s}{(p+j+1)_s\, s!} \left(\frac{-z}{2}\right)^s \\ &\times {}_2F_1\left(n+p+j+s+1, p-n+s+j, p+j+s+1; \frac{1}{2}\right), \end{aligned} \tag{A.25}$$

where we use (A.7) to compute the corresponding derivatives in the series. By formula (A.8) we have that for $N \ge 0$

$${}_2F_1\left(n+p+N+1, p-n+N, p+N+1; \frac{1}{2}\right) = A_N := \begin{cases} \dfrac{\sqrt{\pi}\,\Gamma(p+N+1)}{\Gamma((n+p+N+2)/2)\,\Gamma((p-n+N+1)/2)}, & p-n+N \neq -1, -3, \ldots, \\ 0, & p-n+N = -1, -3, \ldots. \end{cases} \tag{A.26}$$

Then $Q_n^p(z)$, using also (A.3), becomes

$$Q_n^p(z) = \sum_{j=0}^{n-p} \sum_{s=0}^{n-p-j} a_j b_{j+s} A_{j+s} \frac{(-1)^s}{2^s s!} z^{j+s}. \tag{A.27}$$

Now set $N = j+s$:

$$Q_n^p(z) = \sum_{N=0}^{n-p} b_N A_N z^N \sum_{j=0}^{N} a_j \frac{(-1)^{N-j}}{2^{N-j}(N-j)!}. \tag{A.28}$$

Since $(N-j)! = (-1)^j N!/(-N)_j$, for $Q_n^p(z)$, we obtain

$$Q_n^p(z) = \sum_{N=0}^{n-p} {}_2F_1(\beta, -N, 2\beta; 2)\, b_N A_N \frac{(-z)^N}{2^N N!}. \tag{A.29}$$

There are two different cases.

(i) Let $n-p$ Be an Odd Number. In this case (A.29) becomes

$$Q_n^p(z) \equiv 0, \tag{A.30}$$

because

(a) for even indexes N according to (A.26) we have $A_N = 0$;

(b) for odd indexes N Lemma A.1 with $a = \beta$ gives ${}_2F_1(\beta, -N, 2\beta; 2) = 0$.

(ii) Let $n-p$ Be an Even Number. In this case, according to (A.26), we have nonzero coefficients A_N in (A.29) only for even indexes N. Then we set $N = 2k$ and by Lemma A.1 we

have

$$ {}_2F_1(\beta, -2k, 2\beta; 2) = \frac{(1/2)_k}{(1/2+\beta)_k}. \quad \text{(A.31)} $$

Now with (A.4) we calculate

$$ (n+p+1)_{2k} = 2^{2k}\left(\frac{n+p+1}{2}\right)_k\left(\frac{n+p+2}{2}\right)_k, $$
$$ (p-n)_{2k} = 2^{2k}\left(\frac{p-n}{2}\right)_k\left(\frac{p-n+1}{2}\right)_k, $$
$$ (2k)! = 2^{2k}\left(\frac{1}{2}\right)_k k!. \quad \text{(A.32)} $$

Applying the equalities (A.31)-(A.32) into (A.29) with $N = 2k$ and simplifying the derived expression with use of (A.1), we obtain

$$ \begin{aligned} Q_n^p(z) &= \sum_{k=0}^{(n-p)/2} \frac{(1/2)_k}{(1/2+\beta)_k}\frac{b_{2k}A_{2k}}{2^{2k}(2k)!}z^{2k} \\ &= \sqrt{\pi}\Gamma(p+1)\sum_{k=0}^{(n-p)/2}\frac{(1/2)_k}{(1/2+\beta)_k}\frac{(n+p+1)_{2k}(p-n)_{2k}}{\Gamma((n+p+2k+2)/2)\Gamma((p-n+2k+1)/2)}\frac{z^{2k}}{2^{2k}(2k)!} \\ &= c_{n,p}\,{}_2F_1\left(\frac{n+p+1}{2}, \frac{p-n}{2}, \frac{1}{2}+\beta; z^2\right). \end{aligned} \quad \text{(A.33)} $$

