Quasiparticles in condensed matter systems

Peter Wölfle

Institute for Theory of Condensed Matter, Karlsruhe Institute of Technology, 76049 Karlsruhe, Germany
Institute for Nanotechnology, Karlsruhe Institute of Technology, 76031 Karlsruhe, Germany

E-mail: peter.woelfle@kit.edu

Corresponding Editor Professor Gordon Baym

Abstract

Quasiparticles are a powerful concept of condensed matter quantum theory. In this review, the appearence and the properties of quasiparticles are presented in a unifying perspective. The principles behind the existence of quasiparticle excitations in both quantum disordered and ordered phases of fermionic and bosonic systems are discussed. The lifetime of quasiparticles is considered in particular near a continuous classical or quantum phase transition, when the nature of quasiparticles on both sides of a transition into an ordered state changes. A new concept of critical quasiparticles near a quantum critical point is introduced, and applied to quantum phase transitions in heavy fermion metals. Fractional quasiparticles in systems of restricted dimensionality are reviewed. Dirac quasiparticles emerging in so-called Dirac materials are discussed. The more recent discoveries of topologically protected chiral quasiparticles in topological matter and Majorana quasiparticles in topological superconductors are briefly reviewed.

Keywords: quasiparticles, Fermi liquid, stability of quasiparticles, quasiparticles in ordered systems, fractional quasiparticles, topological quasiparticles

1. Introduction

The theoretical description of classical matter—e.g. a classical liquid—is complex, in the sense that concepts known from weakly interacting systems such as dilute gases cannot easily be generalized to apply in the case of dense, strongly interacting systems. The reason is that the collision rate of an atom in a liquid is so high that individual particle-like excitations are not well defined.

In contrast, in quantum matter—i.e. in systems of strongly interacting quantum particles at low temperature—it has been found that well-defined particle-like 'elementary' excitations do exist: so-called quasiparticles (QPs). The quantum numbers of a QP in a translation invariant system are its momentum $\mathbf{p}$ (or crystal momentum in a crystalline solid) and its spin (or pseudospin) projection σ (assuming spin, or pseudospin conservation), occasionally supplemented by additional internal quantum numbers such as energy band index, etc. QPs obey a dispersion relation $\omega = \epsilon_{\mathbf{p}\sigma}$. QPs of generic systems exist in weakly excited states, and for a number of different reasons, as shown below. The existence of QPs allows one to make use of the powerful methods of weak coupling or mean-field-like theories such as Hartree–Fock theory, density functional theory, renormalized perturbation theory, kinetic theory, and more.

The concept of QPs, as developed in the 1940s and 1950s in the examples of the quantum liquids ^{4}He and ^{3}He, has since been used to describe many other types of quantum liquids, like conduction electrons in metals, nucleons in nuclear matter, and systems of bosonic excitations (phonons, plasmons, excitons, …) in solids, with great success. It has been found that at a phase transition into an ordered state, entirely new QPs may appear—while the old QPs change their character. In particular, the conservation of quantum numbers is lost if

the corresponding symmetry is broken in the ordered state. Nonetheless, well-defined QPs may still exist, even though their number, or spin, or momentum is no longer conserved. At a quantum phase transition, it is often found that QPs become critical, i.e. their properties are given by universal scaling laws.

With the discovery of the quantum Hall effects in the early 1980s, an entirely new class of QPs came into view, enabled by the topological properties of these materials. In their simplest form, they are of chiral character—i.e. they have one sense of direction along the edges of a quantum Hall sample, as dictated by the applied magnetic field. Later, it was shown that even in the absence of an external magnetic field, chiral QPs may emerge in semiconductors with 'inverted' band structure and sufficiently strong spin–orbit coupling. In the fractional quantum Hall effect, more exotic QPs may appear, as a consequence of the electron–electron interaction in conjunction with a strong magnetic field. Some of these vortex-like excitations are found to have fractional charge, and may have fractional statistics. Others are found to behave in many respects like fermions moving in a much reduced magnetic field, as if they combined with an even number of flux quanta to form 'composite fermions'. Most recently, zero energy excitations at the edges of a topological superconductor are coming into focus: Majorana zero modes (MZM), or 'half' chiral fermions. These excitations share the property that particle and antiparticle are identical with the Majorana fermions originally proposed in the 1930s. On the other hand, the exchange statistics of MZMs in two dimensions is not that of fermions but—much more sophisticated—that of 'nonAbelian anyons'.

This article aims to give an overview of the many principles and concepts used to demonstrate the existence of QPs, as well as of the character and properties of QPs in different situations. The emphasis is on qualitative aspects rather than a full and detailed mathematical derivation of the QP states in each case.

2. Fermionic and bosonic QPs in a historical perspective

2.1. Fermi systems

The first appearance of fermionic QPs in hindsight may have been in the Sommerfeld theory of metals [1, 2]. Although at that time neither the name 'fermion' nor the notion of 'QPs' was known, the model of a non-interacting electron gas (the 'Fermi gas') has many of the identifying features of what is known nowadays as the 'Fermi liquid phase'. The entirely new feature of the Fermi gas as compared to classical systems is the fact that at low temperatures $T \ll T_{\rm F}$, the Fermi temperature, most of the fermions are frozen in the Fermi sea, and only a fraction $T/T_{\rm F}$ of the particles are in excited states, and take part in thermodynamic and linear response processes at low frequencies and long wave lengths. These may be identified as fermionic QPs. So, for spin 1/2 fermions of mass m, in contact with a particle reservoir at chemical potential μ described by the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} \left(\frac{k^2}{2m} - \mu\right) c^+_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}, \tag{1}$$

where $c^+_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}$ are creation and annihilation operators of fermions in momentum and spin eigenstates $|\mathbf{k},\sigma\rangle$, the equilibrium occupation number is given by the Fermi function $n_{\mathbf{k}\sigma} = [\mathrm{e}^{(k^2/2m-\mu)/T} + 1]^{-1}$. At $T = 0$, there is a sharp boundary separating occupied and non-occupied states in momentum space, called the Fermi surface, defined by the chemical potential $\mu = T_{\rm F} = k_{\rm F}^2/2m$. This defines the Fermi wave number $k_{\rm F}$, which is related to the particle density by $n = k_{\rm F}^3/3\pi^2$, equal to the Fermi volume in momentum space enclosed by the Fermi surface times the spin degeneracy factor, 2. We use units for which Planck's constant $\hbar = 1$ and Boltzmann's constant $k_B = 1$. The QP energy and occupation number are then defined as

$$\begin{aligned} \epsilon^{(0)}_{\mathbf{k}\sigma} &= \frac{k^2}{2m} - \mu, \\ n^{(0)}_{\mathbf{k}\sigma} &= \frac{1}{\mathrm{e}^{\epsilon^{(0)}_{\mathbf{k}\sigma}/T} + 1}. \end{aligned} \tag{2}$$

2.1.1. Landau quasiparticles in Fermi liquids. Even though the Sommerfeld theory of metals appeared to work quite well, it remained a puzzle why one should be allowed to neglect electron–electron interaction effects. After all, the energy scale of the Coulomb interaction between two electrons separated by a lattice spacing is of order eV, and hence is much larger than the excitation energies (thermal energies, electric and magnetic field induced energy shifts) of interest. The answer was given by Landau in a series of fundamental papers [3]. Landau pointed out that even in the presence of interaction, there exist fermionic excitations—nowadays called Landau QPs—which, while not absolutely stable, live long enough to be useful for describing the many-body state of a weakly excited interacting Fermi system. These QPs are in one-to-one correspondence with the bare particles, i.e. they carry the same quantum numbers $(\mathbf{k},\sigma)$ (for a comprehensive account of phenomenological Fermi liquid theory see [4, 5]). Landau even provided a derivation from microscopic theory, later elaborated in a number of pioneering works [6–9]. In the meantime, it has been shown that the Landau Fermi liquid state corresponds to a stable fixed point of a suitably defined renormalization group flow [10].

In microscopic theory, Landau QPs are identified as poles of the single fermion Green's function G in the complex frequency ω plane. In terms of the self-energy Σ we have

$$\begin{aligned} G_{\mathbf{k}\sigma}(\omega + \mathrm{i}0) &= \frac{1}{\omega - \epsilon_{\mathbf{k}\sigma} + \mu - \Sigma_{\mathbf{k}\sigma}(\omega + \mathrm{i}0)} \\ &\approx \frac{Z_{\mathbf{k}\sigma}}{\omega - \epsilon^*_{\mathbf{k}\sigma} + \mathrm{i}\Gamma_{\mathbf{k}\sigma}} + G^{\rm inc}, \end{aligned} \tag{3}$$

where $\epsilon^*_{\mathbf{k}\sigma} = Z_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}\sigma} - \mu + \mathrm{Re}\Sigma_{\mathbf{k}\sigma}(0))$ is the energy, $\Gamma_{\mathbf{k}\sigma} = Z_{\mathbf{k}\sigma}\mathrm{Im}\Sigma_{\mathbf{k}\sigma}(0)$ is the relaxation rate and $Z_{\mathbf{k}\sigma} = \lim_{\omega\to 0} (1 - \partial \mathrm{Re}\Sigma_{\mathbf{k}\sigma}/\partial\omega)^{-1}$ is the weight factor of QPs. The non-QP contributions $G^{\rm inc}$ are largely inaccessible (except by

numerical methods), but are often not needed to calculate observable properties. It is useful to introduce a parametrization of the QP energy in terms of the effective mass m^* as $\epsilon^*_{\mathbf{k}\sigma} = v^*_{\mathrm{F}}(k - k_{\mathrm{F}})$, where the renormalized Fermi velocity is defined as $v^*_{\mathrm{F}} = k_{\mathrm{F}}/m^*$. If the momentum dependence of Σ is negligible, one finds $m/m^* = Z$. Provided the relaxation rate Γ is sufficiently small compared to the excitation energies of interest, e.g. $\Gamma \ll T$, the QPs may be assumed to be stable, i.e. Γ may be put to zero. It follows that the density of states N^*_0 of QPs at the Fermi level is given by that of a free Fermi gas of particles of mass m^*, $N^*_0 = m^* k_{\mathrm{F}}/\pi^2$. Correspondingly, the low temperature specific heat C is given by $C = \frac{\pi^2}{3} N^*_0 T \propto m^*$.

The single particle spectral function $A_{\mathbf{k}\sigma}(\omega) = -\mathrm{Im}\{G_{\mathbf{k}\sigma}(\omega + \mathrm{i}0)\}$ is, within certain limits, directly accessible in angle resolved photo emission experiments (ARPES) [11, 12]. The QP peak is often clearly seen, and its width as a function of frequency ω at fixed momentum $\mathbf{k}$ is equal to the QP relaxation rate $\Gamma_{\mathbf{k}\sigma}$. Conversely, analyzing the peak structure of $A_{\mathbf{k}\sigma}(\omega)$ at fixed ω as a function of $\epsilon_{\mathbf{k}\sigma}$ provides information on $\mathrm{Im}\Sigma_{\mathbf{k}\sigma}(\omega) = \Gamma_{\mathbf{k}\sigma}(\omega)/Z_{\mathbf{k}\sigma}(\omega)$. The distinction between these two QP peak widths will be of importance in so-called non-Fermi liquid situations, when the QP weight factor $Z_{\mathbf{k}\sigma}(\omega)$ may depend on energy ω (see below).

It is useful to consider the QP peak in the spectral function as function of momentum or better, of $\epsilon_{\mathbf{k}}$ (dropping a possible spin dependence for simplicity), instead of as a function of frequency ω. Provided the momentum dependence of $\mathrm{Im}\{\Sigma_{\mathbf{k}\sigma}(\omega + \mathrm{i}0)\}$ is negligible, which is usually the case, the spectral function is a simple narrow Lorentzian of width $\mathrm{Im}\{\Sigma\}$; i.e. there is no incoherent background. This allows one to integrate the spectral function over $\epsilon_{\mathbf{k}}$, with no unknown renormalization factors involved. One may define a QP distribution function labeled 'momentum integrated' as

$$\delta n_{\mathbf{k}\sigma}(\mathbf{r}, t) = \int \frac{\mathrm{d}\epsilon_{\mathbf{k}_1}}{2\pi} \delta G_\sigma(\mathbf{k}_1 = \widehat{\mathbf{k}}|k_1|, \omega = \epsilon^*_{\mathbf{k}\sigma}; \mathbf{r}, t), \quad (4)$$

where

$$\delta G_\sigma(\mathbf{k}, \omega; \mathbf{r}, t) = \int \mathrm{d}\mathbf{r}_1 \mathrm{d}t_1 \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}_1 - \omega t_1)} \delta\langle \psi^\dagger_\sigma(\mathbf{r} + \mathbf{r}_1/2, t + t_1/2)\psi_\sigma(\mathbf{r} - \mathbf{r}_1/2, t - t_1/2)\rangle. \quad (5)$$

2.1.2. Fermi liquid interaction. As the Landau QPs are not eigenstates of the system they do interact. The effect of this Fermi liquid interaction on the QP energy at position $\mathbf{r}$ and at time t may be expressed as

$$\delta\epsilon_{\mathbf{k}\sigma}(\mathbf{r}, t) = \sum_{\mathbf{k}',\sigma'} f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} \delta n_{\mathbf{k}'\sigma'}(\mathbf{r}, t), \quad (6)$$

so that a change in QP distribution $\delta n_{\mathbf{k}'\sigma'}$ is seen to lead to a change in QP energy. For systems with a spherically symmetric Fermi surface, it is useful to expand $f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}$ in terms of Legendre polynomials P_l,

$$f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} = (1/N^*_0) \sum_{l=0}^{\infty} [F^s_l + \sigma\sigma' F^a_l] P_l(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}'), \quad (7)$$

where the dimensionless Landau parameters $F^{s,a}_l$ have been introduced. The above theory was originally developed for electrically neutral systems, such as liquid ^{3}He. It also applies to charged systems where the long-range part of the Coulomb interaction is screened by external charges, as for conduction electrons in metals. For charged system in general the Coulomb interaction should be separated out by replacing $f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} \to f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} + 4\pi e^2/q^2$ [13]. In the quasiclassical limit, the Fourier components of the QP distribution $\delta n_{\mathbf{k}\sigma}(\mathbf{q}, \omega)$ may be shown to obey the Landau–Boltzmann equation

$$(\omega - \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q})\delta n_{\mathbf{k}\sigma} + \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q} \left(\frac{\partial n^{(0)}_{\mathbf{k}}}{\partial \epsilon^*_{\mathbf{k}}} \right) [\delta\epsilon_{\mathbf{k}\sigma} + \delta\epsilon^{\mathrm{ext}}_{\mathbf{k}\sigma}] = I_{\mathrm{c}}, \quad (8)$$

where $n^{(0)}_{\mathbf{k}} = [\exp(\epsilon^*_{\mathbf{k}}/T) + 1]^{-1}$ is the Fermi function, $\mathbf{v}^*_{\mathbf{k}} = \mathbf{k}/m^*$, $\delta\epsilon^{\mathrm{ext}}_{\mathbf{k}\sigma}$ is the QP energy shift induced by an external field, and I_{c} is the QP collision term. A derivation of the Landau–Boltzmann equation from microscopic theory based on the Green's function formalism of Kadanoff and Baym [14] and using the momentum integration technique (for an application in the electron–phonon problem see [15]) may be found in [16].

Alternatively, the spectral function may be integrated over ω, leading to the 'energy integrated' QP representation. In this language, the response functions χ^ρ_l of a Fermi liquid (in the charge/spin channel denoted by $\rho = s, a$, with angular momentum $l = 0,1,..$), describing the change of the density component in the (l, ρ)-channel induced by an external field δh^ρ_l of the same symmetry, may be expressed in terms of QP quantities, amended by non-QP contributions in the form of renormalization constants R^ρ_l, Φ^ρ_l [7, 8]

$$\chi^\rho_l(\mathbf{q}, \nu) = (R^\rho_l)^2 Z^2 \chi^\rho_{\mathrm{QP},l}(\mathbf{q}, \nu) + \Phi^\rho_l$$
$$\chi^\rho_{\mathrm{QP},l} = \sum_\alpha \int \frac{\mathrm{d}\Omega_{\mathbf{k}}}{8\pi} P_l \frac{-(2l+1)Z^2 N^*_0 \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q}}{\nu - \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q}[1 + F]} P_l. \quad (9)$$

Here, $P_l = P_l(\cos\theta)$ is a Legendre polynomial, and $\cos\theta = (\mathbf{k} \cdot \mathbf{q})/kq$. The QP partial susceptibility $\chi^\rho_{\mathrm{QP},l} = \delta n^\rho_l / \delta h^\rho_l$ receives a screening correction caused by the residual Landau Fermi liquid interaction, given by the operator F represented by a matrix in momentum and spin space, $F_{\mathbf{k}\sigma\mathbf{k}'\sigma'} = N^*_0 f_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}$. The static QP susceptibilities are found from equation (9) as

$$\chi^\rho_{\mathrm{QP},l}(\mathbf{q} \to 0, 0) = \frac{N^*_0}{1 + F^\rho_l/(2l+1)}, \quad (10)$$

where F^ρ_l is a corresponding Landau parameter. It is interesting to note that the susceptibilities of the conserved quantities—particle and spin density, and possibly momentum density—are exactly given by the QP contribution; i.e. the non-QP terms are given by $R^\rho_0 = R^s_1 = Z^{-1}$ and $\Phi^\rho_0 = \Phi^s_1 = 0$. The only other case admitting an exact determination of R, Φ is that of the spincurrent density, where $R^a_1 = 1 + F^a_1/3$ and $\Phi^a_1 = (F^s_1 - F^a_1)/[3(1 + F^s_1/3)]$ [8].

