Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

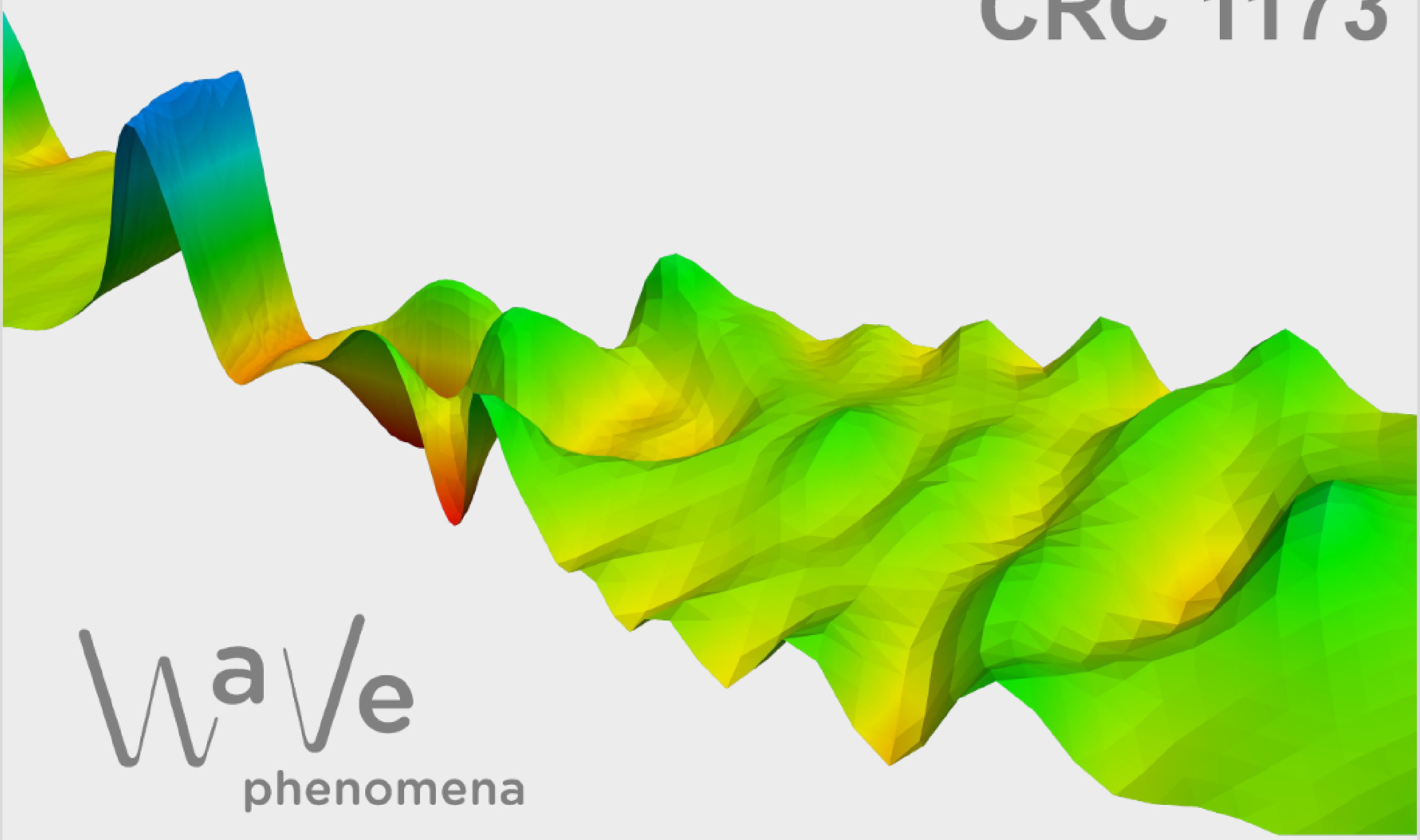
Nonlinear Schrödinger equation, differentiation by parts and modulation spaces

Leonid Chaichenets, Dirk Hundertmark,
Peer Kunstmann, Nikolaos Pattakos

CRC Preprint 2018/1, February 2018

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

Participating universities

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Funded by

DFG

ISSN 2365-662X

NONLINEAR SCHRÖDINGER EQUATION, DIFFERENTIATION BY PARTS AND MODULATION SPACES.

L. CHAICHENETS, D. HUNDERTMARK, P. KUNSTMANN, AND N. PATTAKOS

Abstract. We show the local wellposedness of the Cauchy problem for the cubic nonlinear Schrödinger equation in the modulation space $M^s_{p,q}(\mathbb{R})$ where $1 \leq q < 3$, $2 \leq p < \frac{10q'}{q'+6}$ and $s \geq 0$. This improves [7], where the case $p = 2$ was considered and the differentiation by parts technique was introduced to a problem with continuous Fourier variable. Here the same technique is used, but more delicate estimates are necessary for $p \neq 2$.

1. introduction and main result

We are interested in the nonlinear Schrödinger equation defined by

$$\begin{cases} iu_t - u_{xx} \pm |u|^2 u = 0 & , \ (t, x) \in \mathbb{R}^2 \\ u(0, x) = u_0(x) & , \ x \in \mathbb{R} \end{cases} \tag{1}$$

with initial data u_0 in the modulation space $M_{p,q}(\mathbb{R})$. To state the definition of a modulation space we need to fix some notation. We will denote by $S'(\mathbb{R})$ the space of tempered distributions. Let $Q_0 = [-\frac{1}{2}, \frac{1}{2})$ and its translations $Q_k = Q_0 + k$ for all $k \in \mathbb{Z}$. Consider a family of functions $\{\sigma_k = \sigma_0(\cdot - k)\}_{k \in \mathbb{Z}} \subset C^\infty(\mathbb{R})$ satisfying

(i) $\exists c > 0 : \forall k \in \mathbb{Z} : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,
(ii) $\forall k \in \mathbb{Z} : \operatorname{supp}(\sigma_k) \subseteq \{\xi \in \mathbb{R} : |\xi - k| < 1\} =: B(k, 1)$,
(iii) $\sum_{k \in \mathbb{Z}} \sigma_k = 1$

and define the isometric decomposition operators

$$\Box_k := \mathcal{F}^{(-1)} \sigma_k \mathcal{F}, \quad (\forall k \in \mathbb{Z}) . \tag{2}$$

Then the norm of a tempered distribution $f \in S'(\mathbb{R})$ in the modulation space $M^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, is

$$\|f\|_{M^s_{p,q}} := \Big(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\Box_k f\|_p^q \Big)^{\frac{1}{q}}, \tag{3}$$

with the usual interpretation when the index q is equal to infinity, where we denote by $\langle k \rangle = (1 + |k|^2)^{\frac{1}{2}}$ the Japanese bracket. It can be proved that different choices of the function σ_0 lead to equivalent norms in $M^s_{p,q}(\mathbb{R})$. Later, during the proof of the main

Date: 27th February 2018.

theorem we will make use of this fact. When $s = 0$ we denote the space $M^0_{p,q}(\mathbb{R})$ by $M_{p,q}(\mathbb{R})$. In the special case where $p = q = 2$ we have $M^s_{2,2}(\mathbb{R}) = H^s(\mathbb{R})$ the usual Sobolev spaces. In our calculations we are going to use that for $s > 1/q'$ and $1 \leq p, q \leq \infty$, the embedding

$$M^s_{p,q}(\mathbb{R}) \hookrightarrow C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}/\ f \text{ continuous and bounded}\}, \tag{4}$$

and for $\Big(1 \leq p_1 \leq p_2 \leq \infty,\ 1 \leq q_1 \leq q_2 \leq \infty,\ s_1 \geq s_2\Big)$ or $\Big(1 \leq p_1 \leq p_2 \leq \infty,\ 1 \leq q_2 < q_1 \leq \infty,\ s_1 > s_2 + \frac{1}{q_2} - \frac{1}{q_1}\Big)$ the embedding

$$M^{s_1}_{p_1,q_1}(\mathbb{R}) \hookrightarrow M^{s_2}_{p_2,q_2}(\mathbb{R}), \tag{5}$$

are both continuous and can be found in [3] (Proposition 6.8 and Proposition 6.5). In that paper modulation spaces were introduced for the first time by Feichtinger and since then they have been used extensively in the study of nonlinear dispersive equations. They have become canonical for both time-frequency and phase-space analysis. See [8] for many of their properties such as embeddings in other known function spaces and equivalent expressions for their norm. From [3] (Proposition 6.9) it is known that for $s > 1/q'$ or $s \geq 0$ and $q = 1$ the modulation space $M^s_{p,q}(\mathbb{R})$ is a Banach algebra and therefore an easy Banach contraction principle argument implies that NLS (1) is locally wellposed for $u_0 \in M^s_{p,q}(\mathbb{R})$ with solution $u \in C([0,T]; M^s_{p,q}(\mathbb{R}))$, $T > 0$ (see [2]). In this paper, with a different approach, we are able to cover the remaining cases $0 \leq s \leq 1/q'$, unfortunately not for all values of p, through the differentiation by parts technique that was used in [1] to attack similar problems for the KdV equation but with periodic initial data. In [5] this technique was used to prove unconditional wellposedness of the periodic cubic NLS in one dimension. Our initial data is far from being periodic, and for this reason there are some major differences and some difficulties that do not occur in the periodic setting, which were pointed out in [7] too, where the case $p = 2$ was considered.

The main difference between this paper and [7] is that we are able to obtain estimates on the L^p norm of the operators $R^{J,t}_{T^0,\mathbf{n}}$ (see (62)) for $p \neq 2$ through an L^∞ estimate and an interpolation argument. Another difference is that in [7] an equivalent norm of $M^s_{2,q}$ could be used, namely the norm

$$\Big(\sum_{k\in\mathbb{Z}} \langle k\rangle^{sq} \|\tilde{\Box}_k f\|_2^q\Big)^{\frac{1}{q}},$$

where $\tilde{\Box}_k = \mathcal{F}^{(-1)} 1_{[k,k+1]} \mathcal{F}$, in order to avoid overlaps between two neighbouring σ_n and σ_m which is something that for $p \neq 2$ can not be ignored.

In order to give a meaning to solutions of the NLS in $C([0,T], M_{p,q}(\mathbb{R}))$ and to the nonlinearity $\mathcal{N}(u) := u|u|^2$ we need the following definitions:

Definition 1. *For fixed $1 \leq p \leq \infty$, a sequence of Fourier cutoff operators is a sequence of Fourier multiplier operators $\{T_N\}_{N\in\mathbb{N}}$ with symbols m_N on $\mathcal{S}'(\mathbb{R})$ such that*

- *m_N is compactly supported for all $N \in \mathbb{N}$,*
- $\sup_{N\in\mathbb{N}} \|T_N\|_{p\to p} < \infty$ *and*
- *for every f in a dense subset of $L^p(\mathbb{R})$ we have* $\lim_{N\to\infty} \|T_N f - f\|_p = 0$.

Notice that in our definition a sequence of Fourier cutoff operators depends on the given value of $p \in [1,\infty]$ in $M^s_{p,q}(\mathbb{R})$.

Definition 2. *Let $u \in C([0,T], M^s_{p,q}(\mathbb{R}))$. We say that $\mathcal{N}(u)$ exists and is equal to a distribution $w \in \mathcal{S}'((0,T)\times\mathbb{R})$ if for every sequence $\{T_N\}_{N\in\mathbb{N}}$ of Fourier cutoff operators we have*

$$\lim_{N\to\infty} \mathcal{N}(T_N u) = w, \tag{6}$$

in the sense of distributions on $(0,T)\times\mathbb{R}$.

Definition 3. *We say that $u \in C([0,T], M^s_{p,q}(\mathbb{R}))$ is a weak solution of NLS (1) if*

- *$u(0,x) = u_0(x)$,*
- *the nonlinearity $\mathcal{N}(u)$ exists in the sense of Definition 2,*
- *u satisfies (1) in the sense of distributions on $(0,T)\times\mathbb{R}$, where the nonlinearity $\mathcal{N}(u) = u|u|^2$ is interpreted as above.*

Our main result which guarantees persistent solutions generalises the one in [7] and it is the following:

Theorem 4. *Let $s \geq 0$, $1 \leq q < 3$ and $2 \leq p < \frac{10q'}{q'+6}$. For $u_0 \in M^s_{p,q}(\mathbb{R})$ there exists a weak solution $u \in C([0,T]; M^s_{p,q}(\mathbb{R}))$ of NLS (1) with initial condition u_0 in the sense of Definition 3, where the time T of existence depends only on $\|u_0\|_{M^s_{p,q}}$. Moreover, the solution map is Lipschitz continuous.*

Remark 5. The restriction on the range of p is dictated by the construction of our solution of the NLS. More precisely, we decompose the NLS into countably many "smaller" parts and at the end we sum all of them together. In order for this summation to make sense all the series must by convergent in the appropriate spaces and as a consequence we obtain the restriction $p < \frac{10q'}{q'+6}$ (see the remarks after (79) below).

To conclude this section we need that for $S(t) = e^{it\Delta}$ the Schrödinger semigroup we have the estimate:

$$\|S(t)f\|_{M^s_{p,q}} \lesssim (1+|t|)^{\left|\frac{1}{2}-\frac{1}{p}\right|} \|f\|_{M^s_{p,q}}, \tag{7}$$

where the implicit constant does not depend on f, t. We also need the multiplier estimate (see [8], Proposition 1.9):

Lemma 6. *Let $1 \leq p \leq \infty$ and $\sigma \in C^\infty_c(\mathbb{R})$. Then the multiplier operator $T_\sigma : S'(\mathbb{R}) \to S'(\mathbb{R})$ defined by*

$$(T_\sigma f) = \mathcal{F}^{-1}(\sigma \cdot \hat{f}), \quad \forall f \in S'(\mathbb{R})$$

is bounded on $L^p(\mathbb{R})$ *and*

$$\|T_\sigma\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \lesssim \|\check{\sigma}\|_{L^1(\mathbb{R})}.$$

A useful consequence is that for $1 \leq p_1 \leq p_2 \leq \infty$ the following holds:

$$\|\Box_k f\|_{p_2} \lesssim \|\Box_k f\|_{p_1}, \tag{8}$$

where the implicit constant is independent of k and the function f.

Lastly, let us recall the following number theoretic fact (see [6], Theorem 315) which is going to be used throughout the proof of Theorem 4: Given an integer m, let $d(m)$ denote the number of divisors of m. Then we have

$$d(m) \lesssim e^{c\frac{\log m}{\log\log m}} = o(m^\epsilon), \tag{9}$$

for all $\epsilon > 0$.

2. Proof of the main theorem

The calculations are similar to those presented in [7] where the difference is that instead of using L^2 estimates for the Fourier-space variable we use L^p estimates which is something that will become clearer in the calculations that follow. Nevertheless, there are a lot of new details that need to be taken care of. For this reason, and for the reader's convenience we will be as detailed as possible.

From here on, we consider only the case $s = 0$ in Theorem 4 since for $s > 0$ similar considerations apply. See Remark 20 at the end of the section for a more detailed argument.

Also, since our indices $1 \leq q < 3$ and $2 \leq p < \frac{10q'}{q'+6}$ are fixed, we can find a fixed number $A > 1$ such that

$$2 \leq p < \frac{2q'(2A+3)}{(2A-1)q'+6}. \tag{10}$$

Notice that the function $f(A) = \frac{2q'(2A+3)}{(2A-1)q'+6}$ is decreasing and in the range $A > 1$ it has a global maximum at $A = 1$. From here on, we choose our bump function σ_0 to satisfy the following bounds on its derivatives

$$\left\|\frac{d^J}{dx^J}\,\sigma_0\right\|_\infty \lesssim (J!)^A, \tag{11}$$

for all $J \in \mathbb{Z}_+$. This is crucial for Lemma 17. Notice that $A \leq 1$ can not be true since then our compactly supported function σ_0 would be a real analytic function and therefore, it would be identically zero.