The proof is complete. □

Theorem A.3. *The trace of the function $\Phi^-(\xi,\eta;\xi_0,\eta_0)$ on the line $\{\eta_0 = 1\}$ can be decomposed as follows:*

$$ \Phi^-(\xi,\eta;\xi_0,1) = \Phi_1^-(\xi,\eta;\xi_0) + \Phi_2^-(\xi,\eta;\xi_0), \quad \text{(A.34)} $$

where

$$ \Phi_1^-(\xi,\eta;\xi_0) = (\eta-\xi)^{2\beta}\sum_{k=0}^{[n/2]} c_k^n (1-\xi_0)^{-n+2k}\tilde{E}_k^n(\xi,\eta), \quad \text{(A.35)} $$

c_k^n are nonzero constants, and the function $\Phi_2^-(\xi,\eta;\xi_0)$ satisfies in $D \cap \{\eta < \xi_0\}$ the following estimate:

$$ \left|\Phi_2^-(\xi,\eta;\xi_0)\right| \le k_1\frac{1-\xi_0}{(\xi_0-\eta)^\beta(1-\eta)}, \quad k_1 = const > 0. \quad \text{(A.36)} $$

Proof. For $\Phi^-(\xi,\eta;\xi_0,1)$ from (44) and (49) we obtain

$$ \Phi^-\big|_{\eta_0=1} = \frac{\gamma(\eta-\xi)^{2\beta}}{(\xi_0-\xi)^\beta(1-\eta)^\beta} \cdot \sum_{i=0}^{n}\sum_{j=0}^{\infty}\frac{(\beta)_{j-i}(\beta)_j(-n)_i(n+1)_i}{(-1)^i(2\beta)_j\, i!\, j!} \times \left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1). \quad \text{(A.37)} $$

According to (A.3) and (A.5) we have

$$ \begin{aligned} (\beta-i)_j &= (\beta-i)_{i+(j-i)} = (\beta-i)_i(\beta)_{j-i} \\ &= (-1)^i(1-\beta)_i(\beta)_{j-i} \end{aligned} \quad \text{(A.38)} $$

and, consequently,

$$ \Phi^-\big|_{\eta_0=1} = \frac{\gamma(\eta-\xi)^{2\beta}}{(\xi_0-\xi)^\beta(1-\eta)^\beta} \cdot \sum_{i=0}^{n}\sum_{j=0}^{\infty}d_i b_{i,j}\left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1), \quad \text{(A.39)} $$

where

$$ d_i := \frac{(-n)_i(n+1)_i}{(1-\beta)_i\, i!}, \qquad b_{i,j} := \frac{(\beta-i)_j(\beta)_j}{(2\beta)_j\, j!}. \quad \text{(A.40)} $$

Now we set

$$ \Phi^-\big|_{\eta_0=1} = \frac{\gamma(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\left\{\Psi_1(\xi,\eta;\xi_0) + \Psi_2(\xi,\eta;\xi_0)\right\}, \quad \text{(A.41)} $$

with

$$ \begin{aligned} \Psi_1(\xi,\eta;\xi_0) &:= \left(\frac{1-\xi}{\xi_0-\xi}\right)^\beta\sum_{i=0}^{n}\sum_{j=0}^{i}d_i b_{i,j}\left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1), \\ \Psi_2(\xi,\eta;\xi_0) &:= \left(\frac{1-\xi}{\xi_0-\xi}\right)^\beta\sum_{i=0}^{n}\sum_{j=i+1}^{\infty}d_i b_{i,j}\left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1). \end{aligned} \quad \text{(A.42)} $$

For $0 < \xi < \xi_0 < 1$, using (A.10), we have

$$\left(\frac{\xi_0-\xi}{1-\xi}\right)^{i-j-\beta} = \sum_{q=0}^{\infty}\frac{(j-i+\beta)_q}{q!}\left(\frac{1-\xi_0}{1-\xi}\right)^q, \quad (A.43)$$

and according to this we decompose $\Psi_1(\xi,\eta;\xi_0)$ as follows:

$$\Psi_1(\xi,\eta;\xi_0) = \Psi_{1,1}(\xi,\eta;\xi_0) + \Psi_{1,2}(\xi,\eta;\xi_0), \quad (A.44)$$

