The role of two-point functions in geodesy and their classification

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Abstract
In geodesy, two-point functions appear as covariance functions, convolution kernels like the Green functions, transfer functions of the gravity field functionals and filter kernels. Knowledge of their structure both in the spatial and the spectral domains opens vistas not only for understanding their behaviour, but also enabling their design. Here, we develop the two-point functions in terms of spherical harmonic functions and discuss their structure. We identify homogeneity and isotropy as the two key structural properties of the two-point functions that provide a solid basis for their classification.

1 Introduction

A two-point function $b(\theta, \lambda, \theta', \lambda')$ is one which takes two positions as its input, one *a calculation point* (θ, λ), also known as *evaluation point* and the other *a data point* (θ', λ'), also known as *source point*. Geodesy is replete with such functions, for example, the Stokes and Vening Meinesz kernels (Heiskanen and Moritz, 1967), filters (Pellinen, 1966; Jekeli, 1981), covariance functions (Rummel and Schwarz, 1977), Green functions (Farrell, 1972), Meissl scheme (Meissl, 1971), upward and downward continuation operators (Heiskanen and Moritz, 1967). Many of these functions are similar in their mathematical form in that the function values depend only on the distance between the calculation and data points, and such functions are commonly referred to as (*homogeneous*) *isotropic* functions. The spectral structure and utility of other types of two-point functions were explored by Rummel and Schwarz (1977) in the context of covariance functions for collocation. Later, Jekeli (1981) introduced ideas of filtering with the use of isotropic and *anisotropic* two-point *weight* functions. It is only the advent of the Gravity Recovery and Climate Experiment (GRACE) mission and the need for filtering its data has brought to the fore a variety of two-point functions (Han et al., 2005; Swenson and Wahr, 2006; Kusche, 2007; Klees et al., 2008). Given the importance of the GRACE mission for climate research, and the absolute necessity for filtering GRACE data, provided the right impetus to explore the characteristics of two-point functions. In this contribution, we will discuss the structural characteristics of the two-point functions as they directly influence their spherical harmonic spectrum.

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2 Two-point functions

A general two-point function $b(\cdot,\cdot)$ in terms of a spherical harmonic transform pair is given as (e. g., Varshalovich et al. 1988)

$$b(\theta,\lambda,\theta',\lambda') = \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n,k} B_{lm}^{nk}\, Y_{nk}^{*}(\theta',\lambda'), \tag{2.1a}$$

$$B_{lm}^{nk} = \frac{1}{16\pi^2} \iint_{\Omega,\Omega'} b(\theta,\lambda,\theta',\lambda')\, Y_{lm}^{*}(\theta,\lambda) \times Y_{nk}(\theta',\lambda')\, \mathrm{d}\Omega\, \mathrm{d}\Omega'. \tag{2.1b}$$

$$Y_{lm}(\theta,\lambda) = \begin{cases} N_{lm}\, P_{lm}(\cos\theta)\, e^{im\lambda}, & m \geq 0 \\ (-1)^m\, Y_{l,-m}^{*}(\theta,\lambda), & m < 0 \end{cases}$$

$$N_{lm} = (-1)^m \sqrt{(2l+1)\,\frac{(l-m)!}{(l+m)!}}$$

$$\sum_{l,m} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l}$$

$$\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\lambda$$

where $Y_{lm}(\cdot)$ are the 4π-normalized complex surface spherical harmonics of degree l and order m; $(\theta,\lambda) \in \Omega$ and $(\theta',\lambda') \in \Omega'$ are the coordinates of the calculation and data points, respectively; B_{lm}^{nk} are the spherical harmonic coefficients of the two-point function $b(\cdot,\cdot)$; $P_{lm}(\cos\theta)$ are the associated Legendre functions normalized using the factor N_{lm}.

