Harmonic downward continuation of gravity anomalies

Petr Vaníček

Department of Geodesy and Geomatics Engineering, University of New Brunswick, Canada
E-Mail: Petr.Vanicek@unb.ca

Abstract

The question we would like to address is: Can any of the gravity anomalies (free-air, Fay, Bouguer, Helmert, etc.), multiplied by the radius r, be continued downward, using the (only rigorous) Poisson approach?

1 The discussion of *downward continuation*

(1) When talking about *downward continuation* of a function $f(r,\Omega)$, we understand that we know the functional value $f(r_t,\Omega)$ at the earth surface, i.e., at $r = r_t$, and want to determine the functional value $f(r_g,\Omega)$ at the geoid i.e., at $r = r_g$. Thus the function $f(r,\Omega)$ is to be *downward continued* from the surface of the earth to the geoid, through the topography, i.e., $\tau \equiv \{r_g \leq r \leq r_t\}$.

(2) The Poisson downward continuation is, of course, formulated under the assumption that the downward continued function is, in the region τ, harmonic. In other words, $f(r,\Omega)$ must satisfy the Laplace equation

$$\nabla^2 f(r,\Omega) = 0\,, \tag{1.1}$$

for $r \in \tau$. For other reasons (existence of solution to the BVP of geodesy) we will require that $f(r,\Omega)$ be harmonic for all $r_g \leq r$, i.e., everywhere above the geoid.

(3) We have several choices as how to go about proving that a gravity anomaly, multiplied by radius r, is harmonic above the geoid. To show these choices, let us recall that any gravity anomaly Δg^* can be expressed as a difference (Vaníček et al., 1999)

$$\forall r,\Omega: \quad \Delta g^*(r,\Omega) = g^*(r,\Omega) - \gamma(r-z,\Omega)\,, \tag{1.2}$$

where g^* stands for the gravity that comes from the model pertaining to the particular gravity anomaly (free-air gravity, Bouguer gravity, Helmert gravity, etc.), γ is the normal gravity and z stands for the displacement of the corresponding equipotential surfaces belonging to the two gravities. As harmonicity is associative (a sum of two harmonic functions is a harmonic function) and $r \cdot \gamma(r-z,\Omega)$ is known to be harmonic above the geoid (by definition of the normal gravity) it remains to be proved that also $r \cdot g^*(r,\Omega)$ is harmonic above the geoid.

(4) Alternatively, denoting the potential that generates g^* by V^* and the potential that generates γ by U, we can also express the anomaly Δg^* as

$$\forall r,\Omega: \quad \Delta g^*(r,\Omega) = |\nabla V^*(r,\Omega)|_{r=r} - |\nabla U(r,\Omega)|_{r=r-z}\,. \tag{1.3}$$

Then the proof of harmonicity of $r \cdot \Delta g^*$ reduces to the proof of harmonicity of V^*.

(5) Clearly, if V^* is generated only by masses inside the geoid, as is the case with Helmert anomaly, or any of the isostatically compensated anomalies, the harmonicity of V^* - and thus that of the corresponding Δg^* multiplied by r – above the geoid is assured.

Erschienen bei KIT Scientific Publishing
Schriftenreihe des Studiengangs
Geodäsie und Geoinformatik 2018,1

DOI Einzelbeitrag:
10.5445/KSP/1000080249

Festschrift zur Verabschiedung von
Prof. Dr.-Ing. Dr. h.c. Bernhard Heck
(Schw)Ehre, wem (Schw)Ehre gebührt