It follows from the above that the susceptibility $\chi^{s,a}_{\mathrm{QP},l}(\mathbf{q} \to 0, 0)$ may diverge if the Landau parameter $F^{s,a}_l \to -(2l+1)$. This has been interpreted as indicating the formation of an ordered state if $F^{s,a}_l < -(2l+1)$. The

phenomenon is termed 'Pomeranchuk instability' [17]. Although, from the above, it seems like such instabilities should easily occur, in practice they tend to be elusive. In the $(l = 0; a)$-channel, this is nothing but a ferromagnetic instability of an itinerant interacting fermion system, which is often preëmpted by a first order transition [18, 19]. The situation is similar in the charge channel $(l = 0; s)$, where a diverging compressibility signals phase separation. As for the $l = 1$ channels, it may be shown that the corresponding susceptibilities are bound by a sum rule, and may not diverge [20]. This is obvious in the charge channel, where $\chi_1^s(\mathbf{q} \to 0, 0) = N_0$, the bare density of states, using the effective mass relation [4, 5] $m^*/m = 1 + F_1^s/3$. One may wonder what may happen when $1 + F_1^s/3 \to 0$, or even turns negative. First of all, Fermi liquid theory is no longer applicable in this limit, because the region of validity in $\mathbf{k}$-space shrinks to zero. This follows from the requirement that the QP energy should be less than the microscopic energy scale, set by the bare Fermi energy $k_F|k - k_F|/m^* \ll k_F^2/2m$. Furthermore, by rewriting the effective mass relation in terms of the dimensionful Landau interaction parameter f_1^s, $m^*/m = [1 - N_0 f_1^s/3]^{-1}$ one sees that f_1^s would have to tend to $-\infty$ to let $m^* \to 0$, which is unphysical.

In the spin channel case one also finds $\chi_1^a(\mathbf{q} \to 0, 0) = N_0$, since the non-QP parameters R_1^a and Φ_1^a conspire to remove any dependence on the Landau parameters. In both cases, the pole at $F_l^{s,a} = -(2l + 1)$ is canceled by a corresponding zero in the numerator. The issue here is that although the QP susceptibility may diverge, this does not necessarily imply the divergence of a physical—i.e. observable—susceptibility. The argumentation presented by Pomeranchuk, namely that the QP energy change $\delta E_{\mathrm{QP}} = \sum_{l,r} |\delta n_{l,r\mathbf{k}'\sigma'}|^2/\chi_{\mathrm{QP},l}^{s,a}$ induced by a corresponding change of the QP occupation, $\delta n_{\mathbf{k}\sigma} = \sum_{l,m}(\delta n_{l,s} + \sigma \delta n_{l,a}) Y_{lm}(\widehat{\mathbf{k}})$, does represent the actual energy change, is based on the assumption of the one-to-one correspondence of QPs and particles. This correspondence is not strictly valid. Rather, except for the case of conserved quantities (the $l = 0$ channels), non-QP contributions enter. However, if the QP distribution is interpreted as the momentum integrated Green's function, the resulting kinetic equation captures what would be called non-QP effects in the energy-integrated formulation. This will be shown in the following.

The identical result for χ_1^ρ may be obtained within Fermi liquid theory employing the Landau–Boltzmann equation—equation (8) (and hence the momentum integrated QP distribution function)—by realizing that the physical susceptibilities χ_1^ρ express the response of the respective current densities

$$\mathbf{j}^\rho = \sum_{\mathbf{k},\sigma} s_\sigma^\rho \mathbf{v}_\mathbf{k} [\delta n_{\mathbf{k}\sigma} - (\partial n_\mathbf{k}^{(0)}/\partial \epsilon_\mathbf{k}) \delta \epsilon_{\mathbf{k}\sigma}] \tag{11}$$

(where $s_\sigma^s = 1, s_\sigma^a = \sigma$) to external fields, $\chi_1^\rho = \delta j_i^\rho / \delta A_i^\rho$. Here, δA_i^ρ is the ith component of an external vector field causing the energy shifts $\delta \epsilon_{\mathbf{k}\sigma}^{\mathrm{ext}} = [\sigma \mathbf{k} \cdot \delta \mathbf{A}^a + \mathbf{k} \cdot \delta \mathbf{A}^s]/m$. Note that χ_1^ρ differs from $\chi_{\mathrm{QP},1}^\rho$ by the second term in the square brackets of the expression for $\mathbf{j}^\rho$ [20]. This 'backflow term' provides what has been termed 'non-QP' corrections in the energy integrated formulation presented above. The analogous analysis for higher angular momentum channels, $l > 1$, provides indications that the Pomeranchuk instability may not exist at all [20]. It appears as a very general conclusion of the above discussion that whereas the energy-integrated formulation seems to suggest that the QP picture of linear response only holds for the $l = 0$ channels, the momentum integrated formulation in the form of the kinetic equation equation (8) is also valid for the higher l channels, if proper attention is paid to the collision term.

Application of Fermi liquid theory to systems of reduced symmetry requires introducing further parameters, and therefore weakens its predictive power. For a crystalline solid, the orbital rotation symmetry is restricted to lattice symmetry operations—in principle limiting the usefulness of Fermi liquid theory applied to conduction electrons in metals. However, the usually existing approximate symmetries often suffice to allow interpreting observables such as the specific heat and the spin susceptibility in terms of simple Fermi liquid theory. A further case for which the introduction of many more parameters may be avoided is that of metals containing transition metal, rare-earth or actinide ions with strong atomic spin–orbit interaction. The effect of this interaction may be absorbed into the ionic total angular momentum multiplet structure. Usually, the ground state multiplet is a doublet, characterized by a pseudospin 1/2, which then may take the role of the bare electron spin. The spin–orbit component of the electron–electron interaction is small. A quite different situation arises in the case of neutron matter or nuclear matter, where spin–orbit and tensor forces are known to be strong. The Fermi liquid interaction does then depend on the spin orientations of the incoming and outgoing particle–hole pairs in a general form, $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \to f_{\mathbf{p}\boldsymbol{\sigma},\,\mathbf{p}'\boldsymbol{\sigma}'}$, where $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ denote a 2×2 spin matrix. Keeping the total angular momentum and the total spin of an interacting particle–hole pair as conserved quantities, the Fermi liquid interaction may be parametrized as [21]

$$f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} = f_{\mathbf{p}\mathbf{p}'}^s + f_{\mathbf{p}\mathbf{p}'}^a \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' + h_{\mathbf{p}\mathbf{p}'} S(\widehat{\mathbf{q}}) + k_{\mathbf{p}\mathbf{p}'} S(\widehat{\mathbf{P}}) + l_{\mathbf{p}\mathbf{p}'} A(\widehat{\mathbf{q}}, \widehat{\mathbf{P}}), \tag{12}$$

$$S(\widehat{\mathbf{q}}) = 3(\boldsymbol{\sigma} \cdot \widehat{\mathbf{q}})(\boldsymbol{\sigma}' \cdot \widehat{\mathbf{q}}) - \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}', \tag{13}$$

$$A(\widehat{\mathbf{q}}, \widehat{\mathbf{P}}) = (\boldsymbol{\sigma} \times \boldsymbol{\sigma}') \cdot (\widehat{\mathbf{q}} \times \widehat{\mathbf{P}}), \tag{14}$$

where $\widehat{\mathbf{q}} = (\mathbf{p} - \mathbf{p}')/|\mathbf{p} - \mathbf{p}'|$ and $\widehat{\mathbf{P}} = (\mathbf{p} + \mathbf{p}')/|\mathbf{p} + \mathbf{p}'|$. The additional interaction functions $h_{\mathbf{p}\mathbf{p}'}, k_{\mathbf{p}\mathbf{p}'}, l_{\mathbf{p}\mathbf{p}'}$ may again be expanded in terms of Legendre polynomials. The relative weakness of the nuclear forces (as compared to liquid ^{3}He or strongly correlated electron systems) makes it possible to calculate the extended set of Landau parameters in perturbation theory. There is, however, a further complication: the contribution of multipair processes, which are outside the reach of Fermi liquid theory, may be estimated to be appreciable [21].

2.1.3. QP relaxation rate. As seen above, a further consequence of the QP interaction is a finite decay rate $\Gamma_{\mathbf{k}\sigma}$. At low temperatures $T \ll T_F$, the dominating channel is two-QP scattering. The corresponding scattering amplitude $a(\mathbf{k}_1, \sigma_1; \mathbf{k}_2, \sigma_2; \mathbf{k}_3, \sigma_3; \mathbf{k}_4, \sigma_4)$, describing transitions of

particle 1 scattering off particle 2 and ending in states 3,4 may be taken to be a function of only the two angles θ, ϕ subtended by the momentum vectors, defined as $\cos\theta = \mathbf{k}_1 \cdot \mathbf{k}_2/|\mathbf{k}_1||\mathbf{k}_2|$ and $\cos\phi = (\mathbf{k}_1 \times \mathbf{k}_2) \cdot (\mathbf{k}_3 \times \mathbf{k}_4)/|\mathbf{k}_1 \times \mathbf{k}_2||\mathbf{k}_3 \times \mathbf{k}_4|$ as one may put $|\mathbf{k}_j| = k_\mathrm{F}$, $j = 1, \ldots, 4$. The decay rate $\Gamma_{\mathbf{k}\sigma}$ is then given by the golden rule expression

$$\Gamma_{\mathbf{k}_1\sigma_1} = \sum_{2,3,4} w(1,2;3,4) n_{\mathbf{k}_2\sigma_2}(1 - n_{\mathbf{k}_3\sigma_3})(1 - n_{\mathbf{k}_4\sigma_4})\delta_{\mathbf{k}}\delta_\sigma\delta_\epsilon, \tag{15}$$

where $w = 2\pi|a|^2$, with the scattering amplitude a, and $\delta_{\mathbf{k}}, \delta_\sigma, \delta_\epsilon$ ensure conservation of momentum, spin, and energy. It is useful to introduce the dimensionless scattering amplitude $A = N_0^* a$ and its spin singlet and triplet components $A^{(0,1)}$. The momentum summations may be expressed in terms of three energy integrals and an angular average over θ, ϕ, (see [5, 22]):

$$\begin{aligned}
\Gamma_{\mathbf{k}_1} &= \int \mathrm{d}\epsilon^*_{\mathbf{k}_2} \int \mathrm{d}\epsilon^*_{\mathbf{k}_3} \int \mathrm{d}\epsilon^*_{\mathbf{k}_4} \frac{1}{v^*_{\mathbf{k}_4} k_\mathrm{F}} n_{\mathbf{k}_2}(1 - n_{\mathbf{k}_3})(1 - n_{\mathbf{k}_4})\delta_\epsilon \langle W \rangle \\
&= \frac{\epsilon^{*2}_{\mathbf{k}_1} + (\pi T)^2}{\epsilon^*_\mathrm{F}} \frac{\pi}{64} \langle W \rangle, \\
\langle W \rangle &= \int_0^1 d(\cos\frac{\theta}{2}) \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} W, \quad W = |A^{(0)}|^2 + 3|A^{(1)}|^2.
\end{aligned} \tag{16}$$

The relaxation rate is found to vanish as the square of the temperature or excitation energy, which guarantees that the condition $\Gamma_{\mathbf{k}_1\sigma_1} \ll T, |\epsilon^*_{\mathbf{k}_1}|$ is satisfied in the limit of vanishing energy, independent of the QP interaction strength. As a further consequence the inelastic contribution to the resistivity of weakly disordered metals is found to obey a T^2-law in a temperature window governed by the competition of phonon contributions, electron–electron interaction effects and impurity scattering.

2.1.4. Luttinger theorem. An important relation connecting the microscopic physics with Landau Fermi liquid theory is that the occupation number of plane wave states at $T = 0$, $n_{\mathbf{k}\sigma} = -\int_{-\infty}^0 \mathrm{d}\omega \mathrm{Im} G_{\mathbf{k}\sigma}(\omega + \mathrm{i}0)$, has a discontinuity of size $Z_{\mathbf{k}\sigma}$ at the Fermi energy $\epsilon^*_{\mathbf{k}\sigma} = 0$, which marks the Fermi surface in momentum space. In the presence of anisotropic potentials, such as given in a crystalline solid, $\epsilon^*_{\mathbf{k}\sigma}$ will be anisotropic, and so will be the Fermi surface. As may be expected from the one-to-one correspondence of QPs and bare particles, the volume of the manifold of occupied QP states in momentum space—the Fermi volume V_F—is nonetheless conserved, as stated by Luttinger's theorem [23]. A more recent non-perturbative derivation of Luttinger's theorem points out its topological origin [24].

2.1.5. Microscopic model theories of the Fermi liquid. The Fermi liquid theory provides the correct framework for describing the properties of interacting fermion systems, and enables the connection of various observable quantities in a consistent way. It does, however, not allow one to actually calculate the effective mass and the Landau parameters. Even more than fifty years after its instigation, it is often difficult to obtain reliable results for these parameters in strongly correlated systems. Nonetheless, there are certain qualitative traits that have been identified by a number of approximate treatments, like the Gutzwiller approximation [25–28], dynamical mean field theory (DMFT) [29–32], slave particle theories [33–36], parquet equations [37], and more. For example, a Fermi system with strong short range repulsive interaction near half filling (fermions in the Hubbard model) shows an enhanced effective mass m^*, a strong repulsive density–density interaction, as described by the Landau parameter F_0^s (both diverging at half-filling for a sufficiently strong bare interaction, indicating a Mott–Hubbard metal–insulator transition), and on the other hand a weak ferromagnetic spin exchange interaction (F_0^a).

2.1.6. Heavy fermion systems. The most extreme example of a Fermi liquid is the system of conduction electrons hybridizing with almost localized f-electrons in the so-called heavy fermion metals [38]. In these systems, the local magnetic moments formed by the nearly singly occupied f-states undergo the Kondo effect, i.e. the moments get screened, and a many-body resonance of width T_K (the Kondo temperature) forms close to the Fermi energy [39]. As a result, conduction electrons are scattered resonantly, and spend on average a time of order $1/T_\mathrm{K}$ at each lattice site, before they move on. As a consequence, the conduction electrons acquire a dramatically enhanced effective mass $m^*/m \propto T_\mathrm{F}/T_\mathrm{K}$. At low temperature $T \ll T_\mathrm{K}$, the motion becomes coherent throughout the lattice, and a band of heavy fermions forms. Since T_K is typically a few kelvin, the effective mass ratio may be as large as 10^3. In other words, the QP weight factor is then $Z \sim 10^{-3}$. Nonetheless, Fermi liquid theory is found to be obeyed [38].

2.1.7. Resilient QPs and Hidden Fermi liquid. Recently it has been found, quite unexpectedly, in model calculations for certain strongly correlated systems [40, 41]—where the resistivity obeys the T^2-law at low $T < T_\mathrm{FL}$, but significant deviations from this law are found in a temperature window $T_\mathrm{FL} < T < T_\mathrm{inc}$—that so-called resilient QPs (RQPs) nonetheless appear to be well-defined. Within that temperature window, the specific heat coefficient shows significant temperature dependence, suggesting a temperature dependent effective mass $m^*(T)$. Even more surprisingly, it is found that the relaxation rate of these RQPs approximately follows the law $\Gamma \propto T^2$, as expected for a genuine Fermi liquid. This peculiar state has therefore been termed 'Hidden Fermi liquid'.