An alternative representation of (2.1a) can be obtained by taking the calculation point as the pole of the sphere Ω' (cf. Figure 2.1). This accounts for a rotation of the coordinate system of the sphere Ω' and thereby allowing for the data points on the rotated sphere to be viewed as points at certain spherical distances and azimuths from the calculation point. The rotation of the coordinate system also corresponds to the rotation of the spherical harmonics, which is accomplished by the use of *Wigner-D functions*, for example (Edmonds, 1960).

$$Y_{nq}(\psi,\pi-A) = \sum_{k} D_{nqk}(\lambda,\theta,0)\, Y_{nk}(\theta',\lambda') \tag{2.2a}$$

$$Y_{nk}(\theta',\lambda') = \sum_{q} D_{nkq}(0,-\theta,-\lambda)\, Y_{nq}(\psi,\pi-A) \tag{2.2b}$$

$$= \sum_{q} D_{nkq}^{*}(0,\theta,\lambda)\, Y_{nq}(\psi,\pi-A) \tag{2.2c}$$

where $D_{nkq}(0,-\theta,-\lambda)$ are the 4π-normalized Wigner-D symbols with the three Euler rotation angles ($\alpha = 0$, $\beta = -\theta$, $\gamma = -\lambda$), ψ is the spherical distance and A is the azimuth between (θ,λ) and (θ',λ'). The Wigner-D symbol is defined as

$$D_{nkq}(\alpha,\beta,\gamma) = e^{-ik\gamma}\, d_{nkq}(\beta)\, e^{-iq\alpha} \tag{2.3}$$

For a complete overview on different normalization conventions and the methods of computation used for the Wigner-D functions, consult (Sneeuw, 1991).

Inserting (2.2b) into (2.1a) gives

$$b(\theta,\lambda,\psi,A) = \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n,k} B_{lm}^{nk} \times \sum_{q} D_{nkq}^{*}(0,-\theta,-\lambda) Y_{nq}^{*}(\psi,\pi-A) \tag{2.4a}$$

$$= \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n,q} Y_{nq}^{*}(\psi,\pi-A) \times \sum_{k} B_{lm}^{nk}\, D_{nkq}^{*}(0,-\theta,-\lambda) \tag{2.4b}$$

Such an expression was already presented to the geodetic community by Rummel and Schwarz (1977),

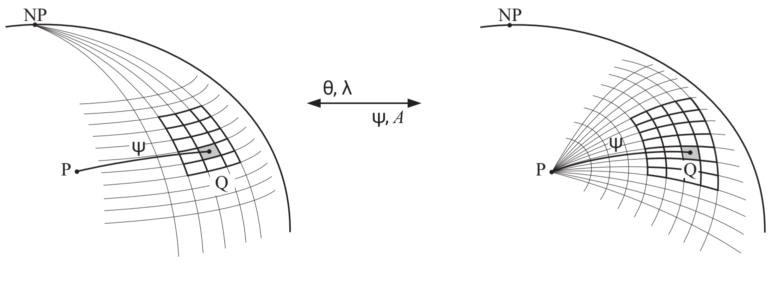


Figure 2.1: Alternative representation of the two-point function.

where they use expression (2.4a) to compute inhomogeneous covariance functions for use in collocation studies. Also, Martinec and Pěč (1985) provide two more expressions for representing the two-point function, which arise by the use of *bipolar spherical harmonics* and *Clebsch-Gordan coefficients* as used in the quantum mechanics and astronomy communities. In this document the expressions and methods of (Rummel and Schwarz, 1977) will be followed.

3 Structural properties of two-point functions

The convenience of the representation shown in (2.4a) is that all the points on the sphere can be referred and/or visualized as points at certain spherical distances and azimuths. This representation also allows an intuitive understanding of the behaviour of the two-point function in terms of the distribution of the function values over the whole sphere.

The values of the two-point function depends on the four arguments – the coordinates of the calculation point and the (spherical) distance and direction (azimuth) of the data point with respect to the calculation point. In the most general case the function value changes with every calculation point given the same values for the ψ and A. In the other extreme, the two-point function only depends on only one argument. For example, the Stokes function depends only on the spherical distance between the calculation and data points, but not on the calculation point and the azimuth.

The independence of the two-point function with respect to its calculation point gives rise to an important property called *homogeneity*. A two-point function is homogeneous if the distribution of the function values over the domain of the data points (ψ, A) remain the same for all the calculation points (θ, λ). For example, a homogeneous covariance function would mean that all the calculation points on the sphere have the same covariance function. Homogeneous functions are also referred to as *translation invariant* functions.

