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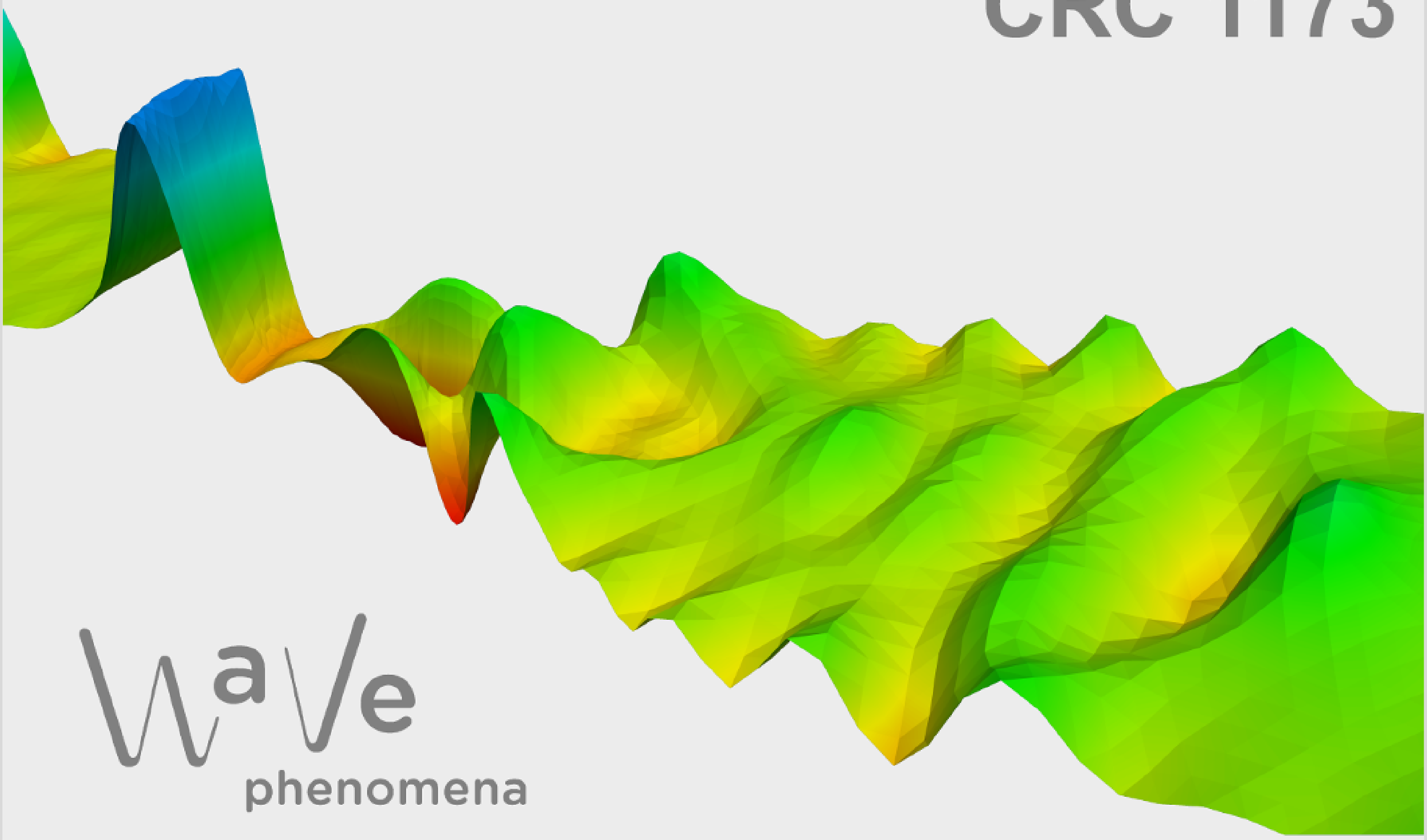

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MODULATION TYPE SPACES FOR GENERATORS OF POLYNOMIALLY BOUNDED GROUPS AND SCHRÖDINGER EQUATIONS

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Abstract. We introduce modulation type spaces associated with the generators of polynomially bounded groups. Besides strongly continuous groups we study in detail the case of bi-continuous groups, e.g. weak*-continuous groups in dual spaces. It turns out that this gives new insight in situations where generators are not densely defined. Classical modulation spaces are covered as a special case but the abstract formulation gives more flexibility. We illustrate this with an application to a nonlinear Schrödinger equation.

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1. Introduction and main results

Classical modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ (or more general on locally compact abelian groups) have been introduced by Feichtinger in the 80s (we refer to[4] and the literatue cited there) by means of the short time Fourier transform (see (19) below for the definition of the latter). There is an equivalent description of these spaces by so-called "uniform" decompositions of the Fourier transforms of their elements ([4]), and we shall rely on that one here. Modulation spaces have been succesfully used in time-frequency analysis and related fields (see, e.g., the survey [5]). In recent years, modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ have been used in the theory of dispersive partial differential equations such as the (nonlinear) Schrödinger equation (see, e.g., [15, 2]). One of the appealing features of modulation spaces in this context is that operators $e^{it\Delta}$, $t \in \mathbb{R}$, are bounded (polynomially in $|t|$) on $M^s_{p,q}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, $1 \le p, q \le \infty$ (see, e.g., [1]), which is in sharp contrast to Lebesgue spaces $L^q(\mathbb{R}^d)$, where this only holds for $q = 2$.

In this paper we propose a different and in a sense more abstract point of view on this phenomenon, within the realm of semigroup theory and with a variant of the Phillips functional calculus as the main technical tool. We start with a general polynomially bounded group $(T(t)) = (e^{-itB})$ in a Banach space X and construct modulation type spaces $M^s_{X,q}(B)$ associated with the "generator" B. Then we study the groups (e^{-itB}) and (e^{-itB^2}) in $M^s_{X,q}(B)$ and apply the results to Schrödinger type equations

$$\begin{cases} i\dot{u}(t) &= B^2 u(t), \quad t \in \mathbb{R}, \\ u(0) &= x. \end{cases}$$

Note that we do not restrict to bounded groups. Moreover, we are particularly interested in the case where B is not densely defined in X. It turns out that the theory of bi-continuous (semi-)groups (see [9]) is well suited to the study of this case, and that this also gives new insight with respect to continuity properties in the case $q = \infty$. In the abstract situation one cannot, in general, resort to some kind of weak*-continuity, and it turns out that, even when this is possible, one might have other possibilities to choose a suitable topology.

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To be more precise, let $T(\cdot) = (T(t))_{t\in\mathbb{R}}$ be a C_0-group in a complex Banach space X. We assume that $(T(t))_{t\in\mathbb{R}}$ is polynomially bounded, i.e. there exist $M \geq 1$ and $\alpha \geq 0$ such that

$$\|T(t)x\|_X \leq M\langle t\rangle^\alpha \|x\|_X, \quad t \in \mathbb{R}, x \in X, \tag{1}$$

where $\langle t\rangle = (1 + |t|^2)^{1/2}$. We denote the generator of $T(\cdot)$ by $-iB$ and write $T(t) = e^{-itB}$, $t \in \mathbb{R}$. We shall use the *Phillips functional calculus* for the operator B (see, e.g. [7, Sect. 3.3]) and denote the space of complex Borel measures μ satisfying $\int_{\mathbb{R}} \langle t\rangle^\alpha \, d|\mu|(t) < \infty$ by $\mathscr{M}_\alpha(\mathbb{R})$. Observe that $\mathscr{M}_\alpha(\mathbb{R})$ is a Banach algebra for convolution (letting $\|\mu\|_{\mathscr{M}_\alpha} := \int_{\mathbb{R}} (1 + |t|)^\alpha \, d|\mu|(t)$ we have $\|\mu * \nu\|_{\mathscr{M}_\alpha} \leq \|\mu\|_{\mathscr{M}_\alpha} \|\nu\|_{\mathscr{M}_\alpha}$). If $\mu \in \mathscr{M}_\alpha(\mathbb{R})$ and

$$\phi(\xi) = \widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} \, d\mu(\tau), \quad \xi \in \mathbb{R},$$

is its Fourier transform then

$$\phi(B)x := \int_{\mathbb{R}} e^{-i\tau B} x \, d\mu(\tau) = \int_{\mathbb{R}} T(\tau)x \, d\mu(\tau), \quad x \in X, \tag{2}$$

defines a bounded operator in X. Moreover, $\phi \mapsto \phi(B)$ is an algebra homomorphism from the space $\mathscr{F}\mathscr{M}_\alpha(\mathbb{R}) := \{\widehat{\mu} : \mu \in \mathscr{M}_\alpha(\mathbb{R})\}$ into $L(X)$, the space of bounded linear operators on X. We denote by $\mathscr{S}(\mathbb{R})$ the Schwartz space, by

$$\widehat{\phi}(\tau) := (\mathscr{F}\phi)(\tau) := \int_{\mathbb{R}} e^{-i\tau r} \phi(r) \, dr, \quad \tau \in \mathbb{R},$$

the Fourier transform of $\phi \in \mathscr{S}(\mathbb{R})$, and by $\mathscr{F}^{-1}\phi$ the inverse Fourier transform of ϕ, given by

$$\mathscr{F}^{-1}\phi(r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau r} \phi(\tau) \, d\tau, \quad r \in \mathbb{R}.$$

We clearly have $\mathscr{S}(\mathbb{R}) \subseteq \mathscr{F}\mathscr{M}_\alpha(\mathbb{R})$ for any $\alpha \geq 0$, and for $\phi \in \mathscr{S}(\mathbb{R})$ the operator $\phi(B)$ is given by

$$\phi(B)x = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(\tau) e^{i\tau B} x \, d\tau = \int_{\mathbb{R}} (\mathscr{F}^{-1}\phi)(\tau) e^{-i\tau B} x \, d\tau, \quad \phi \in \mathscr{S}(\mathbb{R}). \tag{3}$$

The map $\mathscr{S}(\mathbb{R}) \to L(X)$, $\phi \mapsto \phi(B)$, is an algebra homomorphism and we have the estimate

$$\|\phi(B)\|_{L(X)} \leq \frac{M}{2\pi} \|\langle\cdot\rangle^\alpha \widehat{\phi}\|_{L^1}, \quad \phi \in \mathscr{S}(\mathbb{R}), \tag{4}$$

in particular, $\phi \mapsto \phi(B)$ is continuous for the usual topology on $\mathscr{S}(\mathbb{R})$.

