The spectral response of Stokes's integral to modification and truncation

Jianliang Huang and Marc Véronneau

Canadian Geodetic Survey, SGB, Natural Resources Canada, Canada
E-Mail: Jianliang.Huang@Canada.ca, Marc.Veronneau@Canada.ca

NRCan Contribution number: 20170302

Abstract

In this paper, we study the spectral response of Stokes's integral, which is determined by its modification and truncation. Two spectrally modified Stokes's kernel functions are selected and compared to the unmodified Stokes's kernel in terms of the spectral transfer coefficient effectiveness. Stokes's integral is truncated at four spherical cap sizes with spherical radii $\psi_0 = 1^\circ, 3^\circ, 6^\circ, 9^\circ$. The results suggest that the unmodified Stokes's integral is spectrally unstable when being arbitrarily truncated, and a modification to Stokes's kernel is required for a smooth geoid model.

1 Introduction

This paper, part of a special commemorative publication, is a tribute in honor to Dr. Bernhard Heck who is retiring at the end of March 2018 after a long and successful career in geodesy at Karlsruhe Institute of Technology in Germany. It addresses the topic on the modification of Stoke's integral. Heck and Grüninger (1987) studied the combined modification of Stokes's integral. Their study is frequently cited, and their idea is still applied in today's studies (e.g. Featherstone et al., 1998; Sjöberg and Shafiei Joud, 2017). This paper starts with the same generalization of the modified Stokes's kernel function as Heck and Grüninger (1987, Eq. (1.3)), but focuses on characterizing the spectral response of Stokes's kernel and its two spectral modifications.

The goal of the Stokes's kernel modification is to minimize the geoid error. There have been several papers and reports on this topic. Jekeli (1980) provided a comprehensive study of the modifications by Molodenskii et al. (1962), Wong and Gore (1969), and Meissl (1971a,b), etc. in terms of the RMS error. These classical modifications are deterministic in principle and provide basis for further improvement. Vaníček and Kleusberg (1987) re-formulated Molodenskii's modification. Heck and Grüninger (1987) proposed the combined Wong and Gore and Meissl modification, and examined four types of errors. Featherstone et al. (1998) formulated the combined Vaníček and Kleusberg and Meissl modification. Huang and Véronneau (2013) improved Wong and Gore's modification by introducing a transition low-degree band. Considering errors in gravity data and the combination of satellite gravity models and terrestrial gravity data, Wenzel (1982) and Sjöberg (1984) suggested stochastic modifications based on the least-squares principle. However these stochastic modifications require error degree variances for terrestrial gravity data, which are often approximated by the error variance model.

Data obtained from the dedicated satellite gravimetric missions (CHAMP, GRACE, GOCE) have contributed

Erschienen bei KIT Scientific Publishing
Schriftenreihe des Studiengangs
Geodäsie und Geoinformatik 2018,1

DOI Einzelbeitrag:
10.5445/KSP/1000080220

Festschrift zur Verabschiedung von
Prof. Dr.-Ing. Dr. h.c. Bernhard Heck
(Schw)Ehre, wem (Schw)Ehre gebührt

DOI Festschrift:
10.5445/KSP/1000080324

significantly to determining the long wavelength components of the geoid model ($>$ 200 km). It is critical to use the optimum technique for combining satellite and terrestrial gravity data. For regional geoid modelling, the combination is commonly realized by the remove-compute-restore Stokes scheme and the modification to the Stokes kernel. In particular, the Stokes's integration is only carried out regionally within a limited spherical cap around the computational point. The choice of cap size and modification method is mostly empirical or largely based on numerical test and search for the best fit between the resulting geoid model and external validation data such as GNSS-Levelling data on benchmarks. There is a lack of understanding on the spectral response of Stokes's integral to the modification and truncation supporting that choice. In the context of this study, the spectral response is characterized by a set of spherical harmonic degree-dependent transfer coefficients for the corresponding components of the terrestrial gravity data, which will be defined in Section 2.