Another important property of the two-point functions comes from the *directional invariance* of the function values. Here, the functions values are independent of the azimuth, and therefore, they depend only on the spherical distance. Thus, they are axially symmetric around the calculation point, and this property is called *isotropy*.

It must be evident from the description of the structural properties that homogeneity/inhomogeneity is a *global property* as it concerns all the calculation points, while isotropy/anisotropy is a *local property* since it concerns the axial symmetry at a given calculation point.

4 Classifying two-point functions

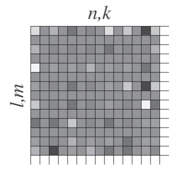
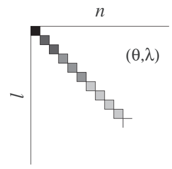
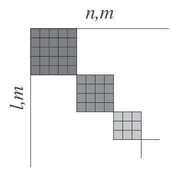
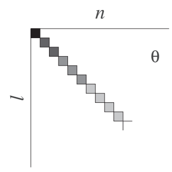
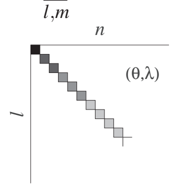
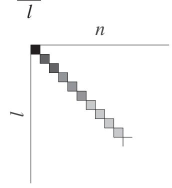
The properties homogeneity and isotropy can be used to classify the two-point functions in terms of their spatial structure. It needs to be ascertained whether such specific spatial structures also correspond to specific spectral structures. One way of identifying the spectral structures is by taking the average of the two-point functions with respect to specific arguments that make them homogeneous and/or isotropic.

$$\text{Homogeneity} \Rightarrow \frac{1}{4\pi} \int_{\Omega} b(\theta, \lambda, \psi, A)\, \mathrm{d}\Omega \;=\; b(\psi, A) \tag{4.1}$$

$$\text{Isotropy} \Rightarrow \frac{1}{2\pi} \int_{A} b(\theta, \lambda, \psi, A)\, \mathrm{d}A \;=\; b(\theta, \lambda, \psi) \tag{4.2}$$

It must be mentioned here that although in the homogeneous case (4.1) there is no explicit reference to the calculation point (θ, λ), it is embedded in the spherical distance (ψ) and the azimuth (A) values. Hence, it is still a two-point function. In the following sections, we will use these two integrals to identify the spectra of the various two-point functions (cf. Table 3.1). Also, we will only show the important results. For a complete discussion and detailed derivations of the different classes of the two-point functions, the reader is referred to (Rummel and Schwarz, 1977; Devaraju, 2015).

Table 3.1: The different classes of two-point functions, their spherical harmonic expansion and spectral structure. Notice that only the anisotropic filters are dependent on the spherical harmonic order, while the isotropic filters are all degree dependent. Also, in the anisotropic case the inhomogeneity is manifest in the structural changes of the spectrum, while in the isotropic case it is manifest in the change of coefficient values depending on the location.

		Anisotropic	Isotropic
Inhomogeneous	Location dependent	$b(\theta,\lambda,\psi,A) = \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n,k} Y^*_{nk}(\theta',\lambda')\, B^{nk}_{lm}$ 	$b(\theta,\lambda,\psi) = \sum_{l} (2l+1)\, P_l(\cos\psi)\, B_l(\theta,\lambda)$ 
	Latitude dependent	$b(\theta,\psi,A) = \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n} B^{nm}_{lm}\, Y_{nm}(\theta',\lambda')$ 	$b(\theta,\psi) = \sum_{l} (2l+1)\, P_l(\cos\psi)\, B_l(\theta)$ 
Homogeneous		$b(\psi,A) = \sum_{l,m} Y^*_{lm}(\psi,\pi-A)\, B_{lm}$ 	$b(\psi) = \sum_{l} (2l+1)\, P_l(\cos\psi)\, B_l$ 

4.1 Homogeneous functions

Convolution (in the classical sense) is a standard operation in signal processing, and homogeneous functions are at the heart of convolution. Convolution can be performed either using an isotropic or an anisotropic kernel. Jekeli (1981) refers to the convolutions with homogeneous isotropic kernels as *convolution of the first kind* and those with homogeneous anisotropic kernels as *convolution of the second kind*. Apart from convolution, homogeneous functions have implications for covariance functions. In the sequel, the general form of the isotropic and anisotropic homogeneous functions will be described and their implications discussed.