with

$$\begin{aligned}\Psi_{1,1}(\xi,\eta;\xi_0) &\coloneqq \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{q=0}^{i-j} d_i b_{i,j}\frac{(j-i+\beta)_q}{q!}Z_{i,j,q}(\xi,\eta;\xi_0),\\ \Psi_{1,2}(\xi,\eta;\xi_0) &\coloneqq \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{q=i-j+1}^{\infty} d_i b_{i,j}\frac{(j-i+\beta)_q}{q!}Z_{i,j,q}(\xi,\eta;\xi_0),\end{aligned} \quad (A.45)$$

where

$$Z_{i,j,q}(\xi,\eta;\xi_0) \coloneqq \left(\frac{1-\xi}{\xi_0-\xi}\right)^{i-j}\left(\frac{1-\xi_0}{1-\xi}\right)^q\left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1). \quad (A.46)$$

(i) Expansion of the Function $\Psi_{1,1}(\xi,\eta;\xi_0)$ in Negative Powers of $1-\xi_0$. According to (45) and (46) we have

$$Z_{i,j,q}(\xi,\eta;\xi_0) = \frac{(-1)^i(1-\xi)^{i-j-q}(1-\eta)^{i-j}(\eta-\xi)^j}{(2-\xi-\eta)^i(1-\xi_0)^{i-j-q}} \quad (A.47)$$

and, in order to extract the negative powers of $1-\xi_0$, we introduce the new index $p = i-j-q$ instead of i. We obtain

$$\begin{aligned}\Psi_{1,1}(\xi,\eta;\xi_0) = &\sum_{p=0}^{n}\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^p\\ &\times\sum_{j=0}^{n-p}\sum_{q=0}^{n-p-j} d_{p+j+q}b_{p+j+q,j}\\ &\cdot\frac{(\beta-p-q)_q}{(-1)^{p+j+q}q!}\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^j\left(\frac{1-\eta}{2-\xi-\eta}\right)^q.\end{aligned} \quad (A.48)$$

Using (A.3) and (A.5) we simplify

$$\frac{(\beta-p-j-q)_j(\beta-p-q)_q}{(1-\beta)_{p+j+q}} = \frac{(-1)^{j+q}}{(1-\beta)_p} \quad (A.49)$$

and we derive

$$\begin{aligned}\Psi_{1,1}(\xi,\eta;\xi_0) = &\sum_{p=0}^{n}(-1)^p d_p\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^p\\ &\times\sum_{j=0}^{n-p}\frac{(p-n)_j(n+p+1)_j}{(p+1)_j}\frac{(\beta)_j}{(2\beta)_j j!}\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^j\\ &\times\sum_{q=0}^{n-p-j}\frac{(p-n+j)_q(n+p+j+1)_q}{(p+j+1)_q q!}\left(\frac{1-\eta}{2-\xi-\eta}\right)^q,\end{aligned} \quad (A.50)$$

which actually gives

$$\Psi_{1,1}(\xi,\eta;\xi_0) = \sum_{p=0}^{n}(-1)^p d_p\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^p Q_n^p\left(\frac{\eta-\xi}{2-\xi-\eta}\right), \quad (A.51)$$

where $Q_n^p(z)$ is the function (A.21) from Lemma A.2.

Now, according to (A.23) we have nonzero terms in the sum only for indexes p of the same parity as n. For this reason let us introduce the new index $k = (n-p)/2$ and by Lemma A.2 we obtain

$$\begin{aligned}\Psi_{1,1}(\xi,\eta;\xi_0) = &\sum_{k=0}^{[n/2]}(-1)^n d_{n-2k}\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^{n-2k}\\ &\times c_{n,n-2k}\,{}_2F_1\left(n-k+\frac{1}{2},-k,\frac{1}{2}+\beta;\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^2\right).\end{aligned} \quad (A.52)$$