Remark 1.1. Letting

$$\mathcal{L}^1_\alpha(\mathbb{R}) := \{f : \langle\cdot\rangle^\alpha \widehat{f} \in L^1(\mathbb{R})\}$$

equipped with the norm $\|f\|_{\mathcal{L}^1_\alpha} := \|(1 + |\cdot|)^\alpha \widehat{f}\|_{L^1}$, we have that $\mathcal{L}^1_\alpha(\mathbb{R})$ is a Banach subalgebra of $\mathscr{F}\mathscr{M}_\alpha(\mathbb{R})$. By (4) we obtain a bounded functional calculus for B on $\mathcal{L}^1_\alpha(\mathbb{R})$. Observe that, by the Riemann-Lebesgue lemma, $\mathcal{L}^1_\alpha(\mathbb{R}) \subseteq C^k_0(\mathbb{R})$ for any $k \in \mathbb{N}_0$ with $k \leq \alpha$.

Nevertheless, we shall mostly work with the functional calculus for Schwartz functions and use the formulae in (3). The following example has to kept in mind throughout the paper and shall give the relation to classical modulation spaces.

Example 1.2. Let $X = L^p(\mathbb{R})$ with $p \in [1, \infty)$ and let $T(t)f := f(\cdot - t)$, $t \in \mathbb{R}$ be the right translation group with generator $-\frac{d}{dx}$. The group is bounded, even isometric, and $M = 1$, $\alpha = 0$ in (1). We have $B = -i\frac{d}{dx}$, $e^{-i\tau B}f = T(\tau)f = f(\cdot - \tau)$ and we see from the second formula in (3) that

$$\phi(B)f = \phi(-i\frac{d}{dx})f = (\mathscr{F}^{-1}\phi) * f = \mathscr{F}^{-1}\big(\xi \mapsto \phi(\xi)\widehat{f}(\xi)\big), \quad \phi \in \mathscr{S}(\mathbb{R}), f \in L^p(\mathbb{R}).$$

Boundedness of the Phillips functional calculus here just means that, for any complex Borel measure μ on $\mathbb{R}$, its Fourier transform $\widehat{\mu}$ is an L^p-Fourier multiplier and that $\widehat{\mu}(B)f = \mu * f$.

We shall associate with B a scale of "modulation type" spaces. The basic idea is to take $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ and put, for $s \in \mathbb{R}$ and $q \in [1, \infty]$,

$$M^s_{X,q,\phi} := \{x : \|x\|_{M^s_{X,q,\phi}} := \Big(\int_{\mathbb{R}} \big(\langle t\rangle^s \|\phi(B - t)x\|_X\big)^q \, dt\Big)^{1/q} < \infty\} \tag{5}$$

with obvious modification in case $q = \infty$ (observe that it makes no difference whether we plug B into $\phi(\cdot - t)$ or $B - t$ into ϕ to obtain $\phi(B - t)$). We shall show below that different ϕ give rise to equivalent norms, so that we can denote this space by $M^s_{X,q}$ (or $M^s_{X,q}(B)$ if we want to specify the operator B in notation). In the context of Example 1.2 this will yield the classical modulation spaces $M^s_{p,q}(\mathbb{R})$. However, we have to be a bit careful about the x we admit in (5). In the classical case we have the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions as an ambient space. For sufficiently large s we can take $x \in X$, but for s small (depending on q) or for negative s we have to resort to a scale X_{-k}, $k \in \mathbb{N}$, of extrapolation spaces, see Section 2.

Just as there is an analogy of classical modulation spaces with classical Besov spaces, our construction can be viewed as a counterpart to the construction of abstract Besov type spaces for sectorial operators (see, e.g., the "intermediate spaces" in [7, Sect. 6.4.1]).

The results in this paper can clearly be extended to finitely many commuting groups, and one can thus recover classical modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ on $\mathbb{R}^d$ for $d > 1$. Moreover, iterating the procedure presented here one can define anisotropic versions of these spaces with different integrability exponents in different directions. We shall not go into further details here.

The paper is organized as follows: In Section 2 we show basic peoperties of the functional calculus for Schwartz functions and introduce the extrapolation scale we shall use. In Section 3 we define the spaces $M^s_{X,q}(B)$ in the case of a strongly continuous group and establish their basic properties. In Section 4 consider the situation of a bi-continuous group with not necessarily densely defined B. Even if the original group in X is strongly continuous, the induced group in $M^s_{X,\infty}(B)$ is not strongly continuous, in general. Resorting to the notion bi-continuous groups allows to study this case in a satisfying setting. Section 5 is devoted to the study of (e^{-itB^2}) in $M^s_{X,q}(B)$, and this relies very much on (extensions of) the Phillips calculus. We give an application of our result to a nonlinear Schrödinger equation in Section 6. In the Appendix we have gathered two auxiliary results and give their proofs for convenience.

2. Basic definitions and properties

We start with fundamental properties of the functional calculus for Schwartz functions.

Lemma 2.1. *Let $\phi \in \mathscr{S}(\mathbb{R})$.*

(1) *For $x \in X$ we have $\phi(B)x \in D(B)$ and $B\phi(B)x = \psi(B)x$ where $\psi(r) := r\phi(r)$.*
(2) *For any $k \in \mathbb{N}$ the operator $\phi(B)$ maps X into $D(B^k)$.*
(3) *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have $R(\lambda, B)\phi(B)x = \psi(B)x$ where $\psi(r) := \phi(r)/(\lambda - r)$.*

(4) *The set* $\bigcup_{\rho\in\mathscr{S}}\rho(B)(X)$ *is dense in* X *and* $\rho(B)x=0$ *for all* $\rho\in\mathscr{S}(\mathbb{R})$ *implies* $x=0$.

Proof. For the proof of (1) let $t>0$. Then

$$T(t)\phi(B)x=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\phi}(\tau)e^{i(\tau-t)B}x\,d\tau=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\phi}(\tau+t)e^{i\tau B}x\,d\tau$$

hence $T(t)\phi(B)x=\big(e^{-it(\cdot)}\phi\big)(B)x$ (in fact, this also follows form the Phillips calculus). We multiply with e^{-t}, integrate $\int_0^\infty\ldots\,dt$ and use Fubini to get

$$R(1,-iB)\phi(B)x=\widetilde{\phi}(B)x,$$

where $\widetilde{\phi}(r)=(1+ir)^{-1}\phi(r)$. This implies (1).
(2) follows from (1) by induction, and (3) also follows from (1).
For $k\in\mathbb{N}$ let $g_k(t):=e^{-t^2/(2k)}$. Then $g_k\in\mathscr{S}(\mathbb{R})$ and $\widehat{g_k}(\tau)=\sqrt{2\pi k}e^{-k\tau^2/2}$. For $x\in X$ and $k\to\infty$ we have

$$g_k(B)x=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g_k}(\tau)e^{i\tau B}x\,d\tau=\int_{\mathbb{R}}\sqrt{\frac{k}{2\pi}}e^{-k\tau^2/2}e^{i\tau B}x\,d\tau\to e^{i\tau B}x\big|_{\tau=0}=x$$

as $k\to\infty$. This proves (4). □

Lemma 2.2. *Let* $\phi\in\mathscr{S}(\mathbb{R})$.

(1) *The map* $\mathbb{R}\to L(X)$, $t\mapsto\phi(B-t)$ *is uniformly continuous in operator norm.*
(2) *If* $x\in X$ *is such that* $\int_{\mathbb{R}}\|\phi(B-t)x\|_X\,dt<\infty$ *then* $\int_{\mathbb{R}}\phi(B-t)x\,dt=\widehat{\phi}(0)x$.

Proof. The estimate (4) implies

$$\begin{aligned}\|\phi(B-t)-\phi(B-s)\|_{L(X)}&\le\frac{M}{2\pi}\int_{\mathbb{R}}\langle\xi\rangle^\alpha\big|e^{-it\xi}-e^{-is\xi}\big||\hat{\phi}(\xi)|\,d\xi\\&=\frac{M}{2\pi}\int_{\mathbb{R}}\langle\xi\rangle^\alpha\big|e^{-i(t-s)\xi}-1\big||\hat{\phi}(\xi)|\,d\xi,\end{aligned}$$

and we can use dominated convergence to prove (1).
To see (2), let again $g_k(t)=e^{-t^2/(2k)}$, $k\in\mathbb{N}$. Then we have

$$\int_{\mathbb{R}}\phi(B-t)x\,dt=\lim_{k\to\infty}\int_{\mathbb{R}}g_k(t)\phi(B-t)x\,dt$$

and

$$\begin{aligned}\int_{\mathbb{R}}g_k(t)\phi(B-t)x\,dt&=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}g_k(t)e^{-i\tau t}\widehat{\phi}(\tau)e^{i\tau B}x\,d\tau\,dt\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g_k}(\tau)\widehat{\phi}(\tau)e^{i\tau B}x\,d\tau.\\&=\int_{\mathbb{R}}\sqrt{\frac{k}{2\pi}}e^{-k\tau^2/2}\widehat{\phi}(\tau)e^{i\tau B}x\,d\tau.\end{aligned}$$

As $k\to\infty$ this tends to $\widehat{\phi}(\tau)e^{i\tau B}x|_{\tau=0}=\widehat{\phi}(0)x$ in $\|\cdot\|_X$. □

The following construction of *extrapolation spaces* is well established (see, e.g., [7, Sect. 6.3]). As $-iB$ generates a C_0-group satisfying (1), B is densely defined and $\pm i$ belong to the resolvent set of B. For $k\in\mathbb{N}$, we denote by X_k the Banach space $D(B^k)$ equipped with the norm $\|x\|_{X_k}:=$

$\|(i+B)^k x\|_X$. We let $X_0 := X$, and for $k \in \mathbb{N}$ we denote by X_{-k} the completion of X for the norm $\|x\|_{X_{-k}} := \|(i+B)^{-k}x\|$. Then

$$\ldots \subseteq X_{k+1} \subseteq X_k \subseteq \ldots \subseteq X_1 \subseteq X_0 = X \subseteq X_{-1} \subseteq \ldots \subseteq X_{-k} \subseteq X_{-(k+1)} \subseteq \ldots$$

with continuous and dense embeddings. For each $k \in \mathbb{Z}$ there is an operator B_k in X_k with $D(B_k) = X_{k+1}$ such that $i + B_k : X_{k+1} \to X_k$ is an isometry, B_k is an extension of B_{k+1} and B_{k+1} is the part of B_k in X_{k+1} (we recall that, for an operator A in a Banach space X and another Banach space $Y \subseteq X$, the *part* A_Y *of* A *in* Y is the operator given by the restriction of A to $D(A_Y) = \{y \in Y \cap D(A) : Ay \in Y\}$).