Vaníček and Featherstone (1998) suggested the spherical harmonic representation of the truncated Stokes's integral, which is useful for studying the spectral response of Stokes's integral to the modification and truncation. They also spectrally compared Stokes's, Wong and Gore's, and Vaníček and Kleusberg's kernels for a fixed truncation cap size.

In this study, we explore the spectral stability of the unmodified and modified Stokes's integrals and how the integration cap size affects the spectral response.

Section 2 gives mathematical formulae. Section 3 provides and discusses numerical results of the spectral response. Section 4 summaries this paper.

2 Mathematical formulae

The kernel function of Stokes's integral (hereafter the Stokes kernel in short form) can be written as (Heiskanen and Moritz, 1967)

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi) \tag{2.1}$$

where P_n is Legendre's polynomial of spherical harmonic degree n; ψ is the angular distance between the computational point and an integration surface element on a sphere.

The modified Stokes kernel can be generalized as (Wenzel, 1982; Heck and Grüninger, 1987; Huang and Véronneau, 2013, Appendix A)

$$S_M(\psi) = \sum_{n=0}^{\infty} \alpha_n(\psi_0) \frac{2n+1}{n-1} P_n(\cos\psi) \tag{2.2}$$

where α_n is the spherical harmonic transfer coefficient of degree n.

In the remove-compute-restore (RCR) Stokes scheme, the gravity anomaly synthesized from a selected global geopotential model (GGM) is first removed from the terrestrial gravity anomaly giving the gravity anomaly residual δg. Then the geoid residual δN is computed from the gravity residual by the Stokes integration over a truncated zone which is often defined as a spherical cap centered at the computational point. Finally, the geoid height synthesized from GGM N_{GGM} is restored. The RCR Stokes scheme can be mathematically expressed as

$$N(\Omega) = N_{GGM}(\Omega) + \delta N(\Omega) \tag{2.3}$$

$$\delta N(\Omega) = \frac{R}{4\pi\gamma} \int_{(\Omega_0')} S_M(\psi) \delta g(\Omega') d\Omega' \tag{2.4}$$

where γ is the normal gravity. Ω_0' stands for the truncation zone. Following Vaníček and Featherstone (1998, Eq. (11)), the modified and truncated Stokes's integral in Equation (2.4) can be generally expressed in a spherical harmonic series as

$$\delta N(\Omega) = \sum_{n=0}^{\infty} \beta_n \delta N_n \tag{2.5}$$

where δN_n is the geoid residual component of degree n. β_n is the corresponding effective spherical harmonic transfer coefficient which can be given by

$$\beta_n^M(\psi) = \alpha_n^M(\psi_0) - \frac{n-1}{2} Q_n^M(\psi_0)\,. \tag{2.6}$$

Q_n^M is called the truncation coefficient (Molodenskii et al., 1962; Heiskanen and Moritz, 1967):

$$Q_n^M(\psi_0) = \int_{\psi_0}^{\pi} S_M(\psi) P_n(\cos\psi) \cos\psi d\psi\,. \tag{2.7}$$

In this study, the truncation coefficients for the Stokes and VK kernels are computed by a FORTRAN program by Martinec (1996).

Equations (2.6) and (2.7) give the transformation between α_n and β_n. The coefficient α_n is derived by either/both minimizing the truncation error, or/and making the spectral combination of GGM and terrestrial gravity data; while the coefficient β_n represents weight which is effectively applied to the corresponding component of gravity anomaly residual. Therefore the kernel modification and truncation to Stokes‘s integral jointly determine the combination method of GGM and the terrestrial gravity data.

In this study, we select the Stokes kernel and two deterministic modifications to the kernel to characterize their spectral response when the Stokes’s integral is truncated to the spherical cap with a radius ψ_0. For the Stokes kernel $\alpha_n = 1$, i.e. the transfer coefficient has the full weight across the whole spectrum.