For $n \in \mathbb{Z}$ let us define

$$u_n(t,x) = \Box_n u(t,x), \tag{12}$$

$$v(t,x) = e^{it\partial_x^2} u(t,x), \tag{13}$$

$$v_n(t,x) = e^{it\partial_x^2} u_n(t,x) = \square_n[(e^{it\partial_x^2} u(t,x)] = \square_n v(t,x). \tag{14}$$

Also for $(\xi, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ we define the function

$$\Phi(\xi, \xi_1, \xi_2, \xi_3) = \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2,$$

which is equal to

$$\Phi(\xi, \xi_1, \xi_2, \xi_3) = 2(\xi - \xi_1)(\xi - \xi_3),$$

if $\xi = \xi_1 - \xi_2 + \xi_3$. Our main equation (1) implies that

$$i\partial_t u_n - (u_n)_{xx} \pm \square_n(|u|^2 u) = 0, \tag{15}$$

and by calculating ($u = \sum_k \square_k u$)

$$\square_n(u\bar{u}u) = \square_n \sum_{n_1,n_2,n_3} u_{n_1}\bar{u}_{n_2}u_{n_3} = \sum_{n_1-n_2+n_3\approx n} \square_n[u_{n_1}\bar{u}_{n_2}u_{n_3}],$$

where by $\approx n$ we mean $= n$, or $= n+1$, or $= n-1$. Next we do the change of variables $u_n(t,x) = e^{-it\partial_x^2} v_n(t,x)$ and arrive at the expression

$$\partial_t v_n = \pm i \sum_{n_1-n_2+n_3\approx n} \square_n\Big(e^{it\partial_x^2}[e^{-it\partial_x^2} v_{n_1} \cdot e^{it\partial_x^2}\bar{v}_{n_2} \cdot e^{-it\partial_x^2} v_{n_3}]\Big). \tag{16}$$

We define the 1st generation operators by

$$Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})(x) = \square_n\Big(e^{it\partial_x^2}[e^{-it\partial_x^2} v_{n_1} \cdot e^{it\partial_x^2}\bar{v}_{n_2} \cdot e^{-it\partial_x^2} v_{n_3}]\Big), \tag{17}$$

and continue with the splitting

$$\partial_t v_n = \pm i \sum_{n_1-n_2+n_3\approx n} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = \sum_{\substack{n_1\approx n\\ or\\ n_3\approx n}} \ldots + \sum_{n_1\not\approx n\not\approx n_3} \ldots, \tag{18}$$

where we define the resonant part

$$R_2^t(v)(n) - R_1^t(v)(n) = \Big(\sum_{n_1\approx n} Q_n^{1,t} + \sum_{n_3\approx n} Q_n^{1,t}\Big) - \sum_{\substack{n_1\approx n\\ and\\ n_3\approx n}} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}), \tag{19}$$

and the non-resonant part

$$N_1^t(v)(n) = \sum_{n_1 \not\approx n \not\approx n_3} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}), \tag{20}$$

which implies the following expression for our NLS (we drop the factor $\pm i$ in front of the sum since they will play no role in our analysis)

$$\partial_t v_n = R_2^t(v)(n) - R_1^t(v)(n) + N_1^t(v)(n). \tag{21}$$

For the resonant part we have the following:

Lemma 7. *For $j = 1, 2$*

$$\|R_j^t(v)\|_{l^q M_{p,q}} \lesssim (1 + |t|)^{4|\frac{1}{2} - \frac{1}{p}|} \|v\|^3_{M_{p,q}},$$

and

$$\|R_j^t(v) - R_j^t(w)\|_{l^q M_{p,q}} \lesssim (1 + |t|)^{4|\frac{1}{2} - \frac{1}{p}|} (\|v\|^2_{M_{p,q}} + \|w\|^2_{M_{p,q}}) \|v - w\|_{M_{p,q}}.$$

Proof. Let us consider R_1^t. By its definition, for fixed n, $R_1^t(n)$ consists of finitely many summands, since $|n - n_1|, |n - n_3| \leq 1$ and $|n - n_2| \leq 3$. We will handle $Q_n^{1,t}(v_n, \bar{v}_n, v_n)$ since the remaining summands can be treated similarly. Since,

$$Q_n^{1,t}(v_n, \bar{v}_n, v_n) = \Box_n \Big(e^{it\partial_x^2} [e^{-it\partial_x^2} v_n \cdot e^{it\partial_x^2} \bar{v}_n \cdot e^{-it\partial_x^2} v_n] \Big)$$

its $M_{p,q}$ norm is bounded from above by

$$\|e^{it\partial_x^2} \Box_n(|u_n|^2 u_n)\|_{M_{p,q}} \lesssim (1 + |t|)^{|\frac{1}{2} - \frac{1}{p}|} \|\Box_n(|u_n|^2 u_n)\|_{M_{p,q}},$$

where we used (7). By estimating this last norm we have

$$\|\Box_n(|u_n|^2 u_n)\|_{M_{p,q}} = \Big(\sum_{m \in \mathbb{Z}} \|\Box_m \Box_n(|u_n|^2 u_n)\|_p^q \Big)^{\frac{1}{q}} \lesssim \Big(\sum_{m \in \mathbb{Z}} \|\Box_m(|u_n|^2 u_n)\|_p^q \Big)^{\frac{1}{q}} =$$

$$\Big(\sum_{l \in \Lambda} \|\Box_{n+l}(|u_n|^2 u_n)\|_p^q \Big)^{\frac{1}{q}} \lesssim \||u_n|^2 u_n\|_p = \|u_n\|^3_{3p},$$

where in the first and last inequalities we used (6) and $\Lambda \subset \mathbb{Z}$ is a finite set. With the use of (8) we have $\|u_n\|_{3p} \lesssim \|u_n\|_p$ and by taking the l^q norm in the discrete variable we arrive at the upper bound

$$(1 + |t|)^{|\frac{1}{2} - \frac{1}{p}|} \Big(\sum_{n \in \mathbb{Z}} \|u_n\|_p^{3q} \Big)^{\frac{1}{q}} \leq (1 + |t|)^{|\frac{1}{2} - \frac{1}{p}|} \|u\|^3_{M_{p,q}},$$

where we used the embedding $l^q \hookrightarrow l^{3q}$. Since $u = e^{-it\partial_x^2} v$ another application of (7) gives us the desired upper bound.

For the R_2^t operator, it suffices to estimate the sum

$$\sum_{\substack{n_1-n_2+n_3\approx n\\ n_1\approx n}} Q^t_n(v_{n_1},\bar{v}_{n_2},v_{n_3})$$

which consists of finitely many sums depending on whether $n_1=n-1$, or $n_1=n$, or $n_1=n+1$. Let us only treat

$$\Box_n\ e^{it\partial_x^2}\ \Big(e^{-it\partial_x^2}v_n\sum_{n_2\in\mathbb{Z}}|e^{-it\partial_x^2}v_{n_2}|^2\Big),$$

since for the remaining sums similar considerations apply. Its $M_{p,q}$ norm by (7) is bounded from above by

$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\Big\|\Box_n u_n\sum_{n_2\in\mathbb{Z}}|u_{n_2}|^2\Big\|_{M_{p,q}}=(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\Big(\sum_{m\in\mathbb{Z}}\Big\|\Box_m\Box_n u_n\sum_{n_2\in\mathbb{Z}}|u_{n_2}|^2\Big\|_p^q\Big)^{\frac{1}{q}},$$

where the last summand is equal to

$$\Big(\sum_{m\in\mathbb{Z}}\Big\|\Box_m\Box_n u_n\sum_{k\in\Lambda'}|u_{n+k}|^2\Big\|_p^q\Big)^{\frac{1}{q}},$$

and $\Lambda'=\{l_1,\dots,l_{k'}\}\subset\mathbb{Z}$ is finite. Again from (6) the sum can be controlled by

$$\Big(\sum_{m\in\mathbb{Z}}\Big\|\Box_m u_n\sum_{k\in\Lambda'}|u_{n+k}|^2\Big\|_p^q\Big)^{\frac{1}{q}}=\Big(\sum_{l\in\Lambda}\Big\|\Box_{n+l} u_n\sum_{k\in\Lambda'}|u_{n+k}|^2\Big\|_p^q\Big)^{\frac{1}{q}}\lesssim\Big\|u_n\sum_{k\in\Lambda'}|u_{n+k}|^2\Big\|_p\le$$

$$\sum_{k\in\Lambda'}\|u_n|u_{n+k}|^2\|_p\le\|u_n\|_{2p}\|u_{n+l_1}\|_{4p}^2+\ldots+\|u_n\|_{2p}\|u_{n+l_{k'}}\|_{4p}^2,$$

and by applying (8) and Hölder's inequality in the discrete variable we have for each individual summand the estimate

$$\|\{\|u_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{2q}}\|\{\|u_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{4q}}^2\le\|\{\|u_n\|_p\}_{n\in\mathbb{Z}}\|_{l^q}^3=\|u\|_{M_{p,q}}^3=(1+|t|)^{3|\frac{1}{2}-\frac{1}{p}|}\|v\|_{M_{p,q}}^3,$$

where we used the embedding $l^q\hookrightarrow l^{2q},l^{4q}$ and the proof is complete.

For the difference part $R^t_1(v)-R^t_1(w)$ we have to estimate terms of the following form $\Box_n e^{it\partial_x^2}(e^{-it\partial_x^2}v_n)^2(e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n)$ in the $l^qM_{p,q}$ norm. As before, from (7) the $M_{p,q}$ norm is bounded above by

$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\|\Box_n(e^{-it\partial_x^2}v_n)^2(e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n)\|_{M_{p,q}},$$

and this last norm is equal to

$$\Big(\sum_{m\in\mathbb{Z}}\|\Box_m\Box_n(e^{-it\partial_x^2}v_n)^2(e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n)\|_p^q\Big)^{\frac{1}{q}}=$$

$$\Big(\sum_{l\in\Lambda}\|\square_{n+l}\square_n(e^{-it\partial_x^2}v_n)^2(e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n)\|_p^q\Big)^{\frac{1}{q}}\lesssim\|(e^{-it\partial_x^2}v_n)^2(e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n)\|_p,$$

where we used (6). Applying Hölder's inequality and (8) we arrive at

$$\|e^{-it\partial_x^2}v_n\|_{4p}^2\|e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n\|_{2p}\lesssim\|e^{-it\partial_x^2}v_n\|_p^2\|e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n\|_p,$$

and by taking the l^q and applying Hölder in the discrete variable with the embedding $l^q\hookrightarrow l^{2q},l^{4q}$ and (7), we have the estimate

$$\|\{\|e^{-it\partial_x^2}v_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{4q}}^2\|\{\|e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{2q}}\leq$$
$$\|\{\|e^{-it\partial_x^2}v_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{q}}^2\|\{\|e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n\|_p\}_{n\in\mathbb{Z}}\|_{l^{q}}=$$
$$\|e^{-it\partial_x^2}v_n\|_{M_{p,q}}^2\|e^{-it\partial_x^2}v_n-e^{-it\partial_x^2}w_n\|_{M_{p,q}}\lesssim(1+|t|)^{3|\frac{1}{2}-\frac{1}{q}|}\|v_n\|_{M_{p,q}}^2\|v_n-w_n\|_{M_{p,q}}.$$

The operator difference $R_2^t(v)-R_2^t(w)$ is treated in a similar way and the proof is complete. □

For the non-resonant part N_1^t we have to split as

$$N_1^t(v)(n)=N_{11}^t(v)(n)+N_{12}^t(v)(n),\tag{22}$$

where

$$N_{11}^t(v)(n)=\sum_{A_N(n)}Q_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3}),$$

and

$$A_N(n)=\{(n_1,n_2,n_3)\in\mathbb{Z}^3:n_1-n_2+n_3\approx n,n_1\not\approx n\not\approx n_3,|\Phi(n,n_1,n_2,n_3)|\leq N\}.\tag{23}$$

The number $N>0$ is considered to be large and will be fixed at the end of the proof. With the use of inequality (9) we estimate N_{11}^t as follows:

Lemma 8.

$$\|N_{11}^t(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|}N^{\frac{1}{q'}+}\|v\|_{M_{p,q}}^3,$$

and

$$\|N_{11}^t(v)-N_{11}^t(w)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|}N^{\frac{1}{q'}+}(\|v\|_{M_{p,q}}^2+\|w\|_{M_{p,q}}^2)\|v-w\|_{M_{p,q}}.$$

Proof. Since $\|N_{11}^t(v)\|_{M_{p,q}}\leq\sum_{A_N(n)}\|Q_n^t(v_{n_1},\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}}$ it suffices to estimate

$$\|Q_n^t(v_{n_1},\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}}=\|e^{it\partial_x^2}\square_n(u_{n_1}\bar{u}_{n_2}u_{n_3})\|_{M_{p,q}}\lesssim(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\|\square_n(u_{n_1}\bar{u}_{n_2}u_{n_3})\|_{M_{p,q}},$$

which by estimating the last norm we have

$$\Big(\sum_{m\in\mathbb{Z}}\|\square_m\square_n(u_{n_1}\bar{u}_{n_2}u_{n_3})\|_p^q\Big)^{\frac{1}{q}}=\Big(\sum_{l\in\Lambda}\|\square_{n+l}\square_n(u_{n_1}\bar{u}_{n_2}u_{n_3})\|_p^q\Big)^{\frac{1}{q}}\lesssim$$
$$\|u_{n_1}\bar{u}_{n_2}u_{n_3}\|_p\leq\|u_{n_1}\|_{3p}\|u_{n_2}\|_{3p}\|u_{n_3}\|_{3p}\lesssim\|u_{n_1}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p,$$

by (6), Hölder and (8), where $\Lambda\subset\mathbb{Z}$ is the same set as in Lemma 7. Therefore, the sum

$$\sum_{A_N(n)}\|u_{n_1}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p\leq\Big(\sum_{A_N(n)}1^{q'}\Big)^{\frac{1}{q'}}\Big(\sum_{A_N(n)}\|u_{n_1}\|_p^q\|u_{n_2}\|_p^q\|u_{n_3}\|_p^q\Big)^{\frac{1}{q}}.$$

Fix n and $\mu\in\mathbb{Z}$ such that $|\mu|\leq N$. From (9) there are at most $o(N^+)$ many choices for n_1 and n_3, and so for n_2 from $n\approx n_1-n_2+n_3$, satisfying

$$\mu=2(n-n_1)(n-n_3).$$

Thus, we arrive at

$$\|N^t_{11}(v)\|_{M_{p,q}}\lesssim(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}N^{\frac{1}{q'}+}\Big(\sum_{A_N(n)}\|u_{n_1}\|_p^q\|u_{n_2}\|_p^q\|u_{n_3}\|_p^q\Big)^{\frac{1}{q}}$$