Moreover, we set $X_\infty := \bigcap_{k\in\mathbb{N}} X_k$ and $X_{-\infty} := \bigcup_{k\in\mathbb{N}} X_{-k}$ (as we do not need topologies on these spaces, we simply consider them as sets). By similarity, each operator $-iB_k$ generates a C_0-group $(e^{-itB_k})_{t\in\mathbb{R}}$ in X_k, and this group satisfies again (4) in X_k. So everything we have done so far can be carried over from X to any of the spaces X_k, $k \in \mathbb{Z}$. Moreover, we have consistency of spaces and operators in the sense of $(X_k)_l = X_{k+l}$ and $(B_k)_l = B_{k+l}$ for all $k, l \in \mathbb{Z}$. For simplicity, we omit in notation the subscript for the different versions of B.

Remark 2.3. *(a) By Lemma 2.1 we have $\phi(B) : X_{-\infty} \to X_\infty$ for any $\phi \in \mathscr{S}(\mathbb{R})$.*
(b) In the case of the translation group on $X = L^p(\mathbb{R})$, $1 < p < \infty$, we obtain the scale of Sobolev spaces $X_k = W^{k,p}(\mathbb{R})$, $k \in \mathbb{Z}$, and $X_\infty = \bigcap_{k\geq 0} W^{k,p}(\mathbb{R})$, which is sometimes denoted by $\mathscr{D}_{L^p}$ and can be equipped with the natural Fréchet space topology. In this notation $X_{-\infty} = (\mathscr{D}_{L^{p'}})'$ would be the dual space of $\mathscr{D}_{L^{p'}}$.

Lemma 2.4. *Let $x \in X_{-\infty}$ and $\phi, \psi \in \mathscr{S}(\mathbb{R})$. Then $\int_{\mathbb{R}} \|\phi(B-t)\psi(B)x\|_X \, dt < \infty$ and $\int_{\mathbb{R}} \phi(B-t)\psi(B)x \, dt = \widehat{\phi}(0)\psi(B)x$.*

Proof. By Lemma 2.2 the integrand is continuous in $\|\cdot\|_X$. For any $k \in \mathbb{N}$ we have

$$\|\phi(B-t)\psi(B)\|_{L(X_{-k},X)} = \|\phi(B-t)(i+B)^k\psi(B)\|_{L(X)}.$$

Thus it suffices to estimate $\|\phi(B-t)\widetilde{\psi}(B)\|_{L(X)}$ where $\widetilde{\psi} := (i+(\cdot))^k\psi \in \mathscr{S}(\mathbb{R})$. We simply use the estimate (4) (for the definition of $V_g f$ we refer to the appendix):

$$\begin{aligned} \|\phi(B-t)\widetilde{\psi}(B)\|_{L(X)} &\leq \frac{M}{2\pi}\int_{\mathbb{R}} \langle\xi\rangle^\alpha |\mathscr{F}\{\phi(\cdot - t)\widetilde{\psi}\}(\xi)| \, d\xi \\ &= \frac{M}{2\pi}\int_{\mathbb{R}} \langle\xi\rangle^\alpha |(V_{\overline{\phi}}\widetilde{\psi})(t,\xi)| \, d\xi. \end{aligned}$$

By Lemma A.1, the integral is finite for each $t \in \mathbb{R}$ and can be integrated with respect to t. $\square$

3. Modulation type spaces associated with B

With the scale of extrapolation spaces from the previous section at hand we can now give the precise definition of the modulation type spaces associated with B.

Definition 3.1. For $s \in \mathbb{R}$, $q \in [1, \infty)$ and $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ we define

$$M^s_{X,q,\phi} := \{x \in X_{-\infty} : \|x\|_{M^s_{X,q,\phi}} := \Big(\int_{\mathbb{R}} \big(\langle t\rangle^s \|\phi(B-t)x\|_X\big)^q \, dt\Big)^{1/q} < \infty\} \tag{6}$$

and

$$M^s_{X,\infty,\phi} := \{x \in X_{-\infty} : \|x\|_{M^s_{X,\infty,\phi}} := \sup_{t\in\mathbb{R}} \langle t\rangle^s \|\phi(B-t)x\|_X < \infty\}. \tag{7}$$

We start with a lemma on simple inclusions.

Lemma 3.2. *We have $X \hookrightarrow M^s_{X,q,\phi}$ for $s < -\frac{1}{q}$ (and for $s \le 0$ if $q = \infty$). If $\widehat{\rho}(0) = 1$ then $M^s_{X,q,\rho} \hookrightarrow X$ for $s > \frac{1}{q'}$ (and $s \ge 0$ if $q = 1$).*

Proof. By (4) we have for $x \in X$ the estimate

$$\|x\|_{M^s_{X,q,\phi}} \le \frac{M}{2\pi} \|\langle\cdot\rangle^\alpha \widehat{\phi}\|_{L^1} \|\langle\cdot\rangle^s\|_{L^q} \|x\|_X \tag{8}$$

where $\|\langle\cdot\rangle^s\|_{L^q} < \infty$ for $s < -\frac{1}{q}$ (and for $s \le 0$ if $q = \infty$). This proves the first assertion.
Take now $\rho \in \mathscr{S}(\mathbb{R})$ with $\widehat{\rho}(0) = 1$ and $s > \frac{1}{q'}$ ($s \ge 0$ if $q = 1$). For $x \in M^s_{X,q,\rho}$ we then have by Hölder

$$\int_{\mathbb{R}} \|\rho(B-t)x\|_X \, dt \le \|\langle\cdot\rangle^{-s}\|_{L^{q'}} \|\langle\cdot\rangle^s \rho(B-\cdot)x\|_{L^q(X)} = \|\langle\cdot\rangle^{-s}\|_{L^{q'}} \|x\|_{M^s_{X,q,\rho}}.$$

Now we use Lemma 2.2 to obtain $\int \rho(B-t)x \, dt = x$ and consequently $\|x\|_X \lesssim \|x\|_{M^s_{X,q,\rho}}$. □

We now prove that different $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ induce equivalent norms. We also show that the continuous type norms involving integrals can be replaced by semi-discrete norms involving sums. We shall study fully discrete norms below.

Theorem 3.3. *Let $s \in \mathbb{R}$ and $q \in [1,\infty]$. Then all norms $\|\cdot\|_{M^s_{X,q,\phi}}$, $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$, are equivalent. Moreover, they are equivalent to the norms given for $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ by*

$$\|x\|_{\widetilde{M}^s_{X,q,\phi}} := \sup_{t\in[0,1]} \Big(\sum_{k\in\mathbb{Z}} \big(\langle k\rangle^s \|\phi(B-k+t)x\|_X \big)^q \Big)^{1/q},$$

again with obvious modification for $q = \infty$.

Proof. It is clear that $\|x\|_{M^s_{X,q,\phi}} \lesssim \|x\|_{\widetilde{M}^s_{X,q,\phi}}$ for all $q \in [1,\infty]$ and that the reverse inequality holds for $q = \infty$. So it rests to show $\|x\|_{\widetilde{M}^s_{X,q,\psi}} \lesssim \|x\|_{M^s_{X,q,\phi}}$ for fixed $\phi, \psi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ and $q \in [1,\infty]$. We give the proof for $q \in [1,\infty)$, the modifications for $q = \infty$ being obvious.
Let $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$, $c := \|\phi\|^2_{L^2}$, and set $\widetilde{\phi} := c^{-1}\overline{\phi}$. Then $\int_{\mathbb{R}} \phi(r)\widetilde{\phi}(r)\, dr = 1$. We let

$$\rho(t) := \int_0^1 \phi(r+t)\widetilde{\phi}(r+t)\, dr, \quad t \in \mathbb{R}.$$

Then $\rho \in \mathscr{S}(\mathbb{R})$, $0 \le \rho \le 1$, and $\int_{\mathbb{R}} \rho(r-t)\, dt = 1$ for all $r \in \mathbb{R}$. For $\psi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ and $t \in [0,1]$ we have, using Lemma 2.4,

$$\begin{aligned}
&\Big(\sum_k \langle k\rangle^{sq} \|\psi(B-k+t)x\|^q_X \Big)^{1/q} \\
= \;& \Big(\sum_k \langle k\rangle^{sq} \|\psi(B-k+t) \int_{\mathbb{R}} \rho(B-k+\sigma)x \, d\sigma\|^q_X \Big)^{1/q} \\
\le \;& \sum_j \Big(\sum_k \langle k\rangle^{sq} \|\psi(B-k+t) \int_0^1 \int_0^1 (\widetilde{\phi}\phi)(B-k+j+\sigma+r)x \, dr\, d\sigma\|^q_X \Big)^{1/q}
\end{aligned}$$

We write $k+j$ for j:

$$= \sum_j \Big(\sum_k \langle k+j\rangle^{sq} \| \int_0^1 \int_0^1 \psi(B-(k+j)+t)(\widetilde{\phi}\phi)(B-k+\sigma+r)x \, dr\, d\sigma\|^q_X \Big)^{1/q}$$

Now we use $\langle k+j\rangle^{sq} \lesssim \langle k\rangle^{sq}\langle j\rangle^{|s|q}$ and Jensen:

$$\begin{aligned}
&\lesssim \sum_j \langle j\rangle^{|s|}\Big(\sum_k \langle k\rangle^{sq}\int_0^1\int_0^1 \|\psi(B-(k+j)+t)(\widetilde{\phi}\phi)(B-k+\sigma+r)x\,dr\,d\sigma\|_X^q\Big)^{1/q}\\
&\le \Big(\sum_j \langle j\rangle^{|s|} \sup_{\sigma,r\in[0,1],k\in\mathbb{Z}} \|\psi(B-(k+j)+t)\widetilde{\phi}(B-k+\sigma+r)\|_X\Big)\\
&\quad \times \sup_{r\in[0,1]}\Big(\sum_k \langle k\rangle^{sq}\int_0^1 \|\phi(B-k+\sigma+r)x\|_X^q\,d\sigma\Big)^{1/q}\\
&= I_1\times I_2.
\end{aligned}$$

For I_1 we use (4) and let $s := \sigma + r \in [0,2]$. As in the proof of Lemma 2.4 we then obtain

$$\begin{aligned}
&\|\psi(B-(k+j)+t)\widetilde{\phi}(B-k+\sigma+r)\|_X\\
\lesssim\ &\|\langle\cdot\rangle^\alpha \mathscr{F}\{\psi(\cdot-j+t)\widetilde{\phi}(\cdot+s)\}\|_{L^1} = \|\langle\cdot\rangle^\alpha\mathscr{F}\{\psi(\cdot-j+t-s)\widetilde{\phi}\}\|_{L^1}\\
\le\ &\int_{\mathbb{R}}\langle\xi\rangle^\alpha|(V_{\overline{\psi}}\widetilde{\phi})(j+s-t,\xi)|\,d\xi,
\end{aligned}$$

where the integrand is $\lesssim \langle\xi\rangle^{-\beta}\langle j\rangle^{-\gamma}$ for all $\beta,\gamma>0$ by Lemma A.1. Hence I_1 is finite. For the estimate of I_2 we fix $r\in[0,1]$ and estimate the term in brackets by

$$\lesssim \Big(\int_{\mathbb{R}}\langle\sigma\rangle^{sq}\|\phi(B-\sigma+r)x\|_X^q\,d\sigma\Big)^{1/q},$$

which is independent of r and equals $\|x\|_{M^s_{X,q,\phi}}$. ☐

Definition 3.4. According to Theorem 3.3 we can define $M^s_{X,q} := M^s_{X,q,\phi}$ for $s\in\mathbb{R}$, $q\in[1,\infty]$ where $\phi\in\mathscr{S}(\mathbb{R})\setminus\{0\}$ is arbitrary.

