Chiral gauge theory and nontrivial spacetime topology: Lorentz and CPT violation

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Abstract

In this dissertation, the effective gauge field action of the chiral gauge theory is calculated, which is defined over a four-dimensional spacetime manifold ($M = \mathbb{R}^3 \times S^1$) with one compactified coordinate. The gauge fields and fermionic fields have periodic boundary conditions over the compact dimension. The Abelian gauge-field configurations $A_\mu(x)$ can depend upon the compactified coordinate $x^4 \in S^1$ and have a vanishing component A_4. The existence of CPT (and Lorentz) violation is derived perturbatively with a generalized Pauli-Villars regularization. For more justification, the CPT anomaly is established nonperturbatively with a lattice regularization scheme based on Ginsparg–Wilson fermions.

Zusammenfassung

In vorliegender Arbeit berechnen wir die effektive Eichfeldwirkung einer chiralen Eichfeldtheorie, welche in einer vierdimensionalen Raumzeitmanigfaltigkeit ($M = \mathbb{R}^3 \times S^1$), mit einer kompaktifizierten Koordinate definiert ist. Die Eich- und Fermionfelder haben in dieser kompaktifizierten Koordinate periodische Randbedingungen. Die Abelschen Eichfeldkonfigurationen $A_\mu(x)$ können im Allgemeinen von dieser Koordinate x^4 abhaengen, haben jedoch eine verschwindende Komponente A_4. Wir zeigen die Verletzung von CPT- (und Lorentz-) Invarianz zunächst mit perturbativen Methoden, unter Verwendung einer generalisierten Pauli-Villars-Regularisierung. Die CPT-Anomalie wird darüber hinaus mit nicht-perturbativer Gittereichtheorie, basierend auf Ginsparg-Wilson-Fermionen, bestätigt.

Publication

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Contents

1 Introduction

In classical field theory, the symmetry transformations of the fields leave the action invariant. But if a quantum field theory does not preserve all the symmetries of the original classical field theory then it is said that this quantum field theory has an anomaly. A well-known example is the triangle anomaly of axial vectors [1, 2]. The most important symmetries of modern particle physics are Lorentz, CPT and gauge invariance. The CPT theorem [3–5] states that every local relativistic quantum field theory is symmetric under the combined operation of C (charge conjugation), P (parity) and T (time reversal). Thus any CPT violation gives a hint of fundamentally new physics for example quantum gravity [6–8] or strings [9].

1.1 Different origins of CPT and Lorentz violation

There are different theoretical models assessing the possibility of CPT and Lorentz violation. One possibility of the origin of CPT and Lorentz violation is introducing a gauge invariant Chern–Simons-like term into the Abelian gauge field Lagrangian, which violates CPT and Lorentz invariance [10]. An example of such term is

$$\mathcal{L}_{\text{CS-Like}} = \frac{1}{4} m \epsilon^{\mu\nu\rho\sigma} k_\mu A_\nu F_{\rho\sigma} \, , \tag{1.1.1}$$

with a real parameter m having dimension of mass and a real symmetry breaking "vector" k_μ of unit length, which may be spacelike or timelike but is fixed once and for all (hence, the quotation marks around the word vector). Here $\epsilon^{\mu\nu\rho\sigma}$ is completely anti-symmetric in four dimension and $F_{\rho\sigma}$ is the field strength tensor with gauge field A_μ. The above theoretical model is known as Maxwell–Chern–Simons (MCS) theory.

Let us consider a classical gauge field interacting with fermions. A CPT and Lorentz violating term can be introduced into the fermionic part of the Lagrangian as follows [11]:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m - \gamma_5 b\!\!\!/)\psi \, , \tag{1.1.2}$$

with the fermionic field ψ. Here the Lorentz and CPT breaking term is proportional to the constant "four-vector" b_μ.

In the above theories CPT and Lorentz violating terms are put in the Lagrangian by hand. Under certain conditions, the CPT and Lorentz violations occur in certain chiral gauge theories defined over a manifold having nontrivial topology. In ref. [12] a possible trade-off between the Z-string global gauge anomaly in topologically nontrivial (3+1)-dimensional spacetime manifold and the CPT and Lorentz violation is described.

In ref. [13] it is shown that CPT and Lorentz violation is present in a class of two-dimensional chiral $U(1)$ gauge theories on the torus. The chiral determinant for periodic fermion fields changes sign under a CPT transformation of the background gauge field.

It is shown in ref. [14] and [15] that the CPT violation occurs in the effective action of the gauge field due to the quantum effects of the chiral fermions in a spacetime manifold with atleast one compactified coordinate, where the gauge fields are assumed to be independent of the compactified coordinate.

1.2 Outline

In this dissertation, we calculate the origin of an anomalous local Chern-Simons-like term in the effective gauge-field action which violates the Lorentz and CPT invariance. We establish our result with two regularization methods, perturbatively, with a generalized Pauli–Villars regularization and nonperturbatively, with lattice regularization based on Ginsparg-Wilson fermions. Finding the same result with two different regularization method suggests that the four dimensional CPT anomaly is not simply an artefact of a particular regularization scheme.

The outline of this dissertation is as follows:

In Chapter 2, an overview of Euclidean fermions is given, paying special attention to the transformation of the fermions. This chapter also contains a summery of local gauge transformations, chiral symmetry and the CPT theorem.

In Chapter 3, the theoretical setup of the problem is described and our notation is established. The calculation is done both perturbatively and nonperturbatively, with appropriate regularization methods.

In Chapter 4, we calculate the anomalous origin of Lorentz and CPT violation with a perturbative approach. In Sec. 4.1, first, we write down the effective gauge-field action for a left-handed chiral fermion. Then, we expanded this effective action perturbatively with an extended version of the generalized Pauli–Villars regularization. In Sec. 4.2, we perform the one-loop calculation of the effective gauge-field action for an Abelian $U(1)$ gauge group to quadratic order. After the regularisation and renormalization a local Chern–Simons-like term is obtained. In Sec. 4.3, we explicitly show that this local Chern–Simons-like term is not invariant under Lorentz and CPT transformations in four spacetime dimensions.

In Chapter 5, we establish the existence of Lorentz and CPT non-invariance with a nonperturbative lattice regularization based on Ginsparg–Wilson fermions. In Sec. 5.1, the basic setup of chiral lattice gauge theory is summarized. In Sec. 5.2, we give a brief review of chiral $U(1)$ gauge theory on the lattice. Then, the fermion action on a regular lattice is written down and the integration measure is described. After that, the action of the discrete transformations on the link variable is described. In Sec. 5.3, we discuss the effective gauge-field action on the lattice and its behavior under a CPT transformation. In Sec. 5.4, we show that the effective action is not invariant under CPT transformation. We consider the possible cases, namely, odd and even $N \equiv L/a$. Here L is the length of the x^4 circle and a is the lattice spacing. In Sec. 5.5, the expression for the CPT-anomaly in the continuum limit ($a \to 0$) is calculated.

In Chapter 6, some important points of our calculations and result are discussed. In Chapter. 7, some concluding remarks are made.

In Appendix A we define the important notations used in our calculation. In Appendix B and C we discuss some detailed description of the diagonalization operators used in nonperturbative lattice calculation.

2 Symmetries in Euclidean space

A quantum field theory always defined over Minkowski spacetime, whereas, most of the calculations in QFT are performed in Euclidean spacetime. In this chapter we shall discuss different symmetries in the Euclidean space, for e.g. the spacetime symmetry, gauge symmetry, chiral symmetry and CPT symmetry. Finally, we shall discuss about the CPT theorem. Most of the conventions and mathematical notations in this chapter is taken from the following Ref. [16].

In mathematics, the Euclidean group $E(n)$, also known as $ISO(n)$, is the symmetry group of n-dimensional Euclidean space. Its elements are the isometries associated with the Euclidean distance, and are called Euclidean isometries, Euclidean transformations or Rigid transformations. On the other hand the Poincaré group is the group of Minkowski spacetime isometries, which is a ten-dimensional noncompact Lie group.

2.1 Spacetime symmetries

The symmetry group for spacetime transformation, in Euclidean space, is called Eucliden group in four spacetime dimensions. The action of this transformation-group on a real vector space leaves the squared distance $ds^2 = \sum_{\mu=1}^{4} dx_\mu^2$ invariant. Let us discuss different symmetries of this group. First one is proper rotations defined as:

$$x_\mu \to x'_\mu = R_{\mu\nu} x_\nu \tag{2.1.1}$$

for $\mu, \nu = 1, 2, 3, 4$, and the rotation matrix $R \in SO(4)$. The rotation matrix R, for an infinitesimal rotation, can be written as:

$$R_{\mu\nu} = \delta_{\mu\nu} + \sum_{\alpha\beta=1}^{4} \omega_{\alpha\beta} (M_{\alpha\beta})_{\mu\nu} + O(\omega^2), \tag{2.1.2}$$

where the parameters

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} \tag{2.1.3}$$

are antisymmetric under α, β and the generators of the rotational group $SO(4)$

$$(M_{\alpha\beta})_{\mu\nu} = (\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \tag{2.1.4}$$

with α, β are the labelling parameters and μ, ν are the indices of the rotation matrix.

The Euclidean transformation group also consists reflections, where the reflection of the x_i-axis is given by

$$x_\mu \to x'_\mu = \begin{cases} x_\mu\,, & \text{for } \mu \neq i, \\ -x_\mu\,, & \text{for } \mu = i\,. \end{cases} \tag{2.1.5}$$

There are also translations

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \tag{2.1.6}$$

with real constant a_μ, which is also a symmetry transformation.

The full Euclidean group $E(4)$ is semi-direct product of the orthogonal group $O(4)$ and translation

$$E(4) = O(4) \ltimes \mathbb{R}^4. \tag{2.1.7}$$

On the other hand, in Minkowski spacetime, the Poincaré group is a semi-direct product of the Lorentz group and translation ,

$$P(1,3) = O(1,3) \ltimes \mathbb{R}^{1,3}. \tag{2.1.8}$$

The transformation behaviour of fermions under translations is same as in Minkowski spacetime. The following subsctions we discuss the transformation behaviour of fermions under rotations and reflections, which differs from that in Minkowski spacetime.

2.1.1 Rotations

Free fermions with mass m in Euclidean spacetime are described by the action

$$S[\bar{\psi},\psi] = \int d^4x \mathcal{L}(\bar{\psi}(x),\psi(x)) = \int d^4x \bar{\psi}(x)(\gamma_\mu\partial_\mu - m)\psi(x). \tag{2.1.9}$$

The anticommuting fermionic fields are Grassman valued. The Dirac-matrices $\gamma_\mu, \mu = 1,2,3,4,$ fulfill the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{2.1.10}$$

Under a rotation of the coordinate axes (2.1.1)the fermions transform according to

$$\psi(x) \to \psi'(x) = S(R)\psi(R^{-1}x) \tag{2.1.11}$$

where the transformation matrix is given by

$$S(R) \equiv \exp\left(\frac{i}{4}\omega_{\mu\nu}(\sigma_{\mu\nu})_{\alpha\beta}\right), \tag{2.1.12}$$

with

$$\sigma_{\mu\nu} \equiv \frac{1}{2i}[\gamma_\mu, \gamma_\nu]. \tag{2.1.13}$$

The matrix $\omega_{\mu\nu}$ is given by (2.1.2). The transformation matrix S transforms the Dirac-matrices γ_μ in the following way:

$$S^{-1}(R)\gamma_\mu S(R) = R_{\mu\nu}\gamma_\mu. \tag{2.1.14}$$

The matrices (2.1.13) are Hermitian and traceless. In contrast to Miinkowski spacetime, the transformation matrices in Euclidean spacetime are always unitary

$$S^\dagger(R) = s^{-1}(R). \tag{2.1.15}$$

The antifermions are defined to transform according to

$$\bar{\psi(x)} \to \bar{\psi}'(x) = \bar{\psi}(R^{-1}x)S^\dagger(R). \tag{2.1.16}$$

2.1.2 Parity

The parity transformation is the simultaneous reflection with respect to the three coordinate axes. At this moment moment, let us follow the usual convention that the $1, 2, 3$-axes are reflected. According to (2.1.5), the transformation matrix $R_{\mathcal{P}} \in O(4)$ has the form

$$R_{\mathcal{P}} = \mathrm{diag}\,(-1, -1, -1, +1)\,. \tag{2.1.17}$$

From equation (2.1.14) the transformation matrix of the spinors $P = S(R_{\mathcal{P}})$ must have the following properties

$$\begin{aligned} P^{-1}\gamma_k P &= -\gamma_k\,, \quad \text{for } k = 1, 2, 3, \\ P^{-1}\gamma_4 P &= \gamma_4\,. \end{aligned} \tag{2.1.18}$$

The parity matrix $P = \eta_{\mathcal{P}}\gamma_4$, $\eta_{\mathcal{P}} \in \mathbb{C}$ respects the above properties. Moreover P has to be unitary so that $\eta_{\mathcal{P}}$ is a phase factor with $|\eta_{\mathcal{P}}| = 1$.

In a nutshell, the parity transformation of fermions and antifermions transform can be written as follows:

$$\psi(x) \to \psi^{\mathcal{P}}(x) = P\psi(R^{-1}{}_{\mathcal{P}}x), \tag{2.1.19}$$

$$\bar{\psi}(x) \to \bar{\psi}^{\mathcal{P}}(x) = \bar{\psi}(R^{-1}{}_{\mathcal{P}}x)P^{-1}. \tag{2.1.20}$$

Under the above transformations the action (2.1.9) remains invariant.

2.1.3 Time reversal

All the directions are equal in Euclidean spacetime, so that there is no difference between the reflection of the time-coordinate and reflection of one of the other coordinates.

However, we should remember that a relativistic quantum field theory always described over the Minkowski spacetime. Therefore, we have to single out one Euclidean coordinate which corresponds to time-coordinate after performing an inverse Wick-rotation. we shall treat this particular coordinate axis differently under reflection.

In this chapter, we assume that the 4-coordinate (x^4) becomes the time axis after an inverse Wick-rotation. But any other axis is equally good for this. Then only changes we should make are the parity transformation and the time reversal. The CPT transformation is the same for all choices of the time axis.

We define a time reversal transformation for fermionic fields by the reflection of the 4-coordinate, followed by Hermitian conjugation. The transformation matrix for the coordinates $R_{\mathcal{T}} \in O(4)$ is defined by

$$R_{\mathcal{T}} = \mathrm{diag}\,(+1, +1, +1, -1)\,. \tag{2.1.21}$$

The fermionic fields transform as

$$\psi(x) \to \psi^{\mathcal{T}}(x) = T\bar{\psi}^t(R^{-1}{}_{\mathcal{T}}x), \tag{2.1.22}$$

$$\bar{\psi}(x) \to \bar{\psi}^{\mathcal{T}}(x) = -\psi^t(R^{-1}{}_{\mathcal{T}}x)T^{-1}. \tag{2.1.23}$$

with the time reversal matrix T. The superscript t denotes the transpose. The negative sign arises in the above equation because the fermions are Grassman valued.

The time-reflected spinors have to transform under rotations as spinors. The rotation, that does not involve the x_4-coordinate, commutes with the time reversal operator. For example, a rotation in the $x_i - x_4$-plane by an angle ϕ followed by a time reflection is equal to a time reflection followed by the rotation in the $x_i - x_4$-plane by an angle $-\phi$. Altogether, the conditions that T has to satisfy is written as:

$$\begin{aligned} &T^{-1}\gamma_k T = -\gamma_k^t\,, \qquad \text{for} \;\; k = 1, 2, 3, \\ &T^{-1}\gamma_4 T = \gamma_4^t\,, \qquad . \end{aligned} \tag{2.1.24}$$

The definition of the time reversal matrix T depends on the representation of the Dirac matrices. For the representation representation of the Dirac matrices given in Appendix A, the time reversal matrix $T = \eta_\tau \gamma_2$, with phase factor η_τ.

Under a time reflection the action (2.1.9) remains invariant.

2.2 Local gauge transformations

One of the most important concepts in the theoretical physics is gauge theory. For example, we can successfully combine three of the four fundamental forces in nature (electromagnetic, strong and weak) into a gauge theory, called standard model. The electroweak theory is described by a $SU(2) \times U(1)$ gauge theory, and the strong interactions are described by a $SU(3)$ gauge theory. We introduce the basic notions for a local gauge transformations in the following.

In a unitary representation, let G be a compact gauge group . The fermion fields $\psi_\alpha(x)$ form a representation space of the group G, with α is the gauge group index (spinor indices are supressed).

A gauge transformation is performed by multiplying a spacetime-dependent unitary matrix $\Lambda(x)$ with the fermionic fields:

$$\psi_\alpha(x) \to \psi^\lambda_\alpha(x) = \sum_\beta (\Lambda(x))_{\alpha\beta}\,\psi_\beta(x), \tag{2.2.1}$$

$$\bar{\psi}_\alpha(x) \to \bar{\psi}^\lambda_\alpha(x) = \bar{\psi}_\beta(x) \sum_\beta \left(\Lambda(x)^\dagger\right)_{\alpha\beta}. \tag{2.2.2}$$

In order to keep the action (2.1.9) to be invariant under the local gauge transformations, the partial derivative ∂_μ has to be replaced by the covariant derivative

$$D_\mu \equiv \partial_\mu + gA^c_\mu(x)T_c, \tag{2.2.3}$$

with real gauge fields $A^c_\mu(x)$ and generators of the gauge group T_c. These generators are are chosen to be traceless anti-Hermitian matrices, normalized to

$$\mathrm{tr}(T_aT_b) = -\frac{1}{2}\delta_{ab}. \tag{2.2.4}$$

The commutation relation of the generators read

$$[T_a, T_b] = f_{abc}T_c. \tag{2.2.5}$$

In the above equation, the above structure constants $f_{abc} \in \mathbb{R}$, which are totally antisymmetric in the indices a, b and c. The constant g is a dimensionless coupling constant.

The gauge field $a_\mu(x) \equiv gA^c_\mu(x)T_c$ transforms under a gauge transformation (2.2.1,2.2.2) as follows:

$$A_\mu(x) \to A^\Lambda_\mu(x) = \Lambda(x)A_\mu(x)\Lambda^{-1}(x) + \Lambda(x)\partial_\mu(x)\Lambda^{-1}(x). \tag{2.2.6}$$

After replacing the partial derivatives by covariant derivatives, the action (2.1.9) becomes

$$S[\bar{\psi}, \psi] = \int d^4x\mathcal{L}(\bar{\psi}(x), \psi(x)) = \int d^4x\bar{\psi}(x)(\gamma_\mu D_\mu - m)\psi(x). \tag{2.2.7}$$

This action remains invariant under gauge transformations (2.2.1,2.2.2,2.2.6).