Then, we take the l^q norm in the discrete variable and apply Hölder's inequality to obtain

$$\|N^t_{11}(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}N^{\frac{1}{q'}+}\Big(\sum_{n\in\mathbb{Z}}\sum_{A_N(n)}\|u_{n_1}\|_p^q\|u_{n_2}\|_p^q\|u_{n_3}\|_p^q\Big)^{\frac{1}{q}},$$

and this final summation is estimated by Young's inequality providing us with the bound $(\|u_n\|_{M_{p,q}}\lesssim(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\|v_n\|_{M_{p,q}})$

$$\|N^t_{11}(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|}N^{\frac{1}{q'}+}\|v\|^3_{M_{p,q}},$$

which finishes the proof. □

In order to continue, we have to look at the N^t_{12} part more closely keeping in mind that we are on $A_N(n)^c$. Our goal is to find a suitable splitting in order to continue our iteration. From (17) we know that

$$\mathcal{F}(Q^{1,t}_n(v_{n_1},\bar{v}_{n_2},v_{n_3}))(\xi)=\sigma_n(\xi)\int_{\mathbb{R}^2}e^{-2it(\xi-\xi_1)(\xi-\xi_3)}\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3)\ d\xi_1d\xi_3,$$

and by the usual product rule for the derivative we can write the previous integral as the sum of the following expressions

$$\partial_t\Big(\sigma_n(\xi)\int_{\mathbb{R}^2}\frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)}\ \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3)\ d\xi_1d\xi_3\Big)-$$
$$\sigma_n(\xi)\int_{\mathbb{R}^2}\frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)}\partial_t\Big(\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3)\Big)\ d\xi_1d\xi_3.$$

Therefore, we have the splitting

$$\mathcal{F}(Q_n^{1,t}) = \partial_t \mathcal{F}(\tilde{Q}_n^{1,t}) - \mathcal{F}(T_n^{1,t}) \tag{24}$$

or equivalently

$$Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = \partial_t(\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})) - T_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}), \tag{25}$$

which allows us to write

$$N_{12}^t(v)(n) = \partial_t(N_{21}^t(v)(n)) + N_{22}^t(v)(n), \tag{26}$$

where

$$N_{21}^t(v)(n) = \sum_{A_N(n)^c} \tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}), \tag{27}$$

and

$$N_{22}^t(v)(n) = \sum_{A_N(n)^c} T_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}). \tag{28}$$

Moreover, we have

$$\mathcal{F}(\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2}\sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\hat{u}_{n_1}(\xi_1)\hat{\bar{u}}_{n_2}(\xi - \xi_1 - \xi_3)\hat{u}_{n_3}(\xi_3)}{(\xi - \xi_1)(\xi - \xi_3)} \, d\xi_1 d\xi_3,$$

and we define

$$\mathcal{F}(R_n^{1,t}(u_{n_1}, \bar{u}_{n_2}, u_{n_3}))(\xi) = \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\hat{u}_{n_1}(\xi_1)\hat{\bar{u}}_{n_2}(\xi - \xi_1 - \xi_3)\hat{u}_{n_3}(\xi_3)}{(\xi - \xi_1)(\xi - \xi_3)} \, d\xi_1 d\xi_3, \tag{29}$$

which is the same as the operator

$$R_n^{1,t}(u_{n_1}, \bar{u}_{n_2}, u_{n_3})(x) = \int_{\mathbb{R}^3} e^{ix\xi}\sigma_n(\xi) \frac{\hat{u}_{n_1}(\xi_1)\hat{\bar{u}}_{n_2}(\xi - \xi_1 - \xi_3)\hat{u}_{n_3}(\xi_3)}{(\xi - \xi_1)(\xi - \xi_3)} \, d\xi_1 d\xi_3 d\xi. \tag{30}$$

At this point we introduce a fattened version of the σ-functions in the following way: Consider a function $\tilde{\sigma}_0$ with the same properties as σ_0 such that $\tilde{\sigma}_0 \equiv 1$ on the support of σ_0, $\text{supp}\tilde{\sigma}_0 \subset B(0, \frac{17}{16})$ and define the tranlations $\tilde{\sigma}_k = \tilde{\sigma}_0(\cdot - k)$, $k \in \mathbb{Z}$.

With this notation, writing out the Fourier transforms of the functions inside the integral in (30) it is not difficult to see that

$$R_n^{1,t}(u_{n_1}, \bar{u}_{n_2}, u_{n_3})(x) = \int_{\mathbb{R}^3} K_n^{(1)}(x, x_1, y, x_3) u_{n_1}(x_1)\bar{u}_{n_2}(y) u_{n_3}(x_3) \, dx_1 dy dx_3, \tag{31}$$

where

$$K_n^{(1)}(x,x_1,y,x_3)=\int_{\mathbb{R}^3} e^{i\xi_1(x-x_1)+i\eta(x-y)+i\xi_3(x-x_3)}$$
$$\frac{\sigma_n(\xi_1+\eta+\xi_3)}{(\eta+\xi_1)(\eta+\xi_3)}\tilde{\sigma}_{n_1}(\xi_1)\tilde{\sigma}_{n_2}(-\eta)\tilde{\sigma}_{n_3}(\xi_3)\ d\xi_1 d\eta d\xi_3=\mathcal{F}^{-1}\tilde{\rho}_n^{(1)}(x-x_1,x-y,x-x_3)$$

and

$$\tilde{\rho}_n^{(1)}(\xi_1,\eta,\xi_3)=\frac{\sigma_n(\xi_1+\eta+\xi_3)}{(\eta+\xi_1)(\eta+\xi_3)}\tilde{\sigma}_{n_1}(\xi_1)\tilde{\sigma}_{n_2}(-\eta)\tilde{\sigma}_{n_3}(\xi_3),\ \rho_n^{(1)}(\xi_1,\eta,\xi_3)=\frac{\sigma_n(\xi_1+\eta+\xi_3)}{(\eta+\xi_1)(\eta+\xi_3)}.$$

The important estimate that the operator $\tilde{Q}_n^{1,t}$ satisfies is described in:

Lemma 9. *For $2\le p\le\infty$*

$$\|R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3})\|_p\lesssim\frac{\|v_{n_1}\|_p\|v_{n_2}\|_p\|v_{n_3}\|_p}{|n-n_1||n-n_3|}, \tag{32}$$

where the implicit constant depends on p.

Proof. First, let us consider the case $p=2$. This repeats the argument of the $M_{2,q}$ case treated in [7]. By duality, let $g\in L^2$, $\|g\|_2\neq 0$, and consider the pairing

$$|\langle R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3}),g\rangle|=\Big|\int_{\mathbb{R}}\mathcal{F}(R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3}))(\xi)\mathcal{F}(g)(\xi)\ d\xi\Big|= \tag{33}$$
$$\Big|\int_{\mathbb{R}^3}\hat{g}(\xi)\ \sigma_n(\xi)\ \frac{\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3)}{(\xi-\xi_1)(\xi-\xi_3)}\ d\xi d\xi_1 d\xi_3\Big|=$$
$$\Big|\int_{\mathbb{R}^3}\hat{g}(\xi_1+\eta+\xi_3)\ \frac{\sigma_n(\xi_1+\eta+\xi_3)}{(\eta+\xi_1)(\eta+\xi_3)}\ \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\eta)\hat{v}_{n_3}(\xi_3)\ d\eta d\xi_1 d\xi_3\Big|=$$
$$\Big|\int_{I_{n_1}}\int_{I_{n_2}}\int_{I_{n_3}}\hat{g}(\xi_1+\eta+\xi_3)\ \rho_n^{(1)}(\xi_1,\eta,\xi_3)\ \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\eta)\hat{v}_{n_3}(\xi_3)\ d\xi_1 d\eta d\xi_3\Big|,$$

where these three intervals are the compact supports of the functions $\hat{v}_{n_1},\hat{\bar{v}}_{n_2},\hat{v}_{n_3}$ (see (14)). By Hölder's inequality we obtain the upper bound

$$\|\rho_n^{(1)}\|_\infty\|v_{n_1}\|_2\|v_{n_2}\|_2\|v_{n_3}\|_2\Big(\int_{I_{n_1}}\int_{I_{n_2}}\int_{I_{n_3}}|\hat{g}(\xi_1+\eta+\xi_3)|^2\ d\xi_1 d\eta d\xi_3\Big)^{\frac{1}{2}},$$

and the last triple integral is easily estimated by

$$\|\hat{g}\|_2\ (|I_{n_2}||I_{n_3}|)^{\frac{1}{2}}=\|g\|_2\ (|I_{n_2}||I_{n_3}|)^{\frac{1}{2}}.$$

Therefore, the following is true

$$\|R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3}))\|_2\lesssim\|\rho_n^{(1)}\|_\infty\|v_{n_1}\|_2\|v_{n_2}\|_2\|v_{n_3}\|_2,$$

and since $\xi_1\approx n_1$, $\eta\approx -n_2$ and $\xi_3\approx n_3$ we obtain

$$\|\rho_n^{(1)}\|_\infty \lesssim \frac{1}{|n-n_1||n-n_3|},$$

which finishes the proof.

Next let us consider the case $p=\infty$. Obviously,

$$\|R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3})\|_\infty = \sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}^3}(\mathcal{F}^{-1}\tilde{\rho}_n^{(1)})(x-x_1,x-y,x-x_3)v_{n_1}(x_1)\bar{v}_{n_2}(y)v_{n_3}(x_3)dx_1dydx_3\Big|,$$

which is bounded by

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}^3}|(\mathcal{F}^{-1}\tilde{\rho}_n^{(1)})(x-x_1,x-y,x-x_3)|dx_1dydx_3\|v_{n_1}\|_\infty\|v_{n_2}\|_\infty\|v_{n_3}\|_\infty =$$

$$\|\mathcal{F}^{-1}\tilde{\rho}_n^{(1)}\|_{L^1(\mathbb{R}^3)}\|v_{n_1}\|_\infty\|v_{n_2}\|_\infty\|v_{n_3}\|_\infty.$$

By the embedding $H^s(\mathbb{R}^3)\hookrightarrow \mathcal{F}L^1(\mathbb{R}^3)$, for $s>3/2$, and the fact that $|\text{supp}(\tilde{\rho}_n^{(1)})|\lesssim 1$, it is sufficient to have an L^∞ bound on the derivatives of $\tilde{\rho}_n^{(1)}$ of order $0,1$ and 2. Trivially,

$$|\tilde{\rho}_n^{(1)}(\xi_1,\eta,\xi_3)| \lesssim \frac{1}{|n-n_1||n-n_3|},$$

since $\xi_1\approx n_1, \eta\approx -n_2$ and $\xi_3\approx n_3$. Then for the first order derivatives we get

$$|\partial_{\xi_j}\tilde{\rho}_n^{(1)}| \lesssim \frac{1}{|\eta+\xi_j|^2|\eta+\xi_{4-j}|} + \frac{\|\sigma_n'\|_\infty+\|\tilde{\sigma}_{n_j}'\|_\infty}{|\eta+\xi_j||\eta+\xi_{4-j}|} \lesssim \frac{1}{|n-n_1||n-n_3|},$$

for $j=1,3$, since $|n-n_1|\geq 1$. For the remaining derivative we observe that

$$|\partial_\eta\tilde{\rho}_n^{(1)}| \lesssim \frac{\|\sigma_n'\|_\infty+\|\tilde{\sigma}_{n_2}'\|_\infty}{|\eta+\xi_1||\eta+\xi_3|} + \frac{|2\eta+\xi_1+\xi_3|}{|\eta+\xi_1|^2|\eta+\xi_3|^2} \lesssim \frac{1}{|\eta+\xi_1||\eta+\xi_3|} + \frac{|\eta+\xi_1|+|\eta+\xi_3|}{|\eta+\xi_1|^2|\eta+\xi_3|^2},$$

which is bounded by

$$\frac{c}{|n-n_1||n-n_3|},$$

since $|\eta+\xi_j|\geq 1$, where $c>0$ is a constant. Similarly we check the 2nd order derivatives of $\tilde{\rho}_n^{(1)}$. Thus,

$$\|R_n^{1,t}(v_{n_1},\bar{v}_{n_2},v_{n_3})\|_\infty \lesssim \frac{\|v_{n_1}\|_\infty\|v_{n_2}\|_\infty\|v_{n_3}\|_\infty}{|n-n_1||n-n_3|}.$$

By interpolating between $p=2$ and $p=\infty$, we arrive at estimate (32) for $2\leq p\leq\infty$. □

Here is the estimate for the N_{21}^t operator:

Lemma 10.

$$\|N^t_{21}(v)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} \|v\|^3_{M_{p,q}},$$

and

$$\|N^t_{21}(v) - N^t_{21}(w)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} (\|v\|^2_{M_{p,q}} + \|w\|^2_{M_{p,q}}) \|v-w\|_{M_{p,q}}.$$

Proof. Starting with the $M_{p,q}$ norm we have the estimate

$$\|N^t_{21}(v)\|_{M_{p,q}} \leq \sum_{A_N(n)^c} \|\tilde{Q}^{1,t}_n(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_{M_{p,q}},$$

and the inner norm is equal to

$$\|\tilde{Q}^{1,t}_n(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_{M_{p,q}} = \Big(\sum_{m\in\mathbb{Z}} \|\square_m \tilde{Q}^{1,t}_n\|^q_p\Big)^{\frac{1}{q}} = \Big(\sum_{m\in\mathbb{Z}} \|\square_m e^{it\partial_x^2} R^{1,t}_n\|^q_p\Big)^{\frac{1}{q}} \lesssim$$
$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} \Big(\sum_{m\in\mathbb{Z}} \|\square_m R^{1,t}_n\|^q_p\Big)^{\frac{1}{q}} = (1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} \Big(\sum_{m\in\mathbb{Z}} \|\mathcal{F}^{-1}\sigma_m \mathcal{F} R^{1,t}_n\|^q_p\Big)^{\frac{1}{q}},$$

from (7). Since the Fourier transform of the operator $R^{1,t}_n$ is supported where σ_n is, the last sum is actually a finite sum, that is

$$\Big(\sum_{l\in\Lambda} \|\square_{n+l} R^{1,t}_n(u_{n_1}, \bar{u}_{n_2}, u_{n_3})\|^q_p\Big)^{\frac{1}{q}} \lesssim \|R^{1,t}_n(u_{n_1}, \bar{u}_{n_2}, u_{n_3})\|_p \lesssim \frac{\|u_{n_1}\|_p \|u_{n_2}\|_p \|u_{n_3}\|_p}{|n-n_1||n-n_3|},$$

by Lemma 9. Then we take the l^q norm in the discrete variable n to arrive at the bound

$$\|N^t_{21}(v)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} \sum_{A_N(n)^c} \frac{\|u_{n_1}\|_p \|u_{n_2}\|_p \|u_{n_3}\|_p}{|n-n_1||n-n_3|},$$

and by Hölder's inequality we are led to the upper bound

$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} \Big(\sum_{A_N(n)^c} \frac{1}{(|n-n_1||n-n_3|)^{q'}}\Big)^{\frac{1}{q'}} \Big(\sum_{A_N(n)^c} \|u_{n_1}\|^q_p \|u_{n_2}\|^q_p \|u_{n_3}\|^q_p\Big)^{\frac{1}{q}}.$$

The first sum (for $\mu = |n-n_1||n-n_3|$) is estimated with the use of (9) from above by

$$\Big(\sum_{\mu=N+1}^{\infty} \frac{\mu^\epsilon}{\mu^{q'}}\Big)^{\frac{1}{q'}} \sim (N^{\epsilon+1-q'})^{\frac{1}{q'}} = N^{\frac{1}{q'}-1+},$$

and then with the use of Young's inequality we arrive at

$$\|N^t_{21}(v)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} \|u\|^3_{M_{p,q}} \lesssim (1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} \|v\|^3_{M_{p,q}},$$

where we used (7) ($u_n = e^{-it\partial_x^2} v_n$) and the proof is complete. □

To the remaining part N_{22}^t we have to make use of equality (21) depending on whether the derivative falls on $\hat{v}_{n_1}$ or $\hat{\bar{v}}_{n_2}$ or $\hat{v}_{n_3}$. Let us see how we can proceed from here:

$$N_{22}^t(v)(n) = -2i \sum_{A_N(n)^c} \Big[\tilde{Q}_n^{1,t}(R_2^t(v)(n_1) - R_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}) + \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3})\Big]$$

plus the corresponding term for $\partial_t \hat{\bar{v}}_{n_2}$ (the number 2 that appears in front of the previous sum is because the expression is symmetric with respect to v_{n_1} and v_{n_3}). Therefore, we can write N_{22}^t as a sum

$$N_{22}^t(v)(n) = N_4^t(v)(n) + N_3^t(v)(n), \tag{34}$$

where $N_4^t(v)(n)$ is the sum with the resonant part $R_2^t - R_1^t$. The following Lemma is true:

Lemma 11.

$$\|N_4^t(v)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{7|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} \|v\|_{M_{p,q}}^5,$$

and

$$\|N_4^t(v) - N_4^t(w)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{7|\frac{1}{2}-\frac{1}{p}|} N^{\frac{1}{q'}-1+} (\|v\|_{M_{p,q}}^4 + \|w\|_{M_{p,q}}^4)\|v-w\|_{M_{p,q}}.$$

Proof. Follows by Lemmata 7 and 10 in the sense that we repeat the proof of Lemma 10 and apply Lemma 7 to the part $R_2^t(v)(n_1) - R_1^t(v)(n_1)$. □

To continue, we have to decompose N_3^t even further. It consists of 3 sums depending on which function the operator N_1^t acts. One of them is the following (similar considerations apply for the remaining sums too)

$$\sum_{A_N(n)^c} \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}), \tag{35}$$

where

$$N_1^t(v)(n_1) = \sum_{m_1 \not\approx n_1 \not\approx m_3} Q_{n_1}^{1,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}),$$

and $n_1 \approx m_1 - m_2 + m_3$. Here we have to consider new restrictions on the frequencies $(m_1, m_2, m_3, n_2, n_3)$ where the "new" triple of frequencies m_1, m_2, m_3 appears as a "child" of the frequency n_1. Thus, for $\mu_1 = \Phi(n, n_1, n_2, n_3)$ and $\mu_2 = \Phi(n_1, m_1, m_2, m_3)$ we define the set

$$C_1 = \{|\mu_1 + \mu_2| \le 5^3 |\mu_1|^{1-\frac{1}{100}}\}, \tag{36}$$

and split the sum in (35) as

(37)
$$\sum_{A_N(n)^c}\sum_{C_1}\dots+\sum_{A_N(n)^c}\sum_{C_1^c}\dots=N^t_{31}(v)(n)+N^t_{32}(v)(n).$$

The following holds:

Lemma 12.
$$\|N^t_{31}(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{8|\frac{1}{2}-\frac{1}{p}|}N^{\frac{2}{q'}-\frac{1}{100q'}-1+}\|v\|^5_{M_{p,q}},$$

and

$$\|N^t_{31}(v)-N^t_{31}(w)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{8|\frac{1}{2}-\frac{1}{p}|}N^{\frac{2}{q'}-\frac{1}{100q'}-1+}(\|v\|^4_{M_{p,q}}+\|w\|^4_{M_{p,q}})\|v-w\|_{M_{p,q}}.$$

Proof. From (9) we know that for fixed n and μ_1, there are at most $o(|\mu_1|^+)$ many choices for n_1 and n_3 and for fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1 and m_3. From (36) we can control μ_2 in terms of μ_1, that is $|\mu_2|\sim|\mu_1|$. In addition, for fixed $|\mu_1|$ there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for μ_2. Also,

$$\|N^t_{31}(v)\|_{M_{p,q}}\leq\sum_{A_N(n)^c}\sum_{C_1}\|\tilde{Q}^{1,t}_n(Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3}),\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}},$$

and by doing the same estimate as in the proof of Lemma (10) for the norm

$$\|\tilde{Q}^{1,t}_n(Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3}),\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}},$$

we arrive at the upper bound

$$\|N^t_{31}(v)\|_{M_{p,q}}\lesssim(1+|t|)^{|\frac{1}{2}=\frac{1}{p}|}\sum_{A_N(n)^c}\sum_{C_1}\frac{\|e^{-it\partial_x^2}Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3})\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p}{|n-n_1||n-n_3|}$$

and the last sum is bounded above by

$$\Big(\sum_{\mu=N+1}^{\infty}\frac{\mu^{1-\frac{1}{100}+}}{\mu^{q'}}\Big)^{\frac{1}{q'}}\Big(\sum_{A_N(n)^c}\sum_{C_1}\|e^{-it\partial_x^2}Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3})\|^q_p\|u_{n_2}\|^q_2\|u_{n_3}\|^q_p\Big)^{\frac{1}{q}}.$$

Now we take the l^q norm and apply Young's inequality for the second expression to arrive at the estimate

$$\|N^t_{31}(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{4|\frac{1}{2}-\frac{1}{p}|}N^{\frac{2}{q'}-\frac{1}{100q'}-1+}\|Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3})\|_{M_{p,q}}\|v\|^2_{M_{p,q}},$$

and we treat the norm $\|Q^{1,t}_{n_1}(v_{m_1},\bar{v}_{m_2},v_{m_3})\|_{M_{p,q}}$ similarly as in Lemma (7) for the operator R^t_1 which finishes the proof. □

For the N_{32}^t part we have to do the differentiation by parts technique which will create the 2nd generation operators. Our first 2nd generation operator $Q_n^{2,t}$ consists of 3 sums

$$\begin{aligned} q_{1,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}), \\ q_{2,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(v_{n_1}, \overline{N_1^t(v)}(n_2), v_{n_3}), \\ q_{3,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, N_1^t(v)(n_3)). \end{aligned}$$

Let us have a look at the first sum $q_{1,n}^{2,t}$ (we treat the other two in a similar manner). Its Fourier transform is equal to

$$\sum_{A_N(n)^c} \sum_{C_1^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{(\xi-\xi_1)(\xi-\xi_3)} \mathcal{F}(N_1^t(v)(n_1))(\xi_1) \hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3)\, d\xi_1 d\xi_3,$$

where

$$\mathcal{F}(N_1^t(v)(n_1))(\xi_1)$$

equals

$$\sum_{\substack{n_1 \approx m_1 - m_2 + m_3 \\ m_1 \not\approx n_1 \not\approx m_3}} \sigma_{n_1}(\xi_1) \int_{\mathbb{R}^2} e^{-2it(\xi_1-\xi_1')(\xi_1-\xi_3')} \hat{v}_{m_1}(\xi_1') \hat{\bar{v}}_{m_2}(\xi_1-\xi_1'-\xi_3') \hat{v}_{m_3}(\xi_3')\, d\xi_1' d\xi_3'.$$

Putting everything together and applying differentiation by parts we can write the integrals inside the sums as

$$\partial_t\Big(\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{e^{-it(\mu_1+\mu_2)}}{\mu_1(\mu_1+\mu_2)} \hat{v}_{m_1}(\xi_1') \hat{\bar{v}}_{m_2}(\xi_1-\xi_1'-\xi_3') \hat{v}_{m_3}(\xi_3') \hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1' d\xi_3' d\xi_1 d\xi_3\Big)$$

minus

$$\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{e^{-it(\mu_1+\mu_2)}}{\mu_1(\mu_1+\mu_2)} \partial_t\Big(\hat{v}_{m_1}(\xi_1') \hat{\bar{v}}_{m_2}(\xi_1-\xi_1'-\xi_3') \hat{v}_{m_3}(\xi_3') \hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3)\Big) d\xi_1' d\xi_3' d\xi_1 d\xi_3,$$

where $\mu_1 = (\xi-\xi_1)(\xi-\xi_3)$ and $\mu_2 = (\xi_1-\xi_1')(\xi_1-\xi_3')$. Equivalently,

$$\mathcal{F}(q_{1,n}^{2,t}) = \partial_t(\tilde{q}_{1,n}^{2,t}) - \mathcal{F}(\tau_{1,n}^{2,t}). \tag{38}$$

Thus, by doing the same at the remaining two sums of $Q_n^{2,t}$, namely $q_{2,n}^{2,t}, q_{3,n}^{2,t}$, we obtain the splitting

$$\mathcal{F}(Q^{2,t}_n) = \partial_t \mathcal{F}(\tilde{Q}^{2,t}_n) - \mathcal{F}(T^{2,t}_n). \tag{39}$$

These new operators $\tilde{q}^{2,t}_{i,n}$, $i = 1, 2, 3$, act on the following "type" of sequences

$$\tilde{q}^{2,t}_{1,n}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}),$$

with $m_1 - m_2 + m_3 \approx n_1$ and $n_1 - n_2 + n_3 \approx n$,

$$\tilde{q}^{2,t}_{2,n}(v_{n_1}, \bar{v}_{m_1}, v_{m_2}, \bar{v}_{m_3}, v_{n_3}),$$

with $m_1 - m_2 + m_3 \approx n_2$ and $n_1 - n_2 + n_3 \approx n$, and

$$\tilde{q}^{2,t}_{3,n}(v_{n_1}\bar{v}_{n_2}, v_{m_1}, \bar{v}_{m_2}, v_{m_3}),$$

with $m_1 - m_2 + m_3 \approx n_3$ and $n_1 - n_2 + n_3 \approx n$.

Writing out the Fourier transforms of the functions inside the integral of $\mathcal{F}(\tilde{q}^{2,t}_{1,n})$ it is not hard to see that

$$\mathcal{F}(\tilde{q}^{2,t}_{1,n}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2}\mathcal{F}(R^{2,t}_{n,n_1}(u_{m_1}, \bar{u}_{m_2}, u_{m_3}, \bar{u}_{n_2}, u_{n_3}))(\xi),$$

where the operator

$$R^{2,t}_{n,n_1}(u_{m_1}, \bar{u}_{m_2}, u_{m_3}, \bar{u}_{n_2}, u_{n_3})(x) = \tag{40}$$

$$\int_{\mathbb{R}^5} K^{(2)}_{n,n_1}(x, x'_1, y', x'_3, y, x_3)u_{m_1}(x'_1)\bar{u}_{m_2}(y')u_{m_3}(x'_3)\bar{u}_{n_2}(y)u_{n_3}(x_3)\ dx'_1 dy' dx'_3 dy dx_3$$

and the Kernel $K^{(2)}_{n,n_1}$ is given by the formula

$$K^{(2)}_{n,n_1}(x, x'_1, y', x'_3, y, x_3) = \tag{41}$$

$$\int_{\mathbb{R}^5} [e^{i\xi'_1(x-x'_1)+i\eta'(x-y')+i\xi'_3(x-x'_3)+i\eta(x-y)+i\xi_3(x-x_3)}]$$

$$\frac{\sigma_n(\xi'_1+\eta'+\xi'_3+\eta+\xi_3)\sigma_{n_1}(\xi'_1+\eta'+\xi'_3)\tilde{\sigma}_{m_1}(\xi'_1)\tilde{\sigma}_{m_2}(-\eta')\tilde{\sigma}_{m_3}(\xi'_3)\tilde{\sigma}_{n_2}(-\eta)\tilde{\sigma}_{n_3}(\xi_3)}{(\eta+\eta'+\xi'_1+\xi'_3)(\eta+\xi_3)[(\eta+\eta'+\xi'_1+\xi'_3)(\eta+\xi_3)+(\eta'+\xi'_1)(\eta'+\xi'_3)]}$$

$$d\xi'_1 d\eta' d\xi'_3 d\eta d\xi_3 =$$

$$(\mathcal{F}^{-1}\tilde{\rho}^{(2)}_{n,n_1})(x-x'_1, x-y', x-x'_3, x-y, x-x_3),$$

and the function $\tilde{\rho}^{(2)}_{n,n_1}$ equals

$$\tilde{\rho}^{(2)}_{n,n_1} = \frac{\sigma_n(\xi'_1+\eta'+\xi'_3+\eta+\xi_3)\sigma_{n_1}(\xi'_1+\eta'+\xi'_3)\tilde{\sigma}_{m_1}(\xi'_1)\tilde{\sigma}_{m_2}(-\eta')\tilde{\sigma}_{m_3}(\xi'_3)\tilde{\sigma}_{n_2}(-\eta)\tilde{\sigma}_{n_3}(\xi_3)}{(\eta+\eta'+\xi'_1+\xi'_3)(\eta+\xi_3)[(\eta+\eta'+\xi'_1+\xi'_3)(\eta+\xi_3)+(\eta'+\xi'_1)(\eta'+\xi'_3)]}.$$

We also define the function

$$\rho^{(2)}_{n,n_1}(\xi_1',\eta',\xi_3',\eta,\xi_3)=\frac{\sigma_n(\xi_1'+\eta'+\xi_3'+\eta+\xi_3)\sigma_{n_1}(\xi_1'+\eta'+\xi_3')}{(\eta+\eta'+\xi_1'+\xi_3')(\eta+\xi_3)[(\eta+\eta'+\xi_1'+\xi_3')(\eta+\xi_3)+(\eta'+\xi_1')(\eta'+\xi_3')]}.$$

By the same calculations we obtain also the operators $R^{2,t}_{n,n_2}$ and $R^{2,t}_{n,n_3}$. They can be treated similarly to $R^{2,t}_{n,n_1}$ and for this reason in order to proceed we state a lemma for the operator $R^{2,t}_{n,n_1}$ as the one we had for $R^{1,t}_n$ (see Lemma 9).

Lemma 13. *For* $2\le p\le\infty$

(42)

$$\|R^{2,t}_{n,n_1}(v_{m_1},\bar v_{m_2},v_{m_3},\bar v_{n_2},v_{n_3})\|_p\lesssim\frac{\|v_{m_1}\|_p\|v_{m_2}\|_p\|v_{m_3}\|_p\|v_{n_2}\|_p\|v_{n_3}\|_p}{|n-n_1||n-n_3||(n-n_1)(n-n_3)+(n_1-m_1)(n_1-m_3)|}.$$

Proof. As in Lemma 9 we use interpolation between L^2 and L^∞, and the only difference is that for the L^∞ estimate we use the embedding of $H^s(\mathbb{R}^5)\hookrightarrow\mathcal{F}L^1(\mathbb{R}^5)$, for $s>5/2$, which means we have to calculate up to the 3rd order derivative of the function $\tilde\rho^{(2)}_{n,n_1}$ in contrast to the function $\tilde\rho^{(1)}_n$ of Lemma 9 where we had to find all derivatives up to order 2. □

Remark 14. The operator $\tilde q^{2,t}_{3,n}$ satisfies exactly the same bound as $\tilde q^{2,t}_{1,n}$ since the only difference between these operators is a permutation of their variables. On the other hand, the operator $\tilde q^{2,t}_{2,n}$ is a bit different, since instead of taking only the permutation we have to conjugate the 2nd variable too. Thus, a similar argument as the one given in Lemma 13 leads to the estimate

(43)

$$\|R^{2,t}_{n,n_2}(v_{n_1},\bar v_{m_1},v_{m_2},\bar v_{m_3},v_{n_3})\|_p\lesssim\frac{\|v_{n_1}\|_p\|v_{m_1}\|_p\|v_{m_2}\|_p\|v_{m_3}\|_p\|v_{n_3}\|_p}{|(n-n_1)(n-n_3)||(n-n_1)(n-n_3)-(n_2-m_1)(n_2-m_3)|}$$

which is not exactly the same as the one we had for the operators $R^{2,t}_{n,n_1}, R^{2,t}_{n,n_3}$ since in the denominator instead of having $\mu_1+\mu_2$ we have $\mu_1-\mu_2$ ($\mu_1=(n-n_1)(n-n_3)$ and in the first case $\mu_2=(n_1-m_1)(n_1-\mu_3)$, m_1,m_3 being the "children" of n_1, whereas in the second case $\mu_2=(n_2-m_1)(n_2-m_3)$, m_1,m_3 being the "children" of n_2). It is readily checked that this change in the sign does not really affect the calculations that are to follow.