As announced above we now give fully discrete norms for special $\rho\in\mathscr{S}(\mathbb{R})\setminus\{0\}$.

Proposition 3.5. *Let $\rho\in\mathscr{S}(\mathbb{R})\setminus\{0\}$ satisfy* $\operatorname{supp}\rho\subseteq[-1,1]$ *and* $\sum_{k\in\mathbb{Z}}\rho(\cdot-k)=1$ *on* $\mathbb{R}$. *Then*

$$\|x\|^{\mathrm{d}}_{M^s_{X,q,\rho}} := \Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{sq}\|\rho(B-k)x\|_X^q\Big)^{1/q}$$

(obvious modification in case $q=\infty$) defines an equivalent norm on $M^s_{X,q}$ for any $s\in\mathbb{R}$ and $q\in[1,\infty]$.

Proof. It is clear that $\|x\|^{\mathrm{d}}_{M^s_{X,q,\rho}} \le \|x\|^{\sim}_{M^s_{X,q,\rho}}$. To see the reverse estimate we observe that, for $t\in[0,1]$, $\rho(\cdot+t)=\sum_{j=-1}^2\rho(\cdot+t)\rho(\cdot+j)$. Hence we have, for $k\in\mathbb{Z}$ and $t\in[0,1]$,

$$\begin{aligned}
\|\rho(B-k+t)x\|_X &= \Big\|\sum_{j=-1}^2(\rho(B-k+t)\rho(B-k+j))x\Big\|_X\\
&\le \sum_{j=-1}^2\|\rho(B-k+t)\rho(B-k+j)x\|_X\\
&\lesssim \sum_{j=-1}^2\|\rho(B-k+j)x\|_X,
\end{aligned}$$

since $\sup_{k\in\mathbb{Z},t\in[0,1]}\|\rho(B-k+t)\|_{L(X)}<\infty$ by (4). Now $\|x\|^{\sim}_{M^s_{X,q,\rho}} \lesssim \|x\|^{\mathrm{d}}_{M^s_{X,q,\rho}}$ follows. ☐

Proposition 3.6 (Lifting property)**.** *The operator $i+B$ is an isomorphism $M^{s+1}_{X,q} \to M^s_{X,q}$ for any $s \in \mathbb{R}$, $q \in [1,\infty]$. Moreover, $M^s_{X,q}(B) = M^{s+1}_{X_{-1},q}(B_{-1})$ with equivalent norms.*

Proof. For $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$, $\phi_1(r) := r\phi(r)$, and x with $t \mapsto \phi_1(B-t)x \in L^q_s$, the basic observation is that $t \mapsto B\phi(B-t)x \in L^q_s$ is equivalent to $t \mapsto t\phi(B-t)x \in L^q_s$. Hence we have $x \in M^{s+1}_{X,q}$ if and only if $x \in M^s_{X,q}(\mathbb{R})$ and $Bx \in M^s_{X,q}(\mathbb{R})$. The assertions now follow easily. □

As a corollary, we can replace $X_{-\infty}$ in Definition 3.1 by any X_{-k}, $k > \frac{1}{q'} - s$ if $q \in (1,\infty]$ and $k \geq -s$ if $q = 1$.

Proposition 3.7 (Completeness)**.** *For $s \in \mathbb{R}$ and $q \in [1,\infty]$ the space $M^s_{X,q,\rho}$ is a Banach space.*

Proof. We give the proof here for $s > \frac{1}{q'}$ (and $s \geq 0$ if $q = 1$) and can exploit $M^s_{X,q,\rho} \hookrightarrow X$. For the general case we then use the lifting property.
Let (x_n) be Cauchy for $\|\cdot\|_{M^s_{X,q,\rho}}$. Then $(\langle\cdot\rangle^s \rho(B-\cdot)x_n)$ is Cauchy in $L^q(\mathbb{R},X)$ and there exists $\langle\cdot\rangle^s f \in L^q(\mathbb{R},X)$ such that $\langle\cdot\rangle^s \rho(B-\cdot)x_n \to \langle\cdot\rangle^s f$ in the norm of $L^q(\mathbb{R},X)$. The assumption on s implies that (x_n) is Cauchy in $\|\cdot\|$ so that $x_n \to x \in X$ in $\|\cdot\|_X$. But then $\rho(B-t)x_n \to \rho(B-t)x$ in $\|\cdot\|_X$ for any $t \in \mathbb{R}$. Thus $f(t) = \rho(B-t)x$ for a.e. $t \in \mathbb{R}$, and dominated convergence yields $x \in M^s_{X,q,\rho}$ and $x_n \to x$ in $\|\cdot\|_{M^s_{X,q,\rho}}$. □

We close this section by the following result on denseness and the group induced by (e^{-itB}) in modulation type spaces associated with B.

Proposition 3.8. *For $q \in [1,\infty)$ and any $s \in \mathbb{R}$, X_∞ is dense in $M^s_{X,q}$. The part of $-iB$ in $M^s_{X,q}$ which has domain $M^{s+1}_{X,q}$ generates a C_0-group in $M^s_{X,q}$ that is consistent with the group $(e^{-itB})_{t\in\mathbb{R}}$ we started with and satisfies again* (1).

Proof. By lifting we resort to $s > 1/q'$ ($s \geq 0$ if $q = 1$). Choose $\rho \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ as in Proposition 3.5. Take $x \in M^s_{X,q}$ and, for $n \in \mathbb{N}$, set $x_n := \sum_{|l|\leq n} \rho(B-l)x$. Then

$$\rho(B-k)x_n = \sum_{|l|\leq n} \rho(B-l)\rho(B-k)x = \begin{cases} \rho(B-k)x & , \ |k| \leq n-1 \\ \\ 0 & , \ |k| \geq n+2 \end{cases},$$

so clearly $x_n \in X_\infty$ and $x_n \in \bigcap_{s\in\mathbb{R}} M^s_{X,q}$. Moreover

$$\|x - x_n\|^q_{M^s_{x,q}} \leq C \sum_{|k|\geq n} \langle k\rangle^{sq} \|\rho(B-k)x\|^q_X \to 0 \qquad (n \to \infty).$$

Clearly, the spaces $M^s_{X,q}$ are invariant under the group operators e^{-itB} and the induced group still satisfies (1). Similarly, $M^s_{X,q}$ is invariant under resolvents of B and norm estimates in X carry over to $M^s_{X,q}$. The resolvent operators induce an operator B in $M^s_{X,q}$ with domain $M^{s+1}_{X,q}$.
By $X_\infty \subseteq M^{s+1}_{X,q}$, the domains $M^{s+1}_{X,q}$ of B is dense in $M^s_{X,q}$. Now Hille-Yosida implies that the induced group (e^{-itB}) is strongly continuous in $M^s_{X,q}$. □

We shall study the case $q = \infty$ in the next section.

4. The case of bi-continuous groups

In our considerations, we also want to include the translation group in spaces such as L^∞ where it is not strongly continuous and the supposed "generator" is not densely defined. However, the translation semigroup is weak*-continuous on L^∞. Also, in our spaces $M^s_{X,\infty}$ the induced group is not densely defined, but it is strongly continuous for a weaker norm (namely in X_{-k} if k is sufficiently large). So we resort to the theory of bi-continuous (semi-)groups and recall basic definitions and properties (cf. [9]).

Assumption 4.1. *Let X be Banach space with norm $\|\cdot\|_X$ and let τ_X be a locally convex topology on X such that*

(i) *Any normbounded τ_X-Cauchy sequence converges in (X, τ_X).*
(ii) *The embedding $(X, \|\cdot\|_X) \hookrightarrow (X, \tau_X)$ is continuous.*
(iii) *The space $(X, \tau_X)'$ is* norming *for X, i.e. for any $x \in X$ we have*

$$\|x\|_X = \sup\{|\langle x, \psi\rangle| : \psi \in (X, \tau_X)', \|\phi\|_{X'} \leq 1\},$$

where $\|\psi\|_{X'} := \sup_{\|z\|_X \leq 1} |\langle z, \psi\rangle|$ is the usual norm on the dual space $(X, \|\cdot\|_X)'$ of $(X, \|\cdot\|_X)$.

By (iii), τ_X is necessarily Hausdorff. In the following we use the notation

$$\Phi_X(\tau_X) := \Phi(\tau_X) := \{\psi \in (X, \tau_X)' : \|\psi\|_{X'} \leq 1\},$$

then we can rewrite (iii) as $\|x\|_X = \sup_{\psi \in \Phi(\tau_X)} |\langle x, \psi\rangle|$ for any $x \in X$. Observe that (iii) is invariant if we multiply $\|\cdot\|_X$ by a positive constant.

We rephrase [9, Def. 3] for our purposes. Observe that here we are only interested in the polynomially bounded case.

Definition 4.2. Suppose that τ_X is as above and that $(T(t))_{t\in\mathbb{R}}$ is a group satisfying (1). Then $(T(t))$ is called *bi-continuous* (with respect to τ_X) if

(i) $\mathbb{R} \to X$, $t \mapsto T(t)x$ is τ_X-continuous for any $x \in X$,
(ii) for any $a > 0$ the set $\{T(t) : |t| \leq a\}$ is *bi-equicontinuous*, i.e. for every $\|\cdot\|$-bounded sequence (x_n) in X that is τ_X-convergent to 0 we have

$$\tau_X - \lim_{n\to\infty} T(t)(x_n) = 0$$

uniformly in $|t| \leq a$.