The partial derivative ∂_μ transforms under rotations like a four-vector, as do the gauge fields. The Euclidean transformations are for rotations

$$A_\mu(x) \to A'_\mu(x)R_{\mu\nu}A_\nu(R^{-1}x). \tag{2.2.8}$$

for the parity transformation

$$A_\mu(x) \to A^{\mathcal{P}}_\mu(x) = \begin{cases} -A_\mu(R^{-1}_{\mathcal{P}}x)\,, & \text{for } \mu = 1, 2, 3, \\ A_\mu(R^{-1}_{\mathcal{P}}x) & \mu = 4, \end{cases} \tag{2.2.9}$$

for the time reversal transformation

$$A_\mu(x) \to A_\mu^{\mathcal{T}}(x) = \begin{cases} -A_\mu^t(R_{\mathcal{T}}^{-1}x), & \text{for } \mu = 1,2,3, \\ A_\mu^t(R_{\mathcal{T}}^{-1}x) & \mu = 4. \end{cases} \tag{2.2.10}$$

The gauge fields themselves have a kinetic term, defined by

$$S[A] = \int d^4x \frac{1}{2} \mathrm{tr}(F_{\mu\nu}(x)F_{\mu\nu}(x)), \tag{2.2.11}$$

with the tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\mu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\mu(x)]. \tag{2.2.12}$$

2.3 Charge conjugation

For a unitary representation $R[U(x)]$ of the gauge group G, the complex conjugated representation $R[U(x)]^*$ is again unitary. The gauge fields transform under a charge conjugation as

$$A_\mu(x) \to A_\mu(x)^* = -A_\mu^t(x) \equiv A^C\mu(x). \tag{2.3.1}$$

For the Abelian U(1) gauge group, this is just a change of the sign in front of the gauge field A_μ, which is equivalent to the interchange of the positive and negative charges.

The fermionic fields transform under a charge conjugation as

$$\psi(x) \to \psi^C(x) = C\bar{\psi}^t(x), \tag{2.3.2}$$

$$\bar{\psi}(x) \to \bar{\psi}^C(x) = -\psi^t(x)C^{-1}, \tag{2.3.3}$$

with charge conjugation matrix C. This matrix, commutes with rotations, respects

$$C^{-1}\gamma_\mu C = \gamma_\mu^t. \tag{2.3.4}$$

The action (2.2.7) remains invariant under charge conjugation:

$$S[\bar{\psi}^C, \psi^C, A^C] = S[\bar{\psi}, \psi, A]. \tag{2.3.5}$$

2.4 Chiral symmetry

In four dimensions the Dirac γ matrices commute with the matrix

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4. \tag{2.4.1}$$

The fermionic field can be decomposed into two orthogonal components

$$\psi(x) = \psi_+(x) + \psi_-(x), \quad \psi_\pm(x) = P_\pm\psi(x) \tag{2.4.2}$$

with the projection operators

$$P_\pm = \frac{1}{2}(\mathbb{I} \pm \gamma_5), \; P_\pm^2 = P_\pm, \; P_+P_- = 0. \tag{2.4.3}$$

In the representation of Dirac matrices in Appendix A, the projected states $\psi_\pm$ are given by

$$\psi_+(x) = \begin{pmatrix} \phi(x) \\ 0 \end{pmatrix}, \; \psi_-(x) = \begin{pmatrix} 0 \\ \xi(x) \end{pmatrix}, \tag{2.4.4}$$

with the two-component Weyl spinors $\phi(x)$ and $\xi(x)$. These two Weyl spinors do not mix under proper rotations.

The Lagrange density of (2.2.7) in the above representation reads

$$\bar{\psi}(x)(\gamma_\mu\partial_\mu - m)\psi(x) = \left(\bar{\phi}(x), \bar{\xi}(x)\right)\begin{pmatrix} -m & (\vec{\sigma}\cdot\vec{\partial} + i\partial_4) \\ (\vec{\sigma}\cdot\vec{\partial} - i\partial_4) & -m \end{pmatrix}\begin{pmatrix} \phi(x) \\ \xi(x) \end{pmatrix} \tag{2.4.5}$$

where

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \quad \vec{\partial} = (\partial_1, \partial_2, \partial_3). \tag{2.4.6}$$

For the massless case, the Weyl spinors ξ and ϕ decouple. This leads to an additional symmetry for massless fermionic fields, the chiral symmetry. The action is invariant under the transformations

$$\psi(x) \to \exp(i\alpha\gamma_5)\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)\exp(i\alpha\gamma_5) \tag{2.4.7}$$

whose infinitesimal form is given by

$$\psi(x) \to \psi(x) + i\epsilon\gamma_5\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x) + i\epsilon\gamma_5. \tag{2.4.8}$$

The reason for decoupling of ξ and ϕ is the anticommutation relation

$$\{\gamma_5, \not{D}\} = 0 \tag{2.4.9}$$

where the massless free Dirac operator

$$\not{D} \equiv \gamma_\mu D\mu. \tag{2.4.10}$$

Thus, one can impose the following constraints on the fermionic fields:

$$\psi_L(x) = P_-\psi_L(x), \quad \bar{\psi}_L(x) = \bar{\psi}_L(x)P_+ \tag{2.4.11}$$

$$\psi_R(x) = P_+\psi_R(x), \quad \bar{\psi}_R(x) = \bar{\psi}_R(x)P_- \tag{2.4.12}$$

Fermions that satisfy these constraints are called chiral fermions, in particular ψ_L creates a left-handed fermion obeying (2.4.11), and ψ_R creates a right-handed fermion obeying (2.4.12).

Inserting the above constraints (2.4.11) in (2.2.7), the action reads

$$
\begin{aligned}
S[\bar{\psi}_L, \psi_L, A] &= \int d^4x \bar{\psi}_L(x)(\gamma_\mu D_\mu)\psi_L(x) \\
&= \bar{\phi}(x)\Big(\sigma_\mu(\partial_\mu + A_\mu)\Big)\xi(x),
\end{aligned} \tag{2.4.13}
$$

with the Pauli matrices

$$
\sigma_\mu = (\sigma_m, i\mathbb{I}_2). \tag{2.4.14}
$$

Because the transformation matrix P anticommutes with γ_5, the action (2.4.13) is not invariant under parity transform

$$
S[\bar{\psi}_L^{\mathcal{P}}, \psi_L^{\mathcal{P}}] \neq S[\bar{\psi}_L, \psi_L], \tag{2.4.15}
$$

where $\bar{\psi}_L^{\mathcal{P}}$ and $\psi_L^{\mathcal{P}}$ are the parity transformed fermionic fields. Also charge conjugation is not a symmetry transformation of (2.4.13). The reason is that $\bar{\psi}_L$ and ψ_L are exchanged. However the combination of parity transformation and charge conjugation

$$
\psi_L(x) \to \psi_L^{C\mathcal{P}}(x) = CP\bar{\psi}_L^t(R_{\mathcal{P}}^{-1}x), \tag{2.4.16}
$$

$$
\bar{\psi}_L(x) \to \bar{\psi}^{C\mathcal{P}}(x) = -\psi_L^t(x)(R_{\mathcal{P}}^{-1}x)P^{-1}C^{-1}, \tag{2.4.17}
$$

is a symmetry transformation for chiral fermions.

Furthermore, time reversal as defined in 2.1.3 is a valid symmetry transformation of the action (2.4.13).

2.5 CPT theorem

One of the most important properties of a local quantum field theory in Minkowski spacetime is CPT invariance. The CPT theorem states that the combined transformation of charge conjugation, parity and time reversal is a symmetry transformation, even though C, P or T separately need not be symmetry transformations.

Unlike the other symmetries, which are put into the theory by hand (e.g., by choosing an action which is invariant under rotations and translations), the CPT symmetry follows from other requirements [[3], [4]]. In a nutshell, the inputs of the CPT theorem in Minkowski spacetime are:

- Lorentz invariance ;
- the standard spin-statistics relation ;
- locality ;
- hermiticity of the Hamiltonian.

It is little surprising that the CPT turns out to be a symmetry, because previously no discrete symmetries are postulated.

In Euclidean spacetime the transformations of the fermionic fields under a CPT transformation is given by

$$\psi(x) \to \psi^{\theta}(x) = CPT\psi(-x) \tag{2.5.1}$$

$$\bar{\psi}(x) \to \bar{\psi}^{\theta}(x) = \bar{\psi}(-x)T^{-1}P^{-1}c^{-1} \tag{2.5.2}$$

From equations (2.1.18), (2.1.24) and (2.3.4) it follows that

$$CPT = \eta_{\theta}\gamma_5, \tag{2.5.3}$$

with a phase factor η_{θ}. The gauge fields transform under CPT, according to (2.2.9), (2.2.10) and, (2.3.1)as follows

$$A_{\mu}(x) \to A^{\theta}_{\mu}(x) = -A_{\mu}(-x). \tag{2.5.4}$$

The actions (2.2.7) and (2.2.12) are invariant under the CPT transformation.

2.5.1 Consequences of CPT theorem

An important consequence of the CPT theorem is that for each particle there exists an antiparticle of equal mass. This is true even if charge conjugation C is not a symmetry of this theory.

In order to preserve CPT symmetry, every violation of the combined symmetry of two of its components (such as CP) must have a corresponding violation in the third component (such as T); in fact, mathematically, these are the same thing. Thus violations in T symmetry are often referred to as CP violations.

3 Physical setup of the problem

We consider a chiral gauge theory which is defined over the following topologically nontrivial four-dimensional spacetime manifold:

$$M = \mathbb{R}^3 \times S^1 , \tag{3.0.1a}$$

with noncompact coordinates

$$x^1, x^2, x^3 \in \mathbb{R} , \tag{3.0.1b}$$

and compact coordinate

$$x^4 \in [0, L] . \tag{3.0.1c}$$

Initially, we consider the spacetime metric to be the flat Euclidean metric,

$$g_{\mu\nu}(x) = [\mathrm{diag}(1,\ 1,\ 1,\ 1)]_{\mu\nu} . \tag{3.0.2}$$

At the end of our calculation, we shall make the Wick rotation from Euclidean metric signature to Lorentzian metric signature, with x^4 corresponding to a compact spatial coordinate, and x^1 or x^2 or x^3 shall be taken as time coordinate t.

We consider the chiral gauge theories those are free of gauge anomalies. Specifically, we take the chiral gauge theory with the following non-Abelian gauge group and representation of left-handed fermions:

$$G = SO(10) , \tag{3.0.3a}$$

$$R_L = 3 \times [16] , \tag{3.0.3b}$$

which contains the $SU(3) \times SU(2) \times U(1)$ Standard Model with 3 families of fermions (and three singlet left-handed antineutrinos).

However, we perform most of our calculations for a chiral $U(1)$ gauge theory consisting of a single gauge boson A and 48 left-handed fermions with $U(1)$ charges q_f, for $f = 1, \ldots, 48$. Specifically, the Abelian gauge group and the left-handed fermion representation (i.e., the set of left-handed charges q_f in units of e, the absolute value of the electron charge) are given by:

$$G = U(1) , \tag{3.0.4a}$$

$$R_L = 3\times\left[6\times\left(\frac{1}{3}\right)+3\times\left(-\frac{4}{3}\right)+3\times\left(\frac{2}{3}\right)+2\times(-1)+1\times(2)+1\times(0)\right]. \tag{3.0.4b}$$

This particular chiral $U(1)$ gauge theory can be embedded in the $SU(2)\times U(1)$ electroweak theory of the Standard Model with $U(1)$ hypercharge $Y \equiv 2\,Q - 2\,T_3$ (the electron has charge $Q = -e$ and the positron has $Q = +e$.) The further embedding in the "safe" $SO(10)$ group with left-handed representation (3.0.3b) explains why the perturbative gauge anomalies cancel out in the chiral $U(1)$ gauge theory considered,

$$\sum_{f=1}^{48} (q_f)^3 = 0\,, \tag{3.0.5}$$

for the charges q_f as given by (3.0.4b). For later use, we also give another sum:

$$\sum_{f=1}^{48} (q_f)^2 = F\,e^2\,, \tag{3.0.6a}$$

$$F = 3\times\left[\frac{40}{3}\right] = 40\,. \tag{3.0.6b}$$

Other chiral $U(1)$ gauge theories give, in general, a different value for the numerical factor F.

We consider the fermion and gauge fields are periodic in the x^4 coordinate with period L,

$$\psi(\vec{x},\, x^4+L) = \psi(\vec{x},\, x^4)\,, \tag{3.0.7a}$$

$$\overline{\psi}(\vec{x},\, x^4+L) = \overline{\psi}(\vec{x},\, x^4)\,, \tag{3.0.7b}$$

$$A_\mu(\vec{x},\, x^4+L) = A_\mu(\vec{x},\, x^4)\,, \tag{3.0.7c}$$

where

$$\vec{x} \equiv (x^1,\, x^2,\, x^3)\,. \tag{3.0.8}$$

Another assumption about the gauge fields is as follows:

$$A_i(x) = A_i(\vec{x},\, x^4)\,, \quad \text{for} \quad i = 1,\, 2,\, 3\,, \tag{3.0.9a}$$

$$A_4(x) = 0\,. \tag{3.0.9b}$$

Such gauge fields can be obtained by a gauge transformation if there are trivial holonomies,

$$h(\vec{x}) \equiv \int_0^L dx^4\, A_4(\vec{x},\, x^4) = 0\,. \tag{3.0.10}$$

The holonomy $h(\vec{x})$ is a gauge-invariant quantity (see the last paragraph of Sec. 4.2).

The background gauge fields A_i are considered to have local support in $\mathbb{R}^3$. We consider, specifically, a ball $B^3 \in \mathbb{R}^3$ with a large fixed radius R. At the end of the calculation radius R can be taken to infinity. We assume the gauge potentials $A_i(x)$, for $i = 1,\, 2,\, 3$, vanish on the boundary of the ball and outside of it,

$$A_i(\vec{x},\, x^4) = 0\,, \quad \text{for} \quad |\vec{x}|^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 \geq R^2\,. \tag{3.0.11}$$

In general, Latin spacetime indices i, j, k, l, etc. run over the coordinate labels $1, 2, 3$, and Greek spacetime indices μ, ν, ρ, etc. over the labels 1, 2, 3, 4. Repeated coordinate (and internal) indices are summed over. Throughout, natural units are used with $\hbar = c = 1$.

For the above setup, we investigate, the non-invariance of the effective gauge-field action $\Gamma[A]$ under Lorentz and CPT transformations. In Secs. 4 and 5, we calculate the effective action $\Gamma[A]$ by integrating out the fermions using, respectively, a perturbative and a nonperturbative method. We perform a CPT transformation of the background gauge field and show $\Gamma[A^{\mathrm{CPT}}] \neq \Gamma[A]$ to establish the CPT anomaly.

First we perform the actual calculation of Sec. 4 for a single left-handed fermion ψ with unit $U(1)$ charge, $q = e$. Only we extend the final result (4.2.34) to all chiral fermions of the theory (3.0.4). We follow the same procedure in Sec. 5.

4 Perturbative approach

4.1 Theory and regularization

4.1.1 Effective action

First we write the Euclidean Weyl action for a left-handed chiral fermion,

$$\begin{aligned} S\left[\overline{\psi},\, \psi,\, A\right] &= \int_M d^4x\, \mathcal{L}\left[\overline{\psi_L}, \psi_L, A\right] \\ &= \int_M d^4x\, i\, \overline{\psi_L}\, \gamma^\mu (\partial_\mu + e\, A_\mu)\, \psi_L\,, \end{aligned} \tag{4.1.1}$$

where A_μ is the anti-Hermitian $U(1)$ gauge field, e the dimensionless electric charge of the fermion ψ, and $\psi_L \equiv \frac{1}{2}(1+\gamma_5)\,\psi$ the left-handed projection of the four-component Dirac spinor ψ. The γ^μ are the 4×4 Dirac matrices and $\overline{\psi} \equiv \psi^\dagger\, \gamma^4$. The Hermitian chirality matrix γ_5 has $\{\gamma_5,\, \gamma^\mu\} = 0$ and $(\gamma_5)^2 = \mathbb{1}_4$.

In this thesis, we are interested to calculate the effective gauge field action for the setup as described in Sec. 3 by integrating out the chiral fermions, while maintaining gauge invariance. In the vacuum, there are virtual fermion-antifermion pairs which interact with the classical background gauge fields. The effective action $\Gamma[A]$ is a functional which takes these interactions into account. The complete action for classical gauge fields interacting with virtual fermion is given by

$$S_{complete}[A] = S[A] + \Gamma[A], \tag{4.1.2}$$

with

$$S[A] = \int d^4x \frac{1}{2} \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}). \tag{4.1.3}$$

The functional $\Gamma[A]$, we consider here, is not the complete effective action, because there are also contributions from the photonic sector such as the classical Maxwell term. But we shall focus solely on the contributions coming from the virtual fermions.

In Feynman's Euclidean path integral formalism, the functional $\Gamma[A]$ is obtained by integrating out the fermionic degrees of freedom,

$$\exp(-\Gamma[A]) = \int \mathcal{D}\overline{\psi_L}(x)\mathcal{D}\psi_L(x)\ \exp\left(-\int_M d^4x\, \mathcal{L}\left[\overline{\psi_L}, \psi_L, A\right]\right). \tag{4.1.4}$$

This is formally equal to the root of the determinant of the operator $\gamma^\mu(\partial_\mu + e A_\mu)$. The above operator has an unbounded spectrum, so that the determinant is infinite, so that the expression (4.1.4) thus needs to be regularized. In the following section we discuss the regularization procedure in details.

4.1.2 Regularization

To find a manifestly gauge-invariant regularization for chiral gauge theories is not straightforward. For example if we consider the usual Pauli-Villars regularization, the mass terms for regularizing fields break the gauge invariance. However a gauge-invariant mass term can be introduced providing a gauge-invariant Pauli-Villars-like regularization. This modified Pauli-Villars-like regularization has been discussed by Frolov and Slavnov [17], which involves an infinite set of bosonic and fermionic Pauli–Villars-type fields ϕ_s and Ψ_s, for $s \in \mathbb{Z}/\{0\}$, with standard (Lorentz-invariant) Dirac-type mass terms $m_s\,\overline{\Psi}_s\,\Psi_s$. The local regularized Lagrangian including an infinite set of Pauli-Villars-like fields is taken in the form:

$$\begin{aligned}\mathcal{L}_{reg} &= \bar{\psi}_L\gamma^\mu(\partial_\mu + A_\mu^{ij}\sigma_{ij})\psi_L + \bar{\psi}_s\gamma^\mu(\partial_\mu + A_\mu^{ij}\sigma_{ij})\psi_s - \frac{1}{2}m_s{\psi_s}^T C_D C\Gamma_{11}\psi_s - \frac{1}{2}m_s\bar{\psi}_s C_D C\Gamma_{11}{\bar{\psi}_s}^T \\ &+ \bar{\phi}_s\gamma^\mu(\partial_\mu + A_\mu^{ij}\sigma_{ij})\phi_s - \frac{1}{2}m_s{\phi_s}^T C_D C\phi_s - \frac{1}{2}m_s\bar{\phi}_s C_D C{\bar{\phi}_s}^T \end{aligned} \tag{4.1.5}$$

Here ψ_L is the $SO(10)$ chiral field, the fields ψ_r are anticommuting and ϕ_r are commuting Pauli-Villars-like fields. For the construction of the spinorial representation of $SO(2n)$ groups it is convenient to introduce $2n$ Hermitian $2^n \times 2^n$ matrices Γ_i which satisfy the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = \delta_{ij}. \tag{4.1.6}$$

The Hermitian matrices

$$\sigma_{ij} = \frac{1}{2}i[\Gamma_i, \Gamma_j] \tag{4.1.7}$$

and

$$\Gamma_{2n+1} = (-i)^n\Gamma_1\Gamma_2...\Gamma_{2n}. \tag{4.1.8}$$

In the above eq.(4.1.5) the repeated index s is summed over, with $s \geq 1$, and $m_s = m|s|$. The matrix C_D is the charge conjugation matrix and the matrix C defined by the relation

$$\sigma_{ij}^T C = -C\sigma_{ij} \tag{4.1.9}$$

The above model can be generalized for the other $SO(2n)$ gauge models.