Next, we transform the hypergeometric function in (A.52) by formula (A.9):

$$\begin{aligned}&{}_2F_1\left(n-k+\frac{1}{2},-k,\frac{1}{2}+\beta;\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^2\right)\\ &= \frac{(-1)^k(1/2-n)_k}{(1/2+\beta)_k}\left(\frac{4(1-\xi)(1-\eta)}{(2-\xi-\eta)^2}\right)^k\\ &\cdot{}_2F_1\left(k-n+\beta,-k,\frac{1}{2}-n;\frac{(2-\xi-\eta)^2}{4(1-\xi)(1-\eta)}\right)\\ &= \frac{(-1)^k(1/2-n)_k}{(1/2+\beta)_k}\\ &\cdot\sum_{i=0}^{k}A_i^k\left(\frac{4(1-\xi)(1-\eta)}{(2-\xi-\eta)^2}\right)^{k-i}.\end{aligned} \quad (A.53)$$

Applying this into (A.52) and defining the following function:

$$\Phi_1^-(\xi,\eta;\xi_0) \coloneqq \frac{\gamma(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\Psi_{1,1}(\xi,\eta;\xi_0), \quad (A.54)$$

we obtain

$$\Phi_1^-(\xi,\eta;\xi_0) = (\eta-\xi)^{2\beta}\sum_{k=0}^{[n/2]}c_k^n(1-\xi_0)^{-n+2k}\widetilde{E}_k^n(\xi,\eta), \quad (A.55)$$

with

$$c_k^n := \frac{2^{2k-n}(1/2-n)_k}{(-1)^{n+k}(1/2+\beta)_k}\gamma c_{n,n-2k}d_{n-2k} \neq 0. \tag{A.56}$$

This establishes the equality (A.35).

(ii) Estimation of the Function $\Psi_{1,2}(\xi,\eta;\xi_0)$ from (A.45). For the function $\Psi_{1,2}(\xi,\eta;\xi_0)$ we have

$$\begin{aligned}\Psi_{1,2}(\xi,\eta;\xi_0) &:= \sum_{i=0}^{n}\sum_{j=0}^{i} d_i b_{i,j} P_{i,j}(\xi,\xi_0)\left(\frac{1-\xi}{\xi_0-\xi}\right)^{i-j} \\ &\cdot \left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1),\end{aligned} \tag{A.57}$$

where

$$P_{i,j}(\xi,\xi_0) := \sum_{q=i-j+1}^{\infty}\frac{(j-i+\beta)_q}{q!}\left(\frac{1-\xi_0}{1-\xi}\right)^q. \tag{A.58}$$

In $P_{i,j}(\xi,\xi_0)$ we set the new index $N = q+j-i-1$ instead of q to obtain

$$\begin{aligned}P_{i,j}(\xi,\xi_0) &= (-1)^{i-j+1}(-\beta)_{i-j+1} \\ &\cdot \sum_{N=0}^{\infty}\frac{(1+\beta)_N}{(1+N)_{i-j+1}N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N+i-j+1} \\ &= (-1)^{i-j}(1-\beta)_{i-j} \\ &\cdot \sum_{N=0}^{\infty}\frac{(\beta+N)(\beta)_N}{(1+N)_{i-j+1}N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N+i-j+1}.\end{aligned} \tag{A.59}$$

Since

$$\frac{\beta+N}{(1+N)_{i-j+1}} < 1, \quad 0<\beta<1, \ j=0,1,\ldots,i, \tag{A.60}$$

for the function $P_{i,j}(\xi,\xi_0)$, it follows the estimate

$$\begin{aligned}&\left|P_{i,j}(\xi,\xi_0)\right| \\ &\leq (1-\beta)_{i-j}\left(\frac{1-\xi_0}{1-\xi}\right)^{i-j+1}\sum_{N=0}^{\infty}\frac{(\beta)_N}{N!}\left(\frac{1-\xi_0}{1-\xi}\right)^N \\ &= (1-\beta)_{i-j}\frac{(1-\xi_0)^{i-j+1}}{(1-\xi)^{i-j+1-\beta}(\xi_0-\xi)^{\beta}},\end{aligned} \tag{A.61}$$

in $D\cap\{\eta<\xi_0\}$. For the last equality, we used (A.10).

Applying this estimate in (A.57) gives that, in $D\cap\{\eta<\xi_0\}$, the following inequality holds:

$$\left|\Psi_{1,2}(\xi,\eta;\xi_0)\right| \leq k_{1,1}\frac{(1-\xi_0)}{(\xi_0-\xi)^{\beta}(1-\xi)^{1-\beta}}, \tag{A.62}$$

where $k_{1,1}$ is a positive constant.