Isotropic

The two-point homogeneous isotropic functions on the sphere depend only on the spherical distance ψ between the calculation and the data points. They are the simplest class of two-point functions defined on the sphere, also the most ubiquitous form. Rummel and Schwarz (1977) provide a detailed derivation of the homogeneous isotropic two-point function derived from the general two-point function. Here, only the final formulae of the spherical harmonic transform pair are given.

$$b(\psi) = \frac{1}{8\pi^2} \int_{\Omega} \int_{A} b(\theta,\lambda,\psi,A)\, \mathrm{d}\Omega\, \mathrm{d}A \tag{4.3a}$$

$$= \sum_{l=0}^{\infty} P_l(\cos\psi) \sum_{m=-l}^{l} B^{lm}_{lm} \tag{4.3b}$$

$$= \sum_{l=0}^{\infty} P_l(\cos\psi)\,(2l+1)\, B_l \tag{4.3c}$$

$$B_l = \frac{1}{2} \int_{0}^{\pi} b(\psi) P_l(\cos\psi) \sin\psi \mathrm{d}\psi \tag{4.3d}$$

where $b(\psi)$ is the homogeneous isotropic function, $P_l(\cos\psi)$ are the unnormalized Legendre polynomials of degree l and B_l is the spectrum of $b(\psi)$. One of the most important use of the homogeneous isotropic functions is in the description of the power spectrum of the gravity field as devised by Kaula (1967).

$$C(\psi) = \sum_l \sigma_l^2 P_l(\cos\psi) \tag{4.4a}$$

$$\text{where} \qquad \sigma_l^2 = \sum_m |K_{lm}|^2 \tag{4.4b}$$

in which K_{lm} are the 4π-normalized complex spherical harmonic coefficients of a given gravity field model. Comparing (4.3a) and (4.4a), it is evident that the power spectrum that is routinely computed is indeed a global average of the covariance function. It tells us about the *average* behaviour of the signal covariance of the given gravity field model.

Anisotropic

The two-point homogeneous anisotropic functions depend both on the spherical distance ψ and the azimuth A. The values of the function can be derived by averaging the general two-point function over all the calculation points as follows:

$$\begin{aligned} b(\psi,A) &= \frac{1}{4\pi}\int_\Omega b(\theta,\lambda,\psi,A)\,\mathrm{d}\Omega \\ &= \sum_{l,m} B_{lm}\,Y_{lm}(\psi,\pi - A) \end{aligned} \tag{4.5}$$

Since co-latitude and longitude are the spherical distance and direction from the pole to any other point on the sphere, the spectrum of the homogeneous anisotropic function resembles that of a function defined on the sphere. This function has very limited (so far) use in geodesy.

4.2 Inhomogeneous functions

Inhomogeneity in its strict sense results in a two-point function that at every calculation point has a unique field of values $b(\cdot,\psi,A)$. In a less restricted sense the two-point function is inhomogeneous only with respect to either the latitude or the longitude. In the following we will discuss only the strict and latitude dependent inhomogeneities.

Isotropic

Inhomogeneous isotropic functions are generated by integrating the general two-point kernel over the azimuth.

$$b(\theta,\lambda,\psi) = \frac{1}{2\pi}\int_0^{2\pi} b(\theta,\lambda,\psi,A)\,\mathrm{d}A \tag{4.6a}$$

$$= \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n,k} B_{lm}^{nk}\, Y_{nk}^*(\theta,\lambda)\,\bar{P}_n(\cos\psi) \tag{4.6b}$$

$$= \sum_n B^n(\theta,\lambda)\,\bar{P}_n(\cos\psi) \tag{4.6c}$$

Equation (4.6c) is the spectrum for a location dependent isotropic function. The inhomogeneity of the two-point function in (4.6a) can be restricted only to the latitude and this results in