For the degree-banded (DB) Stokes kernel, the transfer coefficient is defined as (Huang and Véronneau, 2005)

$$\alpha_n^{DB} = \begin{cases} 0 & n < L+1 \\ 1 & L < n < m_{TG}+1 \\ 0 & n > m_{TG} \end{cases} \tag{2.8}$$

where L represents the modification degree; m_{TG} is the maximum degree of the DB kernel.

For Vaníček and Kleusberg (1987) (VK’s) modification, the transfer coefficient can be written as

$$\alpha_n^{VK}(\psi_0) = \begin{cases} -\frac{n-1}{2} t_n(\psi_0) & n < L+1 \\ 1 & n > L \end{cases} \tag{2.9}$$

where t_n is VK’s modified kernel coefficient of degree n.

3 Numerical examples

Figure 3.1 shows the transfer coefficients α_n for the three kernels. They represent weights on the spherical harmonic components of the gravity anomaly residual if $\psi_0 = 180°$. Differences among the three kernels are in the low degree band from degree 2 to L. The Stokes kernel has a constant weight of 1, while the DB kernel defines them as 0. The VK kernel becomes mathematically undefined in this case.

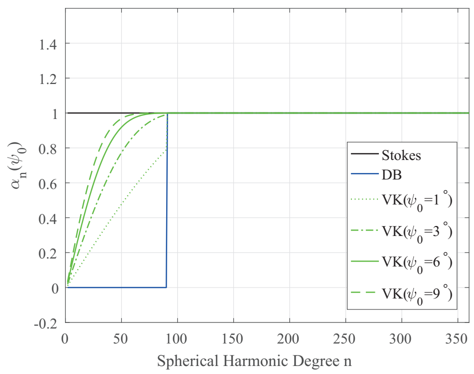

Figure 3.1: The spherical harmonic transfer coefficients of the three Stokes kernels. For the DB kernel, $L = 90$; $m_{TG} = 5400$. For the VK kernel, the modification degree $L = 90$ with the four different cap radii.

Figures 3.2-3.5 show the effective transfer coefficients β_n when Stokes‘s integral is limited to the spherical cap size defined by ψ_0. These coefficients are the most unstable for the Stokes kernel. They distort an individual geoid component by more than 50% at the maximum, even though the sum of the distortions tends to be much smaller due to the cancellation by the oscillation of coefficients with respect to the unit weight. The increase of cap size does not lower the amplitude of the distortion per degree when enhancing the frequency of oscillation with respect to degree. The sum of distortions is equal to the truncation error with an opposite sign. The use of the RCR scheme can significantly reduce the truncation error when an accurate and high-degree GGM is used in the remove step making the magnitude of gravity anomaly residual smaller. Nevertheless, the instability of these coefficients may render a ringing distortion in the resulting geoid model that is dependent of the complexity of gravity field. A spatial modification to the Stokes kernel is required to eliminate the distortion by smoothing the transition of the kernel to zero at the cap edge, and the truncation error is accordingly derived (Meissl, 1971a,b).

The effective transfer coefficients β_n for the DK kernel are relatively more stable than those for the Stokes kernel, but can still introduce 10% distortion per degree at the maximum. Similar to the Stokes kernel, the increase of cap size does not significantly lower the amplitude of distortion when enhancing the frequency of oscillation. However the difference is that the distortion consists of two parts. One is the sum of distortion above degree L which is equal to the truncation error with an opposite sign. The other is the distortion below degree $L+1$ which is considered as the low-degree spectral leakage error. The latter is caused by the spectral discontinuity of the DB kernel from degree L to $L+1$. The truncation error can be minimized by Meissl's modification (Heck and Grüninger, 1987) while the leakage error can be stabilized by introducing a spectrally smooth transition of the transfer coefficients from degree $L+1$ to a lower degree (Huang and Véronneau, 2013). Furthermore, the increase of the cap size fades the low-degree leakage error making the DB kernel approximate the high-pass filter function more closely.