Following [9, Def. 9], the *generator of* $(T(t))$ is the unique operator A on X such that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \quad x \in X, \operatorname{Re}\lambda > 0. \tag{9}$$

The integral here is the limit $\lim_{a\to\infty}$ in operator norm of the integrals $\int_0^a e^{-\lambda t} T(t)x\, dt$ which in turn have to be understood as τ_X-Riemann integrals (we refer also to Proposition A.2 in the appendix and to the arguments in the proof of Proposition 4.4). Then (see [9, Cor. 13]) A is *bi-closed*, i.e. if (x_n) is a norm-bounded sequence, τ_X-convergent to x and such that (Ax_n) is norm-bounded and τ_X-convergent to y, then $x \in D(A)$ and $Ax = y$. Moreover, the domain $D(A)$ of A is *bi-dense* in X, i.e. for any $x \in X$ there is a norm-bounded sequence (x_n) in $D(A)$ that is τ_X-convergent to x (see [9, Cor. 13]).

The following are our main motivations to include the present section.

Examples: 1. The translation group given by $T(t)f = f(\cdot - t)$ is isometric in $X = L^\infty(\mathbb{R}) = \big(L^1(\mathbb{R})\big)'$. It is not strongly continuous, but it is bi-continuous for the weak* topology.
2. More general, if X is not reflexive and $(T(t)) = (e^{-itB})$ is a C_0-group in X satisfying (1) then the dual group $(T(t)')$ in X' satisfies (1) and is bi-continuous for the weak* topology on X'. Its generator is the dual operator $(-iB)'$ of $-iB$ whose domain is bi-dense in X'.
3. The translation group is also not strongly continuous on the space $C_b(\mathbb{R})$ of bounded and continuous functions equipped with the sup-norm. But it is bi-continuous for the topology τ_c of uniform convergence on compact subsets of $\mathbb{R}$ (see [9]).
4. In the situation of Section 3, the induced group (e^{-itB}) is not strongly continuous in $M^s_{X,\infty}$ (unless B is bounded). We shall see below that it is bi-continuous for the norm of X_k where $k < s-1$ and its generator is the operator $-iB$ with domain $M^{s+1}_{X,\infty}$.

In the following we assume

Assumption 4.3. *Let $(X, \|\cdot\|_X)$ be a Banach space and τ_X a topology on X satisfying Assumption 4.1. We assume that $(T(t))_{t\in\mathbb{R}}$ is a group in X that satisfies* (1) *and is bi-continuous with respect to the topology τ_X. As before we denote the generator of $(T(t))$ by $-iB$ and write $T(t) = e^{-itB}$.*

Before we continue we establish the Phillips calculus for the group $(T(t))$ in X. We shall use that, for a τ_X-continuous and norm-bounded function $f : [a,b] \to X$ and a complex Borel measure $\mu \in \mathscr{M}_\alpha(\mathbb{R})$, the integral $\int_{[a,b]} f(t)\, d\mu(t)$ exists in a Riemann sense. For convenience, we prove this in the appendix (cf. Proposition A.2).

Proposition 4.4 (Phillips calculus)**.** *For $\mu \in \mathscr{M}_\alpha(\mathbb{R})$ and $x \in X$ the limit*

$$\widehat{\mu}(B)x := \lim_{a\to-\infty, b\to\infty} \int_{[a,b]} T(t)x\, d\mu(t)$$

exists in operator norm and $x \mapsto \widehat{\mu}(B)$ defines a linear operator $\widehat{\mu}(B) \in L(X)$ satisfying the estimate

$$\|\widehat{\mu}(B)\|_{L(X)} \le M \int_{\mathbb{R}} \langle t\rangle^\alpha\, d|\mu|(t). \tag{10}$$

The map $\mathscr{F}\mathscr{M}_\alpha(\mathbb{R}) \to L(X)$, $\widehat{\mu} \mapsto \widehat{\mu}(B)$ is an algebra homomorphism. Moreover, if $\mathcal{F} \subseteq \mathscr{M}_\alpha(\mathbb{R})$ is a $\|\cdot\|_{\mathscr{M}_\alpha}$-bounded subset satisfying

$$\lim_{a\to\infty} \sup_{\mu\in\mathcal{F}} \int_{|t|>a} \langle t\rangle^\alpha\, d|\mu|(t) = 0 \tag{11}$$

then $\{\widehat{\mu}(B) : \mu \in \mathcal{F}\}$ is bi-equicontinuous.

Proof. Let $\mu \in \mathscr{M}_\alpha(\mathbb{R})$ and $x \in X$. By Proposition A.2 the integral $\int_{[a,b]} T(t)x\, d\mu(t)$ exists in a τ_X-Riemann sense. By Assumption 4.1 (iii) we have

$$\left\| \int_{[a,b]} T(t)x\, d\mu(t) \right\|_X = \sup_{\psi\in\Phi(\tau_X)} \left| \int_{[a,b]} \langle T(t)x, \psi\rangle\, d\mu(t) \right| \le M \int_{[a,b]} \langle t\rangle^\alpha\, d|\mu|(t)\, \|x\|_X.$$

Denoting the operator $x \mapsto \int_{[a,b]} T(t)x\, d\mu(t)$ by $\int_{[a,b]} T(t)\, d\mu(t)$ this estimate implies $\int_{[a,b]} T(t)\, d\mu(t) \in L(X)$. Moreover, it implies that

$$\lim_{a\to-\infty, b\to\infty} \int_{[a,b]} T(t)\, d\mu(t)$$

exists in operator norm. Denoting this limit by $\widehat{\mu}(B)$ as in the assertion, the estimate finally shows (10). For the following we observe that, for $\psi \in \Phi(\tau_X)$, we have

$$\langle \widehat{\mu}(B)x, \psi\rangle = \lim_{\substack{a\to-\infty\\ b\to\infty}} \langle \int_{[a,b]} T(t)x\, d\mu(t), \psi\rangle = \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_{[a,b]} \langle T(t)x, \psi\rangle\, d\mu(t) = \int_{\mathbb{R}} \langle T(t)x, \psi\rangle\, d\mu(t),$$

where the last integral is a Lebesgue integral.

For the algebra property we only have to show multiplicativity, i.e. for $\mu, \nu \in \mathscr{M}_\alpha(\mathbb{R})$ we have to show $\widehat{\mu}(B)\widehat{\nu}(B) = \widehat{\mu * \nu}(B)$. We do this by applying functionals $\psi \in \Phi(\tau_X)$:

$$\begin{aligned} \langle \widehat{\mu * \nu}(B)x, \psi\rangle &= \int_{\mathbb{R}} \langle T(t)x, \psi\rangle\, d(\mu * \nu)(t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \langle T(t+s)x, \psi\rangle\, d\mu(t)\, d\nu(s) \\ &= \int_{\mathbb{R}} \langle T(s) \int_{\mathbb{R}} T(t)x\, d\mu(t), \psi\rangle\, d\nu(s) \\ &= \langle \widehat{\nu}(B)\widehat{\mu}(B)x, \psi\rangle. \end{aligned}$$

Now suppose that $\mathcal{F} \subseteq \mathscr{M}_\alpha(\mathbb{R})$ is $\|\cdot\|_{\mathscr{M}_\alpha}$-bounded and satisfies (11). Let (x_n) be a norm-bounded sequence that is τ_X-convergent to 0. Let p be continuous seminorm and $\varepsilon > 0$. We may assume $p \le \|\cdot\|_X$. By assumption we find $a > 0$ such that

$$\sup_{\mu\in\mathcal{F}} \int_{|t|>a} \langle t\rangle^\alpha\, d|\mu|(t) \le \varepsilon.$$

For $\mu \in \mathcal{F}$ we write

$$p(\widehat{\mu}(B)x_n) \le p\left(\int_{[-a,a]} T(t)x_n\, d\mu(t)\right) + \left\|\int_{|t|>a} T(t)x_n\, d\mu(t)\right\|_X.$$

Using τ_X-continuity of p and (10) we obtain

$$\begin{aligned} p(\widehat{\mu}(B)x_n) &\le \int_{[-a,a]} p(T(t)x_n)\, d|\mu|(t) + M \int_{|t|>a} \langle t\rangle^\alpha\, d|\mu|(t) \\ &\le \sup_{|t|\le a} p(T(t)x_n) \sup_{\mu\in\mathcal{F}} \|\mu\|_{\mathscr{M}_\alpha} + M\varepsilon. \end{aligned}$$

By Assumption 4.3 we have $\sup_{|t|\le a} p(T(t)x_n) \to 0$ as $n \to \infty$, and this implies the assertion. □

The result on bi-equicontinuity can used to show that $\{\langle t\rangle^{-\widetilde{\alpha}} T(t) : t \in \mathbb{R}\}$ is bi-equicontinuous for any $\widetilde{\alpha} > \alpha$. It can also be used to reprove bi-equicontinuity assertions on (powers of) resolvent operators in [9].

Remark 4.5 (Extrapolation scale)**.** Also under Assumption 4.3, it is possible to construct the scale of Banach spaces $(X_k, \|\cdot\|_{X_k})_{k\in\mathbb{Z}}$ such that $i + B : X_k \to X_{k-1}$ is an isometry for every $k \in \mathbb{Z}$, since the construction essentially only uses that $\pm i \in \rho(B)$ (we refer to, e.g., to [7, Sect. 6.3]). Similarly, we can use the operator $i + B$ and its powers to transfer the topology τ_X to the spaces X_k, $k \in \mathbb{Z}$, and we shall denote the induced topologies by τ_{X_k}. In this way $i + B : (X_k, \tau_{X_k}) \to (X_{k-1}, \tau_{X_{k-1}})$ is an isomorphism for each $k \in \mathbb{Z}$. Moreover, in each space X_k we have an induced group (e^{-itB}) which satisfies (1) and is bi-continuous with respect to τ_{X_k}. Its generator is the operator $-iB$ with domain X_{k+1}.