This lemma allows us to move forward with our iteration process and show that the operators

$$N_0^{(3)}(v)(n):=\sum_{A_N(n)^c}\sum_{C_1^c}\tilde Q^{2,t}_n=\sum_{A_N(n)^c}\sum_{C_1^c}\sum_{i=1}^{3}\tilde q^{2,t}_{i,n} \tag{44}$$

and

$$N_r^{(3)}(v)(n):=\sum_{A_N(n)^c}\sum_{C_1^c}\Big(\tilde q^{2,t}_{1,n}(R^t_2(v)(m_1)-R^t_1(v)(m_1),\bar v_{m_2},v_{m_3},\bar v_{n_2},v_{n_3})+ \tag{45}$$

$$\tilde{q}^{2,t}_{1,n}(v_{m_1},\overline{R^t_2(v)(m_2)-R^t_1(v)(m_2)},v_{m_3},\bar{v}_{n_2},v_{n_3})+\ldots+\tilde{q}^{2,t}_{3,n}(v_{n_1}\bar{v}_{n_2},v_{m_1},\bar{v}_{m_2},R^t_2(v)(m_3)-R^t_1(v)(m_3))\Big),$$

are bounded on $l^qM_{p,q}$. The operator $N^{(3)}_r$ appears when we substitute each of the derivatives in the operator $\sum_{i=1}^3\tau^{2,t}_{i,n}$ by the expression given in (21). Notice that the operator $N^{(3)}_0$ has 3 summands and the operator $N^{(3)}_r$ has $3\cdot 5=15$ summands. Here is the claim:

Lemma 15.

$$\|N^{(3)}_0(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{6|\frac{1}{2}-\frac{1}{p}|}N^{-2+\frac{1}{100}+\frac{2}{q'}-\frac{1}{100q'}+}\|v\|^5_{M_{p,q}},$$

and

$$\|N^{(3)}_0(v)-N^{(3)}_0(w)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{6|\frac{1}{2}-\frac{1}{p}|}N^{-2+\frac{1}{100}+\frac{2}{q'}-\frac{1}{100q'}+}(\|v\|^4_{M_{p,q}}+\|w\|^4_{M_{p,q}})\|v-w\|_{M_{p,q}}.$$

$$\|N^{(3)}_r(v)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{9|\frac{1}{2}-\frac{1}{p}|}N^{-2+\frac{1}{100}+\frac{2}{q'}-\frac{1}{100q'}+}\|v\|^7_{M_{p,q}},$$

and

$$\|N^{(3)}_r(v)-N^{(3)}_r(w)\|_{l^qM_{p,q}}\lesssim(1+|t|)^{9|\frac{1}{2}-\frac{1}{p}|}N^{-2+\frac{1}{100}+\frac{2}{q'}-\frac{1}{100q'}+}(\|v\|^6_{M_{p,q}}+\|w\|^6_{M_{p,q}})\|v-w\|_{M_{p,q}}.$$

Proof. Let us start with the operator $N^{(3)}_0$ and for simplicity of the presentation we will consider only the sum with the term $\tilde{q}^{2,t}_{1,n}$. As in the proof of Lemma 12 we have from (9) that for fixed n and μ_1 there are at most $o(|\mu_1|^+)$ many choices for n_1,n_2,n_3 (such that $(n-n_1)(n-n_3)=\mu_1$) and for fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1,m_2,m_3 (such that $(n_1-m_1)(n_1-m_3)=\mu_2$). Since the Fourier transform of the operator $\tilde{q}^{2,t}_{1,n}$ is localised around the interval Q_n, using the same argument as in Lemma 10 together with Lemma 13 we see that

$$\sum_{A_N(n)^c}\sum_{C_1^c}\|\tilde{q}^{2,t}_{1,n}(v_{m_1},\bar{v}_{m_2},v_{m_3},\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}}\lesssim$$
$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\sum_{A_N(n)^c}\sum_{C_1^c}\frac{\|u_{m_1}\|_p\|u_{m_2}\|_p\|u_{m_3}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p}{|n-n_1||n-n_3||(n-n_1)(n-n_3)+(n_1-m_1)(n_1-m_3)|}$$

and the sum of RHS is equal to

$$\sum_{A_N(n)^c}\sum_{C_1^c}\frac{\|u_{m_1}\|_p\|u_{m_2}\|_p\|u_{m_3}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p}{|\mu_1||\mu_1+\mu_2|}$$

which by Hölder's inequality is bounded above by

$$\Big(\sum_{A_N(n)^c}\sum_{C_1^c}\frac{1}{|\mu_1|^{q'}|\mu_1+\mu_2|^{q'}}|\mu_1|^+|\mu_2|^+\Big)^{\frac{1}{q'}}\Big(\sum_{A_N(n)^c}\sum_{C_1^c}\|u_{m_1}\|^q_2\|u_{m_2}\|^q_p\|u_{m_3}\|^q_p\|u_{n_2}\|^q_p\|u_{n_3}\|^q_p\Big)^{\frac{1}{q}}.$$

By a very crude estimate it is not difficult to see that the first sum behaves like the number $N^{-2+\frac{1}{100}+\frac{2}{q'}-\frac{1}{100q'}+}$. Then, by taking the l^q norm and applying Young's inequality for convolutions we are done. For the operator $N_r^{(3)}$ the proof is the same but in addition we use Lemma 7 for the operator $R_2^t - R_1^t$. $\square$

The operator that remains to be estimated is defined as

$$N^{(3)}(v)(n) := \sum_{A_N(n)^c} \sum_{C_1^c} \Big(\tilde{q}^{2,t}_{1,n}(N_1^t(v)(m_1), \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \tag{46}$$

$$\tilde{q}^{2,t}_{1,n}(v_{m_1}, \overline{N_1^t(v)(m_2)}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \ldots + \tilde{q}^{2,t}_{3,n}(v_{n_1}\bar{v}_{n_2}, v_{m_1}, \bar{v}_{m_2}, N_1^t(v)(m_3)) \Big),$$

which is the same as $N_r^{(3)}$ but in the place of the operator $R_2^t - R_1^t$ we have N_1^t. As before, we write

$$N^{(3)} = N_1^{(3)} + N_2^{(3)}, \tag{47}$$

where $N_1^{(3)}$ is the restriction of $N^{(3)}$ onto the set of frequencies

$$C_2 = \{|\tilde{\mu}_3| \leq 7^3 |\tilde{\mu}_2|^{1-\frac{1}{100}}\} \cup \{|\tilde{\mu}_3| \leq 7^3 |\mu_1|^{1-\frac{1}{100}}\}, \tag{48}$$

where $\tilde{\mu}_2 = \mu_1 + \mu_2$ and $\tilde{\mu}_3 = \mu_1 + \mu_2 + \mu_3$. The following is true:

Lemma 16.

$$\|N_1^{(3)}(v)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{10|\frac{1}{2}-\frac{1}{p}|} N^{-2+\frac{1}{100}+\frac{3}{q'}-\frac{2}{100q'}+} \|v\|^7_{M_{p,q}},$$

and

$$\|N_1^{(3)}(v) - N_1^{(3)}(w)\|_{l^q M_{p,q}} \lesssim (1+|t|)^{10|\frac{1}{2}-\frac{1}{p}|} N^{-2+\frac{1}{100}+\frac{3}{q'}-\frac{2}{100q'}+} (\|v\|^6_{M_{p,q}} + \|w\|^6_{M_{p,q}}) \|v-w\|_{M_{p,q}}.$$

Proof. Let us only consider the very first summand of the operator $N_1^{(3)}$, that is the operator $\tilde{q}^{2,t}_{1,n}$ with N_1^t acting on its first variable, since for the other summands similar considerations apply. For the proof we use again the divisor counting argument. From (9) it follows that for fixed n and μ_1 there are at most $o(|\mu_1|^+)$ many choices for n_1, n_2, n_3 ($\mu_1 = (n-n_1)(n-n_3)$, $n = n_1 - n_2 + n_3$). For fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1, m_2, m_3 ($\mu_2 = (n_1 - m_1)(n_1 - m_3)$, $n_1 = m_1 - m_2 + m_3$) and for fixed m_1 and μ_3 there are at most $o(|\mu_3|^+)$ many choices for k_1, k_2, k_3 ($\mu_3 = (m_1 - k_1)(m_1 - k_3)$, $m_1 = k_1 - k_2 + k_3$).

First, let us assume that our frequencies satisfy $|\tilde{\mu}_3| \lesssim |\tilde{\mu}_2|^{1-\frac{1}{100}}$. Since, $\tilde{\mu}_3 = \tilde{\mu}_2 + \mu_3$ we have $|\mu_3| \sim |\tilde{\mu}_2|$. Moreover, for fixed $|\tilde{\mu}_2|$ (equivalently, for fixed μ_1, μ_2) there are at most $O(|\tilde{\mu}_2|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_3$ and hence, for $\mu_3 = \tilde{\mu}_3 - \tilde{\mu}_2$. In addition, $|\mu_2| \lesssim \max(|\mu_1|, |\tilde{\mu}_2|)$ and we should recall that since we are on C_1^c we have $|\tilde{\mu}_2| = |\mu_1 + \mu_2| > 5^3 |\mu_1|^{1-\frac{1}{100}} > 5^3 N^{1-\frac{1}{100}}$. Then by the same localisation argument as in the proof of Lemma 10 together with Lemma 13 we estimate the expression

$$\sum_{A_N(n)^c}\sum_{C_1^c}\sum_{C_2}\|\tilde{q}^{2,t}_{1,n}(Q^{1,t}_{m_1}(v_{k_1},\bar{v}_{k_2},v_{k_3}),\bar{v}_{m_2},v_{m_3},\bar{v}_{n_2},v_{n_3})\|_{M_{p,q}}$$

by

$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\sum_{A_N(n)^c}\sum_{C_1^c}\sum_{C_2}\frac{\|e^{-it\partial_x^2}Q^{1,t}_{m_1}(v_{k_1},\bar{v}_{k_2},v_{k_3})\|_p\|u_{m_2}\|_p\|u_{m_3}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p}{|n-n_1||n-n_3||(n-n_1)(n-n_3)+(n_1-m_1)(n_1-m_3)|}=$$

$$(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}\sum_{A_N(n)^c}\sum_{C_1^c}\sum_{C_2}\frac{\|e^{-it\partial_x^2}Q^{1,t}_{m_1}(v_{k_1},\bar{v}_{k_2},v_{k_3})\|_p\|u_{m_2}\|_p\|u_{m_3}\|_p\|u_{n_2}\|_p\|u_{n_3}\|_p}{|\mu_1||\tilde{\mu}_2|}$$

and by Hölder's inequality we see that the sum is bounded above by

$$\Big(\sum_{\substack{|\mu_1|>N\\|\tilde{\mu}_2|>5^3N^{1-\frac{1}{100}}}}\frac{|\mu_1|^+|\mu_2|^+|\mu_3|^+|\tilde{\mu}_2|^{1-\frac{1}{100}}}{|\mu_1|^{q'}|\tilde{\mu}_2|^{q'}}\Big)^{\frac{1}{q'}}\times \tag{49}$$

$$\Big(\sum_{A_N(n)^c}\sum_{C_1^c}\sum_{C_2}\|e^{-it\partial_x^2}Q^{1,t}_{m_1}(v_{k_1},\bar{v}_{k_2},v_{k_3})\|_p^q\|u_{m_2}\|_p^q\|u_{m_3}\|_p^q\|u_{n_2}\|_p^q\|u_{n_3}\|_p^q\Big)^{\frac{1}{q}}.$$

The first sum is controlled by

$$\Big(\sum_{\substack{|\mu_1|>N\\|\tilde{\mu}_2|>5^3N^{1-\frac{1}{100}}}}\frac{1}{|\mu_1|^{q'-\epsilon}|\tilde{\mu}_2|^{q'-1+\frac{1}{100}-\epsilon}}\Big)^{\frac{1}{q'}}\lesssim\Big(N^{3(1-\frac{1}{100})-q'(2-\frac{1}{100})+\frac{1}{100^2}+}\Big)^{\frac{1}{q'}} \tag{50}$$

and with the use of Young's inequality at the second sum together with an estimate on the norm $\|e^{-it\partial_x^2}Q^{1,t}_{m_1}(v_{k_1},\bar{v}_{k_2},v_{k_3})\|_{M_{p,q}}$ we are done.

On the other hand, if $|\tilde{\mu}_3|\lesssim|\mu_1|^{1-\frac{1}{100}}$, then for fixed μ_1,μ_2 there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_3$ and hence for μ_3. After this observation, the calculations are exactly the same as before but the first sum of (49) becomes

$$\Big(\sum_{\substack{|\mu_1|>N\\|\tilde{\mu}_2|>5^3N^{1-\frac{1}{100}}}}\frac{1}{|\mu_1|^{q'-1+\frac{1}{100}-\epsilon}|\tilde{\mu}_2|^{q'-\epsilon}}\Big)^{\frac{1}{q'}}\lesssim\Big(N^{3-\frac{2}{100}-q'(2-\frac{1}{100})+}\Big)^{\frac{1}{q'}}. \tag{51}$$

Between the two exponents of N in (50) and (51) we see that (51) is the dominating one and the proof is complete. □

To the remaining part, namely $N_2^{(3)}$, we have to apply the differentiation by parts technique again. Note that here we only look at frequencies such that

$$|\tilde{\mu}_3| = |\mu_1 + \mu_2 + \mu_3| > 7^3|\mu_1|^{1-\frac{1}{100}} > 7^3 N^{1-\frac{1}{100}},$$

or equivalently, frequencies that are on the set C_2^c. Instead, we will present the general Jth step of the iteration procedure and prove the required Lemmata. To do this, we need to use the tree notation as it was introduced in [5].

2.1. **The Tree Notation and the Induction Step.** A tree T is a finite, partially ordered set with the following properties:

- For any $a_1, a_2, a_3, a_4 \in T$ if $a_4 \le a_2 \le a_1$ and $a_4 \le a_3 \le a_1$ then $a_2 \le a_3$ or $a_3 \le a_2$.
- There exists a maximum element $r \in T$, that is $a \le r$ for all $a \in T$ which is called the root.

We call the elements of T the **nodes** of the tree and in this content we will say that $b \in T$ is a **child** of $a \in T$ (or equivalently, that a is the **parent** of b) if $b \le a, b \ne a$ and for all $c \in T$ such that $b \le c \le a$ we have either $b = c$ or $c = a$.

A node $a \in T$ is called **terminal** if it has no children. A **nonterminal** node $a \in T$ is a node with exactly 3 children a_1, the left child, a_2, the middle child, and a_3, the right child. We define the sets

$$T^0 = \{\text{all nonterminal nodes}\}, \tag{52}$$

and

$$T^\infty = \{\text{all terminal nodes}\}. \tag{53}$$

Obviously, $T = T^0 \cup T^\infty$, $T^0 \cap T^\infty = \emptyset$ and if $|T^0| = j \in \mathbb{Z}_+$ we have $|T| = 3j + 1$ and $|T^\infty| = 2j + 1$. We denote the collection of trees with j parental nodes by

$$T(j) = \{T \text{ is a tree with } |T| = 3j + 1\}. \tag{54}$$

Next, we say that a sequence of trees $\{T_j\}_{j=1}^J$ is a **chronicle of J generations** if:

- $T_j \in T(j)$ for all $j = 1, 2, \ldots, J$.
- T_{j+1} is obtained by changing one of the terminal nodes of T_j into a nonterminal node with exactly 3 children, for all $j = 1, 2, \ldots, J-1$.

Let us also denote by $\mathcal{I}(J)$ the collection of trees of the Jth generation. It is easily checked by an induction argument that

$$|\mathcal{I}(J)| = 1 \cdot 3 \cdot 5 \ldots (2J-1) =: (2J-1)!!. \tag{55}$$

Given a chronicle $\{T_j\}_{j=1}^J$ of J generations we refer to T_J as an **ordered tree of the Jth generation**. We should keep in mind that the notion of ordered trees comes with associated chronicles. It includes not only the shape of the tree but also how it "grew".