$$b(\theta,\psi) = \frac{1}{2\pi}\int_0^{2\pi} b(\theta,\lambda,\psi)\,\mathrm{d}\lambda \tag{4.7a}$$

$$= \sum_{l,m,n} \bar{P}_{lm}(\cos\theta)\, B_{lm}^{nm}\, \bar{P}_{nm}(\cos\theta)\,\bar{P}_n(\cos\psi) \tag{4.7b}$$

$$= \sum_n B^n(\theta)\,\bar{P}_n(\cos\psi) \tag{4.7c}$$

Although such functions, to the best of our knowledge, have not been used in geodesy, they can be employed for describing location dependent Green functions.

Anisotropic

The general two-point function is a completely inhomogeneous and completely anisotropic kernel (i.e., asymmetric). Such a function can be imagined to have a unique field $f(\psi,A)$ defined at each calculation point (θ,λ). Although Rummel and Schwarz (1977) indicate that the most general of such inhomogeneous functions will not be physically meaningful, Klees et al. (2008) describe an optimal filter for the GRACE data that is completely inhomogeneous and anisotropic (and asymmetric). This is an example of the general two-point function. As in the isotropic case, anisotropic two-point functions can also be made only latitude dependent. Again, it is accomplished by averaging the general two-point function over the longitudes of the calculation points.

$$b(\theta,\psi,A) = \frac{1}{2\pi}\int_0^{2\pi} b(\theta,\lambda,\psi,A)\,\mathrm{d}\lambda \tag{4.8a}$$

$$= \sum_{l,m,n} \bar{P}_{lm}(\cos\theta)\, B_{lm}^{nm}\, Y_{nm}^{*}(\theta',\Delta\lambda) \tag{4.8b}$$

$$= \sum_{l,m} Y_{lm}(\theta,\lambda) \sum_{n} B_{lm}^{nm}\, Y_{nm}^{*}(\theta',\lambda') \tag{4.8c}$$

The spectrum of the latitude dependent two-point function (4.8c) has a clear order-leading block-diagonal structure, because it depends on only one order m instead of two m and k (cf. Table 3.1). The latitude dependent anisotropic two-point functions are an important class of functions for satellite gravimetry, since the covariance derived from the satellite data have an order-leading block diagonal structure (Colombo, 1986; Sneeuw, 2000). For this reason, the most effective filters for the noisy GRACE data have block diagonal structures (Han et al., 2005; Swenson and Wahr, 2006; Kusche, 2007) (cf. Figure 4.1).

A peculiarity of the latitude dependent two-point functions is that they are all isotropic at the poles ($\theta = \{0,\pi\}$) due to convergence (cf. Figure 4.1). Mathematically, it means that the spherical harmonics become order independent at the poles, and hence, the coefficients become degree dependent.

$$b(0,\psi,A) = \sum_{l,n,m} Y_{lm}(0,\lambda)\, B_{lm}^{nm}\, Y_{nm}(\theta',\lambda')$$

$$= \sum_{l,n,m} \sqrt{2l+1}\,\delta_{m0}\, e^{im\lambda}\, B_{lm}^{nm}\, Y_{nm}(\theta',\lambda')$$

$$= \sum_{l,n} \sqrt{2l+1}\,\sqrt{2n+1}\, B_{l0}^{n0}\, P_n(\cos\theta')$$

An interesting case develops when the off-diagonal elements of each of the m blocks of the spectrum of the two-point function become zero. Then the spectrum of the two-point function takes the following form:

$$b(\theta,\psi,A) = \sum_{l,m} Y_{lm}(\theta,\lambda)\, B_{lm}^{lm}\, Y_{lm}^{*}(\theta',\lambda') \tag{4.9a}$$

and the area under the function at each calculation point is

$$\int_{\Omega'} b(\theta,\psi,A)\,\mathrm{d}\Omega' = \sum_{l,m} Y_{lm}(\theta,\lambda)\, B_{lm}^{lm} \int_{\Omega'} Y_{lm}^{*}(\theta',\lambda')\,\mathrm{d}\Omega'$$