The VK kernel is designed to minimize the truncation error. As expected, its effective transfer coefficients β_n are the most stable among the three kernels. On one hand, these coefficients cause the least distortion above degree L, consequently the smallest truncation error. With the increase of cap size, these coefficients approach to the desired unit value reducing the truncation error to a few millimeters. Furthermore, these coefficients show the most stable transition below degree $L+1$ indicating the smoothest combination of GGM and terrestrial gravity data. On the other hand, it introduces greater errors than the DB kernel when the gravity anomaly residual contains the low-degree systematic errors as shown in the North American gravity data (Huang et al., 2008). The increase of cap size leads less modification to the Stokes kernel as shown in Figure 3.1, consequently more contamination from the systematic errors. Considering that the VK kernel is aiming at minimizing the truncation error only, it performs well towards its goal. A further improvement is a Meissl-modified VK kernel which has been formulated by Featherstone et al. (1998).

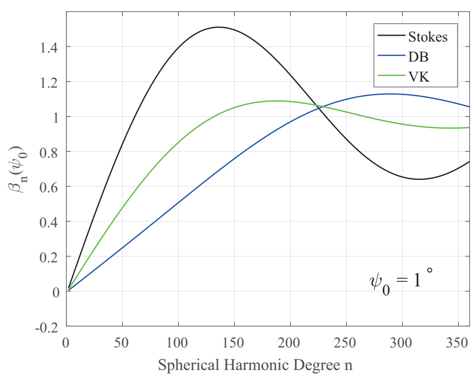

Figure 3.2: The spherical harmonic effective transfer coefficients of the three Stokes kernels with $\psi_0 = 1^\circ$.

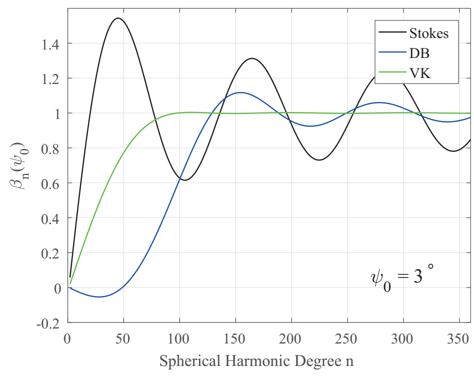

Figure 3.3: Same as Figure 3.2 with $\psi_0 = 3^\circ$.

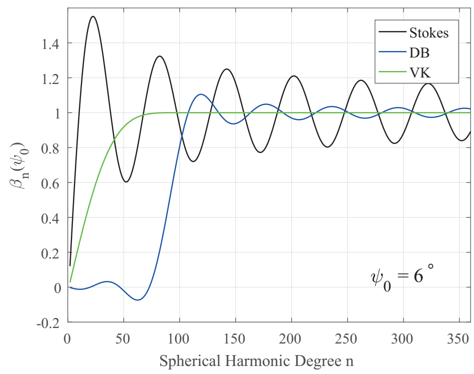

Figure 3.4: Same as Figure 3.2 with $\psi_0 = 6^\circ$.

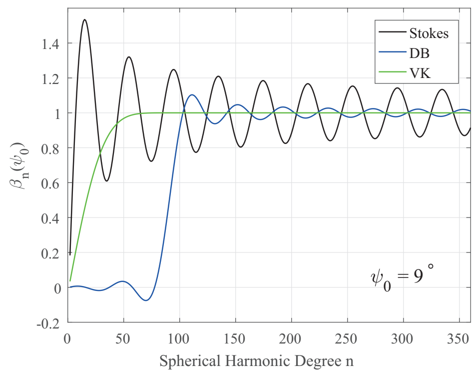

Figure 3.5: Same as Figure 3.2 with $\psi_0 = 9^\circ$.