Considering the expression for $\Gamma_{\mathbf{k}}$ presented above, we note that, adopting the plausible assumption that the QP scattering cross section has negligible energy dependence in the energy range specified by $T_\mathrm{FL} < \epsilon^*_{\mathbf{k}} < T_\mathrm{inc}$, one finds using $v^*_{\mathbf{k}_4} \propto 1/m(\epsilon^*_{\mathbf{k}_4})$ that

$$\begin{aligned}
\Gamma_{\mathbf{k}_\mathrm{F}}(\omega = 0, T) &= \frac{\pi^3}{64} \langle W \rangle \frac{T^2}{\epsilon_\mathrm{F}} \langle \frac{m^*(\epsilon)}{m} \rangle \\
\langle \frac{m^*(\epsilon)}{m} \rangle &= \int \frac{\mathrm{d}\epsilon^*_{\mathbf{k}_2}}{T} \int \frac{\mathrm{d}\epsilon^*_{\mathbf{k}_3}}{T} \frac{m^*(\epsilon^*_{\mathbf{k}_4})}{m} n_{\mathbf{k}_2}(1 - n_{\mathbf{k}_3})(1 - n_{\mathbf{k}_4}),
\end{aligned} \tag{17}$$

where $\epsilon^*_{\mathbf{k}_4} = \epsilon^*_{\mathbf{k}_2} - \epsilon^*_{\mathbf{k}_3}$ follows from energy conservation (we put $\epsilon^*_{\mathbf{k}} = 0$). The double averaging over the energy

dependence of the effective mass may be expected to result in a weakly varying temperature dependence of $\langle m^*(\epsilon)/m\rangle$, such that approximately $\Gamma \propto T^2$. This is to be contrasted with a naive analysis of the FL result $\Gamma \propto T^2/\epsilon_F^* \propto T^2 m^*(T)$, which would suggest strong deviations from the T^2-law. To be clear, resilient QPs may exist if the scattering cross section remains energy independent over a much larger T-regime as expected. In this regime it is, however, not necessary that m^* is T-independent. Hidden Fermi liquid behavior has been identified in the thermodynamic, transport and optical data of several transition metal oxides [42].

2.1.8. Singular interaction: marginal Fermi liquid. The above presentation of Fermi liquid theory assumes the interactions between fermions to be short ranged. Long-ranged interactions may appear as fundamental interactions, such as the interaction between electrical charges induced by exchange of transverse photons of current–current type, or as effective interactions mediated by quantum fluctuations n ear a quantum critical point to be discussed below. Another cause for singular behavior of QPs is reduced dimensionality, in particular in one-dimensional systems. Here, we briefly sketch the case of the interaction of two fermions of momenta $\mathbf{k}+\mathbf{q}/2, \mathbf{k}'-\mathbf{q}/2$ scattering into states $\mathbf{k}-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}/2$ by exchange of a transverse gauge boson of momentum $\mathbf{q}$ and energy ω, which in the limit $q, \omega/q \to 0$ may be shown to take the form [43, 44]

$$V(\mathbf{k},\mathbf{k}';\mathbf{q},\omega) = (\mathbf{v}_\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'} - \frac{(\mathbf{v}_\mathbf{k}\cdot q)(\mathbf{v}_{\mathbf{k}'}\cdot\mathbf{q})}{q^2})\frac{4\pi e^2/c^2}{q^2 - i\gamma|\omega|/q}, \tag{18}$$

where $\mathbf{v}_\mathbf{k} = \mathbf{k}/m$ is the fermion velocity and c is the speed of light. This interaction is unscreened, and therefore singular in the limit $q, \omega/q \to 0$. The so-called Landau damping term $\propto |\omega|/q$ accounts for the decay of a transverse photon into a particle–hole pair. As a consequence of the singular behavior of the interaction in the limit $q, \omega/q \to 0$ the electron self-energy becomes enhanced for low ω

$$\Sigma_{\mathbf{k}\sigma}(\omega + \mathrm{i}0) \propto (v_F/c)^2[\omega \ln|\omega| - \mathrm{i}\pi|\omega|]. \tag{19}$$

It follows that in the limit $\omega \to 0$, the QP weight factor tends to zero $Z(\omega) \propto 1/\ln|\omega|$, rather than approaching a constant value, and the QP relaxation rate $\Gamma(\omega) \propto |\omega|/\ln|\omega|$ is only barely less than the QP energy, rather than vanishing as ω^2. However, since $\Gamma(\omega)/|\omega| \to 0$ for $\omega \to 0$, QPs are still well defined. An interacting Fermi system showing such borderline behavior has been termed a 'Marginal Fermi liquid' (MFL) [45].

At finite temperature, the specific heat $C(T) \propto T\ln(T_0/T)$, and the QP relaxation rate $\Gamma(T) \propto |T|/\ln|T|$ of MFLs is singular for $T \to 0$. In a solid state system, a metal, this singularity is unobservably small, because the prefactor is of order $(v_F/c)^2 \propto 10^{-6}$. The situation is different in high-energy physics, when in a relativistic quark–gluon plasma the process of quarks exchanging transverse gluons leads to non-Fermi liquid behavior of the above type, where now the ratio of quark Fermi velocity and speed of light is of order unity [46, 47]. One finds, then, that the inverse viscosity and the electrical conductivity vary as $T^{5/3}$, and the inverse thermal conductivity as *const* (as compared to T^2 and T for a bona fide Fermi liquid) [47]. Similar effects arise in relativistic electron plasmas as are found in the core of neutron stars [48, 49].

2.2. Bose systems

2.2.1. Phonons and rotons in Bose liquids. Another system which played a leading role in developing the concepts of QPs is superfluid ^{4}He, a system of bosons [50–52]. There one finds two types of well-defined QPs, phonons ($\omega = cq$) and rotons ($\omega = \Delta + q^2/2\mu$), as first postulated by Landau on the basis of the observed thermodynamic properties, and later demonstrated in inelastic neutron scattering experiments. The phonons, first shown to exist in a microscopic model calculation for a weakly interacting gas by Bogoliubov [53], appear actually as a consequence of the broken $U(1)$ gauge symmetry in the superfluid state. The appearence of sound-like excitations is not too surprising. After all, sound waves exist already in the normal phase. In the superfluid, the sound velocity hardly changes with temperature. Additional well-defined QP excitations are the rotons, a kind of optical phonon excitation centered at a finite wave vector. Using variational methods, Feynman and co-workers derived the energy spectrum in good agreement with experiment [54, 55]. The roton minimum Δ shows an abrupt change as one enters the superfluid phase [55]. The build-up of the condensate leads to a two-fluid picture consisting of a superfluid and a normal component of the system—the latter given by the subsystem of thermal excitations. These two components show two collective modes: first and second sound—an in-phase and a counter-phase oscillation of the two components, representing a density wave and a temperature wave respectively.

3. Principles underlying the existence of QPs

In the following, we discuss the various principles that may protect the integrity of QPs.

3.1. The Pauli principle for Fermi systems

The key to the understanding of Fermi liquid theory is to realize the drastic phase space restriction for collision processes between QPs at low T. As mentioned above, on account of the Pauli principle most of the fermions are frozen in the Fermi sea and only a fraction T/T_F may participate in any thermal excitation processes. In addition, the probability for a binary collision process is proportional not only to the density of available collision partners, but also to the density of the available—i.e. empty—final states. Considering energy and momentum conservation in collision processes, only one of the final states may be freely chosen; the other one is then fixed. This means that the QP relaxation rate is proportional to the square of the QP density ($\propto T/T_F$), or [3]

$$\Gamma = \lambda^2 T^2/T_F, \tag{20}$$

where λ is a dimensionless scattering amplitude (see the discussion above). Unless λ acquires critical T-dependence,

i.e. diverges as $T \to 0$, or grows large near a thermal phase transition (see the discussion below),the QP relaxationrate satisfies $\Gamma \ll T$ at low T, implying that Landau QPs are well defined.

3.2. Conservation laws

The longest-known mechanism establishing QP excitations is based on the restriction of the dynamics of a system coming from conservation laws. These lead to the hydrodynamic equations of a fluid or an elastic medium. The one well-defined QP excitation created in this way is longitudinal sound, obeying the dispersion law [56]

$$\omega = cq - \mathrm{i}\alpha q^2. \tag{21}$$

The damping term is seen to vanish faster than the excitation energy $\alpha q^2/cq \to 0$ for $q \to 0$. This is the only well-defined mode in a fluid, since transverse sound and heat waves are overdamped. The distinguishing feature of a solid is its shear stiffness, giving rise to well-defined transverse sound modes. If we adopt the view that a solid is a broken-symmetry phase (continuous translational symmetry is broken down to discrete lattice translations), we may identify the transverse sound mode as the corresponding Goldstone mode (see below).

3.3. Elasticity of the mean field of a quantum liquid

In any liquid, the interaction of the particles causes a mean field in which a given particle is moving. In a classical liquid, this mean field is rapidly fluctuating, and the particle relaxation rate is very fast, so that there is no stable single particle excitation. By contrast, in a quantum liquid such as ^{3}He, the Pauli principle may inhibit rapid fluctuations of the mean field, which forms a kind of elastic medium, described by the Fermi liquid interaction energy. The oscillations of this medium in space and time, known as zero sound [3], constitute a new type of QP. In an isotropic Fermi liquid like ^{3}He, they may be classified according to their angular dependence and spin dependence. The deviation of the QP occupation from the equilibrium value may thus be expressed in an expansion in spherical harmonics $Y_{lm}(\widehat{\mathbf{k}})$ as

$$\delta n_{\mathbf{k}\sigma} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [\delta n^s_{lm} + \sigma \delta n^a_{lm}] Y_{lm}(\widehat{\mathbf{k}}),$$
$$\delta n^{s,a}_{lm}(\mathbf{r}, t) \propto \exp[-\mathrm{i}\omega^{s,a}_{lm}(\mathbf{q})t + \mathrm{i}\mathbf{qr}]. \tag{22}$$

The collective frequencies $\omega^{s,a}_{lm}(\mathbf{q})$ may be found as eigenmodes of the Landau–Boltzmann equation—equation (8)—neglecting the collision term. The effect of collisions is to weakly damp the modes provided $\omega^{s,a}_{lm}\tau \gg 1$, where $\tau = (2\Gamma)^{-1}$ is the QP collision time (the decay time of QP–hole pairs, hence the factor of 2). We recall that, as pointed out below equation (11), the non-QP contributions are included in this formulation. The bosonic zero sound QPs will, however, be overdamped if the excitation energy happens to lie inside the particle–hole continuum, i.e. if energy and momentum conservation allow the decay of a QP of energy $\omega^{s,a}_{lm}(\mathbf{q})$ into a particle–hole pair of fermions of energy $\epsilon^*_{\mathbf{k}+\mathbf{q}} - \epsilon^*_{\mathbf{k}}$. They are well defined if a sufficiently strong interaction or an external field pushes the collective mode dispersion outside the particle–hole continuum. The dispersion of density type zero sound ($l = m = 0$,s) in the case of strong repulsion is given by [4, 5] $\omega^s_{00}(\mathbf{q})/qv_\mathrm{F} = (c_1/v_\mathrm{F})[1 + (2/15)(m^*/m)(1 + F^s_2/5) + O((v_\mathrm{F}/c_1)^4)]$, where c_1 is the velocity of first sound, given by $c_1/v_\mathrm{F} = [(1 + F^s_0)(3 + F^s_1)]^{1/2}$. Here, v_F is the unrenormalized Fermi velocity, and F^s_l are the spin-symmetric Landau parameters. In liquid ^{3}He, longitudinal (ω^s_{00}), transverse (ω^s_{11}) and spin zero sound (ω^a_{00}) have been observed in ultrasound experiments [57–59].

3.4. The Goldstone theorem

The Goldstone theorem [60] states that whenever a continuous symmetry is broken, the system will acquire a new gapless excitation with dispersion $\omega \propto q^\alpha$, $\alpha > 0$, provided the interactions are short-ranged. The most well-known examples are probably the spin-waves in a magnetic system (e.g. ferro- or antiferromagnetic) reflecting the breaking of rotation symmetry in spin space. It has already been mentioned above that transverse sound in a solid may be considered to be a Goldstone mode. In the case of the superfluids ^{4}He and ^{3}He, the breaking of $U(1)$ gauge symmetry leads to the emergence of a new sound mode—which is, however, almost indistinguishable from the usual sound mode. The breaking of rotation symmetry in spin space and orbital space in superfluid ^{3}He gives rise to several more Goldstone modes (see below). It is worth noting that the Goldstone theorem does not guarantee well-defined, i.e. weakly damped Goldstone modes. These modes may be overdamped, depending on the system. For example, orbital waves in superfluid ^{3}He-A, generated by the broken orbital rotation symmetry, have a dispersion $\omega \propto iq^3$ in the limit of low temperature. A general discussion of the requirements to be met in order to ensure well-defined Goldstone modes may be found in [61].

3.5. Elasticity of the order parameter field

A further mechanism generating new bosonic QPs may exist in ordered systems characterized by an order parameter field. Massive collective modes ($\omega(\mathbf{q} \to 0) > 0$) representing bosonic QPs may arise through oscillations of the order parameter field in space and time. For this to happen, the order parameter should have internal structure, in the form of preferred directions in orbital or spin space or some other internal space [62], or else should have a subsidiary pair amplitude in a channel different from the equilibrium pairing channel [63]. The equilibrium form of this structure is determined by the minima of the condensation energy. Oscillations about this fixed structure may occur at frequencies of the order of the condensation energy per particle. Such oscillations have been found to exist in superfluid ^{3}He, where they lead to prominent peaks in the sound absorption as a function of temperature [64]. Even in isotropic superconductors a massive collective mode may be found in principle: the oscillation of the

magnitude of the order parameter about its equilibrium value. It is hard to detect, however, since its frequency, $\omega = 2\Delta$, is at the border of the pair breaking continuum. It has been argued that in special cases, when coupling of the mode to a phonon mode pulls the frequency into the gap, its observation has been possible [65].

In a generalized sense, defects of the order parameter field may also be considered as QPs. Prominent examples are the vortex excitations in the superconductive state and domain walls in a magnetically ordered system.

3.6. Topological constraints

QP excitations may be protected by topological constraints. The first example of this type is a vortex excitation in a Bose superfluid [66] or a type-II superconductor. As mentioned above, a vortex is a defect in the order parameter field of the superfluid. In a Bose superfluid or a conventional superconductor, a vortex carries vorticity of the superfluid flow. Superflow is directly connected with the macroscopic quantum phase ϕ of the many-body wave function since the superfluid velocity is $\mathbf{v}_s \propto \boldsymbol{\nabla}\phi$, and the phase must be unique (modulo 2π). It follows that the vorticity, a line integral around the vortex core, $\int \mathrm{d}\mathbf{s}\mathbf{v}_s$, must be quantized. This implies that a vortex excitation cannot simply disappear. It may do so only at the boundary to a normal-state region where the order parameter vanishes.

In a similar way, vortex-like excitations in the fractional quantum Hall state, Laughlin QPs [67], are protected by the quantization of the magnetic flux they carry (see below).

The above examples are made possible by the strong effect of interparticle interactions leading to a quantum condensed state. Topologically protected QPs may even appear in non-interacting systems, such as in the integer quantum Hall effect, where a strong magnetic field in combination with Anderson localization leads to the existence of conducting edge states of chiral character [68, 69]. In recent years, the possibility of topologically protected QP states has sparked tremendous interest [70, 71], as these states may offer a new route towards quantum information processing devices [72]. Along these lines, it has been proposed that the quantum Hall effect may even be realized without magnetic field in certain semiconductors, as a 'quantum spin Hall effect' (QSHE) (see below). Here the magnetic field is replaced, in some sense, by the spin–orbit interaction (of the Rashba-type), and one needs an inverted band structure (valence and conduction band interchanged) [73, 74].

4. QPs in ordered media

4.1. Superconductors and isotropic Fermi superfluids

As reviewed above, Fermi systems are forced by the Pauli principle to have at most singly occupied single particle states, unlike Bose systems, in which the particles prefer to condense into the lowest single particle state, as far as interaction permits. However, as proposed by Bardeen, Cooper and Schrieffer (BCS) [75], fermions may avoid the Pauli law by forming boson-like entities. This happens in a trivial way when, e.g. protons, neutrons and electrons (all of them fermions!) form ^{4}He-atoms, which are bosons on the scale of energies far below the binding energy of the atom. The conduction electrons of a superconductor form 'Cooper pairs'—loosely bound electron pairs—which may exist only in the presence of the filled Fermi sea. The spatial extension of Cooper pairs, ξ_0, is large compared to the lattice spacing b, such that at any given point $(\xi_0/b)^3 \gg 1$ Cooper pairs overlap. Any other even-number cluster of electrons, e.g. four bound electrons, would serve the purpose equally well, but these higher order clusters are expected to be much less stable. In conventional superconductors, the pairing shows up in that the electric charge of a Cooper pair appears to be $2e$. For Cooper pairs to form, it is necessary that the effective QP interaction has an attractive component. The attraction may originate 'from outside'—e.g. from the coupling to the crystal lattice excitations, as in the initial BCS theory; or 'from inside'—specifically, by exchange of excitations within the system, such as spin fluctuations.

The order parameter of a superconductor below the critical temperature T_c is the amplitude [75, 76]

$$\psi_{\mathbf{k}\sigma\sigma'} = \langle c_{\mathbf{k}\sigma} c_{-\mathbf{k}\sigma'} \rangle \tag{23}$$

of pairs of total momentum zero. The order parameter thus defined has at least two degrees of freedom, a global phase and a modulus, $\psi = |\psi| \mathrm{e}^{\mathrm{i}\phi}$. In an actual system the equilibrium state is characterized by a coherent superposition of states with different phases ϕ, and the defining feature of the superconducting state is the phase stiffness, i.e. changing the phase in space costs energy. Since the absolute value of the phase is not observable (whereas phase differences are), it is sufficient to assume one fixed value of the phase.