For simple Abelian gauge theory the local regularized Lagrangian is written as

$$\mathcal{L} = \sum_{s=-\infty}^{s=\infty} i\bar{\psi}_s\gamma_\mu(\partial_\mu - igA_\mu)\psi_s - m_s\bar{\psi}_s\psi_s \tag{4.1.10}$$

where ψ_s ($s \neq 0$) is an infinite set of Pauli-Villars fields having the following Grassman parity

$$\varepsilon(\psi_s) = (-1)^{s+1}. \tag{4.1.11}$$

Here $\psi_0 \equiv \psi$; $m_0 = 0$; $m_s = m|s|$

For our model we shall extend this regularization, in order to be sensitive to anomalous Lorentz violation. In fact, we will introduce another infinite set of bosonic and fermionic Pauli–Villars-type fields ψ_r, for $r \in \mathbb{Z}/\{0\}$, with Lorentz-violating mass terms $M_r\, \psi_r^\dagger\, \psi_r$.

4.1.2.1 Regularized Lagrangian

The regularized Lagrange density for the chiral $U(1)$ gauge theory including both infinite sets of Pauli–Villars-type fields is written as follows:

$$
\begin{aligned}
\mathcal{L}_{\text{full reg. th.}} &= \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{LI-gen-PV}} + \mathcal{L}_{\text{LV-gen-PV}} \\
&= i\,\overline{\psi_0}(x)\,\gamma^\mu \left(\partial_\mu + e\,A_\mu\right) \psi_0(x) \\
&\quad + \sum_{s\neq 0} \left[i\,\overline{\Psi}_s(x)\,\gamma^\mu \left(\partial_\mu + e\,A_\mu\right) \Psi_s(x) - m_s\,\overline{\Psi}_s(x)\,\Psi_s(x) \right] \\
&\quad + \sum_{r\neq 0} \left[i\,\overline{\psi_r}(x)\,\gamma^\mu \left(\partial_\mu + e\,A_\mu\right) \psi_r(x) - M_r\,\psi_r^\dagger(x)\,\psi_r(x) \right] ,
\end{aligned}
\tag{4.1.12}
$$

with regulator masses,

$$m_s = m\,|s|\,, \tag{4.1.13a}$$

$$M_r = M\,r^2\,, \tag{4.1.13b}$$

$$M \gg m\,. \tag{4.1.13c}$$

The ultraheavy regulator masses M_r are responsible for the violation of Lorentz invariance. In Sec. 4.2, we shall explain the reason why we demand a quadratic r-dependence in (4.1.13b), whereas there is a linear s-dependence in (4.1.13a). Note that, there is no strict need for the inequality in (4.1.13c) for our calculation. Though we introduce the above inequality in order to make sure that the possible Lorentz-violating quantum effects should not dominate over the Lorentz-invariant quantum effects.

The regulator fields Ψ_s in (4.1.12) are four-component Dirac fields. The regulator fields ψ_r, including the original massless field $\psi_0 \equiv \psi_L$, are chiral four-component Dirac fields, obeys the following condition

$$\psi_r \equiv \frac{1}{2}\,(1+\gamma_5)\,\psi_r\,, \quad \text{for} \quad r \in \mathbb{Z}\,. \tag{4.1.14}$$

The regulator fields have the following Grassmann parities:

$$\varepsilon(\Psi_s) = (-1)^{s+1}\,, \quad \text{for} \quad s \in \mathbb{Z}/\{0\}\,, \tag{4.1.15a}$$

$$\varepsilon(\psi_r) = (-1)^{r+1}\,, \quad \text{for} \quad r \in \mathbb{Z}\,. \tag{4.1.15b}$$

Our main goal is searching for the anomalous Lorentz (and CPT) violation. For this reason we only consider the chiral fields ψ_r. We shall explain this in Sec. 4.2.

We now take the Weyl representation of the 4×4 Dirac gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \widetilde{\sigma}^\mu \\ \widetilde{\sigma}^{\mu\dagger} & 0 \end{pmatrix}, \tag{4.1.16}$$

with $\widetilde{\sigma}^\mu \equiv (\sigma^m,\, i\,\mathbb{1}_2)$ in terms of the 2×2 Pauli spin matrices σ^m and the 2×2 identity matrix $\mathbb{1}_2$. As said before, ψ_0 with $M_0 = 0$ in (4.1.12) corresponds to the original four-component chiral field ψ_L and can be written as

$$\psi_0 = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}, \tag{4.1.17}$$

where ξ_0 is an anticommuting two-component spinor field. The $r \neq 0$ fields ψ_r in (4.1.12) constitute an infinite set of Pauli–Villars fields with Grassmann parities (4.1.15b) and regulator masses (4.1.13b). Each chiral regulator field ψ_r $(r \neq 0)$ can also be written as

$$\psi_r = \begin{pmatrix} \xi_r \\ 0 \end{pmatrix}, \tag{4.1.18}$$

with a two-component field ξ_r having the Grassmann parity (i.e., loop-factor in Feynman diagrams)

$$\varepsilon(\xi_r) = (-1)^{r+1}\,, \quad \text{for} \;\; r \in \mathbb{Z}\,. \tag{4.1.19}$$

With the above definitions, the regularized theory is given by

$$\begin{aligned} \mathcal{L}_{\text{trunc. reg. th.}} &= \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{LV-gen-PV}} \\ &= \sum_{r=-\infty}^{\infty} \left[i\,\xi_r^\dagger(x)\,\sigma^\mu \left(\partial_\mu + e\,A_\mu\right) \xi_r(x) - M_r\,\xi_r^\dagger(x)\,\xi_r(x) \right], \end{aligned} \tag{4.1.20}$$

with $\sigma^\mu \equiv (i\sigma^m,\, \mathbb{1}_2)$ and M_r from (4.1.13b).

For the calculation of the next subsection, we define the following quantities

$$\widetilde{\gamma}^1 = i\,\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \widetilde{\gamma}^2 = i\,\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.1.21a}$$

$$\widetilde{\gamma}^3 = i\,\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \widetilde{\gamma}^4 = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.1.21b}$$

and rewrite the standard Weyl action from (4.1.20) as

$$\mathcal{I}_0 = \int d^4x\; \mathcal{L}_{\text{chiral}} = \int d^4x\; i\,\xi_0^\dagger(x)\,\widetilde{\gamma}^\mu(\partial_\mu + e\,A_\mu)\,\xi_0\,, \tag{4.1.22}$$

where ξ_0 is the two-component spinor field. A similar action holds for the chiral regulator fields ξ_r $(r \neq 0)$,

$$\mathcal{I}_{\text{reg}} = \int d^4x\; \mathcal{L}_{\text{LV-gen-PV}} = \int d^4x \sum_{r\neq 0} \left[i\,\xi_r^\dagger(x)\,\widetilde{\gamma}^\mu(\partial_\mu + e\,A_\mu)\,\xi_r - M_r\,\xi_r^\dagger\,\xi_r \right]. \tag{4.1.23}$$

The 2×2 matrices $\widetilde{\gamma}^\mu$ in (4.1.22) and (4.1.23) obey the following relation:

$$\widetilde{\gamma}^i\,\widetilde{\gamma}^j = \tilde{g}^{ij}\,\mathbb{1} - \epsilon^{ijk}\,\widetilde{\gamma}_k\,, \tag{4.1.24}$$

with the three-dimensional Euclidean flat metric $\tilde{g}^{ij} = [\mathrm{diag}(-1,-1,-1)]^{ij}$ and the totally anti-symmetric Levi-Civita symbol ϵ^{ijk}, normalized by $\epsilon^{123} = 1$. From (4.1.24), we have that the anti-commutator of the $\widetilde{\gamma}^i$ matrices has precisely the same structure as the one of Dirac matrices in $\mathbb{R}^3$, namely, $\{\widetilde{\gamma}^i, \widetilde{\gamma}^j\} = 2\,\tilde{g}^{ij}\,\mathbb{1}$. This is, in fact, the reason for using these matrices $\widetilde{\gamma}^\mu$, as will become clear in Sec. 4.2. Note, however, that the matrices $\widetilde{\gamma}^\mu$ do not satisfy the properties of Dirac gamma matrices in four-dimensional spacetime, because $\widetilde{\gamma}^4$ does not anti-commute with the other $\widetilde{\gamma}^i$ matrices. In our calculations, we shall only use relation (4.1.24).

4.2 Calculation

4.2.1 vacuum-polarization kernel

For standard Minkowski spacetime without compactification of the x^4 coordinate, we expand the gauge field A_μ in Fourier modes as follows:

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, A_\mu(p), \tag{4.2.1}$$

and write down the vacuum-polarization kernel

$$\pi^{ij}(p) = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\widetilde{\gamma}^i\, S(k)\, \widetilde{\gamma}^j\, S(k+p)\right]. \tag{4.2.2}$$

In our case, where the x^4 coordinate is compactified, we make the following replacements:

$$\int d^4x \to \int_0^L dx^4 \int_{\mathbb{R}^3} d^3x \tag{4.2.3a}$$

and

$$\int \frac{d^4p}{(2\pi)^4} \to \frac{1}{L}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}. \tag{4.2.3b}$$

The Fourier expansion of the gauge field A_μ is now given by

$$A_\mu(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}\, e^{2\pi i n x^4/L}\, e^{i\vec{p}\cdot\vec{x}}\, A_\mu(p_n), \tag{4.2.4}$$

with the following definitions:

$$p_n \equiv (\vec{p}, \rho_n)\,, \tag{4.2.5a}$$

$$\rho_n \equiv 2\pi n/L\,, \tag{4.2.5b}$$

$$p_n^2 \equiv |\vec{p}|^2 + (\rho_n)^2\,. \tag{4.2.5c}$$

The expression for the perturbatively-expanded effective gauge-field action in three spacetime dimensions with one compactified coordinate has been given in Ref. [18]; see, in particular, Eqs. (22)–(26) of that article. For the action (4.1.22) with the replacement (4.2.3a), we have four spacetime dimensions with one compactified coordinate. Adopting a similar procedure as the one of Ref. [18], we write down the physically relevant factor in the perturbatively-expanded effective gauge-field action,

$$\Gamma[A] = -i\,\frac{e^2}{2}\,\frac{1}{L}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,A_i(-p_n)\,\pi^{ij}(p_n)\,A_j(p_n) + O(e^3), \tag{4.2.6}$$

with the unregularized vacuum-polarization kernel

$$\pi^{ij}(p_n)\Big|^{(\text{unreg.})} = \frac{1}{L}\sum_{m=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\left[\widetilde{\gamma}^i\,S(k_m)\,\widetilde{\gamma}^j\,S(k_m+p_n)\right]. \tag{4.2.7}$$

The propagator $S(k_m)$ is defined as:

$$S(k_m) = \frac{1}{\widetilde{\gamma}^i k_i + \widetilde{\gamma}^4 k_{4m}} = \frac{\widetilde{\gamma}^i\,k_i - \widetilde{\gamma}^4\,k_{4m}}{(\widetilde{\gamma}^i k_i)^2 - k_{4m}^2} = -\frac{\widetilde{\gamma}^i\,k_i - \widetilde{\gamma}^4\,k_{4m}}{(k_i)^2 + k_{4m}^2}. \tag{4.2.8}$$

The ultraviolet divergences of the anomalous terms in (4.2.7) are regularized by the infinite set of Pauli–Villars-type fields $\xi_r(x)$, for $r \neq 0$, from (4.1.23). In addition to the ultraviolet divergences there are also infrared divergences. These infrared divergences are regularized by imposing antiperiodic boundary conditions for the $\xi_r(x)$ fields ($r \in \mathbb{Z}$) on the surface of a large ball B^3, where the gauge potentials $A_i(x)$ vanish according to (3.0.11).

For a particular Fourier mode n of the background gauge field, the regularized two-point function is proportional to the following expression:

$$\pi^{ij}(p_n)\Big|^{(\text{reg.})} = \sum_{r=-\infty}^{\infty}(-1)^r\,\frac{1}{L}\sum_{m=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\,\frac{\mathrm{tr}\left[\widetilde{\gamma}^i(\not{k}+M_r)\widetilde{\gamma}^j(\not{k}+\not{p}+M_r)\right]}{\left(k_m^2+M_r^2\right)\left((k_m+p_n)^2+M_r^2\right)}, \tag{4.2.9}$$

with the short-hand notation $\not{p} \equiv \widetilde{\gamma}^i\,p_i - \widetilde{\gamma}^4\,p_{4n}$ for our matrices (4.1.21), which are Dirac gamma matrices in three spacetime dimensions but not in four. The factor $(-1)^r$ in (4.2.9) comes from the Grassmann parity (4.1.19) of the fields and M_r is given by (4.1.13b). From now on, we drop the superscript 'reg.' as the regularization is manifest from having the sum over r.

4.2.2 Feynman parametrization

We introduce the Feynman parameter x and change the momentum variable k_μ to l_μ, with

$$l_i \equiv k_i + x\,p_i, \quad \text{and} \quad l_4 \equiv k_4\,, \tag{4.2.10}$$

and rewrite our expression for the vacuum-polarization kernel (4.2.9) as

$$\begin{aligned}\pi^{ij}(p_n) \;=\; & \sum_{r=-\infty}^{\infty}(-1)^r\int_0^1 dx\,\frac{1}{L}\sum_{m=-\infty}^{\infty}\int\frac{d^3l}{(2\pi)^3}\\ & \frac{\mathrm{tr}\left[\widetilde{\gamma}^i\left(\widetilde{\gamma}^k\,l_k - x\,\widetilde{\gamma}^k\,p_k - \omega_m + M_r\right)\widetilde{\gamma}^j\left(\widetilde{\gamma}^k\,l_k + (1-x)\,\widetilde{\gamma}^k\,p_k - \omega_m - \rho_n + M_r\right)\right]}{\left(|\vec{l}|^2+\Delta\right)^2}\,,\end{aligned} \tag{4.2.11}$$

with p_n, ρ_n, and ρ_n^2 from (4.2.5) and the further definitions

$$l_m \equiv (\vec{l}, \omega_m)\,, \tag{4.2.12a}$$

$$\omega_m \equiv 2\pi m/L\,, \tag{4.2.12b}$$

$$\Delta \equiv (\omega_m + x\rho_n)^2 + x(1-x)\,p_n^2 + M_r^2\,. \tag{4.2.12c}$$

Note that, we do not shift and redefine the discrete momentum in the 4-direction, which is little different from the regular loop calculation in four dimension with no compactified coordinate.

4.2.3 Anomalous term $\widetilde{T}^{ij}(p_n)$ in the vacuum-polarization kernel

The odd powers of the l_i in the numerator of (4.2.11) vanish by symmetry reasons. The term in (4.2.11) with an odd number of p_n momenta in the numerator of the integrand is written as

$$\widetilde{T}^{ij}(p_n) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{1}{L} \sum_{m=-\infty}^{\infty} (-\omega_m + M_r) \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx\, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4]\, \rho_n}{(|\vec{l}|^2 + \Delta)^2}\,. \tag{4.2.13}$$

Part of the above equation still gives rise to a finite L-independent term with an even number of p_n momenta,

$$\begin{aligned} &\frac{1}{L} \sum_{m=-\infty}^{\infty} (-\omega_m) \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4]\, \rho_n}{(|\vec{l}|^2 + \Delta)^2} \\ &\propto \left(\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, \rho_n\, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^4]\, \rho_n \rho_n\right) \end{aligned} \tag{4.2.14}$$

and we are left with the following term with an odd number of p_n momenta:

$$T^{ij}(p_n) = \frac{1}{L} \sum_{m=-\infty}^{\infty} \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r M_r\, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k - \mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j]\, \rho_n}{(|\vec{l}|^2 + \Delta)^2}\,, \tag{4.2.15}$$

where we have taken care to move the r sum inwards as it must be performed first.

The ρ_n term in the numerator of the integrand of (4.2.15) ultimately gives rise to a term

$$T_{\rho_n} \sim \int_0^L dx^4 \int d^3x\, \delta_{ij}\, A_i\, [\partial_4 A_j]$$

in the effective gauge-field action, which is a total-derivative term and vanishes due to the periodic boundary conditions (3.0.7).

So, we are left with the following potentially CPT-violating term:

$$T^{ij}_{\mathrm{anom}}(p_n) = \frac{1}{L} \sum_{m=-\infty}^{\infty} \int \frac{d^3 l}{(2\pi)^3} \int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r M_r\, \frac{\mathrm{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k}{(|\vec{l}|^2 + \Delta)^2}\,. \tag{4.2.16}$$

At this moment, we mention that there is no contribution comes from the other regulator fields Ψ_s from (4.1.12) to this potentially anomalous term with an odd number of p_n momenta, because the trace of an odd number of Dirac matrices γ^μ vanishes. This is not the case for the trace of $\widetilde{\gamma}^i\, \widetilde{\gamma}^j\, \widetilde{\gamma}^k$, as follows from relation (4.1.24).