Given an ordered tree T we define an **index function** $n : T \to \mathbb{Z}$ such that

- $n_a \approx n_{a_1} - n_{a_2} + n_{a_3}$ for all $a \in T^0$, where a_1, a_2, a_3 are the children of a,
- $n_a \not\approx n_{a_1}$ and $n_a \not\approx n_{a_3}$, for all $a \in T^0$,
- $|\mu_1| := 2|n_r - n_{r_1}||n_r - n_{r_3}| > N$, where r is the root of T,

and we denote the collection of all such index functions by $\mathcal{R}(T)$.

For the sake of completeness, as it was done in [5], given an ordered tree T with the chronicle $\{T_j\}_{j=1}^J$ and associated index functions $n \in \mathcal{R}(T)$, we need to keep track of the generations of frequencies. Fix an $n \in \mathcal{R}(T)$ and consider the very first tree T_1. Its nodes are the root r and its children r_1, r_2, r_3. We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_r, n_{r_1}, n_{r_2}, n_{r_3}).$$

From the definition of the index function we have

$$n^{(1)} \approx n_1^{(1)} - n_2^{(1)} + n_3^{(1)},\ n_1^{(1)} \not\approx n^{(1)} \not\approx n_3^{(1)}.$$

The ordered tree T_2 of the second generation is obtained from T_1 by changing one of its terminal nodes $a = r_k \in T_1^\infty$ for some $k = 1, 2, 3$ into a nonterminal node. Then, the second generation of frequencies is defined by

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Thus, we have $n^{(2)} = n_k^{(1)}$ for some $k = 1, 2, 3$ and from the definition of the index function we have

$$n^{(2)} \approx n_1^{(2)} - n_2^{(2)} + n_3^{(2)},\ n_1^{(2)} \not\approx n^{(2)} \not\approx n_3^{(2)}.$$

This should be compared with what happened in the calculations we presented before when passing from the first step of the iteration process into the second step. Every time we apply the differentiation by parts technique we introduce a new set of frequencies.

After $j-1$ steps, the ordered tree T_j of the jth generation is obtained from T_{j-1} by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a nonterminal node. Then, the jth generation frequencies are defined as

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}),$$

and we have $n^{(j)} = n_k^{(m)} (= n_a)$ for some $m = 1, 2, \ldots, j-1$ and $k = 1, 2, 3$, since this corresponds to the frequency of some terminal node in T_{j-1}. In addition, from the definition of the index function we have

$$n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)},\ n_1^{(j)} \not\approx n^{(j)} \not\approx n_3^{(j)}.$$

Finally, we use μ_j to denote the corresponding phase factor introduced at the jth generation. That is,

$$\mu_j = 2(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)}), \tag{56}$$

and we also introduce the quantities

$$\tilde{\mu}_J = \sum_{j=1}^{J} \mu_j,\ \hat{\mu}_J = \prod_{j=1}^{J} \tilde{\mu}_j. \tag{57}$$

We should keep in mind that everytime we apply differentiation by parts and split the operators, we need to control the new frequencies that arise from this procedure. For this reason we need to define the sets (see (36) and (48)):

$$C_J := \{|\tilde{\mu}_{J+1}| \le (2J+3)^3 |\tilde{\mu}_J|^{1-\frac{1}{100}}\} \cup \{|\tilde{\mu}_{J+1}| \le (2J+3)^3 |\mu_1|^{1-\frac{1}{100}}\}. \tag{58}$$

Let us see how to use this notation and terminology in our calculations. On the very first step, $J = 1$, we have only one tree, the root node r and its three children r_1, r_2, r_3 (sometimes, when it is clear from the context, we will identify the nodes and the frequencies assigned to them, that is, we have the root $n = n_r$ and its three children $n_{r_1} = n_1, n_{r_2} = n_2, n_{r_3} = n_3$) and we have only one operator that needs to be controlled in order to proceed further, namely $\tilde{q}_n^{1,t} := \tilde{Q}_n^{1,t}$.

On the second step, $J = 2$, we have three operators $\tilde{q}^{2,t}_{n,n_1} := \tilde{q}^{2,t}_{1,n}, \tilde{q}^{2,t}_{n,n_2} := \tilde{q}^{2,t}_{2,n}, \tilde{q}^{2,t}_{n,n_3} := \tilde{q}^{2,t}_{3,n}$ that play the same role as $\tilde{q}_n^{1,t}$ did for the first step. Let us observe that for each one of these operators we must have estimates on their L^2 norms in order to be able and continue the iteration. These estimates were provided by Lemmata 9 and 13.

On the general Jth step we will have $|\mathcal{I}(J)|$ operators of the $\tilde{q}^{J,t}_{T^0,\mathbf{n}}$ "type" each one corresponding to one of the ordered trees of the Jth generation, $T \in T(J)$, where $\mathbf{n}$ is an arbitrary fixed index function on T. We have the subindices T^0 and $\mathbf{n}$ because each one of these operators has Fourier transform supported on the cubes with centers the frequencies assigned to the nodes that belong to T^0.

Let us denote by T_α all the nodes of the ordered tree T that are descendants of the node $\alpha \in T^0$, i.e. $T_\alpha = \{\beta \in T : \beta \le \alpha,\ \beta \ne \alpha\}$.

We also need to define the **principal and final "signs" of a node** $a \in T$ which are functions from the tree T into the set $\{\pm 1\}$:

$$\mathrm{psgn}(a) = \begin{cases} +1, & a \text{ is not the middle child of his father} \\ +1, & a = r, \text{ the root node} \\ -1, & a \text{ is the middle child of his father} \end{cases} \tag{59}$$

$$\mathrm{fsgn}(a) = \begin{cases} +1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an even number of middle predecessors} \\ -1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an odd number of middle predecessors} \\ -1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an even number of middle predecessors} \\ +1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an odd number of middle predecessors,} \end{cases} \tag{60}$$

where the root node $r \in T$ is not considered a middle father.

The operators $\tilde{q}^{J,t}_{T^0,\mathbf{n}}$ are defined through their Fourier transforms as

$$\mathcal{F}(\tilde{q}^{J,t}_{T^0,\mathbf{n}}(\{w_{n_\beta}\}_{\beta\in T^\infty}))(\xi) = e^{-it\xi^2}\mathcal{F}(R^{J,t}_{T^0,\mathbf{n}}(\{e^{-it\partial_x^2}w_{n_\beta}\}_{\beta\in T^\infty}))(\xi), \tag{61}$$

where the operator $R^{J,t}_{T^0,\mathbf{n}}$ acts on the functions $\{w_{n_\beta}\}_{\beta\in T^\infty}$ as

$$R^{J,t}_{T^0,\mathbf{n}}(\{w_{n_\beta}\}_{\beta\in T^\infty})(x) = \int_{\mathbb{R}^{2J+1}} K^{(J)}_{T^0}(x,\{x_\beta\}_{\beta\in T^\infty})\Big[\otimes_{\beta\in T^\infty} w_{n_\beta}(x_\beta)\Big]\prod_{\beta\in T^\infty} dx_\beta, \tag{62}$$

and the kernel $K^{(J)}_{T^0,\mathbf{n}}$ is defined as

$$K^{(J)}_{T^0,\mathbf{n}}(x,\{x_\beta\}_{\beta\in T^\infty}) = \mathcal{F}^{-1}(\tilde{\rho}^{(J)}_{T^0,\mathbf{n}})(\{x-x_\beta\}_{\beta\in T^\infty}). \tag{63}$$

Here is the formula for the function $\tilde{\rho}^{(J)}_{T^0,\mathbf{n}}$ with ($|T^\infty| = 2J+1$)-variables, ξ_β, $\beta\in T^\infty$:

$$\tilde{\rho}^{(J)}_{T^0,\mathbf{n}}(\{\xi_\beta\}_{\beta\in T^\infty}) = \Big[\prod_{\beta\in T^\infty}\tilde{\sigma}_{n_\beta}(\xi_\beta)\Big]\Big[\prod_{\alpha\in T^0}\sigma_{n_\alpha}\Big(\sum_{\beta\in T^\infty\cap T_\alpha}\mathrm{fsgn}(\beta)\ \xi_\beta\Big)\Big]\frac{1}{\hat{\mu}_T}. \tag{64}$$

We also define the function

$$\rho^{(J)}_{T^0,\mathbf{n}}(\{\xi_\beta\}_{\beta\in T^\infty}) = \Big[\prod_{\alpha\in T^0}\sigma_{n_\alpha}\Big(\sum_{\beta\in T^\infty\cap T_\alpha}\mathrm{fsgn}(\beta)\ \xi_\beta\Big)\Big]\frac{1}{\hat{\mu}_T}, \tag{65}$$

where we denote by

$$\hat{\mu}_T = \prod_{\alpha\in T^0}\tilde{\mu}_\alpha,\ \tilde{\mu}_\alpha = \sum_{\beta\in T^0\setminus T_\alpha}\mu_\beta, \tag{66}$$

and for $\beta\in T^0$ we have

$$\mu_\beta = 2(\xi_\beta-\xi_{\beta_1})(\xi_\beta-\xi_{\beta_3}), \tag{67}$$

where we impose the relation $\xi_\alpha = \xi_{\alpha_1}-\xi_{\alpha_2}+\xi_{\alpha_3}$ for every $\alpha\in T^0$ that appears in the calculations until we reach the terminal nodes of T^∞. This is because in the definition of the function $\rho^{J,t}_{T^0}$ we need the variables "ξ" to be assigned only at the terminal nodes of the tree T. We use the notation μ_β in similarity to μ_j of equation (56) because this is the "continuous" version of the discrete case. In addition, the variables $\xi_{\alpha_1},\xi_{\alpha_2},\xi_{\alpha_3}$ that appear in the expression (64) are supported in such a way that $\xi_{\alpha_1}\approx n_{\alpha_1},\xi_{\alpha_2}\approx n_{\alpha_2},\xi_{\alpha_3}\approx n_{\alpha_3}$. This is because the functions σ_{n_α} are supported in such a way. Therefore, $|\hat{\mu}_T|\sim|\hat{\mu}_J|$.

For the induction step of our iteration process we need the following lemma which should be compared with Lemmata 9 and 13:

Lemma 17. *For* $2 \le p \le \infty$

$$\|R^{J,t}_{T^0,\mathbf{n}}(\{v_{n_\beta}\}_{\beta\in T^\infty})\|_p \lesssim \Big(\prod_{\beta\in T^\infty}\|v_{n_\beta}\|_p\Big)\frac{\Big((J+1)!^A\ J^{\frac{3J}{2}}\Big)^{1-\frac{2}{p}}}{|\hat{\mu}_T|}, \tag{68}$$

for every tree $T\in T(J)$ *and index function* $\mathbf{n}\in\mathcal{R}(T)$.

Proof. We use interpolation between the L^2 estimate, which is done in exactly the same way as in Lemma 9, and the L^∞ estimate where we use that for $s>\frac{2J+1}{2}$ the embedding $H^s(\mathbb{R}^{2J+1})\hookrightarrow\mathcal{F}L^1(\mathbb{R}^{2J+1})$ is continuous. By Hölder's inequality the embedding constant is bounded above by the quantity

$$|\mathbb{S}^{2J}|^{\frac{1}{2}}\ \Big(\int_0^\infty\frac{r^{2J}}{(1+r)^{2J+2}}\ dr\Big)^{\frac{1}{2}}, \tag{69}$$

where $|\mathbb{S}^{2J}|$ denotes the surface measure of the $2J$-dimensional sphere in $\mathbb{R}^{2J+1}$. It is known that

$$|\mathbb{S}^{2J}|=\frac{2^{J+1}\pi^J}{(2J-1)!!},$$

and the integral part of (69) decays like a polynomial in J, which can be neglected compared to the double factorial decay of the surface measure of $\mathbb{S}^{2J}$. Thus, the embedding constant decays like $1/J^{\frac{J}{2}}$.

Since the function $\tilde{\rho}^{(J)}_{T^0,\mathbf{n}}$ has $2J+1$ variables and consists of $4J+1$ factors and we have to calculate all possible derivatives of order r up to the order $J+1$ we obtain

$$\sum_{r=0}^{J+1}(2J+1)^r(4J+1)^r=\frac{[(2J+1)(4J+1)]^{J+2}-1}{(2J+1)(4J+1)-1}\sim J^{2J}$$

terms in total. Let us notice that the more distributed the derivatives are on the product of functions that consist the function $\tilde{\rho}^{(J)}_{T^0,\mathbf{n}}$ the smaller constants we obtain in terms of growth in J compared to $(J+1)!^A$. The factorial $(J+1)!^A$ appears in the calculations because we take $J+1$ derivatives of the σ-functions. Finally, let us observe that a factorial $(J+1)!$ appears in the calculations too, when all $J+1$ derivatives fall in terms of the form $1/x$, but since $A>1$, $(J+1)!^A$ dominates. □

For the rest of the paper, let us use the notation

$$d_J:=(J+1)!^A\ J^{\frac{3J}{2}}. \tag{70}$$

By Stirling's formula we obtain that d_J has the following behaviour for large J

$$d_J\sim\big(\sqrt{2\pi(J+1)}\big)^A\ \Big(\frac{J+1}{e}\Big)^{A(J+1)}\ J^{\frac{3J}{2}}\sim\frac{J^{\frac{A}{2}}}{e^{AJ}}\ J^{(\frac{3}{2}+A)J}. \tag{71}$$

Given an index function $\mathbf{n}$ and $2J+1$ functions $\{v_{n_\beta}\}_{\beta\in T^\infty}$ and $\alpha \in T^\infty$ we define the action of the operator N_1^t (see (20)) on the set $\{v_{n_\beta}\}_{\beta\in T^\infty}$ to be the same set as before but with the difference that we have substituted the function v_{n_α} by the new function $N_1^t(v)(n_\alpha)$. We will denote this new set of functions $N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta\in T^\infty})$. Similarly, the action of the operator $R_2^t - R_1^t$ (see (19)) on the set of functions $\{v_{n_\beta}\}_{\beta\in T^\infty}$ will be denoted by $(R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta\in T^\infty})$.