The fact that a Cooper pair overlaps in space with a huge number of neighboring Cooper pairs implies that mean field theory works very well. The mean field Hamiltonian, named after Bogoliubov and de Gennes, is found to be [77, 78]

$$H_{BdG} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k},\sigma} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + \frac{1}{2} \sum_{\mathbf{k},\sigma,\sigma'} (\Delta^*_{\mathbf{k},\sigma\sigma'} c_{-\mathbf{k},\sigma'} c_{\mathbf{k},\sigma} + c^\dagger_{\mathbf{k},\sigma} c^\dagger_{-\mathbf{k},\sigma'} \Delta_{\mathbf{k},\sigma'\sigma}) - \frac{1}{2} \sum_{\mathbf{k},\sigma,\sigma'} \Delta^*_{\mathbf{k},\sigma\sigma'} \psi_{\mathbf{k}\sigma\sigma'}. \tag{24}$$

Here, the 'gap parameter' Δ is defined by

$$\Delta_{\mathbf{k},\sigma\sigma'} = \sum_{\mathbf{k}',\sigma_1,\sigma'_1} V_{\mathbf{k},\sigma\sigma';\mathbf{k}'\sigma_1\sigma'_1} \psi_{\mathbf{k}'\sigma_1\sigma'_1}, \tag{25}$$

where V is the pair interaction. The Hamiltonian equation (24) may be diagonalized by a Bogoliubov transformation in the form

$$H_{BdG} = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}\sigma} b^\dagger_{\mathbf{k},\sigma} b_{\mathbf{k},\sigma} + E_{\mathrm{cond}}, \tag{26}$$

where E_{cond} is the energy of the pair condensate. Fermionic excitations, termed 'Bogoliubov QPs' (BQPs) [77], are created by operators

$$b^\dagger_{\mathbf{k},\sigma} = \sum_{\sigma'} (u^*_{\mathbf{k},\sigma\sigma'} c^\dagger_{\mathbf{k},\sigma'} - v^*_{\mathbf{k},\sigma\sigma'} c_{-\mathbf{k},\sigma'}), \tag{27}$$

where the coefficients u,v are given by $u_{\mathbf{k},\sigma\sigma'} = \delta_{\sigma\sigma'}(\xi_{\mathbf{k}\sigma} + E_{\mathbf{k}\sigma})/D_{\mathbf{k},\sigma}$ and $v_{\mathbf{k},\sigma\sigma'} = \Delta_{\mathbf{k},\sigma\sigma'}/D_{\mathbf{k},\sigma}$, with $D_{\mathbf{k},\sigma} = [2E_{\mathbf{k}\sigma}(\xi_{\mathbf{k}\sigma} + E_{\mathbf{k}\sigma})]$ (a unitary state is assumed here). We note that a BQP is a superposition of a particle and a hole. The spectrum of BQPs is found as

$$E_{\mathbf{k}\sigma} = \sqrt{\xi^2_{\mathbf{k}\sigma} + (\Delta_{\mathbf{k}}\Delta^*_{\mathbf{k}})_{\sigma\sigma}}. \tag{28}$$

In the case of an isotropic superconductor, we have $\psi_{\mathbf{k}\sigma\sigma'} = \psi_{-\mathbf{k}\sigma\sigma'} = \psi_{P\mathbf{k}\sigma\sigma'} = i\tau^y_{\sigma\sigma'}\psi_{\mathbf{k}}$, where $P\mathbf{k}$ is the manifold of momentum vectors generated by the operations P of the point group of the lattice, τ^y is the second Pauli matrix, and the last equation follows from the antisymmetry property of fermionic wave functions. Here the gap parameter $\Delta_{\mathbf{k}}$ is diagonal in spin space, but generically depends on momentum—even though the $\mathbf{k}$-dependence is usually neglected for an isotropic superconductor—because $\Delta_{\mathbf{k}}$ is everywhere finite on the Fermi surface, meaning that the energy spectrum has a full gap.

The number of BQPs is not conserved, since two BQPs with quantum numbers $(\mathbf{k}\sigma; -\mathbf{k}, -\sigma)$ may coalesce into a Cooper pair, and vice versa. Actually any pair $(\mathbf{k}+\mathbf{q},\sigma; -\mathbf{k}, -\sigma)$ may coalesce, provided the extra momentum $\mathbf{q}$ (and corresponding energy) will be transferred to phonons. It follows from the above that the character of fermionic QPs changes qualitatively at the transition to the superconducting state. It is worth asking how the Landau QP properties change on approaching T_c. We may expect that the appearence of superconducting fluctuations close above T_c—i.e. superconducting regions of extension of the coherence length $\xi(T) \propto (T - T_c)^{-1/2}$ diverging at T_c, fluctuating in time at a rate $1/\tau_{GL} \propto (T - T_c)$, vanishing at T_c—will lead to an increase in the QP decay rate. Indeed, it may be shown that on the initial approach to T_c, the QP relaxation rate increases [79–81] compared to the T^2-law of a normal Fermi liquid

$$\Gamma(T) \propto \lambda^2 T, \quad \text{for } t_G < T/T_c - 1 < T_c/T_F, \tag{29}$$

where $t_G = (a/\xi_0)^{2d/(4-d)}$ (a = lattice constant; ξ_0 = coherence length at $T = 0$; d = spatial dimension) marks the entrance into the true critical regime governed by interacting fluctuations as given by the Ginzburg criterion [82]. The existence criterion for QPs, $\Gamma(T)/T < 1$, is met here in the case of weak coupling superconductors, considering that the coupling constant, estimated as $\lambda = 1/\ln(T_D/T_c)$, is then less than unity. Here, we have used the expression for the transition temperature $T_c = T_D \exp(-1/\lambda)$, where T_D is the width of the region in energy over which the pair interaction is attractive. Inside the critical regime, $T/T_c - 1 < t_G$, the situation depends on whether the fermions are charged or neutral. In a charged system, the transition to a type I superconductor is likely first order [83]. In strong coupling superconductors, QPs are presumably ill defined in the limit $T \to T_c$.

On the other side of the transition, at $T < T_c$, one may expect fluctuations signaling the breakdown of the ordered state. One distinguishes longitudinal and transverse fluctuations, i.e. fluctuations of the modulus and of the phase of the order parameter. The spectrum of longitudinal fluctuations is gapped, which means they do not contribute much to the increase in QP scattering cross section. The transverse fluctuations are gapless, but in a charged system they couple to the electromagnetic field, and thereby acquire a gap (a 'mass') as well. So, both types of fluctuations (in three dimensions) do not boost the BQP decay rate in a charged system. There is, however, a more important change: the energy gap in the Bogoliubov QP spectrum will be washed out when $\Gamma(T) \gtrsim \Delta(T)$. Here we may estimate [84]

$$\Gamma(T) \approx \lambda^2(T_c^2/T_F)[1 - c_B(\Delta/T_c)^2], \quad 0 < 1 - T/T_c \ll 1. \tag{30}$$

Using $\Delta(T) = c_\Delta T_c(1 - T/T_c)^{1/2}$, the crossover temperature T_x beyond which BQPs are no longer well defined is then found as $(T_x/T_c - 1) \approx (\lambda^2 T_c/c_\Delta T_F)^2$, i.e. T_x is within 10^{-6} of T_c for conventional superconductors. It is worth noting that the QP relaxation rate has a discontinuity in its temperature derivative at T_c. At low $T \ll \Delta$, the number of QPs is exponentially small: $n_{BQP} \propto \exp(-\Delta/T)$, and therefore $\Gamma(T) \propto \exp(-\Delta/T)$.

There are two order parameter collective modes in an isotropic superfluid: oscillations (i) of the phase $\phi(\mathbf{r}, t)$ (the Goldstone mode) and (ii) of the modulus $|\psi(\mathbf{r}, t)|$ in space and time. The equilibrium state of a superconductor is continuously degenerate with respect to changing the global phase ϕ. This implies that in the limit of infinite wavelength, oscillations of the phase should not cost any energy—i.e. these excitations are gapless. However, in an electrically charged system, such as a superconductor, these phase modes are coupled to the electromagnetic field, and in this way both the phase mode and the transverse electromagnetic field modes acquire a gap (Anderson–Higgs mechanism [85–88]). In a neutral superfluid, the phase mode is observable as a density mode (a sound mode) with linear dispersion $\omega = c_{AB}q$, called the Anderson–Bogoliubov mode. The second collective mode is the oscillation of $|\psi(\mathbf{r}, t)|$ about its equilibrium value with dispersion $\omega_{\mathbf{q}} = 2\Delta + c_A q^2 + i\gamma$. This mode is known as the Higgs particle in particle physics [89]. The energy 2Δ is the minimal energy required to break a Cooper pair into its fermionic constituents, and therefore marks the border of the 'pair-breaking continuum' in the pair excitation spectrum. As a consequence, this mode is again overdamped (damping rate γ), and is usually not observed. Actually, the mode is so strongly damped that its decay is non-exponential in time [90, 91]. In a Cooper-paired Fermi system, such as a usual superconductor or superfluid ^{3}He, the equation of motion of the order parameter describes reactive dynamics as expressed by a second order time derivative, leading to the above real-valued mode frequency (leaving aside the damping rate γ for the moment). In contrast, the dynamics of a bosonic superfluid order parameter is relaxational, and the equation of motion of $\psi(\mathbf{r}, t)$ [92, 93] contains a first order time derivative. In that case, the amplitude mode of the order parameter is overdamped: $\omega_{\mathbf{q}} = i\Gamma + c_A q^2$—i.e. is not well defined. The reason for the difference is that the BCS-theory is particle–hole symmetric, prohibiting a first order time derivative. However, particle–hole symmetry may exist even in bosonic systems. Systems of ultracold bosons on a lattice are particle–hole symmetric along a line in the chemical potential–dimensionless

interaction plane, which has led to the search for, and the discovery of evidence of, the Higgs mode in such a system [94]. In certain anisotropic magnetic insulators, when the order parameter (the sublattice magnetization of an $X-Y$-model)) has the symmetry of a complex-valued scalar and obeys reactive dynamics, the equivalent of the Higgs mode has also been identified [89, 95, 96].

4.2. Anisotropic superfluids: Helium 3

One of the best examples of the various types of QP is offered by liquid ^{3}He. In the temperature range from about 100 mK down to a few mK and in the pressure range from ambient pressure up to the melting pressure of about 30 bar it is perfectly described by Fermi liquid theory in the strong coupling regime, i.e. with parameters $m^*/m, F_0^s, F_1^s \gg 1$. The Landau QP weight factor is correspondingly small, $Z \ll 1$. Nonetheless, thermodynamic and transport properties may be described quantitatively within the QP picture.

At temperatures below 2.6(1.0) mK at melting (ambient) pressure a phase transition into superfluid phases occurs [22], [97–100]. Owing to the strong repulsive interaction between ^{3}He-atoms at short distances, pair formation in a relative s-wave state, as in electron pairing in conventional superconductors, may be ruled out. The effective many-body interaction between QPs as inferred from the normal Fermi liquid properties [22] or from a microscopic model using the so-called direct interaction as a starting point [37] may be shown to favor p-wave and, by the Pauli principle, spin triplet pairing. The then available three orbital and three spin substates lead to a complex order parameter $A_{\mu j}$ of 3×3 tensor structure

$$\psi_{\mathbf{k}\sigma\sigma'} = \sum_{\mu j} A_{\mu j} (\mathrm{i}\tau^y \tau^\mu)_{\sigma\sigma'} \widehat{k}_j, \tag{31}$$

where $\tau^\mu, \mu = x, y, z$ are the Pauli matrices, and $\widehat{k}_j, j = 1, 2, 3$ are the three components of the unit vector in k-space. This should be contrasted with a single complex order parameter in the case of conventional superconductors. As a consequence, one has several equilibrium states, depending on pressure, temperature, magnetic field, boundary conditions, etc. The two main phases are the A-phase at elevated pressure and temperature, and the B-phase at low T. The anisotropies of the orbital and spin parts of the pair wave function in the case of non-s wave pairing lead in general to an anisotropic energy gap function $\Delta_{\mathbf{k}\sigma}$ in the Bogoliubov QP dispersion.

In the A-phase, the order parameter matrix is given by $A_{\mu j} = \psi d_\mu (m_j + \mathrm{i} n_j)$, where $\mathbf{m} \perp \mathbf{n}, (\mathbf{d})$ and $\mathbf{l} = \mathbf{m} \times \mathbf{n}$ are unit vectors in orbital (spin) space. The energy gap follows as $|\Delta^A_{\mathbf{k}\sigma\sigma'}| = \delta_{\sigma\sigma'} \Delta_0 \sin\theta_{\mathbf{k}}$, where $\mathbf{k} \cdot \mathbf{l} = k_\mathrm{F} \cos\theta_{\mathbf{k}}$, and has point nodes along the z-axis, assuming $\mathbf{l}$ to point along the z-axis. In the B-phase $A_{\mu j} = \psi R_{\mu j}$, where $R_{\mu j}$ is an arbitrary rotation matrix specifying a relative rotation of spin and orbital space. The balanced superposition of all available substates in this case leads to an isotropic gap $\Delta^B_{\mathbf{k}\sigma\sigma'} = \delta_{\sigma\sigma'}\Delta$.

Whereas the BQP properties in the B-phase are very similar to those of an isotropic superfluid, in the A-phase the zeroes of the gap lead to physical quantities varying as power laws in temperature, rather than exponential as $T \to 0$. The relaxation rate of BQPs depends strongly on the their position on the Fermi surface,

$$\Gamma^A_{\mathbf{k}\|\mathbf{l}} \propto (T^2/T_\mathrm{F})(T/\Delta_0(0))^2, \tag{32}$$

$$\Gamma^A_{\mathbf{k}\perp\mathbf{l}} \propto \exp(-\Delta_0/T). \tag{33}$$

Even more remarkable is the fact that the orbital part of the order parameter is closely coupled to the QP system, since the preferred direction $\mathbf{l} = \mathbf{m} \times \mathbf{n}$ forms the axis of the gap. A rotation of $\mathbf{l}$ therefore implies a change of the energy of the QP system.

Different combinations of symmetries, aside from the gauge symmetry, are broken in the two phases. In the A-phase, the breaking of rotation symmetry separately in spin space and in orbital space generates spin waves and orbital waves as the corresponding Goldstone modes. Here, the spin waves have a linear spectrum $\omega \propto q$, and are weakly damped at low q. Actually liquid ^{3}He is not completely spin rotation invariant, because of the dipole interaction of the nuclear spins, which couples spin and orbital space. In the pair correlated state, the effective dipole interaction is substantially enhanced by the fact that the preferred directions of the Cooper pairs are aligned [99]. This fact introduces gaps in the spin wave spectra. Leaving that aside, the broken rotational invariance in orbital space gives rise to orbital wave modes. These are, however, overdamped, because—as pointed out above—rotation of $\mathbf{l}$ changes the BQP energies and leads to BQP production, such that the dispersion near T_c is found as $\omega \propto i^{1/2} q^2$.

The internal structure of the orbital part of the order parameter of the A-phase, defined by the triad $\mathbf{l}, \mathbf{m}, \mathbf{n}$ of mutually perpendicular unit vectors, offers two possible pair vibration modes [101]: oscillations of (i) the angle $\measuredangle(\mathbf{m}, \mathbf{n})$ (clapping mode), (ii) the angle between $\mathbf{l}$ and $(\mathbf{m}, \mathbf{n})$ (flapping mode). Both modes are reasonably well defined. The clapping mode is stabilized by the minimum in the condensation energy, and consequently $\omega_{cl} \approx 1.2\Delta(T)$, of order of the temperature dependent energy gap. By contrast, the flapping mode is actually an oscillation of the condensate vector $\mathbf{l}$ against the preferred direction $\mathbf{l}_n$ defined by the BQP axis. As the QPs disappear for $T \to 0$, so does the restoring force and therefore $\omega_{fl} \propto \Delta(T)$ near T_c, but $\omega_{fl} \propto T$ at low T. These modes have been detected as peaks in the ultrasound attenuation [98].

In the B-phase, the symmetry of relative rotations in spin and orbital space is broken. The associated Goldstone modes are three spin–orbit waves with linear dispersion. In addition, two types of pair-vibration modes appear, in which the isotropic gap parameter shows a quadrupolar deformation with real or imaginary valued amplitude (called real squashing mode and squashing mode). The frequencies of these modes are $\omega_q = (8/5)^{1/2}\Delta + O(q^2)$ and $\omega_q = (12/5)^{1/2}\Delta + O(q^2)$ respectively [100, 103, 104]. The fact that the squares of the frequencies add up to the square of the pair-breaking frequency $(8/5)\Delta^2 + (12/5)\Delta^2 = (2\Delta)^2$ follows from the supersymmetry of the Hamiltonian [102]. In contrast to the corresponding A-phase modes, the squashing modes are undamped in the limit $T \to 0$. These modes have also been detected as peaks in the ultrasound attenuation [64, 98].