We divide the sum over m in (4.2.16) into two parts, namely, the sum over nonzero m and the single term $m = 0$ [this term has an infrared-divergent momentum integral for the $r = 0$ contribution, which is regularized by antiperiodic boundary conditions as discussed a few lines below (4.2.8)]. The expression then reads

$$T^{ij}_{\text{anom}}(p_n) = T^{ij}_{0}(p_n) + T^{ij}_{\text{rest}}(p_n)\,, \tag{4.2.17}$$

with

$$T^{ij}_{0}(p_n) \;=\; \frac{1}{L}\int \frac{d^3 l}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r\, M_r\, \frac{\text{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k}{(|\vec{l}|^2 + \Delta_0)^2}\,, \tag{4.2.18a}$$

$$\Delta_0 \;\equiv\; x\rho_n^2 + x(1-x)\, p_n^2 + M_r^2\,, \tag{4.2.18b}$$

and

$$T^{ij}_{\text{rest}}(p_n) = \frac{2}{L}\sum_{m=1}^{\infty}\int \frac{d^3 l}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty} (-1)^r\, M_r\, \frac{\text{tr}[\widetilde{\gamma}^i \widetilde{\gamma}^j \widetilde{\gamma}^k]\, p_k}{(|\vec{l}|^2 + \Delta)^2}\,, \tag{4.2.19}$$

First, consider the $m = 0$ contribution (4.2.18). In order to compute the sum over r, we use the following representation (defining $l \equiv |\vec{l}|$):

$$\begin{aligned} S_0 \;&=\; \sum_{r=-\infty}^{\infty} \frac{(-1)^r M_r}{\left(|\vec{l}|^2 + (x\rho_n)^2 + x(1-x)\, p_n^2 + M_r^2\right)^2} \\ &=\; -\frac{1}{2\,l}\frac{\partial}{\partial l}\sum_{r=-\infty}^{\infty} \frac{(-1)^r M_r}{\left(l^2 + (x\rho_n)^2 + x(1-x)\, p_n^2 + M_r^2\right)} \\ &=\; -\frac{1}{2\,l\,M}\frac{\partial}{\partial l}\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{(\tau^2 + r^4)}\,, \end{aligned} \tag{4.2.20a}$$

with

$$\tau^2 \;\equiv\; \left[l^2 + (x\rho_n)^2 + x\,(1-x)\, p_n^2\right]/M^2 \equiv l^2/M^2 + \kappa\,, \tag{4.2.20b}$$

and the following result (for $\tau \neq 0$):

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{\tau^2 + r^4} \;=\; f(\tau)\,, \tag{4.2.21a}$$

$$f(\tau) \;\equiv\; \frac{\pi}{2\sqrt{\tau}}\left(\frac{\exp(i\pi/4)}{\sinh[\,\exp(-i\pi/4)\,\pi\,\sqrt{\tau}\,]} + \frac{\exp(-i\pi/4)}{\sinh[\,\exp(i\pi/4)\,\pi\,\sqrt{\tau}\,]}\right). \tag{4.2.21b}$$

Now if we compare between and our calculation and the Frolov and Slavnov's one in [17], we see that in (4.2.20a), the first sum contains an extra factor M_r in the numerator compared to Eq. (11) of Ref. [17]. In the momentum integrals we need an exponential cutoff which comes from the $1/\sinh$ term in (4.2.21b) as in Eq. (14) of Ref. [17]. In order to get a similar $1/\sinh$ behavior in our calculation, we demand the r^2 behavior in the regulator masses M_r in (4.1.13b).

With result (4.2.21), expression (4.2.18) reduces to

$$
\begin{aligned}
T_0^{ij}(p_n) &= -\frac{1}{4\pi^2 LM}\int_0^1 dx \int_0^\infty l dl\, \frac{\partial}{\partial l}\Big[f(\tau)\Big]\,\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k]\,p_k \\
&= -\frac{1}{4\pi^2 L}\int_0^1 dx \int_0^\infty d\,\eta\,\eta\,\frac{\partial}{\partial\eta}\Big[f(\tau)\Big]\,\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k]\,p_k\,,
\end{aligned}
\tag{4.2.22}
$$

in terms of the dimensionless variable $\eta \equiv l/M$.

In the regularization procedure, we consider the regulator mass scale M to be much larger than a typical momentum component of the gauge field, $M^2 \gg p_n^2$, so that we can take

$$
\kappa \equiv \Big[(x\rho_n)^2 + x\,(1-x)\,p_n^2\Big]\,/M^2 \to 0^+
$$

in the rest of the calculation and the x integral becomes trivial. Using

$$
\mathrm{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k] = 2\,\epsilon^{ijk}\,,
\tag{4.2.23}
$$

we then rewrite (4.2.22) as

$$
T_0^{ij}(p_n) = -\frac{1}{2\pi^2 L}\left(\int_0^\infty d\,\eta\,\eta\,\frac{\partial}{\partial\eta}\Big[f(\eta)\Big]\right)\epsilon^{ijk}\,p_k\,.
\tag{4.2.24}
$$

The η integral in (4.2.24) gives a factor $\pi/2$ and the final result for the $m = 0$ sector reads

$$
T_0^{ij}(p_n) = -\frac{1}{4\pi L}\,\epsilon^{ijk}\,p_k\,.
\tag{4.2.25}
$$

Now turn to the $m \neq 0$ sum (4.2.19),

$$
T_{\mathrm{rest}}^{ij}(p_n) = \frac{1}{L}\sum_{m\neq 0}\int\frac{d^3\eta}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty}(-1)^r\,r^2\,\frac{\mathrm{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\,p_k}{\Big(|\vec{\eta}|^2+\Delta_M\Big)^2}\,,
\tag{4.2.26a}
$$

with

$$
|\vec{\eta}|^2 \equiv |\vec{l}|^2/M^2\,.
\tag{4.2.26b}
$$

and

$$
\Delta_M \equiv \Big[(\omega_m + x\rho_n)^2 + x(1-x)\,p_n^2\Big]/M^2 + r^4 \sim \omega_m^2/M^2 + r^4\,,
\tag{4.2.26c}
$$

for $p_n^2/M^2 \to 0$. With large M, we can treat $\omega_m/M \equiv l_4$ as a continuous variable and rewrite (4.2.26a) as follows:

$$
\begin{aligned}
T_{\mathrm{rest}}^{ij}(p_n) &= \frac{M}{2\pi}\int dl_4 \int\frac{d^3\eta}{(2\pi)^3}\int_0^1 dx \sum_{r=-\infty}^{\infty}(-1)^r\,r^2\,\frac{\mathrm{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\,p_k}{(\lambda^2+r^4)^2} \\
&= M\int\frac{d^4\lambda}{(2\pi)^4}\int_0^1 dx \sum_{r=-\infty}^{\infty}(-1)^r\,r^2\,\frac{\mathrm{tr}[\widetilde{\gamma}^i\widetilde{\gamma}^j\widetilde{\gamma}^k]\,p_k}{(\lambda^2+r^4)^2}\,,
\end{aligned}
\tag{4.2.27}
$$

in terms of the dimensionless variable $\lambda^2 \equiv |\vec{\eta}|^2 + (l_4)^2$.

In order to compute the sum over r in (4.2.27), we again use the following representation:

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r\, r^2}{(\lambda^2+r^4)^2} \;=\; -\frac{1}{2\lambda}\frac{\partial}{\partial\lambda}\sum_{r=-\infty}^{\infty}\frac{(-1)^r\, r^2}{(\lambda^2+r^4)}\,, \tag{4.2.28}$$

where the last sum has the same form as (4.2.21a) and equals $f(\lambda)$ in terms of the function f defined by (4.2.21b). As mentioned above, the x integral in expression (4.2.27) is trivial and the expression reduces to

$$\begin{aligned} T^{ij}_{\text{rest}}(p_n) &= -\frac{M}{16\pi^2}\left(\int_0^{\infty} d\lambda\,\lambda^2\,\frac{\partial}{\partial\lambda}\Big[f(\lambda)\Big]\right)\operatorname{tr}[\widetilde{\gamma}^i\,\widetilde{\gamma}^j\,\widetilde{\gamma}^k]\,p_k \\ &= -\frac{M}{8\pi^2}\left(\int_0^{\infty} d\lambda\,\lambda^2\,\frac{\partial}{\partial\lambda}\Big[f(\lambda)\Big]\right)\epsilon^{ijk}\,p_k\,, \end{aligned} \tag{4.2.29}$$

where the last step uses (4.2.23). The λ integral in (4.2.29) gives the following factor:

$$\xi = 14\,\zeta(3)/\pi^2 \approx 1.70511\,, \tag{4.2.30}$$

and the final expression reads

$$T^{ij}_{\text{rest}}(p_n) = -\xi\,M\,\frac{1}{8\pi^2}\,\epsilon^{ijk}\,p_k\,. \tag{4.2.31}$$

4.2.4 Renormalization

Combining (4.2.25) and (4.2.31) gives the end result for the anomalous vacuum-polarization kernel (4.2.17),

$$T^{ij}_{\text{anom}}(p_n) = -\frac{1}{4\pi L}\,\epsilon^{ijk}\,p_k - \xi\,M\,\frac{1}{8\pi^2}\,\epsilon^{ijk}\,p_k\,, \tag{4.2.32}$$

with the constant ξ given by (4.2.30) and the regulator mass scale M entering the Pauli–Villars-type masses (4.1.13b). The first term in (4.2.32) is L-dependent and finite, whereas the second term is L-independent and divergent as the regulator mass scale M is taken to infinity. As regards the M-dependence of this second term, note that, for four-dimensional quantum electrodynamics, the vacuum polarization from the standard Pauli-Villars regularization also has an M-dependent contribution; cf. Eq. (A.6) in Ref. [17]. A suitable renormalization procedure is to subtract the same result at a reference value L_{ref} and to take $L_{\text{ref}} \to \infty$ corresponding to Minkowski spacetime (cf. Sec. 4.2 of Ref. [19]). This renormalization procedure is used in Casimir energy calculation. After renormalization the second term in (4.2.32) is eliminated and we are left with the first term only,

$$T^{ij}_{\text{anom}}(p_n)\Big|^{\text{(renorm.)}} = -\frac{1}{4\pi L}\,\epsilon^{ijk}\,p_k\,. \tag{4.2.33}$$

4.2.5 Renormalized result for CPT anomalous term

Now the single left-handed fermion ψ_L is replaced by the 48 left-handed fermions of the chiral $U(1)$ gauge theory (3.0.4), with the same regularization for each of these 48 fermions. Using

(4.2.33), the following local expression for the effective gauge-field action (4.2.6) is obtained up to order e^2:

$$\mathcal{T}^{\text{(renorm.)}}_{\text{anom}} = i\,F\,e^2\,\frac{1}{8\pi L}\int_0^L dx^4 \int_{\mathbb{R}^3} d^3x\,\epsilon^{ijk}\,A_i(x)\,\partial_j\,A_k(x)\,. \tag{4.2.34}$$

The overall numerical factor F arises from (3.0.6b), due to the contributions of all chiral fermions of the theory (3.0.4). We get a further factor i in the result (4.2.34) for spacetime metrics with Lorentzian signature and a spatial coordinate $x^4 \in S^1$ (see also the discussion of the last paragraph in Chapter. 7).

For gauge fields $A_\mu(x)$ of local support, the term (4.2.34) is invariant under local Abelian gauge transformations,

$$A_\mu(x) \to A_\mu(x) + i\,\partial_\mu\,\zeta(x)\,, \tag{4.2.35}$$

with arbitrary real gauge parameters $\zeta(x)$ that are x^4-periodic, $\zeta(\vec{x},\,0) = \zeta(\vec{x},\,L)$. As mentioned in chapter 3, the holonomy (3.0.10) is gauge-invariant under these periodic transformations. In principle, we can extend the perturbative calculation of this subsection can to the non-Abelian theory (3.0.3) and we expect a further cubic term in addition to the quadratic term of (4.2.34), in order to maintain invariance under "small" gauge transformations (see Sec. 4 in Ref. [14] for further discussion).

4.3 Lorentz and CPT violation

For arbitrary gauge potentials $A_\mu(x)$ with trivial holonomies (3.0.10) in the chiral $U(1)$ gauge theory (3.0.4) with a Lorentzian metric signature, our result (4.2.34) gives the following term in the effective gauge-field action at the one-loop level:

$$\Gamma_{\text{anom}}[A] \;=\; -2\pi\,F\,e^2\,\Gamma_{\text{CS-like}}[A]\,, \tag{4.3.1a}$$

$$\Gamma_{\text{CS-like}}[A] \;\equiv\; \frac{1}{L}\int_0^L dx^4 \int_{\mathbb{R}^3} d^3x\,\omega_{\text{CS}}[A(\vec{x},x^4)]\,. \tag{4.3.1b}$$

The term $\omega_{\text{CS}}[A(\vec{x},x^4)]$ is called the Chern–Simons density

$$\omega_{\text{CS}}[A(\vec{x},x^4)] \equiv \frac{1}{16\pi^2}\,\epsilon^{ijk}\,A_i(\vec{x},x^4)\,\partial_j\,A_k(\vec{x},x^4)\,. \tag{4.3.2}$$

The numerical factor F in (4.3.1a) is given by (3.0.6b).

The Chern–Simons term, $\Omega_{\text{CS}} = \int \omega_{\text{CS}}$, is topological and defined only for an odd number of spacetime dimensions. However, the action term (4.3.1) is calculated in four spacetime dimensions. Hence, the word "Chern–Simons-like" (abbreviated as "CS-like") used in (4.3.1b) and elsewhere.

The action term (4.3.1) has a nontrivial dependence on the spacetime metric, hence in this sense this is nontopological. This term gives rise to nonstandard effects of the photons in curved spacetime (see Sec. 6.6 of Ref. [20] for further discussion and references).

Observe that the integrand of (4.3.1b) is proportional to $\epsilon^{\mu\nu\rho 4}\, A_\mu(x)\, \partial_\nu\, A_\rho(x)$, which has the spacetime index '4' singled-out. Not every Lorentz index is contracted with a four-vector. Therefore, this term is not Lorentz invariant.

Now, we recall that the CPT transformation of an anti-Hermitian gauge field is given by [14]

$$A_\mu(x) \to A_\mu(-x)\,. \tag{4.3.3}$$

The above term (4.3.1b) changes sign under a CPT transformation (4.3.3). The Lorentz-violating Chern–Simons-like term (4.3.1b) is also CPT-odd. Note that the Lorentz-invariant Maxwell term $(\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\partial^\mu A^\nu - \partial^\nu A^\mu)$ is CPT-even.

5 Nonperturbative approach

In this chapter, we perform our calculation to establish the anomalous origin of CPT violating terms in the effective gauge field action with lattice regularization.

5.1 Lattice setup

In this section we present the basic setup for lattice field theory. Let us consider a chiral gauge theory which is defined over a topologically non trivial four-dimensional spacetime manifold $M = \mathbb{R}^3 \times S^1$, with noncompact coordinates $x^1,\ x^2,\ x^3 \in \mathbb{R}$ and compact coordinate $x^4 \in [0, L]$. Initially, for the calculation purpose, we take the metric to be Euclidean flat metric $g_{\mu\nu} = [\mathrm{diag}(1,\ 1,\ 1,\ 1)]_{\mu\nu}$. The vierbeins (tetrads) are trivial and given by

$$e^a_\mu(x) = \delta^a_\mu\,, \tag{5.1.1}$$

with the Lorentz index $a = 1,\ 2,\ 3,\ 4$ and the Einstein index $\mu = 1,\ 2,\ 3,\ 4$.

We consider, in particular, chiral gauge theories that are free of gauge anomalies. As mentioned in Sec. 3, we can take the $SO(10)$ chiral gauge theory (3.0.3). But, in order to be sure of having a well-defined lattice gauge theory [21], we restrict ourselves to the Abelian $U(1)$ theory (3.0.4). The actual calculation in the rest of this section is performed for a single left-handed fermion ψ_L with unit $U(1)$ charge, $q = e$. Only the final result (5.5.21) is extended to all chiral fermions of the theory (3.0.4).

To regularize the ultraviolet divergences of this gauge theory, we introduce a rectangular hypercubic lattice with lattice spacing a. The lattice points are given by

$$(x^1,\ x^2,\ x^3,\ x^4) \equiv (\vec{x}, x^4) = (\vec{n}\,a, n_4\,a), \tag{5.1.2a}$$

with integers

$$n_1,\ n_2,\ n_3 \in [0,\ N']\,, \quad n_4 \in [0,\ N]\,. \tag{5.1.2b}$$

The fermion fields and link variables are periodic with respect to the x^4 coordinate,

$$\psi(x^1,\ x^2,\ x^3,\ L) = \psi(x^1,\ x^2,\ x^3,\ 0)\,, \tag{5.1.3a}$$

$$\overline{\psi}(x^1,\ x^2,\ x^3,\ L) = \overline{\psi}(x^1,\ x^2,\ x^3,\ 0)\,, \tag{5.1.3b}$$

$$U_\mu(x^1,\ x^2,\ x^3,\ L) = U_\mu(x^1,\ x^2,\ x^3,\ 0)\,, \tag{5.1.3c}$$

with $L \equiv N\,a$. For the other coordinates, the link variables are again periodic but the fermion fields are taken to be antiperiodic, for example,

$$\psi(L', x^2, x^3, x^4) = -\psi(0, x^2, x^3, x^4)\,, \tag{5.1.4a}$$

$$\overline{\psi}(L', x^2, x^3, x^4) = -\overline{\psi}(0, x^2, x^3, x^4)\,, \tag{5.1.4b}$$

$$U_\mu(L', x^2, x^3, x^4) = U_\mu(0, x^2, x^3, x^4)\,, \tag{5.1.4c}$$

and similarly for the other coordinates x^2 and x^3. These antiperiodic boundary conditions are introduced to remove the infrared divergence.

The assumptions (3.0.9) for the continuum gauge fields translates into the following conditions on the link variables of the lattice:

$$U_i(x) = U_i(x^1, x^2, x^3, x^4)\,, \quad \text{for} \;\; i = 1, 2, 3\,, \tag{5.1.5a}$$

$$U_4(x) = \mathbb{1}\,. \tag{5.1.5b}$$

As mentioned before, such link variables can be obtained by a gauge transformation only if there are trivial holonomies,

$$H(x^1, x^2, x^3) \equiv \prod_{\text{links}} U_4(x^1, x^2, x^3, x^4) = \mathbb{1}\,, \tag{5.1.6}$$

where the product runs over all U_4 links in the 4-direction at a fixed value of $\vec{x}$ (for non-Abelian gauge groups, there is a trace of the product matrix).

The anti-Hermitian Abelian gauge field A_μ of the continuum and the $U(1)$ link variable U_μ of the lattice are related as follows [22]:

$$U_\mu(x) = \exp\left[e \int_x^{x+a\widehat{\mu}} dy\, A_\mu(y)\right] \approx \exp\left[e\, a\, A_\mu(x + a\,\widehat{\mu}/2)\right], \tag{5.1.7}$$

where the integration variable y in the second expression runs over a straight line between the spacetime points x and $x+a\widehat{\mu}$, with unit vector $\widehat{\mu}$ in the μ direction. In (5.1.7), e is the dimensionless electric charge of the fermion.

Recall from Sec. 3 that Latin spacetime indices i, j, k, l, etc. run over the coordinate labels 1, 2, 3, and Greek spacetime indices μ, ν, ρ, etc. over the labels 1, 2, 3, 4, and that we use natural units with $\hbar = c = 1$.