The operator of the Jth step, $J \geq 2$, that we want to estimate is given by the formula:

$$N_2^{(J)}(v)(n) := \sum_{T\in T(J-1)} \sum_{\alpha\in T^\infty} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \tilde{q}^{J-1,t}_{T^0}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta\in T^\infty})). \tag{72}$$

Applying differentiation by parts on the Fourier side (keep in mind that from the splitting procedure we are on the sets $A_N(n)^c, C_1^c, \ldots, C_{J-1}^c$) we obtain the expression

$$N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + N_r^{(J+1)}(v)(n) + N^{(J+1)}(v)(n), \tag{73}$$

where

$$N_0^{(J+1)}(v)(n) := \sum_{T\in T(J)} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \tilde{q}^{J,t}_{T^0,\mathbf{n}}(\{v_{n_\beta}\}_{\beta\in T^\infty}), \tag{74}$$

and

$$N_r^{(J+1)}(v)(n) := \sum_{T\in T(J)} \sum_{\alpha\in T^\infty} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \tilde{q}^{J,t}_{T^0,\mathbf{n}}((R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta\in T^\infty})), \tag{75}$$

and

$$N^{(J+1)}(v)(n) := \sum_{T\in T(J)} \sum_{\alpha\in T^\infty} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \tilde{q}^{J,t}_{T^0,\mathbf{n}}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta\in T^\infty})). \tag{76}$$

We also split the operator $N^{(J+1)}$ as the sum

$$N^{(J+1)} = N_1^{(J+1)} + N_2^{(J+1)}, \tag{77}$$

where $N_1^{(J+1)}$ is the restriction of $N^{(J+1)}$ onto C_J and $N_2^{(J+1)}$ onto C_J^c. First, we generalise Lemma 15 by estimating the operators $N_0^{(J+1)}$ and $N_r^{(J+1)}$:

Lemma 18.

$$\|N_0^{(J+1)}(v)\|_{l^q M_{p,q}} \lesssim d_J^{1-\frac{2}{p}} (1+|t|)^{(2J+2)|\frac{1}{2}-\frac{1}{p}|} N^{-\frac{(q'-1)}{q'}J + \frac{(q'-1)}{100q'}(J-1)+} \|v\|^{2J+1}_{M_{p,q}},$$

and

$$\|N_0^{(J+1)}(v) - N_0^{(J+1)}(w)\|_{l^q M_{p,q}} \lesssim$$

$$d_J^{1-\frac{2}{p}}(1+|t|)^{(2J+2)|\frac{1}{2}-\frac{1}{p}|}N^{-\frac{(q'-1)}{q'}J+\frac{(q'-1)}{100q'}(J-1)+}(\|v\|_{M_{p,q}}^{2J} + \|w\|_{M_{p,q}}^{2J})\|v-w\|_{M_{p,q}}.$$

$$\|N_r^{(J+1)}(v)\|_{l^q M_{p,q}} \lesssim d_J^{1-\frac{2}{p}}(1+|t|)^{(2J+5)|\frac{1}{2}-\frac{1}{p}|}N^{-\frac{(q'-1)}{q'}J+\frac{(q'-1)}{100q'}(J-1)+}\|v\|_{M_{p,q}}^{2J+3},$$

and

$$\|N_r^{(J+1)}(v) - N_r^{(J+1)}(w)\|_{l^q M_{p,q}} \lesssim$$

$$d_J^{1-\frac{2}{p}}(1+|t|)^{(2J+5)|\frac{1}{2}-\frac{1}{p}|}N^{-\frac{(q'-1)}{q'}J+\frac{(q'-1)}{100q'}(J-1)+}(\|v\|_{M_{p,q}}^{2J+2} + \|w\|_{M_{p,q}}^{2J+2})\|v-w\|_{M_{p,q}}.$$

Proof. As in the proof of Lemma 15 for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(j)}, n_2^{(j)}, n_3^{(j)}$. In addition, let us observe that μ_j is determined by $\tilde{\mu}_1, \dots, \tilde{\mu}_j$ and $|\mu_j| \lesssim \max(|\tilde{\mu}_{j-1}|, |\tilde{\mu}_j|)$, since $\mu_j = \tilde{\mu}_j - \tilde{\mu}_{j-1}$. Then, for a fixed tree $T \in T(J)$, since the operator $\tilde{q}^{J,t}_{T^0,\mathbf{n}}$ has Fourier transform localised around the interval Q_n, using the same argument as in Lemma 10 together with Lemma 17 we obtain the bound (remember that $|\hat{\mu}_T| \sim |\hat{\mu}_J| = \prod_{k=1}^J |\tilde{\mu}_k|$):

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \|\tilde{q}^{J,t}_{T^0,\mathbf{n}}(\{v_\beta\}_{\beta\in T^\infty})\|_{M_{p,q}} \lesssim (1+|t|)^{|\frac{1}{2}-\frac{1}{p}|} d_J^{1-\frac{2}{p}} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \Big(\prod_{\beta\in T^\infty} \|u_{n_\beta}\|_p\Big)\Big(\prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|}\Big),$$

and by Hölder's inequality the sum is bounded from above by

$$\Big(\sum_{\substack{|\mu_1|>N\\ |\tilde{\mu}_j|>(2j+1)^3N^{1-\frac{1}{100}}\\ j=2,\dots,J}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^{q'}}|\mu_k|^+\Big)^{\frac{1}{q'}}\Big(\sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \prod_{\beta\in T^\infty} \|u_{n_\beta}\|_p^q\Big)^{\frac{1}{q}}. \tag{78}$$

The first sum behaves like $N^{-\frac{(q'-1)}{q'}J+\frac{(q'-1)}{100q'}(J-1)+}$ and for the remaining part we take the l^q norm in n and by the use of Young's inequality we are done.

At this point, let us observe the following: There is an extra factor $\sim J$ when we estimate the differences $N_0^{(J+1)}(v) - N_0^{(J+1)}(w)$ since $|a^{2J+1} - b^{2J+1}| \lesssim (\sum_{j=1}^{2J+1} a^{2J+1-j}b^{j-1})|a-b|$ has $O(J)$ many terms. Also, we have $c_J = |\mathcal{I}(J)|$ many summands in the operator $N_0^{(J+1)}$ since there are c_J many trees of the Jth generation and c_J behaves like a double factorial, namely $(2J-1)!!$ (see (55)). However, these observations do not cause any problem since the constant that we obtain from estimating the first sum of (78) decays like a fractional power of a double factorial in J, or to be more precise we have

(79) $$\frac{c_J\ d_J^{1-\frac{2}{p}}}{\prod_{j=2}^J (2j+1)^{3\cdot\frac{q'-1}{q'}-}} = \frac{d_J^{1-\frac{2}{p}}}{(2J+1)^{3-\frac{3}{q'}-}[(2J-1)!!]^{2-\frac{3}{q'}}} \sim \frac{J^{(\frac{3}{2}+A)(1-\frac{2}{p})J}}{J^{(2-\frac{3}{q'})J}}.$$

In order to maintain the decay in the denominator we must have $2-\frac{3}{q'}-\frac{3}{2}-A+\frac{2A+3}{p}>0$ which is equivalent to the restriction $p<\frac{2q'(2A+3)}{(2A-1)q'+6}$. This is true by the assumptions of Theorem 4 together with (10). For the operator $N_r^{(J+1)}$ the proof is the same but in addition we use Lemma 7 for the operator $R_2^t - R_1^t$. □

The estimate for the operator $N_1^{(J+1)}$, which generalises Lemma 16, is the following:

Lemma 19.

$$\|N_1^{(J+1)}(v)\|_{l^q M_{p,q}} \lesssim d_J^{1-\frac{2}{p}}(1+|t|)^{(2J+6)|\frac{1}{2}-\frac{1}{p}|} N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+} \|v\|^{2J+3}_{M_{p,q}},$$

and

$$\|N_1^{(J+1)}(v)-N_1^{(J+1)}(w)\|_{l^q M_{p,q}} \lesssim d_J^{1-\frac{2}{p}}(1+|t|)^{(2J+6)|\frac{1}{2}-\frac{1}{p}|}$$
$$N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+}(\|v\|^{2J+2}_{M_{p,q}}+\|w\|^{2J+2}_{M_{p,q}})\|v-w\|_{M_{p,q}}.$$

Proof. As before, for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(1)}, n_2^{(1)}, n_3^{(1)}$ and note that μ_j is determined by $\tilde\mu_1,\dots,\tilde\mu_j$.

Let us assume that $|\tilde\mu_{J+1}| = |\tilde\mu_J + \mu_{J+1}| \lesssim (2J+3)^3 |\tilde\mu_J|^{1-\frac{1}{100}}$ holds in (58). Then, $|\mu_{J+1}| \lesssim |\tilde\mu_J|$ and for fixed $\tilde\mu_J$ there are at most $o(|\tilde\mu_J|^{1-\frac{1}{100}})$ many choices for $\tilde\mu_{J+1}$ and therefore, for $\mu_{J+1} = \tilde\mu_{J+1} - \tilde\mu_J$. For a fixed tree $T\in T(J)$ and $\alpha\in T^\infty$, since the operator $\tilde q^{J,t}_{T^0,\mathbf{n}}$ has Fourier transform localised around the interval Q_n, by Lemma 17 we arrive at the upper bound (remember that $|\hat\mu_T| \sim |\hat\mu_J| = \prod_{k=1}^J |\tilde\mu_k|$):

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \|\tilde q^{J,t}_{T^0,\mathbf{n}}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta\in T^\infty}))\|_{M_{p,q}} \lesssim d_J^{1-\frac{2}{p}}(1+|t|)^{|\frac{1}{2}-\frac{1}{p}|}$$
$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \Big(\|e^{-it\partial_x^2}Q^{1,t}_{n_\alpha}(v_{n_{\alpha_1}},\bar v_{n_{\alpha_2}},v_{n_{\alpha_3}})\|_p \prod_{\beta\in T^\infty\setminus\{\alpha\}}\|u_{n_\beta}\|_p\Big)\Big(\prod_{k=1}^J\frac{1}{|\tilde\mu_k|}\Big),$$

and by Hölder's inequality we bound the sum by

(80) $$\Big(\sum_{\substack{|\mu_1|>N\\ |\tilde\mu_j|>(2j+1)^3N^{1-\frac{1}{100}}\\ j=2,\dots,J}} |\tilde\mu_J|^{1-\frac{1}{100}+}\prod_{k=1}^J\frac{1}{|\tilde\mu_k|^{q'}}|\mu_k|^+\Big)^{\frac{1}{q'}}$$

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \Big(\|e^{-it\partial_x^2}Q^{1,t}_{n_\alpha}(v_{n_{\alpha_1}},\bar{v}_{n_{\alpha_2}},v_{n_{\alpha_3}})\|_p^q \prod_{\beta\in T^\infty\setminus\{\alpha\}} \|u_{n_\beta}\|_p^q\Big)^{\frac{1}{q}}.$$

An easy calculation shows that the first sum behaves like $N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+}$ and then by taking the l^q norm with the use of Young's inequality and an estimate on the norm $\|e^{-it\partial_x^2}Q^{1,t}_{n_\alpha}(v_{n_{\alpha_1}},\bar{v}_{n_{\alpha_2}},v_{n_{\alpha_3}})\|_{M_{p,q}}$ we are done.

If $|\tilde{\mu}_{J+1}|\lesssim(2J+3)^3|\mu_1|^{1-\frac{1}{100}}$ holds in (58), then for fixed μ_j, $j=1,\ldots,J$, there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for μ_{J+1}. The same argument as above leads us to exactly the same expressions as in (80) but with the first sum replaced by the following:

$$\Big(\sum_{\substack{|\mu_1|>N\\ |\tilde{\mu}_j|>(2j+1)^3N^{1-\frac{1}{100}}\\ j=2,\ldots,J}} |\mu_1|^{1-\frac{1}{100}}\prod_{k=1}^{J}\frac{1}{|\tilde{\mu}_k|^{q'}}|\mu_k|^+\Big)^{\frac{1}{q'}},$$

which again is bounded from above by $N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+}$ and the proof is complete. □

Remark 20. For $s>0$ we have to observe that all previous Lemmata hold true if we replace the $l^qM_{p,q}$ norm by the $l^q_sM_{p,q}$ norm and the $M_{p,q}(\mathbb{R})$ norm by the $M^s_{p,q}(\mathbb{R})$ norm. To see this, consider $n^{(j)}$ large. Then, there exists at least one of $n_1^{(j)},n_2^{(j)},n_3^{(j)}$ such that $|n_k^{(j)}|\geq\frac{1}{3}|n^{(j)}|$, $k\in\{1,2,3\}$, since we have the relation $n^{(j)}=n_1^{(j)}-n_2^{(j)}+n_3^{(j)}$. Therefore, in the estimates of the Jth generation, there exists at least one frequency $n_k^{(j)}$ for some $j\in\{1,\ldots,J\}$ with the property

$$\langle n\rangle^s\leq 3^{js}\langle n_k^{(j)}\rangle^s\leq 3^{Js}\langle n_k^{(j)}\rangle^s.$$

This exponential growth does not affect our calculations due to the double factorial decay in the denominator of (79).

Remark 21. Notice that all estimates that appear in the previous lemmata of this section are true for all values of $p\in[2,\infty]$, $q\in[1,\infty]$ and $s\geq0$.

2.2. **Existence of Weak Solutions.** In this subsection the calculations are the same as in [5] (and [7]) where we just need to replace the L^2 (or the $M_{2,q}$) norm by the $M_{p,q}(\mathbb{R})$ norm. We will present them for the sake of completion.

Let us start by defining the partial sum operator $\Gamma^{(J)}_{v_0}$ as

$$\Gamma^{(J)}_{v_0}v(t)=v_0+\sum_{j=2}^{J}N_0^{(j)}(v)(n)-\sum_{j=2}^{J}N_0^{(j)}(v_0)(n) \tag{81}$$

$$+\int_0^t R_1^\tau(v)(n)+R_2^\tau(v)(n)+\sum_{j=2}^J N_r^{(j)}(v)(n)+\sum_{j=1}^J N_1^{(j)}(v)(n)\;d\tau,$$

where we have $N_1^{(1)}:=N_{11}^t$ from (22), $N_0^{(2)}:=N_{21}^t$ from (26), $N_1^{(2)}:=N_{31}^t$ from (37) and $N_r^{(2)}:=N_4^t$ from (34) and $v_0\in M_{p,q}(\mathbb{R})$ is a fixed function.

In the following we will denote by $X_T=C([0,T],M_{p,q}(\mathbb{R}))$. Our goal is to show that the series appearing on the RHS of (81) converge absolutely in X_T for sufficiently small $T>0$, if $v\in X_T$, even for $J=\infty$. Indeed, by Lemmata 7, 8, 18, and 19 we obtain (we assume that $T<1$ so that the quantity $(1+T)^{(2J+6)|\frac{1}{2}-\frac{1}{p}|}$ is an exponential in J independent of T which can be neglected by making N possibly larger)

$$\begin{aligned}(82)\qquad \|\Gamma^{(J)}_{v_0}v\|_{X_T}\leq \|v_0\|_{M_{p,q}}+C\sum_{j=2}^J N^{-(1-\frac{1}{q'})(j-1)+\frac{q'-1}{100q'}(j-2)+}(\|v\|^{2j-1}_{X_T}+\|v_0\|^{2j-1}_{M_{p,q}})\\ +CT\Big[\|v\|^3_{X_T}+\sum_{j=2}^J N^{-(1-\frac{1}{q'})(j-1)+\frac{q'-1}{100q'}(j-2)+}\|v\|^{2j+1}_{X_T}\\ +N^{\frac{1}{q'}+}\|v\|^3_{X_T}+\sum_{j=2}^J N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-2)+}\|v\|^{2j+1}_{X_T}\Big].\end{aligned}$$

Let us assume that $\|v_0\|_{M_{p,q}}\leq R$ and $\|v\|_{X_T}\leq\tilde{R}$, with $\tilde{R}\geq R\geq 1$. From (82) we have

(83)

$$\begin{aligned}\|\Gamma^{(J)}_{v_0}v\|_{X_T}\leq R+CN^{\frac{1}{q'}-1+}R^3\sum_{j=0}^{J-2}(N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}R^2)^j+CN^{\frac{1}{q'}-1+}\tilde{R}^3\sum_{j=0}^{J-2}(N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2)^j\\ +CT\Big[(1+N^{\frac{1}{q'}+})\tilde{R}^3+CN^{\frac{1}{q'}-1+}\tilde{R}^5\sum_{j=0}^{J-2}(N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2)^j\\ +N^{\frac{2}{q'}-1-\frac{1}{100q'}+}\tilde{R}^5\sum_{j=0}^{J-2}(N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2)^j\Big].\end{aligned}$$

We choose $N=N(\tilde{R})$ large enough, such that $N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2=N^{99\frac{1-q'}{100q'}}\tilde{R}^2\leq\frac{1}{2}$, or equivalently,

$$(84)\qquad N\geq(2\tilde{R}^2)^{\frac{100q'}{99(q'-1)}},$$

so that the geometric series on the RHS of (83) converge and are bounded by 2. Therefore, we arrive at

$$\|\Gamma_{v_0}^{(J)}v\|_{X_T} \leq R + 2CN^{\frac{1}{q'}-1+}R^3 + 2CN^{\frac{1}{q'}-1+}\tilde{R}^3 \tag{85}$$
$$+CT\Big[(1+N^{\frac{1}{q'}+})\tilde{R}^2 + 2N^{\frac{1}{q'}-1+}\tilde{R}^4 + 2N^{\frac{199-100q'}{100q'}+}\tilde{R}^4\Big]\tilde{R},$$

and we choose $T > 0$ sufficiently small such that

$$CT\Big[(1+N^{\frac{1}{q'}+})\tilde{R}^2 + 2N^{\frac{1}{q'}-1+}\tilde{R}^4 + 2N^{\frac{199-100q'}{100q'}+}\tilde{R}^4\Big] < \frac{1}{10}. \tag{86}$$

With the use of (84) we see that $2CN^{\frac{1}{q'}-1+}\tilde{R}^3 \leq CN^{\frac{1-q'}{100q'}+}\tilde{R}$ and by further imposing N to be sufficiently large such that

$$CN^{\frac{1-q'}{100q'}+} < \frac{1}{10}, \tag{87}$$

we have

$$\|\Gamma_{v_0}^{(J)}v\|_{X_T} \leq R + \frac{R}{10} + \frac{\tilde{R}}{5} = \frac{11}{10}R + \frac{1}{5}\tilde{R}. \tag{88}$$

Thus, for sufficiently large N and sufficiently small $T > 0$ the partial sum operators $\Gamma_{v_0}^{(J)}$ are well defined in X_T, for every $J \in \mathbb{N} \cup \{\infty\}$. We will write Γ_{v_0} for $\Gamma_{v_0}^{(\infty)}$.