4.3. Ferromagnets

In a ferromagnetic metal a ferromagnetic polarization $\mathbf{M}$ of conduction electrons leads to a Zeeman splitting of Landau QP states, such that

$$\epsilon^*_{\mathbf{k}\sigma} = \epsilon^*_{\mathbf{k}} - \sigma h, \tag{34}$$

where $h \propto |\mathbf{M}|$, and the spin quantization axis is chosen to be along $\mathbf{M}$. The broken spin rotation invariance generates Goldstone modes—weakly damped spin waves of quadratic dispersion $\omega \propto q^2$ [105, 106]. The number of fermionic QPs of given spin projection is no longer conserved, as scattering of QPs off spin wave excitations may flip the spin.

We now consider how the Landau QP properties in the paramagnetic state ($T > T_c$) change upon approaching the transition temperature T_c into the ferromagnetic phase. The spectrum of spin fluctuations in three dimensions at low q, ω and near T_c is given by [107]

$$\mathrm{Im}\chi_{\mathrm{FM}}(\mathbf{q}, \omega + \mathrm{i}0) = \mathrm{Im}\frac{N_0}{t^\gamma + (q/k_\mathrm{F})^2 - \mathrm{i}\omega/v_\mathrm{F}q}, \tag{35}$$

where $t = T/T_c - 1$, and γ is a critical exponent ($\gamma = 1$ in mean field approximation). We neglect a possible logarithmic correction factor to the q^2-term [108]. Scattering of QPs off these fluctuations leads to an enhanced relaxation rate Γ, which may be calculated as the imaginary part of the self-energy in one-loop approximation

$$\Gamma_{\mathbf{k}}(T) = Z\mathrm{Im}\Sigma_{\mathbf{k}}(\omega = 0) = Zu^2 \int \frac{\mathrm{d}\nu}{2\pi} \int \frac{\mathrm{d}^d q}{(2\pi)^d} [f(\nu) + b(\nu)] \times \mathrm{Im}\{G_{\mathbf{k}+\mathbf{q}}(\nu + \mathrm{i}0)\}\mathrm{Im}\{\chi_{\mathrm{FM}}(\mathbf{q}, \nu + \mathrm{i}0)\}, \tag{36}$$

where $u \propto N_0^{-1}$ is the interaction, and $f(\nu), b(\nu)$ are the Fermi and Bose functions, respectively (we have dropped the spin label, which is irrelevant in the paramagnetic state). Replacing the spectral function $\mathrm{Im}G$ by the corresponding δ-function, and using that the QP weight factor Z is a constant, one may first do the angular integration for the case of $d = 3$ dimensions to find $\langle \mathrm{Im}G_{\mathbf{k}+\mathbf{q}} \rangle \propto 1/v_\mathrm{F}q$. Further approximating the Fermi and Bose functions by cutoffs, one gets

$$\Gamma_{\mathbf{k}}(T) \propto u^2 \int_0^T \mathrm{d}\nu \int q^2 \mathrm{d}q \frac{1}{v_\mathrm{F}q} \mathrm{Im}\{\frac{N_0}{t^\gamma + (q/k_\mathrm{F})^2 - \mathrm{i}\nu/v_\mathrm{F}q}\}. \tag{37}$$

The remaining integral over q is governed by the singular behavior of the spin fluctuation propagator in the limit $t, \nu \to 0$, controlled by a lower cutoff of the q-integration at $q/k_\mathrm{F} \approx t^{\gamma/2}$, provided $t^\gamma > \nu/v_\mathrm{F}q \to t^{-\gamma/2}T/v_\mathrm{F}k_\mathrm{F}$. The result, after doing the ν-integration, is

$$\Gamma_{\mathbf{k}}(T) \propto \lambda^2 t^{-3\gamma/2} \frac{T^2}{T_\mathrm{F}}, \quad t^{3\gamma/2} > T/T_\mathrm{F}. \tag{38}$$

The QP relaxation rate is seen to grow on approaching T_c until it reaches a plateau value of $\Gamma \approx \lambda^2 T_c$, at $t^{3\gamma/2} < T/T_\mathrm{F}$ (here $\lambda \approx N_0 u$ is a dimensionless coupling constant). The relaxation rate usually still satisfies $\Gamma(T_c) < T_c$, assuming $T_c \ll T_\mathrm{F}$, so that $\lambda \ll 1$. So, QPs remain reasonably well defined at weak coupling.

4.4. Antiferromagnets

In metallic systems, an antiferromagnetic ordered state usually appears below a temperature T_N in the form of an incommensurate spin density wave $\mathbf{M}(\mathbf{Q}) = \mathbf{M}_0 \cos(\mathbf{Q} \cdot \mathbf{r})$, where the wave vector Q is usually large, of the order of a reciprocal lattice vector. Again spin rotation invariance is broken, and the corresponding Goldstone modes are spin waves with dispersion $\omega \propto q$ in the limit of small q [106]. The spectrum of fermionic QPs develops a gap

$$\epsilon^{\mathrm{AFM}}_{\mathbf{k},\sigma} = \frac{1}{2}\{(\epsilon^*_{\mathbf{k}+\mathbf{Q},\sigma} + \epsilon^*_{\mathbf{k},\sigma}) \pm [(\epsilon^*_{\mathbf{k}+\mathbf{Q},\sigma} - \epsilon^*_{\mathbf{k},\sigma})^2 + h_Q^2]^{1/2}\}. \tag{39}$$

The number of QPs of given spin projection is again no longer conserved, as scattering of QPs off spin wave excitations may flip the spin, and simultaneously change the QP momentum by $\mathbf{Q}$.

In the paramagnetic phase, the QP relaxation rate again increases as $T \to T_N$, on account of enhanced scattering off antiferromagnetic spin fluctuations with energy spectrum [107]

$$\mathrm{Im}\{\chi_{\mathrm{AFM}}(\mathbf{q}, \omega)\} = \mathrm{Im}\{\frac{N_0}{t^{\gamma_A} + (\mathbf{q} - \mathbf{Q})^2/k_\mathrm{F}^2 - \mathrm{i}\omega/v_\mathrm{F}Q}\},$$

where $t = T/T_N - 1$, and the exponent $\gamma_A = 1$ in mean field approximation. Since the exchange process involving an antiferromagnetic spin fluctuation necessarily implies a change of momentum of the initial QP on the Fermi surface by $\mathbf{Q}$, the final state will usually be far from the Fermi surface, except if the initial momentum is near a 'hot spot' $\mathbf{k} \approx \mathbf{k}_h$ where $\epsilon^*_{\mathbf{k}_h} = \epsilon^*_{\mathbf{k}_h+\mathbf{Q}} = 0$ (assuming there are 'hot spots'). From the imaginary part of the self energy in one-loop approximation, the initial increase of Γ at a hot spot is found—following the steps described for the ferromagnetic case—as

$$\Gamma_{\mathbf{k}_h} \propto u^2 \int_0^T \mathrm{d}\nu \int \mathrm{d}^3 q \mathrm{Im} \frac{N_0}{t^{\gamma_A} + (q/k_\mathrm{F})^2 - i\nu/v_\mathrm{F}Q} \propto \lambda^2 t^{-\gamma_A/2} \frac{T^2}{T_\mathrm{F}}, \quad t^{\gamma_A} > T/T_\mathrm{F}. \tag{40}$$

(This time, the angular integration produces $\langle \mathrm{Im}G_{\mathbf{k}+\mathbf{q}} \rangle \propto 1/v_\mathrm{F}Q$.) The growth of $\Gamma_{\mathbf{k}_h} \propto t^{-\gamma_A/2}$ is seen to be slower than in the ferromagnetic case (assuming $\gamma_A = 1$). Again, $\Gamma_{\mathbf{k}_h}$ reaches a plateau value of $\Gamma_{\mathbf{k}_h} \approx \lambda^2 T_N^{3/2}/T_\mathrm{F}^{1/2}$, this time at $t^{\gamma_A} = T/T_\mathrm{F}$, where QPs are still reasonably well defined, assuming $T_N \ll T_\mathrm{F}$.

5. The transformation of quasiparticles at a quantum critical point

At a quantum critical point (QCP), i.e. an end point of a phase boundary at $T = 0$, separating the disordered from a continuously forming ordered phase, quantum fluctuations govern the behavior of the system in a wide regime of the phase diagram extending to finite temperature [109, 110]. The system may be tuned through the critical point by variation of a control

parameter (pressure, magnetic field, chemical composition, …). The conventional theory of quantum criticality starts from the assumption that the relevant critical degrees of freedom are the fluctuations of the order parameter, for which a ϕ^4-field theory may be formulated, the Landau–Ginzburg–Wilson theory [111, 112]. In contrast to the classical theory of criticality near a continuous phase transition, which may be phrased in terms of static (time-independent) fields, quantum criticality requires consideration of the dependence of the fluctuation amplitude on imaginary time. As a consequence, the effective dimensionality of the system is larger than the space dimension d, viz. $d_{\rm eff} = d + z$, where the dynamical critical exponent z (controlling the relation of spatial and temporal scaling of the fluctuations, $\omega \propto q^z$) accounts for the dimensionality of time. One frequently finds $d_{\rm eff} > 4$, the upper critical dimension of ϕ^4-field theory, implying that the system is in the Gaussian regime—or, in other words, the interaction between fluctuations scales to zero. The system properties may then be calculated in a controlled way.

Of particular interest are magnetic quantum phase transitions in metals [113], which in principle involve two critical subsystems—that of the order parameter and that of the fermionic QPs. The criticality of fermionic excitations shows up in a critical growth of the effective mass m^* as the QCP is approached, which is experimentally easily detected through a divergence of the electronic specific heat coefficient $C/T \propto m^*$, as the temperature tends to zero and the control parameter approaches its critical value. When a divergence of m^* is not observed, the conventional theory of quantum criticality is applicable. If, however, experiment indicates a diverging m^*, a new theoretical description is called for: one which considers critical bosons (representing the order parameter fluctuations) and critical fermions (the Landau QPs, correspondingly modified). Prominent compounds in this class are the heavy fermion systems $CeCu_{6-x}Au_x$ near a doping concentration of $x = 0.1$ ($m^* \propto T^{-\alpha}$, $\alpha \approx 0.1$, or logarithmic) [114], and $YbRh_2Si_2$ ($m^* \propto T^{-\alpha}$, $\alpha \approx 0.3$) [115]. In these systems, which both show an antiferromagnetic QCP, the observed properties strongly deviate from those of a Fermi liquid—which is why these phases are termed 'Non-Fermi liquids'. It has been argued that Fermi liquid notions are not applicable any more, and a completely different description is called for [116–119]. This may not necessarily be the case, as proposed in the recent work of Abrahams, Schmalian and the present author [120–123]. Rather, in the critical regime, Landau QPs mutate into critical QPs characterized by a scale-dependent effective mass $m^*(\omega) \propto \omega^{-\eta}$, where ω denotes the QP energy, or temperature, whichever is largest. Critical QPs may be shown still to obey the QP criterion $\Gamma(\omega) < \omega$, provided $\eta \lesssim 0.36$. In the following, the main results of that theory will be reviewed.

5.1. Critical Fermi liquid near an antiferromagnetic quantum critical point

We generalize the QP form of the single particle Green's function, equation (3), by allowing the QP weight factor—and therefore the effective mass—to be energy dependent, $Z = Z(\omega), m^* = m^*(\omega)$ (we suppress a possible additional dependence on the position on the Fermi surface, $\mathbf{k}_\parallel$, or on the distance from the Fermi surface, $\mathbf{k}_\perp$), where ω denotes the QP excitation energy. Assuming a power law dependence of the self energy, $\Sigma(\omega) \propto -\mathrm{i}(\mathrm{i}|\omega|)^{1-\eta}$ as obtained from a self-consistent perturbative treatment [120–123] or equivalently a renormalization group study of the spin-fermion model of AFM criticality [123], it is found that the ratio of QP peak width and energy, $\Gamma/\omega = (1-\eta)^{-1}\cot[(1-\eta)\pi/2] < 1$, provided $0 < \eta \lesssim 0.36$. The case $\eta \to 0$ is known as a 'Marginal Fermi liquid' [45]. Therefore, provided the self energy is not too singular, i.e. $\eta \lesssim 0.36$, we may still use the concept of QPs—now in the form of 'critical QPs'—in calculating properties like the self energy $\Sigma(\omega)$, or the antiferromagnetic spin susceptibility $\chi(q,\omega)$. In other words, we may approximate the single particle spectral function at low energy by

$$\mathrm{Im}G(\mathbf{k},\omega) \approx Z(\omega)\delta[\omega - Z(\omega)\epsilon_{\mathbf{k}}]. \tag{41}$$

The spectrum of spin fluctuations is given by [120–123]

$$\mathrm{Im}\chi(\mathbf{q},\omega) = \mathrm{Im}\frac{N_0}{r + (\mathbf{q}-\mathbf{Q})^2/k_{\rm F}^2 - \mathrm{i}\lambda_Q^2\omega/v_{\rm F}Q}, \tag{42}$$

where r is the tuning parameter, $r \propto (H - H_{\rm c})^{2\nu}$, ν is the critical exponent of the correlation length, H is the tuning field, e.g. a magnetic field. We have applied two factors of vertex correction λ_Q to the last term in the denominator—the so-called Landau damping term. In microscopic theory, the vertex corrections, which may be shown to diverge as $\lambda_Q \propto Z^{-1}$ [124], are attached to the two ends of a renormalized bubble diagram.

The QP effective mass m^* and relaxation rate Γ, may be calculated, as defined above, from the self energy Σ. In lowest approximation, so-called one-loop order, the self energy given by a single spin fluctuation exchange process, as in equation (40), is highly anisotropic, showing critical behavior only near so-called 'hot spots'—see the discussion above equation (40). There exists, however, a higher order process, which may contribute critical behavior all over the Fermi surface. It involves exchange of a pair of antiferromagnetic spin fluctuations of small total momentum. The corresponding combined fluctuation χ_E may be viewed as an exchange energy fluctuation. Provided the QP effective mass is singularly enhanced, this contribution gets enhanced by vertex corrections $\lambda_Q \propto Z^{-1}$ at the ends of the AFM spin fluctuation propagators, and additionally by vertex corrections $\lambda_v \propto Z^{-1}$ at the ends of the propagator χ_E. As a result, the vanishing of the bare $\mathrm{Im}\chi_E(\omega) \propto \omega^{d-1/2}$ is overcompensated by a factor $\lambda_Q^{2d+1}\lambda_v^2$, such that the spectrum of the energy fluctuation propagator including the vertex corrections is highly singular [122, 123]. In one-loop order in χ_E, one finds a scale-dependent effective mass:

$$\frac{m^*(\omega)}{m} = Z^{-1}(\omega) \propto \omega^{-\eta},$$
$$\eta = \begin{cases} \frac{1}{4}, & d = 3 \\ \frac{1}{8}, & d = 2 \end{cases}.$$

The dynamical critical exponent z and the correlation length exponent ν follow as $z = 4, (\frac{8}{3})$ and $\nu = \frac{1}{3}, (\frac{3}{7})$ in the case of three-dimensional (two-dimensional) AFM spin fluctuations in a three-dimensional metal. These exponents are in excellent agreement with available experimental data on $YbRh_2Si_2$ ($d = 3$) and $CeCu_{6-x}Au_x$ ($d = 2$) [120–123].

The scale dependence of the QP relaxation rate in the critical regime is found to be linear in both cases ($d = 3$, 2), as anticipated

$$\Gamma(T) = (1-\eta)^{-1}\cot[(1-\eta)\pi/2]T, \quad (H/H_c - 1)^{\nu z} \lesssim T/\epsilon_F^*, \tag{43}$$

whereas the prefactor of the T^2-law in the Fermi liquid regime grows upon approach to the QCP, until the plateau value in the critical regime is reached:

$$\Gamma(T) \propto |H/H_c - 1|^{-z\nu}T^2/\epsilon_F^*, \quad T/\epsilon_F^* \lesssim (H/H_c - 1)^{\nu z}. \tag{44}$$

We thus find, again—similarly to the case of thermal phase transitions in metals considered above—that the QP relaxation rate increases upon approach to the critical point, following a power law in the control parameter (which is now $|H/H_c - 1|$, and was $|T/T_c - 1|$ before), but assumes a control-parameter-independent, though temperature-dependent value $\Gamma \sim T$ (which is $\sim T_c$ in the case of the thermal transition) in the critical regime.

The strong coupling theory described in the above is an example of a mean field theory of transport quantities capturing a dynamical equilibration process [125]. Here, the balance of bosonic and fermionic fluctuations is the physical reason for the emergence of a new strong coupling theory. Similar approaches may be used to describe the Kondo effect in the weak and strong coupling regimes [125] or the Anderson localization problem [126, 127].

6. Quasiparticles with fractional quantum numbers

The QPs based on quasiclassical principles, such as the Landau QPs or collective wave excitations such as spin waves, reviewed above, may be completely destroyed by quantum fluctuations. We now briefly review two cases where this happens, and where new types of QPs appear instead. The first example is a strictly one-dimensional system of fermions interacting via short range interactions (for a textbook presentation see [128]). There, a Landau QP breaks up into two fractional QPs carrying the charge (chargons of charge $Q = -e$ and spin $S = 0$) and the spin (spinons of charge $Q = 0$ and spin $S = 1/2$).