We now comment on the construction of the spaces $M^s_{X,q}$ under the given assumptions. on the other hand it shows that we still have the estimate (4). In particular, the argument which shows continuity of $t \mapsto \phi(B-t)$ in operator norm of X is still valid. We also note that, if we apply $\psi \in \Phi(\tau_X)$, we have

$$\langle \phi(B)x, \psi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\tau)\langle e^{i\tau B}x, \psi\rangle \, d\tau \tag{12}$$

where the integral is a Lebesgue integral. Using this observation we can apply functionals $\psi \in \Phi(\tau_X)$ in the argument of the proof of Lemma 2.1 (1). Thus we obtain that the assertions of Lemma 2.1 (1)-(3) still hold. Concerning the assertion of Lemma 2.1 (4) we take a continuous seminorm p for τ_X and have (with notations as in the proof of Lemma 2.1)

$$p(g_k(B)x - x) \le \int_{\mathbb{R}} \sqrt{\frac{k}{2\pi}}\, e^{-k\tau^2/2} p(e^{i\tau B}x - x)\, d\tau \to 0 \qquad (k \to \infty).$$

Hence the set $\bigcup_{\rho\in\mathscr{S}} \rho(B)(X)$ is bi-dense in X and $\rho(B)x = 0$ for all $\rho \in \mathscr{S}(\mathbb{R})$ still implies $x = 0$. Now it is obvious that the spaces $M^s_{X,q}(B)$ can be defined as before. We summarize all this in

Proposition 4.6. *Under Assumption 4.3 the assertions of Lemma 2.1* (1)-(3) *still hold. In the assertion of Lemma 2.1* (4) *"dense" has to be replaced by "bi-dense". The assertions of Lemma 2.2 and Lemma 2.4 still hold. We can define the spaces $M^s_{X,q,\phi}(B)$ as in Definition 3.1. The assertions of Lemma 3.2, Theorem 3.3, and of the Propositions 3.5, 3.6, and 3.7 are still valid.*

We now study the induced group (e^{-itB}) and its properties in the spaces $M^s_{X,q}$ and give the analog of Proposition 3.8.

Proposition 4.7. *For $q \in [1,\infty)$, the part of $-iB$ in $M^s_{X,q}$ which has domain $M^{s+1}_{X,q}$ generates a C_0-group in $M^s_{X,q}$ that is consistent with the group $(e^{-itB})_{t\in\mathbb{R}}$ we started with and satisfies again* (1)*. For $q = \infty$, the part of $-iB$ in $M^s_{X,\infty}$ which has domain $M^{s+1}_{X,\infty}$ generates a bi-continuous group in $M^s_{X,q}$ w.r.t. the topology τ_{X_k} restricted to $M^s_{X,\infty}$ where $k < s-1$. the group is consistent with the group $(e^{-itB})_{t\in\mathbb{R}}$ we started with and satisfies again* (1)*.*

Observe that the range of k in the assertion is such that $M^s_{X,\infty} \hookrightarrow X_k$.

Proof. As before, the spaces $M^s_{X,q}$ are invariant under the group operators e^{-itB} and the induced group still satisfies (1). Similarly, $M^s_{X,q}$ is invariant under resolvents of B and norm estimates in X carry over to $M^s_{X,q}$. The resolvent operators induce an operator B in $M^s_{X,q}$ with domain $M^{s+1}_{X,q}$. For $q \in [1,\infty)$ the space $M^{s+1}_{X,q}$ is dense in $M^s_{X,q}$ (by the same arguments as in the proof of Proposition 3.8). Again, Hille-Yosida implies that the induced group (e^{-itB}) is strongly continuous in $M^s_{X,q}$ if $q \in [1,\infty)$.

We turn to the case $q = \infty$. For $s \in \mathbb{R}$, the induced group (e^{-itB}) in the space $M^s_{X,\infty}$ is bi-continuous with respect to the topology $\tau_{X_k}|_{M^s_{X,\infty}}$ whenever $k \in \mathbb{Z}$ satisfies $s > k+1$. By the lifting property it suffices to prove the case $k = 0$, $s > 1$. We denote by τ_M the restriction of τ_X to $M^s_{X,\infty}$. First we show that τ_M satisfies Assumption 4.1 in $M^s_{X,\infty}$ in place of X. The inclusion $M^s_{X,\infty} \hookrightarrow (X, \|\cdot\|_X)$ implies (ii). For the proof of (i) let (x_n) be $\|\cdot\|_{M^s_{X,\infty}}$-bounded in $M^s_{X,\infty}$ and τ_M-Cauchy. Then (x_n) is also $\|\cdot\|_X$-bounded in X and τ_X-Cauchy and thus converges in (X,τ_X) by (i) for τ_X in X. We denote the limit by x and have to show $x \in M^s_{X,\infty}$. Let $\rho \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$.

Taking $\psi \in \Phi_X(\tau_X)$ we have, for any $t \in \mathbb{R}$,

$$|\langle \rho(B-t)x, \psi\rangle| = \lim_n |\langle \rho(B-t)x_n, \psi\rangle| \leq \sup_n \|\rho(B-t)x_n\|_X.$$

Hence using (iii) for τ_X in X we get $\|\rho(B-t)x\|_X \leq \sup_n \|\rho(B-t)x_n\|_X$ for every $t \in \mathbb{R}$ which implies in turn

$$\|x\|_{M^s_{X,\infty,\rho}} = \sup_t \langle t\rangle^s \|\rho(B-t)x\|_X \leq \sup_{t,n} \langle t\rangle^s \|\rho(B-t)x_n\|_X = \sup_n \|x_n\|_{M^s_{X,\infty,\rho}} < \infty.$$

So $x \in M^s_{X,\infty,\rho}$, and (i) is proved.

For the proof of (iii) we take $x \in M^s_{X,\infty,\rho}$. For $t \in \mathbb{R}$ we then have, by (iii) for τ_X in X,

$$\|\rho(B-t)x\|_X = \sup_{\psi \in \Phi_X(\tau_X)} |\langle \rho(B-t)x, \psi\rangle|,$$

and for every $\psi \in \Phi_X(\tau_X)$ we have

$$\langle t\rangle^s \langle \rho(B-t)x, \psi\rangle = \lim_{\delta \to 0+} \frac{1}{2\delta} \int_{\mathbb{R}} \langle \rho(B-r)x, \langle r\rangle^s 1_{[t-\delta,t+\delta]}(r)\psi\rangle \, dr.$$

Defining $\psi_\delta : \mathbb{R} \to X'$ by $\psi_\delta(r) = \frac{1}{2\delta}\langle r\rangle^s 1_{[t-\delta,t+\delta]}(r)\psi$ we have $\|\psi_\delta\|_{L^1(X')} \leq 1$. Then we see easily that the functional

$$\Psi_\delta : x \mapsto \int_{\mathbb{R}} \langle \rho(B-r)x, \psi_\delta(r)\rangle \, dr = \langle \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \langle r\rangle^s \rho(B-r)x \, dr, \psi\rangle$$

is τ_M-continuous on $M^s_{X,\infty,\rho}$ and

$$|\langle x, \Psi_\delta\rangle| \leq \|x\|_{M^s_{X,\infty,\rho}} \|\langle\cdot\rangle^{-s}\psi_\delta\|_{L^1(X')} \leq \|x\|_{M^s_{X,\infty,\rho}}.$$

So we have $\Psi_\delta \in \Phi_{M^s_{X,\infty,\rho}}(\tau_M)$. Using the above we thus have

$$\langle t\rangle^s \|\rho(B-t)x\|_X \leq \sup_\delta |\langle x, \Psi_\delta\rangle| \leq \sup\{|\langle x, \Psi\rangle| : \Psi \in \Phi_{M^s_{X,\infty,\rho}}(\tau_M)\}.$$

Taking the sup over $t \in \mathbb{R}$ we obtain (iii) for τ_M in $M^s_{X,\infty,\rho}$. The same arguments can be used for the norms $\|\cdot\|^{\mathrm{d}}_{M^s_{X,\infty,\rho}}$ where ρ satisfies the additional assumptions that are needed.

Now we show that the induced group (e^{-itB}) in $M^s_{X,\infty}$ is bi-continuous with respect to τ_M by checking the conditions in Definition 4.2. Property (i) is obvious from (i) for τ_X in X. Property (ii) also follows from (ii) for τ_X in X and Assumption 4.1 (ii). □

Still under the Assumptions 4.1 and 4.3 we comment on another choice of a topology that makes the induced group on $M^s_{X,\infty}(B)$ bi-continuous.

Remark 4.8. We denote by $X^\flat$ the $\|\cdot\|_X$-closure of $D(B)$ in X and by $\|\cdot\|_{X^\flat}$ the restriction of $\|\cdot\|_X$ to $X^\flat$. Then the restriction $(T(t)^\flat)$ of $(T(t))$ to $X^\flat$ is strongly continuous, and $T(t)^\flat = e^{-itB^\flat}$ where $B^\flat$ is the part of B in $X^\flat$. Denoting by $(X^\flat_k)_{k\in\mathbb{Z}}$ the extrapolation scale w.r.t. $X^\flat$ and $B^\flat$ we clearly have $X_{k+1} \subseteq X^\flat_k \subseteq X_k$ for any $k \in \mathbb{Z}$ which implies $X^\flat_{-\infty} = X_{-\infty}$ and $X^\flat_\infty = X_\infty$. By the results of Section 2 we conclude

$$M^s_{X,\infty}(B) = M^s_{X^\flat,\infty}(B^\flat), \qquad s > 1.$$

Now, Proposition 4.7 tells us that, for $s > 1$, the group (e^{-itB^2}) in $M^s_{X,\infty}(B)$ is bi-continuous w.r.t. to the topology induced by $\|\cdot\|_X$, restricted to $M^s_{X,\infty}(B)$.

We also want to remark that a similar argument applies to, e.g., $M_{X,1}(B)$ in place of $X^\flat$: we have $X_2 \hookrightarrow M_{X,1} \hookrightarrow X$, so $M_{M_{X,1},\infty}(B) \hookrightarrow M_{X,\infty}(B)$. On the other hand, we have, for a suitable ρ and any $k \in \mathbb{Z}$, by an argument as in the proof of Proposition 3.8

$$\langle k\rangle^s \|\rho(B-k)x\|_{M_{X,1}} = \langle k\rangle^s \sum_l \|\rho(B-l)\rho(B-k)x\|_X \leq C\langle k\rangle^s \sum_{l=-1}^{1} \|\rho(B-l)x\|,$$

so that $\|x\|_{M^s_{M_{X,1},\infty}} \lesssim \|x\|_{M^s_{X,\infty}}$. We conclude that

$$M^s_{X,\infty}(B) = M^s_{M_{X,1},\infty}(B), \qquad s > 1,$$

as Banach spaces. Again, Proposition 4.7 tells us that, for $s > 1$, the group (e^{-itB^2}) in $M^s_{X,\infty}(B)$ is bi-continuous w.r.t. to the topology induced by $\|\cdot\|_{M_{X,1}}$, restricted to $M^s_{X,\infty}(B)$. We remark that, in the same way, it can be shown that we have

$$M^s_{M^\sigma_{X,r}(B),q}(B) = M^{s+\sigma}_{X,q}(B)$$

for all $s, \sigma \in \mathbb{R}$ and $r, q \in [1,\infty]$ which yields more choices for an appropriate topology on $M^s_{X,\infty}(B)$.