5.2 Chiral fermions on the lattice

5.2.1 Ginsparg–Wilson relation

It is well known that the lattice Dirac operator have the fermion-doubling problem. To circumvent this doubling of fermion, Wilson proposed an operator, now known as the Wilson–Dirac operator [22]. This operator includes a term which is second order in the difference operators,

$$D_W = \frac{1}{2} \sum_{\mu=1}^{4} \left[\gamma_\mu(\nabla_\mu + \nabla^*_\mu) + s\, a \nabla_\mu \nabla^*_\mu\right], \tag{5.2.1}$$

with 4×4 Dirac matrices γ_μ and a parameter s to be described below. Here, the gauge-covariant derivatives of the continuum are replaced by gauge-covariant forward and backward difference operators on the lattice,

$$\nabla_\mu \psi(x) \equiv \frac{1}{a}\Big(R[U_\mu(x)]\psi(x + a\widehat{\mu}) - \psi(x)\Big), \tag{5.2.2a}$$

$$\nabla_\mu^* \psi(x) \equiv \frac{1}{a}\Big(\psi(x) - R[U_\mu(x - a\widehat{\mu})]^{-1}\psi(x - a\widehat{\mu})\Big), \tag{5.2.2b}$$

where R is a unitary representation of the gauge group.

The Wilson parameter s in (5.2.1) takes the values $s = \pm 1$. For definiteness, we choose

$$s = -1. \tag{5.2.3}$$

The operator in (5.2.1) breaks, however, the chiral invariance. In order to effectively restore the chiral symmetry, Ginsparg and Wilson suggested to implement the following relation [23]:

$$D\gamma_5 + \gamma_5 D = a\, D\gamma_5 D\,, \tag{5.2.4}$$

which is known as the Ginsparg–Wilson relation.

Sixteen years after Ginsparg and Wilson proposed their relation, Neuberger explicitly constructed a corresponding operator [24, 25],

$$D[U] = \frac{1}{a}\Big(\mathbb{1} - V[U]\Big), \tag{5.2.5}$$

in terms of an appropriate unitary operator V. Apart from satisfying the Ginsparg–Wilson relation (5.2.4), the operator V should also be γ_5-Hermitian,

$$V^\dagger = \gamma_5\, V\, \gamma_5\,. \tag{5.2.6}$$

In terms of the Wilson–Dirac operator D_W from (5.2.1) , this operator V reads

$$V = X\,(X^\dagger X)^{-1/2} = \int_{-\infty}^{\infty} \frac{dt}{\pi}\left(t^2 + X^\dagger X\right)^{-1}, \tag{5.2.7a}$$

$$X \equiv \mathbb{1} - a\, D_W\,. \tag{5.2.7b}$$

5.2.1.1 Lattice fermion action

The lattice fermion action with a Ginsparg–Wilson operator $D[U]$ defined by (5.2.5) and (5.2.7),

$$S_F[\overline{\psi}, \psi, U] = a^4 \sum_x \overline{\psi}(x) D[U] \psi(x)]\,, \tag{5.2.8}$$

is invariant under the following infinitesimal transformations [26]:

$$\psi(x) \to \psi(x) + \delta\psi(x)\,, \tag{5.2.9a}$$

$$\overline{\psi}(x) \to \overline{\psi}(x) + \delta\overline{\psi}(x)\,, \tag{5.2.9b}$$

with

$$\delta\psi(x) = i\varepsilon\,\gamma_5 V\,\psi(x) \equiv i\varepsilon\,\widehat{\gamma}_5\,\psi(x)\,, \tag{5.2.10a}$$

$$\delta\overline{\psi}(x) = i\varepsilon\,\overline{\psi}(x)\gamma_5\,, \tag{5.2.10b}$$

where ε is an infinitesimal parameter. The operator $\widehat{\gamma}_5$ from (5.2.10a) is a Hermitian unitary operator with eigenvalues ± 1.

A chiral gauge theory for left-handed fermions on the lattice can be constructed by imposing the following constraints [21]:

$$\psi(x) = \widehat{P}_-\,\psi(x)\,, \tag{5.2.11a}$$

$$\overline{\psi}(x) = \overline{\psi}(x)\,P_+\,, \tag{5.2.11b}$$

with the projection operators

$$\widehat{P}_\pm \equiv \frac{1}{2}\,(1 \pm \widehat{\gamma}_5)\,, \tag{5.2.12a}$$

$$P_\pm \equiv \frac{1}{2}\,(1 \pm \gamma_5)\,, \tag{5.2.12b}$$

where $\widehat{\gamma}_5$ has been defined in (5.2.10a).

Note that, the projectors $\widehat{P}_\pm$ contain the Ginsparg–Wilson operator $D[U]$, so that they depend on the link variables. As a result, the subspaces of fermionic fields, satisfying the condition $\psi_\pm = \widehat{P}_\pm\psi_\pm$, depend on the link variable configuration. So, in short, for every link variable configuration there exist different subsets of left and right-handed fermions.

5.2.1.2 Discrete transformations

In the lattice gauge theory continuous spacetime transformations can not be defined properly, because spacetime itself is discrete. On the hypercubic spacetime lattice, there are certain symmetry transformations. Specifically, these lattice symmetries are

(i) the translations by an integer multiple of the lattice spacing a in the direction of one of the four coordinate axes,

(ii) the rotations by an integer multiple of the angle $\pi/2$ in hyperplanes spanned by two axes,

(iii) the parity transformation,

(iv) the time-reversal transformation,

(v) the charge-conjugation transformation.

Our main focus is CPT transformation, therefore, we now give the parity, time-reversal, and charge-conjugation transformations for the link variable, considering the x^1 coordinate to be the time coordinate for the Lorentzian metric signature and using the notation $x = (x^1, x^2, x^3, x^4) \equiv (x^1, \widetilde{x})$. The parity-transformed link variable is

$$U_\mu{}^{\mathcal{P}}(x^1, \widetilde{x}) = \begin{cases} U_\mu^\dagger(x^1, -\widetilde{x} - a\widehat{\mu}), & \text{for } \mu = 2, 3, 4, \\ U_\mu(x^1, -\widetilde{x}), & \text{for } \mu = 1, \end{cases} \tag{5.2.13a}$$

the time-reflected link variable is

$$U_\mu{}^{\mathcal{T}}(x^1, \widetilde{x}) = \begin{cases} U_\mu^*(-x^1, \widetilde{x}), & \text{for } \mu = 2, 3, 4, \\ U_\mu^t(-x^1 - a, \widetilde{x}), & \text{for } \mu = 1, \end{cases} \tag{5.2.13b}$$

and the charge-conjugated link variable is

$$U_\mu{}^{\mathcal{C}}(x^1, \widetilde{x}) = U_\mu^*(x^1, \widetilde{x}). \tag{5.2.13c}$$

Hence, the combined CPT transformation on a link variable is given by

$$U_\mu{}^{\theta}(x) = U_\mu^\dagger(-x - a\widehat{\mu}). \tag{5.2.14}$$

5.2.1.3 Integration measure

The fermionic integration measure is the product of all integration measures at the sites of the hypercubic lattice,

$$\mathcal{D}\psi(x) = \prod_{x,\alpha} d\psi_\alpha(x), \quad \mathcal{D}\overline{\psi}(x) = \prod_{x,\alpha} d\overline{\psi}_\alpha(x), \tag{5.2.15}$$

with a multi-index α containing the spinor, gauge, and flavor indices.

The fermionic fields can be expanded as follows:

$$\psi(x) = \sum_j v_j(x)\, c_j, \quad \overline{\psi}(x) = \sum_k \bar{c}_k\, \bar{v}_k(x), \tag{5.2.16}$$

where the c_j and $\bar{c}_k$ are Grassmann-valued coefficients and the $v_j(x)$ and $\bar{v}_k(x)$ are two orthonormal bases of complex-valued spinorial functions. The integration measure is then given by

$$\mathcal{D}\psi(x) = \prod_j dc_j, \quad \mathcal{D}\overline{\psi}(x) = \prod_k d\bar{c}_k. \tag{5.2.17}$$

But this integration measure is not unique. Let $\mathcal{U}$ be a unitary operator which diagonalizes the operator $\widehat{\gamma}_5$,

$$\mathcal{U}^\dagger \widehat{\gamma}_5\, \mathcal{U} = \gamma_5, \tag{5.2.18}$$

where γ_5 on the right-hand side is diagonal in the Weyl representation of the Dirac gamma matrices. Then the basis spinors v_j are

$$v_j(x) = \mathcal{U} \chi_j(x), \tag{5.2.19}$$

where the χ_j form a complete canonical spinor basis and satisfy the chirality constraint

$$\widehat{P}_- \chi_j(x) = \chi_j(x). \tag{5.2.20}$$

Now, $\mathcal{U}' = \mathcal{U}Q$ is also a diagonalization operator if Q has the following form:

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad Q_1^\dagger Q_1 = \mathbb{1}, \quad Q_2^\dagger Q_2 = \mathbb{1}\,, \tag{5.2.21}$$

where Q_1 and Q_2 are 2×2 block matrices in spinor space. If the basis vectors change as

$$v'_j(x) = \sum_i v_i(x)\, \mathcal{Q}_{ij}\,, \tag{5.2.22a}$$

with

$$\mathcal{Q}_{ij} \equiv a^4 \sum \chi_i^\dagger(x)\, Q\, \chi_j(x)\,, \tag{5.2.22b}$$

then the measure (5.2.17) changes by a factor $\det \mathcal{Q}$, which is a phase factor since $\mathcal{Q}$ is unitary.

5.3 Effective action and CPT transformation

5.3.1 Effective action

As in Chapter. 4, the effective gauge-field action is obtained by integrating out the chiral fermions, while maintaining gauge invariance. On the lattice, the Euclidean path integral is given by:

$$\exp(-\Gamma[U]) = \frac{1}{Z} \int \prod_x \mathcal{D}\overline{\psi}(x) \prod_x \mathcal{D}\psi(x)\; \exp\left(-S_F[\overline{\psi}, \psi, U]\right), \tag{5.3.1}$$

where S_F is defined by (5.2.8). The normalization constant Z ensures that $\Gamma[\mathbb{1}] = 0$ for the constant link variable configuration $U_\mu(x) = \mathbb{1}$.

The left-handed fermionic fields are now Fourier expanded as follows:

$$\psi(x) = \frac{1}{L} \sum_n \psi_n(x^1, x^2, x^3)\, e^{2\pi i n x^4/L}\,, \tag{5.3.2a}$$

$$\overline{\psi}(x) = \frac{1}{L} \sum_n \overline{\psi}_n(x^1, x^2, x^3)\, e^{-2\pi i n x^4/L}\,, \tag{5.3.2b}$$

where the integer n takes the values

$$-(N-1)/2 \leq n \leq (N-1)/2\,, \quad \text{for odd } \; N \geq 1\,, \tag{5.3.3a}$$

and

$$-(N/2) + 1 \leq n \leq N/2\,, \quad \text{for even } \; N \geq 2\,, \tag{5.3.3b}$$

with $N = L/a$ the number of links in the compact 4-direction. The momentum component in the 4-direction is given by

$$p_4 = 2\pi n_4/L. \tag{5.3.4}$$

Because of the periodic boundary conditions, the fermionic momenta in the 4-direction can vanish. on the other hand, this can not be the case for antiperiodic boundary conditions in 1, 2 and 3-directions.

Using the Fourier expansion (5.3.2) of the fermionic field $\psi(x)$, we expand the operator $X(x)$, defined by (5.2.7b) in terms of D_W from (5.2.1), in the following way:

$$\begin{aligned} X(x)\,\psi(x) &= X\,\frac{1}{L}\sum_n \psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L} \\ &= \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\,X^{(n)}(x)\,\psi_n(x^1,x^2,x^3)\,, \end{aligned} \tag{5.3.5}$$

with

$$X^{(n)} \equiv \cos(2\pi n/N) - a\,\mathbb{D}_W - i\gamma_4\,\sin(2\pi n/N)\,, \tag{5.3.6}$$

and

$$\mathbb{D}_W \equiv \frac{1}{2}\sum_{i=1}^{3}\left[\gamma_i(\nabla_i+\nabla_i^*) + sa\nabla_i\nabla_i^*\right]. \tag{5.3.7}$$

This operator $\mathbb{D}_W$ still contains the standard 4×4 Dirac matrices γ_i.

For the gauge-field configurations (5.1.5), the operator V, defined by (5.2.7), acts on the fermionic field in the following way:

$$\begin{aligned} V\psi(x) &= V\frac{1}{L}\sum_n \psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L} \\ &= \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\int_{-\infty}^{\infty}\frac{dt}{\pi}\,X^{(n)}\left(t^2 + X^{(n)\dagger}X^{(n)}\right)^{-1}\psi_n(x^1,x^2,x^3) \\ &\equiv \frac{1}{L}\sum_n e^{2\pi i n x^4/L}\,V^{(n)}(x)\,\psi_n(x^1,x^2,x^3). \end{aligned} \tag{5.3.8}$$

We now write the fermionic action S_F in terms of the Fourier modes from (5.3.2),

$$\begin{aligned} S_F[\overline{\psi},\psi,U] &= a^4\sum_x \overline{\psi}(x)D[U(x)]\psi(x), \\ &= \frac{1}{L^2}\sum_{m,n} a^4\sum_x \overline{\psi}_m(x^1,x^2,x^3)\,e^{-2\pi i m x^4/L}\,D[U(x)]\,\psi_n(x^1,x^2,x^3)\,e^{2\pi i n x^4/L}, \\ &= \frac{1}{L^2}\sum_{m,n} a^4\sum_x \overline{\psi}_m(x^1,x^2,x^3)\,e^{2\pi i(n-m)x^4/L}\,D^{(n)}[U(x)]\,\psi_n(x^1,x^2,x^3). \end{aligned} \tag{5.3.9}$$

In the last expression of (5.3.9), the quantity $e^{2\pi i n x^4/L}$ is a complex number which commutes with $D^{(n)}[U(x)]$, so that we can rewrite the above equation as follows:

$$S_F[\overline{\psi},\psi,U] \;=\; \frac{1}{L^2}\sum_{n,m} a^4\sum_x$$

$$\left(\overline{\psi}_m(x^1,x^2,x^3)\, e^{-2\pi i m x^4/L}\right) D^{(n)}[U(x)] \left(\psi_n(x^1,x^2,x^3)\, e^{2\pi i n x^4/L}\right). \tag{5.3.10}$$

For each value of m and n, we then redefine our fermion fields as follows:

$$\overline{\psi}_m(x^1,x^2,x^3)\, e^{-2\pi i m x^4/L} \equiv \bar{\phi}'_m(x)\,, \tag{5.3.11a}$$
$$\psi_n(x^1,x^2,x^3)\, e^{2\pi i n x^4/L} \equiv \phi'_n(x)\,, \tag{5.3.11b}$$

and rewrite our lattice fermion action as

$$S_F[\bar{\phi}',\phi',U] = \frac{1}{L^2}\sum_{n,m} a^4 \sum_x \bar{\phi}'_m(x)\, D^{(n)}[U(x)]\, \phi'_n(x)\,, \tag{5.3.12}$$

with the modes of the Ginsparg–Wilson operator $D^{(n)}$ defined by

$$D^{(n)} \equiv \frac{1}{a}\left(\mathbb{1} - V^{(n)}\right)\,, \tag{5.3.13}$$

where $V^{(n)}$ follows from (5.3.8) and (5.3.6).

Redefining the fermionic fields again,

$$\psi'_n(x) \equiv \frac{1}{L}\,\phi'_n(x),\quad \overline{\psi}'_m(x) \equiv \frac{1}{L}\,\bar{\phi}'_m(x)\,, \tag{5.3.14}$$

the final action reads

$$S_F[\overline{\psi}',\psi',U] = \sum_{m,n} a^4 \sum_x \bar{\psi}'_m(x)\, D^{(n)}[U(x)]\, \psi'_n(x)] \equiv \sum_n S_F^{(m,n)}[\overline{\psi}'_m,\psi'_n,U]. \tag{5.3.15}$$

The modes $\overline{\psi}'_m$ and ψ'_n have to satisfy the following constraints:

$$\psi'_n(x) = \widehat{P}_-^{(n)}\psi'_n(x)\,, \tag{5.3.16a}$$
$$\overline{\psi}'_m(x) = \overline{\psi}'_m(x)P_+\,, \tag{5.3.16b}$$

with the usual projection operator P_+ and the modes of the projection operator $\widehat{P}_-$ given by

$$\widehat{P}_-^{(n)} = \frac{1}{2}\left(\mathbb{1} - \gamma_5\, V^{(n)}\right) \equiv \frac{1}{2}\left(\mathbb{1} - \widehat{\gamma}_5^{(n)}\right). \tag{5.3.17}$$

The operators $\widehat{\gamma}_5^{(n)}$ are Hermitian unitary operators. For each n, the operator $V^{(n)}$ is unitary and satisfies

$$V^{(n)\dagger} = \gamma_5 V^{(n)}\gamma_5. \tag{5.3.18}$$

We now expand the Fourier modes of the fermionic fields into the following series:

$$\psi'_n(x) = \sum_j v_j^{(n)}(x)\, c_j^{(n)}\,, \tag{5.3.19a}$$

$$\overline{\psi}'_m(x) = \sum_k \bar{c}_k^{(m)}\, \bar{v}_k^{(m)}(x)\,. \tag{5.3.19b}$$

Here, the $c^{(n)}$ are Grassmann-valued coefficients and the spinor functions $v_j^{(n)}(x)$ and $\bar{v}_k^{(m)}(x)$ form a complete orthogonal basis of complex-valued, antiperiodic spinors, with the following inner products:

$$\left(v_i^{(m)}, v_j^{(n)}\right) \equiv a^4 \sum_x v_i^{(m)\dagger}(x)\, v_j^{(n)}(x) = \delta_{ij}\, \delta_{mn}\,, \tag{5.3.20a}$$

$$\left(\bar{v}_k^{(m)}, \bar{v}_l^{(n)}\right) \equiv a^4 \sum_x \bar{v}_k^{(n)}(x)\, \bar{v}_l^{(m)\dagger}(x) = \delta_{kl}\, \delta_{mn}\,. \tag{5.3.20b}$$

The spinor functions $v_j^{(n)}(x)$ and $\bar{v}_k^{(m)}(x)$ have an x^4-dependence given by, respectively, $e^{2\pi i n x^4/L}$ and $e^{-2\pi i m x^4/L}$, which traces back to the definitions (5.3.11). With these expressions, the effective action for the gauge field can be factorized as follows:

$$\exp\left(-\Gamma[U]\right) = \prod_{m,n} \frac{1}{Z''_{m,n}} \left[\int \prod_k d\bar{c}_k^{(m)} \prod_j dc_j^{(n)} \exp\left(- \sum_{j,k} \bar{c}_k^{(m)} M_{kj}^{(m,n)} c_j^{(n)} \right) \right], \tag{5.3.21}$$

in terms of the matrices

$$M_{kj}^{(m,n)}[U] = a^4 \sum_x \bar{v}_k^{(m)}(x) D^{(n)}[U(x)] v_j^{(n)}(x; U)\,. \tag{5.3.22}$$

The constants $Z''_{m,n}$ in (5.3.21) normalize the integrals, so that $\Gamma[\mathbb{1}] = 0$.