(iii) Estimation of the Function $\Psi_2(\xi,\eta;\xi_0)$ from (A.42). For this function we have

$$\begin{aligned}&\Psi_2(\xi,\eta;\xi_0) \\ &= \left(\frac{1-\xi}{\xi_0-\xi}\right)^{\beta}\sum_{i=0}^{n} d_i Y^i(\xi,\eta;\xi_0,1)\,Q_i(\xi,\eta;\xi_0),\end{aligned} \tag{A.63}$$

where

$$Q_i(\xi,\eta;\xi_0) := \sum_{j=i+1}^{\infty} b_{i,j}X^{-j}(\xi,\eta;\xi_0,1). \tag{A.64}$$

Now, we set $j = N+i+1$ and using (A.3) compute

$$\begin{aligned}Q_i(\xi,\eta;\xi_0) &= (-1)^i(1-\beta)_i \\ &\times \sum_{N=0}^{\infty}\frac{(\beta+N)(\beta)_N(\beta)_N(\beta+N)_{i+1}}{(2\beta)_N(2\beta+N)_{i+1}(1+N)_{i+1}N!} \\ &\times X^{-N-i-1}(\xi,\eta;\xi_0,1).\end{aligned} \tag{A.65}$$

Since

$$\frac{(\beta+N)(\beta+N)_{i+1}}{(2\beta+N)_{i+1}(1+N)_{i+1}} < 1, \quad 0<\beta<1, \tag{A.66}$$

from here it follows the estimate

$$\begin{aligned}\left|Q_i\right| &\leq (1-\beta)_i X^{-i-1}(\xi,\eta;\xi_0,1) \\ &\cdot {}_2F_1\left(\beta,\beta,2\beta;\frac{1}{X(\xi,\eta;\xi_0,1)}\right)\end{aligned} \tag{A.67}$$

in $D\cap\{\eta<\xi_0\}$. By (A.12) with $\alpha=\beta$ we estimate

$$\begin{aligned}&\left|{}_2F_1\left(\beta,\beta,2\beta;\frac{1}{X(\xi,\eta;\xi_0,1)}\right)\right| \\ &\leq c(\beta)\frac{(\xi_0-\xi)^{\beta}(1-\eta)^{\beta}}{(1-\xi)^{\beta}(\xi_0-\eta)^{\beta}}.\end{aligned} \tag{A.68}$$

Applying (A.67)-(A.68) in (A.63) and taking into account the fact that

$$\begin{aligned}&\left|\frac{Y}{X}\right| \leq 1, \\ &\frac{1}{X} \leq \frac{1-\xi_0}{1-\eta}, \quad \eta<\xi_0,\end{aligned} \tag{A.69}$$

we derive that in $D\cap\{\eta<\xi_0\}$ the following inequality holds

$$\left|\Psi_2(\xi,\eta;\xi_0)\right| \leq k_{1,2}\frac{1-\xi_0}{1-\eta}\frac{(1-\eta)^{\beta}}{(\xi_0-\eta)^{\beta}}, \tag{A.70}$$

where $k_{1,2}$ is a positive constant.

Now, to complete the proof, we define

$$\Phi_2^-(\xi,\eta;\xi_0) := \frac{\gamma(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\{\Psi_{1,2}(\xi,\eta;\xi_0)+\Psi_2(\xi,\eta;\xi_0)\}, \quad (A.71)$$

$$k_1 := 2\max\{k_{1,1},k_{1,2}\}.$$

Then from (A.62) and (A.70) we come to the inequality (A.36). This, together with (A.55), gives the statement of the theorem, because

$$\Phi^-(\xi,\eta;\xi_0,1) = \frac{\gamma(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\{\Psi_{1,1}(\xi,\eta;\xi_0) + \Psi_{1,2}(\xi,\eta;\xi_0)+\Psi_2(\xi,\eta;\xi_0)\} = \Phi_1^-(\xi,\eta;\xi_0) + \Phi_2^-(\xi,\eta;\xi_0). \quad (A.72)$$

The proof is complete. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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