A second example is that of fractional QPs in the quantum Hall effect at non-integer fillings ν (see [69, 129]). There, one appears to have Laughlin QPs carrying charge νe, e.g. at fillings of the lowest Landau level of $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$. Near half-filling, $\nu = 1/2$, the picture of 'composite fermions' forming a Fermi liquid in an effective low magnetic field appears to be quite successful.

6.1. Spinons and chargons in one-dimensional systems

The effect of interactions on the properties of a system of particles depends on its dimensionality. Whereas in dimensions $d \geqslant 2$ two particles may avoid interacting by moving past each other at a sufficiently large distance, this is not possible in one dimension. Consequently, interactions have a fundamental effect, even in the weak coupling limit. While in a classical system two particles of low energy may not be able to get past each other at all, quantum particles may tunnel through each other even then.

It has been found early on that the proper excitations in one-dimensional electron systems are bosonic charge and spin excitations, described by density operators

$$\rho_{\nu,r}(q) = \sum_k [c^+_{rk_F+k+q,\uparrow}c_{rk_F+k,\uparrow} + \nu c^+_{rk_F+k+q,\downarrow}c_{rk_F+k,\downarrow}], \tag{45}$$

where $\nu = \rho, \sigma = +, -$ indicate charge/spin, and $r = \pm$ denotes the excitation branch centered at momenta $k = \pm k_F$. Within the Tomonaga–Luttinger model [131–133], including forward scattering interaction processes of fermions on the same branch, $H_4 = \sum_{\nu,r}\sum_q g_{4\nu}\rho_{\nu,r}(q)\rho_{\nu,r}(-q)$ and on different branches $H_2 = 2\sum_\nu \sum_q g_{2\nu}\rho_{\nu,+}(q)\rho_{\nu,-}(-q)$, these excitations are non-interacting. We neglect backward scattering processes and spin-flip scattering ($g_{2\sigma} = 0$), for simplicity. The energies of charge and spin excitations are found to be $\omega = u_\rho q$ and $\omega = u_\sigma q$ at momentum q, generically with different velocities, $u_\nu = [(v_F + g_{4\nu}/\pi)^2 - (g_{2\nu}/\pi)^2]^{1/2}$.

These bosonic excitations show up as power law singularities in the single-particle spectral function, e.g. for $u_\rho > u_\sigma$ at momentum $k - k_F = \kappa > 0$ (k_F is the Fermi momentum) [134]

$$\begin{aligned} A_k(\omega) &\sim |\omega - u_\rho\kappa)|^{(\delta-1)/2}, \quad \omega \approx u_\rho\kappa, \\ &\sim \theta(\omega - u_\sigma\kappa)(\omega - u_\sigma\kappa)^{\delta-1/2}, \ \omega \gtrsim u_\sigma\kappa \\ &\sim \theta(-\omega - u_\rho\kappa)(-\omega - u_\rho\kappa)^{\delta/2}, \quad \omega \lesssim -u_\rho\kappa, \end{aligned} \tag{46}$$

where $\delta = (K_\rho + K_\rho^{-1} - 2)/4$ and the 'Luttinger parameters' are defined as $K_\rho = [(\pi v_F + g_{4,\rho} - g_{2,\rho})/(\pi v_F + g_{4,\rho} + g_{2,\rho})]^{1/2}$. The spectral function equation (46) shows two fractional power law QP peaks at $\omega = u_\rho(k - k_F)$, and $\omega = u_\sigma(k - k_F)$, indicating 'chargon' and 'spinon' excitations, respectively. At the lower peak (here, $\omega = u_\sigma\kappa$), $A_k(\omega)$ shows threshold behavior. Unfortunately, these structures in the spectral function are not easily observed in tunneling or photoemission experiments.

A more accessible observable showing signatures of the above singular structures is the conductance of a quantum wire with a single impurity attached to charge reservoirs. Here, a natural approach is to calculate the conductance G directly, in the framework of a fermionic representation. The way to do this is to first determine the leading scaling contributions to the conductance in perturbation theory, to formulate a renormalization group (RG) scheme for the conductances as functions of scaling variables like the length of the wire,

the temperature, or the bias voltage, to calculate the relevant RG-β-functions, and to find the fixed points of the RG-flow and the full solutions of the RG-equations near the stable fixed points. For a two-lead junction, and in lowest order in the interaction, this program has been first carried out in [135, 136]. It was later found that even at strong coupling, the summation of an infinite class of diagrams of the ladder (or RPA) type makes it possible to obtain results in complete agreement with those obtained from bosonization, conformal field theory, Bethe-ansatz, where available [137–139]. For example, the conductance G of a two-lead junction of spinless TLL-wires as a function of temperature T is obtained in form of the implicit equation [137]

$$\frac{G^K}{1-G} = \frac{G_0^K}{1-G_0}\left(\frac{T}{T_0}\right)^{2(1-K)}, \tag{47}$$

where $K = K_\rho$ is the Luttinger parameter defined above, and $G_0 = G(T_0)$, where T_0 is a sufficiently high reference temperature at which the effect of interactions is weak. It follows that for repulsive interaction, $K < 1$, the conductance vanishes in the limit $T \to 0$ as $G \propto T^{2(1-K)/K} \to 0$, whereas for attractive interaction, the conductance scales to its maximal value $1 - G \propto T^{2(K-1)} \to 0$. The above result is in agreement with results obtained by several different alternative methods [140–142]

6.2. Quasiparticles in the quantum Hall effect

The signature of the quantum Hall effect (QHE) is the quantization of the Hall conductance σ_{xy} of a two-dimensional electron system in a strong magnetic field B perpendicular to the $x-y$ plane, such that $\sigma_{xy}(B)$ forms well-defined plateaus in the vicinity of certain magnetic field values, $\sigma_{xy} = \nu G_Q$ in units of the quantum of conductance $G_Q = \frac{e^2}{2\pi\hbar}$, while the longitudinal conductance vanishes exponentially, $\sigma_{xx} \propto \exp(-\Delta_H/T) \to 0$, in the limit $T \to 0$ [143, 144] (for a review of experiment and theory see [69, 129, 130]). The magnetic field leads to a quantization of the single particle energy into discrete Landau levels $\epsilon_{\mathbf{k}} \to \omega_c(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$ with the cyclotron frequency $\omega_c = \frac{eB}{mc}$. The degeneracy $N_\Phi = L^2B/\Phi_0 = (L/\ell)^2/2\pi$ of the Landau levels is proportional to the sample area L^2, and is equal to the number of magnetic flux quanta $\Phi_0 = \frac{2\pi c}{e}$ threading the sample (here, $\ell = \sqrt{c/eB}$ is the magnetic length). In the expression for the conductance, $\nu = N/N_\Phi$ is the filling factor of the Landau levels (N is the electron number). At high magnetic fields, the Landau level separation is large, such that at low temperatures thermal excitation of electrons into higher Landau levels is completely negligible.

6.2.1. Chiral quasiparticles of the integer quantum Hall effect. The integer quantum Hall effect (IQHE) is characterized by Hall conductance plateaus at integer fillings, $\nu = 1, 2, \ldots$ [143]. The effect may be understood by realizing that ever-present disorder potentials localize all electrons in the bulk of a Hall sample, whereas conducting states exist along the edges (in two dimensions, arbitrarily weak disorder is found to localize all states [145]). These states are strictly 'chiral'—i.e. have a sense of direction in accordance with the cyclotron orbits, such that scattering processes (by impurities, phonons) may not change the sense of direction of an electron, leading to dissipationless transport. The quantization of σ_{xy} is dictated by gauge symmetry [68, 146]. Interaction effects are thought to be negligible in this case, because those require Landau level mixing, which is highly suppressed by the large gap ω_c. This is immediately seen by recognizing that the wave function of the completely filled first Landau level is given by

$$\psi_{\nu=1}(z_1, z_2, \ldots z_N) = \prod_{j<k}(z_j - z_k)\exp[-\frac{1}{2}\sum_l |z_l|^2], \tag{48}$$

where $z_j = (x_j + iy_j)/\ell$ is a complex-valued dimensionless coordinate—(x_j, y_j) denoting the cartesian coordinates of the jth particle and ℓ the magnetic length defined above. The exponential factor expresses the confinement of the particles by the magnetic field (particles are trapped in cyclotron orbits). Any function of the type of equation (48) featuring a polynomial prefactor (analytic function) is inside the Hilbert space spanned by the lowest Landau level states. The wave function $\psi_{\nu=1}$ is unique, i.e. it is not changed by interaction effects, as long as mixing of higher Landau levels is negligible. The state described by $\psi_{\nu=1}$ is incompressible, as addition of an electron requires finite energy ω_c.

One may consider the chiral fermions of the IQHE a new class of QPs protected by topological constraints (here imposed by the external magnetic field). As pointed out by Thouless and co-workers [147], the quantization of the Hall conductance is closely related to the existence of nontrivial topological invariants of the Bloch wave functions of electrons on a lattice in a strong magnetic field. Chiral QPs are not 'fractional' in the narrow sense of the word, and might have been listed in the later section on topological matter as well.

It had been suggested early on that a 'Quantum Anomalous Hall Effect' might occur in certain lattices, even in the absence of an external magnetic field, if time reversal invariance is broken by a magnetic flux distribution of vanishing average or a magnetically ordered subsystem [148, 149]. An experimental realization of this prediction has been found in thin films of chromium-doped $(Bi,Sb)_2\,Te_3$ [150]. More recently, it has been predicted and experimentally verified that the QHE may also exist without an applied magnetic field or magnetically ordered structures, if the magnetic field is effectively provided by the spin–orbit interaction, and if in addition the band structure of the system is inverted, i.e. the valence band is on top of the conduction band in the insulating bulk [73, 74]. In this case, chiral fermions in edge states may allow for counterpropagating dissipationless currents of spin up and down electrons (quantum spin Hall effect). Such systems are termed 'topological insulators': they feature topological conducting states (e.g. chiral states) on the surface of a bulk insulator. The quantum spin Hall effect has been observed in HgTe/CdTe quantum well structures [151]. Below, we will

have a somewhat closer look at these and other examples of topological matter.

6.2.2. Laughlin quasiparticles of the fractional quantum Hall effect. Even more interesting is the fractional QHE, featuring Hall plateaus at rational filling factors $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$ [144]. At the high magnetic fields necessary to achieve $\nu < 1$, the Landau level spacing is the largest energy scale, such that Landau level mixing by the Coulomb interaction and the spin Zeeman splitting may be neglected: electrons occupy states in the lowest Landau level, and are completely spin polarized. In the partially filled lowest Landau level, the kinetic energy is completely quenched, and interaction effects govern the behavior of the system. As shown by Laughlin [67], the ground state of the system is particularly stable at special filling factors—in other words, all excitations then have a large energy gap. For instance, at $\nu = \frac{1}{3}$, Laughlin proposed a trial wave function

$$\psi_{\nu=1/3}(z_1, z_2, ...z_N) = \prod_{j<k}(z_j - z_k)^3 \exp[-\frac{1}{2}\sum_l |z_l|^2]. \quad (49)$$

Here, the polynomial prefactor expresses the mutual repulsion of particles exerted by the Coulomb repulsion, by vanishing rapidly whenever two particles approach each other. The trial wave function may be shown to give an excellent estimate of the ground state energy. The corresponding state is incompressible.

As proposed by Laughlin, QP (quasihole) excitations may be created by the following thought experiment: an infinitely thin solenoid holding a flux quantum Φ_0 is adiabatically passed through the electron fluid at any position z_0. The effect of this operation on the ground state wave function is approximately

$$\psi^q_{\nu=1/3}(z_1, z_2, ...z_N) = Q(\{z_l\}; z_0) \prod_{j<k}(z_j - z_k)^3 \exp[-\frac{1}{2}\sum_l |z_l|^2] \quad (50)$$

$$Q(\{z_l\}; z_0) = \prod_l (z_l - z_0)\,, \quad \text{quasiholes} \quad (51)$$

$$Q(\{z_l\}; z_0) = \prod_l (\frac{\partial}{\partial z_l} - z_0)\,, \quad \text{QPs.} \quad (52)$$

The application of the flux thread generates a local electric charge e^* associated with the QP/hole generated by the screening response of the system. A corresponding countercharge is formed at the edges of the sample, in order to maintain charge neutrality, or else QPs and -holes are created in pairs. An intuitive way to identify the value of e^* is obtained by observing that at $\nu = \frac{1}{3}$ the average flux per electron is $L^2B/N = 3\Phi_0$. An excitation carrying one unit of flux should therefore be expected to have fractional charge $e^* = \pm e/3$. This is exactly what is found in a more detailed analysis. The excitation energy is gapped, and is of order $(e^*)^2/\ell$. These QPs are 'topologically protected', since the magnetic flux carried by a QP is quantized, similar to the vorticity of a vortex excitation in a type-II superconductor. The fractional charges near $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$ have been detected by measuring the shot noise of the back current of the edge channels of a quantum Hall bar setup [154, 155], and by analyzing the excitation spectra of quantum dots observed in scanning tunneling experiments [156, 157].

It has been conjectured [158, 159] that the fractionally charged QP excitations of the Laughlin states at $\nu = \frac{1}{2p+1}$, $p = 1, 2, \ldots$ and similar are anyons with (Abelian) fractional statistical angle $\theta = \pi\nu$. This means that the wave function acquires a phase factor of $e^{i\theta}$ when identical QPs are exchanged in a counterclockwise sense.

In the first generation of experiments, Hall plateaus were identified not only at filling factors $\nu = \frac{1}{2p+1}$, but more generally at $\nu = \frac{p}{qp+1}$, $p = \pm 1, \pm 2, \ldots; q = 2, 4, \ldots$. These fractions may be derived in the framework of the composite fermion picture reviewed below. Later, additional plateaus were found for the lowest Landau level [152], and for higher fillings [153]. These may be interpreted in the framework of the hierarchy picture: changing the filling fraction (by changing the magnetic field or electron density) away from the center of a quantum Hall plateau will lead to the creation of QPs. These are interacting, and may again condense into a Laughlin-like state [160, 161]. By this mechanism, one can generate Hall plateaus at any fractional filling $\nu = p/q$ with an odd integer denominator q [130]. For increasing q one expects these states to be more and more fragile, since the charge of the QP in a state at $\nu = p/q$ is $\pm e/q$, and hence the Coulomb gap is expected to be a decreasing function of q. The finite width of the Hall plateaus arises because if only a few QPs are present (at fillings close to the center of a Hall plateau), they will be pinned to impurities, and will not condense into a new hierarchy state.

6.2.3. Composite fermions. The composite fermion picture of the FQHE invokes an adiabatic mapping of the fractionally filled Landau level system to an integer filled Landau level system [162, 163]. One assumes that each electron binds a vortex containing an even number $2p$ of flux quanta to form a 'composite fermion' (CF). We note that moving quantum objects attached to flux lines containing an even number of flux quanta do not generate an Aharonov–Bohm quantum phase. Suppose the CFs move in a magnetic field B^*, such that the filling factor $\nu^* = N\Phi_0/L^2B^* = q$ is integer. If we assume now that the attached flux is smoothed in space, the magnetic field seen by the electrons would be $B = B^* + 2pN\Phi_0/L^2 = (1 + 2pq)B^*$, corresponding to a filling factor $\nu = q/(1 + 2pq)$. Here $q = \pm 1, \pm 2, \ldots$(allowing positive or negative B^*), and $p = 1, 2, \ldots$ ($p = 0$ corresponds to the integer QHE fillings). We recall that the fractional fillings ν found in this way are in excellent agreement with the experimentally identified Hall plateaus.

The CF picture has been confirmed within a variational wave function approach [163]. The CF energies, obtained as expectation values of the Coulomb interaction with the CF wave function agree with energy values obtained by exact diagonalization of the electron Hamiltonian projected on to the lowest Landau level to within 0.1%. The calculated energy gaps for various FQH states are in good agreement with experimental data.

A microscopic model of the CF picture exists in the form of a Chern–Simons (CS) gauge field theory [164], for which the flux quanta are assumed to be idealized flux lines attached

to each electron. The CS-theory provides a good framework for calculating transport properties, but fails to explain both the size of the energy gap Δ^* protecting the QH states at integer fillings ν^* and the lack of dependence of the CF energies on the bare electron masses.

Of particular interest is the case of a half-filled lowest Landau level, $\nu = \frac{1}{2}$, for which $B^* = 0$ ($\nu^* \to \infty$, $q \to \infty$, $p = 1$), i.e. the composite fermions may be assumed to form a Fermi liquid, barring additional ordering phenomena such as a superconducting phase or a Wigner crystal. According to the Chern–Simons theory, the CFs interact by way of a fictitious gauge field, which is necessarily unscreened, and therefore long-ranged. The single particle properties of CFs are found to be highly singular, while the two-particle properties largely follow the Fermi liquid paradigm [164]. The singularities—e.g. in the density of states or the specific heat—follow from the singular nature of the current–current interaction—equation (18)—induced by the transverse gauge field components [164]. Singular behavior shows up in transport properties (via the phase relaxation rate), once the localizing effect of disorder in these systems is taken into account [165, 166].