$$= 4\pi \sum_{l,m} Y_{lm}(\theta,\lambda)\, B_{lm}^{lm}\, \delta_{l0}\,\delta_{m0}$$

$$= 4\pi\, B_{00}^{00} \tag{4.9b}$$

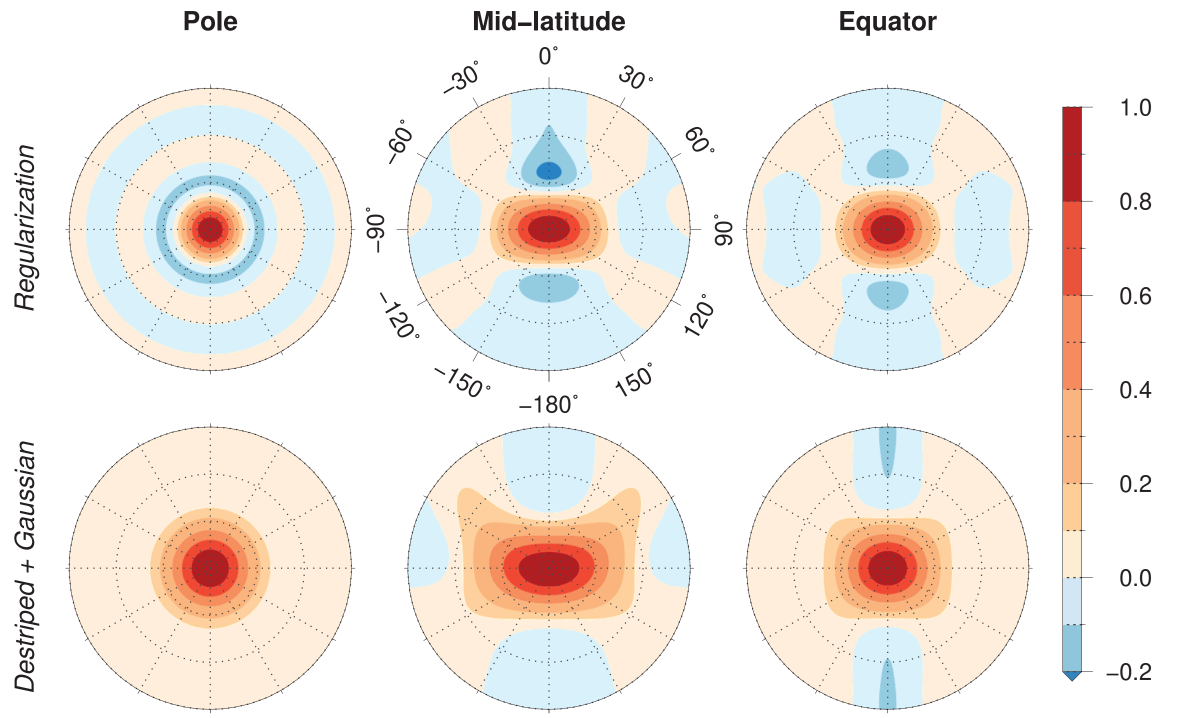


Figure 4.1: Spatial plots of the two of the most commonly used filter kernels in GRACE community – the destriping filter cascaded with a Gaussian filter and the regularization filter – shown here for three different latitudes. Both the filter kernels are anisotropic, but inhomogeneous only in the latitude-direction.

which implies that the area under the two-point function is independent of the latitude. The underlying meaning is that no matter which latitude the function is located, the area under the function must be a constant. This is an important criterion for designing such latitude dependent anisotropic two-point functions.

5 Summary

The two-point function on the sphere is a ubiquitous function in geodesy. It manifests as a transfer function of gravity functionals, as a filter kernel, as a Green function and also as a covariance function. Here, we identified two structural properties of the two-point functions, namely, homogeneity and isotropy, which allowed us to devise a classification scheme. The classification turned out to be meaningful as each structural class had its unique spectrum. We also indicated two classes that play an important role in geodesy, viz. homogeneous isotropic functions and latitude dependent anisotropic functions. In the classification, we did not, however, explore directional two-point functions $b(\cdot, A)$ and longitude dependent two-point functions $b(\lambda, \cdot)$.

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