After the Grassmann integrations in (5.3.21), we obtain the following expression for the effective action:

$$\Gamma[U] = -\sum_{m,n} \ln\left(\frac{1}{Z''_{m,n}} \det M_{kj}^{(m,n)}[U] \right). \tag{5.3.23}$$

5.3.2 Change of the effective action under CPT

the difference between the chiral gauge theory of the continuum and lattice is that, the chiral projector (5.2.12a) for the left-handed fermion in lattice chiral gauge theory depends on the link variables, unlike chiral gauge theory in the continuum. This follows from the definition $\widehat{\gamma_5}[U] \equiv \gamma_5\, V[U]$. If the gauge field is CPT transformed, the basis of the chiral fermions v_j changes. This transformation affects the integration measure and the effective action is CPT noninvariant. The details are as follows.

For the link configurations as considered in (5.1.5), the CPT-transformed link variables are given by

$$U_4^\theta = \mathbb{1}\,, \quad U_i^\theta = U_i^\dagger(x - a\,\widehat{i}\,)\,, \tag{5.3.24}$$

for $i = 1, 2, 3$ and with the unit vector $\widehat{i}$ in the i-direction. Let $\mathcal{R}$ be the coordinate-reflection operator of the three coordinates $\vec{x} \equiv (x^1, x^2, x^3)$,

$$\mathcal{R} : \vec{x} \to -\vec{x}\,, \tag{5.3.25}$$

and let $\mathcal{R}^4$ be the coordinate-reflection operator in the fourth direction,

$$\mathcal{R}^4 : (\vec{x},\, x^4) \to (\vec{x},\, -x^4). \tag{5.3.26}$$

The operator $\mathbb{D}_W$, defined by (5.3.7), has then the following behavior under a CPT transformation:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathbb{D}_W[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \mathbb{D}_W[U]. \tag{5.3.27}$$

The Ginsparg-Wilson-operator modes $D^{(m)}$ from (5.3.9) transform as follows:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, D^{(n)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = D^{(-n)}[U]. \tag{5.3.28}$$

The matrices $M_{k,j}^{(m,n)}[U]$, defined by (5.3.22), now change as follows under the CPT transformation $U \to U^\theta$:

$$\begin{aligned}
M_{k,j}^{(m,n)}[U^\theta] &= a^4 \sum_x \bar{v}_k^{(m)}(x)\, D^{(n)}[U^\theta(x)]\, v_j^{(n)}(x; U) \\
&= a^4 \sum_x \bar{v}_k^{(m)}(x)\, \mathcal{R}\mathcal{R}^4\gamma_5 D^{(-n)}[U(x)]\gamma_5\mathcal{R}^4\mathcal{R}\, v_j^{(n)}(x; U^\theta) \\
&= \sum_{l,i} (\bar{Q}_\theta^{(-m)})_{kl} \left(a^4 \sum_x \bar{v}_l^{(-m)}(x)\, D^{(-n)}[U(x)]\, v_i^{(-n)}(x; U) \right) (Q_\theta^{(-n)})_{ij} \\
&= \sum_{l,i} (\bar{Q}_\theta^{(-m)})_{kl}\, M_{li}^{(-m,-n)}[U]\, (Q_\theta^{(-n)})_{ij}\, .
\end{aligned} \tag{5.3.29}$$

Here, the unitary matrices

$$(Q_\theta^{(-n)})_{ij} = a^4 \sum_x v_j^{(-n)\dagger}(\vec{x}; U)\, \gamma_5\mathcal{R}^4\mathcal{R}\, v_j^{(n)}(x; U^\theta)\, , \tag{5.3.30a}$$

$$(\bar{Q}_\theta^{(-m)})_{kl} = a^4 \sum_x \bar{v}_k^{(m)}(x)\, \mathcal{R}\mathcal{R}^4\gamma_5\, \bar{v}_l^{(-m)}(x)\, , \tag{5.3.30b}$$

are obtained by introducing the projection operator P_+ and making use of the fact that

$$\gamma_5\, D^{(n)} = D^{(n)}\widehat{\gamma}_5^{(n)}\, . \tag{5.3.31}$$

With the completeness of the bases $v_j^{(n)}$ and $\bar{v}_k^{(m)}$, the summation kernels of the projection operators $\widehat{P}_-^{(n)}$ and P_+ are

$$\widehat{P}_-^{(n)}(x, y) = \sum_i v_i^{(n)}(x; U)\, v_i^{(n)\dagger}(y; U) \tag{5.3.32a}$$

and

$$P_+ \frac{1}{a^4}\delta_{xy} = \sum_l \bar{v}_l^{(m)\dagger}(x)\, \bar{v}_l^{(m)}(y). \tag{5.3.32b}$$

The transformation (5.3.29) can be absorbed by a redefinition of the fermionic variables in the multiple integral (5.3.21), but the integration measure picks up a Jacobian factor. Under a CPT transformation, the effective gauge-field action changes to

$$\Gamma[U^\theta] = \Gamma[U] - \sum_{n,m'} \ln\det\left(\sum_l \left(Q_\theta^{(-n)}[U]\right)_{kl}\left(\bar{Q}_\theta^{(-m')}\right)_{lm}\right). \tag{5.3.33}$$

The determinants of the transformation matrices $Q_\theta^{(-n)}$ depend on the link variable $U_i(x)$, which opens up the possibility that the effective action is CPT noninvariant.

5.4 CPT anomaly

In this subsection, we discuss the change of the effective gauge-field action under a CPT transformation. But, in order to calculate the explicit expression for the CPT-violating term, we need to know the explicit form of the bases $v_j^{(n)}$ and $\bar{v}_j^{(m)}$.

5.4.1 Basis spinors

The basis spinors for the antifermions are given by

$$\bar{v}_j^{(m)}(x) = (\bar{\xi}_k^{(m)}(x), 0)\,, \tag{5.4.1}$$

where $\bar{\xi}_k^{(m)}(x)$ form an orthonormal basis of two-spinors in four spacetime dimensions with the explicit x^4-dependence $e^{-2\pi i m x^4/L}$.

The basis vectors $v_j^{(n)}(x;U)$ are more difficult to obtain. We have to find unitary operators $\mathcal{U}^{(n)}$ with the property

$$\mathcal{U}^{(n)\dagger}\,\widehat{\gamma}_5^{(n)}\,\mathcal{U}^{(n)} = \gamma_5, \tag{5.4.2}$$

for

$$\widehat{\gamma}_5^{(n)} \equiv H^{(n)}\left(H^{(n)2}\right)^{-1/2}. \tag{5.4.3}$$

Here, the Hermitian operators $H^{(n)}$ are given by

$$H^{(n)} \equiv \gamma_5\left(\overset{\star}{n} - a\,\mathbb{D}_W - i\gamma_4\,\mathring{n}\right) = \begin{pmatrix} \overset{\star}{n} + \frac{1}{2}\sum_{i=1}^3 w_i[U] & \mathring{n} - \frac{1}{2}\sum_{i=1}^3 \sigma_i\, t_i[U] \\ \mathring{n} + \frac{1}{2}\sum_{i=1}^3 \sigma_i\, t_i[U] & -(\overset{\star}{n} + \frac{1}{2}\sum_{i=1}^3 w_i[U]) \end{pmatrix}, \tag{5.4.4a}$$

with

$$\mathring{n} \equiv \sin(2\pi n/N), \quad \overset{\star}{n} \equiv \cos(2\pi n/N) \tag{5.4.4b}$$

$$t_i[U] \equiv a\,(\nabla_i + \nabla^*{}_i), \quad w_i[U] \equiv a^2\,\nabla_i\nabla^*{}_i\,, \tag{5.4.4c}$$

The four-component basis spinors are then constructed as

$$v_j^{(n)}(x) = \mathcal{U}^{(n)}[U]\,\chi_j^{(n)}(x) \tag{5.4.5a}$$

with

$$\chi_j(x) = \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix}. \tag{5.4.5b}$$

where $\xi_j^{(n)}(x)$ form an orthonormal basis of two-spinors in four spacetime dimensions with the explicit x^4-dependence $e^{2\pi i n x^4/L}$.

For the case of an odd number N of links in the x^4 direction (assuming odd $N \geq 3$), we divide our domain of calculation into three subsets: $n < 0$, $n > 0$, and $n = 0$. A particular property of $\widehat{\gamma}_5^{(n)}$,

$$\widehat{\gamma}_5^{(n)}\,\widetilde{\Gamma}_4 = -\widetilde{\Gamma}_4\,\widehat{\gamma}_5^{(-n)}, \tag{5.4.6}$$

with the definition

$$\widetilde{\Gamma}_4 \equiv i\gamma_4\gamma_5\,, \tag{5.4.7}$$

suggests to impose the following condition:

$$\mathcal{U}^{(-n)}[U] = \widetilde{\Gamma}_4\,\mathcal{U}^{(n)}[U]\,\widetilde{\Gamma}_4\,, \tag{5.4.8}$$

where the link variable U on both sides of this last equation refers to the same configuration.

5.4.2 Fixing the phases

We now obtain the required diagonalization operators for (5.4.2), first for nonzero n and then for $n = 0$.

In the $n \neq 0$ sector, the diagonalization operator $\mathcal{U}^{(n)}$ is of the form (see Appendix B and C)

$$\mathcal{U}^{(n)} = \frac{1}{2}\begin{pmatrix} \mathbb{1} + W^{(n)} & \mathbb{1} - W^{(n)} \\ \mathbb{1} - W^{(n)} & \mathbb{1} + W^{(n)} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathbb{1} + Y^{(n)} & i(\mathbb{1} - Y^{(n)\dagger}) \\ i(\mathbb{1} - Y^{(n)}) & \mathbb{1} + Y^{(n)\dagger} \end{pmatrix} \begin{pmatrix} Q_1^{(n)} & 0 \\ 0 & Q_1^{(-n)} \end{pmatrix}, \tag{5.4.9}$$

with the unitary operators

$$W^{(n)} \equiv \left(\overset{\star}{n} - a\,D_W^{3D}\right)\left[\left(\overset{\star}{n} - a\,D_W^{3D}\right)^\dagger \left(\overset{\star}{n} - a\,D_W^{3D}\right)\right]^{-1/2}, \tag{5.4.10a}$$

$$Y^{(n)} \equiv \left[\left(\overset{\star}{n} - a\,D_W^{3D}\right)^\dagger W^{(n)} + i\mathring{n}\right]\left[\left(\overset{\star}{n} - a\,D_W^{3D}\right)^\dagger \left(\overset{\star}{n} - a\,D_W^{3D}\right) + \mathring{n}^2\right]^{-1/2}, \tag{5.4.10b}$$

and

$$D_W^{3D} \equiv \frac{1}{2}\sum_{i=1}^{3}\left(\sigma_i(\nabla_i + \nabla_i^*) + sa\nabla_i^*\nabla_i\right). \tag{5.4.11}$$

One possible choice of $Q_1^{(n)}$ is

$$Q_1^{(n)}[U] = \begin{cases} \mathbb{1}\,, & \text{for } n > 0\,, \\ W^{(n)}[U]^\dagger\,, & \text{for } n < 0\,. \end{cases} \tag{5.4.12}$$

A change of n to $-n$ gives

$$W^{(-n)} = W^{(n)}, \quad Y^{(-n)} = Y^{(n)\dagger}\,. \tag{5.4.13}$$

In the $n = 0$ sector, the diagonalization operator $\mathcal{U}^{(n)}$ is of the form

$$\mathcal{U}^{(0)} = \frac{1}{2}\begin{pmatrix} \mathbb{1} + W^{(0)^\dagger} & \mathbb{1} - W^{(0)} \\ -\mathbb{1} + W^{(0)^\dagger} & \mathbb{1} + W^{(0)} \end{pmatrix}, \tag{5.4.14}$$

with $W^{(0)}$ defined by (5.4.10a) for $n = 0$. As discussed in App. B of Ref. [15], other possible choices for $\mathcal{U}^{(0)}$ are characterized by an integer $k^{(0)} \in \mathbb{Z}$ and give an additional factor $(2\,k^{(0)} + 1)$ in the final result (5.5.21).

5.4.3 CPT anomaly for odd $N \geq 3$

The diagonalization operators $\mathcal{U}^{(n)}[U]$ are given by (5.4.9) and (5.4.14) and the CPT-violating factor can be calculated as follows.

The operator D_W^{3D} from (5.4.11) transforms under CPT as

$$D_W^{3D}[U^\theta] = \mathcal{R}\mathcal{R}^4\, D_W^{3D}[U]^\dagger\, \mathcal{R}^4\mathcal{R}. \tag{5.4.15}$$

The operators $W^{(n)}$ and $Y^{(n)}$ transform under CPT as follows:

$$W^{(n)}[U^\theta] \;=\; \mathcal{R}\mathcal{R}^4\, W^{(n)^\dagger}[U]\, \mathcal{R}^4\mathcal{R}\,, \tag{5.4.16a}$$

$$Y^{(n)}[U^\theta] \;=\; \mathcal{R}\mathcal{R}^4\, W^{(n)}[U]\, Y^{(n)}[U]\, W^{(n)^\dagger}[U]\, \mathcal{R}^4\mathcal{R}\,. \tag{5.4.16b}$$

With the help of (5.4.16a) and (5.4.16b), we calculate the changes of the diagonalization operators $\mathcal{U}^{(n)}$ under a CPT transformation for $n < 0$, $n > 0$, and $n = 0$. The results are for $n < 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(n)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(n)}[U]\widetilde{\Gamma}_4 \begin{pmatrix} Y^{(n)} & 0 \\ 0 & W^{(n)}Y^{(n)^\dagger}W^{(n)^\dagger} \end{pmatrix}, \tag{5.4.17a}$$

for $n > 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(n)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(n)}[U]\widetilde{\Gamma}_4 \begin{pmatrix} W^{(n)}Y^{(n)}W^{(n)^\dagger} & 0 \\ 0 & Y^{(n)^\dagger} \end{pmatrix}, \tag{5.4.17b}$$

and for $n = 0$:

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(0)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \widetilde{\Gamma}_4\mathcal{U}^{(0)}[U]\widetilde{\Gamma}_4. \tag{5.4.17c}$$

The changed transformation matrices are for $n = 0$:

$$\begin{aligned} \left(Q_\theta^{(0)}[U]\right)_{ij} &= a^4 \sum_x \chi_i^{(0)\dagger}(x)\mathcal{U}^{(0)}[U]^\dagger\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(0)}[U^\theta]\, \chi_j^{(0)}(x) \\ &= a^4 \sum_x \chi_i^{(0)\dagger}(x)\mathcal{U}^{(0)}[U]^\dagger\mathcal{U}^{(0)}[U^\theta]\, \mathcal{R}^4\mathcal{R}\gamma_5\, \chi_j^{(0)}(x)\,, \end{aligned} \tag{5.4.18a}$$

for $n > 0$:

$$\left(Q_\theta^{(n)}[U]\right)_{ij} = a^4 \sum_x \left(0, \xi_i^{(n)\dagger}(x)\right) \begin{pmatrix} W^{(n)}Y^{(n)}W^{(n)\dagger} & 0 \\ 0 & Y^{(n)\dagger} \end{pmatrix} \mathcal{R}\mathcal{R}^4 \gamma_5 \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix}, \tag{5.4.18b}$$

and for $n < 0$:

$$\left(Q_\theta^{(n)}[U]\right)_{ij} = a^4 \sum_x \left(0, \xi_i^{(n)\dagger}(x)\right) \begin{pmatrix} Y^{(n)} & 0 \\ 0 & W^{(n)}Y^{(n)\dagger}W^{(n)\dagger} \end{pmatrix} \mathcal{R}\mathcal{R}^4 \gamma_5 \begin{pmatrix} 0 \\ \xi_j^{(n)}(x) \end{pmatrix}. \tag{5.4.18c}$$

We shall later see that the transformation matrices for the $n < 0$ modes and the $n > 0$ modes do not contribute to the final expression of the anomalous term.

The changed transformation matrices $\bar{Q}_\theta^{(m')}[U]$ are the same for all values of the Fourier index m':

$$\left(\bar{Q}_\theta^{(m')}[U]\right)_{kl} = \left(\bar{\xi}_k^{(m')}(x), 0\right) \mathcal{R}\mathcal{R}^4 \gamma_5 \begin{pmatrix} \bar{\xi}_l^{(m')\dagger}(x) \\ 0 \end{pmatrix}. \tag{5.4.19}$$

The required combinations of transformation matrices give for $n = 0$:

$$\left(Q_\theta^{(0)}[U]\right)_{kl} \left(\bar{Q}_\theta^{(m')}[U]\right)_{lm} = -a^4 \sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\, \delta_{m'0}\,, \tag{5.4.20a}$$

for $n > 0$:

$$\sum_l \left(Q_\theta^{(n)}[U]\right)_{kl} \left(\bar{Q}_\theta^{(m')}[U]\right)_{lm} = -a^4 \sum_x \xi_k^{(n)\dagger}(x) \left(W^{(n)}[U]Y^{(n)}[U]W^{(n)}[U]^\dagger\right) \xi_m^{(n)}(x)\, \delta_{m'n}\,, \tag{5.4.20b}$$

and for $n < 0$:

$$\sum_l \left(Q_\theta^{(n)}[U]\right)_{kl} \left(\bar{Q}_\theta^{(m')}[U]\right)_{lm} = -a^4 \sum_x \xi_k^{(n)\dagger}(x) Y^{(n)}[U]^\dagger \xi_m^{(n)}(x)\, \delta_{m'n}\,. \tag{5.4.20c}$$

For the derivation of (5.4.20), we have used

$$\bar{\xi}_k^{(m')} = \xi_k^{(m')\dagger}(x) \tag{5.4.21a}$$

and the completeness relation of the two-spinor basis $\xi_k^{(n)}(x)$,

$$\sum_k \xi_k^{(m')\dagger}(x) \xi_k^{(n)}(y) = a^{-4}\, \mathbb{1}\, \delta_{xy}\, \delta_{m'n}. \tag{5.4.21b}$$

Because $W^{(n)}$ and $Y^{(n)}$ are unitary, the determinant of (5.4.20b) for $n > 0$ is the inverse of the determinant of (5.4.20c) for $n < 0$, where we have used the relations (5.4.13). This gives

$$\prod_{n>0} \prod_{m'} \det\left(\sum_l \left(Q_\theta^{(n)}[U]\right)_{kl} \left(\bar{Q}_\theta^{(m')}[U]\right)_{lm}\right) \det\left(\sum_l \left(Q_\theta^{(-n)}[U]\right)_{kl} \left(\bar{Q}_\theta^{(m')}[U]\right)_{lm}\right) = 1. \tag{5.4.22}$$

We see from (5.4.22) that the anomalous terms arising from positive frequencies ($n > 0$) are cancelled by the terms arising from negative frequencies ($n < 0$), so that only the $n = 0$ term survives. This $n = 0$ term is given by (5.4.20a), which effectively sets $m' = 0$.