The transport properties of quantum Hall systems near half-filling of the lowest Landau level in the temperature range from about 1 K down to the lowest accessible temperatures are dominated by the scattering of composite fermions at impurities [164, 167, 168]. However, the interaction of CFs with impurities is unusual in two respects. First, in high mobility quantum Hall samples, the donor atoms are doped into a remote δ-layer, such that the random Coulomb potential created by the charged donor ions—which are statistically distributed within the δ-layer—is a long-range correlated, smoothly varying potential ($k_F d \sim 10 - 20$, where d is the spacer distance). Second, the static charge density variation caused by the random potential is associated with a (fictitious) static random magnetic field (RMF), due to the inhomogeneous flux tube distribution following from the inhomogeneity in particle density. The effect of the RMF turns out to dominate the usual potential scattering. The transport properties of composite fermions in the RMF may be described in the classical approximation. This is justified by the large value of the parameter $k_F d$, meaning that the extension of quantum mechanical wavepackets $\sim k_F^{-1}$ is much less than the scale of variation of the magnetic field or vector potential, d. The interaction of the CFs with the RMF is peculiar in that it is characterized by strong forward scattering off (unscreened) transverse gauge fluctuations (see scattering of electrons by the transverse electromagnetic field), leading to a divergence in the integrated cross section. This singularity is removed in the so-called transport cross section by an additional factor $(1 - \cos\phi)$, which weighs a scattering process with respect to the corresponding momentum transfer $q = k_F\sqrt{2(1 - \cos\phi)}$ (here, ϕ is the scattering angle). Consequently, the Born approximation result for the conductivity is finite. Depending on the typical magnitude B_0 of the RMF, one may distinguish two transport regimes: (i) diffusive transport if the cyclotron radius $R_0 \gg d$; (ii) percolative transport for $R_0 \ll d$, when most trajectories are localized around hills and valleys of the RMF, and only 'snake states' along the contours of zero field contribute to transport. In finite effective magnetic field, one finds various regimes of magnetotransport characterized by memory effects, classical localization in strong effective magnetic field, and strong damping of quantum oscillations and resonances. The composite fermion picture of the compressible state of quantum Hall systems near half-filling of the lowest Landau level has been extremely successful in accounting for the observed transport properties. The agreement of classical or quasi-classical transport theory with experiment is in many cases quantitative [167, 168].

The successes of the CF theory as represented in the Chern–Simons theory notwithstanding, there are fundamental problems with the CS theory, in that it does not preserve the particle–hole symmetry of the $\nu = 1/2$ QHE problem [169]. A recent proposal of an effective field theory of CFs as charge-neutral massless particle–hole symmetric Dirac fermions carrying a Berry phase of π, appears to solve the problem, while preserving the established good properties of CF theory [170–173].

Even more exotic states have been predicted to exist at fillings such as $\nu = 5/2$. The QP excitations in these Moore–Read states are thought to obey non-Abelian fractional statistics and have been termed 'nonabelians' [174]. Within the CF theory, the Moore–Read state may be obtained as a spinless p-wave pairing state [175]. Open issues concerning the particle–hole symmetry appear to have been resolved [170].

The FQHE states cannot be classified in the framework of the conventional theory of spontaneous symmetry breaking. Different FQHE states all have the same symmetry, and represent new states of matter that contain a completely new kind of order: topological order. The existence of FQH liquids indicates that there is a whole new universe beyond the paradigm of symmetry breaking, waiting to be explored. The new type of orders represented by FQHE states extend our understanding of quantum phases of condensed matter in new directions. The new phenomena of chiral edge states, fractional charge, fractional statistics, and others encountered in the QHE exemplify the powerful concept of emergence in many-body systems.

7. QPs in Dirac materials

A wide class of conducting materials, like graphene [176–178], topological insulators [74, 180], high temperature d-wave superconductors [181], or ^{3}He-A, share a fundamental property: their low-energy fermionic excitations behave as massless Dirac particles rather than non-relativistic fermions. This emergent phenomenon in condensed matter systems defines the unifying framework for a class of materials [179] called 'Dirac materials' [182]. These seemingly diverse materials exhibit universal properties that are a direct consequence of the Dirac spectrum of QPs. For example, the fermionic specific heat is expected to show similar power law temperature dependence in these systems, as a consequence of the presence of the nodes in the excitation spectrum. Other universal features include transport properties like the dc conductivity,

and the optical conductivity [183]. For example, Dirac QPs are capable of tunneling through arbitrarily high potential barriers (Klein tunneling), leading e.g. to a universal finite conductivity of graphene tuned to the Dirac point by varying the gate voltage.

7.1. Graphene

In graphene, a single layer of carbon atoms tightly packed in a honeycomb crystal lattice, the charge carriers behave as massless relativistic particles (Dirac fermions), and move with little scattering under ambient conditions. This is a consequence of its peculiar electronic band structure, featuring two Dirac cones in the Brillouin zone. Each unit cell of the hexagonal Bravais lattice contains two carbon atoms, which give rise to two sublattices A and B. Obviously, atoms in sublattice A are surrounded by three nearest neighbors in sublattice B, and vice versa. In the neighborhood of the Fermi level, the relevant electronic states are the out-of-plane carbon p_z- orbitals. These form π-bonds with neighboring atoms, and the resulting π-bands can be modeled by a nearest-neighbor tight-binding Hamiltonian:

$$H = \sum_{\mathbf{k}} \sum_{\alpha,\beta=1}^{2} c^{\dagger}_{\mathbf{k},\alpha} H_{\alpha,\beta}(\mathbf{k}) c_{\mathbf{k},\beta}, \tag{53}$$

where α, β are pseudospin labels referring to the two sublattices A, B. The Hamiltonian matrix $H_{\alpha,\beta}$ is off-diagonal, as nearest neighbor hopping on the honeycomb lattice necessarily involves a change of sublattice:

$$H_{\alpha\beta}(\mathbf{k}) = \begin{pmatrix} 0 & \epsilon_{\mathbf{k}} \\ \epsilon^{*}_{\mathbf{k}} & 0 \end{pmatrix} \tag{54}$$

with $\epsilon_{\mathbf{k}} = -t(e^{i\boldsymbol{\tau}_1\mathbf{k}} + e^{i\boldsymbol{\tau}_2\mathbf{k}} + e^{i\boldsymbol{\tau}_3\mathbf{k}})$ and $\boldsymbol{\tau}_j$, $j = 1, 2, 3$ are the vectors connecting to the nearest neighbor lattice sites on the honeycomb lattice. The two bands of energy $E_{\mathbf{k}} = \pm|\epsilon_{\mathbf{k}}|$ touch at the corners of the hexagonal Brillouin zone, $E_{\mathbf{K}} = 0$. Of the six corners, two are inequivalent, and may be chosen as $\mathbf{K}$ and $\mathbf{K}' = -\mathbf{K}$. In the neighborhood of $\mathbf{K}, \mathbf{K}'$, the dispersion is linear:

$$H_{\alpha\beta}(-\mathbf{K} + \mathbf{p}) = v_{\mathrm{F}} \boldsymbol{\sigma}_{\alpha\beta} \cdot \mathbf{p} = H^{*}_{\alpha\beta}(\mathbf{K} + \mathbf{p}), \tag{55}$$

where $\boldsymbol{\sigma}_{\alpha\beta}$ is the vector of Pauli matrices σ^x, σ^y in pseudospin space. This is identical in form to the Hamiltonian of massless Dirac fermions in two dimensions, with the speed of light replaced by the Fermi velocity v_{F}. One may show that a mass term $\propto m v_{\mathrm{F}}^2 \sigma^z$ in the above Hamiltonian is absent as long as inversion symmetry (parity) and time reversal symmetry are preserved.

Very generally, any two-dimensional crystal with trigonal symmetry may host Dirac fermion excitations in the corners, K and $K\prime$, of their hexagonal Brillouin zone. This may be used to design artificial Dirac materials. Successful experimental realizations have been reported for lithographically patterned two-dimensional electron gases in semiconductors, for metal surfaces with hexagonal assemblies of CO molecules, and for ultracold atoms trapped in honeycomb optical lattices (for a review see [184]).

The effect of the Coulomb interaction between the Dirac electrons in general may be taken into account within a Fermi liquid framework [185]. A special situation arises when the chemical potential is tuned to the Dirac point, $\mu = 0$. Since the density of states vanishes linearly in that case, $N(E) \propto |E|/v_{\mathrm{F}}^2$, there are no charges available to screen the Coulomb interaction V, meaning that the static interaction is singular in the limit $q \to 0$, $V(q) = e^2/\epsilon_0 q$. Here, ϵ_0 is the background dielectric constant. The polarization bubble also shows a singularity, $\Pi(\mathbf{q}, \omega + \mathrm{i}0) = -q^2/4\sqrt{v_{\mathrm{F}}^2 q^2 - (\omega + \mathrm{i}0)^2}$, implying threshold behavior of the real and imaginary parts at $\omega = \pm v_{\mathrm{F}} q$. Evaluating the self-energy in one-loop approximation using the dynamically screened Coulomb interaction $V_{sc}(\mathbf{q}, \omega) = V(q)/[1 - V(q)\Pi(\mathbf{q}, \omega)]$ (similar to equation (36)), one finds [185]

$$\mathrm{Im}\Sigma_{\mathbf{k}=0}(\omega + \mathrm{i}0) \propto \alpha^2 |\omega| \tag{56}$$

$$Z_{\mathbf{k}=0} \propto 1/[\alpha^2 \ln(\Lambda/|\omega|)] \tag{57}$$

$$\Gamma_{\mathbf{k}=0} \propto |\omega|/\ln(\Lambda/|\omega|), \tag{58}$$

where $\alpha = e^2/\epsilon_0 v_{\mathrm{F}}$ is a dimensionless coupling constant. This reveals that the system looks like a marginal Fermi liquid [45], caused by the singular nature of the Coulomb interaction. In the context of the discussion above of quantum critical behavior in metallic systems, we note that the velocity should get renormalized, too, as $v_{\mathrm{F}} \to v_{\mathrm{F}}^{*} = Z v_{\mathrm{F}}$. Additional renormalization of v_{F} arises through the momentum dependence of the self energy. Since, however, the coupling constant is inversely proportional to the velocity, it will actually grow when $\omega \to 0$. Inserting the behavior $\alpha \propto Z^{-1}$ into equation (57), one finds a self-consistent equation for Z with the solution $Z \propto \ln(\Lambda/|\omega|)$, indicating a growing Z in the limit $\omega \to 0$, and hence a return to Fermi liquid behavior. A careful analysis of the renormalization group flows of Z and v_{F} leads to the same conclusion: that in the very low energy regime the system becomes a Fermi liquid again ($Z > 0$), although the QP relaxation rate keeps varying linearly with ω [185]. In fact, undoped graphene ($\mu = 0$) is at a quantum critical point, separating two Fermi liquid phases of electrons and holes. The quantum critical regime has been dubbed a 'Dirac liquid' [186]. At finite temperature and at low excitation energy $\omega \ll T$ the scattering rate $\tau^{-1} \approx 2\mathrm{Im}\Sigma \propto \alpha^2 T$ is a linear function of temperature. Nonetheless, contrary to what one might expect, the dc conductivity tends to a nearly temperature independent limit. This behavior arises as a result of cancellation of the T-dependence of τ^{-1} and the compressibility $\kappa \propto T/v_{\mathrm{F}}^2$ when the conductivity is expressed as $\sigma = e^2 \kappa D = (c/\alpha^2) G_Q$, where $D = (v_{\mathrm{F}}^2/2)\tau$ is the diffusion coefficient. Note that the factors v_{F}^2 cancel, but the prefactor $1/\alpha^2$ gets renormalized, depending logarithmically on T [187].

7.2. Weyl semimetals

As demonstrated above, Dirac physics emerges at points in $\mathbf{k}$-space defined by the crossing of bands. Band crossings are also known to exist in three dimensions [188]. In the

neighborhood of such crossing points, the bands are necessarily linear in momentum, which defines a three-dimensional Dirac-like nodal point. Provided the chemical potential may be tuned to lie at the crossing point, and if there is no other band nearby, the system is a three-dimensional semimetal with a linear Dirac spectrum, which may be classified as a Dirac material. If there are just two single bands crossing, the situation resembles that of the Weyl equation of particle physics. The Weyl equation is the massless limit of the Dirac equation, in which case the four components of the Dirac spinor separate into two independent two-component solutions. Materials with no degeneracy in the crossing bands, creating a so-called Weyl point, are therefore often referred to as Weyl semimetals [189]. Apart from solid state systems, superfluid ^{3}He-A may also be considered a Weyl system [190].

8. Topological matter

We have already encountered examples of topological matter in the above, in the form of the quantum Hall systems and in the vortex state of superfluids or superconductors. While these examples have been known for a few decades, more new classes of topological materials have been identified recently.

8.1. Topological insulators

One of the first examples of topological insulators (TI) are the quantum spin Hall effect (QSHE) systems described above, strongly spin–orbit coupled time-reversal invariant (TRI) materials featuring inverted band structure in the bulk, leading to conducting states of nontrivial topology on the surface [73]. As is the case of normal insulators, valence and conduction bands of a TI are separated by an energy gap in the entire Brillouin zone. The conductance and valence bands are, however, inverted with respect to their normal ordering as a function of energy. At the surface of a TI, or at any boundary to a normal insulator, one finds massless Dirac surface states of chiral nature, unless time-reversal symmetry is broken. The effect of interactions in causing decay of these excitations is limited by the energy gap of the bulk insulator, requiring thermal activation processes with exponentially suppressed probability.

Two insulators are defined to be topologically equivalent, if their Hamiltonians can be smoothly deformed into each other without closing the band gap. Insulators can be classified by a Z_2 topological invariant $I = 0,1$ [73]. Insulators which are non-equivalent to the vacuum ($I = 1$) are termed 'topological insulators', while all others are called 'normal insulators' ($I = 0$). In two dimensions, the topological character of the Bloch states $u_{n\mathbf{k}}(\mathbf{r})$ for band n and crystal momentum $\mathbf{k}$ of a lattice Hamiltonian is encoded in the Berry connection $\mathbf{A}(\mathbf{k}) = \mathrm{i}\sum_{n=1}^{n_0}\langle u_{n\mathbf{k}}|\nabla_k|u_{n\mathbf{k}}\rangle$ and the Berry curvature $F(\mathbf{k}) = \nabla_k \times \mathbf{A}(\mathbf{k})$ [191, 192], involving a summation over occupied bands. The invariant I is defined by integrals in momentum space over a volume ($\mathrm{d}\tau$) and having surface ($\mathrm{d}\mathbf{l}$) covering half the two-dimensional Brillouin zone (defined so that $\mathbf{k}$ and $-\mathbf{k}$ are never both included) [193]:

$$I = \frac{1}{2\pi}\left[\oint \mathbf{A}(\mathbf{k})\cdot \mathrm{d}\mathbf{l} - \int F(\mathbf{k})\mathrm{d}\tau\right] \mathrm{mod}2. \qquad (59)$$

This definition may be extended to three-dimensional systems by introducing six invariants by specifying six different integration volumes, of which four are independent [194, 195].

The change of the invariant across an interface separating two materials is intimately related to the occurrence of massless surface states closing the energy gap at the interface [74]. The meaning of topologically distinct insulators and emergent gapless edge states can be demonstrated in the model of a two-component (two bands labeled by a pseudospin index) Dirac Hamiltonian in two spatial dimensions of spinless particles with mass m,

$$H(\mathbf{k}) = v_\mathrm{F}\boldsymbol{\sigma}\cdot\mathbf{k} + mv_\mathrm{F}^2\sigma^z, \qquad (60)$$

leading to energy eigenvalues $E_\mathbf{k} = \pm\sqrt{v_\mathrm{F}^2k^2 + (mv_\mathrm{F}^2)^2}$ (here $\mathbf{k} = (k_x, k_y)$ and $\boldsymbol{\sigma}$ is the vector of pseudospin Pauli matrices). The important observation is now that systems with positive or with negative mass m are topologically inequivalent. A smooth deformation of $H(\mathbf{k})$ from the $m > 0$ case into the $m < 0$ case is not possible without closing the gap ($m \to 0$). Changing positive m into negative m is equivalent to inverting the band structure by exchanging valence and conduction bands. Assuming now that the mass $m = m(y)$ varies as a function of position across an interface located at $y = 0$, from positive at $y > 0$ to negative at $y < 0$, one finds that at given energy inside the gap there exists exactly one solution of the Dirac equation extended along x but localized in y, of the form

$$\Psi_{q_x}(x, y) = \mathrm{e}^{\mathrm{i}q_x x}\exp[-\int_0^y \mathrm{d}y' m(y')v_\mathrm{F}]\begin{pmatrix}1\\1\end{pmatrix}, \qquad (61)$$

with energy $E(q_x) = v_\mathrm{F}q_x$. This band of states has a positive group velocity $\mathrm{d}E(q_x)/\mathrm{d}q_x = v_\mathrm{F}$ and represents a right-moving chiral edge mode. Note that the solution, equation (61), of the Dirac equation— equation (60)—in the case of constant mass (a normal insulator) would diverge exponentially for $y \to -\infty$, and is therefore not acceptable. Rather, for a time-reversal invariant normal insulator, there exist pairs of surface states such that at any energy at least two states with group velocities of opposite sign are available.