5. The group e^{-itB^2} in the spaces $M^s_{X,q}(B)$

In this section we briefly consider a functional calculus of unbounded operators for B in X before we concentrate on the operators e^{-itB^2}, $t \in \mathbb{R}$, and their properties in the modulation type spaces $M^s_{X,q}(\mathbb{R})$. If $F : \mathbb{R} \to \mathbb{C}$ is C^∞, then $F\phi \in C^\infty_c(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$ for any $\phi \in C^\infty_c(\mathbb{R})$ and we define the operator $F(B)$ in X by

$$x \in D(F(B)) \text{ and } F(B)x = y :\Longleftrightarrow \forall \phi \in C^\infty_c(\mathbb{R}) : (F\phi)(B)x = \phi(B)y.$$

An application of Lemma 2.2 (2) yields that $F(B)$ is a well-defined single-valued linear operator. Moreover, $F(B)$ is a closed operator in X and the map $F \mapsto F(B)$ has the properties of an unbounded functional calculus: $C^\infty(\mathbb{R})$ is an algebra and for $F, G \in C^\infty(\mathbb{R})$ it is not hard to see that we have

$$F(B) + G(B) \subseteq (F+G)(B) \qquad \text{and} \qquad F(B)G(B) \subseteq (FG)(B) \tag{13}$$

where $D(F(B)+G(B)) := D(F(B)) \cap D(G(B))$ and $D(F(B)G(B)) := \{x \in D(G(B)) : G(B)x \in D(F(B))\}$ equals $D((FG)(B)) \cap D(G(B))$. Examples of such F are:

(a) If $F(r) = (\lambda - r)^{-1}$ where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $F(B) = R(\lambda, B)$.
(b) If $F(r) = e^{-itr}$ where $t \in \mathbb{R}$, then $F(B) = T(t) = e^{-itB}$.
(c) If $F(r) = r^k$, then $F(B) = B^k$.
(d) We are especially interested in the case $F(r) = e^{-itr^2}$ for $t \in \mathbb{R}$.

In case (a) and (b), $F(B)$ is already in the Phillips calculus: in (a) by the usual representation of the resolvent operators as Laplace transform of the semigroup, and in (b) we have $\mu = \delta_t$ in (2). We shall study (d) in the following.

Proposition 5.1. *Suppose that Assumptions 4.1 and 4.3 hold. There is a constant C_α only depending on α such that, for all $s \in \mathbb{R}$ and all $q \in [1,\infty]$, we have*

$$\|e^{-itB^2}x\|_{M^s_{X,q}} \leq C_\alpha \langle t\rangle^{\alpha+1/2} \|x\|_{M^{s+\alpha}_{X,q}}, \qquad t \in \mathbb{R}, x \in M^{s+\alpha}_{X,q}. \tag{14}$$

In case $\alpha = 0$, (e^{-itB^2}) is a group of bounded operators in $M^s_{X,q}(B)$ for each $s \in \mathbb{R}$. If $q \in [1, \infty)$ then this group is strongly continuous, for $q = \infty$ this group is bi-continuous w.r.t. the restriction of τ_{X_k} to $M^s_{X,\infty}(B)$ where $k < s - 1$.

Proof. We first prove the estimate (14). We take $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ and show

$$\|e^{-itB^2}\phi(B-k)\|_{L(X)} \le C_\alpha \langle t\rangle^{\alpha+1/2}\langle k\rangle^\alpha, \qquad k, t \in \mathbb{R}. \tag{15}$$

This will prove (14) since we then have

$$\|e^{-itB^2}\phi^2(B-k)x\|_X \le C_\alpha \langle t\rangle^{\alpha+1/2}\langle k\rangle^\alpha \|\phi(B-k)x\|_X,$$

which in turn implies

$$\|e^{-itB^2}x\|_{M^s_{X,q,\phi^2}} \le C_\alpha \langle t\rangle^{\alpha+1/2}\|x\|_{M^{s+\alpha}_{X,q,\phi}}.$$

For the proof of (15) we write

$$e^{-itB^2}\phi(B-k) = e^{itk^2}e^{-2iktB}e^{-it(B-k)^2}\phi(B-k)$$

and use (1) and (4) (the latter for $B-k$ in place of B) to obtain

$$\|e^{-itB^2}\phi(B-k)\|_{L(X)} \le \frac{M^2}{2\pi}\langle 2kt\rangle^\alpha \|\langle\cdot\rangle^\alpha \widehat{\psi}_t\|_{L^1},$$

where $\psi_t \in \mathscr{S}(\mathbb{R})$ is given by $\psi_t(r) = e^{-itr^2}\phi(r)$. Then we use

$$\langle 2kt\rangle^\alpha \le c\langle t\rangle^\alpha\langle k\rangle^\alpha.$$

In order to get the right dependence on t and α we calculate $\|\langle\cdot\rangle^\alpha\widehat{\psi}_t\|_{L^1}$ for the special case $\phi(r) = e^{-r^2}$. Then

$$\psi_t(r) = e^{-itr^2}e^{-r^2} = e^{-(1+it)r^2}, \qquad \widehat{\psi}_t(\xi) = \sqrt{\frac{\pi}{1+it}}\, e^{-\xi^2/(4(1+it))}$$

as can be seen by analytic continuation. Hence

$$\|\langle\cdot\rangle^\alpha\widehat{\psi}_t\|_{L^1} = \sqrt{\pi}\,\langle t\rangle^{-1/2}\int_{\mathbb{R}}\langle\xi\rangle^\alpha e^{-\xi^2/(4\langle t\rangle^2)}\,d\xi.$$

For $\alpha = 0$ we substitute $\xi = \langle t\rangle\eta$ and obtain

$$\|\widehat{\psi}_t\|_{L^1} = \sqrt{\pi}\,\langle t\rangle^{1/2}\int_{\mathbb{R}} e^{-\eta^2/4}\,d\eta.$$

For $\alpha > 0$ we use $\langle\xi\rangle^\alpha \le 2^{\alpha/2}(1 + |\xi|^\alpha)$ and get with the same substitution as before

$$\langle t\rangle^{-1/2}\int_{\mathbb{R}}|\xi|^\alpha e^{-\xi^2/(4\langle t\rangle^2)}\,d\xi = \langle t\rangle^{\alpha+1/2}\int_{\mathbb{R}}|\eta|^\alpha e^{-\eta^2/4}\,d\eta.$$

This proves (15). We also infer from Proposition 4.4 that, for any $a, N > 0$ and $\phi \in \mathscr{S}(\mathbb{R})$, the set

$$\{e^{itB^2}\phi(B-k) : |t| \le a, |k| \le N\} \subseteq L(X) \tag{16}$$

is bi-equicontinuous w.r.t. τ_X. Now let $\alpha = 0$. The group property follows from (13). We have to prove continuity properties and that $-iB^2$ with domain $M^{s+2}_{X,q}$ is the generator. We shall use

Case $q < \infty$: We show that $\|e^{-itB^2}x - x\|_{M^s_{X,q}} \to 0$ as $t \to 0$ for any $x \in M^s_{X,q}(B)$. So let $x \in M^s_{X,q}(B)$. For the calculation of the norm we take a $\phi \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ with compact support. Then we have, for any $k \in \mathbb{R}$, $(e^{-it(\cdot)^2} - 1)\phi(\cdot - k) \to 0$ in $\mathscr{S}(\mathbb{R})$, which implies (again by (4))

$$\|(e^{-itB^2} - I)\phi(B-k)x\|_X \to 0 \qquad (t \to 0).$$

Since $q < \infty$ the assertion now follows by dominated convergence. The argument is the same if the original group (e^{-itB}) is just bi-continuous in the sense of Section 4.

Case $q = \infty$: We have to show that, for $s > 1$, the group (e^{-itB^2}) is bi-continuous in $M^s_{X,\infty}(B)$ w.r.t. the restriction of the topology τ_X to $M^s_{X,\infty}(B)$), which we again denote by τ_M. We already know that τ_M satisfies Assumption 4.1 in $M^s_{X,\infty}(B)$. So we only have to show (i) and (ii) of Definition 4.2. We start with (ii) and take a sequence (x_n) in $M^s_{X,\infty}(B)$ such that $\|x_n\|_{M^s_{X,\infty}(B)} \leq C$ and $x_n \to 0$ w.r.t. τ_X, and we let $a, \varepsilon > 0$. We choose $\rho \in \mathscr{S}(\mathbb{R})$ as in Proposition 3.5. Since $s < 1$, $y = \sum_k \rho(B-k)y$ for each $y \in M^s_{X,\infty}$ where the series converges in $\|\cdot\|_X$ and we can find $N > 0$ such that

$$\|\sum_{|k|>N} \rho(B-k)e^{-itB^2}x_n\|_X \leq \sum_{|k|>N} \langle k\rangle^{-s}\langle k\rangle^{s}\|\rho(B-k)e^{-itB^2}x_n\|_X \leq C' \sum_{|k|>N} \langle k\rangle^{-s} \leq \varepsilon$$

for any $n \in \mathbb{N}$, $|t| \leq a$, where $C' := C\sup_{|t|\leq a}\|e^{-itB^2}\|_{L(M^s_{X,\infty})}$. For a τ_X-continuous seminorm $p \leq \|\cdot\|_X$ and $|t| \leq a$ we then have

$$p(e^{-itB^2}x_n) \leq \sum_{|k|\leq N} p\big(e^{-itB^2}\rho(B-k)x_n\big) + \varepsilon,$$

where the sum tends to 0 uniformly in t as $n \to \infty$ by bi-equicontinuity of the set in (16). Now let $x \in M^s_{X,\infty}(B)$. By $M^s_{X,\infty} \hookrightarrow M_{X,1} \hookrightarrow X \hookrightarrow (X, \tau_X)$ we get τ_X-continuity of $t \mapsto e^{-itB^2}x$ from continuity in $\|\cdot\|_{M_{X,1}}$ (the case $q = 1$ is already proved). The generation property is checked via resolvents and the Phillips calculus. $\square$

Remark 5.2. Taking into account Remark 4.8 we also obtain that, for $s > 1$, the group (e^{itB^2}) is bi-continuous in $M^s_{X,\infty}(B)$ w.r.t. the norm-topology of X restricted to $M^s_{X,\infty}(B)$ or w.r.t. the norm-topology of $M_{X,1}(B)$, restricted to $M^s_{X,\infty}(B)$.