4 Summary and discussion

This study numerically analyzed the spectral response of Stokes's integral to the modification of its kernel function and the truncation of integration domain. The results suggest that the unmodified Stokes's integral is spectrally unstable when being truncated to a spherical cap. The degree-banded and Vaníček and Kleusberg's modifications are spectrally more stable, therefore more suitable for the geoid modelling. The choice between them depends on the type of dominant error in terrestrial gravity data. The former can filter out most of the low-degree systematic error enabling satellite global geopotential model to constrain the low-degree geoid components but causes the truncation error which is significant enough to be accounted for. The latter can be greatly affected by the systematic error in the gravity data but causes the truncation error at the *mm* level. A potential improvement on the latter is modifying a narrower low band from degree $L-u$ to L. Huang and Véronneau (2013) applied a cosine modification to the narrower band. It will be worthwhile to study if the latter modification can be applied to the narrower band so that the new modification allows an effective high-pass filtering while minimizing the truncation error.

References

Featherstone, W. E., Evans, J. D., and Olliver, J. G. (1998): A Meissl-modified Vaníček and Kleusberg kernel to reduce the truncation error in gravimetric geoid computations. *Journal of Geodesy* 72(3):154–160. DOI: 10.1007/s001900050157.

Heck, B. and Grüninger, W. (1987): Modification of Stokes's integral formula by combining two classical approaches. In: Proceedings of the IAG Symposia, XIXth General Assembly of the International Union of Geodesy and Geophysics, Vancouver/Canada, 10-22 August 1987, Paris 1988, Tome II, pp. 319–337.

Heiskanen, W. A. and Moritz, H. (1967): Physical geodesy. W. H. Freeman & Co., San Francisco, USA.

Huang, J. and Véronneau, M. (2005): Applications of downward-continuation in gravimetric geoid modeling: case studies in Western Canada. *Journal of Geodesy* 79(1):135–145. DOI: 10.1007/s00190-005-0452-3.

Huang, J., Véronneau, M., and Mainville, A. (2008): Assessment of systematic errors in the surface gravity anomalies over North America using the GRACE gravity model. *Geophysical Journal International* 175(1):46–54. DOI: 10.1111/j.1365-246X.2008.03924.x.

Huang, J. and Véronneau, M. (2013): Canadian gravimetric geoid model 2010. *Journal of Geodesy* 87(8):771–790. DOI: 10.1007/s00190-013-0645-0.

Jekeli, C. (1980): Reducing the error of geoid undulation computations by modifying Stokes‘ function. Report 301. Department of Geodetic Science and Surveying, The Ohio State University, Columbus, USA.

Martinec, Z. (1996). *personal communication* 1996.

Meissl, P. (1971a): On the linearization of the geodetic boundary value problem. Report 152. Department of Geodetic Science and Surveying, The Ohio State University, Columbus, USA.

Meissl, P. (1971b): Preparation for the numerical evaluation of second order Molodensky-type formulas. Report 163. Department of Geodetic Science and Surveying, The Ohio State University, Columbus, USA.

Molodenskii, M. S., Eremeev, V. F., and Yuikina, M. I. (1962): Methods for study of the external gravitational field and figure of the earth. *Translated from Russian (1960) by Israel Program for Scientific Translations, Jerusalem* 1962.

Sjöberg, L. E. (1984): Least-squares modification of Stokes's and Vening Meinesz's formula by accounting for truncation and potential coefficient errors. *Manuscripta Geodaetica* 9:229–248.

Sjöberg, L. E. and Shafiei Joud, M. S. (2017): New Modifications of Stokes' Integral. In: Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 1–8. DOI: 10.1007/1345_2017_17.

Vaníček, P. and Featherstone, W. E. (1998): Performance of three types of Stokes's kernel in the combined solution for the geoid. *Journal of Geodesy* 72(12):684–697. DOI: 10.1007/s001900050209.

Vaníček, P. and Kleusberg, A. (1987): The Canadian geoid – Stokesian approach. *Manuscripta Geodaetica* 12:86–98.

Wenzel, H.-G. (1982): Geoid computation by least-squares spectral combination using integration kernels. In: Nakagawa, I., Kasahara, K., and Tanaka, T. (eds.) Proceedings of IAG general meeting, Tokyo, The Geodetic Society of Japan, pp. 438–453.

Wong, L. and Gore, R. (1969): Accuracy of geoid heights from modified Stokes kernels. *Geophysical Journal of the Royal Astronomical Society* 18(1):81–91.