To summarize, the change in the effective gauge-field action under a CPT transformation is, for odd N, given by

$$\varDelta\varGamma[U] \equiv \varGamma[U^\theta] - \varGamma[U] = -\ln\det\left(a^4 \sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\right), \tag{5.4.23}$$

with the unitary operator

$$W^{(0)}[U] = \left(\mathbb{1} - aD_W^{3D}[U]\right)\left[\left(\mathbb{1} - aD_W^{3D}[U]\right)^\dagger \left(\mathbb{1} - aD_W^{3D}[U]\right)\right]^{-1/2}. \tag{5.4.24}$$

5.4.4 CPT anomaly for even $N \geq 4$

For even N (equal to or larger than 4), we divide the Fourier modes n into four subsets: $-N/2 < n < 0$, $n = 0$, $0 < n < N/2$, and $n = N/2$. The case $N = 2$, for x^4-independent gauge fields, has already been discussed in Ref. [15].

Equation (5.4.6) is also valid for even N, as long as $n \neq N/2$. In fact, we have, for $n = N/2$,

$$\widetilde{\gamma}_5^{(N/2)}\, \widetilde{\Gamma}_4 = -\widetilde{\Gamma}_4\, \widetilde{\gamma}_5^{(N/2)}. \tag{5.4.25}$$

Hence, the results from Sec. 5.4.3 can be used for $n \neq N/2$. But the $n = N/2$ diagonalization operator has to be investigated separately.

For $n = N/2$, we have

$$\mathcal{U}^{(N/2)} = \frac{1}{2}\begin{pmatrix} \mathbb{1} + W^{(N/2)^\dagger} & \mathbb{1} - W^{(N/2)} \\ -\mathbb{1} + W^{(N/2)^\dagger} & \mathbb{1} + W^{(N/2)} \end{pmatrix}, \tag{5.4.26}$$

where the unitary operator $W^{(N/2)}[U]$ is defined as

$$W^{(N/2)}[U] \equiv -\left(\mathbb{1} + aD_W^{3D}[U]\right)\left[\left(\mathbb{1} + aD_W^{3D}[U]\right)^\dagger\left(\mathbb{1} + aD_W^{3D}[U]\right)\right]^{-1/2}. \tag{5.4.27}$$

The total change in effective gauge-field action under a CPT transformation is, for even N, given by

$$\det\left(\sum_l \left(Q_\theta^{(0)}[U]\right)_{kl}\left(\bar{Q}_\theta^{(0)}[U]\right)_{lm}\right)\det\left(\sum_l \left(Q_\theta^{(N/2)}[U]\right)_{kl}\left(\bar{Q}_\theta^{(N/2)}[U]\right)_{lm}\right) =$$
$$\det\left(a^4\sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\right)\det\left(a^4\sum_x \xi_k^{(N/2)\dagger}(x) W^{(N/2)}[U]^\dagger \xi_m^{(N/2)}(x)\right), \tag{5.4.28}$$

with the unitary operators $W^{(0)}$ and $W^{(N/2)}$ given by, respectively, (5.4.24) and (5.4.27).

The expressions (5.4.23) for odd N and (5.4.28) for even N give the change of the effective gauge-field action under a CPT transformation according to (5.3.33) and are the main results of the nonperturbative lattice calculation. In order to better understand the meaning of these expressions, we consider the continuum limit of them in the next subsection.

5.5 CPT anomaly in the continuum limit

As mentioned in Sec. 5.1, first let us consider an Abelian $U(1)$ gauge field coupled to a single unit-charge chiral fermion. The change in the effective gauge-field action under a CPT transformation for an odd number N of links in the 4-direction depends only on $W^{(0)}[U]$, see (5.4.23). For an even number N of links in the 4-direction, the corresponding change is given by (5.4.28).

Consider an even number N of links in the 4-direction and introduce the following short-hand notations:

$$W^{(-)^\dagger} \equiv W^{(0)^\dagger}, \quad W^{(+)^\dagger} \equiv W^{(N/2)\dagger}, \tag{5.5.1}$$

with

$$
\begin{aligned}
W^{(\pm)\dagger} &= \mp(\mathbb{1} \pm a D_W^{3D})^\dagger \left[(\mathbb{1} \pm a D_W^{3D})(\mathbb{1} \pm a D_W^{3D})^\dagger\right]^{-1/2} \\
&= -(D_W^{3D} \pm 1/a)^\dagger \left[(D_W^{3D} \pm 1/a)(D_W^{3D} \pm 1/a)^\dagger\right]^{-1/2}
\end{aligned}
\tag{5.5.2}
$$

for D_W^{3D} from (5.4.11). The change in the effective gauge-field action is calculated from (5.4.28) as

$$
\triangle\Gamma[U] = i\Big(\mathrm{Im}\{\ln\det(D_W^{3D} - 1/a)\} + \mathrm{Im}\{\ln\det(D_W^{3D} + 1/a)\}\Big) \tag{5.5.3a}
$$

$$
\equiv i\,(\mathrm{Im}\{\ln\det(D - m_+)\} + \mathrm{Im}\{\ln\det(D - m_-)\})\ , \tag{5.5.3b}
$$

where, in (5.5.3b), we have introduced further short-hand notations,

$$
D \equiv D_W^{3D}\,, \quad m_+ \equiv 1/a\,, \quad m_- \equiv -(1/a)\,. \tag{5.5.4}
$$

The first operator in (5.5.3a) is a Wilson–Dirac operator with positive mass $1/a$ where as, the second operator is a Wilson–Dirac operator with negative mass $-1/a$. Because of the antiperiodic boundary conditions in the x^1, x^2, x^3 directions, the masses for these operators are effectively increased by a contribution of order $a/(L')^2$. The values of the positive and negative effective masses are now

$$
m_+^{(\mathrm{eff})} = +1/a + c_+\, a/(L')^2\,, \tag{5.5.5a}
$$

$$
m_-^{(\mathrm{eff})} = -1/a + c_-\, a/(L')^2\,, \tag{5.5.5b}
$$

with positive constants $c_\pm$.

The vacuum-polarization kernel of the effective gauge-field action in three dimensions has been calculated in Ref. [27] to second order in the bare coupling constant e. We adopt a similar approach, in order to calculate the change in our effective action under a CPT transformation.

For this purpose, let us consider an auxiliary theory of a (nonchiral) Dirac fermion field $\Psi(x)$ with the following action over the four-dimensional lattice (5.1.2a):

$$
S_F = -a^4 \sum_x \overline{\Psi}(x)\,[D - m]\,\Psi(x)\,, \tag{5.5.6}
$$

where D is the operator from (5.5.4) and m an arbitrary mass. The corresponding effective gauge-field action $\Gamma[A]$ is given by

$$
\Gamma[A] = \ln\det[D - m]\,. \tag{5.5.7}
$$

The fermion propagator $S(x,y)_{\alpha\beta}$ from (5.5.6) is defined by

$$
[(-D + m)S(x,y)]_{\alpha\beta} = \frac{1}{a^4}\,\delta_{\alpha\beta}\,\delta_{xy}\,. \tag{5.5.8}
$$

In momentum space, we have

$$
S(x,y) = \frac{1}{L}\sum_n \int_{-\pi/a}^{\pi/a} \frac{d^3p}{2\pi^3}\, e^{ip(\vec{x}-\vec{y})}\, e^{2\pi i n(x^4-y^4)/L}\, S_n(\vec{p})
$$

$$= \frac{1}{L}\sum_n \int_{-\pi/a}^{\pi/a} \frac{d^3p}{2\pi^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, e^{2\pi i n(x^4-y^4)/L}\, S(p_n)\,, \tag{5.5.9}$$

with, as before,

$$p_n \equiv (\vec{p}, \rho_n)\,, \quad \rho_n \equiv 2\pi n/L. \tag{5.5.10}$$

Let us make comment on the Fourier transforms in (5.5.9). The momentum steps in the fourth direction and those in the other three directions are, respectively, of order $1/L$ and $1/L'$, with $L' \gg L$. Hence, we have kept in (5.5.9) the summation for the momentum in the fourth direction but used an integral for the momenta in the three other directions.

Next, define a quantity $Q(p_n)$ in such a way that

$$S(p_n) = Q(p_n)^{-1}. \tag{5.5.11}$$

This quantity $Q(p_n)$ is a function of $\widehat{p}_{n_\mu}$ and $\widetilde{p}_{n_\mu}$, which are defined as follows:

$$\widehat{p}_{n_\mu} \equiv \frac{2}{a}\sin\left(\frac{1}{2}\,a\,p_{n_\mu}\right), \quad \widetilde{p}_{n_\mu} \equiv \frac{1}{a}\sin(a\,p_{n_\mu}). \tag{5.5.12}$$

We expand the Dirac operator D in powers of the coupling constant e,

$$D = \sum_k^{\infty} e^k D_k\,, \tag{5.5.13}$$

where, for $k \geq 1$, we have

$$D_k\,\Psi(x) = \frac{(ia)^k}{2ak!}\sum_{i=1}^{3}[A_i(x)^k(s+\gamma_i)\Psi(x+a\widehat{i}) + (-1)^k A_i(x-a\widehat{i})^k(s-\gamma_i)\Psi(x-a\widehat{i})]. \tag{5.5.14}$$

For the effective gauge-field action, we have the following expansion in powers of the fermion charge:

$$\Gamma[A] = \sum_k^{\infty} e^k\,\Gamma_k[A]. \tag{5.5.15}$$

With the Fourier transform of the gauge field A_μ, we write the two-point function as

$$\Gamma_2[A] = -i\,\frac{1}{2}\,\frac{1}{L}\sum_n \int_{-\pi/a}^{\pi/a} \frac{d^3q}{2\pi^3}\, A_i(-q_n)\,\widehat{\pi}_{ij}(q_n)\,A_j(q_n)\,, \tag{5.5.16}$$

where we have included the same prefactor $-i/2$ as in (4.2.6) and where the vacuum polarization tensor $\widehat{\pi}_{ij}(q_n)$ is now given by

$$\begin{aligned}\widehat{\pi}_{ij}(q_n) \;=\;& \frac{1}{2}\,\frac{1}{L}\sum_m \int_{-\pi/a}^{\pi/a} \frac{d^3p}{2\pi^3}[1 - T_0(q_n)] \\ &\times \mathrm{tr}\left\{[Q(p_m+q_n/2)]^{-1}\partial_i Q(p_m)\,[Q(p_m-q_n/2)]^{-1}\partial_j Q(p_m)\right\}. \end{aligned}\tag{5.5.17}$$

The symbol $[1 - T_0(q_n)]$ in the above equation stands for a Taylor subtraction at zero momentum. Just as for the perturbative calculation of Sec. 4.2, the anomalous term originates from the $m = 0$

sector of (5.5.17). We now focus on this $m = 0$ sector [denoted by the superscript '(0)'], but will mention later the contribution of the $m \neq 0$ terms.

In the continuum limit, we can use the three-dimensional result from Ref. [27],

$$\widehat{\pi}_{ij}^{(0)\,(\text{cont.})}(q_n) = \lim_{a\to 0} \widehat{\pi}_{ij}^{(0)}(q_n) = \frac{1}{L}\, A(q_n^2)\, \epsilon_{ijk}\, q_n{}^k + \frac{1}{L}\, B(q_n^2)\, (q_n^2\, \delta_{ij} - q_{n\,i}\, q_{n\,j})\,, \tag{5.5.18a}$$

where the amplitudes $A(q_n{}^2)$ and $B(q_n{}^2)$ are given by

$$A(q_n{}^2) \;=\; \frac{1}{2}\, a_0 + \frac{1}{8\pi} \int_0^1 dt \left\{1 - m\,[m^2 + t(1-t)\, q_n{}^2]^{-1/2}\right\}, \tag{5.5.18b}$$

$$B(q_n{}^2) \;=\; \frac{1}{4\pi} \int_0^1 dt \left\{1 - m\,[m^2 + t(1-t)\, q_n{}^2]^{-1/2}\right\}, \tag{5.5.18c}$$

where 'm' is the mass defined by (5.5.6) and not a Fourier component (for the moment, we have Fourier component $m = 0$). Henceforth, we drop the superscript '(cont.)' of (5.5.18a) and focus on the part with an odd number of momenta, containing the Levi–Civita symbol and the $A(q_n{}^2)$ amplitude. With the Wilson parameter $s = -1$, we have the constant $a_0 = -1/(2\pi)$. In the large negative m limit for a fixed value of $q_n{}^2$, the odd-momentum part of the polarization tensor $\widehat{\pi}_{ij}^{(0)}(q)$ vanishes, whereas, in the large positive m limit for fixed $q_n{}^2$, the odd-momentum part of the polarization tensor becomes

$$\lim_{m\to\infty} \widehat{\pi}_{ij}^{(0)\,(\text{odd-mom})}(q) = \frac{1}{L}\,\frac{a_0}{2}\;\epsilon_{ijk}\; q^k = -\frac{1}{4\pi}\,\frac{1}{L}\;\epsilon_{ijk}\; q^k\,. \tag{5.5.19}$$

As mentioned above, the anomalous contribution (5.5.19) originates from the $m = 0$ Fourier sector of (5.5.17). The $m \neq 0$ Fourier terms of (5.5.17) contribute, in addition to the $m = 0$ result (5.5.19), a further term $\propto (1/a)\, \epsilon_{ijk}\, q^k$, which is L-independent and divergent in the continuum limit $a \to \infty$. Just as discussed in Sec. 4.2, this extra term can be removed by a suitable renormalization procedure.

With the results (5.5.18) and (5.5.19) obtained from the auxiliary theory (5.5.6), we now return to the original chiral gauge theory. The first term in (5.5.3) has a positive mass $m = 1/a$ and the second term has a negative mass $m = -1/a$, so that the second term does not contribute to the anomalous change in the effective gauge-field action. The anomalous change in the effective action follows solely from the first term in (5.5.3) and is determined by (5.5.19). Up till now, we have considered an even number N of links in the 4-direction. For an odd number N of links, the second term in (5.5.3) does not appear and the result is the same as for even N.

Changing from momentum space to configuration space, the first term in (5.5.3) gives, using (5.5.19), the following result up to order e^2 in the effective gauge-field action (5.5.16):

$$e^2\, \Gamma_2^{(\text{odd-mom})}[A] = 2\pi i\, e^2\, \frac{1}{L} \sum_{n_4} a \int_{\mathbb{R}^3} d^3x\; \omega_{\text{CS}}[A(\vec{x}, n_4\, a)], \tag{5.5.20}$$

where the Chern–Simons density ω_{CS} has been defined in (4.3.2). The continuum limit has $a \to 0$ and $N \to \infty$, with constant product $Na = L$. Also, we will now change from a Euclidean metric signature to a Lorentzian metric signature and include all fermions of the chiral gauge theory

(3.0.4), with all of these fermions treated equally on the lattice. The expression (5.5.20) then becomes

$$e^2\,\Gamma_2^{\text{(odd-mom)}}[A] = -F\,e^2\,\frac{2\pi}{L}\int_0^L dx^4\,\int_{\mathbb{R}^3} d^3x\;\omega_{\text{CS}}[A(\vec{x},x^4)]\,, \tag{5.5.21}$$

with an extra factor i for the Lorentzian metric signature and an overall numerical factor F from (3.0.6b) due to the contribution of all chiral fermions of the theory (3.0.4). The above result (5.5.21) agrees with the result (4.3.1) obtained from the perturbative calculation.

6 Discussion

In this chapter, we make some general comments to give a better clarification of our calculations performed in chapters 4 and 5.

- **Origin of CPT violation**

We need an explanation for the origin of CPT violation from an apparently CPT-invariant theory. In the perturbative calculation, the ultraheavy regulator masses M_r in (4.1.12) are responsible for the Lorentz and CPT violation. In order to maintain the gauge invariance in the extended version of the generalized Pauli–Villars regularization we sacrifice the Lorentz and CPT invariance. This is discussed in Sec. 6 of Ref. [14]. In the lattice regularization for the nonperturbative calculation, we observe that the gauge-covariant diagonalization operators (5.4.9) and (5.4.14) are not CPT invariant, as shown by (5.4.17).

For an odd number N of links in the 4-direction, in the equation(5.4.22), we have shown that, the changes of the nonperturbative effective gauge-field action under a CPT transformation arising from $n > 0$ sector are cancelled by the corresponding changes arising from $n < 0$ sector. Whereas, the contribution coming from $n = 0$ sector has no counterpart to cancel its change under a CPT transformation. Specifically, the change of the $n = 0$ diagonalization operator is given by

$$\mathcal{R}\mathcal{R}^4\gamma_5\, \mathcal{U}^{(0)}[U^\theta]\, \gamma_5\mathcal{R}^4\mathcal{R} = \mathcal{U}^{(0)}[U]\begin{pmatrix} W^{(0)^\dagger} & 0 \\ 0 & W^{(0)} \end{pmatrix}, \tag{6.0.1}$$

where $W^{(0)^\dagger}$ acts on left-handed fermions and $W^{(0)}$ acts on right-handed fermions. The CPT transformation leads to another theory with different basis spinors [15]. This different theory can be transformed back to the original one by a redefinition of the spinors. But, then, the integration measure picks up a Jacobian factor and the effective gauge-field action $\varGamma[U]$ changes,

$$\varDelta\varGamma[U] \equiv \varGamma[U^\theta] - \varGamma[U] = -\ln\det\left(a^4\sum_x \xi_k^{(0)\dagger}(x)\left(W^{(0)}[U]\right)\xi_m^{(0)}(x)\right). \tag{6.0.2}$$

For an even number N of links in the 4-direction, the argument is same as for an odd number of links. The changes in the measure coming from $0 < n < N/2$ sector are again cancelled by the corresponding changes coming from $n < 0$ sector. The remaining factors are those for $n = 0$ and $n = N/2$ (see equation (5.4.28)). But the additional factor for $n = N/2$ is a lattice artefact which vanishes in the continuum limit.

A further observation is that the CPT anomaly vanishes for Dirac fermions with both left- and right-handed components, because

$$\ln\det W^{(0)^\dagger} + \ln\det W^{(0)} = 0. \tag{6.0.3}$$

- **Conditions on the background gauge field**

Now, we need to discuss the necessary conditions on the background gauge field in our calculation. If the gauge fields depend upon the compactified coordinate x^4, they should not oscillate too rapidly with respect to the x^4 coordinate.

In the perturbative approach, the Fourier expansion of the gauge field A_μ is given by:

$$A_\mu(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\, e^{2\pi i n x^4/L}\, e^{i\vec{p}\cdot\vec{x}}\, A_\mu(p_n)\,. \tag{6.0.4}$$

The discrete momentum component corresponding to the coordinate x^4 is written as

$$\rho_n = 2\pi n/L, \tag{6.0.5}$$

and the frequency of oscillation of the gauge field A_μ with respect to x^4 coordinate is n/L.

In the generalized Pauli–Villars regularization for the perturbative calculation, the regulator mass scale M must be very much larger than the external momentum component $\rho_n = 2\pi n/L$, as discussed on the lines above (4.2.23). So, the condition on the background gauge fields is given by

$$n \ll M\,L\,, \tag{6.0.6}$$

where n is proportional to the dimensionless oscillation frequency of the gauge field A_μ with respect to x^4 and L is the range of the compactified coordinate x^4.