Remark 5.3. Concerning only boundedness of operators $F(B)$, the more general result is, of course, the following (cf. also [6, Thm. 17(1)] for the classical setting): Take $\rho \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ with compact support and suppose that the set

$$\{F(B)\rho(B-k) : k \in \mathbb{R}\} \subseteq L(X)$$

is uniformly bounded in operator norm. Then $F(B)$ acts as a bounded operator in every space $M^s_{X,q}$, $s \in \mathbb{R}$, $q \in [1, \infty]$.

6. Application to a nonlinear Schrödinger equation

We illustrate our results briefly by a simple application. The cubic nonlinear Schrödinger equation (NLS) in one space dimension is

$$(17) \qquad \left\{ \begin{array}{rcl} iu_t + u_{xx} \pm |u|^2 u &=& 0, \quad t \in \mathbb{R}, x \in \mathbb{R} \\ u(0,x) &=& u_0(x), \quad x \in \mathbb{R}. \end{array} \right.$$

We take $T(t)f = f(\cdot - t)$, the right translation group, as in Example 1.2, hence $B = -i\frac{d}{dx}$, and we consider the function space $X = C_b(\mathbb{R})$ of bounded continuous on $\mathbb{R}$, which is an algebra for pointwise multiplication. Denoting by $BUC(\mathbb{R})$ the space of bounded uniformly continuous functions on $\mathbb{R}$ we have

$$L^\infty(\mathbb{R})^\flat = BUC(\mathbb{R}) \subseteq X \subseteq L^\infty(\mathbb{R}).$$

Thus $M^s_{X,\infty}(B)$ equals the classical modulation space $M^s_{L^\infty,\infty}(B) = M^s_{\infty,\infty}(\mathbb{R})$. Since the translation group is bi-continuous on X w.r.t. the topology τ_c of uniform convergence on compact intervals, Proposition 5.1 tells us that, for $s > 1$, the Schrödinger group $(e^{itd_x^2})$ is bi-continuous on $M^s_{\infty,\infty}(\mathbb{R})$ w.r.t. the topology τ_c restricted to $M^s_{\infty,\infty}$. On the other hand, $M^s_{\infty,\infty}(\mathbb{R})$ is an algebra (see, e.g., [4]), moreover pointwise multiplication is τ_c-continuous, at least on $\|\cdot\|_\infty$-bounded sets. We rewrite (17) as a fixed point equation in the space

$$C^b_{\tau_c}([0,T], M^s_{\infty,\infty}(\mathbb{R})) := \{v : [0,T] \to M^s_{\infty,\infty}(\mathbb{R}) : v \text{ is } \|\cdot\|_{M^s_{\infty,\infty}}\text{-bounded and } \tau_c\text{-continuous}\},$$

which equipped with the norm $\|v\|_\infty := \sup_{t\in[0,T]} \|v(t)\|_{M^s_{\infty,\infty}}$ is a Banach space. The fixed point equation is

$$u(t) = e^{itd_x^2}u_0 \pm i\int_0^t e^{i(t-s)d_x^2}\big(|u(s)|^2u(s)\big)\,ds, \quad t \in [0,T]. \tag{18}$$

We call a solution $u \in C^b_{\tau_c}([0,T], M^s_{\infty,\infty}(\mathbb{R}))$ a *mild solution* of (17). The only thing that we now need to solve this via Banach's Fixed Point Theorem in the usual way is the following on the convolution part.

Lemma 6.1. *The convolution with the semigroup* $(e^{itd_x^2})$ *maps* $C^b_{\tau_c}([0,T], M^s_{\infty,\infty})$ *into itself and we have the estimate*

$$\|e^{i(\cdot)d_x^2} * v\|_\infty \le CT\|v\|_\infty, \quad v \in C^b_{\tau_c}([0,T], M^s_{\infty,\infty}).$$

We omit the easy proof. By standard means we then have

Proposition 6.2. *Let* $s > 1$ *and* $u_0 \in M^s_{\infty,\infty}(\mathbb{R})$. *Then there exists* $T > 0$ *only depending on* $\|u_0\|_{M^s_{\infty,\infty}}$ *such that* (17) *has a unique mild solution* $u \in C^b_{\tau_c}([0,T], M^s_{\infty,\infty}(\mathbb{R}))$.

Remark 6.3. Doing the fixed point argument in the space of $\|\cdot\|_{M^s_{\infty,\infty}}$-bounded and $\|\cdot\|_\infty$-continuous or in the space of $\|\cdot\|_{M^s_{\infty,\infty}}$-bounded and $\|\cdot\|_{M_{\infty,1}}$-continuous functions, which is possible since pointwise multiplication is also continuous for these norms, yields *existence* of a more regular solution. However, using the τ_c-topology we get *uniqueness* in a larger space.

Appendix A. Two auxiliary results

Here we provide proofs for a well-known result on the short time Fourier transform and for the contruction of τ_X-Riemann type integrals with respect to measures $\mu \in \mathscr{M}(\mathbb{R})$.

Fix $g \in \mathscr{S}(\mathbb{R})\setminus\{0\}$. The short-time Fourier transform with window g of $f \in \mathscr{S}'(\mathbb{R})$ is the function $V_gf : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is defined by

$$V_gf(x,\xi) := \int_{\mathbb{R}} e^{-iy\xi}f(y)\overline{g(y-x)}\,dy = \langle f, e^{i\xi(\cdot)}g(\cdot - x)\rangle, \tag{19}$$

where the duality bracket extends the usual scalar product in $L^2(\mathbb{R})$. We shall only need that V_g maps $\mathscr{S}(\mathbb{R})$ to rapidly decreasing functions.

Lemma A.1. *For* $g \in \mathscr{S}(\mathbb{R})\setminus\{0\}$, $f \in \mathscr{S}'(\mathbb{R})$ *we have that*

$$(x,\xi) \mapsto \langle x\rangle^k\langle\xi\rangle^l V_gf(x,\xi)$$

is bounded on $\mathbb{R}\times\mathbb{R}$ *for all* $j,k > 0$.

This is well-known but we reprove it here for convenience.

Proof. We have

$$|V_g f(x,\xi)| \leq \int_{\mathbb{R}} |f(y)||g(y-x)|\,dy = |f| * |\sigma g|(x),$$

where $\sigma g(y) = g(-y)$. By $f, g \in \mathscr{S}(\mathbb{R})$ we infer that $\langle x\rangle^k V_g f(x,\xi)$ is bounded for any $k > 0$. For any $l \in \mathbb{N}$ we use integration by parts to obtain

$$(-i\xi)^l V_g f(x,\xi) = (-1)^l \int_{\mathbb{R}} e^{-iy\xi} \frac{d^l}{dy^l}\big(f(y)\overline{g(y-x)}\big)\,dy = (-1)^l \sum_{j=0}^{l} \binom{l}{j} V_{g^{(l-j)}}\big(f^{(j)}\big)(x,\xi).$$

Since $\mathscr{S}(\mathbb{R})$ is invariant under taking derivatives we can combine both arguments to obtain the assertion. $\square$

For the following result on τ_X-Riemann type integrals we suppose that Assumption 4.1 holds.

Proposition A.2. *Let $f : [a,b] \to X$ be a τ_X-continuous and norm-bounded function and let $\mu \in \mathscr{M}(\mathbb{R})$ be a complex Borel measure. For any partition $a = t_0 < t_1 < \ldots < t_n = b$ and any vector $(\xi_1, \ldots, \xi_n)$ with $\xi_j \in [t_{j-1}, t_j)$ we define the Riemann type sum*

$$S(f, t_j, \xi_j) := \sum_{j=1}^{n} f(\xi_j)\mu\big([t_{j-1}, t_j)\big) + f(b)\mu\big(\{b\}\big).$$

Then the τ_X-limit of $S(f, t_j, \xi_j)$ exists in X as $\max_j |t_j - t_{j-1}|$ tends to 0. This limit is denoted $\int_{[a,b]} f(t)\,d\mu(t)$. We have the estimate

$$\left\| \int_{[a,b]} f(t)\,d\mu(t) \right\|_X \leq \sup_{t\in[a,b]} \|f(t)\|_X \,|\mu|([a,b])$$

and for any τ_X-continuous seminorm p we have

$$p\left(\int_{[a,b]} f(t)\,d\mu(t)\right) \leq \int_{[a,b]} p\big(f(t)\big)\,d|\mu|(t) \leq \sup_{t\in[a,b]} p\big(f(t)\big)\,|\mu|([a,b]).$$

Proof. If $\|f(t)\|_X \leq C$ we easily get

$$\|S(f, t_j, \xi_j)\|_X \leq C|\mu|\big([a,b]\big).$$

By Assumption 4.1 (i) it thus suffices to show the Cauchy property. So we let $\delta > 0$ and take two partitions (t_j) and (s_k) such that $\max_j |t_j - t_{j-1}|$ and $\max_k |s_k - s_{k-1}|$ are $\leq \delta$ and we take two corresponding vectors (ξ_j) and (η_k). Then we rewrite

$$S(f, t_j, \xi_j) - S(f, s_k, \eta_k) = \sum_l \big(f(\widetilde{\xi}_l) - f(\widetilde{\eta}_l)\big)\mu\big([u_l, u_{l-1})\big),$$

where (u_l) is a partition obtained by the union of the t_j and the s_k and we have $\{\widetilde{\xi}_l\} = \{\xi_j\}$, $\{\widetilde{\eta}_l\} = \{\eta_k\}$ as sets, but we have to repeat ξ_j according to the splitting of the interval $[t_{j-1}, t_j)$. Then not necessarily $\widetilde{\xi}_l \in [u_{l-1}, u_l)$ but we have at least $|\widetilde{\xi}_l - u_l| \leq \delta$ and $|\widetilde{\xi}_l - u_{l-1}| \leq \delta$. The same holds for the $\widetilde{\eta}_l$ so that $|\widetilde{\xi}_l - \widetilde{\eta}_l| \leq 2\delta$. Taking a τ_X-continuous seminorm p we thus have

$$\begin{aligned} p\big(S(f, t_j, \xi_j) - S(f, s_k, \eta_k)\big) &\leq \sum_l p\big(f(\widetilde{\xi}_l) - f(\widetilde{\eta}_l)\big)|\mu|\big([u_{l-1}, u_l)\big) \\ &\leq \sup_{|\xi-\eta|\leq 2\delta} p\big(f(\xi) - f(\eta)\big)|\mu|\big([a,b)\big). \end{aligned}$$

The assertion thus follows from uniform τ_X-continuity of f on $[a, b]$. The p-estimates are immediate, for the norm estimate we use Assumption 4.1 (iii) and the p-estimates for $p := |\langle \cdot, \psi \rangle|$ where $\psi \in \Phi(\tau_X)$. $\square$

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