Generically, a topological insulator may be modeled by two copies of the Hamiltonian (60) (for example one each for spin up and down) describing pairs of counter-propagating time-reversed states:

$$H_\mathrm{TI}(\mathbf{k}) = \begin{pmatrix}H(\mathbf{k}) & 0\\ 0 & H^*(-\mathbf{k})\end{pmatrix}, \qquad (62)$$

After a unitary transformation, $\sigma^y H^*(-\mathbf{k})\sigma^y = v_\mathrm{F}\boldsymbol{\sigma}\cdot k - mv_\mathrm{F}^2\sigma^z$, the lower right block is equal to $H(\mathbf{k})$ with a sign change of the mass term.

While the quantum spin Hall effect is only realized in two dimensions, the concept of TI can be extended to three-dimensional materials. Using first-principles calculations within density functional theory, the feasibility of converting ternary half-Heusler compounds into a new class of three-dimensional topological insulators (3DTI) has been explored [196]. As a

prototype system, the electronic structure of LaPtBi has been demonstrated to exhibit a distinct band-inversion feature. The 3DTI phase may be realized in LaPtBi by applying uniaxial strain along the [0 0 1] direction, which opens a band gap while preserving the inverted band order. A definitive proof of the strained LaPtBi as a 3DTI is provided by directly calculating the topological Z_2 invariants in such systems without inversion symmetry.

There exist, in particular, so-called strong three-dimensional topological insulators, carrying an odd number of two-dimensional chiral Dirac fermions on each surface [193, 195]. The occurrence of surface states in strong three-dimensional TIs can be understood analogously to the two-dimensional case. A complete symmetry classification of topological quantummatter can be found in [197].

8.2. Topological crystalline insulator

As sketched above, for a TI, a strong spin–orbit interaction is necessary to generate an inverted band structure. The role of spin–orbit coupling may be substituted by point-group symmetries of the crystal [198]. Each spin component may then be considered separately, as a spinless fermion model. In the case of tetragonal crystal symmetry the combined symmetry TU, where T denotes time reversal (which in this case of spinless fermions is complex conjugation), and U denotes the rotation by $\pi/2$ about the c-axis, is found to protect topological surface states in this topological crystalline insulator (TCI) residing in the high symmetry surfaces perpendicular to the c-axis. In contrast to the usual TI surface states, these states have quadratic dispersion [198].

A different point group symmetry, mirror symmetry, has been predicted to give rise to a TCI in the narrow-gap semiconductor $Pb_{1-x}Sn_x$(Se,Te) system [199]. One finds metallic surface states with an even number of Dirac cones on high-symmetry crystal surfaces, such as $\{0\ 0\ 1\}$, $\{1\ 1\ 0\}$ and $\{1\ 1\ 1\}$. Experimental studies of $Pb_{1-x}Sn_xSe$ have indeed shown that it is a TCI for $x = 0.23$ [200]. Temperature-dependent angle-resolved photoelectron spectroscopy demonstrates that the material undergoes a temperature-driven topological phase transition from a trivial insulator to a TCI.

8.3. Majorana fermions in topological superconductors

Majorana fermions were introduced in the early days of quantum field theory as constrained solutions of the Dirac equation [201]. These particles are identical to their antiparticles and are therefore electrically neutral. In the context of condensed matter physics, one may define creation operators of Majorana fermion character $\gamma_{1,2}$ out of the usual electron operators $c^\dagger, c$ as

$$\begin{aligned} \gamma_1 &= c^\dagger + c, \\ \gamma_2 &= \mathrm{i}(c^\dagger - c). \end{aligned} \tag{63}$$

It follows that $\gamma_1^\dagger = \gamma_1$, $\gamma_2^\dagger = \gamma_2$. In some sense, $\gamma_{1,2}$ represent the 'real and imaginary parts' of the electron operator, $c = \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2)$. The operators $\gamma_{1,2}$ indeed obey fermionic anticommutation relations, $[\gamma_{\alpha_1}, \gamma_{\alpha_2}] = 2\delta_{\alpha_1,\alpha_2}$. Majorana fermions are linear superpositions of particles and antiparticles, and as such may only appear in systems where particle number is not conserved. One such system is (as described above) that of Bogoliubov QPs in a superconductor. As may be seen from equation (27), BQPs are linear combinations of particles and holes of momentum $\mathbf{k}$ and $-\mathbf{k}$, and are therefore not equal to their antiparticles—in other words, they are not Majorana fermions. One may, however, define four component spinors in particle–hole and spin space out of two-component BQP operators, essentially doubling the degrees of freedom, which then have the Majorana fermion characteristics [202]. To be sure, this does not mean that BQPs are electrically neutral particles.

Particles and antiparticles in the context of condensed matter are QPs of positive or negative excitation energy. The only way to form Majorana type excitations is, therefore, out of zero energy states, located in the middle of the gap. Such modes may be found in the core of vortices, and at domain walls or boundaries. In fact, in an early paper on vortex motion in superfluid ^{3}He, it was found that the most symmetric A-phase vortex has a QP energy spectrum $E_n = -\omega_0 n$, $n = 0, \pm 1, \ldots$ [203], in contrast to the case of s-wave superconductors (and also ^{3}He-B), where $E_n = -\omega_0(n + 1/2)$, $n = 0, \pm 1, \ldots$ [204]. The momentum dependence of the order parameter may be characterized by a topological invariant N, which for odd integer values guarantees the existence of a zero energy mode [205]. Generally speaking, the existence of zero modes requires that the zero point energy $\omega_0/2$ is compensated by an energy shift induced by a Berry phase [206].

Paired fermion states with broken parity and time-reversal symmetry and the possible appearance of Majorana edge states, called Majorana zero modes (MZM), have been proposed by several authors [175, 207–210]. A single unpaired MZM can exist only in an infinite system. In systems of finite size, Majorana modes always appear in pairs, reflecting the fact that such systems always contain an integral number of electrons. An interesting situation arises when the partners of a pair are spatially separated sufficiently far that their wave functions have little overlap. Local probes may then detect a single MZM. In general, localized BQP states with energy $+E$ inside the gap necessarily have a partner at $-E$. Such finite-energy pairs are not topologically protected, because they can simply be pushed out of the energy gap by perturbing fields. However, a single unpaired bound state at $E = 0$ is protected, because it cannot move away from $E = 0$.

A closer look at the exchange statistics of MZMs in two dimensions reveals that they are not simple fermions. Instead, the superconducting state in the background gives rise to additional phase changes. The zero modes are more correctly described as non-Abelian anyons [174, 175], [208–210]. The associated higher degree of quantum entanglement makes MZMs a promising building block of future quantum computers.

The simplest model system that shows unpaired MZMs is the Kitaev chain [207]. It describes a one-dimensional system of spinless fermions, which may be realized in the presence of spin–orbit coupling (SOC) and a Zeeman field. The Kitaev

model, which is simple and exactly soluble, thus provides a paradigm for MZMs in one spatial dimension. In qualitative terms, the ground state of the system is either given by Majorana fermions bound into electrons at each site (the trivial phase) or two nearest-neighbor Majoranas bound into pairs (topological phase). In the latter phase, one unpaired Majorana fermion is left at each end of the chain, forming an MZM. It is as if one electron were fundamentally delocalized, one half sitting at either end of the chain. It is interesting to ask how one may determine if a given system is a topological superconductor. For one-dimensional lattice models with Hamiltonian

$$H = \frac{\mathrm{i}}{4} \sum_{k,\alpha,\beta} B_{\alpha\beta}(k) \gamma^{\dagger}_{k,\alpha} \gamma_{k,\beta} \tag{64}$$

(comprising all superconducting mean field models), where $\gamma_{k,\beta}$ are the lattice Fourier components of the Majorana fermion operators $\gamma_{j,\beta}$, $\beta = 1, 2$ at each lattice site j, and k is crystal momentum ($-\pi \leqslant k \leqslant \pi$), one may define a topological invariant [207]

$$M = \mathrm{sgn}\left\{ Pf[\mathbf{B}(0)] Pf[\mathbf{B}(\pi)] \right\}. \tag{65}$$

Here, $Pf[\mathbf{B}(k)]$ is the Pfaffian of the antisymmetric matrix $B_{\alpha\beta}$. The labels α, β may include additional quantum numbers, such as spin or orbital quantum numbers. In case $M = -1$, and the spectrum is gapped, the system is a topological superconductor. While this condition may be easily analyzed for any concrete system, there is an even simpler definition available for weak superconductors (gap parameter Δ smaller than all relevant energy scales in the problem, e.g. the band width). In that case, the number ν of Fermi points k_{F} in the interval $[0, \pi]$ is found to determine the sign of the invariant, $M = (-1)^{\nu}$.

Various candidate systems of topological superconductors have been proposed theoretically and studied experimentally as reviewed in [202, 206, 211, 212]. A principle obstacle standing in the way of physical realizations of Kitaev's proposal is the electron spin. In usual realizations of a one-dimensional system, electron spin causes all states to be doubly degenerate, thus leading to an even number of Fermi points. Promising candidate systems are semiconductor nanowires, such as InSb, proximity coupled to superconductors. In these semiconductors the spin Kramers degeneracy is lifted by the strong spin–orbit interaction. A magnetic field applied perpendicular to the wire induces a gap at the Γ-point, leaving a single Fermi point, required to get $M = -1$ in the gap region. There are experimental indications that MZMs have been observed in such systems [202, 213]

9. Summary

Since its first appearance in the 1940s, the idea of QPs in quantum matter has found ubiquitous applications. The notion of QPs allows one to apply concepts of quasiclassical physics to quantum systems, in suitably adapted form. In particular, transport properties—or, more generally, dynamical properties—may be expressed in a transparent form within the framework of the kinetic theory of QPs. From the point of view of microscopic theory, the most astounding fact is the existence of well-defined QP excitations even in strongly interacting systems, with properties very different from those of the bare particles. The energy spectrum may be completely changed, as given by an effective mass very different from the bare mass (Landau QPs in heavy fermion systems), a dispersion changed from quadratic to linear (phonons in superfluid Bose systems), or an energy gap (Bogoliubov QPs in superconductors or superfluids, or QPs in metallic antiferromagnets). The QPs are not energy eigenstates, but interact in a way described by a renormalized scattering amplitude. For the QP states to be well-defined, the effect of interaction should be weak in the sense that their decay rate should be less than their excitation energy. This may be ensured by a number of different reasons, such as (i) limited phase space for interaction processes, (ii) low density of excitations, (iii) weak interaction or (iv) topological constraints.

The overriding simplifying feature in weakly excited (but possibly strongly interacting) systems, where QPs are well defined, is the low number of excitations, allowing a controlled expansion. The time- and space-dependent changes of the QP number induced by external fields may then be obtained by solution of a Boltzmann-type kinetic equation amended by mean field terms (the Landau–Boltzmann equation for Fermi liquids or similar equations for phonons and other bosons). The collision term in these equations, governing the transport properties, may then be restricted to two-QP collisions. In the absence of QP conservation, decay and coalesence processes may dominate.

An important question is how physical properties may be calculated in terms of QP properties. In the case of the Fermi liquid, one may show that the response of conserved quantities may be entirely expressed in terms of QP quantities. This is, however, not the rule: already the response of the currents of the conserved quantities, the particle and spin current densities, is completely different from what is obtained by a naive application of the QP picture as embodied in the Pomeranchuk stability analysis. It is then necessary to amend the QP theory with non-QP correction terms. The way out of this problem is to define a momentum-integrated QP distribution function, which may be shown to obey the Landau–Boltzmann equation, including the above-mentioned non-QP correction terms.

Even if the stability of QP excitations in weakly excited systems is guaranteed by the small number of QPs, one may ask if the small factor of density (T/ϵ_{F} in the case of a Fermi liquid) may be offset by a diverging scattering strength as expected near a continuous phase transition where order parameter fluctuations abound. It turns out that in all cases considered (superfluid, magnetic order) the QP relaxation rate Γ stays finite at the transition, and $\Gamma(T_{\mathrm{c}}) < T_{\mathrm{c}}$ in the weak coupling limit (transition temperature much less than microscopic energy scales such as the Fermi energy). In that case, QPs are still well-defined. One may expect that in strong coupling situations, QPs lose their meaning, although reliable results are necessarily difficult to obtain here.

A noteable exception is a certain strong coupling regime at quantum critical points of metals. In such systems, it is frequently found that not only do the (bosonic) order parameter fluctuations show critical behavior, but so do the fermionic QPs. The latter are then characterized by a diverging effective

mass. Provided that the interaction of the bosonic fluctuations is still sufficiently weak and the effective mass is enhanced by additional quantum fluctuations (e.g. ferromagnetic fluctuations on top of the antiferromagnetic fluctuations of the quantum critical point), the system may enter a strong coupling regime for which the critical properties may be calculated in a controlled way. Fermionic QPs are still well defined in this regime, even though the effective mass becomes a scale-dependent quantity.

A drastic change of the character of fermionic excitations in conducting systems may occur as a consequence of reduced dimensions. For example, in one-dimensional metals, even the weakest interaction leads to a break-up of Landau QPs into fractional excitations carrying charge or spin only (sometimes called chargons and spinons). These excitations show up as two singularities in the single particle spectral function. An observable consequence of this behavior is the emergence of scaling of the conductance of a quantum wire as a function of temperature, bias voltage or the wire length. The corresponding power laws are found to depend on the interaction strength. A further example of the emergence of fractional QPs is a two-dimensional electron system in strong magnetic field showing the quantum Hall effect. In the limit of very strong magnetic field, such that only the lowest Landau level is occupied, one finds Hall plateaus at certain fractional filling factors. As shown by Laughlin, these special many-body states have QP excitations carrying fractional charge, and should obey Abelian fractional exchange statistics.

Yet another type of QPs is present in the so-called Dirac materials, where QPs are found as solutions of the massless Dirac equation. A prominent example is the two-dimensional system graphene. Tuned to the neutrality point (chemical potential $\mu = 0$), graphene is at a quantum critical point, with unusual QP and electrical screening properties. The QPs are found to be well defined in spite of the singular Coulomb interaction, but the QP relaxation rate is significantly enhanced, varying as in the case of a marginal Fermi liquid.

The currently most exciting new development is the discovery of topological materials featuring topologically protected QPs. They may be found at the surface of unusual crystalline solids, which are characterized by a band structure with inverted bands, meaning that the valence band is on top of the conduction band, and there is a full gap; hence, the name 'topological insulator'. The band inversion may be achieved by a strong spin–orbit coupling combined with suitable hopping amplitudes and lattice structure. The role of the spin–orbit potential may be substituted by a special lattice symmetry. As a result, one finds chiral QPs—i.e. QPs moving in one directional sense only, like along the boundary of a two-dimensional system. These QPs may decay only by action of a time reversal symmetry breaking perturbation, such as scattering by magnetic impurities.

A very different type of topological QP may appear as a Majorana zero Mode (MZM) in topological superconductors. This is a Bogoliubov QP (BQP) state in a vortex core residing in a two-dimensional superconductor or at the ends of a superconducting wire, with the unusual property of sitting in the gap center, at zero energy. The lowest BQP state in a vortex core of a usual superconductor is bound to have finite energy forced by zero point fluctuations. In a topological superconductor, the zero point fluctuation effect may be compensated by an unconventional Berry phase contribution. A BQP residing in a state of zero energy is equal to its antiparticle, and hence has the characteristics of a Majorana fermion. There are two different MZMs per electron, which are, loosely speaking, the real and imaginary parts of the electron operator. This indicates that Majoranas are bound to exist only in pairs. The two MZMs making up an electron may be spatially separated, e.g. at the ends of a topological superconductor wire, forming a strongly protected, highly entangled quantum state. Such systems are considered to be promising building blocks of future quantum processing devices.

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ORCID iDs

Peter Wölfle https://orcid.org/0000-0002-4131-1257

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Peter Wölfle is a theoretical physicist at Karlsruhe Institute of Technology, Germany working in the field of electronic correlations in metals, disordered systems and nanostructured materials. He studied at the Technical University of Munich, Germany, where he also received his doctorate in 1969. Subsequently, he worked as a postdoctoral researcher at the Max-Planck-Institute of Physics in Munich and at Cornell University, Ithaca, USA. In 1975 he became a professor of physics at TU Munich. From 1986 to 1989 he was a professor at the University of Florida in Gainesville, USA. In 1989 he returned to Germany as a professor at the University of Karlsruhe, now Karlsruhe Institute of Technology, where he is an Emeritus.