DOI Festschrift:
10.5445/KSP/1000080324

(6) Can the potential V^* for the free-air, Bouguer, or Fay anomalies be found? These anomalies are *defined in the literature only in a two-dimensional sense* as

$$\forall\Omega: \ \Delta g^*(\Omega) = g(r_t,\Omega) - \frac{\partial g^*(r,\Omega)}{\partial r}|_{r=r_g} \cdot H(\Omega) - \gamma(r_e,\Omega)\,, \tag{1.4}$$

where $g(r_t,\Omega)$ is the (observed) gravity on the earth surface, $H(\Omega)$ is the orthometric height of the observed gravity and r_e is the radius of the reference ellipsoid. Can the real gravity $g(r_t,\Omega)$ on the earth surface and the adopted vertical gradient $\partial g^*(r,\Omega)/\partial r|_{r=r_g}$ of g^* be construed as defining either $g^*(r,\Omega)$ or $V^*(r,\Omega)$ in a three-dimensional way?

(7) Another way of approaching the problem is by defining the disturbing potential T^* that corresponds to the specific potential V^* in the following manner:

$$\forall r,\Omega: \ T^*(r,\Omega) = V^*(r,\Omega) - U(r,\Omega)\,. \tag{1.5}$$

Then, applying eqns. (1.2), (1.3) and the Bruns theorem, we get

$$\forall r,\Omega: \ \Delta g^*(r,\Omega) = \frac{\partial T^*(r,\Omega)}{\partial h} + \frac{1}{\gamma} \cdot \frac{\partial \gamma}{\partial h}|_{r=r-z} \cdot T^*(r,\Omega)\,, \tag{1.6}$$

where h is the geodetic height reckoned along the normal to the reference ellipsoid. This equation is often called the *fundamental gravimetric equation*. We note that if $T^*(r,\Omega)$ is harmonic in the desired domain, than

$$\forall r,\Omega: \ \Delta g^{*\prime}(r,\Omega) = \frac{\partial T^*(r,\Omega)}{\partial r} - \frac{2}{r} \cdot T^*(r,\Omega) \tag{1.7}$$

multiplied by r is also harmonic (because harmonicity is associative and because if T^* is harmonic then $r \cdot \partial T^*/\partial r$ is automatically harmonic). It is possible to transform $\Delta g^*(r,\Omega)$ into $\Delta g^{*\prime}(r,\Omega)$ by means of some small correction, as a matter of fact by two smallish ellipsoidal corrections (Vaníček et al., 1999).

(8) But can the disturbing potential $T^*(r,\Omega)$ for the free-air, Bouguer, or Fay anomalies be found? Later on we found out that this cannot be done (Vaníček et al., 2004).

(9) Another way of showing that $r \cdot \Delta g^*(r,\Omega)$ is a harmonic function would be to carry out the Laplacian operation on it and to show that the result is indeed identically equal to 0 in the domain of interest (above the geoid). Can that be done?

Acknowledgements

Dear Bernhard,

When I learnt about your retirement the first thought that occurred to me was: well, he is not that young any more. Then I realised that people do not retire according to the level of wisdom they reach but according to their biological age. Clearly you have reached the level of wisdom that would entitle you to retire at a very young biological age and now your biological age has caught up with you.

In any case I wish you many years of happy retirement when you will be sitting in your rocking chair, twiddling your thumbs and enjoy the company of your grand kids. Just to remind you what we have been worrying about some 20 years ago, when we were both young men, I am enclosing an internal paper, never published of course, that lead eventually to the formulation and publication of our paper "New views of the spherical Bouguer gravity anomaly” in 2004. It is now all an old stuff but sometimes it is good to look at the old stuff to realise what ground we have already covered.

Kindest regards from your old friend

Petr Vaníček

References

Vaníček, P., Huang, J., Novák, P., Véronneau, M., Pagiatakis, S., Martinec, Z., and Featherstone, W. E. (1999): Determination of boundary values for the Stokes-Helmert problem. *Journal of Geodesy* 73:180–192.

Vaníček, P., Tenzer, R., Sjöberg, L. E., Martinec, Z., and Featherstone, W. E. (2004): New views of the spherical Bouguer gravity anomaly. *Geophysical Journal International* 159(2):460–472.