In the lattice regularization for the nonperturbative calculation, in order to be able to calculate the continuum expressions of Sec. 5.5, the momentum in 4-direction ρ_n of the external gauge fields must be very small compared to the regulator scale $1/a$. The corresponding condition (using $\rho_n \sim n/L$) can be written as:

$$\frac{n}{L} = \frac{n}{Na} \ll \frac{1}{a} < m_+\,, \tag{6.0.7}$$

with m_+ is the effective mass (5.5.5a) for the Wilson–Dirac operator. Note that, this effective mass m_+ is similar to the Pauli–Villars regulator mass scale M of the perturbative approach. Since 'n' controls the frequency of oscillation of A_μ with respect to x^4 coordinate, the above condition, (6.0.7), is similar to condition (6.0.6) for the perturbative case.

- **Comparison with the calculations for x^4-independent background gauge fields**

In this paragraph we make some comparative remarks on our present calculations compared with the earlier calculations, where the background background gauge fields are independent of the x^4-coordinate. In the article [14] the UV divergences in the effective action are regularized by standard Pauli–Villars regularization with a single set of regulator fields and a single regulator mass.

Whereas, in perturbative calculation, we use a generalized Pauli–Villars regularization method with an extra infinite set of Pauli–Villars-type fields ψ_r (with regulator masses $M_r = M\,r^2$) which maintains gauge invariance. Note that, after regularization, we also have to renormalize our perturbative result with suitable renormalization procedure unlike [14].

In the nonperturbative lattice calculation, we have explicitly calculated the diagonalization operators $\mathcal{U}^{(n)}$ and have not used an *ad-hoc* phase fixing, unlike the calculation of Ref. [15].

- **Similarity of our result with previous calculations**

In this paragraph we try to explain heuristically why our new result for x^4-dependent background gauge fields is similar to the previous result for x^4-independent background gauge fields. In the nonperturbative approach, in the equation (5.4.22), we see that the CPT-anomalous terms arising from the positive frequency ($n > 0$) are cancelled by the terms arising from the negative frequency ($n < 0$). The only term survives is corresponding to $n = 0$, which contributes to the CPT violation, which also has $m' = 0$ according to (5.4.20a). In (5.3.33) we see how the Fourier modes n and m' enter in the change of the effective action under CPT transformation. The additional mode m' is absent in [15]. This is the reason why there is similarity of the result for the case of x^4-dependent background gauge fields with the case of x^4-independent gauge fields [14, 15]. If we compare the unitary operator in (5.4.23) in our calculation to (5.35) from Ref. [15], we see that both of them are essentially the same unitary operator. Only difference is the dependence on x^4-dependent gauge fields and a sum over (x^1, x^2, x^3, x^4) instead of (x^1, x^2, x^3) in the determinant.

- **Absence of $\partial_4 A_i$ terms in our result**

It sounds surprising that there is no $\partial_4 A_i$ terms in our final result. In the perturbative approach, we have calculated the effective gauge-field action up to two-point functions (second-order in the gauge field A_μ). In the calculation that the CPT-anomalous terms involving the momentum in the fourth direction vanishes due to boundary condition. We discussed this above (4.2.16), where the ρ_n term corresponds to the configuration-space partial derivative ∂_4. Now, if we consider the non-Abelian gauge theory, the CPT-anomalous terms will involve three-point functions (third-order in the gauge field A_μ). Then, it is possible that the CPT-anomalous terms involving ∂_4 will not vanish by symmetry reasons. For the lattice calculation we have considered only Abelian gauge fields. In continuum limit calculation we have expanded the effective gauge field action only up to the two-point function $\Gamma_2[A]$ (second-order in the coupling constant e). For the non-Abelian case, we expect to have higher-order contributions (notably $\Gamma_3[A]$), which may give rise to terms involving the partial derivative ∂_4 acting on the background gauge field.

- **Generalization of our result**

In the lattice calculation there is freedom while calculating the diagonalization operator $\mathcal{U}^{(n)}[U]$ see (5.2.21). This is discussed in Appendix B. in [15]. If we use the freedom our final result can be generalised to an overall factor $(2k_0 + 1)$ with $k_0 \in \mathbb{Z}$. The same freedom occurs in continuum theory also see Ref. [14] due to existence of different universality classes.

- **The mass scale**

The mass scale of the anomalous CPT violating Chern–Simons–like term is of the order $\frac{1}{L}$, where L is the range of the compactified coordinate. The mass scale of the CPT-violating Chern–Simons–like term for the photon field is of the order of

$$\alpha L^{-1} \sim 10^{-35}\mathrm{eV}\left(\frac{\alpha}{1/137}\right)\left(\frac{1.5\times 10^{10}\mathrm{lyr}}{L}\right) \tag{6.0.8}$$

with

$$\alpha \equiv \frac{e^2}{4\pi} \tag{6.0.9}$$

the fine-structure constant (see eq. (6.1) in the article [14]).

7 Conclusion

The topic of this thesis is to establish the existence of a CPT anomaly for a particular chiral gauge theory (described in chapter 3) defined over a topologically nontrivial spacetime manifold both perturbatively (in chapter 4) and nonperturbatively (in chapter 5) for a background gauge field A_μ which depends on the compactified x^4 coordinate and has a vanishing component A_4. In the continuum limit the result calculated via nonperturbative lattice calculation (5.5.21) agrees with the result obtained from the perturbative approach (4.3.1). The final result can carry an overall odd-integer prefactor $(2k_0 + 1)$ with $k_0 \in \mathbb{Z}$ (see Refs. [14, 15] and Sec. 5.4.2). This mechanism of CPT (and Lorentz) violation reminds us the Casimir effect, where spacetime topology plays an important role.

There are possible consequences of the CPT anomaly. The anomalous origin of the local Chern–Simons-like term (4.3.1b) in the effective gauge-field action motivates us to study the phenomenology of the so-called Maxwell–Chern–Simons (MCS) theory [28]. The photonic sector of the MCS theory contains the standard Maxwell term and the local Chern–Simons-like term. The CPT anomalous Chern–Simons-like term in the effective action affects the propagation of light. The Chern-Simons like term rotates the plane of polarization of radiation from distant galaxies and vacuum becomes optical active [28]. This phenomenon is called birefringence.

Secondly a fundamental time asymmetry could arise from the CPT anomaly of certain chiral gauge theories defined over a topologically nontrivial space manifold. This CPT violating effect can be used to determine a “fundamental arrow-of-time". The effect could play a role in determining the initial conditions of the big bang [29].

In the article [30] the couplings between gravity and the Lorentz- and CPT-violating Standard-Model Extension (SME) have been studied, where the influence of an anomolous Chern–Simons-like photon term (4.3.1b) on gravity is described. The presence of Lorentz-violating curvature couplings in the action eq. (58) in [30] leads us to derive some curvature-dependent modifications to the covariant conservation law.

Appendices

A Notation

A.1 Dirac matrices

The following representation of the γ-matrices has been used

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix}, \quad \mu = 1, 2, 3, 4, \tag{A.1.1}$$

with the three (Hermitian) Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1.2}$$

and the fourth (anti-Hermitian) matrix

$$\sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \tag{A.1.3}$$

The corresponding chirality matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ is diagonal:

$$\gamma_5 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{A.1.4}$$

The generators of the spinor transformation matrices (2.1.13) are:

$$\sigma_{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \sigma_{13} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \sigma_{23} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{A.1.5}$$

and

$$\sigma_{14} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \sigma_{24} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \sigma_{34} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{A.1.6}$$

The matrices C, P and T (in this chapter the x_4- coordinate is the time axis) are up to a phase factor,

$$C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \sigma_4 \\ \sigma_4^\dagger & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \tag{A.1.7}$$

For rest of the calculation the x_1-coordinate is the time axis. Then the matrices C, P and T are

$$C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{A.1.8}$$

A.2 Lattice notation

A.2.1 Inner product

Let f and g be two complex valued functions defined on the d-dimensional lattice. The inner product on the lattice is defined by

$$(f, g) = a^d \sum_x f(x)^* g(x), \tag{A.2.1}$$

where the star denotes complex conjugation. This definition is extended to fermions ψ and ϕ by

$$(\psi, \phi) = a^d \sum_x \sum_\alpha \psi^*_\alpha(x)\phi_\alpha(x). \tag{A.2.2}$$

The index α contains all internal parameters, like the spinor and gauge group indices.

A.2.2 Fourier transformation

The fourier transformation on a d-dimensional finite lattice, with lattice spacing a and the length of the lattice L, is defined as

$$\tilde{f}(p) = a^d \sum_x f(x) \exp(-ipx), \tag{A.2.3}$$

and its inverse

$$f(x) = \frac{1}{(L)^d} \sum_p \tilde{f}(p) \exp(ipx), \tag{A.2.4}$$

Here, the lattice points take the values

$$x_\mu \equiv n_\mu a; \quad n_\mu \in [0, N \equiv L/a], \tag{A.2.5}$$

for $\mu = 1, ..., d$. With periodic boundary conditions, the momentum values are

$$p_\mu \equiv \frac{2\pi}{L} n_\nu, \tag{A.2.6}$$

where the integers n_μ take the values

$$-(N-1)/2 \le n_\mu \le (N-1)/2 \ \text{ for odd} N \equiv L/a, \tag{A.2.7}$$

$$-(N/2) - 1 \le n_\mu \le (N/2) \ \text{ for even} N \equiv L/a. \tag{A.2.8}$$

With antiperiodic boundary conditions, the momentum values are

$$p_\mu \equiv \frac{\pi}{L}(2n_\nu + 1), \tag{A.2.9}$$

where the integers n_ν take the same values as in (A.2.7), or (A.2.8).

A.2.3 Difference operators

The derivatives in the continuum are replaced by difference operators on the lattice. There are two in each direction of the coordinate axes, namely the forward difference operator

$$\partial_\mu \psi(x) = \frac{1}{a}\left(\psi(x + a\hat{\mu}) - \psi(x)\right) \tag{A.2.10}$$

$$\partial_\mu^* \psi(x) = \frac{1}{a}\left(\psi(x) - \psi(x - a\hat{\mu})\right) \tag{A.2.11}$$

The gauge-covariant derivatives of the continuum are replaced by gauge-covariant forward and backward difference operators on the lattice,

$$\nabla_\mu \psi(x) \equiv \frac{1}{a}\Big(R[U_\mu(x)]\psi(x + a\widehat{\mu}) - \psi(x)\Big), \tag{A.2.12a}$$

$$\nabla_\mu^* \psi(x) \equiv \frac{1}{a}\Big(\psi(x) - R[U_\mu(x - a\widehat{\mu})]^{-1}\psi(x - a\widehat{\mu})\Big), \tag{A.2.12b}$$

where R is a unitary representation of the gauge group.

B Diagonalization operator

The diagonalization operators $\mathcal{U}^{(n)}$ have the property

$$\mathcal{U}^{(n)}[U]^\dagger \widehat{\gamma}_5^{(-n)} \mathcal{U}^{(n)}[U] = \gamma_5, \tag{B.0.1}$$

for

$$\widehat{\gamma}_5^{(n)} \equiv H^{(n)} \left(H^{(n)2}\right)^{-1/2} . \tag{B.0.2}$$

Here, the Hermitian operators $H^{(n)}$ are given by

$$H^{(n)} \equiv \gamma_5 \left(\overset{\star}{n} - a\,\mathbb{D}_W - i\gamma_4\,\mathring{n}\right) = \begin{pmatrix} \overset{\star}{n} + \frac{1}{2}\sum_{i=1}^3 w_i[U] & \mathring{n} - \frac{1}{2}\sum_{i=1}^3 \sigma_i\, t_i[U] \\ \mathring{n} + \frac{1}{2}\sum_{i=1}^3 \sigma_i\, t_i[U] & -(\overset{\star}{n} + \frac{1}{2}\sum_{i=1}^3 w_i[U]) \end{pmatrix}, \tag{B.0.3a}$$

with

$$\mathring{n} \equiv \sin(2\pi n/N), \quad \overset{\star}{n} \equiv \cos(2\pi n/N), \tag{B.0.3b}$$

$$t_i[U] \equiv a\,(\nabla_i + \nabla_i^*), \quad w_i[U] \equiv a^2\, \nabla_i \nabla_i^* . \tag{B.0.3c}$$

We achieve the block diagonal form of $H^{(n)}$ through the following steps:

First, we make a similarity transform with

$$\bar{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix}. \tag{B.0.4}$$

This gives

$$\bar{U}^\dagger H^{(n)} \bar{U} = \begin{pmatrix} \mathring{n} & \overset{\star}{n} - aD_W^{3D}[U]^\dagger \\ \overset{\star}{n} - aD_W^{3D}[U] & -\mathring{n} \end{pmatrix}, \tag{B.0.5}$$

with

$$D_W^{3D} \equiv \frac{1}{2} \sum_{i=1}^{3} \left(\sigma_i(\nabla_i + \nabla_i^*) + sa\nabla_i^* \nabla_i\right). \tag{B.0.6}$$

Second, we make the following similarity transformation

$$\bar{\mathcal{U}}_1^{(n)}[U] = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & 0\\ 0 & W^{(n)}[U]\end{pmatrix}, \tag{B.0.7}$$

with

$$W^{(n)} \equiv \left(\overset{\star}{n} - a\,D_W^{3D}\right)\left[\left(\overset{\star}{n} - a\,D_W^{3D}\right)^{\dagger}\left(\overset{\star}{n} - a\,D_W^{3D}\right)\right]^{-1/2} \tag{B.0.8}$$

This gives

$$\bar{\mathcal{U}}_1^{(n)}[U]^{\dagger}\begin{pmatrix}\mathring{n} & \overset{\star}{n} - aD_W^{3D}[U]^{\dagger}\\ \overset{\star}{n} - aD_W^{3D}[U] & -\mathring{n}\end{pmatrix}\bar{\mathcal{U}}_1^{(n)}[U] = \begin{pmatrix}\mathring{n} & B[U]\\ B[U] & -\mathring{n}\end{pmatrix}, \tag{B.0.9}$$

with

$$B[U] = \left[\left(\overset{\star}{n} - a\,D_W^{3D}\right)^{\dagger}\left(\overset{\star}{n} - a\,D_W^{3D}\right)\right]^{\frac{1}{2}}. \tag{B.0.10}$$

Third, we make a similarity transformations with

$$\bar{\mathcal{U}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & -i\\ -i & \mathbb{I}\end{pmatrix}, \tag{B.0.11}$$

this gives

$$\bar{\mathcal{U}}_2^{\dagger}\begin{pmatrix}\mathring{n} & B[U]\\ B[U] & -\mathring{n}\end{pmatrix}\bar{\mathcal{U}}_2 = \begin{pmatrix}0 & B[U] - i\mathring{n}\\ B[U] + i\mathring{n} & 0\end{pmatrix}. \tag{B.0.12}$$

Fourth, make a similarity transform with

$$\bar{\mathcal{U}}_3^{(n)}[U] = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & 0\\ 0 & Y^{(n)}[U]\end{pmatrix}, \tag{B.0.13}$$

where,

$$Y^{(n)}[U] = (B[U] + i\mathring{n})\left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}}. \tag{B.0.14}$$

This gives

$$\bar{\mathcal{U}}_3^{(n)}[U]^{\dagger}\begin{pmatrix}0 & B[U] - i\mathring{n}\\ B[U] + i\mathring{n} & 0\end{pmatrix}\bar{\mathcal{U}}_3^{(n)}[U] = \begin{pmatrix}0 & \left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}}\\ \left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}} & 0\end{pmatrix}. \tag{B.0.15}$$

Fifth, and finally, make a similar transform back to the representation of the Dirac matrices of the beginning

$$\bar{U}\begin{pmatrix}0 & \left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}}\\ \left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}} & 0\end{pmatrix}\bar{U}^{\dagger} = \begin{pmatrix}\left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}} & 0\\ 0 & -\left(B[U]^2 + i\mathring{n}^2\right)^{\frac{1}{2}}\end{pmatrix}, \tag{B.0.16}$$

which is the desired result. The diagonalization operator $\mathcal{U}^{(n)}$ is of the form

$$\mathcal{U}^{(n)} = \bar{U}\bar{\mathcal{U}}_3^{(n)}[U]^\dagger \bar{\mathcal{U}}_2^\dagger \bar{U}^\dagger Q^{(n)}[U], \tag{B.0.17}$$

with the freedom $Q^{(n)}[U]$. We can finally write the diagonalization operator $\mathcal{U}^{(n)}$ as follows:

$$\mathcal{U}^{(n)} = \frac{1}{2}\begin{pmatrix} \mathbb{1} + W^{(n)} & \mathbb{1} - W^{(n)} \\ \mathbb{1} - W^{(n)} & \mathbb{1} + W^{(n)} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathbb{1} + Y^{(n)} & i(\mathbb{1} - Y^{(n)\dagger}) \\ i(\mathbb{1} - Y^{(n)}) & \mathbb{1} + Y^{(n)\dagger} \end{pmatrix} \begin{pmatrix} Q_1^{(n)} & 0 \\ 0 & Q_1^{(-n)} \end{pmatrix}. \tag{B.0.18}$$

C Different Phases

As we discussed before, there is a freedom while calculating the diagonalization operator $\mathcal{U}^{(n)}[U]$. In (5.2.21, 5.4.12), there is a choice made for defining the operator $Q_1^{(n)}[U]$. But we can generalize the scenario by defining

$$Q_1^{(n)}[U] = \begin{cases} \left(W^{(n)}[U]\right)^{k_n}, & \text{for } n > 0\,, \\ \left(W^{(n)}[U]^\dagger\right)^{k_n+1}, & \text{for } n < 0\,. \end{cases} \tag{C.0.1}$$

The integers k_n are independent. When $k_n < 0$ the above operator (C.0.1) can be modified as

$$\left(W^{(n)}[U]\right)^{k_n} \equiv \left(W^{(n)}[U]^\dagger\right)^{|k_n|}, \tag{C.0.2}$$

and

$$\left(W^{(n)}[U]^\dagger\right)^{k_n} \equiv \left(W^{(n)}[U]\right)^{|k_n|}. \tag{C.0.3}$$

For $n = 0$ sector the corresponding diagonalization operator $\mathcal{U}^{(0)}[U]$ can be replaced as

$$\mathcal{U}^{(0)\prime} = \mathcal{U}^{(0)} \begin{pmatrix} \left(W^{(0)}\right)^{k_0} & 0 \\ 0 & \left(W^{(0)^\dagger}\right)^{k_0} \end{pmatrix}. \tag{C.0.4}$$

With this above convention we can rewrite the change of the effective action under a CPT transformation

$$\Delta\Gamma[U] \equiv \Gamma[U^\theta] - \Gamma[U] = -(2k_0 + 1) \ln\det\left(a^4 \sum_x \xi_k^{(0)\dagger}(x) W^{(0)}[U]^\dagger \xi_m^{(0)}(x)\right). \tag{C.0.5}$$

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