Our next step is, given an initial datum $v_0 \in M_{p,q}(\mathbb{R})$ to construct a solution $v \in X_T$ in the sense of Definition 3. To this end, let $s > \frac{1}{q'}$ (so that $M_{p,q}^s(\mathbb{R})$ is a Banach algebra that embeds in $M_{p,q}(\mathbb{R}) \cap C_b(\mathbb{R})$) and consider a sequence $\{v_0^{(m)}\}_{m\in\mathbb{N}} \in M_{p,q}^s(\mathbb{R}) \subset M_{p,q}(\mathbb{R})$ whose Fourier transforms are all compactly supported (thus, all $v_0^{(m)}$ are smooth functions) and such that $v_0^{(m)} \to v_0$ in $M_{p,q}(\mathbb{R})$ as $m \to \infty$. Let $R = \|v_0\|_{M_{p,q}} + 1$ and we can assume that $\|v_0^{(m)}\|_{M_{p,q}} \leq R$, for all $m \in \mathbb{N}$. Denote by $v^{(m)}$ the local in time solution of NLS (1) in $M_{p,q}^s(\mathbb{R})$ with initial condition $v_0^{(m)}$. It satisfies the Duhamel formula

$$v^{(m)}(t) = v_0^{(m)} + i\int_0^t N_1^\tau(v^{(m)}) - R_1^\tau(v^{(m)}) + R_2^\tau(v^{(m)})\, d\tau = \tag{89}$$
$$v_0^{(m)} + \sum_{j=2}^\infty N_0^{(j)}(v^{(m)})(n) - \sum_{j=2}^\infty N_0^{(j)}(v_0^{(m)})(n)$$
$$+\int_0^t R_1^\tau(v^{(m)})(n) + R_2^\tau(v^{(m)})(n) + \sum_{j=2}^\infty N_r^{(j)}(v^{(m)})(n) + \sum_{j=1}^\infty N_1^{(j)}(v^{(m)})(n)\, d\tau = \Gamma_{v_0^{(m)}}v^{(m)},$$

and we will show that this holds in X_T for the same time $T = T(R) > 0$ independent of $m \in \mathbb{N}$. Indeed, fix $m \in \mathbb{N}$ and observe that the norm $\|v^{(m)}\|_{X_t} = \|v^{(m)}\|_{C([0,t],M_{p,q})}$ is continuous in t. Since $\|v_0^{(m)}\|_{M_{p,q}} \leq R$ there is a time $T_1 > 0$ such that $\|v^{(m)}\|_{X_{T_1}} \leq 4R$.

Then, by repeating the previous calculations with $\tilde{R} = 4R$ and keeping one of the factors as $\|v^{(m)}\|_{X_{T_1}}$ we get

$$\|v^{(m)}\|_{X_{T_1}} = \|\Gamma_{v_0^{(m)}} v^{(m)}\|_{X_{T_1}} \leq \frac{11}{10}R + \frac{1}{5}\|v^{(m)}\|_{X_{T_1}}, \tag{90}$$

if N and T_1 satisfy (84), (86) and (87). Therefore, we have

$$\|v^{(m)}\|_{X_{T_1}} \leq \frac{19}{10}R < 2R. \tag{91}$$

Thus, from the continuity of $t \to \|v^{(m)}\|_{X_t}$, there is $\epsilon > 0$ such that $\|v^{(m)}\|X_{T_1+\epsilon} \leq 4R$. Then again, from (90) and (91) with $T_1+\epsilon$ in place of T_1 we derive that $\|v^{(m)}\|_{X_{T_1+\epsilon}} \leq 2R$ as long as N and $T_1+\epsilon$ satisfy (84), (86) and (87). By observing that these conditions are independent of $m \in \mathbb{N}$ we obtain a time interval $[0, T]$ such that $\|v^{(m)}\|_{X_T} \leq 2R$ for all $m \in \mathbb{N}$.

A similar computation on the difference, by possibly taking larger N and smaller T leads to the estimate

$$\|v^{(m_1)} - v^{(m_2)}\|_{X_T} = \|\Gamma_{v_0^{(m_1)}} v^{(m_1)} - \Gamma_{v_0^{(m_2)}} v^{(m_2)}\|_{X_T} \leq \tag{92}$$

$$(1 + \frac{1}{10})\|v_0^{(m_1)} - v_0^{(m_2)}\|_{M_{p,q}} + \frac{1}{5}\|v^{(m_1)} - v^{(m_2)}\|_{X_T},$$

which implies

$$\|v^{(m_1)} - v^{(m_2)}\|_{X_T} \leq c\, \|v_0^{(m_1)} - v_0^{(m_2)}\|_{M_{p,q}}, \tag{93}$$

for some $c > 0$ and therefore, the sequence $\{v^{(m)}\}_{m\in\mathbb{N}}$ is Cauchy in the Banach space X_T. Let us denote by v^∞ its limit in X_T and by $u^\infty = S(t)v^\infty$. We will show that u^∞ satisfies NLS (1) in the interval $[0, T]$ in the sense of Definition 3. For convenience, we drop the superscript ∞ and write u, v. In addition, let $u^{(m)} := S(t)v^{(m)}$, where $v^{(m)}$ is the smooth solution to (21) with smooth initial data $v_0^{(m)}$ as described above and note that $u^{(m)}$ is the smooth solution to (1) with smooth initial data $u_0^{(m)} := v_0^{(m)}$. Furthermore, $u^{(m)} \to u$ in X_T because $v^{(m)} \to v$ in X_T and since convergence in the modulation space $M_{p,q}(\mathbb{R})$ implies convergence in the sense of distributions we conclude that $\partial_x u^{(m)} \to \partial_x u$ and $\partial_t u^{(m)} \to \partial_t u$ in $\mathcal{S}'((0, T) \times \mathbb{R})$. Since $u^{(m)}$ satisfies NLS (1) for every $m \in \mathbb{N}$ we have that

$$\mathcal{N}(u^{(m)}) = u^{(m)}|u^{(m)}|^2 = -i\partial_t u^{(m)} + \partial_x^2 u^{(m)},$$

also converges to some distribution $w \in \mathcal{S}'((0, T) \times \mathbb{R})$. Our claim is the following:

Proposition 22. Let w be the limit of $\mathcal{N}(u^{(m)})$ in the sense of distributions as $m \to \infty$. Then, $w = \mathcal{N}(u)$, where $\mathcal{N}(u)$ is to be interpreted in the sense of Definition 2.

Proof. Consider a sequence of Fourier cutoff multipliers $\{T_N\}_{N\in\mathbb{N}}$ as in Definition 1. We will prove that

$$\lim_{N\to\infty} \mathcal{N}(T_N u) = w,$$

in the sense of distributions. Let ϕ be a test function and $\epsilon > 0$ a fixed given number. Our goal is to find $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have

$$|\langle w - \mathcal{N}(T_N u), \phi\rangle| < \epsilon. \tag{94}$$

The LHS can be estimated as

$$\begin{aligned}|\langle w - \mathcal{N}(T_N u), \phi\rangle| \leq |\langle w - \mathcal{N}(u^{(m)}), \phi\rangle| + |\langle \mathcal{N}(u^{(m)}) - \mathcal{N}(T_N u^{(m)}), \phi\rangle| \\ + |\langle \mathcal{N}(T_N u^m) - \mathcal{N}(T_N u), \phi\rangle|.\end{aligned}$$

The first term is estimated very easily since by the definition of w we have that

$$|\langle w - \mathcal{N}(u^{(m)}), \phi\rangle| < \frac{1}{3}\,\epsilon, \tag{95}$$

for sufficiently large $m \in \mathbb{N}$.

To continue, let us consider the second summand for fixed m. By writing the difference $\mathcal{N}(u^{(m)}) - \mathcal{N}(T_N u^{(m)})$ as a telescoping sum we have to estimate terms of the form

$$\Big|\int\int \Big[(I - T_N)u^{(m)}\Big]\ |u^{(m)}|^2\ \phi\ dx\ dt\Big|,$$

where I denotes the identity operator. This integral can be identified with the action of the distribution $\Big[(I - T_N)u^{(m)}\Big]\ |u^{(m)}|^2 \in M^s_{p,q}(\mathbb{R})$ (which is a Banach algebra) onto the test function ϕ, which in its turn can be controlled (Hölder's inequality) by the norms (up to constants)

$$\begin{aligned}\|\phi\|_{L^2_T M_{p',q'}} \|u^{(m)}\|^2_{L^\infty_T M^s_{p,q}} \|(I - T_N)u^{(m)}\|_{L^2_T M^s_{p,q}} \lesssim \\ C_\phi \|u^{(m)}\|^2_{C((0,T), M^s_{p,q})} \|(I - T_N)u^{(m)}\|_{L^2_T M^s_{p,q}} \lesssim C_{\phi,m} \|(I - T_N)u^{(m)}\|_{L^2_T M^s_{p,q}}.\end{aligned}$$

Here we have to observe that for every fixed t the norm $\|(I - T_N)u^{(m)}\|_{M^s_{p,q}} \to 0$ as $N \to \infty$ and an application of Dominated Convergence Theorem in $L^2(0,T)$ implies that there is $N_0 = N_0(m)$ with the property

$$C_{\phi,m} \|(I - T_N)u^{(m)}\|_{L^2_T M^s_{p,q}} < \frac{1}{3}\,\epsilon, \tag{96}$$

for all $N \geq N_0$.

For the last term, we need to observe two things. Firstly, let us consider the sequence $\{\mathcal{N}(T_N u^{(m)})\}_{m\in\mathbb{N}}$, for each fixed N. By applying the iteration process that we described in the previous subsection to $\{S(-t)\mathcal{N}(T_N u^{(m)})\}_{m\in\mathbb{N}}$, which is basically the nonlinearity in equation (21) up to the operator T_N, we see that $\{\mathcal{N}(T_N u^{(m)})\}_{m\in\mathbb{N}}$ is Cauchy in $\mathcal{S}'((0,T)\times\mathbb{R})$, as $m\to\infty$ for each fixed $N\in\mathbb{N}$ since the sequence $u^{(m)}$ is Cauchy in $C((0,T),M_{p,q}(\mathbb{R}))$. Since the operators T_N are uniformly bounded in the L^p norm in N we conclude that this convergence is uniform in N.

Secondly, let us observe that for fixed N, $T_N u$ is in $C((0,T),H^\infty(\mathbb{R}))$ since $u\in M_{p,q}(\mathbb{R})$ and the multiplier m_N of T_N is compactly supported. Hence, $\mathcal{N}(T_N u)=T_N u|T_N u|^2$ makes sense as a function. Therefore, for fixed N we obtain the upper bound

$$\begin{gathered}|\langle\mathcal{N}(T_N u^{(m)})-\mathcal{N}(T_N u),\phi\rangle|\leq\\ \|\phi\|_{L^4_T M_{p',q'}}(\|T_N u^{(m)}\|^2_{L^4_T M_{p,q}}+\|T_N u\|^2_{L^4_T M_{p,q}})\|T_N u^m-T_N u\|_{L^4_T M_{p,q}}\leq\\ C_{\phi,\|u\|_{X_T}}\|u^{(m)}-u\|_{C((0,T),M_{p,q})},\end{gathered}$$

which can be made arbitrarily small. Hence, $\mathcal{N}(T_N u^{(m)})$ converges to $\mathcal{N}(T_N u)$ in $\mathcal{S}'((0,T)\times\mathbb{R})$ as $m\to\infty$ for each fixed N.

From these two observations we derive that $\mathcal{N}(T_N u^{(m)})\to\mathcal{N}(T_N u)$ in $\mathcal{S}'((0,T)\times\mathbb{R})$ as $m\to\infty$ uniformly in N. Equivalently,

$$|\langle\mathcal{N}(T_N u^{(m)})-\mathcal{N}(T_N u),\phi\rangle|<\frac{1}{3}\,\epsilon, \tag{97}$$

for all large m, uniformly in N. Therefore, (94) follows by choosing m sufficiently large so that (95) and (97) hold, and then choosing $N_0=N_0(m)$ such that (96) holds.

□

Finally, we have shown that the function $u=u^\infty$ is a solution to the NLS (1) in the sense of Definition 3. The Lipschitz dependence on the initial data follows from (93) by a limit process.

Acknowledgments: The authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

References

[1] A. Babin, A. Ilyin and E. Titi, *On the regularization mechanism for the periodic Korteweg-de Vries equation.* Comm. Pure Appl. Math. 64(5), 591-648 (2011).

[2] A. Bényi and K.A. Okoudjou, *Local well-posedness of nonlinear dispersive equations on modulation spaces.* Bulletin of the London Mathematical Society, 41(3):549-558, (2009), ISSN 0024-6093.

[3] Feichtinger, *Modulation spaces on locally compact Abelian groups.* Technical Report, University of Vienna, 1983, in: Proc. Internat. Conf. on Wavelet and applications, 2002, New Delhi Allied Publishers, India (2003), 99-140.

[4] S. Guo, *On the 1D Cubic Nonlinear Schrödinger Equation in an Almost Critical Space.* J. Fourier Anal. Appl. 23 (2017), no. 1, 91-124.

[5] Z. Guo, S. Kwon and T. Oh, *Poincaré-Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS.* Comm. Math. Phys. August 2013, Volume 322, Issue 1, 19-48.

[6] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers.* Fifth edition. The Clarendon Press, Oxford University Press, New York, 1979, xvi+426pp.

[7] N. Pattakos, *NLS in the modulation space* $M_{2,q}(\mathbb{R})$. Submitted to the Journal of Fourier Analysis and Applications, arXiv:1802.08274.

[8] B. Wang and H. Hudzik, *The global Cauchy problem for the NLS and NLKG with small rough data.* J. Differ. Equations 232 (2007), 36-73.

Leonid Chaichenets, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
E-mail address: `leonid.chaichenets@kit.edu`

Dirk Hundertmark, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
E-mail address: `dirk.hundertmark@kit.edu`

Peer Kunstmann, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
E-mail address: `peer.kunstmann@kit.edu`

Nikolaos Pattakos, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
E-mail address: `nikolaos.pattakos@kit.edu`